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A STUDY OF THE CHILD'S ABILITY TO UNDERSTAND

THE CONCEPT OF FUNCTION

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DISSERTATION COMMITTEE

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CHAPTER I

INTRODUCTION AND PROBLEM

Mathematical Background

The concept of function has developed intuitively as human beings have had cause for greater and greater amounts of relational thinking. From our present position, it is possible to look back as E. T. Bell and Marshall Stone have done and cite examples to credit people who lived long ago with an intuitive feeling for the concept of functionality. Bell determined that the ancient Babylonians, in about 2000 B. C., devised tables of related numbers and then made use of the correspondences in the tables. He felt it reasonable " . . . to credit the Babylonians with an instinct for functionality." /Bell, 1945; 199/ Stone noted an instance of functional thinking occurring in arithmetic where a pair of numbers were converted into their sum or product and thought this signified an early feeling for functionality. He also felt that Euclid demonstrated an ability for functional thinking for he " . . . had to think in terms of an act converting one (concrete) figure into another, point by point and part by part." /Stone, 1965; 6/

Intuitive functional thinking has been in practice for a long period of time, but the idea of functionality did not achieve prominence until the time of Galileo and Descartes at the end of the sixteenth century and during the first part of the seventeenth century. /Kline, 1972/ In a recent book, Morris Kline stated:

> From the study of motion mathematics derived a fundamental concept that was central to practically all of the work for the next two hundred years--the concept of a function or a relation between variables. One finds this notion almost throughout Galileo's Two New Sciences, the book in which he founded modern mechanics. Galileo expresses his functional relationships in words and in the language of proportion. /Kline, 1972; 338/

Most functions which were presented during this period were originally studied as curves, as the concept was not fully recognized. The clearest definition of the function concept during this period was given by James Gregory in a publication in 1667. He considered a function to be a quantity obtained from other quantities by a series of algebraic operations or by any other operation one might use. Unfortunately, his concept of function was lost and had little influence on his fellows. /Kline, 1972/

In his work on the calculus, beginning in 1665, Newton used the term "fluent" to represent any relationship between variables. However, it was Leibnitz who incorporated the Latin form of the word "function" into his work in 1673. For Leibnitz the term "function" first denoted any quantity connected with a curve, such as the coordinates of a point on the curve or the slope of the curve at a point. However, in 1714 he revised his concept of function to mean quantities that depend on a variable. /Kline, 1972/ The term, function, was accepted by other mathematicians and used by them.

Gradually, the meaning of the concept was refined. Johann Bernoulli, among others, regarded a function as any expression consisting of a variable and some constants. Euler used the term to mean any equation or formula which involved variables and constants, and in 1734 he introduced the familiar f(x) notation to denote a function. /Ēves and Newsom, 1965/ In 1748 he wrote the first book in which the concept of function was a principal idea as well as being the basis for organizing the material. /Kline, 1972/

> In 1755 Euler gave as a new definition of function, "If some quantities depend on others in such a way as to undergo variation when the latter are varied, then the former are called functions of the latter." /Kline, 1972; 506/

Then, at the end of the century Lacroix presented a broader definition with this statement: "Every quantity whose value depends on one or several others is called a function of the latter, whether one knows or one does not know by what operations it is necessary to go from the latter to the first quantity." /Kline, 1972; 949/

Concrete examples of functions accumulated slowly. It was not until the middle of the nineteenth century that there were numerous examples available. /Stone, 1965/ During the first half of the century there was confusion about the function concept in even the best textbooks and all the authors made deductions beyond logical bounds of the definitions. /Kline, 1972/ The concept of function which had been developed, at this point, was neither broad enough nor general enough. Fourier found the need for trigonometric series which involved a more general relationship between variables than had been previously studied.

> In his The Analytical Theory of Heat he says, "In general the function f(x) represents a succession of values or ordinates each of which is arbitrary. . . . We do not suppose these ordinates to be subject to a common law; they succeed each other in any manner whatever. . . ." /Kline, 1972; 949/

In 1837 Dirichlet formulated a general definition of a function which could encompass such relationships.

> A variable is a symbol which represents any one of a set of numbers; if two variables x and y are so related that whenever a value is assigned to x there is automatically

> assigned, by some rule or correspondence, a value to y, then we say y is a (single-valued) function of x. The variable x, to which values are assigned at will, is called the independent variable, and the variable y, whose values depend upon those of x, is called the dependent variable. The permissible values that x may assume constitute the domain of definition of the function, and the values taken on by y constitute the range of values of the function. /Eves and Newsom, 1965; 264/

This definition by Dirichlet was more complete than any previous definition and also more general than any previous definition.

Further refining of the definition of function did not occur for almost a century. According to Stone, ". . . our understanding of the true generality of the function concept clearly grew out of now-classical studies in the theory of real functions, topology, set theory, and logic, dating from around the turn of the century." /Stone, 1965; 7/ Indeed, it was not long after the turn of the century, in 1914, that Hausdorff published his well-known book on set theory, Grundzuge der Mengenlehre. In this book Hausdorff defined two nonempty sets A and B and formed the cross product of the two sets, A X B. A function, he said, was a set P of ordered pairs p = (a, b) where each element a in A was included in one and only one ordered pair in P. In terms of the function notation introduced by Euler, b = f(a). /Hausdorff, 1949; 33/

This definition was different from previous definitions in that it involved a set of ordered pairs rather than two types of variables where one variable is dependent upon the other for its value. The fact that each a in A was included as a first component in only one ordered pair of the function implies that the function defined by Hausdorff was single-valued as was that of Dirichlet. Further refinements in the definition of function involve the concept of a set of ordered pairs with the basic variations being in notation and terminology employed. Let us look at some recent definitions of the function concept.

Halmos gave the following definition of function in Naive Set Theory which appeared almost fifty years after the book by Hausdorff. He defined a relation to be a set of ordered pairs (x, y), where x is an element of a set X and y is an element of a set Y. Then Halmos went on to define a function from one set to another and used the symbol $f: X \longrightarrow Y$ to mean "f is a function from X to Y." /Halmos, 1960; 30/

> If X and Y are sets, a function from (or on) X to (or into) Y is a relation f such that dom f = X and such that for each x in X there is a unique element y in Y with $(x, y) \in f$. The uniqueness condition can be formulated explicitly as follows: if $(x, y) \in f$ and $(x, z) \in f$, then y = z. For each x in X, the unique y in Y such that $(x, y) \in f$ is denoted by f(x). /Halmos, 1960; 30/

Roger Godement, in his book called Algebra which appeared in 1968, defined a function as an ordered triple of sets possessing certain conditions. He stated that " . . . a function is an ordered triple f = (G, X, Y) where G, X, Y are sets satisfying the following conditions: (f 1): $G \subset X \times Y$; and (f 2): for each $x \in X$ there is exactly one $y \in Y$ such that $(x, y) \in G$." /Godement, 1968; 53/ The set G would be the function in terms of Halmos' definition.

Function Concept in Mathematics Education

The impetus for stressing the function concept as an important part of school mathematics originated in a movement to reform school mathematics during the last half of the nineteenth century. The movement began in Europe with many well-known mathematicians advancing its cause. /Henley, 1934/ One of these mathematicians was Felix Klein, a German, who wanted to have the function concept recognized and used as a unifying idea throughout all mathematics. /National Council of Teachers of Mathematics, 1970/ In 1893 Klein addressed the International Congress of Mathematicians in Chicago, stressing that functional thinking be made the central theme of school mathematics. [Henley, 1934]

Klein interpreted the function concept in the broadest sense. He noted that many mathematical ideas depend on the function concept. To illustrate this, he noted that change could be thought of as a translation of points in geometry, allowing one to study geometric figures as continuously varying structures in space. [Henley, 1934]

A speech by E. H. Moore before the American Mathematical Society signaled the beginning of a corresponding reform movement in the United States. While the function concept was not mentioned in his address, many related concepts, such as sets and graphs, were emphasized and the central theme of his address was the unification of school mathematics. [Henley, 1934]

Another International Congress of Mathematicians, meeting in Rome in 1908, appointed a commission to study mathematics teaching. This was the International Commission on the Teaching of Mathematics, authorized to study mathematics curricula and practices throughout the world. Klein was the chairman of this commission. In the reports and recommendations of the commission, the function concept was given an important role in school mathematics. In teaching the concept, emphasis was placed on using concrete illustrations in the early stages to give the student time to become familiar with the new ideas. Opinion favored progressing from this intuitive-experimental approach in the early stages to a strictly logical approach in the last year of school. [Henley, 1934]

In the United States David Eugene Smith and E. R. Hedrick were early supporters of the function concept in school mathematics. Smith presented a report to the Fifth International Congress of Mathematicians, held at Cambridge in 1912. Here he indicated that the function concept should be a part of the school curriculum, so that the student would be familiar with it by the time he progressed to the study of calculus. /Henley, 1934/

The Mathematical Association of America appointed the National Committee on Mathematics Requirements in 1916, and this committee published a report in which the following discussion appeared.

> The one great idea which is best adopted to unify the course is that of the functional relation. The concept of a variable and of the dependence of one variable upon another is of fundamental importance to everyone. It is true that the general and abstract form of these concepts can become significant to the pupil only as a result of very considerable mathematical experience and training. There is nothing in either concept, however, which prevents the presentation of specific concrete examples and illustrations of dependence even in the early parts of the course. /National Committee on Mathematics Requirements, 1921; 10/

Considerable emphasis was placed on the function concept in this report with scattered references to the function concept throughout and a chapter devoted to the function concept in secondary mathematics. This chapter carefully outlined the use of the function concept in algebra, geometry, and trigonometry as well as demonstrating real-life opportunities to use the concept. "Even when no calculation is to be carried out, the problems of real life frequently involve the ability to think correctly about the nature of the relationships which exist between related quantities." /National Committee on Mathematics Requirements, 1921; 60/ Another important statement in the report involved the goals in teaching mathematics. "Training in functional thinking is one of the most fundamental disciplinary aims of the teaching of mathematics." /National Committee on Mathematics Requirements, 1921; 62/

At the time that this committee was appointed in 1916, Hedrick was president of the Mathematical Association of America. Hedrick himself prepared the first draft of the chapter on the function concept. After this report was published textbooks were written which treated the function concept, but usually only in terms of graphing or in an isolated chapter. The first mathematics textbook to have the function concept appear throughout the book was written by John Swenson in the thirties. /National Council of Teachers of Mathematics, 1970/

This trend of recognizing and stressing the importance of the function concept continued. The final report of the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics was published in 1940 under the title, The Place of Mathematics in Secondary Education.

> The ambition to make mathematical instruction more broadly significant through emphasis on concepts has led to stressing the function concept as a unifying element. Inasmuch as it deals with relationships, it is quite true that few concepts have greater universality or importance. A society, all members of which while in school have been given persistent and effective contact with this concept, should view problems and situations more intelligently than a society which has only a certain number of mathematical specialists. /National Council of Teachers of Mathematics, 1970; 40-1/

This report contains a chapter called "Mathematics Curriculum" which has a division listed as "The Field of Relational Thinking." Here success in relational thinking was tied to the understanding of certain basic ideas: constant, variable, independent variable, one-to-one correspondence, function, formula, table, and value of a function. /National Council of Teachers of Mathematics, 1970/

Another important report, Mathematics in General Education, was published in the same year by the Progressive Education Association. In a discussion about the function concept the report stated that while the concept is specifically mathematical it makes a contribution " . . . to the common man's ability to think for himself in all areas of experience."

/Progressive Education Association, 1940; 67/ A special chapter was devoted to the function concept in this book, where it was noted that " . . . by employing the term function in a broad sense much of the system of elementary mathematical concepts may be embraced." /Progressive Education Association, 1940; 141/

The Commission on Mathematics was appointed by the College Entrance Examination Board to make recommendations for high school mathematics. The commission's report in 1959, Program for College Preparatory Mathematics, emphasized balanced preparation in skills, concepts, deductive reasoning, use of mathematical structure and stressed the unifying ideas in mathematics such as set and function. /College Entrance Examination Board, 1959/ A preliminary report of this commission was available to the School Mathematics Study Group (SMSG) when they began in 1958. SMSG was instrumental in curriculum reform and in the writing of materials for use in the schools. Emphasis was placed on understanding, and ideas such as set and function were used frequently. Other curriculum reform groups appeared, writing materials for the schools where emphasis was placed on structure, generalization, proof and abstraction. /National Council of Teachers of Mathematics, 1970/

The Cambridge Conference on School Mathematics met in 1963 and published its recommendations under the title, Goals for School Mathematics. This report included suggested curricula to be followed in the schools. In the section concerning curricula from kindergarten to grades six the report stated the following.

> The ideas of set and function should be introduced as soon as possible. In the earliest grades:
> (1) Number as a property of finite sets
> (2) The comparison of cardinals of finite sets with emphasis on the fact that the result is independent of which mapping function is used.

> (3) Numerical functions determined by very simple formulas. /The Cambridge Conference on School Mathematics, 1963; 34/

The report indicated that the function concept should be introduced at an early level in school, and then as the child progressed through school his knowledge and understanding of the function concept should expand and mature.

In 1967 another conference was held at Cambridge; this one dealt with teacher training. In a report from this conference it was stated:

> Some of the most important items are deliberately omitted from the diagram, not because they are unimportant, but, on the contrary, because they should be nearly ubiquitous. Chief among these are the concepts of function and set which should be used throughout the development wherever natural examples occur. By Grade 6, both the words function and set (and the ideas behind them) should be established firmly and correctly as natural parts of the pupil's mathematical language. /The Cambridge Conference on Teacher Training, 1967; 98/

Background of the Problem

The concept of function holds an important place in mathematics. It is an idea which permeates mathematics and many mathematical ideas are dependent on it. As R. C. Buck stated:

> There is no branch of mathematics whose developments since 1800 can be studied in their present form without an understanding of the general function concept. This is even true when it comes to the applications of mathematics to physical problems. /Buck, 1970; 252/

The function concept has been refined throughout its development to the present state. The previous sections have discussed continual refinement of the function concept from its vague beginnings to its present form. Interest developed toward the end of the nineteenth century in incorporating this concept into the school mathematics curriculum. Various individuals and groups have agreed that the function concept should be a part of the mathematics curriculum in the schools, usually indicating that the concept should be presented early in the child's school experience. In an article on learning the function concept, Marshall Stone stated: "In the teaching of mathematics, it is essential to start at an early stage to lay the groundwork that will enable students, when they reach the stage between 15 and 18 years of age, to study this theory with understanding and master some of its numerous applications in arithmetic, algebra, geometry, and analysis." [Stone, 1965; 5]

Stone pointed out in the same article, "A very important area of study is the stage of maturity that the learner must attain before he is able to grasp these concepts." [Stone, 1965; 9] Buck also had this same concern when he stated: "There seems to be ample evidence that there is something called 'concept readiness' and that refinements of viewpoints may be meaningless to students until a certain threshold has been reached." [Buck, 1970; 256]

Indeed, this is a very significant problem. There is no need to present work on the function concept to children who are not able to understand the material. As Piaget stated,

> It is a great mistake to suppose that a child acquires the notion of number and other mathematical concepts just from teaching. On the contrary, to a remarkable degree he develops them himself, independently and spontaneously. When adults try to impose mathematical concepts on a child prematurely, his learning is merely verbal; true understanding of them comes only with his mental growth. [Piaget, 1953; 74]

Piaget has done a great deal of research concerning the intellectual processes by which children learn mathematical concepts. In an interview several years ago, Piaget said:

> Seven years would be perfectly all right for most operations of set theory because children have their own spontaneous operations that are very akin to those concepts. But when you teach set theory you should use the child's

> actual vocabulary along with activity--making the child do natural things. . . .
> As for teaching children concepts that they have not attained in their spontaneous development, it is completely useless. /Hall, 1970; 30/

Weisman and Safford wrote in an article concerning Piaget, "His developmental findings could aid in curriculum-planning, for example, in determining at what age a child is prepared to assimilate a certain concept." /Weisman and Safford, 1971; 330/ It is interesting to note that Barbel Inhelder, one of Piaget's associates, responded along these same lines when asked in an interview what advice she would give to educators.

> The order of the introduction of the fundamental mathematical concepts might be changed to conform to the developmental laws of the child. . . . Basic concepts in mathematics and in science can be introduced to children from seven to 10, if these are divorced from their traditional mathematical context. /Hall, 1970; 56/

Piaget found learning to be subordinate to development and the order of development to be the same for everyone. He asserted that there appears to be an optimum time to teach an idea and that this time is dependent upon the individual and the subject matter. /Copeland, 1970/ It would seem advantageous to anticipate these optimum times to introduce various ideas into the curriculum. These times would not coincide for all children, but an estimate of optimum time on a collective basis in terms of age might be of assistance.

The idea of interest here is the concept of function. At what point have children progressed far enough in their development to understand the concept of function? Are they able to understand this concept in kindergarten or is it necessary to wait until the children reach a later grade? When should the concept of function be introduced into the curriculum? This study is designed to provide some partial answers to these questions.

The Problem

Statement

What is the child's ability to understand and work with a simple explanation of the concept of function at ages seven, nine and twelve years?

Analysis

Due to the variety in age levels, the subjects did not have a common fund of knowledge. For this reason the first task presented to each subject was a teaching task. After this short period of instruction, the remaining nine tasks were presented to the subject.

Investigation of the problem called for answers to the following questions.

1. What is the child's ability to discriminate instances and non-instances of function?

2. What is the child's ability to work with representations of function in finding images, preimages, domain, range, and sets of images?

3. What stage, if any, has the child attained in understanding the concept of function? /Thomas, 1969/

Limitations

This investigation was limited to the public schools in Marshall, Texas. Sixteen children were selected at each of the three age levels--seven years, nine years and twelve years of age.

Information was gathered by means of individual private interviews during which the children were presented ten tasks by the interviewer.

CHAPTER II

RELATED RESEARCH

Studies on the Concept of Function

A pioneer study on the development of the function concept was done by H. L. Thomas at Columbia University in 1969. In this study Thomas attempted to determine stages in the development of the concept of function. He began by hypothesizing stages and then modified these stages as he obtained results from testing.

His subjects were eighth graders with an average age of thirteen years, ranging in age from 11 years, 6 months to 14 years, 5 months. They were all well above average in ability, having an average IQ of 125. /Lovell, 1971/

His procedure was to give written tests to 201 subjects and then categorize the subjects according to the stages which they had attained in their development of the concept. Using these results, he randomly selected twenty representative subjects to interview personally. The responses to these interviews caused him to again revise the stages in the development of the concept. /Thomas, 1971/

In his final analysis Thomas characterized the stages using three components, which were concept identification, process, and operations.

> Concept identification refers to the ability to discriminate instances and non-instances of function. Process refers to the ability to work with various representations and names of functions in finding images, pre-images, domain, range, and sets of images. Operations refers to the ability to carry out operations

on functions with an indication that the result of the operation is understood again to be a function. /Thomas, 1971; 12/

The final stages were as follows.

I. Subjects are able to carry out processes associated with the concept when they are basically arithmetic in nature or when assignments are given quite specifically as in an arrow diagram or a table. Their interpretation of rules such as "n ⟶ 3n + 5" is as a sequence of arithmetic operations to be performed. Extension to new or less familiar representations of functions, such as an ordered pair point graph, is limited.

II. In addition to the processes described in Stage I, subjects at Stage II are able to find images, preimages, domain, and range in all representations employed with adequate explanations of the processes used. Arithmetic or algebraic rules are stated verbally in quantified form, e.g., "You take any number and you add 23 to it." The differences between Stage I and Stage II are especially marked with respect to the ordered pair point graph representation of a function.

IIIA. Subjects are able to identify relations in the several representational modes employed as functions or not functions and can give an adequate criterion for each such discrimination, whereas subjects at Stages I and II cannot. In addition, their mastery of the process component is at least equivalent to that of Stage I.

IIIB. Subjects show a grasp of the relational aspects of the function concept over all representations employed, in addition to mastery of the concept identification component in these representations. These subjects fall short of the highest level on the operations component.

IIIC. Subjects perform at a high level on the operations component, with intimations that they have begun to treat functions as conceptual entities. Their grasp of the process component is good except for the ordered pair concepts in the geometrical context of the ordered pair point graph. Concept identification has been mastered as in Stage IIIA.

IVA. Performance is at a high level on all components and with a fluency and generality of response that indicates an apparent integration of the subconcepts that remained separated for the subjects at the substages of Stage III. /Thomas, 1971; 15/

Thomas mentioned that the last stage was Stage IVA because the subjects were only at the first level of the stage involving operations on functions.

Another study on the function was done by Orton at Leeds, England, in 1970. The subjects in Orton's study were also students in a modern mathematics curriculum which taught and used the concept of function. Orton worked with 72 subjects who ranged in age from twelve to seventeen years. Eight boys and eight girls were selected from each of the second through the fifth years of school and six boys and two girls (the entire class) were selected from the sixth year. These subjects had been in the curricular program from two to five years, enabling Orton to study their attainment of inverse functions and composition of functions. [Lovell, 1971]

Orton used only private interviews in his testing and, aside from the tasks involving composition of functions, he found stages which closely fit stages I, II, IIIA, and IIIB. In the more advanced material, he identified four stages also.

Stage A. Success with tasks related to the composition of functions and relations, and of their inverses, is limited to finding images by sequencing assignments in the compositions. Domain and range can only be identified in simple cases and by direct reference to a diagram.

Stage B. Subjects are successful with some of the tasks involving composition, and, in particular, are able to define domain and range in simple cases without being restricted to those members contained in a diagram.

Stage C. Pupils can complete tasks involving composition, with some indication that the processes can be thought of as operations on a set of functions. Subjects are able to identify domain and range, including domain and range for composition of inverse relations, but they have difficulty in checking the uniqueness criterion in inverses.

Stage D. At this stage complete mastery over compositions is exhibited and classification of relations as functions or not functions is consistent. Even in the composition of inverse relations the domain is

> correctly defined and the uniqueness criterion checked. [Lovell, 1971; 18]

Other studies involving the learning of the concept of function did not deal with stages in attaining the concept. James Reeves analyzed the function concept in secondary mathematics in 1969. He found that authors of textbooks tended to develop function in abstract form as a set of ordered pairs, the recommendations of the Commission on Mathematics of College Entrance Examination Board were reflected in the current textbooks, there was no significant difference between the treatment given the function concept in the junior high school texts of the present and those of fifteen years before, and there was a significant difference between the treatment of the function concept in the high school texts of the present and those of fifteen years ago. [Reeves, 1969]

In 1968 Leonard Nelson studied the relationship between verbal, visual-spatial, and numerical abilities and learning the function concept. He used four different approaches to a unit on the function concept: verbal treatment, visual treatment, numerical treatment, and eclectic treatment. Each of the first three were devised to depend on the abilities mentioned and the last was a combination of these techniques. Twelve eighth grade mathematics classes, consisting of 284 students, participated, where each treatment group consisted of three classes with two classes being taught by the participating teacher and the third by the experimenter. He found the visual approach achieved significantly better than all of the other groups on the test and the retention test and that this group had the highest mean scores on all tests. There was little evidence that if a student possessed one of these abilities, he would attain, apply or retain the concept better if taught by one of these approaches. Similar findings were noted with students high in two or three of these abilities or low in all three abilities. He concluded that the visual approach was more effective than the others, that there was no interdependence between abilities and approaches, and that numerical ability was more indicative of concept attainment than the other abilities. [Nelson, 1968]

Related Work of Piaget and His Associates

Jean Piaget and his colleagues began work in 1927, concerning themselves with the nature and origin of knowledge. Their work involved studying children and the way the child learns. This type of research was called genetic epistemology.

> Genetic epistemology attempts to explain knowledge, and in particular scientific knowledge, on the basis of its history, its sociogenesis, and especially the psychological origins of the notions and operations upon which it is based. These notions and operations are drawn in large part from common sense, so that their origins can shed light on their significance as knowledge of a somewhat higher level. But genetic epistemology also takes into account, wherever possible, formalization in particular, logical formalizations applied to equilibrated thought structures and in certain cases to transformations from one level to another in the development of thought. [Piaget, 1970; 1]

In his work, Piaget has divided the years of the child into periods: Sensorimotor Period (0 - 2 years), Pre-operational Period (2 - 7 years), Concrete Operational Period (7 - 11 years), and Formal Operations Period (11 - 15 years). Each of these periods was divided into several stages. Different children passed through the stages at different rates. The last two periods pertain more to this study than do the first two. "Concrete, in the Piagetian sense, means that the child can think in a logically coherent manner about objects that do exist and have real properties and about actions that are possible;

he can perform the mental operations involved both when asked purely verbal questions and when manipulating objects." /Sinclair, 1971; 5-6/ On the other hand, formal operations required no objects, for one can deal with abstractions, hypotheses, and propositional logic. /Sinclair, 1971/

In naming these periods of development, Piaget has employed the term operation. In an address he made the following remarks about what he meant by this term.

> To understand the development of knowledge, we must start with an idea which seems central to me--the idea of an operation. Knowledge is not a copy of reality. To know an object, to know an event, is not simply to look at it and make a mental copy, or image, of it. To know an object is to act on it. To know is to modify, to transform the object, and to understand the process of this transformation, and as a consequence to understand the way the object is constructed. An operation is thus the essence of knowledge; it is an interiorized action which modifies the object of knowledge. For instance, an operation would consist of joining objects in a class, to construct a classification. Or an operation would consist of counting, or of measuring. In other words, it is a set of actions modifying the object, and enabling the knower to get at the structures of the transformation.
>
> An operation is an interiorized action. But, in addition, it is a reversible action; that is, it can take place in both directions, for instance, adding or subtracting, joining or separating. So it is a particular type of action which makes up logical structures.
>
> Above all, an operation is never isolated. It is always linked to other operations, and as a result it is always a part of a total structure. /Klausmeier and Prayer, 1970; 24-25/

Piaget continued by stating that it was operational structures, such as seriation, which were the basis of knowledge. He found the central problem to be in the understanding of the way these structures are formed and are organized as well as the way in which they function.

A study was done in 1970 by Donald Sheehan concerning what type of instructional procedure, either concrete or formal mode, was best suited for concrete and formal operational children. He found that the formal operational children had greater achievement than the concrete operational children regardless of the mode of presentation, but that the concrete instruction was more effective with both types of children. Achievement appeared to be more lasting for the formal operational children and they showed an ability to generalize the information. [Sheehan, 1970]

Inhelder worked with young children in a task where the children were asked to draw a picture from memory. The children were shown ten sticks which varied in length and then asked to draw them. If the child was less than four years, he drew a line of roughly equal sticks. At about five years, the child drew three groups of sticks--small, medium and large. It wasn't until a child reached the age of six that his drawing was correct for what he had seen.

After six to eight months had passed the same children were asked to draw the line of sticks again. However, they were not shown the sticks. A majority of the children remembered the arrangement better than just after they had seen the sticks. Their recall seemed to be one stage better. [Hall, 1970; 55]

Piaget has studied the function concept and found that functions do not lead to conservation. He experimented with five year old children by taking a piece of string and attaching it to a spring at one end and a weight at the other end. When the weight on the string was increased, the string was pulled so that the part that was hanging down vertically was increased while the part which was horizontal was decreased. The children were able to grasp the relationships between the lengths of string and the differences in weights, but the sum of the vertical and horizontal parts of the string did not remain constant for them. [Piaget, 1970; 1]

Another task which Piaget used in experimenting with the child's understanding of function dealt with cards and shapes.

> We give children a number of cards, on each of which there is a white part and a red part, and also give them a number of cutouts of different shapes. Their task is to find a cutout that will cover up the red part on the card. It need not correspond exactly, but it simply must cover the red part completely. The interesting thing is that these children understand the relationship many-to-one, since they realize that there are a number of different cutout shapes all of which can completely cover the red, but this does not permit them to construct a good classification system based on the relationship of one-to-many. Here is another case of half of a logical structure. /Piaget, 1970; 51/

Piaget, Szeminska, and Bang experimented with the growth in understanding of a particular kind of function, where the function was a relation between the magnitude of two quantities so that variation in one quantity causes variation in the other quantity in the same proportion. This study developed from the consideration that the thinking of preschool children can be characterized by a number of one-way mappings or functions which are the beginnings of what are called <u>well-formed</u> functions. The mathematical concept of function was more general than this and usually necessitated the child to have attained formal operational thought. /Lovell, 1971/

CHAPTER III

METHOD AND DESIGN

The Study

In order to determine each child's understanding of the concept of function, each child was interviewed privately. There were forty-eight personal interviews with children in three age groups--seven years, nine years, and twelve years of age; sixteen children were interviewed in each age group. These interviews took place in the public schools of Marshall, Texas.

Each subject was presented ten tasks by the interviewer, where the first task in each case was a teaching task. Previous studies on the understanding of the concept of function were done with average and above average subjects who had a common fund of knowledge. The subjects here were selected in a random manner with age being the only required item. Due to this, no common fund of knowledge of the function concept could be presumed. The terminology used in describing the ideas was designed to be basic and easily understood.

The material from the interviews, both the audio-tapes and the written tasks, was rated by the interviewer. Selected interview tapes and materials were rated by two professional educators for comparison with the rating done by the interviewer.

The Sample

The subjects were selected from the school system in Marshall, Texas, in a random manner. The subjects were seven, nine, and twelve years of age. For the purposes of this study, a seven year old subject was in the age range from six years, six months to seven years, six months, a nine year old subject was in the age range from eight years, six months to nine years, six months, and a twelve year old subject was in the age range from eleven years, six months to twelve years, six months. The average ages in the respective age levels were 7 years, 1 month; 9 years, 3 months; and 12 years, 2 months.

The master student index in the office of the superintendent of schools in Marshall was used in the selection of subjects. The names of all students at the proper age levels were taken from the card index and listed and grouped according to age. Then a table of random numbers was used in selecting subjects from these three lists. As it was required by the superintendent that the interviewer have permission from the parents of each subject before the interview could take place, additional subjects were chosen in the event that it would not be possible to interview all of the sixteen subjects in each category. Therefore, the first sixteen approved names were chosen for interviewing in each age level.

All of the public schools in Marshall with students in these age levels participated in this study. Since the city and the schools were approximately 41% black and 59% white, it was interesting to note that the subjects were 48% black and 52% white. This percentage fairly reflects the racial population of Marshall. The following table indicates the racial distribution of the subjects within the age levels.

TABLE 1

DISTRIBUTION OF SUBJECTS ACCORDING TO RACE

Race	Seven Years	Nine Years	Twelve Years
Black	10	7	6
White	6	9	10

The subjects were not evenly divided according to sex. There were more male subjects than female subjects, as 29 male subjects participated. There were a great many more male subjects in the twelve year old level than in the other two levels. This was not planned, for it occurred in the selection process which was described earlier. The following table shows the distribution of the subjects according to their sex.

TABLE 2

DISTRIBUTION OF SUBJECTS ACCORDING TO SEX

Sex	Seven Years	Nine Years	Twelve Years
Female	8	7	4
Male	8	9	12

Another way to compare the subjects was to consider their intelligence quotients, to observe the range of these scores and to determine the way the scores were distributed within the age levels which were tested. However, it was not possible to obtain IQ scores for all of the subjects in the study. Many seven year old subjects had not been tested in this area, and several of the older subjects had evidently missed their opportunities to take the tests. Most of the scores found were from the Otis-Lennon Mental Ability Test, but a few were from the Peabody Picture Vocabulary Test. The subjects ranged in IQ from a low of 71 to a high of 136, averaging 102. The distribution of subjects according to their intelligence quotients is displayed in Table 3.

While not all of the subjects are represented in Table 3, it is possible to see an apparent wide spread in abilities among the subjects at each age level. The subjects in each level clustered between 90 and 110.

TABLE 3

DISTRIBUTION OF SUBJECTS ACCORDING TO IQ

IQ	Seven Years	Nine Years	Twelve Years
Less than 80	-	2	1
80 - 89	1	1	1
90 - 99	4	3	3
100 - 109	-	5	6
110 - 119	2	1	2
More than 119	2	2	1

The Schools

All of the elementary schools in Marshall participated in the study, as well as one junior high school. The elementary schools were George Washington Carver, David Crockett, M. W. Dogan, Sam Houston, Robert E. Lee, J. H. Moore, South Marshall, Grange Hall, and William B. Travis. Three twelve year old subjects attended Pemberton Junior High School.

Most of the elementary schools consisted of grades one through six, with the exception of Carver and Dogan. These schools were in the same area of town. Grades one, two and three were taught at Carver while grades four, five and six were taught at Dogan. Several elementary schools had kindergartens, where admittance was based on financial need.

There were two schools in Marshall which were not part of the Marshall Independent School District. These were private, church-related schools which students attended on a tuition basis. Trinity Episcopal Day School had grades one through seven, and St. Joseph School had grades one through six. Both schools had kindergartens as did several other churches in town. None of the children from the private schools participated in the study.

David Crockett and William B. Travis had students from higher socio-economic levels than did the other schools. Grange Hall was a country school, being at the edge of the city. Robert E. Lee, South Marshall, and Sam Houston had students from middle to lower socio-economic levels, while J. H. Moore, M. W. Dogan and George Washington Carver drew students from the lower socio-economic levels. The student populations of the last three schools were predominately black. Pemberton Junior High School had students with a wide range of backgrounds, as it was entirely devoted to the seventh grade and served the whole city in this capacity.

Table 4 describes how the subjects were distributed throughout the public schools. The greatest number of students interviewed at any one school was nine, while the least number of students interviewed at a school was two.

TABLE 4

SCHOOLS ATTENDED BY THE SUBJECTS

School	Seven Years	Nine Years	Twelve Years	Total
Carver	4	-	-	4
Crockett	1	2	4	7
Dogan	-	-	2	2
Grange Hall	-	2	-	2
Houston	1	1	2	4
Lee	3	1	1	5
Moore	2	5	2	9
Pemberton	-	-	3	3
South Marshall	1	1	1	3
Travis	4	4	1	9

All of the subjects of a particular age level were not in the same grade in school. Most of the seven year old subjects were in the first grade and most of the twelve year old subjects were in the sixth grade. However, the nine year old subjects were divided between the third and fourth grades. Table 5 indicates this distribution of the subjects according to the grade level they have attained in school.

If one looks at Table 5, it is apparent that two nine year old subjects were misplaced in terms of the grade level which they had attained. They had advanced only to the second grade level, each having repeated both grades one and two.

TABLE 5

DISTRIBUTION OF SUBJECTS ACCORDING TO GRADE LEVEL

Grade Level	Seven Years	Nine Years	Twelve Years
1	12	-	-
2	4	2	-
3	-	8	-
4	-	6	-
5	-	-	-
6	-	-	13
7	-	-	3

The Tasks

In previous studies of this type, only average or above average subjects participated. These subjects had a common fund of knowledge, as they had completed a unit of work on the function concept. The interviewers in these studies assumed that the subjects were informed about functions.

In this study no such assumption could be made. For this reason, the first task was a teaching task. Physical objects were provided and both interviewer and subject manipulated them so that the subject could develop an understanding of the function concept. The other tasks were written on paper, usually requiring the subject to write also.

The explanation of the function concept which was given to the subjects was carefully considered in order to assure that the subjects at all age levels would be able to comprehend the terminology. An article by Don Hight in The Mathematics Teacher suggested a good way to approach the explanation. Hight used the term "fickle picker" in discussing functions with a class of college freshmen. "If (a, b) and (a, c) were elements of a relation r and b $\neq$ c, we call a a 'fickle picker.' Since a picks b and a picks c, a is a fickle picker. . . . A function is a relation in which there are no fickle pickers." /Hight, 1968, 578/

Part of Hight's idea was utilized in this study, as each subject was told that the elements of the domain were pickers. The pickers picked elements of the range. If each picker picked and each picker picked just one time, then this was a function.

The word "function" was not used in this study. It was important to have simple terms for the young children to think about and use. Therefore, the term "eff" was substituted for "function."

The task sheets are in Appendix A. Although there was no written material for the first task, diagrams are included to illustrate the positions of the cups and arrows. The investigator used a cover sheet in each interview to note the important details. This sheet is also included in Appendix A.

I. Cup Task

The physical objects used in this task were colored plastic cups. Each interview began with the subject being shown ten cups divided into two sets. There were two red circular cups, two red hexagonal cups, two purple circular cups, two green circular cups and two blue hexagonal cups. Of the cups of the same type, one was large and one was small. All of the smaller cups made up one set and all of the larger cups made up another set. The set of smaller cups was placed on the table to the left of the set of larger cups.

It was explained to the subject that some cups had been placed before him, being divided into two groups. The cups in the set at the left were called pickers. A picker was to pick from the set of cups at the right according to this rule. "Each picker picks according to color and shape." The subject was asked what the red hexagonal cup would pick and then shown how to use the cardboard arrows to point from the picker to the cup it had picked. The subject used the arrows to show how the other pickers picked according to the rule. Two questions were then asked: "Did all of the pickers pick?" and "How many cups did each picker pick?"

The subject was told that this was an "eff" as each picker had picked just one cup. The function concept was explained by saying that there were two sets of things and the things in one of the sets were called pickers. When each picker picked something in the other set and each picker picked only one thing, then that was an "eff."

Cups were added to and removed from both sets and, with each change, the subject was asked if the configuration before him was an "eff." The additional cups were a yellow round cup to the left, a white hexagonal cup to the right, a purple round cup to the left, and a red round cup to the right.

During this time the interviewer gave an example about the subject and his friends visiting an ice cream store. The subject and his friends were called pickers, as they were picking flavors of ice cream for their cones. The cones were paid for with the exact amount of money when they entered the store, so each person had to pick and each person could only pick one. It was pointed out that this was an example of an "eff."

If the subject answered all of the questions correctly, then the interviewer proceeded to the next task. If not, the procedure of adding and removing cups was repeated with the subject answering the same type of questions as before. In each part of this task the subject was told if he was right or wrong. If he was wrong, the correct response was explained to him.

At the end of Task I, the cups and arrows were cleared away and the written tasks were begun.

II. Letter and Word Task

This task consisted of two parts. In both parts the domain was a set of letters and the codomain was a set of simple words. The interviewer explained to the subject that a letter would pick a word which began with that letter, and then she asked the subject to draw lines to show how the letters would pick. After drawing the lines, the subject was asked if this was an "eff."

The second part proceeded in the same way. After the subject had completed both parts he was informed if his answers were correct or not and the reasons why or why not these examples were or were not functions. Since this was the subject's first contact with the function concept where physical objects were not involved, it was important for him to know the correct response.

Then, the subject was requested to consider the set of words as pickers. To ascertain whether or not he understood the picking procedure, he was asked what letter the word "cup" would pick. When the subject understood the procedure, he was then asked whether this would be an "eff."

III. Arrow Task

This task consisted of four parts. The first part involved two sets of numbers. The subject was told which numbers were the pickers and asked to draw lines to show the results of the picking, if each picker picked a number which was one more than itself. After drawing the lines, he was asked if this was an "eff."

The other three parts were alike, as each part consisted of two sets of dots with arrows drawn from the dots of one set to the dots of the other set. The interviewer explained to the subject which dots were the pickers and that the arrows showed how the pickers picked. Then, the subject was asked if this was an "eff."

IV. First Digit Task

This task consisted of four parts. In each part the domain was a set of one digit numbers and the codomain was a set of two digit numbers. The subject was told that the one digit numbers were the pickers and that each picker would pick a number which had the picker as its first digit. The subject was requested to draw lines to show the way the pickers picked and, after this was done, was asked if this was an "eff."

However, after these four parts were completed the subject was asked to think of the last set of two digit numbers as the pickers. Then, he was asked what the number 12 would pick and why.

V. Addition Task

In this task the subject was presented with the domain, which was a set of numbers, and told that the rule was "Add 4." He was asked to find the range. Then, he was asked if this was an "eff."

Another number was then added to the range and the subject was asked if this was still an "eff." In the same way a number was added to the domain and the subject was asked if that was an "eff."

The idea of ordered pairs of numbers was introduced. This was done by showing the subject how to form an ordered pair with the picker as the first component and the picked as the second component. He was then asked to use his result to form ordered pairs and to state whether this set of ordered pairs was a function.

VI. Graph Task

In this task ordered pairs of numbers were represented by points in the plane. The subject was shown a set of points with the corresponding ordered pairs and asked if it represented a function. Then, another point was added and the ordered pair which it represented was written next to it. The subject was asked if this was an "eff."

The interviewer then asked the subject to consider the ordered pairs with the numbers in the opposite order. He was asked if this was an "eff."

VII. Table Task

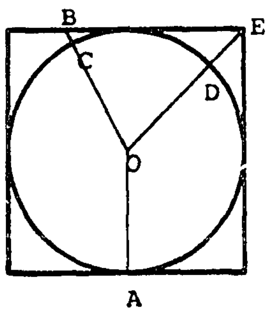
A table of numbers was presented to the subject. The subject was first asked to think of the numbers in the top row as the pickers. Then, the interviewer asked if this was an "eff."

Next, the subject was asked to consider the numbers in the bottom row as the pickers. He was asked if this was an "eff."

In either case, if the subject was unable to determine whether or not this was an "eff," then an arrow diagram was drawn by the interviewer. The subject was then asked each question again with reference to the newly drawn diagram.

VIII. Square-Circle Task

The subject was shown a diagram of a circle inscribed in a square.



He was told that points A, B and E were on the square and that there were many more points on the square which had not been named. He was also told that points A, C and D were on the circle and that there were many more points on the circle which had not been named. O was the center of the circle. The subject was asked to think about the points on the circle as the pickers. These pickers were picking points on the square. It was pointed out that A picked A, C picked B, and D picked E. The subject was asked to give the rule by which he picked. At times, another point was named in order to help the subject determine the rule.

After the subject had explained how the picking was done, he was asked if this was an "eff." Then, the inverse was considered by asking him to think of the points on the square as the pickers where A picked A, B picked C, and E picked D. After the subject explained how this picking was done, he was asked if this was an "eff."

IX. Counting Number Task

The subject was first asked to count to 15 or 20 to be sure he had in mind what the interviewer meant by the counting numbers. In this task the set of counting numbers was the domain and also the codomain.

The rule was $x \longrightarrow x + 1$. The subject was asked to apply the rule to specific cases. Then he was asked to state a large counting number and to show what it would pick. He was asked if this was an "eff."

The next pair involved the inverse. He was asked to think of the other set as the pickers and told that they would pick the numbers that had picked them before. The subject was asked to state the rule used. Then he was asked if this was an "eff." If he did not answer this correctly, he was asked what 1 would pick. In answering this, he should have seen his error and corrected it, as it was pointed out earlier in the task that 0 was not a counting number. Therefore, 1 did not pick.

X. Locker Task

In this task the subject was to consider assigning lockers to the students in a school. This did not seem to be concrete enough, so several examples were provided where the subject could draw lines to assign the lockers.

There were four parts to this task. The parts were differentiated by certain specifications as to how the lockers were to be assigned. In each part the subject was first asked to indicate the locker assignment by drawing lines between names and locker numbers. Then, the subject considered the people as the pickers and determined if this was an "eff." After this was done, he considered the lockers as the pickers and determined whether or not that was an "eff."

Statistical Analysis

Two nonparametric statistical tests, the Kruskal-Wallis one-way analysis of variance and the Mann-Whitney U, were employed in analyzing the information obtained in the interviews with a significance level of .01. The ratings of each task and a composite score were analyzed first by the Kruskal-Wallis one-way analysis of variance using all three age groups. After this, the Mann-Whitney U test was used to analyze the age groups by pairs.

CHAPTER IV

RESULTS OF THE STUDY

The information gathered in this study was used for the purpose of determining the child's ability to understand and work with a simple explanation of the concept of function. The investigator constructed ten multi-item tasks which were presented to the subjects in private, tape-recorded interviews. Forty-eight children were interviewed with sixteen children in each of three age levels--seven years, nine years, and twelve years.

The interviews took place in nine public schools in Marshall, Texas. All of the schools provided a private room for the interviews. The type of room varied from school to school and included clinics, speech rooms, cafeterias, and principals' offices.

Analysis of the Interviews

The tape recording and written material from each interview were rated using a scheme which had been previously used by three different investigators. Almy used this scheme in a study of the young child's understanding of the principle of conservation. [Almy, 1966] Taback also utilized this scheme in studying the child's understanding of the concept of limit. [Taback, 1969] Likewise, Thiessen employed this scheme in a study of the child's understanding of the concept of convexity. [Thiessen, 1971] The scheme consisted of five categories to classify the subjects' responses:

Clear evidence of understanding

Some evidence of understanding

Uncertain evidence of understanding

Clear evidence of not understanding

Evidence lacking.

For the investigator's convenience in organizing and constructing a rating sheet, numbers were assigned to each category in this way: 5--Clear evidence of understanding, 4--Some evidence of understanding, 3--Uncertain evidence of understanding, 2--Clear evidence of not understanding, and 1--Evidence lacking. The rating sheet with these numbers is in Appendix B.

Using this scheme, the investigator rated the responses of each of the subjects with regard to fifty-one items. The last category, "Evidence lacking," was used if the item was not considered or if the response was indeterminable. In addition, each task, with the exception of the Square-Circle Task, was given an over-all rating.

The items to be analyzed were classified as either process items or identification items. Process items were those which involved the "picking" procedure. In these items the subjects were asked to illustrate the procedure by drawing lines from the pickers to their choices. In some cases, the subjects had to use a rule to calculate the choice of a picker before drawing the line indicating the choice. Identification items were those which required the subjects to determine whether or not a relation was an "eff."

<u>Reliability of the Rating</u>

To measure the reliability of the investigator's rating of the responses, six of the interview tapes along with the corresponding written information were rated by two professional educators. In order to assure equal representation, two interview tapes were randomly selected from each of the three age groups.

The investigator and the two professional educators rated the materials independently. The investigator's ratings were in agreement with those of one professional educator on eighty-three percent of the items and with those of the other professional educator on eighty-one percent of the items. As the professional educators were in agreement with each other on eighty-three percent of the ratings, the investigator's ratings were considered to be reliable.

Analysis of the Tasks

I. Cup Task

Almost two-thirds of the subjects (65%) were rated "Clear evidence of understanding" or "Some evidence of understanding" on this task. The rating of the task was based on an overall assessment of the performance on the individual items within the task.

In each item a cup was added to the set of pickers or to the set of picked cups and then the cup was removed before the next item was begun. The subjects were encouraged by the investigator to ask themselves two questions when they were trying to identify instances and non-instances of "effs." These questions were "Do all the pickers pick?" and "How many times does each picker pick?"

In this task, the subjects as a group performed best on the last item, where a yellow round cup was added to the set of pickers. Seventy-one percent of the subjects were rated "Clear evidence of understanding." One twelve year old subject was asked if this was an "eff" and he stated:

"No. There's no yellow cup for this one to pick."

while a nine year old subject said:

"No. It doesn't match. This picker doesn't pick anything."

Several subjects were rated "Some evidence of understanding." These subjects were slower in answering or at times reversed themselves as this nine year old subject did, when asked if this was an "eff."

"Yes, it is. All the pickers pick. No. It ain't. This one doesn't pick."

Eighty-three percent of the subjects were rated "Clear evidence of understanding" or "Some evidence of understanding."

Only thirty-eight percent of the subjects were rated "Clear evidence of understanding" on the first item of this task and no subject received the rating of "Some evidence of understanding" on this part. In this item a white hexagonal cup was added to the set of picked cups, so there was a cup which no picker picked. One subject who received the highest rating responded:

"Yes. All of the pickers picked just one time."

Many subjects gave responses which were rated "Clear evidence of not understanding." Such a response was made by a twelve year old who said:

"No. There's no picker to pick it."

Several subjects suggested that another white hexagonal cup be added to the pickers to pick the one at the right. A few subjects responded in this manner:

"No. One picker didn't pick."

This response indicated a clear misunderstanding as to which cups were the pickers. The investigator found that to some subjects all of the cups were pickers.

Many subjects were confused by these two items. It was difficult for them to distinguish between a picker that didn't pick and a cup that wasn't picked.

The other two items in this task also appeared to have some similarities for the subjects. In one item a purple round cup was added to the set of pickers, so that there would be two pickers picking the same cup. Sixty-seven percent of the subjects were rated "Clear evidence of understanding." A nine year old subject who received this rating responded in this manner:

> "Yes. These two, they pick the same one but they each pick just one time."

A common error in this item is demonstrated by this response:

> "No. One picker picks two."

This type of response was rated "Clear evidence of not understanding." Thirty-one percent of the subjects were in this category.

The other item in this task involved the addition of a red round cup to the set of picked cups, so that one picker picked two cups. Almost one-half of the subjects (48%) were rated "Clear evidence of understanding." A reply rated in this way was

> "No. One picker picks two cups."

which was made by a seven year old subject. The same number of subjects received a rating of "Clear evidence of not understanding." One seven year old in this category insisted that this was an "eff" because "they all pick one time." It was difficult for some subjects to distinguish between two pickers that picked the same thing and one picker which picked two different things.

Table 6 indicates the results of rating the items of the Cup Task. The items of the task are distinguished by the color of the cup which was added for each item. Numbers were used to designate the categories. These are the same numbers which were mentioned earlier as being used on the rating sheets. The number of subjects in each category is listed by age below each item.

TABLE 6

CUP TASK

Age	Rating	White	Purple	Red	Yellow	Task
7	5	3	9	6	10	3
7	4	0	0	1	3	3
7	3	0	0	0	1	8
7	2	13	7	9	1	2
7	1	0	0	0	1	0
9	5	10	11	6	13	7
9	4	0	0	1	2	7
9	3	0	0	0	1	2
9	2	6	5	9	0	0
9	1	0	0	0	0	0
12	5	5	12	11	11	6
12	4	0	1	0	1	5
12	3	1	0	0	3	5
12	2	10	3	5	1	0
12	1	0	0	0	0	0

It is evident from studying the table that the nine and twelve year old subjects had greater success on this task than did the seven year old subjects. It is interesting to note that more nine year old subjects had task ratings of four and five than did the twelve year old subjects.

II. Letter and Word Task

Ninety percent of the subjects were rated in the categories "Clear evidence of understanding" and "Some evidence of understanding" on this task. Five seven year old subjects were the only subjects not included in these categories.

This was the first written task and also the first task in which process items were rated. These process items involved the subject's assigning images (words) to the pickers (letters) by drawing lines. Almost all of the subjects (98%) received the rating "Clear evidence of understanding" on the first process item.

The fact that the word "hat" was not picked was of concern to several subjects. Many suggested that an "h" be included in the set of pickers to pick "hat."

Seventy-nine percent of the subjects correctly identified this as an "eff," and these responses were rated "Clear evidence of understanding." One nine year old subject stated:

"It's an 'eff.' All the pickers pick one time."

Nineteen percent of the subjects received a rating of "Clear evidence of not understanding." A response by a seven year old subject given this rating was

"No. 'Hat' isn't picked."

This identification item was of the same type as the white cup item in the Cup Task, in that an element of the codomain was unassigned.

Only thirty-eight percent of the subjects were rated "Clear evidence of understanding" for this item in the Cup Task while seventy-nine percent of the subjects received this rating in this task.

The second process item was not executed as successfully by the subjects as was the first. Only seventy-nine percent of the subjects were rated "Clear evidence of understanding," while the others were rated "Some evidence of understanding." Apparently this process was more difficult than the previous one, as there was a picker that didn't pick and a picker that picked twice.

Fifty-six percent of the subjects correctly identified this as not an "eff." Some of the responses to the question, "Is this an 'eff'?" were as follows:

"No, it's not. A picker didn't pick."

"No. This picker picked this one and that one."

"No, it isn't. That one picked two and this one didn't pick."

All of these responses were rated "Clear evidence of understanding." Obviously, the last response by a twelve year old subject was the most complete, but the other two nine year old subjects had good reasons for saying that it wasn't an "eff." Twenty-seven percent of the subjects displayed "Clear evidence of not understanding" by indicating that this was an "eff."

This identification item involved the same principles as a combination of the yellow and red cup items in the Cup Task, in that an element of the codomain was unassigned and an element of the domain had two images. In considering those two items, one finds that forty-eight percent of the subjects were rated "Clear evidence of understanding" on the red cup item while seventy-one percent of the subjects received this rating on the yellow cup item. When drawing lines in the process item, some subjects accidentally marked through the letter "i" which was the picker that didn't pick. This caused some difficulty in discerning that "i" had not picked. Another difficulty might have been the similarity of the words "ape" and "age" which "a" picked.

Most of the subjects (92%) said that the word "cup" would pick the letter "c" if the words were the pickers. This type of response was rated "Clear evidence of understanding." All of the twelve year old subjects responded correctly to this item.

Fifty-eight percent of the subjects were able to determine that the inverse was an "eff" and were rated "Clear evidence of understanding." Some were concerned because "i" wasn't picked and others said this was not an "eff" because "ape" and "age" would both pick "a."

Table 7 displays the number of individuals in each rating category for the Letter and Word Task. Each item is listed to the right and the number of subjects in each category is listed by age level below each item. The heading "P1" refers to the first process item and the heading "P2" refers to the second process item. The heading "IIa," "IIb" and "IIc" refer to identification items. The heading "cup" refers to the process item where the subject was asked what "cup" would pick. The number of subjects at each age level rated in each category on their over-all performance on this task appears under the "Task" heading.

The subjects appear to have performed better on this task than they did on the Cup Task. No subject received a task rating of "Clear evidence of not understanding." The subjects performed better on the process items than they did on the identification items.

TABLE 7

LETTER AND WORD TASK

Age	Rating	P1	IIa	P2	IIb	cup	IIc	Task
7	5	16	12	11	7	13	9	9
7	4	0	0	5	1	0	0	2
7	3	0	0	0	3	0	1	5
7	2	0	4	0	5	3	6	0
7	1	0	0	0	0	0	0	0
9	5	15	14	15	10	15	9	11
9	4	1	0	1	0	0	2	5
9	3	0	1	0	1	0	2	0
9	2	0	1	0	5	1	3	0
9	1	0	0	0	0	0	0	0
12	5	16	12	12	10	16	10	9
12	4	0	0	4	3	0	0	7
12	3	0	0	0	0	0	2	0
12	2	0	4	0	3	0	4	0
12	1	0	0	0	0	0	0	0

III. Arrow Task

The subjects performed well on this task with eighty-eight percent of them being rated "Clear evidence of understanding" or "Some evidence of understanding." Only one subject, a seven year old, was rated "Clear evidence of not understanding."

The process item in this task involved the counting numbers. The picking procedure was that each counting number picked its successor.

Most of the subjects (94%) received a rating of "Clear evidence of understanding" on this item. Only one subject received the rating "Clear evidence of not understanding."

The first identification item was to determine if this was an "eff." Ninety-two percent of the subjects correctly said that it was an "eff" and were rated "Clear evidence of understanding." One seven year old subject replied:

"Yes. All the pickers pick one."

Five subjects failed to recognize this as an "eff."

The other identification items involved diagrams of similar types, called arrow diagrams. Two sets of dots were presented in each instance; one set was the domain and the other was the codomain. Arrows had been drawn to indicate the relationship between the pickers and their images. In the first of these, where the correspondence was one-to-one, ninety-four percent of the subjects were rated "Clear evidence of understanding" in recognizing that this was an "eff." The other subjects were all rated "Clear evidence of not understanding."

In the next item two pickers had the same image. Only fifty-two percent of the subjects were rated "Clear evidence of understanding." One subject gave the reason:

"All the pickers pick one time."

Forty-two percent of the subjects received the rating "Clear evidence of not understanding." These subjects claimed that one picker picked twice. This item involved the same principle as the purple cup item in the Cup Task, in that two elements of the domain had the same image. The subjects evidently found the purple cup item easier as sixty-seven percent were rated "Clear evidence of understanding" on that item. Another principle was represented here, in that an element of the codomain was unassigned, similar to the White Cup item. This apparently confused some subjects.

In the last item sixty-nine percent of the subjects discerned correctly that it was not an "eff" because "A picker didn't pick." and were rated "Clear evidence of understanding." Twenty-five percent of the subjects were rated "Clear evidence of not understanding." They did not notice a picker that didn't pick. When asked specifically about that picker, several subjects said that it didn't matter.

Table 8 displays the number of subjects in each rating category for the Arrow Task. Each item is listed to the right with the number of subjects in each category written below it by age. The heading "P3" refers to the process item and the headings "IIIa," "IIIb," "IIIc" and "IIId" refer to the identification items. The number of subjects in each category rated on their over-all performance is under the heading "Task."

More twelve year old subjects were rated "Clear evidence of understanding" on these items than were nine and seven year old subjects. It appears that some subjects had more difficulty in discerning instances and non-instances of "effs" when there was not a one-to-one correspondence between pickers and picked.

IV. First Digit Task

In this task one digit integers picked two digit integers of which they were the first digit, such as 3 ⟶ 34. The task consisted of four process items and three identification items. Ninety-one percent of the subjects received "Clear evidence of understanding" or "Some evidence of understanding" ratings on their over-all performance on the task. All of the twelve year old subjects were in these categories.

On the first process item only one subject did not receive a rating of "Clear evidence of understanding" and that subject was rated

TABLE 8

ARROW TASK

Age	Rating	P3	IIIa	IIIb	IIIc	IIId	Task
7	5	14	14	14	9	8	9
7	4	1	0	0	1	2	4
7	3	0	0	0	0	0	2
7	2	1	2	2	6	6	1
7	1	0	0	0	0	0	0
9	5	16	12	15	8	10	9
9	4	0	0	0	1	0	5
9	3	0	1	0	0	0	2
9	2	0	3	1	7	6	0
9	1	0	0	0	0	0	0
12	5	15	16	16	8	15	14
12	4	1	0	0	1	0	1
12	3	0	0	0	0	1	1
12	2	0	0	0	7	0	0
12	1	0	0	0	0	0	0

"Some evidence of understanding." There was a one-to-one correspondence between pickers and picked. The subject who received the lower rating exhibited some difficulty in understanding the rule by which the pickers picked.

The first identification item was to determine whether or not the correspondence specified by the first process item was an instance of an "eff." Ninety percent of the subjects correctly recognized this as an "eff" and were rated "Clear evidence of understanding." As one nine year old subject said:

> "This one picks that one and this one picks that one and this one picks that one and this one picks that one. So it's an 'eff.'"

Four subjects received the rating of "Clear evidence of not understanding."

The second process item appeared to be more difficult than the first one. The integers used in this item apparently caused some subjects to have more difficulty in applying the rule. Also one picker picked twice. Many subjects required encouragement to draw a second line indicating two choices for the picker "3." Some subjects said the picker "4" would pick "41" and "34." Seventy-one percent of the subjects were rated "Clear evidence of understanding," while twenty-three percent of the subjects were rated "Some evidence of understanding."

In the second identification item, seventy-five percent of the subjects were rated "Clear evidence of understanding." These subjects recognized that the correspondence specified by the second process item was not an "eff" and indicated their conclusion was due to the "3" picking twice. Nineteen percent of the subjects were rated "Clear evidence of not understanding." These subjects did not discern that the "3" picked twice even though they had drawn lines from the "3" to the two numbers it picked. This item involved the same principle as the red cup item in the Cup Task, in that an element of the domain had two images. A considerably greater number of subjects (71%) were rated "Clear evidence of understanding" on this item than on the red cup item (48%).

The third process item involved a picker that didn't pick. Eighty-five percent of the subjects were rated "Clear evidence of understanding on this item. One subject, a twelve year old, tried to stretch the rule so that the "4" would pick "81." Sixty-five percent of the subjects discerned that this was not an "eff" in the third identification item, saying that a "picker didn't pick." and they were rated "Clear evidence of understanding." However, twenty-seven percent of the subjects thought this to be an "eff," saying the fact that the "4" didn't pick "made no difference." This item involved the same principle as the yellow cup item in the Cup Task, in that an element of the domain had no image. It is interesting to note that fewer subjects were rated "Clear evidence of understanding" on this item than on the yellow cup item (71%).

The last item involved the idea of inverse. The subjects were told to consider the set of two digit integers as pickers and then they were asked what "12" would pick. Seventy-five percent of the subjects were rated "Clear evidence of understanding," as one seven year old subject explained:

"12 picks 1 because 12 begins with 1."

Table 9 exhibits the number of subjects in each rating category by age level in the First Digit Task. Each item is listed to the right with the number of subjects in that category written below it according to the age of the subject. The headings "P4," "P5," "P6" and "12" refer to the process items and "IVa," "IVb" and "IVc" refer to the identification items. The number of subjects in each category according to their over-all performance on the task appears in the column headed "Task."

Subjects of all age levels performed well on this task. Most subjects appeared to perform as well or better on the process items than they did on the identification items.

TABLE 9

FIRST DIGIT TASK

Age	Rating	P4	IVa	P5	IVb	P6	IVc	12	Task
7	5	15	15	8	11	13	9	11	10
7	4	1	0	6	1	2	0	0	4
7	3	0	0	1	0	1	0	3	2
7	2	0	1	1	4	0	7	2	0
7	1	0	0	0	0	0	0	0	0
9	5	16	13	14	13	13	9	11	10
9	4	0	1	2	0	2	0	1	2
9	3	0	0	0	0	1	1	1	2
9	2	0	2	0	3	0	6	3	0
9	1	0	0	0	0	0	0	0	0
12	5	16	15	12	12	15	13	14	13
12	4	0	0	3	0	0	0	1	3
12	3	0	0	1	2	1	3	1	0
12	2	0	1	0	2	0	0	0	0
12	1	0	0	0	0	0	0	0	0

V. Addition Task

On this task seventy-nine percent of the subjects received a rating of "Clear evidence of understanding" or "Some evidence of understanding." There were six items in this task: two process items and four identification items.

The first process item involved a procedure which was more complicated than the previous ones. The subjects were asked to consider the set of counting numbers, one through ten, as the set of pickers. The rule "Add 4" was to be employed by the subjects to determine the images of the pickers. Lines were to be drawn to indicate this picking procedure. Many of the seven year old subjects and several nine year old subjects had difficulty doing this. A large part of their problem was evidently due to poor preparation in the necessary arithmetic skills. To make the computations easier, the investigator changed the rule to "Add 1" for these subjects. Eighty-eight percent of the subjects were rated "Clear evidence of understanding." This process item was rated on the subject's ability to perform the computations and indicate the choices of the pickers; the length of time involved was not considered important.

Eighty-eight percent of the subjects correctly identified this as an "eff" and were given the rating "Clear evidence of understanding." The number "16" was added to the set of picked numbers. Only forty-six percent of the subjects recognized this to be an "eff" and received the highest rating. The same percentage were rated "Clear evidence of not understanding," saying as one nine year old subject did in the following:

"There is no number over there to pick it. 12 isn't there."

The number "30" was added to the set of pickers and then the subjects were asked whether or not this was an "eff." Sixty-seven percent were rated "Clear evidence of understanding" in stating that it wasn't an "eff" as "30 doesn't pick." Some subjects, without heeding this rule, determined that "30" would pick "16." This type of response was rated "Clear evidence of not understanding." Seventeen percent of the subjects were in this category as they contended that it was an "eff."

The second process item introduced the idea of ordered pairs of numbers. The subjects arranged the information from the first process item, not including "16" and "30," in ordered pairs. All but three subjects calculated the images of the pickers anew. These three subjects referred to the first diagram to determine the choice of a picker. Eighty-one percent of the subjects evinced "Clear evidence of understanding" while fifteen percent of the subjects were rated "Some evidence of understanding."

The last identification item involved a set of ordered pairs. In the previous process item the subjects had seen how a set of ordered pairs could represent an "eff." Subjects were asked whether a particular set of ordered pairs was an "eff." Twenty-three percent of the subjects received a rating of "Evidence lacking" as they were not asked this question. Thirty-one percent of the subjects were rated "Clear evidence of understanding" on this item.

Table 10 displays the number of subjects rated in each category by age level on the items in the Addition Task. The number of subjects in each category rated on their over-all performance on the task is exhibited under the "Task" heading. The headings "P7" and "P8" refer to process items and headings "eff," "16," "30" and "OP" refer to identification items.

In this task, the subjects appeared to perform as well or better on the process items than they did on the identification items. Due to the large number of subjects rated "Evidence lacking," little information was obtained from the "OP" identification item.

TABLE 10

ADDITION TASK

Age	Rating	P7	eff	16	30	P8	OP	Task
7	5	12	14	4	8	10	5	3
7	4	3	0	0	1	5	2	7
7	3	1	1	0	2	1	2	5
7	2	0	1	12	5	0	1	1
7	1	0	0	0	0	0	6	0
9	5	15	13	8	13	14	5	11
9	4	0	1	1	1	1	0	3
9	3	1	0	2	0	1	1	2
9	2	0	2	5	2	0	1	0
9	1	0	0	0	0	0	9	0
12	5	15	15	10	11	15	5	11
12	4	1	0	0	2	1	1	3
12	3	0	0	0	1	0	0	2
12	2	0	1	5	1	0	2	0
12	1	0	0	1	1	0	8	0

VI. Graph Task

This task consisted of three identification items. The subject was presented a graph where the ordered pairs were written beside the points which represented them. The first item concerned whether or not this was an "eff." Sixty-nine percent of the subjects were rated "Clear evidence of understanding" on this. A nine year old subject replied:

"It's an 'eff.' They all pick one time."

Twenty-five percent of the subjects were rated "Clear evidence of not understanding." The appearance of the graph was similar to the letter "V" with the upper left hand point missing. Some subjects said that it wasn't an "eff" because the missing point was needed to pick the one in the upper right.

With the addition of the point corresponding to the ordered pair (4, 1) only twenty-nine percent of the subjects were rated "Clear evidence of understanding" in identifying this as not being an "eff" because "4 picks 1 and 4 picks 2." Twenty percent of the subjects received a rating of "Uncertain evidence of understanding" by giving the correct answer but not stating a reason. Forty-eight percent of the subjects were rated "Clear evidence of not understanding," as they indicated that this was an "eff."

This rather high percentage can perhaps be explained by the fact that many subjects apparently considered the 4's to be different in the ordered pairs (4, 1) and (4, 2). They had been told in the earlier discussion of ordered pairs that the number on the left was the picker and that it picked the number on the right. So to see if this was an "eff" they evidently checked first to see if there was a number on the right in each case. This showed whether or not each picker picked. Then they checked to see if there was only one number to the right. Of course, this was always the case so they evidently concluded that each picker picked only one time. This is tantamount to assuming that the 4's in the ordered pairs were distinct. Viewed in this way any set of ordered pairs could be considered to represent a function. So it is possible that some subjects clearly understood the routine for determining whether or not it was an "eff" but were still confused about the ordered pair concept.

TABLE 11

GRAPH TASK

Age	Rating	VIa	VIb	VIc	Task
7	5	11	2	1	1
7	4	1	1	0	2
7	3	1	4	5	10
7	2	3	9	10	3
7	1	0	0	0	0
9	5	9	5	2	0
9	4	0	0	0	6
9	3	0	5	8	7
9	2	7	6	5	3
9	1	0	0	1	0
12	5	13	7	4	5
12	4	0	0	1	1
12	3	1	1	1	8
12	2	2	8	8	2
12	1	0	0	2	0

identified this as an "eff" with a statement such as this made by a twelve year old subject:

"They all pick one time."

Those subjects who did not appear to understand the first item fully were asked to consider a diagram similar in appearance to those used in the First Digit Task. When the information was presented in this form forty-six percent of the total number of subjects received the highest rating on this, the second identification item. Twenty-three percent of the subjects were rated "Clear evidence of not understanding" and twenty-nine percent were rated "Evidence lacking" on the second identification item. Some subjects who appeared to understand the first item fully, later experienced difficulty with the third item and were asked to return to the second item which previously they had been allowed to omit. This accounts for the fact that only twenty-nine percent fell in the category "Evidence lacking" on the second item whereas sixty-three percent received the highest rating on the first item.

Only thirteen percent of the subjects were rated "Clear evidence of understanding" on the third identification item when asked to consider the second row of the table as the domain and the first row as the range. Forty-two percent were rated "Clear evidence of not understanding" while the same number were given the rating "Evidence lacking."

All subjects who were asked to consider the second item were also given the fourth identification item. This involved the idea of the inverse but here the information was presented using the diagram format. Fifty-eight percent of the total number of subjects were rated either "Clear evidence of understanding" or "Some evidence of understanding" with only two percent in the latter category. Thirty-one percent were given the rating "Clear evidence of not understanding" and evidence was lacking for eight percent.

It was apparent that some subjects were still confused about the difference between several pickers picking one thing and one picker picking several things. Understanding seemed to be related to the format in which the information was presented to the subject.

On task ratings only eight percent of the subjects were rated "Clear evidence of not understanding." Fifty-four percent received either "Clear evidence of understanding" or "Some evidence of understanding" ratings. The largest percentage in one category (38%) were rated "Uncertain evidence of understanding."

Table 12 exhibits the number of subjects in each of the rating categories on the Table Task. The headings "VIIa," "a/diagram," "VIIb" and "b/diagram" refer to the identification items. The number of students in each category in terms of their over-all performance on the task is listed under the "Task" heading.

Almost all of the seven year old subjects were presented the information in both formats. Several nine and twelve year old subjects were not presented the second item, but most of these were presented the fourth item. Four subjects, three twelve year olds and one nine year old, received "Clear evidence of understanding" on the first and third items and therefore were not asked the second and fourth items.

VIII. Square Circle Task

This task consisted of four items: two process items and two identification items. The diagram of the circle inside the square appears in Appendix A. In the first two items the subject was to consider the set of points of the circle as the domain and the set of points of the square as the range. Several pickers were indicated with their images and the subject was asked to explain the way this picking was done.

Two twelve year old subjects received a rating of "Clear evidence of understanding." These were the only subjects to explain the rule.

Two nine year old subjects and one twelve year old subject were rated "Some evidence of understanding." All but one of the remaining subjects (92%) were rated "Clear evidence of not understanding."

TABLE 12

TABLE TASK

Age	Rating	VIIa	a/diagram	VIIb	b/diagram	Task
7	5	13	10	1	8	3
7	4	1	1	0	0	6
7	3	0	0	1	1	5
7	2	2	4	7	7	2
7	1	0	1	7	0	0
9	5	8	4	1	11	3
9	4	0	0	0	0	5
9	3	3	0	1	0	6
9	2	5	4	7	4	2
9	1	0	8	7	1	0
12	5	9	8	4	8	5
12	4	0	0	0	1	4
12	3	0	0	0	0	7
12	2	7	3	6	4	0
12	1	0	5	6	3	0

If the subject did not understand the picking procedure, he had no way to determine whether or not this was an "eff." Therefore, in most cases this item gave no information when asked. Only three subjects correctly identified this as an "eff" and gave reasons to indicate "Clear evidence of understanding."

The last two items concerned the inverse of the first item. The set of points of the square were the domain and the set of points of the circle were the range. Again, the subject was asked to explain the picking procedure. Four twelve year old subjects were rated "Clear evidence of understanding," and one nine year old was rated "Some evidence of understanding." These same subjects obtained these same ratings on the last item where they were asked if this was an "eff." The other subjects had "Clear evidence of not understanding" or "Evidence lacking" as their ratings except for three subjects in the "Uncertain evidence of understanding" category.

In considering the task as a whole, one finds eighty-eight percent of the subjects who could be rated "Clear evidence of not understanding," while only six percent of the subjects could be rated "Clear evidence of understanding."

Table 13 exhibits the number of subjects in each rating category on the Square Circle Task. The headings "Ability-1" and "Ability-2" refer to the process items and "eff-1" and "eff-2" refer to the identification items.

IX. Counting Task

Eighty-four percent of the subjects were rated either "Clear evidence of understanding" or "Some evidence of understanding" on this task. Only one seven year old subject was rated "Clear evidence of not understanding."

The domain was the set of counting numbers as was the codomain. The rule was that each picker picked its successor, written on the sheet

TABLE 13

SQUARE CIRCLE TASK

Age	Rating	Ability-1	eff-1	Ability-2	eff-2
7	5	0	0	0	0
7	4	0	0	0	0
7	3	0	0	0	0
7	2	16	16	11	11
7	1	0	0	5	5
9	5	0	0	0	0
9	4	2	1	1	1
9	3	0	1	1	1
9	2	14	14	14	14
9	1	0	0	0	0
12	5	2	3	4	4
12	4	1	0	0	0
12	3	1	0	2	1
12	2	12	13	9	10
12	1	0	0	1	1

as $x \longrightarrow x + 1$. The first item was a process item where the subject identified some of the pickers and indicated their images. Eighty-five percent of the subjects were rated "Clear evidence of understanding" and thirteen percent of the subjects were rated "Some evidence of understanding" of the process. Ninety percent of the subjects were given a rating of "Clear evidence of understanding" in identifying this as an "eff."

The inverse was considered next and the subject was asked for the rule whereby a picker would pick the number which had picked it before. Fifty percent of the subjects were rated "Clear evidence of understanding," while nineteen percent were rated "Some evidence of understanding" and seventeen percent of the subjects were rated "Clear evidence of not understanding." In determining whether or not this was an "eff," seventy-eight percent were rated "Clear evidence of understanding" or "Some evidence of understanding."

Table 14 displays the number of subjects in each rating category on the Counting Task. The headings "P9" and "P10" refer to the two process items and "eff-1" and "eff-2" refer to identification items. The number of subjects in each category rated on their over-all performance appears under the "Task" heading.

X. Locker Task

There were eight identification items to classify in this task. Four diagrams (See Appendix A.) were presented to the subjects. Each diagram consisted of a set on the left and a set on the right. The set on the left contained either three or four names of people and the set on the right contained either three or four integers which the subjects were told were locker numbers. The subjects were asked to draw lines to represent locker assignments according to various specified conditions. After the assignments were made the subjects were asked to consider the names as the pickers first and determine whether or not the example was an "eff" and then to consider the numbers as pickers and again determine whether this was an "eff."

TABLE 14

COUNTING TASK

Age	Rating	P9	eff-1	P10	eff-2	Task
7	5	11	13	3	0	2
7	4	4	0	4	10	9
7	3	1	2	2	6	4
7	2	0	1	7	0	1
7	1	0	0	0	0	0
9	5	15	15	9	0	8
9	4	1	0	4	12	6
9	3	0	0	2	4	2
9	2	0	1	1	0	0
9	1	0	0	0	0	0
12	5	15	15	12	1	11
12	4	1	0	1	14	4
12	3	0	0	3	1	1
12	2	0	1	0	0	0
12	1	0	0	0	0	0

Sixty-nine percent of the subjects received ratings of "Clear evidence of understanding" or "Some evidence of understanding." Two subjects, one seven year old and one nine year old, were rated "Clear evidence of not understanding" on this task.

Table 15 displays the number of subjects in each rating category by age level below the appropriate items, including an over-all performance rating under the "Task" heading. Each example was labeled a, b, c, or d with the addition of either p or l to this letter to indicate whether the domain was the set of people or the set of lockers respectively.

TABLE 15

LOCKER TASK

Age	Rating	ap	al	bp	bl	cp	cl	dp	dl	Task
7	5	13	10	3	10	12	4	13	1	2
7	4	0	0	1	0	0	3	0	3	4
7	3	0	1	1	0	0	0	0	1	9
7	2	3	5	11	6	4	9	3	11	1
7	1	0	0	0	0	0	0	0	0	0
9	5	13	14	16	6	12	10	13	10	11
9	4	0	0	0	1	0	1	0	2	2
9	3	1	0	0	1	0	0	0	0	2
9	2	2	2	0	8	4	5	3	4	1
9	1	0	0	0	0	0	0	0	0	0
12	5	15	15	12	11	14	11	11	15	9
12	4	0	0	0	1	0	1	1	0	5
12	3	0	1	0	1	0	1	0	0	2
12	2	1	0	4	3	2	3	4	1	0
12	1	0	0	0	0	0	0	0	0	0

Some of the younger subjects questioned the ability of the lockers to "pick" the people. This procedure was not real to them and they had difficulty accepting the idea. In the first example, there was a one-to-one correspondence between the people and the lockers. Three seven year old subjects who were bothered by this pretense of lockers picking people were able to identify the first item as an "eff" and the second item as not an "eff" because "lockers can't pick."

The principle involved in the third item "bp" was the same as the principle in the purple cup item in the Cup Task, in that two elements of the domain had the same image. This same principle occurred in the seventh item "dp." Sixty-four percent of the subjects were rated "Clear evidence of understanding" in "bp," while sixty-seven percent of the subjects received this rating on the purple cup item and seventy-seven percent of the subjects received this rating on "dp." This indicates considerable fluctuation on the part of some subjects during the course of the tasks; most of the changes were by seven year old subjects.

Items "bl" and "dl" involved the same principle as the red cup item in the Cup Task, in that one element of the domain had two images. While fifty-six percent of the subjects were rated "Clear evidence of understanding" in "bl," fifty-four percent of the subjects received this rating in "dl" and forty-eight percent on the red cup item. Upon looking at Table 15 one sees that many subjects fluctuated in their identifications on these items.

The principles involved in items "cp" and "cl" were like those in the white cup item and the yellow cup item respectively, in that one element of the codomain was not assigned or that one element of the domain had no image. It is interesting to note that seventy-nine percent of the subjects were rated "Clear evidence of understanding" on "cp" and fifty-two percent of the subjects received this rating on "cl" while the percentages on the white cup and yellow cup items were thirty-eight percent and seventy-one percent respectively.

The nine and twelve year old subjects received higher over-all ratings on this task than did the seven year old subjects. Eighty-eight percent of the twelve year old subjects and eighty-one percent of the nine year old subjects were rated in the two highest categories (4 and 5). Only thirty-eight percent of the seven year old subjects were in these categories.

Statistical Analysis

While lengthy statistical analysis was not feasible, the investigator did employ two nonparametric tests, the Kruskal-Wallis one-way analysis of variance and the Mann-Whitney U test, which required data of the ordinal level. The data used in these tests were the numerical rankings of the subjects according to their composite scores or their scores on the individual tasks, whichever was applicable in a given instance. The composite score for each subject was found by adding the subject's score on the individual tasks. It should be noted that one task was not given a task score on the rating sheet. The investigator did assign task scores to this task for the purpose of analyzing this task. However, no value was given to this task in the composite score. This was due to the fact that the majority of the subjects received ratings of "Clear evidence of not understanding" on this task and the addition of the two extra points to each of the composite scores appeared to be of little value.

A similar pattern of analysis was followed for each of the task scores and for the composite scores. First, the Kruskal-Wallis one-way analysis of variance was applied to determine whether the three age groups could be considered as belonging to the same population. Then the Mann-Whitney U test was applied to pairs of age groups to determine whether, pairwise, there was a significant difference in their performance.

The Kruskal-Wallis one-way analysis of variance was found to be significant at the .01 level on the composite score analysis, and therefore the null hypothesis that the samples were all drawn from the same population was rejected. The Mann-Whitney U test was found to be significant at the .01 level for the seven and nine year old subjects. It was concluded that the nine year old subjects had performed significantly better on an over-all basis than had the seven year old subjects. The performance of the seven and twelve year old subjects was also analyzed in this way and a significant difference was found at the .01 level. It was concluded that twelve year old subjects had a significantly better over-all performance than did the seven year old subjects. There was no significant difference between the nine and twelve year old subjects on the Mann-Whitney U test. The sums of rankings computed during the application of the Kruskal-Wallis one-way analysis of variance show higher values for the nine and twelve year old subjects. The sum for the nine year old subjects was considerably higher than for the seven year old subjects. The sum for the twelve year old subjects was higher than that for the nine year old subjects but the difference was not nearly so great as that between the seven and nine year old subjects.

On the Cup Task, the Kruskal-Wallis one-way analysis of variance was significant at the .05 level, which indicated that the samples were drawn from different populations. For the seven and nine year old subjects the Mann-Whitney U test was significant at the .01 level. The conclusion was that the nine year old subjects had performed significantly better than the seven year old subjects on this task. There was also a significant difference between the seven and twelve year old subjects, with the conclusion that the twelve year old subjects performed significantly better than the seven year old subjects on this task. There was no significant difference between the nine year old and twelve year old subjects on this task.

This same pattern of analysis was carried out with the Letter and Word Task, the Arrow Task and the First Digit Task. No significant difference was found on the Kruskal-Wallis one-way analysis of variance for any of these tasks. In examining the calculations made in the process of applying the Kruskal-Wallis one-way analysis of variance on these tasks, it was found that in the Letter and Word Task the nine year old and the twelve year old subjects had higher sums of rankings than did the seven year old subjects. However, the nine year old subjects had a higher sum of rankings than did the twelve year old subjects. In the Arrow Task, the twelve year old subjects had the highest sum of rankings, followed next by the nine year old subjects and last by the seven year old subjects. In the First Digit Task the seven year old and nine year old subjects had exactly the same total of rankings, with the twelve year old subjects having a higher total than theirs. The age level samples were considered by pairs within the tasks and the Mann-Whitney U test was employed with none of these tests indicating significant differences.

There was a significant difference at the .01 level on the Kruskal-Wallis one-way analysis of variance in the Addition Task. It was concluded that the samples were not all drawn from the same population. Between the seven and nine year old subjects there was a significant difference at the .01 level on the Mann-Whitney U test.

It was concluded that the nine year old subjects performed significantly better on this task than did the seven year old subjects. There was also a significant difference between the performances of the seven and twelve year old subjects at the .01 level on the Mann-Whitney U test. It was concluded that the twelve year old subjects had performed significantly better on this task than had the seven year old subjects. There was no significant difference between the nine and twelve year old subjects on this task.

No significant difference was found on the Kruskal-Wallis one-way analysis of variance for the Graph Task, the Table Task and the Square Circle Task. Upon examining the sums of rankings used in applying the Kruskal-Wallis test to each of these tasks the twelve year old subjects usually had the highest sum, the nine year old subjects had the next highest sum and the seven year old subjects had the lowest sum. However, in the Table Task the nine year old subjects had a lower total sum of rankings than did the seven year old subjects. The age level samples were considered by pairs within the tasks and the Mann-Whitney U test was employed with none of these tests indicating significant differences.

The Kruskal-Wallis one-way analysis of variance showed a significant difference at the .01 level in the Counting Task, indicating that all the subjects were not from the same population. The Mann-Whitney U test between the seven and nine year old subjects, where the nine year old subjects performed better than the seven year old subjects, was not significant at the .01 level but it was significant at the .025 level. There was also a significant difference between the seven and twelve year old subjects, with the twelve year old subjects performing significantly better than the seven year old subjects. There was no significant difference between the nine and twelve year old subjects.

This same pattern occurred on the Locker Task. The Kruskal-Wallis one-way analysis of variance was significant at the .01 level indicating that all of the subjects were not from the same population. There was a significant difference at the .01 level between the seven and nine year old subjects on the Mann-Whitney U test. The nine year old subjects performed significantly better than the seven year old subjects. Between the seven and twelve year old subjects, there was a significant difference at the .01 level, indicating that the twelve year old subjects performed better than the seven year old subjects on this task. There was no significant difference between the nine and twelve year old subjects.

In general, there were six tasks on which there were no significant differences between the subjects. There was a significant difference between the subjects on the composite scores and the finding was that the nine and twelve year old subjects performed significantly better than the seven year old subjects. There were also significant differences between the subjects on the Cup Task, the Addition Task, the Counting Task and the Locker Task. Here again it was found that the nine and twelve year old subjects performed significantly better than the seven year old subjects. While no significant differences were found between the performances of the nine and twelve year old subjects, there were indications that the twelve year old subjects performed better than the nine year old subjects.

CHAPTER V

DISCUSSION AND CONCLUSIONS

This study investigated the child's ability to understand and work with the function concept in mathematics. A series of ten multi-item tasks was devised and presented to each of forty-eight subjects during private, personal interviews. There were sixteen subjects in each of three age levels: seven, nine, and twelve years of age. Each interview was tape-recorded, and the tape and written material from each interview were rated by the investigator. Six tapes, selected at random with two from each grade level, were rated by two professional educators who specialize in mathematics education. A comparison was made between the ratings of the professional educators and those of the investigator and agreement was found on over eighty percent of the items.

The subjects were selected in a random manner from the appropriate age levels from the public school population of Marshall, Texas. This sample appeared to represent the school population. There was a wide range in intelligence quotients among the subjects and also a wide range of socio-economic backgrounds among the subjects. The ratio of black subjects to white subjects in the study reflected the ratio in the public schools and also in the town itself. Only the large majority of male subjects at the twelve year level was unusual.

Discussion of the Results

The tasks appeared to vary in their level of difficulty for the subjects. On some tasks, such as the Arrow Task and the First Digit Task, most of the subjects were given ratings of at least "Some evidence of understanding" while on the Square Circle Task, only four subjects were given this rating.

The rating which the subjects received on the Cup Task did not reflect their understanding of the concept of function as additional explanation and work was done, when needed, after the rating was completed. The white cup item caused difficulty for many subjects. The cups were arranged in one-to-one correspondence before the subject with arrows placed to demonstrate this correspondence, when a white cup was added to the set of picked cups. Most of the seven and twelve year old subjects missed this item, while a majority of the nine year old subjects was correct. In item "IIa" a situation similar to this, in that there was an unassigned element in the codomain, existed where the word "hat" was picked. A majority of subjects in all age levels were correct on this item. In the Addition Task the "16" item illustrated this same principle. This item appeared to be more like the white cup item, as the number "16" was added to the set of picked numbers after the one-to-one correspondence had been noted by the subject. The results showed that the majority of the seven year old subjects still did not understand the principle, while the majority of the twelve year old subjects did understand the principle at that point. With many of the seven year old subjects, there appeared to be a consistent error in both the white cup item and the "16" item. Item "IIa" was evidently viewed in a different manner by them.

There was no subject who missed the yellow cup item, where an element of the domain had no image, and consistently missed other items involving this principle. The seven year old subjects who missed the purple cup item, where two elements of the domain had the same image, fluctuated on their responses to later items involving this principle. There were three nine year old subjects and one twelve year old subject who consistently failed to identify instances of functions when this principle was involved. As was pointed out earlier, this principle was evidently difficult for the subjects to discern from the principle in the red cup item, where one element of the domain had two images. Two subjects who missed the red cup item, one seven year old and one twelve year old, failed consistently to recognize instances of functions which exemplified this principle. Several other subjects who missed this item, mainly seven year old subjects, fluctuated in their identifications involving this principle.

Most of the younger subjects appeared anxious to draw lines to show the images of the pickers. Some subjects hesitated on the Addition Task, as they required a second statement by the interviewer on the way a picker would find its image. Several subjects had difficulty because the previous task, the First Digit Task, caused them to think of adding 1 to 1 to obtain 11 rather than 2. The process items appeared easier for the subjects as a whole, and, in general, they were more successful in doing them.

It appeared that the Letter and Word Task, the Arrow Task and the First Digit Task were at a reasonable level of difficulty, as high majorities of the subjects in all three age levels appeared to have at least "Some evidence of understanding" on these tasks. Most of the subjects experienced little trouble with the process items and had reasons to support their identifications.

The Addition Task differed from these tasks, as the subject was presented with the domain and rule and asked to find the range and to show how the assignments were made. Several subjects experienced difficulty with this. They appeared to understand what to do but could not seem to add four to the counting numbers from one to ten. In order to simplify the calculations the investigator changed the rule to "Add 1" for the younger subjects. Aside from difficulties with arithmetic skills subjects were able to do this process item. It took several minutes for some of these calculations with many subjects counting on their fingers. Perhaps some of the subjects needed a written statement such as "2 + 4 = __" to help them.

Most of the subjects were able to write the ordered pairs in this task, although many subjects recalculated the images. Only a few subjects referred to their work at the top of the page. However, their calculations were faster the second time. The investigator pointed out that the left hand number was the picker and the right hand number was the one the picker had picked. In considering the set of ordered pairs, the subject could see that all of the pickers would pick. Some subjects had difficulty when looking at two ordered pairs, such as (3, 1) and (3, 2) as they saw two threes, each of which picked one time rather than seeing that "3" picked both "1" and "2."

This difficulty also occurred in the Graph Task, where the subject was presented with several identification items. In addition, the shape of the graph caused difficulty for some subjects who evidently considered the points on the left side of the graph to be picking points on the right side, and they felt that another point was needed at the upper left.

Many subjects experienced difficulty with this task. A subject's success on this task was dependent upon his understanding of the explanation of ordered pairs in the last part of the Addition Task. Since a majority of the subjects did not have at least "Some evidence of understanding" on this task, it appears reasonable to assume that many subjects did not understand the explanation. It is interesting to observe that two-thirds of the subjects had "Clear evidence of understanding" on the first identification item, which involved a v-shaped graph. The other two items appeared to be more difficult; another point was added to the graph for the second item, and in the last item the subject was to consider the ordered pairs in the opposite order.

Only four subjects demonstrated complete understanding of the Table Task. Twenty-six other subjects identified the first item correctly but they needed a diagram drawn for the next part. The need for a diagram in the second part might have arisen from the fact that a subject did not consider that "0" was picking three times but that three zeros were picking once.

Most subjects achieved a rating of at least "Some evidence of understanding" on the Counting Number Task. The subjects received high ratings on the first two items, but the second process item apparently was more difficult for them, where the subject was to state a rule of correspondence. Even though several of the assignments were indicated, only three seven year old subjects and most of the older subjects were able to state this rule. Some subjects could use the rule, but they couldn't state it. Some of the difficulty, especially for the younger subjects, might have been a failure to understand the term "rule." Only one person clearly understood that this was not an "eff" when asked. Many others could make the correct decision after being asked a helpful question.

The Locker Task was comprised of eight identification items. The fact that the lockers were to pick people caused difficulty for several subjects. Some had to suggest ways that it would seem possible for the lockers to do this picking before proceeding to the identification items involved. The seven year old subjects performed poorly on this task. The nine and twelve year old subjects were much more successful.

All of the principles involved in the cup items were represented in this task as well as two items involving one-to-one correspondences. Since all of the identification principles were included in this task, it appears that the seven year old subjects, as a whole, did not understand, while a majority of the nine and twelve year old subjects did have understanding.

Three twelve year old subjects were rated "Clear evidence of understanding" on the Square Circle Task. One nine year old subject was rated "Some evidence of understanding." The other subjects did not appear to understand the task. All of the twelve year old subjects who received this rating were boys. The nine year old subject who was rated "Some evidence of understanding" was the sister of one of these three twelve year old boys.

In the explanation of the task the investigator told the subjects that each point on the circle would pick a point on the square. The subjects were to explain how the points on the circle would pick the points on the square. Some of the assignments had been made and were indicated by the investigator. The most common response was "They're in alphabetical order." Another common response was "They're close together."

Many subjects thought the lines in the diagram were unimportant. One subject identified the lines as a "Y." When the investigator labeled another point on the circle and asked what point it would pick, many subjects said that it would pick one of the points that had already been picked (usually the nearest one) or that it would pick the center of the circle which was labeled "O" in the diagram. Even when the picking process was demonstrated, many subjects could not understand it. Most subjects did not appear to understand that there were other points on the square and circle which were not labeled.

The subjects evidently noted the point which was the center of the circle, the circle itself and the square. The lines did not have meaning for many of them. The letters which labeled the points were noted. Two subjects drew lines from a letter on the circle to the letter it would pick on the square. This indicated a lack of understanding of the process.

It was not evident why so many subjects saw this figure as a triad. Elkind stated in a study on perception that as a child grows older, he ". . . is better able . . . to integrate parts and wholes. . . ." with a child's perception becoming ". . . increasingly determined by higher-order perceptual and cognitive abilities and decreasingly by the Gestalt properties of the stimulus." /Elkind, 1969; 24/ He also stated that " . . . the growth of perceptual activities that mediate figurative perception is relatively invariant across wide differences in environmental stimulation." /Elkind, 1969; 24/ This inability to see the figure as a whole was related to the age of the subjects. Those successful subjects had attained a higher ability to scan a figure and analyze it. In considering the subjects who were successful on this task, one finds that one twelve year old was in the seventh grade, another was a sixth grader who had the highest IQ in the study, and the other two were brother and sister. Their perception was obviously different from that of the other subjects. They saw the figure as a whole and could determine the relationships of the parts with one another.

While the investigator never mentioned inverse functions to the subjects, several items involved this idea. Many subjects were able to answer these items correctly. One seven year old subject insisted that the pickers had to be the set at the left and could not adapt his thinking for the new idea. Most subjects did not appear to have any difficulty in thinking in this manner.

Many varied factors influenced the performances of the subjects on the tasks. Some of these factors were their intelligence levels, their understanding of the related school work previously presented to them, the atmosphere within their homes, and even their sense of well-being on the day of the interview.

Much attention has been devoted to learning differences in children from different socioeconomic backgrounds. Typically, the homes of people in a low socioeconomic level have been depicted as overcrowded, noisy, disorderly and lacking in items associated with learning skills. Breshnahan and Shapiro noted comparisons of lower and higher class children with the lower class children being less successful than the higher class children in academic and vocational situations, deficient in reading and number concepts and in symbolic representation, more concrete and inflexible in their intellectual functioning, and unresponsive to symbolic or verbal incentive. This statement in their report concerned concept formation.

> It is generally considered that on a concept formation problem the subject selects a hypothesis and retains it until he makes an incorrect response, at which time he rejects his hypothesis and samples a new one. If the hypothesis produces the correct response he stays with it. If the hypothesis produces an incorrect response, he shifts to a new hypothesis. /Breshnahan and Shapiro, 1972; 45/

This procedure of concept formation was referred to as the "win-stay, lose-shift strategy" which was employed by adults and higher class children. The lower class children did not develop this strategy.

The subjects in the present study were reinforced with an "O.K." or "Good" as they progressed through the tasks. These words were meant to encourage the subjects to continue but did not mean he was correct. Only in the first two tasks were the subjects informed if they were right or wrong. This type of reinforcement was of a type to which children from low socioeconomic levels did not respond well. Some of the subjects in this study who didn't answer consistently might have been trying to find a successful strategy of response. A few children from the low socioeconomic levels watched for signs of success from the interviewer.

In a study on classification, Wei found subjects from low socioeconomic levels who appeared to be closer to the higher socioeconomic level subjects after two years of school. /Wei, 1969/ This trend was also found in this study, where the older subjects from low socioeconomic levels performed at a higher level, in general, than did the younger subjects of this type. These were generalizations on the part of the investigator, as no attempt was made to identify particular subjects as being from low socioeconomic levels in this study. These conclusions resulted from judgments made by the investigator based primarily on the district served by the school the subjects attended.

Piaget's theory of equilibration stressed the continual interaction between the individual and the environment. [Furth, 1970] It appears that differences in development would occur in children who come from radically different socioeconomic levels. These differences evidently might be tempered with similar school environments for the children.

Stages in the Attainment of the Function Concept

The stages referred to here were developed by Thomas in his study on the concept of function. The final stages in the attainment of the concept were cited in Chapter II. Thomas distinguished between the processes associated with the function concept, such as assignments in arrow diagrams and rules of the type $x \longrightarrow x + 1$, and the ability to identify instances and non-instances of functions. [Thomas, 1969]

He discerned four stages in the attainment of the function concept. The first two stages dealt with the process aspects of the concept while the other stages dealt with process, identification and operations on functions. Since no work was done with operations on functions in this study, the stages involving that component were not considered in this classification. This eliminated Stages IIIB, IIIC and IVA.

The subjects were classified into four groups: No Stage, Stage I, Stage II, and Stage IIIA. The procedure employed to determine whether a subject had attained a particular stage was to isolate those items relevant to that stage and then note the performance of the subject on those items. If the subject's ratings on these items included some indications of "Clear evidence of understanding" and no indications less than "Some evidence of understanding," then the student was considered to have attained that level.

The basic process items were arrow diagrams, tables and rules such as $x \longrightarrow x + 1$. Items of this type were "P1," "P2," "P3," "P4," "P5," "P6," "P7," and "P9." These process items involved arrow diagrams with either words and letters or numbers. In order for a subject to be classified in Stage I, he must have been rated "Clear evidence of understanding" or "Some evidence of understanding" on these items. A person who did not have these high ratings on these items was classified No Stage.

No person could belong to more than one stage at a time. After classification of the subjects as either Stage I or No Stage, the subjects in Stage I were reviewed to see if any of them were in Stage II or Stage IIIA. Stage II required a higher level of performance on the process component, that is, being able to identify images, preimages, domain, and range and work with the ordered pair point graph representation, than did Stage I. Stage IIIA required that a subject be at the Stage I level in terms of the process component and be able to identify instances and non-instances of functions.

Stage II was characterized by items which involved image and preimage identification and the ordered pair point graph representation. The individual items examined here were "cup," "12," "P8" and "P10." Since there was no process item involved in the Graph Task, those subjects found to be at Stage I before were classified by these items and "VIa" and "VIb" to determine whether or not they were in Stage II. If a subject was found to be in Stage II, then he was classified Stage II, otherwise he remained in Stage I.

Stage IIIA was characterized by the identification items. All of the subjects previously classified in Stage I or Stage II were eligible for Stage IIIA. The investigator considered each of these subject's abilities on the identification items. For a subject to be in Stage IIIA he had to identify all instances and non-instances of functions correctly with appropriate reasons. The items of the Cup Task were not considered in this, nor were the items in the Square Circle Task unless the subject had been successful on the task.

Table 16 indicates the number of subjects in each age group classified as No Stage, Stage I, Stage II or Stage IIIA in the attainment of the function concept.

TABLE 16

STAGES IN THE ATTAINMENT OF THE FUNCTION CONCEPT

Classification	7 years	9 years	12 years	Total
No Stage	6	1	0	7
Stage I	9	12	10	31
Stage II	1	3	3	7
Stage IIIA	0	0	3	3

It is apparent from this table that most of the subjects (64%) were classified in Stage I. Seven subjects were below the level of Stage I and three subjects, all twelve years old, were in Stage IIIA. There were subjects from all three age levels in Stage II. Eighty-five percent of the subjects were at least at the level of Stage I.

This classification differs from previous stage classifications, such as those by Orton, as those classifications involved determining the stage level of each response. However, the subjects themselves were not classified in stages. This manner of classification was selected in order to see if the age of the child appeared to be a factor in the stage he attained.

In addition, it is interesting to note the difference in the attainment of stages according to sex and race. The following tables, Table 17 and Table 18, indicate these differences by showing the number of subjects who have been classified No Stage, Stage I, Stage II and Stage IIIA. Table 17 indicates the number of subjects in these classifications by age group and sex, and Table 18 indicates the number of subjects in these classifications by age group and race. In Table 17 it is apparent that for each age group, the highest stage achieved was by a male subject. In Table 18 it can be seen that the highest stage achieved for each age group was by a white subject.

TABLE 17

ATTAINMENT OF STAGES ACCORDING TO SEX

Classification	7,male	7,female	9,male	9,female	12,male	12,female
No Stage	3	3	1	0	0	0
Stage I	4	5	4	8	7	3
Stage II	1	0	3	0	2	1
Stage IIIA	0	0	0	0	3	0

TABLE 18

ATTAINMENT OF STAGES ACCORDING TO RACE

Classification	7,white	7,black	9,white	9,black	12,white	12,black
No Stage	1	5	1	0	0	0
Stage I	4	5	6	6	6	4
Stage II	1	0	3	0	1	2
Stage IIIA	0	0	0	0	3	0

Conclusions

It is difficult to determine the understanding of a concept when there is no common fund of knowledge involving the concept for all subjects. The teaching task in this study was devised to compensate for this lack of a common fund of knowledge. It was a somewhat lengthy task and some of the subjects responded slowly to the new concept. The tasks were designed to be answerable from the information given. As large a number and variety of process and identification items as employed in previous studies could not be used due to concern with the time factor.

A figure, such as the one presented in the Square Circle Task, evidently is difficult for young children to perceive as a whole. Most of the subjects did not appear to understand the figure. They saw it in three parts: the square, the circle and the point which was the center of the circle. Perception of this figure as a whole and analysis of its parts might require a person to be at the beginning of the Formal Operational Period.

There appeared to be some improvement with age in the ability of the subjects to understand the concept of function. This was not as clearly demonstrated by the subjects' responses to the tasks or by the statistical analysis as it was by the final analysis by stages in the attainment of the function concept. It was clear that the twelve year old subjects as a group achieved a better understanding of the concept than the nine year old subjects did. It was also clear that the nine year old subjects had a better understanding of the concept than the seven year old subjects.

In general, the subjects demonstrated a better understanding of the process items than of the identification items. This was clear in the classification of the subjects by stages, as eighty-five percent of the subjects were at least at the Stage I level with fourteen percent at Stage II level and six percent at the Stage IIIA level. Stages I and II indicated levels of performance on the process component. Stage IIIA required understanding of the identification component in addition to performance at the Stage I level. Therefore, most of the subjects attained some mastery of the process component but only three subjects attained understanding of the identification component.

The subjects had difficulty in recognizing the patterns of assignments which signified instances of functions and not-functions. It appears that consistent recognition of these patterns requires the beginning of Formal Operational Thought. The three twelve year old subjects in Stage IIIA had reached this level. While other subjects correctly responded to some identification items, their responses were not consistent enough for them to be classified in Stage IIIA.

The seven and nine year old subjects in this study appeared to be in the Concrete Operational Period, which was consistent with the findings of Piaget, in that Piaget found the seven year old children to be at the first stages of this period and children of approximately eleven years of age to be entering the Formal Operational Period. In general, the nine year old subjects performed more successfully on the tasks than the seven year old subjects did. The twelve year old subjects, however, appeared to be in the last stages of the Concrete Operational Period or the early stages of the Formal Operational Period. It was apparent that the twelve year old subjects classified in Stage IIIA had reached the Formal Operational Period, while the other twelve year old subjects had not yet reached this period.

Implications for Education

As suggested in Chapter I, studies of ability to understand concepts are useful in curriculum planning. The results of this study show the process component to be more readily understood than the identification component. Since not all of the seven year old subjects could be classified in a stage, the process component should only be introduced to children of this age in simple terms through the use of physical objects and arrow diagrams. Results indicate that the introduction of the identification component should be delayed.

Since all of the nine year old subjects but one were classified in Stages I and II, it appears that these subjects could be introduced to more complex process items through the use of physical objects, arrow diagrams and tables. While most nine year old subjects had more success with the identification items than the seven year old subjects did, it does not appear that nine year old children will have consistent success with the identification component. Some concrete examples of the identification component might be introduced at this time.

Since all of the twelve year old subjects were classified in Stages I, II or IIIA it appears that children of this age could be introduced to the identification component as well as the process component. As many of these subjects were not at the Stage IIIA level, some children of this age will experience difficulty with the identification component.

Since almost all of the subjects experienced difficulty with the diagram in the Square Circle Task, it appears that young children cannot work with this type of diagram. Care must be taken that the examples used in introducing the concept not be too complex.

In summary, it appears that the simple ideas relating to the function concept can be introduced as early as the first grade. However, the more complicated ideas cannot be successfully introduced until some point between the fourth and sixth grades. The abstract concept should be presented even later.

Suggestions for Further Research

The findings in this study suggest several areas needing further study. The children in this study were from a small, southern town of 22,000 people. A similar study in different geographic locations and/or with other age levels would be of value.

Other tasks might be devised to help the investigator in determining a child's understanding of the concept of function. In this study, the investigator presented the subject with as many representations of functions and not-functions as possible, being encumbered by the lack of a common fund of knowledge and the length of time for the interviews.

To circumvent this lack of a common fund of knowledge, young children might be introduced to the function concept in a classroom and then interviewed individually. The presentations might be varied to include the process component or the identification component or both components.

A study utilizing video-tape and audio-tape would allow an investigator to analyze the tasks in more detail. In addition, it might be useful to return to a task utilizing physical objects at the end of each interview to determine any change in the subject's responses with these objects.

Attention might well be directed to the ability of children to understand certain types of functions. Some work has been done in this area concerning functions involving proportion.

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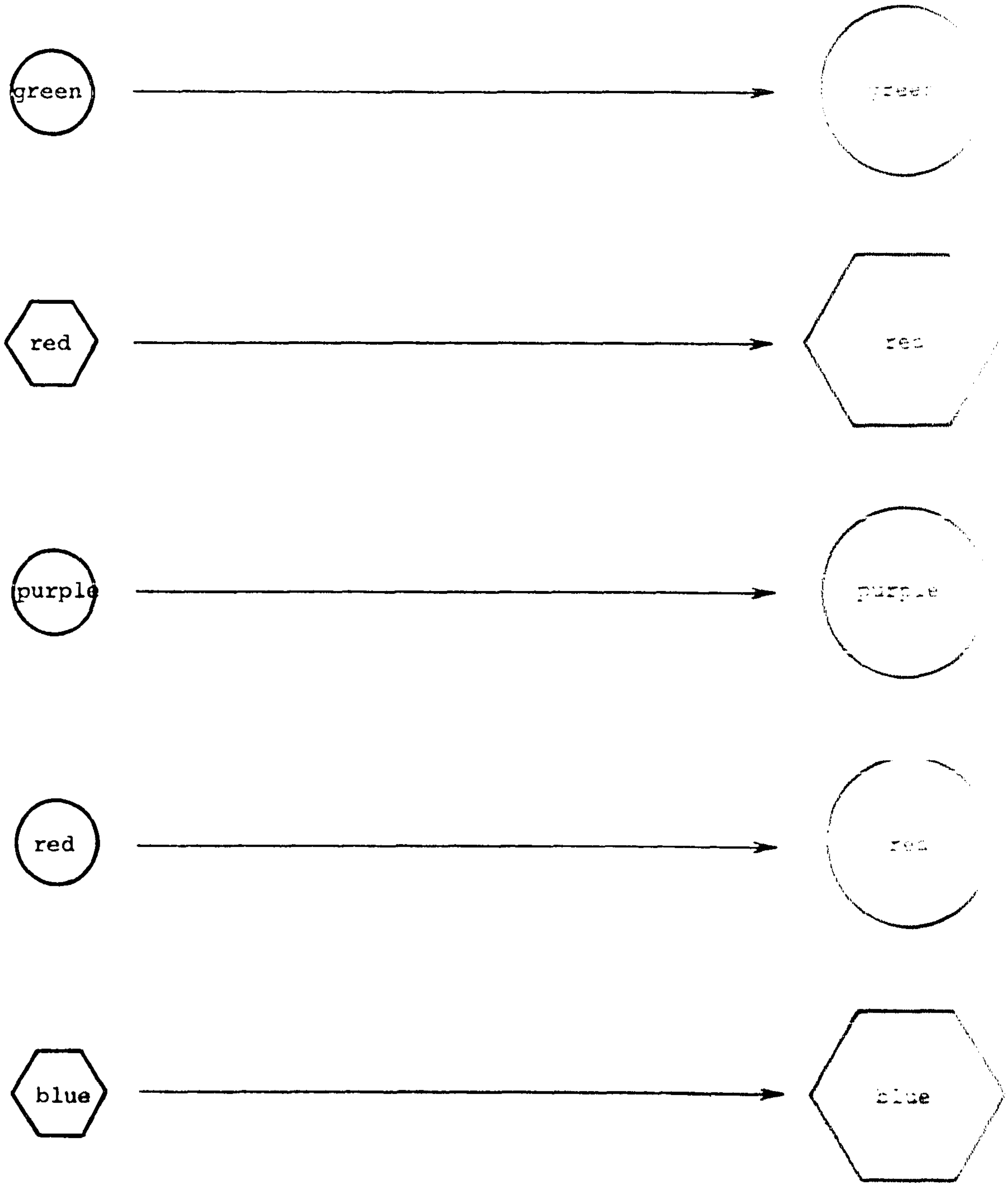
APPENDIXES

APPENDIX A

THE TASKS

I. Cup Task

One-to-One Correspondence



NAME____________________________ SCHOOL________________ CLASS______________

I. CUP TASK--EXPLANATION OF EFF (show with arrows)
   1. 1 to 1 correspondence__________________________
   2. add cup to right. (white)__________________________
   3. add cup to left. (med. purple)______________________
   4. add cup to right. (big round red)____________________
   5. add cup to left. (small yellow)______________________

II. Draw arrows.
   1. eff?____
   2. eff?____
   What if the word picks the letter the word begins with?__________
   Would that be an eff?________________________________

III. Draw arrows with numbers.
   Explain other diagrams without numbers.
   1. eff?____
   2. eff?____
   3. eff?____
   4. eff?____

IV. Draw arrows.
   1. eff?____
   2. eff?____
   3. eff?____
   In last, if rule was reversed, what would 12 pick? Why?__________

V. Fill in arrows and numbers on paper.

VI. Show how ordered pairs on graph like those in previous task.
   1. Is this an eff?___________________________________
   2. What if point (4,1) were added?______________________
   3. What if the ordered pairs were reversed--would it be an eff?_____

VII. Number in top row picks number below it in bottom row.
   1. Eff?____
   What if the rows were reversed, would it be an eff then?__________

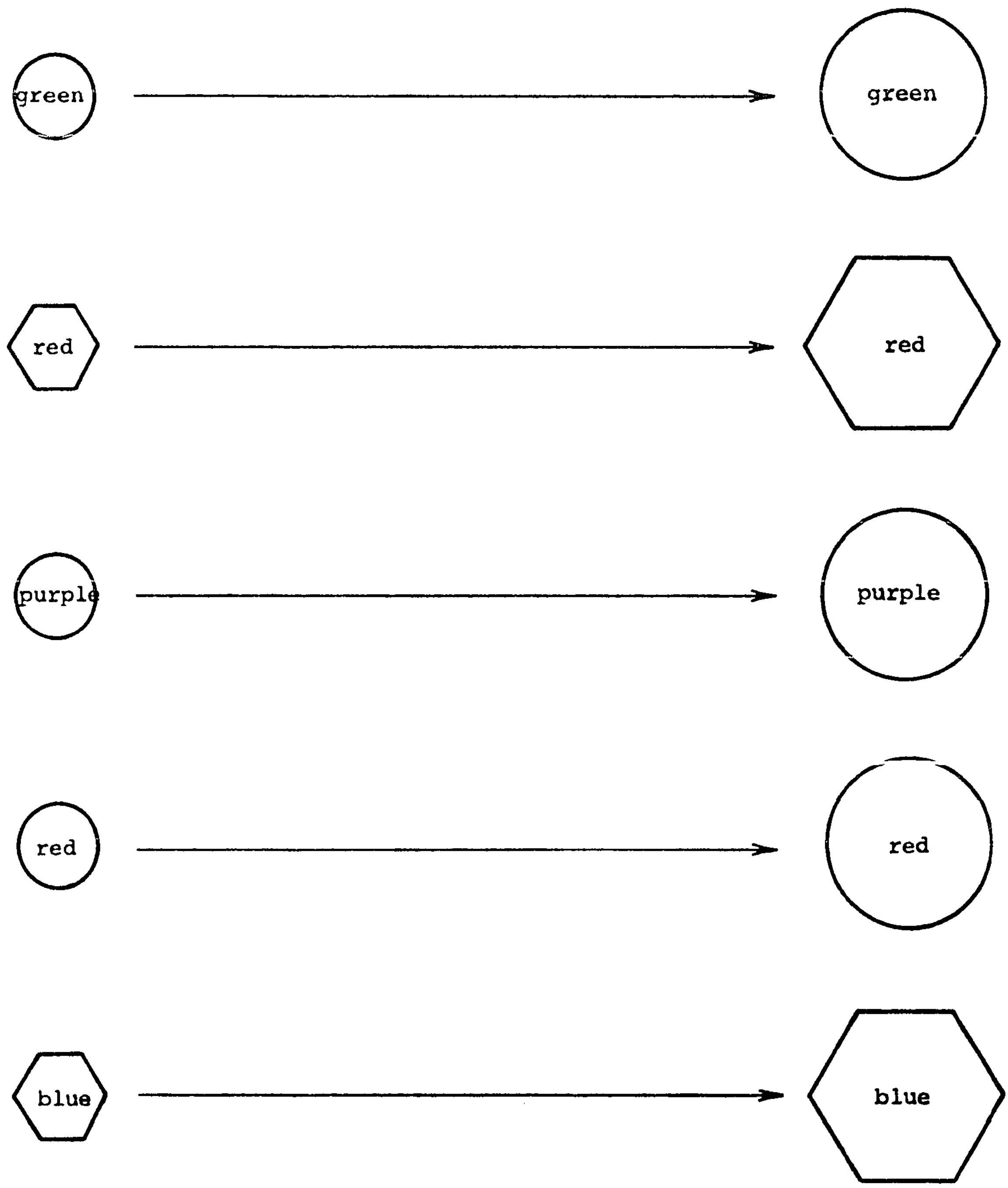
VIII. Circle inside square.
   Can you find the rule by which the points of the circle pick the points of the square?____________________________________
   How does C pick B?___________________________________
   How does D pick E?___________________________________
   How does A pick A?___________________________________
   What would a rule be that allows the points of the square to pick those of the circle?____________________________________
   How could B pick C?__________________________________
   How could E pick D?__________________________________

IX. Is it an eff?______________________________________
   What would be the rule that would "undo" this?_________________
   Would it be an eff?___________________________________

X. 1. There are just enough lockers so that each person can have his own locker.
   a. eff?____________________ b. eff?____________________
   2. There are more lockers than people so that some people have two lockers.
   a. eff?____________________ b. eff?____________________
   3. There are more lockers than people but some are unused.
   a. eff?____________________ b. eff?____________________
   4. There are more people than lockers so that some lockers are assigned to two people.
   a. eff?____________________ b. eff?____________________

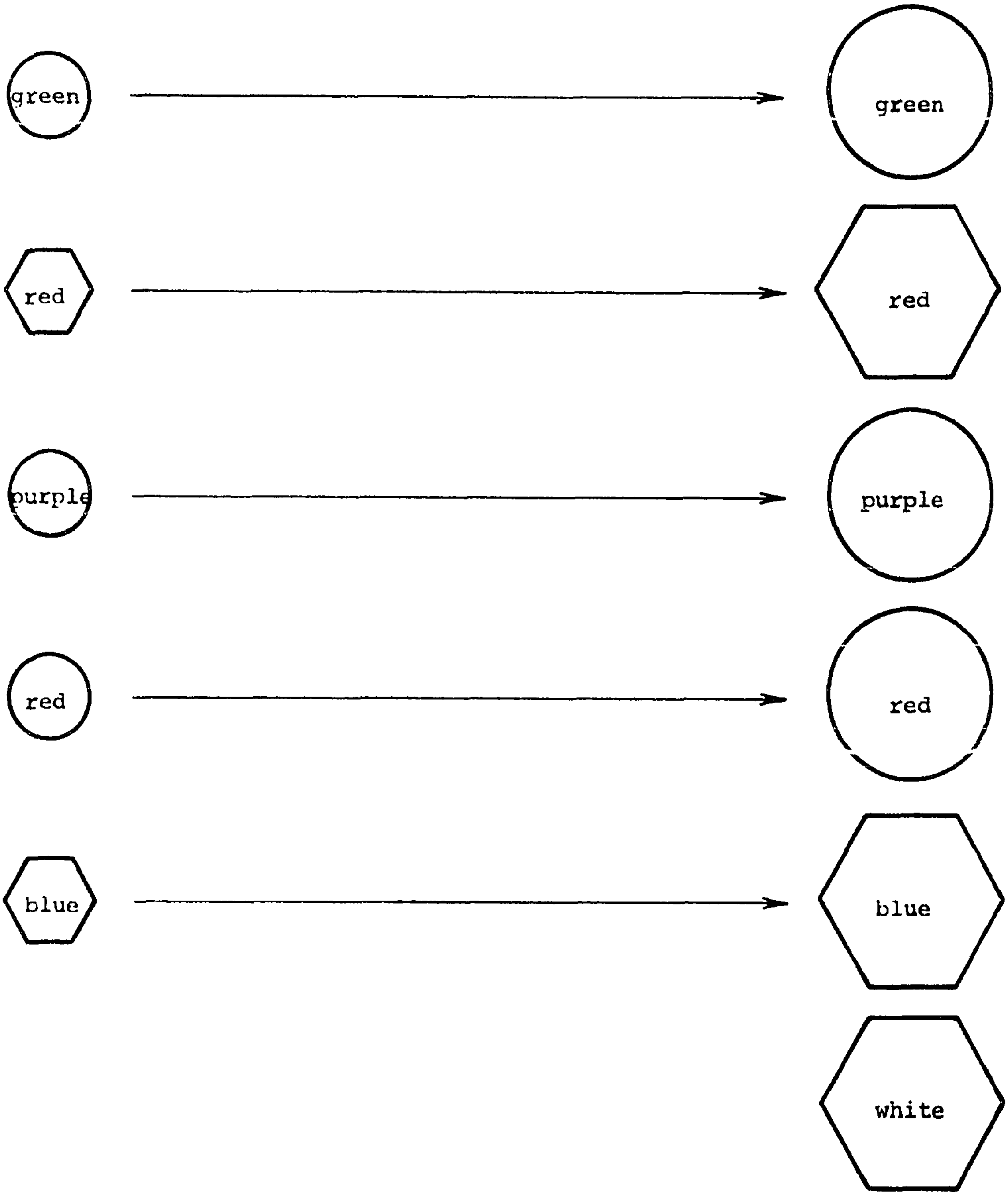
I. Cup Task

One-to-One Correspondence



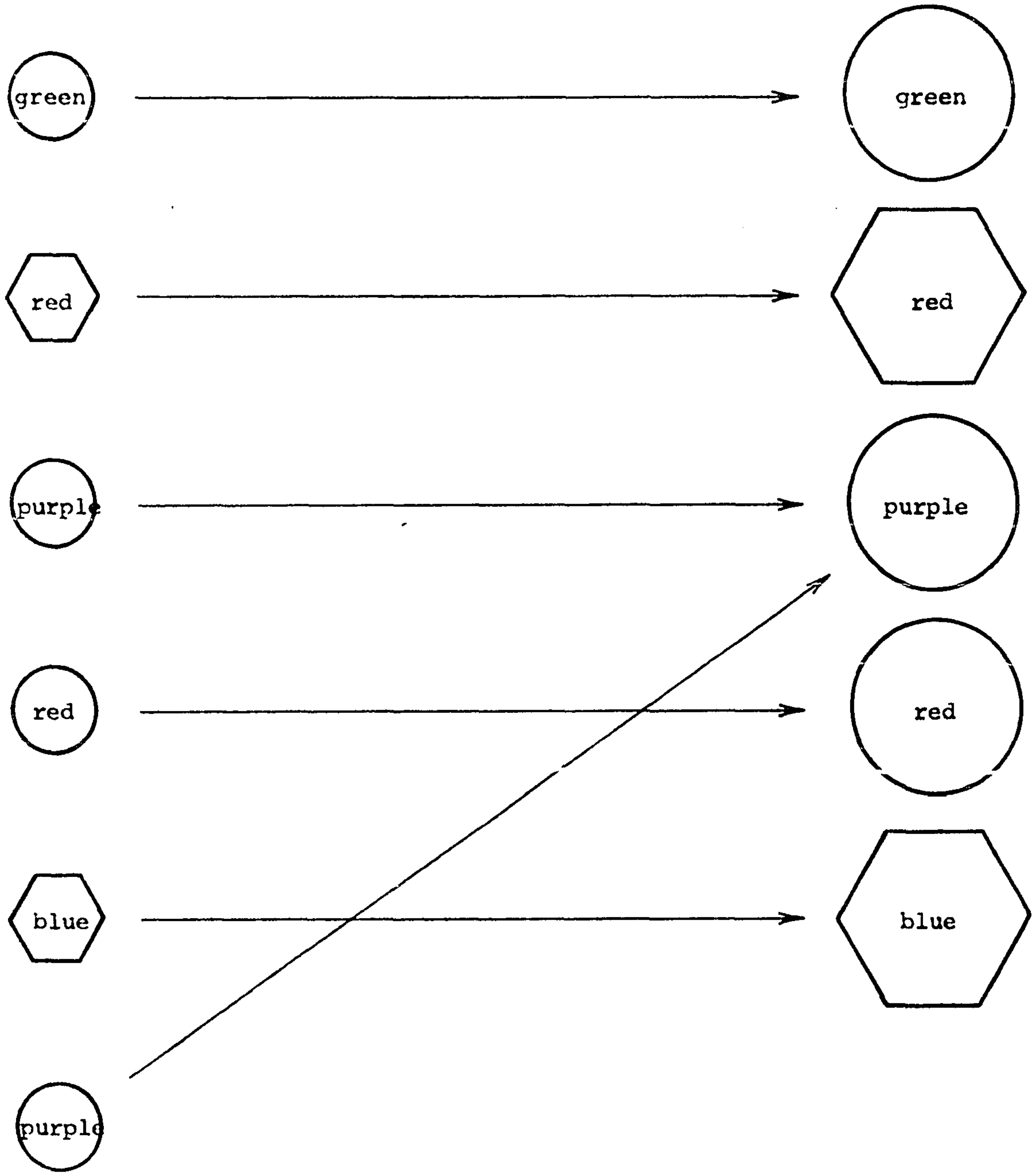
I. Cup Task

White Cup Item



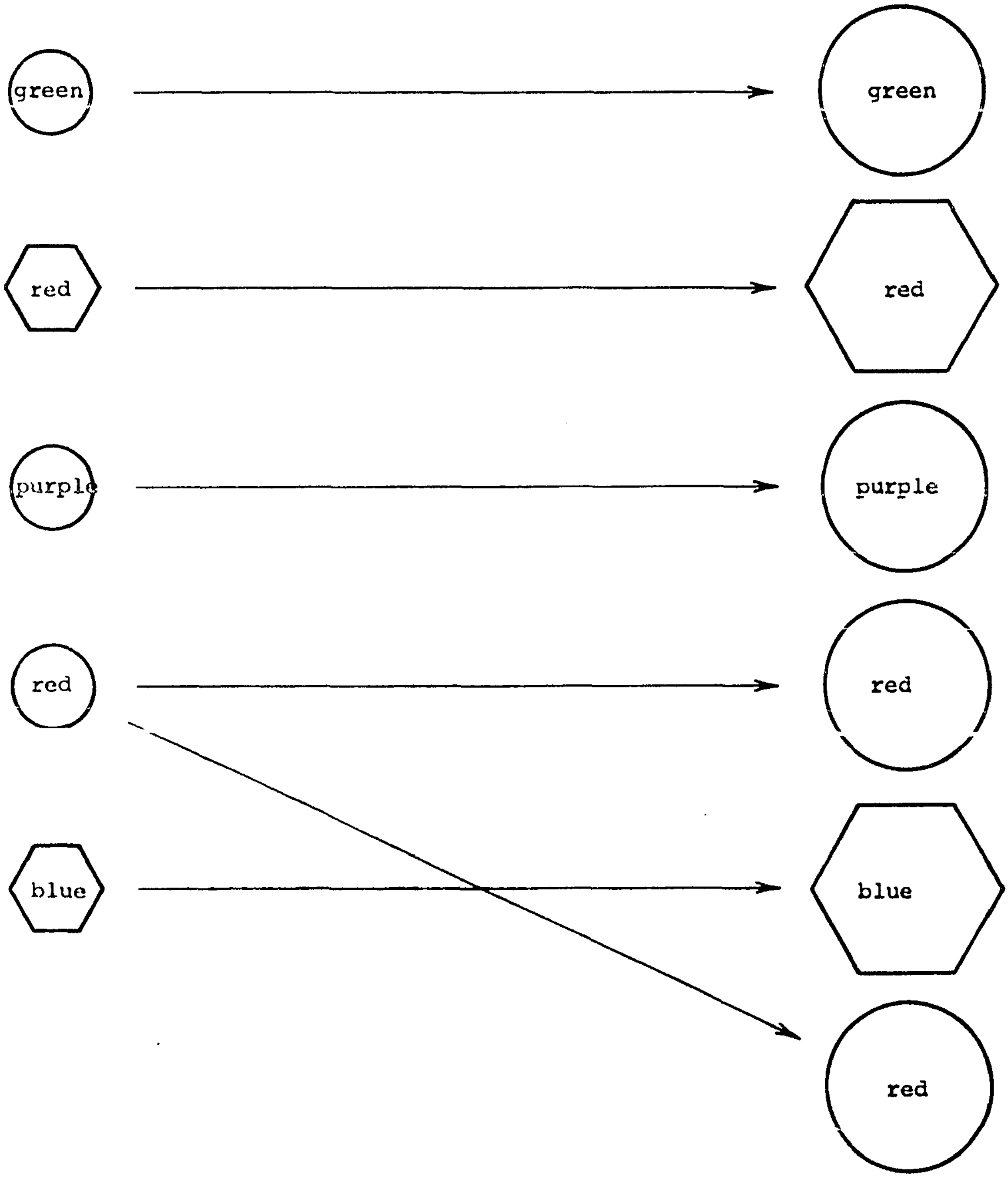
I. Cup Task

Purple Cup Item



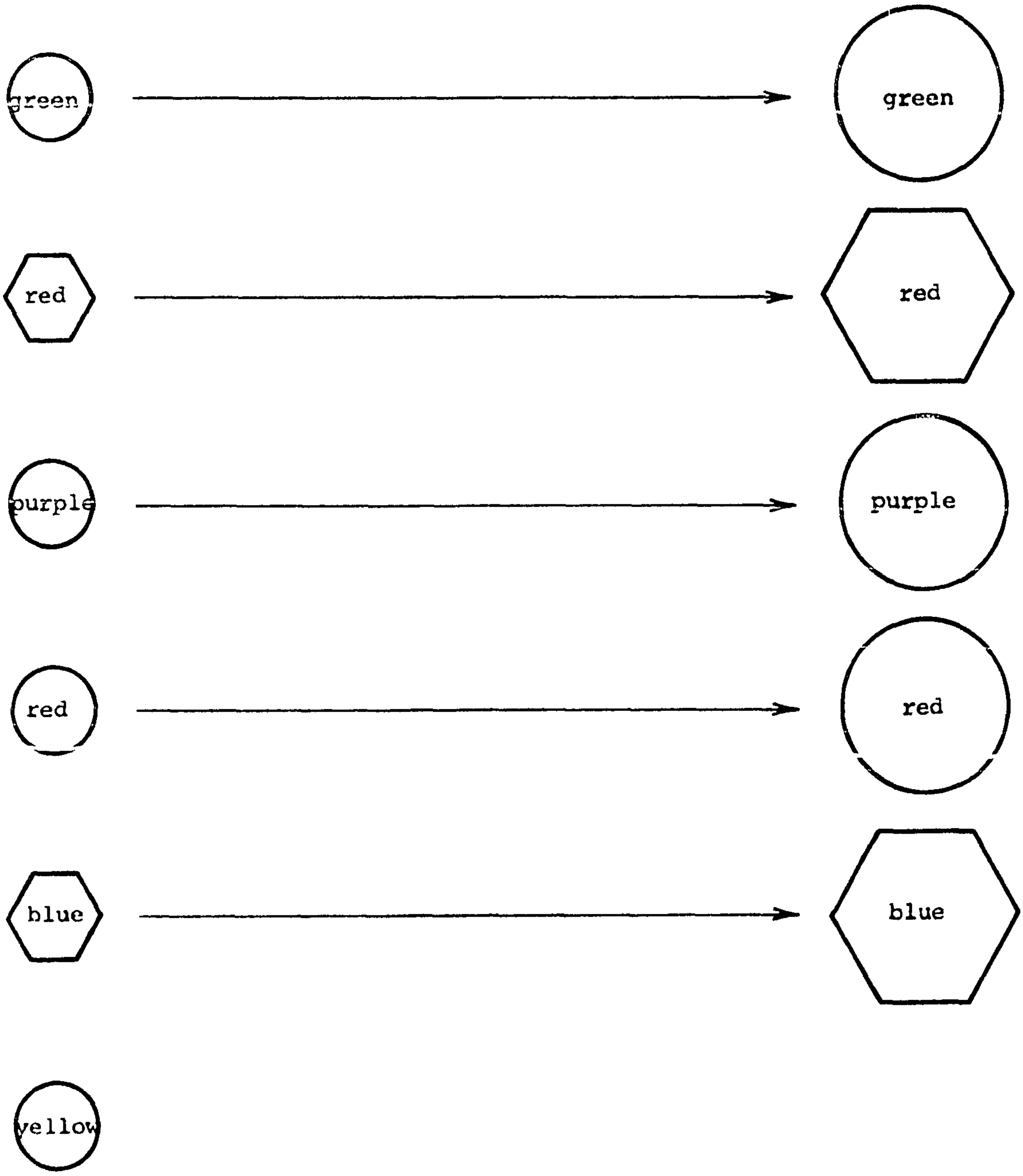
I. Cup Task

Red Cup Item

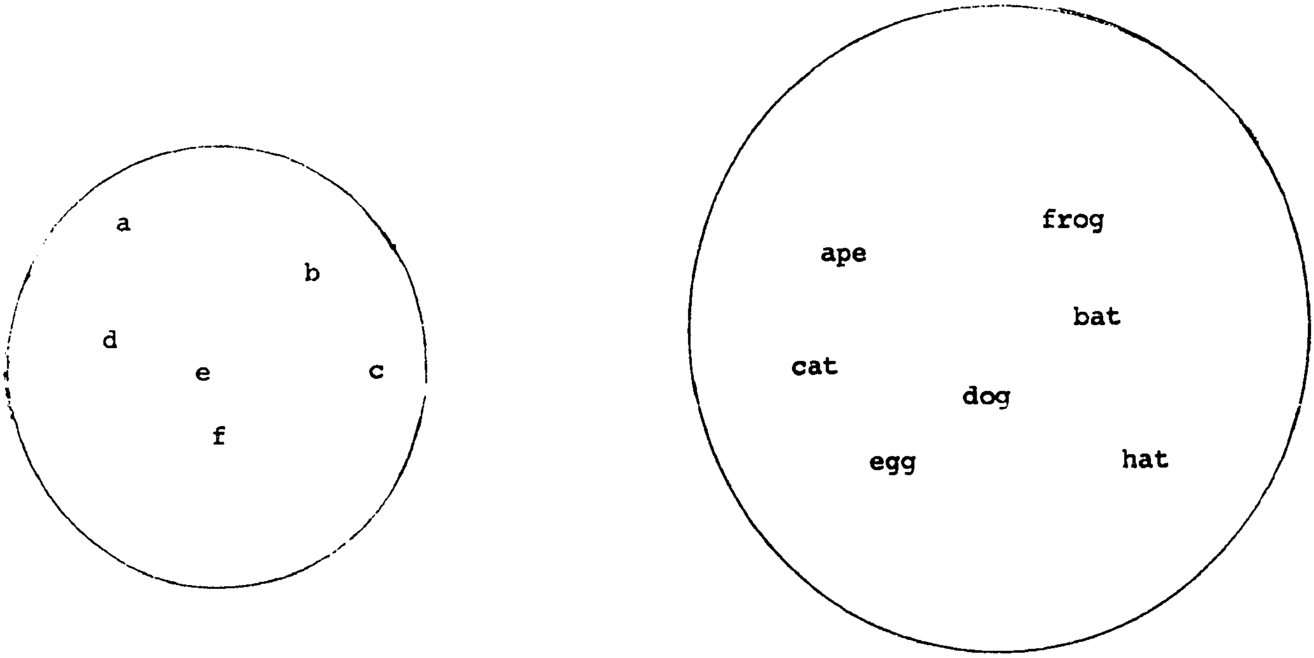


I. Cup Task

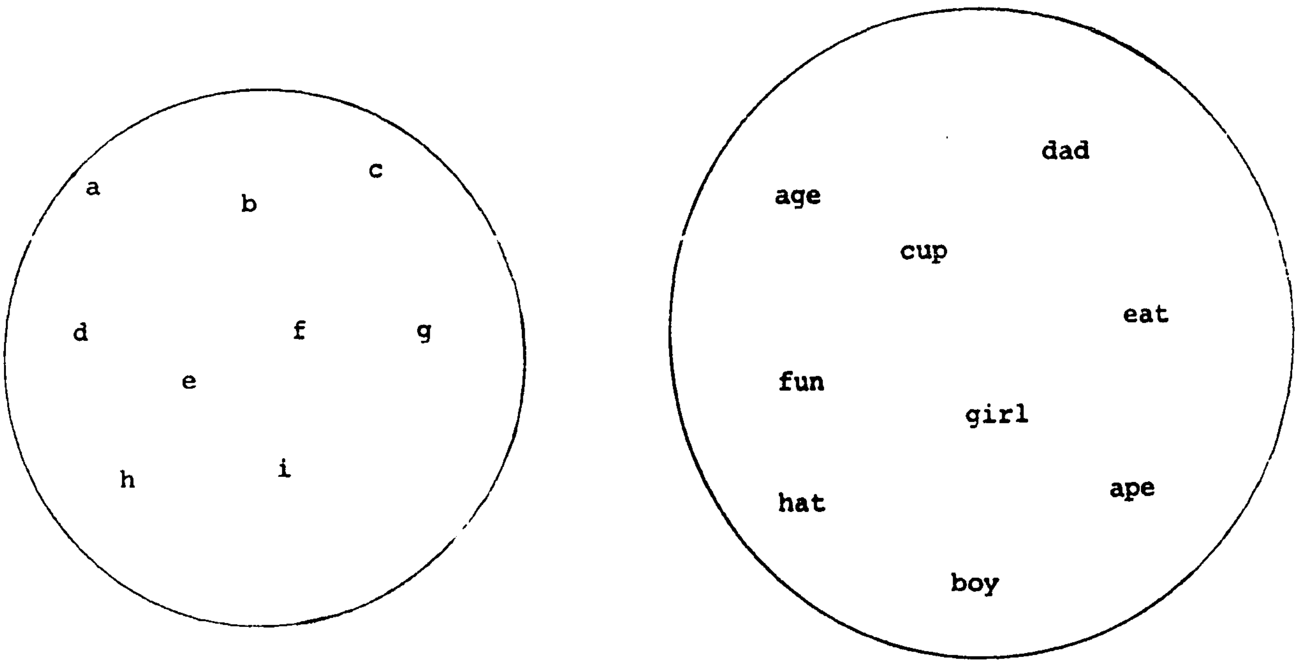
Yellow Cup Item



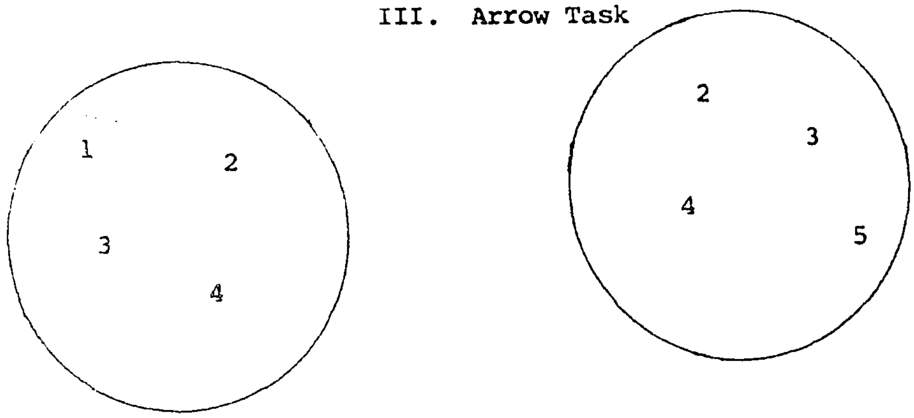
II. Letter and Word Task

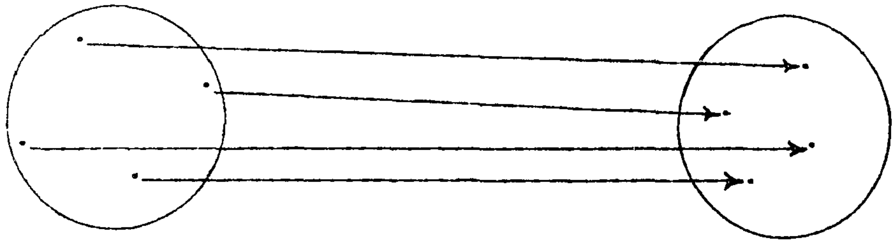


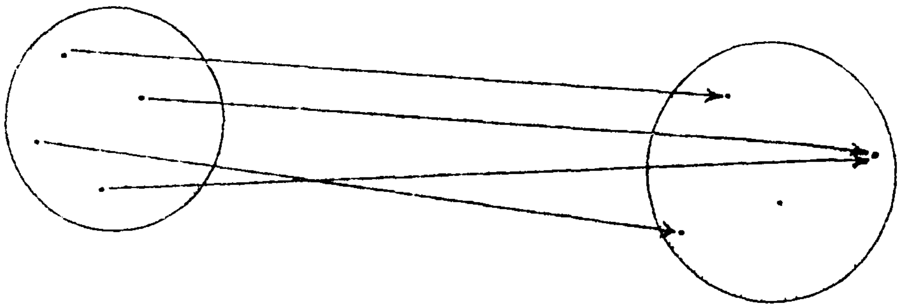
Rule: Each letter picks a word which begins with that letter.

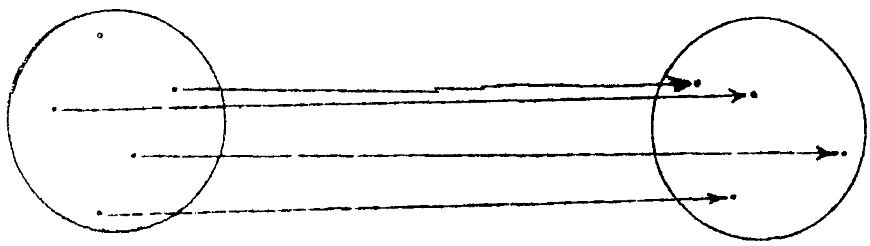


III. Arrow Task

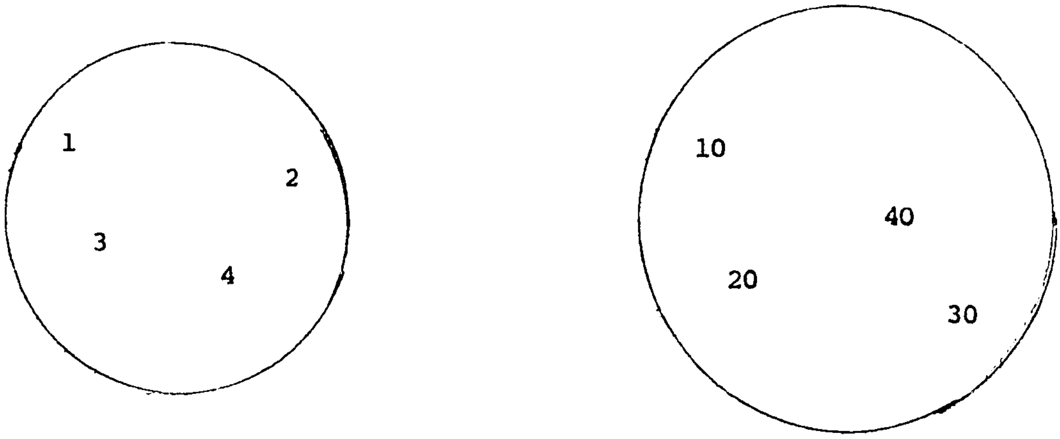




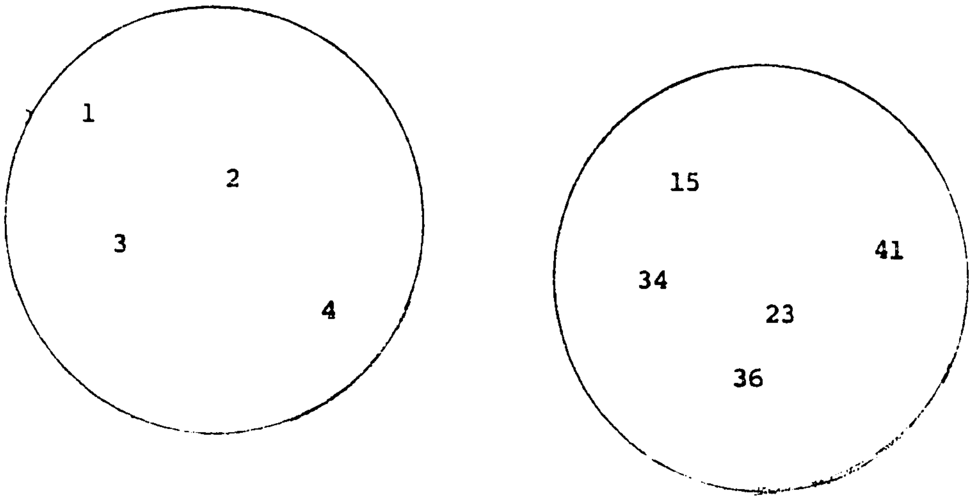


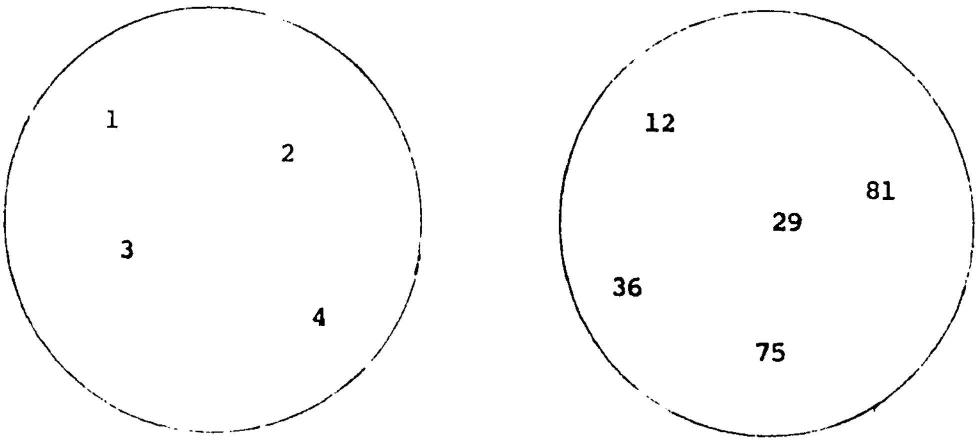


IV. First Digit Task

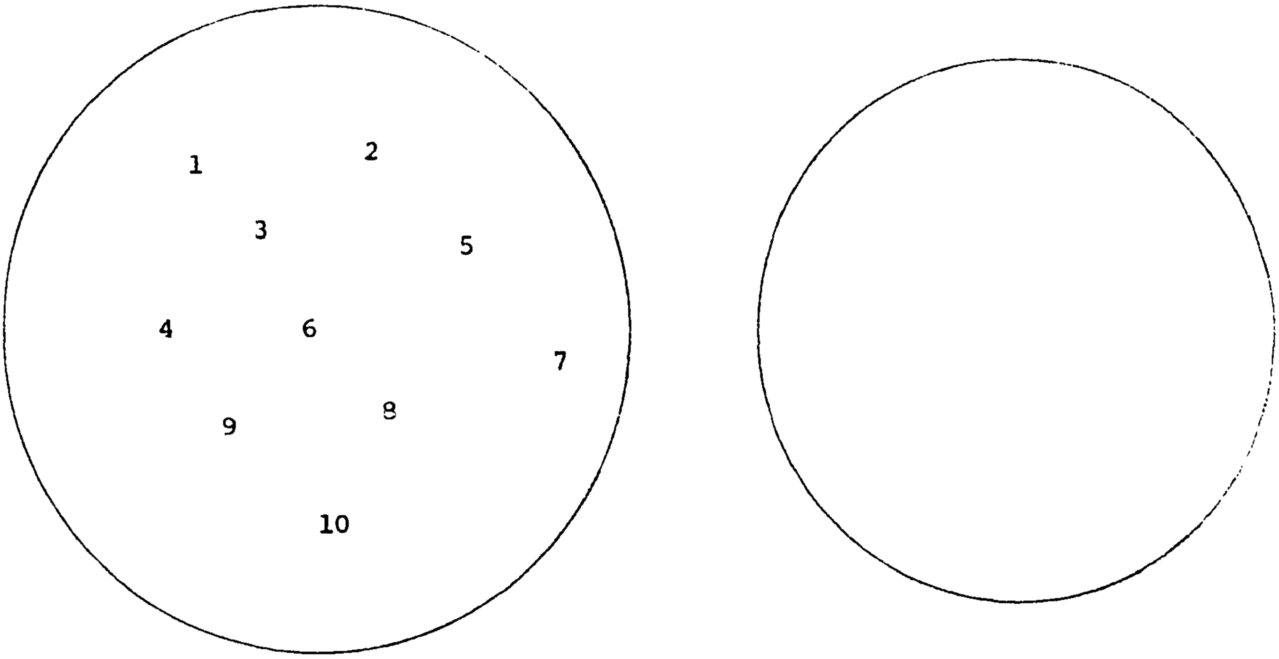


Rule: Each number at the left picks a number with the same first digit.





V. Addition Task



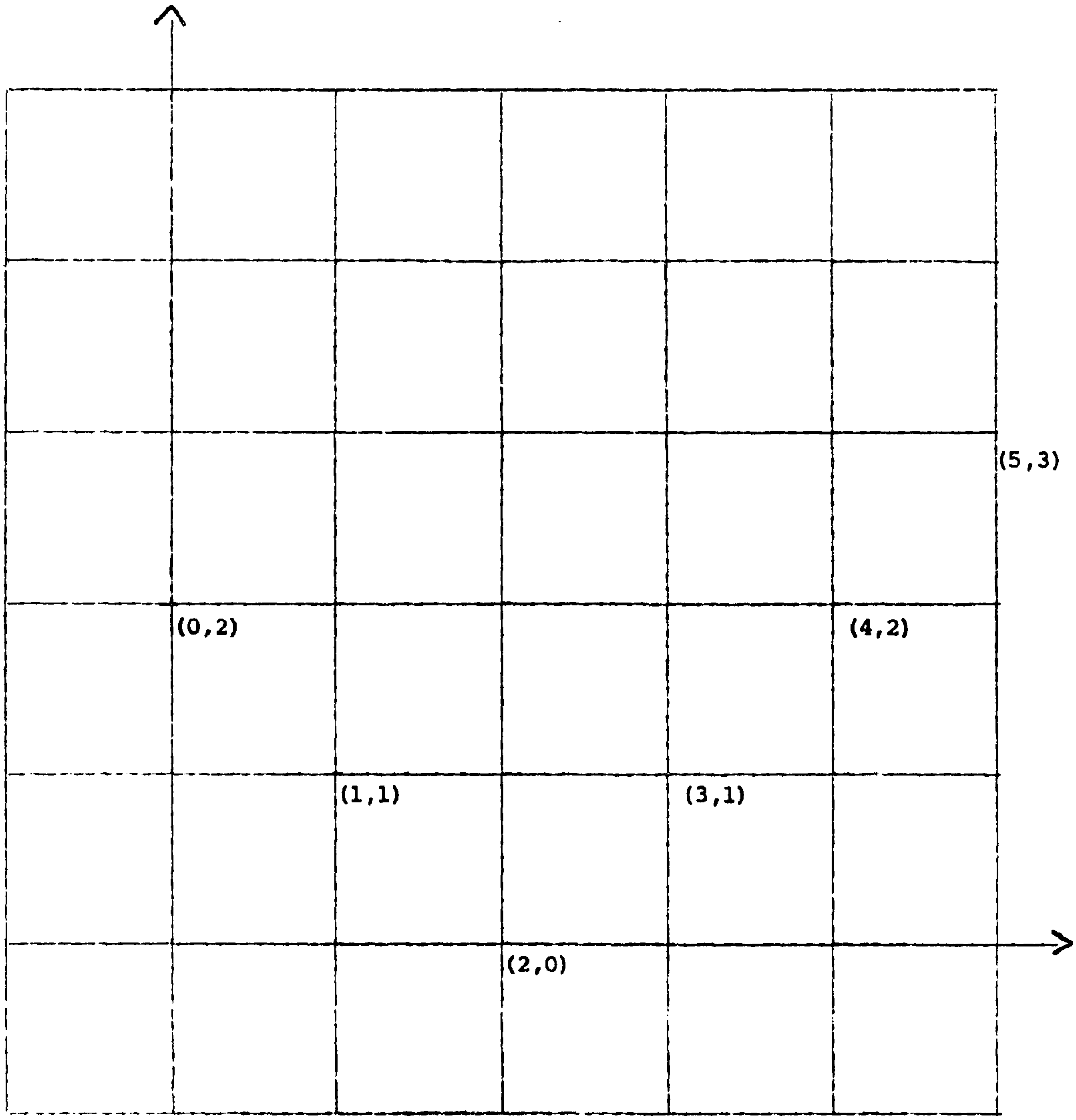
Rule: Add 4

The number 1 picks ____. Let's write it like this (1, ___) with the picker to the left and the picked to the right.

(1,), (2,), (3,), (4,), (5,), (6,), (7,),

(8,), (9,), (10,)

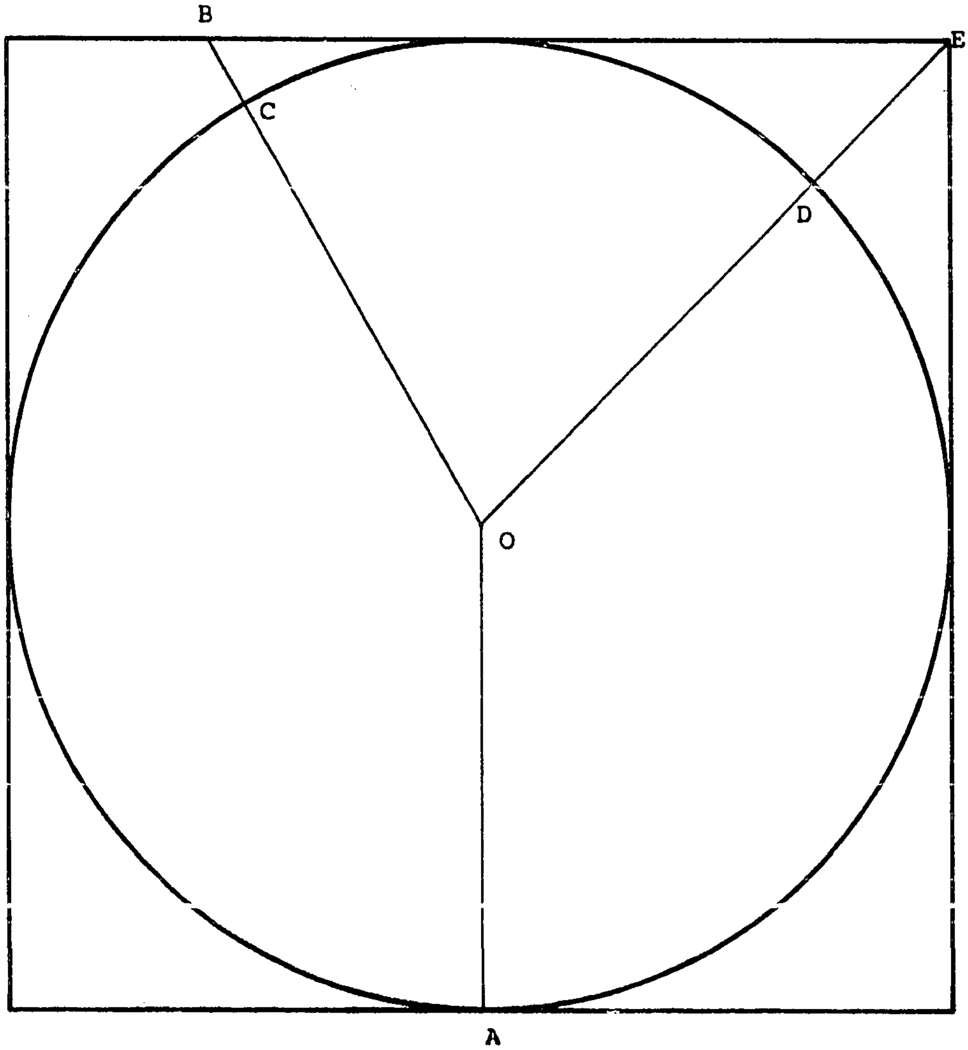
VI. Graph Task



VII. Table Task

x	1	2	3	4	5	6	7	8	9	10	11	12
y	0	3	1	2	0	1	3	2	1	3	0	2

VIII. Square Circle Task



IX. Counting Task

What are the counting numbers?

What is the smallest counting number?

Rule: $x \longrightarrow x + 1$

$1 \longrightarrow$ ___

$15 \longrightarrow$ ___

___ $\longrightarrow 30$

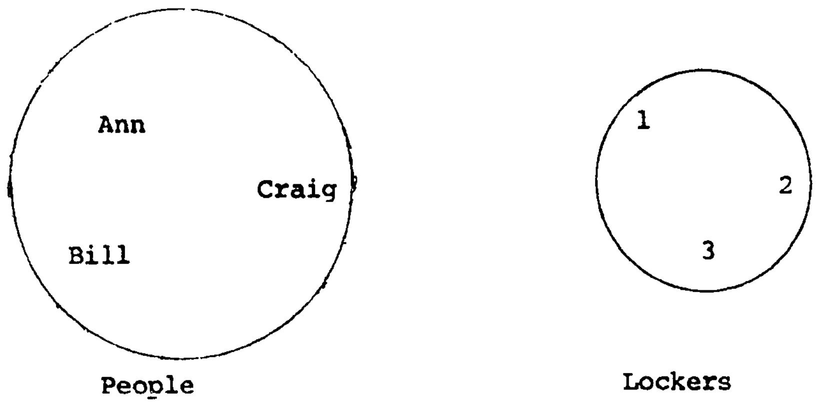
___ $\longrightarrow 1$

What is the biggest counting number you know?

What number would it pick by this rule?

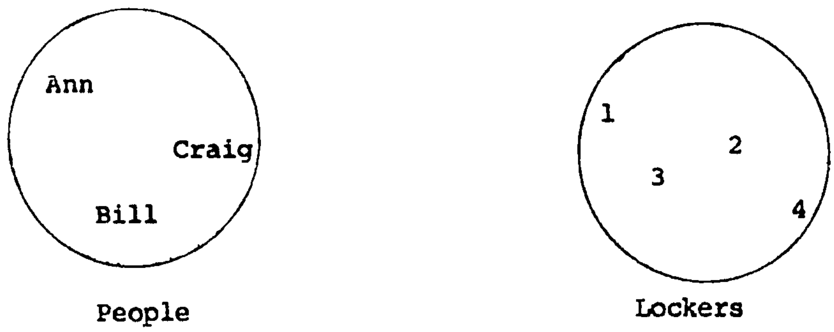
X. Locker Task

1.

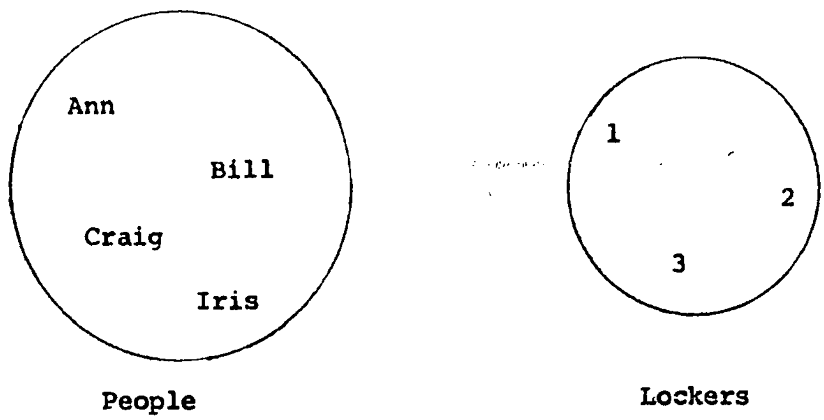


2.

3.



4.



APPENDIX B

THE RATING INSTRUMENT

Name ________________________ Age ________ Grade ____________________

I. Cup Task. In this task, the child receives instruction in the concept of function in terms which are used throughout the tasks.

Left	Right
Small hexagonal red cup	Large hexagonal red cup
Small hexagonal blue cup	Large hexagonal blue cup
Small round red cup	Large round red cup
Small round green cup	Large round green cup
Small round purple cup	Large round purple cup

The group of cups at the left are called pickers. The child is asked to show how the pickers pick (using the arrows provided) if the pickers pick according to color and shape. Then various cups are added to show instances of functions and not-functions.

	1	2	3	4	5
white cup to right					
purple cup to left					
red cup to right					
yellow cup to left					
task					

II. Letter and Word Task. Each letter picks a word which begins with that letter.

	1	2	3	4	5
Process					
IIa					
Process					
IIb					
cup					
IIc					
task					

III. Arrow Task. Identification of functions and not-functions, which are represented by arrow diagrams.

	1	2	3	4	5
Process					
IIIa					
IIIb					
IIIc					
IIId					
task					

IV. First Digit Task. Each number picks another number in which the picker is the first digit.

	1	2	3	4	5
Process					
IVa					
Process					
IVb					
Process					
IVc					
12					
task					

V. Addition Task. Child uses a rule to find the range when he is given the domain.

	1	2	3	4	5
Process					
eff?					
16?					
30?					
Process					
ordered pairs					
task					

VI. Graph Task. Points in the plane are used to represent ordered pairs.

	1	2	3	4	5
VIa					
VIb					
VIc					
task					

VII. Table Task. A table of numbers is presented. In part a, the top row is the domain and the bottom row the range. In part b, the top row is the range and the bottom row is the domain.

	1	2	3	4	5
VIIa					
VIIa with diagram					
VIIb					
VIIb with diagram					
task					

VIII. Square Circle Task.

	1	2	3	4	5
Ability to find a rule by which the points of the circle pick points of the square.					
eff?					
Ability to find a rule by which the points of the square pick points of the circle.					
eff?					

IX. Counting Number Task. Counting numbers pick other counting numbers, x—— x + 1, for x a counting number.

	1	2	3	4	5
Process					
eff?					
Process					
eff?					
task					

X. Locker Task.

a) Same number of people and lockers. Each person is assigned one locker.

b) More lockers than people. Some people are assigned two lockers.

c) More lockers than people. Each person is assigned one locker; some lockers are not used.

d) More people than lockers. Each person is assigned one locker; some people have to share lockers.

	1	2	3	4	5
Xa, People					
Xa, Lockers					
Xb, People					
Xb, Lockers					
Xc, People					
Xc, Lockers					
Xd, People					
Xd, Lockers					
task					