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A STUDY OF THE ASYMPTOTIC RADIATION AND

MOTION OF CLASSICAL CHARGED PARTICLES

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INTRODUCTION

The purpose of this paper is twofold: in Part I, to calculate the rates at which linear and angular momentum are asymptotically radiated from a spinning charged particle; in Part II, to use the calculated radiation rates in the derivation of a linear and an angular equation of motion. The first of these objectives is accomplished by calculating the electromagnetic energy-momentum tensor and integrating the appropriate expressions involving this tensor, while the second objective involves the generalization and extension of a method used by Cohn[1] for the derivation of the equation of motion for a non-spinning particle. As will be seen, the extension of this method determines some, but not all, of the terms in each of the equations of motion for the spinning particle.

Calculations concerning the asymptotic radiation of a spinning particle have been made earlier by Kolsrud and Leer[2]. However, their objective was to calculate the total scalar intensity of the radiation rate, rather than the vector and tensor forms calculated in this paper. The work of Bhabha and Corben[3] parallels this paper to some extent, beginning with some of the same assumptions, but employing the Dirac "world-tube" method to

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[1] J. Cohn, "Derivation of the Equations of Motion of a Classical Radiating Charge," American Journal of Physics, Vol. 35, No. 10 (1967), pp. 949-950.

[2] M. Kolsrud and E. Leer, "Radiation From Moving Dipoles," Physica Norvegica, Vol. 2, No. 3 (1967), pp. 181-188.

[3] H. J. Bhabha and H. C. Corben, "General Classical Theory of Spinning Particles in a Maxwell Field," Proc. Roy. Soc. (London) A178, 273 (1941).

derive the equations of motion. Nyborg[1] summarizes and compares the results of Bhabha and Corben with conflicting results obtained by others using other methods, concluding with a statement concerning the difficulty of reconciling the conflicting results.[2] The equations developed in the present paper do not agree exactly with any of the equations mentioned above, though extensive similarities exist in some cases.

Throughout this paper, the following conventions will be observed: Roman indices should be assumed to run from 1 to 3, while Greek indices will run from 1 to 4. The metric tensor $g^{\mu\nu}$ has the elements (1,1,1,-1) on its main diagonal and zeroes elsewhere, so that $d\tau^2 = - g_{\alpha\beta}dx^\alpha dx^\beta/c^2$. The following definitions will apply unless otherwise stated:

(1) c = velocity of light <u>in vacuo</u>

(2) x^μ = coordinates of retarded field point (observer)

(3) z^μ = coordinates of source point (observer)

(4) $R^\mu = x^\mu - z^\mu$ = 4-vector from particle to observer

(5) ρ = distance (in 3-space) from source point to field point, measured in momentary rest frame of particle

(6) v^μ = particle velocity = $dz^\mu/d\tau$

(7) a^μ = particle acceleration = $\dot{v}^\mu$

(8) $M^{\mu\nu}$ = antisymmetric moment tensor of particle. The space-space components of $M^{\mu\nu}$ in the rest frame of the particle are given by the vector $2m\underline{\mu}/e$, where $\underline{\mu}$ is the magnetic dipole moment of the particle, while the space-time components are given by the vector $2m\underline{\pi}/e$, where $\underline{\pi}$ is the elec-

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[1]P. Nyborg, "On Classical Theories of Spinning Particles," <u>Il Nuovo Cimento</u>, Vol. XXIII, No. 1 (1962), pp. 47-62.

[2]Ibid., p. 62.

tric dipole moment (which is assumed to be zero in the rest frame).

(9) m = mass of the <u>bare</u> particle

(10) $S^{\mu\nu}$ = spin-angular-momentum tensor of bare particle. In the rest frame its space-space components are given by the three-vector $I\underline{\omega}$, where I is the particle moment of inertia and $\underline{\omega}$ is the angular velocity; the space-time components are zero in the rest frame.

Note: The particle is assumed to have a single distinguishable axis. Therefore $\underline{\mu}$ is parallel to $\underline{\omega}$, and as a result the rest-frame expressions for $M^{\alpha\beta}$ and $S^{\alpha\beta}$ are proportional.

(11) A^{μ} = vector 4-potential of particle

(12) $F^{\alpha\beta}$ = field strength tensor of particle = $\partial^{\alpha}A^{\beta} - \partial^{\beta}A^{\alpha}$

(13) $\theta^{\alpha\beta}$ = symmetric electromagnetic energy-momentum tensor

$= (F^{\alpha\lambda}F_{\lambda}{}^{\beta} + \frac{1}{4}g^{\alpha\beta}F_{\delta\varepsilon}F^{\delta\varepsilon})/4\pi$

(14) $J^{\delta\alpha\beta} = -(\theta^{\delta\alpha}x^{\beta} - \theta^{\delta\beta}x^{\alpha})$

For free radiation fields the following definitions are used for linear momentum P^{α} and angular momentum $J^{\alpha\beta}$:

(15) $$P^{\alpha} = \frac{1}{c}\int \theta^{\alpha\beta} d\sigma_{\beta}$$

(16) $$J^{\alpha\beta} = \frac{1}{c}\int J^{\delta\alpha\beta} d\sigma_{\delta}$$

where the surface integration is taken over any space-like surface. Given these definitions, one may in turn define the radiated linear momentum rate (observed at infinity) as

$$(17)\qquad dP^{\alpha} = \lim_{\rho\to\infty} \frac{1}{c}\int_{(\Delta\sigma)} \Theta^{\alpha\beta} d\sigma_{\beta}$$

and the radiated angular momentum rate as

$$(18)\qquad dJ^{\alpha\beta} = \lim_{\rho\to\infty} \frac{1}{c}\int_{(\Delta\sigma)} J^{\delta\alpha\beta} d\sigma_{\delta}$$

where in both cases $\Delta\sigma$ is a spacelike surface segment, dependent on $d\tau$, that is infinitely far away from the location of the charge during $d\tau$.[1]

One other quantity can be defined at this point for the sake of notational simplicity. Since R^{α} is the vector from the source point to the retarded field point, the interval $R^{\alpha}R_{\alpha}$ from source point to retarded field point is equal to zero. This means that the components of R^{α} in the rest frame must be $(\underline{R},\rho)$ since ρ is the magnitude of $\underline{R}$. Also, v^{α} in the rest frame has the components $(\underline{0},c)$. Therefore if one defines the vector u^{α} as

$$(19)\qquad u^{\alpha} = R^{\alpha}/\rho - v^{\alpha}/c$$

the result is that u^{α} in the rest frame has space components forming a unit vector pointing from the source point to the retarded field point and a time component equal to zero. Alternately, $u^{\alpha} = (\underline{R}/\rho,0)$ in the rest frame. This implies that $v^{\alpha}u_{\alpha} = 0$, $u^{\alpha}u_{\alpha} = 1$, and $R^{\alpha}a_{\alpha} = \rho u^{\alpha}a_{\alpha}$, all obtained by considering the appropriate rest-frame expressions for the quantities involved. The expression $u^{\alpha}a_{\alpha}$ occurs frequently enough in the following calculations to warrant defining the shorthand notation $a_u = a^{\alpha}u_{\alpha}$.

[1]Cf. definition of angular radiated momentum given by J. Cohn, "Considerations on the Classical Spinning Electron," Journal of Mathematical Physics, vol. 10, no. 5 (1969), p. 803.

PART I. ASYMPTOTICALLY RADIATED MOMENTA FOR A SPINNING RADIATING CHARGED PARTICLE

1. THE ELECTROMAGNETIC FIELD TENSOR

In the calculation made here, the vector potential is assumed to be the sum of the potential A_p^μ resulting from the point charge and the potential A_d^μ resulting from the (magnetic) dipole moment. The expressions used[1,2] are:

$$A_p^\mu = -\frac{e}{R^\alpha v_\alpha} v^\mu \tag{I.1.1}$$

and

$$A_d^\mu = \frac{e}{2mR^\alpha v_\alpha}\frac{d}{d\tau}\left(\frac{M^{\mu\nu}R_\nu}{R^\beta v_\beta}\right) \tag{I.1.2}$$

Evaluating A_d^μ, one obtains

$$A_d^\mu = \frac{e}{2m(R^\alpha v_\alpha)^2}\left(\dot{M}^{\mu\nu}R_\nu - M^{\mu\nu}v_\nu - \frac{(M^{\mu\nu}R_\nu)(R^\delta a_\delta)}{R^\beta v_\beta} + \frac{(M^{\mu\nu}R_\nu)(v^\delta v_\delta)}{R^\beta v_\beta}\right) \tag{I.1.3}$$

Several simplifications may be made at this point. First, the fact that the electric dipole moment is zero means that $M^{\mu 4} = 0$, and in the rest frame v^4 is the only nonzero component of v^μ. Therefore $M^{\mu\nu}v_\nu = M^{\mu 4}v_4 = 0$. Also, since $R^4 = \rho$ in the rest frame, $R^\alpha v_\alpha = -c\rho$. Finally, also from the rest frame expression for v^μ, $v^\alpha v_\alpha = -c^2$. When these simplifications are incorporated into the expression for A_d^μ, the result is

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[1]Kolsrud and Leer, "Radiation From Moving Dipoles," p. 182.

[2]Unless otherwise specified, retarded field values are used throughout this work.

(I.1.4) $$A_d^{\mu} = \frac{e}{2m\rho^2 c^2}\left(\dot{M}^{\mu\nu}R_\nu + \frac{M^{\mu\nu}R_\nu R^\alpha a_\alpha}{\rho c} + \frac{M^{\mu\nu}R_\nu c}{\rho}\right)$$

Similarly simplified, A_p^μ becomes

(I.1.5) $$A_p^{\mu} = \frac{e}{c\rho}v^{\mu}$$

so that the entire potential A^μ can be written as

(I.1.6) $$A^{\mu} = \frac{e}{c\rho}v^{\mu} + \frac{e}{2mc^2\rho^2}\left(\dot{M}^{\mu\nu}R_\nu + \frac{M^{\mu\nu}R_\nu R^\alpha a_\alpha}{\rho c} + \frac{M^{\mu\nu}R_\nu c}{\rho}\right)$$

The next step is to calculate $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. The following preliminary calculations are helpful: If B^μ is any four-vector used here,

(I.1.7) $$\partial^\nu B^\mu = \frac{\partial B^\mu}{\partial x_\nu} = \frac{dB^\mu}{d\tau}\frac{d\tau}{dx_\nu} = \dot{B}^\mu \partial^\nu \tau$$

From Rohrlich[1] (with corrective factor of 1/c),

(I.1.8) $$\partial^\nu \tau = -\left(u^\nu + \frac{v^\nu}{c}\right)\frac{1}{c} = -\frac{R^\nu}{c\rho}$$

so for the retarded case,

(I.1.9) $$\partial^\nu B^\mu = -\frac{\dot{B}^\mu R^\nu}{c\rho}$$

Also from Rohrlich,[2]

(I.1.10) $$\partial^\mu \rho = u^\mu + a_u R^\mu / c^2$$

Now required derivatives of R^μ, v^μ, u^μ, and a^μ can be obtained:

(I.1.11) $$\partial^\nu R^\mu = \partial^\nu(x^\mu - z^\mu) = g^{\mu\nu} - \partial^\nu z^\mu = g^{\mu\nu} + \frac{v^\mu R^\nu}{c\rho}$$

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[1]F. Rohrlich, Classical Charged Particles, (Reading, Mass.: Addison-Wesley, 1965), p. 83.

[2]Ibid., p. 84.

(I.1.12) $$\partial^\nu v^\mu = -\frac{\dot{v}^\mu R^\nu}{c\rho} = -\frac{a^\mu R^\nu}{c\rho}$$

(I.1.13) $$\partial^\nu u^\mu = \partial^\nu\left(\frac{R^\mu}{\rho} - \frac{v^\mu}{c}\right) = \frac{g^{\mu\nu}}{\rho} + \left(\frac{v^\mu}{c\rho^2} + \frac{a^\mu}{\rho c^2}\right)R^\nu - \frac{R^\mu\lambda^\nu}{\rho^2}$$

(I.1.14) $$\partial^\nu a_u = \frac{a^\nu}{\rho} + \left(\frac{a^2}{\rho c^2} - \frac{\dot{a}^\alpha u_\alpha}{c\rho}\right)R^\nu - \frac{a^\alpha R_\alpha \lambda^\nu}{\rho^2}$$

Defining $\dot{a}^\mu u_\mu = \dot{a}_u$, one has

(I.1.15) $$\partial^\nu a_u = \frac{a^\nu}{\rho} + \left(\frac{a^2}{\rho c^2} - \frac{\dot{a}_u}{c\rho}\right)R^\nu - \frac{a^\alpha R_\alpha \lambda^\nu}{\rho^2}$$

The calculation of $\partial^\nu A_p^\mu$ and $\partial^\nu A_d^\mu$ can now proceed. Using the above expressions, one obtains

(I.1.16) $$\partial^\nu A_p^\mu = \frac{e}{c\rho^2}\left(-\frac{a^\mu R^\nu}{c} - v^\mu \lambda^\nu\right)$$

The calculation of $\partial^\nu A_d^\mu$ is considerably more involved. After lengthy tensor manipulation, the following result is obtained:

(I.1.17) $$\begin{aligned}\partial^\nu A_d^\mu = &-\frac{3e}{2mc\rho^3}M^{\mu\alpha}u_\alpha\lambda^\nu - \frac{e}{2mc^2\rho^3}\dot{M}^{\mu\alpha}u_\alpha R^\nu + \frac{e}{2mc\rho^3}M^{\mu\nu}\\ &-\frac{3ea_u}{2mc^3\rho^2}M^{\mu\alpha}u_\alpha\lambda^\nu - \frac{ea_u}{2mc^4\rho^2}\dot{M}^{\mu\alpha}u_\alpha R^\nu - \frac{e\dot{a}_u}{2mc^4\rho^2}M^{\mu\alpha}u_\alpha R^\nu\\ &+\frac{ea^2}{2mc^5\rho^2}M^{\mu\alpha}u_\alpha R^\nu + \frac{e}{2mc^3\rho^2}M^{\mu\alpha}u_\alpha a^\nu + \frac{ea_u}{2mc^3\rho^2}M^{\mu\nu}\\ &+\frac{ea_u}{2mc^5\rho^2}M^{\mu\alpha}a_\alpha R^\nu - \frac{2e}{2mc^2\rho^3}\dot{M}^{\mu\alpha}R_\alpha\lambda^\nu - \frac{e}{2mc^3\rho^3}\ddot{M}^{\mu\alpha}R_\alpha R^\nu\\ &+\frac{e}{2mc^2\rho^2}\dot{M}^{\mu\nu}\end{aligned}$$

Substituting $u^\nu + \frac{a_u R^\nu}{c^2} = \lambda^\nu$ and combining A^μ_d and A^μ_p, one obtains

$$(I.1.18)\qquad F^{\mu\nu} = \partial^{[\mu}A^{\nu]} = -\frac{3e}{2mc\rho^3} M^{[\nu\alpha}u_\alpha u^{\mu]} - \frac{3ea_u}{2mc^3\rho^3} M^{[\nu\alpha}u_\alpha R^{\mu]}$$

$$- \frac{e}{2mc^2\rho^3}\dot{M}^{[\nu\alpha}u_\alpha R^{\mu]} + \frac{2e}{2mc\rho^3}M^{\nu\mu} - \frac{3ea_u}{2mc^3\rho^2}M^{[\nu\alpha}u_\alpha u^{\mu]}$$

$$- \frac{3e(a_u)^2}{2mc^5\rho^2}M^{[\nu\alpha}u_\alpha R^{\mu]} - \frac{ea_u}{2mc^4\rho^2}\dot{M}^{[\nu\alpha}u_\alpha R^{\mu]} - \frac{e\dot{a}_u}{2mc^4\rho^2}M^{[\nu\alpha}u_\alpha R^{\mu]}$$

$$+ \frac{ea^2}{2mc^5\rho^2}M^{[\nu\alpha}u_\alpha R^{\mu]} + \frac{e}{2mc^3\rho^2}M^{[\nu\alpha}u_\alpha a^{\mu]} + \frac{2ea_u}{2mc^3\rho^2}M^{\nu\mu}$$

$$+ \frac{ea_u}{2mc^5\rho^2}M^{[\nu\alpha}a_\alpha R^{\mu]} - \frac{2e}{2mc^2\rho^3}\dot{M}^{[\nu\alpha}R_\alpha u^{\mu]} - \frac{2ea_u}{2mc^4\rho^3}\dot{M}^{[\nu\alpha}R_\alpha R^{\mu]}$$

$$- \frac{e}{2mc^3\rho^3}\ddot{M}^{[\nu\alpha}R_\alpha R^{\mu]} + \frac{2e}{2mc^2\rho^2}\dot{M}^{\nu\mu} - \frac{e}{c^2\rho^2}a^{[\nu}R^{\mu]}$$

$$- \frac{e}{c\rho^2}v^{[\nu}u^{\mu]} - \frac{ea_u}{c^3\rho^2}v^{[\nu}R^{\mu]}$$

where the notation $A^{[\nu}B^{\mu]} \equiv A^\nu B^\mu - A^\mu B^\nu$ and $M^{[\nu\alpha}A_\alpha B^{\mu]} \equiv M^{\nu\alpha}A_\alpha B^\mu - M^{\mu\alpha}A_\alpha B^\nu$.
Rearranging terms to place those with higher powers of ρ first,

$$(I.1.19)\qquad F^{\mu\nu} = \Big[-\frac{3e(a_u)^2}{2mc^5\rho^2}M^{[\nu\alpha}u_\alpha R^{\mu]} - \frac{ea_u}{2mc^4\rho^2}\dot{M}^{[\nu\alpha}u_\alpha R^{\mu]} - \frac{e\dot{a}_u}{2mc^4\rho^2}M^{[\nu\alpha}u_\alpha R^{\mu]}$$

$$+ \frac{ea^2}{2mc^5\rho^2}M^{[\nu\alpha}u_\alpha R^{\mu]} + \frac{ea_u}{2mc^5\rho^2}M^{[\nu\alpha}a_\alpha R^{\mu]} - \frac{2ea_u}{2mc^4\rho^3}\dot{M}^{[\nu\alpha}R_\alpha R^{\mu]}$$

$$- \frac{e}{2mc^3\rho^3}\ddot{M}^{[\nu\alpha}R_\alpha R^{\mu]} - \frac{e}{c^2\rho^2}a^{[\nu}R^{\mu]} - \frac{ea_u}{c^3\rho^2}v^{[\nu}R^{\mu]}\Big]$$

$$+\left[-\frac{3ea_u}{2mc^3\rho^3}M^{[\nu\alpha}u_\alpha R^{\mu]}-\frac{e}{2mc^2\rho^3}\dot{M}^{[\nu\alpha}u_\alpha R^{\mu]}-\frac{3ea_u}{2mc^3\rho^2}M^{[\nu\alpha}u_\alpha u^{\mu]}\right.$$

$$+\frac{e}{2mc^3\rho^2}M^{[\nu\alpha}u_\alpha a^{\mu]}-\frac{2ea_u}{2mc^3\rho^2}M^{\mu\nu}-\frac{2e}{2mc^2\rho^3}\dot{M}^{[\nu\alpha}R_\alpha u^{\mu]}$$

$$\left.-\frac{2e}{2mc^2\rho^2}\dot{M}^{\mu\nu}-\frac{e}{c\rho^2}v^{[\nu}u^{\mu]}\right]$$

$$+\left[-\frac{3e}{2mc\rho^3}M^{[\nu\alpha}u_\alpha u^{\mu]}-\frac{2e}{2mc\rho^3}M^{\mu\nu}\right]$$

Defining each of the expressions appearing in the large square brackets above as $F^{\mu\nu}_{(-1)}$, $F^{\mu\nu}_{(-2)}$, and $F^{\mu\nu}_{(-3)}$ respectively, one can write

$$F^{\mu\nu}=F^{\mu\nu}_{(-1)}+F^{\mu\nu}_{(-2)}+F^{\mu\nu}_{(-3)} \tag{I.1.20}$$

where the subscript of each term on the right side indicates the power of ρ contained by that term.

It may be noted in passing that (I.1.20) is in agreement with a theorem proved by Goldberg and Kerr.[1] According to this theorem, $F^{\mu\nu}$ must have the ρ-dependence indicated by (I.1.20), and in addition $F^{\mu\nu}_{(-1)}R_\nu = 0$ and $F^{\mu\nu}_{(-2)}R_\nu = AR^\mu$ where A is scalar. Brief calculations using (I.1.19) show that these two latter conditions are indeed satisfied.

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[1] J. N. Goldberg and R. P. Kerr, "Asymptotic Properties of the Electromagnetic Field," Journal of Mathematical Physics, Vol. 5, No. 2 (1964).

2. THE SYMMETRIC ENERGY-MOMENTUM TENSOR

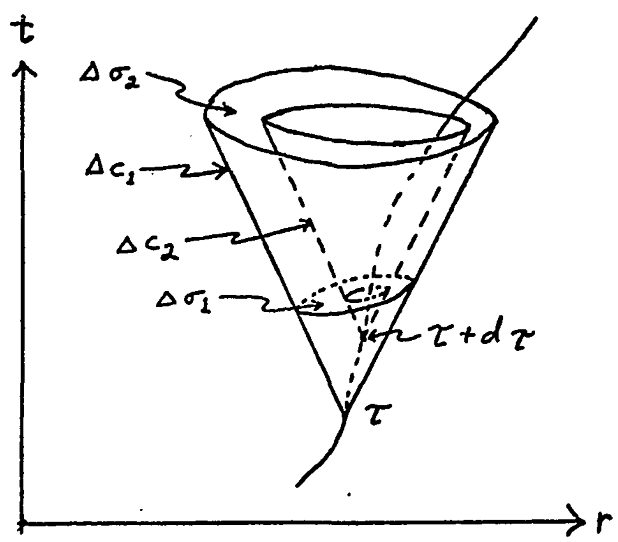
Before proceeding directly with the calculation of $\theta^{\mu\nu}$, it is well to consider the objective of Part I, namely, to calculate the asymptotically radiated linear and angular momenta of a spinning charged particle. The validity of the definitions used for these momenta can be bolstered by showing that the integrals

$$\lim_{\rho\to\infty} \int_{(\Delta\sigma)} \theta^{\mu\nu} d\sigma_\nu \tag{I.2.1}$$

and

$$\lim_{\rho\to\infty} \int J^{\alpha\mu\nu} d\sigma_\alpha \tag{I.2.2}$$

are independent of the particular surface segment $\Delta\sigma$ chosen. To phrase the conditions another way, consider the following drawing of the two light-cones of a particle at points τ and $\tau + d\tau$ on its world line:



By Gauss' Law, the integrals

$$(I.2.3) \quad \int_{(\Delta S)} \Theta^{\mu\nu} d\sigma_\nu$$

and

$$(I.2.4) \quad \int_{(\Delta S)} J^{\alpha\mu\nu} d\sigma_\alpha$$

where ΔS is the surface composed of the surfaces $\Delta\sigma_1$, $\Delta\sigma_2$, Δc_1, and Δc_2, should equal zero; in other words, the radiation entering the volume enclosed by ΔS must equal the radiation leaving it, since ΔS encloses no sources. Therefore if it can be shown that the radiation crossing the light cones is zero in the limit $\rho\to\infty$, then the radiation through $\Delta\sigma_1$ must equal that through $\Delta\sigma_2$ in the limit $\rho\to\infty$. Since $\Delta\sigma_1$ and $\Delta\sigma_2$ are arbitrarily chosen, this would mean the definitions are independent of the surface chosen. As the surface differential on the light cone is $R_\alpha d\omega$ where $d\omega$ is the differential of (three-space) solid angle, the definitions are independent of the surface chosen if

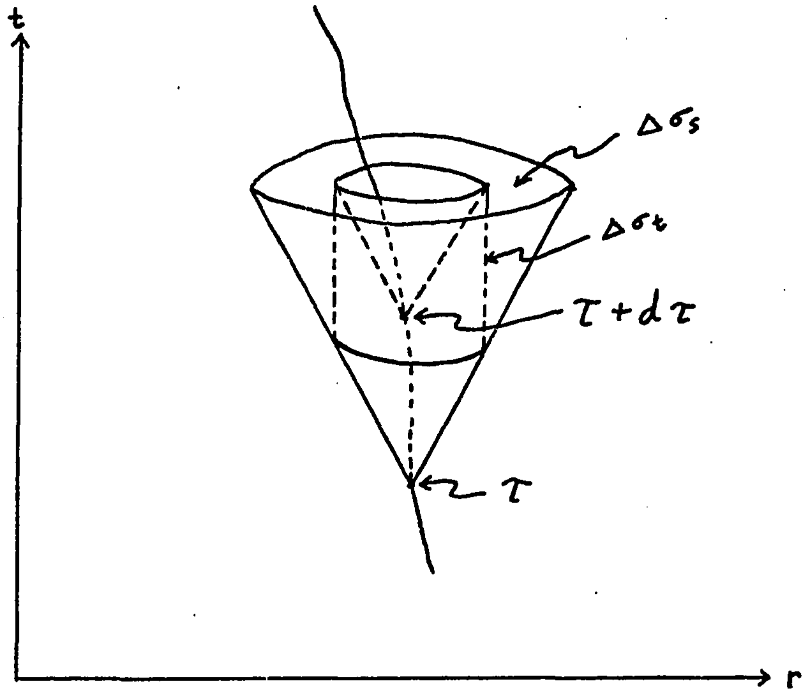
$$(I.2.5) \quad \lim_{\rho\to\infty} \int \Theta^{\mu\nu} R_\nu d\omega = 0$$

and

$$(I.2.6) \quad \lim_{\rho\to\infty} \int J^{\alpha\mu\nu} R_\alpha d\omega = 0$$

The proof of (I.2.5) can be done immediately. (I.1.20) shows that the expression for $F^{\mu\nu}$ contains terms depending on ρ^{-1}, ρ^{-2}, and ρ^{-3}. Since $\theta^{\mu\nu}$ is quadratic in $F^{\mu\nu}$, the highest power of ρ that can appear in <u>any</u> term of $\theta^{\mu\nu}$ is ρ^{-2}. This means that the expression in (I.2.5) approaches zero as ρ^{-1}, and therefore that (I.2.5) is proved and the definition for P^{μ} is independent of the surface segment $\Delta\sigma$ chosen. The proof of (I.2.6) requires additional calculations and is deferred until Section 3.

The expressions for the radiated momenta can be transformed in such a way as to make them somewhat easier to calculate. Applying Gauss' Law to the following diagram,



and remembering that (as has been shown in the linear case, and will be shown in Section 3 for the angular case) the radiation crossing the light cone is zero, one can see that the radiation crossing the spacelike segment of surface $\Delta\sigma_s$ must equal that crossing the timelike surface segment $\Delta\sigma_t$. Since $d\sigma_\alpha = u_\alpha \rho^2 \, d\Omega \, c \, d\tau$[1], the definitions of the radiated momenta may be rewritten as follows:

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[1]Rohrlich, p. 110.

(I.2.7) $$dP^{\mu}_{rad} = -\lim_{\rho\to\infty}\int \theta^{\mu\nu} u_\nu \rho^2 \, d\Omega \, d\tau$$

and

(I.2.8) $$dJ^{\mu\nu}_{rad} = -\lim_{\rho\to\infty}\int J^{\alpha\mu\nu} u_\alpha \rho^2 \, d\Omega \, d\tau$$

Consider in particular the last integral. Rewritten to show its explicit dependence on $\theta^{\mu\nu}$, it becomes

(I.2.9) $$dJ^{\mu\nu}_{rad} = \lim_{\rho\to\infty}\int (\theta^{\alpha\mu} x^{\nu} - \theta^{\alpha\nu} x^{\mu}) u_\alpha \rho^2 \, d\Omega \, d\tau$$

Since $|x^{\mu}| \to \infty$ as $\rho \to \infty$, any term of $\theta^{\mu\nu}$ which contains a power of ρ which is less than -3 will not contribute to the integral in the limit $\rho \to \infty$. When the integrals

(I.2.10) $$\lim_{\rho\to\infty}\int J^{\alpha\mu\nu} R_\alpha \, d\omega = \lim_{\rho\to\infty}\int (\theta^{\alpha\nu} x^{\mu} - \theta^{\alpha\mu} x^{\nu}) R_\alpha \, d\omega$$

and

(I.2.11) $$dP^{\mu}_{rad} = -\lim_{\rho\to\infty}\int \theta^{\mu\nu} u_\nu \rho^2 \, d\Omega \, d\tau$$

are considered, it becomes apparent that any term of $\theta^{\mu\nu}$ which contains a power of ρ which is less than -2 will not contribute in the limit.

In the previous subsection $F^{\mu\nu}$ was written as the sum of three parts $F^{\mu\nu}_{(-1)}$, $F^{\mu\nu}_{(-2)}$, and $F^{\mu\nu}_{(-3)}$, with the subscript denoting the power to which ρ appears in each part. Then

$$\text{(I.2.14)}\quad \Theta^{\mu\nu} = \frac{1}{4\pi}\Big[\left(F^{\mu\alpha}_{(-1)} + F^{\mu\alpha}_{(-2)} + F^{\mu\alpha}_{(-3)}\right)\left(F_{\alpha\;(-1)}^{\;\nu} + F_{\alpha\;(-2)}^{\;\nu} + F_{\alpha\;(-3)}^{\;\nu}\right)$$

$$+ \frac{g^{\mu\nu}}{4}\left(F^{\alpha\beta}_{(-1)} + F^{\alpha\beta}_{(-2)} + F^{\alpha\beta}_{(-3)}\right)\left(F_{\alpha\beta\,(-1)} + F_{\alpha\beta\,(-2)} + F_{\alpha\beta\,(-3)}\right)\Big]$$

Multiplying and regrouping terms to put those with highest ρ-dependence first, one obtains

$$\text{(I.2.15)}\quad \Theta^{\mu\nu} = \frac{1}{4\pi}\Big[\left(F^{\mu\alpha}_{(-1)} F_{\alpha\;(-1)}^{\;\nu} + \frac{g^{\mu\nu}}{4} F^{\alpha\beta}_{(-1)} F_{\alpha\beta\,(-1)}\right) + \Big(F^{\mu\alpha}_{(-1)} F_{\alpha\;(-2)}^{\;\nu}$$

$$+ F^{\mu\alpha}_{(-2)} F_{\alpha\;(-1)}^{\;\nu} + \frac{g^{\mu\nu}}{4}\left(2F^{\alpha\beta}_{(-1)} F_{\alpha\beta\,(-2)}\right)\Big)$$

$$+ \text{lower order terms in } \rho\Big]$$

$$= \Theta^{\mu\nu}_{(-2)} + \Theta^{\mu\nu}_{(-3)} + \text{lower order terms in } \rho$$

where

$$\text{(I.2.16)}\quad \Theta^{\mu\nu}_{(-2)} = \frac{1}{4\pi}\left[F^{\mu\alpha}_{(-1)} F_{\alpha\;(-1)}^{\;\nu} + \frac{g^{\mu\nu}}{4} F^{\alpha\beta}_{(-1)} F_{\alpha\beta\,(-1)}\right]$$

and

$$\text{(I.2.17)}\quad \Theta^{\mu\nu}_{(-3)} = \frac{1}{4\pi}\left[F^{\mu\alpha}_{(-1)} F_{\alpha\;(-2)}^{\;\nu} + F^{\mu\alpha}_{(-2)} F_{\alpha\;(-1)}^{\;\nu} + \frac{g^{\mu\nu}}{2} F^{\alpha\beta}_{(-1)} F_{\alpha\beta\,(-2)}\right]$$

In view of the foregoing observations about the nature of the integrals, one need not be concerned with calculating the terms of $\theta^{\mu\nu}$ whose order in ρ is lower than -3. Thus in place of the full expression for $\theta^{\mu\nu}$, the quantity

$$\text{(I.2.18)} \quad \theta_{\bullet}^{\mu\nu} = \theta_{(-2)}^{\mu\nu} + \theta_{(-3)}^{\mu\nu}$$

may be employed in the calculation of $dJ^{\mu\nu}$, and the quantity $\theta_{(-2)}^{\mu\nu}$ alone may be used in the calculation of dP^{μ} as well as in showing that the definition of angular momentum is valid.

3. VALIDITY OF THE ANGULAR MOMENTUM DEFINITION

As has already been demonstrated, the validity of the definition of linear radiated momentum can be seen without additional calculation. However, the proof of validity for the definition of radiated angular momentum requires the calculation of the integrand

$$(\theta^{\alpha\nu}_{(-2)} x^{\mu} - \theta^{\alpha\mu}_{(-2)} x^{\nu}) R_{\alpha} d\omega \tag{I.3.1}$$

which in the previous section was shown to be equivalent in the limit to the integrand $J^{\alpha\mu\nu}R_{\alpha}d\omega$. The calculation of $\theta^{\mu\nu}_{(-2)}$ requires the evaluation of the tensor products $F^{\mu\alpha}_{(-1)}F_{\alpha(-1)}^{\ \ \nu}$ and $F^{\alpha\beta}_{(-1)}F_{\alpha\beta(-1)}$, which evaluations are routine but lengthy and are therefore not reproduced here. As

$$F^{\alpha\beta}_{(-1)} F_{\alpha\beta(-1)} = 0^{1} \tag{I.3.2}$$

the result for $\theta^{\mu\nu}_{(-2)}$ reduces to

$$\begin{aligned} \Theta^{\mu\nu}_{(-2)} = \frac{1}{4\pi}\Bigg[& \left(\frac{ea^2}{2mc^5} - \frac{e\dot{a}_u}{2mc^4} - \frac{3e(a_u)^2}{2mc^5}\right)^2 M^{\alpha\beta} M_{\beta}^{\ \ \delta} R_{\alpha} R_{\delta} \\ & - 6\left(\frac{ea^2}{2mc^5} - \frac{e\dot{a}_u}{2mc^4} - \frac{3e(a_u)^2}{2mc^5}\right)\frac{ea_u}{2mc^4} M^{\alpha\beta}\dot{M}_{\beta}^{\ \ \delta} R_{\alpha} R_{\delta} \\ & - \frac{2c}{a_u}\left(\frac{ea^2}{2mc^5} - \frac{e\dot{a}_u}{2mc^4} - \frac{3e(a_u)^2}{2mc^5}\right)\frac{ea_u}{2mc^4} M^{\alpha\beta}\ddot{M}_{\beta}^{\ \ \delta} R_{\alpha} R_{\delta} \end{aligned} \tag{I.3.3}$$

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[1]A similar result is noted by Kolsrud and Leer, "Radiation from Moving Dipoles," p. 184.

$$+ \frac{9e^2(a_u)^2}{4m^2c^8}\dot{M}^{\alpha\beta}\dot{M}_\beta{}^\delta R_\alpha R_\delta + \frac{6c}{a_u}\left(\frac{ea_u}{2mc^4}\right)^2 \dot{M}^{\alpha\beta}\ddot{M}_\beta{}^\delta R_\alpha R_\delta$$

$$+ \frac{c^2}{(a_u)^2}\left(\frac{ea_u}{2mc^4}\right)^2 \ddot{M}^{\alpha\beta}\ddot{M}_\beta{}^\delta R_\alpha R_\delta - \rho^2\left(\frac{2e^2a^2}{2mc^7} - \frac{2e^2\dot{a}_u}{2mc^6}\right)M^{\alpha\beta}u_\alpha a_\beta$$

$$+ 6\rho^2\left(\frac{e^2a_u}{2mc^6}\right)\dot{M}^{\alpha\beta}u_\alpha a_\beta + \frac{2c\rho}{a_u}\left(\frac{e^2a_u}{2mc^6}\right)\ddot{M}^{\alpha\beta}R_\alpha a_\beta$$

$$+ \frac{2c\rho}{a_u}\left(\frac{e^2(a_u)^2}{2mc^7}\right)\ddot{M}^{\alpha\beta}R_\alpha v_\beta + \frac{e^2\rho^2}{c^4}\left((a_u)^2 - a^2\right)\Big] \frac{R^\mu R^\nu}{\rho^6}$$

Note that $\theta^{\mu\nu}_{(-2)}$ consists of a scalar quantity times the direct-product tensor $R^\mu R^\nu$. If this scalar quantity is defined as W, the integrand becomes $W(R^\alpha R^\nu x^\mu - R^\alpha R^\mu x^\nu)R_\alpha d\omega = W(R^\nu x^\mu - R^\mu x^\nu)R^\alpha R_\alpha d\omega$. Since the product $R^\alpha R_\alpha = 0$, the integrand equals zero; therefore the limit of the integral is zero and the definition of radiated angular momentum is valid, as is the definition of radiated linear momentum.

4. THE ASYMPTOTICALLY RADIATED LINEAR MOMENTUM

The asymptotic rate of radiation of linear momentum can now be calculated. According to (I.2.7), it may be written as

$$(\text{I.4.1}) \quad dP^{\mu} = -\lim_{\rho\to\infty}\int \theta^{\mu\nu} u_{\nu}\rho^{2}\, d\Omega\, d\tau$$

It has been observed that $\theta^{\mu\nu}_{(-2)}$ as expressed in Equation (I.3.3) has the form $WR^{\mu}R^{\nu}$, where W is a scalar function. Rewriting the above expression, one has

$$(\text{I.4.2}) \quad \frac{dP^{\mu}}{d\tau} = -\lim_{\rho\to\infty}\int W R^{\mu}R^{\nu}u_{\nu}\rho^{2}\, d\Omega$$

$$= -\lim_{\rho\to\infty}\int W R^{\mu}R^{\nu}(R_{\nu}/\rho - v_{\nu}/c)\rho^{2}\, d\Omega$$

$$= \lim_{\rho\to\infty}\int W R^{\mu}R^{\nu}v_{\nu}\frac{\rho^{2}}{c}\, d\Omega$$

$$= -\lim_{\rho\to\infty}\int W R^{\mu}\rho^{3}\, d\Omega$$

Substituting for W from Equation (I.3.3),

$$(\text{I.4.3}) \quad \frac{dP^{\mu}}{d\tau} = -\lim_{\rho\to\infty}\frac{1}{4\pi\rho^{3}}\int\Bigg[\left(\frac{ea^{2}}{2mc^{5}} - \frac{e\dot{a}_{u}}{2mc^{4}} - \frac{3e(a_{u})^{2}}{2mc^{5}}\right)^{2} M^{\alpha\beta}M_{\beta}{}^{\delta}R_{\alpha}R_{\delta}$$

$$-6\left(\frac{ea^{2}}{2mc^{5}} - \frac{e\dot{a}_{u}}{2mc^{4}} - \frac{3e(a_{u})^{2}}{2mc^{5}}\right)\frac{ea_{u}}{2mc^{4}}\, M^{\alpha\beta}\dot{M}_{\beta}{}^{\delta}R_{\alpha}R_{\delta}$$

$$- \frac{2ce}{2mc^4}\left(\frac{ea^2}{2mc^5} - \frac{e\dot{a}_u}{2mc^4} - \frac{3e(a_u)^2}{2mc^5}\right) M^{\alpha\beta}\ddot{M}_\beta{}^\delta R_\alpha R_\delta$$

$$+ \frac{9e^2(a_u)^2}{4m^2c^8}\dot{M}^{\alpha\beta}\dot{M}_\beta{}^\delta R_\alpha R_\delta + \frac{6e^2a_u}{4m^2c^7}\dot{M}^{\alpha\beta}\ddot{M}_\beta{}^\delta R_\alpha R_\delta$$

$$+ \frac{e^2}{4m^2c^6}\ddot{M}^{\alpha\beta}\ddot{M}_\beta{}^\delta R_\alpha R_\delta - \rho^2\left(\frac{2e^2a^2}{2mc^7} - \frac{2e^2\dot{a}_u}{2mc^6}\right) M^{\alpha\beta}u_\alpha a_\beta$$

$$+ \frac{6\rho^2e^2a_u}{2mc^6} M^{\alpha\beta}u_\alpha a_\beta + \frac{2\rho e^2}{2mc^5}\dot{M}^{\alpha\beta}R_\alpha a_\beta$$

$$+ \frac{2\rho e^2 a_u c}{2mc^7}\ddot{M}^{\alpha\beta}R_\alpha v_\beta + \frac{e^2\rho^2}{c^4}\left((a_u)^2 - a^2\right)\Big] R^\mu d\Omega$$

Since the integration with $d\Omega$ involves integrating over all possible orientations of the space components of the unit vector u^α, the substitution $R^\alpha = \rho(u^\alpha + v^\alpha/c)$ must be made before the integration can be carried out. Making this substitution in (I.4.3), factoring out some constants, and remembering that the integral of a direct-product tensor formed from an odd number of u^α-factors is zero over 4π steradians of solid angle (see Appendix A), one obtains

(I.4.4) $$\frac{dP^\mu}{d\tau} = -\lim_{\rho\to\infty}\frac{e^2}{4\pi c^5}\int\Big[\Big[\Big\{\left(-\frac{a^2\dot{a}_u}{2m^2c^4} + \frac{3\dot{a}_u(a_u)^2}{2m^2c^4}\right)u^\mu$$

$$+\left(\frac{(a^2)^2}{4m^2c^5} - \frac{3a^2(a_u)^2}{2m^2c^5} + \frac{(\dot{a}_u)^2}{4m^2c^3} + \frac{9(a_u)^4}{4m^2c^5}\right)\frac{v^\mu}{c}\Big\} M^{\alpha\beta}M_\beta{}^\delta u_\alpha u_\delta$$

$$+\left\{\left(\frac{-3a^2a_u}{2m^2c^4} + \frac{9(a_u)^3}{2m^2c^4}\right)u^\mu + \frac{3\dot{a}_u a_u}{2m^2c^4}v^\mu\right\} M^{\alpha\beta}\dot{M}_\beta{}^\delta u_\alpha u_\delta$$

$$+\left\{\frac{3\dot{a}_u a_u}{2m^2c^4}u^\mu+\left(\frac{9(a_u)^3}{2m^2c^6}-\frac{3a^2a_u}{2m^2c^6}\right)v^\mu\right\}M^{\alpha\beta}\dot{M}_\beta{}^\delta u_\alpha v_\delta$$

$$+\left\{\frac{\dot{a}_u}{2m^2c^2}u^\mu+\left(\frac{3(a_u)^2}{2m^2c^4}-\frac{a^2}{2m^2c^4}\right)v^\mu\right\}M^{\alpha\beta}\ddot{M}_\beta{}^\delta u_\alpha u_\delta$$

$$+\left\{\left(\frac{3(a_u)^2}{2m^2c^4}-\frac{a^2}{2m^2c^4}\right)u^\mu+\frac{\dot{a}_u}{2m^2c^4}v^\mu\right\}M^{\alpha\beta}\ddot{M}_\beta{}^\delta u_\alpha v_\delta$$

$$+\frac{9(a_u)^2}{4m^2c^4}\dot{M}^{\alpha\beta}\dot{M}_\beta{}^\delta u_\alpha u_\delta v^\mu+\frac{9(a_u)^2}{2m^2c^4}\dot{M}^{\alpha\beta}\dot{M}_\beta{}^\delta u_\alpha v_\delta u^\mu$$

$$+\frac{9(a_u)^2}{4m^2c^6}\dot{M}^{\alpha\beta}\dot{M}_\beta{}^\delta v_\alpha v_\delta v^\mu+\frac{3a_u}{2m^2c^2}\dot{M}^{\alpha\beta}\ddot{M}_\beta{}^\delta u_\alpha u_\delta u^\mu$$

$$+\frac{3a_u}{2m^2c^4}\dot{M}^{\alpha\beta}\ddot{M}_\beta{}^\delta u_\alpha v_\delta v^\mu+\frac{3a_u}{2m^2c^4}\dot{M}^{\alpha\beta}\ddot{M}_\beta{}^\delta v_\alpha u_\delta v^\mu$$

$$+\frac{3a_u}{2m^2c^4}\dot{M}^{\alpha\beta}\ddot{M}_\beta{}^\delta v_\alpha v_\delta u^\mu+\frac{1}{4m^2c^2}\ddot{M}^{\alpha\beta}\ddot{M}_\beta{}^\delta u_\alpha u_\delta v^\mu$$

$$+\frac{1}{2m^2c^2}\ddot{M}^{\alpha\beta}M_\beta{}^\delta u_\alpha v^\delta u^\mu+\frac{1}{4m^2c^4}\ddot{M}^{\alpha\beta}\ddot{M}_\beta{}^\delta v_\alpha v_\delta v^\mu$$

$$-\frac{a^2}{mc^2}M^{\alpha\beta}u_\alpha a_\beta u^\mu+\frac{\dot{a}_u}{mc^2}M^{\alpha\beta}u_\alpha a_\beta v^\mu+\frac{3a_u}{mc^2}\dot{M}^{\alpha\beta}u_\alpha a_\beta v^\mu$$

$$+\frac{1}{m}\ddot{M}^{\alpha\beta}u_\alpha a_\beta u^\mu+\frac{1}{mc^2}\ddot{M}^{\alpha\beta}v_\alpha a_\beta v^\mu$$

$$+\frac{a_u}{mc^2}\ddot{M}^{\alpha\beta}u_\alpha v_\beta v^\mu+\left((a_u)^2-a^2\right)v^\mu\Big]\,d\Omega$$

Grouping the factors of u^{α} together in each term and leaving only these factors within the integration signs yields the expression

$$(I.4.5)\quad \frac{dP^{\mu}}{d\tau} = -\lim_{\rho\to\infty}\frac{e^2}{4\pi c^5}\Bigg[\frac{-a^2\dot{a}^{\epsilon}}{2m^2c^4}M^{\alpha\beta}M_{\beta}{}^{\delta}\int u_{\epsilon}u_{\alpha}u_{\delta}u^{\mu}\,d\Omega$$

$$+\frac{3\dot{a}^{\epsilon}a^{\zeta}a^{\eta}}{2m^2c^4}M^{\alpha\beta}M_{\beta}{}^{\delta}\int u_{\epsilon}u_{\zeta}u_{\eta}u_{\alpha}u_{\delta}u^{\mu}\,d\Omega$$

$$+\frac{(a^2)^2}{4m^2c^6}M^{\alpha\beta}M_{\beta}{}^{\delta}v^{\mu}\int u_{\alpha}u_{\delta}\,d\Omega-\frac{3a^2a^{\epsilon}a^{\zeta}}{2m^2c^6}M^{\alpha\beta}M_{\beta}{}^{\delta}v^{\mu}\int u_{\epsilon}u_{\zeta}u_{\alpha}u_{\delta}\,d\Omega$$

$$+\frac{\dot{a}^{\epsilon}\dot{a}^{\zeta}}{4m^2c^4}M^{\alpha\beta}M_{\beta}{}^{\delta}v^{\mu}\int u_{\epsilon}u_{\zeta}u_{\delta}u_{\alpha}\,d\Omega+\frac{9a^{\epsilon}a^{\zeta}a^{\eta}a^{\varpi}}{4m^2c^6}M^{\alpha\beta}M_{\beta}{}^{\delta}v^{\mu}\int u_{\epsilon}u_{\zeta}u_{\eta}u_{\varpi}u_{\alpha}u_{\delta}\,d\Omega$$

$$-\frac{3a^2a^{\epsilon}}{2m^2c^4}M^{\alpha\beta}\dot{M}_{\beta}{}^{\delta}\int u_{\epsilon}u_{\alpha}u_{\delta}u^{\mu}\,d\Omega+\frac{9a^{\epsilon}a^{\zeta}a^{\eta}}{2m^2c^4}M^{\alpha\beta}\dot{M}_{\beta}{}^{\delta}\int u_{\epsilon}u_{\zeta}u_{\eta}u_{\alpha}u_{\delta}u^{\mu}\,d\Omega$$

$$+\frac{3\dot{a}^{\epsilon}a^{\zeta}}{2m^2c^4}M^{\alpha\beta}\dot{M}_{\beta}{}^{\delta}v^{\mu}\int u_{\epsilon}u_{\zeta}u_{\alpha}u_{\delta}\,d\Omega+\frac{3\dot{a}^{\epsilon}a^{\zeta}}{2m^2c^4}M^{\alpha\beta}\dot{M}_{\beta}{}^{\delta}v_{\delta}\int u_{\epsilon}u_{\alpha}u_{\zeta}u^{\mu}\,d\Omega$$

$$+\frac{9a^{\epsilon}a^{\zeta}a^{\eta}}{2m^2c^6}v^{\mu}M^{\alpha\beta}\dot{M}_{\beta}{}^{\delta}v_{\delta}\int u_{\epsilon}u_{\zeta}u_{\eta}u_{\alpha}\,d\Omega-\frac{3a^2a^{\epsilon}}{2m^2c^6}v^{\mu}M^{\alpha\beta}\dot{M}_{\beta}{}^{\delta}v_{\delta}\int u_{\epsilon}u_{\alpha}\,d\Omega$$

$$+\frac{\dot{a}^{\epsilon}}{2m^2c^2}M^{\alpha\beta}\ddot{M}_{\beta}{}^{\delta}\int u_{\epsilon}u_{\alpha}u_{\delta}u^{\mu}\,d\Omega+\frac{3a^{\epsilon}a^{\zeta}}{2m^2c^4}v^{\mu}M^{\alpha\beta}M_{\beta}{}^{\delta}\int u_{\epsilon}u_{\zeta}u_{\alpha}u_{\delta}\,d\Omega$$

$$-\frac{a^2}{2m^2c^4}v^{\mu}M^{\alpha\beta}\ddot{M}_{\beta}{}^{\delta}\int u_{\alpha}u_{\delta}\,d\Omega+\frac{3a^{\epsilon}a^{\zeta}}{2m^2c^4}M^{\alpha\beta}\ddot{M}_{\beta}{}^{\delta}v_{\delta}\int u_{\epsilon}u_{\zeta}u_{\alpha}u^{\mu}\,d\Omega$$

$$-\frac{a^2}{2m^2c^4}M^{\alpha\beta}\ddot{M}_{\beta}{}^{\delta}v_{\delta}\int u_{\alpha}u^{\mu}\,d\Omega+\frac{\dot{a}^{\epsilon}v^{\mu}}{2m^2c^4}M^{\alpha\beta}\ddot{M}_{\beta}{}^{\delta}v_{\delta}\int u_{\epsilon}u_{\alpha}\,d\Omega$$

$$+\frac{9a^{\epsilon}a^{\zeta}}{4m^2c^4}v^{\mu}\dot{M}^{\alpha\beta}\dot{M}_{\beta}{}^{\delta}\int u_{\epsilon}u_{\zeta}u_{\alpha}u_{\delta}\,d\Omega+\frac{9a^{\epsilon}a^{\zeta}}{2m^2c^4}\dot{M}^{\alpha\beta}\dot{M}_{\beta}{}^{\delta}v_{\delta}\int u_{\epsilon}u_{\zeta}u_{\alpha}u^{\mu}\,d\Omega$$

$$+\frac{9a^{\epsilon}a^{\zeta}}{4m^2c^6}\dot{M}^{\alpha\beta}\dot{M}_{\beta}{}^{\delta}v_{\alpha}v_{\delta}v^{\mu}\int u_{\epsilon}u_{\zeta}\,d\Omega+\frac{3a^{\epsilon}}{2m^2c^2}\dot{M}^{\alpha\beta}\ddot{M}_{\beta}{}^{\delta}\int u_{\epsilon}u_{\alpha}u_{\delta}u^{\mu}\,d\Omega$$

$$+\frac{3a^{\epsilon}}{2m^2c^4}\dot{M}^{\alpha\beta}\ddot{M}_{\beta}{}^{\delta}v_{\delta}v^{\mu}\int u_{\epsilon}u_{\alpha}\,d\Omega+\frac{3a^{\epsilon}}{2m^2c^4}\dot{M}^{\alpha\beta}\ddot{M}_{\beta}{}^{\delta}v_{\alpha}v^{\mu}\int u_{\epsilon}u_{\delta}\,d\Omega$$

$$+\frac{3a^{\epsilon}}{2m^2c^4}\dot{M}^{\alpha\beta}\ddot{M}_{\beta}{}^{\delta}v_{\alpha}v_{\delta}\int u_{\epsilon}u^{\mu}\,d\Omega+\frac{1}{4m^2c^2}\ddot{M}^{\alpha\beta}\ddot{M}_{\beta}{}^{\delta}v^{\mu}\int u_{\alpha}u_{\delta}\,d\Omega$$

$$+\frac{1}{2m^2c^2}\ddot{M}^{\alpha\beta}\ddot{M}_{\beta}{}^{\delta}v_{\delta}\int u_{\alpha}u^{\mu}\,d\Omega+\frac{1}{4m^2c^4}\ddot{M}^{\alpha\beta}\ddot{M}_{\beta}{}^{\delta}v_{\alpha}v_{\delta}v^{\mu}\int d\Omega$$

$$-\frac{a^2}{mc^2}M^{\alpha\beta}a_{\beta}\int u_{\alpha}u^{\mu}\,d\Omega+\frac{\dot{a}^{\epsilon}}{mc^2}M^{\alpha\beta}a_{\beta}v^{\mu}\int u_{\epsilon}u_{\alpha}\,d\Omega$$

$$+\frac{3a^{\epsilon}}{mc^2}\dot{M}^{\alpha\beta}a_{\beta}v^{\mu}\int u_{\epsilon}u_{\alpha}\,d\Omega+\frac{1}{m}\ddot{M}^{\alpha\beta}a_{\beta}\int u_{\alpha}u^{\mu}\,d\Omega$$

$$+\frac{1}{mc^2}\ddot{M}^{\alpha\beta}v_{\alpha}a_{\beta}v^{\mu}\int d\Omega+\frac{a^{\epsilon}}{mc^2}\ddot{M}^{\alpha\beta}v_{\beta}v^{\mu}\int u_{\epsilon}u_{\alpha}\,d\Omega$$

$$+a^{\epsilon}a^{\zeta}v^{\mu}\int u_{\epsilon}u_{\zeta}\,d\Omega-a^2v^{\mu}\int d\Omega\Big]$$

Substituting the expressions for the integrals given in Appendix A and simplifying, one has

$$\text{(I.4.6)}\quad \frac{dP^\mu}{d\tau} = \frac{e^2}{4\pi c^5}\Big[\frac{4\pi a^2}{15\cdot 2m^2c^4}\Big(2M^{\mu\beta}M_\beta{}^\delta \dot a_\delta - M^2\dot a^\mu + \frac{M^2a^2}{c^2}v^\mu\Big)$$

$$-\frac{4\pi\cdot 3}{105\cdot 2m^2c^4}\Big(2\dot a^\epsilon a_\epsilon\big(2M^{\mu\beta}M_\beta{}^\alpha a_\alpha - a^\mu M^2\big) + 2\dot a_\alpha M^{\alpha\beta}\big(a^2M_\beta{}^\mu$$

$$+2a^\mu M_\beta{}^\delta a_\delta\big) + \dot a^\mu\big(-a^2M^2 + 2M^{\alpha\beta}M_\beta{}^\delta a_\alpha a_\delta\big) - \frac{a^2v^\mu}{c^2}\big(-a^2M^2$$

$$+2M^{\alpha\beta}M_\beta{}^\delta a_\alpha a_\delta\big)\Big) + \frac{4\pi(a^2)^2M^2v^\mu}{3\cdot 4m^2c^6} + \frac{4\pi\cdot 3a^2v^\mu}{15\cdot 2m^2c^6}\Big(2M^{\alpha\beta}M_\beta{}^\delta a_\alpha a_\delta$$

$$-a^2M^2\Big) + \frac{4\pi v^\mu}{15\cdot 4m^2c^4}\Big(\dot a^2M^2 + \frac{(a^2)^2}{c^2}M^2 - 2M^{\alpha\beta}M_\beta{}^\delta \dot a_\alpha \dot a_\delta\Big)$$

$$+\frac{4\pi\cdot 9v^\mu}{4\cdot 105m^2c^6}\Big(3(a^2)^2M^2 - 12a^2M^{\alpha\beta}M_\beta{}^\delta a_\alpha a_\delta\Big) + \frac{4\pi\cdot 3a^2}{15\cdot 2m^2c^4}\Big(\dot M^{\mu\beta}M_\beta{}^\alpha a_\alpha$$

$$+\frac{v^\mu}{c^2}M^{\alpha\beta}\dot M_\beta{}^\delta a_\alpha v_\delta + M^{\mu\beta}\dot M_\beta{}^\delta a_\delta - a^\mu M^{\alpha\beta}\dot M_{\alpha\beta}\Big) - \frac{4\pi\cdot 9}{105\cdot 2m^2c^4}\Big(3a^2M^{\alpha\beta}\dot M_\beta{}^\mu a_\alpha$$

$$+\frac{3v^\mu a^2}{c^2}M^{\alpha\beta}\dot M_\beta{}^\delta a_\alpha v_\delta + 3a^2M^{\mu\beta}\dot M_\beta{}^\delta a_\delta - 3a^\mu a^2M^{\alpha\beta}\dot M_{\alpha\beta}$$

$$+6a^\mu M^{\alpha\beta}\dot M_\beta{}^\delta a_\alpha a_\delta\Big) - \frac{4\pi\cdot 3v^\mu}{15\cdot 2m^2c^4}\Big(-\dot a^\epsilon a_\epsilon M^{\alpha\beta}\dot M_{\alpha\beta} + M^{\alpha\beta}\dot M_\beta{}^\delta \dot a_\alpha a_\delta$$

$$+M^{\alpha\beta}\dot M_\beta{}^\delta a_\alpha \dot a_\delta - \frac{a^2}{c^2}M^{\alpha\beta}\dot M_\beta{}^\delta a_\alpha v_\delta\Big) - \frac{4\pi\cdot 3}{15\cdot 2m^2c^4}\Big(M^{\mu\beta}\dot M_\beta{}^\delta v_\delta \dot a^\epsilon a_\epsilon$$

$$+a^\mu M^{\alpha\beta}\dot M_\beta{}^\delta \dot a_\alpha v_\delta + \dot a^\mu M^{\alpha\beta}\dot M_\beta{}^\delta a_\alpha v_\delta - \frac{a^2v^\mu}{c^2}M^{\alpha\beta}\dot M_\beta{}^\delta a_\alpha v_\delta\Big)$$

$$-\frac{4\pi\cdot 9v^{\mu}}{15\cdot 2m^2c^6}\left(3a^2 M^{\alpha\beta}\dot{M}_{\beta}{}^{\delta}a_{\alpha}v_{\delta}\right)+\frac{4\pi\cdot 3a^2v^{\mu}}{3\cdot 2m^2c^6}\left(M^{\alpha\beta}\dot{M}_{\beta}{}^{\delta}a_{\alpha}v_{\delta}\right)$$

$$-\frac{4\pi}{15\cdot 2m^2c^2}\left(M^{\alpha\beta}\ddot{M}_{\beta}{}^{\mu}\dot{a}_{\alpha}+\frac{v^{\mu}}{c^2}M^{\alpha\beta}\ddot{M}_{\beta}{}^{\delta}\dot{a}_{\alpha}v_{\delta}+M^{\mu\beta}\ddot{M}_{\beta}{}^{\delta}\dot{a}_{\delta}\right.$$

$$\left.-\frac{a^2}{c^2}M^{\mu\beta}\ddot{M}_{\beta}{}^{\delta}v_{\delta}-\dot{a}^{\mu}M^{\alpha\beta}\ddot{M}_{\alpha\beta}+\frac{a^2v^{\mu}}{c^2}M^{\alpha\beta}\ddot{M}_{\alpha\beta}\right)-\frac{4\pi\cdot 3v^{\mu}}{15\cdot 2m^2c^4}\left(-a^2M^{\alpha\beta}\ddot{M}_{\alpha\beta}\right.$$

$$\left.+2M^{\alpha\beta}\ddot{M}_{\beta}{}^{\delta}a_{\alpha}a_{\delta}\right)-\frac{4\pi\cdot a^2v^{\mu}}{3\cdot 2m^2c^4}M^{\alpha\beta}\ddot{M}_{\alpha\beta}-\frac{4\pi\cdot 3}{15\cdot 2m^2c^4}\left(a^2M^{\mu\beta}\ddot{M}_{\beta}{}^{\delta}v_{\delta}\right.$$

$$\left.+2a^{\mu}M^{\alpha\beta}\ddot{M}_{\beta}{}^{\delta}a_{\alpha}v_{\delta}\right)+\frac{4\pi a^2}{3\cdot 2m^2c^4}M^{\mu\beta}\ddot{M}_{\beta}{}^{\delta}v_{\delta}-\frac{4\pi v^{\mu}}{3\cdot 2m^2c^5}M^{\alpha\beta}\ddot{M}_{\beta}{}^{\delta}\dot{a}_{\alpha}v_{\delta}$$

$$-\frac{4\pi\cdot 9v^{\mu}}{15\cdot 4m^2c^4}\left(-a^2\dot{M}^2+\frac{a^2}{c^2}\dot{M}^{\alpha\beta}\dot{M}_{\beta}{}^{\delta}v_{\alpha}v_{\delta}+2\dot{M}^{\alpha\beta}\dot{M}_{\beta}{}^{\delta}a_{\alpha}a_{\delta}\right)$$

$$-\frac{4\pi\cdot 9}{15\cdot 2m^2c^4}\left(a^2\dot{M}^{\mu\beta}\dot{M}_{\beta}{}^{\delta}v_{\delta}+\frac{a^2}{c^2}\dot{M}^{\alpha\beta}\dot{M}_{\beta}{}^{\delta}v_{\alpha}v_{\delta}v^{\mu}+2a^{\mu}\dot{M}^{\alpha\beta}\dot{M}_{\beta}{}^{\delta}a_{\alpha}v_{\delta}\right)$$

$$-\frac{4\pi\cdot 9a^2}{3\cdot 4m^2c^6}\dot{M}^{\alpha\beta}\dot{M}_{\beta}{}^{\delta}v_{\alpha}v_{\delta}v^{\mu}-\frac{4\pi\cdot 3}{15\cdot 2m^2c^2}\left(\dot{M}^{\alpha\beta}\ddot{M}_{\beta}{}^{\mu}a_{\alpha}+\dot{M}^{\alpha\beta}\ddot{M}_{\beta}{}^{\delta}a_{\alpha}v_{\delta}\frac{v^{\mu}}{c^2}\right.$$

$$\left.+\dot{M}^{\mu\beta}\ddot{M}_{\beta}{}^{\delta}a_{\delta}+\dot{M}^{\alpha\beta}\ddot{M}_{\beta}{}^{\delta}v_{\alpha}a_{\delta}\frac{v^{\mu}}{c^2}-a^{\mu}\dot{M}^{\alpha\beta}\ddot{M}_{\alpha\beta}+a^{\mu}\dot{M}^{\alpha\beta}\ddot{M}_{\beta}{}^{\delta}\frac{v_{\alpha}v_{\delta}}{c^2}\right)$$

$$-\frac{4\pi\cdot 3v^{\mu}}{3\cdot 2m^2c^4}\dot{M}^{\alpha\beta}\ddot{M}_{\beta}{}^{\delta}a_{\alpha}v_{\delta}-\frac{4\pi\cdot 3v^{\mu}}{3\cdot 2m^2c^4}\dot{M}^{\alpha\beta}\ddot{M}_{\beta}{}^{\delta}v_{\alpha}a_{\delta}-\frac{4\pi\cdot 3a^{\mu}}{3\cdot 2m^2c^4}\dot{M}^{\alpha\beta}\ddot{M}_{\beta}{}^{\delta}v_{\alpha}v_{\delta}$$

$$-\frac{4\pi v^{\mu}}{3\cdot 4m^2c^2}\left(-\ddot{M}^2+\frac{1}{c^2}\ddot{M}^{\alpha\beta}\ddot{M}_{\beta}{}^{\delta}v_{\alpha}v_{\delta}\right)-\frac{4\pi}{3\cdot 2m^2c^2}\left(\ddot{M}^{\mu\beta}\ddot{M}_{\beta}{}^{\delta}v_{\delta}\right.$$

$$+\frac{1}{c^2}\ddot{M}^{\alpha\beta}\ddot{M}_\beta{}^\delta v_\alpha v_\delta v^\mu\Big) - \frac{4\pi}{4m^2c^4}\ddot{M}^{\alpha\beta}\ddot{M}_\beta{}^\delta v_\alpha v_\delta v^\mu$$

$$+\frac{4\pi a^2}{3mc^2}M^{\mu\beta}a_\beta - \frac{4\pi v^\mu}{3mc^2}M^{\alpha\beta}\dot{a}_\alpha a_\beta - \frac{4\pi}{3m}\Big(\ddot{M}^{\mu\beta}a_\beta$$

$$+\frac{1}{c^2}\ddot{M}^{\alpha\beta}v_\alpha a_\beta v^\mu\Big) - \frac{4\pi}{mc^2}\ddot{M}^{\alpha\beta}v_\alpha a_\beta v^\mu - \frac{4\pi}{3mc^2}\ddot{M}^{\alpha\beta}a_\alpha v_\beta v^\mu$$

$$-\frac{4\pi}{3}a^2v^\mu + 4\pi a^2 v^\mu\Big]$$

An additional result obtainable from (I.4.6) is the particle's energy radiation rate, which is just the fourth component of (I.4.6) multiplied by a factor of c. Using the rest-frame identities

$$\dot{a}^4 c = a^2 \tag{I.4.7}$$

$$c\,\dot{M}^{\mu 4} = -\dot{M}^{\mu\nu}v_\nu \tag{I.4.8}$$

$$c\,\ddot{M}^{\mu 4} = -\ddot{M}^{\mu\nu}v_\nu \tag{I.4.9}$$

and the fact that in the rest frame $a^4 = M^{\mu 4} = 0$, one obtains

$$\frac{dW}{d\tau} = \frac{e^2}{c^4}\Big[\frac{9(a^2)^2M^2}{140m^2c^5} + \frac{\dot{a}^2M^2}{60m^2c^3} - \frac{a^2M^{\alpha\beta}\ddot{M}_{\alpha\beta}}{15m^2c^3}$$
$$+\frac{3a^2\dot{M}^2}{20m^2c^3} + \frac{\ddot{M}^2}{12m^2c} - \frac{23a^2}{35m^2c^5}M^{\alpha\beta}M_\beta{}^\delta a_\alpha a_\delta \tag{I.4.10}$$

$$- \frac{1}{5m^2c^3} M^{\alpha\beta} M_\beta{}^\delta \dot{a}_\alpha \dot{a}_\delta - \frac{3}{5m^2c^3} M^{\alpha\beta} \dot{M}_\beta{}^\delta \dot{a}_\alpha a_\delta$$

$$- \frac{1}{10m^2c^3} M^{\alpha\beta} \dot{M}_\beta{}^\delta a_\alpha \dot{a}_\delta - \frac{19}{30m^2c^3} \dot{M}^{\alpha\beta} \dot{M}_\beta{}^\delta a_\alpha a_\delta$$

$$+ \frac{3}{10m^2c^3} M^{\alpha\beta} \ddot{M}_\beta{}^\delta a_\alpha a_\delta - \frac{1}{3mc} M^{\alpha\beta} a_\alpha \dot{a}_\beta + \frac{2ca^2}{3} \Bigg]$$

in the particle's rest frame.

Calculation of the radiation from a magnetic dipole (without charge) has been done by Kolsrud and Leer.[1] Even when the charge-dependent terms are removed from (I.4.10), the result differs from the Kolsrud and Leer result. (I.4.10) contains equivalents of all the Kolsrud and Leer terms (although some coefficients differ) plus other terms.

[1] Kolsrud and Leer, p. 187.

5. THE ASYMPTOTICALLY RADIATED ANGULAR MOMENTUM

Using procedures analogous to those used in Section 4, one can now calculate the asymptotic radiation rate of angular momentum. The definition of the angular momentum radiation rate is

$$\text{(I.5.1)} \quad dJ^{\mu\nu} = \frac{1}{c}\int J^{\alpha\mu\nu} d\sigma_\alpha = \frac{1}{c}\int (\theta^{\alpha\nu}x^\mu - \theta^{\alpha\mu}x^\nu)\, d\sigma_\alpha$$

Since $d\sigma_\alpha = -u_\alpha \rho^2 d\Omega\, c\, d\tau$ has a ρ^2 dependence, and x^μ depends on ρ, only those terms of $\theta^{\alpha\mu}$ which depend on at least ρ^{-3} will contribute to the asymptotic limit. The integral can therefore be written as follows:

$$\text{(I.5.2)} \quad dJ^{\mu\nu} = \frac{1}{c}\int (\theta^{\alpha\nu}_{(-2)}x^\mu - \theta^{\alpha\mu}_{(-2)}x^\nu)\, d\sigma_\alpha + \frac{1}{c}\int (\theta^{\alpha\nu}_{(-3)}x^\mu - \theta^{\alpha\mu}_{(-3)}x^\nu)\, d\sigma_\alpha$$

where the second integral varies as ρ^0 and the first seems to vary as ρ^1. These two integrals will be treated separately, with the first one represented as $dJ^{\mu\nu}_{(a)}$ and the second as $dJ^{\mu\nu}_{(b)}$.

The apparent ρ^1-dependence of $dJ^{\mu\nu}_{(a)}$ raises the possibility of divergence in the asymptotic limit; however, it may be shown that divergence does not occur. Using the identity $x^\mu = R^\mu + z^\mu$, one has

$$\text{(I.5.3)} \quad dJ^{\mu\nu}_{(a)} = \frac{1}{c}\int (\theta^{\alpha\nu}_{(-2)}R^\mu + \theta^{\alpha\nu}_{(-2)}z^\mu - \theta^{\alpha\mu}_{(-2)}R^\nu - \theta^{\alpha\mu}_{(-2)}z^\nu)\, d\sigma_\alpha$$

$$= \frac{1}{c}\int (\theta^{\alpha\nu}_{(-2)}R^\mu - \theta^{\alpha\mu}_{(-2)}R^\nu)\, d\sigma_\alpha + \frac{1}{c}\int (\theta^{\alpha\nu}_{(-2)}z^\mu - \theta^{\alpha\mu}_{(-2)}z^\nu)\, d\sigma_\alpha$$

where the first of the above integrals depends on ρ^1 and the second depends on ρ^0. However, as demonstrated in Equation (I.3.3), $\theta^{\mu\nu}_{(-2)}$ has the form $R^\mu R^\nu W$ where W is a scalar. This means the first of the above integrands is equal to $W(R^\alpha R^\nu R^\mu - R^\alpha R^\mu R^\nu)$, which is identically zero. Therefore, the term depending on ρ^1 vanishes, and $dJ^{\mu\nu}_{(a)}$ is actually dependent only on ρ^0:

$$(I.5.4)\quad dJ^{\mu\nu}_{(a)} = -\frac{1}{c}\int \left(\theta^{\alpha\nu}_{(-2)} z^\mu - \theta^{\alpha\mu}_{(-2)} z^\nu\right) u_\alpha \rho^2 d\Omega\, c\, d\tau$$

The possibility of divergence in the limit $\rho \to \infty$ is therefore removed, and evaluation of the integral for $dJ^{\mu\nu}_{(a)}$ may begin with the above expression.

Rewriting $\theta^{\alpha\nu}_{(-2)}$ as $WR^\alpha R^\nu$, one has

$$(I.5.5)\quad dJ^{\mu\nu}_{(a)} = -\frac{1}{c}\int W R^\alpha (R^\nu z^\mu - R^\mu z^\nu) u_\alpha \rho^2 d\Omega\, c\, d\tau$$

or, using the identity $R^\alpha u_\alpha = \rho$ and combining constants,

$$(I.5.6)\quad \frac{dJ^{\mu\nu}_{(a)}}{d\tau} = -\int W (R^\nu z^\mu - R^\mu z^\nu) \quad \rho^3 d\Omega$$

Substituting for W from Equation (I.3.3),

$$(I.5.7)\quad \frac{dJ^{\mu\nu}_{(a)}}{d\tau} = -\frac{1}{4\pi\rho^3}\int (R^\nu z^\mu - R^\mu z^\nu)\Bigg[\left(\frac{ea^2}{2mc^5} - \frac{e\dot{a}_u}{2mc^4} - \frac{3e(a_u)^2}{2mc^5}\right)^2 M^{\alpha\beta} M_\beta{}^\delta R_\alpha R_\delta$$

$$-\left(\frac{ea^2}{2mc^5} - \frac{e\dot{a}_u}{2mc^4} - \frac{3e(a_u)^2}{2mc^5}\right)\frac{6ea_u}{2mc^4} M^{\alpha\beta}\dot{M}_\beta{}^\delta R_\alpha R_\delta$$

$$-\left(\frac{ea^2}{2mc^5} - \frac{e\dot{a}_u}{2mc^4} - \frac{3e(a_u)^2}{2mc^5}\right)\frac{2ce}{2mc^4} M^{\alpha\beta}\ddot{M}_\beta{}^\delta R_\alpha R_\delta$$

$$+\frac{9e^2(a_u)^2}{4m^2c^8}\dot{M}^{\alpha\beta}\dot{M}_\beta{}^\delta R_\alpha R_\delta + \frac{6e^2 a_u}{4m^2c^7}\dot{M}^{\alpha\beta}\ddot{M}_\beta{}^\delta R_\alpha R_\delta$$

$$+\frac{e^2}{4m^2c^6}\ddot{M}^{\alpha\beta}\ddot{M}_\beta{}^\delta R_\alpha R_\delta - \rho^2\left(\frac{2e^2a^2}{2mc^7} - \frac{2e^2\dot{a}_u}{2mc^6}\right) M^{\alpha\beta} u_\alpha a_\beta$$

$$+ \frac{6\rho^2 e^2 a_u}{2mc^2} \dot{M}^{\alpha\beta} u_\alpha a_\beta + \frac{2\rho e^2}{2mc^5} \ddot{M}^{\alpha\beta} R_\alpha a_\beta$$

$$+ \frac{2\rho e^2 a_u}{2mc^6} \ddot{M}^{\alpha\beta} R_\alpha v_\beta + \frac{e^2\rho^2}{c^4}\left((a_u)^2 - a^2\right)\Big] d\Omega$$

In order to make use of the integrals developed in Appendix A, R^μ must be replaced in the above integral with the identity

$$R^\mu = \rho\left(u^\mu + \frac{v^\mu}{c}\right) \tag{I.5.8}$$

Performing this substitution, integrating by means of the formulas from Appendix A, and simplifying, one obtains

$$\frac{dJ_{(a)}^{\mu\nu}}{d\tau} = -\frac{z^{[\mu}}{4\pi}\Bigg\{\frac{e^2(a^2)^2 v^{\nu]}}{4m^2c^{11}}\left(\frac{4\pi}{3}\right)M^{\alpha\beta}M_{\beta\alpha} \tag{I.5.9}$$

$$- \frac{2e^2a^2}{4m^2c^9}\left(\frac{4\pi}{15}\right)\left(2M^{\nu]\beta}M_\beta{}^\delta \dot{a}_\delta + M^{\alpha\beta}M_{\beta\alpha}\left(\dot{a}^{\nu]} + \frac{\dot{a}^\alpha v_\alpha v^{\nu]}}{c^2}\right)\right)$$

$$+ \frac{e^2 v^{\nu]}}{4m^2c^9}\left(\frac{4\pi}{15}\right)\left(\left(\dot{a}^2 + \left(\frac{\dot{a}^\alpha v_\alpha}{c}\right)^2\right)M^{\alpha\beta}M_{\beta\alpha} + 2M^{\alpha\beta}\dot{a}_\alpha M_\beta{}^\delta \dot{a}_\delta\right)$$

$$- \frac{6e^2a^2v^{\nu]}}{4m^2c^{11}}\left(\frac{4\pi}{15}\right)\left(a^2 M^{\alpha\beta}M_{\beta\alpha} + 2M^{\alpha\beta}M_\beta{}^\delta a_\alpha a_\delta\right)$$

$$+ \frac{6e^2}{4m^2c^9}\left(\frac{4\pi}{105}\right)\left(2\dot{a}^\epsilon a_\epsilon\left(2a_\alpha M^{\alpha\beta}M_\beta{}^{\nu]} + a^{\nu]}M^{\alpha\beta}M_{\beta\alpha}\right)\right.$$

$$\left. + 2\dot{a}^\epsilon M_\epsilon{}^\beta\left(a^2 M_\beta{}^{\nu]} + 2a^{\nu]}M_\beta{}^\delta a_\delta\right)\right)$$

$$+\left(\dot{a}^{\nu]}+\frac{\dot{a}^{\epsilon}v_{\epsilon}v^{\nu]}}{c^2}\right)\left(a^2M^{\alpha\beta}M_{\beta\alpha}+2M^{\alpha\beta}M_{\beta}{}^{\delta}a_{\alpha}a_{\delta}\right)\Bigg)$$

$$+\frac{9e^2v^{\nu]}}{4m^2c^{11}}\left(\frac{4\pi}{105}\right)\left(3(a^2)^2M^{\alpha\beta}M_{\beta\alpha}+12a^2M^{\alpha\beta}M_{\beta}{}^{\delta}a_{\alpha}a_{\delta}\right)$$

$$-\frac{6e^2a^2}{4m^2c^9}\left(\frac{4\pi}{15}\right)\Bigg(a_{\alpha}M^{\alpha\beta}\dot{M}_{\beta}{}^{\nu]}+M^{\alpha\beta}\dot{M}_{\beta}{}^{\delta}\frac{a_{\alpha}v_{\delta}v^{\nu]}}{c^2}+M^{\nu]\beta}\dot{M}_{\beta}{}^{\delta}a_{\delta}$$

$$+a^{\nu]}M^{\alpha\beta}\dot{M}_{\beta\alpha}\Bigg)+\frac{6e^2v^{\nu]}}{4m^2c^9}\left(\frac{4\pi}{15}\right)\Bigg(\dot{a}^{\epsilon}a_{\epsilon}M^{\alpha\beta}\dot{M}_{\beta\alpha}$$

$$+M^{\alpha\beta}\dot{M}_{\beta}{}^{\delta}\dot{a}_{\alpha}a_{\delta}+M^{\alpha\beta}\dot{M}_{\beta}{}^{\delta}a_{\alpha}\dot{a}_{\delta}+M^{\alpha\beta}\dot{M}_{\beta}{}^{\delta}\frac{a_{\alpha}v_{\delta}\dot{a}^{\epsilon}v_{\epsilon}}{c^2}\Bigg)$$

$$+\frac{18e^2}{4m^2c^9}\left(\frac{4\pi}{105}\right)\Bigg(3a^2M^{\alpha\beta}a_{\alpha}\dot{M}_{\beta}{}^{\nu]}+\frac{3a^2}{c^2}M^{\alpha\beta}a_{\alpha}\dot{M}_{\beta}{}^{\delta}v_{\delta}v^{\nu]}$$

$$+3a^{\nu]}a^2M^{\alpha\beta}\dot{M}_{\beta\alpha}+3a^2M^{\nu]\beta}\dot{M}_{\beta}{}^{\delta}a_{\delta}+6a^{\nu]}M^{\alpha\beta}\dot{M}_{\beta}{}^{\delta}a_{\alpha}a_{\delta}\Bigg)$$

$$-\frac{6e^2a^2v^{\nu]}}{4m^2c^{11}}\left(\frac{4\pi}{3}\right)\left(M^{\alpha\beta}\dot{M}_{\beta}{}^{\delta}a_{\alpha}v_{\delta}\right)+\frac{6e^2}{4m^2c^9}\left(\frac{4\pi}{15}\right)\Bigg(\dot{a}^{\epsilon}a_{\epsilon}M^{\nu]\beta}\dot{M}_{\beta}{}^{\delta}v_{\delta}$$

$$+a^{\nu]}M^{\alpha\beta}\dot{M}_{\beta}{}^{\delta}\dot{a}_{\alpha}v_{\delta}+\dot{a}^{\nu]}M^{\alpha\beta}\dot{M}_{\beta}{}^{\delta}a_{\alpha}v_{\delta}+v^{\nu]}M^{\alpha\beta}\dot{M}_{\beta}{}^{\delta}a_{\alpha}v_{\delta}\frac{\dot{a}^{\epsilon}v_{\epsilon}}{c^2}\Bigg)$$

$$+\frac{18e^2v^{\nu]}}{4m^2c^{11}}\left(\frac{4\pi}{15}\right)\left(3a^2M^{\alpha\beta}\dot{M}_{\beta}{}^{\delta}a_{\alpha}v_{\delta}\right)-\frac{2e^2a^2v^{\nu]}}{4m^2c^9}\left(\frac{4\pi}{3}\right)\left(M^{\alpha\beta}\ddot{M}_{\beta\alpha}\right)$$

$$+\frac{2e^2}{4m^2c^7}\left(\frac{4\pi}{15}\right)\Bigg(M^{\alpha\beta}\dot{a}_{\alpha}\left(\ddot{M}_{\beta}{}^{\nu]}+\ddot{M}_{\beta}{}^{\delta}\frac{v_{\delta}v^{\nu]}}{c^2}\right)+M^{\nu]\beta}\Bigg(\ddot{M}_{\beta}{}^{\delta}\dot{a}_{\delta}$$

$$+\frac{\ddot{M}_{\beta}{}^{\delta}\dot{a}^{\epsilon}v_{\delta}v_{\epsilon}}{c^2}\Bigg)+\left(\dot{a}^{\nu]}+\frac{\dot{a}^{\epsilon}v_{\epsilon}v^{\nu]}}{c^2}\right)M^{\alpha\beta}\ddot{M}_{\beta\alpha}\Bigg)$$

$$+\frac{6e^2v^{\nu]}}{4m^2c^9}\left(\frac{4\pi}{15}\right)\left(a^2M^{\alpha\beta}\ddot{M}_{\beta\alpha}+2M^{\alpha\beta}\ddot{M}_\beta{}^\delta a_\alpha a_\delta\right)$$

$$-\frac{2e^2a^2}{4m^2c^9}\left(\frac{4\pi}{3}\right)M^{\nu]\beta}\ddot{M}_\beta{}^\delta v_\delta+\frac{2e^2v^{\nu]}}{4m^2c^9}\left(\frac{4\pi}{3}\right)M^{\alpha\beta}\ddot{M}_\beta{}^\delta\dot{a}_\alpha v_\delta$$

$$+\frac{6e^2}{4m^2c^9}\left(\frac{4\pi}{15}\right)\left(a^2M^{\nu]\beta}\ddot{M}_\beta{}^\delta v_\delta+2a^{\nu]}M^{\alpha\beta}\ddot{M}_\beta{}^\delta a_\alpha v_\delta\right)$$

$$+\frac{9e^2v^{\nu]}}{4m^2c^9}\left(\frac{4\pi}{15}\right)\left(a^2\left(\dot{M}^{\alpha\beta}\dot{M}_{\beta\alpha}+\frac{\dot{M}^{\alpha\beta}\dot{M}_\beta{}^\delta v_\alpha v_\delta}{c^2}\right)+2\dot{M}^{\alpha\beta}\dot{M}_\beta{}^\delta a_\alpha a_\delta\right)$$

$$+\frac{18e^2}{4m^2c^9}\left(\frac{4\pi}{15}\right)\left(a^2\left(\dot{M}^{\nu]\beta}\dot{M}_\beta{}^\delta v_\delta+\frac{v^{\nu]}}{c^2}\dot{M}^{\alpha\beta}\dot{M}_\beta{}^\delta v_\alpha v_\delta\right)\right.$$

$$\left.+2a^{\nu]}\dot{M}^{\alpha\beta}\dot{M}_\beta{}^\delta a_\alpha v_\delta\right)+\frac{9e^2v^{\nu]}}{4m^2c^{11}}\left(\frac{4\pi}{3}\right)a^2\dot{M}^{\alpha\beta}\dot{M}_\beta{}^\delta v_\alpha v_\delta$$

$$+\frac{6e^2}{4m^2c^7}\left(\frac{4\pi}{15}\right)\left(\dot{M}^{\alpha\beta}a_\alpha\left(\ddot{M}_\beta{}^{\nu]}+\frac{\ddot{M}_\beta{}^\delta v_\delta v^{\nu]}}{c^2}\right)+\left(\dot{M}^{\nu]\beta}\right.\right.$$

$$\left.\left.+\frac{\dot{M}^{\alpha\beta}v_\alpha v^{\nu]}}{c^2}\right)\ddot{M}_\beta{}^\delta a_\delta+a^{\nu]}\left(\dot{M}^{\alpha\beta}\ddot{M}_{\beta\alpha}+\frac{\dot{M}^{\alpha\beta}\ddot{M}_\beta{}^\delta v_\alpha v_\delta}{c^2}\right)\right)$$

$$+\frac{6e^2v^{\nu]}}{4m^2c^9}\left(\frac{4\pi}{3}\right)\dot{M}^{\alpha\beta}\ddot{M}_\beta{}^\delta a_\alpha v_\delta+\frac{6e^2v^{\nu]}}{4m^2c^9}\left(\frac{4\pi}{3}\right)\dot{M}^{\alpha\beta}\ddot{M}_\beta{}^\delta v_\alpha a_\delta$$

$$+\frac{6e^2}{4m^2c^9}\left(\frac{4\pi}{3}\right)a^{\nu]}\dot{M}^{\alpha\beta}\ddot{M}_\beta{}^\delta v_\alpha v_\delta+\frac{e^2v^{\nu]}}{4m^2c^7}\left(\frac{4\pi}{3}\right)\left(\ddot{M}^{\alpha\beta}\ddot{M}_{\beta\alpha}+\frac{\ddot{M}^{\alpha\beta}\ddot{M}_\beta{}^\delta v_\alpha v_\delta}{c^2}\right)$$

$$+\frac{2e^2}{4m^2c^7}\left(\frac{4\pi}{3}\right)\left(\ddot{M}^{\nu]\beta}\ddot{M}_\beta{}^\delta v_\delta+\frac{v^{\nu]}\ddot{M}^{\alpha\beta}\ddot{M}_\beta{}^\delta v_\alpha v_\delta}{c^2}\right)+\frac{e^2(4\pi)v^{\nu]}}{4m^2c^9}\ddot{M}^{\alpha\beta}\ddot{M}_\beta{}^\delta v_\alpha v_\delta$$

$$-\frac{2e^2a^2}{2mc^7}\left(\frac{4\pi}{3}\right)M^{\nu]\beta}a_\beta+\frac{2e^2v^{\nu]}}{2mc^7}\left(\frac{4\pi}{3}\right)M^{\alpha\beta}\dot{a}_\alpha a_\beta+\frac{2e^2}{2mc^5}\left(\frac{4\pi}{3}\right)\left(\ddot{M}^{\nu]\beta}a_\beta\right.$$

$$+ \frac{v^{\nu]}\ddot{M}^{\alpha\beta}v_\alpha a_\beta}{c^2}\Big) + \frac{2e^2(4\pi)}{2mc^7}\ddot{M}^{\alpha\beta}v_\alpha a_\beta v^{\nu]}$$

$$+ \frac{2e^2 v^{\nu]}}{2mc^7}\ddot{M}^{\alpha\beta}a_\alpha v_\beta\left(\frac{4\pi}{3}\right) + \frac{e^2a^2v^{\nu]}}{c^5}\left(\frac{4\pi}{3}\right) - \frac{e^2a^2v^{\nu]}}{c^5}(4\pi)\Bigg\}$$

From the foregoing it can be seen that the asymptotic angular momentum radiation depends upon the position z^μ of the source particle. Although this may seem strange since any finite z^μ would be negligible at asymptotic observer distances, it can be shown by the following proof (suggested by Cohn[1]) that this dependence exists as long as any energy is asymptotically radiated by the particle. Assume that the asymptotic radiation is unaffected by a change in z^μ. The change in z^μ is equivalent to some change in x^μ, say from x^μ to $x^\mu + x^\mu_\Delta$, so that the preceding sentence may be expressed as

$$\text{(I.5.10)} \quad \lim_{\rho\to\infty}\int\left(\Theta^{\alpha\nu}(x^\mu + x^\mu_\Delta) - \Theta^{\alpha\mu}(x^\nu + x^\nu_\Delta)\right)d\sigma_\alpha = \lim_{\rho\to\infty}\int\left(\Theta^{\alpha\nu}x^\mu - \Theta^{\alpha\mu}x^\nu\right)d\sigma_\alpha$$

This can also be written as

$$\text{(I.5.11)} \quad \lim_{\rho\to\infty}\int\left(\Theta^{\alpha\nu}x^\mu_\Delta - \Theta^{\alpha\mu}x^\nu_\Delta\right)d\sigma_\alpha = 0$$

For convenience assume $x^\mu_\Delta = (1,0,0,0)$, and let $\mu = 1$ and $\nu = 4$. Then one obtains

$$\text{(I.5.12)} \quad \lim_{\rho\to\infty}\int\Theta^{\alpha 4}d\sigma_\alpha = 0$$

But this is proportional to the 4-component of the definition of radiated linear momentum for the particle, which is not zero. Having reached a contradiction, one must accept the fact that the asymptotic angular momen-

[1] J. Cohn, personal letter.

tum radiation does depend on the source's position.

To calculate

$$\text{(I.5.13)} \quad dJ^{\mu\nu}_{(b)} = \frac{1}{c}\int \left(\Theta^{\alpha\nu}_{(-3)} x^{\mu} - \Theta^{\alpha\mu}_{(-3)} x^{\nu}\right) d\sigma_{\alpha}$$

one needs only to calculate

$$\text{(I.5.14)} \quad dJ^{\mu\nu}_{(b)} = \frac{1}{c}\int \left(\Theta^{\alpha\nu}_{(-3)} R^{\mu} - \Theta^{\alpha\mu}_{(-3)} R^{\nu}\right) d\sigma_{\alpha}$$

since when x^{μ} is replaced by $R^{\mu} + z^{\mu}$ and the limit $\rho\to\infty$ is taken, the terms containing z^{μ} approach zero as $1/\rho$. From (I.2.17), one has

$$\text{(I.5.15)} \quad \Theta^{\mu\nu}_{(-3)} = \frac{1}{4\pi}\left(F^{\mu\alpha}_{(-1)} F_{\alpha}{}^{\nu}_{(-2)} + F^{\mu\alpha}_{(-2)} F_{\alpha}{}^{\nu}_{(-1)} + \frac{g^{\mu\nu}}{2} F^{\alpha\beta}_{(-1)} F_{\alpha\beta(-2)}\right)$$

The quantity $F^{\alpha\beta}_{(-1)}F_{\alpha\beta(-2)}$, when calculated, has a value of zero, leaving

$$\text{(I.5.16)} \quad \Theta^{\mu\nu}_{(-3)} = \frac{1}{4\pi}\left(F^{\mu\alpha}_{(-1)} F_{\alpha}{}^{\nu}_{(-2)} + F^{\mu\alpha}_{(-2)} F_{\alpha}{}^{\nu}_{(-1)}\right)$$

When the quantity $F^{\mu\alpha}_{(-1)}F_{\alpha}{}^{\nu}_{(-2)} + F^{\mu\alpha}_{(-2)}F_{\alpha}{}^{\nu}_{(-1)}$ is calculated, the following result is obtained:

$$\text{(I.5.17)} \quad F^{\mu\alpha}_{(-1)} F_{\alpha}{}^{\nu}_{(-2)} + F^{\mu\alpha}_{(-2)} F_{\alpha}{}^{\nu}_{(-1)} = \Bigg[R^{\mu}R^{\nu}\Bigg(\left(\frac{ea^2}{2mc^5} - \frac{e\dot{a}_u}{2mc^4}\right.$$

$$\left. - \frac{3e(a_u)^2}{2mc^5}\right)\frac{3e\,a_u}{mc^3\rho^5} M^{\delta\alpha} M_{\delta}{}^{\beta} u_{\alpha} u_{\beta} + \left(\frac{ea^2}{2mc^5} - \frac{e\dot{a}_u}{2mc^4}\right.$$

$$\left. - \frac{3e(a_u)^2}{2mc^5}\right)\frac{e}{mc^2\rho^5} M^{\delta\alpha} \dot{M}_{\delta}{}^{\beta} u_{\alpha} u_{\beta} - \frac{18e^2(a_u)^2}{4m^2c^7\rho^6} \dot{M}^{\delta\alpha} M_{\delta}{}^{\beta} R_{\alpha} u_{\beta}$$

$$- \frac{6e^2 a_u}{4m^2c^6\rho^6} \dot{M}^{\delta\alpha} \dot{M}_{\delta}{}^{\beta} R_{\alpha} u_{\beta} - \frac{6e^2 a_u}{4m^2c^6\rho^6} \ddot{M}^{\delta\alpha} M_{\delta}{}^{\beta} R_{\alpha} u_{\beta}$$

$$- \frac{2e^2}{4m^2c^5\rho^6} \ddot{M}^{\delta\alpha} \dot{M}_{\delta}{}^{\beta} R_{\alpha} u_{\beta} + \frac{2e^2}{2mc^4\rho^5} \ddot{M}^{\delta\alpha} u_{\delta} R_{\alpha} - \frac{2e^2}{4mc^4\rho^5} \dot{M}^{\delta\alpha} a_{\delta} u_{\alpha}$$

$$+\frac{2e^2 a_u}{2mc^5\rho^5} M^{\delta\alpha} a_\delta u_\alpha \Big) + \left(\dot{M}^{\mu\alpha} R_\alpha R^\nu + \dot{M}^{\nu\alpha} R_\alpha R^\mu\right)\Big(-\frac{3e^2 a_u}{4m^2c^7\rho^5} M^{\delta\beta} u_\delta a_\beta$$

$$+\frac{2e^2}{4m^2c^5\rho^6} \ddot{M}^{\delta\beta} u_\delta R_\beta - \frac{e^2 a_u}{2mc^4\rho^5}\Big) + \Big(M^{\delta\beta} u_\delta a_\beta \Big(-\frac{2e}{2mc^3\rho^4}\Big(\frac{ea^2}{2mc^5}$$

$$-\frac{e\dot{a}_u}{2mc^4} - \frac{3e(a_u)^2}{2mc^5}\Big)\Big)\Big) + \left(\frac{ea^2}{2mc^5} - \frac{e\dot{a}_u}{2mc^4} - \frac{3e(a_u)^2}{2mc^5}\right)\frac{e}{\rho^4}$$

$$-\frac{9e^2(a_u)^2}{4m^2c^8\rho^4} M^{\beta\delta} u_\beta a_\delta + \frac{3e^2 a_u}{4m^2c^7\rho^4} \dot{M}^{\beta\delta} u_\beta a_\delta + \frac{3e^2 a_u}{4m^2c^6\rho^5} \ddot{M}^{\delta\beta} u_\delta R_\beta$$

$$-\frac{e^2}{4m^2c^6\rho^5} \ddot{M}^{\delta\beta} a_\delta R_\beta + \frac{3e^2(a_u)^2}{2mc^5\rho^4} - \frac{e^2 a^2}{2mc^5\rho^4}\Big)\left(\dot{M}^{\mu\alpha} R_\alpha R^\nu + \dot{M}^{\nu\alpha} R_\alpha R^\mu\right)$$

$$+\left(R^\mu u^\nu + R^\nu u^\mu\right)\Big(\Big(\frac{ea^2}{2mc^5} - \frac{e\dot{a}_u}{2mc^4} - \frac{3e(a_u)^2}{2mc^5}\Big)\frac{3ea_u}{2mc^3\rho^4} M^{\delta\alpha} M_\delta{}^\beta u_\alpha u_\beta$$

$$+\left(\frac{ea^2}{2mc^5} - \frac{e\dot{a}_u}{2mc^4} - \frac{3e(a_u)^2}{2mc^5}\right)\frac{2e}{2mc^2\rho^5} M^{\delta\alpha} \dot{M}_\delta{}^\beta u_\alpha R_\beta$$

$$-\frac{9e^2(a_u)^2}{4m^2c^7\rho^5} \dot{M}^{\delta\alpha} M_\delta{}^\beta R_\alpha u_\beta - \frac{6e^2 a_u}{4m^2c^6\rho^6} \dot{M}^{\delta\alpha} \dot{M}_\delta{}^\beta R_\alpha R_\beta$$

$$-\frac{3e^2 a_u}{4m^2c^5\rho^5} \ddot{M}^{\delta\alpha} M_\delta{}^\beta R_\alpha u_\beta - \frac{2e^2}{4m^2c^5\rho^6} \ddot{M}^{\delta\alpha} \dot{M}_\delta{}^\beta R_\alpha R_\beta$$

$$+\frac{e^2 a_u}{2mc^5\rho^4} M^{\delta\alpha} u_\delta a_\alpha - \frac{2e^2}{2mc^4\rho^4} \dot{M}^{\delta\alpha} a_\delta u_\alpha + \frac{e^2 a_u}{c^2\rho^4}\Big)$$

$$+\left(R^\mu v^\nu + R^\nu v^\mu\right)\frac{e^2 a_u}{2mc^6\rho^4} M^{\beta\delta} u_\beta a_\delta$$

$$+\left(R^\mu a^\nu + R^\nu a^\mu\right)\Big(-\frac{e}{2mc^3\rho^4}\Big(\frac{ea^2}{2mc^5} - \frac{e\dot{a}_u}{2mc^4} - \frac{3e(a_u)^2}{2mc^5}\Big) M^{\delta\alpha} M_\delta{}^\beta u_\alpha u_\beta$$

$$+\frac{3e^2 a_u}{4m^2c^7\rho^5}\dot{M}^{\delta\alpha}\dot{M}_\delta{}^\beta R_\alpha u_\beta+\frac{e^2}{4m^2c^6\rho^5}\ddot{M}^{\delta\alpha}M_\delta{}^\beta R_\alpha u_\beta-\frac{e^2}{c^2\rho^4}\Big)$$

$$+\left(\ddot{M}^{\mu\alpha}R_\alpha R^\nu+\ddot{M}^{\nu\alpha}R_\alpha R^\mu\right)\left(\frac{e^2}{4m^2c^6\rho^5}M^{\beta\delta}u_\beta a_\delta-\frac{e^2}{2mc^3\rho^5}\right)$$

$$+\left(\dot{M}^{\mu\delta}a_\delta R^\nu+\dot{M}^{\nu\delta}a_\delta R^\mu\right)\left(\frac{-2e^2}{2mc^4\rho^4}\right)$$

$$+\left(M^{\mu\delta}M_\delta{}^\alpha u_\alpha R^\nu+M^{\nu\delta}M_\delta{}^\alpha u_\alpha R^\mu\right)\frac{2ea_u}{2mc^3\rho^4}\left(\frac{ea^2}{2mc^5}-\frac{e\dot{a}_u}{2mc^4}-\frac{3e(a_u)^2}{2mc^5}\right)$$

$$+\left(\dot{M}^{\mu\delta}M_\delta{}^\alpha u_\alpha R^\nu+\dot{M}^{\nu\delta}M_\delta{}^\alpha u_\alpha R^\mu\right)\frac{2e}{2mc^2\rho^4}\left(\frac{ea^2}{2mc^5}-\frac{e\dot{a}_u}{2mc^4}-\frac{3e(a_u)^2}{2mc^5}\right)$$

$$+\left(M^{\mu\delta}\dot{M}_\delta{}^\alpha R_\alpha R^\nu+M^{\nu\delta}\dot{M}_\delta{}^\alpha R_\alpha R^\mu\right)\left(\frac{-6e^2(a_u)^2}{4m^2c^7\rho^5}\right)$$

$$+\left(\dot{M}^{\mu\delta}\dot{M}_\delta{}^\alpha R_\alpha R^\nu+\dot{M}^{\nu\delta}\dot{M}_\delta{}^\alpha R_\alpha R^\mu\right)\left(\frac{-6e^2a_u}{4m^2c^6\rho^5}\right)$$

$$+\left(M^{\mu\alpha}\ddot{M}_\alpha{}^\beta R_\beta R^\nu+M^{\nu\alpha}\ddot{M}_\alpha{}^\beta R_\beta R^\mu\right)\left(\frac{-2e^2a_u}{4m^2c^6\rho^5}\right)$$

$$+\left(\dot{M}^{\mu\alpha}\ddot{M}_\alpha{}^\beta R_\beta R^\nu+\dot{M}^{\nu\alpha}\ddot{M}_\alpha{}^\beta R_\beta R^\mu\right)\left(\frac{-2e^2}{4m^2c^5\rho^5}\right)\Bigg]$$

To calculate $dJ^{\mu\nu}_{(b)}$, one must use (I.5.17) in (I.5.16) to obtain $\theta^{\mu\nu}_{(-3)}$, and then calculate $(\theta^{\alpha\nu}_{(-3)}x^\mu - \theta^{\alpha\mu}_{(-3)}x^\nu)u_\alpha c\rho^2\, d\Omega\, d\tau$ in terms of u_α. Performing these calculations, and integrating by means of the integration formulas developed in Appendix A, one obtains

$$\text{(I.5.18)}\qquad \frac{dJ^{\mu\nu}_{(b)}}{d\tau}=\frac{-1}{4\pi}\left[\frac{-4\pi\cdot 3e^2}{15\cdot 4m^2c^7}\left(\dot{M}^{[\nu\alpha}M_\alpha{}^\beta a_\beta a^{\mu]}+\dot{M}^{[\nu\alpha}a_\alpha M^{\mu]\beta}a_\beta\right)\right.$$

$$+ \frac{4\pi\cdot 2e^2}{3\cdot 4m^2c^7}\dot M^{[\nu\alpha}\ddot M_\alpha{}^\beta v_\beta v^{\mu]} - \frac{4\pi e^2}{3\cdot 2mc^5}\dot M^{[\nu\alpha}a_\alpha v^{\mu]}$$

$$+ \frac{4\pi\cdot 2e^2}{3m^2\cdot 4c^7}\dot M^{[\nu\alpha}v_\alpha \ddot M^{\mu]\beta}v_\beta - \frac{4\pi e^2}{3\cdot 2mc^5}\dot M^{[\nu\alpha}v_\alpha a^{\mu]}$$

$$+ \frac{4\pi\cdot 2e^2}{15\cdot 4m^2c^7}\Big(M^{[\nu\mu]}M^{\delta\beta}\dot a_\delta a_\beta + M^{[\nu\alpha}M_\alpha{}^\beta a_\beta \dot a^{\mu]}$$

$$+ M^{[\nu\alpha}M_\alpha{}^\beta a_\beta \frac{\dot a^\epsilon v_\epsilon v^{\mu]}}{c^2} + M^{[\nu\alpha}\dot a_\alpha M^{\mu]\beta}a_\beta\Big)$$

$$+ \frac{4\pi\cdot 3e^2}{15\cdot 4m^2c^7}\Big(M^{[\nu\alpha}\dot M_\alpha{}^\delta a_\delta a^{\mu]} + M^{[\nu\alpha}a_\alpha \dot M^{\mu]\delta}a_\delta$$

$$+ \frac{1}{c^2}M^{[\nu\alpha}a_\alpha \dot M^{\beta\delta}v_\beta a_\delta v^{\mu]}\Big) + \frac{4\pi\cdot 3e^2}{15\cdot 4m^2c^7}\Big(M^{[\nu\mu]}\ddot M^{\delta\beta}a_\delta v_\beta$$

$$+ M^{[\nu\alpha}\ddot M_\alpha{}^\beta v_\beta a^{\mu]} + M^{[\nu\alpha}a_\alpha \ddot M^{\mu]\beta}v_\beta\Big)$$

$$- \frac{4\pi e^2}{3\cdot 4m^2c^6}M^{[\nu\mu]}\ddot M^{\delta\beta}a_\delta v_\beta - \frac{4\pi\cdot 2e^2a^2}{3\cdot 4m^2c^9}M^{[\nu\alpha}M_\alpha{}^\beta a_\beta v^{\mu]}$$

$$- \frac{4\pi\cdot 3e^2a^2}{15\cdot 4m^2c^9}M^{[\nu\alpha}M_\alpha{}^\beta a_\beta v^{\mu]} - \frac{4\pi e^2}{15\cdot 4m^2c^7}M^{[\nu\alpha}\ddot M^\delta{}_\alpha a_\delta v^{\mu]}$$

$$- \frac{4\pi e^2}{3\cdot 2mc^5}M^{[\nu\alpha}\dot a_\alpha v^{\mu]} - \frac{4\pi\cdot 3e^2a^2}{15\cdot 4m^2c^9}\Big(v^{[\nu}M^{\delta\mu]}M_\delta{}^\beta a_\beta$$

$$+ v^{[\nu}M^{\delta\alpha}a_\alpha M_\delta{}^{\mu]} + v^{[\nu}a^{\mu]}M^{\delta\alpha}M_{\delta\alpha}$$

$$+ \frac{4\pi\cdot 9e^2}{105\cdot 4m^2c^9}\Big(6v^{[\nu}M^{\delta\mu]}M_\delta{}^\beta a_\beta a^2 + 3v^{[\nu}a^{\mu]}\big(M^{\delta\alpha}M_{\delta\alpha}a^2 + 2M^{\delta\alpha}a_\alpha M_\delta{}^\beta a_\beta\big)\Big)\Big)$$

$$+ \frac{4\pi\cdot 2e^2}{15\cdot 4m^2c^7}\Big(v^{[\nu}M^{\delta\mu]}\dot M_\delta{}^\beta \dot a_\beta + \frac{1}{c^2} v^{[\nu}M^{\delta\mu]}\dot M_\delta{}^\beta v_\beta \dot a^\epsilon v_\epsilon$$

$$+ v^{[\nu}M^{\delta\alpha}\dot a_\alpha \dot M_\delta{}^{\mu]} + v^{[\nu}\dot a^{\mu]}M^{\delta\alpha}\dot M_{\delta\alpha}\Big) - \frac{4\pi\cdot 2e^2a^2}{3\cdot 4m^2c^9} v^{[\nu}M^{\delta\mu]}\dot M_\delta{}^\beta v_\beta$$

$$+ \frac{4\pi\cdot 15e^2}{15\cdot 4m^2c^9}\Big(v^{[\nu}M^{\delta\mu]}\dot M_\delta{}^\beta v_\beta a^2 + 2v^{[\nu}a^{\mu]}M^{\delta\alpha}a_\alpha \dot M_\delta{}^\beta v_\beta\Big)$$

$$+ \frac{4\pi\cdot 6e^2}{15\cdot 4m^2c^7}\Big(v^{[\nu}\dot M^{\delta\mu]}\dot M_\delta{}^\beta a_\beta + v^{[\nu}\dot M^{\delta\alpha}a_\alpha \dot M_\delta{}^{\mu]} + v^{[\nu}a^{\mu]}\dot M^{\delta\alpha}\dot M_{\delta\alpha}$$

$$+ \frac{1}{c^2}v^{[\nu}a^{\mu]}\dot M^{\delta\alpha}v_\alpha \dot M_\delta{}^\beta v_\beta\Big) + \frac{4\pi\cdot 6e^2}{3\cdot 4m^2c^9}v^{[\nu}a^{\mu]}\dot M^{\delta\alpha}\dot M_\delta{}^\beta v_\alpha v_\beta$$

$$+ \frac{4\pi\cdot 3e^2}{15\cdot 4m^2c^7}\Big(v^{[\nu}\ddot M^{\delta\mu]}M_\delta{}^\beta a_\beta + v^{[\nu}\ddot M^{\delta\alpha}a_\alpha M_\delta{}^{\mu]} + v^{[\nu}a^{\mu]}\ddot M^{\delta\alpha}M_{\delta\alpha}\Big)$$

$$+ \frac{4\pi\cdot 2e^2}{3\cdot 4m^2c^7}v^{[\nu}\ddot M^{\delta\mu]}\dot M_\delta{}^\beta v_\beta + \frac{4\pi\cdot 2e^2}{3\cdot 4m^2c^7}v^{[\nu}\ddot M^{\delta\alpha}\dot M_\delta{}^{\mu]}v_\alpha$$

$$+ \frac{4\pi\cdot 2e^2}{3\cdot 2mc^5}v^{[\nu}\dot M^{\delta\mu]}a_\delta - \frac{4\pi\cdot e^2}{3c^3}v^{[\nu}a^{\mu]} + \frac{4\pi e^2}{15\cdot 4m^2c^7}\Big(a^{[\nu}M^{\delta\mu]}M_\delta{}^\beta \dot a_\beta$$

$$+ a^{[\nu}M^{\delta\alpha}M_\delta{}^{\mu]}\dot a_\alpha + a^{[\nu}\dot a^{\mu]}M^{\delta\alpha}M_{\delta\alpha} + \frac{1}{c^2}a^{[\nu}v^{\mu]}\dot a^\epsilon v_\epsilon M^{\delta\alpha}M_{\delta\alpha}\Big)$$

$$+ \frac{4\pi\cdot 3e^2}{15\cdot 4m^2c^7}\Big(a^{[\nu}\dot M^{\delta\mu]}M_\delta{}^\beta a_\beta + \frac{a^{[\nu}v^{\mu]}}{c^2}\dot M^{\delta\alpha}v_\alpha M_\delta{}^\beta a_\beta + a^{[\nu}\dot M^{\delta\alpha}M_\delta{}^{\mu]}a_\alpha\Big)$$

$$+ \frac{4\pi e^2}{3\cdot 4m^2c^7}a^{[\nu}\ddot M^{\delta\alpha}v_\alpha M_\delta{}^{\mu]} - \frac{4\pi e^2a^2}{3\cdot 4m^2c^9}a^{[\nu}v^{\mu]}M^{\delta\alpha}M_{\delta\alpha}$$

$$+ \frac{4\pi\cdot 3e^2}{15\cdot 4m^2c^9}\Big(a^{[\nu}v^{\mu]}(a^2M^{\delta\alpha}M_{\delta\alpha} + 2M^{\delta\alpha}M_\delta{}^\beta a_\alpha a_\beta)\Big)$$

$$+ \frac{4\pi\cdot 3e^2}{3\cdot 4m^2c^9}a^{[\nu}v^{\mu]}\dot M^{\delta\alpha}M_\delta{}^\beta v_\alpha a_\beta + \frac{4\pi e^2}{3\cdot 4m^2c^7}a^{[\nu}v^{\mu]}\ddot M^{\delta\alpha}M_{\delta\alpha}$$

$$-\frac{4\pi e^2}{c^3}\, a^{[\nu} v^{\mu]} + \frac{4\pi e^2}{3\cdot 4m^2c^7}\, \ddot{M}^{[\nu\alpha} M_\alpha{}^\delta a_\delta v^{\mu]}$$

$$+\frac{4\pi e^2}{3\cdot 4m^2c^7}\, \ddot{M}^{[\nu\alpha} v_\alpha M^{\mu]\delta} a_\delta - \frac{4\pi e^2}{3\cdot 2mc^3}\left(\dot{M}^{[\nu\mu]} + \frac{1}{c^2}\ddot{M}^{[\nu\alpha} v_\alpha v^{\mu]}\right)$$

$$-\frac{4\pi e^2}{2mc^5}\, \ddot{M}^{[\nu\alpha} v_\alpha v^{\mu]} - \frac{4\pi\cdot 2e^2}{2mc^5}\, \dot{M}^{[\nu\alpha} a_\alpha v^{\mu]}$$

$$-\frac{4\pi\cdot 2e^2}{15\cdot 4m^2c^7}\Big(M^{[\nu\delta} M_\delta{}^{\mu]} \dot{a}^\epsilon a_\epsilon + M^{[\nu\delta} M_\delta{}^\alpha \dot{a}_\alpha a^{\mu]} + M^{[\nu\delta} M_\delta{}^\alpha a_\alpha \dot{a}^{\mu]}$$

$$+\frac{1}{c^2} M^{[\nu\delta} M_\delta{}^\alpha a_\alpha \dot{a}^\epsilon v_\epsilon v^{\mu]}\Big) + \frac{4\pi\cdot 2e^2a^2}{3\cdot 4m^2c^9}\, M^{[\nu\delta} M_\delta{}^\alpha a_\alpha v^{\mu]}$$

$$-\frac{4\pi\cdot 6e^2}{15\cdot 4m^2c^9}\left(3M^{[\nu\delta} M_\delta{}^\alpha a_\alpha v^{\mu]}\right) + \frac{4\pi\cdot 2e^2a^2}{3\cdot 4m^2c^7}\, \ddot{M}^{[\nu\delta} M_\delta{}^{\mu]}$$

$$-\frac{4\pi\cdot 6e^2}{15\cdot 4m^2c^7}\left(\ddot{M}^{[\nu\delta} M_\delta{}^{\mu]} a^2 + 2\ddot{M}^{[\nu\delta} M_\delta{}^\alpha a_\alpha a^{\mu]}\right)$$

$$-\frac{4\pi\cdot 2e^2}{3\cdot 4m^2c^7}\, \dot{M}^{[\nu\delta} M_\delta{}^\alpha \dot{a}_\alpha v^{\mu]} - \frac{4\pi\cdot 6e^2}{15\cdot 4m^2c^7}\Big(M^{[\nu\delta} \dot{M}_\delta{}^{\mu]} a^2$$

$$+\frac{1}{c^2} M^{[\nu\delta} \dot{M}_\delta{}^\alpha v_\alpha v^{\mu]} a^2 + 2M^{[\nu\delta} \dot{M}_\delta{}^\alpha a_\alpha a^{\mu]}\Big)$$

$$-\frac{4\pi\cdot 6e^2}{3\cdot 4m^2c^9}\, M^{[\nu\delta} \dot{M}_\delta{}^\alpha v_\alpha v^{\mu]} a^2 - \frac{4\pi\cdot 6e^2}{3\cdot 4m^2c^7}\, \dot{M}^{[\nu\delta} \dot{M}_\delta{}^\alpha a_\alpha v^{\mu]}$$

$$-\frac{4\pi\cdot 6e^2}{3\cdot 4m^2c^7}\, \dot{M}^{[\nu\delta} \dot{M}_\delta{}^\alpha v_\alpha a^{\mu]} - \frac{4\pi\cdot 2e^2}{3\cdot 4m^2c^7}\, M^{[\nu\alpha} \ddot{M}_\alpha{}^\beta a_\beta v^{\mu]}$$

$$-\frac{4\pi\cdot 2e^2}{3\cdot 4m^2c^7}\, M^{[\nu\alpha} \ddot{M}_\alpha{}^\beta v_\beta a^{\mu]} - \frac{4\pi\cdot 2e^2}{3\cdot 4m^2c^5}\left(\dot{M}^{[\nu\alpha} \ddot{M}_\alpha{}^{\mu]} + \frac{1}{c^2}\dot{M}^{[\nu\alpha} \ddot{M}_\alpha{}^\beta v_\beta v^{\mu]}\right)$$

$$-\frac{4\pi\cdot 2e^2}{4m^2c^7}\, \dot{M}^{[\nu\alpha} \ddot{M}_\alpha{}^\beta v_\beta v^{\mu]}\Bigg]$$

6. SOME SPECIAL CASES

The expressions developed in the preceding sections are so lengthy as to render their use impractical in many situations. However, if certain approximations and special cases are considered, the length of the expressions decreases significantly. Two special cases are of interest here: the case in which the particle is not spinning at all, and the case in which the spin is sufficiently small so that all second or higher order terms in the spin or its derivatives can be neglected.

For the case of no spin, one obtains the expressions

$$\left(\frac{dP^{\mu}}{d\tau}\right)_{ns} = \frac{2e^2a^2}{3c^5}v^{\mu} \tag{I.6.1}$$

$$\left(\frac{dJ^{\mu\nu}}{d\tau}\right)_{ns} = -\frac{2e^2}{3c^3}\left(a^{[\mu}v^{\nu]} + \frac{a^2}{c^2}v^{[\mu}z^{\nu]}\right) \tag{I.6.2}$$

where the ns notation refers to no spin. Note that the right side of (I.6.2) contains a term depending on the particle's position. The position dependence follows from the fact that an arbitrary origin has been used in the coordinate system. Had the particle position been chosen as origin (equivalent to setting $z^{\mu} = 0$ here), the position dependence would have vanished from (I.6.2), although it would be inherent in the choice of origin.

In the case of small spin, one obtains (after simplifications) the expressions

$$\text{(I.6.3)}\quad \left(\frac{dP^{\mu}}{d\tau}\right)_{ss} = \frac{e^2}{3c^5}\left(\frac{a^2}{mc^2}M^{\mu\beta}a_\beta - \frac{1}{m}\ddot{M}^{\mu\beta}a_\beta - \frac{2}{mc^2}\ddot{M}^{\alpha\beta}v_\alpha a_\beta v^\mu + 2\dot{a}^2 v^\mu\right)$$

$$\text{(I.6.4)}\quad \left(\frac{dJ^{\mu\nu}}{d\tau}\right)_{ss} = -\frac{e^2 z^{[\mu}}{3c^5}\left(\frac{-a^2}{mc^2}M^{\nu]\beta}a_\beta + \frac{1}{m}\ddot{M}^{\nu]\beta}a_\beta + \frac{2}{mc^2}\ddot{M}^{\alpha\beta}v_\alpha a_\beta v^{\nu]}\right.$$

$$\left. - 2a^2 v^{\nu]}\right) - \frac{e^2}{2c^3}\left(\frac{-1}{mc^2}\dot{M}^{[\nu\alpha}a_\alpha v^{\mu]} - \frac{1}{mc^2}\ddot{M}^{[\nu\alpha}v_\alpha v^{\mu]}\right.$$

$$\left. + \frac{1}{3mc^2}M^{[\nu\alpha}a_\alpha a^{\mu]} - \frac{4}{3}a^{[\nu}v^{\mu]} + \frac{2}{3m}\ddot{M}^{\mu\nu}\right)$$

Expressions (I.6.3) and (I.6.4), as well as other expressions derived from them, will be used extensively in the work of Part II.

7. SAMPLE APPLICATION: A "CENTER OF RADIATED ENERGY" THEOREM

As pointed out by Cohn,[1] the fact that a particle may emit angular momentum even if it is not spinning facilitates the development of a theorem concerning the location of the "center of radiated energy" of the (non-spinning) charge. The "center of radiated energy" is defined as[2]

$$R_c^i = \lim_{\rho\to\infty} \frac{\int_{\Delta\sigma} x^i \theta^{44} d^3x}{\int_{\Delta\sigma} \theta^{44} d^3x} \tag{I.7.1}$$

Of the two integrals in the above expression, the one in the denominator is given by

$$\lim_{\rho\to\infty} \int \theta^{44} d^3x = -dW \tag{I.7.2}$$

where dW is the energy radiated by the charge during $d\tau$, and the one in the numerator appears as part of the expression for dJ^{4i} for a non-spinning particle:

$$\begin{aligned} dJ^{4i}_{(no\ spin)} &= \lim_{\rho\to\infty} \frac{1}{c} \int_{\Delta\sigma} (\theta^{44} x^i - \theta^{4i} x^4) d^3x \\ &= \lim_{\rho\to\infty} \left[\frac{1}{c} \int \theta^{44} x^i d^3x - \frac{x^4}{c} \int \theta^{4i} d^3x \right] \end{aligned} \tag{I.7.3}$$

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[1] J. Cohn, "Considerations on the Classical Spinning Electron," Journal of Mathematical Physics, Vol. 10, No. 5 (1969), p. 803.

[2] All quantities in this section are evaluated in the particle rest system.

The last integral in the above line is given by

(I.7.4) $$\lim_{\rho\to\infty}\int \theta^{4i} d^3x = -c\, dP^i = 0$$

since $dP^i = 0$ in the rest frame. Therefore

(I.7.5) $$\lim_{\rho\to\infty}\int \theta^{44} x^i d^3x = c\, dJ^{4i}_{(no\ spin)}$$

Substitution may now be made for the integrals in the definition of R_c^i as follows:

(I.7.6) $$R_c^i = \frac{-c\, dJ^{4i}}{dW} = -c\,\frac{dJ^{4i}/d\tau}{dW/d\tau}$$

$\frac{1}{c}\, dW/d\tau$ is merely the fourth component of radiated linear momentum for a non-spinning particle; from (I.6.1), one has

(I.7.7) $$\frac{dW}{d\tau} = \frac{2e^2a^2}{3c^3}$$

$dJ^{4i}/d\tau$ is obtainable from (I.6.2):

(I.7.8) $$\frac{dJ^{4i}}{d\tau} = \frac{2e^2}{3c^3}\, v^{[4}a^{i]} + \frac{2e^2a^2}{3c^5}\, z^{[4}v^{i]} = \frac{2e^2}{3c^2}\left(a^i - \frac{a^2}{c^2} z^i\right)$$

One then obtains[1]

(I.7.9) $$R_c^i = -c\,\frac{dJ^{4i}/d\tau}{dW/d\tau} = z^i - \frac{c^2}{a^2} a^i$$

so that the "center of radiated energy" depends only on the acceleration of the particle and on its position.

---

[1](I.7.9) agrees with Cohn's result except for a factor of two, a sign, and the presence of the position-dependent term. The factor of two was missing from Cohn's result because of a multiplication error; the sign was omitted from his definition of angular momentum density; the position dependence appears because in this work an arbitrary origin was used in the calculation of angular momentum emission, while in Cohn's work the particle location was used as origin, effectively making the position vector equal to zero.

PART II. EQUATION OF MOTION FOR A PARTICLE WITH SMALL SPIN

1. METHOD OF DERIVATION

The method used in this paper to derive equations of motion for a spinning charged particle is similar to the one used by Cohn[1] for a non-spinning charged particle. It is used here to derive both the linear and the angular equations of motion.

The equations of motion for a charged particle may be written in the forms

$$ma^{\mu} = F^{\mu}_{ext} - \dot{P}^{\mu}_{asm} + D^{\mu}_{lin} \tag{II.1.1}$$

$$\dot{S}^{\mu\nu} = T^{\mu\nu}_{ext} - \dot{J}^{\mu\nu}_{asm} + D^{\mu\nu}_{ang} \tag{II.1.2}$$

where $\dot{S}^{\mu\nu}$ is the rate of change of mechanical angular momentum, F^{μ}_{ext} and $T^{\mu\nu}_{ext}$ are the external force and torque, $\dot{P}^{\mu}_{asm}$ and $\dot{J}^{\mu\nu}_{asm}$ are the asymptotic radiation rates for linear and angular momentum, and D^{μ}_{lin} and $D^{\mu\nu}_{ang}$ are terms which are defined such that the respective equations are true. The problem, then, is to calculate expressions for D^{μ}_{lin} and $D^{\mu\nu}_{ang}$.

Consider the particle as being located at a point p_0 in v^{μ}-, a^{μ}-, $\dot{a}^{\mu}$-, $S^{\mu\nu}$-, $\dot{S}^{\mu\nu}$-, and $\ddot{S}^{\mu\nu}$-space. Now suppose that the particle is moved around an <u>arbitrary</u> path in this space and back to its initial point, designated as p_0. The assumption is made at this point that the initial and final fields differ exactly by the radiated momenta; that is,

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[1] Cohn, "Derivation of the Equations of Motion of a Classical Radiating Charge," pp. 949-950.

(II.1.3) $$\oint_{P_0}^{P_0} F^{\mu}_{ext}\, d\tau = \oint_{P_0}^{P_0} \dot{P}^{\mu}_{asm}\, d\tau$$

(II.1.4) $$\oint_{P_0}^{P_0} T^{\mu\nu}_{ext}\, d\tau = \oint_{P_0}^{P_0} \dot{J}^{\mu\nu}_{asm}\, d\tau$$

A special problem, involving choice of coordinate origin, arises at this point in connection with the angular equation. $S^{\mu\nu}$ and $T^{\mu\nu}_{ext}$ are defined at the particle location; $\dot{J}^{\mu\nu}_{asm}$ was developed using an arbitrary origin. It is a simple matter to move the origin used in determining $\dot{J}^{\mu\nu}_{asm}$ to the particle; however, the particle must be able to to move in an arbitrary manner about the closed path described earlier. This means, of course, that the particle will move away from the origin.

This problem may be resolved by replacing the single coordinate frame used in (II.1.2) and (II.1.4) by a series of consecutive frames, separated by infinitesimal intervals, with origins located on the path followed by the particle. These frames are all at rest with respect to each other, and corresponding axes are parallel. This arrangement allows the integration in (II.1.4) to be carried out using only quantities defined in a coordinate frame using the particle position as origin. Therefore, from this point forward, $\dot{J}^{\mu\nu}_{asm}$ will be considered to have z^{μ} set to zero, thus eliminating all z^{μ}-dependence from $\dot{J}^{\mu\nu}_{asm}$.

It may be noted in passing that this is equivalent to removing from $\dot{J}^{\mu\nu}_{asm}$ all terms which arise from $\dot{J}^{\mu\nu}_{(a)}$ as defined in Section I-5, leaving only those which come from $\dot{J}^{\mu\nu}_{(b)}$. This can also be seen in another way: using the definition $x^\mu = z^\mu + R^\mu$, one can write

$$\text{(II.1.5)} \qquad \dot{J}^{\mu\nu}_{asm} = \lim_{\rho\to\infty}\int(\Theta^{\alpha\nu}x^\mu - \Theta^{\alpha\mu}x^\nu)\, d\sigma_\alpha$$

$$\text{(II.1.6)} \qquad = \lim_{\rho\to\infty}\int(\Theta^{\alpha\nu}z^\mu - \Theta^{\alpha\mu}z^\nu)\, d\sigma_\alpha + \lim_{\rho\to\infty}\int(\Theta^{\alpha\nu}R^\mu - \Theta^{\alpha\mu}R^\nu)\, d\sigma_\alpha$$

$$\text{(II.1.7)} \qquad = (z^\mu \dot{P}^\nu_{asm} - z^\nu \dot{P}^\mu_{asm}) + \lim_{\rho\to\infty}\int(\Theta^{\alpha\nu}R^\mu - \Theta^{\alpha\mu}R^\nu)\, d\sigma_\alpha$$

As is demonstrated in section I-2, the only parts of $\theta^{\mu\nu}$ which will contribute in the asymptotic limit to $\dot{J}^{\mu\nu}_{asm}$ are those defined as $\theta^{\mu\nu}_{(-2)}$ and $\theta^{\mu\nu}_{(-3)}$. Furthermore, $\theta^{\mu\nu}_{(-3)}$ will not contribute in the asymptotic limit to the first integral on the right of (II.1.6); as $\theta^{\mu\nu}_{(-2)}$ has the form $WR^\mu R^\nu$, it will not contribute to the second integral on the right of (II.1.6). Therefore (II.1.6) may be written as

$$\text{(II.1.8)} \qquad \dot{J}^{\mu\nu}_{asm} = \lim_{\rho\to\infty}\int(\Theta^{\alpha\nu}_{(-2)}z^\mu - \Theta^{\alpha\mu}_{(-2)}z^\nu)\, d\sigma_\alpha + \lim_{\rho\to\infty}\int(\Theta^{\alpha\nu}_{(-3)}R^\mu - \Theta^{\alpha\mu}_{(-3)}R^\nu)\, d\sigma_\alpha$$

The first integral on the right of (II.1.8) is seen to be $\dot{J}^{\mu\nu}_{(a)}$ by comparison with (I.5.4); the second is seen to be $\dot{J}^{\mu\nu}_{(b)}$ by comparison with (I.5.14). Finally, the second integral on the right of (II.1.7), which is equal to the second integral of (II.1.6) and therefore to $\dot{J}^{\mu\nu}_{(b)}$, is just the radiated angular momentum when the coordinate origin is taken at the particle.

Thus for the purpose of determining the equation of angular motion,

$$\text{(II.1.9)} \qquad \dot{J}^{\mu\nu}_{asm} = \dot{J}^{\mu\nu}_{(b)}$$

Further simplification of $\dot{J}^{\mu\nu}_{asm}$ will take place in Section II-2.

Integrating both sides of both equations of motion and applying the above assumptions, one obtains

$$\text{(II.1.10)} \quad \oint_{P_0}^{P_0} m a^{\mu} d\tau = \oint_{P_0}^{P_0} F^{\mu}_{ext} d\tau - \oint_{P_0}^{P_0} \dot{P}^{\mu}_{asm} d\tau + \oint_{P_0}^{P_0} D^{\mu}_{lin} d\tau$$

$$\Rightarrow \oint_{P_0}^{P_0} D^{\mu}_{lin} d\tau = \oint_{P_0}^{P_0} m a^{\mu} d\tau = m v^{\mu}\Big|_{P_0}^{P_0} = 0$$

$$\text{(II.1.11)} \quad \oint_{P_0}^{P_0} \dot{S}^{\mu\nu} d\tau = \oint_{P_0}^{P_0} T^{\mu\nu}_{ext} d\tau - \oint_{P_0}^{P_0} \dot{J}^{\mu\nu}_{asm} d\tau + \oint_{P_0}^{P_0} D^{\mu\nu}_{ang} d\tau$$

$$\Rightarrow \oint_{P_0}^{P_0} D^{\mu\nu}_{ang} d\tau = \oint_{P_0}^{P_0} \dot{S}^{\mu\nu} d\tau = S^{\mu\nu}\Big|_{P_0}^{P_0} = 0$$

The above results imply that D^{μ}_{lin} and $D^{\mu\nu}_{ang}$ are the time derivatives of certain vector and tensor quantities C^{μ}_{lin} and $C^{\mu\nu}_{ang}$. If the latter two quantities can be calculated, D^{μ}_{lin} and $D^{\mu\nu}_{ang}$ are obtainable by simple differentiation.

C^{μ}_{lin} and $C^{\mu\nu}_{ang}$ may depend on terms involving the quantities z^{μ}, v^{μ}, a^{μ}, $\dot{a}^{\mu}$, $M^{\mu\nu}$, $\dot{M}^{\mu\nu}$, and $\ddot{M}^{\mu\nu}$, either singly or in combination. Thus they may be represented by expressions of the following type:

$$\text{(II.1.12)} \quad C^{\mu}_{lin} = f_1 V^{\mu}_1 + f_2 V^{\mu}_2 + f_3 V^{\mu}_3 + f_4 V^{\mu}_4 + \ldots$$

$$\text{(II.1.13)} \quad C^{\mu\nu}_{ang} = q_1 \Theta^{\mu\nu}_1 + q_2 \Theta^{\mu\nu}_2 + q_3 \Theta^{\mu\nu}_3 + q_4 \Theta^{\mu\nu}_4 + \ldots$$

where the f_i and q_i are scalar functions, the V^{μ}_i are linearly independent vectors, and the $\Theta^{\mu\nu}_i$ are linearly independent tensors.

The term "linearly independent" is used in a special way here. The V_i^μ and $\Theta_i^{\mu\nu}$ are "linearly independent" vectors and tensors only because these equations must be true for arbitrary motion. Typical examples of the V_i^μ, for instance, would be a^μ and $\dot{a}^\mu$; for any given motion, a^μ and $\dot{a}^\mu$ would of course be related by the equations of motion and would therefore not be independent. It is only in the case of arbitrary motion, where any a^μ and any $\dot{a}^\mu$ must be allowed, that these quantities become independent.

In order to obtain expressions for C_{lin}^μ and $C_{ang}^{\mu\nu}$ which are as general as possible, it is desirable to eliminate from the final expression any reference to the forms of F_{ext}^μ and $T_{ext}^{\mu\nu}$. In the method of solution used here, this is accomplished by contracting the linear equation with v_μ and the angular equation with $M_{\mu\nu}$, which is proportional to $S_{\mu\nu}$. As stated by Cohn,[1] the nature of D_{lin}^μ allows one to choose either F_{ext}^4 or D_{lin}^4 (but not both) arbitrarily, and in this case F_{ext}^4 is defined so that $F_{ext}^\mu v_\mu = 0$. As for $T_{ext}^{\mu\nu}$, the torque on a magnetic dipole is proportional (in three-space) to $\underline{\mu} \times \underline{B}$. Since $\underline{\mu}$ is parallel to $\underline{\omega}$, the triple product $\underline{\omega} \cdot \underline{\mu} \times \underline{B}$ is zero. This means that the part of $T_{ext}^{\mu\nu} M_{\mu\nu}$ resulting from the space-space components of $T_{ext}^{\mu\nu}$ and $M_{\mu\nu}$ must be zero. Since the space-time components of $M_{\mu\nu}$ are zero in the rest frame, the rest of $T_{ext}^{\mu\nu} M_{\mu\nu}$ must also be zero, giving the result that $T_{ext}^{\mu\nu} M_{\mu\nu} = 0$.

The assumption of constant spin magnitude also implies that the contraction $\dot{S}^{\mu\nu} M_{\mu\nu}$ is zero. If $\underline{\omega}^2$ is constant, then $\underline{\omega}$ and $\dot{\underline{\omega}}$ must be perpendicular. This means that the space-space contribution to $\dot{S}^{\mu\nu} S_{\mu\nu}$ must be zero. The space-time contribution is zero because $S_{\mu 4}$ and $S_{4\nu}$ are zero in the rest frame; therefore $\dot{S}^{\mu\nu} S_{\mu\nu}$ and $\dot{S}^{\mu\nu} M_{\mu\nu}$ are both zero.

---

[1] Cohn, "Derivation of the Equations of Motion of a Classical Radiating Charge," p. 949.

Using these results and the fact that $a^{\mu}v_{\mu} = 0$, one obtains by the contractions of the equations on v_{μ} and $M_{\mu\nu}$:

$$0 = 0 - \dot{P}^{\mu}_{asm}v_{\mu} + \dot{C}^{\mu}_{lin} \tag{II.1.14}$$

$$0 = 0 - \dot{J}^{\mu\nu}_{asm}M_{\mu\nu} + \dot{C}^{\mu\nu}_{ang}M_{\mu\nu} \tag{II.1.15}$$

or,

$$\dot{C}^{\mu}_{lin}v_{\mu} - \dot{P}^{\mu}_{asm}v_{\mu} = 0 \tag{II.1.16}$$

$$\dot{C}^{\mu\nu}_{ang}M_{\mu\nu} - \dot{J}^{\mu\nu}_{asm}M_{\mu\nu} = 0 \tag{II.1.17}$$

Both $\dot{C}^{\mu}_{lin}$ and $\dot{P}^{\mu}_{asm}$ are linear combinations of linearly independent vectors; $\dot{C}^{\mu\nu}_{ang}$ and $\dot{J}^{\mu\nu}_{asm}$ are linear combinations of linearly independent second-rank tensors. The above equations may therefore be written as follows:

$$\sum_{j=1}^{m} b_j \Phi^{\mu}_{j} v_{\mu} = 0 \tag{II.1.18}$$

$$\sum_{k=1}^{n} e_k \Lambda^{\mu\nu}_{k} M_{\mu\nu} = 0 \tag{II.1.19}$$

For some of the Φ^{μ}_{j} and $\Lambda^{\mu\nu}_{k}$, $\Phi^{\mu}_{j}v_{\mu} = 0$ and $\Lambda^{\mu\nu}_{k}M_{\mu\nu} = 0$ respectively. In other cases, certain of the $\Phi^{\mu}_{j}v_{\mu}$ may be expressed as linear combinations of other $\Phi^{\mu}_{j}v_{\mu}$, and certain of the $\Lambda^{\mu\nu}_{k}M_{\mu\nu}$ may be expressed as linear combinations of other $\Lambda^{\mu\nu}_{k}M_{\mu\nu}$. However, it is possible to rewrite the above sums as

$$\sum_{j=1}^{m} b'_j \Phi'^{\mu}_{j} v_{\mu} = 0 \tag{II.1.20}$$

$$\sum_{k=1}^{n} e'_k \Lambda'^{\mu\nu}_{k} M_{\mu\nu} = 0 \tag{II.1.21}$$

in which the Φ'^{μ}_{j} and $\Lambda'^{\mu\nu}_{k}$ are linear combinations of the Φ^{μ}_{j} and $\Lambda^{\mu\nu}_{k}$ respectively, the b'_j and e'_k are linear combinations of the b_j and e_k respectively, and none of the $\Phi'^{\mu}_j v_\mu$ or $\Lambda'^{\mu\nu}_k M_{\mu\nu}$ are zero.

At this point it is desirable to assert that the b'_j and e'_k must all be zero for (II.1.20) and (II.1.21) to hold, thus yielding a set of first-order differential equations in the f_i and q_i which make up the b'_j and e'_k. However, this assertion cannot be made without additional statements about the f_i and q_i. As an example, suppose that in (II.1.20), $\Phi'^{\mu}_1 = v^\mu$ and $\Phi'^{\mu}_2 = \dot{a}^\mu$. Part of the sum on the left of (II.1.20) would then consist of the terms $b'_1 v^\mu v_\mu + b'_2 \dot{a}^\mu v_\mu$, or $- b'_1 c^2 - b'_2 a^2$. If $b'_1 = a^2$ and $b'_2 = - c^2$, for example, this part of the sum would be zero even though neither b'_1 nor b'_2 are zero.

To avoid the difficulty mentioned above, the convention is adopted that any kinematic dependence is lumped into the V^μ_i and $\Theta^{\mu\nu}_i$, leaving the f_i and q_i constant. This convention in itself places no limitation on the types of terms that may be present in C^μ_{1in} and $C^{\mu\nu}_{ang}$, although it greatly increases the number of terms which would have to be included if every possible term is to be considered. For example, not only v^μ but $a^2 v^\mu$, $a^\alpha \dot{a}_\alpha v^\mu$, $(a^2)^2 v^\mu$, etc., would have to be included in C^μ_{1in}. As will be seen in the next section, the inclusion of every possible term in an actual solution attempt is an impractical task, and simplifying assumptions and approximations will be required; however, at this point no such approximations have been made.

The procedure from this point, therefore, is to select possible terms for C^μ_{1in} and $C^{\mu\nu}_{ang}$; develop and solve equations for the f_i and q_i from (II.1.20) and (II.1.21); and write out the equations of motion, including the terms of $\dot{C}^\mu_{1in}$ and $\dot{C}^{\mu\nu}_{ang}$ with their coefficients.

2. APPROXIMATIONS AND ASSUMPTIONS

In the method of derivation discussed in the previous section, consideration of all possible independent combinations of the quantities z^μ, v^μ, a^μ, $\dot{a}^\mu$, $M^{\mu\nu}$, $\dot{M}^{\mu\nu}$, and $\ddot{M}^{\mu\nu}$ would give rise to a great many terms (in fact, an infinite number of terms). Assumptions must therefore be made which drastically reduce the number of terms to be considered. The first such assumption made here is the assumption of small spin; the particle's spin (and its derivatives) will be assumed to be sufficiently small so that all terms of quadratic or higher degree in $\underline{M^{\mu\nu}}$, $\underline{\dot{M}^{\mu\nu}}$, and/or $\underline{\ddot{M}^{\mu\nu}}$ are negligible when compared with the first-degree terms in these variables. This assumption greatly reduces the complexity of $\dot{P}^\mu_{asm}$ and $\dot{J}^{\mu\nu}_{asm}$ as well as C^μ_{lin} and $C^{\mu\nu}_{ang}$; $\dot{P}^\mu_{asm}$ becomes the expression given for $\dot{P}^\mu_{ss}$ in (I.6.3), and $\dot{J}^{\mu\nu}_{asm}$ becomes the expression given for $\dot{J}^{\mu\nu}_{ss}$ in (I.6.4) with the z^μ-dependent terms removed:

$$\text{(II.2.1)} \quad \dot{J}^{\mu\nu}_{asm} = \frac{e^2}{2c^3}\left(\frac{1}{mc^2}\dot{M}^{[\nu\alpha}a_\alpha v^{\mu]} + \frac{1}{mc^2}\ddot{M}^{[\nu\alpha}v_\alpha v^{\mu]} - \frac{1}{3mc^2}M^{[\nu\alpha}a_\alpha a^{\mu]} + \frac{4}{3}a^{[\nu}v^{\mu]} - \frac{2}{3m}\ddot{M}^{\mu\nu}\right)$$

Even with the assumption of small spin, several problems remain in the construction of C^μ_{lin} and $C^{\mu\nu}_{ang}$. First, an infinite number of terms would still have to be included in order to exhaust every possibility; further limitations are needed to reduce the problem to a manageable level.

Further problems arise because of the contraction of $\dot{C}^\mu_{lin}$ and $\dot{C}^{\mu\nu}_{ang}$ with v^μ and $M^{\mu\nu}$ respectively in equations (II.1.16) and (II.1.17). One cannot rule out the possibility, for example, of the existence of terms in C^{μ}_{lin} (call them $C^{\mu}_{lin(1)}$, $C^{\mu}_{lin(2)}$, ... , $C^{\mu}_{lin(n)}$), such that $\dot{C}^{\mu}_{lin(1)}v_{\mu} + \dot{C}^{\mu}_{lin(2)}v_{\mu} + \ldots + \dot{C}^{\mu}_{lin(n)}v_{\mu} = 0$. In this case the equations in the f_i belonging to these terms will be homogeneous; these f_i can therefore be determined only to within an arbitrary proportionality constant. In other words, equation (II.1.16) would be satisfied regardless of whether or not these terms were included in C^{μ}_{lin}. Also, there is the problem of terms such as $M^{\mu\alpha}a_{\alpha}$ in C^{μ}_{lin}, whose derivatives contracted with v_{μ} are zero, and whose coefficients are therefore not calculable by this method even in terms of other coefficients. Analogous difficulties occur in the angular case.

From the above considerations it is evident that some terms can neither be included in nor excluded from C^{μ}_{lin} or $C^{\mu\nu}_{ang}$ by the method of solution described in this paper. Certain relatively simple terms, however, are required in C^{μ}_{lin} and $C^{\mu\nu}_{ang}$ if (II.1.16) and (II.1.17) are to be satisfied, since the terms introduced by $\dot{P}^{\mu}_{asm}v_{\mu}$ and $\dot{J}^{\mu\nu}_{asm}M_{\mu\nu}$ must be canceled out by terms in $\dot{C}^{\mu}_{lin}v_{\mu}$ and $\dot{C}^{\mu\nu}_{ang}M_{\mu\nu}$. Since these are the terms whose coefficients are determinable by the method described in this paper, consideration will be restricted to these terms in the determination of coefficients for the terms in the equations of motion.

In order to frame an assumption to select candidates for these terms, one considers each term to be divided into two parts: a part carrying the free index or indices, and a part consisting of various scalar products, as in this example from the linear case:

$$\text{(II.2.2)} \qquad \underline{(a^2)(a^{\alpha}\dot{a}_{\alpha})(a^{\beta}z_{\beta})} \qquad \underline{M^{\mu\delta}z_{\delta}}$$

Terms with like tensor or vector parts (the parts carrying the free indices) are then grouped together so that the expression for C^{μ}_{lin} or $C^{\mu\nu}_{ang}$ consists of a sum of unique vector or tensor parts, each multiplied by a scalar part consisting of a sum of scalar products, as in the following example:

$$\text{(II.2.3)} \quad \left[k_1(a^2) + k_2(a^2)(a^{\alpha}\dot{a}_{\alpha})(a^{\beta}z_{\beta}) + k_3(\dot{a}^2)^3 + \ldots\right] M^{\mu\delta} z_{\delta}$$

The assumption is now made that for each vector or tensor part, the <u>simplest possible</u> scalar part (consistent with satisfying (II.1.16) or (II.1.17)) is the one to be selected. This is equivalent to saying that one assumes C^{μ}_{lin} and $C^{\mu\nu}_{ang}$ to have the simplest possible form.[1] It will be seen that in the equations for linear and angular motion to be developed in the next two sections, constant scalar parts, as defined above, will suffice in all cases. (Actually, since each term is already multiplied by a constant scalar coefficient, this merely means that both the coefficient and the scalar part, and hence their product, will be constant, leaving all variation to the vector or tensor part.)

The above assumptions leave the following vectors as possible vector parts to be included in the expression for C^{μ}_{lin}:

$$\text{(II.2.4)} \quad \begin{array}{llll} z^{\mu} & v^{\mu} & a^{\mu} & \dot{a}^{\mu} \\ M^{\mu\alpha}z_{\alpha} & M^{\mu\alpha}a_{\alpha} & M^{\mu\alpha}\dot{a}_{\alpha} & \\ \dot{M}^{\mu\alpha}z_{\alpha} & \dot{M}^{\mu\alpha}a_{\alpha} & \dot{M}^{\mu\alpha}\dot{a}_{\alpha} & \\ \ddot{M}^{\mu\alpha}z_{\alpha} & \ddot{M}^{\mu\alpha}a_{\alpha} & \ddot{M}^{\mu\alpha}\dot{a}_{\alpha} & \end{array}$$

[1]A similar assumption (preferring simple solutions over more complex solutions) is made by Bhabha and Corben, p. 291.

The number of possible linearly independent tensors to be considered for the tensor parts of $C^{\mu\nu}_{ang}$ remains quite large, even with the foregoing assumptions; further reduction in this number is made when the angular equation of motion is considered. Also, from this point on, it should be remembered that $\dot{P}^{\mu}_{asm}$ and $\dot{J}^{\mu\nu}_{asm}$ have been re-defined to include only those terms considered significant under the small-spin assumption, and in the case of $\dot{J}^{\mu\nu}_{asm}$, to include only those terms which remain after the Section II-1 redefinition of the coordinate frames to be used in (II.1.2) and (II.1.4). These terms are the ones included in the expressions (I.6.3) and (II.2.1). Contracting these with v_{μ} and $M_{\mu\nu}$, one obtains

$$\text{(II.2.5)} \quad \dot{P}^{\mu}_{asm} v_{\mu} = \frac{e^2}{3mc^5} \ddot{M}^{\alpha\beta} v_{\alpha} a_{\beta} - \frac{2e^2a^2}{3c^3}$$

$$\text{(II.2.6)} \quad \dot{J}^{\mu\nu}_{asm} M_{\mu\nu} = -\frac{e^2}{3mc^5} M^{\mu\nu} M_{\nu}{}^{\beta} a_{\mu} a_{\beta} - \frac{e^2}{3mc^3} \ddot{M}^{\mu\nu} M_{\mu\nu}$$

The next two sections are devoted to setting up and solving the systems of equations for the coefficients used in the expressions for C^{μ}_{lin} and $C^{\mu\nu}_{ang}$.

3. THE LINEAR EQUATION OF MOTION

To investigate the linear equation of motion, one must ultimately construct $\dot{C}^{\mu}_{lin}v_{\mu}$ and equate coefficients of like terms in $\dot{C}^{\mu}_{lin}v_{\mu}$ and $\dot{P}^{\mu}v_{\mu}$. This is equivalent to setting up the equations resulting from setting the b_j' and e_k' from (II.1.15) and (II.1.16) to zero. Some of the terms in $\dot{C}^{\mu}_{lin}$ will be removed by contraction with v_{μ} and will therefore remain undetermined by this method; however, the majority of the coefficients will be determined. For the linear case:

$$\text{(II.3.1)} \quad \begin{aligned} C^{\mu}_{lin} &= f_1 z^{\mu} + f_2 v^{\mu} + f_3 a^{\mu} + f_4 \dot{a}^{\mu} \\ &+ f_5 M^{\mu\beta} z_{\beta} + f_6 M^{\mu\beta} a_{\beta} + f_7 M^{\mu\beta} \dot{a}_{\beta} \\ &+ f_8 \dot{M}^{\mu\beta} z_{\beta} + f_9 \dot{M}^{\mu\beta} a_{\beta} + f_{10} \dot{M}^{\mu\beta} \dot{a}_{\beta} \\ &+ f_{11} \ddot{M}^{\mu\beta} z_{\beta} + f_{12} \ddot{M}^{\mu\beta} a_{\beta} + f_{13} \ddot{M}^{\mu\beta} \dot{a}_{\beta} \end{aligned}$$

The term $M^{\mu\beta}v_{\beta}$, as previously stated, is zero because of the lack of an electric dipole moment; the terms $\dot{M}^{\mu\beta}v_{\beta}$ and $\ddot{M}^{\mu\beta}v_{\beta}$ are expressible as $-M^{\mu\beta}a_{\beta}$ and $-2\dot{M}^{\mu\beta}a_{\beta} - M^{\mu\beta}\dot{a}_{\beta}$ respectively. Continuing,

$$\text{(II.3.2)} \quad \dot{C}^{\mu}_{lin} = f_1 v^{\mu} + f_2 a^{\mu} + f_3 \dot{a}^{\mu} + f_4 \ddot{a}^{\mu} + f_5 \dot{M}^{\mu\beta} z_{\beta}$$

$$+ f_8 \left(\ddot{M}^{\mu\beta} z_\beta - M^{\mu\beta} a_\beta\right) + f_6 \left(\dot{M}^{\mu\beta} a_\beta + M^{\mu\beta} \dot{a}_\beta\right)$$

$$+ f_{11} \left(\dddot{M}^{\mu\beta} z_\beta + \ddot{M}^{\mu\beta} v_\beta\right) + f_7 \left(\dot{M}^{\mu\beta} \dot{a}_\beta + M^{\mu\beta} \ddot{a}_\beta\right)$$

$$+ f_9 \left(\dot{M}^{\mu\beta} \dot{a}_\beta + \ddot{M}^{\mu\beta} a_\beta\right) + f_{10} \left(\ddot{M}^{\mu\beta} \dot{a}_\beta + \dot{M}^{\mu\beta} \ddot{a}_\beta\right)$$

$$+ f_{12} \left(\ddot{M}^{\mu\beta} \dot{a}_\beta + \dddot{M}^{\mu\beta} a_\beta\right) + f_{13} \left(\dddot{M}^{\mu\beta} \dot{a}_\beta + \ddot{M}^{\mu\beta} \ddot{a}_\beta\right)$$

(II.3.3)
$$\dot{C}^{\mu}_{lin} v_\mu = - f_1 c^2 - f_3 a^2 + f_4 \ddot{a}^\mu v_\mu - f_5 M^{\mu\beta} a_\mu z_\beta$$

$$+ f_8 \ddot{M}^{\mu\beta} v_\mu z_\beta + f_{11} \dddot{M}^{\mu\beta} v_\mu z_\beta - f_7 \ddot{M}^{\mu\beta} v_\mu a_\beta$$

$$+ \left(f_{10} - \tfrac{1}{2} f_{12}\right) \ddot{M}^{\mu\beta} v_\mu \dot{a}_\beta + \left(f_{12} - f_{10}\right) M^{\mu\beta} a_\mu \ddot{a}_\beta$$

$$+ f_{13} \dddot{M}^{\mu\beta} v_\mu \dot{a}_\beta + f_{13} \ddot{M}^{\mu\beta} v_\mu \ddot{a}_\beta$$

Equating coefficients of like terms with expression (II.2.5), one obtains the following set of equations:

(II.3.4)
$$f_1 = 0$$

(II.3.5)
$$f_3 = 2e^2/3c^3$$

(II.3.6) $$f_4 = 0$$

(II.3.7) $$f_5 = 0$$

(II.3.8) $$f_8 = 0$$

(II.3.9) $$f_{11} = 0$$

(II.3.10) $$f_7 = -e^2/3mc^5$$

(II.3.11) $$f_{10} - \tfrac{1}{2} f_{12} = 0$$

(II.3.12) $$f_{12} - f_{10} = 0$$

(II.3.13) $$f_{13} = 0$$

From equations (II.3.11) and (II.3.12) one can quickly deduce that $f_{10} = f_{12} = 0$. To summarize these results: $f_1 = f_4 = f_5 = f_8 = f_{10} = f_{11} = f_{12} = f_{13} = 0$; $f_3 = 2e^2/3c^3$; $f_7 = -e^2/3mc^5$; f_2, f_6, and f_9 are as yet undetermined. C^{μ}_{lin} may now be written as follows:

(II.3.14) $$C^{\mu}_{lin} = f_2 v^{\mu} + \frac{2e^2}{3c^3} a^{\mu} + f_6 M^{\mu\beta} a_{\beta} - \frac{e^2}{3mc^5} M^{\mu\beta} \dot{a}_{\beta} + f_9 \dot{M}^{\mu\beta} a_{\beta}$$

The linear equation of motion therefore may be written as

$$
\text{(II.3.15)} \qquad m a^{\mu} = F^{\mu}_{ext} - \dot{P}^{\mu}_{asm} + \dot{C}^{\mu}_{lin}
$$

$$
= F^{\mu}_{ext} - \frac{e^2}{3c^5}\left(\frac{a^2}{mc^2} M^{\mu\beta} a_\beta - \frac{1}{m}\ddot{M}^{\mu\beta} a_\beta - \frac{2}{mc^2}\ddot{M}^{\alpha\beta} v_\alpha a_\beta v^{\mu}\right.
$$

$$
\left.+ 2a^2 v^{\mu}\right) + f_2 a^{\mu} + \frac{2e^2}{3c^3}\dot{a}^{\mu} + f_6\left(\dot{M}^{\mu\beta} a_\beta + M^{\mu\beta}\dot{a}_\beta\right)
$$

$$
- \frac{e^2}{3mc^5}\dot{M}^{\mu\beta}\dot{a}_\beta - \frac{e^2}{3mc^5} M^{\mu\beta}\ddot{a}_\beta + f_9\left(\ddot{M}^{\mu\beta} a_\beta + \dot{M}^{\mu\beta}\dot{a}_\beta\right)
$$

The term $f_2 a^\mu$ in the above expression may be transferred to the left side of the equation, giving a left side of $(m - f_2)a^\mu$, where f_2 is a constant. Since m is the <u>bare</u> particle mass, the constant $(-f_2)$ can be interpreted as the mass of the field, and the sum $m + (-f_2)$ as the observable mass of the particle.

If the above transfer is made, the linear equation of motion becomes

$$
\text{(II.3.16)} \qquad m_{obs}\, a^{\mu} = F^{\mu}_{ext} - \frac{e^2}{3c^5}\left(\frac{a^2}{mc^2} M^{\mu\beta} a_\beta - \frac{1}{m}\ddot{M}^{\mu\beta} a_\beta - \frac{2}{mc^2}\ddot{M}^{\alpha\beta} v_\alpha a_\beta v^{\mu}\right.
$$

$$
\left.+ 2a^2 v^{\mu}\right) + \frac{2e^2}{3c^3}\dot{a}^{\mu} + f_6\left(\dot{M}^{\mu\beta} a_\beta + M^{\mu\beta}\dot{a}_\beta\right)
$$

$$
- \frac{e^2}{3mc^5}\dot{M}^{\mu\beta}\dot{a}_\beta - \frac{e^2}{3mc^5} M^{\mu\beta}\ddot{a}_\beta + f_9\left(\ddot{M}^{\mu\beta} a_\beta + \dot{M}^{\mu\beta}\dot{a}_\beta\right)
$$

where f_6 and f_9 are undetermined constants.

4. THE ANGULAR EQUATION OF MOTION

The investigation of the angular equation of motion follows the same procedure in principle as that of the linear equation: however, the additional complexity of the expressions involved makes it expedient to exercise more discrimination in the selection of terms for $C^{\mu\nu}_{ang}$. As was the case in the linear equation, there is a class of possible terms of C_{ang} whose coefficients cannot be determined by contracting $\dot{C}^{\mu\nu}_{ang}$ with $M_{\mu\nu}$ and equating coefficients of like terms with $\dot{J}^{\mu\nu}M_{\mu\nu}$, that is, those terms whose derivatives yield zero when contracted with $M_{\mu\nu}$. Two terms that appear in this class are $q_0 a^{[\mu}v^{\nu]}$ and $q_1 M^{\mu\nu}$. Taking the derivatives and contracting with $M_{\mu\nu}$, one obtains

$$\text{(II.4.1)} \qquad \dot{q}_0 \, a^{[\mu}v^{\nu]}M_{\mu\nu} + q_0 \, \dot{a}^{[\mu}v^{\nu]}M_{\mu\nu} = 0$$

$$\text{(II.4.2)} \qquad \dot{q}_1 \, M^{\mu\nu}M_{\mu\nu} + q_1 \, \dot{M}^{\mu\nu}M_{\mu\nu} = \dot{q}_1 \, M^{\mu\nu}M_{\mu\nu}$$

Thus if the second expression is to be zero, q_1 must be a constant; however, no such restriction need be placed on q_0. This would seem to violate the earlier assumption that all the q_i are constants. That assumption was made in order to be able to solve the set of equations resulting from setting the e'_k in (II.1.16) to zero, which is equivalent to equating like terms in $\dot{J}^{\mu\nu}M_{\mu\nu}$ and $\dot{C}^{\mu\nu}_{ang}M_{\mu\nu}$. However, since q_0 will not appear in those equations, there is no mathematical need for it to be restricted by this assumption. (Physically, one would expect q_0 to be constant if the assumption that the other q_i are constant is to be physically plausible.)

Certain concepts and definitions are useful in further restricting the terms used in the expression for $C^{\mu\nu}_{ang}$. These terms will be composed of factors of z^μ and/or its derivatives, and possibly one factor of $M^{\mu\nu}$ or one of its derivatives. One can define the "degree," in z^μ and/or its derivatives, of such a term as the number of factors of z^μ and/or its derivatives contained in that term. Similarly, one can define the "order" of a term as the total number of time derivatives performed on its factors, considering $M^{\mu\nu}$ and z^μ to be of zero "order." Thus the term $\dot{M}^{\mu\alpha}a_\alpha z^\nu$ would have an "order" of three (one from $\dot{M}^{\mu\alpha}$, two from a_α, none from z^ν) and a "degree" of one in $M^{\mu\nu}$ and two in z^μ.

In most cases, taking the time derivative of a term produces one or more terms which have the same "degree" and one higher "order" than the original term. In some cases, such as those involving a factor $v^\alpha z_\alpha$, this does not hold true, since $d(v^\alpha z_\alpha)/d\tau = a^\alpha z_\alpha - c^2$. These terms are excluded from consideration because the factor $v^\alpha z_\alpha$ would appear in the term's "scalar part" as discussed in Section II-2, and the assumption was made in Section II-2 that the simplest possible scalar parts would be used. It will be demonstrated that constant scalar parts are sufficient to obtain a solution for the angular equation; therefore terms containing $v^\alpha z_\alpha$ are not considered.

If any terms besides $q_0 a^{[\mu}v^{\nu]}$ and $q_1 M^{\mu\nu}$ are to appear in $C^{\mu\nu}_{ang}$, their derivatives contracted with $M_{\mu\nu}$ must match either some term in $\dot{J}^{\mu\nu}_{asm}M_{\mu\nu}$ or some other term in $\dot{C}^{\mu\nu}_{ang}M_{\mu\nu}$. This implies limitations on the "order" and "degree" (as defined above) of these terms. All terms of $\dot{J}^{\mu\nu}_{asm}$ which survive in $\dot{J}^{\mu\nu}_{asm}M_{\mu\nu}$ can be grouped in two classes: those with "degree" 2 in z^μ and "order" 4, and those with "degree" 0 in z^μ and "order" 2.

(All have "degree" 1 in $M^{\mu\nu}$.) Therefore those terms of $\dot{C}^{\mu\nu}_{ang}$ which survive in $\dot{C}^{\mu\nu}_{ang}M_{\mu\nu}$ must fall into one of these two classes if they are to have nonzero coefficients. This in turn implies that those terms in $C^{\mu\nu}_{ang}$ which give rise to the nonzero terms in $\dot{C}^{\mu\nu}_{ang}M_{\mu\nu}$ must fall into one of the following two classes:

(A) "degree" 2 in z^{μ}, "order" 3

(B) "degree" 0 in z^{μ}, "order" 1

Combining all possible terms from classes (A) and (B) above, the initial expression for $C^{\mu\nu}_{ang}$ may be written

(II.4.3)
$$C^{\mu\nu}_{ang} = q_0\, a^{[\mu} v^{\nu]} + q_1 M^{\mu\nu} + q_2 \dot{M}^{\mu\nu} + q_3 M^{[\mu\alpha} z_\alpha \dot{a}^{\nu]} + q_4 M^{[\mu\alpha} a_\alpha v^{\nu]} + q_5 M^{[\mu\alpha} \dot{a}_\alpha z^{\nu]} + q_6 \dot{M}^{[\mu\alpha} z_\alpha a^{\nu]} + q_7 \dot{M}^{[\mu\alpha} a_\alpha z^{\nu]} + q_8 \ddot{M}^{[\mu\alpha} z_\alpha v^{\nu]}$$

Taking the derivative, contracting with $M_{\mu\nu}$, and grouping similar terms, one obtains

(II.4.4)
$$\dot{C}^{\mu\nu}_{ang} M_{\mu\nu} = q_2 \ddot{M}^{\mu\nu} M_{\mu\nu} + 2(q_3 + q_6)\dot{M}^{\mu\alpha} z_\alpha \dot{a}^{\nu} M_{\mu\nu} + 2(q_3 + q_5) M^{\mu\alpha} z_\alpha \ddot{a}^{\nu} M_{\mu\nu}$$

$$+ 2(q_4 - q_6)\, M^{\mu\alpha} a_\alpha a^\nu M_{\mu\nu}$$

$$+ 2(q_5 + q_7)\, \dot{M}^{\mu\alpha} \dot{a}_\alpha z^\nu M_{\mu\nu}$$

$$+ 2(q_6 + q_8)\, \ddot{M}^{\mu\alpha} z_\alpha a^\nu M_{\mu\nu}$$

$$+ 2 q_7\, \ddot{M}^{\mu\alpha} a_\alpha z^\nu M_{\mu\nu}$$

Equating coefficients of like terms with $\dot{J}^{\mu\nu}_{asm} M_{\mu\nu}$, one obtains the following equations:

(II.4.5) $$q_2 = -\frac{e^2}{3mc^3}$$

(II.4.6) $$q_3 + q_6 = 0$$

(II.4.7) $$q_3 + q_5 = 0$$

(II.4.8) $$2(q_4 - q_6) = \frac{e^2}{3mc^5}$$

(II.4.9) $$q_5 + q_7 = 0$$

(II.4.10) $$q_6 + q_8 = 0$$

(II.4.11) $$2 q_7 = 0$$

The solutions are: $q_3 = q_5 = q_6 = q_7 = q_8 = 0$; $q_2 = -e^2/3mc^3$; $q_4 = e^2/6mc^5$. With substitution, differentiation, and simplification, this gives the following expression for $\dot{C}^{\mu\nu}_{ang}$:

$$\dot{C}^{\mu\nu}_{ang} = \dot{q}_0 a^{[\mu} v^{\nu]} + q_0 \dot{a}^{[\mu} v^{\nu]} + q_1 \dot{M}^{\mu\nu} - \frac{e^2}{3mc^3} \dddot{M}^{\mu\nu} + \frac{e^2}{6mc^5} \dot{M}^{[\mu\alpha} a_\alpha v^{\nu]} + \frac{e^2}{6mc^5} M^{[\mu\alpha} \dot{a}_\alpha v^{\nu]} + \frac{e^2}{6mc^5} M^{[\mu\alpha} a_\alpha a^{\nu]} \tag{II.4.12}$$

The angular equation of motion (for small spin) can then be written as

$$\begin{aligned} \dot{S}^{\mu\nu} &= T^{\mu\nu}_{ext} - \dot{J}^{\mu\nu}_{asm} + \dot{C}^{\mu\nu}_{ang} \\ &= T^{\mu\nu}_{ext} - \frac{e^2}{3mc^5} \dot{M}^{[\mu\alpha} a_\alpha v^{\nu]} - \frac{e^2}{3mc^5} M^{[\mu\alpha} \dot{a}_\alpha v^{\nu]} \\ &\quad + \left(\dot{q}_0 + \frac{2e^2}{3c^3}\right) a^{[\mu} v^{\nu]} + q_0 \dot{a}^{[\mu} v^{\nu]} + q_1 \dot{M}^{\mu\nu} \end{aligned} \tag{II.4.13}$$

with $\dot{J}^{\mu\nu}_{asm}$ being obtained from (II.2.1).

The term $q_1\dot{M}^{\mu\nu}$ in the above equation of motion is subject to special interpretation, much as was the term f_2a^μ in the linear equation of motion. Since $S^{\mu\nu}$ and $M^{\mu\nu}$ are proportional, $\dot{S}^{\mu\nu}$ and $q_1\dot{M}^{\mu\nu}$ are also proportional. $\dot{S}^{\mu\nu}$ gives only the rate of change of angular momentum in the bare particle; if $-q_1\dot{M}^{\mu\nu}$ is interpreted as the rate of change of the field angular momentum, the term $q_1\dot{M}^{\mu\nu}$ may be moved to the left side of the equation and this left side rewritten as $\dot{S}^{\mu\nu}_{obs} = \dot{S}^{\mu\nu} + (-q_1\dot{M}^{\mu\nu})$ where $\dot{S}^{\mu\nu}_{obs}$ is the total observed angular momentum change rate of the particle with its field.

The coefficient q_0, like the coefficients f_6 and f_9 in the equation of linear motion, remains undetermined. It is, however, consistent with the assumptions made in this derivation to regard q_0 as a constant. If this is done, and the renormalization mentioned in the previous paragraph is carried out, the equation of angular motion becomes

$$\text{(II.4.14)} \qquad \dot{S}^{\mu\nu}_{obs} = T^{\mu\nu}_{ext} - \frac{e^2}{3mc^5} \dot{M}^{[\mu\alpha} a_\alpha v^{\nu]} - \frac{e^2}{3mc^5} M^{[\mu\alpha} \dot{a}_\alpha v^{\nu]}$$

$$+ \frac{2e^2}{3c^3} a^{[\mu} v^{\nu]} + q_0 \dot{a}^{[\mu} v^{\nu]}$$

5. COMPARISON WITH OTHER EQUATIONS OF MOTION

Some interesting comparisons can be made between the equations of motion developed in this paper and those developed by other methods. Consider first the case of a non-spinning particle. Equation (II.3.16) then becomes

$$m_{obs}\, a^{\mu} = F^{\mu}_{ext} - \frac{2e^2 a^2}{3c^5} v^{\mu} + \frac{2e^2}{3c^3} \dot{a}^{\mu} \tag{II.5.1}$$

which is identical with the equation for a non-spinning particle obtained by Dirac.[1]

When small spin is allowed, comparisons may be made with the results obtained by Bhabha and Corben.[2] In the linear case, in contravariant form and using the small spin assumption, the Bhabha and Corben equation[3] is

$$\begin{aligned} m a^{\mu} + \frac{d}{d\tau}\left\{ I \dot{S}^{\mu\alpha} v_{\alpha} - \frac{g_2}{2} v^{\mu} S_{\alpha\beta} F^{\alpha\beta}_{in} - g_2 S^{\mu}{}_{\alpha} F^{\alpha\beta}_{in} v_{\beta} \right\} \\ = g_1 F^{\mu\alpha}_{in} v_{\alpha} - \frac{g_2}{2} S_{\alpha\beta} \frac{\partial}{\partial x_{\mu}} F^{\alpha\beta}_{in} + g_1^2\left(\frac{2}{3}\right)\left(v^{\mu} a^2 + \dot{a}^{\mu}\right) \\ + g_1 g_2 \Big(\frac{4}{3} v^{\mu} \dot{S}_{\alpha\beta} \dot{a}^{\alpha} v^{\beta} + \frac{2}{3} \dot{S}^{\mu\alpha} v_{\alpha} a^2 - \frac{2}{3} \dddot{S}^{\mu\alpha} v_{\alpha} \\ - 2 \ddot{S}^{\mu\alpha} a_{\alpha} - \frac{2}{3} \dot{S}^{\mu\alpha} \dot{a}_{\alpha} \Big) \end{aligned} \tag{II.5.2}$$

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[1] P. A. M. Dirac, Proc. Roy. Soc. (London) A167, 148 (1938).

[2] H. J. Bhabha and H. C. Corben

[3] Ibid., pp. 298, 310, 313.

where g_1 is the particle charge and g_2 is the particle dipole moment. Four terms in (II.5.2) are seen to involve an external field $F^{\mu\nu}_{in}$. The term $g_1 F^{\mu\alpha}_{in} v_\alpha$ represents the Lorentz force on the charged particle. The term involving $S_{\alpha\beta}\frac{\partial}{\partial x_\mu}F^{\alpha\beta}_{in}$ represents the force exerted on the dipole by a non-homogeneous field. The term involving $\frac{d}{d\tau}\{v^\mu S_{\alpha\beta}F^{\alpha\beta}_{in}\}$ denotes the contribution made by the change of the potential energy of the dipole in the external field because of the rotation of the dipole[1]. Finally, the term containing $\frac{d}{d\tau}\{S^\mu{}_\alpha F^{\alpha\beta}_{in} v_\beta\}$ is not explicitly discussed in the paper by Bhabha and Corben; however, an equation for linear motion of a charged particle developed by Weyssenhof and Raabe[2] contains a similar term[3], which seems to arise from coupling to the space-time components of the torque tensor used in that paper.

These terms have no direct counterpart in the present work because of a fundamental difference in the view taken of external influences on the particle. In the papers mentioned above[4] the external influence was regarded strictly as a Maxwell-Lorentz field; this external field and the field of the particle were considered together. In the present paper, the external influence is viewed simply as a force, F^μ_{ext}, which may originate from a Maxwell-Lorentz field or from some other source. The only specification made about F^μ_{ext} in the present paper is that one can define it in such a way that $F^\mu_{ext} v_\mu = 0$. This is then used to eliminate F^μ_{ext} at the beginning of the calculations, thus separating

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[1]Bhabha and Corben, p. 290.

[2]J. Weyssenhof and A. Raabe, "Relativistic Dynamics of Spin-Fluids and Spin-Particles," Acta Physica Polonica, Vol. IX, Fasc. 1(1947), pp. 7-18.

[3]Ibid., p. 18.

[4]Bhabha and Corben, Weyssenhof and Raabe.

the consideration of external force from the consideration of particle self-effects.

If the four terms discussed above are taken as F^{μ}_{ext} for the Bhabha and Corben equation, one obtains for $F^{\mu}_{ext}v_{\mu}$ the result

$$F^{\mu}_{ext} v_{\mu} = \frac{g_2}{2}\left(\dot{S}_{\alpha\beta} F^{\alpha\beta} + S_{\alpha\beta} \dot{F}^{\alpha\beta} \right) + g_2 \dot{S}^{\mu}{}_{\alpha} F^{\alpha\beta} v_{\beta} v_{\mu} - \frac{g_2}{2} S_{\alpha\beta} v_{\mu} \partial^{\mu} F^{\alpha\beta} \qquad \text{(II.5.3)}$$

Obviously the condition $F^{\mu}_{ext}v_{\mu} = 0$ is not generally true for the Bhabha and Corben equation. Therefore, with the exception of the Lorentz-force term $g_1 F^{\mu\alpha}_{in} v_{\alpha}$, the Bhabha and Corben equation differs from the equation developed in the present paper when a general field is applied.

Rewriting (II.5.2) with no external field and using the notation of the present paper (but keeping a system of units in which c = 1), one has

$$m a^{\mu} + \frac{d}{d\tau}\left\{ I \dot{S}^{\mu}{}_{\alpha} v^{\alpha} \right\} = \frac{2e^2 a^2}{3} v^{\mu} + \frac{2e^2}{3} \dot{a}^{\mu} + \frac{2e^2}{3m} \dot{M}_{\alpha\beta} \dot{a}^{\alpha} v^{\beta} v^{\mu} + \frac{e^2 a^2}{3m} \dot{M}^{\mu\alpha} v_{\alpha} - \frac{e^2}{3m} \dddot{M}^{\mu\alpha} v_{\alpha} - \frac{e^2}{m} \ddot{M}^{\mu\alpha} a_{\alpha} - \frac{e^2}{3m} \dot{M}^{\mu\alpha} \dot{a}_{\alpha} \qquad \text{(II.5.4)}$$

Rearranging terms and making liberal use of the identities of Appendix B as well as the similar identity $\ddot{M}^{\alpha\beta}v_\alpha a_\beta = \dot{M}^{\alpha\beta}\dot{a}_\alpha v_\beta$, one obtains

(II.5.5)
$$
\begin{aligned}
m a^\mu = & \overset{(1)}{-\frac{e^2 a^2}{3m} M^\mu{}_\beta a^\beta} + \overset{(2)}{\frac{2e^2}{3m} \ddot{M}^{\alpha\beta} v_\alpha a_\beta v^\mu} \\
& + \overset{(3)}{\frac{2e^2 a^2}{3} v^\mu} + \overset{(4)}{\frac{2e^2}{3} \dot{a}^\mu} \\
& + \overset{(5)}{I\left(\dot{S}^\mu{}_\alpha a^\alpha + S^\mu{}_\alpha \dot{a}^\alpha\right)} \\
& + \overset{(6)}{\frac{2e^2}{3m} \dot{M}^\mu{}_\alpha \dot{a}^\alpha} + \overset{(7)}{\frac{e^2}{3m} M^\mu{}_\beta \ddot{a}^\beta}
\end{aligned}
$$

The linear equation of motion developed in this paper can be rewritten (also for zero external force) as

(II.5.6)
$$
\begin{aligned}
m_{obs} a^\mu = & \overset{(1)}{-\frac{e^2 a^2}{3mc^7} M^{\mu\beta} a_\beta} + \overset{(1')}{\left(\frac{e^2}{3mc^5} + f_9\right) \ddot{M}^{\mu\beta} a_\beta} \\
& + \overset{(2)}{\frac{2e^2}{3mc^7} \ddot{M}^{\alpha\beta} v_\alpha a_\beta v^\mu} - \overset{(3)}{\frac{2e^2 a^2}{3c^5} v^\mu} \\
& + \overset{(4)}{\frac{2e^2}{3c^3} \dot{a}^\mu} + \overset{(5)}{f_6\left(\dot{M}^{\mu\beta} a_\beta + M^{\mu\beta} \dot{a}_\beta\right)} \\
& + \overset{(6)}{\left(f_9 - \frac{e^2}{3mc^5}\right) \dot{M}^{\mu\beta} \dot{a}_\beta} - \overset{(7)}{\frac{e^2}{3mc^5} M^{\mu\beta} \ddot{a}_\beta}
\end{aligned}
$$

Ignoring the powers of c in the denominators in (II.5.6), one may observe that terms (1), (2), and (4) in (II.5.5) match the corresponding terms in (II.5.6); terms (3) and (7) of (II.5.5) differ from their counterparts in (II.5.6) only by sign; and terms (1'), (5), and (6) in (II.5.6) contain arbitrary constants.

Term (1') in (II.5.6) has no counterpart in (II.5.5). If $f_9 = -e^2/3mc^5$, term (1') vanishes from (II.5.6). This value for f_9 causes term (6) in (II.5.6) to differ from term (6) in (II.5.5) only by sign. Also, if f_6 is chosen so that $f_6 M^{\mu\beta} = IS^{\mu\beta}$, terms (5) in both equations agree.

This leaves terms (3), (6), and (7) differing in sign between the two equations. The sign difference is accounted for by the fact that Bhabha and Corben used a metric with diagonal (-1,-1,-1,1) while in the present paper the metric has a diagonal of (1,1,1,-1). Because of this, every term in (II.5.5) in which an <u>odd</u> number of complete contractions of indices occur should have the opposite sign from the corresponding term in (II.5.6). This condition holds true for (3), (6), and (7) of (II.5.5). Thus, if external fields and forces are excluded, the equation developed in Section II-3 can, by proper choices for f_6 and f_9, be brought into exact agreement with the equation for linear motion produced by Bhabha and Corben.

A quite different situation prevails in the case of the equations for angular motion. Bhabha and Corben's equation,[1] in contravariant form and with the small spin assumption (but again in units such that

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[1] H. J. Bhabha and H. C. Corben, p. 298.

c = 1), is

$$\text{(II.5.7)} \qquad I\dot{S}^{\mu\nu} - I S^{[\mu\alpha} a_\alpha v^{\nu]} = g_2 S^{[\mu}{}_\alpha \left(F_{in}^{\alpha\nu]} - F_{in}^{\alpha\beta} v_\beta v^{\nu]} \right)$$

This equation, as in the equation for linear motion, was developed by considering an external field together with the particle's own field: however, the expression on the right side of (II.5.7) is compatible with a torque such as the $T^{\mu\nu}_{ext}$ used in the present paper. Analysis of the right side of (II.5.7) reveals that its space-space components in the particle rest frame consist of $g_2(\underline{\omega}\times\underline{B})$, while the space-time components are zero. Since $\underline{\mu}$ and $\underline{\omega}$ are parallel in the type of particle under consideration here, and the particle has no electric dipole moment, this tensor is equivalent to the commonly defined torque tensor $\{\underline{\mu}\times\underline{B}:\underline{\pi}\times\underline{E}\}$ in the "six-vector" notation. Furthermore, if the right side of (II.5.7) is identified with $T^{\mu\nu}_{ext}$, the relation $T^{\mu\nu}_{ext}M_{\mu\nu} = 0$ holds true; since $M_{\mu\nu}$ is proportional to $S_{\mu\nu}$, the contraction $T^{\mu\nu}_{ext}M_{\mu\nu}$ is proportional to the triple product $(\underline{\omega}\times\underline{B})\cdot\underline{\omega}$, which is zero. Thus there is no conflict between the Bhabha and Corben equation for angular motion and that of the present paper concerning the terms which represent external torque.

Conflict does arise, however, in the comparison of the force-free terms of the two equations. Aside from the term $I\dot{S}^{\mu\nu}$, the only force-free term found in the Bhabha and Corben angular equation is the term $g_2 S^{[\mu\alpha} a_\alpha v^{\nu]}$. This type of term is not to be found in the angular equation of the present paper, although terms equivalent to the term $g_2 \frac{d}{d\tau}\{S^{[\mu\alpha} a_\alpha\} v^{\nu]}$ are to be found. In addition, terms containing $a^{[\mu}v^{\nu]}$ and $\dot{a}^{[\mu}v^{\nu]}$ occur in the present paper's equation but not in the Bhabha and Corben equation.

Similar or greater conflicts occur when the equations of the present paper are compared with other equations, such as those of Weyssenhof and Raabe[1] mentioned earlier. It may be noted in passing that the Weyssenhof and Raabe equation for linear motion does not reduce, in the non-spinning case, to the Dirac equation. The conflicts mentioned and demonstrated in this section would seem to lend yet greater weight to the comment by Nyborg[2] mentioned in the Introduction.

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[1] J. Weyssenhof and A. Raabe, pp. 17, 18.

[2] Nyborg, p. 62.

APPENDIXES

APPENDIX A. INTEGRALS INVOLVING u^α

Since $u^\alpha = R^\alpha/\rho - v^\alpha/c$, in the particle rest frame $u^\alpha = (\underline{R}/\rho, 0)$. Therefore any integral of the form

$$\int u^\alpha u^\beta \ldots u^\delta \, d\Omega \tag{A.1}$$

is zero in the rest frame if any of the indices equal 4, and attention can be restricted to the spacelike components. One may define

$$u_x = \sin\theta \cos\phi \tag{A.2}$$

$$u_y = \sin\theta \sin\phi \tag{A.3}$$

$$u_z = \cos\theta \tag{A.4}$$

and write

$$\begin{aligned}\int u^\alpha u^\beta \ldots u^\delta \, d\Omega &= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \sin^{i+j+1}\theta \cos^k\theta \cos^i\phi \sin^j\phi \, d\phi \, d\theta \\ &= \int_0^{\pi} \sin^{i+j+1}\theta \cos^k\theta \, d\theta \int_0^{2\pi} \cos^i\phi \sin^j\phi \, d\phi\end{aligned} \tag{A.5}$$

where i, j, and k are the respective numbers of u_x, u_y, and u_z factors present.

Now assume that i is odd. Then the second integral factor in the product above may be written

$$\int_0^{2\pi} \cos^i\phi \sin^j\phi \, d\phi = \int_0^{2\pi} (1-\sin^2\phi)^{\frac{i-1}{2}} \sin^j\phi \cos\phi \, d\phi \tag{A.6}$$

$$= \int_0^{2\pi} P \cos\phi \, d\phi = Q \Big|_0^{2\pi}$$

where P and Q are both polynomials in sin ϕ. Since sin 0 = sin 2π, the integral is zero. By a similar argument the integral can be shown to be zero if j is odd, and by another similar argument the integral

$$\text{(A.7)} \qquad \int_0^{\pi} \sin^{i+j+1}\theta \cos^{k}\theta \, d\theta$$

can be shown to be zero if i and j are even and k is odd. Therefore, the integral

$$\text{(A.8)} \qquad \int u^{\alpha} u^{\beta} \ldots u^{\delta} \, d\Omega$$

is zero unless there are an even number of factors u_x, an even number of factors u_y, and an even number of factors u_z in the integral.

The four versions of this integral needed for the purposes of this paper are evaluated on the following pages. Frequent use is made of the tensor $g^{\mu\nu} + v^{\mu}v^{\nu}/c^2$, which has elements (1,1,1,0) down the main diagonal and zeroes elsewhere (in the rest frame). This tensor is therefore defined (in this appendix only) as $T^{\mu\nu}$. In an analogous manner, $T^{\mu\nu\alpha\beta}$ and $T^{\mu\nu\alpha\beta\gamma\delta}$ are defined to have the value 1 for those components whose indices are all equal, but not equal to 4, and the value 0 for all other components. Although this definition does not in itself insure that $T^{\mu\nu\alpha\beta}$ and $T^{\mu\nu\alpha\beta\gamma\delta}$ are tensors, this does not matter since they do not appear anywhere in the final results.

Version 1: $$\int d\Omega = \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi = 4\pi$$

Version 2: $\int u_\alpha u_\beta \, d\Omega = 0$ if $\alpha \neq \beta$ or if $\alpha = \beta = 4$

$$\text{if } \alpha = \beta \neq 4, \quad \int u_\alpha u_\beta \, d\Omega = \int_0^\pi \cos^2\theta \sin\theta \, d\theta \int_0^{2\pi} d\phi = \frac{4\pi}{3}$$

$$\text{so, } \int u_\alpha u_\beta \, d\Omega = \frac{4\pi}{3} T_{\alpha\beta} = \frac{4\pi}{3}\left(g_{\alpha\beta} + \frac{v_\alpha v_\beta}{c^2}\right)$$

Version 3: $\int u_\alpha u_\beta u_\gamma u_\delta \, d\Omega$

Case 1 for non-zero: 2 different pairs exist, no indices = 4

$$\int u_\alpha u_\beta u_\gamma u_\delta \, d\Omega = \int_0^\pi \sin^3\theta \cos^2\theta \, d\theta \int_0^{2\pi} \cos^2\phi \, d\phi = \frac{4\pi}{15}$$

Case 2 for non-zero: all indices equal, $\neq 4$

$$\int u_\alpha u_\beta u_\gamma u_\delta \, d\Omega = \int_0^\pi \cos^4\theta \sin\theta \, d\theta \int_0^{2\pi} d\phi = \frac{4\pi}{5}$$

So in general,

$$\int u_\alpha u_\beta u_\gamma u_\delta \, d\Omega = \frac{4\pi}{15}\left(T_{\alpha\beta} T_{\gamma\delta} + T_{\alpha\gamma} T_{\beta\delta} + T_{\alpha\delta} T_{\beta\gamma}\right)$$

$$+ \left(\frac{4\pi}{5} - 3\left(\frac{4\pi}{15}\right)\right) T_{\alpha\beta\gamma\delta}$$

$$= \frac{4\pi}{15}\left(\left(g_{\alpha\beta} + \frac{v_\alpha v_\beta}{c^2}\right)\left(g_{\gamma\delta} + \frac{v_\gamma v_\delta}{c^2}\right)\right.$$

$$\left. + \left(g_{\alpha\gamma} + \frac{v_\alpha v_\gamma}{c^2}\right)\left(g_{\beta\delta} + \frac{v_\beta v_\delta}{c^2}\right) + \left(g_{\alpha\delta} + \frac{v_\alpha v_\delta}{c^2}\right)\left(g_{\beta\gamma} + \frac{v_\beta v_\gamma}{c^2}\right)\right)$$

Version 4: $\int u_\alpha u_\beta u_\gamma u_\delta u_\epsilon u_\xi \, d\Omega$

Case 1 for non-zero: 3 different pairs exist, no indices = 4

$$\int u_\alpha u_\beta u_\gamma u_\delta u_\epsilon u_\xi \, d\Omega = \int_0^\pi \sin^5\theta \cos^2\theta \, d\theta \int_0^{2\pi} \sin^2\phi \cos^2\phi \, d\phi = \frac{4\pi}{105}$$

Case 2 for non-zero: one pair, one group of four, none = 4

$$\int u_\alpha u_\beta u_\gamma u_\delta u_\epsilon u_\xi \, d\Omega = \int_0^\pi \sin^3\theta \cdot \cos^4\theta \, d\theta \int_0^{2\pi} \cos^2\phi \, d\phi = \frac{4\pi}{35}$$

Case 3 for non-zero: all indices equal, $\neq 4$

$$\int u_\alpha u_\beta u_\gamma u_\delta u_\epsilon u_\xi \, d\Omega = \int_0^\pi \cos^6\theta \sin\theta \, d\theta \int_0^{2\pi} d\phi = \frac{4\pi}{7}$$

So in general,

$$\begin{aligned}\int u_\alpha u_\beta u_\gamma u_\delta u_\epsilon u_\xi \, d\Omega = \frac{4\pi}{105}\Big[& T_{\alpha\beta}\left(T_{\gamma\delta}T_{\epsilon\xi} + T_{\gamma\epsilon}T_{\delta\xi} + T_{\gamma\xi}T_{\delta\epsilon}\right) \\ & + T_{\alpha\gamma}\left(T_{\beta\delta}T_{\epsilon\xi} + T_{\beta\epsilon}T_{\delta\xi} + T_{\beta\xi}T_{\delta\epsilon}\right) \\ & + T_{\alpha\delta}\left(T_{\beta\gamma}T_{\epsilon\xi} + T_{\beta\epsilon}T_{\gamma\xi} + T_{\beta\xi}T_{\gamma\epsilon}\right) \\ & + T_{\alpha\epsilon}\left(T_{\beta\gamma}T_{\delta\xi} + T_{\beta\delta}T_{\gamma\xi} + T_{\beta\xi}T_{\gamma\delta}\right) \\ & + T_{\alpha\xi}\left(T_{\beta\gamma}T_{\delta\epsilon} + T_{\beta\delta}T_{\gamma\epsilon} + T_{\beta\epsilon}T_{\gamma\delta}\right)\Big] \\ & + \left(\frac{4\pi}{35} - 3\left(\frac{4\pi}{105}\right)\right)\Big[T_{\alpha\beta}T_{\gamma\delta\epsilon\xi} + T_{\alpha\gamma}T_{\beta\delta\epsilon\xi}\end{aligned}$$

$$+ T_{\alpha\delta} T_{\beta\gamma\epsilon\zeta} + T_{\alpha\epsilon} T_{\beta\gamma\delta\zeta}$$

$$+ T_{\alpha\zeta} T_{\beta\gamma\delta\epsilon} + T_{\beta\gamma} T_{\alpha\delta\epsilon\zeta}$$

$$+ T_{\beta\delta} T_{\alpha\gamma\epsilon\zeta} + T_{\beta\epsilon} T_{\alpha\gamma\delta\zeta}$$

$$+ T_{\beta\zeta} T_{\alpha\gamma\delta\epsilon} + T_{\gamma\delta} T_{\alpha\beta\epsilon\zeta}$$

$$+ T_{\gamma\epsilon} T_{\alpha\beta\delta\zeta} + T_{\gamma\zeta} T_{\alpha\beta\delta\epsilon}$$

$$+ T_{\delta\epsilon} T_{\alpha\beta\gamma\zeta} + T_{\delta\zeta} T_{\alpha\beta\gamma\epsilon}$$

$$+ T_{\epsilon\zeta} T_{\alpha\beta\gamma\delta} \Big]$$

$$+ \left(\frac{4\pi}{7} - 15\left(\frac{4\pi}{35}\right) + 30\left(\frac{4\pi}{105}\right)\right) T_{\alpha\beta\gamma\delta\epsilon\zeta}$$

$$= \frac{4\pi}{105}\left[\left(g_{\alpha\beta} + \frac{v_\alpha v_\beta}{c^2}\right)\left\{\left(g_{\gamma\delta} + \frac{v_\gamma v_\delta}{c^2}\right)\left(g_{\epsilon\zeta} + \frac{v_\epsilon v_\zeta}{c^2}\right)\right.\right.$$

$$\left. + \left(g_{\gamma\epsilon} + \frac{v_\gamma v_\epsilon}{c^2}\right)\left(g_{\delta\zeta} + \frac{v_\delta v_\zeta}{c^2}\right) + \left(g_{\gamma\zeta} + \frac{v_\gamma v_\zeta}{c^2}\right)\left(g_{\delta\epsilon} + \frac{v_\delta v_\epsilon}{c^2}\right)\right\}$$

$$+ \left(g_{\alpha\gamma} + \frac{v_\alpha v_\gamma}{c^2}\right)\left\{\left(g_{\beta\delta} + \frac{v_\beta v_\delta}{c^2}\right)\left(g_{\epsilon\zeta} + \frac{v_\epsilon v_\zeta}{c^2}\right)\right.$$

$$\left. + \left(g_{\beta\epsilon} + \frac{v_\beta v_\epsilon}{c^2}\right)\left(g_{\delta\zeta} + \frac{v_\delta v_\zeta}{c^2}\right) + \left(g_{\beta\zeta} + \frac{v_\beta v_\zeta}{c^2}\right)\left(g_{\delta\epsilon} + \frac{v_\delta v_\epsilon}{c^2}\right)\right\}$$

$$+\left(g_{\alpha\delta}+\frac{v_\alpha v_\delta}{c^2}\right)\left\{\left(g_{\beta\gamma}+\frac{v_\beta v_\gamma}{c^2}\right)\left(g_{\epsilon\zeta}+\frac{v_\epsilon v_\zeta}{c^2}\right)\right.$$

$$\left.+\left(g_{\beta\epsilon}+\frac{v_\beta v_\epsilon}{c^2}\right)\left(g_{\gamma\zeta}+\frac{v_\gamma v_\zeta}{c^2}\right)+\left(g_{\beta\zeta}+\frac{v_\beta v_\zeta}{c^2}\right)\left(g_{\gamma\epsilon}+\frac{v_\gamma v_\epsilon}{c^2}\right)\right\}$$

$$+\left(g_{\alpha\epsilon}+\frac{v_\alpha v_\epsilon}{c^2}\right)\left\{\left(g_{\beta\gamma}+\frac{v_\beta v_\gamma}{c^2}\right)\left(g_{\delta\zeta}+\frac{v_\delta v_\zeta}{c^2}\right)\right.$$

$$\left.+\left(g_{\beta\delta}+\frac{v_\beta v_\delta}{c^2}\right)\left(g_{\gamma\zeta}+\frac{v_\gamma v_\zeta}{c^2}\right)+\left(g_{\beta\zeta}+\frac{v_\beta v_\zeta}{c^2}\right)\left(g_{\gamma\delta}+\frac{v_\gamma v_\delta}{c^2}\right)\right\}$$

$$+\left(g_{\alpha\zeta}+\frac{v_\alpha v_\zeta}{c^2}\right)\left\{\left(g_{\beta\gamma}+\frac{v_\beta v_\gamma}{c^2}\right)\left(g_{\delta\epsilon}+\frac{v_\delta v_\epsilon}{c^2}\right)\right.$$

$$\left.\left.+\left(g_{\beta\delta}+\frac{v_\beta v_\delta}{c^2}\right)\left(g_{\gamma\epsilon}+\frac{v_\gamma v_\epsilon}{c^2}\right)+\left(g_{\beta\epsilon}+\frac{v_\beta v_\epsilon}{c^2}\right)\left(g_{\gamma\delta}+\frac{v_\gamma v_\delta}{c^2}\right)\right\}\right]$$

APPENDIX B. IDENTITIES

Numerous identities are used in this work, many of them stemming from the identity $M^{\mu\nu}v_{\nu} = 0$, which is true because the dipole moment under consideration is purely magnetic. Some of the more commonly used identities are presented below.

(B.1) $$M^{\mu\nu} v_{\nu} = 0$$

(B.2) $$\frac{d}{d\tau}\left(M^{\mu\nu} v_{\nu}\right) = M^{\mu\nu} a_{\nu} + \dot{M}^{\mu\nu} v_{\nu} = 0$$

$$\Rightarrow M^{\mu\nu} a_{\nu} = -\dot{M}^{\mu\nu} v_{\nu}$$

(B.3) $$\frac{d^2}{d\tau^2}\left(M^{\mu\nu} v_{\nu}\right) = M^{\mu\nu} \dot{a}_{\nu} + 2\dot{M}^{\mu\nu} a_{\nu} + \ddot{M}^{\mu\nu} v_{\nu} = 0$$

$$\Rightarrow M^{\mu\nu} a_{\mu} \dot{a}_{\nu} + \ddot{M}^{\mu\nu} a_{\mu} v_{\nu} = 0$$

$$\Rightarrow M^{\mu\nu} a_{\mu} \dot{a}_{\nu} = \ddot{M}^{\mu\nu} v_{\mu} a_{\nu}$$

(B.4) $$\frac{d^3}{d\tau^3}\left(M^{\mu\nu} v_{\nu}\right) = M^{\mu\nu} \ddot{a}_{\nu} + 3\dot{M}^{\mu\nu} \dot{a}_{\nu} + 3\ddot{M}^{\mu\nu} a_{\nu} + \dddot{M}^{\mu\nu} v_{\nu} = 0$$

$$\Rightarrow M^{\mu\alpha} a_{\mu} \ddot{a}_{\alpha} + 3\dot{M}^{\mu\alpha} a_{\mu} \dot{a}_{\alpha} + \dddot{M}^{\mu\alpha} a_{\mu} v_{\alpha} = 0$$

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