RARE DECAYS OF THE Z° BOSON

AND THE π^0 MESON

By

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PREFACE

This work is devoted to calculations of some rare decay processes.

We have studied the decay $Z^0 \to gg\gamma$ and the vector current contribution to $Z^0 \to ggg$ for massive quarks in the loop. We have found an interesting coherence effect which results in the $Z^0 \to gg\gamma$ and $Z^0 \to ggg$ decay widths being sensitive functions of the top quark mass.

We have explored an alternative to the standard models for the decays $\pi^0 \to e^+e^-$ and $\eta \to \mu^+\mu^-$. By postulating a once-subtracted dispersion relation for the amplitude, with the subtraction constant fixed to be small, we have achieved excellent agreement with the existing experimental data for these decays.

Finally, we have examined the radiative correction problem in the decay $\pi^0 \to e^+e^-\gamma$. Contrary to expectations, we have found the two-photon-exchange contribution to be non-negligible, with important consequences for the measurements of the π^0 form-factor slope.

I would like to express my appreciation and graditude to my advisor, Dr. Mark A. Samuel, both for his guidance and for several enjoyable collaborations. I also wish to thank my colleague, Dr. Morten Laursen, for his friendship and for many simulating discussions. I am thankful to Dr. Kimball Milton, Dr. N. V. V. J. Swamy, and Dr. John Chandler for serving on my Committee. I would like to acknowledge the support of the physics Department in the form of teaching assistantships and the support provided by the United States Department of Energy through research assistantships. I wish to thank Michele Plunkett for

her excellent typing. Last but not least, I wish to express my deep appreciation to my mother, Gertrude E. Tupper, to whom this thesis is dedicated, for her love, understanding and support through the years.

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CHAPTER I

INTRODUCTION

During the last decade high-energy electron-positron storage rings have come to occupy an important role in elementary particle physics as a proving ground for quantum chromodynamics (QCD)[1], the SU(3) gauge theory[2] of the strong interactions. The principal reason for the special importance of e^+e^- colliders is that the leptons do not participate in the strong interactions while e^+e^- annihilation is well understood within quantum electrodynamics, so allowing for a clean study of the hadronic decays of the virtual photon, with the photon "mass" adjustable by tuning the e^+e^- beam energy. The measurements of $R \equiv \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$ showing the need for color, the discovery of the charmed quark, the detailed studies of J/ψ and upsilon spectroscopy and the studies of three-jet events demonstrating the existence of the gluon, are a few of the triumphs that have been achieved[3].

The future of e^+e^- machines, however, is in doubt, largely due to the rapid rise of synchrotron radiation losses with increasing energy and fixed ring size, resulting in prohibitive trade offs between construction and operating costs[4]. Although this problem may eventually be bypassed through the innovation of the single-pass linear collider[5], there is still one heyday to be had before such a technological leap is required: resonance production of the $Z°$ intermediate vector boson. The

$Z°$ is the netural partner to the $W^{\pm}$ weak bosons which mediate β decay; its mass and couplings are predicted within the enormously successful Weinberg-Salam-Glashow (W-S-G) SU(2) x U(1) model of electro-weak interactions[6], and have been essentially confirmed in $Z°$ production experiments at the CERN $p\bar{p}$ collider[7]. Indeed, the W-S-G model has now come to be commonly accepted on the same footing as QED.

By tuning to the narrow $Z°$ resonance, the new e^+e^- machines LEP at CERN[4] and SLC at SLAC[5] will allow for copious production of the $Z°$ and extensive examination of its hadronic decay modes. Although the decay $Z^0 \rightarrow q\bar{q}g$ is of most immediate interest as a measure of the QCD coupling constant α_s the proposed luminosity $L = 10^{32}\ cm^{-2}\ s^{-1}$ for LEP corresponds to some 1.5×10^5 $q\bar{q}$ events per day, so that one may contemplate studies of rare decay modes, in particular $Z^0 \rightarrow gg\gamma$ and $Z^0 \rightarrow ggg$. These latter decays, which occur through a quark loop, have previously been examined in an approximation where all quark masses are set to zero[8]. In the W-S-G model the cancellation of axial anomalies necessitates the existence of the top quark, but does not fix its mass[9]. Since the as yet unobserved top quark may well have a mass not too far from that of the $Z°$, the massless quark approximation may be invalid; thus in Chapter II we shall study the decays $Z^0 \rightarrow gg\gamma$ and $Z^0 \rightarrow ggg$ for massive quarks in the loop[10].

While the prospect of doing physics at new energy scales is always exciting, there remain interesting problems belonging to what is often described as the "intermediate energy" regime. An example in this regard is the rare decay $\pi^0 \rightarrow e^+e^-$. Although the direct decay $\pi^0 \rightarrow e^+e^-$ was first discussed by Drell[11] some twenty five years ago, it is only within the last six years that any experimental data has been accumulated[12]; moreover, the existing data points to a $\pi^0 \to e^+e^-$ branching ratio far larger than may be explained by the conventional models for this decay. The basic elements of these models were laid down far in advance of the experimental observations and have acquired a certain rigidity with the passage of time. Unlike the standard SU(3) x SU(2) x U(1) theory in the perturbative region, where well defined calculational rules apply, the construction of phenomenological models for $\pi^0 \to e^+e^-$ allows some latitude in defining the computational scheme. This problem is taken up in Chapter III where we examine what may be accomplished by relaxing one's theoretical prejudices and extend our ideas to the decays $\eta \to \mu^+\mu^-$ and $\eta \to e^+e^-$ [13].

Interest in the decay $\pi^0 \to e^+e^-\gamma$ has an even longer history, it having been first discussed by Dalitz[14] in 1951. In this case there has been a wealth of data but a paucity of insight into the problems presented by the experimental measurements. At issue is the π^0 'charge radius', more commonly referred to as the form-factor-slope a_π; as a_π is expected to represent a small effect in the Daliz plot, order α radiative corrections become important. Unfortunately the majority of experiments were performed before there was any serious treatment of the radiative correction problem. While the so called one-photon-exchange corrections were finally understood in a series of papers in the years 1971-72[15], it was still expected that the two-photon-exchange corrections, while contribute to the same order in α, should be small. In Chapter IV we will study in detail the question of whether the two-photon-exchange corrections are in fact small or instead represent a significant effect[16].

A summary of our results and some concluding remarks are given in

Chapter V. In order to not over burden the text, many of the longer and/or more tedious derivations have been consigned to a series of appendicies. The notation and conventions of Bjorken and Drell's "Relativistic Quantum Mechanics"[17] has been followed throughout.

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CHAPTER II

Z° DECAY INTO TWO GLUONS AND A PHOTON FOR MASSIVE QUARKS

A. Introduction

The study of the Z° boson through its decays into two and three jets is of immediate interest due to the fact that the new electron-positron colliders will be able to produce a substantial number of Z°'s in the near future[1].

The rare decays $Z^0 \to gg\gamma$ and $Z^0 \to ggg$ offer an interesting way of testing the standard model of electroweak and strong interactions in higher-order perturbation theory. The differential as well as the total decay rates for these processes have been calculated via the box graph, with all the quark masses in the loop set to zero[2,3].

In this paper, we extend these calculations by including the dependence on the quark masses. We shall mainly concentrate on the process $Z^0 \to gg\gamma$, because the axial-vector coupling is not involved; for the process $Z^0 \to ggg$ we shall report on the vector part only.

Our analysis shows that for a single quark flavor the total decay rate does not change substantially except near the limit $\rho = (2\,m_q/M_Z)^2 \to 1$ (i.e., close to the threshold for producing $q\bar{q}$ pair). As expected, we also find a discontinuity in slope of the decay rate at the point $\rho = 1$.

The single-and double-differential rates we found to be more sensitive to the quark masses. A useful approximation, enabling us to study the quark-mass dependence of various functions, as well as reducing the computational time, has been found by setting all the five quark masses (up to the bottom) equal to m_u and by treating the top quark mass m_t as a free parameter.

B. Details of the Calculations

To investigate the decay $Z^0 \to gg\gamma$ we only have to consider the box diagrams shown in Figure 1. Owing to the symmetry in the color indices ($Tr[T_aT_b] = \frac{1}{2}\delta_{ab}$) only the vector couplings of the Z° to $q_i\bar{q}_i$ have to be included. For the other decay $Z^0 \to ggg$ (see Figure 2) the axial-vector couplings also contribute. In the limit of vanishing quark masses this was not the case, since the axial-vector contribution within each doublet would cancel. Since the contribution from the vector and axial-vector parts add incoherently we can establish a lower bound on the rate for $Z^0 \to ggg$.

The general derivation of the double-differential decay rate has been elegantly described in reference 3, before the $m_q \to 0$ limit was taken, so we only quote the results; after averaging over the initial spins and summing over the helicities one obtains:

$$\frac{1}{\Gamma_o}\,\frac{d^2\Gamma(Z^0\to gg\gamma)}{dx\,dy} = \frac{1}{256}\,[\frac{\alpha_s}{\pi}]^2[\frac{\alpha}{\pi}]\,[\frac{C^{gg\gamma}}{2!}]\frac{[\sum_i a_iQ_i]}{\sum_i(a_i^2+b_i^2)}\,\frac{d^2F}{dxdy} \tag{2-1}$$

We have of course normalized against the total hadronic decay width:

$$\Gamma_o = \sum_i \Gamma(Z^0 \to q_i\bar{q}_i) = \frac{M_Z^3G_F}{2\sqrt{2}\,\pi}\sum_i (a_i^2 + b_i^2) \tag{2-2}$$

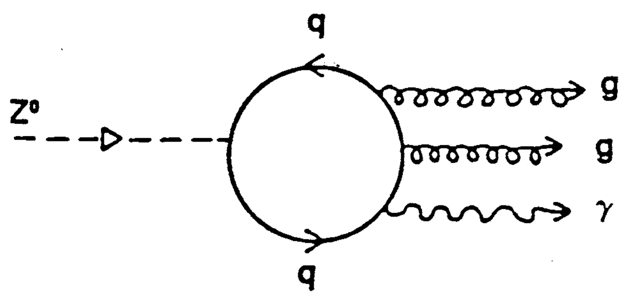

Figure 1. Feynman Diagrams for the Decay
$Z° \rightarrow gg\gamma$

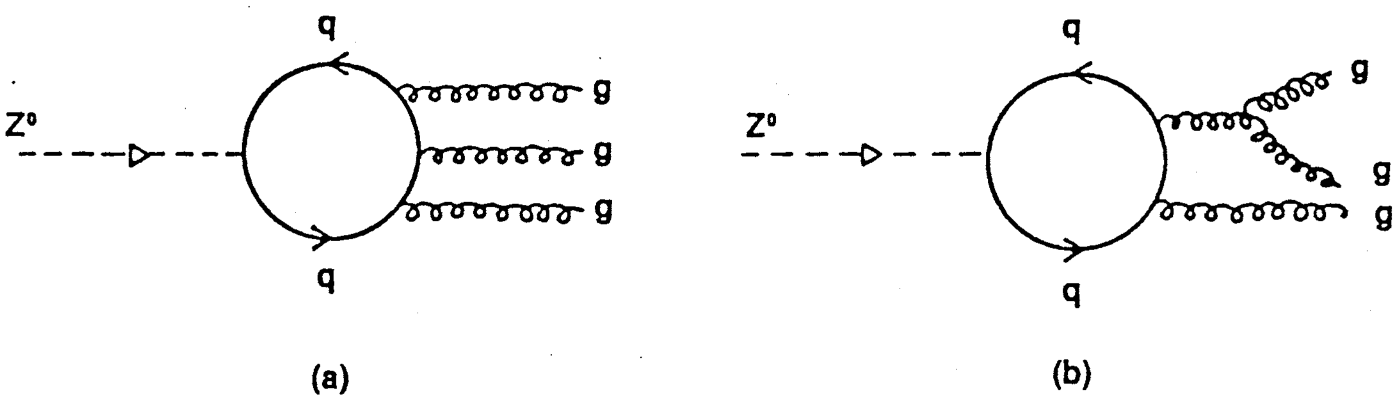

Figure 2. Feynman Diagrams for the Decay Z° → ggg

and we have also introduced a factor of 1/2! since we are below the color threshold. Here Q_i is the electric charge of quark i

$$Q_u = Q_c = Q_t = \frac{2}{3}, \; Q_d = Q_s = Q_b = -\frac{1}{3}$$

and $C^{gg\gamma} = 8$ is a color factor. a_i and b_i are the usual vector and axial-vector couplings of Z° to $q_i\bar{q}_i$.

$$a_i = b_i - 2\, Q_i \sin^2\theta w,$$

$$b_u = -\, b_d = b_c = -\, b_s = b_t = -\, b_b = \frac{1}{2}$$

x and y are the usual scaling variables $x = 2E_a/M_Z$ and $y = 2\, E_b/M_Z$.

The function of $d^2F/dxdy$ is given by

$$\frac{d^2F}{dxdy} = \frac{2}{3}\left\{ \left| M_{+++}(x,y,z) \right|^2 + (x \leftrightarrow y) + (x \rightarrow z) + \left| M_{-++}(x,y,z) \right|^2 \right\} \tag{2-3}$$

where

$$\begin{aligned} \left|M_{\lambda++}(x,y,z)\right|^2 = 8\Big\{ & \frac{y(1-y)}{x(1-x)}\left|E^{(1)}_{\lambda++}(x,y,z)\right|^2 + \frac{z(1-z)}{x(1-x)}\left|E^{(1)}_{\lambda++}(x,z,y)\right|^2 + \\ & - 2\,\frac{(1-y)(1-z)}{x(1-x)}\, R_e\left[E^{(1)}_{\lambda++}(x,y,z)E^{(1)*}_{\lambda++}(x,z,y)\right] + \\ & + \left|E^{(2)}_{\lambda++}(x,y,z)\right|^2 \Big\} \end{aligned} \tag{2-4}$$

and, as usual,

$$x + y + z = 2 \tag{2-5}$$

Whereas in the limit of vanishing quark masses the $E^{(i)}_{\lambda++}(x,y,z)$ where real and comparatively simple functions of the scaling variables, in the case of massive quarks they are given by the coherent sum

$$E^{(i)}_{\lambda++}(x,y,z) = \frac{1}{\sum_i a_i Q_i} \sum_i a_i Q_i \; E^{(i)}_{\lambda++}\left(\frac{1-x}{\rho_i}, \frac{1-y}{\rho_i}, \frac{1-z}{\rho_i}\right) \qquad (2\text{-}6)$$

over the complex helicity amplitudes[4] $E^{(i)}_{\lambda++}$ (r,s,t) as functions of the modified Moller-Mandelstam variables[3] r, s and t satisfying

$$r + s + t = -\mu_1 \geqslant 0 \qquad (2\text{-}7)$$

The expressions for $E^{(i)}_{\lambda++}$ (r,s,t) are given in Appendix A, where we also exhibit the functions $B(r)$, $T(r)$ and $I_0(r,s,\gamma_1)$ on which they depend, in terms of elementary and dilogarithmic[5] functions. In Appendix B we derive various expansions for $B(r)$, $T(r)$ and $I_0(r,s,\mu_1)$, ; finally, in Appendix C, these expansions are used to prove that $|M_{\lambda++}(x,y,z)|^2$ is free of any infrared or collinear ($x \to 0$ or $x \to 1$) type singularities. This is in contrast to the massless case where $d^2F/dxdy$ behaves as $\ln^2(z)$ $[\ln^2(1-z)]$ for $Z \to 0$ $[Z \to 1]$ and is due to the parameter ρ_i which cuts off the logarithmic behavior near the edges of the phase space.

C. Results for a Single-Quark Contribution

As noted above $d^2F/dxdy$ is free of any infrared or collinear singularities so we can safely integrate numerically the single differential rate dF/dx and the total $F(\rho)$ as a function of ρ . For $\rho \to 0$ we should reproduce the answer[3] $F(0) \simeq 80$. The other limit,

$\rho \to 1$, corresponds to closing the channel for the real decay $Z^0 \to q\bar{q}$. A plot of $F(\rho)$ versus $\sqrt{\rho}$ ($\rho < 1$) is exhibited in Figure 3 - it shows a very small rise up to $\rho \leq 0.1$ and as we reach $\rho \to 1$ the limited phase space cuts $F(\rho)$ severely. At $\rho = 1$ the value of F is roughly one-third of F(0). Figure 4 shows the behavior of $F(\rho)$ for $\rho > 1$. For ρ values between 10 and 100 it approaches the asymptotic expression (see Appendix D)

$$F(\rho) = \frac{4352}{30375} \frac{1}{\rho^4} \tag{2-8}$$

As ρ approaches 1 from above, this approximation fails badly, roughly by two orders of magnitude. When the two curves are matched at $\rho=1$ we find a discontinuity in the slope. This of course is expected since for $\rho > 1$ the amplitudes are real functions while for $1 \geq \rho \geq 0$ they have an imaginary part. A similar discontinuity was seen before by De Tollis[4] in the photon-photon scattering process.

For $\rho \leq 1$ our results for $F(\rho)$ are in agreement with Baier et al[6].

D. Results for the Total Contribution:

Dependence on the Mass of the Top Quark

To obtain the total rate for $Z^0 \to gg\gamma$ one has to take into account the coherent contribution of all six quarks in the loop in Figure 1. In most of the region of the phase space $E^{(i)}_{+++}(r,s,t)$ tends to be much larger than $E^{(i)}_{-++}(r,s,t)$. Also, $E^{(i)}_{\lambda++}(r,s,t)$ only deviates from its value for $\rho \to 0$ by about 10% even for the bottom quark, and except near the edges of the phase space. Anyway the contribution to $F(\rho)$ is small

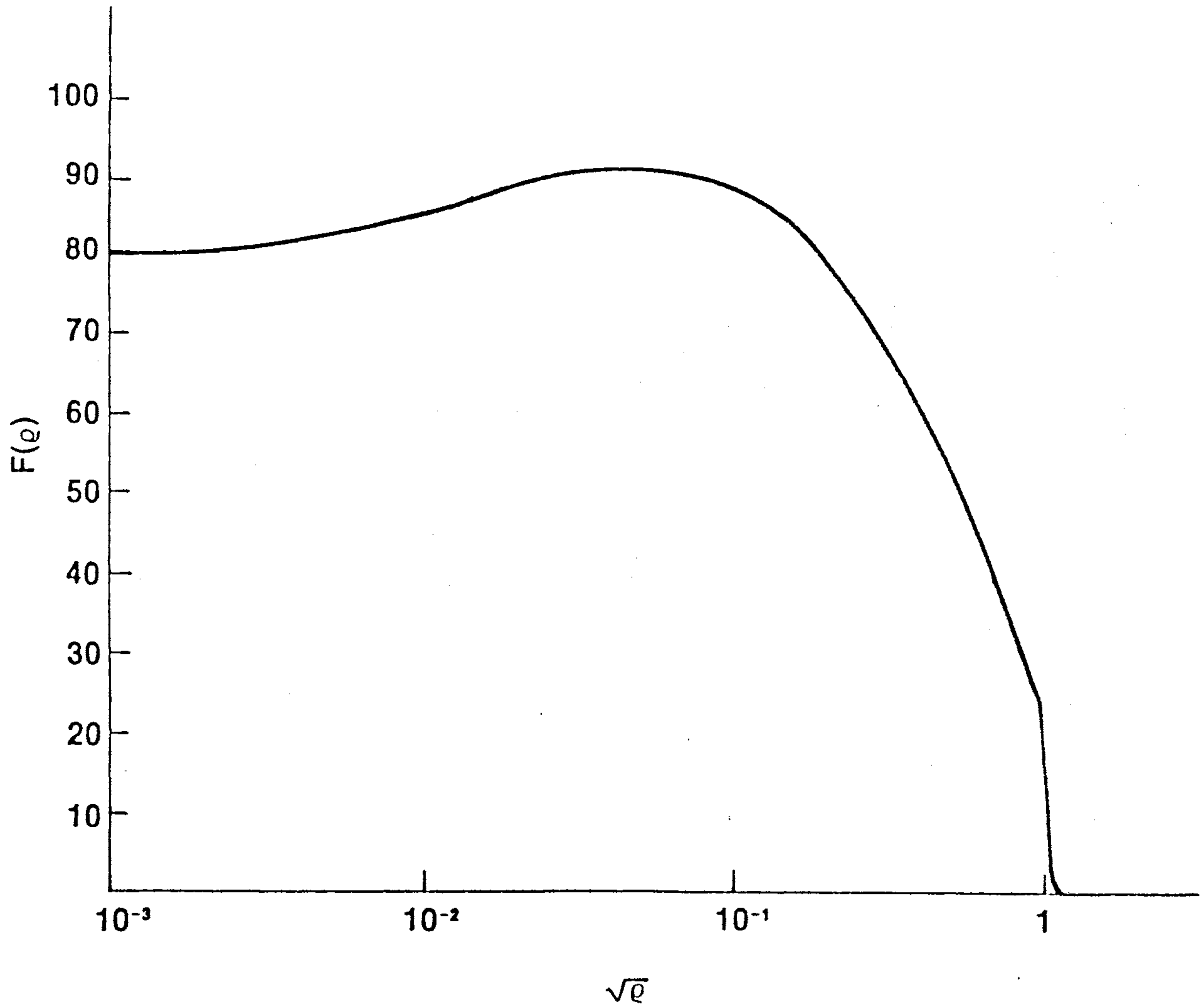

Figure 3. The Function $F(\rho)$ Versus $\sqrt{\rho}$ for $\rho < 1$ (One Quark Flavor)

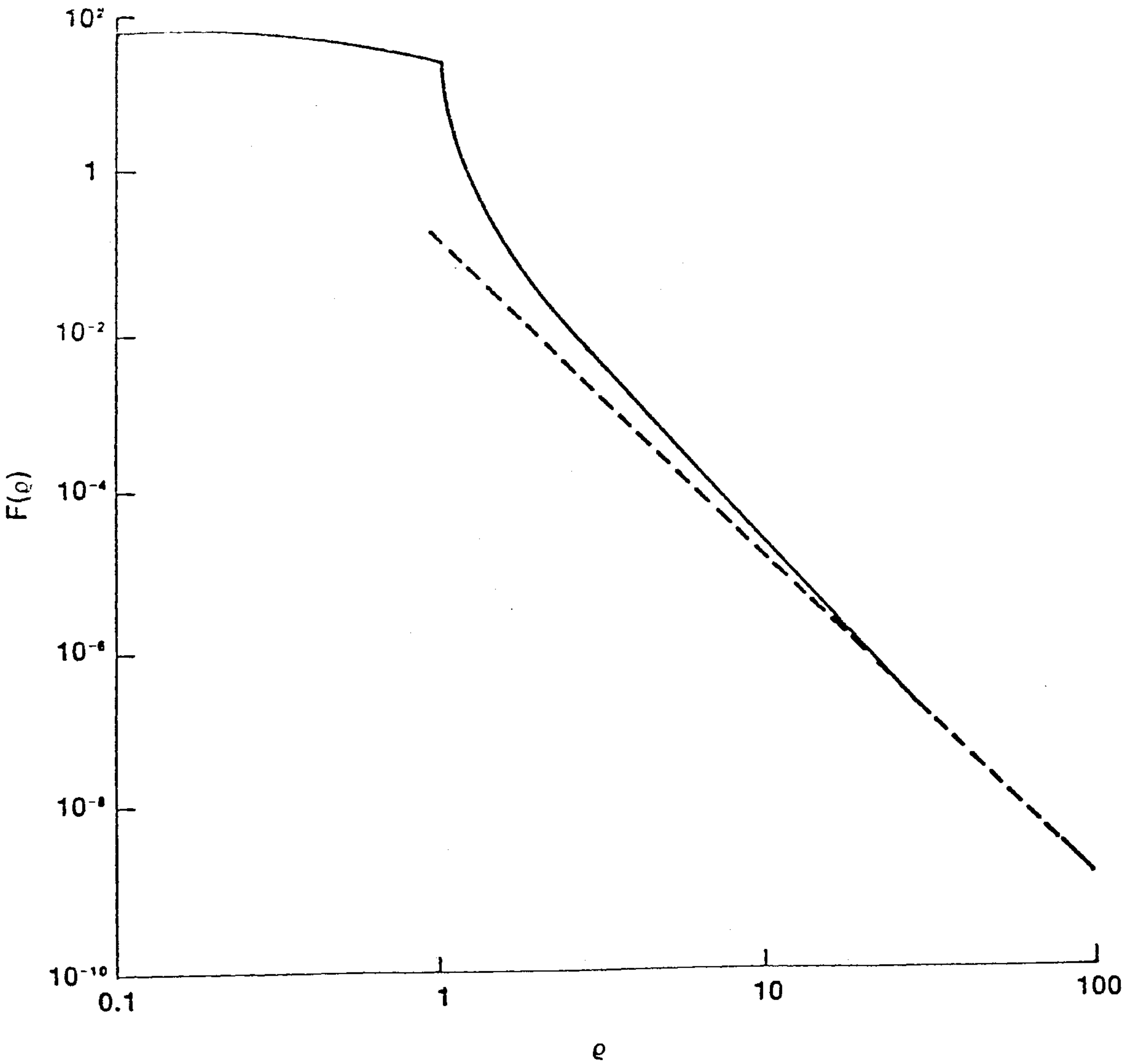

Figure 4. The Function $F(\rho)$ Versus $\sqrt{\rho}$ for $\rho > 1$ (One Quark Flavor)

there due to the cutoff effect of ρ.

We then found it reasonable to use just one $\rho_i = \rho_\mu$ for i = u, d, s, c and b. The top quark mass is treated separately and we have used it as a free parameter. To obtain the vector part for $Z^0 \to ggg$ need make the replacements $a_i Q_i \to a_i$ in Eqns. (2-1) and (2-6), and $C^{gg\gamma}/2! \to C^{ggg}/3!$ in Eqn. (2-1) where the three-gluon color factor is $C^{ggg} = 10/3$. A plot of $F(\rho_t)$ for $Z^0 \to gg\gamma$ as well as the vector part for $Z^0 \to ggg$ is shown in Figure 5. For small ρ_t the value $F(o) \simeq 80$ is obtained. For $\rho_t = 1$ only five quarks should contribute and F goes down to about 70% for the $Z^0 \to gg\gamma$ decay and up to about 200% in the other case. The behavior of $F(\rho_t)$ can qualitatively be understood by looking at the interference patterns in $E^{(i)}_{\lambda++}(r,s,t)$. For the process $Z^0 \to gg\gamma$ all the coupings $a_i Q_i$ are positive. Also, $E^{(i)}_{\lambda++}(r,s,t)$ is negative in most of the phase space for large arguments, which is to say small ρ_i. However, if ρ_i becomes large like ρ_t the amplitude changes sign, thus introducing destructive interference. For the process $Z^0 \to ggg$ the opposite is the case. Here the couplings are only a_i and so alternate sign. Since a_t is positive, a positive value of $E_{\lambda++}$ will increase the total amplitude.

For the top quark mass of m_t = 20 GeV we obtain[7]

$$\frac{\Gamma(Z^0 \to gg\gamma)}{\Gamma_o} = 1.8 \times 10^{-6} \tag{2-9}$$

and

$$\frac{\Gamma(Z^0 \to ggg)}{\Gamma_o} \geqslant 0.8 \times 10^{-5} \tag{2-10}$$

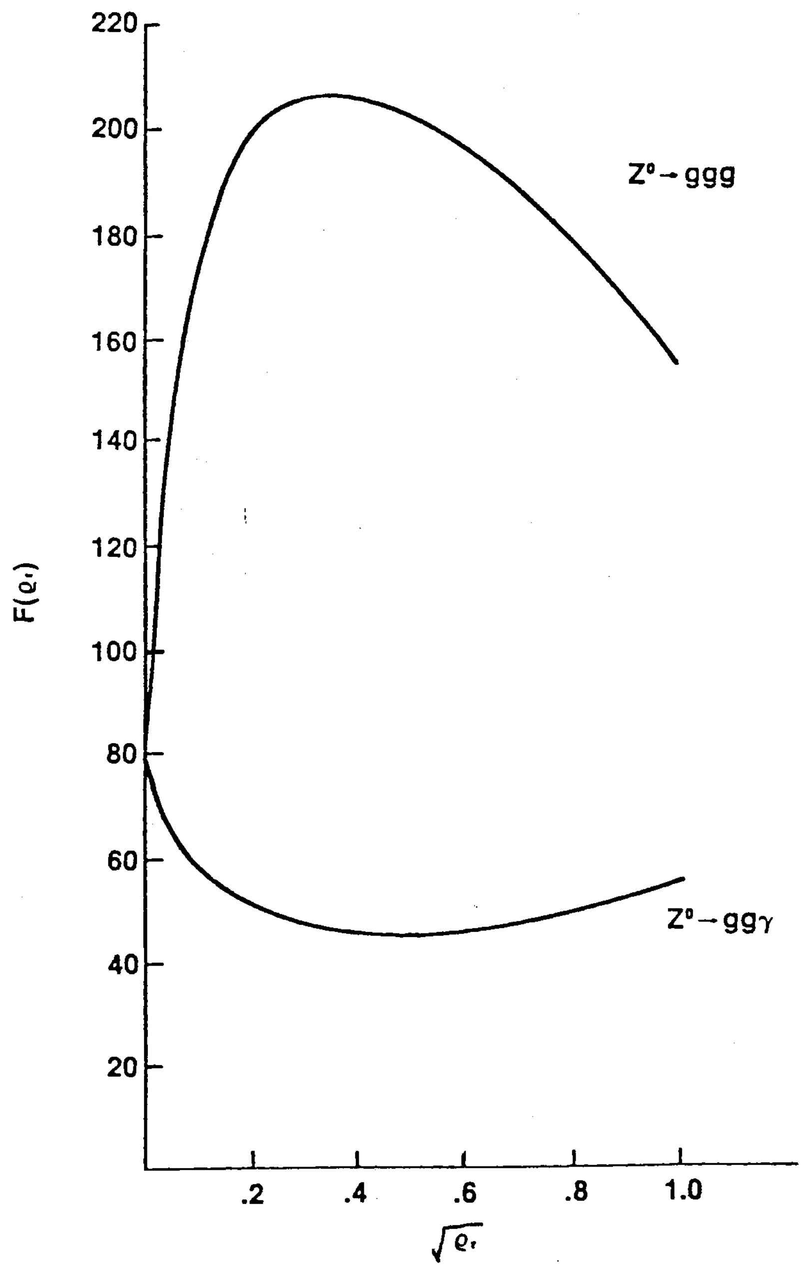

Figure 5. The Function $F(\rho_t)$ Versus $\sqrt{\rho_t}$ for both processes (All Six Quark Flavors Included)

In Figure 6 we also show the double differential function $d^2F/dxdy$ normalized against $F(\rho_t)$ compared to the massless case. Although the shape has changed we still have the symmetries about $x = 1 - \frac{1}{2}y$ satisfied. Finally, in Figure 7 we have displayed the single - differential function dF/dx and compared it with the massless case. For small x they give the same answer, while for $x \to 1$, $dF(\rho_t)/dx$ tends to a finite value and $dF(o)/dx$ goes like $\ln^2(1-x)$.

E. Remarks

(a) As noted above we have assumed that we are below the color threshold, the gluon jets then being indistinguishable. Above the color threshold the results for $Z^0 \to gg\gamma$ and $Z^0 \to ggg$ should be multiplied by 2! and 3! respectively. The agreement with the results of Baier et al[6] is thus obtained.

(b The plot of the function $F(\rho)$ versus $\sqrt{\rho}$ (see Figure 3) is identical in shape to a similar plot exhibited by Baier et al[6]. However we found a discontinuity in the slope of $F(\rho)$ at $\rho = 1$.

(c) We have seen that for the process $Z^0 \to ggg$ that the rate increases by nearly a factor of 3 for a reasonable top quark mass. Since the axial-vector couplings b_i also alternate sign this would suggest that the axial-vector part could give a sizable contribution, thus perhaps lifting the rate even further.

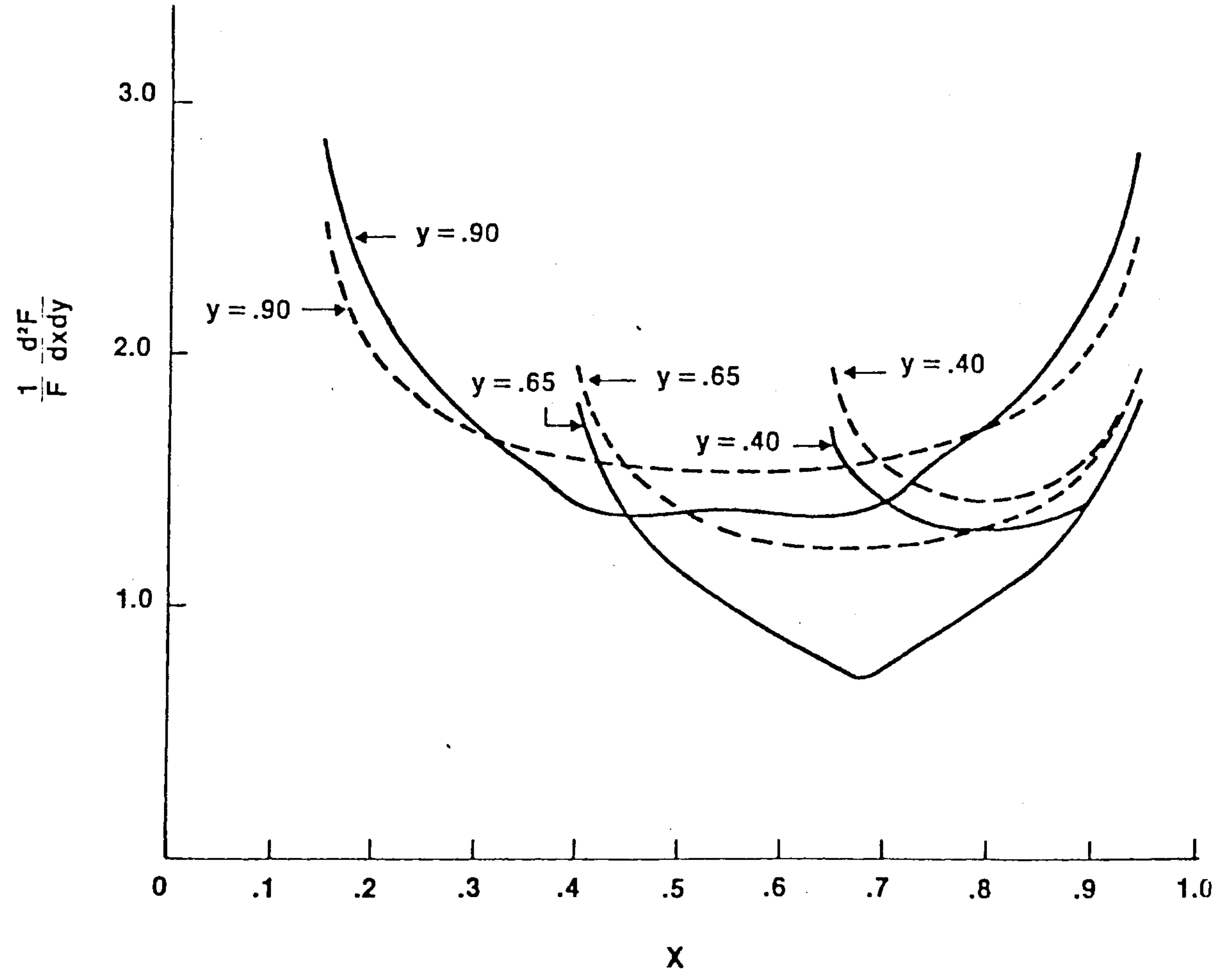

Figure 6. The Double-Differential Function $\frac{1}{F}\frac{d^2F}{dxdy}$ for Several Values of y. Solid Curve Includes Quark Masses. Dashed Curve is the Massless Case

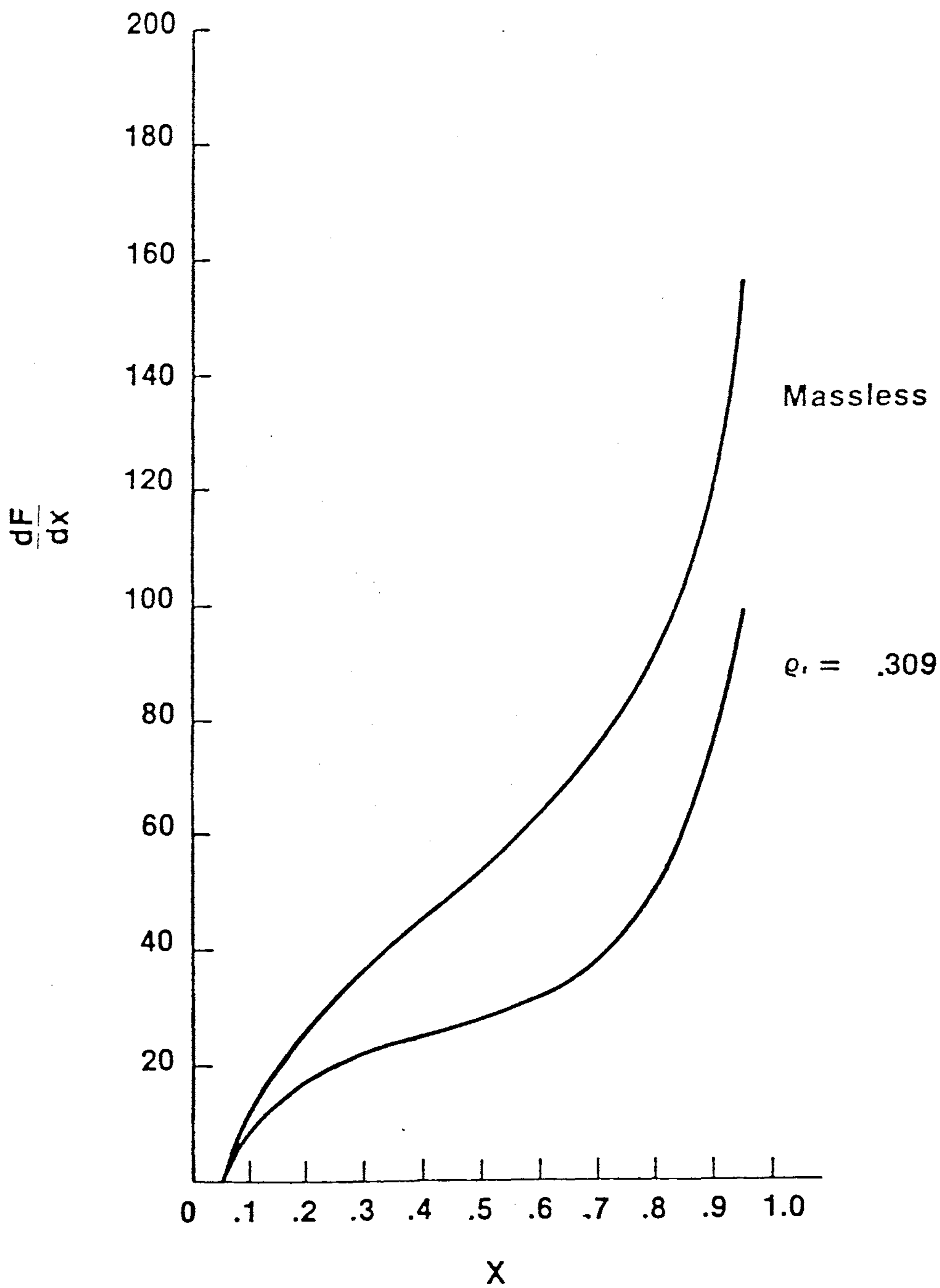

Figure 7. The Single-Differential function dF/dx as a Function of x. The Upper Curve is the Massless Case, Lower Curve Includes Quark Masses. A cut ε=0.025 is Used in the Massless Case

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CHAPTER III

π^0 AND η DECAYS INTO LEPTON PAIRS

A. Introduction

Recently, better experimental data have become available for the rare decays of the neutral pseudoscalar mesons into lepton pairs, which we denote by $P \to \bar{\ell}\ell$; for the decay $\pi^0 \to e^+e^-$, Mischke et al.[1] have reported the branching ratio

$$\Gamma(\pi^0 \to e^+e^-)/\Gamma(\pi^0 \to \mathrm{all}) = (1.8 \pm 0.6) \times 10^{-7} \tag{3-1}$$

which is compatible with the earlier observation by Fischer et al.[2],

$$\Gamma(\pi^0 \to e^+e^-)/\Gamma(\pi^0 \to \mathrm{all}) = (2.23\ {}^{+2.4}_{-1.1}) \times 10^{-7} \tag{3-2}$$

In the case of the decay $\eta \to \mu^+\mu^-$, a new measurement by Djhelyadin et al.[3] gives

$$\Gamma(\eta \to \mu^+\mu^-)/\Gamma(\eta \to \mathrm{all}) = (6.5 \pm 2.1) \times 10^{-6} \tag{3-3}$$

which is two standard deviations below the older result by Hyams et al.[4],

$$\Gamma(\eta \to \mu^+\mu^-)/\Gamma(\eta \to \mathrm{all}) = (2.2 \pm 0.8) \times 10^{-5} \tag{3-4}$$

These results are of considerable interest since the decay $P \to \bar{\ell}\ell$ is thought to be dominated by a two-photon intermediate state [Fig. 8(a)] which probes the pseudoscalar electromagnetic structure at large virtual-photon mass.[5] Indeed, this configuration saturates the absorptive part of the $\pi^0 \to e^+e^-$ amplitude and dominates the absorptive part for $\eta \to \mu^+\mu^-$, from which the model-independent unitarity bounds[6]

$$\Gamma(\pi^0 \to e^+e^-)/\Gamma(\pi^0 \to \text{all}) \geqslant 4.7 \times 10^{-8} \tag{3-5}$$

and[7]

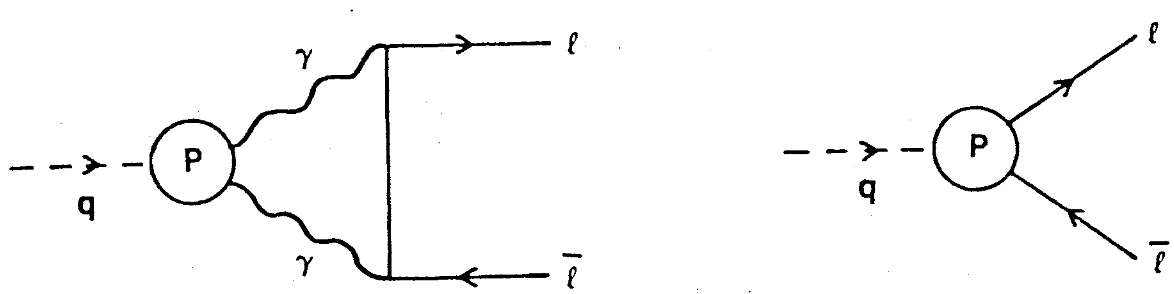
$$\Gamma(\eta \to \mu^+\mu^-)/\Gamma(\eta \to \text{all}) \geqslant 4.1 \times 10^{-6} \tag{3-6}$$

are determined.

In the limit of a pointlike $P(q) \to \gamma(k_1) + \gamma(k_2)$ interaction, the dispersive part of the $P \to \bar{\ell}\ell$ decay amplitude is logarithmically divergent when expressed as an unsubtracted dispersion relation[5] or Feynman integral[6], so that the sensitivity of the branching ratio to the pseudoscalar-meson structure is expressed by the introduction of a form factor $f(k_1^{\,2},\, k_2^{\,2},\, q^2)$ at this vertex. A constraint upon the parametrization of f is provided by the form-factor slope, a_P, which is defined as

$$a_P = m_P^2 \frac{\partial}{\partial k^2} f\,(k^2,\, 0,\, m_P^2)\Big|_{k^2 = 0} \tag{3-7}$$

This quantity is measured in the decay $P \to \bar{\ell}\ell\gamma$; Djhelyadin et al[8] find both a_η and their $\eta \to \mu^+\mu^-$ branching ratio to be in fair agreement with



(a) (b)

Figure 8. Feynman Diagrams for the Decay $P \rightarrow \bar{\ell}\ell$.

the vector-meson-dominance model for f advocated by Quigg and Jackson[7]. In contrast, this same model seriously underestimated both a_π, as measured by Fischer et al.[9], and the $\pi^0 \to e^+e^-$ branching ratio quoted above, while modifications to the model suggested by Q.C.D. asymptotic behavior do nothing to alter this circumstance[10]. Similarly, the single-particle-propagator model for f, first proposed by Berman and Geffen[6], and, more recently, discussed in the context of Q.C.D. by Bergström[11], gives an uncomfortably small $\pi^0 \to e^+e^-$ branching ratio both when a_π is taken as an input, and when a_π is fixed by Q.C.D. asymptotics. Pratap and Smith[12] have calculated f in a nucleon-loop model; when some errors in their work are corrected[13], this model gives $\eta \to \mu^+\mu^-$ and $\pi^0 \to e^+e^-$ branching ratios which are two standard deviations above and below, respectively, the latest experimental values, together with values for a_π and a_η which are 1-2 orders of magnitude smaller than recent observations[8,9] allow. Furthermore, when the nucleon loop is replaced by a sum over quark loops, as in the work of Ametller et al.[13], all of these models agree on the prediction

$$\Gamma(\pi^0 \to e^+e^-)/\Gamma(\pi^0 \to \text{all}) \simeq 6 \times 10^{-8} \tag{3-8}$$

while the predicted $\eta \to \mu^+\mu^-$ branching ratios range from near the unitarity limit for the single-particle-propagator model up to 6.2×10^{-6} for the quark-loop model.

Since the contributions of neutral currents[14] and massive Higgs bosons[15] are expected to be small, the disagreement between theory and experiment for the $\pi^0 \to e^+e^-$ decay constitutes a serious problem. Bergstrom[11] has advanced this discrepancy as a possible indication of new interactions; however, before invoking anomalous couplings we feel that it is important to critically examine the assumption, implicit in the models discussed above, that the $P \to \bar{\ell}\ell$ amplitude is described by an unsubtracted dispersion relation.

B. Notation and the Weak Current Contribution

We define the matrix element for the decay $P \to \gamma\gamma$ by[16]

$$< \gamma(k_1,e_1),\ \gamma(k_2,e_2)|T|p\ (q) > =$$

$$\frac{F_P}{m_P} f(k_1{}^2,\ k_2{}^2,\ q^2)\ \varepsilon_{\mu\nu\rho\sigma} e_1{}^{\mu} e_2{}^{\nu} k_1^{\rho} k_2{}^{\sigma} \tag{3-9}$$

where the form factor f obeys

$$f(k_2{}^2,\ k_1{}^2,\ q^2) = f(k_1{}^2,\ k_2{}^2,\ q^2), \tag{3-10a}$$

$$f(0,0,m_p{}^2) = 1 \tag{3-10b}$$

We also define the invariant matrix element for $P(q) \to \bar{\ell}(P_+) + \ell(P_-)$ by

$$M_{fi} = \bar{u}_\ell(P_-)[i\ \gamma_5(\frac{\alpha}{4\pi})F_p\ \frac{m_\ell}{m_p} K(q^2)]\ v_\ell(P_+) \tag{3-11}$$

Apart from an overall factor, $K(q^2)$ is the effective coupling of P to the $\bar{\ell}\ell$ pair; the factor of m_ℓ in Eqn. (3-11) is required by helicity conservation which forbids the decay of P into a massless fermion-antifermion pair. Using the identity

$$\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\mu\nu\alpha\beta} = -2(g_\rho{}^\alpha g_\nu{}^\beta - g_\rho{}^\beta g_\sigma{}^\alpha) \tag{3-12}$$

Eqn. (3-9) gives

$$\overline{|M_{fi}|^2} = \frac{1}{2}\,|F_p|^2\, m_p{}^2 \tag{3-13}$$

for $P \to \gamma\gamma$, while for $P \to \bar{\ell}\ell$ a simple trace calculation results in

$$\overline{|M_{fi}|^2} = \frac{1}{2}\,(\frac{\alpha}{4\pi})^2\,|F_p|^2\,|K(m_p)|^2 \tag{3-14}$$

Eqns. (3-13) and (3-14), together with the well known phase space[17] for $1 \to 2 + 3$, allow us to write the $P \to \bar{\ell}\ell$ partial width as

$$\Gamma(P \to \bar{\ell}\ell) = \frac{1}{2}\,(\frac{\alpha}{\pi}\,\frac{m_1}{m_p})^2 |K(m_p)|^2\,[1 - 4(\frac{m_1}{m_p})^2]^{1/2}\,\Gamma(P \to \gamma\gamma) \tag{3-15}$$

For $(m_1/m_p) \ll 1$ Eqn. (3-15) reduces to the expression given by Drell[5].

As noted above the standard assumption is that $K(q^2)$ satisfies an unsubtracted dispersion relation in q^2:

$$\mathrm{Re}\, K(q^2) = \frac{1}{\pi}\int_0^\infty \frac{dt}{t-q^2}\,\mathrm{Im}\, K(t) \tag{3-16}$$

To lowest order in the electromagnetic coupling and to all orders in the strong interaction, $\mathrm{Im}\, K(q^2)$ can be written as (see Appendix E)

$$\mathrm{Im}\, K(q^2) = \mathrm{Im}\, K_{\gamma\gamma}(q^2)\theta(q^2) + \sum_x \mathrm{Im}K_x(q^2)\theta(q^2-m_x^2) \tag{3-17a}$$

where the two-photon contribution is

$$\text{Im } K_{\gamma\gamma}(q^2) = \frac{\pi}{\beta} \ln \left(\frac{1-\beta}{1+\beta}\right) f\,(0,0,q^2) \tag{3-17b}$$

$$\beta = [1 - 4\,\frac{m_1^2}{q^2}]^{1/2} \tag{3-17c}$$

and the sum runs over all real hadronic intermediate states. We note that if only the two-photon part of Eqn. (3-17a) is retained and we set Re $K(q^2) = 0$ in Eqn. (3-15), then we reproduce the unitarity bounds given in Eqns. (3-5) and (3-6). Eqn. (3-16) is rendered plausable by noting that the model form factors mentioned above yield sufficient conditions upon $f(0,0,q^2)$ and the remaining terms in Eqn. (3-17a) such that $\text{Im}K(\infty) = 0$ so the dispersion integral converges; this is explicitly demonstrated in Appendix E for the simple vector meson dominance model. Simple convergence, however, is not an a priori guarantee that Eqn. (3-16) is the proper representation of the physical amplitude; we may, for example, consider a once-subtracted dispersion relation for $K(q^2)$:

$$\text{Re } K(q^2) = K_p(0) + \frac{q^2}{\pi}\int_0^\infty \frac{dt}{t(t-q^2)}\,\text{Im } K(t) \tag{3-18}$$

There then remains the problem of fixing the subtraction constant $K_p(0)$. Clearly the choice

$$K_p(0) = \frac{1}{\pi}\int_0^\infty \frac{dt}{t}\,\text{Im } K(t)$$

merely reduces Eqn. (3-18) to Eqn. (3-16) so we must look beyond the electromagnetic interactions in ascribing a value and a physical significance to $K_p(0)$.

Let us recall that the bulk of low energy weak interaction phenomenology is well described by the effective current-current Lagrangian density[18]

$$= \frac{G}{2\sqrt{2}} (J_\lambda^{\dagger} J^\lambda + J^\lambda J_\lambda^{\dagger}), \tag{3-19a}$$

$$J^\lambda = J_\ell^{\ \lambda} + J_h^{\ \lambda} \tag{3-19b}$$

Here G is the Fermi constant,

$$J_\ell^{\ \lambda} = \bar{\nu}_\mu \gamma^\lambda (1-\gamma_5)\mu + \bar{\nu}_e \gamma^\lambda (1-\gamma_5) e \tag{3-19c}$$

is the leptonic current and the hadronic current is

$$J_h^{\ \lambda} = (F_1^{\ \lambda} + i\, F_2^{\ \lambda} - F_1^{\ 5} - i\, F_2^{\ 5\lambda}) \cos\theta c +$$

$$= (F_4^{\ \lambda} + iF_5^{\ \lambda} - F_4^{\ 5\lambda} - iF_5^{\ 5\lambda}) \sin\theta c \tag{3-19d}$$

where θc is the Cabibbo angle. The $F_i^{\ \lambda}$ and $F_i^{\ 5\lambda}$ form, respectively, vector and axial-vector octet representations of SU(3). The current algebra hypothesis rests on the postulate that the leptonic charges

$$W_{\ell +} = \frac{1}{2} \int d^3\, x\, J^o{}_\ell, \tag{3-20}$$

$$W_{\ell-} = W_{\ell+}^{\dagger} \tag{3-20b}$$

and hadronic charges

$$W_{h+} = \frac{1}{2} \int d^3 x\, J_h{}^o, \tag{3-20c}$$

$$W_{h-} = W_{h+}^{\dagger} \tag{3-20d}$$

satisfy the same equal time commutation relations:

$$[W_{\ell+,h+}, W_{\ell-,h-}]_{(-)} = 2W_{\ell3,h3}, \tag{3-21a}$$

$$[W_{\ell3,h3}, W_{\ell\pm,h\pm}]_{(-)} \pm W_{\ell\pm,h\pm} \tag{3-21b}$$

where

$$W_{\ell3} = \frac{1}{2} \int d^3 x\, J_{\ell3}{}^o, \tag{3-22a}$$

$$W_{h3} = \frac{1}{2} \int d^3 x\, J_{h3}{}^o \tag{3-22b}$$

with

$$J_{\ell3}^{\lambda} = \frac{1}{2} [\bar{\nu}_\mu \gamma^\lambda (1-\gamma_5)\nu_\mu - \bar{\mu}\, \gamma^\lambda (1-\gamma_5)\mu +$$

$$+ \bar{\nu}_e\, \gamma^\lambda\, (1-\gamma_5)\, \nu_e - \bar{e}\, \gamma^\lambda (1-\gamma_5) e], \tag{3-23a}$$

$$J_{h3}^{\lambda} = (1 - \frac{1}{2}\sin^2\theta c)(F_3^{\lambda} - F_3^{5\lambda}) + \frac{\sqrt{3}}{2}\sin^2\theta c(F_8^{\lambda} - F_8^{5\lambda}) +$$

$$- \sin\theta c \cos\theta c \,(F_6^{\lambda} - F_6^{5\lambda}) \qquad (3\text{-}23b)$$

If we take this postulate seriously, then we are led to introducing a new interaction term

$$L' = \frac{G}{2\sqrt{2}}(J_{3\lambda}^{\dagger} J_3^{\lambda} + J_3^{\lambda} J_{3\lambda}^{\dagger}), \qquad (3\text{-}23c)$$

$$J_3^{\lambda} = J_{\ell 3}^{\lambda} + J_{h3}^{\lambda} \qquad (3\text{-}23d)$$

and thus, with Eqn. (3-19a) describing the charged current interactions, Eqn. (3-23c) describes the neutral current interactions. This cannot be quite correct since strangeness changing neutral currents, as represented by the third term in Eqn. (3-23b), are not observed[19] at order G; this does not, however, affect our basic argument that L', which has nonvanishing matrix elements between P and $\bar{\ell}\ell$, should set the scale for $P \to \bar{\ell}\ell$ at $q^2 = 0$. The corresponding Feynman diagram is shown in Fig. 8(b). To lowest order in G

$$T_{fi} = - \langle f | \int d^4x \, L_I | i \rangle$$

so we set

$$\bar{u}_e(P_-)[i\,\gamma_5(\frac{\alpha}{4\pi})\,F_\pi \frac{m_e}{m_\pi} K_\pi(0)]\; v_e(P_+) =$$

$$\frac{G}{\sqrt{2}} \bar{u}_e(P_-)[\tfrac{1}{2} \gamma_\lambda(1-\gamma_5)] v_e(P_+)(1 - \tfrac{1}{2} \sin^2\theta c)\langle 0|F_3{}^\lambda - F_3{}^{5\lambda}|\pi^o(q)\rangle\sqrt{2q_o} \quad (3\text{-}24a)$$

$$\bar{u}_\mu(P_-)[i\ \gamma_5(\tfrac{\alpha}{4\pi})F_\eta \frac{m_\mu}{m_\eta} K_\eta(0)] v_\mu(P_+) =$$

$$\frac{G}{\sqrt{2}} \bar{u}_\mu(P_-)[\tfrac{1}{2} \gamma_\lambda(1-\gamma_5) v_\mu(P_+) \frac{\sqrt{3}}{2} \sin^2\theta c\ \langle 0|F_8{}^\lambda - F_8{}^{5\lambda}|\eta(q)\rangle\sqrt{2q_o} \quad (3\text{-}24b)$$

The matrix elements of the vector current vanish by parity; for the π^0 we take[20]

$$\sqrt{2q_o}\ \langle 0|F_3{}^{5\lambda}|\pi^0(q)\rangle = i\ f_\pi\ q^\lambda \quad (3\text{-}24c)$$

and use the anomalous PCAC (partial conservation of axial-vector current) relation

$$\partial_\lambda F_3{}^{5\lambda} = m_\pi^2\ f_\pi\ \pi^0 + \frac{\alpha}{4\pi}\ \varepsilon_{\mu\nu\alpha\beta} F^{\mu\nu} F^{\alpha\beta}$$

which allows us to write

$$\left.\frac{F_\pi}{m_\pi}\right|_{q^2 = 0} = -\frac{\alpha}{\pi f_\pi} \quad (3\text{-}24d)$$

Similarly

$$\sqrt{2q_o}\ \langle 0|F_8{}^{5\lambda}|\eta(q)\rangle = i\ f_\eta q^\lambda \quad (3\text{-}24e)$$

Noting that the charge matrix

$$Q = \frac{1}{2}(\lambda_3 + \frac{1}{\sqrt{3}} \lambda_8)$$

where the λ_i are the usual SU(3) matricies, obeys

$$T_r(Q\ \lambda_8\ Q)\ /\ T_r(Q\ \lambda_3\ Q) = \frac{1}{\sqrt{3}}$$

we also write

$$\partial_\lambda F_8^{5\lambda} = m_\eta^{\ 2}\ f_\eta\ \eta + \frac{1}{\sqrt{3}}\frac{\alpha}{4\pi}\ \varepsilon_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}$$

so

$$\frac{F_\eta}{m_\eta}\Big|\ q^2 = 0 = -\frac{1}{\sqrt{3}}\frac{\alpha}{\pi f_\eta} \tag{3-24f}$$

Finally, consistent with the SU(3) relation[21]

$$(\frac{F_\pi}{m_\pi})(\frac{F_\eta}{m_\eta})^{-1} = \sqrt{3}$$

we put $f_\eta = f_\pi$. Eqns. (3-24a) through (3-24f) then yield

$$K_\pi(0) = -\frac{1}{\sqrt{2}}(1 - \frac{1}{2}\sin^2\theta c)[\frac{2\pi f_\pi}{\alpha}]^2\ G \tag{3-25a}$$

$$K_\eta(0) = -\frac{3}{2\sqrt{2}}\sin^2\theta c\ [\frac{2\pi f_\pi}{\alpha}]^2\ G \tag{3-25b}$$

Now, $G \simeq 10^{-5} M^{-2}_{proton}$, $f_\pi \simeq 93$ MeV and $\cos\theta c \simeq 0.97$ so

$$K_\pi(0) \simeq -0.50 \tag{3-26a}$$

$$K_\eta(0) \simeq -0.046 \tag{3-26b}$$

Although $K_\pi(0)$ and $K_\eta(0)$ are numerically small, Eqn. (3-18), together with these choices for Kp(0), has powerful implications for $\mathrm{Re}K(m_p{}^2)$ as we shall now see.

C. Electromagnetic Contributions

We may immediately note two interesting properties of the representation for $\mathrm{Re}K(q^2)$ given by Eqns. (3-18), (3-25a) and (3-25b): first, the dispersion integral exists for $f(0,0,q^2) = 1$ and $\mathrm{Im}\, K_x(q^2) = 0$ corresponding to pointlike P. Second, for $m_x{}^2 > q^2$ the contribution of the intermediate state x to the dispersion integral is, quite apparently, suppressed by a factor of $q^2/m_x{}^2$. We can thus set $f(0,0,q^2) = 1$ and obtain the leading approximation

$$\mathrm{Re}\, K(m_p^2) \simeq K_p(0) + \mathrm{Re}\, K_{\gamma\gamma}(m_p^2) \tag{3-27a}$$

where

$$\mathrm{Re}\, K_{\gamma\gamma}(m_p^2) \equiv \frac{m_p^2}{\pi}\int_0^\infty \frac{dt}{t(t-m_p^2)}\, [\mathrm{Im}\, K_{\gamma\gamma}(t)]_{f(0,0,q^2)\,=\,1}$$

The integral for $\mathrm{Re}\, K_{\gamma\gamma}(q^2)$ is evaluated in Appendix F; with $q^2 = m_p^2$ the result is

$$\mathrm{Re}\, K_{\gamma\gamma}(m_p^2) = \frac{1}{\beta_o}\Big[\frac{1}{2}\ln^2\Big(\frac{1-\beta_o}{1+\beta_o}\Big) - 2\Phi\Big(\frac{1-\beta_o}{1+\beta_o}\Big) + \frac{\pi^2}{6}\Big], \tag{3-27b}$$

$$\beta_o = [1 - 4 \frac{m_\ell^2}{m_p^2}]^{1/2} \tag{3-27c}$$

Here

$$\Phi(x) = \int_o^x \frac{dt}{t} \ln(1+t) = \sum_{n=1}^{\infty} \frac{(-x)^n}{n^2}, \quad |x| < 1 \tag{3-27d}$$

is the Spence function[22]. It should be noted that the leading logarithm-squared behavior of Re $K(m_p{}^2)$ is typical[6,10,11], what distinguishes our model from previous ones is the absence of unsuppressed terms of the type $\ln(m_\ell / m_x)$ which always reduce the value of Re $K(m_p{}^2)$.

Although by definition Re $K_{\gamma\gamma}(0) = 0$, for the physical masses Re $K_{\gamma\gamma}(m_p^2) >> K_p(0)$. Using Eqns. (3-15) and (3-26a) through (3-27d), our model predicts the approximate branching ratios

$$\frac{\Gamma(\pi^0 \rightarrow e^+e^-)}{\Gamma(\pi^0 \rightarrow \text{all})} \simeq 2.0 \times 10^{-7} \tag{3-28a}$$

$$\frac{\Gamma(\eta \rightarrow \mu^+\mu^-)}{\Gamma(\eta \rightarrow \text{all})} \simeq 6.1 \times 10^{-6} \tag{3-28b}$$

which are in excellent agreement with the latest experimental values cited above. We may ask, however, whether these results are significantly altered by the terms neglected in Eqn. (3-27a). To address this question, as well as to make contact with the form-factor slope data, let us consider a simple vector-meson-dominance model[7] for f. In such a model the matrix element for $P \rightarrow \gamma\gamma$ is given by

$$\langle\gamma(k_1 e_1),\ \gamma(k_2 e_2)\left|T\right|P(q)\rangle =$$

$$[-\ i\ e\ \frac{m_v{}^2}{f_v}\ e_{1\alpha}][-\ i\ e\ \frac{m_v{}^2}{f_v}\ e_{2,\beta}][-i\ \frac{g^{\alpha\mu} - k_1^{\alpha} k_1{}^{\mu}/m_v{}^2}{k_1{}^2 - m_v{}^2}][-i\ \frac{g^{\lambda\mu} - k_2{}^{\beta} k_2{}^{\nu}/m_v{}^2}{k_2{}^2 - m_v{}^2}]$$

$$e^2\ f_{vv}\ \varepsilon_{\mu\nu\rho\sigma}\ k_1{}^{\rho} k_2{}^{\sigma} =$$

$$\frac{m_\nu{}^4}{(m_v{}^2 - k_2{}^2)(m_v{}^2 - k_2{}^2)}\ \frac{e^4 f_{vv}}{f_v{}^2}\ \varepsilon_{\mu\nu\rho\sigma} e_1{}^{\mu} e_2{}^{\nu} k_1{}^{\rho} k_2{}^{\sigma} \tag{3-29}$$

Comparing Eqn. (3-29) to Eqn. (3-9), we set

$$\frac{e^4 f_{vv}}{f_v{}^2} = \frac{F_p}{m_p}$$

so

$$f(k_1{}^2, k_2{}^2, q^2) = \frac{m_v{}^4}{(m_v{}^2 - k_1{}^2)(m_v{}^2 - k_2{}^2)} \tag{3-30}$$

We note that Eqn. (3-30) has the features $f(0,0,q^2) = 1$ and

$$a_p = \frac{m_p{}^2}{m_v{}^2} \tag{3-31}$$

by the definition of Eqn. (3-7). In our model the exact expression for Re $K(m_p{}^2)$ is then

$$\text{Re } K(m_p{}^2) = K_p(0) + \text{Re } K_{\gamma\gamma}(m_p^2) + \text{Re } K_{\gamma v}(m_p^2) + \text{Re } K_{vv}(m_p^2) \tag{3-32a}$$

where Re $K_{\gamma\gamma}(m_p{}^2)$ is as before and

$$\mathrm{Re}\, K_{\gamma v}(m_p^2) \equiv \frac{m_p^2}{\pi} \int_{m_v^2}^{\infty} \frac{dt}{t(t-m_p^2)} \, \mathrm{Im}\, K_{vv}(t)$$

$$\mathrm{Re}\, K_{vv}(m_p^2) \equiv \frac{m_p^2}{\pi} \int_{4m_v^2}^{\infty} \frac{dt}{t(t-m_p^2)} \, \mathrm{Im}\, K_{vv}(t)$$

The imaginary parts of $K_{\gamma v}$ and K_{vv} are determined in Appendix E while the real parts are evaluated in Appendix F; using Eqn. (3-31) and putting $q^2 = m_p^2$ the results are

$$\mathrm{Re}\, K_{\gamma v}(m_p^2) = a_p \sum_{n=o}^{\infty} (a_p)^n [2\, C_n^{(1)} \ln \left(\frac{m_p}{m_l} \frac{1}{\sqrt{a_p}}\right) + C_n^{(2)}] + O\left(\frac{m_\ell^2}{m_v^2}\right) \qquad (3\text{-}32b)$$

$$\mathrm{Re}\, K_{vv}(m_p^2) = -\,4\,\delta_p \sum_{n=0}^{\infty} (-\delta_p)^n\, C_n^{(3)} + O\left(\frac{m_\ell^2}{4m_{v\nu}^2}\right) \qquad (3\text{-}32c)$$

$$\delta_p = \frac{a_p}{4-a_p} \qquad (3\text{-}32d)$$

$$C_n^{(1)} = \frac{2}{(n+1)^2(n+2)^2(n+3)^2} \qquad (3\text{-}32e)$$

$$C_n^{(2)} = \frac{2[3(n+2)^2 - 1]}{(n+1)^2(n+2)^2(n+3)^2} \qquad (3\text{-}32f)$$

$$C_n^{(3)} = \frac{1}{n+2} \sum_{r=1}^{n+2} \frac{1}{2n+5-2r} \qquad (3\text{-}32g)$$

$$\mathrm{Im}\, K_{\gamma v}(m_p^2) = \mathrm{Im} K_{vv}(m_p^2) = 0, \; a_p < 1 \qquad (3\text{-}32h)$$

For $\pi^0 \to e^+e^-$, using the Fischer et al[9] value $a_\pi = 0.10$, we find $\mathrm{Re}\, K_{\gamma\gamma}(m_\pi^2) = 63.8$, $\mathrm{Re}\, K_{\gamma v}(m_\pi^2) = 0.52$ and $\mathrm{Re}\, K_{vv}(m_\pi^2) = -\,0.07$ so the prediction of Eqn. (3-28a) is unchanged. A more substantial effect is seen for $\eta \to \mu^+\mu^-$; Djhelyadin et al give $a_\eta = 0.582$ which yields

Re $K_{\gamma\gamma}(m_\eta^2) = 7.30$, Re $K_{\gamma v}(m_\eta^2) = 1.28$ and Re $K_{vv}(m_\eta^2) = -0.40$. Our prediction for $\eta \to \mu^+\mu^-$ in the simple vector-meson-dominance model is thus

$$\frac{\Gamma(\eta \to \mu^+\mu^-)}{\Gamma(\eta \to \text{all})} = 7.1 \times 10^{-6} \tag{3-33}$$

which represents a 16% enhancement over Eqn. (3-28b) due to vector-meson effects.

Only a large upper limit[23]

$$\frac{\Gamma(\eta \to e^+e^-)}{\Gamma(\eta \to \text{all})} < 3 \times 10^{-4} \tag{3-34}$$

presently exists for the decay $\eta \to e^+e^-$; the unitarity bound is

$$\frac{\Gamma(\eta \to e^+e^-)}{\Gamma(\eta \to \text{all})} \geqslant 1.7 \times 10^{-9} \tag{3-35}$$

Once again using $a_\eta = 0.582$, we have Re $K_{\gamma\gamma}(m_\eta^2) = 99.06$, Re $K_{\eta v}(m_\eta^2) = 3.74$ and Re $K_{vv}(m_\eta^2) = -0.40$. Our model thus predicts

$$\frac{\Gamma(\eta \to e^+e^-)}{\Gamma(\eta \to \text{all})} = 1.1 \times 10^{-8} \tag{3-36}$$

Although small, it is not inconceivable for such a branching ratio to be observed.

D. Remarks

(a) As we have shown, the existing $P \to \bar{\ell}\ell$ data can be accomodated by a model with a two-photon intermediate state dominating if a subtraction is made at $q^2 = 0$, with the subtraction constant small. This is not to say, however, that our particular choice of fixing the subtraction constant via the weak neutral current necessarily represents the underlying physics but rather that the physics does seem to single out Eqn. (3-18) with $K_p(0) \simeq 0$. In this regard it is amusing to note that Eqn. (3-16) and the static quark model yields[11]

$$\mathrm{Re}\; K(m_p^{\;2}) = \mathrm{Re}\; K_{\gamma\gamma}(m_p^{\;2}) - \frac{2}{\beta_o} \ln \left(\frac{1+\beta_o}{1-\beta_o}\right)$$

with the property $\mathrm{Re}\; K(0) = 0$.

(b) It should be cautioned that, although in agreement with one another, the Mischke et al.[1] and Fischer et al.[2] results for $\pi^0 \to e^+e^-$ are not without uncertainties since the former suffers from background subtraction problems while the latter is based on a singularly small number of events.

(c) We wish to emphasize that although we have explicitly evaluated branching ratios for a point-like P and for a simple vector-meson-dominance model we do not particularly advocate either; our model consists not in the choice of the form factor but instead in the definition of the physical amplitude given by Eqns. (3-18), (3-25a) and (3-25b). Indeed, due to the suppression of higher mass intermediate states noted above, we expect Eqn. (3-28a) to hold independent of the choice for $f(k_1^{\;2}, k_2^{\;2}, q^2)$.

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CHAPTER IV

TWO-PHOTON EXCHANGE EFFECT IN RADIATIVE CORRELATIONS TO $\pi^0 \to \gamma e^- e^+$

A. Introduction

The rare decay

$$\pi^0 \to \gamma e^- e^+ \tag{4-1}$$

has been a subject of theoretical and experimental interest for many years. While the earliest calculations[1] of the internal-conversion coefficient

$$\rho \equiv \frac{\Gamma(\pi^0 \to \gamma e^- e^+)}{\Gamma(\pi^0 \to \gamma\gamma)} \tag{4-2}$$

were based upon a pointlike $\pi^0(p_1) \to \gamma(k_1) + \gamma(k_2)$ interaction applied to the lowest-order contribution [Fig. 9(a)], the continuing interest in this decay derives from the fact that, in general, this interaction involves a form factor $f(k_1^2, k_2^2, p_1^2)$, normalized to $f(0,0,m_\pi^2) \equiv 1$, which contains information about the π^0 electromagnetic structure. As first emphasized by Berman and Geffen[2], the decay $\pi^0 (p_1) \to \gamma(k) + e^-(p_2) + e^+(p_3)$ allows a study of this form factor via the lowest-order differential distribution

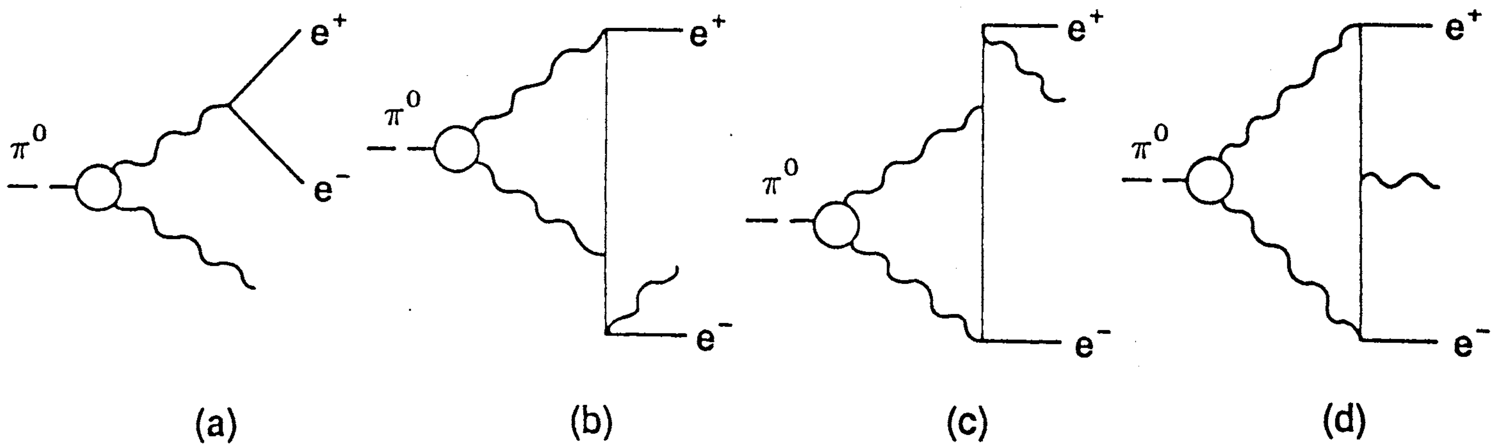

Figure 9. (a) Lowest Order Diagram for the Decay $\pi^\circ \rightarrow \gamma e^+e^-$. (b), (c) and (d). Two-photon Exchange Diagrams.

$$\frac{d\rho_o}{dx} = \frac{2\alpha}{3\pi} \frac{(1-x)^3}{x} [1 + \frac{\beta}{2x}][1 - \frac{\beta}{x}]^{1/2} |f(x)|^2 \tag{4-3a}$$

where

$$x \equiv (p_2 + p_3)^2 / m_\pi^{\ 2} \tag{4-3b}$$

$\beta \equiv (2\ m_e/m_\pi)^2$ and $f(x) \equiv f(x\ m_\pi^{\ 2},\ 0,\ m_\pi^{\ 2})$.

The form factor f(x) is usually discussed in terms of its linear expansion

$$f(x) \simeq 1 + a_\pi x \tag{4-3c}$$

Theoretically, the form-factor slope a_π is expected to be small and positive, with most models[3] giving predictions near the intuitive estimate $a_\pi \simeq (m_\pi/m_\rho)^2 = 0.03$. On the other hand the first three experiments[4-6] to determine a_π from the x distribution reported large negative values. Apparently this conundrum may be traced to an unappreciated subtlety in Joseph's[7] treatment of radiative corrections, as applied to the experimental analysis. Neglecting the identity between decay and bremsstrahlung photons, the one-photon-exchange contribution $\pi^0 \rightarrow \gamma + \gamma^* \rightarrow \gamma + e^- + e^+$ to the decay in Eqn. (4-1) can be generally expressed as[8]

$$\rho^I = \int_\beta^1 dz\ \frac{2(1-z)^3}{\pi\ z} \left|f(z)\right|^2\ \mathrm{Im}\ \pi(z\ m_\pi^{\ 2}) \tag{4-4}$$

where $\pi(q^2)$ is the spectral function for the photon propagator. Using the second-order function $\pi^{(2)}(q^2)$ in Eqn. (4-4) yields ρ_o, while the integrand reproduces Eqn. (4-3a) for $d\,\rho_o/dx$ upon the replacement $z \to x$. However when this is extended to computing $\rho_R{}^I$, the $O(\alpha)$ correction to ρ_o, using $\pi^{(4)}(q^2)$, as in Ref. 7, all direct correlation between z and x is lost due to an implicit integration over inner bremsstrahlung, so that $d\,\rho_R{}^I/dx$ cannot be obtained from the integrand by the same simple substitution[9]. The effect of this distinction between z and x is best seen by noting that the $O(\alpha)$ corrections to diagram (a) of Fig. 9 have been explicitly calculated by Mikaelian and Smith[10], with the result that $d\,\rho_R{}^I/dx$ is negative over most of the range of x, rather than positive as Joseph asserts. Employing Ref. 10's corrections in the analysis, a recent experiment by Fischer et al.[11] has determined the values

$$a_\pi = 0.10 \pm 0.03 \qquad (4\text{-}5a)$$

including radiative corrections and

$$a_\pi = 0.05 \pm 0.03 \qquad (4\text{-}5b)$$

omitting radiative corrections. It is perhaps worth observing that this radiative-corrected value is in agreement with the result of Ref. 6, $a_\pi = 0.11 \pm 0.07$, based upon the total rate, where Eqn. (4-4) can be safely applied.

One interesting aspect of the radiative-correction calculations in Refs. 7 through 10 is the omission of the two-photon-exchange contribution $\pi^0 \to \gamma^* + \gamma^* \to \gamma + e^- + e^+$ given by diagrams (b), (c) and (d) of Fig. 9. One may immediately note that these graphs are obtained from $\pi^0 \to \gamma^* + \gamma^* \to e^-e^+$ by the insertion of a bremsstrahlung photon on the electron line. This close connection with $\pi^0 \to e^-e^+$ has fostered their neglect in the hope that these diagrams share the latters (m_e/m_π) suppression factor[8,10]. Although as noted by Joseph[7] such a factor does appear in the soft-photon limit, $x \to 1$, because the lowest-order contribution is $O(\omega)$ in the photon energy one must generally consider not only the ω^{-1} pole terms, but also all terms up to order $(\omega/m_\pi) = (1-x)/2$. Indeed, as we shall see, the ω^{-1} terms in the interference of diagrams (b) and (c) with diagram (a) of Fig. 9 identically vanish.

Since the argument for suppression of two-photon exchange fails, it is necessary to determine the size of this contribution. The general evaluation of the intereference of diagrams (b), (c) and (d) with diagram (a) of Fig. 9 is rendered intractable by the fact that, as we have noted in Chapter 3, several models for $f(k_1{}^2, k_2{}^2, p_1{}^2)$ exist, each requiring a separate calculation. In addition, we also saw in Chapter 3 that there exist two ways of defining the $\pi^0 \to e^-e^+$ amplitude, which enters as a subgraph in diagrams (b) and (c) of Fig. 9, so doubling the task. Both of these problems can, however, be dealt with by noting that m_e appears through the dimensionless combination $\beta = 5.7 \times 10^{-5}$, so that the massless limit, $m_e \to 0$, should serve as an excellent approximation. Moreover, in this limit, the counter term corresponding to the subracted dispersion relation of Chapter 3 vanishes since it is defined for on-shell fermions and therefore is proportional to m_e. An additional advantage is that, in the absence of counter terms, the total two-photon-exchange contribution must exist for a pointlike π^0. Thus by setting the electron mass to zero and approximating the form factor $f(k_1{}^2, k_2{}^2, p_1{}^2)$, by its on-shell value $f(0, 0, m_\pi{}^2) = 1$ we may obtain expressions valid up to terms of order a_π in the form-factor slope and for x, $(1-x) \gg \beta$, independent of any assumptions.

B. Notation

Following Mikaelian and Smith[10], we define the matrix element for the decay of a pointlike π^0 into two photons by

$$\langle\gamma(k_1 e_1),\gamma(k_2,e_2)|T|\pi^0(p_1)\rangle = \frac{F}{m_\pi}\,\varepsilon_{\mu\nu\rho\sigma}e_1{}^\mu e_2{}^\nu k_1{}^\rho k_2{}^\sigma \tag{4-6a}$$

where F is a dimensionless coupling constant, so that in the π^0 rest frame

$$\Gamma(\pi^0 \to \gamma\gamma) = \frac{m_\pi}{64\pi}|F|^2 \tag{4-6b}$$

We also define the invariant matrix element for $\pi^0(p_1) \to \gamma(k) + e^-(p_2) + e^+(p_3)$ by

$$M_{fi} = e^\lambda\,\bar{u}(p_2)\Gamma_\lambda v(p_3) \tag{4-6c}$$

In evaluating the square of M_{fi} we employ the Mandelstam variables

$$r = (p_1 - k)^2 = (p_2 + p_3)^2 \tag{4-7a}$$

$$s = (p_1 - p_2)^2 = (k + p_3)^2 \tag{4-7b}$$

$$t = (p_1 - p_3)^2 = (k + p_2)^2 \tag{4-7c}$$

$$r + s + t = m_\pi^2 \tag{4-7d}$$

for $m_e = 0$. Ultimately we seek distributions in x, defined in Eqn. (4-3b), and

$$y = \frac{2\, p_1 \cdot (p_3 - p_2)}{m_\pi^2\, (1-x)} \tag{4-8}$$

At intermediate stages, however, it is advantageous to use the fact that x, y and ρ as defined in Eqn. (4-2) are Lorentz invariants and so work in the π^0 rest frame where we may employ the scaling variables

$$x_1 = \frac{2\omega}{m_\pi} \tag{4-9a}$$

$$x_2 = \frac{2E_2}{m_\pi} \tag{4-9b}$$

$$x_3 = \frac{2E_3}{m_\pi} \tag{4-9c}$$

$$x_1 + x_2 + x_3 = 2 \tag{4-9d}$$

The variable sets (x_1, x_2, x_3), (r,s,t) and (x,y) are related by

$$1 - x_1 = \frac{r}{m_\pi^2} = x \tag{4-10a}$$

$$1 - x_2 = \frac{s}{m_\pi^2} = \frac{1}{2}\,(1-x)(1+y) \tag{4-10b}$$

$$1 - x_3 = \frac{t}{m_\pi^2} = \frac{1}{2}\,(1-x)(1-y) \tag{4-10c}$$

Thus, for example, the lowest-order contribution $M_{fi}^{(0)}$ is given by Eqn. (4-6c) with

$$\Gamma_{\lambda}^{(0)} = -\frac{eF}{m_\pi r} e^{\mu} \varepsilon_{\mu\nu\rho\sigma} p_1{}^{\rho} K^{\sigma} \gamma^{\nu} \tag{4-11a}$$

$$K \equiv p_2 + p_3 \tag{4-11b}$$

which yields

$$\overline{|M_{fi}^{(0)}|^2} = -\left(\frac{e}{m_\pi r}\right)^2 |F|^2 \varepsilon_{\mu\nu\rho\sigma} p_1{}^{\rho} K^{\sigma} \varepsilon^{\mu}{}_{\lambda\alpha\beta} p_1{}^{\alpha} K^{\beta} \mathrm{Tr}[\gamma^{\nu} p_3 \gamma^{\lambda} p_2]$$

$$= -\left(\frac{2e}{m_\pi r}\right)^2 |F|^2 \varepsilon_{\mu\nu\rho\sigma} p_1{}^{\rho} K^{\sigma} \varepsilon^{\mu}{}_{\lambda\alpha\beta} p_1{}^{\alpha} K^{\beta} (p_3{}^{\nu} p_2{}^{\lambda} + p_2{}^{\nu} p_3{}^{\lambda} - \frac{r}{2} g^{\nu\lambda})$$

so, using

$$\varepsilon_{\mu\nu\rho\sigma} \varepsilon^{\mu}{}_{\lambda\alpha\beta} = -\det \begin{bmatrix} g_{\nu\lambda} & g_{\nu\alpha} & g_{\nu\beta} \\ g_{\rho\lambda} & g_{\rho\alpha} & g_{\rho\beta} \\ g_{\sigma\lambda} & g_{\sigma\alpha} & g_{\sigma\beta} \end{bmatrix} \tag{4-12a}$$

$$\varepsilon_{\mu\nu\rho\sigma} \varepsilon^{\mu\nu}{}_{\alpha\beta} = -2[g_{\rho\alpha} g_{\sigma\beta} - g_{\rho\beta} g_{\sigma\alpha}] \tag{4-12b}$$

we obtain

$$\overline{|M_{fi}^{(0)}|^2} = \left(\frac{e}{m_\pi r}\right)^2 |F|^2 [r(m_\pi{}^2 - r)^2 - 2\ r\ s\ t]$$

$$= \frac{e^2 |F|^2}{(1-x_1)} [x_1{}^2 - 2(1-x_2)(1-x_3)]$$

$$= \frac{e^2 |F|^2}{2} \frac{(1-x)^2}{x} [1 + y^2] \tag{4-13}$$

Similarly, working in the π^0 rest frame and denoting the $m_e \to 0$ limit by "^" the phase space integral is readily evaluated:

$$\frac{d^2\hat{\rho}}{dx_2 dx_3} = \frac{32\pi}{m_\pi^2 |F|^2} \int \frac{d^3k}{2\omega(2\pi)^3} \int \frac{d^3p_3}{2E_3(2\pi)^3} (2\pi)^4 \delta^4(p_1 - k - p_2 - p_3)$$

$$\overline{|M_{fi}|^2}\ \delta(x_2 - \frac{2E_2}{m_\pi})\ \delta\ (x_3 - \frac{2E_3}{m_\pi})\ \theta(\omega)\theta(E_2)\theta(E_3)$$

$$= \frac{1}{8\pi^2 |F|^2} \int_{-1}^{1} d\cos\theta_{23}\ x_2\ x_3\ \delta(1 - x_2 - x_3 + x_2 x_3 (1 - \cos\theta_{23})/2)$$

$$\overline{|M_{fi}|^2}\ \theta(x_2)\theta(x_3)\theta(2 - x_2 - x_3)$$

$$= \frac{1}{4\pi^2 |F|^2}\ \overline{|M_{fi}|^2}\theta((x_2 + x_3 - 1)(1-x_2)(1-x_3))\theta(x_2)\theta(x_3)\theta(2-x_2-x_3)$$

Noting that the Jacobian for $(x_2, x_3) \to (x,y)$ is

$$\frac{\partial(x_2,x_3)}{\partial(x,y)} = \frac{1}{2}\ (1-x)$$

we thus have

$$\frac{d^2\hat{\rho}}{dxdy} = \frac{(1-x)}{8\pi^2 |F|^2}\ \overline{|M_{fi}|^2}\ \theta(x)\theta(1-x)\theta(1-y^2) \tag{4-14}$$

Eqns. (4-13) and (4-14) give the double-differential distribution in lowest order as

$$\frac{d^2\hat{\rho}_o}{dxdy} = \frac{\alpha}{4\pi}\ \frac{(1-x)^3}{x}\ [1+y^2] \tag{4-15a}$$

and, integrating over y,

$$\frac{d\rho_o}{dx} = \frac{2\alpha}{3\pi}\frac{(1-x)^3}{x} \tag{4-15b}$$

for the single differential distribution.

Before proceeding, let us briefly consider our earlier remark concerning the vanishing interference of the ω^{-1} terms from two-photon exchange with the lowest order; ω^{-1} pole terms arise from external line emission only so, using the free partical Dirac equation and noting that the residue of the ω^{-1} poles in diagrams (b) and (c) of Fig. 9 is just the $\pi^0 \to e^-e^+$ amplitude we have, following the conventions of Chapter 3,

$$\Gamma^{(2\gamma)}_{\lambda,\text{pole}} \sim \left(\frac{p_{2\lambda}}{p_2\cdot k} - \frac{p_{3\lambda}}{p_3\cdot k}\right)[m_e K(m_\pi^{\ 2})]\ \gamma_5$$

in the soft photon limit and for a general form factor. On the other hand, apart from a factor of $f(0,r,m_\pi^{\ 2})$, Eqn. (4-11a) is the correct expression for $\Gamma^{(0)}_\lambda$ when $m_e \neq 0$. Then, using Eqn. (4-6c), we observe that the interference of the ω^{-1} terms with the lowest order contains the factor

$$\text{Tr}\ [\gamma^\nu(p_3 - m_e)\gamma_5(p_2 - m)] = 0$$

by the property

$$\text{Tr}[\gamma_5 a] = \text{Tr}[\gamma_5 ab] = \text{Tr}[\gamma_5 abc] = 0$$

of γ_5.

Returning to the issue at hand, in the $m_e \to 0$/pointlike π^0

approximation the two-photon-exchange contribution $M_{fi}^{(2\gamma)}$ may again be expressed in the form of Eqn. (4-6c) with

$$\Gamma_{\lambda}^{(2\gamma)} = i\, e^3 \frac{F}{m_\pi} \int \frac{d^4\ell}{(2\pi)^4} \frac{\varepsilon_{\mu\nu\alpha\beta}\ell^\alpha p_1^\beta}{\ell^2(\ell - p_1)^2} [\tau^{(b)}{}_\lambda^{\mu\nu} + \tau^{(c)\mu\nu}{}_\lambda + \tau^{(d)\mu}{}_\lambda{}^\nu] \tag{4-16a}$$

$$\tau^{(b)}{}_\lambda^{\mu\nu} = \gamma_\lambda \frac{1}{\not p_1 - \not p_3} \gamma^\mu \frac{1}{\not p_3 - \not \ell} \gamma^\nu \tag{4-16b}$$

$$\tau^{(c)\mu\nu}{}_\lambda = \gamma^\mu \frac{1}{\not p_2 - \not \ell} \gamma^\nu \frac{1}{\not p_2 - \not p_1} \gamma_\lambda \tag{4-16c}$$

$$\tau^{(d)\mu}{}_\lambda{}^\nu = \gamma^\mu \frac{1}{\not p_2 - \not \ell} \gamma_\lambda \frac{1}{\not p_1 - \not p_3 - \not \ell} \gamma^\nu \tag{4-16d}$$

Here the super-scripts (b), (c) and (d) refer to the corresponding graphs in Fig. 9. In turn, Eqns. (4-16a) through (4-16d) contribute an $O(\alpha^2)$ interference term

$$\Delta(2\gamma - 1\gamma) \equiv 2\, \mathrm{Re}[\sum_{pol} M_{fi}^{(o)*} M_{fi}^{(2\gamma)}] \tag{4-17}$$

to $\overline{|M_{fi}|^2}$. Using Eqns. (4-6c), (4-11a) and (4-16a) through (4-16d) we see that Eqn. (4-17) can be expressed as

$$\Delta(2\gamma - 1\gamma) = 2\, \mathrm{Re}\{\frac{1}{r}(\frac{e^2}{m_\pi})^2 |F|^2\, i \int \frac{d4\ell}{(2\pi)^4} \frac{1}{\ell^2(\ell - p_1)^2}$$

$$[\frac{\pi(b;\ell,p_1,p_2,p_3)}{t(p_3 - \ell)^2} + \frac{\pi(c;\ell,p_1,p_2,p_3)}{s(p_2 - \ell)^2} +$$

$$+\frac{\pi(d;\ell,p_1,p_2,p_3)}{(p_2-\ell)^2(p_1-p_3-\ell)^2}]\} \qquad (4\text{-}18a)$$

$$\pi(b;\ell,p_1,p_2,p_3) = \varepsilon_{\mu\nu\alpha\beta}\ell^\alpha p_1{}^\beta \varepsilon^\lambda{}_{\delta\rho\sigma}p_1{}^\rho K^\sigma T_\lambda{}^{\mu\nu\delta}(p_1-p_3,p_3-\ell;p_3,p_2) \qquad (4\text{-}18b)$$

$$\pi(c;\ell,p_1,p_2,p_3) = \varepsilon_{\mu\nu\alpha\beta}\ell^\alpha p_1{}^\beta \varepsilon^\lambda{}_{\delta\rho\sigma}p_1{}^\rho K^\sigma T^{\mu\nu}{}_\lambda{}^\delta(p_2-\ell,p_2-p_1;p_3,p_2) \qquad (4\text{-}18c)$$

$$\pi(d;\ell,p_1,p_2,p_3) = \varepsilon_{\mu\nu\alpha\beta}\ell^\alpha \varepsilon^\lambda{}_{\delta\rho\sigma}p_1{}^\rho K^\sigma T^\mu{}_\lambda{}^{\nu\delta}(p_2-\ell,p_1-p_3-\ell;p_3,p_2) \qquad (4\text{-}18d)$$

where we have defined

$$T^{\mu\nu\alpha\beta}(a,b;p_3,p_2) \equiv T_r[\gamma^\mu \not{a}\gamma^\nu \not{b}\gamma^\alpha \not{p}_3\gamma^\beta \not{p}_2] \qquad (4\text{-}19)$$

Eqns. (4-18b) through (4-18d) are evaluated in Appendix G, with the result that

$$\pi(b;\ell,p_1,p_2,p_3) = \pi(1;\ell,p_1,p_2,p_3)$$

$$\pi(c,\ell,p_1,p_2,p_3) = \pi(1;\ell,p_1,p_3,p_2)$$

$$\pi(d;\ell,p_1,p_2,p_3) =$$

$$\pi(2;\ell,p_1,p_2,p_3) + (p_1-\ell)^2\ \pi(3;\ell,p_1,p_2,p_3) +$$

$$+\ \pi(2;p_1-\ell,p_1,p_3,p_2) + \ell^2\ \pi(3;p_1-\ell,p_1,p_3,p_2)$$

where the $\pi(i;\ell,p_1,p_2,p_3)$ are explicitly given in Eqns. (G-3), (G-6) and (G-7) for i = 1, 2 and 3 respectively. Noting that the loop integrals are at most logarithmically divergent so that a shift $\ell \to p_1 - \ell$ is freely allowed, we see that $\Delta(2\gamma - 1\gamma)$ may also be expressed as

$$\Delta(2\gamma - 1\gamma) = \frac{2}{r}\left(\frac{\alpha}{m_\pi}\right)^2 |F|^2 \, \mathrm{Re}[J(r,s,t) + J(r,t,s)] \tag{4-20a}$$

$$J(r,s,t) = i(4\pi)^2 \int \frac{d4\ell}{(2\pi)^4} \frac{1}{\ell^2} \left[\frac{\pi(1;\ell,p_1,p_2,p_3)}{t(p_1-\ell)^2(p_3-\ell)^2} + \right.$$

$$\left. + \frac{\pi(2;\ell,p_1,p_2,p_3)}{(p_1-\ell)^2(p_2-\ell)^2(p_1-p_3-\ell)^2} + \frac{\pi(3;\ell,p_1,p_2,p_3)}{(p_2-\ell)^2(p_1-p_3-\ell)^2}\right] \tag{4-20b}$$

C. Reduction of $\Delta(2\gamma - 1\gamma)$ to Parametric Integrals

Let us combine the denominators appearing in J(r,s,t) using the general parametric formula

$$[a_1 a_2 \ldots a_n]^{-1} = (n-1)! \int d\rho(n) \; . \; [a_1 y_1 + a_2 y_2 + \ldots + a_n y_n]^{-n} \tag{4-21a}$$

$$\int d\rho(n) \equiv \int_0^1 dy_1 \int_0^1 dy_2 \ldots \int_0^1 dy_n \; \delta(1 - y_1 - y_2 - \ldots - y_n): \tag{4-21b}$$

We first have

$$[(p_1 - \ell)^2 \ell^2 (p_3 - \ell)^2]^{-1} = 2! \int d\rho(3) . \; [(\ell - \ell_1)^2 - d_1(t)]^{-3} \tag{4-22a}$$

with

$$\ell_1 \equiv p_1 y_1 + p_3 y_3 \qquad (4\text{-}22b)$$

$$-d_1(t) \equiv m_\pi^{\ 2} y_1 - \ell_1^{\ 2}$$

$$= m_\pi^{\ 2} y_1 - y_1 [m_\pi^{\ 2} y_1 + (m_\pi^{\ 2} - t) y_3]$$

$$= y_1 [m_\pi^{\ 2} y_2 + t\ y_3] \qquad (4\text{-}22c)$$

similarly

$$[(p_1 - p_3 - \ell)^2 (p_2 - \ell)^2 \ell^2]^{-1} = 2! \int d\rho(3) \quad [\ell - \ell_3)^2 - d_3(t)]^{-3} \qquad (4\text{-}23a)$$

$$\ell_3 \equiv (p_1 - p_3) y_1 + p_2 y_2 \qquad (4\text{-}23b)$$

$$-d_3(t) \equiv t\ y_1 - \ell_3^{\ 2}$$

$$= t\ y_1 - y_1\ [t\ y_1 + (m_\pi^{\ 2} - s - r)\ y_2]$$

$$= t\ y_1 y_3 \qquad (4\text{-}23c)$$

and

$$[(p_1 - \ell)^2 (p_2 - \ell)^2 (p - p_1 - \ell_3)^2 \ell^2]^{-1} = 3! \int d\rho(4) . [(\ell - \ell_2)^2 - d_2(s,t)]^{-4} \qquad (4\text{-}24a)$$

$$\ell_2 \equiv p_1 y_1 + p_2 y_2 + (p_1 - p_3)\, y_3 \tag{4-24b}$$

$$- d_2(s,t) \equiv m_\pi^{\;2}\, y_1 + t\, y_3 - \ell_2^{\;2}$$

$$= m_\pi^{\;2}\, y_1 + t\, y_3 - [m_\pi^{\;2} y_1^{\;2} + t\, y_3^{\;2} + (m_\pi^{\;2} - s)\, y_1 y_2 +$$

$$+ (m_\pi^{\;2} + t)\, y_1 y_3 + t\, y_2 y_3]$$

$$= m_\pi^{\;2}\, y_1 y_4 + s\, y_1 y_2 + t\, y_3 y_4 \tag{4-24c}$$

Shifting loop variables we then obtain $J(r,s,t)$ as

$$J(r,s,t) =$$

$$i(4\pi)^2(2!) \int d\rho(3) . \int \frac{d4\ell}{(2\pi)^4} \left[\frac{\pi(1;\ell+\ell_1,p_1,p_2,p_3)}{t[\ell^2 - d_1(t)]^3} + \frac{\pi(3;\ell+\ell_3,p_1,p_2,p_3)}{[\ell^2 - d_3(t)]^3}\right] +$$

$$+ i(4\pi)^2(3!) \int d\rho(4) . \int \frac{d4\ell}{(2\pi)^4} \frac{\pi(2;\ell+\ell_2,p_1,p_2,p_3)}{[\ell^2 - d_2(s,t)]^4} \tag{4-25}$$

Since the denominators in Eqn. (4-25) are symmetric under $\ell \to - \ell$ we may use

$$\ell_\mu \ell_\nu \to \frac{\ell^2}{4} g_{\mu\nu},\quad \ell_\mu \to 0 \tag{4-26}$$

to write

$$\pi(i,\ell+\ell_i,p_1,p_2,p_3) = \ell^2 N_1(i;r,s,t) + N_2(i;r,s,t) \tag{4-27}$$

The $N_j(i;r,s,t)$ are evaluated in Appendix H; using Eqns. (4-25), (4-27), (H-1) and (H-3) we have the divergent part of $J(r,s,t)$ as

$$J_D(r,s,t) =$$

$$-r(t-s)i(4\pi)^2(2!) \int d\rho(3). \int \frac{d4\ell}{(2\pi)^4} [\frac{3\ell^2}{[\ell^2-d_1(t)]^3} + \frac{\ell^2}{[\ell^2-d_3(t)]^3}] \qquad (4\text{-}28a)$$

while Eqns. (4-25), (4-27), (H-2), (H-4), (H-5), and (H-6) give

$$J_F(r,s,t) =$$

$$-2ri(4\pi)^2(2!) \int d\rho(3). \int \frac{d^4\ell}{(2\pi)^4}[\frac{(t-s)(m_\pi{}^2-t)y_1y_3}{[\ell^2-d_1(t)]^3} + \frac{t^2y_1(y_1+ y_2)}{\ell^2-d_3(t)]^3}] +$$

$$ri(4\pi)^2(3!) \int d\rho(4). \int \frac{d4\ell}{(2\pi)^4} \frac{2\ st[(m_\pi{}^2-t)y_1y_2+ sy_2y_3]-\ell^2[rt-m_\pi{}^2s]}{[\ell^2 - d_2(s,t)]^4} \qquad (4\text{-}28b)$$

Now, using a further parametric formula

$$a^{-m}- b^{-m} = m[b-a] \int_o^1 dz[a(1-z) + bz]^{-m-1} \qquad (4\text{-}29)$$

we observe that

$$J_D(r,s,t) + J_D(r,t,s) =$$

$$-r(t-s)i(4\pi)^2(3!)\int d\rho(3).\int_o^1 dz \int \frac{d4\ell}{(2\pi)^4} \ell^2[\frac{3[d_1(t) - d_1(s)]}{[\ell^2-d_1(t)(1-z)-d_1(s)z]^4} +$$

$$+ \frac{[d_3(t) - d_3(s)]}{[\ell^2-d_3(t)(1-z)-d_3(s)z]^4}] \qquad (4\text{-}30)$$

Eqn. (4-30) is finite by power counting so, carrying out the loop integral followed by the z integral

$$J_D(r,s,t) + J_D(r,t,s) =$$

$$2r(t-s) \int d\rho(3) \, . \, [3 \ \ln \left(\frac{d_1(s)}{d_1(t)}\right) + \ln \left(\frac{d_3(s)}{d_3(t)}\right)] \qquad (4\text{-}31)$$

Carrying out the loop integrals in Eqn. (4-28b) and noting that

$$d_2(s,t) \underset{\substack{y_1 \leftrightarrow y_4 \\ y_2 \leftrightarrow y_3}}{} d_2(t,s)$$

we also obtain

$$J_F(r,s,t) + J_F(r,t,s) =$$

$$-2r(t-s) \int d\rho(3) \, . \, [\frac{(m_\pi{}^2 - t)y_1 y_3}{d_1(t)} - \frac{(m_\pi{}^2 - s)y_1 y_3}{d_1(s)}] +$$

$$+2r(m_\pi{}^2 - r) \int d\rho(3) . \left(\frac{y_1 + y_2}{y_3}\right) + 2r(m_\pi{}^2 - r)^2 \int d\rho(4) . \frac{1}{d_2(s,t)} +$$

$$-2rst \int d\rho(4) . \frac{[(m\ {}^2 - t)y_1 y_2 + (m_\pi{}^2 - s)y_3 y_4 + (m_\pi{}^2 - r)y_2 y_3]}{[d_2(s,t)]^2}] \qquad (4\text{-}31b)$$

Transforming to the scaling variables,

$$d_1(t) = - m_\pi{}^2 y_1 [(y_2 + y_3) - x_3 y_3]$$

$$\equiv - m_\pi{}^2 y_1 \ d(x_3) \qquad (4\text{-}32a)$$

$$d_2(s,t) = - m_\pi^{\ 2}[y_1 y_4 + (1-x_2)y_1 y_2 + (1-x_3)y_3 y_4]$$

$$\equiv - m_\pi^{\ 2}\ D(x_2,x_3) \qquad (4\text{-}32b)$$

and

$$d_3(t) = - m_\pi^{\ 2}(1-x_3)y_1 y_3$$

so, using Eqns. (4-20a) and (4-31a) through (4-32b) we at last have $\Delta(2\gamma-1\gamma)$ expressable in the form

$$\Delta(2\gamma - 1\gamma) = 4\alpha^2 |F|^2\ \mathrm{Re}[I(x_1,x_2,x_3)] \qquad (4\text{-}33a)$$

$$I(x_1,x_2,x_3) \equiv I_1(x_1,x_2,x_3) + I_2(x_1,x_2,x_3) \qquad (4\text{-}33b)$$

$$I_1(x_1,x_2,x_3) \equiv$$

$$(x_2 - x_3) \int d\rho(3)\ .\ [3\ \ell n\ (\frac{d(x_2)}{d(x_3)}) + \ell n\ (\frac{1-x_2}{1-x_3}) +$$

$$+ \frac{x_3 y_3}{d(x_3)} - \frac{x_2 y_3}{d(x_2)}] - x_1^{\ 2} \int d\rho(4)\ .\ [D(x_2,x_3)]^{-1} +$$

$$-(1-x_2)(1-x_3) \int d\rho(4)\ .\ \frac{[x_3 y_1 y_2 + x_2 y_3 y_4]}{[D(x_2,x_3)]^2} \qquad (4\text{-}33c)$$

$$I_2(x_1,x_2,x_3) \equiv$$

$$x_1 \int d\rho(3)\ .\ [\frac{y_1 + y_2}{y_3}] - x_1(1-x_2)(1-x_3) \int d\rho(4)\ .\ \frac{y_2 y_3}{[D(x_2,x_3)]^2} \qquad (4\text{-}33d)$$

D. Results for the Two-Photon-Exchange Contribution

The parametric integrals defining $I_1(x_1,x_2,x_3)$ and $I_2(x_1,x_2,x_3)$ are evaluated in Appendix I and Appendix J, respectively; Eqns. (4-33b), (I-7) and (J-12) give

$$I(x_1,x_2,x_3) =$$

$$-\frac{x_1}{2} - \frac{(1-x_2)(1-x_3)}{2\,x_1}\left[\pi^2 + \ln^2\left(\frac{1-x_2}{1-x_3}\right)\right] - (x_3 - x_2)\,\ln\left(\frac{1-x_2}{1-x_3}\right) +$$

$$- \left[x_1^{\,2} - 2(1-x_2)(1-x_3)\right]\frac{G(x_2,x_3)}{(1-x_1)} \qquad (4\text{-}34a)$$

where

$$G(u,v) = L_{i2}(u) + L_{i2}(v) + \ln(1-u)\ln(1-v) - \frac{\pi^2}{6} \qquad (4\text{-}34b)$$

and

$$L_{i2}(x) = -\int_0^x \frac{dt}{t}\,\ln(1-t) \qquad (4\text{-}34c)$$

is the dilogarithm function[12]. We note that $I(x_1,x_2,x_3)$ is free of mass singularities and imaginary parts in accord with the conjectured connection between these quantities[13].

Transforming to the variables (x,y), Eqns. (4-14), (4-15a), (4-33a) and (4-34a) allow us to write the two-photon-exchange contribution to the double-differential distribution in the form

$$\frac{d^2\rho_R^{II}}{dx\,dy} =$$

$$- \left(\frac{\alpha}{\pi}\right)G(x_+,x_-)\ \frac{d^2\rho_o}{dxdy} - \frac{1}{4}\left(\frac{\alpha}{\pi}\right)^2(1-x)^2\left\{1 + \frac{1}{4}(1-y^2)\left[\pi^2 + \ell n^2\left(\frac{1+y}{1-y}\right)\right] + \right.$$

$$\left. + 2y\ \ell n\ \left(\frac{1+y}{1-y}\right)\ \right\} \tag{4-35a}$$

where

$$x_\pm = 1 - \frac{1}{2}\,(1-x)(1\pm y) \tag{4-35b}$$

Observing that the radiative correction function

$$\delta^{II}\ (x,y)\ \equiv \left(\frac{d^2\rho_R^{\ II}}{dxdy}\right)/\left(\frac{d^2\rho_o}{dxdy}\right) \tag{4-36}$$

is negative definite and symmetric under $y \rightarrow -y$, in Table I we give numerical values for $-\ \delta^{II}(x,y)$.

The single differential distribution is given by

$$\frac{d\rho_R^{\ II}}{dx} = \int_{-1}^{1}\ dy\ \frac{d^2\rho_R^{\ II}}{dxdy}$$

With the change of variable $y = 2z = -1$ several of the integrals are trivial and we have

$$\frac{d\rho_R^{\ II}}{dx} =$$

$$- \frac{\alpha}{\pi}\left[\ell n^2(1-x)\ - \frac{\pi^2}{6}\right]\ \frac{d\rho_o}{dx} - \left(\frac{\alpha}{\pi}\right)^2(1-x)^2\left[\frac{7}{4} - \frac{2}{27} - \frac{5}{9}\ \left(\frac{1-x}{x}\right)\ell n(1-x)\right]\ +$$

TABLE I

$-\delta^{(II)}(x,y)$ Given in Percent for a Range of Values of x and y

X \ Y	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95
0.01	0.0146	0.0146	0.0146	0.0146	0.0147	0.0149	0.0152	0.0157	0.0168	0.0191	0.0217
0.02	0.0295	0.0295	0.0295	0.0295	0.0297	0.0300	0.0306	0.0318	0.0339	0.0385	0.0438
0.03	0.0447	0.0446	0.0446	0.0447	0.0449	0.0454	0.0463	0.0480	0.0513	0.0582	0.0663
0.04	0.0601	0.0601	0.0601	0.0601	0.0604	0.0610	0.0623	0.0646	0.0690	0.0783	0.0891
0.05	0.0758	0.0758	0.0758	0.0758	0.0762	0.0770	0.0786	0.0815	0.0870	0.0987	0.112
0.06	0.0918	0.0918	0.0918	0.0918	0.0922	0.0932	0.0951	0.0987	0.105	0.120	0.136
0.07	0.108	0.108	0.108	0.108	0.109	0.110	0.112	0.116	0.124	0.141	0.160
0.08	0.125	0.125	0.125	0.125	0.125	0.127	0.129	0.134	0.143	0.162	0.185
0.09	0.142	0.142	0.142	0.142	0.142	0.144	0.147	0.152	0.162	0.184	0.210
0.10	0.159	0.159	0.159	0.159	0.160	0.161	0.165	0.171	0.182	0.206	0.235
0.15	0.251	0.251	0.250	0.251	0.252	0.254	0.259	0.269	0.286	0.325	0.369
0.20	0.352	0.352	0.352	0.352	0.353	0.357	0.364	0.377	0.402	0.455	0.517
0.25	0.466	0.466	0.465	0.465	0.467	0.471	0.481	0.498	0.530	0.600	0.682
0.30	0.594	0.593	0.593	0.593	0.595	0.600	0.612	0.633	0.674	0.762	0.865
0.35	0.739	0.738	0.738	0.737	0.740	0.746	0.760	0.786	0.836	0.945	1.07
0.40	0.905	0.904	0.903	0.903	0.905	0.913	0.930	0.961	1.02	1.15	1.31
0.45	1.10	1.10	1.10	1.10	1.10	1.11	1.13	1.16	1.24	1.39	1.58
0.50	1.32	1.32	1.32	1.32	1.32	1.33	1.36	1.40	1.49	1.68	1.90
0.55	1.60	1.59	1.59	1.59	1.59	1.60	1.63	1.68	1.79	2.01	2.27
0.60	1.93	1.93	1.92	1.92	1.92	1.93	1.96	2.03	2.15	2.42	2.73
0.65	2.34	2.34	2.34	2.33	2.33	2.35	2.38	2.46	2.60	2.92	3.30
0.70	2.88	2.88	2.87	2.87	2.87	2.88	2.92	3.01	3.18	3.57	4.03
0.75	3.61	3.61	3.60	3.59	3.59	3.60	3.65	3.76	3.97	4.44	5.01
0.80	4.67	4.67	4.66	4.64	4.63	4.65	4.71	4.84	5.10	5.70	6.42
0.85	6.38	6.37	6.35	6.32	6.31	6.32	6.39	6.56	6.90	7.70	8.67
0.90	9.62	9.61	9.57	9.52	9.49	9.49	9.58	9.81	10.3	11.5	12.9
0.95	18.8	18.8	18.7	18.5	18.4	18.4	18.5	18.9	19.9	22.1	24.8

$$+ \left(\frac{\alpha}{\pi}\right)^2 (1-x)^2 \int_o^1 dz\ z(1-z)\ell n(z)\ell n(1-z) +$$

$$- \left(\frac{\alpha}{\pi}\right)^2 \frac{(1-x)^3}{x} \int_o^1 dz[(1-2z + 2z^2)[2\ L_{i2}(1-(1-x)z) + \ell n(z)\ell n(1-z)]] \quad (4\text{-}37)$$

where $d\rho_o/dx$ is given in Eqn. (4-15b). The remaining integrals may be carried out using the formula given in Appendix K; then, after some simplifications, Eqn. (4-37) yields

$$\frac{d\rho_R^{\ II}}{dx} =$$

$$-\left(\frac{\alpha}{\pi}\right)\ell n^2(1-x)\ \frac{d\rho_o}{dx} - \frac{2}{3}\left(\frac{\alpha}{\pi}\right)^2 (1-x)^2 \left[\frac{5}{2} + \frac{\pi^2}{6} + \frac{2}{(1-x)}\ [\ell n(1-x)-1]\right] +$$

$$- \frac{2}{3}\left(\frac{\alpha}{\pi}\right)^2\ [2x^2 + 3(1-x)]\left[\frac{\pi^2}{6} - L_{i2}(x)\right] \quad (4\text{-}38)$$

In Fig. 10 we show the radiative radiative correction function

$$\delta^{II}(x) \equiv \left(\frac{d\rho_R^{\ II}}{dx}\right)/\left(\frac{d\rho_o}{dx}\right) \quad (4\text{-}39)$$

for $0.1 \leqslant x \leqslant 0.9$. It is readily observed that the two-photon-exchange contribution is non-negligible, particularly for large x where $(d\rho_R^{\ II}/dx)$ falls to zero more slowly than $(d\rho_o/dx)$. A crude estimate of the effect of this contribution on the determination of a_π can be obtained by writing

$$\frac{d\rho_{exp}}{dx} \equiv \left(\frac{d\hat{\rho}_o}{dx}\right)\ (1 + 2a_\pi^o x) \quad (4\text{-}40a)$$

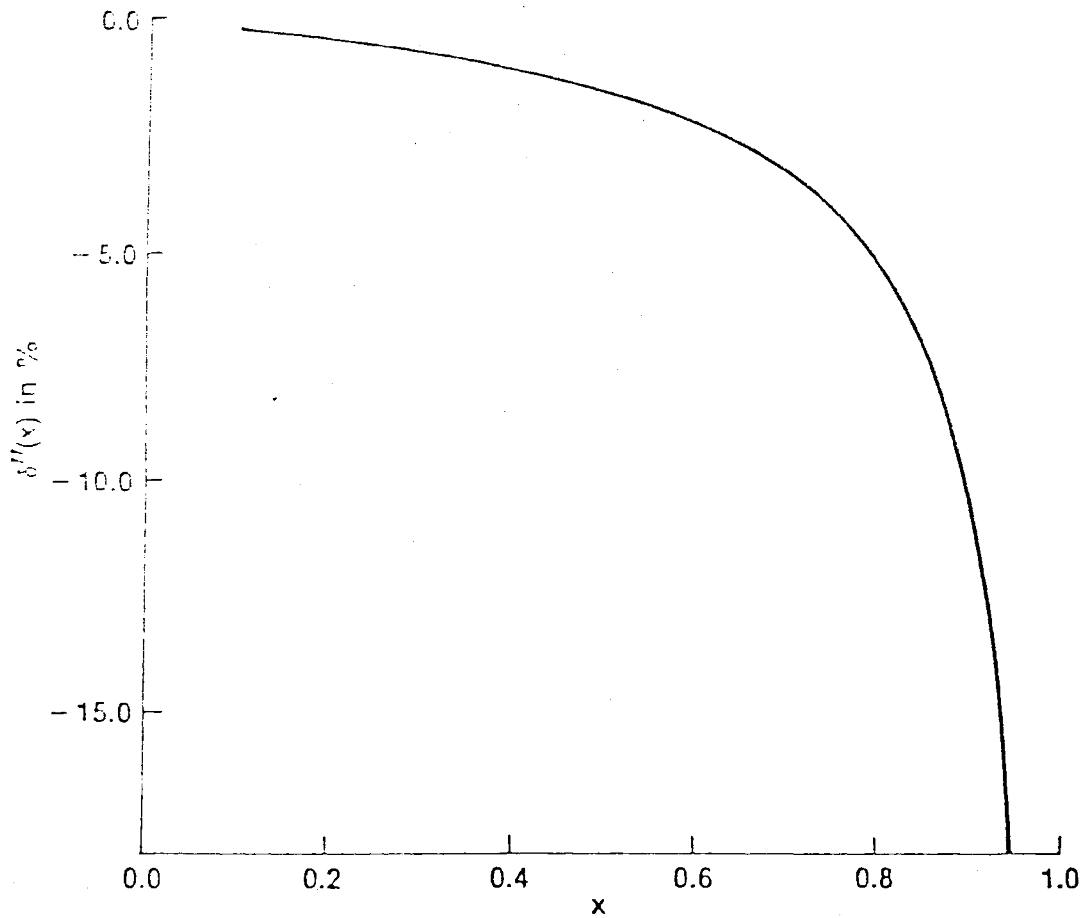

Figure 10. $\delta^{II}(x)$ Given in Percent as a Function of x.

$$\frac{d\rho_{exp}}{dx}\,(1-\delta^{I}(x)) \equiv \left(\frac{d\hat{\rho}_o}{dx}\right)(1 + 2a_\pi^{I}x) \tag{4-40b}$$

$$\frac{d\rho_{exp}}{dx}\,(1-\delta^{I}(x) - \delta^{II}(x)) \equiv \left(\frac{d\hat{\rho}_o}{dx}\right)(1 + 2a_\pi^{II}x) \tag{4-40c}$$

or

$$a_\pi^{II} = a_\pi^{I} - [a_\pi^{o} + \frac{1}{2x}]\,\delta^{II}(x) \tag{4-40d}$$

Using the values[11] $a_\pi^{o} = 0.05 \pm 0.03$ and $a_\pi^{I} = 0.10 \pm 0.03$ as input we find for the one-photon-plus two-photon-exchange corrected value

$$a_\pi^{II} = 0.12\,{}^{+0.05}_{-0.04} \tag{4-41}$$

with $0.1 \leq x \leq 0.8$, consistent with the cuts made in Ref. 11. We emphasize that Eqn. (4-41) is merely an estimate; a precise determination of a_π^{II} requires a reanalysis of the experimental data.

E. Remarks

(a) We have derived our analytical expressions for the two-photon-exchange contribution in the expectation that future experiments will show a_π to be small such that terms of order a_π may be ignored. Should such experiments confirm a value of a_π as large as Eqn. (4-41) indicates, calculation of the model dependent $O(a_\pi)$ corrections to our expressions will become important.

(b) The two-photon-exchange contribution to $\pi^0 \to \gamma e^+e^-$ affects not only the determination of a but also the measurement of the $\pi^0 \to e^+e^-$ branching ratio as it changes the invariant mass $(m_\pi\sqrt{x})$

distribution in the region $x \sim 1$. The feature that $(d\rho_R^{II}/dx)$ is negative definite suggests that the $\pi^0 \to e^+e^-$ branching ratio may be larger than recent experiments[14] indicate, so increasing the disagreement between the standard models and the experiments that was discussed in Chapter III.

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CHAPTER V

SUMMARY AND CONCLUDING REMARKS

In this work we have considered several rare decay processes belonging to the high-energy and intermediate-energy regimes.

In Chapter II we examined the decay $Z^0 \to gg\gamma$ and the contribution of the vector part to $Z^0 \to ggg$. We found that although the amplitudes varied insignificantly from their massless values for quark masses up to the bottom quark, due to a coherence effect the total decay rates for $Z^0 \to gg\gamma$ and the vector part of $Z^0 \to ggg$ showed a dramatic dependence on top quark masses in the range $0 \leqslant m_t \leqslant m_Z$. For a top quark mass of 20 GeV we obtained the branching ratio

$$\frac{\Gamma(Z^0 \to gg\gamma)}{\Gamma_o} = 1.8 \times 10^{-6}$$

and the lower bound

$$\frac{\Gamma(Z^0 \to ggg)}{\Gamma_o} \geqslant 0.8 \times 10^{-5}$$

where Γ_o is the Z° total hadronic width. As we noted there, the same coherence effect may result in a sizeable axial-vector contribution, so raising the prospects for observing this decay mode.

In Chapter III we studied the rare decays $\pi^0 \to e^+e^-$ and $\eta \to \mu^+\mu^-$. After extensively discussing the existing experimental and theoretical situation for these decays, we demonstrated that by relaxing the usual predilection for an unsubtracted dispersion relation in describing the amplitude, and instead choosing a once-subtracted dispersion relation with a small subtraction constant, excellent agreement with the experiments could be achieved. The leading model independent estimates

$$\frac{\Gamma(\pi^0 \to e^+e^-)}{\Gamma(\pi^0 \to \text{all})} \simeq 2.0 \times 10^{-7}$$

and

$$\frac{\Gamma(\eta \to \mu^+\mu^-)}{\Gamma(\eta \to \text{all})} \simeq 6.1 \times 10^{-6}$$

were also shown to be quite stable against the inclusion of model dependent form factor effects. Working in a simple vector meson dominance model for the form factor we found that the $\pi^0 \to e^+e^-$ prediction is unchanged, while the $\eta \to \mu^+\mu^-$ branching ratio is enhanced by 16%. Within this same model we also predicted the as yet unobserved $\eta \to e^+e^-$ decay to have a branching ratio

$$\frac{\Gamma(\eta \to e^+e^-)}{\Gamma(\eta \to \text{all})} = 1.1 \times 10^{-8}$$

In Chapter IV we examined the radiative correction problem for the decay $\pi^0 \to e^+e^-\gamma$. We first discussed how the anomalous results of the early experiments studying the π^0 form factor could be understood in terms of a mistreatment of the one-photon-exchange corrections to the Dalitz-pair spectrum. We then pointed out that the usual argument for suppression of the two-photon-exchange correction fails, and exhibited a complete calculation of the two-photon-exchange contribution in the approximations of a pointlike π^0 and vanishing electron mass. In this manner we arrived at leading model independent analytical expressions for the two-photon-exchange radiative correction functions. We found this correction to be important, with an extimated 20% or so effect on the determination of the π^0 form-factor slope.

We wish to close with one final comment: although the prospect of doing physics at the frontiers of new energy scales is exciting, problems of the sort explored in Chapters II and III should enjoy not less but more attention for the plaguing difficulties and discrepancies they present. Indeed, it is often in the attempt to resolve such problems that we are forced to abandon our preconceptions and so begin on the road to deeper understanding.

APPENDIX A

THE EXPRESSIONS FOR $\hat{E}^{(i)}_{\lambda++}(r,s,t)$

Let

$$u_1 = u + \mu_1, \quad u = r,s,t \tag{A-1}$$

The helicity amplitudes are given by:

$$\begin{aligned}
\hat{E}^{(1)}_{+++}(r,s,t) = & -\frac{2t}{s_1} + \left\{-\frac{4st}{rs_1} - \frac{4\mu_1 t}{s_1{}^2} + \frac{2s}{s_1}\right\} B(s) + \left\{\frac{4t}{r} - \frac{2t}{t_1}\right\} B(t) + \\
& + \left\{-\frac{4\mu_1 t}{rs_1} + \frac{4\mu_1 t}{s_1{}^2} + \frac{2\mu_1}{s_1} - \frac{2\mu_1}{t_1}\right\} B(-\mu_1) + \left\{\frac{1}{r} - \frac{1}{t}\right\} T(r) + \\
& + \left\{\frac{2(s-t)}{r} - \frac{4st}{r^2} - \frac{3}{s_1} - \frac{2t}{s_1{}^2} - \frac{1}{r} - \frac{1}{s} - \frac{1}{t}\right\} T(s) + \\
& + \left\{\frac{2(s-t)}{r} - \frac{4st}{r^2} + \frac{1}{t_1} - \frac{1}{r}\right\} T(t) + \\
& + \left\{ -\frac{2(s-t)}{r} + \frac{4st}{r^2} + \frac{3}{s_1} - \frac{1}{t_1} + \frac{st}{s_1{}^2} + \frac{1}{r} + \frac{1}{t}\right\} T(-\mu_1) + \\
& - \frac{r_1(r-t)}{rst} I_o(r,s,\mu_1) + \frac{r_1}{rt} I_o(r,t,\mu_1) + \\
& + \left\{ -\frac{2(s-t)}{r} + \frac{4st}{r^2} - \frac{1}{t} + \frac{3}{r} + \frac{2}{s}\right\} I_o(s,t,\mu_1),
\end{aligned} \tag{A-2a}$$

$$\hat{E}^{(1)}_{-++}(r,s,t) = \left\{\frac{1}{t} - \frac{1}{r}\right\}[T(r) + T(s) + T(t) - T(-\mu_1)] +$$

$$+ \frac{r_1}{st} I_o(r,s,\mu_1) - \frac{t_1}{rs} I_o(s,t,\mu_1), \tag{A-2b}$$

$$\begin{aligned}\hat{E}^{(2)}_{+++}(r,s,t) = &\{\frac{4s}{r} + \frac{2s}{s_1}\}B(s) + \{\frac{4t}{r} + \frac{2t}{t_1}\}B(t) + \{\frac{4\mu_1}{r} + \frac{2\mu_1}{s_1} + \frac{2\mu_1}{t_1}\}B(-\mu_1) + \\ &+ \{-\frac{1}{r} - \frac{1}{s} - \frac{1}{t}\}T(r) + \{-\frac{4st}{r^2} + \frac{2r_1}{r} + \frac{r}{s_1 t} - \frac{3}{r}\}T(s) + \\ &+ \{-\frac{4st}{r^2} + \frac{2r_1}{r} + \frac{r}{st_1} - \frac{3}{r}\}T(t) + \{\frac{4st}{r^2} - \frac{2r_1}{r} - \frac{r_1}{st} + \\ &+ \frac{1}{s_1} + \frac{1}{t_1} + \frac{3}{r}\}T(-\mu_1) + \\ &+ \{\frac{t}{rs} - \frac{s_1}{rt} + \frac{1}{rs}\}I_o(r,s,\mu_1) + \{\frac{s}{rt} - \frac{t_1}{rs} + \frac{1}{rt}\}I_o(r,t,\mu_1) + \\ &+ \{\frac{4st}{r^2} - \frac{2r_1}{r} - \frac{r_1}{st} + \frac{5}{r} + \frac{1}{st}\}I_o(s,t,\mu_1)\end{aligned} \tag{A-2c}$$

and

$$\begin{aligned}\hat{E}^{(2)}_{-++}(r,s,t) = &- 2 + \{-\frac{1}{r} - \frac{1}{s} - \frac{1}{t}\}[T(r) + T(s) + T(t) - T(-\mu_1)] + \\ &+ \{\frac{1}{t} + \frac{1}{rs}\}I_o(r,s,\mu_1) + \{\frac{1}{s} + \frac{1}{rt}\}I_o(r,t,\mu_1) + \\ &+ \{\frac{1}{r} + \frac{1}{st}\}I_o(s,t,\mu_1)\end{aligned} \tag{A-2d}$$

The functions $B(r)$, $T(r)$ and $I_o(r,s,\ \mu_1)$ appearing in Eqns. (A-2a) through (A-2d) are defined by

$$B(r) = \frac{1}{2} \int_o^1 dy\ \ln(1 - r(1-y^2)), \tag{A-3a}$$

$$T(r) = \int_o^1 \frac{dy}{1-y^2} \ln(1-r(1-y^2)), \tag{A-3b}$$

$$I_o(r,s,\mu_1) = F(r,a) + F(s,a) - F(-\mu_1,a), \tag{A-3c}$$

$$F(r,a) = \int_o^1 \frac{dy}{a^2 - y^2} \ln(1-r(1-y^2)) \tag{A-3d}$$

where

$$a = (1 + \frac{t}{rs})^{1/2} \tag{A-3e}$$

For $r > 1$, $B(r)$, $T(r)$ and $F(r,a)$ are explicitly given by

$$B(r) = \mathrm{Re}\{\frac{b(r)}{2} \ln(\frac{b(r)+1}{b(r)-1}) - 1\} - i \frac{\pi}{2} b(r)$$

$$= b(r) \cosh^{-1}(\sqrt{r}) - 1 - i \frac{\pi}{2} b(r), \tag{A-4a}$$

$$T(r) = \mathrm{Re}[\frac{1}{2} \ln (\frac{b(r)+1}{b(r)-1})]^2 - i \pi \cosh^{-1}(\sqrt{r})$$

$$= [\cosh^{-1}(\sqrt{r})]^2 - \frac{\pi^2}{4} - i \pi \cosh^{-1}(\sqrt{r}), \tag{A-4b}$$

$$F(r,a) = \mathrm{Re}\{\frac{1}{2a}[\ln(r(a^2-b^2(r))) \ln (\frac{a+1}{a-1}) - L_{i2}(\frac{a+1}{a+b(r)}) +$$

$$+ L_{i2} (\frac{a-1}{a+b(r)}) - L_{i2} (\frac{a+1}{a-b(r)}) + L_{i2} (\frac{a-1}{a-b(r)})]\} +$$

$$+ i \frac{\pi}{2a} \ln (\frac{a-b(r)}{a+b(r)}) \tag{A-4c}$$

Here $L_{i2}(x)$ is the dilog-function, defined as

$$L_{i2}(x) = -\int_o^x \frac{dt}{t} \ln(1-t)$$

$$= \sum_{n=1}^{\infty} \frac{x^n}{n^2}, \quad |x| < 1 \qquad \text{(A-5)}$$

and

$$b(r) = (1 - \frac{1}{r})^{1/2} \qquad \text{(A-6)}$$

Note that for $x > 1$, $L_{i2}(x)$ has an imaginary part, however by employing Euler's relation

$$L_{i2}(x) + L_{i2}(1-x) = \frac{\pi^2}{6} - \ln(x)\ln(1-x) \qquad \text{(A-7)}$$

we can make the real part in Eqn. (A-4c) explicit:

$$F(r,a) = \frac{1}{2a}\{\ln (\frac{1+b(r)}{1-b(r)}) \ln (\frac{a+b(r)}{a-b(r)}) - i \pi \ln (\frac{a+b(r)}{a-b(r)}) +$$

$$+ L_{i2} (\frac{b(r)-1}{a+b(r)}) + L_{i2} (\frac{1+b(r)}{b(r)-a}) - L_{i2} (\frac{1+b(r)}{a+b(r)}) +$$

$$- L_{i2} (\frac{1-b(r)}{a-b(r)})\}, \quad r > 1 \qquad \text{(A-8)}$$

The functional forms of $B(r)$, $T(r)$ and $F(r,a)$ for $r \leqslant 1$ can be obtained from those for $r > 1$ by $b(r) \rightarrow i\, d(r)$ where

$$d(r) = (\frac{1}{r} - 1)^{1/2} \qquad \text{(A-9)}$$

We easily have

$$B(r) = d(r)\ \{\frac{\pi}{2} + i\ \cosh^{-1}(\sqrt{r})\} - 1$$

$$= d(r)\ \sin^{-1}(\sqrt{r}) - 1 \qquad \text{(A-10a)}$$

$$T(r) = -[\frac{\pi}{2} + i\ \cosh^{-1}(\sqrt{r})]^2$$

$$= -\ [\sin^{-1}(\sqrt{r})]^2 \qquad \text{(A-10b)}$$

Defining

$$M(r) = \sqrt{a^2 + d^2(r)} \qquad \text{(A-11a)}$$

$$\cos(\theta(r)) = \frac{a}{M(r)} \qquad \text{(A-11b)}$$

we obtain, using Eqns. (A-4c) and (A-5)

$$F(r,a) = \frac{1}{2a}\ \{\ln(r(a^2 + d^2(r)))\ \ln\ (\frac{a+1}{a-1}) - L_{i2}(\frac{a+1}{M(r)}\ e^{-i\theta(r)}) +$$

$$+ L_{i2}(\frac{a-1}{M(r)}\ e^{-i\theta(r)}) - L_{i2}(\frac{a+1}{M(r)}\ e^{i\theta(r)}) + L_{i2}(\frac{a-1}{M(r)}\ e^{i\theta(r)}\}$$

$$= \frac{1}{2a}\ \{\ln(r(a^2 + d^2(r)))\ \ln\ (\frac{a+1}{a-1}) - 2\ \sum_{n=1}^{\infty} \frac{1}{n^2}\ (\frac{a+1}{M(r)})^n \cos(n\theta(r)) +$$

$$+ 2\ \sum_{n=1}^{\infty} \frac{1}{n^2}\ (\frac{a-1}{M(r)})^n \cos(n\theta(r))$$

or

$$F(r,a) = \frac{1}{a}\left\{\frac{1}{2}\ln(r(a^2+d^2(r)))\ \ln\left(\frac{a+1}{a-1}\right) - L_{i2}\left(\frac{a+1}{M(r)},\theta(r)\right) +\right.$$

$$\left. + L_{i2}\left(\frac{a-1}{M(r)},\ \theta(r)\right)\right\}\ ,\ r \leqslant 1 \qquad \text{(A-12)}$$

where $L_{i2}(x,\theta)$, the real part of the dilogarithm of complex argument, is defined by

$$L_{i2}(x,\theta) = -\frac{1}{2}\int_o^x \frac{dt}{t}\ln(1-2t\ \cos\theta + t^2)$$

$$= \sum_{n=1}^{\infty}\frac{x^n}{n^2}\cos(n\theta) \qquad x < 1 \qquad \text{(A-13)}$$

Appendix B

Expansions for B(r), T(r) and $I_o(r,s,\mu_1)$

Here we will derive expansions for B(r), T(r), and $I_o(r,s,\mu_1)$ by appealing directly to the defining integrals, Eqns. (A-3a) through (A-3d).

First consider B(r) and T(r); for r < 1; making a change of variable y = cos θ in Eqns. (A-3a) and (A-3b), and expanding the logarithm:

$$B(r) = -\frac{1}{2}\sum_{n=1}^{\infty}\frac{r^n}{n}\int_o^{\frac{\pi}{2}}d\theta\ [\sin(\theta)]^{2n+1}$$

$$= -\frac{1}{2}\sum_{n=1}^{\infty}\frac{r^n}{n}\frac{(2n)!!}{(2n+1)!!} \qquad \text{(B-1a)}$$

$$T(r) = -\sum_{n=1}^{\infty}\frac{r^n}{n}\int_o^{\frac{\pi}{2}}d\theta\ [\sin(\theta)]^{2n-1}$$

$$= -r-\sum_{n=2}^{\infty}\frac{r^n}{n}\frac{(2n-2)!!}{(2n-1)!!} \qquad \text{(B-1b)}$$

Explicitly

$$B(r) = -\left\{\frac{r}{3}+\frac{2}{15}r^2+\frac{8}{105}r^3\right\}+O(r^4),\ r \ll 1 \qquad \text{(B-2a)}$$

$$T(r) = -\left\{r+\frac{r^2}{3}+\frac{8}{45}r^3\right\}+O(r^4),\ r \ll 1 \qquad \text{(B-2b)}$$

Now consider the combination

$$[T(r) - T(-\mu_1)] - 2r[B(r) - B(-\mu_1)]$$

for $r \to -\mu_1$;using Eqns. (A-3a) and (A-3b) we see that

$$\frac{d}{dr}\{[T(r) - T(-\mu_1)] - 2r[B(r) - B(-\mu_1)]\}\Big|_{r=-\mu_1} =$$

$$\{\frac{d}{dr}T(r) - 2r\frac{d}{dr}B(r)\}\Big|_{r=-\mu_1} =$$

$$-\int_o^1 dy = -1$$

so

$$[T(r) - T(-\mu_1)] - 2r[B(r) - B(-\mu_1)] = -r_1 + O(r_1^{\,2}),\ r_1 \ll 1 \qquad \text{(B-3)}$$

Next consider $I_o(r,s,\mu_1)$. By examination of Eqns. (A-3a), (A-3c) through (A-3e), and (B-1a) we readily observe that

$$I_o(r,s,\mu_1) = 2\frac{rs}{t}[B(s) - B(-\mu_1)] + O(r^2),\ r \ll 1 \qquad \text{(B-4a)}$$

$$I_o(r,s,\mu_1) = 2\frac{rs}{t}[B(r) - B(-\mu_1)] + O(s^2),\ s \ll 1 \qquad \text{(B-4b)}$$

Now consider $I_o(r,s,\mu_1)$ for $t \to 0$; evidently

$$I_o(r,s,\mu_1) = I_o(r,s,\mu_1)\Big|_{a=1} + \frac{t}{2rs}\left[\frac{\partial}{\partial a}I_o(r,s,\mu_1)\right]\Big|_{a=1} + O(t^2),\ t \ll 1$$

by virtue of Eqn. (A-3e). From Eqns. (A-3b) through (A-3d) we have

$$I_o(r,s,\mu_1)\Big|_{a=1} = T(r) + T(s) - T(-\mu_1)$$

Again making the change of variable $y = \cos\theta$ and expanding the logarithm, from Eqn. (A-3d) we obtain for $r < 1$

$$[\frac{\partial}{\partial a} F(r,a)]\Big|_{a=1} = 2\ r \int_o^{\frac{\pi}{2}} \frac{d\theta}{\sin\theta} + 2 \sum_{n=2}^{\infty} \frac{r^n}{n} \frac{(2n-4)!!}{(2n-3)!!}$$

$$= 2\ r \int_o^{\frac{\pi}{2}} \frac{d\theta}{\sin\theta} + \sum_{n=2}^{\infty} r^n\{\frac{1}{n} + \frac{1}{n-1}\} \frac{(2n-2)!!}{(2n-1)!!}$$

Comparing to Eqns. (B-1a) and (B-1b) we see that

$$[\frac{\partial}{\partial a} F(r,a)]\Big|_{a=1} = r\{2 \int_o^{\frac{\pi}{2}} \frac{d\theta}{\sin\theta} - 1\} - T(r) - 2r\ B(r)$$

In this form we can relax the condition $r < 1$; then noting that for $t = 0$

$$r + s + \mu_1 = 0$$

we have

$$[\frac{\partial}{\partial a} I_o(r,s,t)]\Big|_{a=1} = -[T(r) + T(s) - T(-\mu_1)] +$$

$$- 2[r\ B(r) + s\ B(s) + \mu_1 B(-\mu_1)]$$

and thus

$$I_o(r,s,\mu_1) = \left\{1 - \frac{1}{2rs}\right\}[T(r) + T(s) - T(-\mu_1)] + \tag{B-5}$$

$$- \frac{t}{rs}\left\{r[B(r) - B(-\mu_1)] + s[B(s) - B(-\mu_1)]\right\} + O(t^2), \; t \ll 1$$

Finally, let us consider $I_o(r,s,\mu_1)$ for r, s, t and $-\mu_1 \rightarrow 0$. For clarity we first evaluate $F(u, \sqrt{1 + 1/\zeta})$ where ζ/u is order 1; again we proceed from Eqn. (A-3d) with a change of variable $y = \cos\theta$ but now expanding both the logarithm and the denominator:

$$F(u, \sqrt{1 + 1/\zeta}) = -\zeta \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (-1)^m \frac{\zeta^m u^n}{n} \int_o^{\frac{\pi}{2}} d\theta \, [\sin(\theta)]^{2(m+n)+1}$$

$$= -\zeta \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (-1)^m \frac{\zeta^m u^n}{n} \frac{(2m+2n)!!}{(2m+2n+1)!!}; \; \zeta, u < 1 \tag{B-6}$$

If $r, s, t, -\mu_1 \propto 1/\rho$, $\rho \gg 1$, with $\zeta = rs/t$ we have

$$F(r,a) = -\left(\frac{rs}{t}\right)\left[\frac{2}{3} r + \frac{4}{15} r^2 - \frac{8}{15}\left(\frac{rs}{t}\right) r\right] + O(\rho^{-4})$$

so, using Eqns. (2-7) and (A-3c),

$$I_o(r,s,\mu_1) = \frac{2}{3} rs - \frac{4}{15}\left(\frac{rs}{t}\right)[(r+s)^2 - \mu_1^2] + O(\rho^{-4})$$

$$= \frac{2}{3} rs - \frac{4}{15} rst - \frac{8}{15} rs\,\mu_1 + O(\rho^{-4}) \tag{B-7}$$

Appendix C

Proof Of Finiteness For $|M_{\lambda++}(x,y,z)|^2$

In this appendix we will prove that $|M_{\lambda++}(x,y,z)|^2$ is free of any infrared or collinear ($x \to 0$ or $x \to 1$) singularities; in terms of the modified Moller-Mandelstam variables these singularities are associated with $r \to -\mu_1$ and $r \to 0$ respectively. The proof consists of demonstrating that (i) the $\hat{E}^{(i)}_{\lambda++}$ (r,s,t) are individually finite functions, (ii) that $\hat{E}^{(1)}_{\lambda++}$ $(-\mu_1,0,0) = 0$, and (iii) that $\hat{E}^{(1)}_{\lambda++}$ (o,s,t) is a symmetric function of s and t. Note that this last condition suffices since it follows that the leading singular terms in $|M_{\lambda++}(x,y,z)|^2$ reduce to

$$\left\{ \frac{y(1-y)}{x(1-x)} + \frac{z(1-z)}{x(1-x)} - 2\,\frac{(1-y)(1-z)}{x(1-x)} \right\} |E^{(1)}_{\lambda++}(1,y,z)|^2 =$$

$$\left\{ \frac{(1-y)}{x} + \frac{(1-z)}{x} \right\} |E^{(1)}_{\lambda++}\,(1,y,z)|^2 =$$

$$|E^{(1)}_{\lambda++}\,(1,y,z)|^2$$

where we have used Eqn. (2-5). Note also that the proof omits the exceptional case $\rho = 1$ for which the derivatives of B(r), T(r) and F(r,a) with respect to r are undefined when $r = -\mu_1$.

We being with the simplest case, $\hat{E}^{(2)}_{-++}(r,s,t)$ as given in Eqn. (A-2d), which we may alternately write in the form

$$\hat{E}^{(2)}_{-++}(r,s,t) = -2 - \left\{\frac{T(r)}{r} + \frac{T(s)}{s} + \frac{T(t)}{t}\right\} +$$

$$+ \frac{1}{2r} G(s,t,\mu_1) + \frac{1}{2s} G(r,t,\mu_1) + \frac{1}{2t} G(r,s,\mu_1) +$$

$$+ \frac{1}{rs} I_o(r,s,\mu_1) + \frac{1}{rt} I_o(r,t,\mu_1) +$$

$$+ \frac{1}{st} I_o(s,t,\mu_1) \tag{C-1}$$

where we have defined

$$G(r,s,\mu_1) \equiv 2\left\{I_o(r,s,\mu_1) - T(r) - T(s) + T(-\mu_1)\right\} \tag{C-2}$$

Now, by Eqn. (B-1b) we immediately see that the $T(r)/r$ type terms are finite for $r \to 0$. For the $I_o(r,s,\mu_1)/rs$ type terms we have, by Eqns. (2-7), (B-4b) and (B-5):

$$\frac{1}{rs} I_o(r,s,\mu_1) \underset{s\to 0}{\to} -\frac{2}{r_1} [B(r) - B(-\mu_1)], \tag{C-3a}$$

$$\frac{1}{rs} I_o(r,s,\mu_1) \underset{t\to 0}{\to} -\frac{1}{rr_1} [T(r) + T(-r_1) - T(-\mu_1)] \tag{C-3b}$$

while for the $G(r,s,\mu_1)/t$ type terms:

$$\frac{1}{2t} G(r,s,\mu_1) \underset{s\to 0}{\to} \frac{1}{r_1} [T(r) - T(-\mu_1)] \tag{C-3c}$$

$$\frac{1}{2t} G(r,s,\mu_1) \underset{t\to 0}{\to} \frac{1}{2rr_1} [T(r) + T(-r_1) - T(-\mu_1)] +$$

$$+\frac{1}{r_1}\,[B(r) - B(-\mu_1)] + \frac{1}{s_1}[B(s) - B(-\mu_1)] \qquad \text{(C-3d)}$$

where we have further used Eqns. (B-1b) and (B-5). Note that the limiting forms, Eqns. (C-3a) through (C-3d) are finite for $r \to 0$, and $r_1 \to 0$ and $s_1 \to 0$. Thus we see that $\hat{E}^{(2)}_{-++}(r,s,t)$ is a finite function of its arguments.

Next consider $\hat{E}^{(1)}_{-++}(r,s,t)$; using Eqn. (2-7) we rewrite it as

$$\hat{E}^{(1)}_{-++}(r,s,t) = \left\{\frac{T(t)}{t} - \frac{T(r)}{r}\right\} - \left\{\frac{1}{2t}\,G(r,s,\mu_1) - \frac{1}{2r}\,G(s,t,\mu_1)\right\} +$$

$$-\frac{1}{s}\,[I_o(r,s,\mu_1) - I_o(s,t,\mu_1)] \qquad \text{(C-4)}$$

so, by the arguments given above, $\hat{E}^{(1)}_{-++}(r,s,t)$ is clearly a finite function of r, s and t. Now, using Eqns. (B-4b) and (C-3c)

$$\hat{E}^{(1)}_{-++}(r,0,t) = \left\{\frac{T(t)}{t} - \frac{T(r)}{r}\right\} - \left\{\frac{1}{r_1}[T(r) - T(-\mu_1)] - \frac{1}{t_1}\,[T(t) - T(-\mu_1)\right\} +$$

$$-\left\{\frac{2r}{t}\,[B(r) - B(-\mu_1)] - \frac{2t}{r}\,[B(t) - B(-\mu_1)]\right\}$$

$$= \frac{T(t)}{t} - \frac{1}{r_1}\left\{[T(r) - T(-\mu_1)] - 2r[B(r) - B(-\mu_1)]\right\} +$$

$$-\frac{1}{r}\,[T(r) + T(t) - T(-\mu_1)] + \frac{2t}{r}\,[B(t) - B(-\mu_1)]$$

Then, by Eqns. (B-1b) and (B-3) we see that

$$\hat{E}^{(1)}_{-++}(r,0,t) \underset{t\to 0}{\to} -1 - \frac{1}{r_1}\left\{-r_1\right\} + O(t)$$

Thus $\hat{E}^{(1)}_{-++}(-\mu_1,0,0) = 0$. Finally, using Eqns. (B-1b), (B-4a) and (B-5):

$$\hat{E}^{(1)}_{-++}(0,s,t) = \{\frac{T(t)}{t} + 1\} + \{\frac{1}{t}[T(s) - T(-\mu_1)] - \frac{1}{2st}[T(s) +$$

$$+ T(t) - T(-\mu_1)] - \frac{1}{t}[B(s) - B(-\mu_1)] - \frac{1}{s}[B(t) - B(-\mu_1)]\} +$$

$$+ \frac{1}{s}[T(s) + T(t) - T(-\mu_1)]$$

$$= 1 + \{\frac{1}{s} + \frac{1}{t} - \frac{1}{2st}\}[T(s) + T(t) - T(-\mu_1)] +$$

$$- \frac{1}{t}[B(s) - B(-\mu_1)] - \frac{1}{s}[B(t) - B(-\mu_1)]$$

so $\hat{E}^{(1)}_{-++}(o,s,t)$ is indeed a symmetric function of s and t.

In examining $\hat{E}^{(2)}_{+++}(r,s,t)$ it is useful to first define

$$\varepsilon(r,s,t) \equiv \frac{4s}{r}[B(s) - B(-\mu_1)] + \frac{4t}{r}(B(t) - B(-\mu_1)] +$$

$$+ \frac{2st}{r^2} G(s,t,\mu_1) + \frac{2}{r} I_o(s,t,\mu_1) \qquad \text{(C-5a)}$$

and

$$\bar{E}^{(2)}_{+++}(r,s,t) \equiv \hat{E}^{(2)}_{+++}(r,s,t) - \varepsilon(r,s,t) \qquad \text{(C-5b)}$$

Adding and subtracting $\hat{E}^{(2)}_{-++}(r,s,t)$, and using Eqn. (2-7), one easily finds

$$\bar{E}^{(2)}_{+++}(r,s,t) = \frac{1}{s_1}\{2\ S[B(s) - B(-\mu_1)] - [T(s) - T(-\mu_1)]\} +$$

$$+ \frac{1}{t_1}\{2\ t[B(t) - B(-\mu_1)] - [T(t) - T(-\mu_1)]\} - \frac{T(r)}{r} +$$

$$+ \frac{1}{2r} G\ (s,t,\mu_1) + \frac{1}{2s} G(r,t,\mu_1) + \frac{1}{2t} G(r,s,\mu_1) +$$

$$+ (1-r_1)\{\frac{1}{rs} I_o(r,s,\mu_1) + \frac{1}{rt} I_o(r,t,\mu_1) + \frac{1}{st} I_o(s,t,\mu_1) +$$

$$+ \frac{1}{r} G(s,t,\mu_1)\} \qquad \text{(C-5c)}$$

By Eqn. (B-3), and the arguments given above, we clearly see that $\bar{E}^{(2)}_{+++}(r,s,t)$ is finite. Now

$$\varepsilon(r,s,0) = -\frac{4s}{s_1} [B(s) - B(-\mu_1)], \qquad \text{(C-6a)}$$

$$\varepsilon(r,s,t) \underset{r\to 0}{\to} \frac{4s}{r} [B(s) - B(-\mu_1)] + \frac{4t}{r} [B(t) - B(-\mu_1)] +$$

$$+ \{-\frac{2}{r} + \frac{2}{r} [1 - \frac{r}{2st}]\}\ [T(s) + T(t) - T(-\mu_1)]$$

$$+ \{-\frac{4}{r} - \frac{2}{st}\}\{S[B(s) - B(-\mu_1)] + t[B(t) - B(-\mu_1)]\}$$

or

$$\varepsilon(0,s,t) = \frac{1}{s\ s_1} [T(s) + T(-s_1) - T(-\mu_1)] + \frac{2}{s_1} [B(s) - B(-\mu_1)] +$$

$$+ \frac{2}{t_1} [B(t) - B(-\mu_1)] \qquad \text{(C-6b)}$$

Eqns. (C-6a) and (C-6b) have been previously noted to be finite; thus $\hat{E}^{(2)}_{+++}(r,s,t)$ is a finite function of r, s and t.

Finally, let us examine $\hat{E}^{(1)}_{+++}(r,s,t)$; here it is useful to define

$$D(s) \equiv \frac{1}{s_1} + \frac{1}{s_1^2}\,[T(s) - T(-\mu_1)] - \frac{2s}{s_1^2}\,[B(s) - B(-\mu_1)] \tag{C-7a}$$

and

$$\begin{aligned}\bar{E}^{(1)}_{+++}(r,s,t) &\equiv E^{(1)}_{+++}(r,s,t) + 2t\,D(s) + E^{(1)}_{-++}(r,s,t) - \varepsilon(r,s,t)\\ &= \frac{T(t)}{t} - \frac{T(s)}{s} - \frac{4t}{s_1}\,[B(s) - B(-\mu_1)] + \left\{\frac{1}{r} - \frac{(s-t)}{r}\right\}G(s,t,\mu_1) +\\ &+ \frac{3}{s_1}\left\{2\,s[B(s) - B(-\mu_1)] - [T(s) - T(-\mu_1)]\right\} +\\ &- \frac{1}{t_1}\left\{2\,t\,[B(t) - B(-\mu_1)] - [T(t) - T(-\mu_1)]\right\} +\\ &+ r_1\left\{\frac{1}{rs}\,I_o(r,s,\mu_1) + \frac{1}{rt}\,I_o(r,t,\mu_1) + \frac{1}{st}\,I_o(s,t,\mu_1)\right\} +\\ &+ \frac{4}{s}\,I_o(s,t,\mu_1)\end{aligned} \tag{C-7b}$$

The only new feature here is D(s), however by Eqn. (B-3) one readily sees that $D(-\mu_1)$ is finite so $\hat{E}^{(1)}_{+++}(r,s,t)$ is a finite function of r, s and t. Now, by Eqns. (B-1a), (B-1b), (B-4a), (B-4b) and (C-2):

$$\bar{E}^{(1)}_{+++}(-\mu_1,0,0) = \left\{-\frac{2}{\mu_1} + \frac{3}{\mu_1} - \frac{1}{\mu_1}\right\} T(-\mu_1) = 0$$

From Eqn. (C-6a) we have $\varepsilon(-\mu_1,o,o) = 0$, and we have already shown that

$E^{(1)}_{-++}(-\mu_1, o, o) = 0$, so with

$$D(0) = \frac{1}{\mu_1} - \frac{T(-\mu_1)}{\mu_1^{\,2}}$$

we have, by Eqn. (C-7b), $\hat{E}^{(1)}_{+++}(-\mu_1, o, o) = 0$ as promised. Also

$$\hat{E}^{(1)}_{+++}(0,s,t) = \varepsilon(0,s,t) - \hat{E}^{(1)}_{-++}(0,s,t) +$$

$$- 2t\{-\frac{1}{t} + \frac{1}{t^2}[T(s) - T(-\mu_1)] - \frac{2s}{t^2}[B(s) - B(-\mu_1)]\} +$$

$$+ \frac{T(t)}{t} - \frac{T(s)}{s} + 4[B(s) - B(-\mu_1)] - \{\frac{1}{st} - \frac{1}{t} + \frac{1}{s}\}[T(s) +$$

$$+ T(t) - T(-\mu_1)] - 2\{\frac{1}{st} - \frac{1}{t} + \frac{1}{s}\}\{S[B(s) - B(-\mu_1)] +$$

$$+ t[B(t) - B(-\mu_1)]\} - \frac{3}{t}\{2s[B(s) - B(-\mu_1)] - [T(s) - T(-\mu_1)\} +$$

$$+ \frac{1}{s}\{2t[B(t) - B(-\mu_1)] - [T(t) - T(-\mu_1)]\} + \frac{2\mu_1}{t}[B(s) - B(-\mu_1)] +$$

$$+ \frac{2\mu_1}{s}[B(t) - B(-\mu_1)] + \{\frac{\mu_1}{st} + \frac{4}{s}\}[T(s) + T(t) - T(-\mu_1)]$$

$$= \varepsilon(0,s,t) - E^{(1)}_{-++}(0,s,t) + 2 + \{\frac{1}{s} + \frac{1}{t} - \frac{1}{st}\}[T(s) + T(t) +$$

$$- T(-\mu_1)] - \frac{2}{t}(1+s)[B(s) - B(-\mu_1)] - \frac{2}{s}(1+t)[B(t) - B(-\mu_1)] \quad \text{(C-8)}$$

We have already observed that $\hat{E}^{(1)}_{-++}(o,s,t)$ is a symmetric function. If we rewrite Eqn. (C-6b) in the form

From Eqn. (C-6a) we have $\varepsilon(-\mu_1,o,o) = 0$, and we have already shown that $\hat{E}^{(1)}_{-++}(-\mu_1,o,o) = 0$, so with

$$D(0) = \frac{1}{\mu_1} - \frac{T(-\mu_1)}{\mu_1^{\,2}}$$

we have, by Eqn. (C-7b), $\hat{E}^{(1)}_{+++}(-\mu_1,o,o) = 0$ as promised. Also

$$\hat{E}^{(1)}_{+++}(0,s,t) = \varepsilon(0,s,t) - \hat{E}^{(1)}_{-++}(0,s,t) +$$

$$- 2t\{-\frac{1}{t} + \frac{1}{t^2}[T(s) - T(-\mu_1)] - \frac{2s}{t^2}[B(s) - B(-\mu_1)]\} +$$

$$+ \frac{T(t)}{t} - \frac{T(s)}{s} + 4[B(s) - B(-\mu_1)] - \{\frac{1}{st} - \frac{1}{t} + \frac{1}{s}\}[T(s) +$$

$$+ T(t) - T(-\mu_1)] - 2\{\frac{1}{st} - \frac{1}{t} + \frac{1}{s}\}\{S[B(s) - B(-\mu_1)] +$$

$$+ t[B(t) - B(-\mu_1)]\} - \frac{3}{t}\{2s[B(s) - B(-\mu_1)] - [T(s) - T(-\mu_1)\} +$$

$$+ \frac{1}{s}\{2t[B(t) - B(-\mu_1)] - [T(t) - T(-\mu_1)]\} + \frac{2\mu_1}{t}[B(s) - B(-\mu_1)] +$$

$$+ \frac{2\mu_1}{s}[B(t) - B(-\mu_1)] + \{\frac{\mu_1}{st} + \frac{4}{s}\}[T(s) + T(t) - T(-\mu_1)]$$

$$= \varepsilon(0,s,t) - E^{(1)}_{-++}(0,s,t) + 2 + \{\frac{1}{s} + \frac{1}{t} - \frac{1}{st}\}[T(s) + T(t) +$$

$$- T(-\mu_1)] - \frac{2}{t}(1+s)[B(s) - B(-\mu_1)] - \frac{2}{s}(1+t)[B(t) - B(-\mu_1)] \quad \text{(C-8)}$$

We have already observed that $\hat{E}^{(1)}_{-++}(o,s,t)$ is a symmetric function. If we rewrite Eqn. (C-6b) in the form

$$\varepsilon(0,s,t) = -\frac{1}{st}[T(s) + T(t) - T(-\mu_1)] - \frac{2}{t}[B(s) - B(-\mu_1)] + $$

$$- \frac{2}{s}(B(t) - B(-\mu_1)]$$

we see that $\varepsilon(o,st)$ is symmetric; thus by Eqn. (C-8) we indeed find that $\hat{E}^{(1)}_{+++}(o,s,t)$ is a symmetric function of s and t.

Appendix D

Asymptotic Expression For F(p)

In this appendix we will derive an asymptotic expression for F(p) as $\rho \to \infty$.

We first require asymptotic forms for $\hat{E}^{(i)}_{\lambda++}(r,s,t)$ as r, s, t and $-\mu_1 \to 0$; these can be obtained using Eqns. (A-2a) through (A-2d) together with Eqns. (A-3c), (B-1a), (B-1b) and (B-6) with $\zeta = rs/t$. As this is more tedious than instructive we only explicitly exhibit the procedure for the simplest case, $\hat{E}^{(1)}_{-++}(r,s,t)$, where we use Eqns. (B-2b) and (B-7):

$$\hat{E}^{(1)}_{-++}(r,s,t) \simeq -\left\{\frac{1}{t} - \frac{1}{r}\right\}\left[\frac{r^2}{3} + \frac{s^2}{3} + \frac{t^2}{3} - \frac{\mu_1^2}{3} + \frac{8}{45}r^3 + \frac{8}{45}t^3 - \frac{8}{45}(-\mu_1^3)\right] +$$

$$+ \frac{r_1}{st}\left[\frac{2}{3}rs - \frac{4}{15}rst - \frac{8}{15}rs\,\mu_1\right] - \frac{t_1}{rs}\left[\frac{2}{3}st - \frac{4}{15}rst +\right.$$

$$\left. - \frac{8}{15}st\,\mu_1\right]$$

$$= \left[\frac{1}{t} - \frac{1}{r}\right]\left\{\frac{2}{3}[rs + rt + st] + \frac{8}{15}[r^2(s+t) + s^2(r+t) + t^2(r+s)] +\right.$$

$$\left.+ \frac{16}{15}rst\right\} - \left\{\frac{1}{s} + \frac{1}{t}\right\}\left[\frac{2}{3}rs + \frac{4}{15}rst + \frac{8}{15}rs(r+s)\right] +$$

$$+ \left\{\frac{1}{s} + \frac{1}{r}\right\}\left[\frac{2}{3}st + \frac{4}{15}rst + \frac{8}{15}st(s+t)\right]$$

$$= \frac{12}{15}\, s(r-t) + \frac{8}{15}\, [2s(t-r)]$$

$$= \frac{4}{15}\, s(t-r) \qquad \text{(D-1a)}$$

For the remaining helicity amplitudes one obtains

$$\hat{E}^{(1)}_{+++}(r,s,t) = -\,\hat{E}^{(2)}_{+++}(r,s,t) \simeq \frac{22}{45}\, r\, r_1 \qquad \text{(D-1b)}$$

and

$$\hat{E}^{(2)}_{-++}(r,s,t) = -\,\frac{4}{15}\,(rs + rt + st) \qquad \text{(D-1c)}$$

Now, from Eqns. (2-4) through (2-6) and the above, we have for a single-quark:

$$\left|M_{+++}(x,y,z)\right|^2 \simeq 8\left(\frac{22}{45}\right)^2 \frac{x^2(1-x)^2}{\rho^4}\left\{\frac{1}{x(1-x)}[y(1-y) + z(1-z) +\right.$$

$$\left. - 2(1-y)(1-z)] + 1\right\}$$

$$= 16\left(\frac{22}{45}\right)^2 \frac{x^2(1-x)^2}{\rho^4} \qquad \text{(D-2a)}$$

and

$$\left|M_{-++}(x,y,z)\right|^2 \simeq 8\left(\frac{4}{15}\right)^2 \frac{1}{\rho^4}\left\{\frac{1}{x(1-x)}[y(1-y)^3(x-z)^2 + z(1-z)^3(x-y)^2 +\right.$$

$$- 2(1-y)^2(1-z)^2(x-z)(x-y)] + [(1-x)(1-y) +$$

$$+ (1-x)(1-z) + (1-y)(1-z)]^2\}$$

This last expression can be simplified, again using Eqn. (2-5):

$$y(1-y)^3(x-z)^2 + z(1-z)^3(x-y) - 2(1-y)^2(1-z)^2(x-z)(x-y) =$$

$$2x(1-x)(1-y)^2(1-z)^2 + (1-y)^2(1-z)^2[(x-z)(1-y) + (x-y)(1-z) +$$

$$-2(1-y)(1-z)] - (1-x)^2[(x-z)(1-y)^3 + (x-y)(1-z)^3 + 2(1-y)^2(1-z)^2] =$$

$$x(1-x)(1-y)^2(1-z)^2 - (1-x)^2\{x^2(1-y)(1-z) - x(1-x)[(1-y)^2 +$$

$$+ (1-z)^2 - (1-y)(1-z)]\} =$$

$$x(1-x)\{[(1-x)^2(1-y)^2 + (1-x)^2(1-z)^2 + (1-y)^2(1-z)^2 - (1-x)(1-y)(1-z)] =$$

$$x(1-x)\{[(1-x)(1-y) + (1-x)(1-z) + (1-y)(1-z)]^2 - 3(1-x)(1-y)(1-z)\}$$

also

$$(1-x)(1-y) + (1-x)(1-z) + (1-y)(1-z) = x(1-x) + (1-y)(1-z)$$

and thus

$$|M_{-++}(x,y,z)|^2 \simeq 8(\frac{4}{15})^2 \frac{1}{\rho^4} \{2[x(1-x) + (1-y)(1-z)]^2 +$$

$$- 3(1-x)(1-y)(1-z)\} \qquad \text{(D-2b)}$$

Noting that

$$x^2(1-x)^2 + y^2(1-y)^2 + z^2(1-z)^2 =$$

$$2[(1-x)^2(1-y)^2 + (1-x)^2(1-z)^2 + (1-y)^2(1-z)^2 + (1-x)(1-y)(1-z)] =$$

$$2\{[x(1-x) + (1-y)(1-z)]^2 - (1-x)(1-y)(1-z)\}$$

we have, using Eqns. (2-3), (D-2a) and (D-2b):

$$\frac{d^2F}{dxdy} \simeq (\frac{16}{3})(\frac{4}{45})^2 \frac{1}{\rho^4} \{[(11)^2 + 2(3)^2][x(1-x) + (1-y)(1-z)]^2 +$$

$$- [(11)^2 + (3)^3] (1-x)(1-y)(1-z)\}$$

$$= \frac{256}{6075} \frac{1}{\rho^4} \{139[x(1-x) + (1-y)(x+y-1)]^2 +$$

$$- 148(1-x)(1-y)(x+y-1)\} \qquad \text{(D-3)}$$

Finally, $F(\rho)$ is given by the integral

$$F(\rho) = \int_o^1 dx \int_{1-x}^1 dy(\frac{d^2F}{dxdy}) \qquad \text{(D-4)}$$

Using Eqn. (D-3) in Eqn. (D-4) and making a change of variabley = 1 - xν

we have

$$F(\rho) \simeq \frac{256}{6075}\frac{1}{\rho^4}\int_o^1 dx \int_o^1 x\, dv \left\{139\ [x(1-x) + x^2 v(1-v)]^2 + \right.$$

$$\left. - 148\ x^2(1-x)v(1-v)\right\}$$

$$= \frac{256}{6075}\frac{1}{\rho^4}\left\{139\ [\frac{2(3!)}{(6!)} + 2\frac{(4!)}{(3!)(6!)} + \frac{4}{6!}] - \frac{148}{(5!)}\right\}$$

$$= \frac{256}{6075}\frac{1}{\rho^4}\left\{\frac{17}{5}\right\}$$

or

$$F(\rho) = \frac{4352}{30375}\frac{1}{\rho^4} \qquad \text{(D-5)}$$

neglecting terms of order ρ^{-5}

APPENDIX E

EVALUATION OF Im(K(q^2)

We proceed from the unitarity equation for the transition matrix elements T_{fi}:

$$\mathrm{Im}\ T_{fi} = -\frac{1}{2}\sum_n T_{fn}^{+}\ T_{ni}$$

which gives

$$\mathrm{Im}\ \langle\bar{\ell}\ell|T|P\rangle = -\frac{1}{2}\sum_n \langle n|T|\bar{\ell}\ell\rangle^{\dagger}\langle n|T|P\rangle$$

To order α the relevant part of $\langle n|T|\bar{\ell}\ell\rangle$ is that which couples the $\bar{\ell}\ell$ pair to the intermediate state via two photons; thus, separating out the two-photon intermediate state explicitly and expressing in terms of invariant amplitudes we have, in the notation of Eqns. (3-9) and (3-11)

$$\bar{u}_\ell\ (P_-)[i\ \gamma_5(\frac{\alpha}{4\pi})\ F_p\ \frac{m_\ell}{m_p}\ \mathrm{Im}\ K(q^2)]\ v_\ell(p_+) =$$

$$-\frac{1}{2}\sum_{\text{pol.}}\int\{\bar{v}_\ell(p_+)[(-iee_2)\ \frac{i}{p_- - k_2 - m_\ell}\ (iee_1)]u_\ell(p_-)\}^{\dagger}$$

$$\frac{F_p}{m_p}\ f(0,0,q^2)\ \varepsilon_{\mu\nu\rho\sigma}e_1^{\ \mu}e_2^{\ \nu}k_1^{\ \rho}k_2^{\ \sigma}\ d\ \rho_{\gamma\gamma}\ +$$

$$-\frac{1}{2}\sum_x\int\{\bar{v}_\ell(p_+)[(-ie\gamma_\beta)\ \frac{i}{p_1 - k_1 - m_\ell}\ (-ie\gamma_\alpha)]u_\ell(p_-)\ T_x^{\ \alpha\beta}(k_1,k_2)\}^{\dagger}$$

$m_{xp}\ d\ \rho_x$

Here $\int d\ \rho_x$ denotes the integral over the phase space of the intermediate state. Multiplying through by $\bar{v}_\ell(p_+)[i\gamma_5]u_\ell(p_-)$ and summing over spins a simple trace calculation yields

$$\mathrm{Im}K(q^2) = I_m\ k_{\gamma\gamma}(q^2)\theta(q^2) + \sum_x I_m\ K_x(q^2)\theta(q^2 - m_x^2), \tag{E-1a}$$

$$\mathrm{Im}\ K_{\gamma\gamma}(q^2)\theta(q^2) = \frac{(4\pi)^2}{q^2} f(0,0,q^2) \sum_{\mathrm{pol.}} \int d\rho_{\gamma\gamma} \frac{\varepsilon_{\alpha\beta\lambda\omega} e_1^{*\alpha} e_2^{*\beta} k_1^{\lambda} k_2^{\omega}}{[-\ 2\ k_1 \cdot p_-]}$$

$$\varepsilon_{\mu\nu\rho\sigma} e_1^{\mu} e_2^{\nu} k_1^{\rho} k_2^{\sigma}, \tag{E-1b}$$

$$\mathrm{Im}\ K_x(q^2)\theta(q^2 - m_x^2) = \frac{(4\pi)^2}{q^2} \frac{m_p}{F_p} \int d\ \rho_x \frac{\varepsilon_{\alpha\beta\lambda\omega} k_1^{\lambda} k_2^{\omega}}{[k_1^2 - 2k_1 \cdot p_-]} T_x^{*\alpha\beta}\ M_{xp} \tag{E-1c}$$

where we have factored out the θ functions which set the intermediate state thresholds.

Let us now consider $\mathrm{Im}K_{\gamma\gamma}(q^2)$; after summing over polarizations

$$\mathrm{Im}\ K_{\gamma\gamma}(q^2)\theta(q^2) = -\ 4\pi^2 q^2 f(o,o,q^2) \int \frac{d\rho_{\gamma\gamma}}{k_1 \cdot p_-}$$

Then, working in the $\bar{\ell}\ell$ center of momentum frame

$$k_1 \cdot p_- = \frac{q^2}{4}(1 - \beta\ \cos\theta),$$

$$\beta = [1 - 4\frac{m_\ell^2}{E^2}]^{1/2} \tag{E-2a}$$

$$\int d\,\rho_{\gamma\gamma} \equiv \int \frac{d^3k_1}{2\omega_1(2\pi)^3}\,\theta(\omega_1)\int \frac{d^3k_2}{2\omega_2(2\pi)^3}\,\theta(\omega_2)(2\pi)^4\delta^4(p_+ + p_1 - k_1 - k_2)$$

$$= \frac{\theta(q^2)}{16\pi}\int_{-1}^{1} d\,\cos\theta$$

so

$$\mathrm{Im}\,K_{\gamma\gamma}(q^2) = -\,\pi\, f(o,o,q^2)\int_{-1}^{1}\frac{d\,\cos\theta}{1-\beta\,\cos\theta}$$

$$= \frac{\pi}{\beta}\,\ell n\,\left(\frac{1-\beta}{1+\beta}\right)\, f(0,0,q^2) \qquad \text{(E-2b)}$$

Next let us consider the remaining contributions to $\mathrm{Im}\,K(q^2)$ in the context of a simple vector-meson-dominance model for f; the coupling of v to the virtual photon is given by

$$\left[-\frac{ig^{\mu\nu}}{k^2}\right]\left[-\frac{iem_v^{\ 2}}{f_v}\,e_\nu\right]\Big|_{k^2=m_n^{\ 2}} = -\frac{e}{f_v}\,e^\mu \qquad \text{(E-3a)}$$

and we define

$$\langle\gamma(k_1,e_1),V(k_2,e_2)|T|P(q)\rangle = e\, f_{\gamma v}\,\varepsilon_{\mu\nu\rho\sigma}e_1^{\ \mu}e_2^{\ \nu}k_1^{\ \rho}k_2^{\ \sigma} \qquad \text{(E-3b)}$$

$$\langle V(k_1,e_1),\,V(k_2,e_2)|T|P(q)\rangle = e^2 f_{vv}\,\varepsilon_{\mu\nu\rho\sigma}e_1^{\ \mu}e_2^{\ \nu}k_1^{\ \rho}k_2^{\ \sigma} \qquad \text{(E-3c)}$$

such that

$$e^2\,\frac{f_{\gamma v}}{f_v} = e^4\,\frac{f_{vv}}{f_v^{\ 2}} = \frac{F_p}{m_p}\, f(o,o,q^2) = \frac{F_p}{m_p} \qquad \text{(E-3d)}$$

Summing over polarizations we have, for the γv intermediate state

$$\mathrm{Im}\, K_{\gamma v}(q^2)\theta(q^2 - m_v^2) = \frac{4\pi^2}{q^2}\int d\,\rho_{\gamma v}[q^2 - m_v^2]^2\left[\frac{1}{k_1 \cdot p_-} + \frac{2}{2k_2 \cdot p_- - m_v^2}\right]$$

Again we work in the $\bar{\ell}\ell$ center of momentum frame so

$$k_1 \cdot p_- = \frac{(q^2 - m_v^2)}{4}(1-\beta\cos\theta)$$

$$k_2 \cdot p_- - \frac{m_v^2}{2} = \frac{(q^2 - m_v 2)}{4}(1 + \beta\cos\theta)$$

$$\int d\,\rho_{\gamma v} = \frac{q^2 - m_v^2}{16\pi\, q^2}\,\theta(q^2 - m_v^2)\int_{-1}^{1} d\cos\theta$$

and then

$$\mathrm{Im}\, K_{\gamma v}(q^2) = \pi\left[1 - \frac{m_v^2}{q^2}\right]^2 \int_{-1}^{1} d\cos\theta\left[\frac{1}{1-\beta\cos\theta} + \frac{1}{1+\beta\cos\theta}\right]$$

$$= \frac{2\pi}{\beta}\left[1 - \frac{m_v^2}{q^2}\right]^2 \ln\left(\frac{1+\beta}{1-\beta}\right) \qquad \text{(E-4)}$$

For the vv intermediate state, after summing over polarizations

$$\mathrm{Im}\, K_{vv}(q^2)\theta(q^2 - 4m_v^2) = -4\pi^2\int d\rho_{vv}\,\frac{2[q^2 - 4\,m_v^2]}{2k_1 \cdot p_- - m_v^2}$$

and, in the center of momentum frame

$$k_1 \cdot p - \frac{m_v^2}{2} = \frac{(q^2 - 2\,m_v^2)}{4}\left(1 - \frac{\beta}{\lambda}\cos\theta\right),$$

$$\lambda = \left[1 - 2\,\frac{m_v^2}{q^2}\right]\left[1 - 4\,\frac{m_v^2}{q^2}\right]^{-1/2} \qquad \text{(E-5a)}$$

$$\int d\,\rho_{vv} = \frac{1}{16\pi}\,[1 - 4\,\frac{m_\nu^{\,2}}{q^2}]^{1/2}\,\theta\,(q^2 - 4m_v^{\,2})\,\int_{-1}^{1} d\,\cos\theta$$

so

$$\mathrm{Im}\,K_{vv}(q^2) = -\,\pi[1 - 4\,\frac{m_v^{\,2}}{q^2}]\,\lambda^{-1}\,\int_{-1}^{1} \frac{d\,\cos\theta}{1 - \beta\lambda^{-1}\cos\theta}$$

$$= \frac{\pi}{\beta}\,[1 - 4\,\frac{m_v^{\,2}}{q^2}]\,\ell n\,(\frac{1-\beta/\lambda}{1+\beta/\lambda}) \qquad \text{(E-5b)}$$

Finally, for the simple vector meson dominance model we have the complete imaginary part given by the sum of Eqns. (E-2b), (E-4) and (E-5b) with $f(0,0,q^2) = 1$:

$$\mathrm{Im}\,K(q^2) = \frac{\pi}{\beta}\,\ell n\,(\frac{1-\beta}{1+\beta})\theta(q^2) + \frac{2\pi}{\beta}\,[1 - \frac{m_v^{\,2}}{q^2}]^2\,\ell n\,(\frac{1+\beta}{1-\beta})\theta(q^2 - m_v^{\,2}) +$$

$$+ \frac{\pi}{\beta}\,[1 - 4\,\frac{m_v^{\,2}}{q^2}]\,\ell n\,(\frac{1-\beta/\lambda}{1+\beta/\lambda}) \qquad \text{(E-6)}$$

We note that

$$\lambda = 1 + O(q^{-2}),\ q^2 \gg 4\,m_v^{\,2}$$

and then Eqn. (E-6) is seen to yield

$$\mathrm{Im}\,K(q^2) = O(q^{-2}),\ q^2 \gg 4\,m_v^{\,2}$$

Thus $\mathrm{Im}K(\infty) = 0$ as promised.

APPENDIX F

EVALUATION OF THE DISPERSION INTEGRALS FOR Re $K(q^2)$

We begin with

$$\mathrm{Re}\, K_{\gamma\gamma}(q^2) = \frac{q^2}{\pi} \int_o^\infty \frac{dt}{t(t-q^2)} \, \mathrm{Im}\, K_{\gamma\gamma}(t)$$

where Im $K_{\gamma\gamma}(q^2)$ is given in Eqns. (E-2a) and (E-2b). Setting $f(0,0,q^2) = 1$ and making a change of variable $t = 4\, m_\ell{}^2/s$, the integral naturally divides into two parts

$$\mathrm{Re}\, K_{\gamma\gamma}(q^2) = \mathrm{Re}\, K_{\gamma\gamma}^{(1)}(q^2) + \mathrm{Re}\, K_{\gamma\gamma}^{(2)}(q^2) \qquad \text{(F-1a)}$$

$$\mathrm{Re}\, K_{\gamma\gamma}^{(1)}(q^2) = -\,2 \int_1^\infty \frac{d\,s}{1-\beta^2-s} \, \frac{\tanh^{-1}\sqrt{1-s}}{\sqrt{s-1}} \qquad \text{(F-1b)}$$

$$\mathrm{Re}\, K_{\gamma\gamma}^{(2)}\,(q^2) = -\,2 \int_o^1 \frac{d\,s}{1-\beta^2-s} \, \frac{\tanh^{-1}\sqrt{1-s}}{\sqrt{1-s}} \qquad \text{(F-1c)}$$

A further change of variable, $s = 1 + \beta^2\tan^2\theta$, in Eqn. (F-1b) gives

$$\mathrm{Re}\, K_{\gamma\gamma}^{(1)}(q^2) = \frac{4}{\beta} \int_o^{2\pi} \tan^{-1}(\beta \tan\theta)d\theta$$

$$= \frac{2}{\beta}\, [L_{i2}(1+\beta,0) - L_{i2}(1-\beta,0)]$$

$$= \frac{2}{\beta}\,[L_{i2}(1+\beta) - L_{i2}(1-\beta) + i\pi\ \ell n(1+\beta)] \qquad \text{(F-2)}$$

The imaginary part in Eqn. (F-2) arises as

$$L_{i2}(x,0) = L_{i2}(x) + i\ \pi\ \ell n(x)\theta(x-1),\ x > 1$$

Now, let us define

$$\gamma = \frac{1-\beta}{1+\beta} \qquad \text{(F-3)}$$

and make a change of variable $s = 4x/(1+x)^2$ in Eqn. (F-1c) so

$$\text{Re}\ K^{(2)}_{\gamma\gamma}\ I(q^2) = 4(1+\gamma)^2 \int_0^1 \frac{\ell n\ (x)\ dx}{(1-x)^2(1+\gamma)^2 - (1-\gamma)^2(1+x)^2}$$

$$= (1+\gamma)\int_0^1 \frac{\ell n(x)}{1-x}\,[\frac{1}{\gamma-x} + \frac{1}{1-\gamma x}]dx$$

Then, integrating by parts

$$\text{Re}\ K^{(2)}_{\gamma\gamma}\ (q^2) = -\ (\frac{1+\gamma}{1-\gamma}) \int_0^1 \frac{dx}{x}\ \ell n\ (\frac{1-\gamma x}{1-x/\gamma})$$

$$= \frac{1}{\beta}\,[L_{i2}(\gamma) - L_{i2}(\gamma^{-1}) + i\ \pi\ \ell n(\gamma)] \qquad \text{(F-4)}$$

where we have used

$$-\int_0^1 \frac{dt}{t}\ \ell n(1-xt) = L_{i2}(x) - i\ \pi\ \ell n\ x,\ x > 1$$

Schaeffers relation allows us to express Re $K_{\gamma\gamma}^{(1)}(q^2)$ in the form

$$\mathrm{Re}\ K_{\gamma\gamma}^{(1)}(q^2) = \frac{2}{\beta}\left[L_{i2}(-\gamma) - L_{i2}(\gamma) + \frac{\pi^2}{4}\right] \tag{F-5}$$

Finally, noting that

$$L_{i2}(x) + L_{i2}(1/x) = \frac{\pi^2}{3} - \frac{1}{2}\ \ell n^2(x) - i\ \pi\ \ell n(x),\ x > 1$$

Eqns. (F-1a), (F-4) and (F-5) yield

$$\mathrm{Re}\ K_{\gamma\gamma}(q^2) = \frac{1}{\beta}\left[\frac{1}{2}\ \ell n^2(\gamma) - 2\Phi(\gamma) + \frac{\pi^2}{6}\right] \tag{F-6}$$

where $\Phi(x) = -\ L_{i2}(-x)$ is the Spence function.

Next, let us consider

$$\mathrm{Re}\ K_{\gamma v}(q^2) = \frac{q^2}{\pi}\int_{m_v^2}^{\infty} \frac{dt}{t(t-q^2)}\ \mathrm{Im}\ K_{\gamma v}(q^2)$$

where $I_m K_{\gamma v}(q^2)$ is given by Eqns. (E-2a) and (E-4). We are interested in the part of Re $K_{\gamma v}(q^2)$ which does not vanish as $m_\ell \to 0$ so, making a change of variable $t = m_v{}^2/x$ and defining

$$\varepsilon = \frac{q^2}{m_v^2} \tag{F-7}$$

we have

$$\mathrm{Re}\ K_{\gamma v}(q^2) = 2\varepsilon \int_0^1 \frac{(1-x)^2}{1-\varepsilon x}\left[2\ \ell n\left(\frac{m_v}{m_\ell}\right) - \ell n(x)\right]dx + 0\left(\frac{m_\ell^2}{m_v^2}\right) \tag{F-8}$$

For $\varepsilon < 1$ we can express $\mathrm{Re}\, K_{\gamma\nu}(q^2)$ as a series by expanding the denominator in Eqn. (F-8):

$$\mathrm{Re}\, K_{\gamma\nu}(q^2) = 2\varepsilon \sum_{n=0}^{\infty} \varepsilon^n [2C_n^{(1)} \ln(\frac{m_v}{m_\ell}) + C_n^{(2)}] + O(\frac{{m_\ell}^2}{{m_v}^2}) \tag{F-9}$$

where

$$C_n^{(1)} = \int_o^1 (1-x)^2 x^n dx$$

$$= \frac{2}{(n+1)(n+2)(n+3)} \tag{F-9b}$$

$$C_n^{(2)} = - \int_o^1 (1-x)^2 x^n \ln(x) dx$$

$$= \frac{2[3(n+2)^2-1]}{(n+1)^2(n+2)^2(n+3)^2} \tag{F-9c}$$

Finally, let us examine

$$\mathrm{Re}\, K_{vv}(q^2) = \frac{q^2}{\pi} \int_{4{m_v}^2}^{\infty} \frac{dt}{t(t-q)^2} \mathrm{Im}\, K_{vv}(q^2)$$

As with $\mathrm{Re}\, K_{\gamma v}(q^2)$ we are interested in the leading part as $m_\ell \to 0$; in this case we make a change of variable $t = 4\, {m_v}^2/(1-x^2)$ and define

$$\delta = \frac{(\varepsilon/4)}{1 - (\varepsilon/4)} \tag{F-10}$$

to obtain, using Eqns. (E-2a), (E-5a) and (E-5b) for $\mathrm{Im}\, K_{vv}(q^2)$

$$\mathrm{Re}\, K_{vv}(q^2) = 4\, \delta \int_o^1 \frac{x^3 dx}{1 + \delta x^2} \ln (\frac{1-x}{1+x}) + O(\frac{{m_\ell}^2}{4{m_v}^2}) \tag{F-11a}$$

For $\delta < 1$ we may expand the denominator in Eqn. (F-11) and express $\mathrm{Re}\ K_{vv}(q^2)$ as the series

$$\mathrm{Re}\ K_{vv}(q^2) = -\ 4\delta \sum_{n=0}^{\infty} (-\delta)^n C_n^{(3)} + O\left(\frac{m_\ell^{\ 2}}{4m_v^{\ 2}}\right)$$

with

$$C_n^{(3)} = \int_o^1 x^{2n+3} \ell n\left(\frac{1+x}{1-x}\right)\, dx$$

$$= \frac{1}{n+2} \sum_{r=1}^{n+2} \frac{1}{2n-5-2r} \qquad \text{(F-11b)}$$

APPENDIX G

DETERMINATION OF THE $\pi(j;\ell,p_1,p_2,p_3)$

Let us first note some useful identities:

$$\gamma^{\mu}\varepsilon_{\mu\nu\alpha\beta}p_1{}^{\nu}p_2{}^{\alpha}p_3{}^{\beta} = \frac{i}{2}\,[\not{p}_1\not{p}_2\not{p}_3 - \not{p}_3\not{p}_2\not{p}_1]\gamma_5 \tag{G-1a}$$

$$\gamma_{\mu}\not{a}_1\not{a}_2\cdots\not{a}_{2n}\not{a}_{2m+1}\gamma^{\mu} = -2\,\not{a}_{2n+1}\not{a}_{2n}\cdots\not{a}_2\not{a}_1 \tag{G-1b}$$

$$\mathrm{Tr}\,[\not{a}_1\not{a}_2\cdots\not{a}_{2n-1}\not{a}_{2n}] = \mathrm{Tr}\,[\not{a}_{2n}\not{a}_{2n-1}\cdots\not{a}_2\not{a}_1] \tag{G-1c}$$

Eqn. (G-1c) easily yields the relation

$$T^{\mu\nu\alpha\beta}(a,b;p_3,p_2) = T^{\alpha\nu\mu\beta}(b,a;p_2,p_3) \tag{G-1d}$$

for the tensor defined in Eqn. (4-19); further, using Eqns. (4-19b), (4-18c) and (G-1d) we see that

$$\pi(c;\ell,p_1,p_2,p_3) = \pi(b;\ell,p_1,p_3,p_2) \tag{G-2}$$

We begin with $\pi(b;\ell,p_1,p_2,p_3)$ as defined in Eqn. (4-18b); employing Eqns. (G-1a) and (G-1b) we have

$$\pi(b;\ell,p_1,p_2,p_3) =$$

$$- \frac{1}{4}\,Tr[(\gamma_\delta \not{p}_1 \not{K} - \not{K}\not{p}_1 \gamma_2)\gamma_5(\not{p}_1 - \not{p}_3)(\gamma_\nu \not{\ell}\not{p}_1 - \not{p}_1\not{\ell}\gamma_\nu)\gamma_5(\not{p}_3 - \not{\ell})\gamma^\nu \not{p}_3 \gamma^\delta \not{p}_2] =$$

$$- Tr[(\not{p}_1\not{K}\not{p}_2 - \not{p}_2\not{p}_1\not{K})(\not{p}_1 - \not{p}_3)[\not{p}_1\not{\ell}(\not{p}_3 - \not{\ell}) - (\not{p}_3 - \not{\ell})\not{p}_1\not{\ell}]\not{p}_3] =$$

$$-Tr[[r\not{p}_1 - (m_\pi{}^2 - s)\not{K}](\not{p}_1 - \not{p}_3)[2(p_1 . \ell)\not{\ell} - 2\ell^2 p_1 - \not{p}_3\not{p}_1\not{\ell}]\not{p}_3] =$$

$$- (m_\pi{}^2 - s)\not{K}[2(p_1 . \ell)(\not{p}_1 - \not{p}_3)\not{\ell} - 2t\ell^2 - \not{p}_1\not{p}_3\not{p}_1\not{\ell}]\not{p}_3] =$$

$$-Tr[rt[2(\ell . p_1)(\ell . p_3) - \ell^2(m_\pi{}^2 - t)] +$$

$$-(m_\pi{}^2 - s)[2(\ell . p_1)\not{p}_2(\not{p}_1 - \not{p}_3)\not{\ell}\,\not{p}_3 - rt\ell^2 +$$

$$-2(\ell . p_3)\not{p}_2\not{p}_1\not{p}_3\not{p}_1 + (m_\pi{}^2 - t)\not{p}_2\not{p}_1\not{p}_3\,\not{\ell}]] =$$

$$-4[rt[2(\ell . p_1)(\ell . p_3) - \ell^2(m_\pi{}^2 - t)] + rt(m_\pi{}^2 - s)\ell^2 +$$

$$-(m_\pi{}^2 - s)(\ell . p_1)[t(\ell . p_3) + r(p_1 - p_3) . \ell - (m_\pi{}^2 - t)(\ell . p_2)] +$$

$$+ (m_\pi{}^2 - s)(\ell . p_3)[(m_\pi{}^2 - s)(m_\pi{}^2 - t) - r\, m_\pi{}^2] +$$

$$- \frac{1}{2}(m_\pi{}^2 - s)(m_\pi{}^2 - t)[(m_\pi{}^2 - s)(\ell . p_3) + (m_\pi{}^2 - t)(\ell . p_2) - r(\ell . p_1)]]$$

or

$$\pi(b; \ell, p_1, p_2, p_3) \equiv$$

$$\pi(1;\ell,p_1,p_2,p_3) =$$

$$-4[rt[2(\ell\cdot p_1)(\ell\cdot p_3) + \ell^2(t-s)] + st(m_\pi{}^2-s)(\ell\cdot p_3) +$$

$$-(m_\pi{}^2-s)(\ell\cdot p_1)[(t-r)(\ell\cdot p_3) + r(\ell\cdot p_1) - (m_\pi{}^2-t)(\ell\cdot p_2)] +$$

$$-\frac{1}{2}(m_\pi{}^2 r + st)[(m_\pi{}^2-s)(\ell\cdot p_3) + (m_\pi{}^2-t)(\ell\cdot p_2) - r(\ell\cdot p_1)]] \qquad \text{(G-3)}$$

Now, $\pi(d;\ell,p_1,p_2,p_3)$ as defined in Eqn. (4-18d) can be expressed in the form

$$\pi(d;\ell,p_1,p_2,p_3) =$$

$$\varepsilon_{\mu\nu\alpha\beta}\ell^\alpha p_1{}^\beta\varepsilon^\lambda{}_{\delta\rho\sigma}p_1{}^\rho K^\sigma \mathrm{Tr}[(2p_2{}^\mu - \gamma^\mu\not{\ell})\gamma_\lambda[(\not{p}_1-\not{\ell})\gamma^\nu - 2p_3{}^\nu]\not{p}_3\,\gamma^\delta\not{p}_2] =$$

$$-4\,\varepsilon_{\mu\nu\alpha\beta}p_2{}^\mu p_3{}^\nu\ell^\mu p_1{}^\beta\varepsilon^\lambda{}_{\delta\rho\sigma}p_1{}^\rho K^\sigma \mathrm{Tr}[\gamma_\lambda\not{p}_3\gamma^\delta\not{p}_2] +$$

$$+\,2\varepsilon_{\mu\nu\alpha\beta}p_2{}^\nu\ell^\alpha p_1{}^\beta\varepsilon^\lambda{}_{\delta\rho\sigma}p_1{}^\rho K^\sigma \mathrm{Tr}[\gamma_\lambda(\not{p}_1-\not{\ell})\gamma^\lambda\not{p}_3\gamma^\delta\not{p}_2] +$$

$$+\,2\varepsilon_{\mu\nu\alpha\beta}p_3{}^\nu\ell^\alpha p_1{}^\beta\varepsilon^\lambda{}_{\delta\rho\sigma}p_1{}^\rho K^\sigma \mathrm{Tr}[\gamma^\mu\not{\ell}\gamma_\lambda\not{p}_3\gamma^\delta\not{p}_2] +$$

$$-\varepsilon_{\mu\nu\alpha\beta}\ell^\alpha p_1{}^\beta\varepsilon^\lambda{}_{\delta\rho\sigma}p_1{}^\rho K^\sigma \mathrm{Tr}[\gamma^\mu\not{\ell}\gamma_\lambda(\not{p}_1-\not{\ell})\gamma^\nu\not{p}_3\gamma^\delta\not{p}_2]$$

The term in Eqn. (G-4a) containing

$$\mathrm{Tr}[\gamma_\lambda\not{p}_3\gamma^\delta\not{p}_2] = p_{3\lambda}p_2{}^\delta + p_{2\lambda}p_3{}^\delta - \frac{r}{2}\,g_\lambda{}^\delta$$

vanishes by the antisymmetry of $\varepsilon^{\lambda}{}_{\delta\rho\sigma}$; defining

$$\pi(2;\ell,p_1,p,p_3) \equiv$$

$$2\ \varepsilon_{\mu\nu\alpha\beta}p_3{}^{\nu}\ell^{\alpha}p_1{}^{\beta}\ \varepsilon^{\lambda}{}_{\delta\rho\sigma}p_1{}^{\rho}K^{\sigma}\mathrm{Tr}[\gamma^{\mu}\not{\ell}\ \gamma_{\lambda}\not{p}_3\gamma^{\delta}\not{p}_2] \tag{G-4b}$$

we also see that by Eqn. (G-1c)

$$2\ \varepsilon_{\mu\nu\alpha\beta}p_2{}^{\mu}\ell^{\alpha}p_1{}^{\beta}\varepsilon^{\lambda}{}_{\delta\rho\sigma}p_1{}^{\rho}K^{\sigma}\mathrm{Tr}[\gamma_{\lambda}(\not{p}_1-\not{\ell})\gamma^{\lambda}\not{p}_3\gamma^{\delta}\not{p}_2] =$$

$$\pi(2;p_1-\ell,p_1,p_3,p_2)$$

Using Eqns. (G-1a) and (G-1b), we observe that

$$\varepsilon_{\mu\nu\alpha\beta}\ell^{\alpha}p_1{}^{\beta}\mathrm{Tr}[\gamma^{\mu}\not{\ell}\gamma_{\lambda}(\not{p}_1-\not{\ell})\gamma^{\nu}\not{p}_3\gamma^{\delta}\not{p}_2] =$$

$$-\frac{i}{2}\ \mathrm{Tr}[\gamma_5[\gamma_{\nu}\not{\ell}\not{p}_1-\ \not{p}_1\not{\ell}\gamma_{\nu})\not{\ell}\gamma_{\lambda}(\not{p}_1-\not{\ell})\gamma^{\nu}\not{p}_3\gamma^{\delta}\not{p}_2] =$$

$$-\ i\ \mathrm{Tr}[\gamma_5[\not{p}_1\not{\ell}(\not{p}_1-\not{\ell})\gamma_{\lambda}\not{\ell}\ -\ (\not{p}_1-\not{\ell})\gamma_{\lambda}\not{\ell}\not{p}_1\not{\ell}]\not{p}_3\gamma^{\delta}\not{p}_2] =$$

$$i\ \mathrm{Tr}[\gamma_5[\not{\ell}\not{p}_1(\not{p}_1-\not{\ell})\gamma_{\lambda}\not{\ell}\ -\ \ell^2(\not{p}_1-\not{\ell})\gamma_{\lambda}\not{p}_1]\not{p}_3\gamma^{\delta}\not{p}_2] =$$

$$i\ \mathrm{Tr}[\gamma_5[(p_1-\ell)^2\not{\ell}\gamma_{\lambda}\not{\ell}\ +\ \ell^2(p_1-\ \not{\ell})\gamma_{\lambda}(\not{\ell}-\not{p}_1)]\not{p}_3\gamma^{\delta}\not{p}_2] =$$

$$2i\ \mathrm{Tr}[\gamma_5[(p_1-\ell)^2\ell_{\lambda}\ell\ -\ \ell^2(p_1-\ \ell)_{\lambda}(p_1-\ \ell)]p_3\gamma^{\delta}p_2] =$$

$$8(p_1-\ell)^2\ell_{\lambda}\varepsilon_{\mu\alpha}{}^{\delta}{}_{\beta}\ell^{\mu}p_3{}^{\alpha}p_2{}^{\beta}\ +\ 8\ell^2(p_1-\ell)\ {}_{\lambda}\varepsilon_{\mu\alpha}{}^{\delta}{}_{\beta}(p_1-\ell)^{\mu}p_2{}^{\alpha}p_3{}^{\beta}$$

so, defining

$$\pi(3;\ell,p_1,p_2,p_3) \equiv -\,8\,\varepsilon_{\mu\alpha}{}^{\delta}{}_{\beta}\ell^{\mu}p_3{}^{\alpha}p_2{}^{\beta}\varepsilon^{\lambda}{}_{\delta\rho\sigma}\ell_{\lambda}p_1{}^{\rho}k^{\sigma} \qquad \text{(G-4c)}$$

we have

$$\pi(d;\ell,p_1,p_2,p_3) =$$

$$\pi(2;\ell,p_1,p_2,p_3) + (p_1-\ell)^2\pi(3;\ell,p_1,p_2,p_3) +$$

$$\pi(2;p_1-\ell,p_1,p_3,p_2) + \ell^2\,\pi(3;p_1-\ell,p_1,p_3,p_2) \qquad \text{(G-5)}$$

Finally, let us evaluate $\pi(2;\ell,p_1,p_2,p_3)$ and $\pi(3;\ell,p_1,p_2,p_3)$. Employing Eqns. (G-1a) and (G-1b) once again,

$$\pi(2;\ell,p_1,p_2,p_3) =$$

$$-i\,\varepsilon_{\mu\nu\alpha\beta}p_3{}^{\nu}\ell^{\alpha}p_1{}^{\beta}\mathrm{Tr}[\gamma^{\mu}\not{\ell}\gamma_5(\gamma_{\delta}\not{p}_1\not{k}-\not{k}\not{p}_1\gamma_{\delta})\not{p}_3\gamma^{\delta}\not{p}_2] =$$

$$2i\,\varepsilon_{\mu\nu\alpha\beta}p_3{}^{\nu}\ell^{\alpha}p_1{}^{\beta}\mathrm{Tr}[\gamma_5\gamma^{\mu}\not{\ell}(\not{p}_3\not{k}\not{p}_1-\not{k}\not{p}_1\not{p}_3)\not{p}_2] =$$

$$2i\,\varepsilon_{\mu\nu\alpha\beta}p_3{}^{\nu}\ell^{\alpha}p_1{}^{\beta}\mathrm{Tr}[\gamma_5\gamma^{\mu}\not{\ell}(r\,\not{p}_1-(m_{\pi}{}^2-t)\not{k}\,\not{p}_2] =$$

$$8\,\varepsilon_{\mu\nu\alpha\beta}p_3{}^{\nu}\ell^{\alpha}p_1{}^{\beta}\varepsilon^{\mu}{}_{\delta\rho\sigma}[r\ell^{\delta}p_1{}^{\rho}p_2{}^{\sigma}-(m_{\pi}{}^2-t)\ell^{\delta}p_3{}^{\rho}p_2{}^{\sigma}]$$

Using Eqn. (4-12a)

$$\varepsilon_{\mu\nu\alpha\beta}{p_3}^{\nu}\ell^{\mu}{p_1}^{\beta}{\varepsilon^{\mu}}_{\delta\rho\sigma}\ell^{\delta}{p_1}^{\rho}{p_2}^{\sigma} =$$

$$\varepsilon_{\mu\nu\alpha\beta}\ell^{\nu}{p_1}^{\alpha}{p_3}^{\beta}{\varepsilon^{\mu}}_{\delta\rho\sigma}\ell^{\delta}{p_1}^{\rho}{p_2}^{\sigma} =$$

$$- \det \begin{bmatrix} \ell^2 & \ell\cdot p_1 & \ell\cdot p_2 \\ \ell\cdot p_1 & {m_\pi}^2 & \frac{1}{2}({m_\pi}^2-s) \\ \ell\cdot p_3 & \frac{1}{2}({m_\pi}^2-t) & \frac{1}{2}\, r \end{bmatrix} =$$

$$- [\tfrac{1}{4}\,({m_\pi}^2 r - st)\ell^2 - {m_\pi}^2(\ell\cdot p_2)(\ell\cdot p_3) +$$

$$+ \frac{1}{2}\,(\ell\cdot p_1)[({m_\pi}^2-s)(\ell\cdot p_3) + ({m_\pi}^2-t)(\ell\cdot p_2) - r(\ell\cdot p_1)]],$$

$$\varepsilon_{\mu\nu\alpha\beta}{p_3}^{\nu}\ell^{\alpha}{p_1}^{\beta}{\varepsilon^{\mu}}_{\delta\rho\sigma}\ell^{\delta}{p_3}^{\rho}{p_2}^{\sigma} =$$

$$- \varepsilon_{\mu\nu\alpha\beta}\ell^{\nu}{p_1}^{\alpha}{p_3}^{\beta}{\varepsilon^{\mu}}_{\delta\rho\sigma}\ell^{\delta}{p_2}^{\rho}{p_3}^{\sigma} =$$

$$\det \begin{bmatrix} \ell^2 & \ell\cdot p_2 & \ell\cdot p_3 \\ p_1\cdot\ell & \frac{1}{2}({m_\pi}^2-s) & \frac{1}{2}({m_\pi}^2-t) \\ \ell\cdot p_3 & \frac{1}{2}\, r & 0 \end{bmatrix} =$$

$$- \frac{1}{4}\, r({m_\pi}^2 - t)\ell^2 + \frac{1}{2}({m_\pi}^2 - t)(\ell\cdot p_2)(\ell\cdot p_3) +$$

$$+ \frac{1}{2}\ (\ell . p_3)[r(\ell . p_1)\ -\ (m_\pi{}^2 -\ s)(\ell . p_3)]$$

so

$$(\tfrac{1}{8})\ \pi\ (2; \ell, p_1, p_2, p_3)\ =$$

$$-r\ [\tfrac{1}{4}(rt\ -\ m_\pi{}^2 s)\ \ell^2\ -\ m_\pi{}^2(\ell . p_2)(\ell . p_3)\ +$$

$$+ \frac{1}{2}\ (\ell . p_1)[(m_\pi{}^2 - s)(\ell . p_3)\ +\ (m_\pi{}^2 - t)(\ell . p_2)\ -\ r(\ell . p_1)]]\ +$$

$$- \frac{1}{2}\ (m_\pi{}^2 -\ t)[(m_\pi{}^2 -\ t)(\ell . p_2)(\ell . p_3)\ +$$

$$+ \ (\ell . p_3)[r(\ell . p_1)\ -\ (m_\pi{}^2 - s)(\ell . p_3]] \qquad \text{(G-6)}$$

Similarly

$$\varepsilon_{\mu\alpha}{}^{\delta}{}_{\beta\ell}\, p_3{}^{\mu} p_2{}^{\alpha}{}^{\beta}\ \varepsilon^{\lambda}{}_{\delta\rho\sigma}\ell_{\lambda} p_1{}^{\rho} K^{\sigma}\ =$$

$$\varepsilon^{\delta}{}_{\mu\alpha\beta}\,\ell^{\mu} p_2{}^{\alpha} p_3{}^{\beta}\ \varepsilon_{\delta\lambda\rho\sigma}\ell^{\lambda} p_1{}^{\rho} K^{\sigma}\ =$$

$$-\ \det \begin{bmatrix} \ell^2 & \ell . p_1 & \ell . K \\ \ell . p_2 & \frac{1}{2}\ (m_\pi{}^2 - s) & \frac{1}{2}\ r \\ \ell . p_3 & \frac{1}{2}\ (m_\pi{}^2 - t) & \frac{1}{2}\ r \end{bmatrix}\ =$$

$$- \frac{1}{4}\, r(t-s)\ell^2 - \frac{1}{2}\, r\ \ell.(p_2-p_3)(\ell.p_1) +$$

$$+ \frac{1}{2}\, (\ell.K)[(m_\pi{}^2 - t)(\ell.p) - (m_\pi{}^2 - s)(\ell.p_3)]$$

so, noting that the $\ell^2(p_1-\ell)^2$ terms arising from $\pi(3;\ell,p_1,p_2,p_3)$ cancel in Eqn. (G-5), we have

$$(\tfrac{1}{4})(3;\ell,p_1,p_2,p_3) = - r\ \ell.(p_2-p_3)(\ell.p_1) +$$

$$- (\ell.K)[(m_\pi{}^2 - s)(\ell.p_3) - (m_\pi{}^2 - t)(\ell.p_2)] \qquad \text{(G-7)}$$

APPENDIX H

EVALUATION OF THE $N_i(j;r,s,t)$

In this appendix we determine the quantities $N_i(j;r,s,t)$ as defined by Eqn. (4-27).

First consider $\pi(1;\ell+\ell_1, p_1, p_2, p_3)$ where $\pi(1;\ell, p_1, p_2, p_3)$ is given in Eqn. (G-3) and ℓ_1 is defined in Eqn. (4-22b). Using Eqn. (4-26) we immediately see that

$$- N_1(1;r,s,t) = rt[(m_\pi^2 - t) + 4(t-s)] +$$

$$- (m_\pi^2 - s)[\tfrac{1}{2}(t-r)(m_\pi^2 - t) + rm_\pi^2 - \tfrac{1}{2}(m_\pi^2 - t)(m_\pi^2 - s)]$$

$$= rt[(m_\pi^2 - t) + 4(t-s)] - (m_\pi^2 - s)[rt]$$

$$= 3\, rt(t-s) \qquad \text{(H-1)}$$

Now

$$\ell_1 \cdot p_1 = m_\pi^2 y_1 + \tfrac{1}{2}(m_\pi^2 - t)y_3$$

$$\ell_1 \cdot p_2 = \tfrac{1}{2}(m_\pi^2 - s)y_1 + \tfrac{1}{2} r\, y_3$$

$$\ell_1 \cdot p_3 = \tfrac{1}{2}(m_\pi^2 - t)y_1$$

so

$$(t-r)(\ell_1 \cdot p_3) + r(\ell_1 \cdot p_1) - (m_\pi^{\,2} - t)(\ell_1 \cdot p_2) = rt\ y_1$$

$$(m_\pi^{\,2} - s)(\ell_1 \cdot p_3) + (m_\pi^{\,2} - t)(\ell_1 \cdot p_2) - r(\ell_1 \cdot p_1) = st\ y_1$$

and then

$$-\tfrac{1}{4} N_2(1;r,s,t) =$$

$$rt[(m_\pi^{\,2} - t)y_1[m_\pi^{\,2} y_1 + \tfrac{1}{2}(m_\pi^{\,2} - t)y_3] + (t-s)y_1[m_\pi^{\,2} y_1 + (m_\pi^{\,2} - t)y_3]] +$$

$$+ \tfrac{1}{2}\, st(m_\pi^{\,2} - s)(m_\pi^{\,2} - t) - rt(m_\pi^{\,2} - s)y_1[m_\pi^{\,2} y_1 + \tfrac{1}{2}(m_\pi^{\,2} - t)y_3] +$$

$$- \tfrac{1}{2}\, st(m_\pi^{\,2} r + st)y_1$$

or

$$- N_2(1;r,s,t) = 2\ rt(t-s)(m_\pi^{\,2} - t)y_1 y_3 \qquad \text{(H-2)}$$

Next consider $\pi(3;\ell + \ell_3, p_1, p_2, p_3)$ with $\pi(3;\ell, p_1, p_2, p_3)$ given in Eqn. (G-7) and ℓ_3 defined in Eqn. (4-23b). In this case we obtain

$$- N_1(3;r,s,t) = r(t-s) \qquad \text{(H-3)}$$

Noting that

$$\ell_3 \cdot p_1 = \tfrac{1}{2}(m_\pi^2 + t)y_1 + \tfrac{1}{2}(m_\pi^2 - s)y_2$$

$$\ell_3 \cdot p_2 = \tfrac{1}{2}t\ y_1$$

$$\ell_3 \cdot p_3 = \tfrac{1}{2}(m_\pi^2 - t)y_1 + \tfrac{1}{2}r\ y_2$$

so

$$\ell_3 \cdot K = \tfrac{1}{2}m_\pi^2 y_1 + \tfrac{1}{2}r\ y_2$$

$$\ell_3 \cdot (p_2 - p_3) = \tfrac{1}{2}(2t - m_\pi^2)y_1 - \tfrac{1}{2}r\ y_2$$

$$(m_\pi^2 - s)(\ell_3 \cdot p_3) - (m_\pi^2 - t)(\ell_3 \cdot p_2) = \tfrac{1}{2}r\ (m_\pi^2 - t)y_1 + \tfrac{1}{2}r(m_\pi^2 - s)y_2$$

we have

$$- N_2(3;r,s,t) =$$

$$r[(m_\pi^2 + t)y_1 + (m_\pi^2 - s)y_2][(2t - m_\pi^2)y_1 - r\ y_2] +$$

$$+ [r(m_\pi^2 - t)y_1 + r(m_\pi^2 - s)y_2][m_\pi^2 y_1 + r\ y_2] =$$

$$2\ rt\ y_1[(m_\pi^2 + t)y_1 + (m_\pi^2 - s)y_2 - m_\pi^2 y_1 - r\ y_2]$$

or

$$- N_2(3;r,s,t) = 2\ rt^2\ y_1(y_1 + y_2) \qquad \text{(H-4)}$$

Finally, consider $\pi(2;\ell + \ell_2, p_1, p_2, p_3)$ where $\pi(2;\ell, p_1, p_2, p_3)$ is given in Eqn. (G-6) and ℓ_2 is defined in Eqn. (4-24b). We see that

$$- \tfrac{1}{2} N_1(2;r,s,t) =$$

$$r[(rt - m_\pi^2 s) - \tfrac{1}{2} r\ m_\pi^2 + \tfrac{1}{2}[(m_\pi^2 - s)(m_\pi^2 - t) - r\ m_\pi^2]] +$$

$$+ \tfrac{1}{2}(m_\pi^2 - t)[\tfrac{1}{2}(m_\pi^2 - t)r + \tfrac{1}{2} r(m_\pi^2 - t)] =$$

$$r[(rt - m_\pi^2 s) + \tfrac{1}{2}(st - m_\pi^2 r) + \tfrac{1}{2}(m_\pi^2 - t)^2] =$$

$$r[(rt - m_\pi^2) + \tfrac{1}{2}(m_\pi^2 s - rt)]$$

or

$$- N_1(2;r,s,t) = r[rt - m_\pi^2 s] \qquad \text{(H-5)}$$

Then, observing that

$$\ell_2 \cdot p_1 = m_\pi^2\ y_1 + \tfrac{1}{2}(m_\pi^2 - s)\ y_2 + \tfrac{1}{2}(m_\pi^2 + t)\ y_3$$

$$\ell_2 \cdot p_2 = \tfrac{1}{2}(m_\pi^2 - s)\ y_1 + \tfrac{1}{2} t\ y_3$$

$$\ell_2 \cdot p_3 = \tfrac{1}{2}(m_\pi^2 - t)\ y_1 + \tfrac{1}{2} r\ y_2 + \tfrac{1}{2}(m_\pi^2 - t)\ y_3$$

so

$$(m_\pi^2 - s)(\ell_2 \cdot p_3) + (m_\pi^2 - t)(\ell_2 \cdot p_2) - r(\ell_2 \cdot p_1) = st(y_1 + y_3)$$

$$(m_\pi^2 - t)(\ell_2 \cdot p_2) - (m_\pi^2 - s)(\ell_2 \cdot p_3) + r(\ell_2 \cdot p_1) = r\, m_\pi^2\, y_1 + rt\, y_3$$

and thus

$$-\tfrac{1}{2} N_2(2;r,s,t) =$$

$$r[(rt - m_\pi^2 s)[(m_\pi^2 y_1 + ty_3)(y_1 + y_2 + y_3) - s\, y_1 y_2] +$$

$$- m_\pi^2 [(m_\pi^2 - s)y_1 + t\, y_3][(m_\pi^2 - t)(y_1 + y_3) + r\, y_2] +$$

$$st(y_1 + y_3)[2\, m_\pi^2 y_1 + (m_\pi^2 - s)y_2 + (m_\pi^2 + t)\, y_3]] +$$

$$+ (m_\pi^2 - t)[r\, m_\pi^2 y_1 + rt\, y_3][(m_\pi^2 - t)(y_1 + y_3) + r\, y_2] =$$

$$r[(rt - m_\pi^2 s)[(m_\pi^2 y_1 + t\, y_3)(y_1 + y_2 + y_3) - s\, y_1 y_2] +$$

$$+ [m_\pi^2 s\, y_1 - t(m_\pi^2 y_1 + t\, y_3)][(m_\pi^2 - t)(y_1 + y_2 + y_3) - s\, y_2] +$$

$$+ st(y_1 + y_3)[(m_\pi^2 y_1 + t\, y_3) + m_\pi^2 (y_1 + y_2 + y_3) - s\, y_2]] =$$

$$r[- s(m_\pi^2 + t)(m_\pi^2 y_1 + t\, y_3)(y_1 + y_2 + y_3) - rst\, y_1 y_2 +$$

$$+ st(y_1 + y_2 + y_3)(m_\pi^2 y_1 + t\, y_3) + m_\pi^2 s\, (y_1 + y_2 + y_3)(m_\pi^2 y_1 + t\, y_3) +$$

$$- s^2 t(y_1 + y_3)y_2]$$

or

$$N_2(2;r,s,t) = 2\ rst[(m_\pi^2 - t)y_1 y_2 + s\ y_2 y_3] \tag{H-6}$$

APPENDIX I

EVALUATION OF $I_1(x_1,x_2,x_3)$

In this appendix we shall analytically perform the multidimensional integrals appearing in Eqn. (4-33c). This task is rendered less formidable than it appears since the individual terms are finite and may therefore freely choose transformations which simplify the expressions.

First consider

$$\int d\,\rho(3)\;.\;[3\,\ell n(\frac{d(x_2)}{d(x_3)}) + \ell n(\frac{1-x_2}{1-x_3}) + \frac{x_3 y_3}{d(x_3)} - \frac{x_2 y_3}{d(x_2)}]$$

Making a transformation

$$y_1 \to u,\; y_2 \to (1-u)(1-v),\; y_3 \to (1-u)v$$

$$\int d\,\rho(3)\;.\;\to \int_o^1 (1-u)du \int_o^1 dv$$

so that

$$d(x) = (1-u)(1-xv)$$

where d(x) is defined in Eqn. (4-32a), we easily obtain

$$\int d\,\rho(3)\;.\;[3\,\ell n(\frac{d(x_2)}{d(x_3)}) + \ell n(\frac{1-x_2}{1-x_3}) + \frac{x_3 y_3}{d(x_3)} - \frac{x_2 y_3}{d(x_2)}] =$$

$$\tfrac{1}{2}\int_0^1 dv\ [3\ \ell n(\frac{1-x_2 v}{1-x_3 v}) + \ell n(\frac{1-x_2}{1-x_3}) + \frac{1}{1-x_3 v} - \frac{1}{1-x_2 v}] =$$

$$\ell n(\frac{1-x_2}{1-x_3}) - [\frac{1-x_2}{x_2}]\ \ell n(1-x_2) + [\frac{1-x_3}{x_3}]\ell n(1-x_3) \qquad \text{(I-1)}$$

Next consider

$$\int d\ \rho(4)\ .\ [D(x_2,x_3)]^{-1}$$

where $D(x_2.x_3)$ is defined in Eqn. (4-32b). With a transformation

$$y_1 \to uv,\ y_2 \to (1-u)(1-w),\ y_3 \to u(1-v),\ y_4 \to (1-u)w$$

$$\int d\ \rho(4)\ . \to \int_0^1 u(1-u)du \int_0^1 dv \int_0^1 dw$$

we have

$$\int d\ \rho(4)\ .\ [D(x_2,x_3)]^{-1} = \int_0^1 dv \int_0^1 dw[(1-x_1)uw + (1-x_2)v + (1-x_3)w]^{-1}$$

$$= \int_0^1 dv \int_0^1 dw\ [(1-x_1)uw + (1-x_2) + (1-x_3)w]^{-1} +$$

$$+ (x_2 \leftrightarrow x_3)$$

$$= \frac{1}{(1-x_1)} \int_0^1 \frac{dw}{w}\ \ell n\ (\frac{1-x_2 + x_2 w}{1-x_2+(1-x_3)w}) + (x_2 \leftrightarrow x_3)$$

$$= \frac{1}{(1-x_1)} \{L_{i2}(-\frac{1-x_3}{1-x_2}) - L_{i2}(-\frac{x_2}{1-x_2})\} + \{x_2 \leftrightarrow x_3\}$$

$$= \frac{1}{(1-x_1)} \{L_{i2}(-\frac{1-x_3}{1-x_2}) + L_{i2}(-\frac{1-x_2}{1-x_3}) +$$

$$- L_{i2}(-\frac{x_2}{1-x_2}) - L_{i2}(-\frac{x_3}{1-x_3}) \} \tag{I-2}$$

where

$$L_{i2}(x) = - \int_o^x \frac{dt}{t} \ln(1-t)$$

By use of the relations

$$L_{i2}(-x) + L_{i2}(-\frac{1}{x}) = -\frac{\pi^2}{6} - \frac{1}{2} \ln^2(x)$$

$$L_{i2}(x) + L_{i2}(-\frac{x}{1-x}) = -\frac{1}{2} \ln^2(1-x) \tag{I-3a}$$

we see that Eqn. (I-2) can also be expressed as

$$\int d\,\rho(4) \,.\, [D(x_2,x_3)]^{-1} = \frac{G(x_2,x_3)}{(1-x_1)} \tag{I-3b}$$

with

$$G(x_2,x_3) \equiv L_{i2}(x_2) + L_{i2}(x_3) + \ln(1-x_2)\ln(1-x_3) - \frac{\pi^2}{6} \tag{I-3c}$$

The remaining integrals in Eqn. (4-33c) may be carried out implicitly using

$$\int d\rho(4) \,.\, \frac{[x_3y_1y_2 + x_2y_3y_4]}{[D(x_2,x_3)]^2}$$

$$= [x_3(\frac{\partial}{\partial x_2}) + x_2(\frac{\partial}{\partial x_3})] \int d\rho(4) \ . \ [D(x_2,x_3)]^{-1} \tag{I-4}$$

Recalling that, by Eqn. (4-9d), x_1 is an implicit function of x_2, x_3, Eqn. (I-3b) through (I-4) readily yield

$$\int d\rho(4) \ . \ \frac{[x_3y_1y_2 + x_2y_3y_4]}{[D(x_2,x_3)]^2} =$$

$$- \frac{1}{(1-x_1)} \ [[\frac{x_3}{x_2} + \frac{x_2}{1-x_3}] \ \ell n(1-x_2) + [\frac{x_2}{x_3} + \frac{x_3}{1-x_2}] \ \ell n(1-x_3)] +$$

$$- [\frac{x_2 + x_3}{1-x_1}] \ \frac{G(x_2,x_3)}{(1-x_1)} \tag{I-5}$$

Finally, using Eqns. (4-33c), (I-1), (I-3b) and (I-5) we have

$$I_1(x_1,x_2,x_3) =$$

$$(x_2 - x_3)[\ell n(\frac{1-x_2}{1-x_3}) - [\frac{1-x_2}{x_2}]\ell n(1-x_2) + [\frac{1-x_3}{x_3}]\ell n(1-x_3)] +$$

$$- x_1^2 \ \frac{G(x_2,x_3)}{(1-x_1)} + (1-x_2)(1-x_3) \ [\frac{x_2 + x_3}{1-x_1}] \ \frac{G(x_2,x_3)}{(1-x_1)} +$$

$$+ \frac{(1-x_2)(1-x_3)}{(1-x_1)} \ [[\frac{x_3}{x_2} + \frac{x_2}{1-x_3}]\ell n(1-x_2) + [\frac{x_2}{x_3} + \frac{x_3}{1-x_2}]\ell n(1-x_3)] \tag{I-6}$$

Noting that

$$\frac{x_3}{x_2} + \frac{x_2}{1-x_3} = \frac{x_3 + (x_2 + x_3)(x_2 - x_3)}{x_2(1-x_3)}$$

$$= \frac{x_2 + (1-x_1)(x_2 - x_3)}{x_2(1-x_3)}$$

and similarly

$$\frac{x_2}{x_3} + \frac{x_3}{1-x_2} = \frac{x_3 - (1-x_1)(x_2-x_3)}{x_3(1-x_2)}$$

It is apparent that Eqn. (I-6) may be more simply expressed as

$$I_1(x_1,x_2,x_3) =$$

$$(x_2 - x_3)\ln\left(\frac{1-x_2}{1-x_3}\right) + \frac{1}{(1-x_1)}\left[(1-x_2)\ln(1-x_2) + (1-x_3)\ln(1-x_3)\right] +$$

$$- \left[x_1^{\,2} - (1-x_2)(1-x_3) - \frac{(1-x_2)(1-x_3)}{(1-x_1)}\right]\frac{G(x_2,x_3)}{(1-x_1)} \qquad \text{(I-7)}$$

APPENDIX J

EVALUATION OF $I_2(x_1,x_2,x_3)$

In this appendix we evaluate the integrals in Eqn. (4-33d) for $I_2(x_1,x_2,x_3)$. This problem must be approached cautiously since the individual integrals in $I_2(x_1,x_2,x_3)$, unlike those in Eqn. (4-33c) for $I_1(x_1,x_2,x_3)$ are divergent and so undefined. An examination of Eqns. (4-23a) through (4-24c) shows that the $y_3 \to 0$ singularity in the first integral of Eqn. (4-33d) and the y_1, $y_4 \to 0$ singularity in the second integral of Eqn. (4-33d) have a common origin in the vanishing of both photon propagators; guided by this, if we make a transformation

$$y_1 \to (1-u)v, \quad y_2 \to (1-u)(1-v), \quad y_3 \to u$$

$$\int d\rho(4) \; . \to \int_o^1 (1-u)du \int_o^1 dv$$

so

$$\int d\rho(4) \; . \; [\frac{y_1+y_2}{y_3}] \to \int_o^1 \frac{du}{u} (1-u)^2$$

we must use a transformation for

$$\int d\rho(4) \; . \; \frac{y_2 y_3}{[D(x_2,x_3)]^2}$$

such that we may write $I_2(x_1,x_2,x_3)$ as

$$I_2(x_1,x_2,x_3) = x_1 \int du\ H(u;x_1,x_2,x_3) \qquad \text{(J-1)}$$

with $H(0;x_1,x_2,x_3)$ finite. An efficient choice is

$$y_1 \to -u\frac{(1-x_3)(1+v)}{C},\ y_2 \to (1-u)w,\ y_3 \to (1-u)(1-w),\ y_4 \to u\frac{(1-x_2)(1-v)}{C}$$

$$\int d\rho(4)\ .\ \to -2(1-x_2)(1-x_3)\int_o^1 u(1-u)du \int_{-1}^1 \frac{dv}{c^2}\int_o^1 dw$$

where

$$c \equiv (x_3 - x_2) - x_1 v$$

in terms of which

$$D(x_2,x_3) = u\frac{(1-x_2)(1-x_3)}{C}\left[-u\frac{(1-v^2)}{c} - (1-u)(1+v)w + (1-u)(1-v)(1-w)\right]$$

Using

$$\int_o^1 \frac{x(1-x)dx}{[a+bx+c(1-x)]^2} = -\frac{2}{(b-c)^2} + \frac{(2a+b+c)}{(b-c)^3}\ \ell n\ \left[\frac{a+b}{a+c}\right]$$

the above transformations yields, after integrating over w, Eqn. (J-1) with

$$H(u;x_1,x_2,x_3) =$$

$$\frac{(1-u)^2}{u} + \frac{1}{u} \int_1^1 dv \, \{- (1-u) +$$

$$+ \tfrac{1}{2} [u \, \{\frac{4(1-x_2)(1-x_3)}{x_1^{\,2} \, C} + \frac{(x_3-x_2+x_1 v)}{x_1^{\,2}}\} + (1-u)v] \ln(\frac{-(1+v)}{(1-v)}(\frac{b(+)-av}{b(-)-av}))\} \quad (J-2)$$

where

$$a \equiv x_1(1-u) + u$$

$$b(\pm) \equiv x_1(1-u)y \pm u$$

Employing the same transformation and carrying out the w integration, Eqn. (I-3a) gives

$$\frac{G(x_2,x_3)}{(1-x_1)} = \int_o^1 du \int_{-1}^{1} \frac{dv}{c} \ln \, (\frac{-(1+v)}{(1-v)})(\frac{b(+)-av}{b(-)-av})) \quad (J-3)$$

Then noting that

$$\int_{-1}^{1} dv \, \{{v \atop 1}\} \ln \, (\frac{-(1+v)}{(1-v)}) = 2 \, \{{1 \atop i\pi}\}$$

$$u \, \frac{(x_3 - x_2 + x_1 v)}{x_1^{\,2}} + (1-u)v = \frac{u}{x_1} \, (\frac{x_3-x_2}{x_1}) + \frac{av}{x_1}$$

Eqn. (J-1) through (J-3) allow us to express $I_2(x_1,x_2,x_3)$ as

$$\frac{1}{x_1} \, I_2(x_1,x_2,x_3) =$$

$$\frac{2(1-x_2)(1-x_3)}{x_1^{\,2}} \, \frac{G(x_2,x_3)}{(1-x_1)} + \int_o^1 \frac{du}{u} \, \{ \, (1-u)^2 - 2(1-u) + \frac{a}{x_1} +$$

$$+ i\pi \frac{u}{x_1} \left(\frac{x_3-x_2}{x_1}\right) + \frac{1}{2x_1} \int_{-1}^{1} dv \left[av + \left(\frac{x_3-x_2}{x_1}\right)u\right] \ln \left(\frac{b(+)-av}{b(-)-av}\right) \Big\} =$$

$$\frac{2(1-x_2)(1-x_3)}{x_1^{\,2}} \frac{G(x_2,x_3)}{(1-x_1)} - \frac{1}{2} + \frac{1}{x_1} \left[1 + i\pi \left(\frac{x_3-x_2}{x_1}\right)\right] +$$

$$+ \frac{1}{2x_1} \int_0^1 \frac{du}{u} \int_{-1}^{1} dv \left[av + \left(\frac{x_3- x_2}{x_1}\right)u\right] \ln \left(\frac{b(+)-av}{b(-)-av}\right) \qquad \text{(J-4)}$$

We observe that the u^{-1} singularity has vanished; the remaining integrand is finite at $u = 0$.

Now, the integration formula

$$\int_{-1}^{1} dv \ \ln \ (b(\pm)-av) =$$

$$\frac{[a+b(\pm)]\ln(a+b(\pm))}{a} + \frac{[a-b(\pm)]\ln(a-b(\pm))}{a} + i\pi \frac{[a-b(\pm)]}{a} - 2,$$

$$\int_{-1}^{1} vd\ v \ \ln \ (b(\pm)-av) =$$

$$\frac{[a+b(\pm)][a-b(\pm)]}{2\ a^2} \ln \left(\frac{a-b(\pm)}{a+b(\pm)}\right) + i\pi \frac{[a+b(\pm)][a-b(\pm)]}{2\ a^2} - \frac{b(\pm)}{a}$$

together with the relations

$$a + b(\pm) = 2 \left\{ \begin{array}{l} (1-u)(1-x_2) + u \\ (1-u)(1-x_2) \end{array} \right.$$

$$a-b(\pm) = 2 \left\{ \begin{array}{l} (1-u)(1-x_3) \\ (1-u)(1-x_3) + u \end{array} \right.$$

$$a \pm b(\pm) = a \pm b(\ ) + 2u$$

$$\frac{u}{a} = \frac{1}{(1-x_1)} \left[1 - \frac{x_1}{a}\right]$$

and some straight forward, albeit tedious, algebra yield

$$\int_{-1}^{1} dv \left[av + \left(\frac{x_3-x_2}{x_1}\right)u\right] \ell n \left(\frac{b(+)-av}{b(-)-av}\right) =$$

$$\frac{2(1-x_2)(1-x_3)}{x_1} \left[1 + \frac{u}{(1-x_1)a}\right] \ell n \left(\frac{(1-u)^2}{\left(1 + \frac{x_2 u}{1-x_2}\right)\left(1 + \frac{x_3 u}{1-x_3}\right)}\right) +$$

$$- \frac{2\ u(1-x_2)(1-x_3)}{x_1} \left[1 + \frac{1}{(1-x_1)}\right] \ell n \left(\frac{(1-u)^2(1-x_2)(1-x_3)}{(1-x_2+ x_2 u)(1-x_3+ x_3 u)}\right) +$$

$$+ 2\ u \left(\frac{x_3-x_2}{x_1}\right) \ell n \left(\frac{1-x_2+ x_2 u}{1-x_3+ x_3 u}\right) = 2\ u \left[1 + i\pi \left(\frac{x_3-x_2}{x_1}\right)\right]$$

The final integration may be performed with the aid of

$$L_{i2}(x) = - \int_o^x \frac{dt}{t} \ell n\ (1-t)$$

$$= - x \int_o^1 dt \frac{\ln(t)}{1-xt},$$

$$\int_o^x \frac{\ell n[a+bt]}{(c+et)} dt = \frac{1}{2e} \ell n^2\left(\frac{b}{e}(c + ex)\right) - \frac{1}{2e} \left(\frac{bc}{e}\right) +$$

$$+ \frac{1}{e} L_{i2}\left(\frac{bc - ae}{b(c+ex)}\right) - \frac{1}{e} L_{i2}\left(\frac{bc-ae}{bc}\right); \quad \frac{ae-bc}{e} < 0$$

After some simplifications:

$$\frac{1}{2x_1}\int_0^1 \frac{du}{u}\int_{-1}^1 dv[av + (\frac{x_3 - x_2}{x_1})u]\ \ell n(\frac{b(+)-av}{b(-)-av}) =$$

$$- \frac{(1-x_2)(1-x_3)}{x_1^2}[\frac{\pi^2}{3} - L_{i2}(-\frac{x_2}{1-x_2}) - L_{i2}(-\frac{x_3}{1-x_3})] +$$

$$- \frac{(1-x_2)(1-x_3)}{x_1^2(1-x_1)^2}[2\ L_{i2}(1-x_1) + 2\ \ell n(x_1)\ell n(1-x_1)$$

$$- \ell n^2(x_1) - \ell n\ (\frac{x_2 x_3}{(1-x_2)(1-x_3)})\ \ell n(x_1) +$$

$$+ L_{i2}(\frac{1-x_3}{x_2}) - L_{i2}(\frac{1-x_3}{x_1 x_2}) + L_{i2}(\frac{1-x_2}{x_3}) - L_{i2}(\frac{1-x_2}{x_1 x_3})] +$$

$$- \frac{(1-x_2)(1-x_3)}{x_1^2}[\frac{2-x_1}{1-x_1}][\frac{\ell n(1-x_2)}{x_2} + \frac{\ell n(1-x_3)}{x_3}] +$$

$$+ \frac{[(1-x_2) - (1-x_3)]}{x_1^2}[\frac{(1-x_3)\ell n(1-x_3)}{x_3} - \frac{(1-x_2)\ell n(1-x_2)}{x_2}] +$$

$$- \frac{1}{x_1}[1 + i\pi\ (\frac{x_3 - x_2}{x_1})] \qquad \text{(J-5)}$$

By use of Eqns. (I-3a) and (I-3c):

$$\frac{\pi^2}{3} - L_{i2}(\frac{x_2}{1-x_2}) - L_{i2}(-\frac{x_3}{1-x_3}) = \frac{\pi^2}{2} + \frac{1}{2}\ell n^2(\frac{1-x_2}{1-x_3}) + G(x_2,x_3) \qquad \text{(J-6)}$$

Further, Schaeffer's relation

$$L_{i2}(\frac{y(1-x)}{x(1-y)}) = L_{i2}(x) - L_{i2}(y) + L_{i2}(y/x) +$$

$$+ L_{i2}(\frac{1-x}{1-y}) - \frac{\pi^2}{6} + \ell n(x)\ell n(\frac{1-x}{1-y})$$

gives

$$L_{i2}\left(\frac{1-x_3}{x_2}\right) - L_{i2}\left(\frac{1-x_3}{x_1x_2}\right) = -L_{i2}(1-x_2) + L_{i2}(x_1) + L_{i2}\left(\frac{1-x_2}{x_1}\right) - \frac{\pi^2}{6} +$$

$$+ \ell n\left(\frac{1-x_3}{x_1x_2}\right)\ell n\left(\frac{1-x_2}{x_1}\right)$$

or

$$L_{i2}\left(\frac{1-x_3}{x_2}\right) - L_{i2}\left(\frac{1-x_3}{x_1x_2}\right) = L_{i2}(x_2) + L_{i2}(x_1) + L_{i2}\left(\frac{1-x_2}{x_1}\right) - \frac{\pi^2}{3} +$$

$$+ \ell n\left(\frac{1-x_3}{x_1}\right)\ell n\left(\frac{1-x_2}{x_1}\right) + \ell n(x_1)\ell n(x_2) \qquad \text{(J-7)}$$

where we have used

$$L_{i2}(x) + L_{i2}(1-x) = \frac{\pi^2}{6} - \ell n(x)\ell n(1-x) \qquad \text{(J-8)}$$

Similarly:

$$L_{i2}\left(\frac{1-x_2}{x_3}\right) - L_{i2}\left(\frac{1-x_2}{x_1x_3}\right) = L_{i2}(x_3) + L_{i2}(x_1) + L_{i2}\left(\frac{1-x_3}{x_1}\right) - \frac{\pi^2}{3} +$$

$$+ \ell n\left(\frac{1-x_2}{x_1}\right)\ell n\left(\frac{1-x_3}{x_1}\right) + \ell n(x_1)\ell n(x_3) \qquad \text{(J-9)}$$

Also, putting $x = (1-x_2)/x_1$ and $y = 0$ in Schaeffer's relation we have

$$L_{i2}\left(\frac{1-x_2}{x_1}\right) + L_{i2}\left(\frac{1-x_3}{x_1}\right) = \frac{\pi^2}{6} - \ell n\left(\frac{1-x_2}{x_1}\right)\ell n\left(\frac{1-x_3}{x_1}\right) \qquad \text{(J-10)}$$

Thus, employing Eqns. (J-7) through (J-10) and Eqn. (I-3c)

$$2\,L_{i2}(1-x_1) + 2\,\ln(x_1)\ln(1-x_1) + \ln^2(x_1) - \ln(x_1)\ln\left(\frac{x_2 x_3}{(1-x_2)(1-x_3)}\right) +$$

$$+ L_{i2}\left(\frac{1-x_3}{x_2}\right) - L_{i2}\left(\frac{1-x_3}{x_1 x_2}\right) + L_{i2}\left(\frac{1-x_2}{x_3}\right) - L_{i2}\left(\frac{1-x_2}{x_1 x_3}\right) =$$

$$G(x_2,x_3) \tag{J-11}$$

Finally, observing that

$$-\frac{(1-x_2)(1-x_3)}{x_1^{\,2}}\left[\frac{2-x_1}{1-x_1}\right]\left[\frac{\ln(1-x_2)}{x_2} + \frac{\ln(1-x_3)}{x_3} +\right.$$

$$\left. + \frac{[(1-x_2)-(1-x_3)]}{x_1^{\,2}}\left[\frac{(1-x_3)\ln(1-x_3)}{x_3} - \frac{(1-x_2)\ln(1-x_2)}{x_2}\right] =\right.$$

$$-\frac{1}{x_1(1-x_1)}\,[(1-x_2)\ln(1-x_2) + (1-x_3)\ln(1-x_3)],$$

$$\frac{1}{x_1^{\,2}}\left[\frac{2}{(1-x_1)} - 1 - \frac{1}{(1-x_1)^2}\right] = -\frac{1}{(1-x_1)^2}$$

we have, using Eqns. (J-4) through (J-6) and (J-11):

$$I_2(x_1,x_2,x_3) =$$

$$-\frac{x_1}{2} - \frac{x_1(1-x_2)(1-x_3)}{(1-x_1)^2}\,G(x_2,x_3) - \frac{(1-x_2)(1-x_3)}{x_1}\left[\frac{\pi^2}{2} + {}^1\!/_2\,\ln^2\left(\frac{1-x_2}{1-x_3}\right)\right] +$$

$$-\frac{1}{(1-x_1)}\,[(1-x_2)\ln(1-x_2) + (1-x_3)\ln(1-x_3)] \tag{J-12}$$

APPENDIX K

SOME SPECIAL INTEGRALS

Herein we develop some integration formula useful in the evaluation of $d\,\rho_R^{II}/dx$.

First consider

$$\int_o^1 dz\ z^n\ \ell n(z)\ell n(1-z),\ n > -1$$

Integrating by parts

$$\int_o^1 dz\ z^n\ \ell n(z)\ell n(1-z) =$$

$$-\frac{1}{n+1}\int_o^1 dz\ z^{n+1}[\frac{\ell n(1-z)}{z} - \frac{\ell n(z)}{1-z}] =$$

$$\frac{1}{n+1}\int_o^1 dz[(1-z)^{n+1} - z^{n+1}]\ \frac{\ell n(1-z)}{z},\ n > -1$$

so

$$\int_o^1 dz\ \ell n(z)\ell n(1-z) = \int_o^1 dz\ [\frac{1}{z} - 2]\ell n\ (1-z)$$

$$= 2 - \frac{\pi^2}{6}, \qquad (K-1)$$

$$\int_o^1 dz\ z\ \ell n(z)\ell n(1-z) = 1/2\int_o^1 dz\ [\frac{1}{z} - 2]\ell n(1-z)$$

$$= \frac{1}{2}\left[2 - \frac{\pi^2}{6}\right] \tag{K-2}$$

and

$$\int_0^1 dz\ z^2\ \ln(z)\ln(1-z) = \frac{1}{3}\int_0^1 dz(1-2z)[(1-z)^2 + z]\ \frac{\ln(1-z)}{z}$$

$$= \frac{1}{3}\int_0^1 dz\ \left[\frac{\ln(1-z)}{z} - [2 - z + 2z^2]\ln(z)\right]$$

$$= \frac{1}{3}\left[2 - \frac{\pi^2}{6} - \frac{1}{36}\right] \tag{K-3}$$

Now consider

$$\int_0^1 dz\ z^n\ L_{i2}(1-(1-x)z),\ n > -1$$

where

$$L_{i2}(x) = -\int_0^x \frac{dt}{t}\ \ln\ (1-t)$$

$$= -\ x\int_0^1 dt\ \frac{\ln(t)}{1-xt},$$

$$L_{i2}(x) + L_{i2}(1-x) = \frac{\pi^2}{6} - \ln(x)\ln(1-x)$$

With the same trick of integrating by parts

$$\int_0^1 dz\ z^n\ L_{i2}(1-(1-x)z) =$$

$$\frac{1}{n+1}\, L_{i2}(x) - \frac{(1-x)}{n+1} \int_o^1 dz\, \frac{z^{n+1}\ell n((1-x)z)}{1-(1-x)z}$$

so

$$\int_o^1 dz\; L_{i2}(1-(1-x)z) =$$

$$L_{i2}(x) + \int_o^1 dz\; [\frac{-1}{1-(1-x)z} + 1]\ell n((1-x)z) =$$

$$L_{i2}(x) + \ell n(1-x) - 1 + \frac{1}{1-x}\,[\ell n(x)\ell n(1-x) + L_{i2}(x)] =$$

$$\ell n(1-x) - 1 + \frac{1}{1-x}\,[\frac{\pi^2}{6} - x\, L_{i2}(x)], \qquad \text{(K-4)}$$

$$\int_o^1 dz\; z\, L_{i2}(1-(1-x)z) =$$

$$\frac{1}{2}\, L_{i2}(x) + \frac{1}{2}\int_o^1 dz[z + \frac{1}{1-x}[1 - \frac{1}{1-(1-x)z}]]\ell n((1-x)z) =$$

$$\frac{1}{2}\, L_{i2}(x) + \frac{1}{2}\,[\frac{1}{2}\,\ell n(1-x) - \frac{1}{4}] + \frac{1}{2(1-x)}\,[\ell n(1-x)-1] +$$

$$+ \frac{1}{2(1-x)^2}\,[\frac{\pi^2}{6} - L_{i2}(x)] =$$

$$\frac{1}{2}[\frac{1}{2}\,\ell n(1-x) - \frac{1}{4}] + \frac{1}{2(1-x)}\,[\ell n(1-x) - x\, L_{i2}(x) - 1] +$$

$$+ \frac{1}{2(1-x)^2}\,[\frac{\pi^2}{6} - x\, L_{i2}(x)] \qquad \text{(K-5)}$$

and, finally

$$\int_o^1 dz\; z^2\, L_{i2}(1-(1-x)z) =$$

$$\frac{1}{3} L_{i2}(x) + \frac{1}{3} \int_o^1 dz[z^2 + \frac{z}{1-x} + \frac{1}{(1-x)^2}[1 - \frac{1}{1-(1-x)z}]] \ \ell n((1-x)z) =$$

$$\frac{1}{3} L_{i2}(x) + \frac{1}{3}[\frac{1}{3} \ell n(1-x) - \frac{1}{9}] + \frac{1}{3(1-x)} [\frac{1}{2} \ell n(1-x) - \frac{1}{4}] +$$

$$+ \frac{1}{3(1-x)^2} [\ell n(1-x) - 1] + \frac{1}{3(1-x)^3} [\frac{\pi^2}{6} - L_{i2}(x)] =$$

$$\frac{1}{3}[\frac{1}{3} \ell n(1-x) - \frac{1}{4}] + \frac{1}{3(1-x)}[\frac{1}{2} \ell n(1-x) - \frac{1}{4}] + \frac{1}{3(1-x)^2}[\ell n(1-x)-1] +$$

$$- \frac{x}{3}[\frac{1}{1-x} + \frac{1}{(1-x)^2}] \ L_{i2}(x) + \frac{1}{3(1-x)^3} [\frac{\pi^2}{6} - x \ L_{i2}(x)] \qquad \text{(K-6)}$$

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