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THE APPLICATION OF CAPITAL MARKET EQUILIBRIUM THEORY AND MATHEMATICAL PROGRAMMING TO THE CAPITAL BUDGETING PROBLEM UNDER UNCERTAINTY

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THE APPLICATION OF CAPITAL MARKET EQUILIBRIUM THEORY AND MATHEMATICAL PROGRAMMING TO THE CAPITAL BUDGETING PROBLEM UNDER UNCERTAINTY

APPROVED BY

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ABSTRACT

Stock price equilibrium theory and mathematical programming formulation of the capital budgeting problem are reviewed. These two areas are then combined into a unified approach to the capital budgeting problem under uncertainty and culminating with a specific formulation. This formulation seeks to optimize the equilibrium stock price of the company where capital assets are being chosen via mathematical programming. In so doing, the risks associated with covariability with the market portfolio are accounted for automatically. Furthermore, the risks associated with variability of the costs of projects undertaken are explicitly incorporated into the model by computing the expected cost of that variability in terms of its effect upon the equilibrium stock price. The resulting formulation is a mixed integer non-linear programming problem.

Solution procedures developed by Geoffrion as an extension of Benders' earlier work on a decomposition algorithm are discussed since modifications to the budgeting problem makes it a candidate for solution by Generalized Benders Decomposition Algorithm. It is proved that a global optimum may be found by these solution procedures, and the requisite modifications are specified.

Several test problems were constructed and solved by computer using these procedures. The results of the test problems are given and some conclusions are drawn concerning the ability to solve larger problems. Some solution strategies intended to decrease solution difficulties are also discussed.

Sensitivity analysis procedures are specified with some examples of their use and interpretation relative to improving the overall capital budget by changing the rationing of capital.

Finally, examples of alternative formulations that still resemble closely the original specific problems are developed out of changes in the framework of assumptions under which the formulations are applicable to real problems. This topic is then discussed in connection with several suggestions for further research.

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CHAPTER I

INTRODUCTION

The capital budgeting problem referred to hereafter is a generalization of the classic problem originally proposed by Lorie and Savage [27]. It may be stated in general terms as follows:

Management has developed a number of investment opportunities which the firm may undertake providing it possesses or can obtain sufficient resources to support the undertaking. The capital budgeting problem consists of selecting the best combination of investments that comprises some subset of all those available. This selection process is subject to the constraint that the resources required to undertake the selected combination of investments does not exceed those resources made available for such purposes.

The word "best" used to describe a combination of investments is at best a rather obtuse description. However, such a word is necessary in a general statement of a problem to allow an individual formulating a means of solving such a problem the freedom to select his own appropriate definition of "the best combination of investments."

For example, Weingartner [21] chooses to maximize the total of the net present values for all the investments undertaken. An alternative is to maximize the internal rate of return earned on all invested funds, and if one wished to consider the returns as uncertain then the maximum of the total of the expected utilities for all the investments undertaken might be the desired description of the "best combination of investments." This point will be illustrated further in Chapters II and III where discussions of the work done in the area of Portfolio Theory[1] [32 , 26 , 40 , 41] and the applications of mathematical programming techniques to the capital budgeting problem are found. Then in Chapter IV one finds a synthesis of both these areas into a unified approach to the capital budgeting problem under uncertainty, particularly in the development of various definitions of the "best combination of investments."

Portfolio Analysis and Capital Budgeting

In 1952, Harry Markovitz [32] suggested a means of selecting a portfolio of securities that treated risk as a variable to be contended with instead of ignoring it as was commonly done at that time. His work launched two decades of voluminous research into the problems of risk measurement, portfolio selection and performance, market

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[1]The central issues in portfolio theory are the portfolio selection problem and market equilibrium theory, both of which are discussed in Chapter II.

behavior, and various extensions.

Major contributions were made by William F. Sharpe [42] when he used regression theory to reduce the computational effort required to select portfolios by the original Markowitz model. In addition, Sharpe [41], in 1964, and Lintner [26], in 1965, introduced their versions of a capital markets equilibrium theory. These two approaches were later shown to be equivalent by Fama [11]. A summary of these works is found in Chapter II with a discussion of the implications of some of the empirical tests of these theories. The relevance of these theories to the capital budgeting problems of a firm will be fully developed in later chapters, but for now it is sufficient to state that equilibrium theory gives one a relationship between expected return and a measure of risk. If a firm's management makes capital budgeting decisions that somehow upset that relationship, then the market price of the shares of the firm must automatically adjust. The objective, then, is to specify a model which translates a capital budgeting decision into an indication of market pressure on share prices; management can then select assets that will cause the market price of the firm's equity to behave in a desired manner.

Other Valuation Models and Their Exclusion

If one assumes that an investor analyzes financial data concerning a firm in order to determine how much he is willing to pay for a share of the equity of that firm, then there are two distinctively different approaches to defining mathematical relationships in hopes of gaining insight into how the investor arrives at his final decision of price. Both approaches evolved almost simultaneously over the last twenty years.

One approach makes no attempt to determine exactly how this financial data is utilized or even which data is relevant but merely assumes that somehow, based upon available information, the investor is able to formulate (for each stock considered for investment) expectations concerning end of period wealth relatives, variance of these relatives, and covariance of the relatives for every pair of stocks. He then makes his investment decisions based upon his expectations and using one of the portfolio selection models mentioned above. In general, the step of formulating expectations from financial data has been shown to be unnecessary. For example, in a comprehensive empirical test (by Cohen and Pogue [8]) of four portfolio selection models, the performance of portfolios selected on the basis of *ex post* price data were compared, on the basis of *ex ante* data, to each other, to mutual funds, and to randomly selected portfolios. It was not possible to distinguish between the performance of the portfolios selected by the four models, and these portfolios did as well or better than mutual funds and significantly better than the random portfolios. Hence, the basically technical approach which did not require formulation of expectations from financial data appears to work at least as well as the supposedly more fundamental approach of the mutual funds.

The alternative approach, on the other hand, attempts to do exactly what the portfolio models do not accomplish. That is, they seek to identify those financial variables that have a significant effect on the value of the equity of the firm or on its cost of capital and then they attempt to specify a mathematical model which combines those variables in the way investors do in order to evaluate the equity (and/or cost of capital). A number of these models have been developed by Modigliani and Miller [35], Gordon [17], and Lerner and Carleton [24]. It would seem that these models are most relevant to the capital budgeting problem since proforma balance sheets and income statements could be constructed from the expected cash flows generated by proposed capital investments. This provides the necessary expected financial data (and perhaps estimates of the variance of that data) which when applied to one of the valuation models yields an evaluation of these proposed capital expenditures in terms of a favorable or unfavorable change in the equity evaluation or cost of capital. Unfortunately, these equity valuation models have not faired as well under the scrutiny of empirical tests as the portfolio models have. Keenan [23] explained this poor performance by citing a number of difficulties:

1. Financial variables used are of necessity those readily obtainable from company issued financial statements. In particular, balance sheets and income statements, which do not necessarily give a true indication of the firm's actual state of being nor even a state of being once in existence. [23, p. 257]

2. Models are usually constructed so that model parameters are estimatable by least-squares regression techniques. This means that to prevent bias, firms with anomalies in their data are normally excluded from any samples, and that models may be specified in a manner not representative of how investors actually evaluate the financial data. [23, p. 258]

These problems have resulted in estimated coefficients of financial variables that (1) are not significantly different than zero, or (2) are not stable from sample to sample, or (3) are not stable over time [23, p. 243]. Keenan concludes that aside from the theoretical contributions of these models about all that can be shown from the great volume of empirical research is that there is some relationship between equity value and earnings, dividends, retained earnings, growth parameters, capital gain and size (of the firm) although the nature and magnitude of that relationship is as yet unknown [23, p. 244].

Mathematical Programming and Capital Budgeting

The relationship between the capital budgeting problem and the value of a firm's equity has not been explicitly incorporated into the current mathematical programming formulations to solve the capital budgeting problem. By ignoring such a relationship there is an implicit assumption that is common to all such formulations.

That assumption is that management, acting as agents and in the best interest of the firm's owners, is able to apply their judgment and experience to determine the correct rationing of resources so that when the capital budgeting problem is solved, the resulting solution will satisfy the owners. When risk is considered in the problem formulation the assumption is extended to include assuming that management's method of handling risk is a good surrogate for the owner's attitudes towards risk. No attempt will be made to refute these assumptions, however, a formulation will be given in Chapter IV which will deemphasize their necessity. The remaining chapters discuss data requirements for the formulation in Chapter IV, and report on the solution procedures developed and tested with the final chapter mentioning possible extensions and further research.

CHAPTER II

PORTFOLIO AND EQUILIBRIUM THEORY

Portfolio Analysis

One may think of the capital budgeting problem as a portfolio selection problem with some restrictions placed upon the divisibility of assets. With this idea in mind, the portfolio selection problem with infinite divisibility of assets allowed shall be reviewed.

One may suppose that an investor has a total of H dollars to invest in a portfolio of securities and/or government bonds or other risk-free assets, and that the investor is a risk averter and is able to make a choice between alternative portfolios based upon the expected one period return and standard deviation of that return on each alternative. There are some situations where a choice is obvious for risk averters.[1] These situations are as follows:

1. Expected return for two portfolios is the same, but their standard deviations are different. A risk averter would choose the portfolio with the smallest

[1]See page 41 for the conditions under which the choices are obvious.

standard deviation.

2. Expected returns are different, but standard deviations are the same. A risk averter would choose the portfolio with the larger expected return.
3. One portfolio has both a higher expected return and a lower standard deviation than the other portfolio. A risk averter would choose the portfolio with the higher return.

The only instance in which a choice is not clear is when one portfolio has both a higher expected return and a higher standard deviation than the other portfolio. Knowledge of the specific preference or utility function of the risk averter is required to make the choice in this case.

For every possible portfolio, either there exists another portfolio which is a clear or dominant choice over it by virtue of the existence of one of the three situations listed above or no such dominating choice exists. If a portfolio is such that no other portfolio dominates it, then Markowitz [32, p. 81] would refer to this portfolio as efficient and the entire set of efficient portfolios as the efficient frontier. The portfolio selection problem, then, consists of selecting the "best" portfolio from the efficient set of portfolios where "best" is determined by the individual's preference function.

A graph of portfolios consisting of only risky assets is illustrated by Figure 2-1. The heavy dark line represents the efficient frontier and proof of the convexity of the region near the frontier is found in Sharpe [40, p. 52].

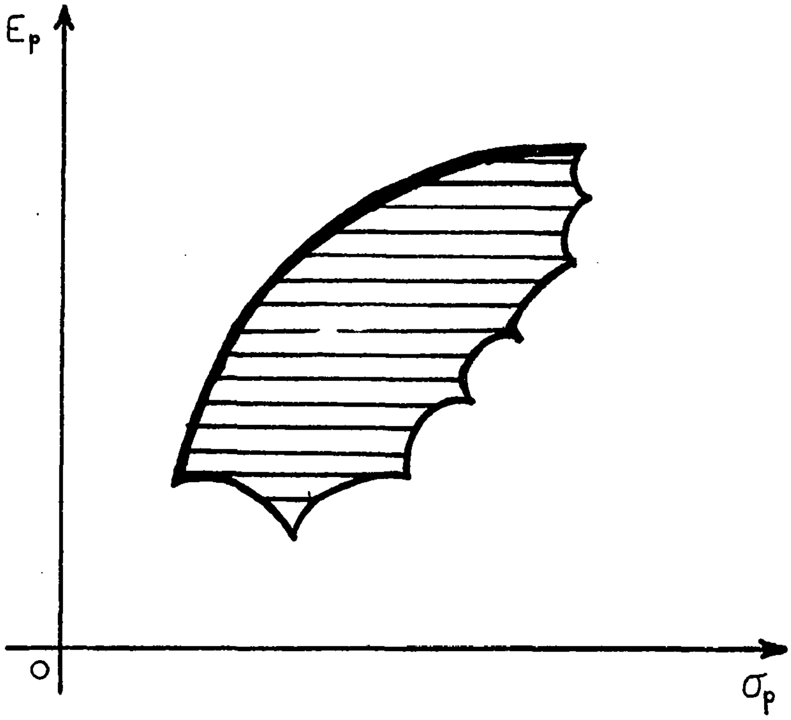

Figure 2-1. Portfolio set plotted on expected return (E_p)--standard deviation (σ_p) coordinate system.

If one assumes that the investor can invest any part of his capital in a risk-free venture returning rate r_f or can borrow any amount at rate r_f and invest the borrowed funds in a portfolio of risky assets, then the entire set of possible portfolios will change to that illustrated in Figure 2-2. The upper heavy line represents the new efficient frontier. It can be shown that any point on the new efficient frontier can be achieved by an investment in

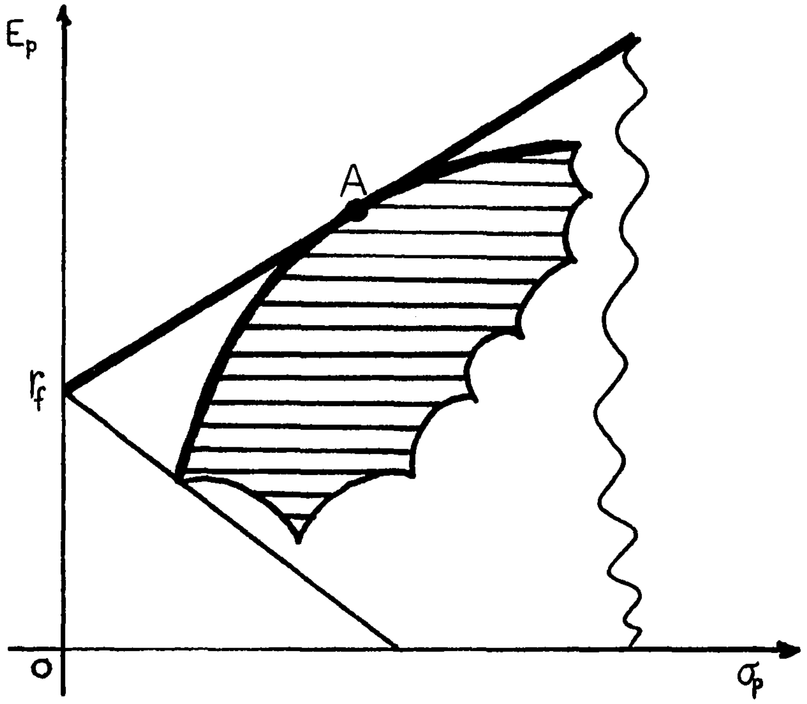

Figure 2-2. Portfolio set modified by inclusion of the risk-free asset.

some linear combination of the risk-free asset and the portfolio, A, found at the point of tangency of the old and new efficient frontiers. This new, linear efficient set is really the only efficient frontier and any risk averter, regardless of the degree of risk aversion, should choose some point contained in this new efficient set. This means that every investor should invest some part of his funds in tangencial portfolio A in Figure 2-2, and the remaining portion of his funds in the risk-free asset. The exact proportions to be invested in each is determined by the individual's preference function. This independence of the problems of selecting the portfolio of risky assets and deciding how much to invest in the risky portfolio is called the separation theorem [40, p. 70] and requires the assumption of equal rates for borrowing and lending to be true. Of course, only if investors completely agree with each other about estimates of return and variance will the tangencial portfolio A be exactly the same portfolio for all investors.

Solving the Portfolio Selection Problem

If one is to select a portfolio for an individual investor he must obtain estimates of the expected return for each security, estimates of the variances of the returns, and estimates of the covariances of every pair of returns. For simplicity, it is assumed that every risky security must be held long if at all. x_i is defined as the proportion of security i held, $\bar{r}_i$ the expected return on security i and σ_{ij} the covariance of r_i and r_j for $i \neq j$ and variance of r_i when $i = j$. Then the expected return on any portfolio p is given by:

$$\bar{r}_p = \sum_{i=1}^{n} x_i \bar{r}_i$$

where:

n -- the total number of securities.

The standard deviation of the portfolio is given by:

$$\sigma_p = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{ij}} \ .$$

An efficient portfolio for any level of return r_o may be found by solving Problem (2.1) below:

$$\begin{aligned} &\min_{x_i} \sigma_p \\ &\text{s.t. } \bar{r}_p \geq r_o \\ &\Sigma x_i = 1 \\ &x_i \geq 0 \quad \forall i \end{aligned} \tag{2.1}$$

or

$$\begin{aligned} &\max_{x_i} \bar{r}_p \\ &\text{s.t. } \sigma_p \leq \sigma_o \\ &\Sigma x_i = 1 \\ &x_i \geq 0 \quad \forall i \end{aligned} \tag{2.2}$$

(2.2) will find an efficient portfolio for any given level of risk σ_o. If the riskless asset is considered in either of the above problems, say i = 1 is the riskless asset, then clearly $r_1 = r_f$; $\sigma_{11} = 0$; $\sigma_{1j} = 0$ for all j. Since borrowing is allowed then the riskless asset may be thought of as shorted and x_1 is unrestricted in the above two problems. Solutions to either Problem (2.1) or Problem (2.2), with the risk free asset included, will always yield points on the linear efficient frontier in Figure 2. Hence, for any σ_o or r_o selected, the risky portfolio may be determined by:

$$x_i' = \frac{x_i^*}{\sum_{i=2}^{n} x_i^*} \quad \text{for } i = 2, 3, \ldots, n$$

where:

x_i' = the proportions of those funds invested in risky assets that are invested in security i, and

x_i^* = an element of the optimal solution to Problem (2.1) or (2.2).

Portfolio Analysis and Capital Budgeting Problem

Portfolio analysis, looked upon as a rational technique for analyzing and selecting an optimal set of risky investments seems a logical procedure for handling risk in the capital budgeting problem. Those taking this approach are Cord [9], VanHorne [48], Mao and Brewster [30], Levy and Sarnat [25], Paine [37] and others.

Some modifications are required. For example, one period rates of return are generally not applicable to the capital budgeting problem, since periods are generally short compared to the length of time a capital project may effect the cash flows of the firm. Therefore, expected internal rates of return and their variances and covariances are sometimes used while if some rate is known then expected net present values and their variances and covariances may be used to avoid the problem of multiple internal rates of return on some projects. Since projects are often not infinitely divisible and cannot be undertaken in greater quantity than their maximum level, then decision variables are frequently restricted to integer values and even more frequently binary 0-1 variables. Funds generally cannot be borrowed or lent in unlimited quantities nor at the same rate and capital is usually rationed. All these differences combine to make the conceptually simple portfolio approach a much more difficult problem. For example, lack of divisibility and the restriction upon borrowing and lending combine to destroy the separation theorem. This means that no single portfolio of risky assets solves the problem independently of preference functions as did portfolio A in the previous section. Therefore, portfolio analysis applied to capital budgeting requires knowledge of someone's utility function. But who's? The president of the company's? The capital budgeting committee's? An aggregate utility function for all stockholders? One can see some of the practical problems with this approach. If one assumes that an appropriate utility function is known and an indifference function for expected return and standard deviation constructed, then such functions are usually nonlinear. The resulting mathematical programming problem is nonlinear and integer and may require a number of solutions to determine the entire set of efficient portfolios of assets. Such a problem may not be easy to solve and quite likely will be expensive to solve. Of course, the portfolio approach just discussed does not represent the only means of handling risk in the capital budgeting problem, but further discussion of these techniques and others will be deferred to the next chapter.

Equilibrium Theory

As mentioned earlier, under certain conditions a single portfolio of risky assets is optimal for risk adverse investors. If all investors agree about predicted expected returns, variances and covariances, then all investors seeking to place some funds in a risky portfolio will seek to obtain the same portfolio. In particular, that portfolio should be optimal, if the investor is rational, and is portfolio A in Figure 2. Using the procedures discussed earlier, A may be found given all estimated expected returns, variances and covariances. However, a useful technique is to apply some necessary and sufficient conditions for optimality of some problem used to determine portfolio A. This yields general relationships between estimated parameters. These relationships must, of necessity, hold in order for the market to be in equilibrium. To clarify, if all investors place their money in risky securities in the proportions suggested by the description of some portfolio A, then the aggregate market value of shares of any security would have the same ratio to the total value of all shares of all securities that is suggested by portfolio A. However, market values are determined in the market place and not by the solution of some mathematical programming problem. One may, therefore, define the market portfolio as that portfolio where the proportion of total portfolio value invested in security i is equal to the proportion of total market value of all securities to the aggregate market value of security i. If this portfolio does not satisfy the necessary conditions for optimality based upon actual investor expectation (i.e., market portfolio is not equal to portfolio A), then the rational investor would be expected to alter his holdings of risky assets until optimality was indicated. Hence, as long as the market portfolio is not equal to portfolio A, alterations of holdings should be occurring which defines a state of disequilibrium. Equilibrium, then, implies that the market portfolio is optimal and necessary conditions for optimality describe the apparent relationships between expected returns, variances, and covariances. Naturally, if expenses are incurred by investors who adjust their portfolios then an optimal policy might be to accept a slightly less optimal portfolio and avoid the expenses of change. Hence, the foregoing explanations presume the absence of transaction costs, taxes, and differences in borrowing and lending rates in addition to the assumed agreement among investors about future expectations already stated.

Basically there are two equilibrium models which differ primarily by development. Lintner's model [26], described below, is developed under the assumptions stated above, as is Sharpe's model [40, 41], discussed later. Lintner approaches the problem by letting θ be the slope of a line determined by the points $(0, r_f)$ and $(\sigma_B, \bar{r}_B)$ where B is any feasible portfolio [26]. Clearly, if one finds a portfolio B such that θ is maximized then B = A and is optimal. Lintner's equilibrium conditions are derived from the necessary conditions for θ maximum. Hence,

$$\theta = \frac{\bar{r}_B - r_f}{\sigma_B} = \frac{\bar{x}}{\sigma_B}$$

where:

$\bar{x}$ = average return on the portfolio in excess of the risk-free rate.

Then for every portfolio with each security i making up a proportion h_i of the total portfolio value, θ becomes:

$$\theta = \frac{\Sigma\, h_i \bar{x}_i}{\sqrt{\sum_i \sum_j h_i h_j \sigma_{ij}}}$$

where:

$\bar{x}_i$ = average excess return (over the risk-free rate) of security i

σ_{ij} = $\mathrm{cov}(x_i, x_j)$ $i \neq j$

σ_{ii} = $\mathrm{var}(x_i)$

If short sales are allowed (i.e., h_i may be less than 0), then h_i may be considered unrestricted for further analysis. Hence:

$$\frac{\partial\theta}{\partial h_i} = \frac{1}{\sqrt{\sum_i \sum_j h_i h_j \sigma_{ij}}} \bar{x}_i - \lambda(h_i\sigma_{ii} + \Sigma h_j\sigma_{ij}) \qquad (2.3)$$

where:

$$\lambda = \frac{\Sigma h_i \bar{x}_i}{\Sigma \Sigma h_i h_j \sigma_{ij}}$$

Letting $z_i = \lambda h_i$, (2.3) becomes:

$$z_i\sigma_{ii} + \sum_j z_j\sigma_{ij} = \bar{x}_i \quad i = 1,2, \ldots, m \qquad (2.4)$$

when set to zero for each i.

Since the matrix (σ_{ij}) is positive definite and θ is a homogeneous function of order zero in the h_i, then equations (2.4) are the necessary and sufficient conditions on the relative values of h_i for a unique global maximum. Hence, since the h_i are to be proportions, $\Sigma |h_i| = 1$ is a requirement. Let h_i' be such that $\Sigma |h_i'| = 1$ then $h_i' = \frac{h_i}{\Sigma |h_i|} = \frac{\lambda h_i}{\lambda\Sigma |h_i|}$, but $\lambda h_i = z_i$ and $\lambda\Sigma |h_i| = \Sigma |\lambda h_i|$ for $\lambda \geq 0$.[2] Consequently, $\frac{\lambda h_i}{\lambda\Sigma |h_i|} = \frac{z_i}{\Sigma |z_i|} = h_i'$.

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[2]Note that $\theta = \frac{\Sigma h_i \bar{x}_i}{\sqrt{\Sigma \Sigma h_i h_j \sigma_{ij}}}$ and $\lambda = \frac{\Sigma h_i \bar{x}_i}{\Sigma \Sigma h_i h_j \sigma_{ij}}$ then $\lambda < 0 \Leftrightarrow \theta < 0$ which implies that no risky portfolio achieves an expected return greater than the risk-free rate. This case may clearly be ruled out.

Therefore, the z_i values may be obtained from system (2.4), the h_i' values as indicated above, and λ by its definition and the h_i' values. However, for equilibrium theory actual solutions are not required. According to the discussion at the beginning of this section, the optimal portfolio (or one that gives max θ) is known and is the market portfolio under conditions of equilibrium. If one defines V_{oi} as the current aggregate market value of security i and T as the total market value of all securities, then the optimal and market portfolio contains a proportion of security i denoted h_i', by the previous notation, where $h_i' = \frac{V_{oi}}{T}$. If one supposes that at the end of the period a dividend of total value D_i is paid and the aggregate value of security i is V_{1i}, then the total expected return on security i is given by $V_{1i} - V_{oi} + D_i$. The excess return $\bar{x}_i$ is $\frac{R_i}{V_{oi}} - r_f$ and σ_{ij} becomes $\frac{1}{V_{oi}V_{oj}} \sigma_{ij}^*$ where σ_{ij}^* is defined as $\mathrm{cov}(R_i, R_j)$ when $i \neq j$ and as $\mathrm{var}(R_i)$ when $i = j$. Equation (2.4) for security i becomes:

$$\lambda \frac{V_{oi}}{T} \frac{1}{V_{oi}V_{oi}} \sigma_{ii}^* + \lambda \sum_j \frac{V_{oi}}{T} \frac{\sigma_{ij}^*}{V_{oi}V_{oj}} = \frac{R_i - r_f V_{oi}}{V_{oi}}$$

or $$(R_i - r_f V_{oi}) = (\sigma_{ii}^* + \sum_j \sigma_{ij}^*) \frac{\lambda}{T}$$

Hence, $$V_{oi} = \frac{R_i - \frac{\lambda}{T}(\sigma_{ii}^* + \sum_j \sigma_{ij}^*)}{r_f}$$

Letting H_i be the expected wealth relative for V_{oi}, one obtains $H_i = V_{1i} + D_i = R_i + V_{oi}$

and
$$V_{oi} = \frac{H_i - V_{oi} - \frac{\lambda}{T}(\sigma_{ii}{}^* + \sum_j \sigma_{ij}{}^*)}{r_f{}^*}$$

or
$$V_{oi}r_f{}^* + V_{oi} = H_i - \frac{\lambda}{T}(\sigma_{ii}{}^* + \sum_j \sigma_{ij}{}^*)$$

$$V_{oi} = \frac{H_i - \frac{\lambda}{T}(\sigma_{ii}{}^* + \sum_j \sigma_{ij}{}^*)}{1 + r_f} \tag{2.5}$$

Lintner then concluded that the current market value of security i is given by the present value (determined at the risk-free rate) of a certainty equivalent for the one period expected wealth relative where the certainty equivalent is determined by reducing the expected wealth relative by an amount dependent upon its risk $[(\sigma_{ii}{}^* + \Sigma\, \sigma_{ij}{}^*)]$ and the market cost of risk $\frac{\lambda}{T}$ which is common to all securities [26, pp. 26, 27].

Equilibrium Theory and the Capital Budgeting Problem

In the same article in which Lintner developed his equilibrium theory [26], he also applied the theory to the corporate problem of selecting projects in which to invest. But first, a number of simplifying assumptions were made. In particular, all assumptions listed earlier in the development of the equilibrium theory terminating with equation (2.5) continued in effect. In addition, Lintner assumed that the investment opportunities available to the company in any time period was independent of the size and selection of projects made in other periods; and that there was not limited liability to corporate stock, nor any institutional or legal restriction on the investment purview of any investor, and that everyone expects the riskless rate r_f to remain constant over time. With all of these restrictions Lintner claimed that:

> These conditions make the present values of the cash flows to any company from its real (and financial) assets and operations equal to the total market value of investors' claims to these flows, i.e., to the sum of aggregate market value of its common (and preferred) stock outstanding and its borrowings (debt). [26, p. 28]

and that the assumptions made are sufficient to establish the Modigliani and Miller Propositions I and II [26, p. 28]. This means the investors will be indifferent to the financing decisions necessary in selecting a set of projects, and hence that capital rationing is not necessary. If, then, a current capital budgeting decision is made that is expected to change the risk-return characteristic of the firm, the current equilibrium aggregate value of the firm's stock should adjust to some new value resulting in a change ΔV_{oi} given below by modification of equation (2.5).

$$\Delta V_{oi} = \frac{\Delta H_i - \frac{\lambda}{T} \Delta (\sigma_{ii}{}^* + \Sigma \sigma_{ij}{}^*)}{1 + r_f} \qquad (2.6)$$

To simplify this expression, Lintner made three additional assumptions:

1. The aggregate market value of all other stocks is unaffected by the capital budgeting decisions of the firm under consideration.
2. σ_{ij}^* is, for all j, invariant of the capital budgeting decision of the i^{th} company.
3. The optimal portfolio of risky assets earns more than the risk-free rate. [26, pp. 28-29]

Lintner justified assumption (3) as obviously reasonable in a universe of risk adverse investors. He claimed assumption (1) was merely a convenience that involved ignoring (generally small) second-order feedback effects and assumption (2), he said, was plausible as a good first approximation [26, p. 29]. Note that Lintner did indicate "approximation" which implied that he did not believe that assumption (2) reflected reality. In order to determine just how good this approximation is one should consider the terms eliminated by assumption 2. Those terms were $\Sigma \Delta\sigma_{ij}^*$, or the sum of the changes in covariances with all other companies resulting from the capital budgeting decision. Surely one would expect that the magnitude of $\Sigma\Delta\sigma_{ij}^*$ in relation to $\Sigma \sigma_{ij}^*$ to be roughly comparable to the magnitude of $\Delta\sigma_{ii}^*$ in relation to σ_{ii}^*. However, Fama [11, p. 36] indicated that there is some evidence that σ_{ii}^* is trivial relative to $\Sigma \sigma_{ij}^*$. It appears, then, that Lintner, in making assumption 2, has discarded the important part of his expression (2.6). The alternative of retaining $\Sigma \Delta\sigma_{ij}^*$ is, perhaps, even more troublesome in terms of solving a real problem. None the less, given

Lintner's three assumptions as stated above, one obtains from equation (2.6):

$$\Delta V_{oi} = \frac{\Delta H_i - \frac{\lambda}{T}\Delta\sigma_{ii}{}^*}{1 + r_f} \tag{2.6a}$$

where:

ΔH_i = change in the expected present value at the end of the first period of cash inflows attributable to the acquired assets.

All present values are computed at the risk-free rate r_f. $\sigma_{ii}{}^*$ was previously given as the variance of R_i, but $H_i = R_i + V_{oi}$ so that $\mathrm{Var}(H_i) = \sigma_{ii}{}^*$. $\Delta\sigma_{ii}{}^*$ is the change in the variance of H_i induced by the new capital budget. It is clear that any set of projects which, if accepted, are such that $\Delta H_i - \frac{\lambda}{T}\Delta\sigma_{ii}{}^* > 0$, then the current value of the stock of the company is enhanced. Since capital need not be rationed, the firm would continue to accept projects until none are left that would increase the value of the stock. Lintner formulates the problem as a nonlinear optimization problem with bounded variables and applies the Kuhn-Tucker conditions to obtain some general results. However, in his own words,

> Perhaps at this point the reader should be reminded of the rather heroic set of simplifying assumptions which were made at the beginning of this section. One consequence of the unreality of these assumptions is, clearly, that the results are not being presented as directly applicable to practical decisions at this stage. [26, p. 32]

Lintner has not claimed a solution to practical capital budgeting problems under uncertainty. An approach has been specified which is seemingly valid and yet still requires development in the realm of reality. Lintner's work, then, provides the motivation and point of departure for this research and the resulting model presented in Chapter IV.

The Sharpe Equilibrium Model

Although the elegance of Lintner's development of his equilibrium model is certainly appealing, the simplicity of Sharpe's model, which is useful without making Lintner's assumption 2, makes it the choice for further application.

Sharpe's development [40] is similar to Lintner's in that he seeks to find a relationship that insures that the market portfolio (same as defined earlier) is at the point of tangency between a straight line through $(0, r_f)$ and the efficient frontier of portfolios composed only of risky securities. The resulting equilibrium relationship is given by:

$$\bar{r}_i = r_f + \mathrm{cov}(r_i, r_m) \frac{\bar{r}_m - r_f}{\sigma_{r_m}^2} \tag{2.7}$$

where:

r_m = the rate of return on the market portfolio.

r_i = the rate of return on stock i, or portfolio i, and expected values are denoted by a bar.

Empirical Evidence

Empirical tests of equilibrium theory are not currently plentiful, but some of those tests published offer encouragement with regard to equilibrium models reflecting reality despite the lack of reality in the assumptions required to develop it. For example, Irwin Friend and Marshall Blume [13] reported on a series of tests of single variable portfolio performance measures which combine expected return and risk into a single unit via equilibrium theory. Friend and Blume's study of one particular measure developed by Jensen [21] to measure the performance of mutual funds is pertinent to understanding the validity of Sharpe's model. Jensen's measure is given by:

$$\eta_i = r_f - \bar{r}_i + \frac{\mathrm{cov}(r_i, r_m)}{\sigma^2_{r_m}}[\bar{r}_m - r_f]$$

which is Equation (2.7) with the additional term η_i added. The index i is for the i^{th} portfolio or equivalent mutual fund i whose assets consist of a portfolio that is conveniently designated as the i^{th} portfolio. Clearly, if Sharpe's equilibrium model holds for individual securities, then it will hold for portfolios and the expected value of η_i is zero. If, however, the market imperfections that exist cause sufficient deviation from Sharpe's model, or a portfolio manager is able to recognize investment opportunities from stocks being underpriced due to a state of disequilibrium or he is able to make better estimates of $\bar{r}_i$, $cov(r_i,r_m)$, $\bar{r}_m$, and $\sigma_{r_m}^2$ than the general investment public; then $E(\eta_i) > 0$ and significant for that particular mutual fund. Jensen found that mutual funds were generally unable to produce significantly positive η_i. However, Friend and Blume's study conducted on Jensen's measure η_i which resulted in an indirect test of Sharpe's equilibrium model is, as stated earlier, more important to this research. Their test consisted of picking random portfolios and obtaining a value η_i which was then regressed on two measures of risk. If η_i is constant with respect to risk, then the equilibrium model fully accounts for how investors regard risk. The opposite was the case, for η_i was found to vary inversely with both risk measures $(cov(r_i,r_m)/\sigma_{r_m}^2;\sigma_{r_i}^2)$ with the relationship being highly significant and linear [13, p. 565]. Friend and Blume concluded that the only discrepancy between reality and the assumptions used to develop equilibrium theory that could account for the bias they found was that borrowing and lending rates are not actually the same [13, p. 569].

In addition to these empirical findings, some work has been done on an assumption used by Sharpe [42], Stapleton [44] and others that simplifies much of the computation with regard to portfolio analysis. This assumption is basically that all securities are related to each other only through each security's individual relationship with a common market effect, i.e., it is assumed that the rate of return on the i^{th} security (r_i) is given by:

$$r_i = \alpha_i + \beta_i I + \epsilon_i$$

where:

$E(\epsilon_i) = 0$

$\mathrm{cov}(I, \epsilon_i) = 0$

$\mathrm{cov}(\epsilon_i, \epsilon_j) = 0 \quad i \neq j$

and where I is an index value of the common market effect. Hence, given this regression equation for each security i simplifies the determination of three important parameters for portfolio analysis. Specifically,

$$\mathrm{cov}(r_i, r_j) = \beta_i \beta_j \sigma_I^2$$

$$\mathrm{var}(r_i) = \beta_i^2 \sigma_I^2 + \sigma_{\epsilon_i}^2$$

and

$$\mathrm{var}(r_p) = \beta_p^2 \sigma_I^2 + \sum_{i=1}^{n} x_i^2 \sigma_{\epsilon_i}^2$$

where r_p is the return on a portfolio composed of proportion x_i of security i and where

$$\beta_p = x_1\beta_1 + x_2\beta_2 + \dots + x_n\beta_n$$

The result of such an assumption is that the covariances implied by this assumption understate the actual covariances. Tests by Cohen and Progue [8] indicated, however, that efficient portfolios selected on the basis of expected return, variances, and covariances estimated from _ex post_ data were equally efficient regardless whether covariances were direct estimates or implied estimates based upon the common index assumption [8, p. 189].

It is not the purpose of this research to validate equilibrium theories nor provide a comprehensive survey of all such validation procedures published. Their mention here has been a means of introducing equilibrium theory which has an integral part in the problem formulation in Chapter IV. The empirical studies serve to provide some valuable information about, at least, Sharpe's equilibrium model so that problems of its use in some practical endeavor may be better understood. It appears that the greatest practical difficulty is the bias of the Sharpe model but which also appears linear and dependent upon risk and therefore correctable.

The next chapter summarizes the literature concerned with the capital budgeting problem which, like the portfolio problem, is an investment problem.

CHAPTER III

MATHEMATICAL PROGRAMMING OF THE CAPITAL BUDGETING PROBLEM

The capital budgeting problem may be formulated as a mathematical programming problem. Since a great deal of effort has been directed at this process, it is convenient to classify the many varied formulations. There are two major classifications:[1]

1. Deterministic--those that do not explicitly handle risk, but instead use parametric analysis to analyze the budget's sensitivity to possible and varied occurrences, or employ special constraints to avoid risks.
2. Probabilistic--those that recognize risk and treat it in some explicit manner.

Much of the work done under either classification certainly provides useful techniques that can be applied to the development of any new formulations, such as the one presented in Chapter IV. For this reason, a review of selected works in both classifications will be presented here.

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[1]See Figure 3 for a summary of those individuals who have published work in the various areas of capital budgeting.

Deterministic Models

The deterministic problem may be written as follows:

$$
\begin{aligned}
\max z_o &= f(x) \\
\text{subject to:}\quad g_1(x) &\le 0 \\
g_2(x) &\le 0 \\
\vdots\quad & \quad\vdots \\
g_n(x) &\le 0 \\
x &\ge 0 \qquad (3.1)
\end{aligned}
$$

where: x is an m-vector.[2]

Problem subclassification may be determined by the form of the functions f and g_i or other restrictions placed upon x. For example, if f is a linear function and each g_i is a linear function and each element of x is restricted to values of 0 or 1, then the above is a very general statement of Weingartner's [50] and Robertson's [39] formulations. Lorie and Savage's [27] problem is similar except that each component of x satisfies $0 \le x_i \le 1$. Reiter's [38] formulation uses a quadratic form for f(x), does not have constraints g_i and the variables are again binary, 0-1 variables. Another of Robertson's [39] formulations uses a linear function f and linear functions g_i except for some special nonlinear constraints designed to limit the debt equity ratio. His variables are of the mixed-integer

[2] See Bernhardt [3] for a more specific general formulation of the capital budgeting problem.

type with integer variables used to accept or reject a project and continuous variables or discrete approximating variables used to determine the level of operation of each project in each period. Robertson offered no solution technique for his formulation which with 20 projects and 12 periods resulted in over 5000 variables and 3000 equations [39, p. 117].

Some formulations might be classified as dynamic programming (D.P.) formulations; however, D.P. is more of a solution technique than a unique formulation. D.P. formulations by Weingartner [49] and Robertson [39] are no more than recastings of the general formulation given above.

One of the contributions of the deterministic models, particularly the integer problems, has been the development of a rather standard set of project interrelationship constraints which may be found in Weingartner's prize-winning dissertation [50]. To exemplify these constraints one may consider a formulation where each project is represented by a variable x_i which takes on a value of zero if the i^{th} project is rejected and a value of one if the i^{th} project is accepted. If J is a set of indices for a mutually exclusive set of projects (i.e., only one project may be selected from the set), then the constraint that guarantees this mutual exclusion becomes:

$$\sum_{i \in J} x_i \leq 1 \quad \text{if one and only one project from set J \underline{may} be selected.} \qquad (3.2)$$

or

$$\sum_{i \in J} x_i = 1 \quad \text{if one and only one project from set } J \text{ \underline{must} be selected.} \tag{3.3}$$

One may also define a set k_i as the set of indices for projects that are contingent upon the selection of project i. For example, the selection of optional air conditioning, power steering, and power brakes for a fleet of company sedans would be contingent upon the purchase of the fleet. The resulting constraint becomes:

$$\sum_{j \in k_i} x_j \leq n x_i \quad \text{if none of the projects in } k_i \text{ are mutually exclusive. } n \text{ is the number of elements in } k_i. \tag{3.4}$$

or

$$\sum_{j \in k_i} x_j \leq x_i \quad \text{if all of the projects in } k_i \text{ are mutually exclusive.} \tag{3.5}$$

If some projects in k_i are mutually exclusive and others are independent, then constraints (3.2) and (3.4) are required simultaneously. One special case of the contingent projects case is of interest. A project may be implemented at any one of n discrete levels. In this case, x_1 represents acceptance or rejection at level one, x_2 represents acceptance or rejection of the incremental investment necessary to achieve level two, x_i represents acceptance or rejection of the incremental investment necessary to achieve level i, etc. Clearly, one cannot accept level k unless k-1 is accepted, hence the constraints:

$$x_n \leq x_{n-1} \leq \ldots \leq x_i \leq x_{i-1} \leq \ldots \leq x_2 \leq x_1. \qquad (3.6)$$

Due to the well-developed state of project interrelationship constraints they will henceforth be referred to by the abbreviation P.I.C. keeping in mind that they are applicable for zero-one variables only.

Resource constraints represent a second major class of constraints. Again, these were discussed by Weingartner [50, p. 125] and take the following general form:

$$\sum_{j=1} a_{ijt} x_j \leq R_{it} \quad i = 1, \ldots, \qquad (3.7)$$

where:

a_{ijt} = the quantity of resource i required by project j in period t

R_{it} = the amount of resource i available in period t.

One of the obvious resources is capital, but since it is of such importance to the capital budgeting problem and treated in ways other than the general expression given by (3.7), it will be considered separately from other resources. In the Lorie and Savage model and in Weingartner's [50, p. 17] integer version of their model the financial constraints take the form given by (3.7) where a_{ijt} is the present value (PV) of the required capital outlay for project j in period t and in this instance the i^{th} resource referred to is capital. R_{it} is the PV of the total available capital for period t. In Weingartner's horizon model [50, p. 141], Robertson's extension of the horizon model [39, p. 55] and in models by Quandt and Baumol [1]; and Moag, Joseph, and Lerner [34]; the financial constraints account for the net of all flows of capital for each project. The horizon model financial constraints may be stated:

$$\Sigma a_{1j}x_j + v_1 - w_1 \leq D_1 \tag{3.8a}$$

$$\Sigma a_{tj}x_j + v_t - (1+r)v_{t-1} + (1+r)w_{t-1} - w_t \leq D_t \tag{3.8b}$$

$$t = 2, \ldots, T$$

where:

a_{tj} = the net cash flows to project j in period t

v_t = the amount loaned at rate r

w_t = the amount borrowed at rate r

D_t = the funds generated by other activities of the firm in period t, and

T = the last period in the planning horizon.

Robertson's modifications [39] to these constraints consisted of using different rates of interest for borrowing and lending and letting $D_t = 0$ for all $t \geq 2$ while D_1 is the total funds the firm is willing to commit to the capital budget. With these modifications the model determines not only which projects but also the sizes of the investment levels in each period by means of lending funds in one period so that they, plus their interest, become available in the next period. The forced carry forward of funds may be accomplished by placing lower bounds on the amounts loaned, v_t.

Quandt and Baumol's model (Q&B) [1] uses financial constraints of the following form:

$$-\Sigma a_{it} x_i + w_t \leq M_t \tag{3.9}$$

where:

a_{it} = the t^{th} period net cash flow for project i

w_t = the funds withdrawn by the owners of the firm

M_t = the money available for use or withdrawal in period t

Moag, Joseph and Lerner (M, J & L) [34] modified the Q&B model by assuming that the cash flows from each project in any period is a non-linear function of the percentage of project i that is undertaken (the percentage is denoted x_i and is one of the decision variables of the problem). To obtain a computationally solvable problem, M,J&L approximate these nonlinear cash flows with segments of linear functions and naturally conclude with constraints that are quite different than that stated above in constraint (3.9).

Not all deterministic models seek to optimize the same function, but in general the maximum of the net present value of all projects undertaken is most popularly used. The Lorie and Savage model and Weingartner's integer version use such a criterion. However, Weingartner goes on to formulate his horizon model which seeks to maximize

$$\sum_j a_j x_j + v_T - w_T \text{ [50, p. 162]} \quad (3.10)$$

where:

a_j = the value of all cash flows occurring for project j subsequent to the horizon period T and discounted to the horizon at rate r, the rate at which funds may be borrowed or lent.

v_T = the outstanding loans at the horizon T

w_T = the outstanding debt for the same period

Hence, Weingartner's horizon model seeks to maximize equation (3.10) subject to expressions (3.8a) and (3.8b) and any dependency constraints of form (3.2), (3.3), (3.4), or (3.5) with x_i restricted to values zero or one. He shows, however, that for independent projects and assuming unlimited amounts may be borrowed or lent at rate r, the horizon model is equivalent to maximizing the net present value of projects determined at rate r and, in fact, the optimal solution will dictate the acceptance of all projects with positive net present values. In effect, then, the horizon model does not consider the capital budgeting problem in the framework of capital rationing. Robertson's extension of the horizon model only embraces capital rationing to the extent forced by unequal borrowing and lending rates. Robertson did go on to consider the effects of absolute ceilings on borrowing.

The objective function used by Quandt and Baumol and again by Moag, Joseph, and Lerner, is the utility of cash withdrawals from the firm by the owners. However, given that an individual (owner) requires a certain rate of return a rational person might prefer a stream of cash flows with a higher present value to a stream with lower present value. In particular, then, the utility function of these models could produce a present value for each flow with the model maximizing the present value of all funds withdrawn from the firm. Their model objective is therefore a more general statement of the same basic concept of maximizing present value.

This discussion of deterministic models has not been exhaustive nor detailed in the discussion of the representative models. It is felt that the discussion is sufficient, as there are already a number of works that provide excellent and detailed descriptions of mathematical programming models. In particular, and most obvious, are the works themselves. In addition, surveys by Weingartner [49], Mao [28], and Bernhard [3], are available. The purpose here has been to highlight some of the more prominent models to provide some background pertinent to the discussion of the probabilistic models. In addition, before one provides yet another formulation for those who come after him to review, some criticism of existing models is needed to provide justification. In the case of the deterministic models, sufficient justification is provided for a probabilistic model by the fact that deterministic models are deterministic. However, some attempts to control risk have been made without going to a probabilistic model. These attempts have culminated with the development of two special constraints, so-called payback and liquidity constraints.

The payback constraints require that the total net present value of all projects undertaken exceeds some value by some time period t' prior to the horizon. Hence, constraints of the form:

$$\sum_{i=1}^{t'} \sum_{j=1}^{m} \alpha^{i} a_{ij} x_j \geq b \tag{3.11}$$

where:

a_{ij} = the net cash flows for project j in period i

α = 1/(1+i) a discount factor

Management specifies t' and b. Clearly, a number of such constraints could be employed with each using different values for t' and b, thus giving management a degree of control over a payback schedule for some specific time span. Since the payback constraint was developed as a means of controlling risk, it is appropriate to analyze constraint (3.11) in terms of the kind of risk that it might control. To this point, risk has been mostly thought of as a variability in return, and it is generally accepted that projects that yield higher returns will also tend to be more risky. One can construct many examples of realistic patterns of cash flows that would seem to indicate that projects with more rapid payback also tend to yield higher return. Hence, inclusion of a payback constraint may press a solution towards some of the riskier, more profitable projects. The risk associated with (3.11) must therefore be the uncertainty about what might be rather than just what cash flows might be. For example, management may wish to avoid committing all normally available funds to a group of long-term projects because a far superior opportunity may present itself in the near future. In a sense, it insures financial flexibility to react to events which are totally uncertain. Totally uncertain means that these events cannot be anticipated much less their likelihood determined. If maintaining this flexibility to react to unknown events is a policy of management, then constraint (3.11) would be appropriate for any capital budgeting formulation regardless whether risk[3] is given explicit consideration or not.

[3]Weston [51] distinguishes between risk and uncertainty by referring to varying degrees of knowledge about the future. Risk applies when outcomes are known and probabilities of outcomes can be assigned. Uncertainty applies when outcomes are known but probabilities not known. Partial ignorance is Weston's label that applies when neither outcomes nor probabilities are known [51, p. 48]. In capital budgeting, only uncertainty and partial ignorance exist, but through much effort in data analysis and assumptions one can shift these up one degree. In this context (3.11) actually protects against partial ignorance.

Liquidity constraints require a commitment of cash, equivalent liquid assets, or unused credit lines to be held in reserve to prevent insolvancy in the event cash-flows unexpectedly turn downward. This is a somewhat explicit means of recognizing risk, but is inefficient in that the same liquidity is maintained regardless whether projects undertaken turn out to be highly risky or ultra conservative. Liquidity requirements usually take the form of tighter rationing of capital, upper bounding of borrowings, lower bounding of lending, increase in interest charges on borrowing as a function of debt-equity ratio, or direct limits on debt-equity ratios.

Probabilistic Models

Models that somehow treat risk in an explicit manner may be further classified into two subclasses: (1) portfolio models and (2) chance-constraint models. The portfolio models have already been briefly discussed in the preceding chapter; however, some elaboration is appropriate. The usual approach is to assume the investment costs are known with certainty and returns are random variables. Additional assumptions are that either:

1. The random variables belong to the same family or distributions possessing two parameters that are independent functions of the mean and variance and the utility function for the decision maker is concave [25,

p. 3].[4] Or,

2. The utility function is quadratic and the distribution parameter used for measuring risk is finite.

Either of these assumptions are sufficient to guarantee that a clear choice between sets of projects is possible given only their expected return and variance of that return.

The capital budgeting problem is then treated as a portfolio selection problem with zero-one variables. The objective function used for the portfolio model may reflect the only real difference from the deterministic models. In particular, the model given by Problem (3.1) is still valid for a general formulation where:

$$f(x) = r(x) - \lambda \sigma_r^{2}(x) \qquad (3.12)$$

and where r(x) is the total expected return from projects undertaken (denoted by vector x of 0-1 values) and $\sigma_r^{2}(x)$ is the variance of the total return and finally λ is a constant reflecting the degree of risk adversion possessed by the individual or group making the investment decision. For a specific case such as returns assumed to be multivariant normal, then:

$$f(x) = \sum_{i=1}^{m} \bar{r}_i x_i - \lambda \sum_{i=1} \sum_{j=1} \sigma_{ij} x_i x_j \qquad (3.12a)$$

[4] Levy and Sarndt reported this finding as a result of an earlier work by Levy and Hanoch, "The Efficient Analysis of Choices Involving Risk," Review of Economic Studies (July, 1969).

where:

$\bar{r}_i$ = the expected return from project i

σ_{ij} = the variance of the returns from project i when j=i and is otherwise the covariance of the returns from projects i and j.

Weingartner demonstrated that Reiter's heuristic solution procedure for quadratic 0-1 problems could be used on such an objective function with some modification to handle mutually exclusive and contingent project relationships, but multi-period financial and resource constraints cannot be handled effectively by the procedure [49]. Most of the literature seems to be more concerned with how to evaluate and choose between two given combinations of assets than it is with the problem of finding that particular combination of assets that maximize the chosen measure.[5] One of the more controversial areas is the question of how to measure return and/or how to measure risk. Van Horne [48] originally proposed the present value, calculated at the risk-free rate, of the stream of expected net cash flows or, more simply, the expected net present value as the measure of expected return and the variance of the distribution of all net present values as a measure of risk. To choose the best combination of assets, Van Horne suggests a slightly more complex form of equation (3.12a) where

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[5]See, for example, Paine [47], Stapleton [44], Van Horne [47], Hamada [18], Levy and Sarnot [25].

non-linear[6] indifference curves between expected return and risk are applied towards finding the combination of projects that are both efficient in the portfolio sense and lie on the highest indifference curve. Levy and Sarnot [25] analyzed the properties of the variance of net present values under various assumptions of project dependence and annual dependence between cash flows. They concluded that the variance of net present value provided an acceptable multiperiod analog to the measure of risk used in portfolio analysis. However, they also concluded that the calculation of the variance rapidly becomes complex as the number of problems and/or project durations is increased. Again, possible computational shortcomings emerge even before consideration was given to the mathematical programming problem of selecting the best combinations of assets.

One alternative to incorporating a complex utility function directly into a programming formulation is to use a simpler programming formulation a number of times to generate the entire efficient set or at least an important segment of the efficient set. Approximate solutions may be obtained by dropping the integer constraint. Hence, some

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[6] Equation (3.12a) is nonlinear in the decision variables but is a linear function of the two variables (1) total expected return and (2) variance of total return. Indifference curves reflect the complete set of all combinations of variables, (1) and (2), that produce identical utility for some given utility function. In effect, equations (3.12) and (3.12a) are linear indifference curves while those constructed from most concave or quadratic utility functions are non-linear in variables (1) and (2).

important results may be computationally obtainable despite the rather discouraging picture given above concerning the solvability of problems arising from the portfolio approach.

At this point one should not be left with the impression that there is complete agreement that the correct decision parameters are expected net present value and variance of net present value. For example, Mao and Brewster [30] have specified a programming model that generates an efficient set defined in terms of expected net-present value and the semi-variance of net present value. They provide some constructed examples of distributions of cash flows where according to the E-V criteria, management would be indifferent between two projects yet using $E\text{-}S_h$[7] criteria management would prefer one project over the other. In another article in which Mao surveys the theory and practice of capital budgeting [28, 31] an interview with a number of corporate executives indicates their primary concern for what is termed "downside risk." The mathematical entity of semi-variance is conceptually more like "downside risk" than is variance [31]. However, most of these arguments supporting semi-variance are academic in that no one has compared and published the actual outcomes of decisions made by the $E\text{-}S_h$ criteria to establish at

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[7]E-V refers to expected value and variance which according to most portfolio theory provides enough information to decide between alternatives if preference functions are known. $E\text{-}S_h$ refers to expected value and semi-variance or $S_h = \int_{-\infty}^{h} (x-h)^2 f(x)dx$ for the continuous density f(x).

least enough benefit to justify the horrendous computational effort inherent in the E-S_h model.

In another example, Fama [12, p. 404] cites some empirical studies that support a proposition that rates of return on securities are not normally distributed, but instead have stable Paretian distributions without finite variances. These distributions have four parameters: (1) characteristic exponent, (2) skewedness parameter which he assumes is zero (symmetric), (3) a location parameter (comparable to mean), and (4) a dispersion parameter (sometimes comparable to variance). The normal distribution is a stable Paretian distribution with characteristic exponent 2 and skewedness parameter zero. Fama then replaces the E-V criteria for portfolio selection with a location parameter-dispersion parameter criteria. Although the author has not found a portfolio formulation for the capital budgeting problem that is comparable to Fama's securities portfolio model with Paretian distributions, such a formulation could be feasible and would bring yet another form of risk and return measurement to the capital budgeting problem.

From the standpoint of this research, Stapleton's work was of particular interest although it could not be considered a programming approach [44].[8] Stapleton contended

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[8] Another work in this area is by R. S. Hamada, "Portfolio Analysis, Market Equilibrium and Corporation Finance," Journal of Finance (March, 1969), pp. 13-31, but Hamada was mostly concerned with substantiating Miller and Modigliani's propositions using equilibrium theory in place of homogeneous risk classes.

that a project should be selected if, and only if, acceptance of it adds more to the stock value of the corporation than it costs the shareholders to make the investment. Clearly, this work will rely upon a stock valuation model. In particular, Stapleton uses Sharpe's equilibrium model discussed earlier and a dividend model.

Using Sharpe's equilibrium model given earlier, one may

$$\bar{r}_j = r_f + \mathrm{cov}(r_j, r_m) \frac{\bar{r}_m - r_f}{\sigma_{r_m}^{\ 2}} \qquad (3.13)$$

assume that one has F dollars with which to purchase some future stream of dividends which have a present value v_j computed at the riskless rate r_f by purchasing shares of the j^{th} stock or equivalently, j^{th} portfolio. Then $v_j - F$ is the excess dollar return earned above the risk-free rate and has expectation $E(v_j - F) = E(v_j) - F$. Total expected dollar return on investment of F dollars is then $E(v_j) - F + r_f F$ with rate of return being $\frac{E(v_j) - F + r_f F}{F}$. Similarly, $\bar{r}_m = \frac{E(v_m) - F + r_f F}{F}$, and from definition of variance and covariance, one has,

$$\sigma_{r_m}^{\ 2} = \frac{\sigma_{v_m}^{\ 2}}{F^2} \text{ and } \mathrm{cov}(r_j, r_m) = \frac{1}{F^2} \mathrm{cov}(v_j, v_m)$$

Making these substitutions into (3.13), simplifying and multiplying by F, one obtains[9]

$$E(v_j) - F = \frac{E(v_m) - F}{\sigma_{v_m}^{\;2}} \operatorname{cov}(v_j, v_m) \tag{3.14}$$

where: index m has referred to the market portfolio in all of the above.

The PV of total dividends paid by company or portfolio j may be denoted by D_j with expectation $E(D_j)$. Then F dollars will purchase a proportion of the aggregate stock of company or portfolio j given by $E(V_j)/E(D_j)$; therefore, the aggregate value of outstanding stock is:

$$P_{oj} = F\frac{E(D_j)}{E(V_j)} = E(D_j) - SR_{jm}\sigma_{D_j} \tag{3.15}$$

where:

$S = \dfrac{E(V_M) - F}{\sigma_{V_M}}$

$R_{jm} = \dfrac{\operatorname{cov}(V_j, V_m)}{\sigma_{V_j}\sigma_{V_m}}$ = correlation coefficient between j and the market portfolio

$\sigma_{D_j} = \dfrac{V_j}{E(V_j)}E(D_j)$ = standard deviation of the PV of all dividends paid by j computed at rate r_f.

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[9](3.14) is exactly Stapleton's equation (12) page 102 although obtained in a somewhat different manner.

Equation (3.15) is a fundamental result of equilibrium theory and the dividend valuation model from which Stapleton develops his investment decision rules. After making an assumption[10] that allows the substitution of the economic index for the market portfolio, Stapleton shows that dividend policy is irrelevant to valuation given the net cash flows of the company. He then provides a valuation model developed from (3.15):

$$P_{oj} = \bar{r}(DVx_j) - S\sigma_{DVx_j'} \qquad (3.16)$$

where:

DVx_j = the discounted values of all future cash flows to the firm

DVx'_j = the expected value of the discounted cash flows given a value for the index mentioned earlier.

Hence, DVx'_j is a function of the index which is a random variable, and DVx'_j is a random variable with standard deviation $\sigma_{DVx'_j}$. Stapleton develops, then, a decision rule of the form:

If $P_{oj}^* - P_{oj} > x_o$, then invest

where P_{oj}^* is the market value of the firm with all cash flows of the project proposal under consideration

---

[10] The correlation coefficient between each firm and the optimal market portfolio is approximately equal to the correlation coefficient between each firm and some economic index, such as Gross National Product.

being included in the valuation model given by (3.16) and P_{oj} is the firm value without the project while x_o is the cost of the project.

This criteria is equivalent to requiring the net present value of projects to be positive when discounted at the appropriate risk adjusted discount rate except that P_{oj} is thought of as the certainty equivalent of the discounted value of all future and uncertain cash flows where the discount rate is not a risk adjusted rate, but instead, the risk-free rate r_f. Stapleton does go on to solve for risk adjusted discount rates so that finding certainty equivalents of discounted values is not necessary. However, the risk adjusted rate is different for each different set of assets held by the firm owing to the different risk posture resulting from different investments. Hence, Stapleton's risk adjusted discount rate is analogous to the conventional hurdle rates or MARR's, except that each project would have its own, possibly unique, hurdle rate as a result of its own, possibly unique, risk characteristics and of how those characteristics correlate with all of the firm's other investments. Stapleton's work then is similar to Lintner's but a somewhat more practical development of investment criteria than Lintner's original work [26] based upon his own equilibrium equations.

This completes the discussion of the portfolio approach to the capital budgeting problem, but some particular points should be illuminated. It should be noted that most of these portfolio models rely upon management to make the decisions without regard to the external world. This seems appropriate until one considers that management may not necessarily be the owners. If the firm is owned by stockholders whose shares are traded in some security market and who do not directly participate in capital budgeting decisions, and if an objective of the firm is stated, as it often is, to maximize the wealth of the stockholders, then decisions should be made with particular regard for the external world. Stapleton, and Lintner, took this approach and to do so requires a known relationship between risk and expectations associated with the financial parameters of the firm and risk and expectations associated with the shareholder's returns.

Chance Constrained Programming

Given any mathematical programming formulation such as (3.1) discussed earlier one can conjecture that all of the functions, or at least some of the functions, are random variables with some joint distribution function. For purposes of exposition one may suppose in formulation (3.1), all functions are linear[11] and random variable coefficients are assumed for functions $f(x)$, and $g_i(x)$, $i = 1, \ldots, k$. $g_i(x)$, $i = k+1, \ldots, n$ are assumed deterministic.

---

[11]Little work can be found concerning chance constraints where the constraints are non-linear.

The approach of chance constraint programming under the above assumptions would be to replace formulation (3.1) with the following (3.1a):

$$
\begin{array}{lll}
\text{Max } E(z_o = f(x)) & & \vdots \\
\text{s.t.} \quad \Pr\{g_1(x) \leq 0\} \geq \alpha_1 & & g_n(x) \leq 0 \\
\qquad \Pr\{g_2(x) \leq 0\} \geq \alpha_2 & & x \geq 0 \\
\qquad \vdots & & x \text{ an } m\text{-vector} \\
\qquad \Pr\{g_k(x) \leq 0\} \geq \alpha_3 & & \\
\qquad\qquad g_{k+1}(x) \geq 0 & &
\end{array}
\tag{3.1a}
$$

α_i clearly denotes a probability which is to be specified by management and is the minimum probability with which management would like to have constraint i satisfied. In order to obtain solutions[12] it is usually hoped that one may assume that the constraints are independent and that the random variables for each constraint have a multivariant normal distribution. Under these assumptions and the linearity of the functions one can express a function in closed form for the following:

$$E(g_i(x))$$

$$\mathrm{Var}(g_i(x))$$

One then knows that $\dfrac{g_i(x) - E(g_i(x))}{\sqrt{\mathrm{Var}(g_i(x))}}$ is a standard normal

[12]See Charnes and Cooper, "Chance-Constrained Programming," Management Science (Oct., 1959) and "Deterministic Equivalents for Optimizing and Satisfying under Chance Constraints," Operations Research II (1963).

random variable. Hence,

$$\Pr\left\{\frac{g_i(x) - E(g_i(x))}{\sqrt{Var(g_i(x))}} \le z_{1-\alpha_i}\right\} = \alpha_i \qquad (3.17)$$

where:

$z_{1-\alpha_i}$ is obtained from a table of standard normal values.

Therefore, since it is desired that $g_i(x) \le 0$ occur with probability greater than α_i, then by requiring that

$$z_{1-\alpha_i} \sqrt{Var(g_i(x))} + E(g_i(x)) \le 0 \qquad (3.17a)$$

one may assert that $\Pr\{g_i(x) \le 0\} \ge \alpha_i$ and the appropriate probability will be guaranteed. Hence, the i^{th} constraint of (3.1a) is replaced by a nonlinear constraint of form (3.17a) for each i = 1, ..., k, and the resulting nonlinear programming problem can be solved. Clearly, solution difficulty arises when each x_i is restricted to values of 0 or 1 as is often the case with capital budgeting problems.

Näslund [36] first applied the chance constraint technique to capital budgeting problems of the form used by Weingartner [50]. He also developed methods for circumventing the problem of zero-one variables. Byrne, Charnes, Cooper, and Kortanek (BCCK) [4] applied the technique to a formulation using payback and liquidity constraints, and their own horizon posture control constraints which Bernhard [3, p. 146] did not accept as being posture control constraints. BCCK solve an example problem with four projects and three periods, which resulted in twelve variables since their model also decided which period to begin each project. Also, Robertson [39] applied the technique to Weingartner's horizon model but experienced some difficulty with declaring constraints independent. He made some approximations but still found the α_i to be conditional probabilities which would be much harder, from an intuitive standpoint, for management to specify in advance. Robertson also suggested chance constraint programming as a means of handling risk in his own deterministic capital-budgeting-operating level programming formulation, but such a suggestion is completely untenable.

To understand the implications of a chance constraint, one may consider a simple example of chance constraint programming applied to a one-period problem constructed for the purpose of illustration:

$$\begin{array}{lll} \text{Max } NPV = \sum \overline{V}_i x_i; & \overline{V}_i & = \text{expected NPV}^{13} \text{ of project } i. \\ \text{s.t. } \sum \overline{C}_i x_i \le F; & \overline{C}_i & = \text{expected cost of project } i \\ 0 \le x_i \le 1 & F & = \text{total funds budgeted} \end{array}$$

Formulated as a chance constraint problem with C_i being normally distributed with means $\overline{C}_i$ and variances and covariances σ_{ii} and σ_{ij}, respectively, the problem becomes:

[13] NPV is used to denote net present value.

$$\max \text{NPV} = \sum_{i=1}^{n} \overline{V}_i x_i$$

$$\text{s.t.} \left[\Pr \sum C_i x_i \le F\right] \ge \alpha$$

$$+ \text{ other constraints}$$

$$0 \le x_i \le 1$$

Let G be a random variable such that:

$$G = \sum C_i x_i .$$

Then G is normally distributed with expected value:

$$E(G) = \sum \overline{C}_i x_i$$

and variance:

$$\text{Var}(G) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{ij}$$

and standard deviation:

$$\text{STD}(G) = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{ij}}$$

Hence,

$$\frac{G - E(G)}{\text{STD}(G)} = \frac{\sum C_i x_i - \sum \overline{C}_i x_i}{\sqrt{\sum \sum x_i x_j \sigma_{ij}}}$$

has a standard normal distribution. Suppose that the α specified by management was .99, then it is known that

$$\Pr\left\{\frac{G - E(G)}{\text{STD}(G)} \le +2.326\right\} = .99$$

or $\Pr\{G \le STD(G)(2.326)\} + E(G) = .99$

and letting $STD(G)(2.326) + E(G) \le F$

then clearly $\Pr\{G \le F\} \ge .99$ and the problem becomes

$$\text{Max NPV} = \Sigma \, \overline{V}_i x_i \tag{17}$$

$$\text{s.t. } \Sigma \, \overline{C}_i x_i \le F - 2.326 \sqrt{\Sigma \, \Sigma \, x_i x_j \sigma_{ij}}$$

$$0 \le x_i \le 1$$

This formulation is precisely the same as the original formulation without a chance constraint except for the term $-2.326 \sqrt{\Sigma \, \Sigma \, x_i x_j \sigma_{ij}}$. Remembering that in the discussion of deterministic models one means of controlling risk was to make constraints, such as the above financial constraint more constraining, thus forcing the withholding of some cash from investments to protect against the risk. The amount withheld was based upon management's subjective opinion of how much should be withheld without really knowing the level of risk that will be present in the final set of projects accepted. The chance constraint technique is essentially the same procedure except that it offers a refinement in that the amount withheld is clearly a function of the risk (measured by standard deviation) of the set of projects accepted. It is also a function of management's abhorrence for events that they consider bad, such as exceeding budgetary limitations. That abhorrence is measured by the specified probability α of the bad event not coming true. However, for the above model, no control is apparent for other risks. For example, projects with very stable costs could also possess highly variable returns so that the final set of projects accepted have little chance of exceeding budgetary constraints but perhaps a much higher chance of not generating enough earnings to pay future expected dividends or support future investment programs. Näslund [36] avoids this problem by applying the technique to a form of Weingartner's [50] horizontal model in which all cash flows appear in each constraint that is made a chance constraint. This limits the probability of unusually low horizon values for the firm. Byrne, Charnes, Cooper and Kornek [4] also handle this problem in that one of their chance constraints is applied to a payback constraint which includes all cash flows for at least some initial period of the project lives. Bernhard [3, p. 152] criticizes both of these models on the basis that it would be difficult to specify meaningful values for the α's, and that in some cases the violation of a financial constraint can be rectified by engaging in short-term borrowing at some cost, the meaning of which is not portrayed by a chance constrained financial constraint. Bernhard also indicated that solutions, particularly integer solutions, are not easily obtained [3, p. 154].

The application of chance constraints to deterministic models certainly appears to be a viable approach to controlling some of the risks involved in capital investment decisions. The value of the technique in real applications has not been substantiated since the model formulations are either not entirely meaningful or are difficult to solve [3, p. 155]. One should neither reject the scheme nor proclaim it unequivocally as the correct means of handling risk.

Summary

The tree structured diagram in Figure 3-1 provides a convenient means of reviewing the various classes of capital budgeting formulations presented in the chapter. Each arc is labeled with its branch of the classification scheme used and the nodes at the end of some chains of arcs give examples of models identified by author's name and date.

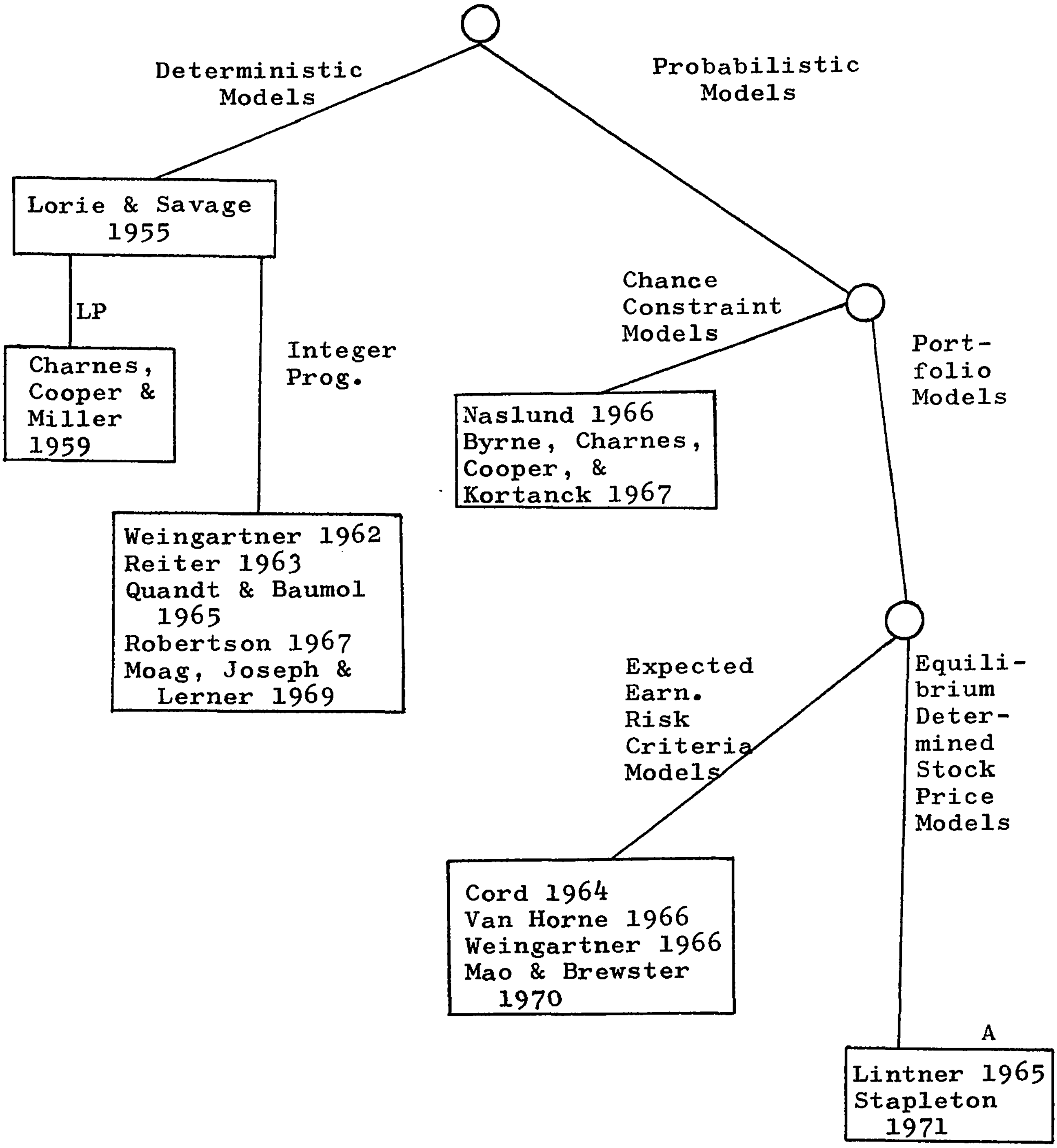

Figure 3-1. Tree-structured classification system.

Note: The model presented in Chapter IV belongs under node A of Figure 3-1.

CHAPTER IV

A MODEL FORMULATION

The ultimate goal of this research is to apply the work done in capital budgeting and portfolio and equilibrium theory to the construction of a mathematical programming formulation that can be solved in a relatively efficient manner.[1] In an effort to make this model more useful as few unrealistic assumptions as possible are made. However, too few assumptions are likely to result in a formulation that is untenable from the standpoint of finding solutions or perhaps more importantly from the standpoint of data requirements.

The procedure for presenting the problem formulation begins with the statement of general assumptions that establish the framework and define the boundaries within which the formulation will be operative. Next, one finds a general development for the objective function with specific objective functions given as examples, followed by a rather specific development of the financial constraints with a computation of the cost for violating a constraint.

[1]See Figure 3 to determine the exact category or class of problem formulation that is being provided herein.

Integrated with these developments, one finds the statement of the specific assumptions required to support the development. Certain assumptions are implicit and remain unstated. For example, if the model uses a particular datum, then obviously it is assumed that it exists and is obtainable.

The resultant programming problem formulation is stated as general functions of pertinent variables defined during the development. This allows management to specify their own functions for replacement of the general ones producing a specific problem formulation tailored to the needs or beliefs of that particular management.

To validate the generality of the problem statement an analysis of the objective function with exact functions specified is provided as a parallel to Stapleton's[2] work, with identical results obtained. The work is then extended into a programming problem context rather than the simple decision rule Stapleton developed.

Finally, a number of specific assumptions are made which allow a complete construction of an example problem stated in specific terms instead of the general terms used for the original formulation. It is this specific problem that is solved in later chapters.

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[2]See Chapter II for a summary of Stapleton's work in this area.

Assumptions

The general framework and boundaries of the problem are defined by the following assumptions.

1. Management determines the budget size and means of acquiring all funds required to support the budget, but automatically relies upon short term borrowing when the planned budget is exceeded. Surplus funds are automatically invested (or loaned) at the risk free rate while the rate the company must pay for borrowed funds is higher, constant during each period, and specified for the problem.
2. Capital markets are not assumed to be perfect.
3. Sharpe's equilibrium equation represents reality.
4. All random variables are normally distributed, but not necessarily independent of each other, i.e., random variables associated with individual projects are correlated with the random variables associated with other companies. Further it is assumed that this correlation may be fully represented by a common relationship with some underlying economic factor. In fact, it is convenient to assume that all correlation between companies is sufficiently approximated by this common relationship with the economic factor.
5. The company is widely held and its stock traded in the security market.

Assumption 1 defines the boundaries of the problem.

It is clear that the only decision left to the model is to select the projects in which to invest and that, of course, will also determine the timing of the use of funds made available by management. This gives management a firm control over the investment budget and allows them the flexibility of exploring a wide variety of means of acquiring funds for investment. Figure 4-1 depicts the problem with its inputs and outputs where the inputs are the results of independent (from the problem) management financing decisions. By parameterizing some of the inputs one may use the model to evaluate the effects of various financing decisions upon the selected projects and the objective function.

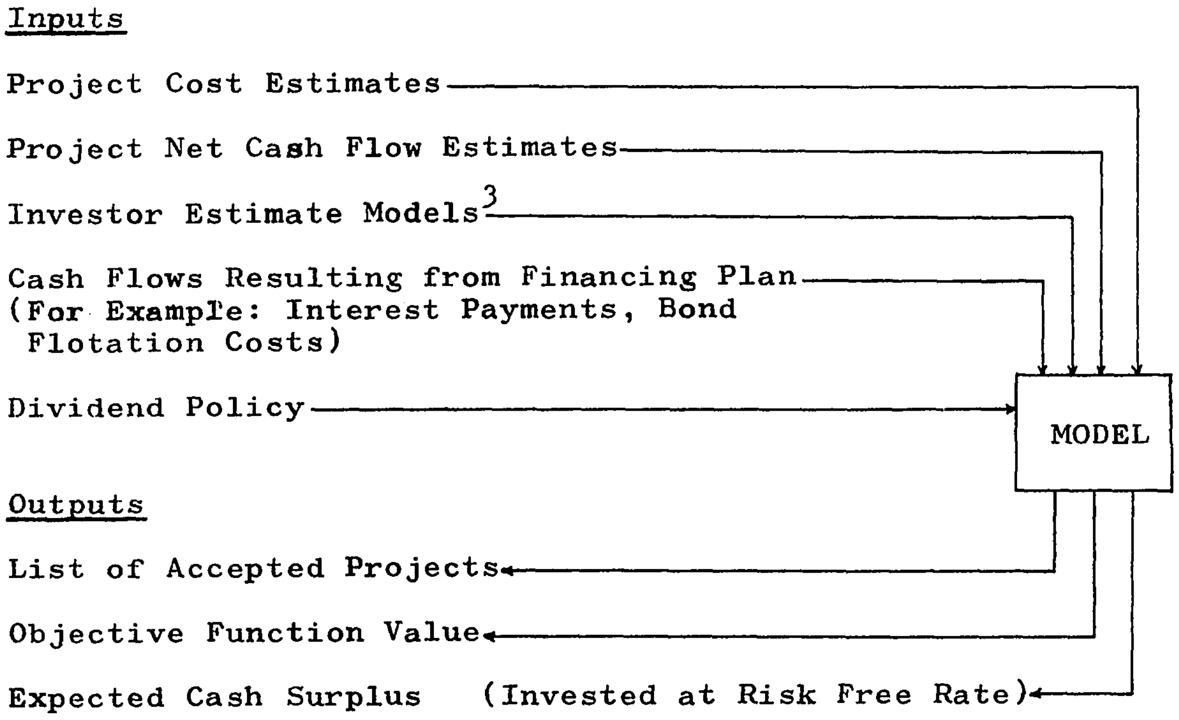

Figure 4-1. Model inputs and outputs.

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[3] Investor estimate models are defined on subsequent pages.

Assumption 2 preserves the reality of the model.

The assumptions behind Sharpe's model that is assumed realistic in assumption 3 are in direct conflict with assumption 2. However, any model developed in a purely theoretical framework may still represent reality if the market imperfections caused by violated assumptions are not too great. As mentioned in Chapter II, Sharpe's model gives a consistently biased picture of reality, but which can be corrected, thus eliminating the disparity between assumptions 2 and 3.

Assumption 4 is both general and restrictive in that it does not specify independent random variables but does specify normality. Normality is not absolutely required provided enough projects are accepted to invoke the Central Limit Theorem. It is, however, convenient and is certainly not uncommonly assumed. Furthermore, most random variables in this research are cash flows and Hillier has indicated that in many cases, one's best subjective probability distribution is one that resembles the normal distribution [20, p. 446].

Since the model selects projects based upon their effects on the equilibrium value of the stock of the company it is necessary to construct a situation which provides an opportunity for the stock to seek an equilibrium price, hence Assumption 5.

The remainder of this chapter is devoted to the development of the mathematical programming project selection model within the environment established by these five general assumptions.

Objective Function Development

$\emptyset_t$ is defined as a vector of financial parameters of the firm at the end of period t. It is assumed that investors predict the future price of the stock of the firm by obtaining an estimate $\hat{\emptyset}_t$ of the vector $\emptyset_t$. Therefore, there exists some relationship between estimated future stock prices and $\hat{\emptyset}_t$. Suppose one may estimate that relationship by

$$\hat{P}_t = f(\hat{\emptyset}_t) + \epsilon \qquad t = 1, 2, \ldots \tag{4.1}$$

where $\hat{P}_t$ is the estimated aggregate value of all outstanding shares at the end of period t, and ϵ is a normally distributed error term $\mu_\epsilon = 0$ and $\text{cov}(\epsilon, \hat{\emptyset}_t) = 0$.[4] $\hat{\emptyset}_t$ is assumed to be a random variable with the same distribution as $\emptyset_t$ and with a mean and variance equal to those estimated for ϕ_t. One should observe that (4.1) is a rather general statement for an equity valuation model. For example, one can consider a dividend model which estimates current price (or value) as the present value of all future dividends:

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[4] $\hat{\emptyset}_t$ may represent either before tax or after tax values. If they are before tax values, then one must assume that the tax situation of the firm is constant and will remain constant after project selection. Under that assumption, $f(\hat{\emptyset}_t)$ should reflect whatever the tax situation happens to be. This gives consideration to tax in an approximate manner.

$$P_t = \sum_{k=0} \frac{\hat{d}_{t+k}}{(1+r)^k} = \hat{d}_t + \sum_{k=1} \frac{\hat{d}_{t+k}}{(1+r)^{k-1}(1+r)}$$

$$\hat{P}_t = \hat{d}_t + \frac{\hat{P}_{t+1}}{1+r}$$

$$\hat{P}_t = f(\hat{d}_t, \hat{P}_{t+1})$$

Hence, the dividend model is exactly equation (4.1) under the following conditions:

1. The model is assumed to always reflect investor estimates exactly and therefore the error term is dropped.
2. The financial variables used are estimates of the dividends paid at the end of period t and an end of period t estimate of the end of period t+1 price.

Equation (4.1) may henceforth be referred to as the investor estimate model.

Invoking assumption three and four, thereby using an economic index m as a surrogate for the market portfolio, one obtains Sharpe's equilibrium model:

$$E(\tilde{R}) = R_f + \operatorname{cov}(\tilde{R}, \tilde{R}_m) \frac{E(\tilde{R}_m) - R_f}{\sigma^2(\tilde{R}_m)} \qquad (4.2)$$

where:

$\tilde{R}$ = the one period return on stock held in the firm

R_f = the riskless rate

$\tilde{R}_m$ = the rate of change in index m

$\sigma^2(\tilde{R}_m)$ = the variance of $\tilde{R}_m$

E = the expected value operator

Equation (4.2) may be referred to simply as the market model. It is proposed that actual market value may be relatively accurately determined by a mathematical combination of the investor estimate model and the market model as a result of assumption 3. Thus, a relationship will be determined between market price of the firm and its financial variables which are affected by its investment decisions.

Clearly, the rate of return for any period t+1, given end of period value P_t, is estimated by:

$$\tilde{R} = \frac{\hat{P}_{t+1} - P_t + \hat{D}_{t+1}}{P_t} \tag{4.2a}$$

where:

$\hat{D}_{t+1}$ = the estimated dividends paid during the period.

The expected rate of return is:

$$E(\tilde{R}) = \frac{E(\hat{P}_{t+1}) - P_t + E(\hat{D}_{t+1})}{P_t} \tag{4.2b}$$

and

$$\mathrm{cov}(\tilde{R},\tilde{R}_m) = E\left\{\left(\frac{\hat{P}_{t+1}-P_t+\hat{D}_{t+1}-E(\hat{P}_{t+1})+P_t-E(\hat{D}_{t+1})}{P_t}\right)\left(\tilde{R}_m-E(\tilde{R}_m)\right)\right\}$$

$$\mathrm{cov}(\tilde{R},\tilde{R}_m) = \frac{1}{P_t}\{\mathrm{cov}(\hat{P}_{t+1},\tilde{R}_m) + \mathrm{cov}(\hat{D}_{t+1},\tilde{R}_m)\} \tag{4.2c}$$

Substituting (4.2b) and (4.2c) into (4.2) and letting

$\lambda = \dfrac{E(\tilde{R}_m) - R_f}{\sigma^2(\tilde{R}_m)}$ which is the market "price of risk" and is assumed constant over time, and solving for P_t one obtains:

$$P_t = \frac{E(\hat{P}_{t+1})+E(\hat{D}_{t+1})-\lambda\{cov(\hat{P}_{t+1},\tilde{R}_m)+cov(\hat{D}_{t+1},\tilde{R}_m)\}}{1 + R_f} \qquad (4.3)$$

From (4.1) and the definition of covariance,

$$cov(\hat{P}_t,\tilde{R}_m) = E\{[f(\hat{\varnothing}_t)+\epsilon-E(f(\hat{\varnothing}_t))][\tilde{R}_m-E(\tilde{R}_m)]\}$$

$$cov(\tilde{P}_t,\tilde{R}_m) = cov(f(\hat{\varnothing}_t),\tilde{R}_m) + cov(\epsilon,\tilde{R}_m) \qquad (4.3a)$$

Substituting (4.3a) into (4.3), (4.3) becomes:

$$P_t = \frac{E(f(\hat{\varnothing}_{t+1}))+E(\hat{D}_{t+1})-\lambda\{cov[f(\hat{\varnothing}_{t+1}),\tilde{R}_m]\}}{1+R_f} + \frac{\lambda\{cov(\epsilon,\tilde{R}_m)+cov(\hat{D}_{t+1},\tilde{R}_m)\}}{1+R_f} \qquad (4.4)$$

Thus, for any set of assets, all that is needed is the dividend policy and values for the estimated financial variables $\hat{\varnothing}_t$ for each period t, and the equilibrium price for the end of each previous period may be determined by equation (4.4). An alternate statement of (4.4) may be obtained if one first defines $\widehat{\Delta P}_t = \hat{P}_{t+1} - P_t$.

Then $$\tilde{R} = \frac{\widehat{\Delta P}_t + \hat{D}_t}{P_t}$$

and

$$\frac{E(\widehat{\Delta P}_t)+E(\hat{D}_{t+1})}{P_t} = R_f + \frac{\lambda}{P_t}\{cov(\widehat{\Delta P}_t,\tilde{R}_m)+cov(\hat{D}_{t+1},\tilde{R}_m)\}$$

$$E(\widehat{\Delta P}_t) = P_tR_f + \lambda\{cov(\widehat{\Delta P}_t,\tilde{R}_m)+cov(\hat{D}_{t+1},\tilde{R}_m)\} - E(\hat{D}_{t+1})$$

$$P_t = \frac{E(\widehat{\Delta P}_t) - \lambda \mathrm{cov}(\widehat{\Delta P}_t, \tilde{R}_m) + [E(\hat{D}_t) - \lambda \mathrm{cov}(\hat{D}_{t+1}, \tilde{R}_m)]}{R_f} \quad (4.4a)$$

Therefore, the current price may be explained as the certainty equivalent of all wealth accruing to the stockholder during period t+1, capitalized at the risk free rate. These certainty equivalences are determined by deducting from expected accrued wealth a penalty for relevant risk[5] (measured by covariance with the economic index) where the amount of that penalty is determined in the capital markets and is sometimes referred to as the "price of risk" [40, p. 34]. Equation (4.4a) may be further modified by recognizing that $\hat{P}_{t+1} - \hat{P}_t = f(\hat{\phi}_{t+1}) - f(\hat{\phi}_t)$ from whence one gets:

$$P_t = \frac{E[f(\hat{\phi}_{t+1})] - E[f(\hat{\phi}_t)] - \lambda\{\mathrm{cov}(f(\hat{\phi}_{t+1})) - \mathrm{cov}(f(\hat{\phi}_t))\} + E(\hat{D}_{t+1})}{R_f}$$

$$\frac{-\lambda \mathrm{cov}(\hat{D}_{t+1}, \tilde{R}_m)}{R_f} \quad (4.4')$$

There is clearly a problem with equations (4.4) and (4.4') and their compatability with the original assumption that all random variables are normally distributed.

---

[5]Relevant risk is not simply variance of return since by diversifying, investors can eliminate most, if not all, variation in returns except that accounted for by covariances with other stocks. Due to the assumption that all covariances are explained by covariances with the economy, hence, the only relevant risk is measured by covariance with the economic index m.

Since dividends tend to be a matter of policy with many corporations attempting to maintain dividends despite lower earnings and then if forced to cut dividends there is a preponderance to hesitate increasing the dividend payout. Thus, dividends will hardly exhibit a normal distribution. Several means of handling this problem are available.

1. For some firms, an exact dividend policy may be established at a conservative enough level that no one would expect any deviation from that policy. For example, some companies do not pay any dividends and have no plans to ever do so. Under this situation a good approximation is achieved by replacing the random variable $\hat{D}_t$ by a deterministic constant D_t and eliminate the covariance term involving D_t.
2. One may assume the existence of an investor estimate model based upon financial variables much as was done in the price estimation model for investors. Thus, $\hat{D}_t$ becomes a random variable representing estimated dividends (by investors). Ideally, one should specify some general model such as $d(\hat{\mathscr{F}}_t) + \epsilon$, however, since dividend policies are executed by firms and investors become aware of such policies, they are likely to influence the estimation model. Realizing that a general model can always be used but would result in a redundant exercise, two specific candidate models are presented.
3. One may assume that investors believe the company uses

a fixed percent times cash flows to determine dividends. Thus, $\hat{D}_t = \alpha\hat{\mathcal{E}}_t$ where α is the fixed percent and $\hat{\mathcal{E}}_t$ is the estimated cash flow for period t, and may be assumed normally distributed.

4. Perhaps a better model is to assume the company attempts to maintain a relatively constant dividend pattern and achieve, on the average, some target fixed percent α. Then, a reasonable model for investors to use to estimate dividends is to find a perpetuity equivalent to the fixed percent times cash flows. Hence, $\hat{D}=R_f \sum_{j=0}^{T} \frac{\alpha(\hat{\mathcal{E}}_j)}{(1+R_f)^j}$

   T is the planning horizon and D is now independent of time. One should observe that with this model, $E(\hat{D}) = R_f \sum_{j=0}^{T} \frac{\alpha E(\hat{\mathcal{E}}_j)}{(1+R_f)^j}$ and the covariance, $\text{cov}(\hat{D},\tilde{R}_m)=R_f \sum_{j=0}^{T} \frac{\alpha\,\text{cov}(\hat{\mathcal{E}}_j,\tilde{R}_m)}{(1+R_f)^j}$ are relatively simple to compute. In general, this fourth method of handling the dividend problem will be used.

The construction of an objective function using stock prices is difficult. The reason for this, as Mao points out, is that very little research has been done in an effort to distinguish between a "good" or "bad" plot or graph of stock prices [28]. Equation (4.4) or (4.4') provides a means of determining a whole series of stock prices given financial parameters, but as just mentioned the process of determining which series is preferable has not yet been developed. Therefore, the intent here will not be to state one objective function and declare it valid, but a number of obvious choices will be presented for possible use. The cases presented will also exemplify the latitude available for other possible formulations.

Case 1:

One may assume it is desirable to maximize the equilibrium price at some horizon T. Then the objective function is:

$$\max_{\emptyset_{T+1}} P_T \quad \text{where } P_T \text{ is given by (4.4).}$$

The shortcoming of this objective function is obvious since it ignores most occurrences prior to T.

Case 2:

One may assume that it is desirable to maximize the present value of the periodic certainty equivalent stock price changes. Clearly, a certainty equivalent price change should equal $R_f . P_t$ which is given by (4.4'). The objective function is:

$$\max_{\substack{\emptyset_t \\ \forall t}} \sum_{t=1}^{T} \frac{R_f P_t}{(1+R_f)^{t+1}}$$

Case 3:

One may assume that it is desirable to maximize the present value of all marginally and periodically acquired capital obtained through the sale of stock. Then the objective function is:

$$\underset{\substack{\hat{\emptyset}_t \\ \forall t}}{\text{Max}} \sum_{t=0}^{T} \frac{P_t}{(1+R_f)^t}$$

Clearly, this objective function accomplishes about the same thing as the one in Case 2.

Case 4:

One may assume that it is desirable to accomplish some very specific objective. Two examples of this are: (1) The firm already plans to raise large amounts of capital by selling stock at the end of the third and fifth periods of the current planning horizon. Furthermore, twice as much capital will be raised in the fifth period as the third and management would like to cause as little dilution as possible. Their objective might be:

$$\underset{\hat{\emptyset}_4, \hat{\emptyset}_6}{\text{Max}} \; P_3 + 2P_5 \quad \text{where } P_3 \text{ and } P_5 \text{ are given by (4.4)}$$

Of course, an objective function such as this ignores most events in all other periods. This may be a dangerous practice. One might compensate by giving at least nominal consideration to prices in other periods with the following objectives:

$$\underset{\hat{\emptyset}_t \text{ all } t}{\text{Max}} \quad P_3 + 2P_5 + \delta P_0 + \delta P_1 + \delta P_2 + \delta P_4 + \sum_{t=6}^{T} \delta P_t$$

where δ is some chosen parameter between zero and one.
(2) In the second example management has established a target growth rate for the equity value of the firm and seeks a consistent price change that is as close to that growth rate as possible. Suppose the target growth rate is R_g, then the objective function is:

$$\underset{\hat{\emptyset} t \forall t}{\text{Min}} \quad \sum_{t=1}^{T} |(1+R_g)^t P_o - P_t|$$

or alternately:

$$\underset{\hat{\emptyset} t \forall t}{\text{Min}} \quad \sum_{t=1}^{T} [(1+R_p)^t P_o - P_t]^2.$$

The second objective function places a higher penalty on large deviations from the target prices.

<u>Case 5</u>:

This case is based upon the assumption that investors estimate prices by using information about financial variables in all future periods instead of just the period for which the price is being estimated. This is accomplished by defining f as a recursion function where one of the financial variables is an estimated future price as was the case in the dividend model example given earlier. Under these circumstances an appropriate objective function is:

$$\max_{\hat{\phi}_{t \forall t}} P_o$$

where P_o is determined by (4.4) and the financial variables resulting from the capital budget under consideration.

The purpose in this section has been to present a generalized framework basic to the construction of various objective functions. The intent has not been to state one objective function and declare it appropriate for all possible considerations, but instead, one fundamental result of equilibrium theory has been proposed (equation (4.4)) with the hope that it will be the only necessary ingredient in the development of at least a class of objective functions.

Financial Constraints

It is assumed that budgeting controls utilized by the model only apply to funds expected to acquire capital assets. Cash flows resulting from operating expenses are not considered in the financial constraints but do appear in the net cash flows computed for each project. Naturally, these cash flows are considered random variables. Funds not used in any period may be carried forward to later periods, but the interest earned on the funds carried forward is not allowed to serve to increase the total size of the capital budget. Any increase in the total amount of capital assets acquired is the direct result of a conscious decision by management and not an automatic spinoff of the problem formulation.[6] One may define $\overline{C}_{it}$ as the expected cost of project i during period t, and x_i as a zero-one decision variable indicating acceptance or rejection of project i. M_t is the amount of funds that management is willing to commit in period t to the current set of opportunities. Then the constraints are as follows:

$$\sum_{i=1}^{m} \overline{C}_{i1} x_i + \overline{S}_1 = M_1 \tag{4.5}$$

$$\sum_{i=1}^{m} \overline{C}_{it} x_i - \overline{S}_{t-1} + \overline{S}_t = M_t \quad t = 2,3, \ldots, T' \tag{4.5a}$$

Since equations (4.5) and (4.5a) are expressed in terms of expected values, they do not take into account any risk due to variability of costs. However, considering the same equations expressed in terms of random variables one obtains:

$$\sum_{i=1}^{m} \tilde{C}_{i1} x_i + \tilde{S}_1 = M_1 \tag{4.5'}$$

and

$$\sum_{i=1}^{m} \tilde{C}_{it} x_i - \tilde{S}_{t-1} + \tilde{S}_t = M_t \tag{4.5a'}$$

At least for the moment it is assumed that the budget sizes

[6] In Weingartner's horizon model, the financial constraints automatically committed all cash flows to the purchase of capital assets and operating budgets for those projects currently under consideration. The attempt here is to formulate a model that would not take any decisions away from management except the decision of which projects to select.

M_t are fixed and that since the costs $\tilde{C}_{il}$ are random variables representing, for example, startup costs, construction costs, and design costs, much of their variability is likely to be caused by acts of God, or technical problems and are therefore considered uncorrelated with the economic index. One may consequently assume $\text{cov}(\tilde{S}_t, \tilde{R}_m) = 0$. However, the expected contribution of unused funds $\tilde{S}_t$ to the expected cash flows of the firm are not as easily dispensed with since by previous assumption $\tilde{S}_t$ is invested in the riskless asset paying R_f per period if it is positive and is borrowed at rate R_{bt} per period if it is negative.[7] From (r.5') and (4.5a') one obtains:

$$\tilde{S}_t = \sum_{k=1}^{t} M_k - \sum_{k=1}^{t} \sum_{i=1}^{m} \tilde{C}_{ik} x_i$$

which implies from assumption 4 that $\tilde{S}_t$ is normally distributed with mean μ_{S_t} given by:

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[7] R_{bt} may instead be considered a per period cost of a budget overrun regardless how the extra funds are actually acquired.

$$\mu_{S_t} = \sum_{k=1}^{t} M_k - \sum_{k=1}^{t} \sum_{t=1}^{m} \overline{C}_{ik} x_i \qquad (4.6)$$

and using the following notation: $\mathrm{Var}(\tilde{C}_{ik}) = \sigma_{iikk}$; $\mathrm{cov}(\tilde{C}_{ik}, \tilde{C}_{jk}) = \sigma_{ijkk}$; $\mathrm{cov}(\tilde{C}_{ik}, \tilde{C}_{i\ell}) = \sigma_{iik\ell}$; $\mathrm{cov}(\tilde{C}_{ik}, \tilde{C}_{j\ell}) = \sigma_{ijk\ell}$ and $\mathrm{Var}(\tilde{S}_t) = \sigma_{S_t}^{2}$ one can compute $\sigma_{S_t}^{2}$ by:

$$\sigma_{S_t}^{2} = \sum_{=1}^{t} \sum_{j=1}^{m} \sum_{k=1}^{t} \sum_{i=1}^{m} \sigma_{ijk\ell}\, x_i x_j \qquad (4.7)$$

Since, all planned borrowing is a management decision outside the programming model a restriction that is coupled with (4.5) and (4.5a) is that $\overline{S}_t \geq 0 \;\forall\; t$. Notice that $\overline{S}_t = \mu_{S_t}$. Clearly, the contribution of unused funds to cash flows is given by the function $g(S_t)$ below:

$$g(\tilde{S}_t) = \begin{cases} R_f & \text{if } \tilde{S}_t \geq 0 \\ R_{bt} & \text{if } \tilde{S}_t < 0 \end{cases}$$

$h(\tilde{S}_t)$ is defined as the normal density function for $\tilde{S}_t$ with mean μ_{S_t} and variance $\sigma_{S_t}^{2}$ given above. Then the expected contribution of unused funds to cash flows is given by:

$$E[g(\tilde{S}_t)\cdot\tilde{S}_t] = \int_{-\infty}^{0} g(\tilde{S}_t)\tilde{S}_t h(\tilde{S}_t)\,d\tilde{S}_t + \int_{0}^{\infty} g(\tilde{S}_t)\tilde{S}_t h(\tilde{S}_t)\,d\tilde{S}_t$$

$$= R_b \int_{-\infty}^{0} \tilde{S}_t h(\tilde{S}_t)\,d\tilde{S}_t + R_f \int_{0}^{\infty} \tilde{S}_t h(\tilde{S}_t)\,d\tilde{S}_t$$

$$= \frac{(R_f - R_b)\sigma_{S_t}}{\sqrt{2\Pi}}\, e^{-\mu_{S_t}^{2}/2\sigma_{S_t}^{2}} + \mu_{S_t} R_f - \mu_{S_t}(R_f - R_b)H(0) \qquad (4.8)$$

where H(0) is the normal distribution function. It is obvious that at least one of the financial variables in the vector $\hat{\phi}_t$ is net cash flows, and equation (4.8) provides a means of determining the expected net cash flows resulting from slack funds.

The Complete Formulation

In order to keep the statement of the objective function general, it will continue to be stated as simply a function Q of $\hat{\phi}_t$, t = 0, 1, ..., T. Keeping the problem statement in this general form allows flexibility for use of various objective functions such as those specified in cases 1 through 5. This is not meant to imply that solution procedures presented in later chapters for the final formulation will solve the problem for any function Q. For that matter, the same is true of the function $f(\hat{\phi}_t)$. In order to utilize all relationships developed thus far it is assumed that one of the financial variables is total predicted cash flows from all accepted projects. The vector $\hat{\phi}'_t$ represents the original vector $\hat{\phi}_t$ with cash flows removed and handled separately. It is also assumed that there exists vector valued functions of the acceptance-rejection variables x_i, i = 1, ..., m or other related variables that defines the financial vector $\hat{\phi}'_t$. These functions are denoted by $\hat{\phi}_t$.

The complete and general formulation may be expressed as follows:[8]

$$\max_{\substack{\hat{\phi}_t \\ t=1,\dots,T}} \quad Q(\hat{\phi}_0, \hat{\phi}_1, \dots, \hat{\phi}_t) \tag{4.9}$$

s.t.:

$$\hat{\phi}'_t = \theta_t \qquad t = 1, \dots, T \tag{4.9a}$$

$$\bar{\mathcal{E}}_t = \sum_{i=1}^{m} \bar{e}_{it} x_i + E[\tilde{S}_t \cdot g(\tilde{S}_t)] \qquad t = 1, \dots, T \tag{4.9b}$$

$$\sum_{i=1}^{m} \bar{C}_1 x_i + \bar{S}_1 = M_1 \tag{4.9c}$$

$$\sum_{i=1}^{m} \bar{C}_{it} x_i + \bar{S}_t - \bar{S}_{t-1} = M_t \qquad t = 2, \dots, T \tag{4.9d}$$

+ other deterministic resource constraints (4.9e)[9]

+ P.I. constraints (4.9f)[10]

$$x_i = 0 \text{ or } 1 \qquad i = 1, 2, \dots, m \tag{4.9g}$$

The variable $\hat{\mathcal{E}}_t$ represents the net cash flow in period t and is a component of $\hat{\phi}_t$, hence $\hat{\phi}_t$ may be written $\begin{bmatrix} \hat{\mathcal{E}}_t \\ \hat{\phi}'_t \end{bmatrix}$. $\hat{e}_{it}$'s are the anticipated net cash flows in period t owing to project i and $g(\tilde{S}_t)$ is as defined before. Constraints (4.9a) and (4.9b) are definitional constraints and

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[8] A specific example is given later.

[9] Such as those found in Weingartner [50].

[10] Such as those found in Weingartner [50].

are required only so that the objective function may be written as a general function of $\hat{\phi}_t$'s while the actual decision variables are the x_i's. In specific problems, these definitional constraints may not be needed, but for convenience in obtaining solutions other definitional constraints may be needed.

A Specific Model

One test of a general model consists of making specific definitions for the general function in an effort to obtain results identical to those found in the literature. In particular, this is accomplished for Stapleton's model with an analysis included to demonstrate some of the consequences of his assumptions. For this development assumption 2 is discarded:

Again α is a dividend fixed percentage defined as dividends per net cash flow, and $\hat{e}_t$ is a random variable representing estimated cash flows to the firm (for all investments) in period t and treated as an end of period t value. $\hat{e}_t$ has mean $\bar{e}_t$ and covariance with the economy denoted by $\mathrm{cov}(\hat{e}_t, \tilde{R}_m)$. It is assumed that an exact investor estimation model is known and it is a dividend model based upon all estimated dividends through the horizon T. Then,

$$\hat{P}_t = f(\hat{D}_{t+1}, \hat{D}_{t+2}, \ldots, \hat{D}_{t+T}) = \sum_{j=1}^{T} \frac{\hat{D}_{t+j}}{(1+R_f)^j}$$

$$\bar{P}_t = \sum_{j=1}^{T} \frac{\bar{D}_{t+j}}{(1+R_f)^j} \quad \& \quad \mathrm{cov}(\hat{P}_t, \tilde{R}_m) = \sum_{j=1}^{T} \frac{\mathrm{cov}(\hat{D}_{t+j}, \tilde{R}_m)}{(1+R_f)^j}$$

From the above and equation (4.4) one can obtain:

$$P_t = \frac{\displaystyle\sum_{j=1}^{T-1} \frac{\bar{D}_{t+1+j}}{(1+R_f)^j} + \bar{D}_{t+1} - \lambda \left(\sum_{j=1}^{T-1} \frac{\mathrm{cov}(\hat{D}_{t+1+j}, \tilde{R}_m)}{(1+R_f)^j} + \mathrm{cov}(\hat{D}_{t+1}, \tilde{R}_m) \right)}{(1+R_f)^j}$$

$$P_t = \sum_{j=1}^{T} \frac{\overline{D}_{t+j}}{(1+R_f)^j} - \lambda \left\{ \sum_{j=1}^{T} \frac{\mathrm{cov}(\hat{D}_{t+j}, \tilde{R}_m)}{(1+R_f)^j} \right\}$$

$$P_t = \sum_{j=1}^{T} (1+R_f)^{-j} \{\overline{D}_{t+j} - \lambda \,\mathrm{cov}(\hat{D}_{t+j}, \tilde{R}_m)\} \qquad (4.10)$$

Two investor dividend estimation models will be considered.

Case 1:

$\hat{D}_t = \alpha \hat{e}_t$ then (4.10) becomes

$P_t = \alpha \sum_{j=1}^{T} (1+R_f)^{-j} \{\overline{e}_{t+j} - \lambda \,\mathrm{cov}(\hat{e}_{t+j}, \tilde{R}_m)\}$ and in particular

when $t = 0$ the above becomes:

$$P_o = \alpha \sum_{j=1}^{T} (1+R_f)^{-j} \{\overline{e}_j - \lambda \,\mathrm{cov}(\hat{e}_j, \tilde{R}_m)\} \qquad (4.11)$$

If the dividend policy is to pay out all net cash flows then $\alpha = 1$ and equation (4.11) yields an identical equation for P_o with that obtained by Stapleton for the same dividend policy.[11]

Case 2:

Management attempts to stabilize dividends causing investors to estimate dividends by $\hat{D} = \dfrac{(1+R_f)^T R_f}{(1+R_f)^T - 1} \displaystyle\sum_{j=1}^{T} \frac{\alpha \hat{e}_j}{(1+R_f)^j}$ which is an annuity equivalent to the present value of cash flows accruing for the purpose of paying dividends. Then equation (4.10) becomes:

[11] See Stapleton's equation (30a) 44, p. 108 .

$$P_t = \{\overline{D} - \lambda \mathrm{cov}(\hat{D},\tilde{R}_m)\} \sum_{j=1}^{T} (1+R_f)^{-j}$$

Defining earnings in a manner similar to Stapleton, $\hat{Y} = \frac{(1+R_f)^T R_f}{(1+R_f)^T - 1} \sum_{j=1}^{T} \frac{\hat{e}_j}{(1+R_f)^j}$ so that $\hat{D} = \alpha\hat{Y}$ one may evaluate the policy of allowing all earnings to be paid out by letting $\alpha = 1$. This gives:

$$P_o = \alpha \sum_{j=1}^{T} (1+R_f)^{-j}\{\overline{Y} - \lambda \mathrm{cov}(\hat{Y},\tilde{R}_m)\} \quad \text{with } x = 1 \qquad (4.12)$$

Equation (4.12) is again a result identical to Stapleton's.[12] Stapleton argued that equations (4.11) and (4.12) are identical and concluded that dividend policy has no effect upon stock evaluation. However, using his definition of $\hat{Y} = R_f \sum_{j=1}^{T} \frac{\hat{e}_j}{(1+R_f)^j}$ which is a perpetuity and not a finite annuity as was defined above, equality of (4.10) and (4.11) cannot be shown unless the horizon T is infinite. Using the annuity definition of $\hat{Y}$, equality can be shown for any horizon T. It appears from equations (4.11) and (4.12) that the value of the equity of the firm may be arbitrarily increased or decreased by increasing or decreasing α. This is contrary to financial theory under perfect capital markets. However, the net cash flows $\hat{e}_j$ is a function of α

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[12] See Stapleton's equation (32a) [44, p. 108].

such that a decrease in α results in an increase in $\hat{e}_j$ and an increase in α results in a decrease in $\hat{e}_j$ given the investment plan of the firm. Furthermore, under perfect capital markets there is no risk associated with that change in $\hat{e}_j$ since it is the direct result of interest paid or not paid as a result of retaining less or more funds respectively, and that interest is computdd at the risk free rate R_f. Therefore, equations (4.11) and (4.12) do not necessarily contradict financial theory.

In the context of the problem formulation (4.9)-(4.9g) and under the assumed investor prediction model given above and assuming perfect capital markets, then it is clear that the correct approach is to assume management has already established dividend policy which investors observe as a historical average α. It is therefore sufficient to consider only $\sum_{j=1}^{T} (1+R_f)^{-j}[\bar{e}_j - \lambda \text{cov}(\hat{e}_j, \tilde{R}_m)]$ and the firm should continue accepting projects so long as the present value of the certainty equivalent of net cash flows is positive. Lintner and Stapleton would both certainly agree with this conclusion [44, p. 110], [26, pp. 29-33]. The formulation would therefore have no financial constraints under assumed perfect capital markets.

Under the assumption of imperfect capital markets there is clear motivation for management to exercise a policy of capital rationing. Mao also found capital rationing to be widely practiced [28]. Given the dividend policy the problem becomes one of selecting that combination of assets, subject to the financial constraints, that will produce cash flows $\hat{e}_j$ such that

$$\sum_{j=1}^{T} (1+R_f)^{-j}[\bar{e}_j - \lambda \operatorname{cov}(\hat{e}_j, \tilde{R}_m)]$$

is maximized. Evaluation of change in dividend policy may only be accomplished by solving the problem for each policy and comparing the total results.

A Specific Formulation

Assumptions and definitions:

1. Assumptions 1, 2, 4, and 5 stated earlier hold.
2. The bias of the Sharpe model is a linear function of covariance with the market and is corrected by an empirically determined adjustment to λ resulting in a new value λ'.
3. Management engages in a dividend stabilizing policy causing investors to estimate dividends by a perpetuity equivalent to a ratio of all cash flows for an investment horizon T. Hence, $\hat{D}_t = \hat{D} = R_f \sum_{j=1}^{T} \frac{\alpha \hat{e}_j}{(1+R_f)^j}$.
4. Investors estimate equity value by a linear function of T periods of retained cash flows.[13] Hence,

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[13]Retained cash flows are net cash flows minus dividends, so that taxes are taken out of retained cash flows, but, of course, taxes are determined before dividends.

$$P = \beta_1\{\hat{e}_1-\hat{D}\} + \beta_2\{\hat{e}_2-\hat{D}\} \ldots \beta_T\{\hat{e}_T-\hat{D}\} + \epsilon.$$

5. The corporate tax situation is constant over time and new investments are not expected to affect it.
6. $\hat{e}_{1t}$ are net cash flows in period t resulting from current investment commitments.
7. $\hat{e}_{it}$ are net cash flows in period t resulting from project i, i = 2,3, ..., n.
8. $g(\tilde{S}_t).\tilde{S}_t$ are net cash flows in period t resulting from slack funds.
9. $\hat{C}_{it}$ are costs incurred in the acquisition of the capital assets required for project i, i = 1,2, ..., n and t = 1,2, ..., T' where T' is the last period requiring capital investment for this particular budget.
10. There is no autocorrelation between investment costs nor any correlation between investment costs and the economic index used to establish $\tilde{R}_m$.
11. The decision variables are x_i, i = 1,2, ..., n with x_i restricted to values of zero or one. Clearly, x_1 has an additional restriction $x_1 \geq 1$.
12. The funds allocated to this project are fixed at M_1, M_2, ..., $M_{T'}$.
13. Management, having already decided upon the means of supporting the dividend policy and the investment schedule in 12, has determined that sufficient short term borrowing is available at rates R_{bt} for t=1,...,T.

Under these conditions, the problem becomes:

$$\max_{\substack{e_t \\ t=1,\ldots,T \\ x_i \\ i=1,\ldots,n \\ S_t \\ t=1,\ldots,T'}} \quad P_o = \frac{\sum_{k=1}^{T} \beta_k \left\{ \bar{e}_k - \alpha R_f \sum_{j=1}^{T} \frac{\bar{e}_j}{(1+R_f)^j} \right\} + \alpha R_f \sum_{j=1}^{T} \frac{\bar{e}_j}{(1+R_f)^j}}{(1+R_f)} +$$

$$\frac{-\lambda' \left\{ \sum_{k=1}^{T} \beta_k \left(\mathrm{cov}(\hat{e}_k, \tilde{R}_m) - \alpha R_f \sum_{j=1}^{T} \frac{\mathrm{cov}(\hat{e}_j, \tilde{R}_m)}{(1+R_f)^j} \right) \right\}}{(1+R_f)} +$$

$$\frac{-\lambda' \left\{ \mathrm{cov}(\epsilon, \tilde{R}_m) + \alpha R_f \sum_{j=1}^{T} \frac{\mathrm{cov}(\hat{e}_j, \tilde{R}_m)}{(1+R_f)^j} \right\}}{(1+R_f)} \qquad (4.13)$$

Subject to

$$\bar{e}_t = \sum_{i=1}^{n} \bar{e}_{it} x_i + E(\tilde{S}_t \cdot g(\tilde{S}_t)) \quad t=1,2,\ldots,T \qquad (4.13a)$$

$$\mathrm{cov}(\hat{e}_t, \tilde{R}_m) = \sum_{i=1}^{n} \mathrm{cov}(\hat{e}_{it}, \tilde{R}_m) x_i \quad t=1,2,\ldots,T \qquad (4.13b)$$

$$E(\tilde{S}_t \cdot g(\tilde{S}_t)) = \frac{(R_f - R_{bt})\sigma_{S_t}}{\sqrt{2\pi}} e^{-\mu_{S_t}^2/2\sigma_{S_t}^2} +$$

$$\mu_{S_t} R_f - \mu_{S_t}(R_f - R_{bt}) H(0) \qquad (4.13c)$$

$$t=1,2,\ldots,T'-1$$

$$E(\tilde{S}_t \cdot g(\tilde{S}_t)) = \frac{(R_f - R_{bT'})\sigma_{S_{T'}}}{\sqrt{2\pi}} e^{-\mu_{S_{T'}}^2/2\sigma_{S_{T'}}^2} + \mu_{S_{T'}}$$

$$+ \mu_{S_{T'}} R_f - \mu_{S_{T'}}(R_f - R_{bT'}) H(0) \qquad (4.13d)$$

$$t=T'$$

$$E(\tilde{S}_t \cdot g(\tilde{S}_t)) = 0 \quad t \geq T' \qquad (4.13d')$$

$$\mu_{S_t} = \sum_{k=1}^{t} M_k - \sum_{k=1}^{t} \sum_{i=1}^{n} \bar{C}_{ik} x_i \qquad t=1,2,\ldots,T' \qquad (4.13e)$$

$$\sigma_{S_t}^{\ 2} = \sum_{k=1}^{t} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \mathrm{cov}(\tilde{C}_{ik},\tilde{C}_{jk}) x_i x_j +$$

$$\sum_{k=1}^{t} \sum_{i=1}^{n} \mathrm{Var}(\tilde{C}_{ik}) x_i^{\ 2} \qquad t=1,2,\ldots,T' \qquad (4.13f)$$

$$\sum_{i=1}^{n} \bar{C}_{i1} x_i + s_1 = M_1 \qquad (4.13g)$$

$$\sum_{i=1}^{n} \bar{C}_{it} x_i + s_t - s_{t-1} = M_t \qquad i=2,\ldots,T' \qquad (4.13h)$$

\+ other deterministic resource constraints (4.13i)

\+ P.I.C. (4.13j)

$$x_i = 0 \text{ or } 1 \qquad i=1,2,\ldots,\ell \text{ and } 0\leq x \leq 1 \quad i=\ell+1,\ \ldots,\ n \qquad (4.13k)$$

$$x_1 \geq 1 \qquad (4.13l)$$

$$s_t \geq 0 \qquad (4.13m)$$

One may observe that $\bar{S}_t$ in equation (4.13g) and (4.13h) equals μ_{S_t} given in equation (4.13e); therefore, (4.13e) may be dropped provided μ_{S_t} in equation (4.13c) and (4.13d) are replaced by $\bar{S}_t$. Also, the objective function is a linear function of $\bar{e}_t$ and $\mathrm{cov}(\hat{e}_t,\tilde{R}_m)$ which are in turn linear functions of x_i. The only nonlinear equations are constraints (4.13c), (4.13d) and (4.13f) with (4.13f) being the only non-linear equation involving the integer variables x_i.

In order to get a better understanding of the problem form, the following is a simplification and restatement of problem (4.13)[14] divorced from the special notation of the capital budgeting problem:

$$\max_{\overline{X},\overline{Z},\overline{S}} \quad P_o = \overline{A}'\overline{X} + \sum_{j=1}^{k} b_j\left[\frac{d_j e^{-s_j^2/2z_j^2}}{\sqrt{2\Pi}} \cdot z_j + R_{fj}s_j - s_j d_j H(0,s_j,z_j)\right] \tag{4.14}$$

$$\text{s.t.} \quad z_j^2 = \overline{X}'\overline{\sigma}_j\overline{X} \qquad j=1,2,\ldots,k \tag{4.14a}$$

$$\overline{C}_1'\overline{X}+s_1 = M_1 \tag{4.14b}$$

$$\overline{C}_j'\overline{X}+s_j-s_{j-1} = M_j \qquad j=2,\ldots,k \tag{4.14c}$$

$$\overline{G}\,\overline{X} \le \overline{q} \;\} \text{ Resource and P.I. Constraints} \tag{4.14d}$$

$\overline{X}$ an n-vector of zero-one variables and/or variables simply bound by zero and one (4.14e)

$\overline{S} \ge 0$ and a k vector (4.14f)

$\overline{Z} \ge 0$ and a k vector (4.14g)

where: $\overline{A}$ is an n-vector of constants

$\overline{\sigma}_j$ is an nxn matrix of constants for each j

$\overline{C}_j$ is an n-vector of constants for each j

$\overline{G}$ is an nxm matrix of constants

$\overline{q}$ is an m-vector of constants

b_j, d_j, e, R_f are all constants

and $H(0;s_j,z_i)$ is:

[14]Appendix C establishes this fact.

$$\frac{1}{\sqrt{2\pi}\, z_j} \int_{-\infty}^{0} e^{-(t-s_j)^2/2z_j^2}\, dt$$

Summary

The primary effort in this chapter has been directed towards the formulation of a mathematical programming model for the capital budgeting problem under conditions of uncertainty. In so doing, knowledge about the deterministic capital budgeting problem and about portfolio investment and equilibrium theory have been synthesized into a unified and generalized model. Its generalized nature has been tested and validated by an analysis of a specific problem structure that produced results identical to those found in the literature. Finally, a specific problem formulation was presented which complies with the general model and some linearity assumptions. It is this specific model whose data requirements and solution procedures have been investigated and presented in the remaining chapters with some specific extensions mentioned in the final chapter.

CHAPTER V

DATA REQUIREMENTS

Before proceeding with the solution techniques for a model one should reflect upon the data requirements and how they may be met. No new and exotic techniques of data estimation are presented herein but rather an attempt is made to identify those techniques that are already available and that will provide the necessary data for the model described by Equations 4.14 through 4.14g. At first glance it appears that those parameters that must be estimated are α (percent of cash flows paid in dividends), β_j (coefficients of the linear investor estimate model), λ' (the empirically determined "market price of risk") as well as the various project parameters such as expected costs, covariances of costs, expected net cash flows, and the covariance of these cash flows with a market index. However, the linear investor estimate model produces an equilibrium stock price objective function that is linear with respect to expected cash flows and covariances; therefore, one will find it convenient to estimate the b_j coefficients directly instead of indirectly via the estimation of α, β_j, and λ'. Hence, only five specific types of data are required: (1) b_j (a coefficient that weights each period's cash flows), (2) expected costs for each project, (3) expected cash flows for each project, (4) covariances of costs, and (5) covariance of cash flows with a market index.

Two of these, Items (2) and (3), are typically required by deterministic capital budget programming formulation, while chance constraint models also require Item (4). The portfolio approach to the capital budgeting problems generally requires Items (2), (3), and a complete covariance matrix for cash flows which is considerably more data than Item (5), and yet does not consider the risks that are handled by chance constraint models and by Formulation (4.14). One may conclude that the only data requirement for Formulation (4.14) that is out of the ordinary is Item (1), the b_j coefficients. The sections that follow discuss ways of meeting the data requirements with particular emphasis on b_j coefficients.

Item (1) b_j

It can be shown under the assumption that led to Formulation (4.14) that the objective function may be written in the following form:[1]

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[1]See Equations C-1 through C-4 of Appendix C.

$$P_o = \sum_{i=1}^{n} \left\{ \sum_{j=1}^{T} b_j(\overline{e}_{ij} - \lambda' \ cov(\hat{e}_{ij}, \tilde{R}_m)) \right\} x_i$$

$$+ \sum_{j=1}^{T} b_j E(\tilde{S}_j \cdot g(\tilde{S}_j)) + \frac{-\lambda'}{1+R_f} cov(\epsilon, \tilde{R}_m) \qquad (5.1)$$

(5.1) is an equation which gives the equilibrium aggregate stock value when investments, expected cash flows, covariance of cash flows with the market portfolio, and expected cash flows from slack funds are known. The x_i variables are there simply to facilitate the calculation of equilibrium values for various combinations of investments. However, past investments are known so that (5.1) may be reduced to the following:

$$P_o = \sum_{j=1}^{T} b_j(\overline{e}_j - \lambda' \ cov(\tilde{e}_j, \tilde{R}_m)) + \frac{-\lambda'}{1+R_f} cov(\epsilon, \tilde{R}_m) \qquad (5.2)$$

where:

$\tilde{e}_j$ = net cash flows in period j from all of the investments including slack (~ denotes a random variable)

$\overline{e}_j$ = expected value of $\tilde{e}_j$.

Finally (5.2) may be reduced to an ordinary multiple regression equation as follows:

$$P_o = b_o + b_1 y_1 + b_2 y_2 + \dots + b_T y_T + \epsilon \qquad (5.3)$$

where:

b_o is taken to be $\frac{-\lambda'}{1+R_f} cov(\epsilon, \tilde{R}_m)$, a constant

y_j is a coded random variable equal to $\overline{e}_j - \lambda' \ cov(\tilde{e}_j, \tilde{R}_m)$, and may be described as a certainty equivalent of cash flows.

There are basically two ways of estimating the coefficients (b_j), (1) subjective estimates of each or (2) mathematical estimations from past data. Subjective estimates are perhaps the least desirable but subjective control over the form of the mathematical estimates should be exercised. For example, it is shown in later chapters that a global optimum to formulation 4.14 can be obtained only when the coefficients b_1, b_2, ..., b_T are non-negative. Furthermore, since the y_j variables are certainty equivalents of cash flows, the restriction that their coefficients be non-negative is both logical and realistic. Hence, if least squares techniques are applied to (5.3), then they should incorporate the added constraints that $b_j \geq 0$ for $j = 1, 2, \ldots, T$. It is also logical that those certainty equivalents of cash flows occurring in the least distant future with respect to the timing of P_o would have the greater effect upon the equilibrium price. This concept may be incorporated by adding constraints of the form $b_j \geq b_{j+1}$, $j = 1, 2, \ldots, T-1$, to the least squares minimization problem.

The intent here has been to suggest a means of establishing numerical values for the coefficients, b_j, that cannot be rejected on logical or theoretical grounds. Statistically, the numerical estimates for each b_j obtained by the above method possesses many difficulties. The most important of these is that the data cannot be obtained in such a manner that it would meet the definition of a random sample, particularly for large T. All this means is that, confidence intervals, or tests of significance, cannot be validly employed. However, one must recognize three facts that keep the above estimation procedures in the proper perspective. (1) Equation (5.3) is the result of equilibrium theory and concepts which produced the formulation in Chapter IV. (2) The intent is not to provide a statistical validation of equilibrium theory[2] but to provide numerical values for the b_j's that (a) comply with theory and logic, and that (b) best fit what has actually happened in the past. (3) Management's confidence in the correctness of the values of b_j, $j = 0,1, \ldots, T$, provides the ultimate determination of the usefulness and application of the solutions obtained from the formulation (4.14).[3]

Items (3) and (5) Data

Items (3) and (5) are values that must be estimated for all projects under consideration. Hertz (19) and Smith (43) have each suggested Monte Carlo type simulation techniques as a means of generating distribution of rates of return or present values of projects being simulated after each component cash flow's distribution parameters have been estimated. Generally, it is better to estimate values for

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[2]This type of research, though incomplete, has already been undertaken by many, see ref. 8, 13, 21, 40, 41.

[3]The effects of various types of incorrect values of b_j $\forall j$ upon solutions obtained for (4.14) are discussed in Chapter VIII.

each component cash flow and then combine the estimates to obtain total cash flows than it is to estimate the total flows directly (see Chapter XIV, ref. 46). The reasons for this are clear: (1) Estimation errors made on component estimates may tend to cancel each other when combined to provide a total estimate, and (2) although a project may be totally new so that management has no experience with it, it will still possess component parts with which management has a great deal of experience and can bring that experience to bear on producing more accurate estimates of those component parts. The components relevant to items (3) and (5) are those that comprise revenues and operating costs. Investment cost data, or items (2) and (4), could also be generated via a simulation routine, but an alternate scheme is also presented in the next section.

The specific simulation schemes presented by Hertz and Smith will not provide the exact data required for the formulation in Chapter IV; however, one must recognize that simulation is a methodology that may easily be adapted to specific needs. It is sufficient, then, to note that acceptable technology does exist that may be used to satisfy the data requirements described by items (3) and (5).

Items (2) and (4) Data

The procedures presented here for obtaining data to satisfy requirements described by items (2) and (4) are combinations of subjective estimation and a use of past data.

They are actually an adaptation of simplifying procedures for handling means, variances, and covariances of stock returns used by Sharpe in a portfolio selection model (see 40, Ch. 7). It is assumed that the expected costs and variances of costs required to implement a project have already been estimated for each period either subjectively or by a simulation routine. The covariances are virtually impossible to estimate subjectively since it is difficult for one to make these estimates in such a way as to guarantee that the covariance matrix will be positive definite. It is possible to generate the covariances in a simulation routine, but does greatly complicate the routine and ultimately must rely upon the same kind of model presented below.

The basic concept is to assume that the deviation of the actual cost of a project from the expected cost is at least partially the responsibility of the management team who specified the estimates, fixed the budgets, and strove to implement the projects within the assigned budgets. Hence, a linear relationship is assumed.

$$\tilde{C}_{ij} = \alpha_{ij} + \beta_{ij}\tilde{I} + \tilde{\epsilon}_{ij} \tag{5.4}$$

where:

~ denotes random variables

$\tilde{C}_{ij}$ = cost of project i in period j and whose mean and variance ($\sigma^2_{C_{ij}}$) is given

α_{ij} = a constant whose value is not needed

β_{ij} = a constant whose value is estimated subjectively

$\tilde{\epsilon}_{ij}$ = an error term with mean 0 and variance $\sigma^2_{\epsilon_{ij}}$

$\tilde{I}$ = an index given by the ratio of actual project costs to estimated expected costs. This is, in effect, an index of management performance and its mean and variance, σ^2_I may be estimated from historical data concerning all known costs for projects previously undertaken by the management team.

Equation (5.4) implies (5.5), a relationship between variances.

$$\sigma^2_{C_{ij}} = \beta^2_{ij}\sigma^2_I + \sigma^2_{\epsilon_{ij}} \tag{5.5}$$

Since $\sigma^2_{C_{ij}}$ and σ^2_I are known, then a subjective estimate for either β_{ij} or $\sigma^2_{\epsilon_{ij}}$ allows the one not estimated to be computed. This subjective estimate may be easier to determine than one might anticipate. To illustrate this, one may consider two extreme examples. In the first example the project is to purchase a tractor and trailor and place it into service. In this case estimated costs have been achieved by contacting prospective sellers. In the event actual costs deviate from estimated costs either positively or negatively it is difficult to imagine how that deviation could have been caused by the efforts of management. Therefore, one might conclude that $\sigma^2_{\epsilon_{ij}}$ = 90% to 100% of $\sigma^2_{C_{ij}}$. In other words, this states that 90% to 100% of the variation of costs is independent of the effects of management. β_{ij} can now be computed. In the second example the project is a completely new production facility based on some new technology. With construction costs, plant layout costs, and potentially heavy start up costs, it is easy to see that the management function would be deep involvement in all phases. This might lead one to conclude that management would be responsible for 80% to 90% of actual cost deviations from estimated costs, or alternatively that 10% to 20% of cost deviations are independent of managerial efforts.

In either of these examples, equation (5.5), together with previous estimates of the overall variance of costs allows one to determine β_{ij}. The computational savings achieved by this linear representation are the result of the standard assumption of regression analysis, specifically that the error terms are independent of the index I, and by Sharpe's additional proposed assumption that all covariability between the dependent random variable of the various regression equations is fully explained by their common relationship with the index I. Under these assumptions it can be shown that the covariance between any two project costs for a period j is given by $\mathrm{Cov}(\tilde{C}_{ij},\tilde{C}_{ki}) = \beta_{ij}\beta_{kj}\sigma_I^2$. (5.6)

One additional convenience of the approach given above occurs when one decides to drop the assumption, made in Chapter IV, that there is no covariance between costs in different periods. In that case the relevant covariances may be computed without making additional estimates. The formulas are given by:

$$\text{Cov}\ (\tilde{C}_{ij},\tilde{C}_{i\ell}) = \beta_{ij}\beta_{i\ell}\sigma_I^2 \tag{5.7a}$$

and

$$\text{Cov}\ (\tilde{C}_{ij},\tilde{C}_{k\ell}) = \beta_{ij}\beta_{k\ell}\sigma_I^2 \tag{5.7b}$$

All covariance matrices constructed by the methods just described will automatically be positive definite; however, as Cohen and Pogue (8) have shown, a defect in Sharpe's assumption will mean that actual covariances are always greater than those computed by (5.6), (5.7a), and (5.7b). This tendency to underestimate the degree of dependence between project costs will result in covariance matrices that tend to underestimate the variance of the total costs associated with any particular combination of projects in the optimization model presented in Chapter IV and solved in later chapters. The consistent bias might be corrected via the use of an appropriate multiplier or more properly by extending the regression model (5.4) to some multiple regression model based on more than one index. A two index model was used for the purpose of constructing covariance matrices for some of the test problems discussed later.

Conclusion

It appears that the formulation in Chapter IV requires parameter values which are obtainable by techniques that are known to exist and that have been used. For every parameter, except those dealing with the investor estimate model, there exists another accepted capital budgeting formulation that also requires that parameter. Those parameters dealing with the investor estimate model (i.e., b_j and λ') are in a sense also required by Stapleton's decision model (see ref. 44) but his assumption of a dividend valuation model is more restrictive than the proposed linear investor estimate model.

It is hereforth assumed for the remaining chapters that the data requirements for the formulation in Chapter IV can be achieved.

CHAPTER VI

SOLUTION PROCEDURES

In Chapter V methods were discussed for meeting the data requirements for a specific form of the problem formulated in Chapter IV. This chapter contains the general solution procedures that can be used after some modifications and transformations have been applied to the problem. Appendix A provides the description of the possible modifications and transformations while Appendix B provides the proofs of the conditions required for these applications. A final form of the problem is then presented in such a state that numerical values may be added and the solution procedures begun directly. Chapter VII presents the results of direct application of these procedures to a number of sample problems.

Generalized Benders

In 1962 J. F. Benders presented a procedure for partitioning semilinear programming problems (2). Although the problem in Chapter IV may be classified as semilinear and therefore of the type that Benders proposed to solve, his technique is not applicable due to the mixed integer nature of the sub-problem derived by his partitioning. However, in 1969, Geoffrion applied some results in nonlinear duality theory (16) that allowed him to generalize Benders' decomposition (or partitioning) procedures, making them applicable to a larger class of problems. For this work, it is not important that a larger class of problems may be solved, but what is important is that the sub-problem need no longer be a linear problem although the subproblem variables must still be continuous. This allows the problem to be decomposed in a reverse manner to that prescribed by Benders resulting in a nonlinear, but continuous, subproblem and a mixed integer, but linear, master problem. Such a problem can be solved by Geoffrion's Generalized Benders provided solution techniques exist for both the subproblem and the master problem and so long as optimal dual variables can be obtained for the subproblem as well as the total problem exhibiting Geoffrion's "Property P."

Because of the integral part that Generalized Benders procedures have in the solution to the problem in Chapter IV, it is completely described below rather than requiring one to refer directly to the original papers. The following description is taken directly from Geoffrion's works (14, 15) but with some notational changes to comply more closely with preceding chapters.

Given the problem:

$$\underset{\overline{X},\overline{Y}}{\text{Max}} \quad f(\overline{X},\overline{Y}) \tag{6.1}$$

$$\text{s.t.:} \quad G(\overline{X},\overline{Y}) \geq 0$$

$$\overline{X} \in X$$
$$\overline{Y} \in Y$$

where:

$f(\overline{X},\overline{Y})$ is a scalar valued function and

$G(\overline{X},\overline{Y})$ is a vector valued function and

X and Y are sets that further constrain the feasible values of the variables. Typically X and Y are used to indicate restraints that may not be or need not be functually stated. For example,

$X = \{\overline{X} \mid \overline{X}$ is an n-vector and $\overline{X} \geq 0\}$

$X = \{\overline{X} \mid \overline{X}$ is an n-vector and $0 \leq \overline{X} \leq 1$ and the first k elements of $\overline{X}$ are integer$\}$

$Y = \{\overline{Y} \mid \overline{Y} \in R^m\}$.

The concept of "partitioning" is to project problem (6.1) into either x-space of y-space. In this case it will be appropriate to project into x-space. The projected problem is as follows:

$$\underset{\overline{X}}{\text{Max}}\ v(\overline{X}) \quad \text{Subject to } \overline{X} \in X \cap V \tag{6.2}$$

where

$$v(\overline{X}) \quad \underset{\overline{Y}}{\text{Supremum}}\ f(\overline{X},\overline{Y}) \tag{6.3}$$

$$\text{s.t.} \quad G(\overline{X},\overline{Y}) \geq 0$$
$$\overline{Y} \in Y$$

and

$$V = \{\overline{X} \mid G(\overline{X},\overline{Y}) \geq 0 \text{ for some } \overline{Y} \in Y\} \tag{6.4}$$

It is clear that for each value of $\overline{X}$ that one wishes to evaluate the objective function in (6.2) one must solve a maximization problem in $\overline{Y}$ given by (6.3) (frequently referred to as the subproblem) and that the set V simply insures that one does not attempt to evaluate the objective function in (6.2) at a value for $\overline{X}$ for which a corresponding feasible solution for the problem indicated by (6.3) does not exist. Intuitively, one can see that (6.2) is equivalent to (6.1); however, Geoffrion has formally shown the equivalence of the projected problem to the original problem. Even so a solution technique certainly does not appear evident from (6.2), (6.3), and (6.4). The major difficulties being the determination and or representation of the set V and the function $v(\overline{X})$ in a computationally useful manner. It is to this purpose that Geoffrion states and proves two theorems which are restated, without proof, below and in a notation that is partially Geoffrion's and partially specialized to fit with the notation in other chapters.

V Representation Theorem: Assume that Υ is a nonempty convex set and that G is concave on Υ for each fixed $\overline{X} \in X$. Assume further that the set $Z_y = \{ \overline{Z} \in R^m \mid G(\overline{X},\overline{Y}) \geq \overline{Z}$ for some $\overline{Y} \in \Upsilon\}$ is closed for each fixed $\overline{X} \in X$. Then a point $\overline{X} \in X$ is also in the set V iff $\overline{X}$ satisfies the (infinite) system:

$$[\underset{\overline{Y} \in \Upsilon}{\text{supremum}}\ \overline{\lambda}^t G(\overline{X},\overline{Y})] \geq 0, \text{ all } \overline{\lambda} \in \Lambda$$

where $\Lambda = \{ \overline{\lambda} \in R^m \mid \overline{\lambda} \geq 0 + \sum_{i=1}^{m} \lambda_i = 1\}.$

<u>v Representation Theorem</u>: Assume that Y is a nonempty convex set and that f and G are concave on Y for each fixed $\overline{X} \in X$. Assume further that, for each fixed $\overline{X}^k \in X \cap V$, at least one of the following three conditions holds:

(a) $v(\overline{X}^k)$ is finite and the problem indicated by (6.3) (i.e., the subproblem) possesses an optimal multiplier vector;

(b) $v(\overline{X}^k)$ is finite, $G(\overline{X}^k,\overline{Y})$ and $f(\overline{X}^k,\overline{Y})$ are, continuous on Y, Y is closed, and the ϵ-optimal solution set of the subproblem is nonempty and bounded for some $\epsilon \geq 0$;

(c) $v(\overline{X}^k) = +\infty$.

Then the optimal value of the subproblem equals that of its dual on $X \cap V$, that is

$$v(\overline{X}) = \underset{\overline{U} \geq 0}{\text{Infimum}} \, [\underset{\overline{Y} \in Y}{\text{Supremum}}(f(\overline{X},\overline{Y}) + \overline{U}^t G(\overline{X},\overline{Y}))]$$

for all $\overline{X} \in X \cap V$.

One should recognize that the only important parts of these two theorems are the results and not the assumptions required to prove those results for particular cases. What this means specifically is that any particular problem may not meet the assumptions of the v Representative Theorm and yet the results of that theorem may be true for that problem. Therefore, in making applications in specific instances one needs to check only the results of this theorem for validity by whatever means available and not simply by varifying the assumption of the theorem. Furthermore, the V Representative Theorem is not even necessary so long as one has a useful means of being able to represent the set V as defined earlier. The theorem merely suggests one such means that is known to work under the conditions specified in the theorem. Remembering that problem (6.2) was

$$\underset{\overline{X}\epsilon X\cap V}{\text{Maximize}}\ v(\overline{X})$$

and assuming the results of the v Representation Theorem to be true one has the following:

$$\underset{\overline{X}\epsilon X\cap V}{\text{Maximize}}[\underset{\overline{U}\geq 0}{\text{Infimum}}[\underset{\overline{Y}\ \epsilon\ Y}{\text{Supremum}}(f(\overline{X},\overline{Y}) + \overline{U}^t G(\overline{X},\overline{Y}))]] \qquad (6.5)$$

or, using infinum as the greatest lower bound,

$$\underset{\substack{\overline{X}\epsilon X\cap V\\ r}}{\text{Maximize}}\quad r \qquad (6.6)$$

Subject to:

$$r \leq \underset{\overline{Y}\ \epsilon\ Y}{\text{Supremum}}\{f(\overline{X},\overline{Y}) + \overline{U}^t G(\overline{X},\overline{Y})\},\ \text{for all } \overline{U}\geq 0$$

and if the V Representation Theorem is true one obtains:

$$\underset{\substack{\overline{X}\ \epsilon\ X\\ r}}{\text{Maximize}}\quad r \qquad \begin{matrix}(6.7)\\(6.7a)\end{matrix}$$

$$\text{s.t.:}\quad r \leq \underset{\overline{Y}\epsilon\ Y}{\text{Supremum}}\{f(\overline{X},\overline{Y}) + \overline{U}^t G(\overline{X},\overline{Y})\},\ \text{for all } \overline{U} \geq 0 \qquad (6.7b)$$

$$\underset{\overline{Y}\ \epsilon\ Y}{\text{Supremum}}\{\overline{\lambda}^t G(\overline{X},\overline{Y})\} \geq\ 0,\ \text{for all } \overline{\lambda}\ \epsilon \Lambda \qquad (6.7c)$$

One might also note at this point that if some particular constraint, $g_i(\overline{X},\overline{Y}) \geq 0$, is actually of the form $g_i(\overline{X}) \geq 0$, (i.e., independent of $\overline{Y}$) then for any $\overline{\lambda}\ \epsilon \Lambda$ the following is true:

$$\underset{\overline{Y} \in Y}{\text{Supremum}} \{\lambda_i g_i(\overline{X})\} = \lambda_i g_i(\overline{X})$$

and since $\lambda_i \geq 0$, the i^{th} element in the system (6.7c) may be reduced to $g_i(\overline{X}) \geq 0$, the original constraint.

The problem form given by (6.7a), (6.7b), and (6.7c) is very close to a form that can be solved by a relaxation technique. Actually, the only step remaining is to be able to express the right sides of the system (6.7b) and the left sides of the system (6.7c) as some mathematical function of the variable vector $\overline{X}$. This is possible providing the original problem (6.1) exhibits a property which Geoffrion has called "Property P," stated below:

> Property P: For every $\overline{U} \geq 0$, the supremum of $f(\overline{X},\overline{Y}) + \overline{U}^t G(\overline{X},\overline{Y})$ over Y can be taken essentially independently of $\overline{X}$, and for every $\overline{\lambda} \in \Lambda$, the supremum of $\overline{\lambda}^t G(\overline{X},\overline{Y})$ over Y can be taken essentially independently of $\overline{X}$.

As long as the problem exhibits "Property P" then any algorithm which finds the optimal $\overline{Y}$ and $\overline{U}$ for the subproblem and its dual may be used and the right sides of system (6.7b) and left side of system (6.7c) are generally expressable in a functional form of $\overline{X}$. One notable example of this is the semilinear programming problem which, because all functions of $\overline{X}$ and $\overline{Y}$ are linearly separable, always possess "Property P." Furthermore, if $\overline{Y}^k$, $\overline{U}^k$ are optimal primal and dual variables of the subproblem for some fixed $\overline{X}^k$, then the right side of one constraint in system (6.7b) is given by

$$f(\overline{X},\overline{Y}^k) + (\overline{U}^k)^t G(\overline{X},\overline{Y}^k) \tag{6.7b'}$$

which is strictly a function of $\overline{X}$ and is in the proper form.

If no feasible solution exists for the subproblem when $\overline{X}$ is fixed at a specific value $\overline{X}^k$ then at least one constraint in the system (6.7c) is violated by $\overline{X}^k$ so that optimal multipliers $\overline{\lambda}^k$ and optimal values $\overline{Y}^k$ are needed. The left side of the constraint in system (6.7c) then becomes

$$(\overline{\lambda}^k)^t G(\overline{X},\overline{Y}^k)$$

which again is a function of $\overline{X}$ and is in a useful form. Geoffrion uses the term L/dual adequate to refer to algorithms which can solve the subproblem in such a way as to produce primal and dual variables and that will also produce functional forms for the right and left sides of the system (6.7b) and (6.7c). The L refers to the existence of the appropriate functional form which is guaranteed when "Property P" is present and a dual-adequate algorithm exists. One may note that the L may also properly refer to the Lagrangean of the subproblem.

One may now suppose that all conditions are met that allow one to express problem (6.7) such that (6.7b) is a system of ordinary mathematical constraints (one for each $\overline{U} \geq 0$) and (6.7c) is a system of ordinary mathematical constraints (one for each $\overline{\lambda} \in \Lambda$). Clearly, such a supposition is not helpful since there are conceivably infinitely many $\overline{U} \geq 0$ and infinitely many $\overline{\lambda} \in \Lambda$. This means that problem (6.7) may have infinitely many constraints. Relaxation appears to be the only means of solving such a formidable problem. To investigate this further one may assume that he has found a value for the vector $\overline{X}$, call it $\overline{X}^k$, and a value for r, call it r^k, which he feels is a candidate solution to problem (6.7).

Exactly one of two possibilities is true:

1. $\exists\ \overline{Y} \in \Upsilon \ni G(\overline{X}^k, \overline{Y}) \geq 0$

or 2. $\not\exists\ \overline{Y} \in \Upsilon \ni G(\overline{X}^k, \overline{Y}) \geq 0$

If 1. is true then the entire system (infinite or not (6.7c) is satisfied by definition. Furthermore, the entire system (6.7b) is also satisfied unless $\exists\ \overline{U} \geq 0 \ni r^k > \underset{\overline{Y} \in \Upsilon}{\text{Supremum}}$ $\{f(\overline{X}^k, \overline{Y}) + \overline{U}^t G(\overline{X}^k \overline{Y})\}$ and such a $\overline{U} \geq 0$ exists if and only if

$$\underset{\overline{U} \geq 0}{\text{Infimum}}\ \underset{\overline{Y} \in \Upsilon}{\text{Supremum}}\{f(\overline{X}^k, \overline{Y}) + \overline{U}^t G(\overline{X}^k, \overline{Y})\} < r^k. \tag{6.8}$$

But this implies under the hypothesis of the v Representative Theorem that one need only to solve the subproblem:

$$\underset{\overline{Y} \in \Upsilon}{\text{Max}}\ z = f(\overline{X}^k, \overline{Y}) \tag{6.9}$$

$$\text{s.t.: } G(\overline{X}^k, \overline{Y}) \geq 0$$

and compare the optimal value z^* with r^k. If $z^* \geq r^k$ then no $\overline{U} \geq 0$ exists for which a constraint in the system (6.7b) is violated and $(\overline{X}^k, r^k)$ is an optimal solution to (6.7). If $z^* < r^k$ then for at least the optimal dual variables of (6.9) a constraint in (6.7b) is violated and may therefore be added to problem (6.7).

If 2. is true then that fact becomes evident while attempting to solve (6.9). If the algorithm used to solve (6.9) is a Two Phase method then hopefully at the end of Phase I multipliers $\overline{\lambda} \in \Lambda$ may be obtained and the constraint in system (6.7c) that is violated by $(\overline{X}^k, r^k)$ may be generated and added to problem (6.7). One should recognize that continuous repetition of the process being described is a relaxation procedure which will increase the size of problem (6.7) by one constraint each time one of the many constraints not included is violated. The specific solution steps are found in Appendix A and again in Chapter VII.

Application of Generalized Benders

Under the assumptions in Chapter IV a specific capital budgeting formulation was derived and expressed primarily in matrix form in problem (4.14). The problem is restated below with the following notational simplification: ($\overline{S},\overline{Z}$ denotes vectors)

$$f(\overline{S},\overline{Z}) = \sum_{j=1}^{k} b_j\left[\frac{d_j e^{-s_j^2/2z_j^2}}{\sqrt{2\Pi}} \cdot z_j + R_f s_j - s_j d_j \int_{-\infty}^{o} \frac{1}{\sqrt{2\Pi} z_j} e^{-(t-s_j)^2/2z_j^2} dt\right]$$

where from the discussion of Chapter V one concludes that b_j, d_j, and R_f are all known and constant quantities. Problem (4.14) then becomes

$$\underset{\overline{X},\overline{Z},\overline{S}}{\text{Maximize}}\ P_o = \overline{A}^t\overline{X} + f(\overline{S},\overline{Z}) \tag{6.10}$$

s.t.:

$$z_j = \sqrt{\overline{X}^t\overline{\sigma}_j\overline{X}} \qquad j=1, \ldots, k \tag{6.10a}$$

$$\overline{C}_1^t\overline{X} + s_1 = M_1 \tag{6.10b}$$

$$\overline{C}_j^t\overline{X} + s_j - s_{j-1} = M_j \quad j=2, \ldots, k \tag{6.10c}$$

$$\overline{G}\ \overline{X} \le \overline{q} \tag{6.10d}$$

$$\overline{S} \ge 0 \tag{6.10e}$$

$$\overline{Z} \ge 0 \tag{6.10f}$$

$$0 \le \overline{X} \le 1 \text{ (mixed integer vector)} \qquad (6.10g)$$

By applying the conjugate matrix transformations discussed in Appendix A to the system of constraints (6.10a) one may replace that system with the following expanded system:

$$z_j = \sqrt{\overline{Y}_j^t \overline{D}_j \overline{Y}_j} \qquad j = 1,2, \ldots, k \qquad (6.10a')$$

$$\overline{Y}_j = \overline{E}_j^{-1} \overline{X} \qquad j = 1,2, \ldots, k \qquad (6.10a'')$$

Clearly all z_j in $f(\overline{S},\overline{Z})$ may be replaced by functions of y_{ij} which are, themselves, linear functions of $\overline{X}$. Furthermore, by successive substitution the system (6.10b) and (6.10c) becomes:

$$s_j = \sum_{i=1}^{j} M_i - \sum_{i=1}^{j} \overline{C}_i^t \overline{X} \quad j = 1, \ldots, k \qquad (6.10bc)$$

Therefore, $f(\overline{S},\overline{Z})$ is a function of linear functions of $\overline{X}$ as required for the manipulations and substitutions in Appendix A.

Moreover, a slightly different linear transformation than the one used in (6.10a") may be used to simplify (6.10a'). Since the expression $\overline{Y}_j^t \overline{D}_j \overline{Y}_j$ [1] may be written

$$\sum_{i=1}^{n} y_{ij}^2 d_{iij} = \sum_{i=1}^{n} (\sqrt{d_{iij}}\, y_{ij})^2$$

and since $\overline{Y}_j = \overline{E}_j^{-1} \overline{X}$ may be written $y_{ij} = \sum_{k=1}^{n} e_{ikj} x_k$, where e_{ikj} is the i,k^{th} element of the square matrix $\overline{E}_j^{-1}$,

---

[1] $\sqrt{d_{iij}}$ will always be real since a covariance matrix is always a positive definite matrix and a conjugate matrix transformation will always produce positive diagonal elements when applied to a positive definite matrix.

then it is clear that defining $y'_{ij} = \sqrt{d_{iij}}\,y_{ij} = \sqrt{d_{iij}} \sum_{k=1}^{n} e_{ikj}x_k$ simplifies (6.10a') to $z_j = \sqrt{(\overline{Y}'_j)^t\overline{Y}'_j} = \sqrt{\sum_{i=1}^{n} y'^{2}_{ij}}$. To simplify notation the (') will be dropped and (6.10a') and (6.10a") are restated in their final forms:

$$z_j = \sqrt{\overline{Y}_j^t\overline{Y}_j} \qquad j = 1,2,\ \ldots,\ k \qquad (6.10a''')$$

$$\overline{Y}_j = \overline{E}_j^t\overline{X} \qquad j = 1,2,\ \ldots,\ k \qquad (6.10a'^{v})$$

where $\overline{E}_j$ is redefined as the appropriate linear transformation matrix as described in the preceding discussions.

In Appendix B it is proved that $f(\overline{S},\overline{Z})$ is a monotonic decreasing function of each z_j therefore making it possible to replace (6.10a''') with the following:

$$z_j \geq \sqrt{\overline{Y}_j^t\overline{Y}_j} \qquad (6.10a^{v})$$

without any loss in generality.

This leads to the final form of the problem to be solved given below by (6.10').

$$\underset{\overline{X},\overline{Y},\overline{Z},\overline{S}}{\underline{\text{Maximize}}}\ P_o = \overline{A}^t\overline{X} + f(\overline{S},\overline{Z}) \qquad (6.10')$$

$$\text{s.t.:} \quad \left.\begin{aligned} z_j &\geq \sqrt{\sum_{i=1}^{n} y_{ij}^2} && (6.10a^{v}) \\ \overline{Y}_j &= \overline{E}_j^{-1}\,\overline{X} && (6.10a'^{v}) \\ \overline{S}_j &= \sum_{i=1}^{j} M_i - \sum_{i=1}^{j} \overline{C}_i^t\overline{X} && (6.10c) \end{aligned}\right\} j=1,\ldots,k$$

$$\overline{G}\,\overline{X} \le \overline{q} \tag{6.10d}$$

$$\overline{S} \ge 0 \tag{6.10e}$$

$$\overline{Z} \ge 0 \text{ (actually unnecessary)} \tag{6.10f}$$

$$0 \le \overline{X} \le 1 \text{ (mixed integer)} \tag{6.10g}$$

This formulation is sufficiently general to apply generalized Benders without making the additional manipulations suggested in Appendix A. One may observe also that all functions of $\overline{X}$ are linearly separable from functions of $\overline{Y}$, $\overline{Z}$, and $\overline{S}$. This means that if one wishes to partition between these two sets of variables then the problem will exhibit "Property P." With respect to the problem of V representation, Geoffrion's theorem is not really needed since for every $\overline{X}$ there exists feasible $\overline{Y}_j$ and z_j, $j = 1, \ldots, k$ and so long as $\overline{X}$ is selected to satisfy the following

$$\sum_{i=1}^{j} M_i - \sum_{i=1}^{j} \overline{C}_i^t \overline{X} \ge 0 \qquad j = 1, \ldots, k \tag{6.11}$$

which does not unrealistically restrict $\overline{X}$, then there will always exist a feasible $\overline{S}$. Therefore, in the master problem for this specific problem, the infinite system of constraints (6.7c) may be replaced by the finite and well defined system (6.11) thus eliminating the need for relaxation procedures on that portion of the problem.[2]

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[2]Geoffrion (15) indicated possible computational difficulties arising from relaxation of the system (6.7c) whose purpose is to keep the algorithm feasible while system (6.7b) works towards optimality. That potential problem is eliminated by the fortunate circumstances allowing the use of (6.11) to replace (6.7c).

The subproblem which is the original problem except that $\overline{X}$ is fixed to a particular value $\overline{X}^{\ell}$ may be written down as follows:

Subproblem:

$$\underset{\overline{Y},\overline{Z},\overline{S}}{\text{Maximize}}\ P_o = \overline{A}^t\overline{X}^{\ell} + f(\overline{S},\overline{Z}) \qquad (6.10'\ \text{sub})$$

$$\text{s.t.:}\quad \left.\begin{aligned} & z_j - \sqrt{\sum_{i=1}^{n} y_{ij}^2} \geq 0 \\ & \overline{Y}_j - \overline{E}_j^{-1}\,\overline{X}^{\ell} = 0 \\ & s_j - \sum_{i=1}^{j} M_i + \sum_{i=1}^{j} \overline{C}_i^t\overline{X}^{\ell} = 0 \end{aligned}\right\} j=1,2,\ldots,k$$

$$\overline{S} \geq 0$$

$$\overline{Z} \geq 0$$

All other constraints involve $\overline{X}$ only and are therefore eliminated from the subproblem. One will find in Appendix B a proof that the hypothesis of the v-Representation Theorem is true for (6.10' sub). That fact, together with linear separability and "Property P" allows one to express the right side of any constraint in system (6.7b) of the master problem in the usual manner given by (6.7b'). Hence, using the following notation:

u_j denotes the dual multiplier for constraints

$$z_j - \sqrt{\sum_{i=1}^{n} y_{ij}^2} \geq 0$$

$\overline{V}_j$ denotes the dual multiplier vectors for the constraints $\overline{Y}_j - \overline{E}_j^{-1}\,\overline{X}^{\ell} = 0$

w_j denotes the dual multiplier for the constraints

$$s_j - \sum_{i=1}^{j} M_i + \sum_{i=1}^{j} \overline{C}_i^t \overline{X}^{\ell} = 0$$

the master problem may be written down thusly:

Master Problem:

$$\underset{r,\overline{X}}{\text{Maximize}}\ r \qquad (6.10' \text{ master})$$

$$\text{s.t.:} \quad r \leq \overline{A}^t \overline{X} + f(\overline{S}^{\ell}, \overline{Z}^{\ell}) + \sum_{j=1}^{k} u_j^{\ell} \left(z_j^{\ell} - \sqrt{\sum_{i=1}^{n} (y_{ij}^{\ell})^2} \right)$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{k} v_{ij}^{\ell} (y_{ij}^{\ell} - \sum_{p=1}^{n} e_{ipj} x_p)$$

$$+ \sum_{j=1}^{k} w_j^{\ell} (s_j^{\ell} - \sum_{i=1}^{j} M_i + \sum_{i=1}^{j} \overline{C}_i^t \overline{X}) \quad \ell = 1, 2, \ldots$$

$$\overline{G}\ \overline{X} \leq \overline{q}$$

$$\sum_{i=1}^{j} \overline{C}_i^t \overline{X} \leq \sum_{i=1}^{j} M_i \qquad j = 1, \ldots, k$$

$$0 \leq \overline{X} \leq 1$$

This problem is simply a mixed integer linear programming problem for which solutions are obtainable by existing techniques.

It is this master problem that is solved by relaxation where the constraint set restraining r is initially dropped. This constraint set is rebuilt one constraint at a time by a process that successively solves, in turn, the relaxed master problem and the subproblem.

Summary

In Chapter IV a rather general formulation of the capital budgeting problem was given. Certain restrictive assumptions were made which allowed a much more specific model to be stated. In a later chapter one will find that different assumptions will cause a somewhat different specific model to be derived from the general formulation. In fact, a whole family of specific models can be generated by systematic alteration of assumptions. In this chapter a relatively new solution technique for nonlinear programming problems was explained and together with the modifications, transformations, and conditions developed in the appendices it was shown that the solution technique can be applied to the first specific model. Indeed, the real value of the technique is that it is applicable to an entire family of such problems.[3]

The next chapter reports on computational experience derived from the solution of several sample problems after decomposition of the problem as described above.

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[3]Although the technique is applicable as a systematic means of obtaining solutions to the entire family, the global optimality achieved with specific formulation (4.14) has been proved only for (4.14).

CHAPTER VII

SOLUTIONS TO SAMPLE PROBLEMS

In order to obtain some idea of the computational efficiency of the generalized Benders' solution techniques and to determine if solutions of the formulation in Chapter IV conform logically with investment theory, a computer program was developed and run with three basic problem structures whose parameters were varied to produce the results cited herein. Two important terms are used to describe these results. These terms are defined as follows:

1. Project contribution--each project contributes, in a linear manner, to the size of the objective function and the amount of the contribution is measured by its coefficient in the vector $\bar{A}$ which appears in the objective function of the original problem and as part of the technological coefficients in the Benders constraints of the master problem.
2. Slack contribution--since funds not spent for projects are automatically invested at the risk free rate, they will contribute to the size of the objective function. Furthermore, negative slack funds will have a negative contribution since it is assumed that these funds must be

borrowed at some rate higher than the risk free rate. The net contribution of slack funds, then, is the expected contribution of these invested or borrowed slack amounts and is computed by the nonlinear term in the objective function of the original problem and therefore by the Lagrangean of the subproblem.

The general solution procedures may be stated in a stepwise fashion as follows:

1. Determine a set of integer feasible projects.
2. Optimize the subproblem over all variables except project variables which are held fixed to the values just determined. Optimization must determine both primal and dual variables.
3. Construct a Benders constraint from the Lagrangean of the subproblem and add that constraint to the master problem.
4. Optimize the master problem for a new set of projects using all previously generated Benders constraints.
5. If the current objective function value has not changed from the value obtained in the previous iteration, stop. Otherwise, return to step 2.

The actual solution procedure steps utilized were modifications of the above listing and are fully discussed with rationale for their use given in the next section.

Solution Code

The subproblem is solved analytically using the Kuhm-Tucker conditions; therefore, the mathematical expressions giving these solutions were programmed directly into several subroutines to be used by the main computer program.

The master problem is a mixed integer (0-1) linear programming problem for which a number of solution procedures may be applicable. Among these are:

1. Gomory cutting plane algorithms.
2. Benders type cutting plane algorithms.
3. Branch and bound or Balas type partial enumeration algorithms.
4. Group theoretic algorithms.

Of these the Gomory cutting plane algorithm using the stronger Gomory cuts as given in Taha (45) was selected for the following reasons:

1. The Gomory cutting plane algorithm is easy to program.
2. The final solution yields an optimal simplex tableau which lends itself to more thorough interpretation.
3. Since the Gomory cutting plane algorithm is a relaxation procedure it is more compatible with the overall relaxation procedures specified in the introduction. Furthermore, it was hoped that those Gomory cuts added at each iteration of the Benders procedures (i.e., those added between the addition of Benders cuts) would be relevant after the addition of more Benders cuts and would tend to hold the master problem at or near integer feasible solutions, thus minimizing the need for additional Gomory cuts.

4. It was felt that since the integer variables could take on only two values (o or 1) then pathological cases experienced with Gomory's algorithm were unlikely, particularly with the relatively small problem sizes attempted. The pathological problems often encountered are inefficiency due to the necessity of a large number of cuts or due to errors created by cumulating truncation errors which may lead to a large number of cuts and/or solutions that are actually infeasible or nonoptimal.

Although Benders solution procedures generally call for complete solution of the master problem before generating a new constraint by solving the subproblem, a modification which consists of solving the total problem completely without any integer requirements being incorporated (i.e., the problem is further relaxed by eliminating temporarily all integer requirements) was utilized. The integer requirements were then implemented via the addition of Gomory cuts and if necessary, more Benders cuts. The reasons for employing this strategy were:

1. Recent research (McDaniel, 33) indicates that when Benders algorithm is applied to mixed-integer linear programming problems some computational improvement is achieved by employing the strategy of relaxing the integer requirements until an initial solution is obtained. This causes several Benders constraints to be added to the master problem before an integer solution is attempted. The

major savings comes from solving fewer integer problems. Although, the problem in Chapter IV is not linear, it was hoped that similar computational efficiencies could be obtained by employing the same strategy.

2. By obtaining a complete solution to the original problem without regard for the integer requirements, one obtains valuable marginal slack and project contribution values. These values are useful in comparing solutions of the formulation with the results expected from established investment theory. Once integer requirements are implemented, these marginal values change and their interpretation is no longer clear.

With this solution strategy applied to the five steps stated earlier one obtains the modified procedures listed below.

1. Obtain any set of feasible[1] projects (i.e., a value for the vector $\overline{X}$).
2. Optimize the subproblem over all variables $(\overline{Y},\overline{Z},\overline{S})$ except project variables $(\overline{X})$ which are held fixed to the values just determined. Optimization must determine both primal $(\overline{Y},\overline{Z},\overline{S})$ and dual $(\overline{U},\overline{V},\overline{W})$ variables.
3. Construct a Benders constraint from the Lagrangean of the subproblem and add that constraint to the master problem. Such a constraint is of the form

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[1]Feasible refers to solutions that satisfy constraints while integer feasible refers to solutions that satisfy constraints and all integer requirements.

$r \leq L(\overline{X},\overline{Y}^*,\overline{Z}^*,\overline{S}^*,\overline{U}^*,\overline{V}^*,\overline{W}^*)$ where the notation (*) indicates the values that optimized the subproblem, and L is the Lagrangean.

4. Optimize the master problem for a new set of projects ($\overline{X}$) and objective function variable (r) using all previously generated Benders constraints.
5. If r has not changed from the value obtained in the previous iteration, go to step 6. Otherwise, go to step 2.
6. If the solution satisfies all integer requirements, stop. Otherwise, go to step 7.
7. Generate a Gomory cut and add it to the master problem.
8. Optimize the master problem using all previously generated Benders constraints and Gomory cuts.
9. If the solution satisfies all integer requirements, go to step 2. Otherwise, go to step 7.

A flowchart of these procedures is given in Figure 7-1. To shorten the description of these operations the terms Benders iterations, Gomory iterations, and cycles will henceforth refer to the performance of steps two through 5, 7 through 9, and two through 9, respectively. Hence, each time a Benders constraint is added and an LP solution obtained that is a Benders iteration, each time a Gomory cut is added and an LP solution obtained that is a Gomory iteration, and each time both types of iterations have been performed leading to an integer and feasible solution,[2] then that is a cycle.

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[2]Not necessarily feasible with respect to Benders constraints not yet added.

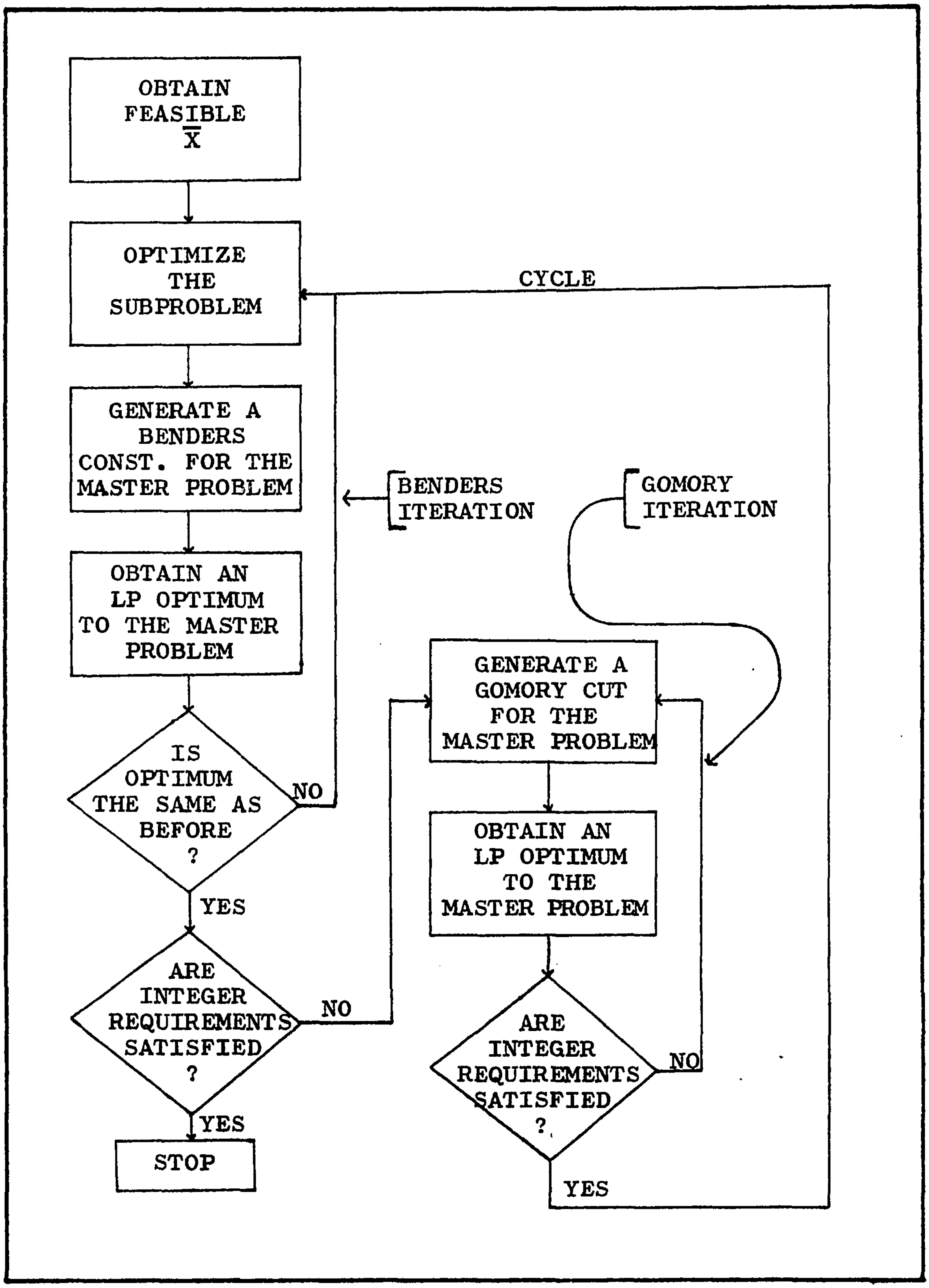

Figure 7-1. Flowchart.

Computer Code

Since the subproblem may be optimized analytically, the only numerical optimization procedures required are those for an ordinary linear programming problem. However, the primal simplex method is required once while the dual simplex method is needed each time a Benders constraint or Gomory cut is added to the previously optimal tableau. Furthermore, since the project decision variables are bounded by one, then either a bounded variables routine or several additional constraints are necessary. In an effort to keep the simplex tableau as small as possible the bounded variables routine was chosen. The basic optimization program may therefore be described as a primal and dual, bounded variable simplex algorithm. This algorithm was written as subroutines for a main program which read in all data and then called these subroutines in the sequence necessary to accomplish the procedures depicted in the flowchart of Figure 7-1.

Additional subroutines required were those used to compute the subproblem primal and dual variables according to the formulas determined by the analytical solution to the subproblem (see formulas page B-6). Subroutines were also needed to compute the new technological coefficients for all Benders constraints and Gomory cuts as well as a subroutine to integrate numerically the term

$$\int_{-\infty}^{-s_j/z_j} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$$

for each period j and each set of values s_j and z_j. Simpson's Tropezoidal Rule was used for this purpose with $-\infty$ replaced by -10.0 with virtually no loss in accuracy.

One may observe that the master problem without any Benders constraints added takes the form,

$$\underset{\overline{X},r}{\text{Max}} \quad r$$

$$\text{s.t.} \quad \overline{G}\overline{X} \leq \overline{q}$$

$$\sum_{i=1}^{j} \overline{C}_i^t \overline{X} \leq \sum_{i=1}^{j} M_i \qquad j = 1,2, \ldots, k(\# \text{ of periods})$$

so that the addition of a surrogate constraint, such as $r \leq \overline{A}^t\overline{X} + M$, where M is a very large number, makes the master problem suitable for generating a good initial set of projects as required by step 1 of the procedure described earlier. This was precisely the method used to accomplish step 1.

The pathological difficulties mentioned earlier did, unfortunately, arise in some of the test problems. For the most part, the difficulties became serious as a result of cumulating truncation errors which became more serious when ineffective Gomory cuts were being generated in some of the test problems. To minimize these problems all simplex operations were converted to a double precision mode of operation. This meant that the 17^{th} digit was truncated instead of the 9^{th} digit. The increased accuracy was obtained at the expense of storage and computation time but did eliminate most problems arising from cumulation of truncation errors.

The difficulty with ineffective Gomory cuts is unavoidable and something one must tolerate when it occurs.[3] It should be noted that no difficulties occurred as a result of Benders constraints; only Gomory cuts caused significant problems. For large scale capital budgeting problems (i.e., number of projects > 50) one of the other aforementioned mixed integer solution techniques may be applicable with greater computational efficiency than the Gomory Cut Algorithm used to produce the example results reported in this chapter.

Feeder Program

One feeder program was utilized to construct the covariance matrix for costs for each period and then to compute the conjugate vectors matrix and invert that matrix to provide data for the optimization program. This was used for problems 2 and 3 while the covariance matrix for problem 1 was found in an article by Mao and Brewster (see ref. 30). The method used to construct the covariance matrix for problems 2 and 3 is essentially the method used by Cohens and Pogue (see ref. 8) and is similar to the technique described in Chapter V.

The regression equation used for this purpose was as follows:

$$\tilde{c}_{ij} = \alpha_{ij} + \beta_{ij}\tilde{I}_1 + \gamma_{ij}\tilde{I}_2 + \tilde{\epsilon}_{ij}$$

[3]If one cannot tolerate this problem then some alternative solution technique should be used in place of the Gomory cutting plane method.

where $\tilde{I}_1$ and $\tilde{I}_2$ are two management performance indices with correlation coefficient ρ_{12}, and variance σ_1^2, and σ_2^2. $\tilde{c}_{ij}$ is the cost of project i in period j. It was assumed that all covariance between project costs could be explained by their common relationship with the two indices. The values for α, β, γ, σ_ϵ^2, σ_1^2, σ_2^2, and ρ_{12} were selected largely at random except with respect to a single objective of creating variety. This means that an attempt was made to insure that some costs had high variability while others had low variability, and some were highly correlated with other projects while others were largely independent of other project costs.

The procedures for constructing conjugate vectors were found in Zangwill (see ref. 52, Ch. 6) and a simplex type pivot operation was used for matrix inversion.

All of these operations; covariance generation, conjugate matrix construction, and matrix inversion, required very little computation time and would not create a computational burden for relatively large problems.

Test Problems

As mentioned earlier there were three basic test problems used. The first of these, called problem 1, may be described as an eight project two period problem in which all eight projects were treated by (0-1) integer variables. The second, called problem 2, consisted of 12 (0-1) integer projects and three periods. The third, called problem 3, consisted of 16 (0-1) integer projects and three periods. All three problem types had some mutually exclusive and contingent project sets. Problems are henceforth designated with numbers and letters. The number indicating which of the three basic forms it fits and the letter distinguishing similar problems with differing parameters. The pertinent data for all of the problems may be found in the tables of Appendix D. All parameters were initially affixed to randomly chosen values within reasonable ranges with some parameters systematically altered to produce new problems designed to test the computational performance of the solution procedures and compliance of the formulation to investment theory.

Computation Results

Solution times by computer are often stated as an indication of efficiency. That practice will not be executed here because of the large variations of these times that can be caused by differences in computer equipment, differences in programming technique, and differences in the amount of intermediate output. Furthermore, since no results on solutions to similar formulations exist, there is nothing to compare these results with. However, since most of the test problems did not require excessive computer time, one may conclude that relatively large problems (50-60 projects and virtually any number of periods) may be solved. Perhaps more indicative of actual efficiency (or lack of it) is the number of constraints and integer solutions required at intermediate steps before a global optimum is found, but even these values are drastically affected by stopping at epsilon-optimal solutions rather than adding Benders constraints that have very small effects upon the objective function values. However, the number of Benders and Gomory iterations and number of cycles are reported with some general comments about various solution strategies that were applied to modify the theoretical solution procedures.

The results, as measured by number of cycles and number of Benders and Gomory iterations, are given in Table 7-1. Problems 3c, 3d, 3e, and 3f are the only problems that required more than one complete cycle to obtain the solution. However, employing a stopping rule whereby an epsilon optimal solution was acceptable with $\epsilon = 2.5$[4] eliminated all but the first cycle for 3d, 3e, and 3f and limited 3c to two cycles. The values of epsilon given in the table are closer to the true values because the above stopping rule was only applied at the end of cycles so that much better solutions than required were obtained between successive applications of the stopping rule.

There are four general conclusions that may be drawn from the experience of solving the 17 test problems.

1. It is as easy or easier to obtain integer solutions via the addition of Gomory cuts after the addition of Benders

[4]With $\epsilon = 2.5$ the maximum possible discrepancy between the solution and the optimum is less than 5/100 of 1% of the objective function value.

TABLE 7-1

NUMBER OF REQUIRED CONSTRAINTS AND CYCLES

Problem No.	Cycle 1		Cycle 2		Partial Cycle 3	Total		
	Benders Constraints*	Gomory Cuts*	Benders Constraints*	Gomory Cuts*	Benders Constraints*	B	G	All
1a	2	0	$\epsilon = 0$	--	--	2	0	2
1b	NA	2	NA	NA	NA	NA	2	2
1c	3	0	$\epsilon = 0$	--	--	3	0	3
1d	NA	13	NA	NA	NA	NA	13	13
2a	2	1	1	0	$\epsilon = 0$	3	1	4
2b	NA	15	NA	NA	NA	NA	15	15
2c	7	4	$\epsilon = 0$	--	--	7	4	11
2d	2	0	$\epsilon = 0$	--	--	2	0	2
2e	2	1	1	0	$\epsilon = 0$	3	1	4
2f	2	1	1	0	$\epsilon = 0$	3	1	4
2g	2	1	1	0	$\epsilon = 0$	3	1	4
3a	2	0	$\epsilon = 0$	--	--	2	0	2
3b	9	4	$\epsilon = 0$	--	--	9	4	13
3c	9	21	4	20	$\epsilon = .03236$	13	41	54
3d	8	20	$\epsilon = 1.51002$	--	--	8	20	28
3e	4	7	$\epsilon = .20630$	--	--	4	7	11
3f	10	3	$\epsilon = .26367$	--	--	10	3	13

Note: Completion of the solution procedures is indicated by citing a value for ϵ. If ϵ is equal to zero this indicates the procedures were operated until an absolute optimum was found. If ϵ is not zero then an epsilon-optimum was found and the max. value for ϵ is indicated.

* Or iterations

constraints than it is before their addition.

2. Once an integer solution is found one cannot expect the Gomory cuts to hold a solution at or near integer values during the subsequent addition of Benders constraints.
3. For risk free rates in the neighborhood of 6% and the rate of short term borrowing in the neighborhood of 10%, the integer solution at the end of the first cycle will probably be the global optimum with subsequent cycles accomplishing nothing more than adjustments to the objective function value.
4. Practical solution procedures for the formulation in Chapter IV are readily available.

The discussion of the specific experience that leads one to these four conclusions follows:

Conclusion 1. Problems 1b, 1d, and 2b utilized the same input data as 1a, 1c, and 2a, respectively; however, 1b, 1d, and 2b sought to obtain an integer solution to only the linear parts of the formulation. Although 1a and 1b required the same total number of constraints (2) to obtain an integer solution, 1c required 10 fewer than 1d and 2a required 11 fewer than 2b. Furthermore, original attempts at solving problems 3c through 3f met with difficulty because of a large number of cycles. However, all solution modifications that reduced the number of Benders constraints that were added prior to the addition of any Gomory cuts resulted in complete failure because in each case over 70 Gomory cuts were added, exceeding both time and storage limitations and still not obtaining the first integer solution. The results finally obtained and reported in Table 7-1 were achieved by employing the epsilon optimal stopping rule at the end of each cycle.

Conclusion 2. At the beginning of this chapter it was stated that Gomory cut solution procedures were chosen for a number of reasons. One of those reasons was that it was hoped that Gomory cuts already added to the tableau would tend to hold variables to integer values during subsequent Benders iterations. Such a hope was justified for problems 1a through 3b, but failure of that hope was the root of all of the computational difficulties experienced with problems 3c, 3d, 3e and 3f. In each case the optimal integer solution was found at the end of one cycle but was not held integer during subsequent cycles. Table 7-2 shows the specific results obtained with an early attempt to solve problem 3c. It was these results that led to the use of the epsilon stopping rule mentioned previously.

Conclusion 3. The integer solution found at the end of the first cycle was indeed the global optimum for all problems attempted. Furthermore, this particular result has been reported by McDaniel (33) in his solutions to mixed integer linear programming problems. Of course, these test problems are not linear by virtue of the fact that the marginal contribution of slack is not constant for all values of slack, nor for all combinations of projects. The degree of nonlinearity

TABLE 7-2

ATTEMPTED SOLUTIONS TO PROBLEM 3c

Cycle No.	No. of Bend. Iter.	No. of Gom. Iter.	Solutions: Projects 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	Objective Function
1	9	--	0	1	1	1	1	0	0	1	1	.08	.31	1	1	0	1	0	7242.10
1	--	21	0	1	0	1	1	0	0	1	1	0	1	1	1	0	0	0	7239.85*
2	4	--	0	1	.21	1	1	0	0	1	1	.21	.79	1	1	0	0	0	7239.84
2	--	20	0	1	0	1	1	0	0	1	1	0	1	1	1	0	0	0	7238.84*
3	5	--	0	1	0	1	1	0	0	1	1	0	1	1	1	0	.002	0	7238.82
3	--	4	0	1	0	1	1	0	0	1	1	0	1	1	1	0	0	0	7238.816*
4	1	--	0	1	.03	1	1	0	0	1	1	0	.97	1	1	0	.03	0	7238.81
4	--	6	No solution --time and storage limitations exceeded.																
4	19	51	Totals																

*Denotes identical solutions

is restricted by the fact that although the marginal contribution of slack is not constant, it is bounded above and below with its range being less than one half of the difference between the risk free rate and the rate of borrowing. For two and three period models this turns out to be a small range in relation to the midrange magnitude of the marginal contribution. Therefore, one would not expect the required number of integer solutions to differ significantly from those found in the solution of mixed integer linear problems.

Conclusion 4. It is shown in Appendix B that the procedures in Chapter VI produce a global optimum. The only issue then is the practicality of those procedures. The test problems showed that computational difficulties can certainly be encountered, but in every case those difficulties were the direct result of the algorithm used to obtain the mixed integer solutions to the master problem. Indeed, the hopes expressed earlier with regard to the use of Gomory's cutting plane algorithm were justified for most test problems, but not all. This lack of reliability is not good enough, for when these hopes failed practical solutions became impossible for even small problems. Although it was possible to salvage all of the test problems by using double precision in the computer program and employing an effective epsilon optimal stopping rule, the possible inability of Gomory's cutting plane algorithm to obtain even the first mixed integer solution does not predict a bright future for the solution of larger problems. However, the encouraging points are that (a) the Benders constraints appear strong and predictable, (b) the strategy of relaxing the integer requirements to build several Benders constraints appears to limit the need of obtaining more than one mixed integer solution.

Consequently, conclusion 4 is contingent upon the existence of a practical mixed integer linear programming code and some supportive evidence of point (a) above. With respect to the code, there are such codes that have been applied to rather large problems with success[5] although research is continuing to develop procedures that are even more practical.[6] Point (a) may be investigated by observing the performance of the algorithm during the Benders iterations of each cycle for the test problems. Some problems were such that the marginal project contribution was, for each project, either above or below every possible value for marginal slack contribution. In these cases no more than two Benders constraints were required in the first cycle and one constraint in the second cycle which did not change the previous integer solution. The more interesting problems are those that have at least one project whose marginal contribution is within the range of all marginal slack contribution values. These were, specifically, problems 1c, 2c,

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[5]Principally, branch and bound and Balas type partial enumeration algorithms.

[6]Much work is being directed at Benders type partitioning algors.

3c, 3d, 3e, and 3f and required the greatest number of Benders constraints.

One may recall that each time a Benders constraint is added to the master problem, a new solution is found which is feasible although not necessarily feasible for unadded constraints since all relevant Benders constraints may not have been added. However, after each new constraint is added to the tableau, the right hand side of that constraint (before the dual pivots are initiated) has an absolute value that is equal to the maximum difference between the current solution and the optimal solution. This may be stated mathematically when RHS is the right hand side of the constraint just added and is negative, r_c is the current value of the objective function and r_o is the optimal but unknown value of the objective function. Such a statement follows:

$$r_c + \text{RHS} \leq r_o \leq r_c$$

It is clear then that for the process to work properly RHS must converge to zero as Benders constraints are added and that RHS must converge to zero quickly if the process is to be efficient. Figure 7-2 is a graph of the percent change in RHS plotted as a function of the Benders constraint number. Only those problems that required more than two Benders constraints are displayed in the figure. Since the first solution to the master problem (before any constraints have been added) always yields an objective function value greater than M where M is the arbitrarily large number in the surrogate

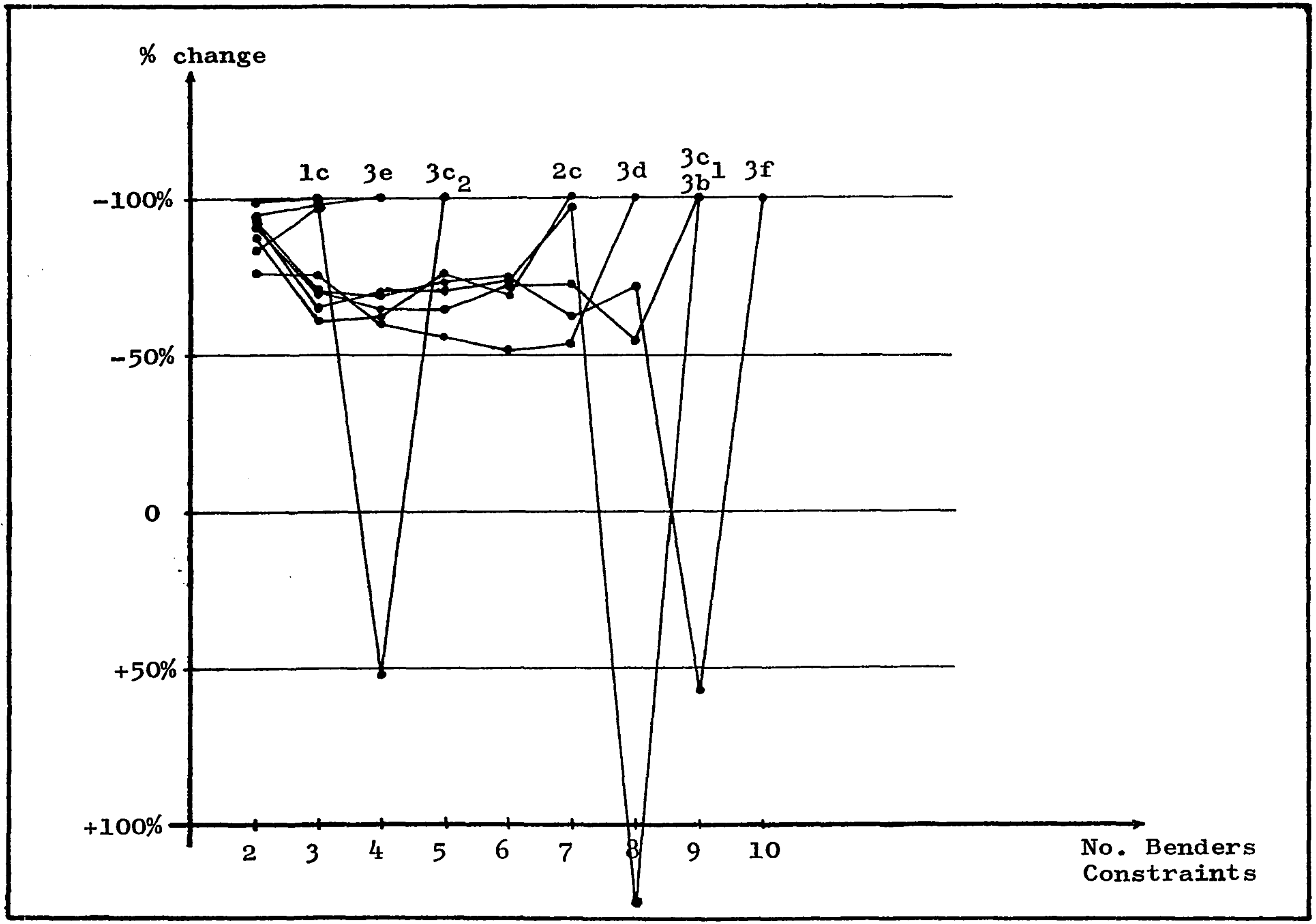

Figure 7-2. Percent change from previous optimum versus number of Benders constraints.

constraint discussed earlier, then the first Benders constraint always cuts the objective function value to some value greater than but in the neighborhood of the optimum. The percent change in this case may be made arbitrarily close to -100% by starting with an arbitrarily large value of M. Therefore, the change is not graphed for the first Benders constraint. The termination point for each problem has been labeled with the problem number and indexed with the cycle number if applicable.

One may observe that one of the most rapid convergences occurred with problem 3e while one of the slowest was problem 3d. These two problems differed only by the amount of available funds, leading one to conclude that the rate of convergence is more parameter sensitive than it is problem sensitive since all other convergence patterns, regardless of which problem, fell between these two patterns. One may also note the three anomalies where the next to the last constraint in the sequence for $3c_2$, 3b, and 3f produced an increase in the RHS rather than the expected decrease. However, in each case the RHS was less than .001 in magnitude so that the increase may be assumed to be the result of the accumulation of truncation errors that had occurred in previous iterations, rather than an indication that the process does not converge.

The average percent change in RHS for all Benders constraints displayed in Figure 7-2 was 64% while an average of 77% was obtained when the three anomalies were excluded.

This means that the RHS may be cut to ½% of its original size by the addition of from fewer than 8 to 11 Benders constraints. These numbers are certainly not excessive. This fact, coupled with a high probability of not needing more than one cycle indicates that if an efficient mixed integer algorithm is used to complete a cycle after Benders constraints have been added, then practical solutions to the formulation in Chapter IV are possible.

Compliance with Theory

Basic theory implies that optimality occurs when the marginal contributions of competing alternatives are equal. Problems 2a, 2c, and 2d were constructed to test specifically compliance with this theory. When the competing alternatives are projects, compliance is automatic since linear programming techniques are used for their selection. However, in the formulation in Chapter IV, there is also competition between projects and slack funds. Problems 2a, 2c and 2d were constructed to isolate that competition between project 9 and the slack funds. As discussed earlier the marginal contribution of slack is bounded above and below. Therefore, problem 2a was constructed so that the marginal contribution of each project was greater than the upper bound for the corresponding marginal contribution of slack. Problem 2c was identical to 2a except that the contribution of project 9 was adjusted so that its marginal contribution was within the range for the corresponding marginal contribution of slack.

Problem 2d was again identical to 2a except that the marginal contribution of Project 9 was lowered still further so that it was below the lower bound for the corresponding marginal contribution of slack. Project 9 was selected for this purpose because these changes in its objective function coefficient did not affect its ability to compete with other projects.

If the theory is complied with, then one would expect the selection of projects that comprise the optimal solution to the master problem before and after the addition of Benders constraints to exhibit the following:

2a. The before and after selections should be identical since there can never be an improvement by substituting slack for projects.

2c. The slack should be substituted for portions of project 9 until their marginal contributions are equal. This assumes $x_9 > 0$ in the initial solution.

2d. The slack should be substituted for all of project 9 since its marginal contribution can never be as large as the marginal contribution of slack. Again this assumes $x_9 > 0$ in the initial solution.

Table 7-3 gives the results of the appropriate non-integer solutions.

It remains to verify that the marginal contributions are equal after the addition of the 7 Benders constraints in problem 2c. To make these calculations one may consider a small change $\Delta x_9 = .01$. From Appendix D it may be seen that a change in x_9 of .01 results in a change of the objective function of .01(16600) = 166.00 and a change in the slack for each period of .01(7000) = 70.00 for period 1,

TABLE 7-3

SOLUTIONS TO VARIATIONS OF PROBLEM 2a

Project	Initial Solution for Problems 2a, 2c, 2d	Solution After 2 BC Added Problem 2a	Solution After 7 BC Added Problem 2c	Solution After 2 BC Added Problem 2d
1	1.0	1.0	1.0	1.0
2	1.0	1.0	1.0	1.0
3	0.0	0.0	0.0	0.0
4	1.0	1.0	1.0	1.0
5	0.0	0.0	0.0	0.0
6	1.0	1.0	1.0	1.0
7	1.0	1.0	1.0	1.0
8	1.0	1.0	1.0	1.0
9	0.57949	0.57949	0.31904	0.0
10	1.0	1.0	1.0	1.0
11	0.0	0.0	0.0	0.0
12	0.0	0.0	0.0	0.0

.01(7000+6500) = 135.00 for period 2 and, .01(7000+6500+6000) = 195.00 for period 3. From Appendix B one finds that the derivative of the objective function with respect to the slack of each period is given by

$$\frac{\partial(\mathrm{OBJ})}{\partial s_j} = b_j(R_{fj} - d_j\int_{-\infty}^{-s_j/z_j} \frac{1}{\sqrt{2\Pi}} e^{-t^2/2} dt) \qquad (7.1)$$

From problem 2c and the second solution given in Table 7-3 for problem 2c the following values are known:

$b_1 = .91 \quad R_{f1} = .06 \quad d_1 = -.04 \quad s_1 = 4866.72 \quad z_1 = 2895.50$

$b_2 = .83 \quad R_{f2} = .06 \quad d_2 = -.04 \quad s_2 = 11192.96 \quad z_2 = 2998.95$

$b_3 = .75 \quad R_{f3} = 1.06 \quad d_3 = -.04 \quad s_3 = 5078.72 \quad z_3 = 3085.64$

Therefore, the three marginal values given by (7.1) are .05629 for period 1, .04980 for period 2, and .7965 for period 3. The marginal contribution of slack corresponding to a change of .01 in project 9 may be computed

$$.05629(70) + .04980(135) + .7965(195) = 165.98$$

Although, 165.98 is not exactly equal to 166.00 the difference is less than 2/100 of one percent and easily accounted for by truncation error rather than a defect in the compliance with investment theory.

Conclusion

This chapter has presented evidence that the Generalized Benders solution procedures, modified in particular ways for the formulation presented in Chapter IV, provide a viable solution technique whose efficiency is both predictable and encouraging. In addition, analysis was provided to verify the logic and compliance with theory for the formulation in Chapter IV. The next chapter is devoted entirely to the subject of sensitivity analysis based upon the solution techniques presented in this and the preceding chapter.

CHAPTER VIII

SENSITIVITY ANALYSIS

An important exercise that follows the solution of a mathematical programming problem is the sensitivity analysis. Two motivations for performing this analysis may be stated:

1. To determine the effect upon the overall problem solution of a problem parameter that may be poorly estimated.
2. Decisions external to the problem formulation may have been made which fixed problem parameters to certain values. Sensitivity analysis can determine the effect of those decisions and perhaps the marginal improvement achieved by relaxing them.

The basic parameters of the problem in Chapter IV may be listed:

1. Project costs
2. Project cash flows
3. λ', the market price of risk
4. R_{ft}, the risk free rate for period t
5. R_{bt}, the borrowing rate for period t
6. b_t, weighting factor for each period's cash flows
7. M_t, funds available in period t

As seen in Appendix C items two through six determine specifically the value for a_i, the objective function coefficient for project i in formulation (4.14). Sensitivity analysis with respect to these values will be discussed relative to parametric changes in the a_i's. Sensitivity analysis with respect to item 1 is handled identically to the methods used for changes in the a_i's so that items 1 through 6 are essentially covered together. Perhaps the most important parameter is item (7), the available funds, whose sensitivity analysis is discussed in the next section.

Sensitivity Analysis with Respect to M_t

The most important aspects of sensitivity analysis are those dealing with changes in the funds made available for investment in each period of the planning horizon. There are basically two possible results that may occur as a result of changes in available funds.

1. A different combination of projects than the previous optimal combination becomes optimal along with the attendant changes in slack funds for each period.
2. The optimal combination of projects remains optimal but slack funds change for each period that available funds change.

In the second case, the objective function changes only as a result of a change in the contribution of slack which is a function of the expected slack. Sensitivity analysis in this case is simply a direct application of the dual multipliers for the subproblem that corresponds to the optimal solution. These multipliers are denoted by w_j. One should also observe that if M_j is changed by an amount Δ_j and the optimal combination of projects is not altered, then the slack in period j and each period thereafter is also changed by Δ_j since slack funds are passed on from period to period. Therefore, change in the contribution of slack to the objective function value as a result of changes in allocated funds with the optimal projects held constant may be computed directly for small Δ_j as follows:

$$\sum_{j=1}^{k} w_j \sum_{i=1}^{j} \Delta_i = \text{Objective Function Change} \qquad (8.1)$$

This may be illustrated with test problems 2a, 2e, 2f, and 2g where the problems are identical except as indicated in Table 8-1.

TABLE 8-1

SOLUTIONS OF VARIATIONS OF PROBLEM 2a

Problem	M_1	M_2	M_3	Objective Function Value*	Change From 2a	Predicted Change**
2a	50000	20000	5000	78742.87		
2e	50010	20000	5000	78751.88	9.01	8.996
2f	50000	20010	5000	78751.32	8.45	8.448
2g	50000	20000	5010	78750.82	7.95	7.950

*Obtained by independent solution of each problem.

**Obtained by application of (8.1) where w_j is obtained from the optimal solution of problem 2a.

It should be clear that the only advantage of small increases of funds available in any given period is that it lowers the probability of needing to borrow on a short term basis and increases the expected amount loaned at the risk free rate thereby increasing the total expected contribution of the slack funds. If some specific alternative source of additional funds is available, such as long term debt, then it is of interest to note that one may compute how much these funds can cost so that obtaining them is preferable to not obtaining them. One may suppose that R is such a rate for long term debt, then if \$1 is obtained via long term debt a negative cash flow of \$1·R will result in each period thereafter. The effect upon the objective function will be as follows for \$1 borrowed at the beginning of period j:

$$\$1 \cdot R \sum_{i=j}^{T} b_i \tag{8.2}$$

Equating this to (8.1) one obtains

$$\$1 \cdot R \sum_{i=j}^{T} b_i = \$1 \sum_{\ell=j}^{k} w_\ell$$

or

$$R = \frac{\sum_{\ell=j}^{k} w_\ell}{\sum_{i=j}^{T} b_i} \tag{8.3}$$

Therefore, if long term debt can be acquired in the capital markets for a rate less than R (given by 8.3), then it should be acquired for the purpose of providing protection against the risk of forced borrowing at higher rates.

Perhaps a more relevant question than the one above is: Given the rate at which long term debt can be obtained, how much should be obtained? This can be answered by fixing R in equation (8.3) and solving for the amount of slack that produces equality. (One should recognize that w_{ℓ} is a function of the amount of slack involved in period ℓ.) This procedure represents the application of the concept that marginal costs should equal marginal returns at the optimum. In Appendix B the relationship between w_{ℓ} and s_{ℓ} is given as follows:

$$w_{\ell} = b_{\ell}(R_{f\ell} - d_{\ell}\int_{-\infty}^{-s_{\ell}/z_{\ell}} \frac{1}{\sqrt{2\Pi}} e^{-t^2/2}\,dt)$$

so that for problem 2a with R = .096, management would like to know how much long term debt to acquire in period 1, then the following equation may be solved to answer that question:

$$.096 = \frac{.91(.06+.04\int_{-\infty}^{-\frac{7100+\Delta}{2816.8}} \frac{1}{\sqrt{2\Pi}} e^{-t^2/2}\,dt)}{10}$$

$$+ \frac{.83(.06+.04\int_{-\infty}^{-\frac{15500+\Delta}{2901.91}} \frac{1}{\sqrt{2\Pi}} e^{-t^2/2}\,dt)}{10}$$

$$+ \frac{.75(1.06+.04\int_{-\infty}^{-\frac{11300+\Delta}{2973.88}} \frac{1}{\sqrt{2\Pi}} e^{-t^2/2}\,dt)}{10}$$

Solving the equation for Δ gives Δ = -12500 which indicates that, if possible, management should eliminate 12500 of previously acquired debt if the company is paying 9.6%. This indication that long term debt should be paid off in this problem occurs for every rate greater than 9%. The reason seems to be that the company has, after making all investments, a great deal of excess cash in each period upon which it can only earn 6%.

Another example is problem 1d. Although this problem was solved without regard for the nonlinear term in the objective function and its related subproblem, the subproblem may still be used to determine how much long term debt might be acquired profitably to protect against cost overruns. In this particular problem $b_1 = .9383$, $b_2 = .8554$, $\Sigma b_i = 12.2$ and at the optimum $s_1 = 700$, $s_2 = 230$, and $z_1 = 2348.67$, $z_2 = 2366.87$. Also, $R_f = .05$, $R_b = .10$ for every period.

Therefore, if R = 8%, then the equation becomes

$$.08 = \frac{.9383(.05+.05\int_{-\infty}^{-\frac{700+\Delta}{2348.67}} \frac{1}{\sqrt{2\Pi}} e^{-t^2/2} dt)}{12.2}$$

$$+ \frac{.8554(1.05+.05\int_{-\infty}^{-\frac{230+\Delta}{2366.87}} \frac{1}{\sqrt{2\Pi}} e^{-t^2/2} dt)}{12.2} .$$

Solving the equation for Δ gives $\Delta = 500$. In this case the borrowed funds would decrease the objective function by 488 while the additional slack increases it by 494.35 from 5800.19 to 6294.53. Assuming the 500 is obtained, then the total objective function value becomes 5806.54 when all cash flows are considered. Again this analysis is under the assumption that project selections do not change as a result of the increased funds.

The more difficult sensitivity problem comes when the optimal integer solution changes as a result of changing the parameter being analyzed. However, it may be possible to incorporate the external decisions into the original model. Should this fail, then enumeration of possible parametric values and subsequent solution of each problem would give the desired information. An example of incorporating outside decisions is provided by sample problem 3f, while enumerated values are exemplified by problems 3c, 3d, and 3e.

In problem 3f there were 13 projects that could be selected within a total financial constraint of 7500 in funds provided in the first period only. However, an additional 1000 in funds could be obtained in any one of the three periods by way of a debt issue costing 9% per period. Since 1000 in face value of a debt issue may bring more or less than 1000, then the revenues from the three possible debt issues were treated as probabilistic values, and the negative cash flows they generate (9%(1000)) were treated as certain. By using the formulation in Chapter IV, incorporating these "outside" variables, particularly with respect to financing decisions, is easily accomplished by treating them as "negative projects"[1] in the manner exemplified by problem 3f. The specific parameter values for these three "negative projects" used in problem 3f, are given in Appendix D. The optimal solution to 3f indicated that the 1000 units of funds should be obtained in period 3 so that an additional project could be afforded with the given financial constraints. As problem 3f exemplifies, the need for sensitivity analysis was alleviated by being able to incorporate the "outside" decision variable directly into the problem. The current alternative to this method is to enumerate various pertinent values of the parameter being analyzed and to then solve the mathematical programming problem for each of these values. As stated earlier this technique is exemplified by problems 3c, 3d, and 3e. In each of these problems the only funds available were those made available in the first period. For 3c, that amount was 7500 units; for 3d, it was 8500 units; and for 3e, it was 9500 units. Clearly, if the only funds the company actually have available is 7500 units, then the extra 1000 or 2000 units would have to be acquired through the capital markets. If debt is considered, then the sensitivity analysis is not necessary as the debt instruments may be treated as negative projects as described above. However, if the sale of stock is being

[1]The objective function contribution is negative while ordinary projects are positive and project costs per period are negative while ordinary project costs are positive costs.

considered, then the enumerative sensitivity analysis provides relevant information. The use of this information is best demonstrated by an example. For this example it is assumed that there are 1000 shares outstanding, which, if the budget obtained in problem 3c is accepted, will have an equilibrium price per share of \$7238.82/1000, or \$7.24 where the monetary units are taken to be dollars and the \$7238.82 is the objective function value for problem 3c. The question is: If enough additional shares are sold to obtain \$1000 in new capital, what is the minimum price that should be obtained for each share? To answer this question one may let x be that minimum price. Therefore, to obtain an additional \$1000 an additional \$1000/x shares must be sold. If that is done, then the budget in the solution to problem 3d can be accepted and the aggregate equilibrium price becomes 3d's objective function, or \$8160.96. Hence, the equation

$$7.24 = \frac{8160.96}{1000 + \frac{1000}{x}}$$

which, if satisfied, will insure no change in the equilibrium price per share as a result of the stock sale. In this case the solution is $x = \$7.861$. Clearly, if the stock were sold for more than \$7.861 the equilibrium price per share would increase above \$7.24 and if sold for less than \$7.861 the equilibrium price per share would decrease below \$7.24; therefore, management must receive, as a minimum, \$7.861 per share.

If an additional \$2000 in new capital is sought, then the results of problem 3c and 3e are used in a similar equation given by

$$7.24 = \frac{9083.39}{1000 + \frac{2000}{x}}$$

which produces a minimum price x = \$7.855.

A second case for the use of information obtained from an enumerative analysis is when the company actually has \$9500 available and is considering cutting the budget to either \$8500 or \$7500. In the event either action is taken then something must be done with the extra \$1000 or \$2000 that would be made available. If it is being considered to use those funds to purchase stocks, bonds, government securities, or even to retire some debt, these are just ordinary investments that can and should be incorporated directly into the model by adding some appropriate decision variables. However, if these additional funds are used to declare an extra dividend, then again the enumerative type of sensitivity analysis provided by sample problems 3c, 3d, and 3e provides the necessary information. Again, an example is used to illustrate this process. Presumably, the hope is to maximize the equilibrium value of stock holder equity plus any other wealth stockholders accrue as a result of stock ownership.

For this example it is assumed that the average stockholder must pay 15% income tax on all dividends. The three alternatives are:

1. Keep $9500 and invest it according to the solution to 3e.
2. Keep $8500 and invest it according to the solution to 3d and declare an additional dividend of $1000.
3. Keep $7500 and invest it according to the solution to 3c and declare an additional dividend of $2000.

The respective benefits to the stockholders of these three alternatives are:

1. 9083.39
2. 8160.96 + 1000(1-.15) = 9010.96
3. 7238.82 + 2000(1-.15) = 8938.82

In this case, alternative 1. is the preferable alternative.

In the event one wishes to deal with the possible uncertainty surrounding the financial parameter M_j, then rather that use sensitivity analysis, the appropriate technique would be to estimate the mean and variance of the parameter and integrate its uncertainty into the formulation explicitly. This is easy to accomplish by minor modification of the formulation in Chapter IV and for which the solution technique stated in Chapter VI is still applicable.

Sensitivity Analysis with Respect to Project Parameters (a_i, c_{ij})

At the completion of the solution procedures described in Chapter VI an optimal simplex tableau is obtained for the master problem. Although that tableau was produced via relaxation procedures and only after the addition of many constraints that were not originally there, it does represent the solution of one original problem that could have been solved without relaxation procedures if the Benders and Gomory constraints that were ultimately added had been known in advance. The form of that problem, assuming all subsequently added constraints are known and included may be written as follows:

$$\underset{r,\overline{X}}{\text{Maximize}}\ r \qquad (8.4)$$

$r - \sum_{i=1}^{n} a_{ik}x_i \leq b_k$	$k = 1,2,\ldots,p\}$	Benders constraints	(8.4a)
$\sum_{j=1}^{\ell} \sum_{i=1}^{n} c_{ij}x_i \leq \sum_{j=1}^{\ell} M_j$	$\ell = 1,2,\ldots,T\}$	Financial constraints	(8.4b)
$\sum_{i=1}^{n} g_{ik}x_i \leq q_k$	$k = 1,2,\ldots,s\}$	Gomory constraints	(8.4c)
$\sum_{i=1}^{n} h_{ik}x_i \leq d_k$	$k = 1,2,\ldots,u\}$	Project interrelationship constraints	(8.4d)
$0 \leq x_i \leq 1$	$i = 1,2,\ldots,n\}$	Bounds	(8.4e)

where 1. a_{ik} is a constant function of the a_i coefficients defined earlier and of c_{ij} cost coefficients and of dual multipliers obtained from the subproblem

2. all other values except x_i and r are constant.

Considering the problem of form (8.4) any change in the values of parameters a_i, and c_{ij} will result only in a calculable change in the technological coefficients a_{ik} and c_{ij} found in systems (8.4a) and (8.4b). Furthermore, if these changes in technological coefficients occur for an i such that x_i is nonbasic in the optimal solution to (8.4), and that change is small enough that x_i remains nonbasic then:

1. The current basic solution remains optimal and it will continue to satisfy the integer requirements that forced the introduction of the system of Gomory constraints (8.4c)
2. The current basic solution remains optimal for the non-linear problem (i.e., no more than the p Benders constraints can be added).

If the change in technological coefficients is large enough that the optimality condition indicates that x_i should become a basic variable, and if x_i is made basic, then the new solution will probably not satisfy the integer requirements and more Gormory constraints will be needed. If the addition of these Gomory constraints yields a new mixed integer solution, then additional Benders constraints may also be needed. It is possible, however, that the addition of the Gomory cuts will force the solution back to the same integer solution as before thereby not actually changing to a new solution as a result of the changes in the technological coefficients.

The important point is that standard parametric analysis techniques[2] concerning changes in technological coefficients of nonbasic variables for linear programming problems is indirectly applicable to the nonlinear formulation

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[2]See Taha, Chapters 4 and 9 (45).

of Chapter IV via direct application to the linear master problem created by a Generalized Benders Decomposition. Specifically, what can be determined by the application of those standard techniques is a minimum range for the changes in these technological coefficients over which no change in the solution will occur. At each end of these ranges are so called critical values which in ordinary LP problems are values that if exceeded a change in the solution will occur. However, in a mixed integer program these critical values may force the addition of more Gomory cuts and may not actually cause a change in the mixed integer solution. The only way to determine that is to actually change the technological coefficients to something exceeding their critical values, reoptimize the last tableau and then restart the Gomory and Benders algorithms described in Chapter VI.

CHAPTER IX

EXTENSIONS AND FURTHER RESEARCH

An appropriate direction for future research into the particular problem area of capital budgeting under uncertainty is provided by the general formulation in Chapter IV, the data generation techniques discussed in Chapter V, and the solution procedures presented in Chapter VI and applied in Chapter VII. It is anticipated that any new formulation derived from the general formulation in Chapter IV but exhibiting differences from the specific formulation (4.14) in Chapter IV as a result of different assumptions may, in general, be decomposed in a manner described in Chapter VI and Appendix A. However, whether the generalized Benders solution procedures produce a global optimum for any specific case will depend upon whether the resultant subproblems have the required properties. Any proof of such properties should follow closely the method of proof developed and presented in Appendix B.

To exemplify the process of developing refined models via the alteration or refinement of the assumptions made in Chapter IV some example cases are discussed in the next section. The final sections are devoted to an outline and classification of further research efforts, and some general conclusions are drawn.

Extensions

When one formulates a mathematical programming model the usual preparatory steps are to specify the assumptions under which the model is to be valid. When one applies the model to a real problem it must be determined that the assumptions are true or that those assumptions that are violated are not violated by an amount significant enough to cause difficulties if they are ignored. If the deviation from an assumption is significant, then the assumption must be changed and the model also changed to reflect the new assumption. Hence, a formulation may be expanded into a family of related formulations via a systematic relaxation, generalization or modification of the original assumptions. When this occurs in such a way as to extend the applicability of a basic formulation to not only those problems that fit the original assumptions, but to a larger set of problems as well, then one has accomplished an extension. In this section, three original assumptions that are the most vulnerable with respect to criticism of their realism are considered together with the attendant changes in the specific formulation. These three assumptions are:

1. That unlimited short term borrowing is available at some constant rate above the risk free rate (see Assumption 13 for models 4.13 and 4.14 in Chapter IV)

2. That all random variables are normally distributed. (See Assumption 1 for the general model in Chapter IV)
3. That there is no autocorrelation between investment costs nor any correlation between those costs and the economic index used to establish $\tilde{R}_m$. (See Assumption 10 for specific models 4.13 and 4.14)

Case 1: One may wish to assume that the rate of interest that must be paid on short term borrowings is a piecewise linear and constant but increasing function of the amount borrowed, rather than a simple constant function as implied by 1. above. Hence, the company may borrow from 0 to some amount a_1 at rate R_{b0t} in period t, or from a_1 to some amount a_2 at rate R_{b1t}, or from a_j to some amount a_{j+1} at rate R_{bjt} where $R_{b(j+1)t} \geq R_{bjt}$ $\forall$ j and t. For computational reasons it should also be assumed that $\exists\, R_{bkt}$ such that the supply of capital at that rate is unlimited. In this case the resulting change to the formulation is easy to carry out since all that is required is a change in the way the expected cash flows from slack funds are computed.

One may recall from Chapter IV that under the original assumptions the expected cash flows from slack funds in period t were given by:

$$E(\tilde{S}_t \cdot g(\tilde{S}_t)) = \int_{-\infty}^{0} \frac{qR_{bt}}{\sqrt{2\Pi}\,\sigma_{s_t}} e^{-(q-\mu_{s_t})^2/2\sigma_{s_t}^{2}} dq$$

$$+ \int_{0}^{\infty} \frac{qR_{ft}}{\sqrt{2\Pi}\,\sigma_{s_t}} e^{-(q-\mu_{s_t})^2/2\sigma_{s_t}^{2}} dq \qquad (9.1)$$

These terms may be replaced by:

$$E(\tilde{S}_t \cdot g(\tilde{S}_t)) = \int_{-\infty}^{-a_k} \frac{qR_{bkt}}{\sqrt{2\Pi}\sigma_{s_t}} e^{-(q-\mu_{s_t})^2/2\sigma_{s_t}^2} dq$$

$$+ \sum_{j=1}^{k-1} \int_{-a_{j+1}}^{-a_j} \frac{qR_{bjt}}{\sqrt{2\Pi}\sigma_{s_t}} e^{-(q-\mu_{s_t})^2/2\sigma_{s_t}^2} dq$$

$$+ \int_{-a_1}^{o} \frac{qR_{bot}}{\sqrt{2\Pi}\sigma_{s_t}} e^{-(q-\mu_{s_t})^2/2\sigma_{s_t}^2} dq$$

$$+ \int_{o}^{\infty} \frac{qR_{ft}}{\sqrt{2\Pi}\sigma_{s_t}} e^{-(q-\mu_{s_t})^2/2\sigma_{s_t}^2} dq \qquad (9.2)$$

The remaining parts of the model are unchanged.

Case 2: The assumption that all random variables are normally distributed affects the formulation only with respect to the calculation of the expected return from slack funds. Making such an assumption implies that the slacks are also normally distributed since they are linear combinations of normally distributed project costs and the constant M_j (available funds in period j). The purpose of the assumption was to allow the determination of the slack's distribution. However, if a large number of projects are invested in, an assumption that the slacks are normally distributed may be valid even when the distribution of individual project costs are not normal. In this case the original assumption is replaced by the new assumption and no change in the formulation occurs. Actually, it is sufficient to assume some approximate distribution for slacks and make no assumptions about individual project cost distributions.

Two specific generalizations may occur in this situation.

1. Slack funds will, regardless which projects are selected, exhibit some probability distribution which is fully defined by its two parameters, mean and variance. Hence, slack funds have density functions $f(q;\mu_{s_t},\sigma_{s_t})$

This assumption affects only a change in the way the contributions of slack funds are computed. That change is from the form reviewed by equation (9.1) to (9.3)

$$E(\tilde{S}_t \cdot g(\tilde{S}_t)) = \int_{-\infty}^{0} qR_{bt}f(q;\mu_{s_r}\ \sigma_{s_t})dq + \int_{0}^{\infty} qk_{ft}f(q;\mu_{s_t},\sigma_{s_t})dq \quad (9.3)$$

2. Slack funds will, regardless which projects are selected, exhibit some probability distribution which is defined by k parameters $\delta_{1t}, \delta_{2t}, \ldots, \delta_{kt}$. Hence, slack funds have density function $f(q; \delta_{1t}, \delta_{2t}, \ldots, \delta_{kt})$

This assumption again causes a change in the way the contributions of slack funds are computed, thus yielding

$$E(\tilde{S}_t \cdot g(\tilde{S}_t)) = \int_{-\infty}^{0} qR_{bt}f(q;\delta_{1t},\delta_{2t},\ldots,\delta_{kt})dq$$

$$+ \int_{0}^{\infty} qR_{ft}f(q;\delta_{1t},\delta_{2t},\ldots,\delta_{kt})dq \quad (9.4)$$

An additional change also occurs in that the resultant formulation does not necessarily possess the specific constraints in formulation (4.13) given by (4.13e) and (4.13f) but do certain constraints of a general form as given by,

$$\delta_{it} = h_i(x_1,x_2,\ldots,x_n;c_{11},c_{12},\ldots,c_{1t};c_{21},c_{22},\ldots,c_{2t};\ldots; c_{nt};M_1,M_2,\ldots,M_t),$$

where h_i is some function that relates the individual project costs and available funds to the i^{th} parameter of the distribution for the slack in period t.

In either of these two generalizations the parts of formulation (4.13) not mentioned specifically remain unchanged.

Case 3: The assumption of no autocorrelation between investment costs may be eliminated with no change in the formulation. The only change is in the way input data are prepared (see Chapter VI). However, to assume correlation between project costs and the economic index or more directly to assume correlation between the slack and the economic index causes a change in the formulation. The change is derived from the fact that with the original assumption one could ignore the covariance between the return on slack funds and the market index. Under the generalized assumption the covariance must be treated explicitly. Specifically, this means an addition of two more systems of equality constraints to formulation (4.13) which serve to calculate the required covariance. This covariance term may then be incorporated into the objective function in the usual manner. Returning to the normality assumption yields a joint bivariant normal distribution for $\tilde{S}_t$ and $\tilde{R}_m$ with parameters μ_{s_t}, σ_{s_t}, $\bar{R}_m$, σ_m, and $\text{cov}(\tilde{S}_t,\tilde{R}_m)$.

Denoting the distribution by $f(q,r;\mu_{s_t},\sigma_{s_t},\overline{R}_m,\sigma_m,\mathrm{cov}(\tilde{S}_t,\tilde{R}_m))$ allows the immediate specification of the first additional system as follows:

$$\mathrm{cov}(\tilde{S}_t\cdot g(\tilde{S}_t),\tilde{R}_m) = \int_{-\infty}^{\infty}\int_{-\infty}^{0} (qR_{bt}-E(\tilde{S}_t\cdot g(\tilde{S}_t))(r-\overline{R}_m)\cdot f(q,r,\mu_{s_t},\sigma_{s_t},\overline{R}_m,\sigma_m,\mathrm{cov}(\tilde{S}_t,\tilde{R}_m))\,dq\,dr$$
$$+\int_{-\infty}^{\infty}\int_{0}^{\infty}(qR_{ft}-E(\tilde{S}_t\cdot g(\tilde{S}_t))(r-\overline{R}_m)\cdot f(q,r;\mu_{s_t},\sigma_{s_t},\overline{R}_m,\sigma_m,\mathrm{cov}(\tilde{S}_t,\tilde{R}_m))\,dq\,dr \quad (9.4)$$

The second required system states a relationship between project cost covariances with the market portfolio and slack covariance with the market portfolio. In effect it serves to define $\mathrm{cov}(\tilde{S}_t,\tilde{R}_m)$. Hence,

$$\mathrm{cov}(\tilde{S}_t,\tilde{R}_m) = -\sum_{k=1}^{t}\sum_{i=1}^{n} \mathrm{cov}(\tilde{C}_{ik},\tilde{R}_m)x_i \quad t=1,2,\ldots,T'.$$

This system is linear and causes no additional solution problems; however, the system (9.4) causes a good deal more nonlinearity in the overall formulation. The advantage of this generalization is that it makes a further generalization that allows available funds to be random variables with a covariance with $\tilde{R}_m$ easy to incorporate into the formulation.

The three cases stated above provide a basis for demonstrating the flexibility of the Chapter IV formulations. Any number of different specific formulations may be generated by altering the framework of assumptions with which each formulation must comply. For at least the three cases stated above, the solution procedures applied in Chapter VII appear to be the only possible means of solving the modified formulations. What remains is to accomplish the research required to demonstrate the applicability of those procedures. The next section is devoted to precisely that matter.

Further Research

The research reported in the preceding chapters has opened a number of opportunities for continued research. These opportunities may be categorized according to the special interests of the researcher.

For those whose interest is in the translation of theory into application there is the review and validation of the assumptions used to construct the framework within which the formulation (4.14) was spawned. If the framework proves inadequate for application to a particular real problem, then new formulations may be developed as exemplified by the three cases described earlier in this chapter.

For those whose interest is mathematical programming there is the determination of the ability of the procedures in Chapter VI to obtain a global optimum for formulations that may be produced by altering the framework of assumptions. Much of this work may be patterned after the developments and proofs in Appendix A and B.

For those interested in efficiency of mathematical programming algorithms the obvious course is to integrate a more efficient means of solving the mixed integer master problem in Chapter VI than by the Gomory Cutting Plane method.

For those interested in statistics and specifically estimation, whether subjective or not, there is the determination of error associated with data estimation procedures given in Chapter V and the investigation into ways of improving the estimates.

For those interested in any problem where the risk of violating a constraint is real, one may consider adding, to the objective function, a term which calculates the expected cost of violating a constraint. This particular approach appears to result in a solvable formulation and offers a new alternative to chance constraint programming.

General Conclusions

Although formulation (4.14) was developed under a restrictive set of assumptions, it encompasses more realism than those formulations developed under assumed certainty, single interest rates, or those formulations that consider only one of the two basic risks, (1) variability of income, or (2) variability of costs. In addition, it offers an alternative to chance constrained programming techniques in the way the risk of the variability of costs are incorporated into the problem. Including all of these things necessitated a formulation that is not straight forward. However, despite this and the mixed integer nature of the problem, the solution techniques are relatively straight forward, workable, and as the computational experience showed, efficient enough for one to expect to be able to solve larger problems. The data requirements, though considerable, are not excessive, and are basically the same kinds of data that one might generate for any large project regardless what means of project analysis are used. Finally, sensitivity analysis of solutions to formulation (4.14) provides valuable information to assist corporate managers with the long term financing decisions.

Now that the formulation has been specified, solution procedures developed along with proofs that those procedures will produce a global optimum, tests of the performance of these procedures have been performed, and methods of employing sensitivity analysis to broaden the applicability of the formulation have been specified; then one may conclude that the work presented in previous chapters is at least of academic interest. The ultimate question is: Can it be applied to real problems? At this point, any answer to this question is only a matter of opinion. The final answer can come only after attempts at implementation have been made and if successful, then only after several years of use can its total value be specified. It is hoped that the management of some corporations will see enough value in this work to seek to find the answer to the ultimate question.

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APPENDIX A

The Problem

The solution procedures presented in this appendix are the result of a direct application of Generalized Benders Decomposition Algorithm developed by Geoffrion (14,15) and applied to a class of problems of which the problem in Chapter IV is a member.

Given the problem

$$\max_{0 \le \overline{X} \le 1} \quad \overline{q}^t \overline{X} + f(L_1(\overline{X}),\ L_2(\overline{X}),\ \ldots,\ L_n(\overline{X}))$$

$$\text{s.t.:} \quad A\overline{X} \le \overline{b}$$

$$\overline{X} \text{ vector of mixed integer variables (VMIV)} \qquad \text{(A.1)}$$

where f is a function that is monotonic[1] in each of its variables and $L_i(\overline{X})$ is a linear function of $\overline{X}$. By introducing a variable vector $\overline{Y}$ where $y_i = L_i(\overline{X})$, problem (A.1) may be rewritten:

$$\max_{\substack{0 \le \overline{X} \le 1 \\ \overline{Y}}} \quad \overline{q}^t \overline{X} + f(\overline{Y})$$

---

[1]Monotonacity over the entire range is slightly stronger than necessary. The weaker requirements are stated in preliminary developments to the statement of equation (A.3b) appearing later.

$$\begin{aligned} \text{s.t.:}\quad & y_1 = L_1(\overline{X}) \\ & y_2 = L_2(\overline{X}) \\ & \vdots \qquad \vdots \\ & y_n = L_n(\overline{X}) \\ & \overline{A}\ \overline{X} \leq \overline{b} \\ & \overline{X} \quad \text{(VMIV)} \end{aligned} \tag{A.2}$$

If $f(\overline{Y})$ is monotonic increasing in its i^{th} variable for all values of that variable and since in the total problem, y_i has no effect upon y_j $(i \neq j)$, then the following replacement is possible without materially affecting the problem:

$$y_i = L_i(\overline{X}) \text{ replaced by } y_i \leq L_i(\overline{X}) \tag{A.3}$$

If the above condition holds except that $f(\overline{Y})$ is monotonic decreasing in its i^{th} variable then:

$$y_i = L_i(\overline{X}) \text{ is replaced by } y_i \geq L_i(\overline{X}) \tag{A.3a}$$

If $f(\overline{Y})$ is monotonic decreasing in its i^{th} variable for all values of $y_i \geq a$ and monotonic increasing for $y_i < a$ and is symmetric about "a" in the $f(\overline{Y})$-y_i plane, then:

$y_i = L_i(\overline{X})$ is replaced by:

$y_i - a \geq |L_i(\overline{X}) - a|$ or equivalently

$y_i - a \geq L_i(\overline{X}) - a$ and $y_i - a \geq -L_i(\overline{X}) + a$

$$\text{or}\quad y_i \geq L_i(\overline{X}) \text{ and } y_i \geq 2a - L_i(\overline{X}) \tag{A.3b}$$

If $f(\overline{Y})$ is monotonic increasing for $y_i \geq a$ and monotonic decreasing for $y_i < a$ then the problem cannot be handled by these techniques.

The following substitutions and manipulations are accomplished in order to generalize the statement of the problem to be decomposed by the Generalized Benders technique:

1. For all i for which (A.3a) replacement is appropriate one may use an equivalent form $-y_i \leq -L_i(\overline{X})$ and replace y_i by $-y_i$. Then $y_i \leq -L_i(\overline{X})$.

2. Let $\overline{q}_i$ be a row vector $\ni$ either $L_i(\overline{X}) = \overline{q}_i\overline{X}$ or $-L_i(\overline{X}) = \overline{q}_i\overline{X}$ dependent upon whether substitution 1. above has taken place. Then, one may define a matrix Q where

$$Q = \begin{bmatrix} \overline{q}_1 \\ \overline{q}_2 \\ \cdot \\ \cdot \\ \cdot \\ \overline{q}_n \end{bmatrix}$$

3. Where replacement (A.3b) applies, a manipulation similar to 1. above can be accomplished yielding

   $y_i \leq -L_i(\overline{X})$ and $y_i \leq L_i(\overline{X}) - 2a$

   The first of these constraints is used to help construct the matrix Q in 2., and the second constraint is used to construct a similar matrix Q' as follows:

   $\overline{q}_i{}'$ is a row vector $\ni L_i(\overline{X}) = \overline{q}_i{}'\overline{X}$

   if replacement (A.3b) occurred for i or

   $\overline{q}_i{}'$ is a row vector of zeros if (A.3b) did not occur for i. Then

$$Q' = \begin{bmatrix} \bar{q}_1{}' \\ \bar{q}_2{}' \\ \cdot \\ \cdot \\ \cdot \\ \bar{q}_n{}' \end{bmatrix}$$

4. $\bar{d}$ is a vector composed of elements a_i where a_i is the value denoted by "a" in (A.3b) or a_i = large negative value -M if (A.3b) does not apply for i.

Problem (A.2) may now be stated as a result of all necessary replacements (A.3), (A.3a), or (A.3b) and substitutions and manipulations 1, 2, 3, and 4 given above.

$$\begin{aligned} &\underset{\bar{X},\bar{Y}}{\text{Max}} && \bar{q}^t\bar{X} + f(\bar{Y}) \\ &\text{s.t.:} && \bar{Y} \le \bar{Q}\,\bar{X} \\ & && \bar{Y} \le \bar{Q}'\bar{X} - 2\bar{d} \\ & && \bar{A}\,\bar{X} \le \bar{b} \\ & && \bar{X} \ \ (\text{VMIV}) \end{aligned} \qquad (A.4)$$

Generalized Benders Decomposition

$\bar{X}$ is defined as the "complicating variables" (15, p. 1). One should observe that all functions of $\bar{X}$ and $\bar{Y}$ are linearly separable which insures compliance with Geoffrion's "property P"[2] (15, p. 11). The sub-problem obtained by a Benders decomposition is essentially problem (A.4) except that the variable $\bar{X}$ is fixed at constant values

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[2] Property "P" states in essence that the supremum over $\bar{Y}$ of the Lagrangean of the problem may be found independently of $\bar{X}$ for given dual multipliers.

$\overline{X}^k$ and the problem is solved for optimal $\overline{Y}$. Hence, the subproblem:

$$\begin{aligned} &\max_{\overline{Y}} && f(\overline{Y}) \\ &\text{s.t.:} && \overline{Y} \leq \overline{Q}\,\overline{X}^k \\ & && \overline{Y} \leq \overline{Q}'\overline{X}^k - 2\overline{d} \end{aligned} \tag{A.4a}$$

Geoffrion requires that the subproblem be solved in such a way that a global optimum is obtained and optimal dual multipliers are also obtained. Since the constraints of this problem are simple bounds on the variables y_i (half of these bounds are redundant) and since the bounds were constructed a certain way dependent upon the monotonacity of f, then the solution to this problem is trivial. Furthermore, simple bound constraints will always satisfy the "constraint qualifications" (52, p. 39) so that optimal dual multipliers are easily obtained from the Kuhn-Tucker conditions[3] providing the Lagrangean of problem (A.4a) possesses a saddle-point. If one defines $\overline{Y}^k$ as the optimal $\overline{Y}$ obtained from the subproblem constructed with assumed values $\overline{X}^k$; and $\overline{U}^k$, $\overline{V}^k$ as the corresponding optimal vectors of dual multipliers, then the master problem may be stated.

$$\begin{aligned} &\max_{r,\overline{X}} && r \\ &\text{s.t.:} && r \leq \overline{q}^t\overline{X} + f(\overline{Y}^k) - (\overline{U}^k)^t(\overline{Y}^k - \overline{Q}\,\overline{X}) - (\overline{V}^k)^t(\overline{Y} - \overline{Q}'\overline{X} + 2\overline{d}) \\ & && k = 1, 2, \ldots, P \end{aligned}$$

---

[3]If the problem does not exhibit the conditions required to replace "=" constraints by "≤" constraints, then unique optimal dual multipliers may not be obtainable.

$$\bar{A}\,\bar{X} \leq \bar{b}$$
$$\bar{X} \quad \text{(VMIV)} \tag{A.4b}$$

where P is the number of different feasible $\bar{X}^k$ for which a corresponding subproblem has been solved. There are two important points that one should observe.

1. Problem (A.4b) is a mixed-integer <u>linear</u> programming problem in $\bar{X}$ and r.
2. $\exists$ a feasible solution $\bar{Y}$ to problem (A.4a) for every $\bar{X}^k$ that is feasible in problem (A.4b). This situation precludes the necessity of using some of the constraints described by Geoffrion, and prevents some of the potential computational difficulty reported by him. (15, p. 8, Eq. 10b; p. 17)

<u>Example Problem</u>

The problem decomposed here will be the classic quadradic programming problem. The resulting technique for solving such a problem is new and is applicable regardless whether the variables $\bar{X}$ are continuous, integer or mixed integer. Initially, it is assumed that $\bar{X}$ is continuous. The problem:

$$\max_{\bar{X}} \quad \bar{q}^t\bar{X} - \bar{X}^t\bar{F}\,\bar{X}$$
$$\text{s.t.:} \quad \bar{A}\,\bar{X} \leq \bar{b} \tag{A.5}$$

where $\bar{F}$ is a positive definite or semidefinite symetric matrix.

A linear transformation is required and is developed as follows:

$\exists$ a matrix $\overline{E}$ of $\overline{F}$-conjugate vectors $\ni$

$\overline{E}^t\overline{F}\ \overline{E} = \overline{D}$ (a diagonal matrix of non-negative elements)

Introducting new variables $\overline{Y} \ni \overline{Y} = \overline{E}^{-1}\overline{X}$

$$\text{then}\quad \overline{Y}^t\overline{D}\ \overline{Y} = \overline{X}^t(\overline{E}^{-1})^t\overline{D}\ \overline{E}^{-1}\overline{X} = \overline{X}^t(\overline{E}^{-1})^t\overline{E}^t\overline{F}\ \overline{E}\ \overline{E}^{-1}\overline{X}$$
$$= \overline{X}^t\overline{I}\overline{F}\overline{I}\overline{X} = \overline{X}^t\overline{F}\ \overline{X}$$

Applying these results to problem (A.5) one obtains (A.6):

$$\begin{aligned} \underset{\overline{X},\overline{Y}}{\text{Max}}\quad & \overline{q}^t\overline{X} + f(\overline{Y}) \\ \text{s.t.:}\quad & \overline{Y} = \overline{E}^{-1}\overline{X} \\ & \overline{A}\ \overline{X} \leq \overline{b} \end{aligned} \tag{A.6}$$

where $f(\overline{Y}) = -\overline{Y}^t\overline{D}\ \overline{Y} = -\Sigma y_i{}^2 d_{ii}$ and $d_{ii} \geq 0\ \ \forall\ i.$

Clearly, $f(\overline{Y})$ is a monotonic increasing function of y_i for $y_i < 0$ and monotonic decreasing function of y_i for $y_i \geq 0.$[4] Therefore, one may apply replacement (A.3b) and obtain the problem (A.7) below.

$$\begin{aligned} \underset{\overline{X},\overline{Y}}{\text{Max}}\quad & \overline{q}^t\overline{X} + f(\overline{Y}) \\ \text{s.t.:}\quad & \overline{Y} \geq \overline{E}^{-1}\overline{X} \\ & \overline{Y} \geq -\overline{E}^{-1}\overline{X} \\ & \overline{A}\ \overline{X} \leq b \end{aligned} \tag{A.7}$$

By performing manipulation 1. on the greater than

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[4]Note that $f(\overline{Y})$ is also symmetric about 0 in the $f(\overline{Y})\text{-}y_i$ plane.

or equal to constraints the problem becomes

$$\underset{\overline{X},\overline{Y}}{\mathrm{Max}} \quad \overline{q}^t\overline{X} + f(\overline{Y})$$

$$\text{s.t.:} \quad \overline{Y} \le \overline{E}^{-1}\overline{X}$$

$$\overline{Y} \le -\overline{E}^{-1}\overline{X}$$

$$\overline{A}\ \overline{X} \le \overline{b} \tag{A.8}$$

This problem is in the same form as (A.4) with $\overline{d} = 0$, and therefore decomposes into the subproblem:

$$\underset{\overline{Y}}{\mathrm{Max}} \quad -\Sigma y_i^{\,2} d_{ii}$$

$$\text{s.t.:} \quad y_i \le e_{i1}x_1^{\,k} + e_{i2}x_2^{\,k} + \dots + e_{in}x_n^{\,k} \quad i = 1, \dots, n$$

$$y_i \le -e_{i1}x_1^{\,k} - e_{i2}x_2^{\,k} - \dots - e_{in}x_n^{\,k} \quad i = 1, \dots, n$$

where $\overline{X}^k$ is fixed, and the master problem:

$$\underset{r,\overline{X}}{\mathrm{Max}} \quad r$$

$$\text{s.t.:} \quad r \le \overline{q}^t\overline{X} - \Sigma(y_i^k)^2 d_{ii} - (\overline{U}^k)^t(\overline{Y}^k - \overline{E}^{-1}\overline{X}) - (\overline{V}^k)^t(\overline{Y}^k + \overline{E}^{-1}\overline{X}) \quad k = 1, \dots, P$$

$$\overline{A}\ \overline{X} \le \overline{b}\ .$$

The master problem is linear in $\overline{X}$ and r so that any additional requirements necessary (such as $\overline{X}$ being integer or mixed integer) may be added without causing an inability to solve the problem.

As indicated earlier the subproblem is trivial so that given $\overline{X}^k$ the global maximum is clearly,

$$y_i^* = -\left| e_u x_1^k + e_{12} x_2^k + \ldots + e_{1n} x_n^k \right| \tag{A.9}$$

which will exactly satisfy one of the bounds on each y_i and amply satisfy the other bound since it is redundant. From the Kuhn-Tucker conditions one has the following

1. $\overline{Y}^*$ is clearly feasible
2. $-2d_{ii}y_i^* - u_i^* - v_i^* = 0 \qquad i = 1, \ldots, n$
3. $$\left.\begin{aligned} u_i^*(y_i^* - \sum_j e_{ij}x_j^k) &= 0 \\ v_i^*(y_i^* + \sum_j e_{ij}x_j^k) &= 0 \end{aligned}\right\} \qquad i = 1,2, \ldots, n \tag{A.10}$$

From (A.9) one sees that if $\sum_j e_{ij}x_j^k < 0$, then $y_i^* = \sum e_{ij}x_j^k$ $\therefore y_i^* + \sum_j e_{ij}x_j^k \neq 0$ so that by K-T condition 3., $v_i^* = 0$ hence $u_i^* = -2d_{ii}y_i^*$. Clearly, $u_i^* \geq 0$. Also, if $\sum_j e_{ij}x_j^k > 0$ then $y_i^* = -\sum e_{ij}x_j^k$. $\therefore y_i^* - \sum_j e_{ij}x_j^k \neq 0$, hence from 3., $u_i^* = 0$ and from 2, $v_i^* = -2d_{ii}y_i^*$. Clearly, $v_i^* \geq 0$.

The stepwise solution procedures may now be stated:

1. $r = +\infty$, $k = 1$, find any $\overline{X}^k \ni \overline{A}\ \overline{X}^k \leq \overline{b}$ and $\overline{X}^k$ satisfies whatever integer requirements there are.
2. Use (A.9) and (A.10) to determine $\overline{Y}^*$, $\overline{U}^*$ and $\overline{V}^*$. Index the values with k.
3. Construct a constraint for the master problem and add that constraint to the master problem.
4. Solve the master problem for optimal $\overline{X}^*$, r^*. Set $k = k + 1$ and index $\overline{X}^*$, and r^* with k.

5. If $|r^k - r^{k-1}| > \epsilon$ return to step 2. Otherwise stop because the current solution is ϵ-optimal for the original problem.

Conclusions

The solution technique presented in this appendix was chosen to be applied to the problem developed in Chapter IV for the following reasons:

1. It can be shown that the problem in Chapter IV possesses the correct monotonacity characteristics and is receptive to the type of linear transformation discussed herein so that the technique will succeed. Furthermore, these are the only known solution procedures for a mixed integer problem with the type and degree of nonlinearity possessed by this problem.
2. Some of the assumptions used to formulate the problem may be relaxed producing more complex but more realistic formulations. However, these problems will possess essentially the same characteristics and may be solvable using the same procedures.
3. A certain amount of sensitivity analysis may be performed in a direct manner from the solutions obtained by these procedures.

APPENDIX B

This appendix contains the proof of the property of problem (4.14) and (6.10) that is required to apply the substitutions and transformations in Appendix A. The proof of the hypothesis of the v Representation Theorem for the subproblem (6.10'sub) is also contained herein.

Property I: The function,

$$f(\overline{S},\overline{Z}) = \sum_{j=1}^{k} b_j\left[\frac{d_j e^{-s_j^2/2z_j^2}}{\sqrt{2\Pi}} \cdot z_j + R_f s_j - s_j d_j \int_{-\infty}^{0} \frac{e^{-(t-s_j)^2/2z_j^2}}{\sqrt{2\Pi} z_j} dt\right]$$

where: $b_j \geq 0 \;\forall\; k$

$d_j \leq 0 \;\forall\; j$

$R_f > 0$

$s_j \geq 0 \;\forall\; j$

$z_j \geq 0 \;\forall\; j$; is a monotonic decreasing function of z_j.

<u>Proof</u>: The proof consists of showing that the partial derivative of f with respect to z_j is less than or equal to zero for every s_j and $z_j \geq 0$. Taking this derivative is somewhat simpler if the term

$$\int_{-\infty}^{0} \frac{1}{\sqrt{2\Pi} z_j} e^{-(t-s_j)^2/2z_j^2} dt$$

is modified by a change of variables to $q = \dfrac{t-s_j}{z_j}$ thus producing

$$\int_{-\infty}^{-s_j/z_j} \frac{1}{\sqrt{2\Pi}} e^{-q^2/2} dq.$$ [1] Hence,

$$\frac{\partial f(\overline{S},\overline{Z})}{\partial z_j} = b_j \left\{ \frac{d_j e^{-s_j^2/2z_j^2}}{\sqrt{2\Pi}} + d_j z_j \left(\frac{s_j^2}{z_j^2}\right) \frac{e^{-s_j^2/2z_j^2}}{\sqrt{2\Pi}} - s_j d_j \left(\frac{s_j}{z_j^2}\right) \frac{e^{-s_j^2/2z_j^2}}{\sqrt{2\Pi}} \right\}$$

$$= b_j \left\{ \frac{d_j e^{-s_j^2/2z_j^2}}{\sqrt{2\Pi}} + \frac{d_j s_j^2}{z_j^2} \frac{e^{-s_j^2/2z_j^2}}{\sqrt{2\Pi}} - \frac{d_j s_j^2}{z_j^2} \frac{e^{-s_j^2/2z_j^2}}{\sqrt{2\Pi}} \right\}$$

$$= \frac{b_j d_j e^{-s_j^2/2z_j^2}}{\sqrt{2\Pi}}$$

since $b_j \geq 0$; $d_j \leq 0$; $e^{-s_j^2/2z_j^2} \geq 0$, then $\frac{\partial f(\overline{S},\overline{Z})}{\partial z_j} \leq 0$

$\forall\ z_j$ and s_j, in particular $\forall\ z_j$ and $s_j \geq 0$.

Property II: For the problem

$$\underset{\overline{Y},\overline{S},\overline{Z}}{\text{Maximize}}\ P_o = \overline{A}^t\ \overline{X}^{\ell} + f(\overline{S},\overline{Z}) \tag{B.1}$$

$$\text{s.t.:} \quad \left. \begin{aligned} & z_j - \sqrt{\sum_{i=1}^{n} y_{ij}^2} \geq 0 \\ & \overline{Y}_j - \overline{E}_j^{-1}\ \overline{X}^{\ell} = 0 \\ & s_j - \sum_{i=1}^{j} M_i + \sum_{i=1}^{j} \overline{C}_i^{\ t}\ \overline{X}^{\ell} = 0 \end{aligned} \right\} \ j=1,2,\ldots,k$$

$\overline{S} \geq 0$

$\overline{Z} \geq 0$; The optimal value P_o^* equals that of

[1] Derivative under an integral may be found in Kaplan (page 220) (22).

the dual for every value of $\overline{X}^{\ell}$ that satisfies the system

$$\sum_{i=1}^{j} M_i - \sum_{i=1}^{j} \overline{C}_i^{\,t} \, \overline{X}^{\ell} \geq 0. \quad j=1,\ldots,k \tag{B.2}$$

<u>Proof</u>: Any point $(s_1, s_2, \ldots, s_k, z_1, z_2, \ldots, z_k, y_{11}, y_{12}, \ldots, y_{1k}, y_{21}, y_{22}, \ldots, y_{2k}, \ldots, y_{n1}, y_{n2}, \ldots, y_{nk})^t$ in $E^{n.k+2k}$ (n.k+2k dimensional Real Euclidean Space) may be a candidate for solution to problem (B.1) but since $\overline{X}^{\ell}$ must satisfy system (B.2) in the master problem and, all y_{ij} are real then no generality is lost by restricting the solution to (B.1) to a convex subset of $E^{n.k+2k}$. In particular, the subset, called F, is defined as follows:

$$F = \{ \overline{X} \in E^{n.k+2k} \mid x_i \geq 0, \; i=1,2,\ldots,2k \}$$ [2]

the problem that will be proved to exhibit dual equality is given by:

$$\underset{\overline{S},\overline{Z},\overline{Y}\in F}{\text{Maximize}} \quad P_o = \overline{A}^t \overline{X}^{\ell} + f(\overline{S},\overline{Z}) \tag{B.3}$$

$$\text{s.t.:} \quad \left.\begin{aligned} z_j - \sqrt{\sum_{i=1}^{n} y_{ij}^{\,2}} &\geq 0 \\ \overline{Y}_j - \overline{E}_j^{-1} \, \overline{X}^{\ell} &= 0 \\ s_j - \sum_{i=1}^{j} M_i + \sum_{i=1}^{j} \overline{C}_i^{\,t} \overline{X}^{\ell} &= 0 \end{aligned}\right\} \; j=1,2,\ldots,k$$

The proof will consist of finding a solution to (B.3) and finding dual multipliers such that the solution to (B.3)

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[2] Notice that the only values restricted by this definition are those values for s_j and z_j where the values for y_{ij} remain unrestricted.

maximizes the Lagrangean function over F. First, further modification must be made to (B.3). Since $\overline{X}^{\ell}$ is a constant each time the subproblem is solved all functions containing $\overline{X}^{\ell}$ may be evaluated, and replaced, by a constant value thus simplifying notation. Furthermore, the equality constraints may be replaced by two inequality constraints. These modifications yield the following form of problem (B.3).

$$\underset{(\overline{S},\overline{Z},\overline{Y})\epsilon F}{\text{Maximize}} \quad P_o = c_{oo} + f(\overline{S},\overline{Z}) \tag{B.3'}$$

				Dual Variables
s.t.	$z_j - \sqrt{\sum_{i=1}^{n} y_{ij}^2}$	≥ 0	$j=1,2,\ldots,k$	u_j
	$\overline{Y}_j - \overline{C}_j$	≥ 0	$j=1,2,\ldots,k$	v_{ij}^+
	$\overline{Y}_j - \overline{C}_j$	≤ 0	$j=1,2,\ldots,k$	v_{ij}^-
	$s_j - c_{oj}$	≥ 0	$j=1,2,\ldots,k$	w_j^+
	$s_j - c_{oj}$	≤ 0	$j=1,2,\ldots,k$	w_j^-

where $\overline{C}_j$ are constant vectors with elements c_{ij} and c_{oj} are scalars for $j=0,1,2,\ldots,k$.

The Lagragean is,

$$L(\overline{S},\overline{Z},\overline{Y},\overline{U},\overline{V}^+,\overline{V}^-,\overline{W}^+,\overline{W}^-) = c_{oo} + f(\overline{S},\overline{Z}) + \sum_{j=1}^{k} u_j(z_j - \sqrt{\sum_{i=1}^{n} y_{ij}^2})$$

$$+ \sum_{j=1}^{k} \sum_{i=1}^{n} v_{ij}^+ (y_{ij}-c_{ij})$$

$$- \sum_{j=1}^{k} \sum_{i=1}^{n} v_{ij}^- (y_{ij}-c_{ij}) + \sum_{j=1}^{k} w_j^+ (s_j-c_{oj})$$

$$- \sum_{j=1}^{k} w_j^- (s_j-c_{oj}) \tag{B.4}$$

It is obvious that the optimal primal solution to (B.3') is given by (B.5).

$$s_j^* = c_{oj} \qquad j=1,2,\ldots,k \qquad \text{(B.5)}$$

$$y_{ij}^* = c_{ij} \qquad i=1,2,\ldots,n;\; j=1,2,\ldots,k$$

$$z_j^* = \sqrt{\sum_{i=1}^{n} (y_{ij}^*)^2} \qquad j=1,2,\ldots,k$$

Applying the Kuhn-Tucker conditions provides the following

K-T Condition 1. (The proposed primal solution must be feasible.)

Solution (B.5) clearly satisfies this condition.

K-T Condition 2. ($\exists$ non-negative dual multipliers such that the product of the dual multipliers and their corresponding constraint function must equal zero at the proposed primal solution.)

Solution (B.5) satisfies all constraints exactly which means the constraint function's values at that solution are all zero and hence any set of dual non-negative multipliers will satisfy this condition.

K-T Condition 3. (The gradient of the Lagrangean evaluated at the proposed solution must be equal to zero.)

This condition is used to determine values for the dual multipliers and the resulting equations may be summarized as follows:

$$\frac{\partial f(\overline{S}^*,\overline{Z}^*)}{\partial z_j} + u_j = 0 \qquad j=1,\ldots,k \qquad \text{(B.6a)}$$

$$\frac{\partial f(\overline{S}^*,\overline{Z}^*)}{\partial s_j} + w_j^+ - w_j^- = 0 \qquad j=1,\ldots,k \qquad \text{(B.6b)}$$

$$-u_j \frac{y_{ij}^*}{\sqrt{\sum_{i=1}^{n}(y_{ij}^*)^2}} + v_{ij}^+ - v_{ij}^- = 0 \qquad \begin{array}{l} i=1,\ldots,n \\ j=1,\ldots,k \end{array} \qquad \text{(B.6c)}$$

Therefore,

$$u_j^* = \frac{-\partial f(\overline{S}^*,\overline{Z}^*)}{\partial z_j} = \frac{-b_j d_j e^{-c_{oj}^2/2\sum_{i=1}^{n} c_{ij}^2}}{\sqrt{2\Pi}} \geq 0.$$

The non-negativity of u_j^* is guaranteed by Property I stated earlier. Also,

$$(w_j^+)^*-(w_j^-)^* = -\frac{\partial f(\overline{S}^*,\overline{Z}^*)}{\partial s_j} = -b_j\left[R_f-d_j\int_{-\infty}^{-c_{oj}^2/\sqrt{\sum_i c_{ij}^2}} \frac{1}{\sqrt{2\Pi}}e^{-q^2/2}dq\right]$$

$$\leq 0,$$ [3]

and one may assume that $(w_j^+)^* = 0$ which guarantees that $(w_j^-)^* \geq 0$ $\therefore$ satisfying the non-negativity requirement.

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[3] $$\frac{\partial f(\overline{S},\overline{Z})}{\partial s_j} = b_j\left[\frac{d_j z_j}{\sqrt{2\Pi}}e^{-s_j^2/2z_j^2}\left(\frac{-s_j}{z_j^2}\right)+R_f-d_j\int_{-\infty}^{-s_j/z_j}\frac{e^{-q^2/2}}{\sqrt{2\Pi}}dq\right.$$

$$\left.-\frac{s_j d_j}{\sqrt{2\Pi}}e^{-s_j^2/2z_j^2}\left(-\frac{1}{z_j}\right)\right] = b_j\left(R_f-d_j\int_{-\infty}^{-s_j/z_j}\frac{e^{-q^2/2}}{\sqrt{2\Pi}}dq\right)$$

since $b_j \geq 0$; $R_f \geq 0$; $d_j \leq 0$; and probability is ≥ 0 then $\frac{\partial f(\overline{S},\overline{Z})}{\partial s_j} \geq 0 \;\forall\; s_j$ and z_j.

Finally

$$(v^+_{ij})^* - (v^-_{ij})^* = \frac{u^*_j \; c_{ij}}{\sqrt{\sum_{i=1}^{n} c_{ij}{}^2}} = - \frac{b_j d_j c_{ij} e^{-c_{oj}^2/2\sum_{i=1}^{n} c_{ij}{}^2}}{\sqrt{2\Pi \sum_{i=1}^{n} c_{ij}{}^2}}$$

and if $c_{ij} \geq 0$ then $(v^-_{ij})^* = 0$

or if $c_{ij} < 0$ then $(v^+_{ij})^* = 0$, then $(v^+_{ij})^*$ and $(v^-_{ij})^*$ will satisfy the non-negative requirement of K-T condition 2.

It will now be shown that

$$L(\overline{S}^*, \overline{Z}^*, \overline{Y}^*, \overline{U}^*, (\overline{V}^+)^*, (\overline{V}^-)^*, (\overline{W}^+)^*, (\overline{W}^-)^*) =$$

$$\underset{(\overline{S},\overline{Z},\overline{Y})\epsilon F}{\text{Maximum}} \; L(\overline{S}, \overline{Z}, \overline{Y}, \overline{U}^*, (\overline{V}^+)^*, (\overline{V}^-)^*, (\overline{W}^+)^*, (\overline{W}^-)^*)$$

and to simplify notation the symbol for the Lagrangean will be written $L(\overline{X}, \overline{\lambda})$ so that the above may be written $L(\overline{X}^*, \overline{\lambda}^*) = \underset{\overline{X}\epsilon F}{\text{Max}} \; L(\overline{X}, \overline{\lambda}^*)$. The function whose maximum over F is to be found is given by (B.6)

$$L(\overline{X},\overline{\lambda}^*) = c_{oo} + f(\overline{S},\overline{Z}) \tag{B.6}$$

$$- \sum_{j=1}^{k} \frac{b_j d_j e^{-c_{oj}^2/2\sum_{i=1}^{n} c_{ij}^2}}{\sqrt{2\Pi}} \left(z_j - \sqrt{\sum_{i=1}^{n} y_{ij}{}^2} \right)$$

$$- \sum_{j=1}^{k} \sum_{i=1}^{n} \frac{b_j d_j c_{ij} e^{-c_{oj}^2/2\sum_{i=1}^{n} c_{ij}^2}}{\sqrt{2\Pi \sum_i c_{ij}^2}} \cdot (y_{ij} - c_{ij})$$

$$- \sum_{j=1}^{k} b_j \left(R_f - d_j \int_{-\infty}^{-c_{oj}/\sqrt{\sum_{i=1}^{n} c_{ij}^2}} \frac{1}{\sqrt{2\Pi}} e^{-q^2/2} dq \right) (s_j - c_{oj})$$

One may observe that the terms containing s_j or z_j are linearly separable from those containing y_{ij} so that their maximums may be determined independently. In addition, some terms in (B.6) are constant with respect to s_j, z_j, and y_{ij} and, of course, need not be considered while locating an optimum. Hence, (B.6) simplifies to,

$$f(\overline{S},\overline{Z}) - \sum_{j=1}^{k} \frac{b_j d_j e^{-c_{oj}^2/2\,\Sigma c_{ij}^2}}{\sqrt{2\Pi}} \cdot z_j \qquad - \sum_{j=1}^{k} s_j b_j \left(R_f - d_j \int_{-\infty}^{-c_{oj}/\sqrt{\Sigma c_{ij}^2}} \frac{e^{-q^2/2}}{\sqrt{2\Pi}} dq \right)$$

(B.6a)

and

$$\sum_{j=1}^{k} \frac{b_j d_j \sqrt{\sum_{i=1}^{n} y_{ij}^2}}{\sqrt{2\Pi}} e^{-c_{oj}^2/2 \sum_{i=1}^{n} c_{ij}^2}$$

$$- \sum_{j=1}^{k} \frac{b_j d_j \sum_{i=1}^{n} c_{ij} y_{ij}}{\sqrt{2\Pi \sum_{i=1}^{n} c_{ij}^2}} e^{-c_{oj}^2/2 \sum_{i=1}^{n} c_{ij}^2} \qquad \text{(B.6b)}$$

Again it may be noted that in (B.6a), s_j and z_j are contained in terms that are linearly separable from all terms containing s_ℓ and z_ℓ so long as $\ell \neq j$. Therefore, to maximize (B.6a) over all s_j and z_j, one needs only to maximize the terms containing s_j and z_j for each i or to maximize (B.6aj).

$$\frac{b_j d_j e^{-s_j^2/2z_j^2}}{\sqrt{2\Pi}} \cdot z_j + b_j R_f s_j - b_j d_j s_j \int_{-\infty}^{-s_j/z_j} \frac{1}{\sqrt{2\Pi}} e^{-q^2/2} dq$$

$$- \frac{b_j d_j e^{-c_{oj}^2/2\Sigma c_{ij}^2}}{\sqrt{2\Pi}} \cdot z_j$$

$$- s_j b_j \left(R_f - d_j \int_{-\infty}^{-c_{oj}/\sqrt{\Sigma c_{ij}^2}} \frac{e^{-q^2/2}}{\sqrt{2\Pi}} dq \right) \qquad \text{(B.6aj)}$$

Because functions of y_{ij} and $y_{i\ell}$ are linearly separable for $j \neq \ell$ then (B.6b) may also be maximized for each j independently so that (B.6b) becomes (B.6bj)

$$A\sqrt{\sum_{i=1}^{n} y_{ij}^{\,2}} - A \frac{\sum_{i=1}^{n} c_{ij} y_{ij}}{\sqrt{\sum_{i=1}^{n} c_{ij}^{\,2}}} \qquad \text{(B.6bj)}$$

where $A = \dfrac{b_j d_j}{\sqrt{2\Pi}} e^{-c_{oj}^{\,2}/2\Sigma c_{ij}^{\,2}}$ and $A < 0$.

To maximize (B.6aj) one may hold s_j fixed and investigate the properties of the partial with respect to z_j.

$$\frac{\partial (\text{B.6aj})}{\partial z_j} = \frac{b_j d_j e^{-s_j^2/2z_j^2}}{\sqrt{2\Pi}} \left(\frac{s_j^2}{z_j^2} \right) + \frac{b_j d_j e^{-s_j^2/2z_j^2}}{\sqrt{2\Pi}}$$

$$- \frac{b_j d_j}{\sqrt{2\Pi}} e^{-s^2/2z_j^2} \left(\frac{s_j^2}{z_j^2}\right) - \frac{b_j d_j e^{-c_{oj}^2/2\Sigma c_{ij}^2}}{\sqrt{2\Pi}}$$

$$\frac{\partial (B.6aj)}{\partial z_j} = - \frac{b_j d_j}{\sqrt{2\Pi}} \left(e^{-c_{oj}^2/2\Sigma c_{ij}^2} - e^{-s_j^2/2z_j^2}\right)$$

Since $\frac{b_j d_j}{\sqrt{2\Pi}} < 0$ then $\frac{-b_j d_j}{\sqrt{2\Pi}} > 0$ so that it becomes clear that when

$$-c_{oj}{}^2/2\Sigma c_{ij}{}^2 > -s_j{}^2/2z_j{}^2 \text{ then } \frac{\partial (B.6aj)}{\partial z_j} > 0$$

and when $-c_{oj}{}^2/2\Sigma c_{ij}{}^2 < -s_j{}^2/2z_j{}^2$ then $\frac{\partial (B.6aj)}{\partial z_j} < 0$

and when $-c_{oj}{}^2/2\Sigma c_{ij}{}^2 = -s_j{}^2/2z_j{}^2$ then $\frac{\partial (B.6aj)}{\partial z_j} = 0$.

These three conditions lead to the following conclusions:

1. When $z_j < \frac{s_j}{c_{oj}} \sqrt{\Sigma c_{ij}{}^2}$, $\frac{\partial (B.6aj)}{\partial z_j} > 0$

2. When $z_j > \frac{s_j}{c_{oj}} \sqrt{\Sigma c_{ij}{}^2}$, $\frac{\partial (B.6aj)}{\partial z_j} < 0$

3. When $z_j = \frac{s_j}{c_{oj}} \sqrt{\Sigma c_{ij}{}^2}$, $\frac{\partial (B.6aj)}{\partial z_j} = 0$

Since $c_{oj} > 0$ and since in order for $\overline{X} \in F$ then $s_j \geq 0$; therefore, for every $s_j \geq 0 \; \exists \; \overline{X} \in F$ s_j is a component of $\overline{X}$ and such that $\overline{X}$ satisfies condition 3. Therefore, for given $s_j^1 \geq 0$ (B.6aj) may be maximized over z_j by

$z_j^1 = \frac{s_j^1}{c_{oj}\sqrt{\sum_{i=1}^{n} c_{ij}^2}}$ with the certainty that $(\bar{S}^1, \bar{Z}^1, \bar{Y}) \in F$.

Substitution of the expression $z_j = \frac{s_j}{c_{oj}\sqrt{\sum_{i=1}^{n} c_{ij}^2}}$ into (B.6aj) and maximization over $s_j \geq 0$ will complete the maximization of part of the Lagrangean over F. Hence, (B.6aj) becomes:

$$\frac{b_j d_j e^{-s_j^2 c_{oj}^2 / 2 s_j^2 \sum_{i=1}^{n} c_{ij}^2}}{\sqrt{2\Pi}} \left(\frac{s_j}{c_{oj}\sqrt{\sum_{i=1}^{n} c_{ij}^2}} \right)$$

$$- b_j d_j s_j \int_{-\infty}^{-s_j c_{oj}/s_j\sqrt{\sum_{i=1} c_{ij}^2}} \frac{1}{\sqrt{2\Pi}} e^{-q^2/2}\, dq$$

$$- \frac{b_j d_j e^{-c_{oj}^2/2\Sigma c_{ij}^2}}{\sqrt{2\Pi}} \left(\frac{s_j}{c_{oj}} \sqrt{\Sigma c_{ij}^2} \right)$$

$$+ b_j d_j s_j \int_{-\infty}^{-c_{oj}/\sqrt{\Sigma c_{ij}^2}} \frac{1}{\sqrt{2\Pi}} e^{-q^2/2}\, dq,$$

which simplifies exactly to zero. Therefore, no matter what non-negative value for s_j is chosen there exists a non-negative value for z_j such that all terms containing s_j and z_j are maximized and reduced to zero at the same time. In particular, when $s_j^1 = s_j^*$ then $z_j^1 = \frac{s_j^*}{c_{oj}}\sqrt{\Sigma c_{ij}^2} = \frac{c_{oj}}{c_{oj}}\sqrt{\Sigma c_{ij}^2}$

$= \sqrt{\Sigma c_{ij}^2} = z_j^*$.

One may now consider (B.6bj) and its maximum over all y_{ij}. It suffices to prove that (B.6bj) ≤ 0.[4]
Suppose (B.6bj) > 0, then

$$A\sqrt{\sum_i y_{ij}^2} - A\frac{\sum_i c_{ij}y_{ij}}{\sqrt{\sum_i c_{ij}^2}} > 0; \text{ or since } A < 0,$$

then
$$\frac{\sum_{i=1}^n c_{ij}y_{ij}}{\sqrt{\sum_{i=1}^n c_{ij}^2}} - \sqrt{\sum_{i=1}^n y_{ij}^2} > 0$$

and
$$\Sigma c_{ij}y_{ij} > \sqrt{\sum_{i=1}^n c_{ij}^2}\ \sqrt{\sum_{i=1}^n y_{ij}^2}\ . \tag{B.7}$$

If the left side of (B.7) is less than or equal to zero then clearly (B.7) cannot be true. Suppose then, the left side is positive, thereby making it possible to square both sides without affecting the inequality (B.7) becomes:

$$\left(\sum_{i=1}^n c_{ij}y_{ij}\right)^2 > \left(\sum_{i=1}^n c_{ij}^2\right)\left(\sum_{i=1}^n y_{ij}^2\right)$$

or
$$\sum_{i=1}^n c_{ij}^2 \sum_{j=1}^n y_{ij}^2 - \left(\sum_{i=1}^n c_{ij}y_{ij}\right)^2 < 0$$

or
$$\sum_{i=1}^n \sum_{\ell=1}^n c_{ij}^2 y_{\ell j}^2 - \sum_{i=1}^n \sum_{\ell=1}^n c_{ij}y_{ij}c_{\ell j}y_{\ell j} < 0 \tag{B.8}$$

[4] If (B.6bj) ≤ 0, then obviously, its maximum is 0 which is easily achieved when $y_{ij}=c_{ij}$. Hence, $y_{ij}=c_{ij}=y_{ij}^*$ will maximize (B.6bj).

For every pair of integers s and t each between 1 and n and not equal, there are four terms in (B.8). Specifically, these terms are:

$$c_{sj}^2 y_{tj}^2 - c_{sj}y_{sj}c_{tj}y_{tj} - c_{tj}y_{tj}c_{sj}y_{sj} + c_{tj}^2 y_{sj}^2 \qquad \text{(B.9)}$$

$$= c_{sj}^2 y_{tj}^2 - 2c_{sj}y_{tj}c_{tj}y_{sj} + c_{tj}^2 y_{sj}^2 = (c_{sj}y_{tj} - c_{tj}y_{sj})^2 \geq 0.$$

When s = t one obtains two terms:

$$c_{tj}^2 y_{tj}^2 - c_{tj}^2 y_{tj}^2 \text{ or } c_{sj}^2 y_{sj}^2 - c_{sj}^2 y_{sj}^2 = 0. \qquad \text{(B.10)}$$

Since the left side of (B.8) is equal to the sum of all possible terms of form (B.9) plus those of form (B.10), then it is clear that (B.8) has been contradicted, which proves that (B.6bj) $\leq$ 0.

To summarize what has been found and shown, the following items are listed:

1. A vector $\overline{X}^* = (\overline{S}^*, \overline{Z}^*, \overline{Y}^*)$ was found and is given by equation (B.5). It was shown that $\overline{X}^* \epsilon$ F and $\overline{X}^*$ was feasible for problem (B.3').
2. Dual multiplier vector $\overline{\lambda}^* = (\overline{U}^*, \overline{V}^{+*}, \overline{V}^{-*}, \overline{W}^{+*}, \overline{W}^{-*})$ was found and is given by equation (B.6a), (B.6b), and (B.6c). It was shown that $\overline{\lambda}^* \geq 0$ and that the product of each multiplier vector component and its respective primal constraint is zero.
3. It was shown that $L(\overline{X}^*, \lambda^*) = \max_{\overline{X} \epsilon F} L(\overline{X}, \lambda^*)$.

Lemma 1:[5] For any problem

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[5]This lemma is an adaptation of Lemma 2.17 and Theorem 2.19 found in Zangwill (52).

$$\max f(\overline{X})$$
$$\text{s.t. } g_i(\overline{X}) \geq 0 \quad i = 1,\ldots,m$$
$$\overline{X} \in F$$

Given:

1. $\exists\ \overline{X}^{-1} \ni \overline{X}^{-1}$ is feasible for the problem.
2. $\exists$ multiplier vector $\overline{\lambda}^{1} = \{\lambda_i^1\} \ni \lambda_i^1 g_i(\overline{X}^1) = 0$ and $\lambda_i^1 \geq 0\ \forall\ i$.
3. $L(\overline{X}^1,\overline{\lambda}^1) = \max\limits_{\overline{X}\in F} L(\overline{X},\overline{\lambda}^1)$.

Then a saddlepoint exists.

Proof: Since $\overline{X}^1$ is feasible, then $\min\limits_{\overline{\lambda}\geq 0} L(\overline{X}^1,\overline{\lambda}) = f(\overline{X}^1)$, and from parts 2. and 3. one has

$$f(\overline{X}^1) = f(\overline{X}^1) + \sum_{i=1}^{m} \lambda_i^1 g_i(\overline{X}^1) = L(\overline{X}^1, \overline{\lambda}^1) = \max_{\overline{X}\in F} L(\overline{X},\overline{\lambda}^1)$$

Therefore, $\min\limits_{\overline{\lambda}\geq 0} L(\overline{X}^1,\overline{\lambda}) = L(\overline{X}^1,\overline{\lambda}^1) = \max\limits_{\overline{X}\in F} L(\overline{X},\overline{\lambda}^1)$ which is the definition of a saddlepoint.

Application of this lemma to the preceding developments summarized above indicates that a saddlepoint does exist and is easy to find. This, of course, implies dual equality for the subproblem which in this particular case guarantees a global optimum can be found for the problem in Chapter IV by the application of Generalized Benders Algorithm.

APPENDIX C

This appendix contains the exact relationships between the variables of the specific model developed in Chapter IV and the matrix representation of the same formulation. Formulation (4.13-4.13m) was written in a manner that retained the notation and equations of previous developments in Chapter IV. However, simplification is both possible and desirable in order to more clearly understand the mathematical structure of the problem. One may first of all note that μ_{S_t} is exactly the same as $\overline{S}_t$. This is easily shown by solving the system (4.13g) and (4.13h) for the individual $\overline{S}_t$ and then noting that they are equivalent to the (4.13e) definition of μ_{S_t}. Hence, the first simplification consists of eliminating the redundant equations (4.13e) and replacing the notation μ_{S_t} by $\overline{S}_t$ in equations (4.13c) and (4.13d). Secondly, the system (4.13a) and (4.13b) may be eliminated by direct substitution into the objective function (4.13) thus producing the following:

$$P_o = \frac{\sum_{k=1}^{T} \beta_k \left\{ \sum_{i=1}^{n} \bar{e}_{ik} x_i + E(\tilde{S}_k \cdot g(\tilde{S}_k)) - \alpha R_f \sum_{j=1}^{T} \frac{\sum_{i=1}^{n} \bar{e}_{ij} x_i + E(\tilde{S}_j \cdot g(\tilde{S}_j))}{(1+R_f)^j} \right\}}{(1+R_f)}$$

$$+ \frac{\alpha R_f \sum_{j=1}^{T} \frac{\sum_{i=1}^{n} \bar{e}_{ij} x_i + E(\tilde{S}_j \cdot g(\tilde{S}_j))}{(1+R_f)^j}}{(1+R_f)}$$

$$+ \frac{-\lambda' \left\{ \sum_{k=1}^{T} \beta_k \left(\sum_{i=1}^{n} \mathrm{cov}(\hat{e}_{ik}, \tilde{R}_m) x_i - \alpha R_f \sum_{j=1}^{T} \frac{\sum_{i=1}^{n} \mathrm{cov}(\hat{e}_{ij}, \tilde{R}_m) x_i}{(1+R_f)^j} \right) \right\}}{(1+R_f)}$$

$$+ \frac{-\lambda' \left\{ \mathrm{cov}(\epsilon, \tilde{R}_m) + \alpha R_f \sum_{j=1}^{T} \frac{\sum_{i=1}^{n} \mathrm{cov}(\hat{e}_{ij}, \tilde{R}_m) x_i}{(1+R_f)^j} \right\}}{(1+R_f)}$$

Rearranging terms and changing the order of summation the above equation becomes:

$$P_o = \frac{\sum_{i=1}^{n} x_i}{(1+R_f)} \left\{ \sum_{k=1}^{T} \beta_k \bar{e}_{ik} - \left(\alpha R_f \left[\sum_{k=1}^{T} \beta_k - 1 \right] \right) \sum_{i=1}^{T} \frac{\bar{e}_{ij}}{(1+R_f)^j} \right\}$$

$$+ \left(\frac{-\lambda' \sum_{i=1}^{n} x_i}{1+R_f} \right) \left\{ \sum_{k=1}^{T} \beta_k \mathrm{Cov}(\hat{e}_{ik}, \tilde{R}_m) - \left(\alpha R_f \left[\sum_{k=1}^{T} \beta_k - 1 \right] \right) \sum_{j=1}^{T} \frac{\mathrm{Cov}(\hat{e}_{ij}, \tilde{R}_m)}{(1+R_f)^j} \right\}$$

$$+ \left(\frac{1}{1+R_f} \right) \left\{ \sum_{k=1}^{T} \beta_k E(\tilde{S}_k \cdot g(\tilde{S}_k)) - \left(\alpha R_f \left[\sum_{k=1}^{T} \beta_k - 1 \right] \right) \sum_{j=1}^{T} \frac{E(\tilde{S}_j \cdot g(\tilde{S}_j))}{(1+R_f)^j} \right\}$$

$$+ \left(\frac{-\lambda'}{1+R_f} \right) \mathrm{cov}\ (\epsilon, \tilde{R}_m)$$

And this equation may be further simplified to:

$$P_o = \frac{1}{1+R_f} \sum_{i=1}^{n} \left\{ \sum_{j=1}^{T} \left(\beta_j - \frac{\alpha R_f \sum_{i=1}^{T} {}_k - 1)}{(1+R_f)^j} \right) \left(\bar{e}_{ij} - \lambda' \operatorname{cov}(\hat{e}_{ij}, \tilde{R}_m) \right) \right\} x_i$$

$$+ \frac{1}{1+R_f} \sum_{j=1}^{T} \left(\beta_j - \frac{\alpha R_f \left(\sum_{k=1}^{T} \beta_k - 1 \right)}{(1+R_f)^j} \right) E(\tilde{S}_j \cdot g(\tilde{S}_j)) + \left(\frac{-\lambda'}{1+R_f} \right) \operatorname{cov}(\epsilon, \tilde{R}_m)$$

(C-1)

A change of variable name from σ_{S_t} to z_t and substitution of equations (4.13c), (4.13d) and (4.13d') into the above equation produces

$$P_o = \sum_{i=1}^{n} x_i \frac{\sum_{j=1}^{T} \left(\beta_j - \frac{\alpha R_f (\Sigma \beta_k - 1)}{(1+R_f)^j} \right) (\bar{e}_{ij} - \lambda' \operatorname{Cov}(\hat{e}_{ij}, \tilde{R}_m))}{(1+R_f)}$$

$$+ \sum_{j=1}^{T'} \left[\frac{\left(\beta_j - \frac{\alpha R_f (\sum_{k=1}^{T} \beta_k - 1)}{(1+R_f)^j} \right)}{(1+R_f)} \right] \left(\frac{R_f - R_{bj}}{\sqrt{2\Pi}} z_j e^{-\bar{S}_j^2 / 2 z_j^2} + \bar{S}_j R_f - \bar{S}_j (R_f - R_{bj}) H(0, \bar{S}_j, z_j) \right)$$

$$+ \frac{\beta_{T'} - \frac{\alpha R_f (\sum_{k=1}^{T} \beta_k - 1)}{(1+R_f)^{T'}}}{1+R_f} \cdot \bar{S}_{T'} + \left(\frac{-\lambda'}{1+R_f} \right) \operatorname{cov}(\epsilon, \tilde{R}_m).$$

Since β_j, α, R_f, R_{bj}, $\bar{e}_{ij}$, $\operatorname{cov}(\hat{e}_{ij}, \tilde{R}_m)$, and λ' are all constants, then the following notation for known constants may be defined:

$$b_j = \frac{\beta_j - \dfrac{\alpha R_f(\sum_{k=1}^{T} \beta_k - 1)}{(1+R_f)^j}}{1+R_f} \qquad j = 1, \ldots, T \qquad \text{(C-2)}$$

$$d_j = R_f - R_{bj} \text{ which is always less than zero} \qquad j = 1, \ldots, T' \qquad \text{(C-3)}$$

$$a_i = \sum_{j=1}^{T} b_j(\bar{e}_{ij} - \lambda' \mathrm{Cov}(\hat{e}_{ij}, \tilde{R}_m)) \quad i = 1, \ldots, n \qquad \text{(C-4)}$$

Throughout most of Chapter IV and this appendix a bar (—) notation over a symbol has represented an expected value. Since a conversion to matrix notation is desirable, a change in notation is necessary. Henceforth, all bar (—) notations are used to represent vectors or matrices as it is no longer necessary (from a mathematical point of view) to distinguish between variable names that are ordinary variables, expected values, random variables, or standard deviations as there will always be a mathematical expression which defines them correctly. Furthermore, the number of periods over which the budget is planned is henceforth designated k instead of the previous T'. Finally, one term may be eliminated by indexing R_f so that the following is true: $R_{fj} = R_f$ for $j = 1, \ldots, k-1$ and $R_{fk} = 1+R_f$.

One may now define the vectors $\bar{A} = \{a_i\}$ an n-dimensional vector, $\bar{X}$ also an n-vector, $\bar{S}$ a k-vector of slacks, $\bar{Z}$ a k-vector of standard deviation. Therefore, P_o is a function of $\bar{X}$, $\bar{S}$, and $\bar{Z}$ and may be written

$$P_o = \overline{A}^t\overline{X} + \sum_{j=1}^{k} b_j\left(d_j z_j e^{-s_j^2/2z_j^2}/\sqrt{2\Pi} + R_{fj}s_j - s_j d_j H(0, s_j, z_j)\right) + \text{const}$$

where const $= \frac{-\lambda'}{1+R_f} \text{cov}(\epsilon, \tilde{R}_m)$ which may be dropped for purposes of optimization since it is constant with respect to the variables. With the const term dropped this expression for P_o is precisely objective function (4.14). One should remember that substitution into (4.13) allowed the elimination of (4.13a), (4.13b), (4.13c), (4.13d), (4.13d'), and (4.13e); therefore, no equivalent expressions appear in formulation (4.14). Equation (4.13f) are clearly the quadratic products of the decision vector $\overline{X}$ with some covariance matrix for each period $\leq$ k. If each of those covariance matrices is written $\overline{\sigma}_j$ and remembering that $\sigma_{S_j}^{\ 2} = z_j^{\ 2}$ then (4.13f) may be written

$$z_j = \sqrt{\overline{X}^t \overline{\sigma}_j \overline{X}} \qquad j = 1, \ldots, k$$

which is exactly how (4.14a) is expressed. Systems (4.14b), (4.14c), (4.14d), (4.14e), (4.14f), and (4.14g) are clearly just matrix representations of (4.13g through 4.13m).

Therefore, the equivalency of formulations (4.13) and (4.14) is established.

APPENDIX D

The three basic problem types are distinguished by their sizes:

Type 1: Eight project, two period model. (Summarized in Table D-1)

Type 2: Twelve project, three period model. (Summarized in Table D-2)

Type 3: Sixteen project, three period model. (Summarized in Table D-3)

All problems of each size utilized the same covariance matrices for the project costs. However, a different covariance matrix was used for each period within any given problem. All projects were represented by 0-1 integer variables.

Table D-4 gives the solutions to all test problems.

TABLE D-1

PROBLEMS OF TYPE 1

1a:

Project	Objective Coefficient	Cost Pd 1	Cost Pd 2	Mutually Exclusive Sets 1	Mutually Exclusive Sets 2	Contingent Projects
1	367	700	65	*		
2	641	1000	120	*		
3	547	900	80			
4	1389	1500	800		*	
5	797	1100	100		*	
6	733	1200	200			*
7	377	600	40			
8	402	500	25			*

Periods	Funds Available	b_j Coefficient	R_f	R_b
1	5500	.94	.05	.10
2	700	.86	.05	.10
		$\sum_{j=1}^{T} b_j = 12.2$		

1b: Same as 1a except that an integer solution to only the master problem was obtained.

1c and 1d are similar to 1a and 1b respectively with the only change being the objective function coefficient for each project. These values were changed to the following:

Project	Coefficient
1	725
2	1100
3	900
4	2200
5	1150
6	1200
7	525
8	601

TABLE D-2

PROBLEMS OF TYPE 2

2a:

Project	Objective Coeffic.	Cost Pd 1	Cost Pd 2	Cost Pd 3	Mutually Exclusive Sets 1	2	3	Contingent Projects
1	8000	5000	500	0				
2	8000	5500	600	0	*			
3	7900	5750	700	0	*			*
4	7500	5900	800	100				
5	7000	6000	800	200		*		
6	7500	6200	800	200		*		
7	8100	6300	900	900				
8	9500	6500	1000	1000				
9	17000	7000	6500	6000				*
10	20000	7500	7000	7000			*	
11	25000	8500	8000	8000			*	
12	29800	10000	10000	10000			*	

Periods	Funds Available	b_j Coefficient	R_f	R_b
1	50000	.91	.06	.10
2	20000	.83	.06	.10
3	5000	.75	1.06	1.10

$$\sum_{j=1}^{T} b_j = 10.0$$

2b: Same as 2a except that an integer solution to only the master problem was obtained.

2c: Same as 2a except that the objective function coefficient for project 9 is 16600.

2d: Same as 2a except that the objective function coefficient for project 9 is 16300.

2e: Same as 2a except that the funds in period 1 are 50010.

2f: Same as 2a except that the funds in period 1 are 20010.

2g: Same as 2a except that the funds in period 3 are 5010.

TABLE D-3

PROBLEMS OF TYPE 3

3a:

Project	Objective Coeffic.	Cost Pd 1	Cost Pd 2	Cost Pd 3	Mutually Exclusive Sets 1	Mutually Exclusive Sets 2	Contingent Sets 1	Contingent Sets 2
1	460	500	0	0	*			
2	540	600	0	0			*	
3	600	650	0	0				*
4	675	700	0	0				
5	700	750	0	0				
6	1250	650	800	0				
7	1300	700	700	0		*		
8	1000	900	100	0		*		
9	990	1000	50	0			*	
10	1100	250	1000	0				*
11	1480	250	500	1000				
12	525	100	200	300				
13	1100	900	50	50				
14	1100	500	500	400				
15	435	0	500	0	*			
16	402	0	0	600	*			

Periods	Funds Available	b_j Coefficient	R_f	R_b
1	8500	.91	.06	.10
2	3100	.83	.06	.10
3	900	.75	1.06	1.10

$$\sum_{j=1}^{T} b_j = 10.0$$

3b: Same as 3a except that available funds are 6100, 3100 and 650.

3c: Same as 3a except that available funds are 7500, 0, and 0.

3d: Same as 3a except that available funds are 8500, 0, and 0.

3e: Same as 3a except that available funds are 9500, 0, and 0.

3f: Same as 3c except that projects 1, 15, and 16 are replaced by the options of acquiring additional funds through long term debt at 9% interest rate. Each option will represent the acquisition of 1000 units of funds, but in different periods. Since cash flows will be -9%(1000) in all periods following the acquisition then the objective coefficients may be computed by

$$-9\%(1000) \sum_{i=p+1}^{T} b_i$$ where p is the period of acquisition.

Thus, the parameters for these projects become:

Project	Objective Coeffic.	Cost Pd 1	Cost Pd 2	Cost Pd 3
1	-818	-1000	0	0
15	-743	0	-1000	0
17	-676	0	0	-1000

The mutual exclusive constraint for these three alternatives was retained to reflect an assumed management desire to approach the capital markets only once during the three periods in the planning horizon. It was further assumed that the actual amount of funds acquired are subject to some variability so that the covariance matrices for problem 3c were retained.

TABLE D-4

SOLUTIONS TO TEST PROBLEMS

Problem No.	Projects 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	Expected Slack Funds Period 1	Expected Slack Funds Period 2	Expected Slack Funds Period 3	Slack Std Dev Period 1	Slack Std Dev Period 2	Slack Std Dev Period 3	Objective Function**
1a																	5500	6200		0	0		5826.68
1b		*		*		*	*	*									700	215		2310.12	2323.25		3730.94
1c		*		*													3000	2780		1038.80	1046.04		5937.57
1d	*		*	*		*		*									700	230		2348.67	2366.87		5800.25
2a	*	*		*		*	*	*		*							7100	15500	11300	2816.80	2901.91	2973.88	78742.87
2b	*	*		*	*			*	*	*							6600	9400	100	2953.17	3201.32	3415.98	77868.10
2c	*	*		*		*	*	*		*							7100	15500	11300	2816.80	2901.91	2973.88	78742.87
2d	*	*		*		*	*	*		*							7100	15500	11300	2816.80	2901.91	2973.88	78742.87
2e	*	*		*		*	*	*		*							7110	15510	11310	2816.80	2901.91	2973.88	78751.88
2f	*	*		*		*	*	*		*							7100	15510	11310	2816.80	2901.91	2973.88	78751.32
2g	*	*		*		*	*	*		*							7100	15500	11310	2816.80	2901.91	2973.88	78750.82
3a		*	*	*	*			*	*	*	*	*	*		*		2400	3100	2650	596.04	656.79	700.75	11537.17
3b		*	*	*	*			*	*	*	*	*	*		*		0	700	0	596.04	656.79	700.75	9161.20
3c		*		*	*			*	*		*	*	*				2300	1400	50	508.98	525.60	579.57	7238.82
3d		*	*	*	*			*	*	*		*	*		*		2650	750	400	571.62	620.76	623.26	8160.96
3e		*	*	*	*			*	*	*	*	*	*				3400	1500	150	596.04	638.75	683.87	9083.39
3f		*	*	*	*			*	*		*	*	*			*	1650	750	400	524.42	540.57	609.11	7375.93

Notes: * Indicates which projects are accepted in the final budget.

** 1. Objective function values computed in the course of finding a solution were subject to a cumulation of truncation errors. These errors never exceeded 1/100 of 1% of the total objective function value; however, the values reported are corrected values.

2. Epsilon optimal solutions exceeded 1/100 of 1% error in the objective function value, but never exceeded 1/10% error. These values have likewise been corrected.

3. Problems 1b, 1d, and 2b were maximized only with respect to project contribution; however, the reported objective function value includes the contribution of slack.

APPENDIX E

I. Major Program Documentation

A. Title: Generalized Benders Algorithm

Programmer: C. A. Mount-Campbell

Advisor: R. P. Lutz

Date Completed: March 1974

Machine Used: IBM 360

Language Used: FORTRAN IV

Compiler Used: GCL

Compilation Time: 102 seconds

Computation Time: Variable

Lines of Output: Variable

Approx Core Required: 190K Bites

B. Purpose: This program was written to obtain solutions to test problems of the form given in the last section of Chapter VI. It may be made to output optimal simplex tableaus after the end of each Benders or Gomory iteration. It may also be made to output subproblem optimal primal and dual variables. The linear programming algorithm may be either primal or dual and will accommodate the use of bounded variables.

C. Restrictions: (The restrictions may be circumvented by changing the variable dimensions.)

1. The maximum allowed number of periods in the capital budget planning horizon is five (5).
2. The simplex Tableau for the master problem is restricted to 75 rows (including the x_o row, original constraints, and Benders and Gomory constraints added during execution). It is also restricted to 150 columns (including the "right-hand side," and all slack variables)
3. The maximum allowed number of project variables is 20. Any number of them may be continuous or integer.

D. List of Subroutines and Their Function:

1. MAIN--This routine reads the input data and directs the major sequence of operations as given in the figure 7-1 of Chapter VII.
2. LABEL--When this subroutine is called it is directed to print a specific label that will identify subsequently printed output.
3. SETBAK--The last Benders iteration always adds a Benders constraint that is satisfied by the previous solution; therefore, the subroutine is called to eliminate the constraint and its slack variable from the simplex tableau.
4. RESET--The Gomory algorithm is designed to

terminate when it has obtained near integer solutions (max error is .00001) whereupon this subroutine is called to roundoff the appropriate "right-hand-side" to the nearest exact integer value.

5. GOMORY--This subroutine directs a group of subroutines to accomplish the Gomory iterations.

   a. FINCK--In order to prevent long computer times as a result of pathological difficulties a criteria that if any set of integer variables were basic and remain basic during 5 successive Gomory iterations then termination would occur. This subroutine reviews a set of indexing values to determine if 1. any remaining basic integer variables are currently non integer or if 2. the above criteria has been violated. It also maintains the aforementioned set of indices.

   b. CKINT--Identifies the basic integer variable whose value is the most distant from an integer value, and assists in maintaining the indexing system reviewed by FINCK. A Gomory cut will be generated from the row corresponding to the variable identified by this subroutine.

c. INTCST--Generates the required Gomory cut and creates an additional row and column for it in the simplex tableau.

6. REBASE--Any constraint newly added to an optimal tableau must be brought current by substituting the current values of the basic variables. This subroutine accomplishes that function.

7. CALRHS--Given the solution to the subproblem this routine calculates the right-hand-side of a Benders constraint.

8. BLDCST--Given the solution to the subproblem, this subroutine builds a Benders constraint (in conjunction with CALRHS) and adds it to the previously obtained optimal tableau.

9. SUBPPV--This subroutine uses the formulas given in Appendix B and calculates the optimal primal variable values for the subproblem.

10. SUBPDV--This subroutine uses the formulas given in Appendix B and calculates the optimal dual variable values for the subproblem.

11. PRIALG--This subroutine directs a group of subroutines to perform the bounded variable, primal simplex algorithm procedures.

    a. PNEGCK--Identifies the column which has the most negative value in the objective function row of the tableau. (Optimality Criteria)

b. PFEAS--Identifies the row in which a pivot should occur to satisfy the feasibility criteria.

12. DUALG--This subroutine directs a group of subroutines to perform the bounded variable, dual simplex algorithm procedures.

    a. DUBCK--Determines which row corresponds to the most infeasible variable.

    b. DENTCK--Determines which column corresponds to the variable that will enter the basis.

13. PIV--Given the indices for any row and any column, this subroutine will perform a simplex type pivot operation on the element lying at the intersection of the given row and column.

14. PSUB--In a bounded variable algorithm it is often necessary to fix a variable to its upper bound and replace it in the problem with an expression as follows: x_i = (UPPER BOUND FOR x_i) - x_i'. This operation removes x_i from the problem and replaces it with a ficticious variable x_i'. If x_i' then becomes positive x_i will automatically be prevented from exceeding its upper bound. This subroutine accomplishes the required substitution.

15. TRACE--This subroutine traces through an optimal

tableau and determines the solution value for each variable in the problem.

16. OUTPUT--This subroutine is called to output any single precision array from one to three dimensions. The arrays are outputted in blocks to facilitate reading.
17. DUTPUT--This subroutine has the same purpose as 16 except it was designed for double precision arrays.
18. PROB--This subroutine determines the probability under the standard normal curve from $-\infty$ to any specific value $A \leq 0$. This is accomplished via a combination of table look-up and application of Simpson's rule for numerical integration.

E. List of undimensioned variables

1. M--number of rows in simplex tableau
2. N--number of columns in simplex tableau
3. KSP--number of periods in the budget planning horizon
4. NSP--number of projects under consideration
5. IST--unused variable name
6. ISLAK--index for the first column in which a slack variable appears corresponding to the first financial constraint
7. IPRIT--option variable that controls amount of

intermediate output. 0 implies minimum output and 1 implies maximum output.

8. I21
9. I30
10. I20
11. I5

parameters that specify dimensioning sizes for use by subroutines OUTPUT, DUTPUT. Although values for these parameters are read in from cards they should always have the following values unless dimensions are changed within the program.

I21 = 75 = max number of tableau rows.

I30 = 150 = max number of tableau columns.

I20 = 20 = max number of projects.

I5 = 5 = max number of periods.

12. IBET--execution option variable with options as follows:

1 causes integer solutions to the total problem

2 causes integer solutions to the master problem (i.e., no Benders iterations)

3 causes continuous solutions to the total problem

4 causes continuous solutions to the master problem (i.e., no Benders or Gomory iterations)

13. CLOSE--the value of epsilon when epsilon optimal solutions are desirable.

14. SATIN--used to save the previously obtained optimal solution.

F. List of dimensiond variables

1. LABLE(10,15)--used to store alphameric data

inputted on cards and printed during execution in order to label output. The exact input for this array is given later.

2. TABLE(75,150)--the simplex tableau with 75 rows (the first of which is the x_o row) and 150 columns (the first of which is the right hand side of all constraints).
3. IBAS(75)--array of indices of basic variables.
4. IUP(150) = 0 if the column indicated by the index value of IUP has not been altered in the manner described by paragraph D.14 above.

   = 1 if such an alteration has taken place.
5. UPPER(150)--array of upper bounds of the variables corresponding to the columns of the tableau (lower bounds are assumed to be zero).
6. INTEG(150) = 0 if the variable indicated by the index value of INTEG corresponds to a variable that is not required to be integer.

   = 1 if the corresponding variable is integer.

   = 2 if the corresponding variable is integer by nature rather than by requirement; this is often true of slacks when all projects are integer.
7. S(5)--corresponds to s_j (slack funds) in the

notations of other chapters.

8. Z(5)--corresponds to z_j (standard deviation of slack funds) in the notation of other chapters.

9. B(5)--corresponds to b_j in the notation of other chapters (see Appendix C).

10. D(5)--corresponds to $d_j = R_{fj} - R_{bj}$ in the notation of other chapters.

11. U(5)--dual variables for constraints of the form $z_j \geq \sqrt{\overline{Y}_j^t \overline{Y}}$ found in the notation of other chapters (see 6.10' sub).

12. W(5)--dual variables for the financial constraints (see 6.10' sub).

13. Y(20,5)--corresponds to y_{ij} in the notation of other chapters (see 6.10' sub).

14. V(20,5)--dual variables for the constraints that relate variables y_{ij} to the variables x_i (see 6.10' sub).

15. A(20)--corresponds to the matrix $\overline{A}$ of project contributions.

16. E(20,20,5)--linear transformation matrix between $\overline{X}$ and $\overline{Y}$ which is constructed via conjugate directions and inputted to this program.

17. RHSM(5)--corresponds to the available funds (M_j).

18. C(20,5)--corresponds to c_{ij} (expected cost for project i in period j).

19. X(150)--decision variables including all slacks for the master problem.

G. Organization of input data

1. The first READ statement exactly requires the following data cards:

CARD #	COLUMN 1 ↓
1	VALUES OF X ARE
2	VALUES OF S ARE
3	VALUES OF Z ARE
4	VALUES OF Y ARE
5	VALUES OF U ARE
6	VALUES OF V ARE
7	VALUES OF W ARE
8	OPTIMAL SIMPLEX TABLEAU
9	RAW DATA
10	SIMPLEX TABLEAU FROM GOMORY
11	BENDERS CUT IN TABLEAU IS
12	ORIGINAL BENDERS CUT IS
13	THE GOMORY CUT IS
14	(BLANK CARD)
15	THE STARTING TABLEAU

2. The second READ statement requires one card with values as listed in listing that follows documentation.

3. The third READ statement reads in the starting simplex tableau for the master problem. The master problem begins with the form given on page 126 and includes the surrogate constraint mentioned on that page. It must then be placed in the standard form and arranged as follows.

Row 1--Objective function row (data should include sign change)
Row 2--surrogate constraint
Row 3--first interrelationship constraint

⋮

Row M-KSP last interrelationship constraint
Row M-KSP+1 financial constraint for period 1

⋮

Row M--financial constraint for period KSP

The first value in each row should be the value for the right hand side of the respective constraint.

The second value in each row should be the value for the coefficients of the variable r.

The third through NSP+2 value in each row are the coefficients for the NSP projects.

The NSP+3 through N value in each row are the coefficients for the slack variables used to form the standard form.

4. The fourth READ statement reads in the upper bounds for the variables corresponding to tableau columns 2 through N.

5. The fifth READ statement reads in the integer requirement code for the variables corresponding to tableau columns 1 through N. Since column 1 is a right hand side of a constraint the first integer requirement code is meaningless and may be left blank.

6. The sixth READ statement reads in the KSP values for b_j, j=1, ..., KSP. Remember that these values are ≥ 0.

7. The seventh READ statement reads in the KSP

values for d_j, j=1, ..., KSP. Remember that these values are ≤ 0.

8. The eighth READ statement reads in the KSP values for R_{fj}, j=1, ..., KSP. These are the risk free rates of return.

9. Remember that $\overline{E}_j$ was a transformation such that $\overline{Y}_j^t \overline{D}_j \overline{Y}_j = \overline{X}^t \overline{\sigma}_j \overline{X}$ where $\overline{Y}_j = \overline{E}_j^{-1} \overline{X}$ and $\overline{D}_j$ a diagonal matrix. The ninth READ statement reads one diagonal element of $\overline{D}_j$ while the tenth READ statement reads the corresponding row of $\overline{E}_j^{-1}$. These two statements are repeated in like order until all diagonal elements of $\overline{D}_j$ and all rows of $\overline{E}_j^{-1}$ are read, and the process is then repeated for a different j. j stands for the planning period number and the card blocks should be arranged in ascending order by period number. The modification discussed on pages 112 and 113 is automatically accomplished by the program.

H. Example input deck: (The 15 label cards are deleted from this example)

The example is taken from problem 2f and has the following form for the master problem:

Max r

st $\left.\begin{array}{l} r-8000x_1-8000x_2-7900x_3-7500x_4 \\ -7000x_5-7500x_6-8100x_7-9500x_8 \\ \text{(constraint continued)} \end{array}\right]$ surrogate constraint with large M = 75000

$$-17000x_9-20000x_{10}-25000x_{11}-29800x_{12}$$
$$\leq 75000$$

$$\left.\begin{array}{ll} x_1 - x_{10} & \leq 0 \\ x_2 + x_3 & \leq 1 \\ x_5 + x_6 & \leq 1 \\ x_{10} + x_{11} + x_{12} & \leq 1 \end{array}\right] \text{Project interrelationship constraints}$$

$$\text{Financial Const. Period 1} \left[\begin{array}{l} 5000x_1 + 5500x_2 + 5750x_3 + 5900x_4 + 6000x_5 \\ \quad + 6200x_6 + 6300x_7 + 6500x_8 + 7000x_9 \\ \quad + 7500x_{10} + 8500x_{11} + 10000x_{12} \leq 50000 \end{array}\right.$$

$$\text{Financial Const. Period 2} \left[\begin{array}{l} 5500x_1 + 6100x_2 + 6450x_3 + 6700x_4 + 6800x_5 \\ \quad + 7000x_6 + 7200x_7 + 7500x_8 + 13500x_9 \\ \quad + 14500x_{10} + 16500x_{11} + 20000x_{12} \leq 70010 \end{array}\right.$$

$$\text{Financial Const. Period 3} \left[\begin{array}{l} 5500x_1 + 6100x_2 + 6450x_3 + 6800x_4 + 7000x_5 \\ \quad + 7200x_6 + 8100x_7 + 8500x_8 + 19500x_9 \\ \quad + 21500x_{10} + 24500x_{11} + 30000x_{12} \leq 75010 \end{array}\right.$$

For an example of how this problem was coded refer to Figures E-1 through E-7.

CARD COLUMN NUMBER

12345678	1111111 90123456	11122222 78901234	22222333 56789012	33333334 34567890	44444444 12345678	45555555 90123456	55566666 78901234	66666777 56789012	77777778 34567890
9	22 3	12	20	1 75	150	1 20	5 ←	PARAMETER CODE	
TABLEAU DATA:									
0	-1.	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0								
75000.	1.	-8000.	-8000.	-7900.	-7500.	-7000.	-7500.	-8100.	-9500.
-17000.	-20000.	-25000.	-29800.	1.	0	0	0	0	0
0	0								
0	0	1.	0	0	0	0	0	0	0
0	-1.	0	0	0	1.	0	0	0	0
1.	0	0	1.	1.	0	0	0	0	0
0	0	0	0	0	0	1.	0	0	0
0	0								
1.	0	0	0	0	0	1.	1.	0	0
0	0	0	0	0	0	0	1.	0	0
0	0								
1.	0	0	0	0	0	0	0	0	0
0	1.	1.	1.	0	0	0	0	1.	0
0	0								
50000.	0	5000.	5500.	5750.	5900.	6000.	6200.	6300.	6300.
7000.	7500.	8500.	10000.	0	0	0	0	0	1.
(TABLEAU CONTINUED ON NEXT PAGE)									

Figure E-1.

CARD COLUMN NUMBER

12345678	1111111 90123456	11122222 78901234	22222333 56789012	33333334 34567890	44444444 12345678	45555555 90123456	55566666 78901234	66666777 56789012	77777778 34567890
70010.	0	5500.	6100.	6450.	6700.	6800.	7000.	7200.	7500.
13500.	14500.	16500.	20000.	0	0	0	0	0	0
1.	0								
75010.	0	5500.	6100.	6450.	6800.	7000.	7200.	8100.	8500.
19500.	21500.	24500.	30000.	0	0	0	0	0	0
0	1.								
UPPER BOUND DATA:									
1000000.	1.	1.	1.	1.	1.	1.	1.	1.	1.
1.	1.	1.	1000000.	1000000.	1000000.	1000000.	1000000.	1000000.	1000000.
1000000.	1000000.								
INTEGER CODE DATA:									
0	0 1	1	1 1	1 1	1	1 1	1	1 1	0 2
2	2 2	2	2 2						
b_j DATA:									
.91	.83	.75							
d_j DATA:									
-.04	-.04	-.04							
R_{fj} DATA:									
.06	.06	1.06							

Figure E-2.

CARD COLUMN NUMBER

12345678	1111111 90123456	11122222 78901234	22222333 56789012	33333334 34567890	44444444 12345678	45555555 90123456	55566666 78901234	66666777 56789012	77777778 34567890
DIAGON	AL ELEME	NTS OF $\overline{D}_j$	ARE AL	TERNATED	WITH RO	WS OF $\overline{E}_j^{-1}$			
DATA F	OR PERIO	D j=1:							
146575.									
1.0000	1.0822	1.1651	2.3716	0.8596	0.7737	0.6670	0.6017	0.4951	0.4298
0.3646	0.2993								
2810.									
0.0	1.0000	1.8902	4.2434	2.1860	1.9674	1.5173	1.5302	1.0801	1.0930
1.1058	1.1187								
616.									
0.0	0.0	1.0000	1.7378	0.9541	0.8589	0.6508	0.6678	0.4599	0.4770
0.4941	0.5111								
654.									
0.0	0.0	0.0	1.0000	0.5184	0.4670	0.3519	0.3634	0.2481	0.2595
0.2705	0.2817								
100.									
0.0	0.0	0.0	0.0	1.0000	0.2182	0.1469	0.1713	0.1000	0.1231
0.1480	0.1710								
215.									
0.0	0.0	0.0	0.0	0.0	1.0000	0.0468	0.0546	0.0316	0.0389
0.0465	0.0536								
25007.									
0.0	0.0	0.0	0.0	0.0	0.0	1.0000	0.0003	0.0002	0.0002
0.0002	0.0003								
100008.									

Figure E-3.

CARD COLUMN NUMBER

12345678	11111111 90123456	11122222 78901234	22222333 56789012	33333334 34567890	44444444 12345678	45555555 90123456	55566666 78901234	66666777 56789012	77777778 34567890
0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	0.0	0.0001
0.0001	0.0001								
1000003.									
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	0.0
0.0	0.0								
400004.									
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000
0.0	0.0								
900006.									
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1.0000	0.0								
2500007.									
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	1.0000								
DATA F	OR PERIOD	j=2:							
147744.									
1.0000	1.0804	1.1621	2.3586	0.8580	0.7749	0.6686	0.6033	0.5168	0.4470
0.3772	0.3074								
3052.									
0.0	1.0000	1.7981	4.0746	2.0514	1.8302	1.4105	1.4206	0.8440	0.8903
0.9366	0.9830								
1176.									
0.0	0.0	1.0000	1.7338	0.9291	0.8273	0.6254	0.6417	0.3543	0.3904
0.4266	0.4628								

Figure E-4.

CARD COLUMN NUMBER

12345678	1111111 90123456	11122222 78901234	22222333 56789012	33333334 34567890	44444444 12345678	45555555 90123456	55566666 78901234	66666777 56789012	77777778 34567890
2219.									
0.0	0.0	0.0	1.0000	0.2104	0.0223	-0.0301	-0.0231	-1.2381	-0.9627
-0.6876	-0.4123								
515.									
0.0	0.0	0.0	0.0	1.0000	0.9337	0.7969	0.8160	2.7350	2.2796
1.8241	1.3685								
731.									
0.0	0.0	0.0	0.0	0.0	1.0000	0.4317	0.4471	0.6991	0.6514
0.6037	0.5561								
30373.									
0.0	0.0	0.0	0.0	0.0	0.0	1.0000	0.0059	0.0049	0.0054
0.0058	0.0063								
104186.									
0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	0.0010	0.0013
0.0015	0.0018								
2003890.									
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	0.0015
0.0010	0.0005								
802240.									
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000
1801081.									
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1.0000	0.0003								

Figure E-5.

CARD COLUMN NUMBER

12345678	1111111 90123456	11122222 78901234	22222333 56789012	33333334 34567890	44444444 12345678	45555555 90123456	55566666 78901234	66666777 56789012	77777778 34567890
5000416.									
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	1.0000								
DATA FOR	PERIOD	j=3:							
147744.									
1.0000	1.0804	1.1621	2.3586	0.8580	0.7749	0.6686	0.6033	0.5168	0.4470
0.3772	0.3074								
3052.									
0.0	1.0000	1.7981	4.0746	2.0514	1.8302	1.4105	1.4206	0.8440	0.8903
0.9366	0.9830								
1176.									
0.0	0.0	1.0000	1.7338	0.9291	0.8273	0.6254	0.6417	0.3543	0.3905
0.4266	0.4628								
2583.									
0.0	0.0	0.0	1.0000	0.2819	0.1194	0.1527	0.1493	-0.8757	-0.6391
-0.4027	-0.1662								
767.									
0.0	0.0	0.0	0.0	1.0000	0.8616	0.9632	0.9525	2.5363	2.1737
1.8109	1.4481								
1029.									
0.0	0.0	0.0	0.0	0.0	1.0000	0.4468	0.4524	0.8048	0.7334
0.6620	0.5906								
33430.									
0.0	0.0	0.0	0.0	0.0	0.0	1.0000	0.0203	0.0245	0.0246

Figure E-6.

CARD COLUMN NUMBER

	1111111	11122222	22222333	33333334	44444444	45555555	55566666	66666777	77777778
12345678	90123456	78901234	56789012	34567890	12345678	90123456	78901234	56789012	34567890
0.0246	0.0246								
108625.									
0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	0.0065	0.0066
0.0067	0.0068								
3005117.									
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	0.0013
0.0010	0.0006								
1203247.									
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0020	0.0014								
2701921.									
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1.0000	0.0005								
7501142.									
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	1.0000								

Figure E-7.

I. Flow Charts

For the basic flow charts refer to Figures E-8 through E-10.

J. Program Listing

The program listing follows Figure E-10.

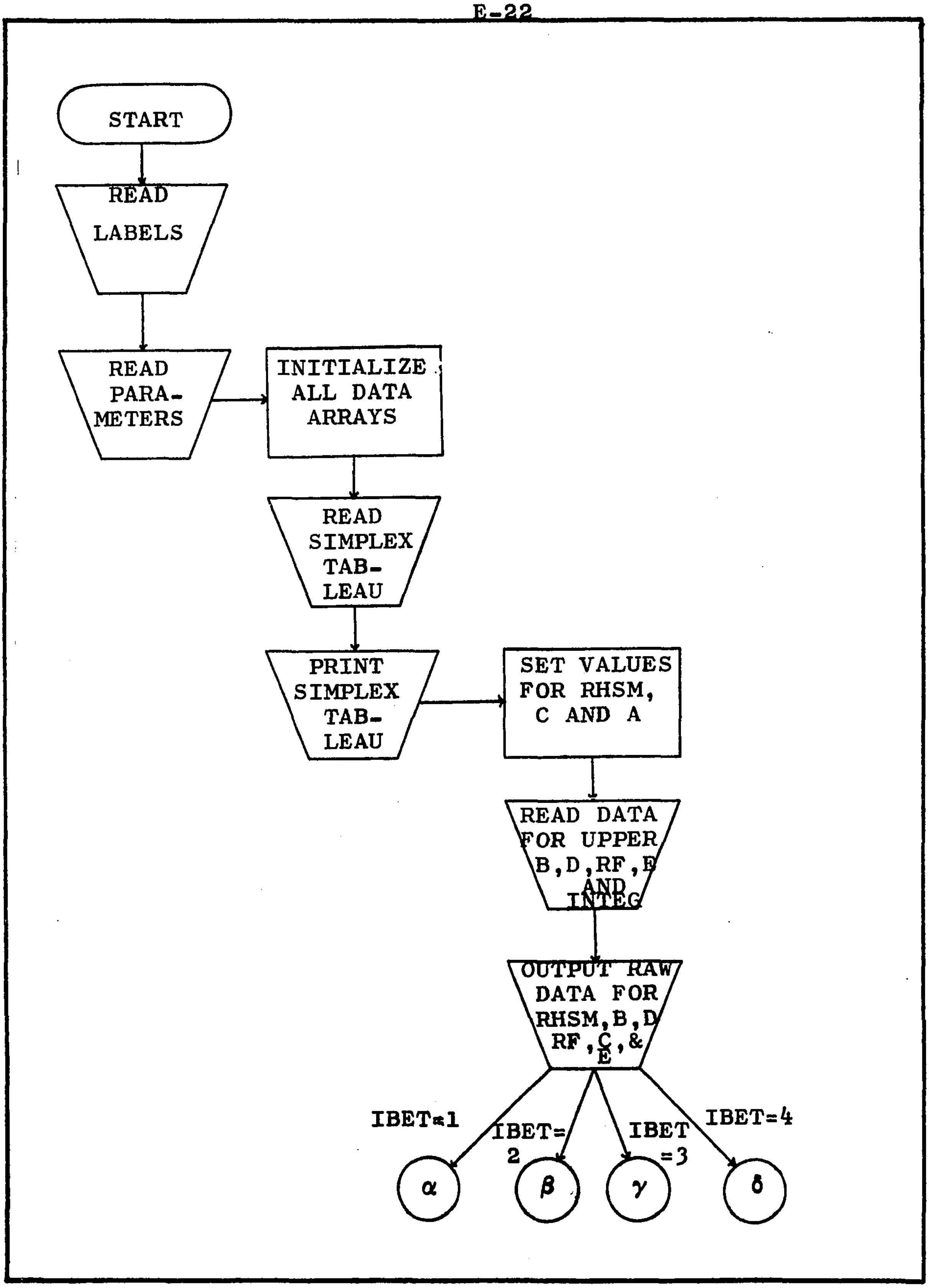
START
READ LABELS
READ PARA-METERS
INITIALIZE ALL DATA ARRAYS
READ SIMPLEX TAB-LEAU
PRINT SIMPLEX TAB-LEAU
SET VALUES FOR RHSM, C AND A
READ DATA FOR UPPER B,D,RF,E AND INTEG
OUTPUT RAW DATA FOR RHSM,B,D RF,C,& E
IBET=1
IBET=2
IBET=3
IBET=4
α
β
γ
δ

Figure E-8.

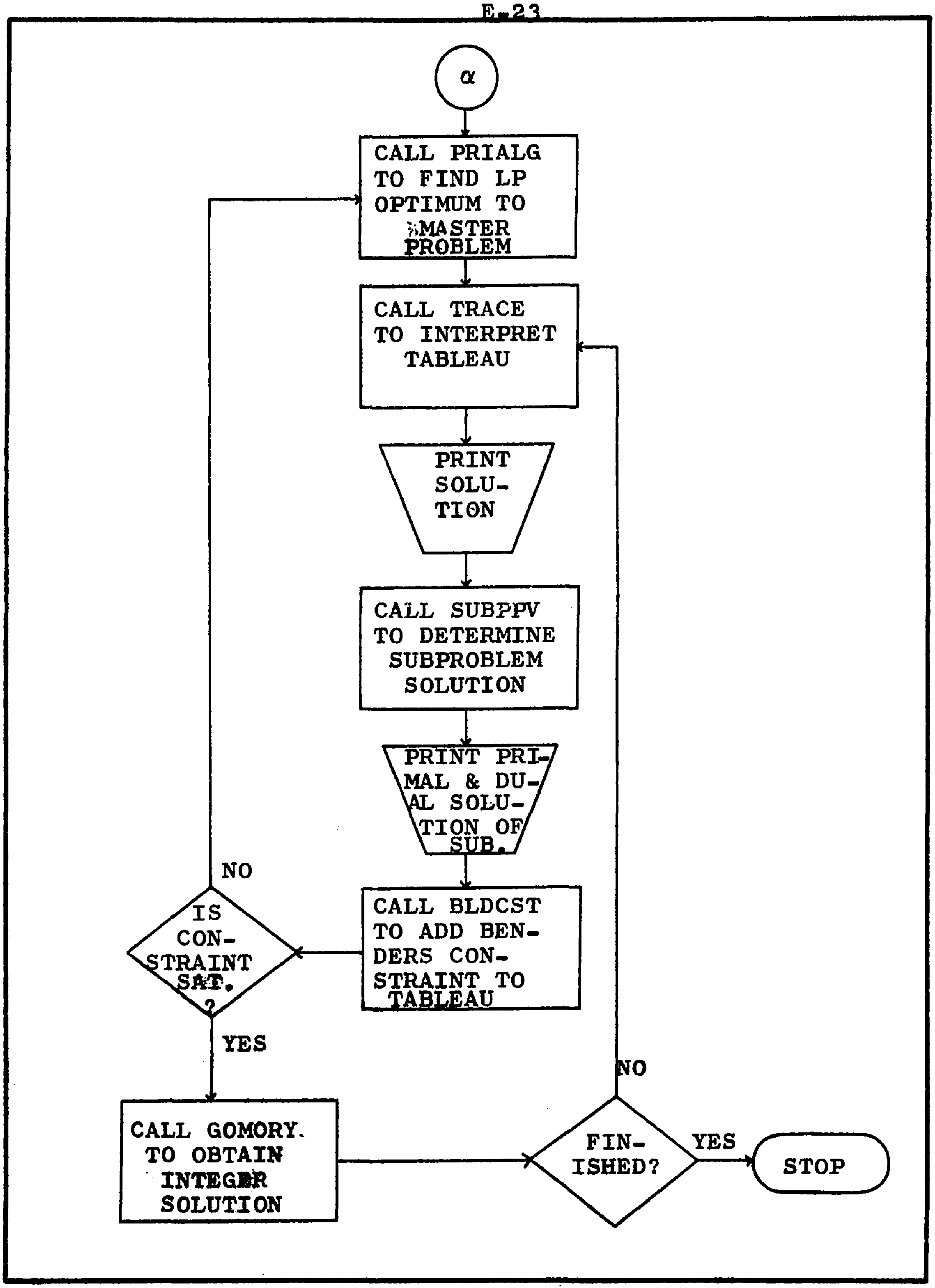
α
CALL PRIALG TO FIND LP OPTIMUM TO MASTER PROBLEM
CALL TRACE TO INTERPRET TABLEAU
PRINT SOLU- TION
CALL SUBPPV TO DETERMINE SUBPROBLEM SOLUTION
PRINT PRI- MAL & DU- AL SOLU- TION OF SUB.
CALL BLDCST TO ADD BEN- DERS CON- STRAINT TO TABLEAU
IS CON- STRAINT SAT. ?
NO
YES
CALL GOMORY TO OBTAIN INTEGER SOLUTION
FIN- ISHED?
NO
YES
STOP

Figure E-9.

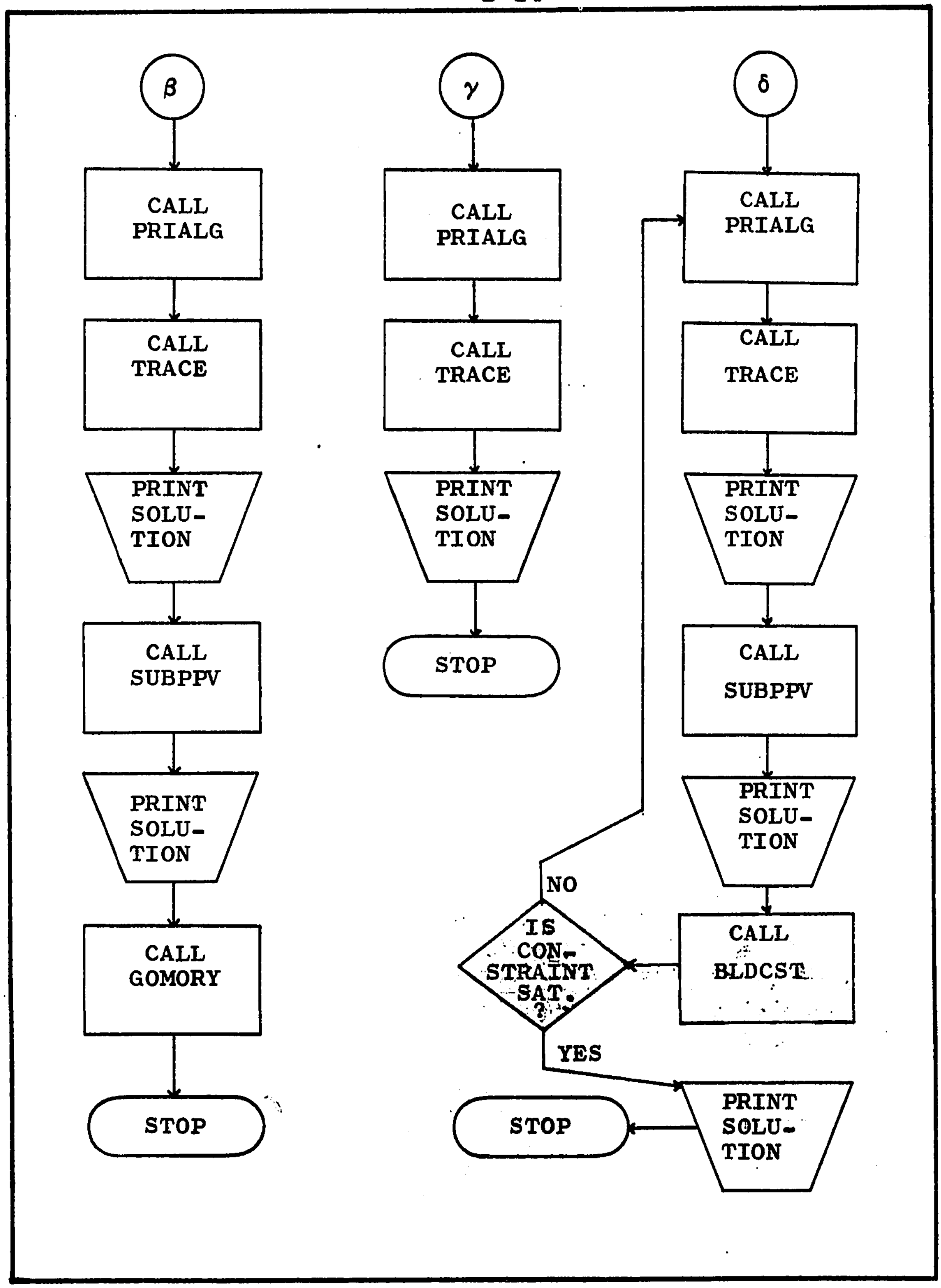
β
γ
δ
CALL PRIALG
CALL TRACE
PRINT SOLU-TION
CALL SUBPPV
PRINT SOLU-TION
CALL GOMORY
STOP
CALL PRIALG
CALL TRACE
PRINT SOLU-TION
STOP
CALL PRIALG
CALL TRACE
PRINT SOLU-TION
CALL SUBPPV
PRINT SOLU-TION
CALL BLDCST
IS CON-STRAINT SAT.?
NO
YES
PRINT SOLU-TION
STOP

Figure E-10.

```
      DOUBLE PRECISION TABLE
      DOUBLE PRECISION SATIN
      COMMON TABLE(75,150),IBAS(75),IUP(150),UPPER(150),INTEG(150)
      COMMON M,N,KSP,NSP,IST,ISLAK,IPRIT,I21,I30
      COMMON S(5),Z(5),B(5),D(5),U(5),W(5),RF(5),V(20,5),A(20)
      COMMON E(20,20,5),RHSM(5),C(20,5),X(150),LABLE(10,15),Y(20,5)
      IONE=1
      ITW=2
      READ(5,102) LABLE
  102 FORMAT(10A4)
   41 READ(5,100) M,N,KSP,NSP,IST,ISLAK,IPRIT,I21,I30,IBET,I20,I5,CLOSE
  100 FORMAT(12I5,F10.0)
      IF(M.EQ.1000) GO TO 42
      WRITE(6,200)
  200 FORMAT(1X,5('*******NEW PROBLEM*******'))
C  THE FOLLOWING DESCRIBES THE EFFECT OF VARIOUS VALUES OF IBET
C     1= INTEGER SOLUTIONS TO THE TOTAL PROBLEM
C     2 = INTEGER SOLUTIONS TO THE MASTER PROBLEM
C     3 = CONTINUOUS SOLUTIONS TO THE MASTER PROBLEM
C     4 = CONTINUOUS SOLUTIONS TO THE TOTAL PROBLEM
      DO 110 I=1,KSP
      S(I)=0.
      Z(I)=0.
      B(I)=0.
      D(I)=0.
      U(I)=0.
      W(I)=0.
      RF(I)=0.
      RHSM(I)=0.
      DO 110 J=1,NSP
      C(J,I)=0.
      Y(J,I)=0.
      V(J,I)  =0.
      DO 110 K=1,NSP
  110 E(J,K,I)=0.
      DO 120 I=1,I30
      IUP(I)=0.
      UPPER(I)=110000.
      INTEG(I)=0.
      DO 120 J=1,I21
  120 TABLE(J,I)=0.
      DO 121 I=1,I21
  121 IBAS(I)=0.
      DO 10 I=1,M
   10 READ(5,101) (TABLE(I,J),J=1,N)
      LTDIM=I21*I30
      CALL OUTPUT(TABLE,M,N,IONE,I21,I30,LTDIM,LABLE(1,15))
  101 FORMAT(10F8.0)
      JEF=ISLAK-NSP-1
      JES=JEF+KSP-1
      SUMP=0
      DO 40 I=JEF,JES
      RHSM(I-JEF+1)=TABLE(I,1)
      DO 40 JCOST=1,NSP
   40 C(JCOST,I-JEF+1)=TABLE(I,JCOST+2)
      READ(5,101)(UPPER(J),J=2,N)
      NOP=NSP+2
      DO 31I=3,NOP
```

```
   31 A(I-2)=-TABLE(2,I)
      IQ=N-M
      DO 30 I=1,M
   30 IBAS(I)=IQ+I
      IBAS(1)=1
      READ(5,103)(INTEG(I),I=1,N)
  103 FORMAT(16I5)
      READ(5,101)(S(I),I=1,KSP)
      READ(5,101)(D(I),I=1,KSP)
      READ(5,101)(RF(I),I=1,KSP)
      DO 25 I=1,KSP
      DO 25 K=1,NSP
      READ(5,101) DIA
      READ(5,101)(E(K,J,I),J=1,NSP)
      DIA=SQRT(DIA)
      DO 35 J=1,NSP
   35 E(K,J,I)=DIA*E(K,J,I)
   25 CONTINUE
      LTDIM=KSP*IONE*IONE
      CALL OUTPUT(RHSM,KSP,IONE,IONE,KSP,IONE,LTDIM,LABLE(1,9))
      CALL OUTPUT(B    ,KSP,IONE,IONE,KSP,IONE,LTDIM,LABLE(1,9))
      CALL OUTPUT(D    ,KSP,IONE,IONE,KSP,IONE,LTDIM,LABLE(1,9))
      CALL OUTPUT(RF   ,KSP,IONE,IONE,KSP,IONE,LTDIM,LABLE(1,9))
      LTDIM=I20*I5
      CALL OUTPUT(C,NSP,KSP,IONE,I20,I5 ,LTDIM,LABLE(1,9))
      LTDIM=I20*I20*I5
      CALL OUTPUT(E    ,NSP,NSP,KSP,I20,I20,LTDIM,LABLE(1,9))
      GO TO (1,2),IST
    1 CONTINUE
    2 CALL PRIALG
    3 LTDIM=I21*I30*IONE
   33 LTDIM=I21*I30
      INITT=1
      SATIN=TABLE(1,1)
      CALL OUTPUT(TABLE,M,N,IONE,I21,I30,LTDIM,LABLE(1,8))
   34 CALL TRACE
      LTDIM=N*IONE*IONE
      CALL OUTPUT(X,N,IONE,IONE,N,IONE,LTDIM,LABLE(1,1))
      RSAVE=X(1)
      CALL SUBPPV
      LTDIM=KSP*IONE*IONE
      CALL OUTPUT(S    ,KSP,IONE,IONE,KSP,IONE,LTDIM,LABLE(1,2))
      CALL OUTPUT(Z    ,KSP,IONE,IONE,KSP,IONE,LTDIM,LABLE(1,3))
      LTDIM=I20*I5
      CALL OUTPUT(Y,NSP,KSP,IONE,I20,I5 ,LTDIM,LABLE(1,4))
      CALL SUBPDV
      LTDIM=KSP*IONE*IONE
      CALL OUTPUT(U    ,KSP,IONE,IONE,KSP,IONE,LTDIM,LABLE(1,5))
      LTDIM=I20*I5
      CALL OUTPUT(V,NSP,KSP,IONE,I20,I5,LTDIM,LABLE(1,6))
      LTDIM=KSP*IONE*IONE
      CALL OUTPUT(W    ,KSP,IONE,IONE,KSP,IONE,LTDIM,LABLE(1,7))
      GO TO (11,12,13,11),IBET
   11 CALL BLDCST
      IBAS(M)=N
      IF(M.GT.I21) STOP
      CALL LABEL(LABLE(1,11))
  300 FORMAT(20(//,4X,8F14.5))
```

MAIN DATE = 74099

```
      WRITE(6,300)(TABLE(M,IKE),IKE=1,N)
  104 FORMAT('1',5A4)
      GO TO(151,152),INITT
  151 IF(TABLE(M,1)) 39,17,17
  152 IF(TABLE(M,1)+CLOSE) 39,17,17
   39 CALL DUALG
      GO TO 33
   17 CALL SETBAK
      LTDIM=I21*I30
      CALL OUTPUT(TABLE,M,N,IONE,I21,I30,LTDIM,LABLE(1,8))
      GO TO (12,12,13,13),IBET
   12 CALL GOMORY
      INITT=2
      CALL RESET
      IF(TABLE(1,1).EQ.SATIN) GO TO 41
      SATIN=TABLE(1,1)
      GO TO 34
   18 CONTINUE
   13 GO TO 41
   42 WRITE(6,201)
  201 FORMAT(1X,5('*******END OF RUN*******'))
      STOP
      END
```

LABEL DATE = 74099

```
      SUBROUTINE LABEL(LA)
      DIMENSION LA(10)
      WRITE(6,LA)
      RETURN
      END
```

SETBAK DATE = 74099

```
      SUBROUTINE SETBAK
      DOUBLE PRECISION TABLE
      COMMON TABLE(75,150),IBAS(75),IUP(150),UPPER(150),INTEG(150)
      COMMON M,N,KSP,NSP,IST,ISLAK,IPRIT,I21,I30
      COMMON S(5),Z(5),B(5),D(5),U(5),R(5),RF(5),V(20,5),A(20)
      COMMON E(20,20,5),RHSM(5),C(20,5),X(150),LABLE(10,15),Y(20,5)
      WRITE(6,100)
      DO 10 I=1,M
   10 TABLE(I,N)=0
      DO 20I=1,N
   20 TABLE(M,I)=0
      M=M-1
      N=N-1
  100 FORMAT(' THE LAST BENDERS CUT IS BEING DROPPED DUE TO REDUNDANCY')
      RETURN
      END
```

RESET DATE = 74099

```
      SUBROUTINE RESET
      DOUBLE PRECISION TABLE
      COMMON TABLE(75,150),IBAS(75),IUP(150),UPPER(150),INTEG(150)
      COMMON M,N,KSP,NSP,IST,ISLAK,IPRIT,I21,I30
      COMMON S(5),Z(5),B(5),D(5),U(5),W(5),RF(5),V(20,5),A(20)
      COMMON E(20,20,5),RHSM(5),C(20,5),X(150),LABLE(10,15),Y(20,5)
      DO 10I=2,M
      J=IBAS(I)
      KQEO=INTEG(J)+1
      GO TO (10,1,10),KQEO
    1 IB=TABLE(I,1)+.5
      TABLE(I,1)=IB
   10 CONTINUE
      RETURN
      END
```

GOMORY DATE = 74099

```
      SUBROUTINE GOMORY
      DOUBLE PRECISION TABLE
      DIMENSION ICK(150)
      COMMON TABLE(75,150),IBAS(75),IUP(150),UPPER(150),INTEG(150)
      COMMON M,N,KSP,NSP,IST,ISLAK,IPRIT,I21,I30
      COMMON S(5),Z(5),B(5),D(5),U(5),W(5),RF(5),V(20,5),A(20)
      COMMON E(20,20,5),RHSM(5),C(20,5),X(150),LABLE(10,15),Y(20,5)
      ION=I21*I30
      IONE=1
      ICK(1)=1
      ICK(2)=2
      CALL FINCK(ICK)
    1 CALL CKINT(I,ICK)
      CALL FINCK(ICK)
      IF(I.GE.1000) GO TO 2
      IF(ICK(1).EQ.100) GO TO 2
      CALL INTCST(I)
      CALL LABEL(LABLE(1,13))
  300 FORMAT(20(//,4X,5F14.5))
      WRITE(6,300)(TABLE(M,IKE),IKE=1,N)
      CALL DUALG
      IF(IPRIT.EQ.0) GO TO 1
      CALL OUTPUT(TABLE,M,N,IONE,I21,I30,ION ,LABLE(1,10))
      IF(IPRIT.EQ.999) STOP
      GO TO 1
    2 WRITE(6,100) I,ICK(1)
  100 FORMAT(' MASTER PROBLEM IS NOW INTEGER/OPTIMAL',2I10)
      RETURN
      END
```

FINCK DATE = 74099

```
      SUBROUTINE FINCK(ICK)
      DOUBLE PRECISION TABLE
      DIMENSION ICK(150)
      COMMON TABLE(75,150),IBAS(75),IUP(150),UPPER(150),INTEG(150)
      COMMON M,N,KSP,NSP,IST,ISLAK,IPRIT,I21,I30
      DO 20I=1,N
      IF(ICK(I).EQ.0) GO TO 20
      IA=ICK(I)
      GO TO 2
   20 CONTINUE
      ICK(1)=100
      RETURN
    2 DO 30 J=I,N
      IF(ICK(J).EQ.0) GO TO 30
      IF(ICK(J).EQ.IA) GO TO 30
      GO TO 40
   30 CONTINUE
      IF(IA.GE.5) ICK(1)=100
      RETURN
   40 DO 50 I=1,150
   50 ICK(I)=0
      RETURN
      END
```

CKINT DATE = 74099

```
      SUBROUTINE CKINT(ISAVE,ICK)
      DOUBLE PRECISION TABLE
      DIMENSION ICK(150)
      COMMON TABLE(75,150),IBAS(75),IUP(150),UPPER(150),INTEG(150)
      COMMON M,N,KSP,NSP,IST,ISLAK,IPRIT,I21,I30
      COMMON S(5),Z(5),B(5),D(5),U(5),W(5),RF(5),V(20,5),A(20)
      COMMON E(20,20,5),RHSM(5),C(20,5),X(150),LABLE(10,15),Y(20,5)
      IF(IPRIT.EQ.0) GO TO 99
      WRITE(6,100) M,N
  100 FORMAT(' CKINT',2I10)
   99 ISAVE=1000
      TEST=.00001
      DO 10 I=2,M
      J=IBAS(I)
      KQEO=INTEG(J)+1
      GO TO (10,1,10), KQEO
    1 IB=TABLE(I,1)
      ICK(J)=ICK(J)+1
      TAB=TABLE(I,1)
      T1=ABS(IB-TAB)
      T2=ABS(IB-TAB+1)
      IF(T1.GT.T2) T1=T2
      IF(T1.LT.TEST) GO TO 10
      ISAVE=I
      TEST=T1
   10 CONTINUE
      IF(IPRIT.EQ.0) RETURN
      WRITE(6,100) ISAVE
      RETURN
      END
```

INTCST DATE = 74099

```
      SUBROUTINE INTCST(I)
      DOUBLE PRECISION TABLE
      DOUBLE PRECISION FK,FKJ
      COMMON TABLE(75,150),IBAS(75),IUP(150),UPPER(150),INTEG(150)
      COMMON M,N,KSP,NSP,IST,ISLAK,IPRIT,I21,I30
      COMMON S(5),Z(5),B(5),D(5),U(5),W(5),FF(5),V(20,5),A(20)
      COMMON E(20,20,5),RHSM(5),C(20,5),X(150),LABLE(10,15),Y(20,5)
      IF(IPRIT.EQ.0) GO TO 99
      WRITE(6,100) I,M,N
  100 FORMAT(' INTCST',3I10)
   99 IB=TABLE(I,1)
      FK=IB
      FK=TABLE(I,1)-FK
      TABLE(M+1,1)=-FK
      IBAS(M+1)=N+1
      DO 10 J=2,N
      IF(TABLE(I,J).EQ.0.) GO TO 10
      IF(IBAS(I).EQ.J) GO TO 10
      IF(TABLE(I,J).GE.0..AND.INTEG(J).EQ.0.) TABLE(M+1,J)=-TABLE(I,J)
      IF(TABLE(I,J).LT.0..AND.INTEG(J).EQ.0.) TABLE(M+1,J)=-(FK/(FK-1.))
     1*TABLE(I,J)
      KQEO=INTEG(J)+1
      GO TO (10,1),KQEO
    1 JB=TABLE(I,J)
      IF(TABLE(I,J).LT.0.) JB=JB-1
      FKJ=JB
      FKJ=TABLE(I,J)-FKJ
      IF(FKJ.LE.FK) TABLE(M+1,J)=-FKJ
      IF(FKJ.GT.FK) TABLE(M+1,J)=-(FK*(1.-FKJ))/(1.-FK)
   10 CONTINUE
      TABLE(M+1,N+1)=1.
      M=M+1
      N=N+1
      RETURN
      END
```

REBASE DATE = 74099

```
      SUBROUTINE REBASE
      DOUBLE PRECISION TABLE
      COMMON TABLE(75,150),IBAS(75),IUP(150),UPPER(150),INTEG(150)
      COMMON M,N,KSP,NSP,IST,ISLAK,IPRIT,I21,I30
      COMMON S(5),Z(5),B(5),D(5),U(5),W(5),FF(5),V(20,5),A(20)
      COMMON E(20,20,5),RHSM(5),C(20,5),X(150),LABLE(10,15),Y(20,5)
      IF(IPRIT.EQ.0.) GO TO 99
      WRITE(6,100) M,N
  100 FORMAT(' REBASE',2I10)
   99 CONTINUE
      MM=M-1
      DO 20I=2,N
      IF(IUP(I).EQ.0) GO TO 20
      TABLE(M,1)=TABLE(M,1)-UPPER(I)*TABLE (M,I)
      TABLE(M,I)=-TABLE(M,I)
   20 CONTINUE
      DO 10 I=2,MM
      J=IBAS(I)
   10 CALL PIV(I,J)
      RETURN
      END
```

```
      SUBROUTINE CALRHS(VALUE)
      DOUBLE PRECISION TABLE
      COMMON TABLE(75,150),IBAS(75),IUP(150),UPPER(150),INTEG(150)
      COMMON M,N,KSP,NSP,IST,ISLAK,IPRIT,I21,I30
      COMMON S(5),Z(5),B(5),D(5),U(5),W(5),RF(5),V(20,5),A(20)
      COMMON E(20,20,5),RHSM(5),C(20,5),X(150),LABLE(10,15),Y(20,5)
      IF(IPRIT.EQ.0.) GO TO 99
      WRITE(6,100) KSP,NSP
100   FORMAT( ' CALPHS',2I10)
99    CONTINUE
      SQ=SQRT(2.*3.1415927)
      SUM=0
      DO 10J=1,KSP
      BB=S(J)/Z(J)
      CALL PROB(BB,PRO)
      BB=-(BB**2)/2.
      TSUM=(D(J)*EXP(BB)/SQ)*Z(J)
      TSUM=TSUM+RF(J)*S(J)
      TSUM=TSUM-S(J)*D(J)*PRO
10    SUM=SUM+B(J)*TSUM
      VALUE=SUM
      SUM=0.
      DO 20J=1,KSP
      DO 20I=1,NSP
20    SUM=SUM+V(I,J)*Y(I,J)
      VALUE=VALUE+SUM
      SUM=0.
      DO 30J=1,KSP
30    SUM=SUM-W(J)*(RHSM(J)-S(J))
      VALUE=VALUE+SUM
      WRITE(6,101) VALUE
101   FORMAT(' CALRHS2',E16.7)
      RETURN
      END
```

```
      SUBROUTINE BLDCST
      DOUBLE PRECISION TABLE
      COMMON TABLE(75,150),IBAS(75),IUP(150),UPPER(150),INTEG(150)
      COMMON M,N,KSP,NSP,IST,ISLAK,IPRIT,I21,I30
      COMMON S(5),Z(5),P(5),D(5),U(5),R(5),FF(5),V(20,5),A(20)
      COMMON E(20,20,5),RHSM(5),C(20,5),X(150),LABLE(10,15),Y(20,5)
      IF(IPRIT.EQ.0.) GO TO 99
      WRITE(6,100) KSP,NSP,M,N
  100 FORMAT(' BLDCST',4I10)
   99 CONTINUE
      MM=M+1
      NN=N+1
      TABLE(MM,2)=1.
      TABLE(MM,NN)=1.
      IBAS(MM)=NN
      DO 10L=1,NSP
      SUM=0.
      DO 20J=1,KSP
      TSUM=0.
      DO 30I=1,NSP
   30 TSUM=TSUM+E(I,L,J)*V(I,J)
   20 SUM=SUM+TSUM-W(J)*C(L,J)
   10 TABLE(MM,L+2)=SUM-A(L)
      CALL CALPHS(VALUE)
      TABLE(MM,1)=VALUE
      M=MM
      N=NN
      CALL LABEL(LABLE(1,12))
      WRITE(6,300)(TABLE(M,IKE),IKE=1,N)
  300 FORMAT(20(//,4X,8F14.5))
      CALL REBASE
      RETURN
      END
```

SUBPPV DATE = 74099

```
      SUBROUTINE SUBPPV
      DOUBLE PRECISION TABLE
      COMMON TABLE(75,150),IBAS(75),IUP(150),UPPER(150),INTEG(150)
      COMMON M,N,KSP,NSP,IST,ISLAK,IPRIT,I21,I30
      COMMON S(5),Z(5),B(5),D(5),U(5),W(5),RF(5),V(20,5),A(20)
      COMMON E(20,20,5),RHSM(5),C(20,5),X(150),LABLE(10,15),Y(20,5)
      KO=ISLAK
      IF(IPRIT.EQ.0.) GO TO 99
      WRITE(6,100) KSP,NSP
  100 FORMAT(' SUBPPV',2I10)
   99 CONTINUE
      DO 10J=1,KSP
   10 S(J)=X(KO-1+J)
      DO 30I=1,NSP
      DO 30J=1,KSP
      SUM=0.
      DO 40K=1,NSP
   40 SUM=SUM+E(I,K,J)*X(K+2)
   30 Y(I,J)=SUM
      DO 50J=1,KSP
      SUM=0.
      DO 60I=1,NSP
   60 SUM=SUM+Y(I,J)**2
   50 Z(J)=SQRT(SUM)
      RETURN
      END
```

SUBPDV DATE = 74099

```
      SUBROUTINE SUBPDV
      DOUBLE PRECISION TABLE
      COMMON TABLE(75,150),IBAS(75),IUP(150),UPPER(150),INTEG(150)
      COMMON M,N,KSP,NSP,IST,ISLAK,IPRIT,I21,I30
      COMMON S(5),Z(5),B(5),D(5),U(5),W(5),RF(5),V(20,5),A(20)
      COMMON E(20,20,5),RHSM(5),C(20,5),X(150),LABLE(10,15),Y(20,5)
      IF(IPRIT.EQ.0.) GO TO 99
      WRITE(6,100) KSP,NSP
  100 FORMAT(' SUBPDV',2I10)
   99 CONTINUE
      DO 10IJ=1,KSP
      J=KSP+1-IJ
      TERM=S(J)/Z(J)
      CALL PROB(TERM,VALUE)
      W(J)=-B(J)*(RF(J)-D(J)*VALUE)
      TERM=-(TERM**2)/2.
      VALUE=SQRT(2.*3.1415927)
      U(J)=-B(J)*D(J)*EXP(TERM)/VALUE
      DO 10 I=1,NSP
   10 V(I,J)=U(J)*Y(I,J)/Z(J)
      RETURN
      END
```

PRIALG DATE = 74099

```
      SUBROUTINE PRIALG
      DOUBLE PRECISION TABLE
      COMMON TABLE(75,150),IBAS(75),IUP(150),UPPEF(150),INTEG(150)
      COMMON M,N,KSP,NSP,IST,ISLAK,IPRIT,I21,I30
      IF(IPRIT.EQ.0.) GO TO 99
      WRITE(6,100) M,N
  100 FORMAT(' PRIALG',2I10)
   99 CONTINUE
    5 CALL PNEGCK(J)
      IF(J.GT.N) GO TO 6
      CALL PFEAS(I,J)
      IF(I.EQ.1000) GO TO 1
      IF(TABLE(I,J)) 3,2,2
    2 CALL PIV(I,J)
      GO TO 5
    3 ISAVE=IBAS(I)
      CALL PIV(I,J)
      CALL PSUB(ISAVE)
      GO TO 5
    1 CALL PSUB(J)
      GO TO 5
    6 RETURN
      END
```

PNEGCK DATE = 74099

```
      SUBROUTINE PNEGCK(J)
      DOUBLE PRECISION TABLE
      DOUBLE PRECISION TEST
      COMMON TABLE(75,150),IBAS(75),IUP(150),UPPEF(150),INTEG(150)
      COMMON M,N,KSP,NSP,IST,ISLAK,IPRIT,I21,I30
      IF(IPRIT.EQ.0.) GO TO 99
      WRITE(6,100) N
  100 FORMAT(' PNEGCK',I10)
   99 CONTINUE
      TEST=90000000
      JSAVE=1000
       DO 10J=2,N
      IF(TABLE(1,J).GE.TEST) GO TO 10
      TEST=TABLE(1,J)
      JSAVE=J
   10 CONTINUE
      J=N+1
      IF (TEST.LT.0.) J=JSAVE
      WRITE(6,100) J
      RETURN
      END
```

PFEAS DATE = 74099

```
      SUBROUTINE PFEAS(I,J)
      DOUBLE PRECISION TABLE
      DOUBLE PRECISION A,THETA
      COMMON TABLE(75,150),IBAS(75),IUP(150),UPPER(150),INTEG(150)
      COMMON M,N,KSP,NSP,IST,ISLAK,IPRIT,I21,I30
      IF(IPRIT.EQ.0.) GO TO 99
      WRITE(6,100) J,M,N
  100 FORMAT(' PFEAS',3I10)
   99 CONTINUE
      THETA=UPPER(J)
      ISAVE=1000
      DO 10I=2,M
      IF(TABLE(I,J)) 1,10,2
    2 A=TABLE(I,1)/TABLE(I,J)
    5 IF(A.GE.THETA) GO TO 10
      THETA=A
      ISAVE=I
      GO TO 10
    1 A=(TABLE(I,1)-UPPER(IBAS(I)))/TABLE(I,J)
      GO TO 5
   10 CONTINUE
      I=ISAVE
      WRITE(6,100) I
      RETURN
      END
```

DUALG DATE = 74099

```
      SUBROUTINE DUALG
      DOUBLE PRECISION TABLE
      COMMON TABLE(75,150),IBAS(75),IUP(150),UPPER(150),INTEG(150)
      COMMON M,N,KSP,NSP,IST,ISLAK,IPRIT,I21,I30
      IF(IPRIT.EQ.0.) GO TO 99
      WRITE(6,101) M,N
  101 FORMAT(' DUALG',2I10)
   99 CONTINUE
    4 CALL DUBCK(I)
      IF(I.GT.M) GO TO 7
      CALL DENTCK(I,J)
       IF(J.EQ.1000) GO TO 6
      IF(TABLE(I,1).GE.0.) GO TO 5
      IF((TABLE(I,1)/TABLE(I,J))-UPPER(J)) 1,1,3
    1 CALL PIV(I,J)
      GO TO 4
    3 CALL PSUB(J)
      GO TO 4
    5 ISAVE=IBAS(I)
      CALL PIV(I,J)
      CALL PSUB(ISAVE)
      GO TO 4
    6 JJJ=IBAS(I)-1
      WRITE(6,100) JJJ
  100 FORMAT(1X,'X',I3,'   CANNOT BE MADE FEASIBLE (DUALG)')
       IPRIT=999
    7 RETURN
      END
```

DUBCK DATE = 74099

```
      SUBROUTINE DUBCK(I)
      DOUBLE PRECISION TABLE
      DOUBLE PRECISION A,THETA
      COMMON TABLE(75,150),IBAS(75),IUP(150),UPPER(150),INTEG(150)
      COMMON M,N,KSP,NSP,IST,ISLAK,IPRIT,I21,I30
      IF(IPRIT.EQ.0.) GO TO 99
      WRITE(6,100) M
  100 FORMAT(' DUBCK',I10)
   99 CONTINUE
      THETA=90000000
      ISAVE=1000
      DO 10I=2,M
      IF(TABLE(I,1)) 1,10,2
    1 A=TABLE(I,1)
   15 IF(A.GE.THETA) GO TO 10
      THETA=A
      ISAVE=I
      GO TO 10
    2 A=UPPER(IBAS(I))-TABLE(I,1)
      GO TO 15
   10 CONTINUE
      I=M+1
      IF(THETA.LT.0.) I=ISAVE
      WRITE(6,100) I
      RETURN
      END
```

DENTCK DATE = 74099.

```
      SUBROUTINE DENTCK(I,J)
      DOUBLE PRECISION TABLE
      DOUBLE PRECISION B,TEST,A
      COMMON TABLE(75,150),IBAS(75),IUP(150),UPPER(150),INTEG(150)
      COMMON M,N,KSP,NSP,IST,ISLAK,IPRIT,I21,I30
      IF(IPRIT.EQ.0.) GO TO 99
      WRITE(6,100) I,M,N
  100 FORMAT(' DENTCK',3I10)
   99 CONTINUE
      A=1
      IF(TABLE(I,1).GE.0.)    A=-1
      TEST=90000000
      JSAVE=1000
      DO 10J=2,N
      IF(IBAS(I).EQ.J) GO TO 10
      IF(A*TABLE(I,J).GE.0.) GO TO 10
      B=DABS(TABLE(1,J)/(A*TABLE(1,J)))
      IF(B.GE.TEST) GO TO 10
      TEST=B
      JSAVE=J
   10 CONTINUE
      J=JSAVE
      WRITE(6,100) J
      RETURN
      END
```

PIV DATE = 74099

```
      SUBROUTINE PIV(I,J)
      DOUBLE PRECISION TABLE
       DOUBLE PRECISION A
      COMMON TABLE(75,150),IBAS(75),IUP(150),UPPEF(150),INTEG(150)
      COMMON M,N,KSP,NSP,IST,ISLAK,IPRIT,I21,I30
      IF(IPRIT.EQ.0.) GO TO 99
      WRITE(6,100) I,J,M,N
  100 FORMAT(' PIV',4I10)
   99 CONTINUE
      A=TABLE(I,J)
      DO 10K=1,N
   10 TABLE(I,K)=TABLE(I,K)/A
      DO 20L=1,M
      IF(L.EQ.I) GO TO 20
      A=-TABLE(L,J)
      IF(A) 21,20,21
   21 DO 30K=1,N
   30 TABLE(L,K)=TABLE(L,K)+A*TABLE(I,K)
   20 CONTINUE
      DO 50K=1,M
   50 TABLE(K,J)=0
      TABLE(I,J)=1.0D00000
      IBAS(I)=J
      RETURN
      END
```

PSUB DATE = 74099

```
      SUBROUTINE PSUB(J)
      DOUBLE PRECISION TABLE
      COMMON TABLE(75,150),IBAS(75),IUP(150),UPPEF(150),INTEG(150)
      COMMON M,N,KSP,NSP,IST,ISLAK,IPRIT,I21,I30
      IF(IPRIT.EQ.0.) GO TO 99
      WRITE(6,100) J,M,N
  100 FORMAT(' PSUB',3I10)
   99 CONTINUE
      DO 10I=1,M
      TABLE(I,J)=-TABLE(I,J)
   10 TABLE(I,1)=TABLE(I,1)+UPPEF(J)*TABLE(I,J)
       IUP(J)=IUP(J)+1
      IF(IUP(J).EQ.2)   IUP(J)=0
      RETURN
      END
```

TRACE DATE = 74099

```
      SUBROUTINE TRACE
      DOUBLE PRECISION TABLE
      COMMON TABLE(75,150),IBAS(75),IUP(150),UPPER(150),INTEG(150)
      COMMON M,N,KSP,NSP,IST,ISLAK,IPRIT,I21,I30
      COMMON S(5),Z(5),B(5),D(5),U(5),W(5),RF(5),V(20,5),A(20)
      COMMON E(20,20,5),FMSM(5),C(20,5),X(150),LABLE(10,15),Y(20,5)
      IF(IPRIT.EQ.0.0.) GO TO 99
      WRITE(6,101) M,N
  101 FORMAT(' TRACE',2I10)
   99 CONTINUE
      DO 20I=1,N
   20  X(I)=0.
      DO 10I=1,M
      IM=IBAS(I)
      IF(IM.GT.N.OR.IM.LE.0) GO TO 11
      X(IM)=TABLE(I,1)
      GO TO 10
   11 WRITE(6,100) IM
  100 FORMAT(' ERROR IN TRACE',I7)
      STOP
   10 CONTINUE
      DO 30I=1,N
      IF(IUP(I)) 30,30,31
   31 X(I)=UPPER(I)-X(I)
   30 CONTINUE
      RETURN
      END
```

OUTPUT DATE = 74099

```
      SUBROUTINE OUTPUT(O,N1,N2,N3,NN1,NN2,NN4,LA)
      DIMENSION O(NN4)
      DIMENSION LA(10)
      NN3=NN4/(NN1*NN2)
      NO=(N2-1)/8
      NO=NO*8+1
      IF(N1.GT.1) WRITE(6,100)
      DO 50K=1,N3
      DO 50I=1,NO,8
      IF(I.GT.N2) GO TO 10
  100 FORMAT(' ')
      I1=I+7
      IF(I1.GT.N2) I1=N2
      WRITE(6,LA)
      IF(NN3.GT.1) WRITE(6,105) K
  105 FORMAT(1X,'MATRIX NO.',I2)
      IF(NN2.GT.1) WRITE(6,106) (KJ,KJ=I,I1)
  106 FORMAT(3X,'|J',6X,I2,7(12X,I2))
      WRITE(6,107)
  107 FORMAT(2X,'I',1X,116(1H_))
      DO 60 J=1,N1
  102 FORMAT(2X)
   11 WRITE(6,103) J   ,(O(J+(KJ-1)*NN1+(K-1)*NN1*NN2),KJ=I,I1)
  103 FORMAT(1X,I2,'|',8F14.5)
   60 CONTINUE
    2 WRITE(6,102)
   50 CONTINUE
   10 RETURN
      END
```

```
      SUBROUTINE OUTPUT(D,N1,N2,N3,NN1,NN2,NN4,LA)
      DOUBLE PRECISION D
      DIMENSION D(NN4)
      DIMENSION LA(10)
      NN3=NN4/(NN1*NN2)
      NO=(N2-1)/8
      NO=NO*8+1
      IF(N1.GT.1) WRITE(6,100)
      DO 50K=1,N3
      DO 50I=1,NO,8
      IF(I.GT.N2) GO TO 10
  100 FORMAT(' ')
      I1=I+7
      IF(I1.GT.N2) I1=N2
      WRITE(6,LA)
      IF(NN3.GT.1) WRITE(6,105) K
  105 FORMAT(1X,'MATRIX NO.',I2)
      IF(NN2.GT.1) WRITE(6,106) (KJ,KJ=I,I1)
  106 FORMAT(5X,'|J',6X,I2,7(12X,I2))
      WRITE(6,107)
  107 FORMAT(2X,'I',1X,116(1H_))
      DO 60 J=1,N1
  102 FORMAT(2X)
   11 WRITE(6,103) J  ,(D(J+(KJ-1)*NN1+(K-1)*NN1*NN2),KJ=I,I1)
  103 FORMAT(1X,I2,'|',8F14.5)
   60 CONTINUE
    2 WRITE(6,102)
   50 CONTINUE
   10 RETURN
      END
```

PROB DATE = 74099

```
      SUBROUTINE PROB(A,SUM)
      WRITE(6,101) A
  101 FORMAT(' PROB',F20.10)
   99 CONTINUE
      IF(A.LT.0.) A=-A
      ACQ=.04
      SUM=0.
      IF(A.EQ.0.) GO TO 4
      IF(A.GT.5.) GO TO 2
      TSUM=.49999827
      IF(A.GT.4.5) GO TO 3
      TSUM=.49997
      IF(A.GT.4.) GO TO 3
      TSUM=.49977
      IF(A.GT.3.5) GO TO 3
      TSUM=.49865
      IF(A.GT.3.0) GO TO 3
      TSUM=.49379
      IF(A.GT.2.5) GO TO 3
      TSUM=.47725
      ACQ=.01
      IF(A.GT.2.0) GO TO 3
      TSUM=.43319
      ACQ=.004
      IF(A.GT.1.5) GO TO 3
      TSUM=.34134
      IF(A.GT.1.0) GO TO 3
      TSUM=.19146
      IF(A.GT..5) GO TO 3
      ACQ=.01
      TSUM=0.
    3 I=(A*10.)/5.
      Q=I*5
      Q=Q*.1
      IF(Q.GE.A) Q=Q-.5
      N=((A-Q)/ACQ)+.5
      N=2*N
      IF(N.LE.2) N=4
      H=N
      H=(A-Q)/H
      M=N-1
      DO 10I=1,N,2
      X=Q+H*I
      Y=-(X**2)/2.
   10 SUM=SUM+4.*EXP(Y)
      DO 20I=2,M,2
      X=Q+H*I
      Y=-(X**2)/2.
   20 SUM=SUM+2.*EXP(Y)
      X=-(Q**2)/2.
      Y=-(A**2)/2.
      SUM=SUM+EXP(X)+EXP(Y)
      SQ=(2.*3.1415927)**.5
      SUM=(H/(3.*SQ))*SUM
      SUM=SUM+TSUM
    4 SUM=.50000000-SUM
    2 CONTINUE
      WRITE(6,101) SUM
      RETURN
      END
```

II. Feeder Program Documentation

A. Title: Conjugate Matrix Generation

Programmer: C. A. Mount-Campbell

Advisor: R. P. Lutz

Date Completed: Jan. 1974

Machine Used: IBM 360

Language Used: FORTRAN IV

Compiler Used: WAT FIVE

Compilation Time: 3.76 seconds

Computation Time: Variable with problem size

Lines of Output: Variable with problem size

Approx Core Required: 15K Bites

B. Purpose: This program was written to generate a covariance matrix for the computation of variance of slack funds for each period under the assumption of no autocorrelation. These covariances matrices are then used to generate a matrix of conjugate vectors with the first vector being $(1,0,0,\ldots,0)^t$ in all cases. The quality of results are checked by multiplying the original covariance matrix first by the transpose of the conjugate matrix and then by the original conjugate matrix. Finally the conjugate matrix is inverted with the result being output on cards in the format required by the previous program. The diagonal elements of the resulting diagonal matrix are also output on cards.

C. Restrictions: Current dimensions limit the program to the generation of 25 x 25 covariance matrices.

D. List of subroutines and their function:

1. MAIN--This routine reads the input data and directs the major calling sequence of the other subroutines to accomplish the stated purpose.
2. PIV--This subroutine is the same as the one used in the first program but is used for matrix inversion in this program.
3. CONJ--This subroutine generates the matrix of conjugate vectors.
4. CHECK--This subroutine performs the matrix multiplication.
5. OUTPUT--This subroutine is used for printing matrices.
6. OUTPUN--This subroutine is used for punching matrices.
7. INPT--This subroutine reads the data for the linear relationship given on page 127 and generates the equivalent covariance matrices.

E. List of undimensioned variables:

1. N--number of projects (i.e., covariance matrix is N x N).
2. SIG1= $\sigma^2_{I_1}$ for equation on page 127.
3. SIG2= $\sigma^2_{I_2}$ for same equation.
4. SIG12= $\mathrm{Cov}(I_1, I_2)$ for the same equation.

F. List of dimensioned variables:

1. A(25)--used to store project variances.
2. SAVE(25,25)--used for temporary storage of a matrix.
3. LA(10,10)--used to store formats and labels for controlling and identifying output.
4. B(25,25)--used for temporary storage of correlation matrix, covariance matrix, and conjugate matrix.
5. TABLE(25,50)--used for matrix inversion.
6. E(25)--used to store diagonal elements of diagonal matrix.
7. DUM(25,25)--summary matrix used for temporary storage during the calculation of conjugate vectors, and during matrix multiplication.
8. BETA(25)--equivalent to β_{ij} of the equation on page 127 for fixed j.
9. GAMMA(25)--equivalent to γ_{ij} of the equation on page 127 for fixed j.
10. EP(25)--equivalent to the error variance for the equation on page 127 for fixed j.
11. COV(25,25)--matrix of project covariance generated by subroutine INPT.

NOTE: Other dimensioned variable names were used by subroutines but appear in COMMON statements with those listed above. Therefore, their description is also given above.

G. Organization of Input:

1. The first READ statement exactly requires the following data cards:

Card #	Column #1
1	(' MATRIX TO BE DIAGONALIZED')
2	(' RESULTING DIAGONAL ELEMENTS')
3	(' DIAGONALIZED MATRIX')
4	(' INVERSE OF CONJUGATE DIR. MATRIX')
5	(' MATRIX OF CONJUGATE DIRECTIONS')
6	(10F8.0)
7	(10F8.1)
8	(10F8.2)
9	(10F8.3)
10	(10F8.4)

2. The next group of input cards is repeated for each period in the planning horizon. These groups are read by the second READ statement and by subroutine INPT.

   a. First card should hold a value for N in the first two columns.

   b. Second card should hold the following starting in column 1: (' COV GENERATE')

   c. The next N cards should hold the value for β_{ij}, γ_{ij}, $\sigma^2_{\epsilon_{ij}}$ where i represents the project and j the period. These data appear respectively in columns 1-10, 11-20, 21-30. Decimal points should be punched.

   d. The last card for a period should hold values for $\sigma^2_{I_1}$, $\sigma^2_{I_2}$, and $Cov(I_1,I_2)$ using the same format as c.

H. Flow Chart:

For the basic flowchart refer to Figure E-11.

I. Program Listing:

The program listing follows Figure E-11.

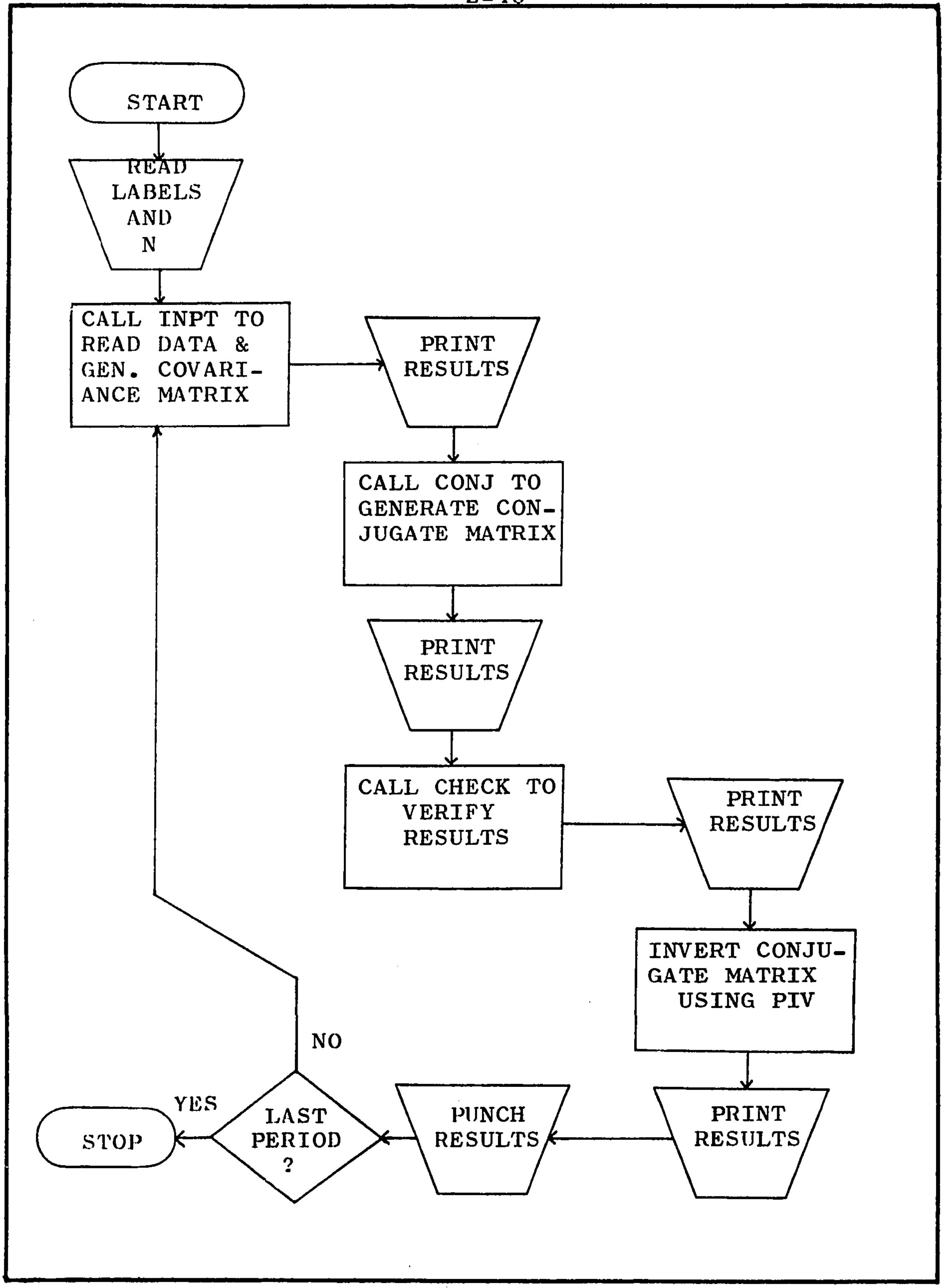
START
READ LABELS AND N
CALL INPT TO READ DATA & GEN. COVARIANCE MATRIX
PRINT RESULTS
CALL CONJ TO GENERATE CONJUGATE MATRIX
PRINT RESULTS
CALL CHECK TO VERIFY RESULTS
PRINT RESULTS
INVERT CONJUGATE MATRIX USING PIV
PRINT RESULTS
PUNCH RESULTS
LAST PERIOD ?
NO
YES
STOP

Figure E-11.

```
//JOB            ,FP=20,TI---120,PUNCH,PAGES=100
      DIMENSION A(25),SAVE(25,25),LA(10,10)
      COMMON B(25,25),TABLE(25,50),E(25),DUM(25,25)
      READ(5,102) LA
  102 FORMAT(10A4)
      READ(5,100) N
  100 FORMAT(I2)
      DO 5 I=1,N
      DO 5 J=1,N
    5 SAVE(I,J)=0.0
      IO=1
      I10=25
       I15=25
      I30=50
  333 CONTINUE
      CALL INPT(B,A,N)
      DO 10I=1,N
      DO 10J=1,N
   10 TABLE(I,J)=B(I,J)
      DO 20 I=1,N
      DO 20 J=1,N
      TABLE(I,J)=TABLE(I,J)*A(I)*A(J)
      SAVE(I,J)=SAVE(I,J)+TABLE(I,J)
   20 TABLE(I,J)=SAVE(I,J)
      IZ=I10*I10
      CALL OUTPUT(SAVE,N,N,IO,I10,I10,IZ,LA(1,1))
      CALL CONJ(N)
      M=N
      IZ=I10*IO
      CALL OUTPUT(E,N,IO,IO,I10,IO,IZ,LA(1,2))
      K=10
      DO 110 I=1,N
      IF(E(I).GT.999.9999) K=9
      IF(E(I).GT.9999.999) K=8
      IF(E(I).GT.99999.99) K=7
      IF(E(I).GT.999999.9) K=6
  110 CONTINUE
      CALL OUTPUN(E,N,IO,IO,I10,IO,IZ,LA(1,K))
      IZ=I10*I10
      CALL OUTPUT(B,N ,N,IO,I10,I10,IZ,LA(1,5))
      CALL CHECK(N)
      CALL OUTPUT(DUM,N,N,IO,I10,I10,IZ,LA(1,3))
    3 DO 40I=1,N
      DO 40J=1,N
   40 TABLE(I,J)=B(I,J)
      DO 25 I=1,N
      DO 25 J=1,N
      K=N+1
      TABLE(J,K)=0.0
      IF(I.EQ.J) TABLE(J,K)=1.0
   25 CONTINUE
      M=2*N
      DO 30I=1,N
   30 CALL PIV(I,I,N,M)
      IK=I30-N
      IZ=I15*IK
       CALL OUTPUT(TABLE(1,N+1),N,N,IO,I15,IK,IZ,LA(1,4))
      CALL OUTPUN(TABLE(1,N+1),N,N,IO,I15,IK,IZ,LA(1,10))
      GO TO 333
      STOP
```

```
      END

      SUBROUTINE PIV(I,J,M,N)
      COMMON P(25,25),TABLE(25,50),E(25),DUM(25,25)
      A=TABLE(I,J)
      DO 10K=1,N
   10 TABLE(I,K)=TABLE(I,K)/A
      DO 20L=1,M
      IF(L.EQ.I) GO TO 20
      A=-TABLE(L,J)
      DO 30K=1,N
   30 TABLE(L,K)=TABLE(L,K)+A*TABLE(I,K)
   20 CONTINUE
      DO 50K=1,M
   50 TABLE(K,J)=0
      TABLE(I,J)=1.0000000
      RETURN
      END

      SUBROUTINE CONJ(N)
      COMMON D(25,25),Q(25,50),DQD(25),E(25,25)
      DO 10I=1,N
   10 D(I,1)=0
      D(1,1)=1.0
      DO 20K=1,N
      DQD(K)=0
      DO 30I=1,N
      E(I,K)=0
      DO 40J=1,N
   40 E(I,K)=E(I,K)+D(J,K)*Q(J,I)
   30 DQD(K)=DQD(K)+E(I,K)*D(I,K)
      IF(K.EQ.N) GO TO 20
      DO 50I=1,N
      SUM=0
      DO 60J=1,K
   60 SUM=SUM+(E(K+1,J)/DQD(J))*D(I,J)
      Z=0.0
      IF(I.EQ.K+1) Z=1.0
   50 D(I,K+1)=Z-SUM
   20 CONTINUE
      RETURN
      END

      SUBROUTINE CHECK(N)
      COMMON D(25,25),Q(25,50),E(25),DUM(25,25)
      DIV=1.
      DO 20I=1,N
      DO 20J=1,N
      DUM(I,J)=0
      DO 30K=1,N
   30 DUM(I,J)=DUM(I,J)+D(K,I)*Q(K,J)
   20 CONTINUE
      DO 40I=1,N
      DO 50J=1,N
      E(J)=0
      DO 50K=1,N
   50 E(J)=E(J)+DUM(I,K)*D(K,J)
      DO 40J=1,N
   40 DUM(I,J)=E(J)*DIV
```

```
      RETURN
      END

      SUBROUTINE OUTPUT(D,N1,N2,N3,NN1,NN2,NN4,LA)
      DIMENSION D(NN4)
      DIMENSION LA(10)
      NN3=NN4/(NN1*NN2)
      LINES=54
      NO=(N2-1)/8
      NO=NO*8+1
      WRITE(6,100)
      DO 50 K=1,N3
      DO 50 I=1,NO,8
      IF(I.GT.N2) GO TO 10
  100 FORMAT('1')
      I1=I+7
      IF(I1.GT.N2) I1=N2
      WRITE(6,LA)
      IF(NN3.GT.1) WRITE(6,105) K
  105 FORMAT(1X,'MATRIX NO.',I2)
      IF(NN2.GT.1) WRITE(6,106) (KJ,KJ=I,I1)
  106 FORMAT(3X,' J',6X,I2,7(12X,I2))
      WRITE(6,107)
  107 FORMAT(2X,'I',1X,116(1H ))
      DO 60 J=1,N1
  102 FORMAT(2X)
   11 WRITE(6,103) J  ,(D(J+(KJ-1)*NN1+(K-1)*NN1*NN2),KJ=I,I1)
  103 FORMAT(1X,I2,' ',8F14.5)
   60 CONTINUE
      LINES=LINES-N1-5
      IF(LINES-N1-4) 1,2,2
    1 IF(I1.EQ.N2) GO TO 50
      WRITE(6,100)
      LINES=54
    2 WRITE(6,102)
   50 CONTINUE
   10 RETURN
      END

      SUBROUTINE OUTPUN(D,N1,N2,N3,NN1,NN2,NN4,LA)
      DIMENSION D(NN4)
      DIMENSION LA(10)
      NN3=NN4/(NN1*NN2)
      NO=(N2-1)/10
      NO=NO*10+1
      DO 50 K=1,N3
      DO 50 I=1,NO,10
      IF(I.GT.N2) GO TO 10
      I1=I+9
      IF(I1.GT.N2) I1=N2
      DO 60 J=1,N1
   11 WRITE(7,LA)(D(J+(KJ-1)*NN1+(K-1)*NN1*NN2),KJ=I,I1)
   60 CONTINUE
   50 CONTINUE
   10 RETURN
      END

      SUBROUTINE INPT(COV,BETA,N)
      DIMENSION BETA(25),GAMMA(25),EP(25),COV(25,25),LA(10)
      READ(5,101) LA
```

```
 101 FORMAT(10A4)
     KA=25
     KO=1
     DO 10 I=1,N
  10 READ(5,100) BETA(I),GAMMA(I),EP(I)
 100 FORMAT(3F10.0)
     READ(5,100) SIG1,SIG2,SIG12
     DO 20 I=1,N
     DO 20 J=1,N
     IF(I.EQ.J) GO TO 1
     COV(I,J)=BETA(I)*BETA(J)*SIG1+BETA(I)*GAMMA(J)*SIG12
     COV(I,J)=COV(I,J)+BETA(J)*GAMMA(I)*SIG12+GAMMA(I)*GAMMA(J)*SIG2
     GO TO 20
   1 COV(I,J)=(BETA(I)**2)*SIG1+(GAMMA(J)**2)*SIG2
     COV(I,J)=COV(I,J)+2.*BETA(I)*GAMMA(J)*SIG12+EP(I)
  20 CONTINUE
     DO 30 I=1,N
     DIV=SQRT(COV(I,I))
     BETA(I)=DIV
     DO 30 J=1,N
     COV(I,J)=COV(I,J)/DIV
  30 COV(J,I)=COV(J,I)/DIV
     KB=KA**2
     CALL OUTPUT(COV,N,N,KO,KA,KA,KB,LA)
     RETURN
     END
$EXEC
```