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JOHN BRUCE BENNETT

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VOLTERRA INTEGRAL EQUATIONS AND

FRÉCHET DIFFERENTIALS

APPROVED BY

W. T. Reid

Gene Levy

George M. Ewing

John C. Brixey

T. K. Pan

DISSERTATION COMMITTEE

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VOLTERRA INTEGRAL EQUATIONS AND FRÉCHET DIFFERENTIALS

INTRODUCTION

In Chapter 1 we will be concerned with the nonlinear Volterra integral equation,

$$(E[f,g]) \qquad x(t) = f(t) + \int_a^t g(t,s,x(s))ds,$$

where f, g and x are functions whose definitions will be made precise in the theorems that follow. In particular, variational equations for E[f,g] will be found by computing the Fréchet differential of the solution of E[f,g] with respect to the functions f and g. We will follow the work of Bliss [2], who carried out the same program for ordinary differential equations.

In Chapter 2 we will study certain solutions of the integral equation

$$(E[f,k,g]) \qquad x(t) = f(t) + \int_0^t k(t,s)g(s,x(s))ds$$

which exist on the interval $[0,\infty)$. An implicit function theorem of Hildebrandt and Graves, [8], will be used to establish the existence, uniqueness and Fréchet differentiability of these solutions. As a result, these solutions will possess a stability property with respect to changes in the function f. The Banach and Schauder-Tychonoff fixed point theorems will be used to establish existence and stability of solutions of perturbed equations corresponding to E[f,k,g]. Also, a nonlinear variation of constants formula will be presented.

CHAPTER 1

DIFFERENTIABILITY OF SOLUTIONS OF VOLTERRA INTEGRAL EQUATIONS

1. Notation and conventions. Let R denote the set of real numbers and $R^n = \{x = (x_1, \ldots, x_n);\ x_i \in R \text{ for } i = 1, \ldots, n\}$. The symbol $|a|$ will mean the absolute value of the real number a and if $x \in R^n$, the norm of x is defined to be $||x|| = \Sigma_{i=1}^n |x_i|$. If $A = [a_{ij}]$ is an $n \times n$ matrix with real elements, define the norm of A by $||A|| = \Sigma_{i=1}^n \Sigma_{j=1}^n |a_{ij}|$. If $B = [b_{ij}]$ is an $n \times n$ matrix function defined on a set U with real valued functions as elements, then B(u) denotes the $n \times n$ matrix $[b_{ij}(u)]$. Thus, the meaning of the symbol $||B(u)||$ is $\Sigma_{i=1}^n \Sigma_{j=1}^n |b_{ij}(u)|$. Let $D \subset R \times R^n$ be a domain (connected open set) and represent elements of D by (s,x) where $s \in R$ and $x \in R^n$. Let $I = [a, a + p]$ be a nondegenerate closed interval of real numbers. We consider I and D to be fixed throughout Chapter 1. If $f\colon I \to R^n$ and $g\colon I \times D \to R^n$, denote their respective component functions by f_i and g_i for $i = 1, \ldots, n$. We will use the symbol g_x to represent the $n \times n$ matrix function $[\partial g_i/\partial x_j]$, where i is the row index and j is the column index. Let

(1.1.1) $\quad C(I,R^n) = \{f\colon I \to R^n;\ f \text{ is continuous on } I\}$,

(1.1.2) $\quad C(I \times D,R^n) = \{g\colon I \times D \to R^n;\ g \text{ is continuous on } I \times D\}$

and

(1.1.3) $\quad BC^1(I \times D,R^n) = \{g\colon I \times D \to R^n;\ g_x$ exists and g_i and

$\partial g_i/\partial x_j$ are bounded and continuous on $I \times D\}$.

The sets $C(I,R^n)$ and $BC^1(I \times D,R^n)$ become Banach spaces upon introducing the following norms:

(1.1.4) $$||f|| = \sup\{||f(t)||;\ t \in I\},$$

and

(1.1.5) $$||g||_1 = \sup\{||g(t,s,x)||;\ (t,s,x) \in I \times D\} + \sup\{||g_x(t,s,x)||;\ (t,s,x) \in I \times D\}$$

where $f \in C(I,R^n)$ and $g \in BC^1(I \times D,R^n)$.

If X and Y are Banach spaces and $(x,y) \in X \times Y$, define the norm of (x,y) by $||(x,y)|| = ||x|| + ||y||$, where $||x||$ and $||y||$ are the respective norms of $x \in X$ and $y \in Y$. It is well known that $X \times Y$ is a Banach space with the above definition of the norm. If $F\colon X \to Y$ is a bounded linear operator, denote the norm (uniform) of F by $||F||$.

If x is a member of a Banach space, then $B(x,\delta)$ represents the open ball of radius δ centered at x. Let $\overline{A}$ be the closure of the subset A of a Banach space.

A solution of the integral equation,

(E[f,g]) $$x(t) = f(t) + \int_a^t g(t,s,x(s))ds,$$

is defined to be a continuous function ξ with domain $[a,b]$, $(a < b)$, such that $(t,s,\xi(s))$ is in the domain of g for $a \leq s \leq t \leq b$ and

$$\xi(t) = f(t) + \int_a^t g(t,s,\xi(s))ds.$$

2. Preparatory results. Two theorems of Sato [13] will be stated since they will be used in the sequel.

THEOREM 1.2.1 (Sato). *Let* $f \in C(I,R^n)$ *and* $\Delta(a + p,f,b) = \{(t,s,x);\ x \in R^n,\ a \leq s \leq t \leq a + p$ *and* $||x - f(t)|| \leq b(b > 0)\}$. *Suppose* $g\colon \Delta(a + p,f,b) \to R^n$ *is continuous. Then* E[f,g] *has at least one solution defined on* $[a,a + \alpha]$, *where* $\alpha = \min\{p,b/M\}$ *and*

$$M = \sup\{||g(t,s,x)||;\ (t,s,x) \in \Delta(a+p,f,b)\}.$$

Furthermore, if there exists a non-negative real number L *such that if* (t,s,x) *and* (t,s,y) *belong to* $\Delta(a+p,f,b)$ *then*

(1.2.1) $$||g(t,s,x) - g(t,s,y)|| \leqslant L||x - y||,$$

then $E[f,g]$ *has a unique solution.*

THEOREM 1.2.2 (Sato). *Let* $f \in C(I,R^n)$ *and* $g \in C(I \times D,R^n)$. *If a solution,* ξ, *of* $E[f,g]$ *exists on* $[a,t_0]$ $(a < t_0 < a + p)$, *then there is an* $\varepsilon > 0$ *such that* ξ *is defined on* $[a,t_0 + \varepsilon]$.

LEMMA 1.2.1. *Let* $f^o \in C(I,R^n)$ *and* $g^o \in C(I \times D,R^n)$ *and assume that* $E[f^o,g^o]$ *has a solution,* ξ^o, *defined on* I. *Then there is a* $\delta > 0$ *such that if* (f,g) *belongs to* $B(f^o,\delta) \times C(I \times D,R^n)$, *then* $E[f,g]$ *has a solution.*

Since ξ^o: $I \to R^n$ is a solution of $E[f^o,g^o]$, it follows that $gr(\xi^o) = \{(s,\xi^o(s));\ s \in I\} \subset D$. In particular, $(a,f^o(a)) \in D$. Since D is open, there exists a $\beta > 0$ such that $\overline{B}(a,\beta) \times \overline{B}(f^o(a),\beta) \subset D$. Define a real valued function h on I by $h(t) = \beta/3 - ||f^o(t) - f^o(a)||$ and observe that h is continuous. If h has a zero on I, let t_0 be the smallest one; otherwise, set $t_0 = a + p$. Also, observe that $0 \leqslant h(t) \leqslant \beta/3$ on $[a,t_0)$, so that $||f^o(t) - f^o(a)|| < \beta/3$ on $[a,t_0)$.

Let $T = \min\{t_0 - a,\beta/3\}$. It will be verified that $\Delta(T,f^o,\beta/3) = \{(t,s,x);\ a \leqslant s \leqslant t \leqslant T,\ ||x - f^o(t)|| \leqslant \beta/3\}$ is contained in $I \times D$. If $(t,s,x) \in \Delta(T,f^o,\beta/3)$, then $t \in I$, $|s - a| \leqslant T \leqslant \beta/3$ and $||x - f^o(t)|| \leqslant \beta/3$. It follows that $||x - f^o(a)|| \leqslant ||x - f^o(t)|| + ||f^o(t) - f^o(a)|| \leqslant 2\beta/3 < \beta$. Hence, $(t,s,x) \in I \times \overline{B}(a,\beta) \times \overline{B}(f^o(a),\beta) \subset I \times D$, and so $\Delta(T,f^o,\beta/3) \subset I \times D$.

If $f \in B(f^o,\beta/3)$, then the set $\Delta(T,f,\beta/3)$ is also in $I \times D$. For, as before, if $(t,s,x) \in \Delta(T,f,\beta/3)$, then $t \in I$, $|s - a| \leqslant T \leqslant \beta/3$ and

$||x - f(t)|| \leq \beta/3$, and $||x - f^o(a)|| \leq ||x - f(t)|| + ||f(t) - f^o(t)|| + ||f^o(t) - f^o(a)|| \leq \beta/3 + \beta/3 + \beta/3 = \beta$. Consequently, $(t,s,x) \in I \times \overline{B}(a,\beta) \times \overline{B}(f^o(a),\beta)$ and so $\Delta(T,f,\beta/3) \subset I \times D$.

With $(f,g) \in B(f^o,\beta/3) \times C(I \times D,R^n)$, an application of Theorem 1.2.1 yields the existence of a solution of $E[f,g]$ defined on some subinterval of I. The result follows on setting δ equal to $\beta/3$.

If $g^o \in C(I \times D,R^n)$ and δ is a positive real number, define $V(g^o,\delta)$ as $\{ g \in C(I \times D,R^n);\ \sup \{ ||g(t,s,x) - g^o(t,s,x)||;\ (t,s,x) \in I \times D\} < \delta\}$.

THEOREM 1.2.3. _Let_ $g^o \in C(I \times D,R^n)$ _satisfy_ (1.2.1) _on_ $I \times D$ _with_ $L > 0$. _Assume that_ $f^o \in C(I,R^n)$ _and that_ $E[f^o,g^o]$ _has a solution_, ξ^o, _defined on_ I. _Then for each_ $\varepsilon > 0$ _there is a_ $\delta_\varepsilon > 0$, _such that if_ (f,g) _belongs to_ $B(f^o,\delta_\varepsilon) \times V(g^o,\delta_\varepsilon)$, _then_ $E[f,g]$ _has a solution; and if_ ξ _is a solution of_ $E[f,g]$, _then_ ξ _is defined on_ I _and_ $\xi \in B(\xi^o,\varepsilon) \subset C(I,R^n)$.

Let $\varepsilon > 0$ be fixed. As $gr(\xi^o) \subset D$, the distance between the sets ∂D, the boundary of D, and $gr(\xi^o)$ is positive, provided ∂D is not empty. Let β^o be less than this distance and also less than ε. Define a neighborhood of $gr(\xi^o)$ in D by $N(\xi^o,\beta^o) = \{ (s,x);\ a \leq s \leq a + p,\ ||x - \xi^o(s)|| < \beta^o\}$. If ∂D is empty, then $D = R^{n+1}$, so that in either case there is a β^o, satisfying $0 < \beta^o < \varepsilon$, such that $I \times N(\xi^o,\beta^o) \subset I \times D$.

It follows from Lemma 1.2.1 that there is a δ such that if $(f,g) \in B(f^o,\delta) \times V(g^o,\delta)$, then at least one solution of $E[f,g]$ exists. Let $\delta_\varepsilon = \min \{ \delta, \beta^o L/[2((1 + L)\exp(pL) - 1)]\}$, where L is the Lipschitz constant of (1.2.1), and suppose that (f,g) is fixed in $B(f^o,\delta_\varepsilon) \times V(g^o,\delta_\varepsilon)$. If ξ is a solution of $E[f,g]$, let $T = \sup \{ t \in I;\ \xi \text{ exists on } [a,t]\}$. We will show that ξ is defined at T and that $T = a + p$.

If $T < a + p$, then for $t \in [a,T)$ we have that,

$$\begin{aligned} ||\xi(t) - \xi^o(t)|| &\leq ||f(t) - f^o(t)|| \\ &+ \int_a^t ||g(t,s,\xi(s)) - g^o(t,s,\xi(s))||ds \\ &+ \int_a^t ||g^o(t,s,\xi(s)) - g^o(t,s,\xi^o(s))||ds \\ &\leq \delta_\varepsilon + \delta_\varepsilon(t - a) + L\int_a^t ||\xi(s) - \xi^o(s)||ds. \end{aligned}$$

An application of the Gronwall-Reid inequality yields

(1.2.2) $||\xi(t) - \xi^o(t)|| \leq \delta_\varepsilon[((1 + L)\exp(pL) - 1)/L] \leq \beta^o/2 < \beta^o < \varepsilon$

for $t \in [a,T)$.

Consequently, $gr(\xi)$ is contained in the compact set $\overline{N}(\xi^o,\beta^o/2)$.

It will be established that $\lim_{t \to T-} \xi(t)$ exists. We note that f is uniformly continuous on I and that g is uniformly continuous on $I \times \overline{N}(\xi^o,\beta^o/2)$. Let $\epsilon > 0$ be fixed. There is a δ_ϵ such that if $t_2 < t_1$, $0 < T - t_1 < \delta_\epsilon$ and $0 < T - t_2 < \delta_\epsilon$, then $||f(t_1) - f(t_2)|| < \epsilon/3$ and $||g(t_1,s,\xi(s)) - g(t_2,s,\xi(s))|| < \epsilon/[3(T - a)]$. Also, select $\delta_\epsilon < \epsilon/[2(3M + 1)]$ where $M = \sup\{ ||g(t,s,x)||;\ (t,s,x) \in I \times \overline{N}(\xi^o,\beta^o/2)\}$. Then it follows that,

$$\begin{aligned} ||\xi(t_1) - \xi(t_2)|| &\leq ||f(t_1) - f(t_2)|| \\ &+ \int_a^{t_2} ||g(t_1,s,\xi(s)) - g(t_2,s,\xi(s))||ds \\ &+ \int_{t_2}^{t_1} ||g(t_1,s,\xi(s))||ds. \end{aligned}$$

As a result we have that

(1.2.3) $||\xi(t_1) - \xi(t_2)|| < \epsilon/3 + \epsilon(t - a)/[3(T - a)] + M\epsilon/[3M + 1] < \epsilon.$

Hence, $\lim_{t \to T-} \xi(t)$ exists by Cauchy's condition for convergence.

Let $\xi(T)$ be defined as $\lim_{t \to T-} \xi(t)$ and note that $(T,\xi(T)) \in \overline{N}(\xi^o,\beta^o/2) \subset D$. Also, for $a \leq t < T$, it follows that,

$$\begin{aligned} ||\xi(T) - f(T) - \int_a^T g(T,s,\xi(s))ds|| &\leq ||\xi(T) - \xi(t)|| + ||f(t) - f(T)|| \\ &+ \int_a^t ||g(t,s,\xi(s)) - g(T,s,\xi(s))||ds \\ &+ \int_t^T ||g(T,s,\xi(s))||ds. \end{aligned}$$

That the right hand side of the above inequality is arbitrarily small for $|t - T|$ sufficiently small, follows by the same type of estimates used to obtain (1.2.3). Consequently, ξ is a solution of $E[f,g]$ on $[a,T]$. By Theorem 1.2.2, ξ may be extended to the right of T. This contradicts the definition of T. Hence, $T = a + p$ and so ξ is a solution of $E[f,g]$ on I. Since the inequality (1.2.2) also holds with $t = T$, it follows that $\xi \in B(\xi^o,\varepsilon)$.

In the proof of the above theorem it was shown that if $\eta^o \in C(I,R^n)$ and $gr(\eta^o) \subset D$, then there is a neighborhood $N(\eta^o,\beta)$ of $gr(\eta^o)$ such that $N(\eta^o,\beta) \subset D$. Note that if $\eta \in C(I,R^n)$, then $gr(\eta) \subset N(\eta^o,\beta)$ if and only if $\eta \in B(\eta^o,\beta) \subset C(I,R^n)$. Consequently, if $\eta^o \in C(I,R^n)$ and $gr(\eta^o) \subset D$, then there is a $\beta > 0$ such that if $\eta \in B(\eta^o,\beta)$, then $gr(\eta) \subset D$. In what follows, use will be made of the convexity of $B(\eta^o,\beta)$.

We digress to discuss linear integral equations. Let $\Delta(p) = \{(t,s);\ a \leq s \leq t \leq a + p\}$ and $f \in C(I,R^n)$. If k is a continuous $n \times n$ matrix function defined on $\Delta(p)$, then the equation

(1.2.4) $$x(t) = f(t) + \int_a^t k(t,s)\, x(s)ds$$

has a unique solution on I. It is well known, [5; p. 125], that the solution is given by

(1.2.5) $$\xi(t) = f(t) + \int_a^t r(t,s)\, f(s)ds$$

where r, the reciprocal kernel associated with k, is a continuous $n \times n$ matrix function defined on $\Delta(p)$ and

(1.2.6) $$r(t,s) = \sum_{m=1}^{\infty} k^{(m)}(t,s).$$

The matrices $k^{(m)}(t,s)$ are defined by

$$k^{(1)}(t,s) = k(t,s)$$
$$k^{(m)}(t,s) = \int_s^t k(t,u)k^{(m-1)}(u,s)du$$

for $m = 2,3,\ldots.$ Also well-known is the fact that if $|k(t,s)| \leq M$

on $\Delta(p)$, then

(1.2.7) $$||k^{(m)}(t,s)|| \leq M^m(t - s)^{m-1}/(m - 1)!$$

for $(t,s) \in \Delta(p)$ and $m = 2,3,\ldots.$ The inequality (1.2.7) together with the expression (1.2.6), yields the inequality

(1.2.8) $$||r(t,s)|| \leq M \exp(Mp) \text{ for } (t,s) \in \Delta(p).$$

Perhaps less well-known are the inequalities which will be established in the next lemma.

LEMMA 1.2.2. _If_ k _and_ $\hat{k}$ _are continuous_ $n \times n$ _matrix functions defined on_ $\Delta(p)$ _such that_ $||k(t,s)|| \leq M$, $||\hat{k}(t,s)|| \leq M$ _and_ $||\hat{k}(t,s) - k(t,s)|| \leq \delta$ _on_ $\Delta(p)$, _then_

$$||\hat{k}^{(m)}(t,s) - k^{(m)}(t,s)|| \leq m\delta M^{m-1}(t - s)^{m-1}/(m - 1)!$$

_on_ $\Delta(p)$ _for_ $m = 2,3,\ldots.$ _Furthermore, if_ r _and_ $\hat{r}$ _are the kernels reciprocal to_ k _and_ $\hat{k}$ _respectively, then_ $||\hat{r}(t,s) - r(t,s)|| \leq \delta(1 + Mp) \exp(Mp)$ _on_ $\Delta(p)$.

By hypothesis, $||\hat{k}^{(1)}(t,s) - k^{(1)}(t,s)|| \leq \delta$. If

$$||\hat{k}^{(m)}(t,s) - k^{(m)}(t,s)|| \leq m\delta M^{m-1}(t - s)^{m-1}/(m - 1)!,$$

then

$$\begin{aligned}
&||\hat{k}^{(m+1)}(t,s) - k^{(m+1)}(t,s)|| \\
&\quad\leq \int_s^t ||\hat{k}(t,u)\hat{k}^{(m)}(u,s) - k(t,u)k^{(m)}(u,s)||du \\
&\quad\leq \int_s^t ||\hat{k}(t,u)|| \, ||\hat{k}^{(m)}(u,s) - k^{(m)}(u,s)||du \\
&\quad\quad + \int_s^t ||\hat{k}(t,u) - k(t,u)|| \, ||k^{(m)}(u,s)||du \\
&\quad\leq m\delta M^m/(m - 1)! \int_s^t (u - s)^{m-1}du + \delta M^m/(m - 1)! \int_s^t (u - s)^{m-1} du \\
&\quad\leq (m + 1)\delta M^m(t - s)^m/m!.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
||\hat{r}(t,s) - r(t,s)|| &\leq \Sigma_{m=1}^{\infty} ||\hat{k}^{(m)}(t,s) - k^{(m)}(t,s)|| \\
&\leq \Sigma_{m=1}^{\infty} m\delta M^{m-1}(t - s)^{m-1}/(m - 1)! \\
&\leq (1 + M(t - s)) \exp[M(t - s)]
\end{aligned}$$

$\leq \delta(1 + Mp)\exp(Mp)$.

We recall that if $g^o \in BC^1(I \times D,R^n)$, then $B(g^o,\delta) = \{g \in BC^1(I \times D,R^n);\ ||g - g^o||_1 < \delta\}$ where $BC^1(I \times D,R^n)$ and $||\cdot||_1$ were defined in (1.1.3) and (1.1.5) respectively.

LEMMA 1.2.3. _Let_ (f^o,g^o) _be an element of_ $C(I,R^n) \times BC^1(I \times D,R^n)$ _for which_ ξ^o, _the solution of_ $E[f^o,g^o]$, _is defined on_ I. _Then with each_ $\varepsilon > 0$ _there is associated a_ $\delta_\varepsilon > 0$ _such that if_ $(f,g) \in B(f^o,\delta_\varepsilon) \times B(g^o,\delta_\varepsilon)$, _then_:

(i) _the solution_, ξ, _of_ $E[f,g]$ _is defined on_ I,

(ii) _if_ r^o, r^1 _and_ r^2 _are the reciprocal kernels associated with_ $g^o_x(t,s,\xi^o(s))$, $\int_0^1 g_x(t,s,\xi^o(s) + \alpha(\xi(s) - \xi^o(s)))d\alpha$ _and_ $g_x(t,s,\xi(s))$, _respectively_, _then_:

(a) $||r^1(t,s) - r^o(t,s)|| < \varepsilon$ _for_ $(t,s) \in \Delta(p)$,

(b) $||r^2(t,s) - r^o(t,s)|| < \varepsilon$ _for_ $(t,s) \in \Delta(p)$.

Let $\varepsilon > 0$ be fixed. As in Theorem 1.2.3, there exists a $\beta > 0$ such that $N(\xi^o,2\beta) = \{(s,x);\ t \in I,\ ||x - \xi^o(t)|| < 2\beta\}$ is a subset of D. Also, it follows from Theorem 1.2.3 that there is a $\delta_1 > 0$ such that if $(f,g) \in B(f^o,\delta_1) \times B(g^o,\delta_1)$ then the solution of $E[f,g]$ has its graph in $N(\xi^o,\beta)$. Let $M = \sup\{||g||_1;\ g \in B(g^o,\delta_1)\}$ and set $\varepsilon_1 = \varepsilon/[4(1 + Mp)\exp(Mp)]$.

The function g^o_x is continuous on the compact set $I \times \overline{N}(\xi^o,\beta)$, so there is a $\delta_2 > 0$ such that if (t',s',x') and (t,s,x) belong to $I \times \overline{N}(\xi^o,\beta)$ and $||(t',s',x') - (t,s,x)|| < \delta_2$, then $||g^o_x(t',s',x') - g^o_x(t,s,x)|| < \varepsilon_1$. Once again it follows from Theorem 1.2.3 that there is a $\delta_3 > 0$ such that if $(f,g) \in B(f^o,\delta_3) \times B(g^o,\delta_3)$, then the solution ξ of $E[f,g]$ satisfies $\xi \in B(\xi^o,\delta_2)$. In particular, it is to be noted that if $\xi \in B(\xi^o,\delta_2)$, then $||(t,s,\xi(s)) - (t,s,\xi^o(s))|| < \delta_2$ for $a \leq s \leq t \leq a + p$.

Set $\delta_\varepsilon = \min\{\delta_1, \delta_3, \varepsilon_1\}$, and suppose that $(f,g) \in B(f^o, \delta_\varepsilon) \times B(g^o, \delta_\varepsilon)$. Let ξ be the solution of $E[f,g]$ and denote $\xi - \xi^o$ by $\Delta\xi$. Then

$$||\int_0^1 g_x(t,s,\xi^o(s) + \alpha\Delta\xi(s))d\alpha - g_x^o(t,s,\xi^o(s))||$$
$$\leq \int_0^1 ||g_x(t,s,\xi^o(s) + \alpha\Delta\xi(s)) - g_x^o(t,s,\xi^o(s) + \alpha\Delta\xi(s))||d\alpha$$
$$+ \int_0^1 ||g_x^o(t,s,\xi^o(s) + \alpha\Delta\xi(s)) - g_x^o(t,s,\xi^o(s))||d\alpha,$$

and so

$$||\int_0^1 g_x(t,s,\xi^o(s) + \alpha\Delta\xi(s))d\alpha - g_x^o(t,s,\xi^o(s))|| < \delta_\varepsilon + \varepsilon_1 < 2\varepsilon_1.$$

Also,

$$||g_x(t,s,\xi(s)) - g_x^o(t,s,\xi^o(s))|| \leq ||g_x(t,s,\xi(s)) - g_x^o(t,s,\xi(s))||$$
$$+ ||g_x^o(t,s,\xi(s)) - g_x^o(t,s,\xi^o(s))||,$$

and we have that $||g_x(t,s,\xi(s)) - g_x^o(t,s,\xi^o(s))|| < 2\varepsilon_1$. Since $2\varepsilon_1 < \varepsilon/[2(1 + Mp)\exp(Mp)]$, it follows from Lemma 1.2.2 that $||r^1(t,s) - r^o(t,s)|| < \varepsilon$ and $||r^2(t,s) - r^o(t,s)|| < \varepsilon$ for $(t,s) \in \Delta(p)$.

3. <u>Equations of variation</u>. Recall the definition of the Fréchet differential. Let X and Y be normed linear spaces and let A be an open set containing $x \in X$. A mapping $F\colon A \to Y$ is differentiable at x if there exists a bounded linear operator $dF(x; \cdot)\colon X \to Y$ such that $||F(x + \xi) - F(x) - dF(x; \xi)||/||\xi|| \to 0$ as $||\xi|| \to 0$. The linear operator $dF(x; \cdot)$ is called the Fréchet differential of F at x. If such a linear operator exists, then it is unique.

If $g \in BC^1(I \times D, R^n)$, then the boundedness of $||g_x||$ implies that g satisfies a Lipschitz condition with respect to x. Also, if $(f^o, g^o) \in C(I,R^n) \times BC^1(I \times D,R^n)$ is such that the solution of $E[f^o,g^o]$ is defined on I, then it follows from Theorem 1.2.3 that there is a $\delta > 0$ such that if $(f,g) \in B(f^o,\delta) \times B(g^o,\delta)$, then the solution of $E[f,g]$ is also defined on I. Consequently, a function $F\colon B(f^o,\delta) \times B(g^o,\delta) \to C(I,R^n)$ may be defined by $\xi = F(f,g)$ where ξ

is the unique solution of $E[f,g]$. That ξ exists and is unique follows from Theorem 1.2.1.

THEOREM 1.3.1. Let $(f^o,g^o) \in C(I,R^n) \times BC^1(I \times D,R^n)$ be an element for which the solution ξ^o of $E[f^o,g^o]$ exists on I. Then F is Fréchet differentiable at (f^o,g^o).

There exists a $\beta > 0$ such that any element of $B(\xi^o,\beta)$ has graph in D. There exists a $\delta_1 > 0$ such that if $(f,g) \in B(f^o,\delta_1) \times B(g^o,\delta_1)$, then $F(f,g)$, the solution of $E[f,g]$, is in $B(\xi^o,\beta)$.

For $(f,g) \in B(f^o,\delta_1) \times B(g^o,\delta_1)$, define the difference function $\Delta\xi$ by $\Delta\xi = \xi - \xi^o = F(f,g) - F(f^o,g^o)$. Then,

$$\begin{aligned}\Delta\xi(t) &= f(t) - f^o(t) + \int_a^t g(t,s,\xi(s))ds - \int_a^t g^o(t,s,\xi^o(s))ds \\ &= f(t) - f^o(t) + \int_a^t [g(t,s,\xi^o(s)) - g^o(t,s,\xi^o(s))]ds \\ &\quad + \int_a^t [g(t,s,\xi(s)) - g(t,s,\xi^o(s))]ds.\end{aligned}$$

Making use of Taylor's theorem with integral form of the remainder, and denoting $f - f^o$ by Δf and $g - g^o$ by Δg, we have that

$$\begin{aligned}\Delta\xi(t) &= \Delta f(t) + \int_a^t \Delta g(t,s,\xi^o(s))ds \\ &\quad + \int_a^t \left(\int_0^1 g_x(t,s,\xi^o(s) + \alpha\Delta\xi(s))d\alpha\right)\Delta\xi(s)ds.\end{aligned}$$

It follows from (1.2.5) that

$$\begin{aligned}\Delta\xi(t) &= \Delta f(t) + \int_a^t \Delta g(t,s,\xi^o(s))ds + \int_a^t r(t,s)f(s)ds \\ &\quad + \int_a^t r(t,s)\left(\int_a^t \Delta g(s,u,\xi^o(u))du\right)ds\end{aligned}$$

where $r(t,s)$ is the reciprocal kernel associated with $\int_0^1 g_x(t,s,\xi^o(s) + \alpha\Delta\xi(s))d\alpha$. The equation

$$x(t) = \Delta f(t) + \int_a^t \Delta g(t,s,\xi^o(s))ds + \int_a^t g_x^o(t,s,\xi^o(s))x(s)ds$$

is linear, so it has a solution

$$\begin{aligned}dF(f^o,g^o;\ \Delta f,\Delta g)(t) &= \Delta f(t) + \int_a^t \Delta g(t,s,\xi^o(s))ds \\ &\quad + \int_a^t r^o(t,s)\Delta f(s)ds \\ &\quad + \int_a^t r^o(t,s)\left(\int_a^s \Delta g(s,u,\xi^o(u))du\right)ds\end{aligned}$$

where $r^o(t,s)$ is the reciprocal kernel for the kernel $g_x^o(t,s,\xi^o(s))$.

The difference $||\Delta\xi(t) - dF(f^o,g^o;\ \Delta f,\Delta g)(t)||$ may be bounded as follows:

$$\begin{aligned}
&||\Delta\xi(t) - dF(f^o,g^o;\ \Delta f,\Delta g)(t)|| \\
&\qquad \leqslant \int_a^t ||r(t,s) - r^o(t,s)||\ ||\Delta f(s)||ds \\
&\qquad + \int_a^t ||r(t,s) - r^o(t,s)||\left(\int_a^s ||\Delta g(s,u,\xi^o(u))||du\right)ds \\
&\qquad \leqslant ||(\Delta f,\Delta g)||\int_a^t ||r(t,s) - r^o(t,s)||(1 + (s - a))ds \\
&\qquad \leqslant ||(\Delta f,\Delta g)||(1 + p)\int_a^t ||r(t,s) - r^o(t,s)||ds.
\end{aligned}$$

For fixed $\varepsilon > 0$, we will establish the existence of a $\delta_\varepsilon > 0$ such that, if $(f,g) \in B(f^o,\delta_\varepsilon) \times B(g^o,\delta_\varepsilon)$, then $\sup\{||\Delta\xi(t) - dF(f^o,g^o;\ \Delta f,\Delta g)(t)||;\ t \in I\} < \varepsilon||(\Delta f,\Delta g)||$. It follows from Lemma 1.2.3 that there is a $\delta_\varepsilon > 0$ satisfying $\delta_\varepsilon < \delta_1$, and such that if $(f,g) \in B(f^o,\delta_\varepsilon) \times B(g^o,\delta_\varepsilon)$, then $||r(t,s) - r^o(t,s)|| <$ $\varepsilon/[2p(1 + p)]$ for $a \leqslant s \leqslant t \leqslant a + p$. So, if $(f,g) \in B(f^o,\delta_\varepsilon) \times B(g^o,\delta_\varepsilon)$, then $||\Delta\xi(t) - dF(f^o,g^o;\ \Delta f,\Delta g)(t)|| < \varepsilon||(\Delta f,\Delta g)||/2$. It follows that $||\Delta\xi - dF(f^o,g^o;\ \Delta f,\Delta g)|| = \sup\{||\Delta\xi(t) - dF(f^o,g^o;\ \Delta f,\Delta g)(t)||;$ $t \in I\} < \varepsilon||(\Delta f,\Delta g)||$.

It is clear that if $(\phi,\gamma) \in C(I,R^n) \times BC^1(I \times D,R^n)$, then

$$\begin{aligned}
(1.3.1)\qquad dF(f^o,g^o;\ \phi,\gamma)(t) &= \phi(t) + \int_a^t \gamma(t,s,\xi^o(s))ds + \int_a^t r^o(t,s)\phi(s)ds \\
&\quad + \int_a^t r^o(t,s)\left(\int_a^s \gamma(s,u,\xi^o(u))du\right)ds
\end{aligned}$$

is an element of $C(I,R^n)$. Consequently, a mapping

$$dF(f^o,g^o;\ \cdot,\cdot):\ C(I,R^n) \times BC^1(I \times D,R^n) \to C(I,R^n)$$

is defined by (1.3.1).

It is easily verified that $dF(f^o,g^o;\ \cdot,\cdot)$ is linear on $C(I,R^n) \times BC^1(I \times D,R^n)$. The boundedness of $dF(f^o,g^o;\ \cdot,\cdot)$ as a linear operator will now be established. If $(\phi,\gamma) \in C(I,R^n) \times BC^1(I \times D,R^n)$, then

$$\begin{aligned}||dF(f^o,g^o;\ \phi,\gamma)(t)|| &\leq ||\phi(t)|| + \int_a^t||\gamma(t,s,\xi^o(s))||ds \\ &+ \int_a^t||r^o(t,s)||\ ||\phi(s)||ds \\ &+ \int_a^t||r^o(t,s)||(\int_a^s||\gamma(s,u,\xi^o(u))||du)ds \\ &\leq ||(\phi,\gamma)||[1 + (t - a) + \int_a^t||r^o(t,s)||ds \\ &+ \int_a^t||r^o(t,s)||(s - a)ds].\end{aligned}$$

It follows from (1.2.8) that $||r^o(t,s)|| \leq M \exp(Mp)$ where M is an upper bound for $||g_x^o(t,s,x)||$ on $I \times D$. Moreover,

$||dF(f^o,g^o;\ \phi,\gamma)(t)|| \leq (1 + p + Mp \exp(Mp) + Mp^2 \exp(Mp))||(\phi,\gamma)||$

for $t \in I$, and hence,

$\sup\{||dF(f^o,g^o;\ \phi,\gamma)(t)||;\ t \in I\} \leq ((1 + p)(1 + Mp \exp(Mp))||(\phi,\gamma)||$.

As (ϕ,γ) is arbitrary in $C(I,R^n) \times BC^1(I \times D,R^n)$, it follows that $dF(f^o,g^o;\ \cdot,\cdot)$ is a bounded linear operator and thus it is the Fréchet differential of F at (f^o,g^o).

As was brought out in Theorem 1.3.1 above, $dF(f^o,g^o;\ \phi,\gamma)$ is the solution of

(1.3.2) $\quad x(t) = \phi(t) + \int_a^t\gamma(t,s,\xi^o(s))ds + \int_a^t g_x^o(t,s,\xi^o(s))x(s)ds$

where $\xi^o = F(f^o,g^o)$. Consequently, equation (1.3.2) is termed the <u>variational equation</u> associated with $E[f^o,g^o]$.

Since F is differentiable at (f^o,g^o), it is well known [9; p. 153] that F has two partial differentials,

$$d_1F(f^o,g^o;\ \phi)(t) = \phi(t) + \int_a^t r^o(t,s)\phi(s)ds$$

and

$$d_2F(f^o,g^o;\ \gamma)(t) = \int_a^t\gamma(t,s,\xi^o(s))ds + \int_a^t r^o(t,s)(\int_a^s\gamma(s,u,\xi^o(u))du)ds.$$

Moreover, the differentials $d_1F(f^o,g^o;\ \phi)$ and $d_2F(f^o,g^o;\ \gamma)$ are solutions of the respective equations

(1.3.3) $\qquad x(t) = \phi(t) + \int_a^t g_x^o(t,s,\xi^o(s))x(s)ds$

and

$$(1.3.4)\qquad x(t) = \int_a^t \gamma(t,s,\xi^o(s))ds + \int_a^t g_x^o(t,s,\xi^o(s))x(s)ds.$$

Equation (1.3.3) is the variational equation associated with $E[f^o,g^o]$ when only f is allowed to vary, and equation (1.3.4) is the variational equation associated with $E[f^o,g^o]$ when g alone varies.

THEOREM 1.3.2. _Let_ (f^o,g^o) _be an element of_ $C(I,R^n) \times BC^1(I \times D,R^n)$ _for which_ ξ^o, _the solution of_ $E[f^o,g^o]$, _is defined on_ I. _Then_ dF _is continuous at_ (f^o,g^o).

It follows from Theorem 1.2.3 that there is a $\beta > 0$ such that if $(f,g) \in B(f^o,\beta) \times B(g^o,\beta)$, then the solution of $E[f,g]$ exists on I. By Theorem 1.3.1, the function F is defined and differentiable on $B(f^o,\beta) \times B(g^o,\beta)$.

Suppose $\varepsilon > 0$ is fixed and for $M = \sup\{||g||_1;\ g \in B(g^o,\beta)\}$ let $\varepsilon_1 = \varepsilon/[2(2p + p^2(1 + Mp)\exp(Mp))]$. Also, let $\varepsilon_2 > 0$ be a number for which any element of $B(\xi^o,\varepsilon_2)$ has graph in D and choose $\varepsilon_3 = \min\{\varepsilon_1,\varepsilon_2\}$.

It follows from Theorem 1.2.3 and Lemma 1.2.3 that there exists a $\delta_\varepsilon > 0$ such that if $(f,g) \in B(f^o,\delta_\varepsilon) \times B(g^o,\delta_\varepsilon)$, then $\xi = F(f,g)$ belongs to $B(\xi^o,\varepsilon_3)$ and $||r(t,s) - r^o(t,s)|| < \varepsilon_1$ where $r(t,s)$ and $r^o(t,s)$ are the kernels reciprocal to $g_x(t,s,\xi(s))$ and $g_x^o(t,s,\xi^o(s))$, respectively.

Let $(\phi,\gamma) \in C(I,R^n) \times BC^1(I \times D,R^n)$ and $(f,g) \in B(f^o,\delta_\varepsilon) \times B(g^o,\delta_\varepsilon)$. Then, in view of (1.3.1)

$$\begin{aligned}
&||dF(f,g;\ \phi,\gamma)(t) - dF(f^o,g^o;\ \phi,\gamma)(t)|| \\
&\quad\leq ||\int_a^t(\gamma(t,s,\xi(s)) - \gamma(t,s,\xi^o(s)))ds \\
&\quad+ \int_a^t(r(t,s) - r^o(t,s))\phi(s)ds \\
&\quad+ \int_a^t(r(t,s) - r^o(t,s))\int_a^s\gamma(s,u,\xi(u))duds \\
&\quad+ \int_a^t r^o(t,s)\int_a^s(\gamma(s,u,\xi(u)) - \gamma(s,u,\xi^o(u)))duds||
\end{aligned}$$

$$\leq \int_a^t (\int_0^1 ||\gamma_x(t,s,\xi^o(s) + \alpha\Delta\xi(s))||d\alpha)||\Delta\xi(s)||ds$$
$$+ \int_a^t ||r(t,s) - r^o(t,s)|| \; ||\phi(s)||ds$$
$$+ \int_a^t ||r(t,s) - r^o(t,s)||\int_a^s ||\gamma(s,u,\xi(u))||duds$$
$$+ [\int_a^t ||r^o(t,s)||\int_a^s (\int_0^1 ||\gamma_x(s,u,\xi^o(u) + \alpha\Delta\xi(u))||d\alpha)$$
$$||\Delta\xi(u)||duds].$$

It follows from (1.2.8) that $||r^o(t,s)|| \leq M\exp(Mp)$. Consequently,

$$||dF(f,g;\ \phi,\gamma)(t) - dF(f^o,g^o;\ \phi,\gamma)(t)||$$
$$\leq ||\gamma||_1||\xi - \xi^o||p + ||\phi||\varepsilon_1 p + ||\gamma||_1\varepsilon_1 p^2$$
$$+ ||\gamma||_1 M\exp(Mp)||\xi - \xi^o||p^2$$
$$\leq ||(\phi,\gamma)||(\varepsilon_3 p + \varepsilon_1 p + \varepsilon_1 p^2 + \varepsilon_3 p^2 M\exp(Mp))$$
$$\leq ||(\phi,\gamma)||(2p + p^2(1 + M\exp(Mp))\varepsilon_1.$$

In view of the definition of ε_1 we have that

$$||dF(f,g;\ \phi,\gamma) - dF(f^o,g^o;\ \phi,\gamma)|| \leq \varepsilon||(\phi,\gamma)||/2.$$

Since (ϕ,γ) is arbitrary in $C(I,R^n) \times BC^1(I \times D,R^n)$, it follows that $||dF(f,g;\ \cdot,\cdot) - dF(f^o,g^o;\ \cdot,\cdot)|| < \varepsilon$. Hence, dF is continuous at (f^o,g^o).

CHAPTER 2

STABILITY PROPERTIES OF A VOLTERRA INTEGRAL EQUATION

1. Notation and conventions. Some further notation will be introduced although the notation of Chapter 1 remains in force unless there is a specific statement to the contrary. If $R^+ = [0,\infty)$, let

$$BC(R^+) = \{ f: \; R^+ \to R^n; \; f \text{ is bounded and continuous on } R^+\}.$$

The function defined by

$$||x|| = \sup \{||x(t)||; \; t \in R^+\}$$

establishes a norm for $BC(R^+)$ and $(BC(R^+),||\cdot||)$ is a Banach space. In the sequel we will use $BC(R^+)$ to denote $(BC(R^+),||\cdot||)$.

Denote $\{ (t,x) \in R \times R^n; \; 0 \leq t < \infty \text{ and } ||x|| < q\}$ by $D(q)$, where q is a positive real number. Let $B(0,q)$ be the ball of radius q centered at 0 in $BC(R^+)$. The ball of radius q_m centered at 0 in R^m is denoted by $B(0,q_m)$. We observe that if $x \in B(0,q)$, then $gr(x)$ is in $D(q)$. Let h be a function defined on $D(q) \times B(0,q_m)$ with range in R^n. If the component functions of h are h_i, let h_x denote the $n \times n$ matrix $[\partial h_i/\partial x_j]$ of partial derivatives of the h_i with respect to the components x_j of $x \in R^n$. Let h_c denote the $n \times m$ matrix $[\partial h_i/\partial c_s]$ of partial derivatives of the h_i with respect to the components c_s of $c \in R^m$. The $n \times (n + m)$ matrix $[h_{ix_j}, h_{ic_s}]$ whose first n columns consist of h_x and whose last m columns are h_c is denoted by h_z.

Assume that h satisfies the following:

(A_1^o) h *is continuous on* $D(q) \times B(0,q_m)$ *and* $h(t,0,0) = 0$ *for* $t \in R^+$;

(A_2^o) *the component functions of* h *have continuous first order partial derivatives with respect to the components of* $x \in R^n$ *and* $c \in R^m$, *and the* $n \times (n + m)$ *matrix* $||h_z(t,x,c)||$ *is bounded above by* P^o *for* $(t,x,c) \in D(q) \times B(0,q_m)$;

(A_3^o) *for each* $\varepsilon > 0$ *there is a* $\delta_\varepsilon > 0$ *such that if* (t,x,c) *and* (t',x',c') *belong to* $D(q) \times B(0,q_m)$ *and* $||x - x'|| < \delta_\varepsilon$ *and* $||c - c'|| < \delta_\varepsilon$, *then* $||h_z(t,x,c) - h_z(t',x',c')|| < \varepsilon$.

Let $\Delta = \{ (t,s) \in R^2;\ 0 \leq s \leq t < \infty\}$ and for $b > 0$ define $\Delta(b)$ as $\{(t,s) \in R^2;\ 0 \leq s \leq t \leq b\}$. Suppose that k is a continuous $n \times n$ matrix function defined on Δ with the property that there exists an $M > 0$ such that

$$\int_0^t ||k(t,s)||ds \leq M, \text{ for } t \in R^+. \tag{2.1.1}$$

We will denote the integral equation

$$x(t) = f(t) + \int_0^t k(t,s)h(s,x(s),c)ds$$

by $E[f,k,h]$.

2. *Existence and differentiability of solutions of* $E[f,k,h]$. With f, x and c denoting arbitrary elements of $BC(R^+)$, $B(0,q)$ and $B(0,q_m)$ respectively, define a function G^o: $BC(R^+) \times B(0,q_m) \times B(0,q) \to BC(R^+)$ by

$$G^o(f,c,x)(t) = f(t) + \int_0^t k(t,s)h(s,x(s),c)ds - x(t). \tag{2.2.1}$$

Since f, k, h and x are continuous, $G^o(f,c,x)$ is a continuous function of t. Also,

$$\int_0^t k(t,s)h(s,x(s),c)ds = \int_0^t k(t,s)[h(s,x(s),c) - h(s,0,0)]ds$$
$$= \int_0^t \left(\int_0^1 k(t,s)h_z(s,\alpha x(s),\alpha c)d\alpha\right)z(s)ds$$

where Taylor's theorem with remainder has been applied and z(s) is the transpose of the n + m dimensional row vector $[x(s),c]$. In view of hypothesis (A_2^o) and the inequality (2.1.1) we obtain

$$\begin{aligned}||G^o(f,c,x)(t)|| \leq\ & ||f(t)|| \\ & + \int_0^t (\int_0^1 ||k(t,s)||\ ||h_z(s,\alpha x(s),\alpha c)||d\alpha)||z(s)||ds \\ & + ||x(t)|| \\ \leq\ & ||f|| + P^o(||c|| + ||x||)\int_0^t ||k(t,s)||ds + ||x|| \\ \leq\ & ||f|| + P^o M(||c|| + ||x||) + ||x||.\end{aligned}$$

That is, $||G^o(f,c,x)(t)||$ is bounded, so that $G^o(f,c,x) \in BC(R^+)$.

In [1; p. 517], R. Bellman established a result which contains the following theorem as a special case.

THEOREM 2.2.1. _A necessary and sufficient condition for_ $\int_0^t k(t,s)x(s)ds \in BC(R^+)$ _whenever_ $x \in BC(R^+)$ _is that the inequality_ (2.1.1) _be satisfied_.

LEMMA 2.2.1. _The partial differential of_ G^o _with respect to the first variable exists and is continuous on_ $BC(R^+) \times B(0,q_m) \times B(0,q)$.

With $(f,c,x) \in BC(R^+) \times B(0,q_m) \times B(0,q)$ and $\phi \in BC(R^+)$, we have that $G^o(f + \phi,c,x) - G^o(f,c,x) = \phi(t)$, and hence

(2.2.2) $$d_1G^o(f,c,x;\ \phi) = \phi.$$

Since (f,c,x) is arbitrary in $BC(R^+) \times B(0,q_m) \times B(0,q)$ and d_1G^o is the identity map on $BC(R^+)$, the result follows.

LEMMA 2.2.2. _The partial differential of_ G^o _with respect to the second variable exists and is continuous on_ $BC(R^+) \times B(0,q_m) \times B(0,q)$.

If $(f,c,x) \in BC(R^+) \times B(0,q_m) \times B(0,q)$ and $\lambda \in R^m$ are such that $c + \lambda \in B(0,q_m)$, then

$$\begin{aligned}& G^o(f,c + \lambda,x) - G^o(f,c,x) \\ & \qquad = \int_0^t (\int_0^1 k(t,s)h_c(s,x(s),\alpha\lambda)d\alpha)ds\lambda.\end{aligned}$$

We proceed to show that

(2.2.3) $$d_2G^o(f,c,x;\ \lambda)(t) = \int_0^t k(t,s)h_c(s,x(s),c)ds\lambda.$$

Let $\varepsilon > 0$ be fixed. From hypothesis (A_3^o), there is a $\delta_\varepsilon > 0$ such that if (t,u,c) and (t,u,c') belong to $D(q) \times B(0,q_m)$ and $||c - c'|| < \delta_\varepsilon$, then $||h_c(t,u,c) - h_c(t,u,c')|| < \varepsilon/(2M)$, where M is defined by (2.1.1). Let $\lambda \in B(0,q_m)$ be such that $c + \lambda \in B(0,q_m)$ and $0 < ||\lambda|| < \delta_\varepsilon$. Then $||h_c(s,x(s),c + \alpha\lambda) - h_c(s,x(s),c)|| < \varepsilon/(2M)$ for each $\alpha \in [0,1]$. It follows that

$$\begin{aligned} &||G^o(f,c + \lambda,x)(t) - G^o(f,c,x)(t) - d_2G^o(f,c,x;\lambda)(t)|| \\ &\leq \int_0^t(\int_0^1||k(t,s)||\ ||h_c(s,x(s),c + \alpha\lambda) - h_c(s,x(s),c)||d\alpha)ds\lambda \\ &\leq ||\lambda||\varepsilon/2. \end{aligned}$$

Hence,

$$||G^o(f,c + \lambda,x) - G^o(f,c,x) - d_2G^o(f,c,x;\ \lambda)||/||\lambda|| < \varepsilon.$$

With λ an arbitrary element of R^m, it is clear that $d_2G^o(f,c,x;\ \lambda)$ is linear in λ. Also,

$$||d_2G^o(f,c,x;\ \lambda)(t)|| \leq \int_0^t||k(t,s)||\ ||h_c(s,x(s),c)\lambda||ds,$$

and in view of (2.2.1) and the fact that $h_c(t,x(t),c)\lambda \in BC(R^+)$ for each $\lambda \in R^m$, it follows from the above Theorem 2.2.1 that $d_2G^o(f,c,x;\ \cdot)$ is a bounded linear map of R^m into $BC(R^+)$. Hence, $d_2G^o(f,c,x;\ \cdot)$ is the required partial differential.

In order to establish the continuity of d_2G^o, let $\varepsilon > 0$ be fixed and select a $\delta_\varepsilon > 0$ such that if (t,u,c) and (t,u',c') are elements of $D(q) \times B(0,q_m)$, while $||u - u'|| < \delta_\varepsilon$ and $||c - c'|| < \delta_\varepsilon$, then $||h_c(t,u,c) - h_c(t,u',c')|| < \varepsilon/(2M)$. Let (f,c,x) and (f',c',x') belong to $BC(R^+) \times B(0,q_m) \times B(0,q)$ and be such that $||(f,c,x) - (f',c',x')|| < \delta_\varepsilon$. Then, with $\lambda \in R^m$ we have that

$$\begin{aligned} &||d_2G^o(f,c,x;\ \lambda)(t) - d_2G^o(f',c',x';\ \lambda)(t)|| \\ &\leq \int_0^t||k(t,s)||\ ||h_c(s,x(s),c) - h_c(s,x'(s),c)||ds||\lambda|| \\ &\leq ||\lambda||\varepsilon/2. \end{aligned}$$

Consequently, $||d_2G^o(f,c,x;\ \lambda) - d_2G^o(f',c',x';\ \lambda)|| \leqslant \varepsilon||\lambda||/2$, and since λ is an arbitrary element of R^m, it follows that $||d_2G^o(f,c,x;\ \cdot) - d_2G^o(f',c',x';\ \cdot)|| < \varepsilon$, and hence, d_2G^o is continuous on $BC(R^+) \times B(0,q_m) \times B(0,q)$.

LEMMA 2.2.3. <u>The partial differential of</u> G^o <u>with respect to the third variable exists and is continuous on</u> $BC(R^+) \times B(0,q_m) \times B(0,q)$.

If $(f,c,x) \in BC(R^+) \times B(0,q_m) \times B(0,q)$ and $\xi \in B(0,q)$ are such that $x + \xi \in B(0,q)$, then

$$G^o(f,c,x+\xi)(t) - G^o(f,c,x)(t)$$
$$= \int_0^t\left(\int_0^1 k(t,s)h_x(s,x(s) + \alpha\xi(s),c)d\alpha\right)\xi(s)ds - \xi(t).$$

We shall proceed to establish that

$$(2.2.4) \qquad d_3G^o(f,c,x;\ \xi)(t) = \int_0^t k(t,s)h_x(s,x(s),c)\xi(s)ds - \xi(t).$$

Let $\varepsilon > 0$ be fixed. From hypothesis (A_3^o), there exists a $\delta_\varepsilon > 0$ such that if (t,u,c) and (t,u',c) belong to $D(q) \times B(0,q_m)$ and $||u - u'|| < \delta_\varepsilon$, then $||h_x(t,u,c) - h_x(t,u',c)|| < \varepsilon/(2M)$. In particular, if $\xi \in B(0,q)$ with $x + \xi \in B(0,q)$ and $0 < ||\xi|| < \delta_\varepsilon$, then $||h_x(s,x(s) + \alpha\xi(s),c) - h_x(s,x(s),c)|| < \varepsilon/(2M)$ for $\alpha \in [0,1]$. It follows that if $d_3G^o(f,c,x;\ \xi)$ is defined by (2.2.4), then

$$||G^o(f,c,x+\xi)(t) - G^o(f,c,x)(t) - d_3G^o(f,c,x;\ \xi)(t)||$$
$$\leqslant \int_0^t\left(\int_0^1 ||k(t,s)||\ ||h_x(s,x(s) + \alpha\xi(s),c) - h_x(s,x(s),c)||d\alpha\right)||\xi(s)||ds$$
$$\leqslant \varepsilon||\xi||/2.$$

Hence, we have that

$$||G^o(f,c,x+\xi) - G^o(f,c,x) - d_3G^o(f,c,x;\ \xi)||/||\xi|| < \varepsilon.$$

With $\xi \in BC(R^+)$, it is clear that $d_3G^o(f,c,x;\ \xi)$ is linear in ξ. Also, we have the inequality

$$||d_3G^o(f,c,x;\ \xi)(t)|| \leqslant \int_0^t ||k(t,s)h_x(s,x(s),c)||\ ||\xi(s)||ds,$$

and since $\int_0^t ||k(t,s)h_x(s,x(s),c)||ds \leq MP^o$ it follows from the above Theorem 2.2.1 that $d_3G^o(f,c,x;\ \cdot)$ is a bounded linear transformation on $BC(R^+)$. Hence, $d_3G^o(f,c,x;\ \cdot)$ is the required partial differential.

An argument similar to the one given in Lemma 2.2.2 serves to establish the continuity of d_3G^o as a function of (f,c,x).

THEOREM 2.2.2. _Suppose that hypotheses_ (A_1^o), (A_2^o), (A_3^o) _and_ (2.1.1) _are satisfied and that_ G^o _is the function defined by_ (2.2.1). _Then_ G^o _is continuously differentiable on_ $BC(R^+) \times B(0,q_m) \times B(0,q)$. _Furthermore, if_ $(f,c,x) \in BC(R^+) \times B(0,q_m) \times B(0,q)$ _and_ $(\phi,\lambda,\xi) \in BC(R^+) \times R^m \times BC(R^+)$, _then_

$$dG^o(f,c,x;\ \phi,\lambda,\xi)(t) = \phi(t) + (\int_0^t k(t,s)h_c(s,x(s),c)ds)\lambda$$
$$+ \int_0^t k(t,s)h_x(s,x(s),c)\xi(s)ds - \xi(t).$$

From Lemmas 2.2.1, 2.2.2 and 2.2.3, we have that $d_1G^o(f,c,x;\ \phi)$, $d_2G^o(f,c,x;\ \lambda)$ and $d_3G^o(f,c,x;\ \xi)$ exist and are continuous on $BC(R^+) \times B(0,q_m) \times B(0,q)$. It follows from a well known theorem [9; p. 154] that G^o is continuously differentiable on $BC(R^+) \times B(0,q_m) \times B(0,q)$ and

$$dG^o(f,c,x;\ \phi,\lambda,\xi) = d_1G^o(f,c,x;\ \phi) + d_2G^o(f,c,x;\ \lambda)$$
$$+ d_3G^o(f,c,x;\ \xi).$$

LEMMA 2.2.4. _Let the hypotheses of Theorem_ 2.2.2 _be satisfied. Suppose there exists a real number_ $M_{r^o} \geq 0$ _such that_ $r^o(t,s)$, _the reciprocal kernel associated with_ $k(t,s)h_x(s,0,0)$, _satisfies_

$$\int_0^t ||r^o(t,s)||ds \leq M_{r^o} \text{ for } t \in R^+.$$

Then, $d_3G^o(0,0,0;\ \cdot)$ _is a linear homeomorphism of_ $BC(R^+)$ _onto_ $BC(R^+)$.

For each $f \in BC(R^+)$ the linear equation

$$\xi(t) = -f(t) + \int_0^t k(t,s)h_x(s,0,0)\xi(s)ds \tag{2.2.5}$$

has exactly one continuous solution which is given by

$$\xi(t) = -f(t) - \int_0^t r^o(t,s)\xi(s)ds.$$

Theorem 2.2.1 shows that $\xi \in BC(R^+)$ and since (2.2.5) is equivalent to $d_3G^o(0,0,0;\ \xi) = f$, it follows that $d_3G^o(0,0,0;\ \cdot)$ is bijective.

It was established in Lemma 2.2.3 that $d_3G^o(0,0,0;\ \cdot)$ is bounded and linear. Since $BC(R^+)$ is a Banach space, it is well known [14; p. 180] that $(d_3G^o(0,0,0;\ \cdot))^{-1}$ exists and is a bounded linear operator on $BC(R^+)$. Therefore, $d_3G^o(0,0,0;\ \cdot)$ is a linear homeomorphism of $BC(R^+)$ onto $BC(R^+)$.

THEOREM 2.2.3. Let the hypotheses of Lemma 2.2.4 be satisfied. Then there exist open balls $B((0,0),\hat{r}_1) \subset BC(R^+) \times B(0,q_m)$ and $B(0,\hat{r}_2) \subset BC(R^+)$ and a function F^o: $B((0,0),\hat{r}_1) \to B(0,\hat{r}_2)$ with the following properties:

(i) the point $((f,c),\ F^o(f,c)) \in B((0,0),\hat{r}_1) \times B(0,\hat{r}_2)$ is a solution of $G^o(f,c,x) = 0$ for each $(f,c) \in B((0,0),\hat{r}_1)$, and there is no other solution with the same (f,c) in $B(0,\hat{r}_2)$;

(ii) the partial differential $d_3G^o(f,c,F^o(f,c);\ \cdot)$ is invertible for each $(f,c) \in B((0,0),\hat{r}_1)$;

(iii) F^o is continuously differentiable on $B((0,0),\hat{r}_1)$.

The proof is immediate for we observe that $G^o(0,0,0) = 0$ and that the hypotheses of the implicit function theorem of Hildebrandt and Graves [8; p. 150] are satisfied.

THEOREM 2.2.4. Assume that the hypotheses of Theorem 2.2.3 are satisfied and that F^o: $B((0,0),\hat{r}_1) \to B(0,\hat{r}_2)$ is the corresponding implicit function. Also, suppose $(f,c) \in B((0,0),\hat{r}_1)$ and $x \in B(0,\hat{r}_2)$ satisfy $x = F^o(f,c)$. Then,

$$\begin{aligned}(2.2.6)\qquad dF^o(f,c,;\ \phi,\lambda)(t) &= \phi(t) + \int_0^t r^o(t,s,c)\phi(s)ds \\ &+ \left(\int_0^t k(t,s)h_c(s,x(s),c)ds\right)\lambda \\ &+ \left(\int_0^t r^o(t,s,c)\left[\int_0^s k(s,u)h_c(u,x(u),c)du\right]ds\right)\lambda\end{aligned}$$

where $r^o(t,s,c)$ *is the kernel reciprocal to* $k(t,s)h_x(s,x(s),c)$. *Consequently*, $dF^o(f,c;\ \phi,\lambda)$ *is the solution of the variational equation*

$$\xi(t) = \phi(t) + \int_0^t k(t,s)h_c(s,x(s),c)ds\lambda + \int_0^t k(t,s)h_x(s,x(s),c)\xi(s)ds. \tag{2.2.7}$$

By Lemma 2.2.4, $(d_3G^o(f,c,x;\ \cdot))^{-1}$ exists and is a bounded linear transformation on $BC(R^+)$. If $d_3G^o(f,c,x;\ \xi) = \phi$, then

$$\xi(t) = -\phi(t) + \int_0^t k(t,s)h_x(s,x(s),c)\xi(s)ds.$$

The solution of this linear equation is

$$\xi(t) = -\phi(t) - \int_0^t r^o(t,s,c)\phi(s)ds,$$

so

$$(d_3G^o(f,c,x;\ \cdot))^{-1}(\phi)(t) = -(\phi(t) + \int_0^t r^o(t,s,c)\phi(s)ds).$$

Using the chain rule, we obtain

$$\begin{aligned} dF^o(f,c;\ \phi,\lambda)(t) &= [-(d_3G^o(f,c,x;\ \cdot))^{-1}\circ (d_1G^o(f,c,x;\ \cdot) + d_2G^o(f,c,x;\ \cdot))](\phi,\lambda)(t) \\ &= [-(d_3G^o(f,c,x;\ \cdot))^{-1}(d_1G^o(f,c,x;\ \phi) + d_2G^o(f,c,x;\ \lambda))](t) \\ &= [-(d_3G^o(f,c,x;\ \cdot))^{-1}(\phi + \int_0^{\cdot} k(\cdot,s)h_c(s,x(s),c)ds\lambda)](t) \\ &= \phi(t) + \int_0^t r^o(t,s,c)\phi(s)ds \\ &\quad + [\int_0^t k(t,s)h_c(s,x(s),c)ds]\lambda \\ &\quad + [\int_0^t r^o(t,s,c) (\int_0^s k(s,u)h_c(u,x(u),c)du)ds]\lambda. \end{aligned} \tag{2.2.8}$$

Since the right hand side of (2.2.8) is the solution of (2.2.7), the result follows.

Since F^o is differentiable at $(f,c) \in B((0,0),\hat{r}_1)$, it follows [9; p. 153] that the two partial differentials of F^o at (f,c) are given by

$$d_1F^o(f,c;\ \phi)(t) = \phi(t) + \int_0^t r^o(t,s,c)\phi(s)ds$$

and

$$d_2F^o(f,c;\ \lambda)(t) = \int_0^t k(t,s)h_c(s,x(s),c)ds\lambda + \int_0^t r^o(t,s,c)\int_0^s k(s,u)h_c(u,x(u),c)duds\lambda.$$

We observe that $d_1F^o(f,c;\ \phi)$ is the solution of the variational equation

$$\xi(t) = \phi(t) + \int_0^t k(t,s)h_x(s,x(s),c)\xi(s)ds$$

and $d_2F^o(f,c;\ \lambda)$ is the solution of

$$\xi(t) = \int_0^t k(t,s)h_c(s,x(s),c)ds\lambda + \int_0^t k(t,s)h_x(s,x(s),c)\xi(s)ds.$$

In the following sections we wish to suppress the dependence of h on the parameter c. With this in mind we make the following definitions and hypotheses.

Let g be a function defined on D(q) with range in R^n that satisfies the following conditions:

(A_1) g *is continuous on* D(q) *and* $g(t,0) = 0$ *for* $t \in R^+$;

(A_2) *the components* g_i *of* g *have continuous partial derivatives with respect to the components* x_j *of* $x \in R^n$ *and the* $n \times n$ *matrix function* $||g_x(t,x)||$ *is bounded above by* P *for* $(t,x) \in D(q)$;

(A_3) *for each* $\varepsilon > 0$, *there is a* $\delta_\varepsilon > 0$ *such that if* (t,x) *and* (t,y) *are in* D(q) *and* $||x - y|| < \delta_\varepsilon$, *then* $||g_x(t,x) - g_x(t,y)|| < \varepsilon$.

In order to facilitate further discussion we will list two more hypotheses.

(A_4) *Let* k *be a continuous* $n \times n$ *matrix function defined on* Δ *for which there exists an* $M > 0$ *such that*

$$\int_0^t ||k(t,s)||ds \leq M \text{ for } t \in R^+.$$

(A_5) *Let* $r(t,s)$ *be the kernel reciprocal to* $k(t,s)g_x(s,0)$ *and suppose that there exists an* $M_r > 0$ *such that*

$$\int_0^t ||r(t,s)||ds \leq M_r \text{ for } t \in R^+.$$

Even though the hypothesis (A_4) is already in force we list it again so that in the sequel we may refer to the above hypotheses collectively as hypotheses (A).

Let $G\colon\ BC(R^+) \times B(0,q) \to BC(R^+)$ be defined by

$$G(f,x) = f(t) + \int_0^t k(t,s)g(s,x(s))ds - x(t). \tag{2.2.9}$$

Arguments similar to those used to establish that the range of G^o is in $BC(R^+)$ suffice to show that the range of G is also in $BC(R^+)$.

THEOREM 2.2.4. *If hypotheses* (A_1) *through* (A_4) *are satisfied, then* G *is continuously differentiable on* $BC(R^+) \times B(0,q)$. *Furthermore, if* $(f,x) \in BC(R^+) \times B(0,q)$ *and* $(\phi,\xi) \in BC(R^+) \times BC(R^+)$, *then*

$$dG(f,x;\ \phi,\xi)(t) = \phi(t) + \int_0^t k(t,s)g_x(s,x(s))\xi(s)ds - \xi(t). \tag{2.2.10}$$

Let $h(t,x,c) = g(t,x)$ for $(t,x) \in D(q)$ and $c \in B(0,q_m)$. In view of Theorem 2.2.2, the result follows.

THEOREM 2.2.5. *If hypotheses* (A) *are satisfied, then there exist open balls* $B(0,r_1) \subset BC(R^+)$ *and* $B(0,r_2) \subset B(0,q)$ *and a function* $F\colon\ B(0,r_1) \to B(0,r_2)$ *with the following properties:*

(i) *the point* $(f,F(f)) \in B(0,r_1) \times B(0,r_2)$ *is a solution of* $G(f,x) = 0$ *for each* $f \in B(0,r_1)$, *and there is no other solution with the same* f *in* $B(0,r_2)$;

(ii) *the partial differential* $d_2G(f,F(f);\ \cdot)$ *is invertible for each* $f \in B(0,r_1)$;

(iii) F *is continuously differentiable on* $B(0,r_1)$.

The result follows from Theorem 2.2.3 on setting $h(t,x,c) = g(t,x)$ for $(t,x) \in D(q)$ and $c \in B(0,q_m)$.

THEOREM 2.2.6. *Assume hypotheses* (A) *are satisfied and let* $F\colon\ B(0,r_1) \to B(0,r_2)$ *be the implicit function of Theorem* 2.2.5. *Also, suppose that* $(f,x) \in B(0,r_1) \times B(0,r_2)$ *satisfies* $x = F(f)$. *Then*

(2.2.11) $$dF(f;\ \phi) = \phi(t) + \int_0^t r(t,s;\ x)\phi(s)ds,$$

where $r(t,s;\ x)$ *is the kernel reciprocal to* $k(t,s)g_x(s,x(s))$. *Consequently,* $dF(f;\ \phi)$ *is the solution of the variational equation*

($V[\phi,k,g,x]$) $$\xi(t) = \phi(t) + \int_0^t k(t,s)g_x(s,x(s))\xi(s)ds.$$

Once more we set $h(t,x,c) = g(t,x)$ for $(t,x) \in D(q)$ and $c \in B(0,q_m)$. In view of Theorem 2.2.2, (2.2.11) follows from (2.2.6), since $h_c = 0$. Also, it follows from (2.2.8) that $dF(f;\ \phi)$ satisfies $V[\phi,k,g,x]$.

If $f \in B(0,r_1)$, Theorem 2.2.5 guarantees the existence of exactly one solution of

($E[f,k,g]$) $$x(t) = f(t) + \int_0^t k(t,s)g(s,x(s))ds$$

in $B(0,r_2)$. However, the possibility of the existence of solutions not in $B(0,r_2)$ remains. Consequently, the following theorem is appropriate.

THEOREM 2.2.7. *Let hypotheses* (A) *be satisfied. If* $f \in B(0,r_1)$, *then* $E[f,k,g]$ *has exactly one continuous solution and this solution is in* $B(0,r_2)$.

Let f be fixed in $B(0,r_1)$ and $x = F(f)$ be the solution of $E[f,k,g]$ given by the implicit function ofTheorem 2.2.5. Assume y is a continuous solution of the same equation and that y is defined on a non-degenerate interval J. The interval J is closed on the left with left hand end point 0.

Suppose there exists a $t_1 \in J$ with $y(t_1) \neq x(t_1)$. The set $\{t \in J;\ ||y(t) - x(t)|| > 0\}$ is bounded below by 0 since $y(0) = f(0) = x(0)$. Set $t_2 = \inf\ \{t \in J;\ ||y(t) - x(t)|| > 0\}$. Let $t_0 = \sup\ \{t \in J;\ t \leq t_2$ and $||y(t) - x(t)|| = 0\}$. If $t_0 < t_2$, it follows from the definition of t_0 that there exists a $t \in (t_0,t_2)$ such that $||y(t) - x(t)|| > 0$, which contradicts the definition of t_2. We conclude that $t_0 = t_2$. If

$t_0 > 0$, then $y(t_0) = x(t_0)$ because $||y(t) - x(t)||$ is continuous on $[0,t_0]$. Note that t_0 is not the right hand end point of J because $t_1 \in J$ and $||y(t_1) - x(t_1)|| > 0$.

Since $x \in B(0,r_2)$ and $r_2 \leq q$, it follows that $\{(t,x(t));\ 0 \leq t \leq t_0\} \subset D(q)$. As $(t_0,y(t_0)) = (t_0,x(t_0)) \in D(q)$, the real number $q - ||y(t_0)||$ is positive and, due to the continuity of $q - ||y(t)||$, there exists a $\beta > 0$ such that $[t_0,t_0 + \beta] \subset J$ and $q - ||y(t)|| > 0$ on $[t_0,t_0 + \beta]$. With $b = t_0 + \beta$, it follows that $\{(t,y(t));\ t \in [0,b]\} \subset D(q)$.

The set $\Delta(b) = \{(t,s);\ 0 \leq s \leq t \leq b\}$ is compact and so there exists an $N \geq 0$ such that $||k(t,s)|| \leq N$ on $\Delta(b)$. For $t \in [0,b]$, we have the inequalities

$$\begin{aligned}
||y(t) - x(t)|| &\leq \int_0^t ||k(t,s)||\ ||g(s,y(s)) - g(s,x(s))||ds \\
&\leq \int_0^t\int_0^1 (||k(t,s)||\ ||g_x(s,x(s) + \alpha(y(s) - x(s))|| \\
&\qquad\qquad ||y(s) - x(s)||)d\alpha ds \\
&\leq PN\int_0^t ||y(s) - x(s)||ds
\end{aligned}$$

where P is a bound for $||g_x(t,x)||$. It follows from Gronwall's inequality that $||y(t) - x(t)|| = 0$ on $[0,b]$. From the definition of t_0, there exists a $t \in (t_0,b)$ such that $||y(t) - x(t)|| > 0$. So, the assumption of the existence of a $t \in J$ such that $y(t) \neq x(t)$ leads to a contradiction. Hence, $x = F(f)$ is the only continuous solution of $E[f,k,g]$ and it is in $B(0,r_2)$.

3. Stability and linearization of $E[f,k,g]$. With the exception of Theorem 2.3.2, the theorems of this and the next section are generalizations of results concerning the Liapunov stability of solutions of differential equations of the type $\dot{x} = w(t,x)$, which may be written in the form $x(t) \doteq x_0 + \int_0^t w(s,x(s))ds$. In our theorems the above equation

is replaced by E[f,k,g], and f plays the role of the initial vector x_0.

A common technique used in studying perturbed linear equations of the form $\dot{x} = A(t)x + w(t,x)$ is to express the solution of the equation in the form

$$x(t) = \Phi(t)\Phi^{-1}(0)x_0 + \int_0^t \Phi(t)\Phi^{-1}(s)w(s,x(s))ds$$

where Φ is a fundamental matrix for the system $\dot{x} = A(t)x$. This is accomplished by the method of variation of parameters. Various authors (see [11] and [12]) have adapted this technique to the study of integral equations of the form

$$x(t) = f(t) + \int_0^t k(t,s)[x(s) + g(s,x(s))]ds.$$

In our work, the implicit function of Theorem 2.2.5 will play a role similar to the variation of constants formula.

THEOREM 2.3.1. _Assume that hypotheses_ (A) _hold and let_ $F\colon B(0,r_1) \to B(0,r_2)$ _be the implicit function of Theorem_ 2.2.5. _Then, for each_ $\varepsilon > 0$ _there is a_ $\delta_\varepsilon > 0$ _such that if_ $||f|| < \delta_\varepsilon$, _then the unique solution_ x _of_ E[f,k,g] _satisfies_ $||x|| < \varepsilon$.

Let $\varepsilon > 0$ be fixed and suppose that $\varepsilon \leqslant r_2$. The function F is continuous and, as a result, $F^{-1}(B(0,\varepsilon))$ is open in $B(0,r_1)$. Since $0 \in F^{-1}(B(0,\varepsilon))$, there exists a $\delta_\varepsilon > 0$ with $\delta_\varepsilon < r_1$ and such that $B(0,\delta_\varepsilon) \subset F^{-1}(B(0,\varepsilon))$. So, if $f \in B(0,\delta_\varepsilon)$, then $x = F(f) \in B(0,\varepsilon)$.

In the language of differential equations, the above theorem might be stated as follows: The zero solution of E[0,k,g] is stable.

The next result concerns the linearization of E[f,k,g].

THEOREM 2.3.2. _Assume that hypotheses_ (A) _hold and let_ $F\colon B(0,r_1) \to B(0,r_2)$ _be the implicit function of Theorem_ 2.2.5. _Let_ x _and_ ξ _denote solutions of_ E[f,k,g] _and_ $V[\phi,k,g,0]$ _respectively. Then, for each_ $\varepsilon > 0$, _there exists a_ $\delta_\varepsilon > 0$ _such that if_ f _and_ ϕ _belong to_ $B(0,\delta_\varepsilon)$, _then_

$||x - \xi|| < \varepsilon$.

Let $\varepsilon > 0$ be fixed. For each $\phi \in BC(R^+)$ the solution ξ of $V[\phi,k,g,0]$ is given by

$$\xi(t) = \phi(t) + \int_0^t r(t,s)\phi(s)ds,$$

where $r(t,s)$ is the kernel reciprocal to $k(t,s)g_x(s,0)$. From (A_5), we have that $\int_0^t ||r(t,s)||ds \leq M_r$ for $t \in R^+$. With $\delta_1 = \varepsilon/[2(1 + M_r)]$, we have that, if $||\phi|| < \delta_1$, then

$$\begin{aligned} ||\xi(t)|| &\leq ||\phi(t)|| + \int_0^t ||r(t,s)||\ ||\phi(s)||ds \\ &\leq (1 + M_r)||\phi||, \end{aligned}$$

and so $||\xi|| < \varepsilon/2$. The function F is continuous and satisfies $F(0) = 0$. As a result, there exists a $\delta_2 > 0$, satisfying $\delta_2 < r_1$, and such that if $||f|| < \delta_2$, then $||x|| < \varepsilon/2$, where $x = F(f)$. Let $\delta_\varepsilon = \min\{\delta_1,\delta_2\}$ and f and ϕ belong to $B(0,\delta_\varepsilon)$; then $||x - \xi|| \leq ||x|| + ||\xi|| < \varepsilon$.

We digress in order to examine the restrictive nature of the hypothesis $g(t,0) = 0$ for $t \in R^+$. Let $x^o \in BC(R^+)$ be a solution of the equation

$$(E[f^o,k,w]) \qquad y(t) = f^o(t) + \int_0^t k(t,s)w(s,y(s))ds,$$

where $f^o \in BC(R^+)$, and w is defined on $D(x^o,q) = \{(t,x) \in R^{n+1};\ t \in R^+$ and $||x - x^o(t)|| < q\}$ with range in R^n. The function $w^o(t,z) = w(t,x^o(t) + z) - w(t,x^o(t))$ is defined on $D(q) = \{(t,z) \in R^{n+1};\ t \in R^+$ and $||z|| < q\}$ and satisfies $w^o(t,0) = 0$ for $t \in R^+$.

Assume the zero solution of

$$(E[0,k,w^o]) \qquad z(t) = \int_0^t k(t,s)w^o(s,z(s))ds$$

has the following property: <u>for each</u> $\varepsilon > 0$ <u>there exists a</u> $\delta_\varepsilon > 0$ <u>such that if</u> $f \in B(0,\delta_\varepsilon)$, <u>then</u> $E[f,k,w^o]$ <u>has a solution in</u> $B(0,\varepsilon)$.

Let $\varepsilon > 0$ and corresponding $\delta_\varepsilon > 0$ be fixed and suppose $f \in B(f^o,\delta_\varepsilon)$. Then $\Delta f = f - f^o \in B(0,\delta_\varepsilon)$, and so a solution, Δx, of

$$z(t) = \Delta f(t) + \int_0^t k(t,s)w^o(s,z(s))ds$$

exists and satisfies $\Delta x \in B(0,\varepsilon)$. With $x(t) = x^o(t) + \Delta x(t)$, we have that

$$\begin{aligned} x(t) - x^o(t) &= \Delta x(t) \\ &= \Delta f(t) + \int_0^t k(t,s)w^o(s,\Delta x(s))ds \\ &= \Delta f(t) + \int_0^t k(t,s)[w(s,x(s)) - w(s,x^o(s))]ds \end{aligned}$$

and so,

$$\begin{aligned} x(t) &= \Delta x(t) + x^o(t) \\ &= f(t) + \int_0^t k(t,s)w(s,x(s))ds. \end{aligned}$$

Hence, x is a solution of $E[f,k,w]$ and since $\Delta x \in B(0,\varepsilon)$, it follows that $x \in B(x^o,\varepsilon)$.

Therefore, if one wishes to examine the change in a solution of $E[f^o,k,w]$ induced by a change in f^o, it is sufficient to consider the zero solution of $E[0,k,w^o]$.

In the event that w^o satisfies the same hypotheses as the function g in our previous theorems, it follows that the solution, ξ, of $V[\Delta f,k,w^o,0]$ satisfies $||\Delta x - \xi|| < \varepsilon$ so that ξ approximates Δx when Δf is small.

4. Stability of perturbed equations. In this section we direct our attention to perturbed equations corresponding to $E[f,k,g]$.

THEOREM 2.4.1. *Assume that hypotheses* (A) *hold and let* $F\colon B(0,r_1) \to B(0,r_2)$ *be the implicit function of Theorem* 2.2.5. *Let* $B(0,q_1) \subset BC(R^+)$ *and suppose* $H\colon B(0,q_1) \to BC(R^+)$ *satisfies* $H(0) = 0$ *and* $||H(u) - H(v)|| \leq M_1||u - v||$ *for some* $M_1 > 0$ *and all* u *and* v *that belong to* $B(0,q_1)$. *If* $||dF(0;\cdot)||M_1 < 1$, *then for each* $\varepsilon > 0$ *there is a* $\delta_\varepsilon > 0$ *such that if* $||f|| < \delta_\varepsilon$, *then the integral equation*

$$(E[f,k,g,H]) \quad x(t) = f(t) + \int_0^t k(t,s)g(s,x(s))ds + H(x)(t)$$

has a solution in $B(0,\varepsilon)$. *Furthermore, if* $E[f,k,g,H]$ *has a unique solution for each* f *in* $B(0,\delta_\varepsilon)$, *then the solution* $x(\cdot,f)$, *corresponding to* f, *is a continuous function of* f.

Let $\varepsilon > 0$ be fixed and select $\varepsilon_1 > 0$ so that

$$(\varepsilon_1 + ||dF(0;\cdot)||)M_1 < 1. \tag{2.4.1}$$

Since F is continuously differentiable on $B(0,r_1)$, there is a δ_1 satisfying $0 < \delta_1 < r_1$, and such that if $f \in B(0,\delta_1)$, then $||dF(f;\cdot) - dF(0;\cdot)|| < \varepsilon_1$. Alternately,

$$||dF(f;\cdot)|| < \varepsilon_1 + ||dF(0;\cdot)||. \tag{2.4.2}$$

Choose $\delta_2 < \min\{\delta_1, M_1q_1, M_1\varepsilon\}$. It follows from the mean value theorem [9; p. 149] that if f and f' belong to $\overline{B}(0,\delta_2)$, then

$$||F(f) - F(f')|| \leq (\varepsilon_1 + ||dF(0;\cdot)||)||f - f'||. \tag{2.4.3}$$

Let λ be a real number that satisfies

$$(\varepsilon_1 + ||dF(0;\cdot)||)M_1 = 1 - \lambda \tag{2.4.4}$$

and note that $0 < \lambda < 1$. Set

$$\rho = (1 - \lambda)\delta_2/M_1 \tag{2.4.5}$$

and define $\hat{H}$: $B(0,\lambda\delta_2) \times \overline{B}(0,\rho) \to BC(R^+)$ by

$$\hat{H}(f,u)(t) = f(t) + H(u)(t). \tag{2.4.6}$$

Fix $f \in B(0,\lambda\delta_2)$ and let $u \in \overline{B}(0,\rho)$; then it follows from (2.4.6) and the hypotheses on H that

$$\begin{aligned} ||\hat{H}(f,u)(t)|| &\leq ||f(t)|| + ||H(u)(t) - H(0)(t)|| \\ &\leq ||f|| + ||H(u) - H(0)||. \end{aligned}$$

Since $f \in B(0,\lambda\delta_2)$ and $u \in \overline{B}(0,\rho)$, it follows from the hypotheses on H and (2.4.5) that

$$||f|| + ||H(u) - H(0)|| \leq ||f|| + M_1||u|| < \lambda\delta_2 + M_1\rho = \delta_2$$

and therefore,

$$||\hat{H}(f,u)|| < \delta_2. \tag{2.4.7}$$

With $f \in B(0,\lambda\delta_2)$ fixed, the composite function $F \circ \hat{H}(f,\cdot)$ is defined on $\overline{B}(0,\rho)$ since $\hat{H}(f,u) \in B(0,\delta_2)$ for $u \in \overline{B}(0,\rho)$.

We will verify that $F \circ \hat{H}(f,\cdot)$ is a contraction map on $\overline{B}(0,\rho)$. Since $F(0) = 0$,

$$||F(\hat{H}(f,u))|| = ||F(\hat{H}(f,u)) - F(0)||$$

and, in view of (2.4.3) and (2.4.7), we have

$$||F(\hat{H}(f,u)) - F(0)|| \leqslant (\varepsilon_1 + ||dF(0;\ \cdot)||)\delta_2.$$

In view of (2.4.4) and (2.4.5), we have that

$$(\varepsilon_1 + ||dF(0;\ \cdot)||)\delta_2 = (1 - \lambda)\delta_2/M_1 = \rho;$$

so, $||F(\hat{H}(f,u))|| \leqslant \rho$ for each $u \in \overline{B}(0,\rho)$. It follows that $F \circ \hat{H}(f,\cdot)$: $\overline{B}(0,\rho) \to \overline{B}(0,\rho)$. Also, if u and v belong to $\overline{B}(0,\rho)$, then, from (2.4.6), $||\hat{H}(f,u) - \hat{H}(f,v)|| = ||H(u) - H(v)||$. By hypothesis, $||H(u) - H(v)|| \leqslant M_1||u - v||$ and so, $||\hat{H}(f,u) - \hat{H}(f,v)|| \leqslant M_1||u - v||$. In view of (2.4.3) and (2.4.4), it follows that

$$\text{(2.4.8)} \quad \begin{aligned} ||F(\hat{H}(f,u)) - F(\hat{H}(f,v))|| &\leqslant (\varepsilon_1 + ||dF(0;\ \cdot)||)M_1||u - v|| \\ &\leqslant (1 - \lambda)||u - v||. \end{aligned}$$

Since $0 < 1 - \lambda < 1$, we have that $F \circ \hat{H}(f,\cdot)$ is a contraction map on $\overline{B}(0,\rho)$, and it follows from the Banach fixed point theorem that there exists a unique $x \in \overline{B}(0,\rho)$ such that $x = F(\hat{H}(f,x))$. Since $\hat{H}(f,x)(t) = f(t) + H(t)$, the above statement is equivalent to the existence of a solution of $E[f,k,g,H]$ in $\overline{B}(0,\rho)$. As $\rho = (1 - \lambda)\delta_2/M_1 < \varepsilon$, the first conclusion follows on setting $\delta_\varepsilon = \lambda\delta_2$.

Now suppose the solutions of $E[f,k,g,H]$ are unique for $f \in B(0,\delta_\varepsilon)$. This hypothesis is necessary because it is possible that $E[f,k,g,H]$ has solutions not in $\overline{B}(0,\rho)$ even though $f \in B(0,\delta_\varepsilon)$. As we have seen in (2.4.8), $||F(\hat{H}(f,u)) - F(\hat{H}(f,v))|| \leqslant (1 - \lambda)||u - v||$ for each $f \in B(0,\delta_\varepsilon)$, and hence, $F \circ \hat{H}$: $B(0,\delta_\varepsilon) \times \overline{B}(0,\rho) \to \overline{B}(0,\rho)$ is a contraction in u, uniformly over f. That is, the contraction constant, $1 - \lambda$, is independent of f. Also, if f and f' belong to $B(0,\delta_\varepsilon)$ and $u \in \bar{B}(0,\rho)$, then it follows from (2.4.6) that $||\hat{H}(f,u) - \hat{H}(f',u)|| = ||f - f'||$ and consequently, $\hat{H}$ is continuous in f for each fixed u. Since F is continuous, $F \circ \hat{H}$ is continuous in f for each fixed u and it follows from a well known theorem [9; p. 230] that $x(\cdot,f) = F \circ \hat{H}(f,x(\cdot,f))$ is a continuous function of f.

COROLLARY 2.4.1. *Assume that hypotheses* (A) *hold and let* $F\colon B(0,r_1) \to B(0,r_2)$ *be the implicit function of Theorem* 2.2.5. *Suppose that* k^1 *is a continuous function defined on* $\Delta \times B(0,q_1) \subset R^{n+2}$ *with range in* R^n. *Assume further that* k^1 *satisfies the following conditions:*

(i) $k^1(t,s,0) = 0$, *for* $(t,s) \in \Delta$;

(ii) *there exists a positive continuous function* $k^o\colon \Delta \to R$ *and a real number* $M^o > 0$ *such that* $||k^1(t,s,x) - k^1(t,s,y)|| \leq k^o(t,s)||x - y||$ *and* $\int_0^t k^o(t,s)ds \leq M^o$ *for* $t \in R^+$ *and* (t,s,x) *and* (t,s,y) *in* $\Delta \times B(0,q_1)$.

If $||dF(0;\ \cdot)||M^o < 1$, *then for each* $\varepsilon > 0$ *there exists a* $\delta_\varepsilon > 0$ *such that if* $||f|| < \delta_\varepsilon$, *then*

$$(E[f,k,g,k^1]) \qquad x(t) = f(t) + \int_0^t k(t,s)g(s,x(s))ds + \int_0^t k^1(t,s,x(s))ds$$

has at least one solution in $B(0,\varepsilon)$.

With $u \in B(0,q_1)$, define $H\colon B(0,q_1) \to BC(R^+)$ by $H(u)(t) = \int_0^t k^1(t,s,u(s))ds$. Since k^1 and u are continuous, H(u) is a continuous function of t. Also, if u and v belong to $B(0,q_1)$, then

$$\begin{aligned} ||H(u)(t) - H(v)(t)|| &\leq \int_0^t ||k^1(t,s,u(s)) - k^1(t,s,v(s))||ds \\ &\leq ||u - v||\int_0^t k^o(t,s)ds \\ &\leq M^o||u - v||. \end{aligned}$$

Since $H(0) = 0$, it follows on replacing v with 0 that $H(u) \in BC(R^+)$.

An application of Theorem 2.4.1 establishes the result.

LEMMA 2.4.1. _Under the hypotheses of Corollary_ 2.4.1, _there exists a_ $\delta > 0$ _such that if_ $f \in B(0,\delta)$, _then the solution of_ $E[f,k,g,k^1]$ _is unique_.

Recall that g is defined on D(q) and that k^1 is defined on $\Delta \times B(0,q_1)$. Let ε be a real number that satisfies $0 < \varepsilon < \min\{q,q_1\}$. In view of Corollary 2.4.1, there exists a $\delta > 0$ such that if $||f|| < \delta$, then $E[f,k,g,k^1]$ has a solution, x, in $B(0,\varepsilon)$.

Assume that y is another solution of $E[f,k,g,k^1]$ defined on a non-degenerate interval J. The interval J has left hand end point 0 and is closed on the left.

Suppose there exists a t in J such that $y(t) \neq x(t)$. As in Theorem 2.2.7, this implies the existence of a $t_0 \in J$ such that t_0 is not the right hand end point of J and $y(t) = x(t)$ for $t \in [0,t_0]$. Also, if $t_1 \in J$ and $t_1 > t_0$, then there exists a $t \in (t_0,t_1)$ such that $y(t) \neq x(t)$.

Since $(t,x(t)) \in D(q)$ for $t \in R^+$ and $(t,s,x(s)) \in \Delta \times B(0,q_1)$, it follows that $\{(t,y(t));\ t \in [0,t_0]\} \subset D(q)$ and $\{(t,s,y(s));\ (t,s) \in \Delta(t_0)\} \subset \Delta \times B(0,q_1)$. Consequently, there are positive numbers β_1 and β_2 such that $[0,t_0 + \beta_1] \subset J$, $[0,t_0 + \beta_2] \subset J$ while $\{(t,y(t));\ t \in [t_0,t_0 + \beta_1]\} \subset D(q)$ and $\{(t,s,y(s));\ (t,s) \in \Delta(t_0 + \beta_2)\} \subset \Delta \times B(0,q_1)$.

Set $b = t_0 + \min\{\beta_1,\beta_2\}$ and note that $\Delta(b)$ is compact. As a result, there exist real numbers $N \geq 0$ and $N^o \geq 0$ such that $||k(t,s)|| \leq N$ and $k^o(t,s) \leq N^o$ on $\Delta(b)$. In view of hypothesis (ii) of Corollary 2.4.1 and hypotheses (A_2) and (A_3) we have the following inequalities:

$$||y(t) - x(t)|| \leq \int_0^t ||k(t,s)||\ ||g(s,y(s)) - g(s,x(s))||ds$$

$$+ \int_0^t ||k^1(t,s,y(s)) - k^1(t,s,x(s))||ds$$
$$\leq (NP + N^o) \int_0^t ||y(s) - x(s)||ds.$$

It follows from Gronwall's inequality that $y(t) - x(t) = 0$ on $[0,b]$. This contradicts the assumption of the existence of $t \in J$ such that $y(t) \neq x(t)$, and hence the lemma is established.

COROLLARY 2.4.2. _Let the hypotheses of Corollary 2.4.1 be satisfied. Then for each_ $\varepsilon > 0$ _there is a_ $\delta_\varepsilon > 0$ _such that if_ $f \in B(0,\delta_\varepsilon)$, _then_ $E[f,k,g,k^1]$ _has a unique solution in_ $B(0,\varepsilon)$ _and the solution is a continuous function of_ f.

With $\varepsilon > 0$ fixed, it follows from Corollary 2.4.1 and Lemma 2.4.1 that there exists a $\delta_1 > 0$ such that if $||f|| < \delta_1$, then $E[f,k,g,k^1]$ has a unique solution in $B(0,\varepsilon)$.

From Theorem 2.4.1, there is a $\delta_2 > 0$ such that if $||f|| < \delta_2$ and solutions of $E[f,k,g,k^1]$ are unique, then the solution of $E[f,k,g,k^1]$ is a continuous function of f.

The result follows on setting $\delta_\varepsilon = \min\{\delta_1,\delta_2\}$.

COROLLARY 2.4.3. _Assume that hypotheses (A) hold and let_ $F: B(0,r_1) \to B(0,r_2)$ _be the implicit function of Theorem 2.2.5. Suppose_ k^o _is an_ $n \times n$ _matrix function with corresponding real number_ M^o, _that satisfies hypothesis_ (A_4). _Assume_ g^o _is a continuous function defined on_ $D(q_2) = \{(t,x) \in R^{n+1}: t \in R^+$ _and_ $||x|| < q_2\}$ _with range in_ R^n. _Assume further that_ g^o _satisfies the following conditions:_

(i) $g^o(t,0) = 0$ _for_ $t \in R^+$;

(ii) _for each_ $\zeta > 0$ _there exists an_ $\eta > 0$ _such that if_ (t,x) _and_ (t,y) _belong to_ $D(q_2)$ _with_ $||x|| < \eta$ _and_ $||y|| < \eta$, _then_ $||g^o(t,x) - g^o(t,y)|| \leq \zeta||x - y||$.

Then, for each $\varepsilon > 0$ _there is a_ $\delta_\varepsilon > 0$ _such that if_ $||f|| < \delta_\varepsilon$, _then_

$$x(t) = f(t) + \int_0^t k(t,s)g(s,x(s))ds + \int_0^t k^o(t,s)g^o(s,x(s))ds$$

has a unique solution in $B(0,\varepsilon)$. Furthermore, the solution is a continuous function of f.

We will verify the hypotheses of Corollary 2.4.1. Let $\zeta^o > 0$ satisfy $||dF(0;\ \cdot)||M^o\zeta^o < 1$; then from the above hypothesis (ii), there exists an η^o, with $0 < \eta^o \leqslant q_2$, such that if (t,x) and (t,y) belong to $D(q_2)$ and $||x|| < \eta^o$ and $||y|| < \eta^o$, then $||g^o(t,x) - g^o(t,y)|| \leqslant \zeta^o||x - y||$.

Since $\eta^o \leqslant q_2$, g^o is defined on $D(\eta^o) = \{(t,x) \in R^{n+1};\ t \in R^+$ and $||x|| < \eta^o\}$. Define k^1 by $k^1(t,s,x) = k^o(t,s)g^o(s,x)$ for $(t,s,x) \in \Delta \times B(0,\eta^o)$. Then k^1 satisfies the conditions:

(a) $k^1(t,s,0) = k^o(t,s)g^o(s,0) = 0$ for $(t,s) \in \Delta$;

(b) $$||k^1(t,s,x) - k^1(t,s,y)|| \leqslant ||k^o(t,s)||\ ||g^o(s,x) - g^o(s,y)|| \leqslant \zeta^o||k^o(t,s)||\ ||x - y||,$$

for all (t,s,x) and (t,s,y) that belong to $\Delta \times B(0,\eta^o)$.

Also, $\int_0^t \zeta^o||k^o(t,s)||ds \leqslant \zeta^o M^o$ for $t \in R^+$.

In view of Corollary 2.4.2, the result follows.

COROLLARY 2.4.4. *Assume that hypotheses* (A) *hold and let* $F\colon B(0,r_1) \to B(0,r_2)$ *be the implicit function of Theorem* 2.2.5. *Suppose* k^o *is a continuous* $n \times n$ *matrix function defined on* $R^+ \times R^+$ *that satisfies the following conditions:*

(i) *there exists an* $M^o > 0$ *such that* $\int_0^\infty ||k^o(t,s)||ds \leqslant M^o$ *for* $t \in R^+$;

(ii) *for each* $\eta > 0$ *and* $T > 0$, *there exists a* $\zeta > 0$ *such that if* t *and* t' *belong to* $[0,T]$ *and* $|t - t'| < \zeta$, *then*

$$\int_0^\infty ||k^o(t,s) - k^o(t',s)||ds < \eta.$$

If $||dF(0;\ \cdot)||M^o < 1$, *then for each* $\varepsilon > 0$ *there is a* $\delta_\varepsilon > 0$ *such that*

<u>if</u> $||f|| < \delta_\varepsilon$, <u>then there exists at least one solution of</u>

$$x(t) = f(t) + \int_0^t k(t,s)g(s,x(s))ds + \int_0^\infty k^o(t,s)x(s)ds$$

<u>in</u> $B(0,\varepsilon)$.

Define H: $BC(R^+) \to BC(R^+)$ by $H(u)(t) = \int_0^\infty k^o(t,s)u(s)ds$. We will verify that $H(u) \in BC(R^+)$ whenever $u \in BC(R^+)$. Let $\eta > 0$, $u \in BC(R^+)$, and $t_0 \in R^+$ be fixed. Set $T = t_0 + 1$ and choose $\zeta > 0$ so that if $t \in [0,T]$ and $|t - t_0| < \zeta$, then

$$\int_0^\infty ||k^o(t,s) - k^o(t_0,s)||ds < \eta/(1 + ||u||).$$

Then,

$$||H(u)(t) - H(u)(t_0)|| \leqslant \int_0^\infty ||k^o(t,s) - k^o(t_0,s)|| \; ||u(s)||ds$$
$$\leqslant ||u||\int_0^\infty ||k^o(t,s) - k^o(t_0,s)||ds,$$

and so $||H(u)(t) - H(u)(t_0)|| \leqslant ||u||/(1 + ||u||) < \eta$. Since t_0 is arbitrary in R^+, it follows that $H(u)$ is a continuous function on R^+.

If u and v belong to $BC(R^+)$, then,

$$||H(u)(t) - H(v)(t)|| \leqslant \int_0^\infty ||k^o(t,s)|| \; ||u(s) - v(s)||ds$$
$$\leqslant M^o||u - v||.$$

Consequently, $||H(u) - H(v)|| \leqslant M^o||u - v||$. Since $H(0) = 0$, we may set $v = 0$ and conclude that $H(u) \in BC(R^+)$.

In conclusion, observe that we have verified those hypotheses of Theorem 2.4.1 which guarantee the existence of a solution of $E[f,k,g,H]$ in $B(0,\varepsilon)$ whenever $||f|| < \delta_\varepsilon$.

Theorem 2.4.1, Corollary 2.4.2 and Corollary 2.4.4 are similar to theorems established by Corduneanu [6]. The principal difference is that our results pertain to perturbations of nonlinear systems while Corduneanu confined his attention to perturbed linear systems.

The hypotheses placed on the function g^o in Corollary 2.4.3 are similar to those found in Miller, Nohel and Wong [11].

For completeness, we state a version of the Schauder-Tychonoff Fixed Point Theorem that will be used in the sequel. This form of the theorem may be found in [4; p. 9].

SCHAUDER-TYCHONOFF THEOREM. *Let $C(R^+,R^n)$ denote the set of continuous functions on R^+ with range in R^n. Let S be the subset of $C(R^+,R^n)$ that consists of those functions x such that $||x(t)|| \leq \mu(t)$ for $t \in R^+$, where μ is a fixed positive continuous function defined on R^+. Let T be a mapping of S into itself with the following properties:*

(i) *T is continuous, in the sense that if (x_m) is a sequence in S and $x_m \to x$ uniformly on every compact subinterval of R^+, then $T(x_m) \to x$ uniformly on every compact subinterval of R^+;*

(ii) *the functions in the image set T(S) are equicontinuous and bounded at every point of R^+.*

Then T has at least one fixed point in S.

LEMMA 2.4.2. *Assume hypotheses (A) hold and let F: $B(0,r_1) \to B(0,r_2)$ be the implicit function of Theorem 2.2.5. Suppose (f_m) is a sequence in $\overline{B}(0,\hat{r})(0 < \hat{r} < r_1)$ that converges to f uniformly on each compact subinterval of R^+. Then, $F(f_m)$ converges to F(f) uniformly on each compact subinterval of R^+.*

It is clear that if $f_m \to f$ uniformly on each compact subinterval of R^+, then $f \in \overline{B}(0,\hat{r}) \subset B(0,r_1)$.

With $x = F(f)$, $x_m = F(f_m)$ and $\varepsilon > 0$ fixed, let J be a compact subinterval of R^+. There exists a $T > 0$ such that $J \subset [0,T]$. It follows from the compactness of $\Delta(T)$ that there is a positive real number K such that $||k(t,s)|| \leq K$ on $\Delta(T)$. Since $f_m \to f$ uniformly on [0,T], there is a positive integer N such that if $m > N$, then

$$||f_m(t) - f(t)|| < \varepsilon \exp(-KPT)/2$$

where P is an upper bound for $||g_x||$ on D(q). Taking note of the fact that g is Lipschitzian in x with Lipschitz constant P, we have that

$$\begin{aligned} ||x_m(t) - x(t)|| &\leq ||f_m(t) - f(t)|| \\ &\quad + \int_0^t ||k(t,s)||\, ||g(s,x_m(s)) - g(s,x(s))||ds \\ &\leq \varepsilon \exp(-KPT) + KP\int_0^t ||x_m(s) - x(s)||ds \end{aligned}$$

for $m > N$ and $t \in [0,T]$. An application of Gronwall's inequality yields

$$||x_m(t) - x(t)|| \leq \varepsilon \exp(-KPT)\exp(KPT)/2 < \varepsilon,$$

and so $F(f_m) \to F(f)$ uniformly on $[0,T]$. Therefore, $F(f_m) \to F(f)$ uniformly on J. Since J is an arbitrary compact subinterval of R^+, the result follows.

THEOREM 2.4.2. *Assume that hypotheses (A) hold and let* $F: B(0,r_1) \to B(0,r_2)$ *be the implicit function of Theorem 2.2.5. Let* γ^o *be a positive number and suppose* $H^o: B(0,\gamma^o) \to BC(R^+)$ *satisfies* $||H^o(u)|| \leq M^o||u||$ *for some* M^o *and all* $u \in B(0,\gamma^o)$. *Suppose further that if* $0 < \hat{r} < \gamma^o$, *then the functions in* $H^o(\overline{B}(0,\hat{r}))$ *are equicontinuous at each* $t \in R^+$; *and if* (u_m) *is a sequence in* $\overline{B}(0,\hat{r})$ *such that* $u_m \to u$ *uniformly on each compact subinterval of* R^+, *then* $H^o(u_m) \to H^o(u)$ *uniformly on each compact subinterval of* R^+.

If $||dF(0;\cdot)||M^o < 1$, *then for each* $\varepsilon > 0$ *there exists a* $\delta_\varepsilon > 0$ *such that if* $f \in B(0,\delta_\varepsilon)$, *then*

$$(E[f,k,g,H^o])\ x(t) = f(t) + \int_0^t k(t,s)g(s,x(s))ds + H^o(x)(t)$$

has at least one solution in $B(0,\varepsilon)$.

Let $\varepsilon > 0$ be fixed and select ε_1 so that

$$(\varepsilon_1 + ||dF(0;\cdot)||)M^o = 1 - \lambda \tag{2.4.9}$$

where $0 < \lambda < 1$. Since F is continuously differentiable on $B(0,r_1)$, there is a δ_1 satisfying $0 < \delta_1 < r_1$, and such that if $f \in B(0,\delta_1)$, then

(2.4.10) $$||dF(f;\ \cdot)|| < \varepsilon_1 + ||dF(0;\ \cdot)||.$$

Choose a $\delta_2 > 0$ such that

(2.4.11) $$\delta_2 < \min\{\delta_1, M^o\gamma^o, M^o\varepsilon\}.$$

It follows from the mean value theorem that if f and f' belong to $B(0,\delta_2)$, then

(2.4.12) $$||F(f) - F(f')|| \leq (\varepsilon_1 + ||dF(0;\ \cdot)||)||f - f'||.$$

Set

(2.4.13) $$\rho^o = (1 - \lambda)\delta_2/M^o$$

and, for fixed $f \in B(0,\lambda\delta_2)$, define $\hat{H}^o\colon\ B(0,\rho^o) \to BC(R^+)$ by

(2.4.14) $$\hat{H}^o(u)(t) = f(t) + H^o(u)(t).$$

If $u \in B(0,\rho^o)$, then from (2.4.11), (2.4.13), (2.4.14) and the hypotheses on H^o, it follows that $||\hat{H}^o(u)|| \leq ||f|| + M^o||u||$. However, $(||f|| + M^o||u||) < (\lambda\delta_2 + M^o\rho^o)$ and from (2.4.13) it follows that $(\lambda\delta_2 + M^o\rho^o) = (\lambda\delta_2 + (1 - \lambda)\delta_2) = \delta_2$. So, $||\hat{H}^o(u)|| < \delta_2$ and since $\overline{B}(0,\delta_2) \subset B(0,r_1)$, the function $F \circ \hat{H}^o$ is well defined. Also, $||F(\hat{H}^o(u))|| = ||F(\hat{H}^o(u)) - F(0)||$ and, on using (2.4.12) and (2.4.9) one obtains

$$\begin{aligned} ||F(\hat{H}^o(u)) - F(0)|| &\leq (\varepsilon_1 + ||dF(0;\ \cdot)||)||\hat{H}^o(u)|| \\ &\leq (\varepsilon_1 + ||dF(0;\ \cdot)||)\delta_2 \\ &\leq (1 - \lambda)\delta_2/M^o. \end{aligned}$$

It follows from (2.4.13) that $F \circ \hat{H}^o\colon\ \overline{B}(0,\rho^o) \to \overline{B}(0,\rho^o)$.

We will verify the hypotheses of the Schauder-Tychonoff Theorem for the set $\overline{B}(0,\rho^o)$ and the mapping $F \circ \hat{H}^o$.

Clearly, $u \in \overline{B}(0,\rho^o)$ if and only if $||u(t)|| \leq \rho^o$. So with $\mu(t) = \rho^o$, one of the hypotheses is verified.

Suppose (u_m) is a sequence in $\overline{B}(0,\rho^o)$ and $u_m \to u$ uniformly on compact subintervals of R^+. Then $u \in \overline{B}(0,\rho^o)$ and, it follows from (2.4.14) that $||\hat{H}^o(u_m)(t) - \hat{H}^o(u)(t)|| = ||H^o(u_m)(t) - H^o(u)(t)||$. Since $H^o(u_m) \to H^o(u)$ uniformly on compact subintervals of R^+, the statement also holds for $\hat{H}^o$. It follows from Lemma 2.4.2 that $F \circ \hat{H}^o(u_m) \to F \circ \hat{H}^o(u)$ uniformly on compact subintervals of R^+.

Since $F \circ \hat{H}^o$: $\overline{B}(0,\rho^o) \to \overline{B}(0,\rho^o)$, the functions in the image set are bounded at each point.

In order to establish that the functions in $\overline{B}(0,\rho^o)$ are equicontinuous at each point, we suppose that $\eta > 0$ is fixed, $t_0 \in R^+$, $T = t_0 + 1$, $u \in \overline{B}(0,\rho^o)$ and $x = F(\hat{H}^o(u))$. The set $\Delta(T)$ is compact and so there exists a $K > 0$ such that $||k(t,s)|| \leq K$ on $\Delta(T)$. If $t > t_0$ and $t \in [0,T]$, then, on taking note of the relation

$$||g(t,s,x(s))|| \leq \int_0^1 ||g_x(s,\alpha x(s))||d\alpha||x(s)|| \leq P||x||,$$

we obtain the following inequalities:

$$(2.4.15)\quad \begin{aligned} ||x(t) - x(t_0)|| \leq\ & ||f(t) - f(t_0)|| \\ &+ \int_0^{t_0} ||k(t,s) - k(t_0,s)||\ ||g(s,x(s))||ds \\ &+ \int_{t_0}^{t} ||k(t,s)||\ ||g(s,x(s))||ds \\ &+ ||H^o(u)(t) - H^o(u)(t_0)|| \\ \leq\ & ||f(t) - f(t_0)|| \\ &+ P\rho^o \int_0^{t_0} ||k(t,s) - k(t_0,s)||ds \\ &+ KP\rho^o|t - t_0| + ||H^o(u)(t) - H^o(u)(t_0)||. \end{aligned}$$

There exists a $\zeta_1 > 0$ such that if $|t - t_0| < \zeta_1$, then $||f(t) - f(t_0)|| < \eta/4$. There exists a $\zeta_2 > 0$ such that if $|t - t_0| < \zeta_2$, then $||k(t,s) - k(t_0,s)|| < \eta/[4(PT\rho^o + 1)]$. There is a $\zeta_3 > 0$ such that if $|t - t_0| < \zeta_3$, then $||H^o(u)(t) - H^o(u)(t_0)|| < \eta/4$ for all $u \in \overline{B}(0,\rho^o)$. If $\zeta = \min\{\zeta_1,\zeta_2,\zeta_3,1/(KP\rho^o + 1)\}$, then, in view of (2.4.15), if $|t - t_0| < \zeta$, then $||x(t) - x(t_0)|| < \eta$. Since ζ does not depend on u, the functions in $F \circ \hat{H}^o(\overline{B}(0,\rho^o))$ are equicontinuous at each point.

A similar argument holds if $t < t_0$. As the hypotheses of the Schauder-Tychonoff Theorem have been verified, if $||f|| < \lambda\delta_2$, then $E[f,k,g,H^o]$ has a solution in $\overline{B}(0,\rho^o)$. Also, in view of (2.4.13) and (2.4.11), $\rho^o < \varepsilon$ and the theorem follows on setting $\delta_\varepsilon = \lambda\delta_2$.

COROLLARY 2.4.5. Assume that hypotheses (A) hold and let $F\colon B(0,r_1) \to B(0,r_2)$ be the implicit function of Theorem 2.2.5. Suppose k^o is an $n \times n$ matrix function with corresponding real number M^o, that satisfies hypothesis (A_4). Let g^o be a continuous function defined on $D(q_1) = \{(t,x) \in R^{n+1};\ t \in R^+$ and $||x|| < q_1\}$ with range in R^n that satisfies:

(i) $g^o(t,0) = 0$ for $t \in R^+$;

(ii) for each $\beta > 0$ there is a $\gamma > 0$ such that if $(t,x) \in D(q_1)$ and $||x|| < \gamma$, then $||g^o(t,x)|| \leq \beta||x||$.

Then for each $\varepsilon > 0$ there is a $\delta_\varepsilon > 0$ such that if $||f|| < \delta_\varepsilon$, then

$$x(t) = f(t) + \int_0^t k(t,s)g(s,x(s))ds + \int_0^t k^o(t,s)g^o(s,x(s))ds$$

has at least one solution in $B(0,\varepsilon)$.

Let $\beta^o > 0$ be such that $||dF(0;\ \cdot)||\beta^o M^o < 1$. There exists a $\gamma^o > 0$, satisfying $\gamma^o < q_1$, and such that if $||x|| < \gamma^o$, then

(2.4.16) $$||g^o(t,x)|| < \beta^o||x||.$$

Define $H^o\colon\ B(0,\gamma^o) \to BC(R^+)$ by

$$H^o(u)(t) = \int_0^t k^o(t,s)g^o(s,x(s))ds.$$

Then

$$||H^o(u)(t)|| \leq \beta^o\int_0^t ||k^o(t,s)||\ ||u(s)||ds \leq \beta^o M^o||u||$$

and, since $H^o(u)$ is a continuous function of t, it follows that $H^o(u) \in BC(R^+)$.

Let ρ^o be a real number that satisfies $0 < \rho^o < \gamma^o$, and suppose that (u_m) is a sequence in $\overline{B}(0,\rho^o)$ that converges to u uniformly on compact subintervals of R^+. Assume that $J \subset R^+$ is a compact interval and suppose $T > 0$ is such that $J \subset [0,T]$. Let $K^o = \sup\{||k^o(t,s)||;\ (t,s) \in \Delta(T)\}$. Then for $t \in [0,T]$, we have

$$\begin{aligned} ||H^o(u_m)(t) - H^o(u)(t)|| &\leq \int_0^t ||k^o(t,s)||\ ||g^o(s,u_m(s)) \\ &\qquad - g^o(s,u(s))||ds \\ &\leq TK^o \sup\{||g^o(s,u_m(s)) \\ &\qquad - g^o(s,u(s))||;\ s \in [0,T]\}. \end{aligned}$$

Since g^o is continuous and $u_m \to u$ uniformly on $[0,T]$, it follows that $H^o(u_m) \to H^o(u)$ uniformly on $[0,T]$, and therefore on J. Hence, $H^o(u_m) \to H^o(u)$ uniformly on compact subintervals of R^+.

If $u \in \overline{B}(0,\rho^o)$, $t_0 \in R^+$ and $t > t_0$ (a similar argument holds if $t < t_0$), then

$$\begin{aligned} ||H^o(u)(t) - H^o(u)(t_0)|| &\leq \int_0^t ||k^o(t,s) - k^o(t_0,s)||\ ||g^o(s,u(s))||ds \\ &\quad + \int_{t_0}^t ||k^o(t,s)||\ ||g^o(s,u(s))||ds. \end{aligned}$$

In view of (2.4.16), the continuity of k^o and the boundedness of $\int_0^t ||k(t,s)||ds$, we have that the functions in $H(\overline{B}(0,\rho^o))$ are equicontinuous at t_0.

As the hypotheses of Theorem 2.4.2 are satisfied, the result follows.

COROLLARY 2.4.6. <u>Assume that hypotheses</u> (A) <u>hold and let</u> $F\colon B(0,r_1) \to B(0,r_2)$ <u>be the implicit function of Theorem</u> 2.2.5. <u>Suppose that</u> k^o <u>is a continuous</u> $n \times n$ <u>matrix function defined on</u> $R^+ \times R^+$ <u>and that</u> M^o <u>is a positive real number which together satisfy the following conditions</u>:

(i) $\int_0^\infty ||k^o(t,s)||ds \leq M^o$ <u>for</u> $t \in R^+$;

(ii) $\lim_{h \to 0} \int_0^\infty ||k^o(t + h,s) - k^o(t,s)||ds = 0$ *for* $t \in R^+$.

Let g^o *satisfy the hypotheses of Corollary* 2.4.5.

Then for each $\varepsilon > 0$ *there is a* $\delta_\varepsilon > 0$ *such that if* $||f|| < \delta_\varepsilon$, *then*

$$x(t) = f(t) + \int_0^t k(t,s)g(s,x(s))ds + \int_0^\infty k^o(t,s)g^o(s,x(s))ds$$

has at least one solution in $B(0,\varepsilon)$.

Let $\beta^o > 0$ satisfy $||dF(0; \cdot)||\beta^o M^o < 1$. There exists a $\gamma^o > 0$, satisfying $\gamma^o < q_1$, such that if $||x|| < \gamma^o$, then $||g^o(t,x)|| < \beta^o||x||$.

Define H^o: $B(0,\gamma^o) \to BC(R^+)$ by

$$H^o(u)(t) = \int_0^\infty k^o(t,s)g^o(s,u(s))ds.$$

Then, if $u \in B(0,\gamma^o)$,

$$||H^o(u)(t)|| \leqslant \int_0^\infty ||k^o(t,s)||\ ||g^o(s,u(s))||ds \leqslant \beta^o M^o||u||.$$

Consequently, $H^o(u) \in BC(R^+)$ and $||H^o(u)|| \leqslant \beta^o M^o||u||$.

Let ρ^o be a real number satisfying $0 < \rho^o < \gamma^o$ and suppose (u_m) is a sequence in $\overline{B}(0,\rho^o)$ that converges to u uniformly on compact subintervals of R^+. If T is a positive real number, then, with $t \in [0,T]$,

$$\begin{aligned} ||H^o(u_m)(t) - H^o(u)(t)|| &\leqslant \int_0^\infty ||k^o(t,s)||\ ||g^o(s,u_m(s)) \\ &\qquad - g^o(s,u(s))||ds \\ &\leqslant M^o \sup\{||g^o(s,u_m(s)) \\ &\qquad - g^o(s,u(s))||;\ s \in [0,T]\}. \end{aligned}$$

It follows that $H^o(u_m) \to H^o(u)$ uniformly on $[0,T]$. As we have seen before, this is sufficient to insure that $H^o(u_m) \to H^o(u)$ uniformly on compact subintervals of R^+.

If $u \in \overline{B}(0,\rho^o)$ and $t_0 \in R^+$, then

$$||H^o(u)(t) - H^o(u)(t_0)|| \leqslant \beta^o\rho^o\int_0^\infty||k^o(t,s) - k^o(t_0,s)||ds.$$

By hypothesis, $\int_0^\infty||k^o(t,s) - k^o(t_0,s)||ds \to 0$ as $t \to t_0$ and so it follows that the functions in $H^o(\overline{B}(0,\rho^o))$ are equicontinuous at each point.

An application of Theorem 2.4.2 yields the result.

Corollaries 2.4.5 and 2.4.6 are similar to results obtained by Nohel [12]. The result of Nohel corresponding to our Corollary 2.4.6 is strictly nonlinear in the sense that Nohel's hypotheses exclude the case where $g(t,x) = x$. However, our hypotheses admit the possibility that $g(t,x) = x$.

5. <u>A nonlinear variation of constants formula</u>. In this section we will make use of the notions of derivative and integral of a function defined on an interval of real numbers with values in a Banach space. A full discussion of these topics may be found in [7; Ch. 8]. However, we will list those definitions and results that will be used in the sequel.

Let $[a,b]$ be a non-degenerate compact interval of real numbers, and suppose that X and Y are Banach spaces. Let $A \subset X$ be an open set and let $h\colon [a,b] \to A$ and $H\colon A \to Y$ be functions.

For fixed $s_0 \in [a,b]$, $h'(s_0)$ is defined by $h'(s_0) = \lim_{s \to s_0} [(h(s) - h(s_0))/(s - s_0)]$ provided the limit exists. The vectors $h'(a)$ and $h'(b)$ are defined by one-sided limits. If H is Fréchet differentiable on A and h' exists on $[a,b]$, then it follows from the discussion in [7; pp. 149, 150] that $(H \circ h)'$ exists and

$$(H \circ h)'(s) = dH(h(s); h'(s)) \tag{2.5.1}$$

for each $s \in [a,b]$. It is also true that if h' is continuous on $[a,b]$, then

$$h(b) - h(a) = \int_a^b h'(s)ds, \tag{2.5.2}$$

where the preceding integral and formula are respectively defined and deduced in [7; pp. 160, 161].

Recall that $E[f,k,g]$ denotes the equation

$$x(t) = f(t) + \int_0^t k(t,s)g(s,x(s))ds.$$

THEOREM 2.5.1. _Assume that hypotheses_ (A) _hold and let_ $F\colon\ B(0,r_1) \to B(0,r_2)$ _be the implicit function of Theorem_ 2.2.5. _Suppose that_ f^o _and_ f _belong to_ $B(0,r_1)$ _and that_ x _and_ y _are the respective solutions of_ $E[f^o,k,g]$ _and_ $E[f,k,g]$. _Then_ y _satisfies_

$$y(t) = x(t) + f(t) - f^o(t) + \int_0^1\int_0^t r(t,s,\sigma)(f(s) - f^o(s))ds\,d\sigma \tag{2.5.3}$$

where $r(t,s,\sigma)$ _is the kernel reciprocal to_ $k(t,s)g_x(s,x(s,\sigma))$, _and_ $x(t,\sigma)$ _is the solution of_ $E[f^o + \sigma(f - f^o),k,g]$ _for each_ $\sigma \in [0,1]$.

Theorem 2.2.5 and Theorem 2.2.7 guarantee the existence of a unique solution, $x = x(t,\sigma)$, in $B(0,r_2)$ of $E[f^o + \sigma(f - f^o),k,g]$ for each $\sigma \in [0,1]$.

Define $\lambda\colon\ [0,1] \to B(0,r_1)$ by

$$\lambda(\sigma) = f^o + \sigma(f - f^o), \tag{2.5.4}$$

and note that

$$\lambda'(\sigma) = f - f^o \tag{2.5.5}$$

for each $\sigma \in [0,1]$. Also, define $\Lambda\colon\ [0,1] \to B(0,r_2)$ by $\Lambda(\sigma) = (F \circ \lambda)(\sigma)$. It follows from Theorem 2.2.5 that $x(\cdot,\sigma) = (F \circ \lambda)(\sigma) = \Lambda(\sigma)$ is the solution of $E[f^o + \sigma(f - f^o),k,g]$ and that F is continuously differentiable on $B(0,r_1)$. Since λ is continuously differentiable on $[0,1]$, it follows that Λ is continuously differentiable on $[0,1]$. Also, in view of (2.5.1) and Theorem 2.2.6, we have

$$\begin{aligned}\Lambda'(\sigma) &= dF(\lambda(\sigma);\ \lambda'(\sigma)) \\ &= \lambda'(\sigma) + \int_0^{\cdot} r(\cdot,s,\sigma)\lambda'(\sigma)(s)ds.\end{aligned}$$

Taking note of (2.5.2), we obtain

$$\Lambda(1) - \Lambda(0) = \lambda(1) - \lambda(0) + \int_0^1\int_0^{\cdot} r(\cdot,s,\sigma)\lambda'(\sigma)(s)ds\,d\sigma$$

or

(2.5.6) $x(\cdot,1) - x(\cdot,0) = f - f^o + \int_0^1\int_0^{\cdot} r(\cdot,s,\sigma)(f(s) - f^o(s))dsd\sigma$.

On evaluating each side of (2.5.6) at $t \in R^+$, we obtain

(2.5.7) $x(t,1) - x(t,0) = f(t) - f^o(t) + \int_0^1\int_0^t r(t,s,\sigma)(f(s) - f^o(s))dsd\sigma$.

Since $y(t) = x(t,1)$ and $x(t) = x(t,0)$, the result is established.

COROLLARY 2.5.1. _Assume that hypotheses_ (A) _hold and let_ F: $B(0,r_1) \to B(0,r_2)$ _be the implicit function of Theorem_ 2.2.5. _If_ C _is a subset of_ $BC(R^+)$ _and_ H: $C \to B(0,r_1)$, _suppose_ $y \in C$ _and_ $f \in B(0,r_1)$ _are such that_ $f + H(y) \in B(0,r_1)$. _Assume further that_ y _satisfies_

(E[f,k,g,H]) $y(t) = f(t) + \int_0^t k(t,s)g(s,y(s))ds + H(y)(t)$

_on_ R^+. _Then, if_ x _is the solution of_ E[f,k,g],

(2.5.8) $y(t) = x(t) + \int_0^1\int_0^t r(t,s,\sigma)H(y)(s)dsd\sigma + H(y)(t)$.

Since $f + H(y) \in B(0,r_1)$, the equation

$$z(t) = f(t) + \int_0^t k(t,s)g(s,z(s))ds + H(y)(t),$$

has a unique solution which we will call z. Hence,

$$y(t) - z(t) = \int_0^t k(t,s)[g(s,y(s)) - g(s,z(s))]ds$$

and the argument given in Theorem 2.2.7, concerning uniqueness, establishes that $y(t) = z(t)$ for $t \in R^+$. An appeal to Theorem 2.5.1 establishes the result.

We will call equation (2.5.3) a _variation of constants formula_.

We note that in the case $g(t,x) = x$, the difference in the solutions of $y(t) = f(t) + \int_0^t k(t,s)y(s)ds$ and $x(t) = f^o(t) + \int_0^t k(t,s)x(s)ds$ may be computed directly and it is found to be

(2.5.9) $y(t) - x(t) = f(t) - f^o(t) + \int_0^t r(t,s)(f(s) - f^o(s))ds$,

where $r(t,s)$ is the kernel reciprocal to $k(t,s)$. In this case, $k(t,s)g_x(t,x(t,\sigma)) = k(t,s)$ so that $r(t,s,\sigma) = r(t,s)$. It follows that (2.5.3) reduces to (2.5.9).

LEMMA 2.5.1 _Assume that hypotheses_ (A) _hold and let_ F: $B(0,r_1)$

$\to B(0,r_2)$ _be the implicit function of Theorem_ 2.2.5. _Suppose_ f _and_ f^o _belong to_ $B(0,r_1)$, $x(\cdot,\sigma)$ _is the solution of_ $E[f^o + \sigma(f - f^o),k,g]$ _for each_ σ _in_ $[0,1]$ _and_ $r(t,s,\sigma)$ _is the kernel reciprocal to_ $k(t,s)g_x(s,x(s,\sigma))$. _Then, for_ T _fixed in_ R^+ _and_ $(t,s,\sigma) \in \Delta(T) \times [0,1]$, _the kernel_ $r(t,s,\sigma)$ _is continuous in_ (t,s,σ).

The map $\sigma \mapsto f^o + \sigma(f - f^o)$ from $[0,1]$ to $B(0,r_1)$ is continuous; and since F is continuous, $x(\cdot,\sigma) = F(f^o + \sigma(f - f^o))$ is a continuous function of σ in the topology of $BC(R^+)$.

Let $\varepsilon > 0$ and (t_0,s_0,σ_0) be fixed in R and $\Delta(T) \times [0,1]$ respectively. Then with $(t,s,\sigma) \in \Delta(T) \times [0,1]$,

$$\begin{aligned} &||k(t,s)g_x(s,x(s,\sigma)) - k(t,s)g_x(s,x(s,\sigma_0))|| \\ &\quad \leq \sup\{||k(t,s)||;\ (t,s) \in \Delta(T)\} \\ &\qquad ||g_x(s,x(s,\sigma)) - g_x(s,x(s,\sigma_0))||. \end{aligned}$$

From hypothesis (A_3) and Lemma 1.2.2, it follows that there is a $\delta_1 > 0$ such that if $|\sigma - \sigma_0| < \delta_1$ and $\sigma \in [0,1]$, then $||r(t,s,\sigma) - r(t,s,\sigma_0)|| < \varepsilon/2$ uniformly in (t,s) on $\Delta(T)$.

For each fixed σ, $r(t,s,\sigma)$ is a continuous function of (t,s). So there is a $\delta_2 > 0$, depending on (t_0,s_0,σ_0), such that $||r(t,s,\sigma_0) - r(t_0,s_0,\sigma_0)|| < \varepsilon/2$ provided $||(t,s) - (t_0,s_0)|| < \delta_2$ and $(t,s) \in \Delta(T)$. If $\delta_\varepsilon = \min\{\delta_1,\delta_2\}$, then it follows that if $(t,s,\sigma) \in \Delta(T) \times [0,1]$ and $||(t,s,\sigma) - (t_0,s_0,\sigma_0)|| < \delta_\varepsilon$, then

$$\begin{aligned} ||r(t,s,\sigma) - r(t_0,s_0,\sigma_0)|| &\leq ||r(t,s,\sigma) - r(t,s,\sigma_0)|| \\ &\quad + ||r(t,s,\sigma_0) - r(t_0,s_0,\sigma_0)|| \\ &< \varepsilon. \end{aligned}$$

The result is established.

COROLLARY 2.5.2. _Under the hypotheses of Theorem_ 2.5.1, _the formula_ (2.5.3) _may be written_

$$(2.5.10) \qquad y(t) = x(t) + f(t) - f^o(t) + \int_0^t[\int_0^1 r(t,s,\sigma)d\sigma](f(s) - f^o(s))ds.$$

Let t be fixed in R^+. Making use of Lemma 2.5.1, we see that the map, $(s,\sigma) \mapsto r(t,s,\sigma)(f(s) - f^o(s))$ from $[0,t] \times [0,1]$ into R^n, is continuous. It follows from elementary calculus that

$$\int_0^1[\int_0^t r(t,s,\sigma)(f(s) - f^o(s))ds]d\sigma = \int_0^t[\int_0^1 r(t,s,\sigma)d\sigma](f(s) - f^o(s))ds,$$

and consequently equation (2.5.10) holds.

THEOREM 2.5.2. _Assume that hypotheses_ (A) _hold and let_ $F\colon B(0,r_1) \to B(0,r_2)$ _be the implicit function of Theorem_ 2.2.5. _If_ $f \in B(0,r_1)$ _and_ x _is the solution of_ $E[f,k,g]$, _then_

$$(2.5.11) \qquad x(t) = f(t) + \int_0^t[\int_0^1 r(t,s,\sigma)d\sigma]f(s)ds$$

where $r(t,s,\sigma)$ _is the kernel reciprocal to_ $k(t,s)g_x(s,x(s,\sigma))$, _and_ $x(t,\sigma)$ _is the solution of_ $E[\sigma f,k,g]$ _for each_ σ _in_ $[0,1]$.

Furthermore, if the following additional conditions hold,

(i) $\lim_{t \to \infty} ||f(t)|| = 0,$

(ii) $\lim_{t \to \infty} \int_0^T||\int_0^1 r(t,s,\sigma)d\sigma||ds = 0$ _for each_ $T > 0$,

(iii) _there exists a real number_ $M_1 > 0$ _such that_

$$\int_0^t||\int_0^1 r(t,s,\sigma)d\sigma||ds \leq M_1 \text{ for } t \in R^+,$$

then $\lim_{t \to \infty} ||x(t)|| = 0.$

Since 0 is the solution of $E[0,f,g]$, it follows from Theorem 2.5.1 and Corollary 2.5.2 that (2.5.11) holds.

Fix $\varepsilon > 0$ and choose a real number $T_1 > 0$ so that if $t \geq T_1$, then

$$(2.5.12) \qquad ||f(t)|| < \max\{\varepsilon/3, \varepsilon/(3M_1)\}.$$

Select a real number T_2 satisfying $T_2 > T_1$ such that if $t > T_2$, then

$$(2.5.13) \qquad \int_0^{T_1}||\int_0^1 r(t,s,\sigma)d\sigma||ds < \varepsilon/[3(1 + ||f||)].$$

Then, with $t > T_2$ we have that

$$||x(t)|| \leq ||f(t)|| + \int_0^{T_1}||\int_0^1 r(t,s,\sigma)d\sigma|| \; ||f(s)||ds$$

$$+ \int_{T_1}^{t} ||\int_0^1 r(t,s,\sigma)d\sigma||\ ||f(s)||ds.$$

In view of (2.5.12) and (2.5.13) it follows that $||x(t)|| < \varepsilon$ and therefore $\lim_{t\to\infty} ||x(t)|| = 0$.

COROLLARY 2.5.3. Assume that hypotheses (A) hold and let $F\colon B(0,r_1) \to B(0,r_2)$ be the implicit function of Theorem 2.2.5. If f and f^o belong to $B(0,r_1)$ and x and x^o are the respective solutions of $E[f,k,g]$ and $E[f^o,k,g]$, then $\lim_{t\to\infty} ||x(t) - x^o(t)|| = 0$ provided $\lim_{t\to\infty} ||f(t) - f^o(t)|| = 0$ and $r(t,s,\sigma)$ satisfies the hypotheses of Theorem 2.5.2.

In view of Theorem 2.5.1 and Corollary 2.5.2, we have

$$x(t) - x^o(t) = f(t) - f^o(t) + \int_0^t (\int_0^1 r(t,s,\sigma)d\sigma)(f(s) - f^o(s))ds.$$

An application of Theorem 2.5.2 yields the result.

The hypotheses placed on the kernel $\int_0^1 r(t,s,\sigma)d\sigma$ are similar to those of Nohel [12], which considered the case of kernels not depending on σ.

In a recent paper Brauer [3] established a variation of constants formula for nonlinear Volterra integral equations under different hypotheses and by entirely different methods. The connection between these two formulas is unknown to this author. However, our work does permit us to give a formal derivation of a variation of constants formula which Brauer [3] attributes to Miller [10].

In Corollary 2.5.1, let $g(t,x) = x$ and

$$H(y)(t) = \int_0^t k(t,s)w(y(s))ds$$

where w is a suitably defined function. Then

$$y(t) - x(t) = \int_0^t k(t,s)w(y(s))ds + \int_0^1\int_0^t r(t,s)\int_0^s k(s,u)w(y(u))du\,ds\,d\sigma \tag{2.5.14}$$

where $r(t,s)$ is the kernel reciprocal to $k(t,s)$. As in [5; p. 125], one may verify that

$$r(t,s) - k(t,s) = \int_s^t r(t,u)k(u,s)du. \tag{2.5.15}$$

On using Dirichlet's formula for interchanging the order of integration, then exchanging the dummy variables s and u and finally, invoking (2.5.15), the equation (2.5.14) may be written

$$\begin{aligned} y(t) - x(t) &= \int_0^t k(t,s)w(y(s))ds \\ &\quad + \int_0^t \left(\int_s^t r(t,u)k(u,s)du\right)w(y(s))ds \\ &= \int_0^t r(t,s)w(y(s))ds, \end{aligned}$$

which is Miller's formula.

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