73-15,315

BEER, Carl Neil, 1935-
POLYHEDRAL SUBSPACES AND THE SELECTION OF OPTIMAL BASES IN PROBABILISTIC LINEAR PROGRAMMING.

The University of Oklahoma, Ph.D., 1972
Engineering, industrial

University Microfilms, A XEROX Company, Ann Arbor, Michigan

THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

POLYHEDRAL SUBSPACES AND THE SELECTION OF OPTIMAL BASES IN PROBABILISTIC LINEAR PROGRAMMING

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

BY

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Norman, Oklahoma

1972

POLYHEDRAL SUBSPACES AND THE SELECTION OF OPTIMAL

BASES IN PROBABILISTIC LINEAR PROGRAMMING

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This dissertation is dedicated to the
memory of my father, Carl R. Beer, and
to his many years of total commitment
to excellence in education.

ABSTRACT

For the dis^ribution problem in probabilistic linear programming, no method exists for ranking the various feasible bases in accordance with their respective probabilities of being optimal; hence, no suitable approximation of the cumulative distribution of the maximum value of the objective function is possible. Present techniques generate the bases only in a contiguous manner and require extensive probability calculations for each basis. For a class of multivariate density functions, this dissertation provides two solution methods for ranking the optimal probabilities of the bases without performing any probability computations. A fringe benefit of these methods is that for PLP problems requiring a constant strategy, rather than an approximation of the CDF, the solution is rapidly obtained.

Both solution methods are geometrical and based on an embedded hypersphere concept. Both methcds are exact for $n = 2$ and when certain conditions are satisfied, the first method is also exact for $n > 2$. While sufficiency cannot be shown for the second method for $n > 2$, it has withstood several attempts to find a counterexample. Either method can be applied in a heuristic sense to any size problem to achieve near optimal

results. In addition to those PLP problems having appropriate distributions, these solution methods will also provide usable results over a wide range of joint probability density functions. After appropriate theory and the two methods have been developed, both techniques are then applied to the same example problem.

ACKNOWLEDGEMENTS

To my committee chairman, Dr. Bob L. Foote, I extend my deep appreciation for his expert guidance and advice. But more importantly, I am indebted for his consistent support and expressions of confidence throughout my academic program.

To the members of my excellent committee, Dr. Michael D. Devine, Dr. LaVerne L. Hoag, Dr. Hillel J. Kumin, and Dr. Robert A. Shapiro, I offer my sincere thanks for their encouragement and cooperation.

A special acknowledgement is due my employer, the United States Air Force; my doctoral program was made possible only by their sponsorship. In particular, I wish to thank the USAF Academy and the Department of Mathematics, my unit of assignment.

Finally, I must express very special thanks to my wife, Jan. I hope her superb techniques for treating my natural inclination toward passive activity will eventually have a permanent effect.

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CHAPTER I

INTRODUCTION

The assumption made in linear programming that the coefficients are constants is seldom satisfied in practice. Since adequate statistical information about these coefficients generally exists and will support an assumption of random variation, much effort has been expended in designing algorithms which incorporate this non-deterministic behavior. These random coefficients are generally regarded as independent and may or may not have identical distributions. This type of problem will hereafter be referred to as probabilistic linear programming, thereby saving the stochastic description for those cases where the probability distribution function of the coefficients may change with time. The literature is somewhat lacking in that this distinction between probabilistic and stochastic programming is usually not made.

Various probabilistic programming problems have been investigated, each with the obvious intent of achieving a mathematical statement regarding the optimal value of the objective function. In order to illuminate existing techniques and areas of interest, consider the standard linear programming problem:

$$\text{maximize } z(x) = \sum_{j=1}^{n} c_j x_j \qquad (1)$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i, \quad i = 1,2, \ldots, m \qquad (2)$$

$$x_j \geq 0, \quad j = 1,2, \ldots, n, n+1, \ldots, n+m \qquad (3)$$

In the above problem some, or all, of the c_j's, b_i's, or a_{ij}'s may be probabilistic. It is noted that if only the c_j's or b_i's are random, then the dual problem has essentially the same form as the primal and the treatment would be quasi-analogous. In the event a significant number of a_{ij}'s are random variables, attempts to establish solution methods are generally unsuccessful since the convex space will have a positive probability of being empty or unbounded [8].

The initial efforts in probabilistic linear programming were those of Dantzig and Ferguson [12], under the condition that a discrete demand distribution existed. Later efforts by Elmaghraby [15,16] treated the case where the distribution of the b_i's was continuous for the same multiperiod allocation problem. Other applications include management decision problems [29], capital budgeting [27], and transportation problems [36,42].

In the event the random coefficients consist only of a_{ij}'s or b_i's, the term chance-constrained programming has been used to describe the PLP problem. For this case the constraints in (2) become

$$\Pr\left[\sum_{j=1}^{n} a_{ij}x_j \le b_i\right] \ge \alpha_i, \quad i = 1,2, \ldots, m$$

where either the a_{ij} or the b_i are given by some probability distribution and $0 \le \alpha_i \le 1$. This type of problem was first introduced by Charnes and Cooper [10] and later expanded by others [11, 20,25,35]. Recently, Seppala [33] showed how to construct sets of uniformly tighter linear constraints to replace a chance constraint, in order to solve the problem by the simplex technique. Another subordinate type of PLP is called aspiration criterion programming [19], and in this instance (1) becomes

$$\max \Pr[z(x) = \sum_{j=1}^{n} c_jx_j] \ge z_o \quad \text{where } z_o \text{ is constant.}$$

In general, the solution of any PLP problem will be in accordance with one of two approaches, active or passive. The terminology here is due to Tintner [38]. The same two categories are described by Madansky [22] as the "here-and-now" or "wait-and-see" approaches. The active approach is characterized by a basis being selected and fixed before the values of the random coefficients are observed or known. Thus, a basic solution may become nonoptimal or infeasible, depending on whether the c_j's or b_i's are probabilistic. The passive approach allows the basis to change with variations in the random values of the c_j's or b_i's, thereby assuring that an optimal basis is selected.

The active approach usually leads to the development of nonlinear programming problems, which can be described as "deterministic equivalents" and serve as approximations to the original

PLP problem. The solution technique involves the selection of some criterion, usually optimizing the expected value of the objective function, and then including a penalty function that represents the cost associated with making an incorrect decision [32]. In the event some a_{ij} may be probabilistic, this approach may lead to a form for (1) and (2) such as:

$$\max \sum_{j=1}^{k} c_j x_j + E_y[F(y)] \;,$$ where $E_y[F(y)]$ implies the expected value of $F(y)$ wrt y.

subject to

$$\sum_{j=1}^{k} a_{ij} x_j + \sum_{j=k+1}^{n} a_{ij} y_j \le b_i, \quad i = 1,2, \ldots, m,$$

where the a_{ij}, $j = k+1, \ldots, n$, or the b_i's may be random variables. For examples using the active approach, see [13,14,23,39, 40,41,43,44,46].

The passive approach gives rise to a specific type of problem which consists of determining the distribution of the optimal value of the objective function when the distributions of the random variables are known. The distribution problem was first introduced by Tintner [38] as "passive stochastic programming". The DP does not belong exclusively to the passive approach since, in either case, the objective is to find a mathematical expression for max z(x). Sengupta, Tintner, and Morrison [31] give relationships between max z(x) for the active and max z(x) for the passive approaches. Also the distribution problem is stated for the nonstationary stochastic case by Bereanu [8].

The realization of the distribution of objective function values in closed form has been very elusive. Approximation techniques ranging from the "method of sample points" [31] to enumeration of extreme points (coupled with simulation) by Bracken and Soland [9] have been offered. Closed forms have been obtained for severely restricted cases by Babbar [1] and Prekopa [28]. The most significant contributions were by Bereanu, who in [4] obtained a closed form expression for the distribution of max z(x) for PLP problems having a single random variable, and in [6] utilizes the Laplace transform to find the approximate distribution of max z(x). In a recent paper [7], Bereanu offers a rather unrestricted solution method and includes an example using three random variable coefficients. The technique employed is based on a Cartesian multidimensional quadrature formula (see Stroud and Secrest [34]), which in the limit becomes an exact expression for the iterated integrals it represents. Since the limit is apparently not approached, there is some doubt regarding the exactness of the closed form distribution thus obtained.

Two recent dissertations by Zinn [47] and Ewbank [17] have attacked the DP directly: Zinn describes an algorithm that generates only the optimal simplex bases, which (in the case of random c_j's) means those feasible bases having a positive probability of being optimal, and Ewbank shows how to calculate the exact objective function cumulative distribution when given those "optimal" bases. Thus, Ewbank has accomplished a significant breakthrough in that the iterated integral probability statements, which have prevented determination of the exact distribution heretofore, can now be solved using the Jacobian transformation of variables technique. Neither paper treats the case of random a_{ij} and Ewbank's method is weak for those problems involving both random c_j's and b_i's, but a solid beginning has finally been made.

Statement of Problem

The problem to be treated in this thesis is an outgrowth of the work by Zinn [47] and Ewbank [17]. Assume the same passive probabilistic programming problem in the following form:

$$\max z(x) = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i, \quad i = 1,2, \ldots, m$$

$$x_j \geq 0, \quad j = 1,2, \ldots, n, n+1, \ldots, n+m$$

where the values of either the c_j's or b_i's are given in terms of random variables, and the distribution of max $z(x)$ as a function of all values of the random coefficients is desired. The feasible region is assumed non-empty and bounded, or in the case of random b_i's, the probability of such must be positive. Also, the joint density function of the c_j's or b_i's is defined everywhere over appropriate domain and assumed to be piece-wise continuous. Ewbank's method must then be provided the distinct, "optimal" bases and the cumulative distribution function of max $z(x)$ can be calculated. Zinn's technique can be used to enumerate those required bases, but in the iterative simplex algorithm, many probabilities involving the optimality criterion must be calculated. Furthermore, the iterative algorithm produces the bases in a contiguous manner [47], instead of by a priority determination in accordance with their respective probabilities of being optimal.

Consider the case of random c_j's and let $\underline{c}$ be the row vector $(c_1, c_2, \ldots, c_n)$ and $\underline{c}' = (\underline{c}, 0, 0, \ldots, 0_{n+m})$. Define $\underline{c}_B$ as the vector $(c_1, c_2, \ldots, c_m)$ such that $c_i = c_j$ if x_j is the $i^{\underline{th}}$ element of the basis for $j \le n$ and $c_i = 0$ if x_j is in the basis for $j > n$. Denote by $\underline{x}^t$ the extreme point corresponding to basis t, where $t = 1, 2, \ldots, h$ for $h \le \binom{m+n}{m}$ and $\underline{x}$ is defined as the column vector $(x_1, x_2, \ldots, x_n, x_n+1, \ldots, x_{n+m})'$, where at least n elements are zeros and the remaining m x_j elements form the basis $\underline{x}_B$. The extreme points $\underline{x}^t$ are fixed in the case of probabilistic c_j's and the distribution of max z(x) is a function of $\underline{c}$ over the solution space. It is expressed as:

$$z(\underline{c}) = \max\left[z(x)\,|\,\underline{c}\right] ,$$

and for the $t^{\underline{th}}$ basis becomes

$$z_t(\underline{c}) = \left[z(\underline{x}^t)\,|\,\underline{c}\right] .$$

Associated with each basis is a probability P_{k_t} that the $t^{\underline{th}}$ basis is optimal (optimality and feasibility criterion satisfied). Then the distribution of the objective function is given by

$$z(\underline{c}) = \sum_{t=1}^{q} P_{k_t}\, z_t(\underline{c}),$$ where q represents the number of feasible bases, and where $\sum_{t=1}^{q} P_{k_t} = 1$.

Since the number of bases is finite, this sum presumably exists and is likewise finite.

In order to generate the q bases, [47] uses the simplex technique and requires extensive probability calculations. Also, there is no assurance that those bases having the greater P_k will not be generated last. Thus, this paper will develop techniques to provide [17] the required bases by utilizing a more efficient method, and more importantly, to generate first those bases having the largest P_k. For many problems where a small number of bases comprise the major portion of the optimal probability space, it is highly desirable to determine the CDF of max $z(x)$ based on just those few.

CHAPTER II

CONCEPTUAL REQUIREMENTS

Geometry of Polyhedral Subspaces

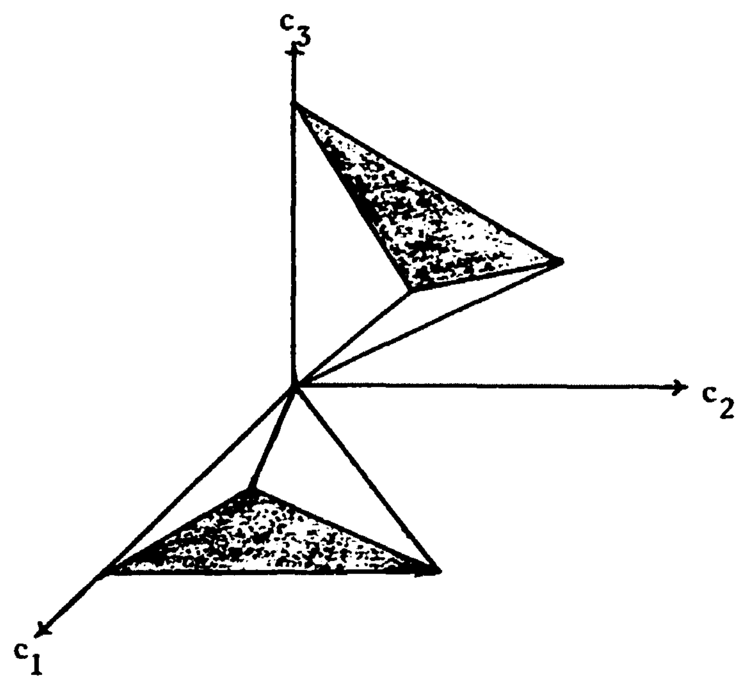
Define P_j as the column vector $(a_{1j}, a_{2j}, \ldots, a_{mj})'$ associated with the decision variable x_j, $j = 1, 2, \ldots, n, n+1, \ldots, n+m$. Denote as B the mxm basis matrix containing the ordered P_j associated with $\underline{x}_B$. For basis t, define A_t as the mxn matrix containing those P_j columns corresponding to the non-basic x_j, and denote as $\underline{c}_{t'}$ the 1xn row vector of their objective function coefficients. An "unprimed" t denotes quantities associated with basic x_j. Then, for deterministic linear programming, two conditions must be met in order to realize the optimal solution: (1) the product $B_t^{-1}\underline{b}$ must be $\geq \underline{0}$ for the basic solution $\underline{x}_B$ to be feasible, and (2) the optimality criterion (for best $\underline{x}^t$) requires the quantities $(\underline{c}_{B_t} B_t^{-1} A_{t'} - \underline{c}_{t'})$ to be non-negative for the maximization problem.

Consider the case of probabilistic c_j's and let $f(\underline{c})$ be the continuous joint probability density function of the c_j's. Assuming a basis t such that $B_t^{-1}\underline{b} \geq \underline{0}$, $\underline{x}_{B_t}$ is an optimal solution if and only if $\{\underline{c}_{B_t} B_t^{-1} A_{t'} - \underline{c}_{t'}\} \geq \underline{0}$. But since the c_j's are random, the nonnegativity of the optimality criterion can only be measured in a probabilistic sense. Define $S_t = \left\{\underline{c} \middle| \{\underline{c}_{B_t} B_t^{-1} A_{t'}, -\underline{c}_{t'}\} \geq \underline{0}\right\}$, then assuming feasibility, the probability that the $t^{\underline{th}}$ basis is optimal is given by

$$P_{k_t} = P\left[\{\underline{c}_{B_t} B_t^{-1} A_{t'}, -\underline{c}_{t'}\} \geq \underline{0}\right] = \iint_{S_t} \cdots \int f(\underline{c}) dc_1, dc_2, \ldots dc_n. \qquad (5)$$

Thus, the problem becomes that of finding those bases associated with largest P_k, without using Zinn's simplex algorithm which must iterate over all feasible bases, and wherein probabilities similar to (5) must be calculated each time the basis is changed to determine the entering x_j. To develop such a technique requires first an investigation of the geometry of the space over which (5) is integrated to obtain the $\{P_{k_t}\}$. In (4) it was stated that the $\sum_{t=1}^{q} P_{k_t} = 1$; this is only true if the "optimal" probabilities associated with the feasible bases represent mutually exclusive and exhaustive events. For the $t^{\underline{th}}$ basis the n quantities given by $\{\underline{c}_{B_t} B_t^{-1} A_{t'}, -\underline{c}_{t'} = \underline{0}\}$ constitute hyperplanes in R^n space, where $n = \dim \underline{c} = \dim\left\{\underline{c}_{B_t} \bigcup \underline{c}_{t'}\right\}$, since all elements in $\underline{c}$ are contained in $\left\{\underline{c}_{B_t} \bigcup \underline{c}_{t'}\right\}$. For n=3 the resulting three hyperplanes, regardless of the size of m, can be regarded as a convex polyhedron in R^3 which consists of three sides plus possible additional ones imposed by $D[f(\underline{c})] = D(c_1) \times D(c_2) \times D(c_3)$, the Cartesian product of the given domains of the random variables. Since the hyperplanes defined by $\{\underline{c}_{B_t} B_t^{-1} A_{t'}, -\underline{c}_{t'} = \underline{0}\}$ are homogeneous, i.e., subspaces of R^n, a description as "polyhedral subspace" is appropriate. We are assured the subspace is convex since $\{\underline{c}_{B_t} B_t^{-1} A_t, -\underline{c}_t, \geq \underline{0}\}$ are linear inequalities (half-spaces), and the intersection of halfspaces is convex [45, p. 26].

It is assumed that c_j, $j = 1,2, \ldots, n$, may have vector representation. For example, if dim $\underline{c} = 3$, then the standard basis unit vectors will be used to span R^3; thus, the direction c_1 would be denoted by the column vector $(1,0,0)'$, c_2 by $(0,1,0)'$, and c_3 by $(0,0,1)'$, and a right-hand coordinate system would be employed. For illustrative purposes, the case for n=3, with non-negative domains on the c_j's and with two distinct subspaces formed by appropriate hyperplanes, could (for a specific problem) appear as shown below:



The possibility of a degenerate polyhedron would exist only for those cases where the subspace generated in R^n conflicts with the D[f(c)]. At least two of the three hyperplanes, $\{\underline{c}_{B_t} B_t^{-1} A_{t'}, -\underline{c}_{t'} = \underline{0}\}$, for each basis t, contain exclusive c_j's and are therefore linearly independent. In fact, by using vector techniques, the hyperplanes associated with any basis (regardless of feasibility) will be shown as linearly independent and thus spanning $R^{\dim \underline{c}}$ (see Theorem 5). If the basis contains the nonslack variables, the argument for linear independence is based on the fact that $A_{t'}$ is made up of unit basis vectors, so that the product $B_t^{-1} A_{t'}$ cannot yield proportional vectors. However, "trivial" cases may be constructed where a degenerate polyhedron does exist. For example, given the "starting solution" made up of all slack variables then $\{\underline{c}_{B_t} B_t^{-1} A_{t'}, -\underline{c}_{t'}\}$ becomes $\{-c_1, -c_2, -c_3\}$ and if the D[f($\underline{c}$)] is restricted to nonnegative values, then the subspace formed is $\{\underline{c} | -c_1, -c_2, -c_3 \geq 0\}$, which contains only the point (0,0,0).

The mutual exclusiveness of the q polyhedrons is stated as a theorem in Chapter III. This does not mean exclusive in the strict sense; to be noted is that a common boundary, either a hyperplane or the intersection of edges, is shared by at least every two subspaces. But these mutual points are of probability measure zero since a subspace cannot exist or be defined if any $\underline{c}_B B^{-1} P_j - c_j$ is identically equal to zero. As previously defined, let $S_r = \left\{\underline{c} | \{\underline{c}_{B_r} B_r^{-1} A_{r'}, -\underline{c}_{r'}\} \geq \underline{0}\right\}$ and $S_s = \left\{\underline{c} | \{\underline{c}_{B_s} B_s^{-1} A_{s'}, -\underline{c}_{s'}\} \geq \underline{0}\right\}$; then it must be shown that S_r and S_s intersect only in sets of zero content [3], implying those common boundary points for adjacent subspaces (see Theorem 3). Zero content means that R^n will not be spanned by $S_r \cap S_s$, which means the dim $(S_r \cap S_s) < n$. To demonstrate this exclusiveness property, consider the following example:

$$\max z(x) = c_1x_1 + c_2x_2 + c_3x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 &\le 430 \\ 3x_1 \qquad + 2x_3 &\le 460 \\ x_1 + 4x_2 \qquad &\le 420 \end{aligned} \qquad \begin{aligned} &\text{where } x_1,x_2,x_3 \ge 0, \\ &\text{and } c_1,c_2,c_3 \in R. \end{aligned}$$

The initial tableau becomes

	x_1	x_2	x_3	x_4	x_5	x_6	
	$-c_1$	$-c_2$	$-c_3$	0	0	0	
x_4	1	2	1	1	0	0	430
x_5	3	0	2	0	1	0	460
x_6	1	4	0	0	0	1	420

For the feasible basis $(x_4,x_3,x_6)'$, the polyhedral subspace is defined by the linear inequalities

$$(3/2c_3-c_1,-c_2,1/2c_3 \ge 0) \tag{6}$$

For the feasible basis $(x_3,x_5,x_2)'$, the polyhedral subspace is defined by the linear inequalities

$$(1/4c_2-c_1+1/2c_3,c_3,1/4c_2-1/2c_3 \ge 0) \tag{7}$$

Note that (6) defines a subspace exclusive of the subspace defined when the basis is $(x_4,x_5,x_6)'$, except for boundary points when $c_3 = 0$. Consider now the subspaces (6) and (7) and note that the bases are not sequential by the simplex generation technique. To show these subspaces intersect only in sets of zero content, let the vector representation of the lines of intersection for each set be expressed as $(0,0,0)' + \tau_i(\underline{d})$, where $\tau_i \in R^+$. To obtain these vectors requires the simultaneous solution of each pair of intersecting hyperplanes to determine the vectors (of arbitrary length) which represent the lines of intersection. The appropriate direction for each vector is determined by first combining the hyperplane normal vectors to obtain a point satisfying the linear inequalities, and then equating this point with a linear combination of the "intersection" vectors having nonzero scalar coefficients:

$$\text{(6)} \qquad \left.\begin{aligned} 3/2c_3 - c_1 &\geq 0 \\ -c_2 &\geq 0 \end{aligned}\right\} \text{ implies } \alpha\begin{pmatrix}3\\0\\2\end{pmatrix}$$

$$\left.\begin{aligned} 3/2c_3 - c_1 &\geq 0 \\ 1/2c_3 &\geq 0 \end{aligned}\right\} \text{ implies } \gamma\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\left.\begin{aligned} -c_2 &\geq 0 \\ 1/2c_3 &\geq 0 \end{aligned}\right\} \text{ implies } \beta\begin{pmatrix}1\\0\\0\end{pmatrix}$$

Let P_N denote a point satisfying the inequalities as determined by the normal vectors; then a linear combination yields

$$P_N = 1/3\begin{pmatrix}-1\\0\\3/2\end{pmatrix} + 1/3\begin{pmatrix}0\\-1\\0\end{pmatrix} + 1/3\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}-1/3\\-1/3\\5/6\end{pmatrix}$$

so that

$$\alpha\begin{pmatrix}3\\0\\2\end{pmatrix} + \gamma\begin{pmatrix}0\\1\\0\end{pmatrix} + \beta\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}-1/3\\-1/3\\5/6\end{pmatrix} \text{ implies } \begin{cases}\alpha \in R^+\\ \gamma \in R^-\\ \beta \in R^-\end{cases}$$

and the vectors representing of the lines of intersection for (6) are given as

$$\tau_1\begin{pmatrix}3\\0\\2\end{pmatrix},\ \tau_2\begin{pmatrix}0\\-1\\0\end{pmatrix},\ \tau_3\begin{pmatrix}-1\\0\\0\end{pmatrix}.$$

(7) In an analogous manner the subspace vectors associated with the feasible basis $(x_3,x_5,x_2)'$ are obtained as

$$\tau_1\begin{pmatrix}1\\4\\0\end{pmatrix},\ \tau_2\begin{pmatrix}1\\2\\1\end{pmatrix},\ \tau_3\begin{pmatrix}-1\\0\\0\end{pmatrix}.$$

If an interior point in subspace$_7$ can be $\subset$ subspace$_6$, then any point in subspace$_7$ must be expressible as a linear combination of vectors defining subspace$_6$. Such an arbitrary but fixed interior point is $(1/3,2,1/3)'$ and considering

$$\tau_1\begin{pmatrix}3\\0\\2\end{pmatrix} + \tau_2\begin{pmatrix}0\\-1\\0\end{pmatrix} + \tau_3\begin{pmatrix}-1\\0\\0\end{pmatrix} = \begin{pmatrix}1/3\\2\\1/3\end{pmatrix},$$

we find that $\tau_2 < 0$, indicating that the intersection of polyhedral subspaces (6) and (7) is empty relative to interior points since $\tau_i \in R^+$.

Consider now the linear inequalities for the infeasible basis $(x_5,x_3,x_6)'$, which are $(c_3-c_1, 2c_3-c_2, c_3 \geq 0)$. The subspace they define includes interior points in subspace$_6$; for example, the point $(-1,-1,1)'$ is interior to both. Therefore, the exclusive property does not exist when infeasible bases are permitted, even though they may have a positive probability of the optimality criterion being satisfied. For $\sum_t P_{k_t} = 1$, t can only index over subspaces associated with feasible bases.

Even though the x_j's are normally restricted to nonnegative values, this is not necessarily the case for the c_j's. The given joint density function $f(\underline{c})$ must state the appropriate domain, and when accumulated over all $\underline{c}$ values, the sum will equal unity. In the preceding example problem there were subspaces generated by feasible bases which included both positive and negative domains for c_j. Thus, for a specific problem, there may be some feasible bases with associated $P_{k_t} = 0$; but those remaining will define the desired exclusive subspaces over $D[f(\underline{c})]$ which have associated positive probability of being optimal.

<u>Subspaces and Optimality Probabilities</u>

Since we are now directly interested in obtaining those bases having the largest probability of being optimal, recall that for the $t^{\underline{th}}$ feasible basis

$$P_{k_t} = P\left[\{\underline{c}_{B_t} B_t^{-1} A_{t'}, -\underline{c}_{t'}\} \geq 0\right] = \int\int_{S_t}\cdots\int f(\underline{c})dc_1 \ldots dc_n .$$

And it is obvious that the effect of $f(\underline{c})$ must now be considered jointly with the well defined polyhedral subspaces. It would be highly desirable to develop a general algorithm which would generate these bases independently of the given joint density function. But it is shown in Chapter IV that such an algorithm is only possible for a particular class of probability density functions. In the calculation of $\{P_{k_t}\}$ using Ewbank's transformation of variables technique, the limits on the integrals become complicated if $f(\underline{c})$ changes functional form over different portions of R^n, thereby causing multiple integrations over each of the regions. Most desirable (if indeed not necessary) is a single functional form for $f(\underline{c})$ over $D[f(\underline{c})]$. This is not to be interpreted as highly restrictive since realistic problems meet this requirement, e.g., the multivariate normal over all of R^n, or the negative exponential over $\{\underline{c} \in R^n \mid \underline{c} \geq 0\}$. But difficulties do arise, even for $f(\underline{c})$ of a single functional form, when the $D[f(\underline{c})]$ is given upper bounds such as $\{\underline{c} \in R^n \mid \underline{0} \leq \underline{c} \leq \underline{u}\}$. The effects of various $f(\underline{c})$ and $D[f(\underline{c})]$ will be made clearer in the example problem which will now be investigated.

For ease of illustration, consider the following simple two variable PLP problem solved by the technique of [47]:

$$\max z(\underline{x}) = (c_1,c_2)\begin{pmatrix}x_1\\x_2\end{pmatrix}, \text{ where: } \quad f(c_1) = 1/10\, e^{-1/10c_1}$$

$$f(c_2) = 1/10\, e^{-1/10c_2}$$

$$c_1,c_2 \geq 0$$

$$\text{s.t.} \quad \begin{pmatrix}1 & 2\\2 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} \leq \begin{pmatrix}10\\10\\6\end{pmatrix}$$

$$x_1,x_2 \geq 0$$

x_1	x_2	x_3	x_4	x_5
$-c_1$	$-c_2$	0	0	0
1	2	1	0	0
2	1	0	1	0
1	1	0	0	1
$P_1 = P[-c_1,-c_2 \geq 0] = 0$				
0	$c_{1/2}-c_2$	0	$c_{1/2}$	0
0	3/2	1	-1/2	0
1	1/2	0	1/2	0
0	1/2	0	-1/2	1

$$P_2 = P[c_{1/2}-c_2, c_{1/2} \geq 0]$$

$$P_2 = \int_0^\infty f(c_2) \int_{2c_2}^\infty f(c_1)dc_1dc_2 = 1/3$$

x_1	x_2	x_3	x_4	x_5
0	0	0	c_1-c_2	$2c_2-c_1$
0	0	1	1	-3
1	0	0	1	-1
0	1	0	-1	2

$$P_3 = P[c_1-c_2, 2c_2-c_1 \geq 0]$$

$$P_3 = \int_0^\infty f(c_2) \int_{c_2}^{2c_2} f(c_1)dc_1dc_2 = 1/6$$

0	0	c_2-c_1	0	$2c_1-c_2$
0	0	1	1	-3
1	0	-1	0	2
0	1	1	0	-1

$$P_4 = P[c_2-c_1, 2c_1-c_2 \geq 0]$$

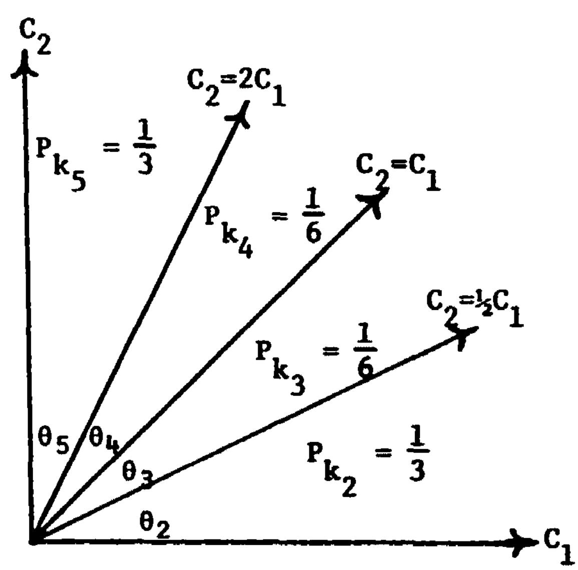
$$P_4 = \int_0^\infty f(c_1) \int_{c_1}^{2c_1} f(c_2)dc_2dc_1 = 1/6$$

$c_{2/2}-c_1$	0	$c_{2/2}$	0	0
3/2	0	-1/2	1	0
1/2	0	-1/2	0	1
1/2	1	1/2	0	0

$$P_5 = P[c_{2/2}-c_1, c_{2/2} \geq 0]$$

$$P_5 = \int_0^\infty f(c_1) \int_{2c_1}^\infty f(c_2)dc_2dc_1 = 1/3$$

Note the entering x_j for each iteration was determined as the one having the greatest probability of its $z(\underline{x})$ coefficient being nonpositive. This is a relatively simple problem in that the optimality probabilities (and the cumulative distribution of max $z(\underline{x})$) can be calculated without need of [17]. We also know there are no other feasible bases since $\sum_{t=1}^{5} P_{k_t} = 1$. In accordance with the theorems of Chapter III, the polyhedral subspaces, which in this example problem are planar areas, can be depicted on the given $D[f(\underline{c})]$ as follows:



To satisfy the problem stated by this thesis, the solution technique to be developed would first generate bases 2 and 5, since they have the greatest probability of being optimal. Since larger values of P_k can presumably be obtained from the subspaces having the larger $D[f(\underline{c})]$ regions, we first determine that basis for which the subspace angle is a maximum. The measurement technique is the scalar product wherein the hyperplanes (lines) are represented as vectors. For the two variable problem, we are concerned with one angle θ_i for each subspace.

Let $e_1 = \binom{1}{0}$ and $e_2 = \binom{0}{1}$; the five vectors for the example problem are then stated as e_1, $2e_1+e_2$, e_1-e_2, e_1+2e_2, and e_2. The θ_i are calculated to be

$$\theta_2 = \theta_5 = \cos^{-1} 2/\sqrt{5} \approx 26.5°$$

$$\theta_3 = \theta_4 = \cos^{-1} 3/\sqrt{10} \approx 18.5°$$

And, at least for this particular PLP problem, the larger angles do correspond to the larger P_k. To see what factors control such an outcome, consider the same basic problem, only this time, let the density functions be changed so they are no longer identical. To get a pronounced effect, let $f(c_1) = 1/2e^{-1/2c_1}$, while the density function of c_2 remains $f(c_2) = 1/10e^{-1/10c_2}$. The feasible bases remain the same and therefore the subspaces are also unchanged. But the $\{P_{k_t}\}$ is altered significantly:

$$P_{k_2} = \frac{1}{11},\ P_{k_3} = \frac{5}{66},\ P_{k_4} = \frac{5}{42},\ \text{and } P_{k_5} = \frac{5}{7};\quad \sum_{t=2}^{5} P_{k_t} = 1.$$

Thus, we see that the largest θ_i (θ_2 and θ_5) no longer correspond to the largest P_k (P_{k_4} and P_{k_5}). The intervals of integration still have their "fullest extent" for P_{k_2} and P_{k_5}, but the new density function $f(c_1)$ now has more probability concentrated for small values of c_1 than it did previously. (Note: All probabilities are calculated assuming the random variables, c_1 and c_2, are independent.)

Combining dissimilar density functions is not the only way this same effect is produced. Changing the $D[f(\underline{c})]$ can create the same shift in optimal probabilities. To see this, we change the domain $D(c_1)$ from $\{c_1 \in R \mid c_1 \geq 0\}$ to $\{c_1 \in R \mid 0 \leq c_1 \leq 10\}$, so the density function becomes $f(c_1) = 0.158e^{-1/10c_1}$. The probability that basis 2 is optimal (P_{k_2}) now decreases from 0.333 ot 0.185, and again, P_{k_4} is greater than P_{k_2} even though θ_2 is greater than θ_4. In fact, by restricting c_1 to $\{c_1 \in R \mid 0 \leq c_1 \leq 5\}$, for which $f(c_1) = 0.415e^{-1/10c_1}$, the probability that basis 2 is optimal is zero.

CHAPTER III

DEVELOPMENT OF BASIC THEORY

As previously defined, assume $S_r = \left\{\underline{c} \mid \{\underline{c}_{B_r} B_r^{-1} A_r, -\underline{c}_{r'}\} \geq \underline{0}\right\}$ and $S_s = \left\{\underline{c} \mid \{\underline{c}_{B_s} B_s^{-1} A_s, -\underline{c}_{s'}\} \geq \underline{0}\right\}$ exist as two subspaces associated with feasible bases r and s. Then, providing S_r and S_s are nonempty, the basic solutions represented by $\underline{x}_r$ and $\underline{x}_s$ will have a positive probability of being optimal. Investigation of several pertinent areas is necessary prior to proving existence, nonemptiness, and that $S_r \cap S_s = \phi$ for all interior points.

An <u>unbounded feasible region</u> may exist for a specific problem, but depending on the $D[f(\underline{c})]$, the optimal value for z(x) may still be finite. Recall that the attitude numbers of the objective function hyperplane determine if z(x) is bounded in the event the solution space is not.

<u>Theorem 1:</u> If z(x) is unbounded for some particular $\underline{c}$, z(x) need not be unbounded for all $\underline{c}$.

<u>Proof</u>: Suppose z(x) is unbounded for a selection of c_j's forming the vector $\underline{c}$. Then associated with a non-basic x_j is a z(x) coefficient $\underline{c}_B B^{-1} P_j - c_j < 0$ and the column $B^{-1}P_j \leq \underline{0}$. Let there be a change in $\underline{c}$, changing $\underline{c}$ to $\underline{c}'$. Then the coefficient $\underline{c}'_B B^{-1} P_j - c'_j$ may become

nonnegative, while the column $B^{-1}P_j$ remains nonpositive since its value does not depend on $\underline{c}$. Therefore, if the solution space is unbounded z(x) need not be for all values of $\underline{c}$. q.e.d.

The treatment for this case is straightforward. If for any feasible basis t, $B_t^{-1}P_j \leq \underline{0}$ for some j then the probability of $\underline{c}_{B_t} B_t^{-1}P_j - c_j$ being nonpositive must be calculated. If $P[(\underline{c}_{B_t} B_t^{-1}P_j - c_j) \leq 0] > 0$, then the associated value of z(x) would be unbounded. However, if the probability equals zero, the constraints on the domain of $f(\underline{c})$ have intervened and the iteration of feasible bases may continue.

Degeneracy for deterministic LP creates no special difficulties since the optimal value of z(x) remains constant even though the "optimal" basis is not unique. However, in PLP uniqueness must exist else the exclusiveness of $S_r \cap S_s = \phi$ would not occur. Thus, if more than m constraints pass through the same point, not all elements of $B^{-1}\underline{b}$ will be nonzero. To prevent this, some of the b_i's, $(b_{i_1}, b_{i_2}, \ldots, b_{i_k})$, k < m, must be changed by arbitrarily small numbers $(\varepsilon_{i_1}, \varepsilon_{i_2}, \ldots, \varepsilon_{i_k})$, so that each optimal basis t will yield $\underline{x}_{B_t} \underline{\geq 0}$. Proof of the sufficiency of this technique is obvious. Since the values of the $x_j \in \underline{x}_{B_t}$ depend directly on the magnitude of the $b_i \in \underline{b}$, changing some b_i values will provide a $B_t^{-1}\underline{b}'$ with nonzero values for $\underline{x}_{B_t}$.

Alternative optimal solutions also prevent the desired uniqueness of the feasible bases. However, unlike degeneracy, there is no convenient technique for preventing such occurrence. For a given selection of c_j's which form the vector $\underline{c}$, if z(x) is then parallel to an active constraint, the basis may be changed without affecting the optimal value of z(x). Thus, the subspaces defined by S_r and S_s would both contain such points $\underline{c}$. Since the c_j's are random variables, it is not possible to modify their values by arbitrarily small numbers as in the degenerate case. As shown by the following theorem, the location of such points relative to the subspaces is most fortunate.

<u>Theorem 2</u>: If for feasible basis r, a particular selection of $\underline{c}$ results in an alternative optimal z(x) over feasible basis s, then such points $\underline{c} \subset S_r \cap S_s$ are confined to the common boundaries of the subspaces defined by S_r and S_s.

<u>Proof</u>: Assume a particular value for $\underline{c}$ which yields identical values for z(x) over feasible bases r and s. Since the bases are distinct, one or more $x_j \in \underline{x}_{B_r}$ are distinct from those in $\underline{x}_{B_s}$. Then

$$z_r(x) = \underline{c}_{B_r} \underline{x}_{B_r} = \underline{c}_{B_s} \underline{x}_{B_s} = z_s(x)$$

so that

$$\underline{c}_{B_r} B_r^{-1} \underline{b} = \underline{c}_{B_s} B_s^{-1} b \ .$$

Consider the change in z(x) when bases r and s are adjacent and non-basic variable x_j is selected for entry:

$\underline{c}_{B_s} B_s^{-1}\underline{b} = \underline{c}_{B_r} B_r^{-1}\underline{b} - \delta(\underline{c}_{B_r} B_r^{-1}P_j - c_j)$, where $\delta \in R^+$ and chosen to assure feasibility.

So only if there exists some nonbasic x_j for which $\underline{c}_{B_r} B_r^{-1}P_j - c_j = 0$ would the variable x_j be able to enter the basis, thereby assuring that the condition $\underline{c}_{B_r} B_r^{-1}\underline{b} = \underline{c}_{B_s} B_s^{-1}\underline{b}$ is satisfied.

Therefore, for an assumed value of $\underline{c}$, if there exists in one basis one or more nonbasic x_j for which $c_B B^{-1}P_j - c_j = 0$, then there exists a different basis having the same $z(x)$ value, and $\{\underline{c} | \underline{c}_{B_k} B^{-1}A_{k'} - \underline{c}_{k'} \geq \underline{0}\}$ will include only those points $\underline{c}$ located on the boundaries. q.e.d. (Because these points have probability measure zero, their inclusion in more than one subspace is of no pertinent consequence.)

The two preceding theorems, coupled with the prescribed treatment for degeneracy, enable the proof of the exclusiveness theorem.

Theorem 3: If S_r and S_s represent the subspaces associated with feasible bases r and s, then $S_r \cap S_s = \phi$ for all interior points $\underline{c}$.

Proof: Assume the solution space is bounded over all feasible $\underline{x}$. The proof is by contradiction: assume for a particular value of $\underline{c} \in S_r \cap S_s$, that the optimality conditions (linear inequalities) for both S_r and S_s are satisfied, so that $z_r(x) = z_s(x)$. But by Theorem 2, this equality implies alternative optimal solutions, and $\underline{c}$ would be simply a common boundary point in that event.

Let $\bar{\underline{x}}_r$ be an m+n component vector of the specific values of $\underline{x}$ associated with basis r. And, similarly define $\bar{\underline{x}}_s$ for basis s. Then, $z_r(x) = z_s(x)$ if and only if $\underline{c}' \bar{\underline{x}}_r = \underline{c}' \bar{\underline{x}}_s$. This requires that $\bar{\underline{x}}_r = \bar{\underline{x}}_s$ (except for alternative optimals), which is not possible since r and s are distinct bases, and thus, $\underline{x}_{B_r}$ and $\underline{x}_{B_s}$ contain different nonzero components.

For $\underline{c} \varepsilon S_r \cap S_s$, a third possibility is that $\bar{\underline{x}}_r$ and $\bar{\underline{x}}_s$ each contain <m nonzero elements, i.e., $\underline{x}_{B_r}$ and $\underline{x}_{B_s}$ each contain at least one zero component. But this is the degenerate case which is prevented by changing $\underline{b}$ as previously shown.

The only remaining alternative is that $\underline{x}_{B_r}$ and $\underline{x}_{B_s}$ do not represent extreme points, but rather a coincident convex point within the feasible region. The proof that the optimum must occur at an extreme point for random c_j's is given in [47, p. 7].

Therefore, since $\underline{c}$ is arbitrary but fixed, and since r and s are arbitrary feasible bases, the original assumption that $\underline{c} \varepsilon S_r \cap S_s$ is contradicted except in the case of boundary points; thus, $S_r \cap S_s = \phi$ for all interior points. q.e.d.

An equivalent statement of the preceding result is found in [5], and utilizing the notation of this thesis, it would be expressed as $P[\{\underline{c}|\underline{c} \in S_r \cap S_s\}] = 0$. This statement is correct since the set of boundary points in common has zero content. The exclusiveness of $S_r \cap S_s$ proved in Theorem 3 tacitly assumed that dim $\underline{c} = n$. The following corollary can also be stated:

<u>Corollary 3.1</u>: Let $S_I = \{\underline{c}|\underline{c} \in S_r \cap S_s\}$ where dim $\underline{c} = n$; then all interior points $\subset S_I$ will have dimension $< n$ or S_I will be empty.

<u>Proof</u>: By Theorem 3, $S_r \cap S_s = \phi$ for R^n space, but $S_r \cap S_s$ may be nonempty for $R^{<n}$. Therefore, since $S_r \cap S_s$ cannot contain an n-cube, $S_I = \phi$ or constitutes a set of zero content, that is, dim $(S_r \cap S_s) < n$. q.e.d.

A more useful result of the exclusiveness theorem is provided by the next corollary:

<u>Corollary 3.2</u>: Since the convex subspaces associated with the unique feasible bases are exclusive, the P[an optimal basis exists] $= \sum_{t=1}^{q} P_{k_t}$.

<u>Proof</u>: Define O_t as the event the $t^{\underline{th}}$ basis is optimal, so that $P_{k_t} = P(O_t)$. The probability of at least one optimal basis, in accordance with existing laws of probability, is given as

$$P[\bigcup_{t=1}^{q} O_t] \leq \sum_{t=1}^{q} P(O_t) \qquad [21, p. 7].$$

By assumption (Theorem 3) the subspaces over t are exclusive so

$$P[\bigcap_{t=1}^{q} O_t = \phi] = 1,$$

and therefore, the equality holds and we have

$$P[\text{an optimal basis exists}] = P[\bigcup_{t=1}^{q} O_t] = \sum_{t=1}^{q} P_{k_t}. \quad \text{q.e.d.}$$

The events of optimal bases are exhaustive since the number of possible feasible bases is finite and will occur. Also, the polyhedral subspaces in R^n have contiguous locations over $D[f(\underline{c})]$. Thus, $\sum_{t=1}^{q} P_{k_t} = 1 - P[\text{no basis is optimal}] = 1 - \overline{P}$. The quantity $\overline{P}$ could be positive for a specific problem, due to either the restrictions on $D[f(c)]$ or the lack of any feasible bases. Since such probability is primarily fiducial and would always be zero for a practical problem, no extended treatment of $\overline{P}$ will be undertaken. In order that $\sum_{t=1}^{q} P_{k_t} = 1$, the summation is restricted to include only those P_k associated with feasible bases. This requirement is due to the following theorem:

<u>Theorem 4</u>: If bases r and s are feasible and infeasible, respectively, then $\{\underline{c} | \underline{c} \in S_r \cap S_s\}$ may be nonempty relative to interior points.

<u>Proof</u>: By example (see pp. 13-15).

Let S_t be the set of points $\underline{c}$ contained in the polyhedral subspace associated with basis t, so that $S_t = \left\{\underline{c} | (\underline{c}_{B_t} B_t^{-1} A_t, -\underline{c}_t,) \geq \underline{0}\right\}$. Define the probability subspace for basis t to be degenerate (does not exist) if S_t is empty or contains only boundary points. The fact that S_t may be an open set due to infinite domains on c_j does not constitute degeneracy, and we consider those problems having empty S_t due to a prohibitive domain $D[f(\underline{c})]$ as special cases. For an S_t containing only boundary points, the polyhedral subspace would be defined by at least two linearly dependent inequalities (of opposite signs) $\subseteq \{\underline{c}_{B_t} B_t^{-1} A_{t'}, -\underline{c}_{t'} \geq \underline{0}\}$. Thus, to prove the existence of the subspaces for any basis t, we must show that the defining hyperplanes in R^n are linearly independent and that the resulting subspace is nonempty. The following existence theorem can now be stated:

<u>Theorem 5</u>: If the domain $D(c_j) = R$, $j=1, \ldots, n$, $\exists$ a nondegenerate polyhedral subspace $\forall$ basis.

<u>Proof</u>: (Part I, Linear Independence): Let the set of hyperplanes for the current basis t be given as $H_t = \{\underline{c}_{B_t} B_t^{-1} A_{t'}, -\underline{c}_{t'} = \underline{0} \mid t=1,2,\ldots,h\}$. It must be shown that each H_t is a linearly independent set, and then, since all $c_j \subset \{\underline{c}_{B_t} \cup \underline{c}_{t'}\}$, we know that R^n will be spanned for any basis t. Let $\{c_j \mid j=1,\ldots,n\}$ be regarded as unit basis vectors, a maximal linearly independent set in R^n.

For any basis t, let the mxn matrix $A_{t'}$ and the corresponding 1xn row vector $\underline{c}$ be ordered, $\ni$ the initial columns $\subset A = \{P_j \mid j$ represents a nonbasic, nonslack $x_j\}$, so that the initial elements $\subset \underline{c}_{t'}$ will be the c_j coefficients of the same x_j. Then, for a "starting solution," $A_{1'} = (P_1, P_2, \ldots, P_n)$ and $\underline{c}_{1'} = (c_1, c_2, \ldots, c_n)$, and if m=n (for a basis t containing all nonslacks), $A_{t'} = I_{mxm}$ and $\underline{c}_{t'}$ would be a row vector of zeroes.

In general, let j_i and s_i denote nonslack and slack variables, respectively. Then H_t will be given as

$$\underline{c}_{B_t} B_t^{-1}(P_{j_1}, P_{j_2}, \ldots, P_{j_k}, I_{s_{j_{k+1}}}, \ldots, I_{s_n}) -$$
$$(c_{j_1}, c_{j_2}, \ldots, c_{j_k}, 0_{s_{j_{k+1}}}, \ldots, 0_{s_n}),$$

where P_{j_i} ≡ mxl column vector associated with nonbasic, nonslack x_j,

I_{s_i} ≡ mxl identity vector associated with nonbasic, slack x_j,

and $\sum_i 1 = n$.

Since $\{c_{j_i} \mid i=1,2,\ldots,k\} \cap \{c_j \mid c_j \in \underline{c}_{B_t}\} = \phi$, the hyperplanes represented by $\{\underline{c}_{B_t} B_t^{-1} P_{j_i} - c_{j_i} = 0 \mid i=1,2,\ldots,k\}$ will be linearly independent due to the exclusive c_j's. Thus, we need to investigate only those n-k hyperplanes given as $\{\underline{c}_{B_t} B_t^{-1} I_{s_i} = 0 \mid i=k+1,k+2,\ldots,n\}$, all of which have identical c_j's since $\underline{c}_t$, consists of zero elements for $i > k$.

We know that the m columns of matrix B_t are linearly independent $\forall$ selected basis. Also since B_t and $B_t B_t^{-1}$ are nonsingular, so is B_t^{-1} nonsingular [18, p. 90]. Clearly, all hyperplanes $\subset \{\underline{c}_{B_t} B_t^{-1} I_{s_i} \mid i=k+1,k+2,\ldots,n\}$ will have c_j coefficients provided by distinct columns of B_t^{-1}, which is sufficient for linear independence if $\underline{c}_{B_t}$ is composed of all nonzero elements. Difficulty is suspected if $\underline{x}_{B_t}$ contains ≥ 1 and $<m-1$ slack variables. To see this, consider the following example for $n = m = 3$ and let $\underline{x}_B = (x_1,x_4,x_3)'$, so that $\underline{c}_B = (c_1,0,c_3)$. Then

$$\underline{c}_B B^{-1}A-\underline{c} = (c_1,0,c_3)\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}\begin{pmatrix} a_{12} & 0 & 0 \\ a_{22} & 1 & 0 \\ a_{32} & 0 & 1 \end{pmatrix} - (c_2,0,0),$$

where the b_{ij} matrix denotes B^{-1} and the matrix operations yield two hyperplanes with the same c_j, $b_{12}c_1 + b_{32}c_3$ and $b_{13}c_1 + b_{33}c_3$. Now if $b_{12} = -b_{13}$ and $b_{32} = -b_{33}$, the polyhedral subspace would have zero content (consist only of boundary points). The opposite signs may occur, but due to the nonsingularity of B, given as

$$\begin{pmatrix} a_{11} & 1 & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{pmatrix}, \quad | b_{12} | \neq | b_{13} | \text{ and } | b_{32} | \neq | b_{33} |.$$

Checking; $|b_{12}| = |b_{13}|$ implies $a_{33} = a_{23}$, and $|b_{32}| = |b_{33}|$ implies $a_{31} = a_{21}$, which is prevented since the last two rows of B cannot be proportional. Thus, for n=3, we are assured of linear independence for $\{H_t\}$ due to the nonsingularity of the B matrix.

This same result holds for n=4 (where $\underline{x}_{B_t}$ contains ≥ 1 and <3 slack variables), but the calculations are tedious and prohibit this method of proof for the general case. So for n>4, the argument is similar to that of Theorem 8 (see p. 47). Assuming the extreme point $(\underline{x}^t)$

exists, it is defined by a particular set of the original linearly independent constraints. These constraints act as bounds on the magnitude of riS_t if mapped onto the $\underline{c}$ space. When certain conditions and operations are satisfied (see pp. 48-50), these original constraints can define a space equivalent to S_t. Since these sets of constraints are independent over t for arbitrary n, the supporting hyperplanes for $\{S_t\}$ are also. Therefore, the H_t are linearly independent and R^n is spanned for any basis t.

(Part II, Nonemptiness): We know that S_t is convex and, from Part I, that the hyperplanes H_t are linearly independent. Let $\{\underline{c}_i \mid i=1,\ldots,n\}$ be selected points satisfying $H_t \ni \{\underline{c}_i\}$ constitutes a basis for S_t. Let $\bar{\underline{c}} = \sum_{i=0}^{n} \lambda_i \underline{c}_i$, where $\sum_{i=0}^{n} \lambda_i = 1$ and $\lambda_i > 0$ for all i. To prove that S_t is nonempty, we must now show that $\bar{\underline{c}}$ lies in the interior of S_t. (A similar proof is required by the exercises on p. 17 of B. Grünbaum, Convex Polytopes, Wiley, 1967).

Since S_t is a subspace by definition, let $\underline{c}_0 = 0$ and the $\underline{c}_i$ for $i \geq 1$ will then constitute the basis. Hence every point $\underline{c}$ in S_t is a unique linear combination of the $\underline{c}_i$'s. Suppose $\bar{\underline{c}}$ is not in int S_t (interior of S_t); then there is a sequence $\{z_k\}$ $(k=1,2,\ldots)$ of points not interior to $S_t \ni \lim_{k\to\infty} z_k = \bar{\underline{c}}$. Expressing

$z_k = \sum_{i=1}^{n} \alpha_i^k \underline{c}_i$ implies that $\{\alpha_i^k\}_k \to \lambda_i$ for each $1 \le i \le n$. Since z_k is not in S_t, some $\alpha_i^k = 0$ for infinitely many k. This implies the contradiction that some $\lambda_i = 0$. Thus, $\underline{\bar{c}} \in \text{int } S_t$.

Therefore, since the H_t are linearly independent and since S_t is nonempty, we conclude that a nondegenerate polyhedral subspace exists $\forall$ basis. q.e.d.

CHAPTER IV

PROBABILITY SPACE SOLUTION METHOD

The intent of this chapter is to develop a solution technique when the polyhedral subspaces are defined by the known linear inequalities associated with each feasible basis. A polyhedral convex set is defined as the intersection of a finite collection of closed half-spaces. (In the event this set is bounded, the points contained in the intersection of half-spaces would define a convex polytope [24].) Since the half-spaces we are concerned with are the set of linear inequalities $\{\underline{c}_{B_t} B_t^{-1} A_{t'} - \underline{c}_{t'} \geq \underline{0}\}$, the term polyhedral subspace is most appropriate. And since there is a distinct set of linear inequalities associated with each basis t, the probability space denoted by R^n is a finite collection of polyhedral subspaces.

The subspaces will be unbounded provided there are no domain $D[f(\underline{c})]$ restrictions. This will be the general framework for the following development since the typical joint probability density functions, e.g., the multi-variate normal or exponential, have either no bounds or simply a nonnegativity requirement on the random variable domains. (In the case of the normal distribution, the domain is R^n, while the domain of the exponential is restricted to R^{n+}.) Since the vertex of the polyhedral subspace is the origin in all cases, the nonnegative domains of the exponential density function could affect only the geometry of the subspace, not the unboundedness.

In accordance with the concepts stated in Chapter II, the objective is to obtain that subspace having the largest relative interior, denoted by riS_t, and then determine the class of density functions for which this particular subspace also has the largest probability (P_{k_t}) of being optimal. (The term relative interior is used since a convex set does not have an interior in the sense of the whole metric space, but rather an interior relative to its convex hull.) Thus, the objective is to rank the various P_k by order of magnitude without requiring any probability calculations.

Formulation of Subspace Hyperspheres

As before, let H_t denote the set of supporting hyperplanes for riS_t, the relative interior of the polyhedral subspace associated with feasible basis t, where $H_t = \{\underline{c}_{B_t} B_t^{-1} A_{t'} - \underline{c}_{t'} = \underline{0}\}$ and $S_t = \{\underline{c} \mid \underline{c}_{B_t} B_t^{-1} A_{t'} - \underline{c}_{t'} \geq \underline{0}\}$. The set H_t consists of n homogeneous equations in the n random variables. Let the c_j coefficients be denoted by b_{ij}; then the linear inequalities may be expressed as $\sum_{j=1}^{n} b_{ij} c_j \geq 0$, i=1,...,n. Define $\underline{b}_i$ as the 1xn row vector $(b_{i1}, b_{i2}, \ldots, b_{in})$, so that in terms of the scalar product [30], the linear inequalities become $\langle \underline{c}, \underline{b}_i \rangle \geq 0$, i = 1, ..., n. The subspace thus defined is an area for $n = 2$, volume for $n = 3$, and hypervolume for $n > 3$.

Assuming $c_j \varepsilon R$, $j = 1, \ldots, n$, the subspaces have infinite relative interiors; consequently, a relative volume measure between subspaces must be defined. The unit sphere with center at the origin is given by $\sum_{j=1}^{n} c_j^2 = 1$. Let A_n denote the surface area of the sphere and A_{s_t} denote the sphere surface area for the now bounded S_t. Then the relative volume measure for S_t is defined as $\frac{A_{s_t}}{A_n}$ for a particular n and basis t. This ratio will be referred to as relative portion of R^n for a particular subspace and denoted by rpR^n. Thus, for example, if $rpR^n = 1$ for basis t, then $riS_t = R^n$.

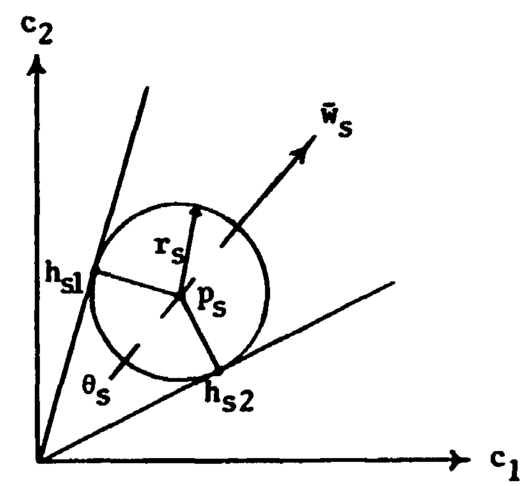
To be noted is that the quantity rpR^n is theoretical since its value cannot be calculated. A_{s_t} is analogous to the solid angle and would provide a comparative measure over t of the various rpR^n, but again, it cannot be determined since the differential element (dA), say for n=3, would be triangular rather than four-sided as required for the solid angle. In what follows, it is convenient to consider riS_t as synonymous with rpR^n for any basis t.

As previously stated then, the basis t having associated subspace with largest riS_t also defines the largest rpR^n. The approximate measure of A_{s_t} to be used is r_t, the radius of the hypersphere contained by the polyhedral subspace relative to the vertex. Then the S_t with largest r_t will be the subspace having largest riS_t and the feasible bases may be ranked accordingly. For comparative purposes, the various r_t must be determined at the same fixed distance from the vertex for all subspaces.

Theorem 6: Under the conditions stated by Part II of the proof, then given $\{S_t | t=1,\ldots,q\}$ such that $\exists \{r_t\}$, if the radius $r_s \geq \sup\{r_t | t=1,\ldots,q,\ t\neq s\}$, then $riS_s \geq \sup\{riS_t | t=1,\ldots,q,\ t\neq s\}$.

Proof: Geometrically inscribe a closed hypersphere in polyhedral subspace S_s. Let $\bar{w}_s$ be the vector from the vertex passing through the center of the sphere. Let p_s be the point on $\bar{w}_s$ one unit from the vertex and let $H_{s'} \varepsilon H_s$. Then $r_s = ||H_{s'} - p_s||$, the $\perp$ distance from the point p_s to the points of tangency with H_s. Let the tangent points be denoted by h_{si}, $i = 1, \ldots, n$. The proof is in two parts.

Part I (n=2): Let $\{\theta_i | i = 1, \ldots, q\}$ be the vertex angles formed by the two hyperplanes (lines) associated with each basis. The hypersphere for n=2 is simply a circle as shown below:

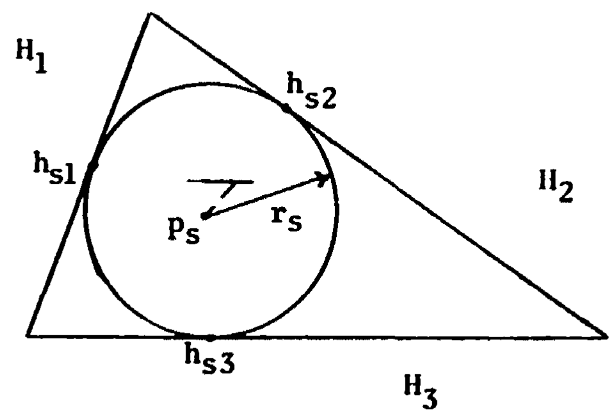


The riS_s is directly proportional to the magnitude of θ_s for each basis s. The vector $\bar{w}_s$ bisects θ_s and the radius of the hypersphere is determined by $r_s = ||h_{s1,2} - P_s||$. For the 90° triangle, $\overline{0p_s h_{s1}}$, r_s is the length of the side opposite the vertex angle $\theta_s/2$. Thus any change in the magnitude of θ_s is reflected by corresponding change in r_s, and since riS_s is proportional to θ_s, the larger r_s implies the larger riS_s. Therefore, if $r_s \geq \sup\{r_t | t=1,\ldots,q,\ t \neq s\}$, then $riS_s \geq \sup\{riS_t | t=1,\ldots,q,\ t \neq s\}$. q.e.d.

Part II (n > 2): As previously defined, $\{\underline{b}_i | i=1,\ldots,n\}$ are the normal vectors to the associated set of supporting hyperplanes for the subspace S_s. Let $\{\alpha_{ii'} \mid i \neq i'$ and $ii' = i'i\}$ be the $\binom{n}{2}$ angles determined by the scalar products of the normal vectors for basis s. The following three cases may arise: (1) the $\binom{n}{2}$ angles remain constant over t; (2) any $\binom{n}{2} - 1$ of $\{\alpha_{ii'}\}$ remain constant over t; or (3) less than $\binom{n}{2} - 1$ of $\{\alpha_{ii'}\}$ remain constant. Note that rotation of the subspaces in R^n is permitted by these cases; the angular requirements only affect the geometry of $\{S_t\}$.

The first case is trivial since all S_t would have equal riS_t. Due to the linear independence of each H_t, a convenient check exists: for a particular t, determine $\{\alpha_{ii'}\}$ and then, if any n-1 of these angles repeat over t where either the i or i' value appears in all n-1 angles for each t, the riS_t remains constant.

The second case permits only one of $\{\alpha_{ii'}\}$ to vary over t, where the distinct $\alpha_{ii'}$ in each set has arbitrary i,i'. Consider a cross-section passing thru the hypersphere, $\{h_{si}\}$, and bounded by the intersection with H_s. It could appear as follows for n=3:



When only one $\alpha_{ii'}$ varies, the cross-section vertex angles remain constant. Since the areas of the cross-sections over t are a direct measure of $\{riS_t\}$, and since the magnitude of r_s changes directly with the cross-section area for fixed vertex angles, if $r_s \geq \sup\{r_t\}$, then $riS_s \geq \sup\{riS_t\}$.

For the third case, when less than $\binom{n}{2} - 1$ of the $\alpha_{ii'}$ are constant, we cannot show in general that increasing r_t implies increasing riS_t over t. The changes now permitted in the geometry of the subspaces make attempts at quantification most difficult. It must be assumed by conjecture that the theorem as stated is true in this case so that if

$r_s \geq \sup\{r_t \mid t = 1, \ldots, q, t \neq s\}$ then $riS_s \geq \sup\{riS_t \mid t = 1, \ldots, q, t \neq s\}$.

Applicable Joint Probability Density Functions

Since the polyhedral subspaces can be ranked according to riS_t, $t=1,\ldots,q$, we now determine under what conditions the largest riS_t corresponds to the basis t having the largest probability of being optimal. The domain $D[f(\underline{c})]$ is identically $\{S_t\}$, so that the larger riS_t implies greater domain for the values of the random variables.

The probability that the feasible basis t is optimal is given by

$$P_{k_t} = \iint_{S_t} \cdots \int f(\underline{c})dc_1 \ldots dc_n$$

where the product $dc_1 \ldots dc_n$ is a differential element of volume and S_t defines the subspace over which dV is accumulated. Since the subspaces are contiguously located with vertices at the origin in R^n, the subspace with greater rpR^n would always have larger P_k only if the product $f(\underline{c})dc_1 \ldots dc_n$ is constant for a given distance from the origin. To illustrate this consider the case for n=3, and since the subspaces are unbounded, transform the problem into the equivalent one using spherical coordinates. Then the probability that $\underline{c} \subset S_t$ is given alternatively by

$$P_{k_t} = \iiint_{vol_t} f(r,\theta,\phi) r^2 \sin\theta d\theta d\phi dr \ ,$$

where $r^2\sin\theta d\theta d\phi dr$ is the differential volume in spherical coordinates. We see that for the larger P_{k_t} to correspond to the subspace having larger vol_t, the functional values of $f(r,\theta,\phi)$ must be independent of the angles (θ,ϕ) and thus a function of r only. Then the subspace with largest integration intervals on (θ,ϕ) will yield the largest P_k. For n>3, wherein an additional angle represents each additional dimension, the argument is analogous. The following theorem can now be stated:

<u>Theorem 7</u>: For unbounded polyhedral subspaces, if $f(\underline{c})$ is independent of the content of S_t, which implies $f(\underline{c})$ constant for a fixed distance from the origin, then largest P_{k_t} corresponds to that basis t having largest riS_t.

<u>Proof</u>: Assume a joint density function of the form $f(\underline{c}) = k_1 e^{-k_2(c_1^2 + c_2^2 + \ldots + c_n^2)^p}$, where k_1, k_2 are nonzero constants, $p \in R^+$, and $c_j \in R$, then $f(\underline{c})$ is spherically symmetric since it can be expressed as $f(r,\theta,\phi, \ldots) = f(r) = k_1 e^{-k_2(r^2)^p}$. Thus, since the density functional values depend only on r, $dV = r^2 \sin\theta d\theta d\phi dr$ and $dP_{k_t} = f(r) r^2 \sin\theta d\theta d\phi dr$, where the values of the domain D(r) are identical over all bases t. The differential solid angle is given by $d\Omega = \frac{dA}{r^2} = \sin\theta d\theta d\phi$. This implies that the largest riS_t will be that subspace having largest domain $D_t(\theta,\phi)$, since the difference in values of the various P_k is dependent only on the differences in their respective domains $D(\theta,\phi)$. Therefore, the largest P_{k_t} corresponds to

largest riS_t for unbounded subspaces. q.e.d.

Notice that the joint density function in Theorem 7 cannot be allowed utilizing Zinn's technique [47], and even though it is theoretically covered by Ewbank's solution method [17], it is computationally infeasible (see p. 54). As previously discussed, Zinn's technique requires probability calculations in order to generate the bases in a contiguous manner. Variable limits are required on some or all of the integrals at each iteration, and for the multivariate normal distribution, this prevents exact evaluation. The same difficulty exists using Ewbank's method, even though the Jacobian transformation enables constant limits on the integrals. The resulting integrand then contains not only squared terms but also cross product terms in most cases. Numerical integration techniques would be required for evaluation, but since infinite limits are involved, the usefulness of any such approximation would be highly questionable. Thus, this dissertation enables a new class of PLP problems to be treated.

As indicated by the preceding discussion, Theorem 7 (for p=1) holds for the symmetrical multivariate normal distribution wherein the c_j are independent. Thus, we have a well quantified joint density function for which the P_{k_t} may always be ranked according to magnitude without making any probability calculations. Also the standard exponential density function, where the $c_j \geq 0$, could easily be adapted to take advantage of Theorem 7, since the basic difference between $f(\underline{c}) = e^{-(c_1 + c_2 + \ldots + c_n)}$ and $f(\underline{c}) = \frac{1}{(\pi/2)^{n/2}} e^{-(c_1^2 + c_2^2 + \ldots + c_n^2)}$ is only the rate of decay for increasing c_j values. Consequently, the requirements of Theorem 7 are not viewed as restrictive providing the random variables have no upper bounds. So in lieu of extensive probability computations we can use the convenient geometrical technique of determining the embedded hypersphere radius r_t.

Solution Method I

Prior to stating the general algorithm, the case for n=2 is treated separately. The supporting hyperplanes for a given subspace (basis t) become $H_t = \{<\underline{c},\underline{b}_1> \ , \ <\underline{c},\underline{b}_2> = 0\}$. The rpR^2 between subspaces is simply measured by the angle θ_t formed by the two linear inequalities defining each S_t, so that larger θ_t corresponds to larger riS_t. Denote the vector representations of the two lines (H_t) as $\underline{u}_t$ and $\underline{v}_t$, where the magnitude is arbitrary and thus $\underline{u}_t$ and $\underline{v}_t$ may be regarded as extreme rays (half-lines emanating from the origin). We then calculate θ_t for each basis t by

$$\theta_t = \cos^{-1} \frac{<\underline{u}_t,\underline{v}_t>}{||\underline{u}_t|| \ ||\underline{v}_t||}$$

and rank the subspaces accordingly (see example, pp. 17-20).

For the general case, define $\bar{\underline{c}}_t$ as a particular point ε S_t and define $h_t^{\ i}$, i = 1, ..., n, as the $i^{\underline{th}}$ hyperplane ε H_t. As before, denote $\underline{b}_{i_t}$, i = 1, ..., n, as in the $i^{\underline{th}}$ row vector of the c_j coefficients. Suppressing the subscript t, define $d_{\bar{\underline{c}}h}i$ as the distance from the point $\bar{\underline{c}}$ to the hyperplane h^i. This distance is given by

$$d_{\underline{\bar{c}}h^i} = \frac{\langle \underline{\bar{c}}, \underline{b}_i \rangle}{||\underline{b}_i||}, \quad \text{where } ||\underline{b}_i|| = \sqrt{\sum_{j=1}^{n} b_{ij}^2} \tag{8}$$

Note that $\{\underline{b}_i \mid i = 1, \ldots, n\}$ are the respective normal vectors for $\{h^i \mid i = 1, \ldots, n\}$. Define $\underline{\bar{c}}_e$ as the point equidistant from all h^i and located one unit from the origin. As before, let $\bar{w}$ be the ray from the origin which represents the locus of points equidistant from all h^i, so that $\underline{\bar{c}}_e \in \bar{w}$. Then the desired radius of the unit hypersphere is the distance from $\underline{\bar{c}}_e$ to any h^i, that is, $r_t = d_{\underline{\bar{c}}_e h_t^i}$ for basis t.

Let $\bar{c}_t = (\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_n) \in S_t$ be a particular point equidistant from $\{h_t^i \mid i = 1, \ldots, n\}$ for basis t. Then for each feasible basis, the steps of the algorithm can be stated as:

<u>Step 1</u>: To determine the point $\bar{c}$, equate $\{d_{\bar{c}h^i} \mid i = 1, \ldots, n\}$ given by (8) as follows:

$$\frac{\langle \bar{c}, \underline{b}_1 \rangle}{||\underline{b}_1||} = \frac{\langle \bar{c}, \underline{b}_2 \rangle}{||\underline{b}_2||} = \cdots = \frac{\langle \bar{c}, \underline{b}_n \rangle}{||\underline{b}_n||} = y$$

This yields n equations in the unknowns $(\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_n, y)$. The number of unknowns exceeds the number of equations; however, since $\bar{c}$ is contained by the locus of points equidistant from all h^i, a particular $\bar{c}_i$ may be assigned a value consistent with the set of linear inequalities. If this choice is not obvious, a linear combination of $\{\underline{b}_i\}$ will yield a point $\in$ s, from which an arbitrary $\bar{c}_i$ may be chosen. Thus, there are n nonhomogeneous equations in n unknowns, which can be solved by numerous existing techniques. In the event a solution of this system does not exist, one or more of the n equations would be ignored, thereby causing the method to become heuristic. (Dropping an equation would correspond to removing one of the hyperplanes defining the subspace S.)

Step 2: The ray $\bar{w} = \tau(\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_n)$, so that $\bar{\underline{c}}_e$, the point one unit from the origin, is given (for $\tau=1$) by

$$\bar{\underline{c}}_e = \frac{\bar{w}}{||\bar{w}||} = \frac{1}{\left(\sum_{j=1}^{n} \bar{c}_i^{\,2}\right)^{1/2}} (\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_n)$$

Step 3: The comparative measure of riS, the radius r of the hypersphere one unit from the origin, is now calculated. Since $\bar{\underline{c}}_e$ is equidistant from all h^i, r is identical for all h^i; thus

$$r = \frac{\langle \bar{\underline{c}}_e, \underline{b}_i \rangle}{||\underline{b}_i||} \quad \text{for an arbitrary } i.$$

Step 4: Repeat steps 1, 2, and 3 for each basis t and then rank the r_t by order of magnitude. As previously indicated, this ranking will have the same order as would $\{P_{k_t}\}$, the probabilities that the various bases are optimal.

The degree of difficulty in performing the calculations required by the algorithm is very low; only the first step presents some involved algebraic operations. Programming this algorithm to handle large problems will be a relatively easy task. These same comments apply to Solution Method II presented in Chapter V.

CHAPTER V

FEASIBLE SPACE SOLUTION METHOD

This chapter develops a solution technique utilizing the supporting hyperplanes for the feasible space, rather than the hyperplanes which define the probability space. The premise is that for $P_{k_t} = \iint\limits_{S_t} \ldots \int f(\underline{c})dc_1 \ldots dc_n$, and for appropriate $f(\underline{c})$, the basis t having largest P_{k_t} will be that basis for which the intervals of integration are a maximum. Such an event would relate to that extreme point, $\underline{x}^t$, where the objective function has maximum freedom of angular rotation prior to a different extreme point becoming optimal. The angular degrees of freedom for the linear functional rotation equals n, since the number of constraint hyperplanes necessary to define an extreme point in R^n is exactly n. (The context will make it clear whether R^n refers to the probability or feasible space.)

Analogous to Solution Method I, the technique now being developed is based on the same rpR^n concept, and therefore, applies to those problems having an $f(\underline{c})$ in accordance with the requirements of Theorem 7. However, the possible exactness of Solution Method I, due to the exclusiveness of $\{S_t\}$, can only be shown for the present method for n=2. Perhaps this technique should be titled as a heuristic, but since no counterexample has been found, this technique will be referred to as Solution Method II.

The linear inequalities we are now concerned with are given by $-\sum_{j=1}^{n} a_{ij}x_j + b_i \geq 0$, $i = 1, \ldots, m$, and $x_j \geq 0$, $j = 1, \ldots, n$. It is assumed the feasible space thus defined is nonempty and closed. Since each $\underline{x}^t$ is considered independently, the intersection of n linear inequalities with vertex $\underline{x}^t$ constitutes a polyhedral space in R^n providing the domain $D(x_j) = R$ for all j. Thus, the nonnegative restrictions on the x_j are dropped, and since we are interested in the geometry of $\underline{x}^t$ and not its location, we perform a translation to the origin by setting all b_i to zero. Therefore, each set of n linear inequalities associated with each $\underline{x}^t$ may now be described as a polyhedral subspace, unbounded and containing the appropriate points $\underline{x}$. Let the set of n hyperplanes with vertex $\underline{x}^t$ be defined by $G_t = P_s \cup Q_s$, where

$P_s = \{-\sum_{j=1}^{n} a_{i_s j}x_j = 0 | s=1,\ldots,k\}$ and $Q_s = \{x_{j_s} = 0 | s=k+1,\ldots,n\}$.

Then define $F_t = \{\underline{x} | \underline{x} \in G_t \geq 0\}$, the unbounded convex subspace associated with basis t wherein the $\underline{x}$ have no requirement for being feasible.

Recalling that $S_t = \{\underline{c} | \underline{c}_{B_t} B_t^{-1} A_{t'} - \underline{c}_{t'} \geq \underline{0}\}$, consider the $\mathrm{ri}F_t$ in the same sense as the previously defined $\mathrm{ri}S_t$, that is, the measure of the relative interior being the radius of the embedded hypersphere located one unit from the vertex. The following theorem can be stated:

<u>Theorem 8</u>: Given S_t and F_t associated with feasible basis t, then $\mathrm{ri}S_t$ is inversely proportional to $\mathrm{ri}F_t$.

Proof: Let the n angular degrees of freedom about $\underline{x}^t$ for the linear functional, $z_t(\underline{c}) = [z(\underline{x}^t)|\underline{c}] = c_1x_1^t + \ldots + c_nx_n^t$, be represented by $\phi_t = \{\phi_i \mid i = 1, \ldots, n\}$, where ϕ_i is the interior angle between $z_t(\underline{c})$ and $g_i \in G_t$ and $0 \leq \phi_i < \Pi$. (The domain $D[z_t(\underline{c})]$ would be R^n for $\sum_{i=1}^{n} \phi_i = n\Pi$, since F_t would then have zero content.)

The magnitude of $\sum_{i=1}^{n} \phi_i$ for basis t is bounded by G_t, the supporting hyperplanes for F_t. Thus, if riF_t is large, the $D[z_t(\underline{c})]$ is limited since $\sum_{i=1}^{n} \phi_i$ will be restricted. In fact, if $\sum_{i=1}^{n} \phi_i = 0$, then $\underline{x}^t$ vanishes. Therefore, since $D[z_t(\underline{c})]$ is a direct measure of riS_t, the quantities riF_t and riS_t are inversely proportional. q.e.d.

An extension of the preceding theorem is necessary. Assuming we have a suitable mapping function $\sigma:R^n_{\underline{c}} \rightarrow R^n_{\underline{x}}$, so that riS_t can be measured on the same space as riF_t, S_t can be exactly stated. The angular rotation of $z(\underline{c})$ about $\underline{x}^t$ is bounded by G_t; thus, there would be no region allotted S_t contained by either F_t or its mirror image. Let $F_t' = \{\underline{x} \mid \underline{x} \in G_t \leq \underline{0}\}$, then the equivalent riS_t would be exactly equal to $\frac{1}{2}ri(\overline{F_t \cup F_{t'}})$, denoted hereafter as riS_t^e.

Since riS_t and riF_t are inversely related, it would appear a ready technique is available for ranking the various P_{k_t}. Let r_t' denote the hypersphere radius associated with F_t in the present development. Then, where decreasing r_t over feasible t in Solution Method I corresponds to decreasing P_{k_t}, the same ranking should now be obtained by arranging the values of r_t' in increasing order. This is not the case since $\{F_t\}$ are obviously not exclusive, i.e., $\bigcap_{t=1}^{q} F_t \neq \phi$. Thus, Solution Method II concentrates on ranking $\{S_t^e\}$, the set equivalent of $\{S_t\}$. The ranking of the P_{k_t} will then be in accordance with decreasing values of the r_t belonging to $\{S_t^e\}$. And as exhibited by the example in Chapter VI, there is no requirement that $\{S_t^e\}$ be necessarily exclusive.

Formulation of Extreme Point Hyperspheres

Consider the two variable problem where $\{\underline{x}^t\}$ are defined by the intersection of two constraints. Then for a particular $\underline{x}^t$, G_t will contain two hyperplanes (lines) and riF_t will be an unbounded planar subspace. If either one of the linear inequalities defining F_t is then reversed, the resulting subspace is the desired S_t^e. For this simple case, we see that $riS_t^e = riS_t$; however, for n>2, the technique to be used is more complex.

Define $\underline{a}_i^+$ as the normal vector to the $i^{\underline{th}}$ linear inequality associated with $\underline{x}^t$, so that (suppressing the t notation) $\underline{a}_N = \{\underline{a}_i \mid i=1,\ldots,n\}$. Thus $\underline{a}_i = (-a_{i_s 1}, \ldots, -a_{i_s n})$, $s=1,\ldots,k$, and $\underline{a}_i = (1_{i_s})$ for $s = k+1, \ldots, n$. Let $\{\theta_{ii'} \mid i \neq i' \text{ and } ii' = i'i\}$ be the $\binom{n}{2}$ angles formed by the normal vectors and calculated by the scalar product $\langle \underline{a}_i, \underline{a}_{i'} \rangle$. The procedure again is to reverse one of the n linear inequalities to obtain S_t^e, but the choice of which one is determined by the following sequential rules:

Rule 1: If any $<\underline{a}_i, \underline{a}_{i'}>$ yields $\theta_{ii'} = \Pi/2$, reverse the $g_i, g_{i'} \in G_t \geq 0$ of greater dimension. If no $\theta_{ii'} = \Pi/2$, select Rule 2 or 3 as appropriate.

Rule 2: If $\Sigma\theta_{ii'} > \binom{n}{2}\frac{\Pi}{2}$, reverse the $g_i, g_{i'} \in G_t \geq 0$ of greater dimension associated with largest $\theta_{ii'}$.

Rule 3: If $\Sigma\theta_{ii'} \leq \binom{n}{2}\frac{\Pi}{2}$, reverse the $g_i, g_{i'} \in G_t \geq 0$ of greater dimension associated with smallest $\theta_{ii'}$.

*Using the appropriate rule, if a tie exists regarding dimension, the choice of which inequality to reverse is arbitrary.

Using the preceding selection rules, the choice of which $g_i \in G_t \geq 0$ to reverse is determined and S_t^e is obtained. Thus, if g_s is reversed, then $S_t^e = \{\underline{x} \mid g_i \geq 0,\ i=1, \ldots s-1, s+1, \ldots, n \text{ and } -g_s \geq 0\}$.

Since S_t^e and S_t are analogous as polyhedral subspaces, and since $\dim S_t^e = \dim S_t = n$ for all feasible bases, the development of the embedded hypersphere concept is the same as before (see Ch. 4, pp. 35-37). The measure for riS_t^e will be r_t, the radius of the hypersphere with center on the locus of points equidistant from G_t, the supporting hyperplanes. Justification of this technique requires only trivial modification of Theorem 6.

Solution Method II

Since the case for n=2 involves only the comparison of t angles, it will not be stated separately (see p.43) for analogous treatment). Define $\underline{\overline{x}}_t$ as a particular point $\in S_t^e$; as previously defined let $\underline{a}_i^t$ denote the $i^{\underline{th}}$ row vector of the x_j coefficients. Let h_t^i denote the $i^{\underline{th}}$ supporting hyperplane $\in G_t \geq 0$, then suppressing the subscript t, the distance from the point $\bar{\underline{x}}$ to h^i is given by

$$d_{\underline{\bar{x}}h^i} = \frac{\langle \underline{\bar{x}}, \underline{a}_i \rangle}{||\underline{a}_i||}, \quad \text{where } ||\underline{a}_i|| = \sqrt{\sum_{j=1}^{n} a_{ij}^2} \tag{9}$$

Define $\tau\bar{x}$, $\tau \varepsilon R^+$, as the ray equidistant from all h^i, where $\bar{x} \varepsilon S_t^e = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$, and $\bar{\underline{x}}_e \varepsilon \tau\bar{x}$, $\ni \bar{\underline{x}}_e$ is located one unit from the origin. Then $r_t = d_{\bar{\underline{x}}_e h_t^i}$ for basis t and arbitrary i.

The steps of the algorithm for each basis t may be stated as follows:

Step 1: $\underline{x}_{B_t}$ implies the modified active constraints $\subset G_t \geq 0$. Reverse the $g_i \; \varepsilon \; G_t \geq 0$, where selection of the particular g_i is in accordance with the previously stated rules (see p. 50). Let G_t' denote the set of inequalities after the reversal; then $\{h^i\} = G_t' \overset{set}{=} 0$.

Step 2: To determine the ray, $\tau\bar{x}$, equate $\{d_{\bar{x}h^i} \mid i=1,\ldots,n\}$ as given by (9), where $\bar{x}_i$ is chosen to satisfy $G_t' \geq 0$. (See Step 1, p. 44.)

Step 3: Determine the point $\bar{\underline{x}}_e$ by obtaining $\bar{x}$ as a unit vector. (See Step 2, p. 45.)

Step 4: Calculate r_t, the relative measure of riS_t^e, by

$$d_{\bar{\underline{x}}_e h^i} = \frac{\langle \bar{\underline{x}}_e, \underline{a}_i \rangle}{||\underline{a}_i||},$$

where i is now arbitrary. (See Step 3, p. 45.)

Step 5: Repeat steps 1, 2, 3, and 4 for each basis t and then rank the r_t by order of magnitude. As in Solution

Method I, this ranking should yield the same ordering of $\{P_{k_t} \mid t=1,\ldots,q\}$, the probabilities the various bases are optimal.

CHAPTER VI

JACOBIAN TRANSFORMATION AND EXAMPLE PROBLEM

The two solution methods (Chapters IV and V) are now illustrated by application to an example problem where the distribution of the random variables is given by the multivariate normal density function. The results are verified by simulation to obtain estimates of the various P_{k_t}. Simulation is required since exact evaluation of the iterated integrals representing $\{P_{k_t}\}$ is not possible. Since, in general, some of the limits of integration for a particular S_t are variable, we obtain constant limits by using the Jacobian transformation of integrals (see [3, pp. 335-336]). Expressing $\sum_{j=1}^{n} b_{ij}c_j \geq 0$, $i=1,\ldots,n$, in matrix notation, we have

$$B\underline{c} = \underline{s} \geq \underline{0}$$

so that

$$\underline{c} = B^{-1}\underline{s}.$$

Recalling that $$P_{k_t} = \iint_{S_t} \cdots \int f(\underline{c})dc_1dc_2\ldots dc_n \tag{10}$$

the transformation yields

$$P_{k_t} = \iint_{\underline{s}\geq\underline{0}} \cdots \int f(B^{-1}\underline{s}) \mid J_s \mid ds_1ds_2\ldots ds_n \tag{11}$$

where

$$J_s = \det \begin{vmatrix} \frac{\partial c_1}{\partial s_1} & \cdots & \frac{\partial c_1}{\partial s_n} \\ \vdots & & \vdots \\ \frac{\partial c_n}{\partial s_1} & \cdots & \frac{\partial c_n}{\partial s_n} \end{vmatrix} = \det(B^{-1})$$

Thus, if possible, (11) would be used for each basis t for exact verification of the solution methods. Note in the following example problem that evaluation of the integral expressions is not possible. Therefore, it is typical of a class of problems which can now be solved due to the techniques of this dissertation. It is shown that the obtained rankings of the various P_k are identical using either solution method. The PLP problem is stated as:

$$\max z(x) = c_1x_1 + c_2x_2 + c_3x_3$$

$$\begin{aligned} \text{s.t.} \quad x_1 + 2x_2 + x_3 &\leq 430 \\ 3x_1 \qquad + 2x_3 &\leq 460 \\ x_1 + 4x_2 \qquad &\leq 420 \end{aligned}$$

$x_j \geq 0$; c_j random independent variables (NML;0,σ)

The joint probability density function is given by [26, p. 211] as

$$f(\underline{c}) = \frac{1}{|V|^{1/2}(2\pi)^{3/2}} e^{-1/2\, \underline{c}^t V^{-1} \underline{c}} ,$$

where V is the covariance matrix, which, for independent c_j and symmetrical $f(\underline{c})$, implies $\sigma_{11} = \sigma_{22} = \ldots = \sigma_{nn}$ and $\sigma_{ij} = 0$ for $i \neq j$. Thus, for our example problem

$$V = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{11} & 0 \\ 0 & 0 & \sigma_{11} \end{bmatrix},$$

and (10) becomes

$$P_{k_t} = \frac{1}{(2\Pi\sigma_{11})^{3/2}} \iiint_{S_t} e^{-\frac{1}{2\sigma_{11}}(c_1^2+c_2^2+c_3^2)} dc_1 dc_2 dc_3 ,$$

and (11) becomes

$$P_{k_t} = \frac{1}{(2\Pi\sigma_{11})^{3/2}} \iiint_{\underline{s}\geq\underline{0}} e^{-\frac{1}{2\sigma_{11}}\left|\sum_{i=1}^{3}(B_i^{-1}\underline{s})^2\right|} |J_s| ds_1 ds_2 ds_3$$

The pertinent data for each of the eight feasible bases are as follows where $a = \frac{1}{(2\Pi\sigma_{11})^{3/2}}$ and $b = \frac{1}{2\sigma_{11}}$:

$$\underline{x}_{B_1} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 430 \\ 460 \\ 420 \end{bmatrix} \qquad S_1 = \{\underline{c}\} \ni \begin{Bmatrix} -c_1 & & & \geq 0 \\ & -c_2 & & \geq 0 \\ & & -c_3 & \geq 0 \end{Bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -s_1 \\ -s_2 \\ -s_3 \end{bmatrix} \qquad |J_1| = 1$$

$$P_{k_1} = a \int_0^\infty \int_0^\infty \int_0^\infty e^{-b(s_1^2+s_2^2+s_3^2)} ds_1 ds_2 ds_3 \approx 0.118$$

$$\underline{x}_{B_2} = \begin{bmatrix} x_2 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 105 \\ 220 \\ 460 \end{bmatrix} \qquad S_2 = \{\underline{c}\} \ni \begin{Bmatrix} -4c_1 + c_2 & & \geq 0 \\ & - c_3 & \geq 0 \\ c_2 & & \geq 0 \end{Bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4}s_1 + \frac{1}{4}s_3 \\ s_3 \\ -s_2 \end{bmatrix} \qquad |J_2| = 1/4$$

$$P_{k_2} = \frac{a}{4}\int_0^\infty\int_0^\infty\int_0^\infty e^{-b[(-\frac{1}{4}s_1+\frac{1}{4}s_3)^2+s_3^2+s_2^2]}\,ds_1ds_2ds_3 \approx 0.162$$

$$\underline{x}_{B_3} = \begin{pmatrix} x_1 \\ x_4 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1533/10 \\ 2767/10 \\ 2667/10 \end{pmatrix} \qquad S_3 = \{\underline{c}\} \ni \left\{\begin{array}{lll} 2c_1 & -3c_3 & \geq 0 \\ & -c_2 & \geq 0 \\ c_1 & & \geq 0 \end{array}\right\}$$

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} s_3 \\ -s_1 \\ -1/3s_2+2/3s_3 \end{pmatrix} \qquad |J_3| = 1/3$$

$$P_{k_3} = \frac{a}{3}\int_0^\infty\int_0^\infty\int_0^\infty e^{-b[s_3^2+s_1^2+(2/3s_3-1/3s_2)^2]}\,ds_1ds_2ds_3 \approx 0.175$$

$$\underline{x}_{B_4} = \begin{pmatrix} x_3 \\ x_4 \\ x_6 \end{pmatrix} = \begin{pmatrix} 230 \\ 200 \\ 420 \end{pmatrix} \qquad S_4 = \{\underline{c}\} \ni \left\{\begin{array}{llll} -2c_1 & & +3c_3 & \geq 0 \\ & -c_2 & & \geq 0 \\ & & c_3 & \geq 0 \end{array}\right\}$$

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}s_1+3/2s_3 \\ -s_2 \\ s_3 \end{pmatrix} \qquad |J_4| = 1/2$$

$$P_{k_4} = \frac{a}{2}\int_0^\infty\int_0^\infty\int_0^\infty e^{-b[(3/2s_3-1/2s_1)^2+s_2^2+s_3^2]}\,ds_1ds_2ds_3 \approx 0.206$$

$$\underline{x}_{B_5} = \begin{pmatrix} x_2 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} 100 \\ 230 \\ 20 \end{pmatrix} \qquad S_5 = \{\underline{c}\} \ni \left\{\begin{array}{llll} -4c_1 & -\ c_2 & +\ 6c_3 & \geq 0 \\ & -\ c_2 & +\ 2c_3 & \geq 0 \\ & c_2 & & \geq 0 \end{array}\right\}$$

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4}s_1+\frac{1}{2}s_2+3/4s_3 \\ s_2 \\ \frac{1}{2}s_2+\frac{1}{2}s_3 \end{pmatrix} \qquad |J_5| = 1/8$$

$$P_{k_5} = \frac{a}{8}\int_0^\infty\int_0^\infty\int_0^\infty e^{-b[(-\frac{1}{4}s_1+\frac{1}{2}s_2+3/4s_3)^2+s_2^2+(\frac{1}{2}s_2+\frac{1}{2}s_3)^2]}\,ds_1ds_2ds_3 \approx 0.146$$

$$\underline{x}_{B_6} = \begin{pmatrix} x_2 \\ x_3 \\ x_5 \end{pmatrix} = \begin{pmatrix} 105 \\ 220 \\ 20 \end{pmatrix} \qquad S_6 = \{\underline{c}\} \ni \begin{Bmatrix} -4c_1 + c_2 + 2c_3 \geq 0 \\ c_2 - 2c_3 \geq 0 \\ c_3 \geq 0 \end{Bmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4}s_1 + s_2 + \frac{1}{4}s_3 \\ 2s_2 + s_3 \\ s_2 \end{pmatrix} \qquad |J_6| = 1/4$$

$$P_{k_6} = \frac{a}{4} \int_0^\infty \int_0^\infty \int_0^\infty e^{-b[(-\frac{1}{4}s_1+s_2+\frac{1}{4}s_3)^2+(2s_2+s_3)^2+s_2^2]} ds_1 ds_2 ds_3 \approx 0.042$$

$$\underline{x}_{B_7} = \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1533/10 \\ 667/10 \\ 1434/10 \end{pmatrix} \qquad S_7 = \{\underline{c}\} \ni \begin{Bmatrix} 4c_1 - c_2 - 6c_3 \geq 0 \\ 4c_1 - c_2 \geq 0 \\ c_2 \geq 0 \end{Bmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{4}s_2 + \frac{1}{4}s_3 \\ s_3 \\ -1/6s_1 + 1/6s_2 \end{pmatrix} \qquad |J_7| = 1/24$$

$$P_{k_7} = \frac{a}{24} \int_0^\infty \int_0^\infty \int_0^\infty e^{-b[(\frac{1}{4}s_2+\frac{1}{4}s_3)^2+s_3^2+(\frac{1}{6}s_2-\frac{1}{6}s_1)^2]} ds_1 ds_2 ds_3 \approx 0.142$$

$$\underline{x}_{B_8} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 205/2 \\ 215 \end{pmatrix} \qquad S_8 = \{\underline{c}\} \ni \begin{Bmatrix} -4c_1 + c_2 + 6c_3 \geq 0 \\ 4c_1 - c_2 - 2c_3 \geq 0 \\ 4c_1 + c_2 - 6c_3 \geq 0 \end{Bmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{4}s_1 + 3/8s_2 + 1/8s_3 \\ \frac{1}{2}s_1 + \frac{1}{2}s_3 \\ \frac{1}{4}s_1 + \frac{1}{4}s_2 \end{pmatrix} \qquad |J_8| = 1/32$$

$$P_{k_8} = \frac{a}{32} \int_0^\infty \int_0^\infty \int_0^\infty e^{-b[(\frac{1}{4}s_1+\frac{3}{8}s_2+\frac{1}{8}s_3)^2+(\frac{1}{2}s_1+\frac{1}{2}s_3)^2+(\frac{1}{4}s_1+\frac{1}{4}s_2)^2]} ds_1 ds_2 ds_3$$

$$\approx 0.009$$

The values of $\{P_{k_t} | t=1,\ldots,8\}$ listed above were obtained by simulation, wherein 2000 random triples belonging to the given density functions were generated. Boundary points were not counted in order that the 2000 triples would represent only interior points $\subset \{S_t\}$. The ranking of the subspaces by the magnitude of their respective P_{k_t} yields S_4, S_3, S_2, S_5, S_7, S_1, S_6, S_8. Note that the exact probability for P_{k_1} is 0.125, whereas the simulation produced 0.118. In the two solution methods which follow, S_1 is placed in the fourth position and this represents the only change from the simulation ranking.

Solution Method I

Basis 1

Step 1: $\{d_{\bar{c}h^i}\}$ under equality implies $-\bar{c}_1 = -\bar{c}_2 = -\bar{c}_3$

Let $\bar{c}_1 = -1$, then $\bar{c}_2 = \bar{c}_3 = -1$.

Step 2: $\bar{w} = \tau\bar{c} = \tau(-1,-1,-1)$

$\underline{\bar{c}}_e = (-0.576,-0.576,-0.576)$.

Step 3: $r_1 = d_{\underline{\bar{c}}_e h^i} = 0.576$.

Basis 2

Step 1: $\{d_{\bar{c}h^i}\}$ under equality implies $\dfrac{-4\bar{c}_1+\bar{c}_2}{\sqrt{17}} = -\bar{c}_3 = \bar{c}_2$

Let $\bar{c}_2 = 1$, then $\bar{c}_3 = -1$, $\bar{c}_1 = -0.78$.

Step 2: $\bar{w} = \tau\bar{c} = \tau(-0.78,1,-1)$

$\underline{\bar{c}}_e = (0.482,0.62,-0.62)$.

Step 3: $r_2 = d_{\underline{\bar{c}}_e h^i} = 0.62$.

Basis 3

Step 1: $\{d_{\bar{c}h^i}\}$ under equality implies

$$\frac{2\bar{c}_1-3\bar{c}_3}{\sqrt{13}} = -\bar{c}_2 = \bar{c}_1.$$

Let $\bar{c}_1 = 1$, then $\bar{c}_2 = -1$, $\bar{c}_3 = -0.535$.

Step 2: $\bar{w} = \tau\bar{c} = \tau(1,-1,-0.535)$

$\underline{\bar{c}}_e = (0.66,-0.66,-0.353)$.

Step 3: $r_3 = d_{\underline{\bar{c}}_e h^i} = 0.66$

Basis 4

Step 1: $\{d_{\bar{c}h^i}\}$ under equality implies

$$\frac{-2\bar{c}_1+3\bar{c}_3}{\sqrt{13}} = -\bar{c}_2 = \bar{c}_3.$$

Let $\bar{c}_3 = 1$, then $\bar{c}_2 = -1$, $\bar{c}_1 = -0.303$.

Step 2: $\bar{w} = \tau\bar{c} = \tau(-0.303,-1,1)$

$\underline{\bar{c}}_e = (-0.21,-0.692,0.692)$.

Step 3: $r_4 = d_{\underline{\bar{c}}_e h^i} = 0.692$.

Basis 5

Step 1: $\{d_{\bar{c}h^i}\}$ under equality implies

$$\frac{-4\bar{c}_1-\bar{c}_2+6\bar{c}_3}{\sqrt{53}} = \frac{-\bar{c}_2+2\bar{c}_3}{\sqrt{5}} = \bar{c}_2$$

Let $\bar{c}_2 = 1$, then $\bar{c}_3 = 1.618$, $\bar{c}_1 = 0.355$.

Step 2: $\bar{w} = \tau\bar{c} = \tau(0.355,1,1.618)$

$\underline{\bar{c}}_e = (0.183,0.516,0.835)$.

Step 3: $r_5 = d_{\underline{\bar{c}}_e h^i} = 0.516.$

<u>Basis 6</u>

Step 1: $\{d_{\bar{c}h^i}\}$ under equality implies

$$\frac{-4\bar{c}_1+\bar{c}_2+2\bar{c}_3}{\sqrt{21}} = \frac{\bar{c}_2-2\bar{c}_3}{\sqrt{5}} = \bar{c}_3$$

Let $\bar{c}_3 = 1$, then $\bar{c}_2 = 4.236$, $\bar{c}_1 = 0.414$.

Step 2: $\bar{w} = \tau\bar{c} = \tau(0.414, 4.236, 1)$

$\underline{\bar{c}}_e = (0.095, 0.97, 0.229).$

Step 3: $r_6 = d_{\underline{\bar{c}}_e h^i} = 0.229.$

<u>Basis 7</u>

Step 1: $\{d_{\bar{c}h^i}\}$ under equality implies

$$\frac{4\bar{c}_1-\bar{c}_2-6\bar{c}_3}{\sqrt{53}} = \frac{4\bar{c}_1-\bar{c}_2}{\sqrt{17}} = \bar{c}_2$$

Let $\bar{c}_2 = 1$, then $\bar{c}_1 = 1.28$, $\bar{c}_3 = -2.23$.

Step 2: $\bar{w} = \tau\bar{c} = \tau(1.28, 1, -2.23)$

$\underline{\bar{c}}_e = (0.464, 0.362, -0.808).$

Step 3: $r_7 = 0.362.$

<u>Basis 8</u>

Step 1: $\{d_{\bar{c}h^i}\}$ under equality implies

$$\frac{-4\bar{c}_1+\bar{c}_2+6\bar{c}_3}{\sqrt{53}} = \frac{4\bar{c}_1-\bar{c}_2-2\bar{c}_3}{\sqrt{21}} = \frac{4\bar{c}_1+\bar{c}_2-6\bar{c}_3}{\sqrt{53}}$$

Let $\bar{c}_3 = 20$, then $\bar{c}_1 = 30$, $\bar{c}_2 = 49$.

*Since an arbitrary value for a particular $\bar{c}_i$ is not obvious in this case, the value of 20 for $\bar{c}_3$ was determined by forming a linear combination of the vectors normal to $\{h_8^i\}$. The point obtained in this manner $(\underline{\bar{c}}_8)$ must be ε S_8.

Step 2: $\bar{w} = \tau\bar{c} = \tau(30,49,20)$

$\underline{\bar{c}}_e = (0.494, 0.805, 0.329)$.

Step 3: $\bar{r}_8 = 0.112$.

The ranking of the subspaces by decreasing magnitude of their respective r_t is S_4, S_3, S_2, S_1, S_5, S_7, S_6, S_8. And the position of S_1, relative to the various P_{k_t}, is the fourth rather than the sixth position as predicted by simulation of the probability functions. Theorem 6 does not resolve this discrepancy, rather we are only assured that $\{S_1,S_2,S_3,S_4\}$ is correctly ranked by S_4, S_3, S_2, S_1.

Solution Method II

Basis 1

Step 1: $\underline{x}_{B_1} = \begin{pmatrix} x_4 \\ x_4 \\ x_6 \end{pmatrix}$ implies $G_1 = \begin{Bmatrix} x_1 \geq 0 \\ x_2 \geq 0 \\ x_3 \geq 0 \end{Bmatrix}$

$\theta_{12} = \theta_{13} = \theta_{23} = \Pi/2$, so reversal by Rule 1 is arbitrary and yields

$$G_1' = \begin{Bmatrix} x_1 \geq 0 \\ x_2 \geq 0 \\ -x_3 \geq 0 \end{Bmatrix}.$$

Step 2: $\{d_{\bar{x}h^i}\}$ under equality implies $\bar{x}_1 = \bar{x}_2 = -\bar{x}_3$.

Let $\bar{x}_3 = -1$, then $\bar{x}_1 = \bar{x}_2 = 1$.

Step 3: $\tau\bar{x} = \tau(1,1,-1)$; $\underline{\bar{x}_e} = (0.576, 0.576, -0.576)$.

Step 4: $r_1 = d_{\underline{\bar{x}_e} h^i} = 0.576$.

<u>Basis 2</u>

Step 1: $\underline{x}_{B_2} = \begin{pmatrix} x_2 \\ x_4 \\ x_5 \end{pmatrix}$ implies $G_2 = \left\{\begin{matrix} -x_1 & - 4x_2 & & \geq 0 \\ x_1 & & & \geq 0 \\ & & x_3 & \geq 0 \end{matrix}\right\}$

$\theta_{13} = \Pi/2$, so reversal by Rule 1 yields

$$G_2' = \left\{\begin{matrix} x_1 & + 4x_2 & & \geq 0 \\ x_1 & & & \geq 0 \\ & & x_3 & \geq 0 \end{matrix}\right\}$$

Step 2: $\{d_{\bar{x}h^i}\}$ under equality implies

$$\frac{\bar{x}_1 + 4\bar{x}_2}{\sqrt{17}} = \bar{x}_1 = \bar{x}_3$$

Let $\bar{x}_1 = 1$, then $\bar{x}_2 = 0.78$, $\bar{x}_3 = 1$.

Step 3: $\tau\bar{x} = \tau(1,0.78,1)$; $\underline{\bar{x}_e} = (0.62, 0.483, 0.62)$

Step 4: $r_2 = d_{\underline{\bar{x}_e} h^i} = 0.62$.

<u>Basis 3</u>

Step 1: $\underline{x}_{B_3} = \begin{pmatrix} x_1 \\ x_4 \\ x_6 \end{pmatrix}$ implies $G_3 = \left\{\begin{matrix} -3x_1 & & - 2x_3 & \geq 0 \\ & x_2 & & \geq 0 \\ & & x_3 & \geq 0 \end{matrix}\right\}$

$\theta_{12} = \Pi/2$, so reversal by Rule 1 yields

$$G_3' = \left\{\begin{matrix} 3x_1 & & + 2x_3 & \geq 0 \\ & x_2 & & \geq 0 \\ & & x_3 & \geq 0 \end{matrix}\right\}$$

Step 2: $\{d_{\bar{x}h^i}\}$ under equality implies

$$\frac{3\bar{x}_1+2\bar{x}_3}{\sqrt{13}} = \bar{x}_2 = \bar{x}_3$$

Let $\bar{x}_2 = 1$, then $\bar{x}_3 = 1$, $\bar{x}_1 = 0.535$.

Step 3: $\tau\bar{x} = \tau(0.535,1,1)$; $\underline{\bar{x}}_e = (0.353,0.66,0.66)$

Step 4: $r_3 = d_{\underline{\bar{x}}_e h^i} = 0.66.$

Basis 4

Step 1: $\underline{x}_{B_4} = \begin{pmatrix} x_3 \\ x_4 \\ x_6 \end{pmatrix}$ implies $G_4 = \begin{Bmatrix} -3x_1 & & -\ 2x_3 \geq 0 \\ x_1 & & \geq 0 \\ & x_2 & \geq 0 \end{Bmatrix}$

$\theta_{13} = \Pi/2$, so reversal by Rule 1 yields

$$G_4' = \begin{Bmatrix} 3x_1 & & +\ 2x_3 \geq 0 \\ x_1 & & \geq 0 \\ & x_2 & \geq 0 \end{Bmatrix}$$

Step 2: $\{d_{\bar{x}h^i}\}$ under equality implies

$$\frac{3\bar{x}_1+2\bar{x}_3}{\sqrt{13}} = \bar{x}_1 = \bar{x}_2$$

Let $\bar{x}_1 = 1$, then $\bar{x}_2 = 1$, $\bar{x}_3 = 0.3025$.

Step 3: $\tau\bar{x} = \tau(1,1,0.3025)$; $\underline{\bar{x}}_e = (0.692,0.692,0.209)$

Step 4: $r_4 = d_{\underline{\bar{x}}_e h^i} = 0.692.$

Basis 5

Step 1: $\underline{x}_{B_5} = \begin{pmatrix} x_2 \\ x_3 \\ x_6 \end{pmatrix}$ implies $G_5 = \begin{Bmatrix} -\ x_1 - 2x_2 - \ x_3 \geq 0 \\ -3x_1 \qquad - 2x_3 \geq 0 \\ x_1 \qquad\qquad \geq 0 \end{Bmatrix}$

$\theta_{12} = 55.5°$, $\theta_{13} = 114°$, $\theta_{23} = 146.4°$;

$\Sigma\theta = 315.9° > 270°$, so reversal by Rule 2 yields

$$G_5' = \begin{Bmatrix} -x_1 - 2x_2 - x_3 \geq 0 \\ 3x_1 + 2x_3 \geq 0 \\ x_1 \geq 0 \end{Bmatrix}$$

Step 2: $\{d_{\bar{x}h^i}\}$ under equality implies

$$\frac{-\bar{x}_1 - 2\bar{x}_2 - \bar{x}_3}{\sqrt{6}} = \frac{3\bar{x}_1 + 2\bar{x}_3}{\sqrt{13}} = \bar{x}_1$$

Let $\bar{x}_1 = 1$, then $\bar{x}_3 = 0.3025$, $\bar{x}_2 = -1.874$.

Step 3: $\tau\bar{x} = \tau(1,-1.874, 0.3025)$; $\underline{x}_e = (0.466,-0.872,0.141)$.

Step 4: $r_5 = d_{\underline{x}_e h^i} = 0.466$.

<u>Basis 6</u>

Step 1: $\underline{x}_{B_6} = \begin{pmatrix} x_2 \\ x_3 \\ x_5 \end{pmatrix}$ implies $G_6 = \begin{Bmatrix} -x_1 - 2x_2 - x_3 \geq 0 \\ -x_1 - 4x_2 \geq 0 \\ x_1 \geq 0 \end{Bmatrix}$

$\theta_{12} = 27°$, $\theta_{13} = 114°$, $\theta_{23} = 104°$;

$\Sigma\theta = 245° < 270°$, so reversal by Rule 3 yields

$$G_6' = \begin{Bmatrix} x_1 + 2x_2 + x_3 \geq 0 \\ -x_1 - 4x_2 \geq 0 \\ x_1 \geq 0 \end{Bmatrix}$$

Step 2: $\{d_{\bar{x}h^i}\}$ under equality implies

$$\frac{\bar{x}_1 + 2\bar{x}_2 + \bar{x}_3}{\sqrt{6}} = \frac{-\bar{x}_1 - 4\bar{x}_2}{\sqrt{17}} = \bar{x}_1$$

Let $\bar{x}_1 = 1$, then $\bar{x}_2 = -1.28$, $\bar{x}_3 = 4.01$

Step 3: $\tau\bar{x} = \tau(1,-1.28,4.01)$; $\underline{x}_e = (0.231,-0.295,0.926)$.

Step 4: $r_6 = d_{\underline{\bar{x}}_e h^i} = 0.231$

Basis 7

Step 1: $\underline{x}_{B_7} = \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix}$ implies $G_7 = \begin{Bmatrix} -3x_1 & & -2x_3 \geq 0 \\ -x_1 & -4x_2 & \geq 0 \\ & & x_3 \geq 0 \end{Bmatrix}$

$\theta_{23} = \Pi/2$, so reversal by Rule 1 yields

$$G_7' = \begin{Bmatrix} -3x_1 & & -2x_3 \geq 0 \\ x_1 & +4x_2 & \geq 0 \\ & & x_3 \geq 0 \end{Bmatrix}$$

Step 2: $\{d_{\bar{x}h^i}\}$ under equality implies

$$\frac{-3\bar{x}_1 - 2\bar{x}_3}{\sqrt{13}} = \frac{\bar{x}_1 + 4\bar{x}_2}{\sqrt{17}} = \bar{x}_3$$

Let $\bar{x}_3 = 1$, then $\bar{x}_2 = 1.497$, $\bar{x}_1 = -1.868$.

Step 3: $\tau\bar{x} = \tau(-1.868, 1.497, 1)$; $\underline{\bar{x}}_e = (-0.72, 0.576, 0.386)$

Step 4: $r_7 = d_{\underline{\bar{x}}_e h^i} = 0.386.$

Basis 8

Step 1: $\underline{x}_{B_8} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ implies $G_8 = \begin{Bmatrix} -x_1 & -2x_2 & -x_3 \geq 0 \\ -3x_1 & & -2x_3 \geq 0 \\ -x_1 & -4x_2 & \geq 0 \end{Bmatrix}$

$\theta_{12} = 55.5°$, $\theta_{13} = 27°$, $\theta_{23} = 77.6°$;

$\Sigma\theta = 160.1° < 270°$, so reversal by Rule 3 yields

$$G_8' = \begin{Bmatrix} x_1 & +2x_2 & +x_3 \geq 0 \\ -3x_1 & & -2x_3 \geq 0 \\ -x_1 & -4x_2 & \geq 0 \end{Bmatrix}$$

Step 2: $\{d_{\bar{x}h^i}\}$ under equality implies

$$\frac{\bar{x}_1+2\bar{x}_2+\bar{x}_3}{\sqrt{6}} = \frac{-3\bar{x}_1-2\bar{x}_3}{\sqrt{13}} = \frac{-\bar{x}_1-4\bar{x}_2}{\sqrt{17}}$$

Let $\bar{x}_1 = -1$, then $\bar{x}_2 = 0.087$, $\bar{x}_3 = 1.215$.

Step 3: $\tau\bar{x} = \tau(-1,0.087,1.215)$; $\underline{\bar{x}}_e = (-0.635,0.551,0.451)$

Step 4: $r_8 = d_{\underline{\bar{x}}_e h^i} = 0.1006$

The ranking of the subspaces, $\{S_t^e\}$, by decreasing magnitude of r_t yields the solution S_4, S_3, S_2, S_1, S_5, S_7, S_6, S_8, which is the same ranking obtained by Solution Method I. Thus, without any probability calculations, the $\{P_{k_t}\}$ may be ordered accordingly.

SELECTION OF THE SOLUTION METHOD

There are three primary factors affecting the choice of which solution method to use for a particular problem:

1. The joint probability density function of the random variables.
2. The size of the problem, implying the relative magnitude of m+n.
3. The size of the m relative to n, regardless of the magnitude of m+n.

When the given density function belongs to the class stated by Theorem 7, the last two factors will determine the solution method to be used. However, even when the density function does not belong, the choice may still be arbitrary. The example problem (p. 17), for which the probability function is the joint exponential, is correctly solved by either method. (This may be slightly misleading since the joint exponential is essentially symmetric and could be classified as a quasi-member of the set in Theorem 7.) Regardless of such cases, though, it is noted that since Solution Method I is shown to be exact under certain conditions for $n \geq 2$, it would be the best choice when a particular probability distribution does not belong to the class covered by Theorem 7.

Using either technique for large problems, say $m>6$, $n>12$, a computer program should be used to generate the required data (which is similar for both methods). Balinski [2] describes such a program, which, for an LP problem of 65 variables and 35 constraints, required only 496 pivot operations to generate the 31 extreme points. Thus, if the problem is large and a computer is available, the possible exactness of Solution Method I makes it most desirable. If the size of the problem is relatively small, however, Solution Method II is best since fewer numerical calculations are required to determine the n linear inequalities; that is, the inequalities are immediate for Method II, while in Method I they are obtained from the matrix operations required by $\underline{c}_B B^{-1} A - \underline{c} \geq \underline{0}$.

Considering the third factor, there is a clear choice if m is significantly less than n, say $m \leq \frac{1}{2}n$, regardless of the size of the problem. The reference here is to the dimension of the n linear inequalities. For Solution Method II in this case, at least half of the inequalities for each basis would be of dimension one, thereby greatly simplifying the calculations to obtain the radii of the unit hyperspheres.

CONCLUSION AND RECOMMENDATIONS FOR FURTHER RESEARCH

The theory and solution methods developed in this dissertation apply to those PLP problems having random objective function coefficient vectors ($\underline{c}$), or random resource vectors ($\underline{b}$). (The primal-dual equivalency properties would be used to restate the problem in case of random $\underline{b}$.) Further research to extend existing concepts should be directed toward a solution method for those problems having c_j and b_i as simultaneous random variables, and finally to permit the inclusion of some or all random a_{ij}.

Further research to enhance the strength of this dissertation should concentrate on the ramifications of Theorems 6 and 7. Existing mathematical theory regarding the intent of Theorem 6 is severely limited. The embedded hypersphere technique should always be a good method for providing an ordinal ranking of the interiors of polyhedral spaces in R^n, but intensive research may disclose a method for which a stronger result could be proved. At the very least, this research should extend the cases covered by Theorem 6.

Regarding Theorem 7, initial research efforts should be directed toward inclusion of the general case of the multivariate normal density function. Assuming only that the c_j are independent,

$\{c_j | j=1,\ldots,n\}$ equals $\{\sigma_j z_j + \mu_j | j=1,\ldots,n\}$, where z_j is the standardized normal deviate with $\mu_{z_j} = 0$, $\sigma_{z_j} = 1$, and the z_j are independent. The objective function is then the maximization of

$\sum_{j=1}^{n} (\sigma_j z_j + \mu_j)x_j$, and S_t (analogous to Chapter IV), given by

$\sum_{j=1}^{n} b_{ij}c_j \geq 0$, $i=1,\ldots,n$, or $B_t\underline{c} \geq \underline{0}$ in matrix notation, becomes

$B_tV'\underline{z} \geq \underline{K}_t$. B_t is the mxm matrix unchanged by the z_j substitutions,

$$V' = \begin{bmatrix} \sigma_1 & 0 & \cdot & \cdot & 0 \\ 0 & \sigma_2 & \cdot & \cdot & 0 \\ \cdot & 0 & & & \cdot \\ \cdot & \cdot & & & \cdot \\ 0 & 0 & \cdot & \cdot & \sigma_n \end{bmatrix} , \quad z = (z_1,z_2,\ldots,z_n)' ,$$

and $\underline{K}_t$ is a nx1 column vector of linear combinations of the various μ_j.

To orientate this discussion with the theory in Chapter III, we know that $B_tV'\underline{z} \geq \underline{K}_t$ is a convex space in R^n and interest now concerns the rpR^n of each space. To ease the calculation of the hypersphere radius, and to form the subspace, translate the vertex of the polyhedron to the origin by setting $\underline{K}_t = \underline{0}$ to obtain $B_tV'\underline{z} \geq \underline{0}$.

Considering V' as a linear map: $R^n \rightarrow R^n$, if $\sigma_1 = \sigma_2 = \ldots = \sigma_n$, then V' is called a homothety (stretching); thus $B_t\underline{c} \geq \underline{0}$ and $B_tV'\underline{z} \geq 0$ define identical subspaces. In this case $\{S_t\}$ are unchanged by the transformation from the $\underline{c}$ to $\underline{z}$ random vector space. Therefore, the theory developed regarding $\{S_t\}$ applies and $\{B_tV'\underline{z}\}$ are exclusive. If $\sigma_i \neq \sigma_j$ for $i \neq j$, then V' is still constant over all t, but B_t is transformed and not necessarily equivalent to B_tV'. Even though the transformation is invariant on t, $\{B_tV'\underline{z}\}$ will not necessarily be exclusive and the magnitude of the relative interiors may have changed; hence, we have identified the first area for investigation.

Then notice that

$$P_{k_t} = \iint_{S_t \geq \underline{0}} \cdots \int f(\underline{c})dc_1 \ldots dc_n = \iint_{B_tV'\underline{z} \geq \underline{K}_t} \int f(\underline{z})dz_1 \ldots dz_n ,$$

where $f(\underline{z})$ satisfies the requirements of Theorem 7 since $\mu_{z_j} = 0$ and $\sigma_{z_j} = 1$ for $j=1,\ldots,n$. However, for the largest P_{k_t} to correspond to that basis t having largest $ri(B_tV'\underline{z} \geq \underline{K}_t)$, the theorem requires that the subspace be unbounded. Since $\underline{K}_t$ may effect bounds on $\{\underline{z}\}$, we have the second area for investigation. Research should disclose the conditions necessary for Solution Method I to provide optimal or near optimal results for this general case of the multivariate normal.

Finally, further investigation should be conducted to enumerate the density functions for which this dissertation will provide useable results. In this context, a measure of efficiency should then be devised to give some assurance regarding the accuracy of the rankings obtained.

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