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A LINEAR RIEMANN-STIELTJES INTEGRAL EQUATION SYSTEM

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A LINEAR RIEMANN-STIELTJES INTEGRAL EQUATION SYSTEM

CHAPTER I

INTRODUCTION

1. Introduction. The system treated here is a type of linear vector Riemann-Stieltjes integral equation. Under certain conditions the system reduces to the classical second-order linear differential system.

Chapter II is concerned with existence, uniqueness, and related basic properties of solutions. Chapter III is concerned with the determination of the adjoint, compatibility of the system, and basic properties of conjoined solutions. Necessary and sufficient criteria for solutions to satisfy certain boundary conditions are given in Chapter IV. Also, the relationship between the given system and an associated Riccati integral system is considered; in particular, some results concerning principal solutions are given. For self-adjoint systems, it is shown in Chapter V that there are criteria of oscillation and non-oscillation which are direct generalizations of known criteria for the classical self-adjoint differential system, while Chapter VI is devoted to the extension to such systems of the oscillation, separation, and comparison theorems occurring in the generalization of the classical Sturmian theory due to Morse ([2], [3; Chs. III, IV]).

2. The system. We shall be concerned with the system

$$\text{(E)} \qquad u(t) = u_o + \int_a^t [dN]v,$$

$$v(t) = v_o + \int_a^t [dM]u, \qquad \text{for } t \in [a,b],$$

where M and N are $n \times n$ dimensional complex valued matrix functions, while u and v are n-dimensional complex valued vector functions. By a solution we shall mean $(u(t);v(t))$ which satisfy (E) for some values of u_o and v_o. <u>We shall assume that</u> M <u>and</u> N <u>satisfy</u> H <u>which is given by</u>

H. M <u>and</u> N <u>are of bounded variation and</u> N <u>is continuous on</u> [a,b].

At various other times we shall assume M and N satisfy the following hypothesis.

H_h. M(t) <u>and</u> N(t) <u>are hermitian for</u> $t \in [a,b]$.

H^+. N <u>is strictly increasing; that is,</u> N(t) <u>is hermitian for</u> $t \in [a,b]$ <u>and</u> $N(t) - N(s)$ <u>is positive definite for</u> $s,t \in [a,b]$, $s < t$.

H_N. <u>System</u> (E) <u>is identically normal on</u> [a,b]; <u>that is, the only vector function</u> v(t), <u>such that</u> (0;v(t)) <u>is a solution of</u> (E) <u>on any interval</u> $[c,d] \subset [a,b]$ <u>is</u> $v(t) \equiv 0$.

We will also be interested in the general matrix system

$$(E_q) \qquad U(t) = U_o + \int_a^t [dN]V,$$

$$V(t) = V_o + \int_a^t [dM]U, \quad \text{for} \quad t \in [a,b],$$

where U and V are $n \times q$ matrix functions. If M and N are absolutely continuous functions, (E) may be reduced to the classical second order differential system, while if N is absolutely continuous and M of bounded variation, system (E) may be reduced to the system found in Reid [4].

Matrix notation is used throughout; in particular, matrices of one column are called vectors, all $n \times n$, $n \geq 1$, identity matrices are denoted by the symbol E, and 0 is used indiscriminately for the zero matrix of any dimensions. Let C^n denote the set of n-dimensional complex valued vectors. If $u = (u_1,\cdots,u_n)$ and $v = (v_1,\cdots,v_n)$ are elements of C^n, then the inner product of u and v, (u,v), will be the usual inner product $\sum_\alpha u_\alpha \bar{v}_\alpha$; the norm of u, $|u|$, will be the usual norm $(u,u)^{1/2}$; and the symbol (u;v) will denote the 2n-dimensional vector $y = (y_1,\cdots,y_{2n})^T$ such that $y_i = u_i$ and $y_{i+n} = v_i$ for $i = 1,\cdots,n$. The conjugate transpose of a matrix H is denoted by H^*, and H is called hermitian whenever $H^* = H$. The symbol $\int_a^b [dN]u$ will denote the n-dimensional vector whose i-th component is $\sum_\alpha \int_a^b [dN_{i\alpha}]u_\alpha$ and if U is an $n \times q$ matrix function $\int_a^b [dN]U$ will denote the $n \times q$ matrix whose ij-th component is $\sum_\alpha \int_a^b [dN_{i\alpha}]U_{\alpha j}$. The integrals are Riemann-Stieltjes and the variable of integration normally will be omitted. N is called non-decreasing on [a,b], if N(t) is hermitian for t in this interval and $N(t) - N(s)$ is non-negative definite for $s,t \in [a,b]$, $s < t$. The norm of an $n \times d$ matrix N is $|N| = \sup\{|N\xi| \mid \xi \in C^d, |\xi| = 1\}$.

The sets BV[a,b], C[a,b], C^1[a,b], BC[a,b], and CB[a,b] are the n-dimensional vector functions of bounded variation on [a,b], the n-dimensional vector functions which are continuous on [a,b], the subset of C[a,b] of functions with continuous first derivative, BV[a,b] $\times$ C[a,b], and $C[a,b] \times BV[a,b]$, respectively. If a subscript of 0 is used on any set, it will indicate that the functions are restricted to be zero at the endpoints; for example, $\eta \in C_o[a,b]$ if $\eta \in C[a,b]$ and $\eta(a) = 0 = \eta(b)$. The symbol (1.1) will refer to the statement numbered 1.1 in the chapter it is given, while (II.1.1) will refer to statement 1.1 of Chapter II, and will be used in chapters other than II. Theorems, lemmas, and corollaries will also be numbered in this manner.

CHAPTER II

EXISTENCE AND UNIQUENESS OF SOLUTIONS

1. Existence. We first wish to examine the existence of solutions to our system. If F is an n $\times$ n matrix function of bounded variation on [a,b], let $h = h_F$ be defined as follows:

$$\text{(1.1)}\quad h(a) = 0,$$

$$h(t) = h_F(t) = \sup\{\textstyle\sum_{j=1}^{k} |F(s_j) - F(s_{j-1})| \,\big|\, a \le s_o \le s_1 \le \cdots \le s_k \le t\}$$

$$\text{for } t \in (a,b].$$

Then h is monotone non-decreasing on [a,b], and continuous at $t \in [a,b]$ if F is continuous at t.

If $w(t) = (w_\alpha(t))$ is a vector function on [a,b] which is such that $\int_a^b [dF]w$ exists and ϕ is a real valued function such that $\int_a^b \phi dh_F$ exists and $|w(t)| \le \phi(t)$ for $t \in [a,b]$, then it follows readily that

$$\left|\int_a^b [dF]w\right| \le \int_a^b \phi dh_F.$$

Using these conclusions, we get the following results.

THEOREM 1.1. If M and N satisfy H, system (E) has a solution for arbitrary n-dimensional u_o, v_o.

We have that (u;v) is a solution of (E) if and only if

$$\text{(1.2)}\qquad u(t) = g(t) + \int_a^t [dN(s)]\{\int_a^s [dM(r)]u(r)\}$$

with $g(t) = u_o + [N(t) - N(a)]v_o$. In particular, $g(t)$ is continuous.

Let

$$(1.3) \quad u_o(t) \equiv 0,$$

$$u_{m+1}(t) = g(t) + \int_a^t [dN(s_1)] \int_a^{s_1} [dM(s_2)]u_m(s_2) \ , \ m = 0,1,2,\cdots.$$

Then

$$u_2(t) - u_1(t) = \int_a^t [dN(s_1)] \int_a^{s_1} [dM(s_2)]g(s_2) \ .$$

Let $h_M = \mu$, $h_N = \nu$, where for general matrix functions h_F is defined as above. Then ν is continuous and μ,ν are monotone non-decreasing on $[a,b]$. If k is a constant such that $|g(t)| \leq k$ on $[a,b]$, then we have

$$(1.4) \qquad \left|\int_a^s [dM]g\right| \leq k\int_a^s d\mu = k\mu(s), \text{ for } s \in [a,b],$$

so that

$$(1.5) \qquad |u_2(t) - u_1(t)| \leq \int_a^t k\mu d\nu = k\int_a^t \mu d\nu \text{ for } t \in [a,b].$$

Suppose that the inequality

$$(1.6) \qquad |u_{m+1}(t) - u_m(t)| \leq \frac{k}{m!}\left[\int_a^t \mu d\nu\right]^m, \text{ for } t \in [a,b]$$

holds for $m = r$. Then we have

$$\begin{aligned} |u_{r+2}(t) - u_{r+1}(t)| &= \left|\int_a^t [dN(s_1)] \int_a^{s_1} [dM(s_2)][u_{r+1}(s_2) - u_r(s_2)]\right| \\ &\leq \int_a^t \left[\int_a^{s_1} \frac{k}{r!}\left[\int_a^{s_2} \mu(s_3)d\nu(s_3)\right]^r d\mu(s_2)\right] d\nu(s_1) \\ &\leq \frac{k}{r!}\int_a^t \int_a^{s_1} \left[\int_a^{s_2} \mu(s_3)d\nu(s_3)\right]^r d\mu(s_2)\ d\nu(s_1) \end{aligned}$$

$$\leq \frac{k}{r!} \int_a^t \left[\int_a^{s_1} \mu(s_2)d\nu(s_2)\right]^r d\left[\int_a^{s_1} \mu(s_2)d\nu(s_2)\right]$$

$$\leq \frac{k}{(r+1)!} \left[\int_a^t \mu d\nu\right]^{r+1}.$$

Hence, we have that (1.6) holds for all $m = 1,2,\cdots$ and

$$|u_{m+1}(t) - u_m(t)| \leq \frac{k}{m!} \left[\int_a^b \mu d\nu\right]^m, \text{ for } m = 1,2,\cdots.$$

Consequently, the series $\sum_j [u_j(t) - u_{j-1}(t)]$ converges uniformly on $[a,b]$, and

$$(1.7) \qquad |\textstyle\sum_j [u_j(t) - u_{j-1}(t)]| \leq \sum_j \frac{k}{j!} \left[\int_a^b \mu d\nu\right]^j$$

$$\leq ke^{\int_a^b \mu d\nu}.$$

Since the convergence is uniform, we get that

$$(1.8) \qquad u(t) = \textstyle\sum_{j=1}^{\infty} [u_j(t) - u_{j-1}(t)]$$

is continuous and $u(t)$, $v(t) = v_o + \int_a^t [dM]u$ is a solution to our system (E).

2. Uniqueness.

THEOREM 2.1. *The solution of* (E) *for given values of* u_o *and* v_o *is unique*.

This is equivalent to showing that if $(u;v)$ is a solution of (E) with $u_o = 0 = v_o$ then $u(t) \equiv 0 \equiv v(t)$. But $u(t)$ is continuous so there exists a $k \geq 0$ such that $|u(t)| \leq k$ for $t \in [a,b]$. Moreover,

$$u(t) = \int_a^t [dN(s_1)] \int_a^{s_1} dM(s_2)u(s_2)$$

so that

$$|u(t)| \le k\int_a^t \mu d\nu \le k\int_a^b \mu d\nu.$$

If we assume $|u(t)| \le \frac{k}{r!}\left[\int_a^t \mu d\nu\right]^r$ for $t \in [a,b]$ we have

$$|u(t)| \le \int_a^t \int_a^{s_1} |u(s_2)|d\mu(s_2)\ d\nu(s_1)$$

$$\le \frac{k}{r!}\int_a^t \left[\int_a^s \mu d\nu\right]^r d\left[\int_a^s \mu d\nu\right]$$

$$\le \frac{k}{(r+1)!}\left[\int_a^t \mu d\nu\right]^{r+1}.$$

Hence, $|u(t)| \le [k/(r+1)!][\int_a^t \mu d\nu]^{r+1} \le [k/(r+1)!][\int_a^b \mu d\nu]^{r+1}$ for arbitrary integers r and thus $u(t) \equiv 0$, $v(t) = \int_a^t [dN]u \equiv 0$.

In the above argument, we could have used any point $t_o \in [a,b]$ as the initial point and obtained the inequality

$$(2.1) \qquad |u(t)| \le ke^{\left|\int_{t_o}^t \mu d\nu\right|}, \quad t \in [a,b].$$

If we let $u = u(t,t_o,u_o,v_o)$, $v = v(t,t_o,u_o,v_o)$ be the solutions of (E) with initial conditions $u(t_o) = u_o$, $v(t_o) = v_o$, we find that u and v are not necessarily continuous in t_o. However, we may choose k uniformly for a set of the type

$$(2.2) \qquad D_j = \{(t_o,u_o,v_o) \mid t_o \in [a,b],\ |u_o| \le j,\ |v_o| \le j\},$$

so that we get the following result directly from the uniform convergence of the sequence (1.8).

COROLLARY 2.1. *For a given* $j > 0$ *there exists a* $c_j > 0$ *such that* $|u(t,t_o,u_o,v_o)| \le c_j$, $|v(t,t_o,u_o,v_o)| \le c_j$ *for* $(t_o,u_o,v_o) \in D_j$, $t \in [a,b]$;

<u>moreover</u>, <u>uniformly for</u> $t_o \in [a,b]$ <u>the vector function</u> $u(t,t_o,u_o,v_o)$ <u>is continuous in</u> (t,u_o,v_o) <u>on</u> D_j, <u>and uniformly for</u> $t,t_o \in [a,b]$ <u>the vector function</u> $v(t,t_o,u_o,v_o)$ <u>is continuous in</u> (u_o,v_o) <u>on</u> $|u_o| \leq j$, $|v_o| \leq j$.

CHAPTER III

PRELIMINARY RESULTS

1. <u>The adjoint system.</u> In this section we shall consider the adjoint system and some of its properties. To do this, system (E) shall be changed into a 2n-dimensional system by the following substitutions. Let

$$J = \begin{bmatrix} 0 & E_n \\ -E_n & 0 \end{bmatrix}, \quad \mathfrak{M} = \mathrm{diag}\{-M,N\}, \quad y = (u;v).$$

It is to be noted that $J^{-1} = J^* = -J$, and that $\mathfrak{M}$ is hermitian if and only if M and N are hermitian. Thus system (E) is equivalent to

$$(1.1) \qquad L[y](t) \equiv J[dy] + [d\mathfrak{M}]y = d\Psi_y$$

for $y \in CB[a,b]$ and $d\Psi_y = 0$. However, we may study L[y] for $\Psi_y \in BC[a,b]$, and will do so. Let $\mathcal{D}(L) = CB[a,b]$. Since solutions to $L[y](t) = 0$ are uniquely determined by initial values $y(t_o)$ at a given $t_o \in [a,b]$ we obtain the following result for the corresponding matrix system

$$(1.1_{2n}) \qquad L[Y](t) \equiv J[dY] + [d\mathfrak{M}]Y = 0.$$

LEMMA 1.1. <u>If</u> Y(t) <u>is a solution of</u> (1.1_{2n}), <u>then the rank of</u> Y <u>is constant</u>.

We wish to show that the system defined by

$$(1.2) \qquad L^*[z](t) \equiv J[dz] + [d\mathfrak{M}^*]z = 0$$

is the adjoint system of (1.1). To do this we need the following result.

LEMMA 1.2. *If* z *is a solution of* $L^{*}[z] = 0$ *and* y *is a solution of* $L[y] = 0$, *then there exists a constant* k *such that* $z^{*}Jy \equiv k$.

We have that if $y = (u;v)$ and $z = (\eta;\zeta)$ are solutions of the respective equations $L[y] = 0$ and $L^{*}[z] = 0$,

$$v(s) - v(s^{-}) = [M(s) - M(s^{-})]u(s) \text{ and}$$

$$\zeta(s) - \zeta(s^{-}) = [M^{*}(s) - M^{*}(s^{-})]\eta(s);$$

so we may verify by direct substitution that

$$z^{*}(s)Jy(s) = z^{*}(s^{-})Jy(s^{-}), \quad s \in (a,b].$$

In a similar fashion we may show that

$$z^{*}(s)Jy(s) = z^{*}(s^{+})Jy(s^{+}), \quad s \in [a,b),$$

and thus $z^{*}Jy$ is a continuous function on $[a,b]$.

Let $\eta \in C_{o}^{1}[a,b]$. Then we wish to examine

$$\int_{a}^{b} z^{*}Jy\eta' dt = \int_{a}^{b} z^{*}Jyd\eta. \tag{1.3}$$

Now y and z are elements of $CB[a,b]$ so the integrals of (1.3) equal

$$\int_{a}^{b} [dz^{*}Jy\eta] - \int_{a}^{b} [dz^{*}]Jy\eta - \int_{a}^{b} z^{*}J[dy]\eta. \tag{1.4}$$

But $[dz^{*}]J = z^{*}[d\mathcal{M}]$, $J[dy] = -[d\mathcal{M}]y$, and $\eta(a) = 0 = \eta(b)$, so that (1.4) becomes

$$-\int_{a}^{b} z^{*}[d\mathcal{M}]y\eta + \int_{a}^{b} z^{*}[d\mathcal{M}]y\eta = 0. \tag{1.5}$$

Consequently, by the fundamental lemma of the calculus of variations there exists a constant k such that $z^{*}Jy$ is equal to k at every point of the interval $[a,b]$.

If $Y(t)$ is a fundamental matrix solution of (1.1), and $Z(t)$ is a fundamental matrix solution of (1.2), then $Z^*JY \equiv C$ where C is a constant, non-singular matrix. Thus $JY^{*-1}(t) = Z(t)C^{*-1}$, and $Z(t)C^{*-1}$ is a fundamental matrix solution of (1.2). Also, $Y^{-1}(t) = C^{-1}Z^*(t)J$ is of the form $[P \quad Q]$ where Q is continuous and P and Q are $2n \times n$ matrix functions of bounded variation on $[a,b]$. In particular, the integrals

$$\int_a^b [dY^{-1}]Y \quad \text{and} \quad \int_a^b Y^{-1}[dY]$$

exist. Moreover,

$$\int_a^b [dY^{-1}J^{-1}]JY \quad \text{and} \quad \int_a^b Y^{-1}J^{-1}[dJY].$$

also exist.

Let ϕ be a function on $[a,b]$ such that ϕ^*y is constant on $[a,b]$ for all y which are solutions of $L[y](t) = 0$. If Y is a fundamental matrix solution of (1.1) and $z(t) = J\phi(t)$, then $\gamma^* = z^*JY$ where γ is a constant vector. Moreover, if $\phi = J^*z$ with $z = JY^{*-1}\gamma$, then $\phi^*y = z^*Jy = \gamma^*Y^{-1}J^{-1}Jy = \gamma^*Y^{-1}y$ which is constant on $[a,b]$. Now, for $t \in [a,b]$, we have

$$0 = \int_a^t dE = \int_a^t [dY^{-1}J^{-1}]JY + \int_a^t Y^{-1}J^{-1}d[JY]$$

$$= \int_a^t \{[dY^{-1}J^{-1}]J - Y^{-1}J^{-1}[d\eta]\}Y$$

so that

$$0 = \int_a^t \{[dY^{-1}J^{-1}]J - Y^{-1}J^{-1}[d\eta]\}$$

$$= \int_a^t \{J^*[dJ^*Y^{*-1}] - [d\eta^*]J^{*-1}Y^{*-1}\},$$

and since $J^* = -J$ it follows that $z = J^{*-1}Y^{*-1}\gamma$ is a solution of (1.2).

We let

$$(1.6)\qquad L^{\star}[z](t) \equiv Jdz + [d\mathcal{M}^{*}]z = d\Psi^{\star}_{z},\quad \mathcal{D}(L^{\star}) = CB[a,b].$$

It is to be noted that if M and N satisfy hypothesis H_h, then $L^{\star}[z](t) = 0$ is exactly $L[y](t) = 0$. If we let $u_y(t) = Jy(t)$ and $U_Y(t) = JY(t)$, we get the following result.

THEOREM 1.1. (i) <u>If</u> y <u>and</u> z <u>are solutions of</u> (1.1) <u>and</u> (1.2), <u>respectively, then</u> $z^*(t)Jy(t)$ <u>is constant on</u> [a,b]; (ii) <u>if</u> Y(t) <u>and</u> Z(t) <u>are solutions of the matrix equations for</u> (1.1) <u>and</u> (1.2) <u>respectively, then there is a constant matrix</u> C <u>such that</u> $Z^*(t)JY(t) = C$ <u>on</u> [a,b]; (iii) <u>if</u> Y(t) <u>is a fundamental matrix for</u> (1.1) <u>and</u> Z(t) <u>is defined by</u> $Z^*(t)JY(t) = C$ <u>where</u> C <u>is a constant matrix, then</u> Z <u>is a solution of</u> (1.2); <u>moreover,</u> Z(t) <u>is a fundamental matrix for</u> (1.2) <u>if and only if</u> C <u>is non-singular</u>.

In view of (1.1) and (1.6), we have the identity

$$(1.7)\qquad (d\Psi_y, z) - (y, d\Psi^{\star}_z) = d(z^*Jy),$$

so that

$$(1.8)\qquad \int_a^b (d\Psi_y, z) - \int_a^b (y, d\Psi^{\star}_z) = z^*Jy\Big|_a^b,$$

for $y \in \mathcal{D}(L)$, $z \in \mathcal{D}(L^{\star})$.

In particular, from (1.8) it follows that

$$(1.9)\qquad \int_a^b (d\Psi_y, z) = \int_a^b (y, d\Psi^{\star}_z) \text{ for } y \in \mathcal{D}_o(L),\ z \in \mathcal{D}(L^{\star}),$$

where it is to be recalled that $\mathcal{D}_o(L) = \{y \mid y \in \mathcal{D}(L),\ y(a) = 0 = y(b)\}$. Moreover, if $z \in CB[a,b]$ and there exists an $\Psi^{\star}_z \in BC[a,b]$ such that

$$(1.10)\qquad \int_a^b (d\Psi_y, z) - \int_a^b (y, d\Psi^{\star}_z) = 0 \text{ for } y \in \mathcal{D}_o(L),$$

we can rewrite (1.10) as

$$(1.11) \qquad 0 = \int_a^b (dy, J^* z) + \int_a^b (y, [d\eta^*]z - d\Psi_z^*), \text{ for } y \in \mathcal{D}_o(L),$$

so that

$$(1.11') \qquad 0 = \int_a^b (dy, J^* z - \int_a^t [d\eta^*]z + \Psi_z^*), \text{ for } y \in \mathcal{D}_o(L).$$

Consequently, by the fundamental lemma of the calculus of variations, there exists a constant vector γ such that

$$Jz(t) + \int_a^t [d\eta^*]z + \gamma = \Psi_z^*(t)$$

or

$$J[dz] + [d\eta^*]z = d\Psi_z^*.$$

That is, we have the following result.

THEOREM 1.2. <u>The class $\mathcal{D}(L^*)$ is characterized as the set of vector functions</u> $z \in CB[a,b]$ <u>such that there exists a corresponding</u> $\Psi_z^* \in BC[a,b]$ <u>for which</u> (1.10) <u>holds, and for</u> $z \in \mathcal{D}(L^*)$ <u>the corresponding</u> $d\Psi_z^*$ <u>is uniquely determined as</u> $L^*[z]$.

2. <u>Compatibility</u>. We now wish to consider the operator L with domain D(L), a manifold between $\mathcal{D}_o(L)$ and $\mathcal{D}(L)$, and examine those functions $y \in D(L)$ that satisfy the system

$$(2.1) \qquad L[y](t) = 0, \quad y \in D(L).$$

Clearly the solutions of (2.1) form a vector space. If there are non-trivial solutions, then there is a uniquely determined integer k, $(1 \le k \le n)$ such that $y^{(1)}, y^{(2)}, \cdots, y^{(k)}$ are linearly independent and span the space of solutions. If $k > 0$, we say system (2.1) is <u>compatible</u>

and has <u>index</u> k. If there are no non-trivial solutions, we will say the system is <u>incompatible</u>.

Let

$$D(L;a,b) = \{\hat{u}_y \mid \hat{u}_y = (Jy(a), Jy(b)) \text{ for all } y \in D(L)\},$$

where J is defined in the preceding section. Now, D(L;a,b) specifies D(L) since the condition that $\mathcal{D}_0(L) \subset D(L) \subset \mathcal{D}(L)$ implies that $y \in D(L)$ if and only if $y \in \mathcal{D}(L)$ and $\hat{u}_y \in D(L;a,b)$. Thus we can examine D(L) by considering D(L;a,b). Let P be a matrix whose column vectors form a basis for D(L;a,b). By examining the various cases as in Reid [6; pp. 127-128] we can obtain the following result.

THEOREM 2.1. <u>The system</u> (2.1) <u>has index</u> 2n <u>if</u> dim D(L;a,b) = 4n; <u>it has index</u> 0 <u>if</u> dim D(L;a,b) = 0. <u>If</u> dim D(L;a,b) = 4n-m, $1 \leq m \leq 4n-1$, <u>and the index is</u> k, <u>then the rank of the</u> $4n \times (6n-m)$ <u>matrix</u> $[\hat{U}_Y \;\; P]$ <u>is</u> $6n - m - k$.

Now we wish to examine the conditions under which we get a solution to the differential system

$$(2.5) \qquad L[y](t) = 0, \quad \hat{u}_y - \omega \in D(L;a,b),$$

for ω some 4n-dimensional vector. To do this we need the adjoint system

$$(2.6) \qquad L^*[z](t) = 0, \quad \hat{z} \in D(L^*;a,b),$$

where $D(L^*;a,b)$ is the set of all vectors orthogonal to the space spanned by vectors of the set

$$(2.7) \qquad T = \{\tau \mid \tau = Q\hat{u}_y \text{ for } y \in D(L)\}$$

with $Q = \operatorname{diag}\{-E_{2n}, E_{2n}\}$.

To consider (2.5), we shall examine

(2.8) $\qquad L[y](t) = d\Psi(t)$, for $t \in [a,b]$, $y - w \in D(L)$,

where $w \in D(L)$ and $\Psi \in BC[a,b]$. If ω is an arbitrary 4n-dimensional vector, there is a $w \in D(L)$ such that $\hat{u}_w = \omega$, so system (2.8) is equivalent to

(2.8') $\qquad L[y](t) = d\Psi(t)$, for $t \in [a,b]$, $\hat{u}_y - \omega \in D(L;a,b)$.

If (2.8) has a particular solution $y_p(t)$, then the general solution for (2.8) is the sum of $y_p(t)$ and the general solution for (2.1).

By examining the various cases we can obtain the following result.

THEOREM 2.2. <u>If</u> $y_p(t)$ <u>is a particular solution of the nonhomogeneous system</u> (2.8), <u>and</u> Y <u>is a fundamental matrix of the corresponding homogeneous system, then</u>: (i) <u>if</u> $\dim D(L;a,b) = 4n$, (2.8) <u>has a solution of the form</u> $y(t) = y_p(t) + Y(t)\xi$, <u>for arbitrary</u> ξ; (ii) <u>if</u> $\dim D(L;a,b) = 0$, (2.8) <u>has a solution if and only if</u> $[\hat{U}_Y \;\; \hat{u}_w - \hat{u}_{y_p}]$ <u>has rank</u> 2n, <u>and the solution is unique</u>; (iii) <u>if</u> $\dim D(L;a,b) = 4n-m$, $1 \le m \le 4n-1$, (2.8) <u>has a solution if and only if the matrices</u> $[\hat{U}_Y \;\; P]$ <u>and</u> $[\hat{U}_Y \;\; P \;\; \hat{u}_w - \hat{u}_{y_p}]$ <u>have the same rank</u>. <u>The general solution is</u> $y(t) = y_p(t) + y^{(1)}(t)\alpha_1 + y^{(2)}(t)\alpha_2 + \cdots + y^{(k)}(t)\alpha_k$, <u>where</u> α_i <u>is arbitrary</u>, $i = 1,\cdots,k$, <u>and</u> $\{y^{(1)}(t),\cdots,y^{(k)}(t)\}$ <u>is a basis for the set of solutions of</u> (2.8).

The symbol $D(L^{\star})$ will denote the manifold of all 2n-dimensional vector functions $z \in CB[a,b]$, for which there is a corresponding $\Psi_z^{\star} \in BC[a,b]$ such that

$$\int_a^b (d\Psi_y, z) - \int_a^b (y, d\Psi_z^{\star}) = 0 \text{ for all } y \in D(L).$$

Since $\mathcal{D}_o(L) \subset D(L)$, Theorem 1.2 implies that $D(L^{\star}) \subset \mathcal{D}(L^{\star})$ and

$d\Psi_z^{\star}(t) = L^{\star}[z](t)$ for $t \in [a,b]$, $z \in D(L^{\star})$. Thus $D(L^{\star})$ has the characterization

$$D(L^{\star}) = \{z \mid z \in \mathcal{D}(L^{\star}),\ \hat{z}^{*}Q\hat{u}_y = 0 \text{ for } y \in D(L)\}.$$

Consequently, the system

$$L^{\star}[z](t) = 0, \text{ for } t \in [a,b],\ z \in D(L^{\star}) \tag{2.6'}$$

is equivalent to (2.6) and is called the <u>adjoint</u> of the system (2.1).

LEMMA 2.1. <u>If</u> k <u>is the index of</u> (2.1) <u>and</u> k^* <u>is the index of</u> (2.6), <u>then</u> $2n + k^* = m + k$.

The proof if $m = 0$ or $m = 4n$ is obvious, so we need only examine the case that $1 \leq m \leq 4n-1$. If k is the index of the system (2.1), the matrix $[\hat{U}_Y \quad P]$ has rank $6n - m - k$. Therefore, if Z is a fundamental matrix solution of $L^{\star}[z](t) = 0$ such that $Z^{*}JY = E$ then

$$\begin{bmatrix} Z^{*}(a) & Z^{*}(b) \\ 0 & Z^{*}(b) \end{bmatrix} Q[\hat{U}_Y \quad P] \tag{2.10}$$

has rank $6n - m - k$. But this matrix is of the form

$$\begin{bmatrix} 0 & \hat{Z}^{*}QP \\ E_{2n} & X \end{bmatrix}, \tag{2.11}$$

and thus $P^{*}Q\hat{Z}$ has rank $4n - m - k$. But Theorem 2.1 applied to (2.6) yields the result that $P^{*}QZ$ has rank $2n - k^{*}$. Hence $4n - m - k = 2n - k^{*}$ or $2n + k^{*} = m + k$.

Using this result, we may establish the following result by examining the various cases.

THEOREM 2.3. <u>System</u> (2.8) <u>has a solution if and only if</u>

$$\int_a^b z^{*}d\Psi = \hat{z}^{*}Q\hat{u}_w \tag{2.12}$$

for all solutions of the homogeneous adjoint system (2.6).

3. *Conjoined solutions*. We have shown that $L^*[z] = L[z]$ if H_h is satisfied, so Theorem 1.1 implies the following result.

LEMMA 3.1. *If hypothesis* H_h *is satisfied, while* $(u_1;v_1)$ *and* $(u_2;v_2)$ *are solutions of* (E), *then*

(3.1) $\{u_1;v_1|u_2;v_2\}(t) = v_2^*(t)u_1(t) - u_2^*(t)v_1(t) \equiv k$ *for* $t \in [a,b]$,

where k *is some constant*.

Since this result will be used heavily in the remainder of the chapter, *we shall assume* H_h *is satisfied in the remainder of this chapter*.

If $(U_\alpha;V_\alpha)$, $(\alpha = 1,2)$ are solutions of (E_n), then Lemma 1.2 implies that

$$\{U_1;V_1|U_2;V_2\} = V_2^*U_1 - U_2^*V_1$$

is constant on $[a,b]$.

If $(u_\alpha;v_\alpha)$, $(\alpha = 1,2)$, are solutions of (E) such that the constant function $\{u_1;v_1|u_2;v_2\}$ is zero, these solutions are said to be (*mutually*) *conjoined*. If $(u;v)$ is a solution of (E) such that $\{u;v|u;v\}$ is zero, we say that $(u;v)$ is *self-conjoined*; in particular, all real solutions $(u;v)$ are self-conjoined.

If $Y(t) = (U(t);V(t))$ is a $2n \times r$ matrix whose column vectors are r linearly independent solutions of (E) which are mutually conjoined, these solutions form a basis for a *conjoined family of solutions of dimension* r, consisting of the set of all solutions of (E) which are linear combinations of these vectors. As in Reid [6, p. 306] we have the following result.

THEOREM 3.1. *The maximal dimension of a conjoined family of* *solutions of* (E) *is* n; *moreover, a given conjoined family of solutions of dimension* $r < n$ *is contained in a conjoined family of dimension* n.

If $Y(t) = (U(t);V(t))$ is a solution of (E_n) on $[a,b]$ whose vectors form a basis for an n-dimensional conjoined family of solutions, then for brevity we shall say that $Y(t)$ is a *conjoined basis* of (E). In particular, if $c \in [a,b]$, we shall denote by $Y(t,c) = (U(t,c);V(t,c))$ the solution of (E_n) satisfying the initial conditions $Y(c,c) = (0;E)$. As $\{U(c,c);V(c,c)|U(c,c);V(c,c)\} = 0$ it follows that $Y(t,c)$ is a conjoined basis for (E). Correspondingly, if $Y_o(t,c) = (U_o(t,c);V_o(t,c))$ is the solution of (E_n) satisfying the initial condition $Y_o(c,c) = (E,0)$, then $Y_o(t,c)$ is also a conjoined basis for (E).

The following result is of basic importance for the study of systems (E).

THEOREM 3.2. *Suppose* $Y_1(t) = (U_1(t);V_1(t))$ *is a solution of* (E_n) *with* $U_1(t)$ *non-singular on* $[c,d] \subset [a,b]$ *and* $K = -\{U_1;V_1|U_1;V_1\}$. *The matrix function* $Y(t) = (U(t);V(t))$ *is a solution of* (E_n) *on* $[c,d]$ *if and only if on this interval*

$$(3.2) \qquad U(t) = U_1(t)H(t), \quad V(t) = V_1(t)H(t) + U_1^{*-1}(t)[K_1 - KH(t)],$$

where K_1 *is a constant matrix and* $H(t)$ *is a solution of the matrix equation*

$$(3.3) \qquad \int_c^t dH(s) = \int_c^t U_1^{-1}(s)[dN(s)]U_1^{*-1}(s)[K_1 - KH(s)], \text{ for } t \in [c,d].$$

If T *is the solution of the matrix system*

$$(3.4) \qquad \int_c^t dT(s) = -\int_c^t U_1^{-1}(s)[dN(s)]U_1^{*-1}(s)KT(s), \quad T(c) = E,$$

then $H(t)$ *is of the form*

$$(3.5) \qquad H(t) = T(t,c|U_1)[K_o + S(t,c|U_1)K_1], \text{ for } t \in [c,d],$$

<u>where</u>

$$(3.6) \qquad S(t,c|U_1) = \int_c^t T^{-1}(s,c|U_1)U_1^{-1}(s)[dN(s)]U_1^{*-1}(s) \; \underline{\text{for}} \; t \in [c,d],$$

<u>and</u> $K_1 = -\{U;V|U_1;V_1\}$.

We may write $U(t) = U_1(t)H(t)$, $V(t) = V_1(t)H(t) + A(t)$. Then

$$\int_c^t [dU_1]H + \int_c^t U_1[dH] = \int_c^t [dU] = \int_c^t [dN]V_1H + \int_c^t [dN]A$$

$$= \int_c^t [dU_1]H + \int_c^t [dN]A$$

so that

$$\int_c^t U_1[dH] = \int_c^t [dN]A, \text{ for } t \in [c,d].$$

Also,

$$\int_c^t [dV_1]H + \int_c^t V_1[dH] + \int_c^t [dA] = \int_c^t [d(V_1H + A)] = \int_c^t [dV]$$

but

$$\int_c^t [dV] = \int_c^t [dM]U_1H = \int_c^t [dV_1]H$$

so that

$$\int_c^t V_1[dH] + \int_c^t [dA] = 0, \text{ for } t \in [c,d].$$

Consequently, we have

$$\int_c^t U_1^*[dA] = -\int_c^t K[dH] - \int_c^t [dU_1^*]A \text{ for } t \in [c,d].$$

Thus

$$U_1^*(t)A(t) - U_1^*(c)A(c) = - KH(t) + KH(c),$$

and we have

$$A(t) = U_1^{*-1}(t)[U_1^*(c)A(c) + KH(c) - KH(t)], \text{ for } t \in [c,d],$$

but

$$U_1^*(c)A(c) + KH(c) = K_1,$$

so that

$$A(t) = U_1^{*-1}(t)[K_1 - KH(t)] \quad \text{for} \quad t \in [c,d].$$

Now

$$\int_c^t U_1[dH] - \int_c^t [dN]U_1^{*-1}[K_1 - KH] = 0$$

which implies (3.3).

Since (3.3) has a unique solution, we may verify (3.5) by substituting the stated value of H(t) into (3.3). The proper use of integration by parts shows that this value satisfies (3.3).

CHAPTER IV

NORMALITY AND ABNORMALITY

1. Definitions. For a nondegenerate subinterval I_o, let $\Lambda(I_o)$ denote the set of all functions v such that $(u(t) \equiv 0; v(t))$ is a solution of (E). It is to be noted that if $v \in \Lambda(I_o)$, then $v(t)$ is a constant vector function such that $N(t)v(t)$ is also constant on I_o. If the dimension of $\Lambda(I_o)$ is $d = d(I_o)$ and $d > 0$ we say (E) is *abnormal of order* d, while if $d = 0$ we shall say (E) is *normal*. If $I_o \subset I_o^1$, then $d(I_o) \geq d(I_o^1)$. Moreover, if N satisfies hypothesis H^+ we have that $d = 0$ and thus (E) is normal.

Two points $c,d \in [a,b]$ are said to be *(mutually) conjugate* with respect to (E) if there is a solution $y(t) = (u(t);v(t))$ of (E) such that $u(c) = 0 = u(d)$ and $u(t) \not\equiv 0$ on the subinterval with c and d as endpoints. The system is called *disconjugate* on $[c,d]$ if no two points of this subinterval are conjugate. If there exists an interval of the form (c,∞) for which no two points are conjugate, then (E) is said to be *disconjugate for large* t.

We shall let the vector space $\Omega_o[a,b]$ be the space of all functions $(u(t);v(t))$ which are solutions of (E) such that $u(a) = 0 = u(b)$, and denote by $k[a,b]$ the dimension of $\Omega_o[a,b]$. It is to be noted that $k[a,b] \geq d[a,b]$ and $k[a,b] > d[a,b]$ if and only if a and b are mutually conjugate. The number $k[a,b] - d[a,b]$ is the *order of* b{a} *as a conjugate* *point to* a {b}.

LEMMA 1.1. *If* $[c,d] \subset [a,b]$ *and* u^c, u^d *are n-dimensional vectors, then there exists a solution* $y_o(t) = (u_o(t); v_o(t))$ *of* (E) *such that* $u_o(c) = u^c$, $u_o(d) = u^d$ *if and only if*

$$(1.1) \qquad v^*(c)u^c - v^*(d)u^d = 0 \text{ for arbitrary } (u(t);v(t)) \in \Omega_o[a,b].$$

This is a direct application of Theorem III.2.2, since (E) together with the boundary conditions $u(c) = 0 = u(d)$ is self-adjoint.

Let $\mathcal{D}(I_o)$ be the set of all functions η such that

$$(1.2) \qquad \int_a^t d\eta = \int_a^t [dN]\zeta$$

where ζ is any function which is integrable with respect to N. Then we have for $\rho \in \Lambda[a,b]$,

$$(1.3) \qquad 0 = \int_a^t \rho^*\{d\eta(s) - [dN(s)]\zeta(s)\} + \int_a^t [d\rho^*]\eta(s)$$

$$= \int_a^t \{\rho^*[d\eta(s)] + [d\rho^*]\eta(s)\} - \int_a^t \rho^*[dN(s)]\zeta(s)$$

$$= \int_a^t [d\rho^*\eta(s)].$$

This relation, together with Lemma 1.1, gives the following result.

LEMMA 1.2. *If* η *satisfies* (1.2) *for* $\zeta(t)$, *then for* $\rho \in \Lambda[a,b]$ *the function* $\rho^*\eta(t)$ *is constant on* $[a,b]$. *Moreover, if* $[c,d] \subset [a,b]$ *and* c *and* d *are not mutually conjugate, then there is a solution of* (E) *satisfying* $u(c) = u^c$, $u(d) = 0$, $\{u(c) = 0,\ u(d) = u^d\}$, *if and only if* $\rho^*u^c = 0$ $\{\rho^*u^d = 0\}$ *for all* $\rho \in \Lambda[c,d]$.

If c is a point of [a,b] such that (E) is normal for every interval containing c as an end point, and $Y(t,c) = (U(t,c);V(t,c))$ is a solution of (E) satisfying $U(c,c) = 0$, $V(c,c) = E$, then a value d distinct from c is conjugate to c if and only if $U(d,c)$ is singular. Moreover, if $U(d,c)$ has rank r then the order of d as a conjugate point to c is $n - r$.

If $[e,f] \subset [a,b]$ and $d[e,f] > 0$, then we can find an $n \times d$ matrix Δ such that the column vectors form a basis for $\Lambda[e,f]$.

LEMMA 1.3. *Suppose that* $[e,f] \subset [a,b]$, *and* c *is a point of* $[e,f)$ *such that* $d[e,x] = d[e,f] = d$ *for* $x \in (c,f]$, Δ *is as above, while* R *is an* $n \times (n-d)$ *matrix such that* $[\Delta \quad R]$ *is nonsingular. Let*

$$Y_\alpha(t) = (U_\alpha(t);V_\alpha(t)),\ \alpha = 0,1,2,3,$$

be the solutions of (E_q) *satisfying the respective initial conditions*

$$Y_0(e) = (0;\Delta) \qquad Y_1(e) = (0;R)$$

$$Y_2(e) = (\Delta;0) \qquad Y_3(e) = (R;0).$$

Then a value $t_1 \in (c,f]$ *is conjugate to* $t = e$ *relative to* (E) *if and only if one of the following conditions is satisfied:*

1° $U_1(t_1)$ *has rank less than* $n - d$;

2° *the* $n \times n$ *matrix* $[U_2(t_1) \quad U_1(t_1)]$ *is singular;*

3° *the* $2n \times (2n-d)$ *matrix*

$$\begin{bmatrix} U_1(e) & U_2(e) & U_3(e) \\ U_1(t_1) & U_2(t_1) & U_3(t_1) \end{bmatrix}$$

has rank less than $2n-d$.

In particular, if $R^*\Delta = 0$, *then* $([U_2(t) \quad U_1(t)];\ [V_2(t) \quad V_1(t)])$ *is a conjoined basis for* (E).

In order to prove the conclusion involving 1°, note that if $U_1(t_1) = [u^{(1)}(t) \cdots u^{(n-d)}(t)]$ is such that $U_1(t_1)$ has rank less than $n-d$, then there exist constants $\xi_1,\cdots,\xi_{n-d}$, not all zero, and such that $u(t) = \sum_{i=1}^{n-d} \xi_i u^{(i)}(t)$ satisfies $u(t_1) = 0$. Then $u(t) \not\equiv 0$ on $[e,t_1]$, and since $u(e) = 0$ we have that t_1 is conjugate to e. Now if $t_1 \in (c,f]$ and $(u(t);v(t))$ is any solution of (E) satisfying $u(e) = 0$, then there exist constants $\xi_1,\cdots,\xi_{n-d}$ and $\xi_1',\cdots,\xi_d'$ such that

$$u(t) = \sum_{i=1}^{n-d} \xi_i u^{(i)}(t),\ v(t) = \sum_{i=1}^{n-d} \xi_i v^{(i)}(t) + \sum_{j=1}^{d} \xi_j' \delta^{(j)},$$

where $V(t) = [v^{(1)}(t) \cdots v^{(n-d)}(t)]$ and $\Delta = [\delta^{(1)} \cdots \delta^{(d)}]$, and consequently if $t = t_1$ is conjugate to e then the constants $\xi_1,\cdots,\xi_{n-d}$ are not all zero and the $n \times (n-d)$ matrix $U_1(t_1)$ must be singular.

In order to prove the conclusion involving condition 2°, note that if $U_1(t_1)$ has rank less than $n-d$, then the $n \times n$ matrix $[U_2(t_1) \quad U_1(t_1)]$ is singular. Conversely, if $[U_2(t_1) \quad U_1(t_1)] = [u^{(1)}(t_1) \cdots u^{(n)}(t_1)]$ is singular, then we can find constants $\xi_1,\cdots,\xi_n$, not all zero, such that $u(t) = \sum_i \xi_i u^{(i)}(t)$ and $u(t_1) = 0$. By Lemma 1.1, $\rho^* u^{(i)}(t)$ is constant on $[e,f]$ for all $\rho \in \Lambda[e,f]$ and we can assume $\Delta = [\rho^{(1)} \cdots \rho^{(d)}]$ is such that $\Delta^*\Delta = E_d$, so that $\rho^{(i)*}u(e) = \xi_i$ for $i = 1,\cdots,d$. Thus, ξ_i must be zero for $i = 1,\cdots,d$, so that $U_1(t_1)$ has rank less than $n-d$.

In view of the above, condition 3° is true if and only if $U_1(t_1)$ has rank less than $n-d$.

2. <u>Endpoint behavior of solutions</u>. We shall now turn our attention to the behavior of solutions of (E) in a neighborhood of an end-point of

a non-compact interval of existence. The following result can be shown in the same manner as in Reid [6, pp. 315-316].

THEOREM 2.1. _Suppose that_ $Y_1(t) = (U_1(t);V_1(t))$ _is a solution of_ (E_n) _on an interval_ $[a,b]$, _with_ $U_1(t)$ _non-singular on_ $[c,d] \subset [a,b]$ _and let_ $S(t,e|U_1)$ _denote the matrix function defined by_ (3.6) _in the statement of Theorem_ III.3.2.

(i) _If_ $e \in [c,d]$ _is such that_ (E) _is normal on every subinterval of_ $[c,d]$ _with_ e _as an end-point, and_ $t_1 \in [c,d]$ _and distinct from_ e _then_ $S(t_1,e|U_1)$ _is singular if and only if_ t_1 _is conjugate to_ e, _relative to_ (E).

(ii) _If_ I _is an open interval_ (a_o,b_o), $(-\infty \leq a_o < b_o \leq +\infty)$, _on which_ (E) _is identically normal, while_ (E) _is disconjugate on a subinterval_ $I_o = (c_o,b_o)$ _of_ I _and_ $Y_1(t) = (U_1(t);V_1(t))$ _is a solution of_ (E_n) _with_ $U_1(t)$ _nonsingular on_ I_o, _then for_ $c \in I_o$ _the matrix_ $S(t,c|U_1)$ _is non-singular for_ $t \in I_o$, $t \neq c$. _Moreover, if there exists a_ $c \in I_o$ _such that_ $S^{-1}(t,c|U_1) \to 0$ _as_ $t \to b_o$, _then_ $S^{-1}(t,b|U_1) \to 0$ _as_ $t \to b_o$ _for all_ $b \in I_o$.

Conclusion (ii) of Theorem 2.1 implies that if (E) is identically normal on an interval (c_o,d_o) and disconjugate on (e_o,d_o) with $U_1(t)$ non-singular on $(e_1,d_o) \subset (e_o,d_o)$ and $S^{-1}(t,e|U_1) \to 0$ as $t \to d_o$ for some $e \in (e_1,d_o)$, then $S^{-1}(t,e|U_1) \to 0$ as $t \to d_o$ for all $e \in (e_1,d_o)$. We shall call such a solution a _principal solution_ after Reid [6] and Hartman [1]. In the same manner as is employed by Reid [6; pp. 316-317], we may obtain the following result about principal solutions.

THEOREM 2.2. _Suppose that_ (E) _is identically normal on an open interval_ $I = (a_o,b_o)$, $(-\infty \leq a_o < b_o \leq +\infty)$. _If_ (E) _is disconjugate on a subinterval_ $I_o = (c_o,b_o)$ _of_ I, _then a solution_ $Y_1(t) = (U_1(t);V_1(t))$ _of_ (E_n) _is a principal solution of_ (E_n) _at_ b_o _if_ $U_1(t)$ _is nonsingular on_ some subinterval $I\{Y_1\} = (c_1,b_o)$ of I, and there exists a solution $Y_2(t) = (U_2(t);V_2(t))$ of (E_n) with $U_2(t)$ nonsingular on some subinterval $I\{Y_2\} = (c_2,b_o)$, and such that for some $c \in (c_1,b_o)$

$$U_2^{-1}(t)U_1(t)T(t,c|U_1) \to 0 \text{ as } t \to b_o; \tag{2.1}$$

moreover, $\{U_2;V_2|U_1;V_1\}$ is nonsingular for any such Y_2. Conversely, if (E) is disconjugate on a subinterval (c_o,b_o), and $Y_1(t) = (U_1(t);V_1(t))$ is a principal solution with $U_1(t)$ nonsingular on (c_1,b_o), then any solution $Y_2(t) = (U_2(t);V_2(t))$ of (E_n) with $\{U_2;V_2|U_1;V_1\}$ nonsingular is such that $U_2(t)$ is nonsingular on some subinterval (c_2,b_o) and (2.1) holds for arbitrary $c \in (c_1,b_o)$.

3. The Riccati equation. In this section we will assume that M and N satisfy hypothesis H_h. Under this assumption, we wish to study the relationship of solutions of (E) and solutions of the corresponding Riccati equation,

$$\mathcal{R}[W] \equiv \int_a^t dW + \int_a^t W[dN]W - \int_a^t dM = 0. \tag{3.1}$$

The basic relationship is given by the following result which can be shown by direct substitution.

THEOREM 3.1. There is a solution $Y_1(t) = (U_1(t);V_1(t))$ of (E) on an interval (a_o,b_o) with $U_1(t)$ non-singular on this interval if and only if there is a solution $W = W_1(t)$ of (3.1) on (a_o,b_o) with $W_1(t) = V_1(t)U_1^{-1}(t)$. Moreover, $U = U_1(t)$ is a fundamental matrix solution of the matrix equation

$$\int_a^t dU = \int_a^t [dN]WU. \tag{3.2}$$

For $W_1(t)$, a solution of (3.1) on (a_o,b_o) and $c \in (a_o,b_o)$, let us take $H = H(t,c|W_1)$ and $G = G(t,c|W_1)$ to be solutions of the respective systems

(3.3) $$\int_c^t [dH] + \int_c^t H[dN]W_1 = 0, \quad H(c) = E,$$

(3.4) $$\int_c^t [dG] + \int_c^t W_1[dN]G = 0, \quad G(c) = E.$$

We may obtain existence and uniqueness of solutions of (3.2), (3.3), and (3.4) in a manner similar to the proofs of Theorems II.1.1 and II.2.1. Thus, the solutions U, H, and G of these systems are continuous, of bounded variation, and nonsingular on (a_o,b_o). If $W_1(t) = V_1(t)U_1^{-1}(t)$, then $H(t) = U_1(c)U_1^{-1}(t)$ and we have that

(3.5) $$W_1^* - W_1 = U_1^{*-1}V_1^* - V_1U_1^{-1} = -U_1^{*-1}KU_1^{-1}$$

where $K = -\{U_1;V_1|U_1;V_1\}$. Consequently,

(3.6) $$\int_c^t [dG] + \int_c^t W_1^*[dN]G = \int_c^t [W_1^* - W_1][dN]G.$$

Let $G(t,c|W_1) = H^*(t,c|W_1)P(t,c|W_1)$ and substitute this into (3.6). Then we get

(3.7) $$\int_c^t [dH^*P] + \int_c^t W_1^*[dN]G = -\int_c^t U_1^{*-1}KU_1^{-1}[dN]G,$$

so that

(3.8) $$\int_c^t H^*[dP] = -\int_c^t U_1^{*-1}KU_1^{-1}[dN]H^*P,$$

and consequently

$$\int_c^t U_1^* H^* [dP] U_1^{*-1}(c) = -\int_c^t KU_1^{-1}[dN] H^* P U_1^{*-1}(c).$$

Simplifying, we obtain

$$\int_c^t U_1^*(c)[dP]U_1^{*-1}(c) = -\int_c^t KU_1^{-1}[dN]U_1^{*-1}U_1^*(c)PU_1^{*-1}(c). \tag{3.9}$$

If we let $F(t,c|W_1) = F = U_1^*(c)P(t,c|W_1)U_1^{*-1}(c)$ we have that F satisfies the system

$$\int_c^t dF = -\int_c^t KU_1^{-1}[dN]U_1^{*-1}F,\ F(c) = E. \tag{3.10}$$

We have that $K = -K^*$ and by the definition of $T(t,c|U_1)$ in the statement of Theorem III.3.2 we have

$$\int_c^t dT^* = \int_c^t T^* KU_1^{-1}[dN]U_1^{*-1}$$

so that

$$-\int_c^t [dT^{*-1}] = \int_c^t KU_1^{-1}[dN]U_1^{*-1}T^{*-1},$$

and this implies $F(t,c|W_1) = T^{*-1}(t,c|U_1)$. Thus

$$G(t,c|W_1) = H^*(t,c|W_1)U_1^{*-1}(c)T^{*-1}(t,c|U_1)U_1^*(c). \tag{3.11}$$

If the matrix function $Z(t,c|W_1)$ is defined as

$$Z(t,c|W_1) = \int_c^t H[dN]G, \tag{3.12}$$

we find that

$$Z(t,c|W_1) = U_1(c)S^*(t,c|U_1)U_1^*(c), \tag{3.13}$$

where $S(t,c|U_1)$ is as defined in the statement of Theorem III.3.2.

Using the results developed above we can obtain the following result.

LEMMA 3.1. _If_ $W_1(t)$ _is a solution of_ (3.1) _on_ (a_o,b_o), _then_ $W(t)$ _is a solution of_ (3.1) _on_ (a_o,b_o) _if and only if the constant matrix_ $\Gamma = W(c) - W_1(c)$ _is such that_ $E + Z(t,c|W_1)\Gamma$ _is nonsingular on_ (a_o,b_o) _and_

$$W(t) = W_1(t) + G(t,c|W_1)\Gamma[E + Z(t,c|W_1)\Gamma]^{-1}H(t,c|W_1). \tag{3.14}$$

We have that if W and W_1 are $n \times n$ matrix functions which are of bounded variation, then

$$\mathcal{R}[W] - \mathcal{R}[W_1] = \int_c^t d\Psi + \int_c^t \Psi[dN]W_1 + \int_c^t W_1[dN]\Psi + \int_c^t \Psi[dN]\Psi, \tag{3.15}$$

where $\Psi(t) = W(t) - W_1(t)$. Suppose $\mathcal{R}[W_1] = 0$. Now, if $Q(t)$ is defined by

$$\Psi(t) = G(t,c|W_1)Q(t)H(t,c|W_1),$$

then Q is of bounded variation and

$$\int_c^t d\Psi = \int_c^t d[GQH]$$

$$= -\int_c^t W_1[dN]W + 2\int W_1[dN]W_1 + \int G[dQ]H - \int W[dN]W_1.$$

It then follows that $\mathcal{R}[W] = 0$ if and only if

$$\int_c^t G[dQ]H = -\int_c^t [W-W_1][dN][W-W_1],$$

or, equivalently, if and only if Q satisfies the equation

$$\int_c^t dQ = -\int_c^t QH[dN]GQ, \quad Q(c) = W(c) - W_1(c) = \Gamma. \tag{3.16}$$

If $Q_1(t)$ is defined by

$$Q_1(t) = Q(t)[E + Z(t,c|W_1)\Gamma] - \Gamma \tag{3.17}$$

then $Q_1(c) = 0$ and

$$\int_c^t dQ_1 = \int_c^t d\{Q(t)[E + Z(t,c|W_1)\Gamma] - \Gamma\}$$

$$= \int_c^t QH[dN]GQ_1.$$

Thus Q_1 is a solution of

$$\int_c^t dQ_1 = -\int_c^t QH[dN]GQ_1, \quad Q_1(c) = 0, \tag{3.18}$$

which implies $Q_1(t) \equiv 0$.

Suppose η is an n-dimensional vector such that $[E + Z(t,c|W_1)\Gamma]\eta = 0$. As $Q_1(t) \equiv 0$ we have $Q_1(t)\eta \equiv 0$ so that $\Gamma\eta = 0$ and therefore $\eta = 0$. This implies $E + Z(t,c|W_1)\Gamma$ is nonsingular, and thus (3.14) holds.

If Γ is such that $E + Z(t,c|W_1)\Gamma$ is nonsingular on (a_o,b_o), we can let $Q(t) = \Gamma[E + Z(t,c|W_1)\Gamma]^{-1}$ and thus $Q(t)$ satisfies (3.16).

If $W_1(t)$ is a solution of (3.1) such that $Z^{-1}(t,c|W_1) \to 0$ as $t \to b_o$ for some $c \in (a_o,b_o)$, we shall say $W_1(t)$ is a <u>distinguished solution of (3.1) at</u> b_o. The concept of a distinguished solution of (3.1) at a_o is defined in a similar fashion.

If (E) is an identically normal system, we may obtain results similar to those of Reid [5] concerning the relationship of distinguished solutions of (3.1) and principal solutions of (E). Moreover, a method for obtaining a principal solution, or a distinguished solution, may be demonstrated in a fashion entirely analogous to that used for differential equations.

CHAPTER V

AN ASSOCIATED FUNCTIONAL

1. Definitions. In this chapter we shall assume that M and N satisfy H_h. Let us take the following sets:

$\mathfrak{L}[a,b] = \{\zeta | \zeta$ is an n-dimensional vector function which is integrable with respect to N$\}$;

$\mathfrak{D}[a,b] = \{\eta |$ there exists a function $\zeta \in \mathfrak{L}[a,b]$ such that $L_2[\eta,\zeta] = 0\}$,

where [a,b] is a compact interval and

$$(1.1) \qquad L_2[\eta,\zeta] = d\eta - [dN]\zeta.$$

The relationship between η and ζ will be indicated by $\eta \in \mathfrak{D}[a,b]:\zeta$.

If $(\eta_\alpha;\zeta_\alpha) \in \mathfrak{D}[a,b] \times \mathfrak{L}[a,b]$, $(\alpha = 1,2)$, let $J[\eta_1:\zeta_1,\eta_2:\zeta_2;a,b]$ denote the functional defined by

$$(1.2) \qquad J[\eta_1:\zeta_1,\eta_2:\zeta_2;a,b] = \int_a^b \zeta_2^*[dN]\zeta_1 + \int_a^b \eta_2^*[dM]\eta_1.$$

If M and N satisfy H_h, then (1.2) defines an hermitian form on $\mathfrak{D}[a,b] \times \mathfrak{L}[a,b]$; that is, if $(\eta_\alpha;\zeta_\alpha) \in \mathfrak{D}[a,b] \times \mathfrak{L}[a,b]$, $(\alpha = 1,2,3)$, then

a) $J[\eta_1:\zeta_1,\eta_2:\zeta_2;a,b] = \overline{J[\eta_2:\zeta_2,\eta_1:\zeta_1;a,b]}$,

b) $J[c\eta_1:c\zeta_1,\eta_2:\zeta_2;a,b] = cJ[\eta_1:\zeta_1,\eta_2:\zeta_2;a,b]$,

c) $J[\eta_1 + \eta_2:\zeta_1 + \zeta_2,\eta_3:\zeta_3;a,b]$

$$= J[\eta_1:\zeta_1,\eta_3:\zeta_3;a,b] + J[\eta_2:\zeta_2,\eta_3:\zeta_3;a,b].$$

In general, for a given η the corresponding vector function ζ is not unique. However, the value of (1.2) is independent of the choice of ζ satisfying $\eta \in \mathcal{D}[a,b]:\zeta$; for this reason, we shall write (1.1) as

$$J[\eta_1,\eta_2;a,b] = \int_a^b \zeta_2^*[dN]\zeta_1 + \int_a^b \eta_2^*[dM]\eta_1. \tag{1.3}$$

Also, for brevity we write $J[\eta_1;a,b]$ for $J[\eta_1,\eta_1;a,b]$.

If we let

$$L_1[\eta,\zeta] = -d\zeta + [dM]\eta,$$

the following result is a ready consequence of the above definitions.

LEMMA 1.1. *If* $\eta_\alpha \in \mathcal{D}[a,b]:\zeta_\alpha$, $(\alpha = 1,2)$, *then*

$$J[\eta_1,\eta_2;a,b] = \eta_2^*\zeta_1\Big|_a^b + \int_a^b \eta_2^* L_1[\eta_1,\zeta_1]; \tag{1.4'}$$

$$J[\eta_1;a,b] = \eta_1^*\zeta_1\Big|_a^b + \int_a^b \eta_1^* L_1[\eta_1,\zeta_1]; \tag{1.4''}$$

$$\int_a^b \eta_2^* L_1[\eta_1,\zeta_1] - \int_a^b (L_1[\eta_2,\zeta_2])^*\eta_1 = \{\zeta_2^*\eta_1 - \eta_2^*\zeta_1\}\Big|_a^b = \{\eta_1;\zeta_1|\eta_2;\zeta_2\}\Big|_a^b. \tag{1.4'''}$$

From this we see that if $t_1,t_2 \in [a,b]$ are conjugate and $(u;v)$ is a solution of (E) with $u(t_1) = 0 = u(t_2)$ and $u \not\equiv 0$ on $[t_1,t_2]$, then $(\eta(t);\zeta(t))$ defined by $(u(t);v(t))$ on $[t_1,t_2]$ and identically zero elsewhere, are functions such that $\eta \in \mathcal{D}_o[a,b]:\zeta$ and (1.4") implies

$$J[\eta;a,b] = J[u;t_1,t_2] = 0.$$

Thus we have the following result.

COROLLARY 1.1. *There are no points* $t_1,t_2 \in [a,b]$ *which are conjugate if the only* $\eta \in \mathcal{D}_o[a,b]$ *such that* $J[\eta;a,b] = 0$ *is* $\eta(t) \equiv 0$.

THEOREM 1.1. *If* u *is continuous and of bounded variation on* $[a,b]$, *then there exists a* v *such that* $(u;v)$ *is a solution of* (E) *on* $[a,b]$ *if and only if there exists a* $v_1 \in \mathcal{L}[a,b]$ *such that* $u \in \mathcal{D}[a,b]:v_1$ *and*

$$\text{(1.5)} \qquad J[u:v_1,\eta:\zeta;a,b] = 0 \quad \textit{for all} \quad \eta \in \mathcal{D}_o[a,b].$$

If $(u;v)$ is a solution of (E) on $[a,b]$ and $\eta \in \mathcal{D}_o[a,b]$, then $u \in \mathcal{D}[a,b]:v$ and (1.5) is a consequence of (1.4') for $(\eta_1;\zeta_1) = (u;v)$, $(\eta_2;\zeta_2) = (\eta;\zeta)$.

On the other hand, suppose $u \in \mathcal{D}[a,b]:v_1$ and (1.5) holds. If $v_o(t)$ is a solution of the equation

$$\text{(1.6)} \qquad -\int_a^t dv_o + \int_a^t [dM]u = 0,$$

then (1.5) becomes

$$\int_a^b \{[d\eta^*]v_1 + \eta^*[dv_o]\} = 0.$$

But $\int_a^b \{[d\eta^*]v_o + \eta^*[dv_o]\} = \int_a^b [d\eta^* v_o] = 0$ since $\eta \in \mathcal{D}_o[a,b]$, so that we have

$$\text{(1.7)} \qquad \int_a^b \zeta^*[dN][v_1 - v_o] = 0, \text{ if } \zeta \in \mathcal{L}[a,b] \text{ and } \int_a^b \zeta^*[dN] = 0.$$

By a well known result of functional analysis (see, for example, Taylor

[7, p. 138]), if we restrict ζ to be a continuous function, we have that there exists a constant vector λ such that

$$\int_a^b \zeta^*[dN][v_1 - v_o] = \int_a^b \zeta^*[dN]\lambda, \text{ for } \zeta \text{ continuous.}$$

That is, we have that $\int_a^b \zeta^* d\{\int_a^t [dN][v_1 - v_o - \lambda]\} = 0$, for ζ an arbitrary continuous vector function, and consequently $\int_a^t [dN][v_1 - v_o - \lambda] \equiv 0$ on $[a,b]$. If $v(t) = v_o(t) + \lambda$, then since $\int_a^t dv_o = \int_a^t dv$, we have that $\int_a^t dv = \int_a^t [dM]u$, $t \in [a,b]$ and $u \in \mathfrak{D}[a,b]:v$, so that $(u;v)$ is a solution of (E).

COROLLARY 1.2. *If* $J[\eta;a,b]$ *is non-negative definite on* $\mathfrak{D}_o[a,b]$, *and* u *is an element of* $\mathfrak{D}_o[a,b]$ *satisfying* $J[u;a,b] = 0$, *then there exists a* $v \in BV[a,b]$ *such that* $(u;v)$ *is a solution of* (E) *on* $[a,b]$. *In particular, if* $u(t) \not\equiv 0$, a *and* b *are conjugate.*

If $\eta \in \mathfrak{D}_o[a,b]$, we have that $u + \sigma\eta \in \mathfrak{D}_o[a;b]$ for arbitrary σ, so that

$$\begin{aligned} 0 &\le J[u + \sigma\eta;a,b] \\ &= J[u;a,b] + \bar{\sigma}J[u,\eta;a,b] + \sigma J[\eta,u;a,b] + |\sigma^2|J[\eta;a,b]. \end{aligned}$$

As $J[u;a,b] = 0$, we can make the right-hand side negative unless $J[u,\eta;a,b] = 0$. Thus $J[u,\eta;a,b] = 0$ for all $\eta \in \mathfrak{D}_o[a,b]$.

COROLLARY 1.3. *If* $J[\eta;a,b]$ *is non-negative definite on* $\mathfrak{D}_o[a,b]$ *and* $(u;v)$ *is a solution of* (E), *while* $u_o \in \mathfrak{D}[a,b]$ *with* $u_o(a) = u(a)$, $u_o(b) = u(b)$, *then* $J[u_o;a,b] \ge J[u;a,b]$; *moreover, if* $J[\eta;a,b]$ *is positive definite on* $\mathfrak{D}_o[a,b]$ *the inequality holds with equality only if* $u_o(t) \equiv u(t)$.

Preliminary to the study of necessary and sufficient conditions for the system (E) to be disconjugate on a subinterval of [a,b], the following result will be stated without proof, as it may be established by direct substitution.

LEMMA 1.2. <u>Suppose that</u> $U(t)$, $V(t)$ <u>are</u> $n \times r$ <u>matrix functions of bounded variation on</u> [a,b] <u>with</u> U <u>continuous. If</u> η_α <u>is continuous and of bounded variation,</u> $\zeta_\alpha \in \mathcal{L}[a,b]$, <u>for</u> $\alpha = 1,2$, <u>and there exists an</u> r-<u>dimensional vector function</u> $h_\alpha(t)$, <u>such that</u> h_α <u>is of bounded variation and continuous on</u> [a,b] <u>while</u> $\eta_\alpha(t) \equiv U(t)h_\alpha(t)$, <u>then on this interval we have the identity</u>

$$\begin{aligned}\int_a^t \{\zeta_2^*[dN]\zeta_1 + \eta_2^*[dM]\eta_1\} = \int_a^t \{[\zeta_2 - Vh_2]^*[dN][\zeta_1 - Vh_1] \\ - h_2^*V^*L_2[\eta_1,\zeta_1] - (L_2[\eta_2,\zeta_2])^*Vh_1 \\ + h_2^*(V^*L_2[U,V] + U^*L_1[U,V])h_1 \\ - h_2^*[U^*V - V^*U][dh_1] + d[h_2^*U^*Vh_1]\}.\end{aligned}$$

COROLLARY 1.4. <u>If the column vectors of</u> $Y(t) = (U(t);V(t))$ <u>form a basis for an</u> r-<u>dimensional conjoined family of solutions of</u> (E), <u>while</u> $\eta \in \mathcal{D}[a,b]:\zeta$ <u>and there exists a function</u> $h(t)$ <u>which is continuous and of bounded variation such that</u> $\eta(t) = U(t)h(t)$ <u>for</u> $t \in [a,b]$, <u>then</u>

$$J[\eta;a,b] = \eta^*Vh\Big|_a^b + \int_a^b [\zeta - Vh]^*[dN][\zeta - Vh].$$

THEOREM 1.2. <u>If</u> $J[\eta;a,b]$ <u>is non-negative definite on</u> $\mathcal{D}_o[a,b]$, <u>then</u> $N(t)$ <u>is a non-decreasing function.</u>

Suppose N is not non-decreasing. Then there exists an interval [c,d] and a vector ξ, with $|\xi| = 1$, and such that $\xi^*[N(d) - N(c)]\xi = \int_c^d \xi^*[dN]\xi < 0$. Also, there exists a $k_1 > 0$ such that

$$\int_c^d \xi^*[dN]\xi = -k_1\nu[c,d], \tag{1.8}$$

where $\nu[c,d]$ is the variation function h_N of N as defined by II.1.1. For any $\delta > 0$ there must exist an interval $[e,f] \subset [c,d]$ with $|e-f| < \delta$, and such that

$$\int_e^f \xi^*[dN]\xi = -k_1^*\nu[e,f] \leq -k_1\nu[e,f], \quad k_1^* \geq k_1.$$

If not, there is a $\delta > 0$, such that any interval $[e,f] \subset [c,d]$ with $|e - f| < \delta$ is such that $\int_e^f \xi^*[dN]\xi > -k_1\nu[e,f]$. We can partition $[c,d]$ into a finite number of non-overlapping intervals $[e_i,f_i]$, with $|e_i - f_i| < \delta$ and such that $\bigcup_i [e_i,f_i] = [c,d]$. Then

$$\int_c^d \xi^*[dN]\xi = \sum_i \int_{e_i}^{f_i} \xi^*[dN]\xi > \sum_i(-k_1\nu[e_i,f_i]) = -k_1\nu[c,d],$$

a contradition to (1.8).

Let m be the first positive integer such that $2^m > n$, and $[c,d]$ an interval such that (1.8) holds and

$$\nu[c,d] < k_1/(2^m V[M]) \tag{1.9}$$

where $V[M]$ is the variation of M on $[a,b]$. In particular, $[c,d]$ may be chosen to satisfy (1.9) since ν is continuous on $[a,b]$. If we consider the functions $\int_c^t \xi^*[dN]\xi$ and $\int_t^d \xi^*[dN]\xi$, then there exists a $g \in (c,d)$ such that $\int_c^g \xi^*[dN]\xi = \int_g^d \xi^*[dN]\xi = -k_1\nu[c,d]/2$. The intervals $[c,g]$

and $[g,d]$ may then be subdivided in the same manner, until this operation has been repeated m times. There is thus obtained a partition of $[c,d]$, $c = t_0 < t_1 < \cdots < t_{2^m} = d$ such that

$$(1.10) \qquad \int_{t_{i-1}}^{t_i} \xi^*[dN]\xi = -k_1\nu[c,d]/2^m \quad i = 1,\cdots,2^m.$$

Then, if $\chi_{[t_{i-1},t_i]}$ is the characteristic function of $[t_{i-1},t_i]$, we define $\eta(t) = \int_c^t [dN]\xi\phi$, where $\phi(t) = \sum_i c_i\chi_{[t_{i-1},t_i]}(t)$ with the c_i chosen so that $\sum_i |c_i|^2 = 1$ and $\eta(d) = \bar{0}$. If $u(t) = \eta(t)$ for $t \in [c,d]$ and zero elsewhere, then we have that u is such that $u(t) \in \mathcal{D}_o[a,b]:\xi\phi$. Moreover, $|\eta(t)| \leq \int_c^t [d\nu(s)]|\xi||\phi(s)| \leq \int_c^t d\nu(s) = \nu[c,t]$, so that

$$\int_a^b \bar{\phi}\xi^*[dN]\xi\phi + \int_a^b \eta^*[dM]\eta \leq \sum_{i=1}^{2^m} c_i{}^2(-k_1\nu[c,d]/2^m) + (\nu[c,d])^2V[M] < 0,$$

in view of (1.9). Consequently, the assumption that N is not non-decreasing has led to a contradiction of the non-negative definiteness of J.

In the same manner as Reid [6, pp. 326-328], the following result can be established.

THEOREM 1.3. <u>Suppose</u> $J[\eta;a,b]$ <u>is positive definite on</u> $\mathcal{D}_o[a,b]$. <u>If</u> $d[a,b] = d$, Δ <u>is a basis for</u> $\Lambda[a,b]$ <u>with</u> $\Delta^*\Delta = E_d$, <u>and</u> R <u>is an</u> $n \times (n-d)$ <u>matrix such that</u> $R^*\Delta = 0$ <u>and</u> $[\Delta \quad R] = 0$ <u>is nonsingular, then there exists a unique solution</u> $Y_b(t) = (U_b(t);V_b(t))$ <u>of</u> (E_{n-d}) <u>such that</u>

$$(1.11) \qquad U_b(a) = R, \quad U_b(b) = 0, \quad V_b^*(a)\Delta = 0.$$

The column vectors of $Y_b(t)$ *form a basis for a conjoined family of solutions of (E) of dimension* $n - d$, *and if* $Y_4(t) = (U_4(t);V_4(t))$ *is a second solution of* (E_{n-d}) *whose column vectors form a basis for a conjoined family of solutions of (E) of dimension* $n - d$, *and satisfying*

$$(1.12) \qquad U_4(a) = R, \quad V_4^*(a)\Delta = 0, \quad U_4^*(a)V_4(a) > U_b^*(a)V_b(a),$$

then $U_4(t)$ *is of rank* $n - d$ *on* $[a,b]$. *Moreover, if* $Y_2(t) = (U_2(t);V_2(t))$ *is the solution of* (E_d) *satisfying the initial conditions* $U_2(a) = \Delta$, $V_2(a) = 0$, *then*

$$Y(t) = ([U_2(t) \quad U_4(t)];[V_2(t) \quad V_4(t)]) = (U(t);V(t))$$

is a conjoined basis for (E) with $U(t)$ *nonsingular on* $[a,b]$.

THEOREM 1.4. *The form* $J[\eta;a,b]$ *is positive definite on* $\mathcal{D}_o[a,b]$ *if and only if* $N(t)$ *is a non-decreasing matrix function on* $[a,b]$ *and there exists a conjoined basis* $Y(t) = (U(t);V(t))$ *for (E) with* U *nonsingular on* $[a,b]$.

Since $J[\eta;a,b]$ is positive definite on $\mathcal{D}_o[a,b]$, Theorems 1.2 and 1.3 imply that $N(t)$ is non-decreasing on $[a,b]$, and the existence of a conjoined basis $Y(t) = (U(t);V(t))$ with $U(t)$ nonsingular on $[a,b]$. Conversely, if such a basis exists, then in view of Lemma 1.2 we have for $\eta \in \mathcal{D}_o[a,b]:\zeta$ that

$$J[\eta;a,b] = \int_a^b [\zeta - Vh]^*[dN][\zeta - Vh],$$

with $h(t) = U^{-1}(t)\eta(t)$. But N being a non-decreasing hermitian matrix function implies that

$$K[\alpha;a,b] = \int_a^b \alpha^*[dN]\alpha$$

is a non-negative definite hermitian form on the vector space of functions α which are N-integrable. Thus, if

$$\int_a^b [\zeta - Vh]^*[dN][\zeta - Vh] = 0$$

we must have

$$\int_a^t [dN][\zeta - Vh] \equiv 0 \quad \text{for } t \in [a,b].$$

As $L_2[\eta,\zeta] = 0$ and $L_2[U,V] = 0$, it follows that

$$\int_a^t Udh = \int_a^t [dN][\zeta - Vh] \equiv 0.$$

Also, since $\int_a^t Udh \equiv 0$ implies $\int_a^t dh \equiv 0$, and the condition $\eta(a) = 0$ implies that $h(a) = 0$, it follows that $h(t) \equiv 0$, and $\eta(t) \equiv 0$. Consequently, $J[\eta;a,b]$ is positive definite on $\mathfrak{D}_0[a,b]$.

THEOREM 1.5. <u>The form</u> $J[\eta;a,b]$ <u>is positive definite on</u> $\mathfrak{D}_0[a,b]$ <u>if and only if</u> $N(t)$ <u>is non-decreasing on</u> $[a,b]$ <u>and there is no point</u> $t_1 \in (a,b]$ <u>conjugate to</u> a.

Corollary 1.1 and Theorem 1.2 imply (E) is disconjugate and $N(t)$ is non-decreasing whenever $J[\eta;a,b]$ is positive definite on $\mathfrak{D}_0[a,b]$.

Conversely, suppose $N(t)$ is non-decreasing and a has no conjugate point on $(a,b]$. Let $c = \sup\{t \in [a,b]: J[\eta;a,t]$ is positive definite on $\mathfrak{D}_0[a,t]\}$. We know $c > a$ since, if we take $(U(t);V(t))$ the solution of (E_n) such that $(U(a);V(a)) = (E;0)$, we have U is nonsingular on some nondegenerate subinterval $[a,t]$, and Theorem 1.4 implies that $J[\eta;a,t]$ is positive definite on $\mathfrak{D}[a,t]$. We will first show that $J[\eta;a,c]$ is non-negative definite on $\mathfrak{D}_0[a,c]$. Suppose $\eta_1 \in \mathfrak{D}_0[a,c]:\zeta_1$. Let

$Y_1(t) = (U_1(t);V_1(t))$ be such that $(U_1(c);V_1(c)) = (E;0)$; also, for $d_1 = \lim_{t\to c} d[t,c]$, let $\varepsilon_1 > 0$ be such that $0 < \varepsilon_1 < c - a$, $d[c - \varepsilon_1,c] = d_1$ and $U_1(t)$ is nonsingular on $[c - \varepsilon_1,c]$. Let Δ be such that $\Delta^*\Delta = E_{d_1}$, and Δ is a basis for $\Lambda[c - \varepsilon_1,c]$; moreover, let ε_o be such that $0 < \varepsilon_o < \varepsilon_1$, with $d[c - \varepsilon_1,c - \varepsilon_o] = d_1$. Corollary 1.1 and Theorem 1.4 imply that (E) is disconjugate on $[c - \varepsilon_1,c]$. Also, Lemma IV.1.2 implies that for any ε satisfying $0 \le \varepsilon < \varepsilon_1$ there exists a solution $(u_\varepsilon(t);v_\varepsilon^o(t))$ of (E) satisfying

$$u_\varepsilon(c - \varepsilon_1) = \eta(c - \varepsilon_1),\ u_\varepsilon(c - \varepsilon) = 0.$$

The general form of $v_\varepsilon(t)$ is $v_\varepsilon^o(t) + \Delta\gamma$ where γ is a d-dimensional constant vector. Thus, there is unique solution satisfying $\Delta^*v_\varepsilon(c) = 0$. Moreover, since the matrix in criterion 3^o of Lemma IV.1.3 has rank 2n-d and encompasses all solutions with $\Delta^*v(c) = 0$, we have that $(u_\varepsilon(t);v_\varepsilon(t))$ tends to $(u_o(t);v_o(t))$ uniformly on $[c - \varepsilon_1,c]$ as $\varepsilon \to 0$. For $0 \le \varepsilon \le \varepsilon_o$ define

$$\begin{aligned}(\eta_\varepsilon(t);\zeta_\varepsilon(t)) &= (\eta_1(t);\zeta_1(t)),\ t \in [a,c - \varepsilon_1];\\ &= (u_\varepsilon(t);v_\varepsilon(t)),\ t \in [c - \varepsilon_1,\ c - \varepsilon];\\ &= (0,0), \quad t \in [c - \varepsilon,c].\end{aligned}$$

Then $\eta_\varepsilon \in \mathcal{D}_o[a,c]:\zeta_\varepsilon$ and $\eta_\varepsilon \in \mathcal{D}_o[a,c-\varepsilon]:\zeta_\varepsilon$, so that $J[\eta_\varepsilon;a,c] = J[\eta_\varepsilon;a,c-\varepsilon] > 0$, and consequently upon letting $\varepsilon \to 0$ we obtain $J[\eta_o;a,c] \ge 0$. Theorem 1.4 implies that $J[u_o;c-\varepsilon_1,c] \ge 0$ and Corollary 1.3 implies that $J[\eta_1;c-\varepsilon_1,c] \ge J[u_o;c-\varepsilon_1,c]$. Thus $J[\eta_1;a,c] \ge 0$, so that $J[\eta;a,c]$ is non-negative definite for $\eta \in \mathcal{D}_o[a,c]$. If $\eta \in \mathcal{D}_o[a,c]$ and $J[\eta;a,c] = 0$, then Corollary 1.2 implies there is a $v \in \mathcal{L}[a,c]$ such that $(\eta;v)$ is a solution of (E) satisfying $\eta(a) = 0 = \eta(c)$ so that $\eta \equiv 0$. Thus $J[\eta;a,c]$ is positive definite on $\mathcal{D}_0[a,c]$. But Theorem 1.4 gives the existence of a conjoined basis $Y(t) = (U(t);V(t))$ on $[a,c]$, and since $U(c)$ is nonsingular, we have a conjoined basis with $U(t)$ non-singular on $[a,c+\delta]$, $(\delta > 0)$. Thus J is positive definite on $\mathcal{D}_0[a,c+\delta]$ and we have a contradiction to our choice of c unless $c = b$, and $J[\eta;a,b]$ is positive definite on $\mathcal{D}_0[a,b]$.

If the roles of $t = a$ and $t = b$ are interchanged, one may establish the following result.

COROLLARY 1.5. *The form* $J[\eta;a,b]$ *is positive definite on* $\mathcal{D}_0[a,b]$ *if and only if* $N(t)$ *is non-decreasing on* $[a,b]$, *and there is no value on* $[a,b)$ *which is conjugate to* $t = b$.

2. *Disconjugacy criteria*. The results of the preceding section will be compressed here for ready reference.

THEOREM 2.1. *If* $N(t)$ *is non-decreasing for* $t \in [a,b]$, *then the following conditions are equivalent*.

i) (E) *is disconjugate on* $[a,b]$.

ii) $J[\eta;a,b]$ *is positive definite on* $\mathcal{D}_0[a,b]$.

iii) *There is no point on* $(a,b]$ *conjugate to* $t = a$.

iv) *There is no point on* $[a,b)$ *conjugate to* $t = b$.

v) *There exists a conjoined basis* $Y(t) = (U(t);V(t))$ *for* (E) *with* $U(t)$ *nonsingular on* $[a,b]$.

vi) *There exists an* $n \times n$ *hermitian matrix function* $W(t)$, $t \in [a,b]$, *which is a solution of the Riccati matrix equation*

$$\text{(2.1)} \qquad \mathcal{R}[W](t) \equiv \int_a^t [dW] + \int_a^t W[dN]W - \int_a^t dM = 0, \quad t \in [a,b].$$

Suppose that for $\alpha = 1,2$ the matrix functions M_α and N_α satisfy hypotheses H and H_h. The corresponding classes $\mathcal{D}[a,b]$ and $\mathcal{D}_o[a,b]$ will be denoted by $\mathcal{D}_\alpha[a,b]$ and $\mathcal{D}_{\alpha o}[a,b]$. If we have

$$N_1(t) \equiv N_2(t) \tag{2.2}$$

then $\mathcal{D}_1[a,b] = \mathcal{D}_2[a,b]$ and $\mathcal{D}_{10}[a,b] = \mathcal{D}_{20}[a,b]$. However, these relations may occur without (2.2) holding. For $\alpha = 1,2$ we have the corresponding systems

$$\begin{aligned} L_1^\alpha[u,v](t) &= -dv(t) + [dM_\alpha(t)]u(t) = 0 \\ L_2^\alpha[u,v](t) &= du(t) - [dN_\alpha(t)]v(t) = 0 \end{aligned} \tag{2.3_α}$$

and corresponding functionals

$$J_\alpha[\eta,\zeta;a,b] = \int_a^b \{\zeta^*[dN_\alpha]\zeta + \eta^*[dM_\alpha]\eta\}. \tag{2.4_α}$$

In particular, if $\mathcal{D}_1[a,b] = \mathcal{D}_2[a,b] = \mathcal{D}[a,b]$ then the difference functional

$$J_{12}[\eta;a,b] = J_1[\eta;a,b] - J_2[\eta;a,b] \tag{2.5}$$

is well defined for $\eta \in \mathcal{D}[a,b]$.

THEOREM 2.2. <u>Suppose that for</u> $\alpha = 1,2$, <u>the</u> $n \times n$ <u>matrix functions</u> $N_\alpha(t)$, $M_\alpha(t)$ <u>satisfy hypotheses</u> H <u>and</u> H_h <u>and</u> $N_2(t)$ <u>is non-decreasing</u>. <u>Also suppose</u> $\mathcal{D}_1[a,b] = \mathcal{D}_2[a,b]$ <u>and</u> $J_{12}[\eta;a,b]$ <u>is non-negative definite on</u> $\mathcal{D}_o[a,b] = \mathcal{D}_{10}[a,b] = \mathcal{D}_{20}[a,b]$. <u>If</u> (2.3_2) <u>is disconjugate on</u> $[a,b]$, <u>then</u> (2.3_1) <u>is also disconjugate on</u> $[a,b]$. <u>Moreover, if</u> $J_{12}[\eta;a,b]$ <u>is positive definite on</u> $\mathcal{D}_o[a,b]$ <u>then the solutions of</u> (2.3_2) <u>oscillate more rapidly than the solutions of</u> (2.3_1) <u>in the following sense:</u> <u>if</u> t_1 <u>and</u> t_2 <u>are mutually conjugate with respect to</u> (2.3_1) <u>then any conjoined</u>

basis $Y(t) = (U(t);V(t))$ for (2.3_2) is singular at least once on (t_1,t_2).

If (2.3_2) is disconjugate on $[a,b]$, then (ii) of Theorem 2.1 implies that $J_2[\eta;a,b]$ is positive definite on $\mathcal{D}_o[a,b]$ so that $J_1[\eta;a,b]$ is positive definite on $\mathcal{D}_o[a,b]$. Thus Theorem 1.2 implies that $N_1(t)$ is non-decreasing. Hence (2.3_1) is disconjugate on $[a,b]$.

Now, in a manner similar to the proof of Theorem 1.5, it can be shown that if there is a conjoined basis $Y(t) = (U(t);V(t))$ of (2.3_2) with $U(t)$ nonsingular on (a,b) and $N_2(t)$ is non-decreasing on this interval, then $J_2[\eta;a,b]$ is non-negative definite on $\mathcal{D}_o[a,b]$. Let $u(t)$ be a solution of (2.3_1) with $u(t_1) = 0 = u(t_2)$, and $u(t) \not\equiv 0$ on $[t_1,t_2]$, where $a \le t_1 < t_2 \le b$. If $\eta(t) = u(t)$ for $t \in [t_1,t_2]$, $\eta(t) \equiv 0$ on $[a,t_1] \cup [t_2,b]$, then $\eta \in \mathcal{D}_o[a,b]$ and $J_1[\eta;a,b] = J_1[u;t_1,t_2] = 0$, so that $J_2[\eta;a,b] < 0$. Hence, any conjoined basis $Y(t) = (U(t);V(t))$ of (2.3_2) must have at least one point on (t_1,t_2) where $U(t)$ is singular.

THEOREM 2.3. If $N(t)$ is non-decreasing on $[a,b]$, then (E) is disconjugate on $[a,b]$ if and only if one of the following conditions holds:

(i) there exists on $[a,b]$ a nonsingular $n \times n$ matrix function $U(t) \in \mathcal{D}[a,b]:V$ with V of bounded variation on $[a,b]$ while $\{U;V|U;V\}(t) \equiv 0$, and $\int_a^t U^* L_1[U,V]$ is non-decreasing for $t \in [a,b]$;

(ii) there exists an $n \times n$ hermitian matrix function $W(t)$ of bounded variation on $[a,b]$ which is such that

$$\mathcal{R}[W](t) \equiv \int_a^t [dW] + \int_a^t W[dN]W - \int_a^t [dM]$$

is non-increasing for $t \in [a,b]$.

If (E) is disconjugate on $[a,b]$ then there is a conjoined basis $Y(t) = (U(t);V(t))$ of (E) with $U(t)$ non-singular on $[a,b]$; also, $U(t)$

satisfies (i) and $W(t) = V(t)U^{-1}(t)$ satisfies (ii).

On the other hand, if $U(t)$ satisfies (i) then let $P(t) = \int_a^t U^*L_1[U,V]$. Since $U(t)$ is continuous, the integral exists and defines a matrix function of bounded variation on $[a,b]$. If we take the system (2.3_2) to be such that

$$dN_2(t) \equiv dN(t) \qquad dM_2(t) \equiv dM(t) - U^{*-1}(t)[dP(t)]U^{-1}(t),$$

then $(U;V)$ is a conjoined basis for (2.3_2). If (2.3_1) is system (E), then

$$J_{12}[\eta;a,b] = \int_a^b \eta^* U^{*-1}[dP]U^{-1}\eta \geq 0$$

for $\eta \in \mathfrak{D}_0[a,b]$, so that Theorem 2.2 implies that (E) is disconjugate on $[a,b]$.

Under the condition (iii) if $\Psi(t) = \mathfrak{R}[W](t)$, then $\Psi(t) \in BV[a,b]$ and Ψ is non-increasing. If we take $U(t)$ to be the solution of the system

$$\int_a^t dU = \int_a^t [dN(s)]W(s)U(s), \qquad U(a) = E,$$

and $V(t) = W(t)U(t)$, then U and V are $n \times n$ matrix functions on $[a,b]$ with V of bounded variation on $[a,b]$; moreover, $U \in \mathfrak{B}[a,b]:V$, U is nonsingular on $[a,b]$, while $\{U;V|U;V\}(t) \equiv 0$, $t \in [a,b]$, and

$$\int_a^t U^*L_1[U,V] = -\int_a^t U^*[d\Psi]U$$

which is non-decreasing on $[a,b]$, so we have reduced case (ii) to case (i).

Results may be obtained corresponding to the results of Reid [6; pp. 341-344] concerning sufficient conditions for the existence of principal solutions and properties of solutions when a principal solution exists.

3. Focal points. We shall denote by $\mathcal{D}_{*0}[a,b]$ the class of all $\eta \in \mathcal{D}[a,b]$ with $\eta(b) = 0$ and by $\mathcal{D}_{0*}[a,b]$ the class of all $\eta \in \mathcal{D}[a,b]$ with $\eta(a) = 0$. Then $\mathcal{D}_0[a,b] = \mathcal{D}_{*0}[a,b] \cap \mathcal{D}_{0*}[a,b]$. We shall also consider the functional

$$(3.1) \qquad \hat{J}[\eta_1:\zeta_1,\eta_2:\zeta_2;a,b] = \eta_2^*(a)\Gamma\eta_1(a) + J[\eta_1,\eta_2;a,b].$$

If M and N satisfy H_h and Γ is a hermitian matrix then $J[\eta_1:\zeta_1,\eta_2:\zeta_2;a,b]$ is a hermitian form on $\mathcal{D}[a,b] \times \mathcal{L}[a,b]$. As in the case of the functional $J[\eta;a,b]$, if $\eta_\alpha \in \mathcal{D}[a,b]:\zeta_\alpha$, $(\alpha = 1,2)$, then the value of (3.1) is independent of the value of ζ_α so that we will abbreviate to $\hat{J}[\eta_1,\eta_2;a,b]$ or $\hat{J}[\eta_1;a,b]$ if $\eta_1 = \eta_2$.

Using the results of Theorem 1.1 and Corollary 1.2 we can obtain the following results.

THEOREM 3.1. <u>There exists a solution</u> $(u;v)$ <u>of</u> (E) <u>such that</u>

$$(3.2) \qquad \Gamma u(a) - v(a) = 0$$

<u>if and only if there exists a</u> $v_1 \in \mathcal{L}[a,b]$ <u>such that</u> $u \in \mathcal{D}[a,b]:v_1$ <u>and</u> $\hat{J}[u;v_1,\eta:\zeta;a,b] = 0$ <u>for</u> $\eta \in \mathcal{D}_{*0}[a,b]:\zeta$.

COROLLARY 3.1. <u>If</u> $\hat{J}[\eta;a,b]$ <u>is non-negative definite on</u> $\mathcal{D}_{*0}[a,b]$, <u>and there exists a</u> $u \in \mathcal{D}_{*0}[a,b]$ <u>satisfying</u> $\hat{J}[u;a,b] = 0$, <u>then there exists a</u> v <u>such that</u> (u,v) <u>is a solution of</u> (E) <u>on</u> [a,b] <u>which satisfies the condition</u>

$$(3.3) \qquad \Gamma u(a) - v(a) = 0, \quad u(b) = 0.$$

Since Γ is hermitian, the solution $Y(t) = (U(t);V(t))$ which satisfies $Y(a) = (E,\Gamma)$ is a conjoined basis. The following result is proved in a manner similar to that used to establish Theorem 1.5.

THEOREM 3.2. <u>The functional $\hat{J}[\eta;a,b]$ is positive definite on $\mathcal{D}_{*o}[a,b]$ if and only if $N(t)$ is non-decreasing on $[a,b]$, and the conjoined basis $Y(t) = (U(t);V(t))$ for (E) satisfying $Y(a) = (E;\Gamma)$ is such that $U(t)$ is nonsingular on $[a,b]$.</u>

Relative to the functional (3.1), or relative to system (E) with initial condition (3.2), a value $\tau \in [a,b]$ is a <u>right-hand</u> {<u>left-hand</u>} <u>focal point to</u> $t = a$ if $\tau > a$ $\{\tau < a\}$ and there is a solution $(u(t);v(t))$ of (E) which satisfies (3.2), has $u(\tau) = 0$, and $u(\tau) \not\equiv 0$ on the interval with a and τ as endpoints.

The following result can be established by an argument similar to that occurring in the proof of Theorem 2.2, and using the result of Theorem 3.2.

THEOREM 3.3. <u>Suppose that for $\alpha = 1,2$ the $n \times n$ matrix functions $M_\alpha(t)$ and $N_\alpha(t)$ satisfy hypotheses H and H_h, while $N_2(t)$ is non-decreasing on $[a,b]$. Moreover, for arbitrary $[c,d] \subset [a,b]$ we have $\mathcal{D}_1[c,d] = \mathcal{D}_2[c,d] = \mathcal{D}[c,d]$, and Γ_α $(\alpha = 1,2)$ are hermitian matrices such that</u>

$$\begin{aligned}\hat{J}_{12}[\eta;a,b] &= \hat{J}_1[\eta;a,b] - \hat{J}_2[\eta;a,b] \\ &= \eta^*(a)[\Gamma_1 - \Gamma_2]\eta(a) + J_{12}[\eta;a,b]\end{aligned}$$

<u>is non-negative definite on $\mathcal{D}_{*o}[a,b]$. If relative to $\hat{J}_2[\eta;a,b]$ there is no right-hand focal point to a on $(a,b]$, then relative to $\hat{J}_1[\eta;a,b]$ there is also no right-hand focal point to a on $(a,b]$.</u>

CHAPTER VI

MORSE FUNDAMENTAL FORMS

1. *Focal points.* The results of this section correspond to the results found in Reid [6, pp. 356-366] and the proofs of the results are in most cases the direct analog of Reid's proofs. We wish to examine the relationship of the Morse Quadratic Form and the idea of focal points as defined in Section 3 of the last chapter. That is, if hypothesis H_N is satisfied and $Y(t) = (U(t);V(t))$ is a conjoined basis for (E) on $[a,b]$, then c is a *focal point of the family of order* k if $U(c)$ is singular and of rank $n-k$. The following lemma is basic to the study of these points.

LEMMA 1.1. *Suppose hypothesis* H_N *holds and* (E) *is disconjugate on* $[a,b]$. *If* $Y(t) = (U(t);V(t))$ *is a conjoined basis for* (E), *then on* $(a,b]$ *and* $[a,b)$ *there are at most* n *focal points, each point being counted a number of times equal to its order. Moreover, the focal points of a conjoined basis are isolated.*

Throughout the remainder of this chapter we will assume that H_N *holds.*

A partition

$$a = t_o < t_1 < \cdots < t_m < t_{m+1} = b \tag{1.1}$$

will be called a *fundamental partition* if (E) is disconjugate on each of the subintervals $[t_{i-1},t_i]$, $i = 1,\cdots,m+1$. Such a partition exists since, in view of the results of Corollary II.2.1 and Theorem V.2.1, there exists a $\delta > 0$ such that if $|c-d| < \delta$, $[c,d] \subset [a,b]$, then (E) is disconjugate on $[c,d]$. Moreover, if $T = \{t_o, t_1, \cdots, t_m, t_{m+1}\}$ is a fundamental partition, then any refinement is also a fundamental partition.

If T is a fundamental partition, then in view of the condition H_N of identical normality and the result of Lemma IV.1.1 we have a unique solution $u = u_{\xi j}$, $v = v_{\xi j}$ of (E) such that $u_{\xi j}(t_{j-1}) = \xi_{j-1}$, $u_{\xi j}(t_j) = \xi_j$ $(j = 1,2,\cdots,m+1)$, where the ξ_j are arbitrary n-dimensional vectors. If ξ is defined to be the $n(m+1)$ vector

$$\xi = (\xi^{(\rho)}) \quad \rho = 1,2,\cdots,n(m+1)$$

with $\xi^{(nj+\alpha)} = \xi_{\alpha j}$, $(\alpha = 1,\cdots,n,\ j = 0,\cdots,m)$, then the corresponding vector function

$$u_\xi(t) = u_{\xi j}(t), \quad t_{j-1} \le t \le t_j, \quad (j = 1,\cdots,m+1),$$

is continuous on $[a,b]$ and linear in the components of ξ. We shall denote by $\mathcal{S}(\Pi)$, the set of all vectors ξ. If $\xi_{m+1} = 0$ we shall say $\xi \in \mathcal{S}_{*o}(\Pi)$, and if $\xi_o = 0$ we shall say $\xi \in \mathcal{S}_{o*}(\Pi)$. Moreover, set $\mathcal{S}_o(\Pi) = \mathcal{S}_{o*}(\Pi) \cap \mathcal{S}_{*o}(\Pi)$. If G is an $n \times n$ hermitian matrix, the form

$$Q^o_*[\xi^1,\xi^2|\Pi] = \xi^{2*}_o G\xi^1_o + J[u_{\xi^1},u_{\xi^2};a,b] \tag{1.2}$$

is hermitian on $\mathcal{S}_{*o}[\Pi]$ since J is a hermitian functional. Thus, there is an $n(m+1)$ dimensional, hermitian matrix Q^o_* such that

$$Q^o_*[\xi^1,\xi^2|\Pi] = \xi^{2*}Q^o_*\xi^1.$$

THEOREM 1.1. <u>If</u> G <u>is</u> <u>an</u> $n \times n$ <u>hermitian matrix and</u> T <u>and</u> $u_\xi(t)$ <u>are specified as above, then</u> Q^o_* <u>is of rank</u> $n(m+1)-r$ <u>if and only if</u> $t = b$ <u>is a focal point of order</u> r <u>of the conjoined family of solutions</u>

$Y(t) = (U(t);V(t))$ _of_ (E) _with_ $Y(a) = (E;G)$. _Moreover, the elements of_ Q_*^o _are continuous functions of the elements of_ G _and of_ $t_1, t_2, \cdots, t_m$.

For the systems of ordinary differential equations considered in [6; Chapter VII, Section 7] the proof of a result corresponding to that of the above theorem uses the continuity of the vector functions u_ξ and v_ξ as functions of $t, t_o, t_1, \cdots, t_{m+1}$. In the present situation the functions u_ξ are continuous functions, but the functions v_ξ are not necessarily continuous. However, the type of argument used by Reid [5; pp. 716-717] to establish the stated result for a system (E) where N(t) is absolutely continuous is still valid for the more general problem considered here.

The dimension of the null space

$$\{\xi \mid Q_*^o \xi = 0\}$$

is called the _nullity of_ Q_*^o, and the dimension of the largest subspace on which Q_*^o is negative definite is called the _(negative) index of_ Q_*^o. We can now obtain the following results.

THEOREM 1.2. _If_ Π _is a fundamental partition of_ [a,b], _then the index of_ $Q_*^o[\xi|\Pi]$ _is equal to the number of points on the open interval_ (a,b) _which are right-hand focal points to_ t = a _relative to the functional_ $\hat{J}[\eta;a,b]$ _where each focal point is counted a number of times equal to its order._

THEOREM 1.3. _If_ Π _is a fundamental partition of_ [a,b], _then the index,_ {_index plus nullity_}, _of_ $Q_*^o[\xi|\Pi]$ _is equal to the largest non-negative definite integer_ k _such that there exists a_ k-_dimensional manifold in_ $\mathcal{D}_{*o}[a,b]$ _on which_ $\hat{J}[\eta;a,b]$ _is negative definite,_ {_non-positive definite_}.

For a conjoined basis $Y_o(t) = (U_o(t);V_o(t))$ of (E), the designation of a point c where $U_o(c)$ is singular as a focal point is consistent with the characterization of a focal point in terms of the functional $\hat{J}$. If $t = a$ is a point such that $U_o(a)$ is non-singular then $W_o(a) = V_o(a)U_o^{-1}(a)$ is hermitian, and $(U(t);V(t)) = (U_o(t)U_o^{-1}(a);V_o(t)U_o^{-1}(a))$ is a conjoined solution which satisfies $U(a) = E$, $V_o(a) = V_o(a)U_o^{-1}(a)$. If we let $\Gamma = W_o(a)$, then a value $c > a$ will be a focal point of $\hat{J}[\eta;a,b]$ of order k if and only if $U(c)$ is singular of order $n-k$.

For a given $c \in [a,b]$, the points of $[a,b]$ which are right-hand focal points to $t = c$ relative to the functional $\hat{J}[\eta;a,b]$ will be ordered as a sequence $\tau_p^+(\Gamma)$, $(p = 1,2,\cdots)$, and numbered so that $\tau_p^+(\Gamma) \le \tau_{p+1}^+(\Gamma)$, with each repeated a number of times equal to its order as a focal point. For focal points we have the following basic separation theorem.

THEOREM 1.4. _Suppose that_ (E) _satisfies hypothesis_ H_N, _and for_ $\alpha = 1,2$ _let_

$$\hat{J}_\alpha[\eta;a,b] = \eta^*(a)\Gamma_\alpha\eta(a) + \int_a^b \{\zeta^*[dN_\alpha]\zeta + \eta^*[dM_\alpha]\eta\},$$

where Γ_1 _and_ Γ_2 _are_ $n \times n$ _hermitian matrices. Moreover, let_ $\mathcal{P}$ _and_ $\mathcal{N}$ _denote the number of positive and negative proper values of the hermitian matrix_ $\Gamma_1 - \Gamma_2$, _where each proper value is repeated a number of times equal to its multiplicity. If for a positive integer_ p _the focal point_ $\tau_{p+\mathcal{P}}^+(\Gamma_2)$ _exists, then_ $\tau_p^+(\Gamma_1)$ _exists and_ $\tau_p^+(\Gamma_1) \le \tau_{p+\mathcal{P}}^+(\Gamma_2)$; _if_ $\tau_{p+\mathcal{N}}^+(\Gamma_1)$ _exists then_ $\tau_p^+(\Gamma_2)$ _exists and_ $\tau_p^+(\Gamma_2) \le \tau_{p+\mathcal{N}}^+(\Gamma_1)$.

2. _Conjugate points_. If we take fundamental partitions as in the last section, and $\xi_i \in \mathcal{S}_o(\Pi)$, $(i = 1,2)$, then we again obtain a form

$Q^o[\xi^1,\xi^2|\Pi]$ which is fundamental to the study of conjugate points. Using the same techniques as in Section 1, we may establish results corresponding to Theorems 1.1, 1.2, 1.3, and 1.4, along with the following additional results.

THEOREM 2.1. *The number of points on* (a,b), $\{(a,b]\}$, *conjugate to* a *is the same as the number of points on* (a,b), $\{[a,b)\}$ *conjugate to* b, *where each point is counted a number of times equal to its order as a conjugate point.*

If we let $t_p^+(a)$ and $t_p^-(a)$ be the p-th right and left conjugate point of a, respectively, again with the usual order and numbering convention, we get the following results.

THEOREM 2.2. *If* $t_p^+(c)$, $\{t_p^-(c)\}$, *exists for* $c = c_o$, *then there exists a* $\delta > 0$ *such that* $t_p^+(c)$, $\{t_p^-(c)\}$ *exists for* $c \in (c_o - \delta, c_o + \delta)$; *moreover,* $t_p^+(c)$, $\{t_p^-(c)\}$ *is continuous at* c_o.

THEOREM 2.3. *If* $a_\alpha \in [a,b]$, $(\alpha = 1,2)$, *and* $a_1 < a_2$, *then whenever* $t_p^+(a_2)$, $\{t_p^-(a_1)\}$ *exists, the conjugate point* $t_p^+(a_1)$, $\{t_p^-(a_2)\}$ *also exists and* $t_p^+(a_2) > t_p^+(a_1)$, $\{t_p^-(a_2) > t_p^-(a_1)\}$.

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