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PRODUCTS AND DUALS OF GENERALIZED LINEAR SPACES.

The University of Oklahoma, Ph.D., 1974
Mathematics

THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

PRODUCTS AND DUALS OF GENERALIZED LINEAR SPACES

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

BY

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Bloomington, Illinois

1974

PRODUCTS AND DUALS OF GENERALIZED LINEAR SPACES

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DISSERTATION COMMITTEE

ACKNOWLEDGMENTS

Dr. David C. Kay, for his strong guidance and advice, inspiration, patience, understanding, and invigorating and penetrating questions.

Illinois Wesleyan University, for moral encouragement and financial support.

Cathy and Erik, for the inevitable family sacrifices made.

PRODUCTS AND DUALS OF GENERALIZED LINEAR SPACES

CHAPTER I

INTRODUCTION

Much of mathematics is synthesized from the properties of a set of objects or of the objects themselves, and the properties of a family of mappings on those sets or the mappings themselves. For this dissertation a set X with a family $\mathcal{L}$ of "lines" in X and a family $\mathcal{F}$ of real-valued ($\mathbb{R}$ ≡ reals) functions is considered, which has as its prototype any usual convex subset of E^n with usual lines and the family of usual linear functions restricted to the subset. Initially $(X, \mathcal{L})$ is a "generalized linear space", (defined in Chapter II and symbolized G.L.S.), as studied by Cantwell [1]. A definition of "linear" is considered via a "line" structure $\mathcal{L}'$ induced upon $X \times \mathbb{R}$ and the graphs of functions. The term "dual" space is used to indicate the set of "linear" functions, which is shown to be a real module under the usual operations on functions. The conditions under which X is isomorphic to its "dual" space has been investigated thoroughly in classical works when X is a real module. With the line structure $\mathcal{L}'$ induced on $X \times \mathbb{R}$ in a natural manner, $(X, \mathcal{L})$ is shown to be isomorphic to a real module precisely when $(X \times \mathbb{R}, \mathcal{L}')$ is a G.L.S. More generally a line structure $\mathcal{L}''$ may be induced on X × Y where X and Y are G.L.S., and X, Y are shown to be isomorphic to real modules if and only if $(X \times Y, \mathcal{L}'')$ is a G.L.S.

Certain properties of the family $\mathfrak{F}$ and its members are found to be sufficient to show that X must be a G.L.S. under the imposed convexity and line structures. The imposed structures on X are shown to possess the property that inverse-images of singletons for non-trivial functions are hyperplanes in X (and conversely if X is of finite dimension).

Considering these properties of $\mathfrak{F}$, together with a G.L.S. X of finite dimension, a topology τ is induced upon X under which $\mathfrak{F}$ is a family of continuous functions. This topology is shown to be equivalent to the topology considered by Cantwell, so that a result of Cantwell's that (X, τ) is homeomorphic to E^n for some n holds. (X, τ) is shown to be a locally convex generalized linear space, so that classical results such as the Krein-Milman theorem can be obtained, (as shown by Shirley [7]).

CHAPTER II

LINEAR FUNCTIONS VIA PRODUCTS OF GENERALIZED LINEAR SPACES

Let X be a set, and $\mathcal{L}$ a family of subsets of X bearing certain properties in common with lines in geometry. Traditional language is used: Members of X and $\mathcal{L}$ are called "points" and "lines" respectively. Each line is assumed to have a total ordering from which, as shown by Prenowitz [3], an interval convexity structure can be extracted. If $x,y \in L \in \mathcal{L}$, $x \neq y$ then let (xy) denote the set of points strictly between x and y. Correspondingly: $(xy = (xy) \cup \{y\}$, $xy) = (xy) \cup \{x\}$, and $xy = (xy) \cup \{x,y\}$. The notation xyz means $y \in xz$ and (xyz) means $y \in xz$. We freely use this notation for real numbers. If $x,y,z \in L \in \mathcal{L}$, then x, y, z are collinear. A non-collinear triple is called a *triangle*. We consider a class of spaces which was studied by Shirley [7] and by Cantwell [1] before him.

2.1. DEFINITION. The pair $(X,\mathcal{L})$ is called a *generalized linear space* (G.L.S.) if the following three axioms are satisfied:

A. Each line is order-isomorphic to the reals.

B. Each pair of distinct points belong to a unique line.

C. If $x, y, z \in X$, (xuy), (uvz) then there exists w such that (xwz) and (wvy).

The unique line containing $x \neq y$ as given by axiom B will be denoted by $L(x,y)$. The order-isomorphism given by axiom A shall be denoted by π; that is, $\pi: L \to \mathbb{R}$. This isomorphism depends upon the particular line L, so that at times additional subscripts will be necessary. Since infinitely many isomorphisms would be available; we shall assume that for each line there is, up to an arbitrary constant, a fixed isomorphism. Thus when we say that $(X,\mathcal{L})$ is a G.L.S., a particular family of isomorphisms from the members of $\mathcal{L}$ to $\mathbb{R}$ will be assumed.

2.2. DEFINITION. A directed distance function d on X is a function $d: X \times X \to \mathbb{R}$ such that

$$d(x,y) = 0 \quad \text{if } x = y,$$

$$d(x,y) = \pi(y) - \pi(x) \quad \text{if } x \neq y,$$

where $\pi = \pi_{L(x,y)}$ is the isomorphism guaranteed from $L(x,y)$ to $\mathbb{R}$.

2.3. PROPERTIES OF d. In view of Definition 2.2. we have:

(i) $d(x,y) = 0$ if and only if $x = y$

(ii) $d(x,y) = -d(y,x)$

(iii) Given $x \in L$ with d defined on L, to each $\lambda \in \mathbb{R}$ there corresponds a unique $y \in L$ such that $d(x,y) = \lambda$.

(iv) If x, y, z are any three collinear points then $d(x,y) = d(x,z)$ if and only if $y = z$.

(v) If x, y, z are any three collinear points

$$(1) \quad d(x,y) + d(y,z) = d(x,z).$$

(vi) Point y is between x and z if and only if x,y, and z are distinct, collinear points and

$$(2) \quad |d(x,y)| + |d(y,z)| = |d(x,z)|$$

or equivalently, if and only if x, y, z are collinear and

$$(3) \quad d(x,y)d(y,z) > 0.$$

We note that d does not necessarily satisfy the triangle inequality:

$$|d(x,y)| \leq |d(x,z)| + |d(y,z)|.$$

Proof (i). Suppose $d(x,y) = 0$ and $x \neq y$. Then if $L = L(x,y)$ $\pi_L(y) - \pi_L(x) = 0$ or $\pi_L(y) = \pi_L(x)$, from which we obtain $y = x$ (since π_L is an isomorphism), a contradiction. The second half is immediate from the definition of d.

Proof (ii). Clear from definition of d.

Proof (iii),(iv). Clear since π_L is an isomorphism from L to $\mathbb{R}$.

Proof (v). If $x = y = z$, or $x = y \neq z$, or $x \neq y = z$ then (1) is clearly satisfied. If $x = z \neq y$ then

$$d(x,y) + d(y,z) = d(x,y) - d(x,y) = 0 = d(x,z).$$

If x, y, z are distinct then

$$\begin{aligned} d(x,y) + d(y,z) &= [\pi_L(y) - \pi_L(x)] + [\pi_L(z) - \pi_L(y)] \\ &= \pi_L(z) - \pi_L(x) = d(x,z) \end{aligned}$$

and (1) is satisfied, completing the proof.

Proof (vi). Now y is between x and z if and only if (xyz) which implies x, y, z are distinct collinear points. Since π is an order-isomorphism, we have $\pi(x) < \pi(y) < \pi(z)$ or $\pi(x) < \pi(y) < \pi(z)$ if and only if $|\pi(y) - \pi(x)| + |\pi(z) - \pi(y)| = |\pi(z) - \pi(x)|$ which is equivalent to (2). Similarly for (3).

We now turn to a consideration of space $X \times \mathbb{R} = \{(x,\alpha) \mid x \in X, \alpha \in \mathbb{R}\}$ with a view to making it a G.L.S. The motivation for this is to formulate a definition for linear functions from X to $\mathbb{R}$.

2.3. DEFINITION. A line in $X \times \mathbb{R}$ is a set of one of the following two types, for each $m \in X$, $\mu, \lambda \in \mathbb{R}$, $L \in \mathcal{L}$:

(i) $\{m\} \times \mathbb{R}$

(ii) $\{(x, \mu\pi(x) + \lambda) \mid x \in L\}$.

Observe that the lines in $X \times \mathbb{R}$ may be made order-isomorphic to $\mathbb{R}$ under the two isomorphisms π', π'' where $\pi'\colon \{m\} \times \mathbb{R} \to \mathbb{R}$ is defined by $\pi'(m,\alpha) = \alpha$ and $\pi''\colon \{(x, \mu\pi(x) + \lambda)\} \to \mathbb{R}$ is defined by $\pi''((x, \mu\pi(x) + \lambda)) = \pi(x)$. Thus axiom A of a G.L.S. is satisfied.

Let (a,α), (b,β) be two distinct points in $X \times \mathbb{R}$. If $a = b$, then a line in $X \times \mathbb{R}$ containing (a,α), (a,β) has the form $\{a\} \times \mathbb{R}$ or $\{(x, \mu\pi(x) + \lambda) \mid x \in L\}$ for some μ, $\lambda \in \mathbb{R}$ and line L thru a. But in the latter case, $x = a$ implies $\alpha = \mu\pi(a) + \lambda$ and $\beta = \mu\pi(a) + \lambda$ yielding $\alpha = \beta$ a contradiction, and the line is of the form $\{a\} \times \mathbb{R}$. If $a \neq b$ then there exists $L \in \mathcal{L}$ such that $L = L(a,b)$. Thus the corresponding line in $X \times \mathbb{R}$, namely

$$\ell = \{(x, \mu\pi(x) + \lambda) \mid x \in L\}$$

will contain (a,α), (b,β) if and only if

$$d(a,b)\mu = \beta - \alpha\,, \qquad d(a,b)\,\lambda = \alpha\pi(b) - \beta\pi(a)$$

since $d(a,b) \neq 0$ so that μ,λ are unique. Hence for any two distinct points in $X \times \mathbb{R}$, there is a unique line joining them and axiom B of a G.L.S. is satisfied.

We state a simple lemma, whose proof is obvious, which will be useful later.

2.4. LEMMA. For $a \neq b$ the line $L((a,\alpha), (b,\beta))$ is the set of all points (x,ξ), $x \in L(a,b)$ where

$$\xi = \left(1 - \frac{d(a,x)}{d(a,b)}\right)\alpha + \frac{d(a,x)}{d(a,b)}\,\beta = \frac{d(a,x)}{d(a,b)}\,\beta + \frac{d(x,b)}{d(a,b)}\,\alpha\,.$$

2.5. DEFINITION. Let (a,α), (b,β), and (c,γ) be any three collinear points in $X \times \mathbb{R}$. Then (b,β) is said to be between (a,α) and (c,γ) if and only if either b is between a and c in X or $a = b = c$ and β is between α and γ in $\mathbb{R}$.

Before investigating axiom C as it applies to $X \times \mathbb{R}$, we consider "flats" in $X \times \mathbb{R}$. Following the usual manner in which flats are defined in terms of lines, we adopt the following.

2.6. DEFINITION. A flat in $X \times \mathbb{R}$ is a set F such that for all $p,q \in F$, $L(p,q) \subset F$. A flat H is called a hyperplane in $X \times \mathbb{R}$ if and only if H is a maximal proper flat in $X \times \mathbb{R}$. If $A \subset X \times \mathbb{R}$ the flat spanned by A is $fl(A) = \{F \mid A \subset F,\ F \text{ a flat in } X \times \mathbb{R}\}$. (Similar definitions are made if $(Y,\mathcal{Y})$ is any pair satisfying axiom B, or, in particular, a G.L.S.)

2.7. DEFINITION. A function $f\colon X \to \mathbb{R}$ is called linear if and only if graph $f = \{(x,f(x)) \mid x \in X$ is a flat in $X \times \mathbb{R}\}$.

REMARK. Linear functions can be thought of as those functions which are either trivial on lines or preserve the ratios determined by the directed distance function d between points on the lines of X. For, if L is a line in X with $x \neq y \in L$, a point on the line $L((x,f(x)),(y,f(y)))$ in $X \times \mathbb{R}$ has the form

$$\left(m,\frac{d(x,m)}{d(x,y)}\, f(y) + \frac{d(m,y)}{d(x,y)}\, f(x)\right) = (m,f(m)), \qquad m \in L(x,y)$$

which yields

$$\frac{f(x) - f(m)}{f(y) - f(y)} = \frac{d(x,m)}{d(x,y)}\,.$$

2.8. LEMMA. If $m \in fl(x,y,z)$, then $(m,\delta) \in fl((x,\delta),(y,\delta),(z,\delta))$.

Proof. If x,y,z are not distinct, then $z \in L(x,y)$ and $m \in L(x,y)$. By Lemma 2.4. $(m,\delta) \in L((x,\delta),(y,\delta))$. Suppose x, y, z are distinct. Since $m \in fl(x,y,z)$ and X is a G.L.S., it follows from properties of flats, that either $L(z,m) \cap L(x,y) \neq \emptyset$, $L(x,m) \cap L(y,z) \neq \emptyset$ or $L(y,m) \cap L(x,z) \neq \emptyset$; wolog, $L(z,m) \cap L(x,y) = \{u\}$. Then $(u,\delta) \in L((x,\delta),(y,\delta))$ and $(m,\delta) \in L((z,\delta), (u,\delta))$. Hence $(m,\delta) \in fl((x,\delta), (y,\delta), (z,\delta))$.

<u>2.9. COROLLARY</u>. Let f be a linear function such that $f(x)=f(y)=f(z)=\delta$ If $m \in fl(x,y,z)$ then $f(m) = \delta$.

<u>2.10. THEOREM</u>. The graph of every linear function f on X is a hyperplane in $X \times \mathbb{R}$.

<u>Proof</u>. The set F = graph f is clearly proper in $X \times \mathbb{R}$. Thus suppose that F is not a maximal flat; that is, suppose there exists H, a flat of $X \times \mathbb{R}$, such that $F \subsetneq H \subset X \times \mathbb{R}$. We shall show $H = X \times \mathbb{R}$. Let $(y,\eta) \in X \times \mathbb{R} \setminus H$. If $\{y\} = X$ then $(y,\delta) \in H$ for some $\delta \neq f(y)$. But

$$(y,\eta) \in \{y\} \times \mathbb{R} = L((y,\delta), (y,f(y))) \subset H$$

so that $H = X \times \mathbb{R}$. If $X \neq \{y\}$ then there exists $x \neq y \in X$, $\delta \in \mathbb{R}$ such that $(x,\delta) \in H \setminus F$. We can assume that $f(x) - \delta \neq \eta - f(y)$ for if not, then let $\delta' = (\delta + f(x))/2$ so that $(x,\delta') \in H \setminus F$ and $f(x) - \delta' \neq \eta - f(y)$. Since $(y,\eta) \in H$, $x \neq y$, there exists $m \in L(x,y)$ such that

$$\pi(m) = \frac{\pi(y)(f(x) - \delta) + \pi(x)(f(y) - \eta)}{f(x) - \delta + f(y) - \eta}$$

from which follows the equation

$$\frac{d(x,m)}{d(x,y)}\eta + \frac{d(m,y)}{d(x,y)}f(x) = \frac{d(x,m)}{d(x,y)}f(y) + \frac{d(m,y)}{d(x,y)}\delta = \xi .$$

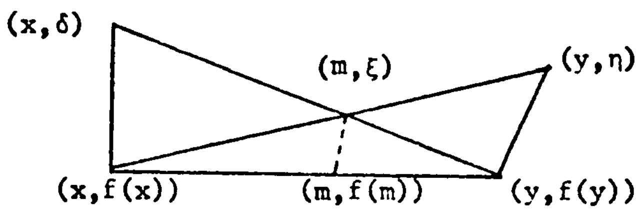

Figure 1.

By Lemma 2.4., $(m,\xi) \in L((x,f(x)),(y,\eta)) \cap L((x,\delta),(y,f(y)))$ (see fig. 1). Hence $(m,\xi) \in H$ so that $(y,\eta) \in H$ and $H = X \times \mathbb{R}$.

We now investigate the "dual" of X, denoted X* where

$X^* = \{f \mid f:X \to \mathbb{R},\ f \text{ is linear}\}$.

2.11. THEOREM. X* is a module over $\mathbb{R}$, under the usual definitions of function addition and scalar multiplication.

Proof. We need only show that for all $\lambda \in \mathbb{R}$, $f, g \in X^*$,

(1) graph $(\lambda\cdot f)$ is a flat.

(2) graph $(f + g)$ is a flat.

Proof of (1). Let $x \neq y \in X$; then two points in graph $(\lambda\cdot f)$ are

$$(x, (\lambda\cdot f)(x)) = (x, \lambda f(x)),\ (y, (\lambda\cdot f)(y)) = (y, \lambda f(y)).$$

The line joining these two points has points of the form

$$\left(m, \frac{d(x,m)}{d(x,y)}\lambda f(y) + \frac{d(m,y)}{d(x,y)}\lambda f(x)\right)$$

$$= \left(m, \lambda\left[\frac{d(x,m)}{d(x,y)} f(y) + \frac{d(m,y)}{d(x,y)} f(x)\right]\right), \quad m \in L(x,y),$$

and since graph f is a flat this point is

$$(m, \lambda f(m)) = (m, (\lambda\cdot f)(m)).$$

Thus $(m, \lambda\cdot f(m)) \in$ graph $(\lambda\cdot f)$ for all $m \in L(x,y)$. Hence $\lambda\cdot f$ is a linear function.

Proof of (2). Let $x \neq y \in X$, then

$$(x, (f+g)(x)),\ (y, (f+g)(x)) \in \text{graph } (f+g).$$

The line joining these two points has points of the form

$$\left(m, \frac{d(x,m)}{d(x,y)}(f+g)(y) + \frac{d(m,y)}{d(x,y)}(f+g)(x)\right)$$

$$= (m, \frac{d(x,m)}{d(x,y)} f(y) + \frac{d(m,y)}{d(x,y)} f(x) + \frac{d(x,m)}{d(x,y)} g(y) + \frac{d(m,y)}{d(x,y)} g(x))$$

where $m \in L(x,y)$, and since graph f and graph g are flats

$$(m, f(m) + g(m)) = (m, (f + g)(m)) \in \text{graph } (f + g).$$

Thus $f + g$ is a linear function, completing the proof.

Now X^* is a real module, so that X^* is convex-isomorphic to $(Y,\mathcal{B})$ where $\mathcal{B}$ is a convexity stru~ture on Y if and only if Y is also a real module (Shirley [7]). Hence, the following result is established.

2.12. THEOREM. X^{**} is convex-isomorphic to X if and only if X is a real module.

Thus we see that the conditions in the definition of "linear", which at first seem relatively general, are indeed quite restrictive, as we obtain the classical situation.

We observe examples of a G.L.S. X with regard to axiom C in $X \times \mathbb{R}$.

EXAMPLE (1). Let X be a real module with usual lines. If the order-isomorphisms are induced by the usual coordinate projections, then $X \times \mathbb{R}$ satisfies axiom C.

EXAMPLE (2). Let X be a moulton plane. If the order-isomorphisms are again induced by the usual coordinate projections which on the broken lines will be piecewise linear, then $X \times \mathbb{R}$ does not satisfy axiom C on triangles whose corresponding triangles in X have at least one broken edge.

As a specific example, let $a = (0,0)$, $b = (4,1)$, $c = (0,-2)$ be three distinct points in X, $u \in bc$, $v \in au$ be such that $u = (2,0)$, $v = (1,0)$. Let $w = (0,-2/3)$ so that $v \in wb$ and $w \in ac$ in order to satisfy axiom C in X. For L(a,c) let π be first coordinate projection; for the others let π be second coordinate projection. Consider three points in $X \times \mathbb{R}$

$a' = (a,0)$, $b' = (b,4)$, $c' = (c,-2)$. The line containing b', c' has points of the form $(m, (3/2)\pi(m) - 2)$ for $m \in L(b,c)$ so that $u' = (u,1) \in L(b',c')$. Similarly $v' = (v,1/2) \in L(a',u')$. Now the line containing v', b' has points of the form $(m, (7/6)\pi(m) - 2/3)$ for $m \in L(v,b)$. Thus $w' = (w,-13/9) \in L(v',b')$. But $w' \notin L(a',c')$ as w would correspond to the point $(w,-2/3) \in L(a',c')$. Hence axiom C is not satisfied by triangle $(a,0)$, $(b,4)$, $(c,-2)$.

<u>EXAMPLE (3)</u>. If X is an open convex set in E^2 with usual lines where the order-isomorphisms are natural homeomorphisms between open intervals and the reals, then $X \times \mathbb{R}$ does not satisfy axiom C.

As a specific example let X be the interior of the square whose vertices are $(\pm 4, \pm 4)$. Let $a = (0,0)$, $b = (2,2)$, and $c = (2,0)$ with $u = (2,1)$, $v = (1,1/2)$ and $w = (2/3,0)$ so that $u \in bc$, $v \in au$ and $v \in wb$, $w \in ac$. For $L(b,c)$ let π be second coordinate projection followed by $\tau:(-4,4) \to \mathbb{R}$ where $\tau(t) = t/\sqrt{16 - t^2}$; for the others let π be first coordinate projection followed by τ. Consider three distinct points in $X \times \mathbb{R}$, $a' = (a,0)$, $b' = (b,2/\sqrt{3})$, and $c' = (c,1/\sqrt{3})$. The corresponding u', v' are then $u' = (u, (1 + \sqrt{5})/\sqrt{15})$ and $v' = (v, (1 + \sqrt{5})/(2\sqrt{15}))$. On the line $L(u',v')$ the correspondent to w is $w' = (w, 2/(3\sqrt{15}))$ whereas on $L(a',c')$, w corresponds to $w'' = (w, 1/\sqrt{35})$. Hence axiom C is not satisfied by triangle a',b',c'.

We now consider what bearing axiom C in $X \times \mathbb{R}$ has on properties of X. As we shall see, quite strong consequences can be developed. We shall assume that axiom C holds throughout the results 2.13. through 2.17. below.

2.13. LEMMA. Let x,y,z be non-collinear points in X, $\lambda \in \mathbb{R}$ where $u \in (xy)$, $v \in (uz)$, and $m \in (yz)$ such that $v \in (xm)$.

(a) If $d(u,y) = \lambda d(x,y)$, $d(v,u) = \lambda d(z,v)$ then $d(v,m) = \lambda d(x,m)$ and $d(m,y) = \lambda d(z,y)$.

(b) If $d(u,y) = \lambda d(x,y)$, $d(m,y) = \lambda d(z,y)$ then $d(v,u) = \lambda d(z,v)$ and $d(v,m) = \lambda d(x,v)$.

Proof. Consider the points $(x,0)$, $(y,1)$, $(z,2)$ in $X \times \mathbb{R}$. Then by Lemma 2.4. we have the $(u,1-\lambda) \in (x,0)(y,1)$ and $(v,1) \in (u,1-\lambda)(z,2)$. Now by axiom C in $X \times \mathbb{R}$, there exists μ such that $(m,\mu) \in (y,1)(z,2)$ and $(v,1) \in (x,0)(m,\mu)$.

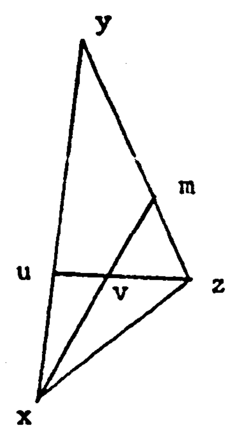

Figure 2.

Thus by definition of lines in $X \times \mathbb{R}$ we have for $L(y,z)$ and $L(x,v)$ respectively

$$(*) \qquad \begin{aligned} \mu &= \frac{d(z,m)}{d(z,y)}\cdot 1 + \frac{d(m,y)}{d(z,y)}\cdot 2 \\ \mu &= \frac{d(x,m)}{d(x,v)}\cdot 1 + \frac{d(m,v)}{d(x,v)}\cdot 0. \end{aligned}$$

Considering the points $(x,0)$, $(y,1/(1-\lambda))$, $(z,1)$ in $X \times \mathbb{R}$ we have by Lemma 2.4. that

$$(u,1) \in (x,0)(y, \frac{1}{1-\lambda}), \quad (v,1) \in (u,1)(z,1).$$

By axiom C in $X \times \mathbb{R}$, there exists μ' such that

$$(m,\mu') \in (y, \frac{1}{1-\lambda})(z,1), \quad (v,1) \in (m,\mu')(x,0).$$

Thus for $L(y,z)$ and $L(x,v)$ respectively we have

$$(**) \quad \begin{aligned} \mu' &= \frac{d(z,m)}{d(z,y)} \cdot \frac{1}{1-\lambda} + \frac{d(m,y)}{d(z,y)} \cdot 1 \\ \mu' &= \frac{d(x,m)}{d(x,v)} \cdot 1 + \frac{d(m,v)}{d(x,v)} \cdot 0 \end{aligned}$$

Now the second equations in (*) and (**) refer to $L(x,v)$ and yield that $\mu = \mu'$. The first equations refer to $L(y,z)$ and yield

$$d(z,m) + 2d(m,y) = \frac{1}{1-\lambda} d(z,m) + d(m,y)$$

or equivalently

$$d(m,y) = \lambda d(z,y).$$

Thus $\mu = (1-\lambda) + 2\lambda = 1 + \lambda$ from equation 1 in (*), so by the second equation in (**)

$$1 + \lambda = \frac{d(x,m)}{d(x,v)}$$

or equivalently

$$d(v,m) = \lambda d(x,v).$$

Hence (a) is completed.

(b) We shall prove the left equation; symmetry will yield the right equation. Choose $v' \in L(u,z)$ such that $d(v',u) = \lambda d(z,v')$. Since $0 \leq \lambda \leq 1$, $v' \in uz$, so by axiom C there exists $m' \in yz$ such that $v' \in xm'$. By (a) $d(v',m') = \lambda d(x,v')$ and $d(m',y) = \lambda d(z,y)$. But $d(m,y) = \lambda d(z,y)$ so that $d(m',y) = d(m,y)$ or $m = m'$. Since lines joining two points are unique $v = v'$. Hence, $d(v,u) = \lambda d(z,v)$ and

$d(v,m) = \lambda d(x,v)$, (b) is completed.

REMARK: We note that in (a) for $\lambda = 1/2$, (that is when u is the midpoint of x and y and v is 1/3 the distance from u to z) then v is 1/3 the distance from m to x and m is the midpoint of y and z. In (b) for $\lambda = 1/2$ we obtain the following familiar geometric theorem.

2.14. THEOREM. The medians of a triangle are concurrent at a point which is 1/3 the distance on the median from the midpoint of a side to the vertex opposite that side.

2.15. LEMMA. Let x, y, z be non-collinear points , $x', y', m, m' \in X$ such that $x' \in (xy)$, $z' \in (zy)$, $m' \in (x'y')$, $m \in (xz)$, and $m' \in (my)$. If for $\lambda, \alpha \in \mathbb{R}$, $d(x',y) = \lambda d(x,y)$, $d(z',y) = \lambda d(z,y)$ and $d(x',m') = \alpha d(x',y')$, then $d(x,m) = \alpha d(x,z)$ and $d(m',y) = \lambda d(m,y)$.

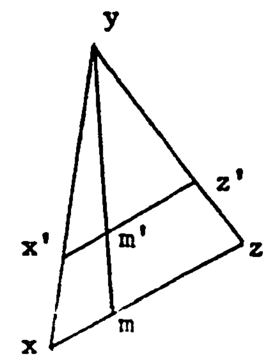

Figure 3.

Proof. Consider the points $(x,1)$, $(y,0)$, (z,β) in $X \times \mathbb{R}$ where $\beta = \frac{\alpha - 1}{\alpha}$. Then by Lemma 2.4. $(x',\lambda) \in (x,1)(y,0)$, $(z', \lambda\beta) \in (y,0)(z,\beta)$, $(m',0) \in (x',\lambda)(z', \lambda\beta)$ and $(m',0) \in (y,0)(m,0)$. But since $X \times \mathbb{R}$ is a G.L.S. $(m,0) \in (x,1)(z,\beta)$ so that by Lemma 2.4. $d(x,m) = \alpha d(x,z)$.

For the second equation we consider the points $(x,1)$, $(y,0)$, $(z,1)$ in $X \times \mathbb{R}$. By Lemma 2.4. we have $(x',\lambda) \in (x,1)(y,0)$, $(z',\lambda) \in (y,0)(z,1)$. $(m',\lambda) \in (x',\lambda)(z',\lambda)$ and $(m,1) \in (x,1)(z,1)$. But since $X \times \mathbb{R}$ is a G.L.S. $(m',\lambda) \in (m,1)(y,0)$ so that by Lemma 2.4. $d(m',y) = \lambda d(m,y)$.

We are now able to define two operations $+ : X \times X \to X$ and $\cdot : \mathbb{R} \times X \to X$ so that X becomes a vector space over $\mathbb{R}$ with respect to $+,\cdot$.

Let θ be a fixed element of X. If $x = \theta$ let d be the distance function of an arbitrary line through θ so that $d(\theta,x) = 0$. If $x \neq \theta$ then d is as in Definition 2.2., $d(\theta,x) = \pi(x) - \pi(\theta)$ where π is an order-isomorphism of $L(\theta,x)$ to $\mathbb{R}$.

2.16. DEFINITION. For each $x, y \in X$ define the midpoint m of xy to be the unique point m on $L(x,y)$ such that $d(x,m) = (1/2)d(x,y)$ if $x \neq y$ and $m = x$ if $x = y$. Define $x + y$ to be the unique point z such that $z \in L(\theta,m)$ and $d(\theta,z) = 2d(\theta,m)$, where m is midpoint of xy.

2.17. DEFINITION. For each $x \in X$, $\lambda \in \mathbb{R}$ define $\lambda \cdot x$ to be the unique $y \in L(\theta,x)$ such that $d(\theta,y) = \lambda d(\theta,x)$.

We need to show that $(X; +, \cdot)$ is a left $\mathbb{R}$ module. The abelian group properties for $(+)$ are clear except for associativity; thus, we shall check the following for all $\lambda, \mu \in \mathbb{R}$, x, y, z $\in$ X:

(i) $\lambda\cdot(\mu\cdot x) = (\lambda\mu)\cdot x$

(ii) $1 \cdot x = x$

(iii) $(\lambda + \mu) \cdot x = \lambda \cdot x + \mu \cdot x$

(iv) $\lambda \cdot (x + y) = \lambda \cdot x + \lambda \cdot y$

(v) $(x + y) + z = x + (y + z)$.

Proof of (i). Now $\mu \cdot x = y$ if $y \in L(\theta,x)$ such that $d(\theta,y) = \mu d(\theta,x)$, and $\lambda \cdot(\mu\cdot x) = \lambda \cdot y = z$ if $z \in L(\theta,y) = L(\theta,x)$ such that $d(\theta,z) = \lambda d(\theta,y)$. Thus $d(\theta,z) = \lambda(\mu d(\theta,x)) = (\lambda\mu)d(\theta,x)$ or

$$\lambda\cdot (\mu\cdot x) = z = (\lambda\mu) \cdot x.$$

Proof of (ii). Now $1 \cdot x = y$ if $y \in L(\theta,x)$ such that $d(\theta,y) = 1d(\theta,x)$. But by properties of d, $d(\theta,y) = d(\theta,x)$ implies $y = x$. Thus

$1 \cdot x = x.$

Proof of (iii). $(\lambda + \mu)\cdot x = \lambda\cdot x + \mu\cdot x.$ We first note that $(\lambda + \mu)\cdot x,\ \lambda\cdot x,\ \mu\cdot x \in L(\theta,x)$. Now $\lambda \cdot x + \mu \cdot x = z$ if $z \in L(\theta,x)$ such that $d(\theta,z) = 2d(\theta,m)$ where m is midpoint of $(\lambda\cdot x)(\mu\cdot x)$. That is, $d(\lambda x,m) = (1/2)d(\lambda\cdot x,\mu\cdot x)$. But

$$d(\lambda\cdot x,\mu\cdot x) = d(\lambda\cdot x,\theta) + d(\theta,\mu\cdot x)$$

and
$$d(\theta,m) = d(\theta,\lambda\cdot x) + d(\lambda\cdot x,m)$$

so that

$$\begin{aligned} d(\theta,z) = 2d(\theta,m) &= 2d(\theta,\lambda\cdot x) + 2d(\lambda\cdot x,m) \\ &= 2d(\theta,\lambda\cdot x) + \left[d(\lambda\cdot x,\theta) + d(\theta,\mu\cdot x)\right] \\ &= d(\theta,\lambda\cdot x) + d(\theta,\mu\cdot x) \\ &= (\lambda + \mu)d(\theta,x). \end{aligned}$$

Thus $z = (\lambda + \mu)\cdot x = \lambda \cdot x + \lambda \cdot \mu$.

Proof of (iv). $\lambda\cdot(x + y) = \lambda\cdot x + \lambda\cdot y$. Let m, m' be midpoints of x and y , $\lambda\cdot x$ and $\lambda\cdot y$ respectively. If $\lambda = 0$ or $x = y$, the assertion is clear. By Lemma 2.15. $d(\theta,m') = \lambda d(\theta,m)$ and $m' \in L(\theta,m)$. Thus $\lambda\cdot(x + y)$, $\lambda\cdot x + \lambda\cdot y \in L(\theta,m)$ and $d(\theta,\lambda\cdot(x + y)) = \lambda d(\theta, x + y) = \lambda\cdot 2d(\theta,m) = 2d(\theta,m') = d(\theta,\lambda\cdot x + \lambda\cdot y)$.

Hence $\lambda\cdot(x + y) = \lambda\cdot x + \lambda\cdot y$.

Proof of (v). $(x + y) + z = x + (y + z)$. The proof is the same as the proof given in Kay [4] in a different setting, but will be included for the purpose of completeness. Assume none of $x, y, z = \theta$, for otherwise, the assertion is trivial.

Case 1. $x = y = z$. $x + (x + x) = (x + x) + x$ by commutativity.

Case 2. If x, y, z, θ are collinear then assertion follows from (iii).

Case 3. $x \neq y = z$. Let m be the midpoint of xy, m' the midpoint of y and

$x + y$, and m_1 the midpoint of x and $2y$. Let $r \in L(\theta, m') \cap L(x,y)$, $s \in L(\theta, m_1) \cap L(x,y)$. Considering θ, $2y$, x we have by Lemma 2.13. $d(y,s) = (1/3)d(y,x)$. For θ, $x + y$, y we have by Lemma 2.13. $d(m,r) = (1/3)d(m,y)$. Since $d(y,m) = (1/2)d(y,x)$ we have $d(y,s) = (1/3)d(y,x) = (1/3)\cdot 2d(y,m)$ or equivalently $d(m,s) = (1/3)d(m,y)$. Hence $r = s$, and $m_1 = m'$ by Lemma 2.3. and assertion follows.

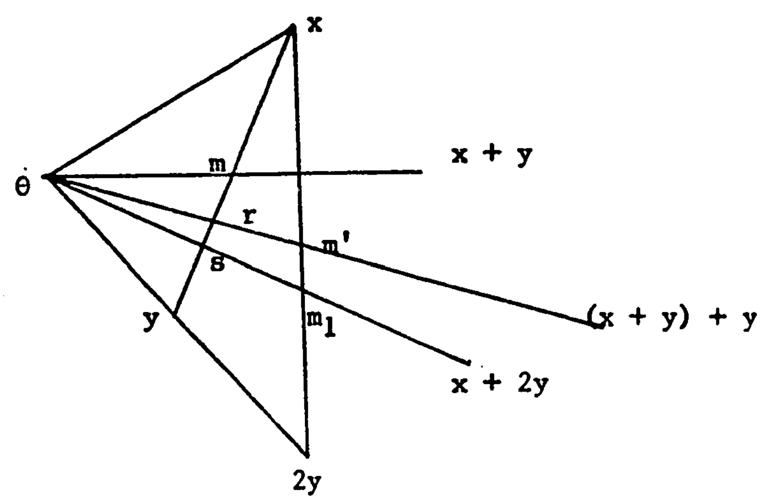

Figure 4.

<u>Case 4</u>. $x = y \neq z$

$$(x + y) + z = z + (x + y) = z + (y + x) = (z + y) + x$$
$$= x + (z + y) = x + (y + z).$$

<u>Case 5</u>. $x = z \neq y$ Similar to (4).

<u>Case 6</u>. Suppose x, y, z are distinct, including when x, y, z pairwise collinear with θ. Let m be the midpoint of xy, m_1 the midpoint of $x + y$ and z, m' the midpoint of y and z, and m_1' the midpoint of x and $y + z$.

Considering θ, x, $y + z$, we have by Lemma 2.13. that $2d(r,m') = d(x,r)$. Considering θ, z, $x + y$, we have $2d(s,m) = d(z,s)$. Considering x, y, z we have $2d(t,m) = d(z,t)$ and $2d(t,m') = d(x,t)$. Combining the first and the last we have $r = t$ and the middle two yield $s = t$. Thus

$r = t = s$ and by Lemma 2.13. $m_1 = m_1'$ so that $x + (y + z) = (x + y) + z$.

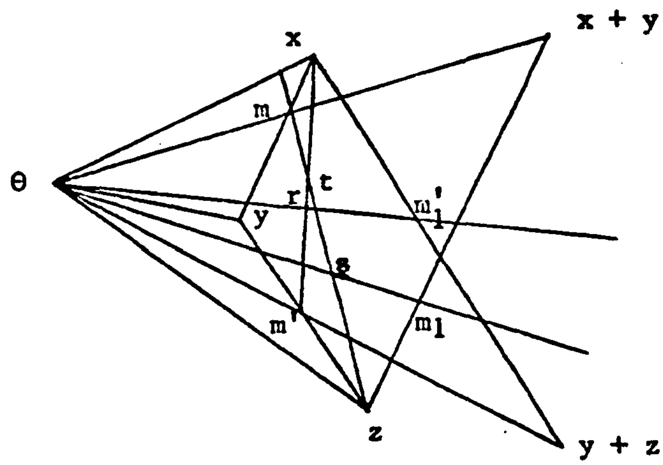

Figure 5.

Thus under the assumption that X and $X \times \mathbb{R}$ are G.L.S.'s we see that (X; +, ·) is a real vector space. However, if X is a real vector space, then $X \times \mathbb{R}$ is a real vector space. So we have:

<u>2.18. THEOREM.</u> $X \times \mathbb{R}$ is a G.L.S. if and only if X is a vector space.

Since $X \times \mathbb{R}$ is a real vector space whenever X is a vector space, we have:

<u>2.19. COROLLARY.</u> Axiom C in $X \times \mathbb{R}$ is equivalent to the vector space axioms in $X \times \mathbb{R}$.

We now investigate a generalization of products of the type $X \times \mathbb{R}$. It is possible to generalize all the results for $X \times \mathbb{R}$ to $X \times V$ where V is any vector space. But a further generalization is possible from which the results for $X \times V$ can be obtained as a special case. Consider a product $X \times Y$ for two G.L.S.'s X and Y with families of lines $\mathcal{L}$, $\mathcal{L}'$ respectively and d, d' the respective distance functions.

2.20. DEFINITION. A line in $X \times Y$ is any set of one of the following three types $(x, y \in X,\ x', y' \in Y)$:

$$\{x\} \times L'(x',y'),\quad L(x,y) \times \{x'\}$$

$$\{ (m,m') \mid m \in L(x,y),\ m' \in L'(x',y'),\ \frac{d(x,m)}{d(x,y)} = \frac{d'(x',m')}{d'(x',y')} \}.$$

REMARK. By the order-isomorphisms of X and Y, for each $m \in L(x,y)$, $x \neq y$, there exists $m' \in L(x',y')$, $x' \neq y'$ such that

$$\frac{d(x,m)}{d(x,y)} = \frac{d'(x',m')}{d'(x',y')} \quad \text{and conversely.}$$

Let us now consider the structure of $X \times Y$ as it pertains to the axioms of a G.L.S.

The lines of $X \times Y$ have a natural order as follows: For lines of the form $\{m\} \times L'$ define $(m,x') \leq (m,y')$ if and only if $d'(y',x') \geq 0$. For lines of the form $L \times \{m'\}$ define $(x,m') \leq (y,m')$ if and only if $d(y,x) \geq 0$. For lines of the third type $L((x,x'),(y,y'))$ define $(z,z') \leq (m,m')$ if and only if $d(m,z) \geq 0$. From the properties of d and d' these lines are then order-isomorphic to the reals. Hence axiom A is satisfied.

Let $(x,x'), (y,y') \in X \times Y$. If $x = y$ then the line $\{x\} \times L'(x',y')$ is a line joining (x,x') and (y,y'). By definition of the other types of lines this is the only possibility, and it is unique since $L'(x',y')$ is unique . If $x' = y'$ then then above applies to the line $L(x,y) \times \{x'\}$. Thus suppose $x \neq y$ and $x' \neq y'$. The line joining (x,x') and (y,y') has points of the form (m,m') where $m \in L(x,y)$, $m' \in L'(x',y')$ and $\frac{d(x,m)}{d(x,y)} = \frac{d'(x',m')}{d'(x',y')}$. Since $L(x,y)$ and $L'(x',y')$ are both unique, $L((x,x'), (y,y'))$ is unique. Hence axiom B is satisfied.

By our previous examples we know that $X \times Y$ need not satisfy axiom C.

2.21. LEMMA. Let $(m,m') \in L((x,x'),(y,y'))$ $x \neq y$, $x' \neq y'$. Then

$$\frac{d(x,m)}{d(x,y)} = \frac{d'(x',m')}{d'(x',y')} \text{ if and only if } \frac{d(m,y)}{d(x,y)} = \frac{d'(m',y')}{d'(x',y')}.$$

Proof. Immediate from properties of d and d'.

2.22. LEMMA. Let $(m,m') \in L((x,x'),(y,y'))$ $x \neq y$, $x' \neq y'$, $\lambda \in \mathbb{R}$. Then $d(x,m) = \lambda d(x,y)$ if and only if $d'(x',m') = \lambda d'(x',y')$.

Proof. Since $L((x,x'),(y,y'))$ is of type three, the equivalence is immediate from definition.

2.23. DEFINITION. Let (x,x'), (y,y'), and (z,z') be any three collinear points in $X \times Y$. Then (y,y') is said to be *between* (x,x') and (z,z') if and only if either y is between x and z in X, or $x = y = z$ and y' is between x' and z' in Y.

Following the usual manner in which flats are defined in terms of lines, we adopt the following.

2.24. DEFINITION. A *flat* in $X \times Y$ is a set F such that for all $p,q \in F$ then $L(p,q) \subset F$. A flat H is called a *hyperplane in* $X \times Y$ if and only if H is a maximal proper flat in $X \times Y$. If $A \quad X \times Y$ then the *flat spanned* by A is $fl(A) = \bigcap\{F \mid A \subset F,\ F \text{ is a flat in } X \times Y\}$.

2.25. DEFINITION. A mapping $f: X \to Y$ is called *linear* if and only if the graph $f = \{(x,f(x)) \mid x \in X\}$ is a flat in $X \times Y$.

2.26. THEOREM. The graph of every linear function f from X to Y is a hyperplane in $X \times Y$.

Proof. The proof follows the argument of Theorem 2.10. Suppose there exists H a flat in $X \times Y$ such that graph $f \subsetneq H \subset X \times Y$. Let

$(x,x') \in X \times Y \setminus H$. We will show $(x,x') \in H$. Let $(y,y') \in H \setminus \text{graph } f$ such that $\alpha = d'(x',f(x)) + d'(f(y),y') \neq 0$. Let $z \in L(x,y)$ such that

$$\pi(z) = \frac{d'(x',f(x))}{\alpha}\pi(y) + \frac{d'(f(y),y')}{\alpha}\pi(x).$$

If $x = y$, then let $z' \in L'(y',f(x))$. If $x \neq y$ then there exists z' such that

$$\frac{d(x,z)}{d(x,y)} = \frac{d'(x',z')}{d'(x',f(y))} = \frac{d'(f(x),z')}{d'(f(x),y')}$$

or $(z,z') \in L((x,x'),(y,f(y))) \cap L((x,f(x)),(y,y'))$. Since $L((x,f(x)),(y,y')) \subset H$ we have $(z,z') \in H$. Hence $L((z,z'),(y,y')) \subset H$ so that $(x,x') \in H$. Thus $H = X \times Y$.

The definition of linear function from X to Y did not require that X and Y both satisfy axiom C. We consider then two theorems relating the concept of linearity and axiom C.

<u>2.27. THEOREM.</u> Let X be a G.L.S. and Y satisfy only axioms A and B. If f is a linear onto mapping from X to Y, then Y is a G.L.S.

<u>Proof.</u> Let $x',y',z',u',v' \in Y$ such that $u' \in (x'y')$, $v' \in (u'z')$. Since F is onto there exist $x,y,z,u,v \in X$ such that $f(x) = x'$, $f(y) = y'$, $f(z) = z'$, $f(u) = u'$, $f(v) = v'$ and by Lemma 2.22. $u \in (xy)$, $v \in (uz)$. Now by axiom C in X there exists $w \in (xz)$ such that $v \in (wy)$. Since graph f is a flat

$$\frac{d(x,w)}{d(x,z)} = \frac{d'(x',f(w))}{d'(x',z')} \quad \text{and} \quad \frac{d(y,w)}{d(y,v)} = \frac{d'(y',f(w))}{d'(y',v')}.$$

Thus $f(w) \in (x'z')$ and $v' \in (f(w)y')$. Now Y clearly satisfies the special cases of axiom C, so Y is a G.L.S.

<u>2.28. THEOREM.</u> Let X satisfy only axioms A and B, Y be a G.L.S. If f is linear, one-to-one mapping from X to Y, then X is a G.L.S.

<u>Proof</u>. Let $x, y, z, u, v \in X$ such that $u \in (xy)$, $v \in (uz)$. Then for distinct x, y, z, f being linear and one-to-one yields $f(x)$, $f(y)$, $f(u)$, $f(v)$, $f(z)$ distinct. By Lemma 2.22. and graph f is flat we obtain $u' \in (x'y')$, $v' \in (u'z')$. But axiom C in Y yields $w' \in (x'z')$ such that $v' \in w'y'$. By Lemma 2.22. and graph f is a flat, we have there exists $w \in (xz)$ such that $f(w) = w'$, and $v \in (wy)$. Since X satisfies the special cases of axiom C, we have that X is a G.L.S.

We now investigate the structure of X and Y when X, Y, and $X \times Y$ are G.L.S.

<u>2.29. LEMMA</u>. Let x, y, z be non-collinear points in X, and $u,v,w,m \in X$ such that $u \in (xy)$, $v \in (yz)$, $m \in (uv)$, $w \in (xz)$, and $m \in (yw)$. If $\frac{d(u,y)}{d(x,y)} = \lambda = \frac{d(v,y)}{d(z,y)}$, and $d(u,m) = d(m,v)$ then $d(x,w) = d(w,z)$ and $d(m,y) = \lambda d(w,y)$.

<u>Proof</u>. Consider the points (x,y'), (y,y'), (z,z'), $y' \neq z'$ in $X \times Y$. Now by properties of d' in Y, there exists $v' \in (y'z')$ such that $d'(v',y') = d'(z',y')$, so that by definition of lines $(v,v') \in (y,y')(z,z')$. Also there exists $m' \in (y'v')$ such that $d'(y',m') = d'(m',v')$. Since $(u,y') \in (x,y')(y,y')$, we have by definition of lines that $(m,m') \in (u,y')(v,v')$. $X \times Y$ is a G.L.S. so that there exists $(w,w') \in L((x,y'),(z,z')) \cap L((m,m'),(y,y'))$ such that

$$(*) \quad \frac{d(x,w)}{d(x,z)} = \frac{d'(y',w')}{d'(y',z')} \quad \text{and} \quad \frac{d(y,m)}{d(y,w)} = \frac{d'(y',m')}{d'(y',w')}.$$

Considering the points (x,z'), (y,y'), (z,y'), we obtain in like manner $(w,w'') \in L((x,z'),(z,y')) \cap L((m,m'),(y,y'))$ such that

$$(**) \quad \frac{d(x,w)}{d(x,z)} = \frac{d'(z',w'')}{d'(z',y')} \quad \text{and} \quad \frac{d(y,m)}{d(y,w)} = \frac{d'(y',m')}{d'(y',w'')}.$$

Comparing the second two equations of (*), (**) respectively we have that $w' = w''$. Then the first two equations yield that $d'(y',w') = d'(w',z')$ from which, by Lemma 2.22, we have that $d(x,w) = d(w,z)$.

To establish $d(m,y) = \lambda d(w,y)$ we consider the points (x,x'), (y,y'), (z,z'), $x' \neq y'$. Now there exists $u' \varepsilon (x'y')$ such that $d'(u',y') = \lambda d'(x',y')$. Since $d(u,y) = \lambda d(x,y)$ we have by definition of lines that $(u,u') \varepsilon (x,x')(y,y')$ and since $d(v,y) = \lambda d(z,y)$ we have $(v,u') \varepsilon (y,y')(z,x')$. Now $(m,u') \varepsilon (u,u')(v,u')$ and $(w,x') \varepsilon (x,x')(z,x')$ so that since $X \times Y$ is a G.L.S. $(m,u') \varepsilon (w,x')(y,y')$. By definition of lines $\frac{d(m,y)}{d(w,y)} = \frac{d'(u',y')}{d'(x',y')}$ but $d'(u',y') = \lambda d'(x',y')$ so that $d(m,y) = \lambda d(w,y)$, completing the proof.

We observe that the same arguments hold for Y so we have:

2.30. COROLLARY. Let x',y',z' be non-collinear points in Y, and $u',v',w',m' \varepsilon Y$ such that $u' \varepsilon (x'y')$, $v' \varepsilon (y'z')$, $m' \varepsilon (u'v')$, $w' \varepsilon (x'z')$ and $m' \varepsilon (y'w')$. If $\frac{d'(u',y')}{d'(x',y')} = \lambda = \frac{d'(v',y')}{d'(z',y')}$, and $d'(u',m') = d'(m',v')$ then $d'(x',w') = d'(w',z')$, and $d'(m',y') = \lambda d'(w',y')$.

2.31. LEMMA. Let x,y,z be non-collinear points in X, $u,m,v \varepsilon X$ such that $u \varepsilon (xy)$, $v \varepsilon (yz)$, and $m \varepsilon (xv) \cap (uz)$. If $d(x,u) = d(u,y)$ and $d(z,v) = d(v,y)$ then $d(u,m) = (1/2)d(m,z)$ and $d(v,m) = (1/2)d(m,x)$.

Proof. Consider the points (x,x'), (y,y'), (z,z') in $X \times Y$ where $y' \neq z'$ and x' is midpoint of $y'z'$. Then $(v,x') \varepsilon (y,y')(z,z')$ and $(m,x') \varepsilon (x,x')(v,x')$. Since $X \times Y$ is a G.L.S. there exists u' such that $(u,u') \varepsilon (x,x')(y,y') \cap (m,x')(z,z')$. By Lemma 2.22.

$$(*) \quad \frac{d(u,m)}{d(m,z)} = \frac{d'(u',x')}{d'(x',z')} \quad \text{and} \quad \frac{d(x,u)}{d(x,y)} = \frac{d'(x',u')}{d'(x',y')} = \frac{1}{2}$$

Now $d'(x',z') = d'(y',x')$ so that

$$\frac{1}{2} = \frac{d'(x',u')}{d'(x',y')} = \frac{d'(x',u')}{-d'(x',z')} = \frac{d'(x',u')}{d'(x',z')} = \frac{d(u,m)}{d(m,z)} .$$

By symmetry we have $d(v,m) = (1/2)d(m,x)$, completing the proof.

2.32. COROLLARY. Let x', y', z' be non-collinear points in Y, u', m', $v' \in Y$ such that $u' \in (x'y')$, $v' \in (y'z')$ and $m' \in (u'z')(x'v')$. If $d'(x',u') = d'(u',y')$ and $d'(z',v') = d'(v',y')$ then $d'(u',m') = (1/2)d'(m',z')$ and $d'(v',m') = (1/2)d'(m',x')$.

Since the definitions 2.16. and 2.17. made use of only the distance function on a G.L.S., we have definitions for $+$, $\cdot$ on X and Y respectively, and can prove the following.

2.33. THEOREM. If X, Y, $X \times Y$ are each a G.L.S. then X and Y are real vector spaces.

2.34. COROLLARY. If V is a real vector space, X and $X \times V$ are each a G.L.S., then X is a real vector space.

The discussion of the product of X and Y each of which is a G.L.S. was motivated by considering the structure of $X \times \mathbb{R}$ in formulating a definition of linear function. We conclude with the following.

CONJECTURE. Let f be a linear mapping from X to Y. If X, Y, and $X \times Y$ are each a G.L.S. then f is onto or trivial.

CHAPTER III

GENERALIZED LINEAR SPACE VIA PROPERTIES ON THE GENERALIZED DUAL

We now consider the question of what type of properties on a set of real-valued functions are necessary or desirable in order to describe geometric properties in the domain set. Ky Fan [2] considered a similar situation, but was concerned with properties which would require the functions to be lower-semi continuous in reference to a convexity structure.

Throughout the discussion X will be an arbitrary non-empty set and $\mathfrak{F}$ will be a non-empty subset of the generalized dual of X, the family of all functions from X to $\mathbb{R}$, with usual operations. We shall consider the following properties of X and $\mathfrak{F}$ and their effect on X:

P1. $\mathfrak{F}$ is element distinguishing; that is, for all $x, y \in X$ $x \neq y$ there exists $f \in \mathfrak{F}$ such that $f(x) \neq f(y)$.

P2. $\mathfrak{F}$ is a real module under usual function addition and real multiplication.

P3. For each $f \in \mathfrak{F}$, if for some $x, y \in X$ and $\alpha \in \mathbb{R}$, $f(x) < \alpha < f(y)$, then there exists $z \in f^{-1}(\alpha)$ such that $z \in g^{-1}[g(x),g(y)]$ for each $g \in \mathfrak{F}$.

P4. For all $x, y \in X$, there exists $z \neq x, y$ such that $y \in g^{-1}[g(x),g(z)]$ for each $g \in \mathfrak{F}$.

P5. For $x, y \in X$ and $\alpha \in \mathbb{R}$ $\cap\{f^{-1}(\alpha) \mid f \in \mathfrak{F}$ and $f(x) = f(y) = \alpha\} \neq X$.

NOTATION. For $\lambda, \mu \in \mathbb{R}$, $[\lambda,\mu] \equiv \{\xi \mid \lambda \leq \xi \leq \mu$ or $\lambda \geq \xi \geq \mu\}$.

REMARK. By way of summary we could say that the first two properties above pertain to the family of functions $\mathfrak{F}$, whereas the next two properties are combined properties of $\mathfrak{F}$ and X. The last merely requires X to be at least 2-dimensional.

EXAMPLES. Let X be a real vector space. Examples of $\mathfrak{F}$ are:

(1) $\mathfrak{F}_1$, the family of linear functionals,

(2) $\mathfrak{F}_2$, the family of convex functionals.

We now consider a set mapping σ which is shown to be a segment operator; that is, for $x, y \in X$

(i) $\{x,y\} \subset \sigma(x,y)$,

(ii) $\{u,v\} \subset \sigma(x,y)$ implies $\sigma(u,v) \subset \sigma(x,y)$.

3.1. DEFINITION. Define the map $\sigma : X \times X \to P(X)$ as follows: $(x,y) \mapsto (x,y) = \cap\{f^{-1}[f(x),f(y)] \mid f \in \mathfrak{F}\}$.

3.2. LEMMA. $\{x,y\} \subset \sigma(x,y)$.

Proof. Clear since $\{x,y\} \subset f^{-1}[f(x),f(y)]$ for all $f \in \mathfrak{F}$.

3.3. LEMMA. $\{u,v\} \subset \sigma(x,y)$ implies $\sigma(u,v) \subset \sigma(x,y)$.

Proof. Let $w \in \sigma(u,v)$ so that $f(w) \in [f(u),f(v)]$ for all $f \in \mathfrak{F}$. Now since $\{u,v\} \in \sigma(x,y)$ we have $\{f(u),f(v)\} \subset [f(x),f(y)]$ for all $f \in \mathfrak{F}$. Thus $[f(u),f(v)] \subset [f(x),f(y)]$ for all $f \in \mathfrak{F}$. Hence

$$f(w) \in [f(u),f(v)] \subset [f(x),f(y)] \qquad \text{for all } f \in \mathfrak{F},$$

or $$w \in f^{-1}[f(x),f(y)] \qquad \text{for all } f \in \mathfrak{F}.$$

Thus $w \in \sigma(x,y)$ and $\sigma(u,v) \subset \sigma(x,y)$.

Since σ is a segment operator, we can obtain a natural convexity structure on X.

3.4. DEFINITION. Let $A \subset X$. A set A is called convex if and only if $\sigma(x,y) \subset A$ for all $x, y \in A$. $\mathcal{C} = \{A \subset X \mid A \text{ is convex}\}$ is called a convexity structure on X.

REMARK. We note that as a result of property (ii) above for segment operators, each segment $\sigma(x,y)$ is convex. Moreover, $\mathcal{C}$ is a T_1 convexity structure (that is, $\{x\} \in \mathcal{C}$ for all $x \in X$) since $\mathfrak{F}$ is element distinguishing and $\sigma(x,x) = \{x\}$ for each x.

Now $\mathcal{C}$ has an associated hull-operator H, defined by

$$H(C) = \bigcap \{A \mid A \in \mathcal{C}, C \subset A\}, \qquad C \subset X.$$

3.5. THEOREM. $H(\{x,y\}) = \sigma(x,y)$.

Proof. Since $H(\{x,y\}) = \bigcap\{A \in \mathcal{C} \mid \{x,y\} \subset A\}$ and $\{x,y\} \subset \sigma(x,y) \in \mathcal{C}$, we have that $H(\{x,y\}) \subset \sigma(x,y)$. Now $\sigma(x,y) \subset A$ for all $A \in \mathcal{C}$ such that $\{x,y\} \subset A$, so that $\sigma(x,y) \subset H(\{x,y\})$. Hence we have the desired equality.

NOTATION. From this point on we use more standard notation:

$\sigma(x,y) = xy$, $(xy = \sigma(x,y) \setminus \{x\}$

xyz if and only if $y \in \sigma(x,z)$.

3.6. DEFINITION. A function is called convex if for $C \in \mathcal{C}$ then $f(C)$ is convex as a set of real numbers. A function is called preconvex if

$f^{-1}(C) \in \mathcal{C}$ for each convex subset C of $\mathbb{R}$.

3.7. LEMMA. Each $f \in \mathfrak{F}$ is preconvex and convex.

Proof. Let C be a convex subset of $\mathbb{R}$. Let $x, y \in f^{-1}(C)$, let $z \in xy$. We need to show $z \in f^{-1}(C)$. Now by definition of σ, $z \in \bigcap g^{-1}[g(x),g(y)] \subset f^{-1}[f(x),f(y)]$. Since $[f(x),f(y)] \subset C$, we have $z \in f^{-1}[f(x),f(y)] \subset f^{-1}(C)$. Hence $f^{-1}(C)$ is convex under $\mathcal{C}$ so that f is preconvex.

Let $C \in \mathcal{C}$, $x' \neq y' \in f(C)$, and choose $\alpha \in [x',y']$. Now there exists $x, y \in C$ such that $f(x) = x'$, $f(y) = y'$. Since $\alpha \in [f(x), f(y)]$ there exists by P3 $z \in f^{-1}(\alpha)$ such that $z \in g^{-1}[g(x),g(y)]$ for all $g \in \mathfrak{F}$. Thus $z \in xy \subset C$ and $f(z) = \alpha \in f(C)$. Hence $f(C)$ is convex.

We now derive three important properties of segments will be useful in the definition of lines.

3.8. LEMMA. For each $x, y \in X$ and $f \in \mathfrak{F}$, f is either trivial or one-to-one on xy.

Proof. We assume $x \neq y$, since $xx = \{x\}$. Suppose f is not trivial and not one-to-one on xy. Thus there exist $u \neq v \in xy$ such that $f(u) = f(v)$. Now $u \in f^{-1}[f(x),f(y)]$ so we may assume $f(x) \leq f(u) = f(v) \leq f(y)$. Since f is not trivial on xy, we have either $f(x) \neq f(u)$ or $f(v) \neq f(y)$; in particular $f(x) < f(y)$. By P1 there exists $g \in \mathfrak{F}$ such that $g(u) \neq g(v)$. Now $u, v \in g^{-1}[g(x), g(y)]$ and we assume $g(x) > g(y)$ (if not, replace g with $-g$). Define $h = \alpha g + f$ where α is the unique real such that $\alpha(g(x) - g(y)) = f(y) - f(x)$. Since $g(x) > g(y)$ and $f(y) > f(x)$ we have $\alpha > 0$. Also for this choice of α, we have $h(x) = h(y)$, for

$$h(x) = \frac{f(y) - f(x)}{g(x) - g(y)} \cdot g(x) + f(x) = \frac{f(y)g(x) - f(x)g(y)}{g(x) - g(y)}$$

$$h(y) = \frac{f(y) - f(x)}{g(x) - g(y)} \cdot g(y) + f(y) = \frac{-f(x)g(y) + f(y)g(x)}{g(x) - g(y)}.$$

Now $h(u) = \alpha g(u) + f(u) \neq \alpha g(v) + f(v) = h(v)$ since $\alpha > 0$, $g(u) \neq g(v)$ and $f(u) = f(v)$. But $u, v \in h^{-1}[h(x), h(y)]$ and since $h(x) = h(y)$, we have $h(x) = h(v) = h(u) = h(y)$. Thus we have a contradiction.

<u>3.9. LEMMA</u>. If $z \in xy$, $z \neq xy$ then $xz \cap zy = \{z\}$.

<u>Proof</u>. Now there exists $g \in \mathfrak{F}$ such that $g(x) \neq g(y)$. Hence g is one-to-one on xy and

$$\begin{aligned}
\{z\} &\subset (xz \cap zy) \cap xy \\
&\subset \bigcap_{f \in \mathfrak{F}} f^{-1}[f(x), f(z)] \cap \bigcap_{f \in \mathfrak{F}} f^{-1}[f(z), f(y)] \cap xy \\
&\subset g^{-1}[g(x), g(z)] \cap g^{-1}[g(z), g(y)] \cap xy \\
&= g^{-1}(g(z)) \cap xy = \{z\}.
\end{aligned}$$

<u>3.10. LEMMA</u>. If $z \in xy$, then $xz \cup zy = xy$

<u>Proof</u>. By Lemma 3.3. we have $xz \cup zy \subset xy$. To avoid trivialities, let $x \neq y$, and let $u \in xy \setminus zy$. Thus

$$h(z) \in [h(x), h(y)] \qquad \text{for all } h \in \mathfrak{F}$$

$$h(u) \in [h(x), h(y)] \qquad \text{for all } h \in \mathfrak{F}$$

and there exists $g \in \mathfrak{F}$ such that

$$g(u) \notin [g(z), g(y)] .$$

Since $g(z) \in [g(x), g(y)]$, we have $[g(x), g(y)] = [g(x), g(z)] \cup [g(z), g(y)]$. But $g(u) \in [g(x), g(y)]$ so that $g(u) \notin [g(z), g(y)]$ yields $g(u) \in [g(x), g(z)]$. Now $f(u) \in [f(x), f(z)]$ for functions trivial on xy, so that we consider when f is not trivial (that is, one-to-one) on xy.

<u>Claim</u>. $f(u) \in [f(x),f(z)]$ for $u \neq x, z$.

Suppose not. Since $f(u) \in [f(x),f(y)] = [f(x),f(z)] \cup [f(z),f(y)]$, we have $f(u) \in [f(z),f(y)]$. Define a new function $h \in \mathfrak{F}$ by setting $h = \alpha f + g$ where α is the unique real such that $\alpha(f(x) - f(y)) = g(y) - g(x)$. Since $f(x) \neq f(y)$, α is well defined and is such that $h(x) = h(y) = \lambda$. Also from $z \in xy$ it follows that $\lambda = h(z)$. Similarly $\lambda = h(u)$. Now we have $\lambda = h(x) = h(y) = h(u) = h(z)$, which yields

$$\lambda = \alpha f(x) + g(x) = \alpha f(y) + g(y) = \alpha f(u) + g(u) = \alpha f(z) + g(z),$$

and since f is one-to-one on xy and $u \neq y$, $u \neq z$, we have

$$\alpha = \frac{g(y) - g(u)}{f(u) - f(y)} = \frac{g(z) - g(u)}{f(u) - f(z)}.$$

Now $g(z) - g(u) \neq 0$ (otherwise, $g(u) \in [g(z),g(y)]$) so we have

$$\frac{g(y) - g(u)}{g(z) - g(u)} = \frac{f(y) - f(u)}{f(z) - f(u)}.$$

But this is impossible for $g(u) \notin [g(z),g(y)]$ and $f(u) \in [f(z),f(y)]$. Hence the claim is established.

Thus $f(u) \in [f(x),f(z)]$ for all $f \in \mathfrak{F}$ and $u \in xz$. It follows that $xy \subset xz \cup zy$ and $xy = xz \cup zy$.

We now define "lines" in X and show that they possess all of the natural geometric properties.

<u>3.11. DEFINITION</u>. For all $x,y \in X$, the <u>line</u> joining x and y is taken to be the set

$$L(x,y) \equiv \cap\{f^{-1}(\alpha) \mid f \in \mathfrak{F} \text{ and } f(x) = f(y) = \alpha\}.$$

<u>REMARK</u>. If $x = y$ then by P1, $L(x,x) = \{x\}$.

<u>3.12. LEMMA.</u> $xy \subset L(x,y) = L(y,x)$.

<u>Proof.</u> $xy = \sigma(x,y) = \cap\{f^{-1}[f(x),f(y)] \mid f \in \mathfrak{F}\}$

$$\subset \cap\{f^{-1}(\alpha) \mid f \in \mathfrak{F} \text{ and } f(x) = f(y) = \alpha\}$$

$$= L(x,y) = L(y,x).$$

<u>3.13. LEMMA.</u> If $x \neq u \in L(x,y)$, then $L(x,y) \subset L(x,u)$.

<u>Proof.</u> Suppose not. Let $z \in L(x,y)\setminus L(x,u)$. Since $z \notin L(x,u)$ and by P5 $L(x,y) \neq X$, there exists $g \in \mathfrak{F}$ such that $g(x) = g(u) \neq g(z)$. By P1 there exists $f \in \mathfrak{F}$ such that $f(x) \neq f(u)$. If $g(x) = g(y)$ then $z \in L(x,y)$ implies $g(x) = g(y) = g(z)$, a contradiction of function g. Thus we assume $g(x) \neq g(y)$ and define h as follows: $h = \alpha g + f$ where α is the unique real such that $\alpha(g(y) - g(x)) = f(x) - f(y)$. If $f(x) = f(y)$, then we have a contradiction of $u \in L(x,y)$, since $f(x) \neq f(u)$. Thus α is non-zero. Now α was chosen so that $h(x) = h(y)$. We thus obtain $h(x) = \alpha g(x) + f(x)$ and $h(u) = \alpha g(u) + f(u)$, so that $\alpha \neq 0$, $g(x) = g(u)$, and $f(x) \neq f(u)$ imply $h(x) \neq h(u)$. But $u \in L(x,y)$ so that $h(x) = h(y) = h(u)$. The contradiction then establishes the result.

<u>3.14. LEMMA.</u> If $u \neq v \in L(x,y)$ with $x \neq u$, $y \neq v$ then $L(x,y) \subset L(u,v)$.

<u>Proof.</u> If $u = y$ then Lemma 3.13. applies to $u \neq y \in L(x,y)$. If $v = x$ then Lemma 3.13. applies to $u \neq x \in L(x,y)$. Thus assume $u \neq y$ and $v \neq x$. Now Lemma 3.13. applied to $u \in L(x,y)$ implies $L(x,y) \subset L(x,u)$. Also, $v \in L(x,y) \subset L(x,u)$, so that since $v \neq x,u$ we apply Lemma 3.13. to obtain $L(x,u) \subset L(u,v)$. Thus $L(x,y) \subset L(u,v)$.

<u>3.15. THEOREM.</u> If $u \neq v \in L(x,y)$ with x,u,v,y distinct then $L(x,y) = L(u,v)$.

<u>Proof.</u> By Lemma 3.14. $L(x,y) \subset L(u,v)$. Now $x,y \in L(x,y)$ and

x, y, u, v distinct so that Lemma 3.14. applies to $x, y \in L(u,v)$, yielding $L(u,v) \subset L(x,y)$. Hence $L(x,y) = L(u,v)$.

3.16. COROLLARY. Each pair of distinct points has a unique line containing them.

We now show that the lines could have been defined in terms of the segment operator σ.

3.17. THEOREM. For $x \neq y$, $L(x,y) = \{z \mid xzy, zxy, \text{ or } xyz\}$.

Proof. If (xzy), then by Lemma 3.12. $z \in xy \subset L(x,y)$. If (zxy) then $x \in L(z,y)$ so by Lemma 3.13. $L(z,y) \subset L(x,y)$ or $z \in L(x,y)$. If (xyz) then by Lemma 3.13. $y \in L(x,z) \subset L(x,y)$ so that $z \in L(x,y)$. If $x = z$ or $z = y$ then $z \in L(x,y)$. Hence $\{z \mid xzy, zxy, \text{ or } xyz\} \subset L(x,y)$.

For the second part, suppose $z \in L(x,y)$, $z \notin xy$, $x \notin zy$, and $y \notin xz$. Then there exist $f, g, h \in \mathfrak{F}$ such that

$$f(x) \notin [f(z),f(y)]$$

$$g(y) \notin [g(x),g(z)]$$

$$h(z) \notin [h(x),h(y)] .$$

Now define $G = \beta g + h + f$ where $\beta(g(x) - g(y)) = f(y) - f(x) + h(y) - h(x)$. Since $g(x) \neq g(y)$, β is well defined and $G(x) = G(y)$. If $G(x) \neq G(z)$ then we have a contradiction of $z \in L(x,y)$. Thus $G(x) = G(z) = G(y)$ and

$$\beta g(x) + h(x) + f(x) = \beta g(z) + h(z) + f(z) = \beta g(y) + h(y) + f(y)$$

from which follows

$$\beta(g(z) - g(y)) = f(y) - f(z) + h(y) - h(z).$$

Let $g(z) - g(y) = \alpha(g(x) - g(y))$ where $\alpha \neq 0$ since $g(z) \neq g(y)$. Using the definition of β we have

$$\beta(g(z) - g(y)) = f(y) - f(z) + h(y) - h(z)$$
$$= \alpha\beta(g(x) - g(y)) = \alpha(f(y) - f(x) + h(y) - h(x)).$$

Thus we obtain

$$(*) \quad (1 - \alpha)h(y) + \alpha h(x) - h(z) = -((1 - \alpha)f(y) + \alpha f(x) - f(z)).$$

Now $g(y) \notin [g(x),g(z)]$ so that $\alpha > 0$. If $0 < \alpha \leq 1$ then $(1-\alpha)h(y) + \alpha h(x) - h(z) \neq 0$, for otherwise $h(z) \in [h(x),h(y)]$, a contradiction. Hence $\alpha > 1$, and $(1-\alpha)f(y) + \alpha f(x) - f(z) \neq 0$, for otherwise $f(x) \in [f(z),f(y)]$, a contradiction. Thus $(*)$ is an equality of non-zero quantities. If we replace h by 2h then $2h(z) \notin [2h(x),2h(y)]$ since $h(z) \notin [h(x),h(y)]$, and $(*)$ will no longer be an equality. Thus we have a contradiction and $L(x,y) = \{z \mid xzy,\ zxy, \text{ or } xyz\}$.

<u>3.18. LEMMA</u>. For all $x, y \in X$, there exist z, u distinct from x and y such that xyz and uxy.

<u>Proof</u>. By P4 there exists $z \neq x, y$ such that $g(y) \in [g(x),g(z)]$ for all $g \in \mathfrak{F}$. Thus $y \in xz$ or equivalently xyz. Similarly there exists u such that $g(x) \in [g(u),g(y)]$ for all $g \in \mathfrak{F}$ so that uxy.

<u>3.19. LEMMA</u>. Each $f \in \mathfrak{F}$ is either trivial or one-to-one on lines of X.

<u>Proof</u>. Immediate from $L(x,y) = \{z \mid xzy,\ zxy, \text{ or } xyz\}$ and from Lemma 3.8, which states that each function is either trivial or one-to-one on xy for all x,y.

Each line in X can now be given an order via the segments on that line as shown by Shirley [7]. We define for each line in X an order via non-trivial functions on that line. A relationship between that ordering and segments is then established yielding a connection to Shirley's work.

3.20. DEFINITION. Let $x \neq y \in X$, f be a non-trivial function on $L(x,y)$. For all $u, v \in L(x,y)$, define $u \leq_f v$ if and only if $f(u) \leq f(v)$. Define $u \geq_f v$ if and only if $u \not\leq_f v$ or $u = v$.

3.21. LEMMA. $xy = \{z \in L(x,y) \mid x \leq_g z \leq_g y\}$ where g is a one-to-one function on $L(x,y)$ such that $g(x) \leq g(y)$.

Proof. If $x = y$ then both sides are $\{x\}$. Thus suppose $x \neq y$. If $z \in xy \subset L(x,y)$, then $f(z) \in [f(x),f(y)]$ for all $f \in \mathfrak{F}$ so certainly $g(z) \in [g(x),g(y)]$. Since $g(x) < g(y)$ we have $g(x) \leq g(z) \leq g(y)$ and z belongs to the right hand side.

Let $z \in L(x,y)$, $x \leq_g z \leq_g y$ and $f \in \mathfrak{F}$. Define $h = \alpha g + f$ where $\alpha(g(y) - g(x)) = f(x) - f(y)$. Now $g(x) \neq g(y)$ since $x \neq y$ and g is one-to-one; thus α is well defined. For this choice of α, $h(x)=h(y)$; hence, h is trivial on $L(x,y)$. Thus $h(x) = h(z) = h(y)$ or

$$(1) \qquad \alpha g(x) + f(x) = \alpha g(z) + f(z)$$

$$(2) \qquad = \alpha g(y) + f(y).$$

If $\alpha \geq 0$, then since $g(x) - g(z) < 0$, $g(z) - g(y) < 0$ we have (1) implies

$$\alpha(g(x) - g(z)) = f(z) - f(x) \leq 0 \quad \text{or} \quad f(x) \geq f(z)$$

and (2) implies

$$\alpha(g(z) - g(y)) = f(y) - f(z) \leq 0 \quad \text{or} \quad f(z) \geq f(y).$$

Hence $f(x) \geq f(z) \geq f(y)$. If $\alpha < 0$ then (1) yields $f(x) < f(z)$ and (2) yields $f(z) < f(y)$, so that $f(x) < f(z) < f(y)$. Thus in either case $f(z) \in [f(x),f(y)]$. Since f was an arbitrary element of $\mathfrak{F}$, we have that $z \in xy$.

3.22. COROLLARY. $\leq_g$ is equivalent to either $\leq_f$ or $\geq_f$ for g, f non-trivial functions on L.

We can now combine Theorem 3.17., Lemmas 3.18., 3.21. to obtain:

3.23. THEOREM. If f is a non-trivial function on $L(x,y)$, then f is an order isomorphism from $L(x,y)$ to $f(L(x,y))$ and $f(L(x,y))$ is an open interval on $\mathbb{R}$ (possibly $= \mathbb{R}$).

The set X has unique lines joining any two distinct points and the lines are order-isomorphic to an open subset of the reals, hence, order-isomorphic to the reals. X will thus be a G.L.S. after the following.

3.24. Theorem. Let x, y, z be distinct points in X. If $u \in xy$, $v \in uz$ then there exists $w \in xz$ such that $v \in wy$.

Proof. If $z \in L(x,y)$ then the result is trivial, so suppose $z \notin L(x,y)$. If $u = x$, let $w = v$; if $u = y$, let $w = z$; if $v = u$, let $w = x$; if $v = z$, let $w = z$. Thus assume $u \in (xy)$ and $v \in (uz)$. Let $f \in \mathfrak{F}$ such that $f(x) \neq f(y) = f(v)$ which exists since $x \notin L(y,v)$. Then $u \in (xy)$, $v \in (uz)$ yield $f(u) \in [f(x),f(y)]$ and $f(v) \in [f(u),f(z)]$. Now f is trivial or one-to-one on lines of X so that $f(x) \neq f(y)$ implies $f(u) \neq f(y) = f(v)$ which in turn implies $f(z) \neq f(v)$. Now $f(z) \neq f(x)$ as $f(x) < f(u) < f(y) = f(v)$ for example and $f(v) \in [f(u),f(z)]$ imply $f(u) < f(v) < f(z)$ so that $f(x) < f(v) < f(y)$. Since f is convex on xz, there exists $w \in (xz)$ such that $f(w) = f(v) = f(y)$.

Claim: $v \in (wy)$. Suppose not, then there exists $g \in \mathfrak{F}$ such that $g(v) \notin [g(w),g(y)]$. Define h as follows: $h = \alpha f + g$ where $\alpha(f(v) - f(u)) = g(u) - g(v)$ which is well defined since $f(v) \neq f(u)$. With this choice of α, $h(u) = h(v)$. Also $h(u) = h(v) = h(z)$ since $v \in (uz)$ and h is trivial on $L(u,v)$. If $h(x) = h(u)$ then h is trivial on all lines under consideration, in particular $h(y) = h(v)$ so that

$$\alpha f(y) + g(y) = \alpha f(v) + g(v).$$

But $f(y) = f(v)$ so that $g(y) = g(v)$ and we have a contradiction of $g(v) \notin [g(w),g(y)]$. Thus $h(x) \neq h(u)$. Suppose $h(x) < h(u) < h(y)$. Then since $h(u) = h(z)$ and $w \in (xz)$ we have $h(x) < h(w) < h(z)$. But $h(u) = h(z) = h(v)$ so we have

$$h(x) < h(w) < h(v) < h(y)$$

or

$$\alpha f(w) + g(w) < \alpha f(v) + g(v) < \alpha f(y) + g(y).$$

But $f(w) = f(v) = f(y)$ so that

$$g(w) < g(v) < g(y)$$

which is a contradiction of $g(v) \notin [g(w),g(y)]$. If $h(x) > h(u) > h(y)$ we arrive at the same contradiction. Thus we have an h such that $h(u) \notin [h(x),h(y)]$. Hence $u \notin (xy)$ which is a contradiction and result is established.

It is of interest to note that we have used P4 in showing lines are order-isomorphic to an open interval of reals and only then. Property P3 implies that functions are convex, first used in the preceeding theorem. The following lemma was proved by Cantwell using simple geometric considerations such as Theorems 3.23., 3.24. We include a proof which does not use P4 indicating that Cantwell's work up to the separation theorems can be duplicated using P1, P2, P3.

<u>3.25. LEMMA</u>. Let a, b, c be distinct points. If $x \in (ab)$, $z \in (ac)$ then there exists $y \in (zb) \cap (xc)$.

<u>Proof</u>. Let f be such that $f(x) = f(c) > f(a)$. Then $f(z) < f(c)$ and $f(b) > f(x)$. Since f is convex on zb there exists $y \in (zb)$ such that $f(y) = f(z) = f(c)$.

<u>Claim</u>. $y \in (xc)$. Suppose not, then there exists $g \in \mathfrak{F}$ such that $g(y) \notin [g(x),g(c)]$. Define $h \in \mathfrak{F}$ as follows: $h = \alpha f + g$ where

$\alpha(f(b) - f(z)) = g(z) - g(b)$ which is well defined since $f(z) < f(c) = f(x) < f(b)$. Then $h(z) = h(y) = h(b)$. If $h(x) = h(y)$ then $f(x) = f(y)$ yielding $g(x) = g(y)$, a contradiction. Thus assume $h(x) < h(y)$. Then $x \in (ab)$, $z \in (ac)$ yield

$$h(a) < h(x) < h(y) = h(b) = h(z) < h(c)$$

or
$$\alpha f(x) + g(x) < \alpha f(y) + g(y) < \alpha f(c) + g(c).$$

But $f(x) = f(y) = f(c)$ so that $g(x) < g(y) < g(c)$, a contradiction. Thus claim is verified and proof is completed.

Recall that a flat was previously defined as a set F with the property that for each $x, y \in F$, $L(x,y) \subset F$, and a hyperplane was any maximal proper flat. In the classical situation hyperplanes correspond to linear functionals; we obtain only part of that correspondence.

<u>3.26. LEMMA</u>. If $f \in \mathfrak{F}$ is a non-trivial function, $\alpha \in$ Range f, then $[f:\alpha] \equiv f^{-1}(\alpha)$ is a hyperplane.

<u>Proof</u>. If $u, v \in [f:\alpha]$, then $f(u) = f(v) = \alpha$. But $w \in L(u,v)$ iff $g(w) = g(u) = g(v)$ whenever $g(u) = g(v)$. Thus $f(w) = \alpha$ or $w \in [f:\alpha]$. So $L(u,v) \subset [f:\alpha]$ and $[f:\alpha]$ is a flat.

Suppose there exists a flat H such that $[f:\alpha] \subsetneq H \subset X$. Let $z \in X$, $y \in H \setminus [f:\alpha]$. Now $f(y) > \alpha$ or $f(y) < \alpha$, so we assume $f(y) > \alpha$. If $f(z) < \alpha$ then since f is convex there exists $x \in (yz)$ such that $f(x) = \alpha$. Thus $x \in [f:\alpha] \subset H$. Since $y \in H$ and H is a flat $z \in L(x,y) \subset H$. If $f(z) > \alpha$, then by P4 there exists y' such that $f(y') < \alpha$. We repeat the above argument for z and y'. Thus in either case $z \in H$. Hence $X \subset H$ and $H = X$.

We follow Cantwell's work in the following.

<u>3.27. DEFINITION.</u> $x_0,\dots,x_h \in X$ are <u>independent</u> if $x_i \notin fl(x_0,\dots,\hat{x}_i,\dots,x_h)$ for $i = 0,\dots,h$. If F is a flat, dim F = $\sup\{h \mid x_0,\dots,x_h \in F,$ with $x_0,\dots,x_h$ independent$\}$.

<u>3.28. LEMMA.</u> For each set of n+1 points $x_0,\dots,x_n \in X$ with $x_0 \notin fl(x_1,\dots,x_n)$ there exists $f \in \mathfrak{F}$ such that $f(x_0) \neq \alpha = f(x_i)$ $i \geq 1$.

<u>Proof.</u> Let A(n) be the statement of lemma; we proceed by induction.

<u>Part 1.</u> n = 1,2. A(1) is true by axiom P1, and A(2) is true as we are assuming X is not a line.

<u>Part 2.</u> Assume A(k) is true for all k < n. Let $x_0,\dots,x_n$ be such that $x_0 \notin fl(x_0,\dots,x_n)$. Then $x_0 \notin fl(x_2,\dots,x_n)$ so by hypothesis there exists $f \in \mathfrak{F}$ such that $f(x_0) \neq \alpha = f(x_i)$ $i \geq 2$. If $f(x_i) = \alpha$ we are finished so suppose $f(x_1) \neq \alpha$. Two cases occur: $L(x_0,x_1)$ meets $[f:\alpha]$ or $L(x_0,x_1) \cap [f:\alpha] = \emptyset$.

<u>Case 1.</u> $L(x_0,x_1) \cap [f:\alpha] = \{v\}$ for some v. Now by induction hypothesis there exists $g \in \mathfrak{F}$ such that $g(v) \neq \beta$ and $x_i \in [g:\beta]$ $i \geq 2$, since $v \notin fl(x_2,\dots,x_n)$ as $x_0 \notin fl(x_1,\dots,x_n)$. Let $h = \lambda f + g$ where $\lambda(f(x_2) - f(x_1)) = g(x_1) - g(x_2)$, which is well defined since $\alpha = f(x_2) \neq f(x_1)$. Clearly $h(x_i) = \alpha\lambda + \beta$ for all $i \geq 1$.

<u>Claim.</u> $h(x_0) \neq \alpha\lambda + \beta$. Suppose that $h(x_0) = \alpha\lambda + \beta = h(x_1)$. Then h is constant on $L(x_0,x_1)$ and $h(v) = \alpha\lambda + \beta = f(v)\lambda + g(v) = \alpha\lambda + g(v)$. That is, $g(v) = \beta$, a contradiction. Thus h is the required function.

<u>Case 2.</u> $L(x_0,x_1) \cap [f:\alpha] = \emptyset$. Assume wolog $f(x_0) > \alpha$. Since $f(x_1) \neq \alpha$, either $f(x_1) > \alpha$ or $f(x_1) < \alpha$. Now $f(x_1) < \alpha$ implies by the convexity of f that there is a $v \in [f:\alpha]$ such that $x_0 v x_1$ or $v \in L(x_0,x_1) \cap [f:\alpha]$ so we again have case 1. Hence consider $f(x_0) > \alpha$ and $f(x_1) > \alpha$. There exists u such that $x_0 x_2 u$ and since f is one-to-one on $L(x_0,x_2)$

we have $f(u) < f(x_2) < f(x_0)$ yielding $f(u) < \alpha < f(x_1)$. By convexity of f there exists v such that x_1vu and $f(v) = \alpha$. Let $w \in x_0v \cap x_1x_2$. Now $L(x_0,w) \cap [f:\alpha] = \{v\}$ so by case 1 there is an $h \in \mathfrak{F}$ such that $h(x_0) \neq h(w) = h(x_i)$ $i \geq 2$. But $w \in x_1x_2$ so $h(x_1) = h(w) = h(x_2)$. Thus $h(x_0) \neq h(x_i)$ $i \geq 1$.

3.29. COROLLARY. If X is finite dimensional, then for each hyperplane $H \subset X$, there exist $f \in \mathfrak{F}$, $\alpha \in \mathbb{R}$ such that $H = [f:\alpha]$.

Proof. Let $x_1,\ldots,x_n$ be maximal independent in H so that $fl(x_1,\ldots,x_n) = H$, and let $x_0 \notin H$. By Lemma 3.28. there is an $f \in \mathfrak{F}$ such that $f(x_0) \neq \alpha = f(x_i)$ $i \geq 1$. Hence $[f:\alpha]$ is a flat containing $x_1,\ldots,x_n$ so that $H \subset [f:\alpha] \subsetneq X$. Since H is a maximal proper flat, $H = [f:\alpha]$.

We now include some results which were proved by Cantwell and will be needed later. The proof of the basic separation theorem will be omitted since Cantwell's proof of that is actually an easy extension of Valentine's proof in [8] of the classical separation theorem to a G.L.S. Since all later results apply only to finite dimensional spaces and we have a convenient functional representation for hyperplanes in that case, we shall now assume that $\dim X = n < \infty$.

3.30. DEFINITION. Let $H = [f:\alpha]$ be a hyperplane. The sides of H are defined to be the sets

$$H^+ \equiv \{x \mid f(x) > \alpha\} \quad \text{and} \quad H^- \equiv \{x \mid f(x) < \alpha\}.$$

The sets H^+ and H^- are also called open half spaces, while

$$cl\ H^+ \equiv H^+ \cup H \quad \text{and} \quad cl\ H^- \equiv H^- \cup H$$

are the closed half spaces corresponding to H. Two sets A and B in X are said to be <u>separated</u> by H if $A \subset \text{cl } H^+$ and $B \subset \text{cl } H^-$, and the separation is <u>strong</u> if $A \subset H^+$ and $B \subset H^-$.

We note that for any hyperplane H the space X is partitioned by the triple (H,H^+,H^-). A result which is easy to establish, owing to the preconvexity of each $f \in \mathfrak{F}$ is that H^+ and H^-, as are cl H^+ and cl H^-, are convex.

3.31. LEMMA. For any hyperplane $H=[f:\alpha]$ both types of half spaces determined by H are convex. Moreover, if $x \in H^+$ and $y \in H^-$ then

$$(xy) \cap H = \{z\}$$

for some $z \in X$.

Proof. Let $R = \{\lambda \mid \lambda > \alpha\}$ and $\text{cl } R = \{\lambda \mid \lambda \geq \alpha\}$, clearly convex subsets of $\mathbb{R}$. Then $f^{-1}(R) = H^+$ and $f^{-1}(\text{cl } R) = \text{cl } H^+$ are convex (similarly for H^- and cl H^-). Since $f(x) > \alpha$ and $f(y) < \alpha$ we have that f is convex and one-to-one on $L(x,y)$. Hence there is a unique $z \in xy$ such that $f(z) = \alpha$; and since $z \neq x$ and $z \neq y$, $z \in (xy) \cap H$.

Another property of half spaces requires a definition.

3.32. DEFINITION. A set $A \subset X$ is called <u>convex-open</u> iff it is convex and $\forall\ a \in A,\ x \in X\ \exists\ y \in (ax \ni ay) \subset A$.

3.33. LEMMA. Open half spaces are convex-open.

Proof. Let $a \in H^+$ where H is a hyperplane and $x \in X$, $x \neq a$. If $x \in H^+$, then $ax) \subset H^+$. If $x \in H$ and $y \in ax)$ where $y \notin H^+$ then by Lemma 3.31. either $y \in H$ or there is $y' \in (ax)$ such that $y' \in H$ so that $a \in L(x,y) \subset H$ or $a \in L(x,y') \subset H$, which is a contradiction since $a \in H^+$. Thus, for each $x \in H$, $ax) \subset H^+$. If $x \in H^-$ then since $a \in H^+$ let

$(ax) \cap H = \{y\}$ so that $ay) \subset H^+$. Hence H^+ is convex-open. (A similar proof holds for H^-.)

<u>3.34. BASIC SEPARATION THEOREM.</u> For each pair of convex sets $A, B \subset X$ where $A \cap B = \emptyset$ and A contains a nonempty convex-open subset, there exists a hyperplane separating them.

We also state without proof another result of Cantwell which will be used later.

<u>3.35. STRONG SEPARATION THEOREM.</u> Let A and B be any two finite sets such that $H(A) \cap H(B) = \emptyset$. Then H(A) and H(B) may be strongly separated by a hyperplane.

Some easily proved results of Cantwell on independence of points will also be needed.

<u>3.36. LEMMA.</u> If $b_0, \ldots, b_k$ are independent points and $b_{k+1} \notin fl(b_0, \ldots, b_k)$ then $b_0, \ldots, b_k, b_{k+1}$ are independent.

<u>3.37. LEMMA.</u> If $b_0, \ldots, b_k$ are maximal independent in A then

$$fl(A) = fl(b_0, \ldots, b_k).$$

<u>3.38. EXISTENCE OF DIMENSION THEOREM.</u> If $b_0, \ldots, b_k$ and $b_0', \ldots, b_\ell'$ are both maximal independent in a flat F then $k = \ell$.

Recall that dim F was defined as the unique number of elements in a maximal independent subset of F. To show how these results can be used in the theory, we prove a result not given by Cantwell.

3.39. COROLLARY. A set H in a flat F is a hyperplane in F if and only if H is a flat of dimension dim F - 1.

Proof. Let H be a hyperplane in F, where dim F = k, and let $b_0,\ldots,b_\ell$ be maximal independent in H. Thus $\ell \leq k$. If $\ell = k$ then by Lemma 3.37. $H = fl(b_0,\ldots,b_\ell) = F$. If $\ell \leq k - 2$ then let $b_0,\ldots,b_\ell,\ldots,b_k$ be maximal independent in F. Now by Lemma 3.36.

$$H = fl(b_0,\ldots,b_\ell) \subsetneq fl(b_0,\ldots,b_{k-1}) \subsetneq fl(b_0,\ldots,b_k) = F.$$

Thus if $\ell \neq k - 1$ we contradict that H is a maximal proper flat contained in F.

Let dim F = k and let $b_0,\ldots,b_{k-1}$ be any k independent points in F such that $H = fl(b_0,\ldots,b_{k-1})$. Suppose $H \subsetneq G \subset F$ for some flat G. Consider $b_k \in G \setminus H$. Now since dim F = k, we have by Lemma 3.36. that $b_0,\ldots,b_k$ are maximal independent in F; hence, also in G. Then by Lemma 3.37. $F = fl(b_0,\ldots,b_k) = G$. Thus H is a maximal proper flat contained in F.

We end the chapter by establishing that if $b_0,\ldots,b_n$ are maximal independent in X, then the set

$$(*) \qquad H(b_0,\ldots,b_n) \setminus \bigcup_{i=0}^{n} H(b_0,\ldots,\widehat{b_i},\ldots,b_n)$$

is convex-open for $n \geq 1$. We shall use the fact noted by Cantwell that if A is convex then the join of x and A $xA \equiv \bigcup\{xa \mid a \in A\}$ is convex (proved by applying Axiom C). Then it follows that $xA = H(x \cup A)$ for all $x \in X$ and $A \in X$, a property referred to as join-hull commutativity in Kay-Womble [5].

First, the above set (*) is shown to be nonempty, by applying induction on $n = \dim X \geq 1$.

Case 1. Let $b_0 \neq b_1$ and choose $x \in (b_0b_1) = H(b_0,b_1) \setminus \{b_0,b_1\}$ so that (*) is nonempty for $n = 1$.

Case 2. Suppose that (*) is onempty for $\dim X < n$. Consider $X' = fl(b_1,\ldots,b_n)$ which has dimension $n-1$ so by the induction hypothesis there exists a point

$$x' \in H(b_1,\ldots,b_n) \setminus \cup_{i=1}^{n} H(b_1,\ldots,\hat{b}_i,\ldots,b_n).$$

Now $x' \neq b_0$ by independence of $b_0,b_1,\ldots,b_n$, and $H(b_0,\ldots,b_n) = b_0H(b_1,\ldots,b_n)$ so choose $x \in b_0x'$. Thus $x \in H(b_0,\ldots,b_n)$, so $x \notin \cup_{i=0}^{n} H(b_0,\ldots,\hat{b}_i,\ldots,b_n)$ remains to be shown. Suppose $x \in H(b_0,\ldots,\hat{b}_i,\ldots,b_n)$ for some i. Then $i > 0$ for otherwise $b_0 \in L(x,x') \subset fl(b_1,\ldots,b_n)$ contradicting the independence of $b_0,\ldots,b_n$. But since $i > 0$ $H(b_0,\ldots,\hat{b}_i,\ldots,b_n) = b_0H(b_1,\ldots,\hat{b}_i,\ldots,b_n)$ and there is $x'' \in H(b_1,\ldots,\hat{b}_i,\ldots,b_n)$ such that $x \in (b_0x'')$. Since

$$x' \notin \cup_{i=1}^{n} H(b_1,\ldots,\hat{b}_i,\ldots,b_n)$$

we have $x' \neq x''$ so that $x \in (b_0x')$ and $x \in (b_0x'')$ yield $L(b_0,x) = L(x',x'')$. Thus $b_0 \in L(x',x'') \subset fl(b_1,\ldots,\hat{b}_i,\ldots,b_n)$ a contradiction. Hence (*) is nonempty.

For notational purposes we introduce the ray $R(x,y)$ from x to y as the set $R(x,y) \equiv \{ z \mid xzy \text{ or } xzy \}$, $x \neq y$. We note the fact that if $z \in R(x,y)$ and $z \neq x$ then $R(x,y) = R(x,z)$.

3.4. LEMMA. Let H be a hyperplane with $x \notin H$, $z \in H$, (xyz) and $R(y,w)$ any ray from y not passing through x and not meeting H. Then there exists a point $q \in H$ such that xq meets $R(y,w)$ at a point $s \neq y$.

Proof. Choose p such that (xzp). Then x and p lie on opposite sides of H, likewise for w and p. Hence let $(pw) \cap H = \{q\}$ and apply

Lemma 3.25. to the points x,y,p,q,w with (xyp) and (pqw) so that there exists $s \in (xq) \cap (yw)$, the desired point.

<u>3.41. LEMMA.</u> Let $b_0,\ldots,b_n$ be maximal independent in X and let $x \in H(b_0,\ldots,b_n) \setminus \cup_{i=0}^{n} H(b_0,\ldots,\hat{b}_i,\ldots,b_n)$, $n \geq 1$. Then every ray from x meets $H(b_0,\ldots,\hat{b}_i,\ldots,b_n)$ for some i.

<u>Proof.</u> Apply induction on n, the case $n = 1$ being obvious. Suppose $R(x,y)$ does not meet any $H(b_0,\ldots,\hat{b}_i,\ldots,b_n)$ $0 \leq i \leq n$. In particular, $R(x,y)$ does not contain b_0 and does not meet $H(b_1,\ldots,b_n)$.

<u>Claim.</u> $R(x,y)$ does not meet the hyperplane $X' = fl(b_1,\ldots,b_n)$. Suppose $z' \in fl(b_1,\ldots,b_n) \cap R(x,y)$ and let x' be that point in $H(b_1,\ldots,b_n)$ such that $x \in (b_0x')$. Now by induction hypothesis $R(x',z')$ meets $H(b_1,\ldots,\hat{b}_i,\ldots,b_n)$ for some i say at w'. By applying Lemma 3.25. to b_0,x,x',w',z' with (b_0xx') and $(x'w'z')$ as $z' \in R(x,y)$ which does not meet $H(b_1,\ldots,b_n)$, we obtain $w \in (xz') \cap (b_0w') \subset R(x,y) \cap H(b_0,\ldots,\hat{b}_i,\ldots,b_n)$ a contradiction. Hence claim is established.

By Lemma 3.40. there is $z' \in X' = fl(b_1,\ldots,b_n)$ such that b_0z' meets $R(x,y)$ at $z \neq x$. With x' as above, by induction hypothesis $R(x',z')$ meets $H(b_1,\ldots,\hat{b}_i,\ldots,b_n)$ for some $1 \leq i \leq n$ say at w'. In the case $(x'w'z')$, Lemma 3.25. applies to obtain $w \in (xz) \cap (b_0w') \subset R(x,y) \cap H(b_0,\ldots,\hat{b}_i,\ldots,b_n)$, a contradiction. If $w' = z'$ then $z \in R(x,y) \cap H(b_0,\ldots,\hat{b}_i,\ldots,b_n)$, a contradiction. If $(x'z'w')$, consider the 2-dimensional flat $fl(b_0,x',z')$ (where lines are hyperplanes). Then b_0,x',w' are not in $L(x,z)$ so either x' or b_0 lie on the opposite side of $L(x,z)$ from w'. But x' cannot do so for otherwise $L(x,y)$, and hence $R(x,y)$ would meet $x'z' \cap H(\hat{b}_0,b_1,\ldots,b_n)$, a contradiction. Thus b_0 is opposite of w' so that $L(x,z)$ and $R(x,y)$ meet b_0w' at a point $w \in H(b_0,\ldots,\hat{b}_i,\ldots,b_n)$, a contradiction.

3.42. THEOREM. If $b_0,\ldots,b_n$ are maximal independent in X, $n \geq 1$, then the set

$$H(b_0,\ldots,b_n) \setminus \bigcup_{i=0}^{n} H(b_0,\ldots,\hat{b}_i,\ldots,b_n)$$

is a nonempty convex-open set.

Proof. Let $x \in H(b_0,\ldots,b_n) \setminus \bigcup_{i=0}^{n} H(b_0,\ldots,\hat{b}_i,\ldots,b_n)$ and let $y \in X$, $y \neq x$. By Lemma 3.41., ray $R(x,y)$ meets $H(b_0,\ldots,\hat{b}_i,\ldots,b_n)$ at some point z where $z \neq x$ and $xz \subset H(b_0,\ldots,b_n)$. Now $fl(b_0,\ldots,\hat{b}_i,\ldots,b_n)$ is a hyperplane for each i so since $R(x,y) \not\subset fl(b_0,\ldots,\hat{b}_i,\ldots,b_n)$ for any i, we have at most a finite number of such z. Thus there exists $w \in (xz) \cap (xy)$ such that

$$(xw) \cap \bigcup_{i=0}^{n} H(b_0,\ldots,\hat{b}_i,\ldots,b_n) = \emptyset.$$

Hence

$$xw) \subset H(b_0,\ldots,b_n) \setminus \bigcup_{i=0}^{n} H(b_0,\ldots,\hat{b}_i,\ldots,b_n)$$

which completes the proof.

CHAPTER IV

A TOPOLOGICAL STRUCTURE ON GENERALIZED LINEAR SPACES

In this chapter we expand on the five properties P1-P5 of Chapter 3 to include the derivation of a topological structure, induced by specifying the sub-basic elements, which will turn out to be equivalent to the topology considered by Cantwell.

We assume throughout that X is a non-empty set with a family $\mathfrak{F}$ of real-valued functions satisfying properties P1-P5. Let the sub-basic elements for a topology τ be the subsets of X of the form $f^{-1}(R)$, where $f \varepsilon \mathfrak{F}$ and $R \subset \mathbb{R}$ is an open ray. We note that if $R = (-\infty,\alpha)$ is an open ray on the reals $\mathbb{R}$ then $f^{-1}(R) = (-f)^{-1}(R^+)$, where $R^+ = (\alpha,\infty)$. Thus we may always assume that rays are of the form (α,∞) for some $\alpha \varepsilon \mathbb{R}$. We observe two immediate properties:

(1) (X,τ) is T_2.

(2) τ is the smallest topology on X for which each f is continuous.

<u>4.1. LEMMA</u>. The relative topology on each line L is equivalent to the order topology on L which is induced on L by the order isomorphisms of L with $\mathbb{R}$.

<u>Proof</u>. Let $x \varepsilon f^{-1}(R) \cap L$. If f is trivial on L, then for all $u,v \varepsilon L$ such that (uxv), we have $x \varepsilon (uv) \subset f^{-1}(R) \cap L$. If f is non-trivial on L, then there exist $u,v \varepsilon L$ such that (uxv) and $f(x) \varepsilon (f(u),f(v)) \subset R$.

But by definition of segments $x \in (uv) \subset f^{-1}(f(u),f(v)) \cap L \subset f^{-1}(R) \cap L$. Thus the sub-basic open sets are order-topology-open. Hence any open set in the relative topology on L is order-topology-open.

Let $x \in (uv) \subset L$. Let f be any non-trivial function on L such that $f(u) < f(v)$. Let R_u, R_v be open rays such that $R_u = (f(u),\infty)$ and $R_v = (-\infty,f(v))$. Thus $(f(u),f(v)) = R_u \cap R_v$ and $f^{-1}((f(u),f(v))) = f^{-1}(R_u) \cap f^{-1}(R_v)$. But by definition of segments $f^{-1}((f(u),f(v))) \cap L \subset (uv)$ so that $x \in f^{-1}(R_u) \cap f^{-1}(R_v) \cap L \subset (uv)$. Hence order-topology open sets are open in the relative topology.

Since segments are order isomorphic to closed and bounded real intervals, we have an obvious consequence of Lemma 4.1.

4.2. COROLLARY. Segments are compact.

Let $\mathcal{C}$ define the family of convex sets in X (as in Chapter 2), with H the convex hull operator.

4.3. DEFINITION. For $C \in \mathcal{C}$, define

$$\text{c-int } C = \{a \in C \mid \forall x \in C, \exists\, y \in X \ni a \in xy) \subset C\}.$$

4.4. LEMMA. Let $b_0,\ldots,b_k$ be independent points and $b'_{k+1} \notin fl(b_0,\ldots,b_k)$. Further, let $b'_0,\ldots,b'_k$ be such that $b_i \in (b'_i b'_{k+1})$ for $i = 0,\ldots,k$. Then $b'_0,\ldots,b'_{k+1}$ are independent and

$$\text{c-int } H(b_0,\ldots,b_k) \subset \text{c-int } H(b'_0,\ldots,b'_{k+1}).$$

Proof. If $b'_i \in fl(b'_0,\ldots,\widehat{b'_i},\ldots,b'_{k+1}) \equiv F$ for $0 \le i \le k$, then $b_j \in L(b'_j,b'_{k+1}) \subset F$ for $0 \le j \le k$ and $fl(b_0,\ldots,b_k) \subset F$. If $b_0,\ldots,b_k,\ldots,b_l$ $(l \ge k)$ are maximal independent in F, then by Theorem 3.38. $l = k$. Thus $b_0,\ldots,b_k$ are maximal independent in F and we have by Lemma 3.37. $fl(b_0,\ldots,b_k) = F$. But this provides the contradiction $b'_{k+1} \in fl(b_0,\ldots,b_k)$. Similarly, if $b'_{k+1} \in fl(b'_0,\ldots,b'_k)$ then

$\mathrm{fl}(b_0,\ldots,b_k) \subset \mathrm{fl}(b_0',\ldots,b_k')$ which implies the contradiction

$$b_{k+1} \in \mathrm{fl}(b_0',\ldots,b_k') = \mathrm{fl}(b_0,\ldots,b_k).$$

Hence, $b_0',\ldots,b_{k+1}'$ are independent.

Now let $x \in \text{c-int } H(b_0,\ldots,b_k)$. By hypothesis $b_0,\ldots,b_k \in H(b_0',\ldots,b_{k+1}')$ so that $x \in H(b_0,\ldots,b_k) \subset H(b_0',\ldots,b_{k+1}')$. By join-hull commutativity there exists $y \in H(b_0',\ldots,b_k')$ such that $x \in yb_{k+1}$. Since $H' \equiv \mathrm{fl}(b_0,\ldots,b_k)$ is a hyperplane relative to $X' \equiv \mathrm{fl}(b_0',\ldots,b_{k+1}')$ and $b_{k+1}' \notin H'$, we may assume that $b_{k+1}' \in H'^{+}$. Then $b_0',\ldots,b_k' \in H'^{-}$ and $y \in H(b_0',\ldots,b_k') \subset H'^{-}$. Since $x \in H'$ we have $x \neq y$ and $x \notin H(b_0',\ldots,b_k')$. If $x \in H(b_0',\ldots,\widehat{b_i'},\ldots,b_{k+1}')$ for $i \neq k+1$ then consider $x \in b_i H(b_0,\ldots,\widehat{b_i},\ldots,b_k)$ which implies $x \in b_i z$ for some $z \in H(b_0,\ldots,\widehat{b_i},\ldots,b_k)$. Since $x \neq z$ and $b_j \in \mathrm{fl}(b_0',\ldots,\widehat{b_i'},\ldots,b_{k+1}')$ for $j \neq i,k+1$, we have $z \in \mathrm{fl}(b_0,\ldots,\widehat{b_i},\ldots,b_k) \subset \mathrm{fl}(b_0',\ldots,\widehat{b_i'},\ldots,b_{k+1}')$ so

$$b_i \in L(x,z) \subset \mathrm{fl}(b_0',\ldots,\widehat{b_i'},\ldots,b_{k+1}').$$

Then

$$b_i' \in L(b_i,b_{k+1}') \subset \mathrm{fl}(b_0',\ldots,\widehat{b_i'},\ldots,b_{k+1}')$$

denying the independence of $b_0',\ldots,b_{k+1}'$. Therefore,

$$x \in H(b_0',\ldots,b_{k+1}') \subset \bigcup_{i=0}^{k+1} H(b_0',\ldots,\widehat{b_i'},\ldots,b_{k+1}') \equiv S,$$

and since, relative to X', S is convex-open by Theorem 3.42, we have

$$x \in \text{c-int } H(b_0',\ldots,b_{k+1}').$$

<u>4.5. COROLLARY.</u> If A is convex open, $x \in A$, and $\dim X = n$ then there exist $n+1$ independent points $b_0,\ldots,b_n$ in A such that

$$x \in \text{c-int } H(b_0,\ldots,b_n).$$

Proof. (By induction on n). If $n = 1$ then by definition of convex-open there exist $b_0, b_1 \in A$ such that $x \in (b_0 b_1) \subset A$. Assume the result for all dimensions $< n$, and consider any hyperplane H through x. Now $A \cap H = A'$ is convex-open relative to $H = X'$ and by Corollary 3.39. $\dim X' = n - 1$. By induction hypothesis there exist independent points $b_0, \ldots, b_{n-1}$ in $A' \subset A$ such that $x \in \text{c-int}H(b_0, \ldots, b_{n-1})$. Since A is convex open $A \not\subset H$ so choose $b_n' \in A \setminus H$ and $b_i' \in A$ such that $b_i \in (b_i', b_n')$ for $0 \le i \le n - 1$. Now by Lemma 4.4., $b_0', \ldots, b_n'$ are independent and

$$x \in \text{c-int } H(b_0, \ldots, b_{n-1}) \subset \text{c-int } H(b_0', \ldots, b_n').$$

4.6. LEMMA. If $b_0, \ldots, b_k$ are independent points in X, then

$$\text{c-int } H(b_0, \ldots, b_k) = H(b_0, \ldots, b_k) \setminus \bigcup_{i=0}^{k} H(b_0, \ldots, \widehat{b_i}, \ldots, b_k).$$

Proof. Since the set on the right of the equality sign is convex-open relative to $fl(b_0, \ldots, b_k)$ we have that

$$H(b_0, \ldots, b_k) \setminus \bigcup_{i=0}^{k} H(b_0, \ldots, \widehat{b_i}, \ldots, b_k) \subset \text{c-int } H(b_0, \ldots, b_k).$$

For the other inclusion, suppose $x \in \text{c-int } H(b_0, \ldots, b_k)$ and that $x \in H(b_0, \ldots, \widehat{b_i}, \ldots, b_k)$ for some i, $0 \le i \le k$. Hence $x \ne b_i \in H(b_0, \ldots b_k)$ and there is $y \in H(b_0, \ldots, b_k)$ such that $x \in b_i y)$. Also $y \in b_i H(b_0, \ldots, \widehat{b_i}, \ldots, b_k)$ implies there is $z \in H(b_0, \ldots, \widehat{b_i}, \ldots, b_k)$ such that $y \in b_i z$. Thus $x \in b_i z)$ and since $z \in H(b_0, \ldots, \widehat{b_i}, \ldots, b_k)$ we have $b_i \in L(x,z) \subset fl(b_0, \ldots, \widehat{b_i}, \ldots, b_k)$ contradicting the independence of $b_0, \ldots, b_k$. Therefore

$$x \in H(b_0, \ldots, b_k) \setminus \bigcup_{i=0}^{k} H(b_0, \ldots, \widehat{b_i}, \ldots, b_k),$$

and proof is completed.

<u>4.7. LEMMA.</u> If $b_0,\ldots,b_n$ are maximal independent, then there exist h_i, R_i, where $0 \leq i \leq n$ such that

$$\text{c-int } H(b_0,\ldots,b_n) \subset \cap h_i^{-1}(R_i) \subset H(b_0,\ldots,b_n).$$

<u>Proof.</u> Let $a \in$ c-int $H(b_0,\ldots,b_n)$. By Lemma 3.28. for each i, there exist h_i, α_i such that $h_i(b_i) > h_i(b_j) = \alpha_i$ for all $j \neq i$. Let R_i be the open ray, $R_i = (\alpha_i,\infty)$ for each i, $0 \leq i \leq n$. Now for each i there is $b \in H(b_0,\ldots,\hat{b}_i,\ldots,b_n)$ such that $a \in bb_i$ by join-hull commutativity; since $a \in$ c-int $H(b_0,\ldots,b_n)$, (bab_i) holds. Since $h_i(b) = \alpha_i$ and h_i is preconvex, we have $h_i(b) < h_i(a) < h_i(b_i)$ or $a \in h_i^{-1}(R_i)$ for all i. Thus $a \in \cap h_i^{-1}(R_i)$ or

$$\text{c-int } H(b_0,\ldots,b_n) \subset \cap h_i^{-1}(R_i).$$

Let $x \in \cap h_i^{-1}(R_i)$. Since $a \in$ c-int $H(b_0,\ldots,b_n) =$ $H(b_0,\ldots,b_n) \setminus \cup_{i=0}^{k} H(b_0,\ldots,\hat{b}_i,\ldots,b_n)$, by Theorem 3.42. we may take $b \in H(b_0,\ldots,b_n)$, $b \neq a$, such that $x \in R(a,b)$. Suppose $x \notin H(b_0,\ldots,b_n)$. Then $R(a,b) \cap H(b_0,\ldots,b_n) = ac$ and (xca) where $c \in H(b_0,\ldots,\hat{b}_i,\ldots,b_n)$ for some i. But $h_i(a) \in R_i$, $h_i(c) = \alpha_i = \text{glb } R_i$ and since h_i is one-to-one on $L(a,x)$, we have $x \notin h_i^{-1}(R_i)$, a contradiction. Thus $x \in H(b_0,\ldots,b_n)$ and $\cap h_i^{-1}(R_i) \subset H(b_0,\ldots,b_n)$.

The topology introduced by Cantwell was defined as the topology whose basic open sets were convex-open.

<u>4.8. THEOREM.</u> If $\dim X = n$, the topology on X whose basic open sets are convex-open sets is equivalent to the topology whose sub-basic open sets are $f^{-1}(R)$ where $f \in \mathfrak{F}$ and $R \subset \mathbb{R}$ is an open ray.

<u>Proof.</u> Consider $f^{-1}(R)$ where $R = (\alpha,\infty)$ for some α. Let $a \in f^{-1}(R)$ and $x \in X$. If $f(x) = f(a)$ then choose and b such that $xab)$. Since f would be constant on $L(x,a)$, we have $f(ab) \subset R$ yielding $ab) \subset ab \subset f^{-1}(R)$.

If $f(x) < f(a)$, choose b such that $xab)$. If $b \in f^{-1}(R)$, then $ab) \subset f^{-1}(R)$. If $b \notin f^{-1}(R)$, then since f is convex there is a c, (bca) such that $f(c) \in R$. Thus $a \in xc)$ and $ac) \subset f^{-1}(R)$. Hence for each f, $f^{-1}(R)$ is convex-open.

Let A be convex-open and suppose $a \in A$. Then by Corollary 4.5. there exist $b_0,\ldots,b_n \in A$ independent points such that $a \in$ c-int $H(b_0,\ldots,b_n)$. By Lemma 4.7. there exist h_i, R_i such that

$$a \in \text{c-int } H(b_0,\ldots,b_n) \subset \cap h_i^{-1}(R_i) \subset H(b_0,\ldots,b_n) \subset A.$$

Hence A is open.

REMARK. Hence the two topologies are equivalent. Cantwell shows that his topology makes X homeomorphic to E^n if dim $X = n$, so we would have the same result for (X,τ).

We continue to investigate the properties of this topology by showing that the convex hull-operator is continuous on the power set (with Hausdorff limit topology). Certain results of Shirley[8] then follow since we have representation of closed hyperplanes by continuous functions. In particular the Krein-Milman Theorem will hold.

4.9. DEFINITION. Let $\{A_\lambda\}_{\lambda\in D}$ be a net of subsets of X (D is a directed set), and define the sets

$$\liminf A_\lambda \equiv \{x \mid \text{Each neighborhood of } x \text{ eventually meets } A_\lambda\},$$

$$\limsup A_\lambda \equiv \{x \mid \text{Each neighborhood of } x \text{ frequently meets } A_\lambda\}.$$

The Hausdorff limit of $\{A_\lambda\}$ is said to exist whenever $\liminf A_\lambda = \limsup A_\lambda$, and in that case we write $\lim A_\lambda$ for the common set.

We note that we always have $\liminf A_\lambda \subset \limsup A_\lambda$ so that $\lim A_\lambda = A$ if and only if

$$\limsup A_\lambda \subset A \subset \liminf A_\lambda.$$

Moreover, if $\{A_\lambda\}$ has limit A then any subnet has limit A, and the limit structure defined for $\mathcal{P}(X)$ in this manner determines a unique Hausdorff topology for $\mathcal{P}(X)$, given a topology for X.

The continuity of the convex hull operator $H: X^m \to \mathcal{P}(X)$ for each integer $m \geq 1$ is now investigated where X has the topology introduced earlier. Let $\{x_i{}^\lambda\}$ be a net in X for each $i = 1,\ldots,m$, and put

$$A_\lambda = H(x_1{}^\lambda,\ldots,x_k^\lambda), \quad \lambda \in D$$
$$A = H(x_1,\ldots,x_k).$$

We shall prove that

$$\lim\sup A_\lambda \subset A \subset \lim\inf A_\lambda$$

which will establish

$$\lim H(x_1{}^\lambda,\ldots,x_m{}^\lambda) = H(x_1,\ldots,x_m).$$

Let $x \in \lim\sup A_\lambda$. If $x \notin A$ then by the strong separation theorem (Theorem 3.35.) there is a hyperplane $H = [f:\alpha]$ strongly separating x and A, with $A \subset H^+$ and $x \in H^-$.

$$H^+ = \{x \mid f(x) > \alpha\} = f^{-1}(R)$$

where $R = (\alpha,\infty)$, so H^+ is an open convex set containing A as well as all the points $x_1,\ldots,x_m$, so that $\{x_i{}^\lambda\}$ is eventually in H^+ for all i. Hence $H(x_1{}^\lambda,\ldots,x_m{}^\lambda) = A_\lambda$ is eventually in $H(H^+) = H^+$. But H^- is also an open set containing $x \in \lim\sup A_\lambda$; thus, H^- meets frequently many A_λ. Thus there is a $\lambda \in D$ such that $A_\lambda \subset H^+$ and $A_\lambda \cap H^- \neq \emptyset$, an impossibility since $H^+ \cap H^- = \emptyset$. Therefore $x \in A$.

Next we consider $x \in A$. Let U be a neighborhood of a and let B be a basic open subset of U containing x, with

$$B = \cap_{j=1}^k f_j^{-1}(R_j),$$

where $R_j = (\alpha_j,\infty)$ for each j. Consider $x \in B \subset f_j^{-1}(R_j)$ so that $f_j(x) > \alpha_j$.

If some subnet $\{A_\mu\}$ of $\{A_\lambda\}$ exists such that $f_j(x_j^\mu) \leq \alpha_j$ for all $i = 1,\ldots,m$ then by continuity of f_j

$$f_j(x_i) \leq \alpha_j$$

for $i = 1,\ldots,m$ which by the convexity of A and the preconvexity of f_j yields

$$f_j(x) \leq \alpha_j$$

a contradiction. Hence for each j, eventually all A_λ meet $f_j^{-1}(R_j)$ and since there are only finitely many $f_j^{-1}(R_j)$'s eventually all A_λ meet $\bigcap_{j=1}^k f_j^{-1}(R_j)$. Thus, A_λ eventually meets U, so that $x \in \lim\inf A_\lambda$. This proves:

<u>4.10. THEOREM</u>. The convex hull operator H is continuous from each finite product X^m to the power set of X.

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