PERT TIME ESTIMATES: A LOGICAL ALTERNATIVE WITH IMPROVED ACCURACY

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PREFACE

This study is concerned with the estimate of the mean and the standard deviation of the stochastic task times in Project Evaluation and Review Technique (PERT). The primary objective is to develop an alternative to the PERT time estimate procedure, so that the estimate formulae is valid for a wide range of beta distributions, the time elicitation is consistent with the probability elicitation literature, and the accuracy of the time estimate is improved. An alternative is proposed based on theoretical analysis, and improvement in accuracy is examined.

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CHAPTER I

INTRODUCTION

Program Evaluation and Review Technique (PERT) is a network model widely employed to aid management in planning and controlling large-scale projects. Malcolm *et al.* (1959) developed PERT in the late 1950s in an effort to speed up the Polaris missile project. PERT stresses probabilistic activity time estimates and is suitable for an environment typified by high uncertainty. The PERT technique has received widespread interest and has been used for many types of projects.

The PERT is based on a network model with stochastic activity times. The basic objective of PERT is to obtain a probability distribution of the completion time of a specific project, accomplished by breaking the project down into sub-parts or activities, estimating their distributions, and then summing these smaller distributions to obtain the total project distribution. Based on this total distribution, the project can be effectively monitored, analyzed, and controlled.

1.1 Representing Projects as Networks

A project is represented by a network or by a precedence diagram to depict major project activities and their sequential relationships. The program is composed of arrows, representing project activities, and nodes, representing points in time when the activities represented by incoming arrows are completed and the activities represented by outgoing arrows can be started (this is referred to as an "event"). There are only one starting node and one ending node for a project. The activities and the nodes in a network form various paths. A path is a continuous chain of activities from the starting node to the ending node, via various nodes in the network. The nodes linked by the chain of activities also form a chain. Each path will then be identified by this chain of nodes. A network can have many paths. It is the objective of PERT to find the "Critical Path," which is the path with the longest duration among all the paths. The activities on such a path are called "critical activities." The critical path determines the duration of the whole project, and thus receives the maximum attention of management. The task of management is to try to shorten activity times of critical activities, therefore shortening the duration of the whole project.

A simple example of such a network is shown in Figure 1-1.

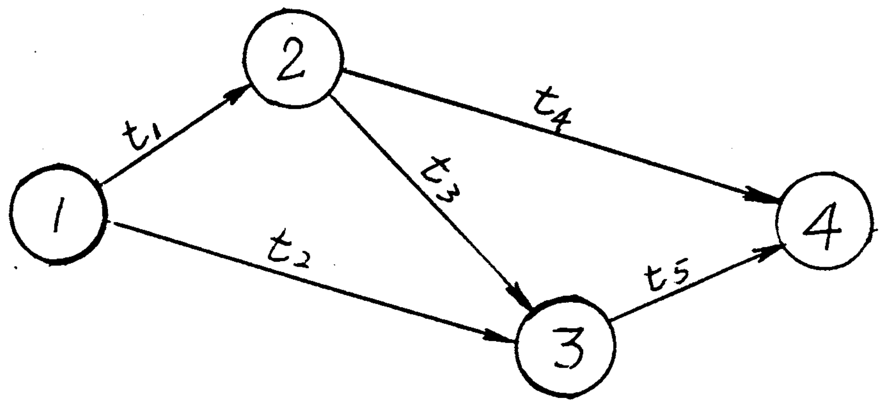


Figure 1-1 A Simple Network Example

In the network shown in Figure 1-1, there are five activities: activity 1-2, activity 1-3, activity 2-3, activity 2-4, and activity 3-4. Denote the stochastic time durations of these activities (known as activity times or task times) t_1, t_2, t_3, t_4, and t_5, respectively. The activities, their durations, and their precedence relationship are shown in the table below.

Table 1-1 Network Activities, Their Durations, and Precedence Relationship

Activity	1-2	1-3	2-3	2-4	3-4
Duration	t_1	t_2	t_3	t_4	t_5
Precedes	2-3, 2-4	3-4	3-4	None	None

In the network represented by Figure 1-1 and Table 1-1, there are three paths: 1-2-4, 1-3-4, and 1-2-3-4. We denote these three paths as P_1, P_2, and P_3, respectively. It is intuitively obvious that the three paths have different durations. The purpose of network analysis is to determine the critical path and its length, and find out ways to shorten this path.

The length of a path is determined by the tasks consisting this path. Since task times t_i's are stochastic, they have mean and standard deviations, which are denoted as $\mu(t_i)$ and $\sigma(t_i)$ respectively. Let p_i represent the stochastic path time of path P_i, $\mu(p_i)$ its expected value and $\sigma(p_i)$ the standard deviation. Then, for this example, $p_1 = t_1 + t_4$, $p_2 = t_2 + t_5$, and $p_3 = t_1 + t_3 + t_5$. The project completion time or the path time of the longest path ("critical path") will then be

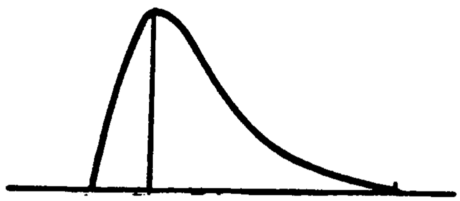
$$\text{Max}\{p_1,\ p_2,\ p_3\} = \text{Max}\{(t_1+t_4),(t_2+t_5),(t_1+t_3+t_5)\}.$$

The project completion time will be, of course, a random variable with its own mean and standard deviation. The basic OR problems in PERT are: (1) to determine the probability distributions of the individual activities; and (2) given the distributions of the individual activities, find the distribution of the project completion time, its mean, and standard deviation. From this we can see that to determine the probability distributions of the individual activities is the basis of all the further analyses. The PERT originators (Malcolm *et al.*, 1959) assumed that the task times are beta distributed random variables. Supposedly based on this assumption (which will be discussed in Chapter II), they developed a procedure to elicit several time estimates, and to convert these estimates to the most useful parameters of a distribution: the mean and the standard deviation. This procedure of time estimation has been the major interest of many research papers for the past thirty years, and it is the focus of the current study.

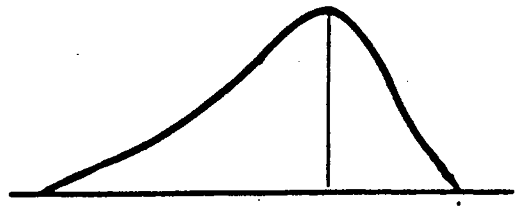
1.2 Basic PERT Methodology

The PERT procedure begins with eliciting time estimates of an activity. According to the PERT originators, an "expert" (such as the project manager or an engineer) will be asked to first estimate the "most likely time" (denoted as "m"), which the PERT originators believed would be perceived by the "expert" as the mode of the distribution. The "expert" will then be asked to estimate the "extreme times" or the "optimistic time" and the "pessimistic time" (denoted as "a" and "b" respectively).

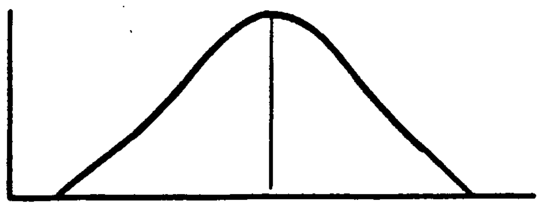
The PERT originators assumed that the distribution of the task times will be uni-modal, with two positive abscissa intercepts. They chose the beta distribution to represent distributions of the above features (a brief description of the beta distribution will be given in Chapter II. More detailed discussions can be found in, e.g., Johnson and Kotz, 1970). Three examples of the beta distribution are illustrated in Figure 1-2 below. They may be symmetrical [Figure 1-2 (c)] or skew in either direction [Figure 1-2 (a) and (b)].



(a)



(b)



(c)

Figure 1-2 Examples of Beta Distribution

In the PERT procedure, the mode of the beta distribution is equated to the "most likely time" m, and the two abscissa intercepts are equated to the optimistic and the pessimistic times a and b respectively.

The PERT originators then developed two formulae to convert the a, m, and b to the mean μ and the standard deviation σ of the distribution:

$$\mu = (a + 4m + b)/6 \quad (1\text{-}1a)$$

$$\sigma = (b - a)/6 \quad (1\text{-}1b)$$

PERT then finds the critical path through a network algorithm (which is not our focus in this study), using the expected activity time $\mu(t_i)$ in the algorithm. The mean project duration will be the sum of the $\mu(t_i)$'s of the activities on the critical path, and the variance of the project time will be the sum of the $[\sigma(t_i)]^2$ of the activities on the critical path. Based on the critical path found, together with the mean and standard deviation of the duration of the critical path, the project manager can control the project more effectively.

1.3 Problems in the PERT Time Estimation Procedure and Our Proposal for Improvement

Grubbs (1962) studied the PERT formulae (1-1) to convert the "optimistic", "most likely", and "pessimistic" times a, m, and b to the mean μ and the standard deviation σ of the distribution, and found that the PERT formulae are based on a very restricted subset of beta distribution, rather than having great versatility as claimed by the PERT originators. Since then, many researchers (for example, MacCrimmon and Ryavec, 1963; Moder and Rogers, 1968; Swanson and Pazer, 1971) studied this problem. Some researchers also performed error analyses on the PERT formulae and pointed out the numerical errors they have. A few remedial methods were proposed (more detailed discussion can be found in Chapter II of this study). Unfortunately, the few remedial methods were largely based on experiments only, and failed to propose a correct method through theoretical analysis. In addition, these remedial methods only proposed adjustments in individual stages of the PERT procedure, without making corresponding corrections in the other stages, thus reducing the significance and weakening the justification of these adjustments. Even these to-be-improved remedial methods, however, have not received significant attention in the MS/OR community: numerous OR textbooks are still teaching students the incorrect and inaccurate PERT formulae.

Based on the brief discussion above, our objectives in the current study are:

(1) to further point out the logical shortcomings of the PERT time estimate procedure (including the elicitation of the probabilistic times a, m, and b and the conversion from a, m, and b to μ and σ);

(2) to develop alternative formulae which are logical and accurate, and which are based on the solid ground of

probability elicitation literature (see next point);

(3) to propose the correct method in probability elicitation based on existing literature on this issue; and

(4) to show the extent of improvement the proposed method can have over the PERT procedure in the accuracy of activity time estimates.

The ultimate effort of this study is to raise the attention of the MS/OR community on this long-neglected yet very important issue, to provide a correct and effective tool for activity time estimates, and to provide the project network analysis with a solid ground in theory, which will finally enable true cost savings in large-scale projects.

1.4 Organization of the Dissertation

The dissertation is organized into six chapters, with the first chapter being the introduction. In Chapter II, we will perform literature review on the field of the PERT time estimates, focusing on two aspects: the definitions and estimates of the PERT "basic times" (a, m, and b), and the PERT time conversion formulae which converts the basic times a, m, and b to the mean (μ) and the standard deviation (σ). Since the PERT procedure is based on the elicitation of a, m, and b, and our proposed alternative will also be based on the elicitation of fractiles of a distribution, we will survey the literature on probability elicitation, which is the ground of the probabilistic time elicitation.

In Chapter III, we will further discuss the shortcomings of the PERT time estimate procedure (including the elicitation of a, m, and b and the conversion of a, m, and b to μ and σ), and will propose an alternative whose conversion formulae will be based on the properties of the beta distribution and whose time elicitation will be based on the probability elicitation literature. The general procedure of mathematical inference leading to our alternative formulae and a linear regression to determine the coefficients in those formulae will be described.

Chapter IV is the major part of this dissertation, in which the mathematical inference of our alternative formulae will be shown, and the linear regression to obtain the coefficients for the formulae will be discussed and described in details. Results of the linear regression will be obtained, and the alternative formulae for time estimate will be presented.

With the results obtained in Chapter IV, we will conduct error analysis in Chapter V. Performance of the PERT formulae and that of the proposed alternative will be compared in terms of accuracy. Numerical examples on the discrepancy between the "absolute endpoints" and the "inner fractiles" will also be shown, which will support our argument that the simple substitution of the "absolute endpoints" with the "inner fractiles" can lead to substantial errors.

We will conclude this dissertation in Chapter IV, with discussions on some issues emerging in the procedure of this study, and on the directions for future studies.

CHAPTER II

LITERATURE REVIEW

The PERT time-estimate procedure literature is focused mainly on the following phases: the PERT formulae to convert the "basic times" a, b, and m to mean and standard deviation, and the determination of the "basic times" a, b, and m. We will survey the literature on these two phases. In addition, the probability elicitation literature will be surveyed, since this is the theoretical ground of the elicitation of the "basic times" on which the conversion is based.

2.1 Errors in PERT Time-Estimate Formulae

2.1.1 PERT formulae implicitly lead to a restricted subset of the beta distribution

Grubbs (1962) pointed out that the PERT formulae are valid only for a small subset of the beta distribution. His reasoning is as follows:

A beta distribution with end points U and V is defined as

$$f(t) = (t-U)^{\alpha}(V-t)^{\beta}/[(V-U)^{\alpha+\beta+1} B(\alpha+1,\beta+1)], \quad (U<t<V)$$
$$f(t) = 0, \quad \text{otherwise} \qquad (2-1)$$

here α and β are the parameters of the beta distribution governing its "shape" (skewness and kurtosis), and B(x,y) is the Beta function value given independent variables x and y (see, for example, Johnson and Kotz, 1970).

With the transformation of $t = U + (V-U)x$, we can obtain the standardized beta distribution

$$f(x) = x^{\alpha}(1-x)^{\beta}/[B(\alpha+1,\beta+1)], \quad (0<x<1)$$
$$f(x) = 0. \quad \text{otherwise} \qquad (2-2)$$

The mathematical expectation (mean) and the standard deviation of the distribution are (Johnson and Kotz, 1970)

$$\mu_x = E(x) = (\alpha+1)/(\alpha+\beta+2), \qquad (2-3a)$$

$$\sigma_x^2 = (\alpha+1)(\beta+1)/[(\alpha+\beta+3)(\alpha+\beta+2)^2]. \qquad (2-3b)$$

Transforming the x variable back to the t variable, we have

$$\mu_t = E(t) = U + (V-U)[(\alpha+1)/(\alpha+\beta+2)], \qquad (2-4a)$$

$$\sigma_t^2 = (V-U)^2(\alpha+1)(\beta+1)/[(\alpha+\beta+3)(\alpha+\beta+2)^2]. \qquad (2-4b)$$

It is also known from properties of the beta distribution that the expression for m, the mode of the distribution, is

$$m = (U\beta+V\alpha)/(\alpha+\beta). \qquad (2-5)$$

Expressing μ_t in terms of U, V, and m:

$$\mu_t = [U + (\alpha+\beta)m + V]/(\alpha+\beta+2). \qquad (2-6)$$

Note that the PERT formula of mean is

$$\mu_t = (U + 4m + V)/6, \quad (2\text{-}7)$$

The equivalence between (2-6) and (2-7) means that we must have

$$\alpha + \beta = 4, \quad (2\text{-}8)$$

That is, the PERT mean formula is only valid for those beta distributions whose shape parameters (α,β) satisfy (2-8). Sasieni (1986) replicated this result.

It is worth noticing that in the two previous studies not much was said about the σ formula. Actually, when we substitute (2-8) into (2-4b), we have

$$\sigma_t^2 = (V-U)^2(\alpha+1)\cdot(\beta+1)/[7\times6^2] \quad (2\text{-}9)$$

Compare (2-9) with the σ formula (1-1b) in PERT (here we use U and V to denote the two "endpoints"):

$$\sigma_t^2 = (V-U)^2/6^2 \quad (2\text{-}10)$$

we can see that to equate (2-9) and (2-10), we must have

$$(\alpha+1)(\beta+1) = 7 \quad (2\text{-}11)$$

Solve for α and β by solving the following simultaneous equations:

$$\alpha + \beta = 4, \quad (2\text{-}8)$$

$$(\alpha+1)(\beta+1) = 7, \quad (2\text{-}11)$$

we have

$$\alpha = 2 \pm \sqrt{2}$$
$$\beta = 2 \mp \sqrt{2} \qquad (2\text{-}12)$$

The preceding shows that there are only two points in the α-β space which can satisfy both of the PERT formulae. The preceding also shows that if we only restrict the α-β value with "$\alpha + \beta = 4$", the μ formula of PERT (1-1a) is an exact relationship (on this very restricted subset) but the σ formula in PERT is only an approximation (even on this already restricted subset).

Swanson and Pazer (1971) started from a standard beta distribution

$$f(x) = (1/K)x^{\alpha}(1-x)^{\beta}, \quad 0 \le x \le 1 \qquad (2\text{-}13)$$

where $K = B(\alpha+1, \beta+1)$ (see equations (2-1) and (2-2)).

The mean μ in the above distribution is the same as (2-3): $\mu = (\alpha+1)/(\alpha+\beta+2)$. But with the PERT formulae, $\mu = (4m+1)/6$ (when a=0 and b=1). Equate these two expressions of μ:

$$(\alpha+1)/(\alpha+\beta+2) = (4m+1)/6. \qquad (2\text{-}14)$$

The mode of a standard beta distribution is (Johnson and Kotz, 1970):

$$m = \alpha/(\alpha+\beta). \qquad (2\text{-}15)$$

From (2-14) and (2-15) they showed that $\alpha + \beta = 4$, the same result as (2-8) obtained by Grubbs (1962).

Swanson and Pazer (1971) also studied the σ formula of PERT. They implied in their paper that when the σ formula of PERT (1-1b) holds, the relationship between α and β is not exactly "$\alpha + \beta = 4$" (Swanson and Pazer, 1971, Figure 3, p. 470). This shows that when the σ is "fixed" (or restricted) at (V-U)/6, the PERT formula for μ is only an approximation, rather than an exact relationship.

Gallagher (1987) extended Sasieni's work (1986) and summarized the new findings and the result of Littlefield and Randolph's (1987) as follows:

The PERT formulae can be obtained in two ways:

1. **restrict** the set of possible beta distributions to those for which the **standard deviation** is EXACTLY 1/6 the range, then **approximate the mean** (Gallagher, 1987, p.1360; Littlefield and Randolph, 1987, p.1358); or

2. **restrict** the set of beta distribution to those for which α **+** β **= 4**, then **approximate the variance** (Gallagher, 1987, p.1360).

The inference from (2-9) to (2-12) above is a support to the conclusion of Gallagher's (1987).

All of the above studies point out that the PERT formulae are NOT valid for all beta distributions. It is valid for only a very small subset of beta distributions, contrary to what is claimed by the PERT originators and what is perceived by the PERT users.

2.1.2 PERT formulae result in gross numerical errors

MacCrimmon and Ryavec (1963) studied the worst absolute errors resulting from the PERT formulae. They expressed the mode, the mean, and the standard deviation in terms of α and β based on the properties of the beta distribution, as formulae (2-3) and (2-15). They solved for the mean and the standard deviation as functions of α and m, and obtained the worst absolute error in the mean as

$$|(1/6)(4m+1) - m(\alpha+1)/(\alpha+2m)|, \qquad (2\text{-}16)$$

and the worst absolute error in the standard deviation as

$$|1/6 - \sqrt{m^2(\alpha+1)(\alpha-\alpha m+m)/[(\alpha+2m)^2(\alpha+3m)]}|. \qquad (2\text{-}17)$$

They reported that the worst absolute error in the mean can be 33%, and in the standard deviation 17%.

MacCrimmon and Ryavec (1963) also studied the errors of mean and standard deviation resulting from PERT formulae based on possible errors from the estimates of a, b, and m. They suggested that the errors will be substantial for activities with narrower range (i.e., when a is close to b).

2.2 Questions on the Meanings and Estimates of a, b, and m

2.2.1 The meaning of a, b, and m

The PERT originators (Malcolm *et al.*, 1959) did not give explicit definitions for a, b, and m. They named a "optimistic time," b "pessimistic time," and m "most likely time." But they did imply that a and b are "two extremes," that is, the absolute endpoints. This is implied in Figure 3 in their paper (1959).

Swanson and Pazer (1971) did a survey on several popular OR textbooks, and found that the definitions of a and b are grossly different in those books. In some of the textbooks, a and b are defined as "absolute end points," while in the others they are defined as the two points between which the project has a 98% or 95% probability of being finished. This inconsistency ("a and b are endpoints" and "a and b embraces 95% of probability") can lead to difference in the values of a and b.

As for m, although most of the authors correctly defined it as mode, an author defined it as "a figure the planner felt he had a 50-50 chance of hitting," which is not consistent with the definition for the mode. Misconceptions like this can lead to gross errors.

2.2.2 The estimate of a, b, and m

Grubbs (1962) pointed out that "estimating end points may be tricky and hazardous business!", suggesting that it is very difficult for one to accurately estimate the end points of a probabilistic distribution.

Moder and Rogers (1968) pointed out that it is very rare, if not impossible, for a manager to experience the "absolute end points". Therefore, it can lead to very little reliability to ask managers to estimate a and b as "absolute end points". With the erroneous or unreliable a and b, one can expect gross errors in estimating mean and variance using PERT formulae based on the values of a, b, and m.

Swanson and Pazer (1971) pointed out that "it is admittedly easier for the estimator to conceive of a value which can be exceeded 1 per cent of the time rather than one which cannot be exceeded at all." They pointed out at the same time that "the use of the t_e(i.e., μ) and σ formulae will not yield an adequate transformation without a further compounding of the errors, and hence a new transformation is in order."

In summary, the PERT procedure of estimating the mean and the standard deviation of the task times in a project network has the following problems:

1. The PERT formulae cannot be inferred directly from the beta distribution without imposing extra restrictions or extra conditions. These extra restrictions or conditions make PERT formulae base only on a very small subset of the beta distributions, instead of general beta distributions as the PERT originators claimed.

2. Given a, b, and m as defined in PERT, there can be gross errors in the estimate of task times.

3. The meanings of a, b, and m have been ambiguous.

4. It is difficult, if not impossible, to estimate the a and b values.

2.3 Some Remedial Efforts

Some efforts have been made to remedy the aforementioned problems.

Moder and Rogers (1968), based on the "adjustmental" definition of a and b as "near end points (1 and 99 or 5 and 95 percentiles)" by some of the then practitioners ("Various PERT practitioners have taken liberty, and rightly so, with these definitions and changed them, for example, to 1 and 99 or 5 and 95 percentiles," Moder and Rogers, 1968), wisely perceived a possible relationship between the differences of these "paired percentiles (5 and 95)" and the standard deviation. They proposed that there might be a "robust" relationship between the differences of the percentiles and the standard deviations of the distributions. They studied five families of distributions: triangular, beta, uniform, normal, and exponential distributions. They found that the ratios of the differences between the paired percentiles to the standard deviations of the distributions are relatively more "robust" when the pairs of 5 and 95 are used, than when the pair of 0 and 100 are used. They rightly pointed out that if the 5 and 95 percentiles are to be used, the PERT formulae should be changed. But they failed to propose a new set of formulae based on theoretical analysis. Their formula for calculating μ is exactly the PERT version, except that they simply substituted the 5th and the 95th percentiles for the 0th and the 100th percentiles in the formula.

Perry and Greig (1975) proposed that the 5 and 95 percentiles be used instead of 0 and 100 percentiles. They proposed a set of empirical formulae in place of the PERT formulae. But their formulae are obtained "by experiments" instead of being based on theoretical analysis.

The studies mentioned above have largely been ignored by the textbooks. These studies, however, motivated the current research.

2.4 The Probability Elicitation Literature

The main logic of PERT formulae is to transform the three time estimates (a, b, and m) into the mean and standard deviation of the distribution. The first step of this procedure is to elicit fractiles for the (subjective) probability distribution, so that the values of a and b (no matter whether they are defined as the 0th and the 100th fractiles or as some "inner fractiles", such as the 5th and the 95th fractiles) can then be estimated. Large literature exists on this topic.

There is a key concept in this topic: fractiles. A "fractile" is also known as a "quantile" or a "percentile".

For a random variable T (such as the stochastic activity times in the PERT), we define T_α as T's α fractile, if

$$Prob(T < T_\alpha) = \alpha.$$

For example, $T_{0.1}$ is the 0.1 fractile of the random variable T, i.e.,

$$Prob(T < T_{0.1}) = 0.1.$$

There are many ways to estimate a subjective probability distribution, among which is the **fractile method**. In the fractile method, a number of required fractile levels α_i are specified, and a subject is asked to estimate the fractiles corresponding to these fractile levels. For example, if α_i are a set of 0.01, 0.1, 0.9, then the subject is asked to estimate $T_{0.01}$, $T_{0.1}$, and $T_{0.9}$.

Hampton *et al.* (1973) conducted a comprehensive study on the elicitation of subjective probability distributions. They studied the following six groups of methods: (1) Direct fractile approach, (2) Judgmental curve fitting, (3) Smoothing of historical data, (4) Psychometric ranking, (5) Hypothetical future samples, and (6) Equivalent prior sample. They concluded that, among the six groups of methods to elicit subjective probability distributions, the direct fractile approach, which is the same as the "fractile method" stated in the previous paragraph, is "the most useful method."

In a survey published in 1975, Chesley (1975) compared two groups of techniques:

(1) Direct Methods: direct estimation, odds estimation, graphical, hypothetical future sample, distribution parameter estimation, etc.

(2) Inference techniques: betting.

The "direct estimation" in the group "Direct Methods" is the same as the "fractile method" introduced above. Chesley reported that direct estimation (or fractile method) is "the simplest technique in view of question construction" (among the direct techniques), and he implied that more consistent distributions can be expected with this method.

Winterfeldt and Edwards (1986) also regarded the fractile method as the best known method in eliciting probability distributions. They further pointed out that the fractile method facilitates the consistency check, which improves the validity of this method.

Given its merit, the fractile method has been further studied and its operational features probed by later researchers. Selvidge (1980) studied the procedure of handling fractile method in probability elicitation, with an emphasis on the assessment of the extremes of probability distributions, which is one of our major concerns in PERT probabilistic activity time estimate. In his study, Selvidge compared four procedures with different features: number of fractiles asked to assess, order of fractiles assessed, and whether or not the assessment procedure is divided into stages. Selvidge's study has the following findings:

1. The process performs better when the subjects are asked to assess seven fractiles than when they are asked to assess five fractiles. In the seven-fractile case, in addition to the three "central fractiles" $T_{0.25}$, $T_{0.50}$, and $T_{0.75}$, the "extremes" $T_{0.01}$, $T_{0.10}$, $T_{0.90}$, and $T_{0.99}$ are assessed; while in the five-fractile case, the "extremes" assessed are $T_{0.10}$ and $T_{0.90}$. Selvidge has found that with the seven-fractile method, the subjects can assess the "extreme" fractiles more accurately.

2. The process performs better when the "central" fractiles $T_{0.25}$, $T_{0.50}$, and $T_{0.75}$ are assessed first.

Selvidge's findings provide a strong methodological background for the current study at the stage of probability elicitation, which is what the converting formulae (from fractiles to μ and σ) are based on.

CHAPTER III

RESEARCH METHODOLOGY

3.1 Objectives of the Current Study

The PERT procedure has two components: the elicitation of subjective estimates related to probabilistic times, and the conversion of these estimates to the mean and standard deviation of the stochastic activity time. Charles E. Clark (1962), one of the PERT originators indicated that the mean and standard deviation of a distribution are "too complex for immediate appraisal," and proposed that the mode of the distribution be first estimated and then the "extreme times" are estimated. One can then convert the information available (the mode, and the two extreme times) into expected value and variance of the stochastic time (Clark, 1962).

We believe that the basic objectives behind a procedure like that of PERT are:

Objective 1, elicit subjective time estimates from an "expert" (a manager or engineer, who has the expertise in the activities of the project);

Objective 2, convert these estimates into the mean and standard deviation of the time, **recognizing that the distribution of the time can have a wide variety of shapes** (as does the beta distribution).

Although the PERT originators were aware of the above objectives (explicitly or implicitly), the procedure they developed does not really achieve the objectives. The objectives of the current study are, therefore, to further study the PERT procedure and find out the shortcomings of the existing procedure in handling the above two objectives, and to develop an alternative which can effectively achieve the objectives. Specifically, the current study will

1. study and point out the PERT procedure's shortcomings in defining, using, and eliciting the values a and b;

2. study and point out the PERT procedure's shortcomings in using and eliciting the value m;

3. study and point out the PERT procedure's logical shortcomings in the conversion formulae;

4. propose a set of alternative formulae, which are valid for a wide range of shapes of beta distributions and are more accurate than the PERT conversion formulae

(Objective 2), and which are based on the data obtained in consistence with the probability elicitation literature (Objective 1);

5. conduct simulations on the PERT formulae and the proposed alternative, calculate the mean and the standard deviation from the data generated, show the advantage of the alternative over the PERT procedure.

3.2 On Making Subjective Time Estimates

3.2.1 The fractile method

Estimating a, m and b in PERT is the initial step for obtaining the "subjective probability distribution" of the stochastic task time T. A large body of literature exists on the elicitation of subjective probability distributions (for example, Hampton *et al.*, 1973; Chesley, 1975; Wallsten and Budescu, 1983; Winterfeldt and Edwards, 1986). From this literature, it is apparent that the most common method of eliciting T's subjective probability distribution is the "fractile method" as discussed in Section 2.4.

With reference to the subjective probability literature, the PERT procedure of estimating a, m, and b has shortcomings as discussed in the following subsections.

3.2.2 Ambiguity of defining and inadequacy of using "a" and "b"

According to the original PERT developers (Malcolm *et al.*, 1959), a and b are the "absolute endpoints" T_0 and T_1, respectively. Many widely used OR/MS textbooks (e.g., Hillier and Lieberman, 1980, p.252; Gould, Eppen, and Schmidt, 1991, p.444; Taylor III, 1993, p.626) state or imply (in graphs) that a and b are the absolute endpoints or "upper and lower bounds" of the distributions of the task times of interest. However, the probability elicitation literature (e.g., Alpert and Raiffa, 1969; Selvidge, 1980) indicates that it is difficult for a person to estimate accurately the absolute endpoints (T_0 and T_1) of a stochastic quantity. Our common sense also suggests that it is difficult to locate the two extremes of a stochastic quantity. **But a and b being the "absolute endpoints" is the necessity for the PERT formulae to hold.** This is stated by the PERT developers, and is studied and confirmed by many researchers (Sasieni, 1986; Littlefield and Randolph, 1987; Gallagher, 1987). As mentioned in Chapter II, recent studies point out that PERT formulae hold only for a restricted subset of the beta distribution and **only when a and b are "absolute endpoints" of the distribution.** This brings up a dilemma: PERT formulae hold only when a and b are absolute endpoints, but absolute endpoints cannot be actually estimated by human being.

On the other hand, many other MS/OR textbooks state that a and b should be T's 0.01 and 0.99 fractiles (e.g., Buffa and Miller, 1979, p.624; Lee, Moore, and Taylor, 1990, p.299). The probability elicitation literature indicates that it is more appropriate for one to estimate the 0.01 and the 0.99 fractiles than the absolute endpoints, and a more accurate result can be expected from the estimate of the "inner fractiles" (i.e., not the extremes T_0 and T_1). **However**, these "inner fractiles" are inconsistent with the justifications of the PERT formulae ("PERT formulae are valid ONLY when a and b are endpoints") given in Malcolm et al. (1959), Littlefield and Randolph (1987) and Gallagher (1987), as seen in the previous paragraph. This brings up the second dilemma: it is more reasonable to elicit $T_{0.01}$ and $T_{0.99}$ from the viewpoint of the probability elicitation literature under the criteria of the ease and the accuracy of the estimate, but the a and b so defined are not consistent with the justification of the PERT procedure.

One might try to overcome the two dilemmas and justify the substitution of the inner fractiles into the PERT formulae by claiming that $T_{0.01}$ (or $T_{0.99}$) is very close to T_0 (or T_1), and therefore the discrepancy is negligible. We will illustrate later (in section 5.1.3) that this discrepancy can be very substantial and therefore the justification for the substitution is on a shaky ground.

3.2.3 Shortcomings of using "m"

According to the PERT developers, the value m is the mode of the time distribution. It is defined as "the most likely time" of finishing the activity of interest. We know that the mode in different distributions corresponds to very different fractiles. On the other hand, a and b are both prescribed fractiles, whether they are defined as absolute endpoints T_0 and T_1 or as "inner fractiles" $T_{0.01}$ and $T_{0.99}$. Then we have the following problem: the "expert" (engineer or manager) is asked to estimate two fractiles and the mode, which is **NOT** a **PRESCRIBED** fractile. It is very likely that the "expert" will be confused in estimating two prescribed fractiles and another value which is not a prescribed fractile. In this case, the "expert" may confuse the mode with the median (**which is a fractile ($T_{0.5}$)**). Trout (1989) and his reviewer raised a plausible supposition: most managers are not clear about the distinction between a mode and a median. Therefore, when a manager is asked to make three estimates (i.e., a, b, and m) where two (i.e., a and b) are prescribed fractiles but one is not, there is little assurance that the manager will not end up with estimating the median (a fractile) instead of the mode for "m". Actually, we even have a ready example of such an error, made not by a manager, but by a person in academia: Timms (1966) defined "m" in his textbook as follows:

> "... a figure the planner felt he had a 50-50 chance of hitting" (**which is NOT the mode!** -- words between the parentheses are by the author of this dissertation)

The "50-50 chance of hitting (a target due date)" here indicates that this is a date which the project will last longer with a 50% chance, and which the project will be able to be finished on or by with a 50% chance. The above definition is an example that there is a good chance for a manager to confuse the median for the mode.

It would not be a big concern in practice if the mode and the median were not so far apart (although confusing the median for the mode is already not a trivial conceptual error). Unfortunately, the actual difference between the mode and the median can be very substantial for the type of asymmetrical distributions that the PERT and the beta distribution are explicitly designed to handle. So, asking the managers or the "experts" to estimate a, b, and m can be a major source of both conceptual and numerical errors.

Another shortcoming of using the mode is, while the existing probability elicitation literature provides various methods to check and adjust the consistency of the estimate of fractiles (including the median) (see, e.g., Lichtenstein *et al.*, 1982), we have not seen such "check and adjust" methods established for estimating the mode of a probability distribution. Without a "consistency check", the validity of the estimate of the mode and the accuracy of the resulting value is in question.

3.3 The Logical Inadequacy of the PERT Formulae

It is widely accepted that the reason of the PERT's employing the beta distribution is to convert the three time estimates (a, m, and b) to the (supposedly true) mean and variance (or standard deviation) of the task time **which can have a wide range of shapes** (e.g., Clark, 1962; Grubbs, 1962; Swanson and Pazer, 1971). Unfortunately, the PERT formulae and their basis on the three time estimates (a, m, and b) fail to achieve the above purpose.

One of the logical shortcomings of the PERT procedure is that the PERT formulae are valid only for a very restricted subset of the beta distribution, which has been pointed out by many researchers (see the related discussion in Chapter II).

In addition to the above shortcoming, the PERT procedure actually cannot define a beta distribution. A beta distribution with range (U,V) has the following form:

$$f(t) = (t-U)^{p-1}(V-t)^{q-1}/[B(p,q)(V-U)^{p+q-1}] \qquad (3\text{-}1)$$

This distribution has four parameters: U, V, p, and q (in comparison with the formula for beta distribution in Chapter II, we use (p,q) to denote the beta distribution parameter hereafter, instead of $(\alpha+1, \beta+1)$ as used in that chapter. The transformations between the two are obvious: $p = \alpha + 1$, and $q = \beta + 1$). The first two parameters (U and V) determine the two endpoints of the distribution, therefore determining the "location" of the distribution; while the latter two (p and q) determine the skewness and kurtosis (the "shape" of the distribution). It is, therefore, natural to reason that it takes at least four parameters to determine or to specify a beta distribution. But in the PERT procedure, only three values (a, m, and b) are estimated, and they are then used to "determine" a **FOUR-PARAMETER** beta distribution. In this case, with one "free" parameter, the beta distribution is actually not determined.

To determine a four-parameter beta distribution with only three parameters is another logical shortcoming of the PERT procedure.

In addition, with the definition of a and b being the "absolute endpoints" (which is the condition for the PERT formulae to be correct), the PERT procedure does not use all the available information about a specific distribution, because a and b so defined do not reflect the "shape" of a distribution, and the determination of the "shape" of the distribution is solely on the value of m, which is not the case in the beta distribution. This issue will be further discussed in Chapter IV.

3.4 A Logical Alternative

We discussed above the PERT procedure's shortcomings in both the subjective probability elicitation (objective 1) and the conversion of a, m, and b to the mean and standard deviation of the task time distribution (objective 2). We will develop a logical alternative to the PERT time estimating procedure, to improve the procedure and to achieve both of the objectives.

3.4.1 Some basic properties of the beta distribution

For the f(t) given in (3-1), t's mean and standard deviation are:

$$\mu = U + (V-U)p/(p+q),$$
$$\sigma = (V-U)\sqrt{pq/[(p+q)^2(p+q+1)]}. \qquad (3\text{-}2)$$

The parameters(U,V) in f(t) are the distribution's two endpoints, and the parameters (p,q) control the distribution's "shape" (skewness and kurtosis). The distribution is symmetrical when the ratio $p/q = 1$; its skewness and kurtosis increase as the ratio p/q deviates from 1. If $p<1$ and/or $q<1$, the distribution is J- or U-shaped. As p and q increase from 1, the distribution evolves from a uniform distribution and tends to a normal distribution as p and q become large. Therefore, for practical purposes we will consider only f(t) with $1<(p,q)<100$ (say).

The mean and standard deviation of a beta distribution can be calculated using formulae (3-2), once the p and q values are known. We will use this property of the beta distribution to generate the data sets of mean and standard deviation for the linear regression to determine the linear expressions of μ and σ as functions of fractiles, which is discussed in the next subsection.

3.4.2 μ and σ as linear combination of fractiles

Moder and Rogers (1968) proposed that the 5 and 95 fractiles be used in the places of 0 and 100 fractiles. But they did not modify the PERT formulae correspondingly (which they should, because the PERT formulae are correct only when $a = T_0$ and $b = T_1$). Instead, they simply substituted the T_0

(lower endpoint a, or U) and T_1 (upper endpoint b, or V) with $T_{0.05}$ and $T_{0.95}$. As we have pointed out in the previous section, $T_{0.05}$ can be far from T_0, and $T_{0.95}$ can be far from T_1. The substitution of the "inner fractiles" without changing the conversion formulae accordingly can lead to substantial errors, as will be illustrated later with numerical examples. In addition, the Moder and Rogers' method was from experiments and needs theoretical support.

Pearson and Tukey, in a study not intended for PERT (1965), suggested that a distribution's mean and standard deviation may be approximated by linear functions of the distribution's fractiles. They suggested the following formula as the approximation of the mean:

$$\mu = T_{0.5} + 0.185\Delta,$$

where $\Delta = T_{0.95} + T_{0.05} - 2T_{0.50}$.

They also proposed an iterative procedure for approaching the value of standard deviation. Their results motivated our study to find μ and σ as linear combinations of fractiles.

Many of the earlier empirical works on estimating subjective probability distributions recommend that, except for the median, fractiles should be estimated in symmetrical pairs (e.g., Hampton *et al.*, 1973; Selvidge, 1980; Solomon, 1982). Pearson and Tukey (1965) also used "paired" fractiles in their formulae ($T_{0.95}$ and $T_{0.05}$). It is plausible for the symmetrical fractile pairs to be used, since people would tend to perceive, for example, $T_{0.25}$ and $T_{0.75}$ (or $T_{0.20}$ and $T_{0.80}$) better than they do $T_{0.20}$ and then $T_{0.60}$, the latter two being asymmetrical. All of the above enlightened us to employ paired fractiles in our linear functions to estimate μ and σ.

We tried to express the mean of a standardized beta distribution as follows:

$$\mu(t) = a + bt_{0.1} + ct_{0.9} + dt_{0.5}, \qquad (3\text{-}3)$$

here a, b, c, and d are constants.

We can similarly obtain the mean for a generalized beta distributed random variable T:

$$\mu(T) = a + bT_{0.1} + cT_{0.9} + dT_{0.5} \qquad (3\text{-}4)$$

Through the transformation $T = U + (V-U)t$, we can determine the value for some of the coefficients.

Similarly, we can construct the function for estimating σ, and determine some of the coefficients through the comparison of the functions for standardized and general beta distributions.

To obtain the remaining coefficients of the linear combinations, we will conduct linear regression on the means (standard deviations) and the fractiles. We will generate various p-q values (for various beta distributions) and obtain the specified fractiles (say, $T_{0.10}$, $T_{0.25}$, $T_{0.5}$, $T_{0.75}$, $T_{0.90}$) for the beta distributions with those various p-q values. We will then calculate the μ and σ of these distributions based on the properties of the beta distribution (formulae (3-2)), using the p-q values generated. Linear regression will be conducted on the data sets (the μ/σ data sets and the fractile data set) so generated. Coefficients of the linear combination of fractiles to estimate μ and σ will thus be obtained.

Since the simplicity of the PERT formulae (in their mathematical form) has been one of the strong arguments of the PERT proponents, we will try to find a simple form for our improved formulae, so that numerical and logical improvements are achieved without incurring an extra mathematical difficulty or operational burden. In a similar consideration, we try to keep as much as possible the fundamental and reasonable PERT assumptions such as the stochastic time being beta distributed, so that the improved method will be accepted and implemented with relatively little change to the *status quo*.

3.4.3 A "clean" fractile method

In Section 3.2 we pointed out PERT's shortcomings in using a, m, and b, two of them being fractiles and one being not (a prescribed fractile). We believe that one should employ a "clean" fractile method in which only fractiles (that is, $T_{\alpha i}$'s, e.g., $T_{0.01}$, $T_{0.1}$, $T_{0.5}$, etc.) are used. In such a method, only fractiles are to be estimated, and the fractile levels (i.e., the α_i for the $T_{\alpha i}$'s) should be clearly specified.

The next question needs to answer is: how many and which fractiles should be estimated? Selvidge (1980) showed that the following fractile estimation procedure performed best, in terms of the accuracy of the estimates of the values of the "extreme fractiles":

1. Assess seven fractiles. That is, the three central fractiles: the 0.25, 0.50, and 0.75 fractiles; and the four extreme fractiles: the 0.01, 0.10, 0.90, and 0.99 fractiles.

2. Assess the central fractiles first.

We will employ this procedure in our study.

3.5 Error Analysis to Compare the Proposed Alternative against the PERT Procedure

Error analysis will be conducted to compare the performance of the proposed method with that of the PERT procedure. The error analysis will be conducted in the following four aspects:

1. Comparison of the accuracy of the proposed alternative with the PERT procedure over a wide range of beta distribution. Since it is the objective of the PERT developers for their method to cover a wide range of shapes of the task time distributions, it is reasonable to compare the performance of the two methods on a wide range of shapes of the beta distributions (i.e., beta distributions with wide range of p-q parameters).

2. Error analysis of the PERT formulae over a specified range of shapes of beta distributions. As pointed out by the existing studies, the PERT formulae only hold for a restricted subset of the beta distributions. Although this is already a shortcoming of the PERT formulae, we try to further conduct analysis over the cases in which the PERT formulae are supposed to be "correct." We will conduct error analyses on this already restricted subset, to see

whether the PERT formulae can lead to satisfactory accuracy over this (restricted) subset.

3. Comparison of the simplified formulae of the proposed alternative with the PERT formulae. As proposed in Section 3.4.2, we will try to seek a simple form for the formulae in the proposed alternative to keep the simplicity of the time estimating procedure. We foresee that it may be necessary to sacrifice accuracy (to a certain extent) in order for this simplicity to be pursued. We will thus compare the simplified formulae with the PERT formulae, to examine the improvement of (even) the simplified formulae over the PERT formulae.

4. Numerical examples to indicate the discrepancy of substitution of the endpoints with the "inner fractiles" in the PERT formulae. It has been believed by some researchers and practitioners that the "inner fractiles" (e.g., $T_{0.01}$, $T_{0.99}$; $T_{0.05}$, $T_{0.95}$) are not so far from the "absolute endpoints" (T_0 and T_1), and therefore it is acceptable to substitute the endpoints with the "inner fractiles." We will show that the "inner fractiles" can be very far from the endpoints. Gross errors may occur when the "inner fractiles" are used in the places of the endpoints.

CHAPTER IV

CONSTRUCTING LINEAR FUNCTIONS OF FRACTILES FOR APPROXIMATING μ AND σ

4.1 Derivation of μ Function and σ Function

4.1.1 Derivation of μ function

Let T be any beta variable with endpoints U and V, and t be the corresponding standardized beta variable:

$$T = U + (V-U)t \tag{4-1}$$

Let W = V - U, then

$$T = U + Wt \tag{4-2}$$

For simplicity's sake, assume that only the symmetrical fractile pair $t_{0.1}$ and $t_{0.9}$, and median $t_{0.5}$ will be used to construct a linear function for estimating μ (as is indicated in the probability elicitation literature and the study of Pearson and Tukey, 1965. See section 3.4.2 of this dissertation).

Assume that a desired μ function is of the form

$$\mu(t) = a + bt_{0.1} + ct_{0.9} + dt_{0.5} \tag{4-3}$$

where a, b, and c are constants, and the $t_{\alpha i}$'s are t's α_i fractiles.

For generalized beta variable T, we have

$$\mu(T) = a + bT_{0.1} + cT_{0.9} + dT_{0.5} \tag{4-4}$$

Take the mathematical expectation of (4-2), we have

$$\mu(T) = U + W\mu(t), \tag{4-5}$$

since both U and W are constants.

Combining (4-3) and (4-5):

$$\begin{aligned} \mu(T) &= U + W\,(a + bt_{0.1} + ct_{0.9} + dt_{0.5}) \\ &= U + aW + (bt_{0.1} + ct_{0.9} + dt_{0.5})W \end{aligned} \tag{4-6}$$

However, combining (4-2) and (4-4) gives

$$\begin{aligned} \mu(T) &= a + bT_{0.1} + cT_{0.9} + dT_{0.5} \\ &= a + b(U+Wt_{0.1}) + c(U+Wt_{0.9}) + d(U+Wt_{0.5}) \\ &= a + (b+c+d)U + (bt_{0.1} + ct_{0.9} + dt_{0.5})W \end{aligned} \tag{4-7}$$

Since (4-6) and (4-7) must be equivalent, we must have

$$\begin{aligned} a &= 0 \\ b + c + d &= 1 \end{aligned} \tag{4-8}$$

In (4-8) we can see that the coefficients of the various fractiles must add up to one; or, the "weights" attached to the fractiles must add up to one. Please note that we did not pre-specify these as conditions of the linear combination; yet they emerged in the procedure of the inference as properties of the combination.

Consider now a special case in which t is symmetric with mean $\mu = t_{0.5} = 0$. In (4-3), in order for $\mu = 0$, we must have $b = c$, whereby the two symmetrical fractiles $t_{0.1}$ and $t_{0.9}$ (which have same absolute value yet opposite signs) can cancel. From this we know that μ functions should contain only the sums of symmetrical fractile pairs. In the case of the general beta distributed random variable T, we define the "inter-fractile sum" S10 as:

$$S10 \equiv T_{0.10} + T_{0.90} \tag{4-9a}$$

We can similarly define other "inter-fractile sums" S01 and S25 as follows:

$$S01 \equiv T_{0.01} + T_{0.99} \tag{4-9b}$$

$$S25 \equiv T_{0.25} + T_{0.75} \tag{4-9c}$$

With the inter-fractile sums defined above, the linear function for the μ of the distribution should have the form:

for seven fractiles:

$$\mu = k_1(S01) + k_2(S10) + k_3(S25) + k_4(T_{0.5}) \tag{4-10}$$

for five fractiles:

$$\mu = c_1(S10) + c_2(S25) + c_3(T_{0.5}) \tag{4-11}$$

here the k_i's and the c_i's are constants to be determined.

4.1.2 Derivation of σ function

Apply the preceding μ function logic to the σ functions. We again start with the standardized beta variable t.

Assume that a desired σ function is of the form

$$\sigma(t) = a + bt_{0.1} + ct_{0.9}. \qquad (4\text{-}12)$$

(4-12) should also be valid for T. So $\sigma(T)$ can be computed as

$$\sigma(T) = a + bT_{0.1} + cT_{0.9} \qquad (4\text{-}13)$$

Combining (4-2) and (4-13) gives

$$\sigma(T) = a + (b+c)U + (bt_{0.1} + ct_{0.9})W \qquad (4\text{-}14)$$

However, $\sigma(T) = W\sigma(t)$ must also hold, because W is a constant. Substituting (4-12) into this gives:

$$\sigma(T) = aW + (bt_{0.1} + ct_{0.9})W \qquad (4\text{-}15)$$

Since (4-14) and (4-15) must be equivalent, we must have

$$\begin{aligned} a &= 0 \\ b + c &= 0 \end{aligned} \qquad (4\text{-}16)$$

Formulae (4-16) indicate that the coefficients of the two symmetrical fractiles $t_{0.1}$ and $t_{0.9}$ are of opposite signs. It is also true for their counterparts $T_{0.1}$ and $T_{0.9}$ in generalized beta distribution. We can therefore combine

$T_{0.1}$ and $T_{0.9}$ into a single term which is referred to as the "inter-fractile difference":

$$D10 \equiv T_{0.90} - T_{0.10} \tag{4-17a}$$

Similarly, we can define other "inter-fractile differences" D01 and D25 as follows:

$$D01 \equiv T_{0.99} - T_{0.01} \tag{4-17b}$$

$$D25 \equiv T_{0.75} - T_{0.25} \tag{4-17c}$$

With the inter-fractile differences defined above, the linear function for the σ of the distribution should have the form:

for seven fractiles:

$$\sigma = k_5(D01) + k_6(D10) + k_7(D25) \tag{4-18}$$

for five fractiles:

$$\sigma = c_4(D10) + c_5(D25) \tag{4-19}$$

here k_i's and c_i's are constants to be determined.

We now list the four formulae for the μ and the σ functions from this and the previous subsections and renumber them for clarity's sake as follows:

for seven fractiles:

$$\mu = k_1(S01) + k_2(S10) + k_3(S25) + k_4(T_{0.5}) \tag{4-20a}$$

$$\sigma = k_5(D01) + k_6(D10) + k_7(D25) \tag{4-20b}$$

for five fractiles:

$$\mu = c_1(S10) + c_2(S25) + c_3(T_{0.5}) \quad (4\text{-}21a)$$

$$\sigma = c_4(D10) + c_5(D25) \quad (4\text{-}21b)$$

4.2 General Description of Estimating the Coefficients k_i's and c_i's

4.2.1 The objective

Formulae (4-20) and (4-21) have given the general form for the μ and the σ of a beta distributed random variable to be expressed as linear combinations of fractiles of that distribution. We still need to know the coefficients k_i's and c_i's, in order to use these formulae to estimate the μ and σ of the beta distribution.

The objective can be expressed as:

To determine the values of the k_i's and the c_i's in formulae (4-20) and (4-21) that will estimate μ and σ **accurately** for all beta distributions (i.e., for all combinations of "shape parameters" p and q).

4.2.2 Using standard beta distributions

In Section 4.1 we showed the derivation of μ function and σ function without restricting the value of the endpoints U and V. In other words, values of k_i and c_i

applicable for one set of (U,V) should also be applicable to all other sets of (U,V). Therefore, we only need to consider the standardized beta distributions in which U = 0 and V = 1. This will not only simplify the procedure but also make the illustration clearer.

The following linear regression procedure is used to estimate the k_i's and c_i's.

4.3 Linear Regression to Determine k_i's and c_i's

To determine the coefficients in formulae (4-20) and (4-21), we should obtain the data sets of μ's, σ's, and the "inter-fractile sums" and the "inter-fractile differences", or the fractiles. We then perform linear regression on these data sets and obtain the coefficients for the linear functions with μ/σ as dependent variables and the "inter-fractile sums/differences" as independent variables. The following subsection will show the procedure of the linear regression.

4.3.1 Generating the data sets

To construct the data sets, we first need to obtain various beta distributions. The beta "shape parameters" (p,q) are first generated randomly with a uniform random number generator, in the ranges $1 < p < 100$ and $1 < q < 100$, so that the resulting beta distributions are not restricted in some specific, biased subset. For each pair of (p,q), the required fractiles (e.g., $T_{0.01}$, $T_{0.10}$, $T_{0.25}$, $T_{0.50}$, $T_{0.75}$, $T_{0.90}$, $T_{0.99}$) for the standardized beta distribution with these (p,q) parameters are computed using subroutine BETIN (meaning "beta inverse") in IMSL (1987) Library. For example, assume that we randomly generate

$$(p,q) = (61.98,\ 20.62). \qquad (4\text{-}22)$$

The fractiles of a standardized beta distribution with this pair of (p,q) parameters are then computed by BETIN as:

$$\begin{aligned}&T_{0.01} = 0.6322,\ \ T_{0.10} = 0.6882,\\ &T_{0.25} = 0.7194,\ \ T_{0.50} = 0.7524,\ \ T_{0.75} = 0.7835, \qquad (4\text{-}23)\\ &T_{0.90} = 0.8098,\ \ T_{0.99} = 0.8507\end{aligned}$$

Based on the $T_{\alpha i}$'s above, we can then obtain the "inter-fractile sums/differences" through arithmetic operations using formulae (4-9) and (4-17):

$$\begin{aligned}
S01 &= T_{0.01} + T_{0.99} = 1.4829, \\
S10 &= T_{0.10} + T_{0.90} = 1.4890, \\
S25 &= T_{0.25} + T_{0.75} = 1.5029, \\
D01 &= T_{0.99} - T_{0.01} = 0.2185, \\
D10 &= T_{0.90} - T_{0.10} = 0.1216, \\
D25 &= T_{0.75} - T_{0.25} = 0.0641.
\end{aligned} \tag{4-24}$$

Repeating (4-22) to (4-24) for, say, 2000 times gives 2000 sets of values "observations" which are the "independent variables" in formulae (4-20) and (4-21).

We still need the "dependent variables" to perform the linear regression. To obtain these "dependent variables" (μ's and σ's), we use formulae (3-2) which is redisplayed below:

$$\begin{aligned}
\mu &= U + (V-U)p/(p+q) \\
\sigma &= (V-U)\sqrt{pq/[(p+q)^2(p+q+1)]}
\end{aligned} \tag{3-2}$$

Since U = 0 and V = 1 now, we have

$$\begin{aligned}
\mu &= p/(p+q) \\
\sigma &= \sqrt{pq/[(p+q)^2(p+q+1)]}
\end{aligned} \tag{4-25}$$

With the (p,q) randomly generated as described earlier in this section, we can then have the corresponding μ's and σ's using formulae (4-25). For our (p,q) = (61.98, 20.62) in

(4-22), the μ and σ are respectively

$$\mu = 0.7503,$$

$$\sigma = 0.0473.$$

Repeating this procedure for 2000 times, we obtain the data sets of the "dependent variables" (μ and σ).

We employ the SAS package to perform the linear regression using these 2000 sets of "observations" and the models stated in formulae (4-20) and (4-21). The linear regression determines the coefficients c_i's and k_i's.

4.3.2 Results of the linear regression

Linear regression is performed on 2000 sets of μ, σ, and S01, S10, S25, D01, D10, D25 obtained from the $T_{\alpha i}$'s (seven fractiles). The coefficients for the μ function and the σ function are obtained as follows:

Table 4-1. Estimates of Coefficients in (4-20)

μ function					σ function			
k_1	k_2	k_3	k_4	R^2	k_5	k_6	k_7	R^2
0.0375	0.1187	0.2230	0.2415	1.000	0.1934	-0.5505	1.1227	0.9996

From the coefficients obtained above, we have our μ function and σ function as follows:

$$\mu = 0.0375 \times S01 + 0.1187 \times S10 + 0.2230 \times S25 + 0.2415 \times T_{0.5}$$
$$\sigma = 0.4219 \times D01 - 0.4390 \times D10 + 1.0341 \times D25 \qquad (4\text{-}26)$$

The above regression procedure is repeated with another 2000 sets of "observations" μ, σ, S01, S10, S25, D01, D10, D25, and $T_{0.5}$ (seven fractiles). The second set of coefficients are obtained as follows:

Table 4-2. Estimates of Coefficients in (4-20) (Data Set 2)

μ function					σ function			
k_1	k_2	k_3	k_4	R^2	k_5	k_6	k_7	R^2
0.0384	0.1130	0.2335	0.2301	1.000	0.2047	-0.6321	1.2390	0.9994

Regression for the five-fractile model are than performed on the two data sets. The coefficients obtained are presented in the following table.

Table 4-3. Estimates of Coefficients in (4-21)
(Five-fractile Models)

	μ function				σ function		
	c_1	c_2	c_3	R^2	c_4	c_5	R^2
Data Set 1	0.4219	-0.4390	1.0341	1.000	0.6819	-0.5532	0.9995
Data Set 2	0.3945	-0.3429	0.8967	1.000	0.7442	-0.6709	0.9994

We can see that the R^2 values in the above functions are all very high (0.9994 to 1.000). This confirms our conjecture before that a beta distribution's μ and σ can be accurately estimated by linear functions of the distribution's fractiles.

There is a possibility that the high R^2 value might be coincident with a specific set of data, as illustrated in Draper and Smith (1966, P. 63). In our case, however, high R^2 values were obtained for regressions performed on many different sets. This has ruled out the possibility of coincidence. We can, of course, find a smaller data set to provide a satisfactory confidence interval for the regression, but since the computer time spent on our current regressions is a matter of a few seconds, we did not investigate the question further.

4.3.3 Features of the coefficients

We note that, the μ functions generated in the linear regression are

for seven fractiles:

$$\mu_e = k_1(S01) + k_2(S10) + k_3(S25) + k_4(T_{0.5}) \qquad (4\text{-}10)$$

for five fractiles:

$$\mu_e = c_1(S10) + c_2(S25) + c_3(T_{0.5}). \qquad (4\text{-}11)$$

We rewrite formulae (4-10) and (4-11), substituting the "inter-fractile sums" S01, S10, and S25 with the sum of fractiles $T_{0.01}$, $T_{0.10}$, $T_{0.25}$, $T_{0.75}$, $T_{0.90}$, and $T_{0.99}$ explicitly:

for seven-fractiles:

$$\mu_e = k_1(T_{0.01}+T_{0.99}) + k_2(T_{0.10}+T_{0.90}) + k_3(T_{0.25}+T_{0.75}) + + k_4(T_{0.5}) \qquad (4\text{-}27)$$

for five-fractiles:

$$\mu_e = c_1(T_{0.10}+T_{0.90}) + c_2(T_{0.25}+T_{0.75}) + c_3(T_{0.5}) \qquad (4\text{-}28)$$

According to (4-8) in our inference of the μ functions, we expect that the coefficients for the fractiles sum up to 1, that is:

for seven-fractile models:

$$k_1 + k_1 + k_2 + k_2 + k_3 + k_3 + k_4 = 1 \qquad (4\text{-}29)$$

for five-fractile models:

$$c_1 + c_1 + c_2 + c_2 + c_3 = 1. \qquad (4\text{-}30)$$

Our μ functions resulting from the linear regression confirm this anticipation:

from Table 4-1 (seven-fractile model based on Data Set 1):

$$2\times0.0375 + 2\times0.1187 + 2\times0.2230 + 0.2415 = 0.9999$$

from Table 4-2 (seven-fractile model based on Data Set 2):

$$2\times0.0384 + 2\times0.1130 + 2\times0.2335 + 0.2301 = 0.9999$$

from Table 4-3:

$$2\times(0.4219 - 0.4390) + 1.0341 = 0.9999$$

$$2\times(0.3945 - 0.3429) + 0.8967 = 0.9999$$

Formulae (4-29) and (4-30) and the above example show that, the μ functions generated in the linear regression always turn out to be a "weighted average" of the fractiles. Interestingly, this is not pre-imposed when the regression is performed. The result confirms the theoretical prediction of formulae (4-8).

4.3.4 Searching for "simple" coefficients

An observation of the results of the regressions performed on the 2000-observation data sets is that the coefficients k_i's and c_i's can be quite different from data set to data set. For example, the c_2 from data set 1 is -

0.4390 but the c_2 from data set 2 is -0.3429. In order to probe this phenomenon, we generated three more data sets of 1000, 3000, and 4000 "observations" of fractiles and μ/σ, respectively, and performed regression on these data sets. The results are as follows:

Table 4-4. Estimates of Coefficients, Seven-fractile Models

Num. of obs.	μ function					σ function			
	k_1	k_2	k_3	k_4	R^2	k_5	k_6	k_7	R^2
1000	0.0388	0.1050	0.2575	0.1976	1.0000	0.3416	-1.5410	2.4954	0.9996
3000	0.0388	0.1024	0.2660	0.1855	1.0000	0.2706	-1.1214	1.9414	0.9995
4000	0.0388	0.1036	0.2626	0.1901	1.0000	0.2448	-0.8334	1.4863	0.9994

Table 4-5. Estimates of Coefficients (Five-fractile Models)

Num.of obs.	μ function				σ function		
	c_1	c_2	c_3	R^2	c_4	c_5	R^2
1000	0.308	-0.048	0.481	1.000	0.667	-0.530	0.9996
3000	0.303	-0.033	0.460	1.000	0.810	-0.796	0.9993
4000	0.368	-0.264	0.792	1.000	0.672	-0.538	0.9994

Again, we can see that values for the same coefficients are different among data sets. This fact is seemingly disturbing. If the models are valid, should the same coefficient not have the same (or close) value among different data sets? This is a legitimate question. Before we performed in-depth study, however, we found that the feature found in the previous subsection that "weights for fractiles in μ functions sum up to 1" still holds. For example, for seven-fractile μ functions:

1000 "observations":

$$2\times(0.0388 + 0.1036 + 0.2626) + 0.1901 = 1.0001$$

3000 "observations":

$$2\times(0.0388 + 0.1024 + 0.2660) + 0.1855 = 0.9999$$

4000 observations:

$$2\times(0.0388 + 0.1049 + 0.2574) + 0.1975 = .09997$$

In order to verify the validity of our models, we used the μ and σ functions developed with data set 1 to estimate the μ and σ in data set 2, and vice versa, and found that the "switched" functions give the same high R^2 values with the other data set. We also repeated this experiment with the data sets with 1000, 3000, and 4000 "observations", and obtained the same high R^2 values.

The fact that (1) high R^2 values are obtained on the "switched" data sets and (2) same coefficient (e.g., c_2) can have different values in the linear functions obtained from different data sets suggests that the different sets of the values for the coefficients are equally good. In other words, the differences in the k_i's and c_i's between various data sets are due to the existence of **wide bands of near-optimal values** for the k_i's and c_i's. The fact that the relationship "weights for the fractiles in the μ functions sum up to 1" provides a supporting evidence for the robustness of the relationship in the models (despite the change of values of the coefficients, their relationship remains). This feature of the coefficients in the regression models will be studied further in the later sections.

The property pointed out above is a very useful one: since the regression models with different values for the coefficients perform almost equally well, we are given the chance to adjust the values of the coefficients, hopefully ending up with combinations of some "simpler" or "cleaner" coefficients. In other words, the fact that there are wide bands of near-optimal values existing for the coefficients encourages us to search for formulae with "simple" or "clean" coefficients by testing various round-off modifications of the values shown in Tables 4-1 to 4-5. After many trials, we obtained the following simple formulae:

for seven fractiles:

$$\mu_e = 0.04\times S01 + 0.11\times S10 + 0.23\times S25 + 0.24\times T_{0.5} \quad (4\text{-}31a)$$

$$\sigma_e = 0.2\times D01 - 0.6\times D10 + 1.2\times D25 \quad (4\text{-}31b)$$

for five fractiles:

$$\mu_e = 0.4\times (S10\text{-}S25) + T_{0.5} \quad (4\text{-}32a)$$

$$\sigma_e = 0.7\times D10 - 0.59\times D25 \quad (4\text{-}32b)$$

A less accurate but simpler alternative to (4-31a) is

for seven fractiles:

$$\mu_e = 0.05\times S01 + 0.10\times S10 + 0.25\times S25 + 0.2\times T_{0.5}, \quad (4\text{-}33a)$$

which can be written as:

$$\mu_e = (S01 + 2\times S10 + 5\times S25 + 4\times T_{0.5})/20 \quad (4\text{-}33b)$$

This is a fairly simple formula.

This section has developed μ and σ formulae using fractiles at α levels of 0.01, 0.10, 0.25, 0.50, 0.75, 0.90, and 0.99 (seven fractiles), and at α levels of 0.10, 0.25, 0.50, 0.75, and 0.90 (five fractiles). The resulting μ functions and σ functions are simple linear functions of the fractiles of the distributions.

Regarding the fractiles used in the functions, we do not claim that these seven (five) fractiles are the best to use on this purpose based on any mathematical-statistical evidence. These fractiles are used because the literature (such as Selvidge, 1980, and Solomon, 1982) recommends that these fractiles be elicited from human estimators.

4.4 Explanation of the Power of the Proposed Formulae in Estimating μ and σ

In the previous section we have seen that the μ and σ of a beta distribution can be expressed as linear functions of the fractiles of the distribution. In this section, we will provide further theoretical inference for this relationship between the fractiles and the mean/standard deviation. It will be shown that the power of the proposed formulae in estimating μ and σ is based on solid theoretical ground.

4.4.1 $S(\alpha)$'s as perfect μ-estimators for symmetrical distributions

Define a symmetrical inter-fractile sum for a random variable T as

$$S(\alpha) \equiv T_{\alpha} + T_{1-\alpha}, \qquad (4\text{-}34)$$

where T_{α} is T's α fractile.

For any symmetrical distribution (e.g., normal, uniform), it is obvious that T's mean is

$$\mu = S(\alpha)/2 \qquad (4\text{-}35)$$

Define $R(\alpha)$ as

$$R(\alpha) \equiv S(\alpha)/\mu,$$

then from (4-35), we have

$$R(\alpha) = S(\alpha)/\mu = 2 \qquad (4\text{-}36)$$

for a symmetrical distribution at any α value. Therefore, $S(\alpha)/2$ is a perfect estimator of μ (at any α value). Naturally, any linear combination of $S(\alpha_i)$'s is a perfect estimator of μ. Therefore, we have

$$\mu = c_1 S(\alpha_1) + c_2 S(\alpha_2) + \ldots + c_n S(\alpha_n), \qquad (4\text{-}37)$$

where c_i can have any value as long as the c_i's satisfy

$$2[\Sigma_n (c_i)] = 1$$

We investigate below the behavior of $R(\alpha) = S(\alpha)/\mu$ for asymmetrical beta distribution. We will show that (4-36) remains approximately valid for asymmetrical beta distributions.

4.4.2 R(α) remains approximately constant for beta distributions

We will consider a standardized beta-distributed X with (p,q) being in a wide range. Let us first see an example.

When (p,q) = (61.98, 20.62), we have

$$T_{0.01} = 0.6322, \quad T_{0.10} = 0.6882,$$

$$T_{0.25} = 0.7194, \quad T_{0.50} = 0.7524, \quad T_{0.75} = 0.7835, \quad (4\text{-}23)$$

$$T_{0.90} = 0.8098, \quad T_{0.99} = 0.8507$$

From the definition of S:

$$S(0.01) = 1.4829, \; S(0.10) = 1.4980,$$

$$S(0.25) = 1.5029.$$

We know in section 4.3.1 that for this specific beta distribution with (p,q) = (61.98, 20.62), μ = 0.7503. So, by definition,

$$R(0.01) = S(0.01)/\mu = 1.4829/0.7503 = 1.976. \quad (4\text{-}38a)$$

Similarly, we can obtain the other R(α)'s as

$$R(0.10) = 1.997, \; R(0.25) = 2.003. \quad (4\text{-}38b)$$

The R(α) values remain fairly close to 2.

We randomly generated 2000 sets of (p,q) in the range of $1<p<20$ and $1<q<20$; and another 2000 sets of (p,q) in the range of $1<p<100$ and $1<q<100$. We generated the $T_{\alpha i}$'s of these beta distributions and then calculated the $S(\alpha_i)$'s,

μ's, and $R(\alpha_i)$'s correspondent to these distributions. The results are shown below:

Table 4-6 Mean and Standard Deviation of R(α) for Beta Distributions

p,q range	1<(p,q)<20		1<(p,q)<100	
α	Mean of R(α)	S.D. of R(α)	Mean of R(α)	S.D. of R(α)
0.01	2.111	0.355	2.060	0.241
0.02	2.084	0.269	2.044	0.179
0.05	2.047	0.151	2.024	0.098
0.10	2.019	0.062	2.009	0.038
0.20	1.993	0.022	1.996	0.015
0.25	1.986	0.045	1.993	0.032
0.30	1.981	0.062	1.990	0.041
0.40	1.975	0.082	1.987	0.053

From Table 4-6, we can see that R(α)'s means are close to 2, and their standard deviations are quite small for all α's. Table 4-6 shows that S(0.01), S(0.10), and S(0.25) can each individually serve as a very good μ estimator.

The above results indicate that equations (4-36) and (4-37) hold fairly well for random variables of beta distributions, and this explains the phenomenon depicted in

Tables 4-1 to 4-5 and formulae (4-31) to (4-33):

1. Since any single $S(\alpha)$ can serve as a very good estimator of μ, it is a natural consequence that a μ-predicting formula as linear combinations of $S(\alpha)$'s will have $R^2 = 1.000$. One should note that $R^2 = 1.000$ does not mean that the prediction is error-free; it simply means that the errors are small compared to the variation of the 2000 μ_e's.

2. Since the coefficients c_i's can have any value in (4-37) as long as they satisfy $2[\Sigma_n(c_i)] = 1$, we can see the reason why c_i's can have a wide band of near-optimal values.

CHAPTER V

ERROR ANALYSIS

We have developed the formulae for estimating μ and σ (formulae (4-31), (4-32), and (4-33)) from the fractiles of a beta distributed random variable. We will perform error analysis on the accuracy of these formulae, in comparison with that of the traditional PERT formulae. We will also show the difference between $T_{0.1}$ ($T_{0.9}$) and $T_{0.01}$ ($T_{0.99}$) for some beta distributions to support our arguments in Chapter III, Section 3.2.1 that $T_{0.1}$ ($T_{0.9}$) can be very far from $T_{0.01}$ ($T_{0.99}$).

5.1 Error Analysis for the Proposed Formulae

5.1.1 Absolute error (AE) and absolute percentage error (APE)

To evaluate the accuracy of (4-31)-(4-33), we will employ the criteria "Absolute Error" and "Absolute Percentage Error", with the definition of each criterion as follows:

for μ:

$$AE = |\mu_e - \mu|, \quad (5\text{-}1a)$$

$$APE = 100AE/\mu; \quad (5\text{-}1b)$$

for σ:

$$AE = |\sigma_e - \sigma|, \quad (5\text{-}2a)$$

$$APE = 100AE/\sigma, \quad (5\text{-}2b)$$

where

μ = the true value of the mean of standardized beta distribution (with the randomly generated (p,q) pair), obtained from (4-25),

μ_e = the mean of the distribution given by (4-31)-(4-33) with the fractiles obtained from the same (p,q) pair above,

σ = the true value of the standard deviation of standardized beta distribution (with the same (p,q) pair above), obtained from (4-25),

σ_e = the standard deviation of the distribution given by (4-31)-(4-33) with the fractiles obtained from the same (p,q) pair above.

5.1.2 Error analysis and its results

We used the procedure described in Section 4.3.1 (formulae (4-22)-(4-24)) to generate three new data sets (numbers 3, 4, and 5), each with 2000 sets of (p,q), the corresponding T_α's based on the sets of (p,q), and the corresponding μ (actual value) and σ (actual value) calculated with formulae (4-25) based on the same (p,q) values.

Whereas data sets 1 and 2 were both generated with (p,q) in the range of 1 to 100, the (p,q) ranges in data sets 3, 4, and 5 are (1,100), (1,50), and (1, 500), respectively. This is done to ensure that our study is not restricted by the (p,q) ranges and, therefore, our conclusions are not dependent on any specific (p,q) ranges of the data sets.

The μ_e and σ_e values computed with formulae (4-31)-(4-33) are compared with the actual μ and σ values obtained through (4-25). The result of the comparison is listed in the following table.

Table 5-1. Statistics of Errors Incurred by Formulae (4-31)-(4-33)

Formula	Data Set	(p,q) Range	Average μ or σ	Abs. Err. (AE)			Abs.% Err. (APE)		
				Avg.	99%	Max	Avg.	99%	Max
(4-31a)	3	1-100	0.504	0.000027	0.0001	0.0001	0.009	0.03	0.36
(for μ,	4	1- 50	0.498	0.000029	0.0001	0.0001	0.009	0.09	0.23
7 frac.)	5	1-500	0.497	0.000027	0.0001	0.0001	0.011	0.12	0.80
(4-31b)	3	1-100	0.046	0.000451	0.0012	0.0057	1.035	4.50	6.9
(for σ,	4	1- 50	0.065	0.000640	0.0024	0.0046	1.059	5.61	7.6
7 frac.)	5	1-500	0.020	0.000229	0.0005	0.0015	1.121	1.94	6.9
(4-32a)	3	1-100	0.504	0.000091	0.0004	0.0005	0.041	0.57	1.3
(for μ,	4	1- 50	0.498	0.000141	0.0005	0.0006	0.060	0.73	1.2
5 frac.)	5	1-500	0.497	0.000044	0.0001	0.0003	0.020	0.35	1.8
(4-32b)	3	1-100	0.046	0.000199	0.0016	0.0067	0.571	6.54	9.0
(for σ,	4	1- 50	0.065	0.000383	0.0033	0.0061	0.718	7.41	10.0
5 frac.)	5	1-500	0.020	0.000056	0.0003	0.0022	0.328	2.71	10.5
(4-33)	3	1-100	0.504	0.000174	0.0008	0.0029	0.087	1.39	2.6
(for μ,	4	1- 50	0.498	0.000321	0.0016	0.0023	0.143	1.82	2.7
7 frac.)	5	1-500	0.497	0.000046	0.0002	0.0009	0.027	0.50	2.0

In the above table, we listed AE and APE data based on the three data sets (data sets 3, 4, and 5). In the columns under the header "Absolute Error (AE)", we listed three data: the average absolute error, the 99th percentile of the 2000 AE's in each data set, and the maximum AE in each data set. The data in the columns under the header "Absolute Percentage Error" have the same interpretations.

From the table, we can see that the absolute percentage errors from all the formulae based on all the data sets are lower than 1.2%, with average APE's being lower than 0.8% in 12 out of the 15 cases. Actually, average APE's are lower than or at 0.1% in 9 out of the 15 cases.

Looking at the maximum APE's, we have maximum APE's lower than 3% in 9 out of the 15 cases. We note that there are six cases in which the maximum APE's are greater than 5%, but even in those cases, the 99th percentiles of the APE's are only between 2-7.5%. We should note that the 99th percentiles of the APE's for the rest of the μ's (or σ's) are all below 2% (most of them are FAR BELOW).

5.1.3 Some numerical examples

Consider the exact fractiles in (4-23) for a standardized beta distribution with (U,V) = (0,1) and (p,q) = (61.98, 20.62). For the convenience of comparison, (4-23) is redisplayed below:

$$T_{0.01} = 0.6322, \quad T_{0.10} = 0.6882,$$

$$T_{0.25} = 0.7194, \quad T_{0.50} = 0.7524, \quad T_{0.75} = 0.7835, \qquad (4\text{-}23)$$

$$T_{0.90} = 0.8098, \quad T_{0.99} = 0.8507.$$

We have the following observations:

1. The discrepancy between T_0 (T_1) and $T_{0.01}$ ($T_{0.99}$) is substantial. From (4-23) we can see that while $T_0 = 0$ and

$T_1 = 1$, $T_{0.01} = 0.6322$ and $T_{0.99} = 0.8507$. This illustrates our earlier statement: one cannot assume that $T_{0.01}$ (or $T_{0.99}$) is usually close to T_0 (or T_1). We can also see that one cannot assume that $T_{0.10}$ (or $T_{0.90}$) is usually close to $T_{0.01}$ (or $T_{0.99}$). The two pairs ($T_{0.01}$ and $T_{0.10}$; $T_{0.90}$ and $T_{0.99}$) have percentage deviations ranging from 5% to 8.8%.

2. The PERT formulae are very inaccurate while the proposed formulae (4-31)-(4-33) are very accurate. Based on the data given in (4-23), if one defines $a = T_{0.01}$, and $b = T_{0.99}$, then the σ obtained from the PERT formula (1-1b) will be:

$$\sigma_1 = (b-a)/6 = (0.8507-0.6322)/6 = 0.0364.$$

However, if one defines $a = T_0$ and $b = T_1$, then

$$\sigma_2 = (b-a)/6 = (1-0)/6 = 0.1667,$$

which differs from σ_1 by **358%**.

From (4-25), the **true** σ value of the beta distribution with parameters $(p,q) = (61.98, 20.62)$ will be

$$\sigma = \sqrt{pq/[(p+q)2(p+q+1)]} = 0.04734.$$

So, both σ_1 and σ_2 are poor estimates of the correct σ.

In contrast, using (4-31b) and (4-32b) with the figures in (4-23) gives

$$\sigma_3 = 0.2\times0.2185 - 0.6\times0.1216 + 1.2\times0.0641$$
$$= 0.04766 \qquad (5\text{-}3a)$$
$$\sigma_4 = 0.7\times0.1216 - 0.59\times0.0641 = 0.04730 \qquad (5\text{-}3b)$$

σ_3 is within 1% of the exact σ, and σ_4 is practically identical to the exact σ.

5.2 Error Analysis for the PERT Formulae

It may be intuitively obvious, from the discussion in the previous section, that formulae (1-1) are substantially less accurate than (4-31)-(4-33). In order to show this in a more systematical way, we present results of an error analysis of (1-1b) in Table 5-2, which is a counterpart of Table 5-1 for (1-1b). In the analysis leading to Table 5-2, two versions of (1-1b) are considered: version A uses T_0 and T_1 ("absolute endpoints") for "a" and "b", whereas version B uses $T_{0.01}$ and $T_{0.99}$ ("inner fractiles"). The results are as follows:

Table 5-2. Statistics of Errors Incurred by Formula (1-1b)

				Abs. Err.(AE)			Abs.% Err.(APE)		
Formula	Data Set	(p,q) Range	Average μ or σ	Avg.	99th Percentile	Max	Avg.	99th Percentile	Max
Ver. A	3	1-100	0.046	0.1209	0.150	0.155	301.7	880	1304
(using	4	1- 50	0.065	0.1020	0.137	0.147	178.1	471	731
T_0 & T_1)	5	1-500	0.020	0.1462	0.160	0.162	61.8	2504	3697
Ver. B	3	1-100	0.046	0.0106	0.025	0.051	23.2	25.2	29.9
(using	4	1- 50	0.065	0.0155	0.038	0.076	23.7	27.1	34.7
$T_{0.01}/T_{0.99}$)	5	1-500	0.020	0.0046	0.010	0.025	22.6	23.5	24.6

In Table 5-2, we can see that the errors of using either version are substantial as we anticipated (average APE's range from 22.6% to 301.7%). Looking into the data more closely, however, we can find that version A is more inaccurate than version B: under all the three measurements of the APE (average, 99th percentile, and maximum), version A performs worse than version B. This may be a little beyond expectation at first, because version A uses the "correct" interpretation of "a" and "b" ("absolute endpoints"), and one can speculate that version A should not be more inaccurate than version B. This seemingly anti-intuitive result can be explained as follows: the interpretation of a and b as absolute endpoints is "correct" only when the restrictive condition of "$p + q = 6$" (or "$\alpha + \beta = 4$") holds. But in Table 5-2, we have used (1-1b) to handle beta distributions in data sets 1 to 3, which contain very wide range of bell-shaped beta distributions; and under these circumstances, version A does not have advantage over version B.

One may argue that since (1-1b) is meant to handle the subset of beta distributions in which the relationship "$p + q = 6$" holds, it seems not a reasonable comparison if formula (1-1b) is used to handle distributions it is not supposed to handle. As pointed out by Gallagher (1987), (1-1b) is applicable to two types of beta distributions:

(1) those with $\sigma = (b-a)/6$, with which (1-1b) is supposed to be error-free;

(2) those with $p + q = 6$, with which (1-1b) is an acceptable approximation.

To evaluate the accuracy of (1-1b) with distributions satisfying "$p + q = 6$", we consider standardized beta distributions with p from 1.01 to 4.99 in steps of 0.01 and q = 6 - p; the resultant 399 distributions constitute data set 6. The performance of versions A and B of (1-1b) is then tested on data set 6 and the results are summarized in Table 5-3 below.

Table 5-3. Statistics of Errors Incurred by Formulae (1-1b), on Data Sets Satisfying p + q = 6

				Abs. Err. (AE)			Abs.% Err. (APE)		
Formula	Data Set	(P,q) Range	Average μ or σ	Avg.	99th Percentile	Max	Avg.	99th Percentile	Max
A	6	1.1-4.9	0.174	0.0141	0.024	0.025	17.85	16.9	17.9
B	6	1.1-4.9	0.174	0.0525	0.058	0.058	29.05	30.4	30.4

From Table 5-3, we can see that version A now performs better than version B. But they are still much less accurate than (4-31)-(4-33), even for this very restricted subset of beta distributions on which (1-1) is supposed to be applicable.

Looking back to Table 5-2, it is interesting to note that, if one insists on using (1-1b) to estimate σ, one might as well also use the "wrong" definitions:

$$a = T_{0.01} \text{ and } b = T_{0.99},$$

since the accuracy of version A drops sharply when it is applied to beta distributions outside the restricted subset (for example, data sets 3 to 5).

5.3 Conclusions of the Results of the Error Analysis

From the error analysis conducted in this chapter, we can see that the proposed alternative formulae (4-31)-(4-33) have outperformed the PERT formulae (1-1) in the following aspects:

1. The general performance of (4-31)-(4-33) is much better than that of the PERT formulae (1-1). Table 5-4 on the next page shows the comparison of performance of formulae (4-31)-(4-33) with that of the PERT formulae (1-1).

From Table 5-4, we can see that the performance of the proposed alternative formula is 33 to 291 times better than the PERT formula in terms of average APE, 3.66 to 1290 times better in terms of the 99th percentile, and 2.46 to 536 times better in terms of the maximum APE.

Table 5-4
Comparison of Performance
of Formulae (4-31)-(4-33) and Formulae (1-1)

Formula	Data Set	(p,q) Range	Abs.% Err.(APE) Avg.	99th Percentile	Max
(4-31b)	3	1-100	1.035	4.50	6.9
(for σ,	4	1- 50	1.059	5.61	7.6
7 frac.)	5	1-500	1.121	1.94	6.9
(4-32b)	3	1-100	0.571	6.54	9.0
(for σ,	4	1- 50	0.718	7.41	10.0
5 frac.)	5	1-500	0.328	2.71	10.5
Ver. A	3	1-100	301.7	880	1304
(using	4	1- 50	178.1	471	731
T_0 & T_1)	5	1-500	61.8	2504	3697
Ver. B	3	1-100	23.2	25.2	29.9
(using	4	1- 50	23.7	27.1	34.7
$T_{0.01}/T_{0.99}$)	5	1-500	22.6	23.5	24.6
Ver.A	6	1.1-4.9	17.85	16.9	17.9
Ver.B	6	1.1-4.9	29.05	30.4	30.4

We should note that the alternative formulae in the above comparison are those with rounded coefficients ("simple" or "clean" formulae). More significant improvement in performance can be expected if the before-rounding proposed alternative formulae are introduced in this comparison.

2. The PERT formulae perform poorly in accuracy. Even if the performance is measured on the basis of the restricted data sets where the PERT formulae are supposed to perform well, the performance of the **PERT formulae in** these

"favorable situations" is still **lower than** that of the **proposed alternative in general (and "neutral") situations.**

CHAPTER VI

CONCLUSIONS AND DISCUSSIONS

6.1 Conclusions

This research studies the objectives and the procedure of the PERT time estimation, points out the shortcomings in the PERT procedure in handling the objective of eliciting subjective probabilities and in handling the objective of converting the "basic times" a, m, and b to the mean and the standard deviation of the distribution of the task times of interest. This study, based on the probability elicitation literature and the properties of the beta distribution, proposes an alternative to the PERT time estimate procedure and has accomplished the following:

1. The proposed alternative is based on the established probability elicitation literature, which avoids the difficulties (inaccuracy and possible confusion) the PERT procedure encounters when the "basic times" a, b, and m are to be estimated. The proposed alternative enables the researchers and practitioners to take advantage of the existing probability elicitation literature and the existing

practices in probability elicitation procedure, which provides a solid ground for the improvement on the validity and accuracy of time estimation. This has accomplished the Objective 1 set in Section 3.1 (which the PERT procedure failed to).

2. The proposed alternative can handle a wide range of shapes of beta distributions, therefore avoiding the shortcoming of the PERT procedure which is only valid on a very restricted subset of the beta distributions. This improvement, accordingly, makes the proposed time estimate procedure more versatile, "robust", and valid in the real-world applications of project management. This has successfully accomplished the Objective 2 set in Section 3.1 (which the PERT procedure failed to).

3. The proposed alternative has improved the accuracy of time estimation substantially, in comparison with the PERT procedure. This is very important for real-world project management tasks, especially where large-scale projects are involved. Great economic benefits can be expected from the improvement of time estimation in project management.

4. The proposed alternative retains the simplicity of

the PERT procedure while accomplishing all of the above. After the fractiles are obtained, all the calculations needed to be performed are just plain arithmetic operations and can easily be performed in a short time, without demanding complex computing facilities. In addition, the formulae in the proposed alternative are simple enough not to intimidate the practitioners. The retention of simplicity will be an important feature for this new alternative to be accepted by the real-world management.

In general, the proposed alternative is a logical, more accurate, and simple procedure in estimating the mean and the standard deviation of a stochastic time duration, in comparison with the traditional PERT procedure. The alternative is free from the restrictions imposed by the PERT procedure on the range of shapes the distributions can take, and can therefore handle a wider range of beta distributions which the PERT originators intended to but did not achieve. The proposed alternative is based on theoretical inference instead of "trial and error" type of experiments. It has its inputs (the fractiles of a stochastic time distribution) based on the solid ground of the probability elicitation literature. The time estimates obtained from the proposed alternative are much more accurate than that from the PERT procedure, which is the strongest justification of the introduction of the alternative. The proposed alternative is simple, which facilitates its acceptance by the practitioners.

In our research, we also study the shortcomings of the PERT procedure, and, specifically, point out and measure the inaccuracy of the PERT formulae through numerical examples. The results of our study show that PERT formulae are inaccurate and should be replaced. Students and practitioners in MS/OR should be made aware of the existence of better alternative(s) when they are taught or performing project management.

6.2 On the Number of Fractiles Used in the Alternative

A question may arise that if one can elicit sufficient number of fractiles, then one can precisely compose the whole probability distribution. In that case, one will be able to accurately calculate the mean and the standard deviation (or median, or any other statistics) and be free from the approximation of estimating the mean or the standard deviation. This speculation is theoretically correct. But to obtain a satisfactorily accurate probability distribution, we may need to elicit, say, 100 or more fractiles. This "prelude" to a real-world project time analysis (with a project network) can be intimidating to the managers and/or engineers working on the project. Reluctance or even resistance can be expected if this tedious duty is to be imposed on the personnel who are responsible for the network analysis for the project management. We believe that, for a certain new method to be accepted by potential users, the (perceived) potential workload to be imposed on the concerned parties is an important issue to consider before the method is introduced. From this point of view, eliciting 100 or more fractiles for composing a complete probability distribution function curve may impose too much work on the project managers/engineers and may not be able to be actually carried out.

One may be interested in the possibility of a human subject's ability to estimate the mean and/or the standard deviation directly. This direct estimate is often very difficult (as pointed out by Clark, 1962) or even impossible. In some cases, the mean of a distribution may have an obvious value when it actually does not even exist! A classical example is shown with the Cauchy distribution (Zehna, 1970, pp 84-85, p. 100; Mood *et al.*, 1974, p. 117). In figure 6-1 below, it seems to be obvious that the mean of the distribution shown is zero, yet it does not exist, because the integral

$$\int_{-\infty}^{+\infty} f(x)\,dx$$

does not converge.

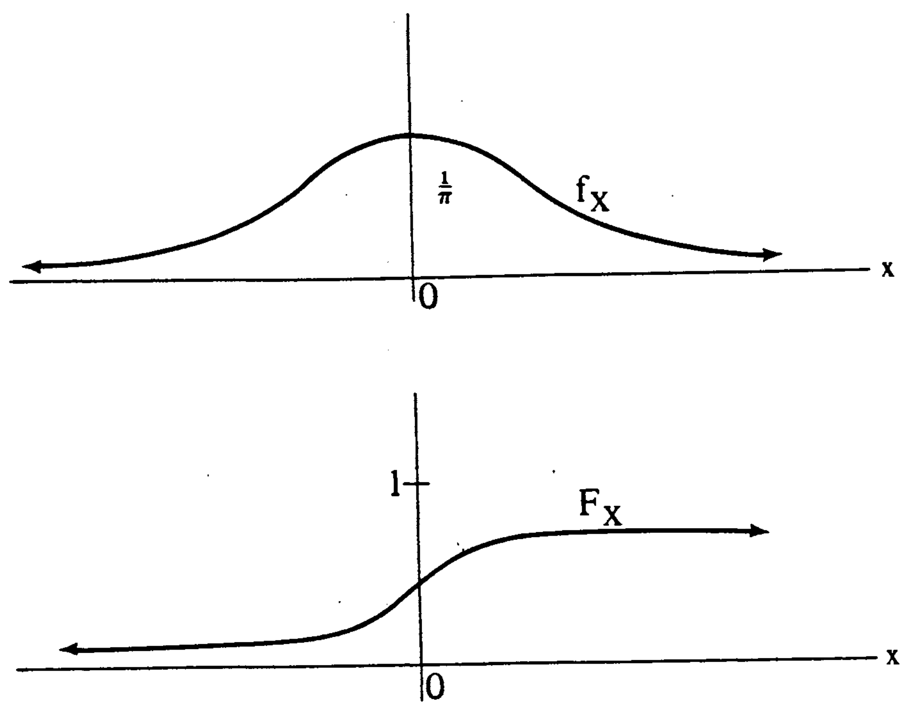


Figure 6-1 Graph of the Cauchy Density and Distribution Function

There may be, on the other hand, another set of questions that "Why should we elicit seven fractiles? Isn't it too much?" These questions, if asked, would most probably come from the practitioners. In answering the first question, we would like to note that there are two separate issues in the use of fractiles in estimation of the mean and the standard deviation of a probability distribution: 1. eliciting the fractiles for the estimations; and 2. computing the mean and the standard deviation with the estimated fractiles. When dealing with issue 2, one assumes that the fractiles obtained are error-free and treats the "inputting" fractiles as perfect. However, fractiles usually cannot be estimated error-free. Selvidge's (1980) results suggest that when a person is required to estimate the seven fractiles ($T_{0.01}$, $T_{0.10}$, $T_{0.25}$, $T_{0.50}$, $T_{0.75}$, $T_{0.90}$, and $T_{0.99}$) instead of other sets of fractiles, the person tends to make more accurate estimates, especially those estimates regarding extreme fractiles. Thus, the reason for our using seven fractiles is in handling issue 1 (accuracy of fractile estimation), but not in handling issue 2 (converting fractiles to means and standard deviations).

In answering the second question ("seven fractiles are too much"), we are presenting an example of successful usage of the seven-fractile method in a field study. Solomon (1982) reported his studies on probability assessment conducted in seven of the "Big Eight" accounting firms and one other large national accounting firm. The subjects in his studies were a mix of audit staff, audit seniors, and managers/supervisors. The method he employed was the fractile method with seven fractiles, the same method studied by Selvidge (1980). In his studies, he asked the subjects to estimate prior probability distributions (PPDs), defined as the quantified subjective beliefs held by an individual auditor or team of auditors prior to collecting objective evidence through the performance of subjective audit tests. His "choice was made because of the simplicity of the technique and the ease with which subjects can be trained to used it." Solomon's study (1982) provides a strong evidence supporting the use of the seven-fractile method in eliciting subjective probability from a wide range of personnel in the real world. There should not be major concern regarding the ease of use of the seven-fractile method by managers or professionals, as those "experts" in the PERT.

In addition, if we look at major projects with a perspective expense of millions of dollars, the cost of making the estimates of seven fractiles instead of three numbers (as in the PERT procedure) will well be offset by the great amount of saving one can expect from obtaining a much more accurate estimate of the mean and the standard deviation of the task times. In this case, making estimates of seven fractiles is very well justified by economic benefits.

6.3 One More Advantage of The Alternative

As we showed in section 4.4, beta distribution has a special feature that

$$R(\alpha) = 2, \tag{4-36}$$

where $R(\alpha) \equiv S(\alpha)/\mu$,

$$S(\alpha) \equiv T_\alpha + T_{1-\alpha},$$

where α is fractile level of random variable T.

We have seen that $S(\alpha)/2$ is a perfect estimator of μ. Consequently, any linear combination of $S(\alpha)$ is also a perfect estimator of μ. Unfortunately, the traditional PERT procedure does not take advantage of this important feature of the beta distribution and totally wastes the information contained in the fractiles. When one uses the PERT formulae (1-1) to estimate μ with endpoints a=0 and b=1, the sum of a and b is always 1 regardless of the true mean. In this case, a and b do not actually contribute to the determination of the mean. Instead, the mean is solely determined by the mode m, which is not correct. This error is obvious when we have a symmetrical bell-shape distribution where mean = mode. The mean calculated by the PERT formula, however, is $(a+4m+b)/6 = (1+4m)/6$, which is not m as it should be. When one uses the PERT formulae (1-1) to estimate σ with endpoints a=0 and b=1, the difference of a and b is always 1, and the σ will **always** be 1/6 regardless of the true σ. Therefore, for the standardized variates, the mean μ will be determined only by the mode, and the standard deviation σ will always be the same (=1/6), regardless of the actual shape of the distribution of interest.

We can see from the discussion above that the PERT procedure has lost some important information which could have been included in the time estimation procedure. The PERT procedure, therefore, has missed the opportunity of improvement in the accuracy of this estimation which could have been achieved. This is a critical shortcoming of the PERT procedure. The proposed alternative avoids this shortcoming of the PERT and makes full use of the information contained in the fractiles in the estimation of the mean and the standard deviation of the stochastic time.

6.4 Recommendations for Future Studies

6.4.1 On the generation of the beta shape parameter (p,q)

In our study, we generate the beta distribution shape parameter (p,q) with a uniform random variable generator. This procedure serves to generate the (p,q) pairs randomly and "evenly" in a wide range of the (p,q) values. But with more careful examination, we find that to generate the (p,q) pairs uniformly may not be as "evenly" as it seems.

Referring to the "$(\beta_1-\beta_2)$-diagram" (adapted from Hahn and Shapiro 1967) on the following page, where $\beta_1 = \mu_3^2/\mu_2^3$, $\beta_2 = \mu_4/\mu_2^2$, and the μ_i's are the μth central moments of the probabilistic distributions, we can see that it is the $(\beta_1-\beta_2)$ pair which govern the transformation (or transition) from one type of distributions to another, and among the

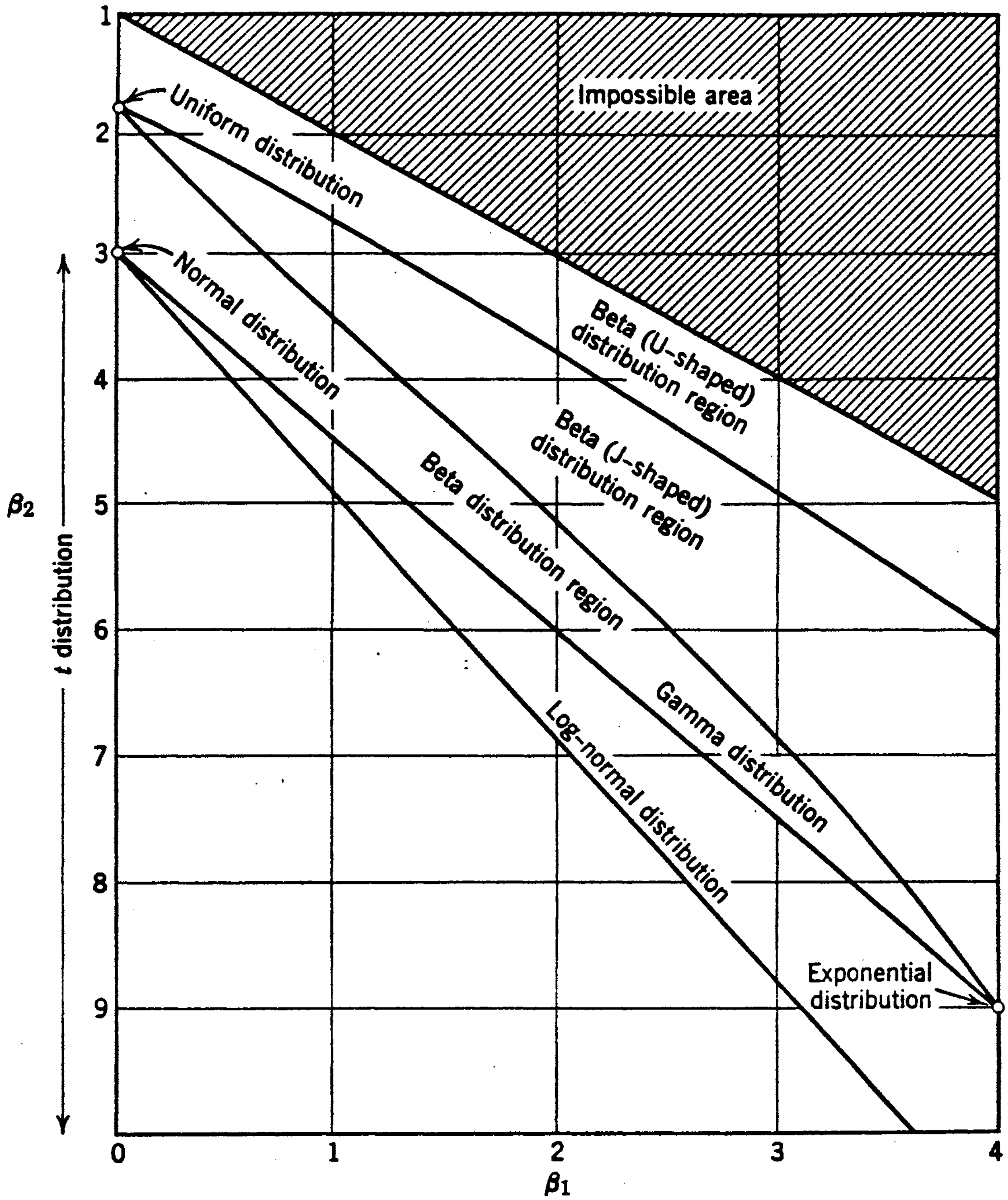


Figure 6-2 Regions in (β_1, β_2) Plane for Various Distributions

various shapes of distributions within one group or category of distributions. Therefore, it will be more logical to "evenly" generate the (β_1-β_2)-pair in order to cover the beta distributions "evenly". From the above point of view, it is advisable to probe in that direction for the future research. In that case, different beta distributions with various (β_1-β_2)-values will be generated evenly, and research can be done on the data sets generated this way.

6.4.2 Considerations of using distributions other than the beta distribution

Taking one more step from our discussion in the previous subsection, based on the (β1-β2)-diagram, we further notice that, the bell-shaped beta distributions consist but one relatively small area in the areas occupied by all the bell-shaped distributions on the (β1-β2)-diagram. To allow the (β1-β2)-values to vary within the category of the bell-shaped beta distributions is nothing but moving within the small area standing for, or occupied by, the bell-shaped beta distributions, which is just a part of all the bell-shaped distributions. Therefore, to truly allow the shape of a probabilistic distribution to change in a wide range, one should allow the ($\beta1$-$\beta2$)-values to vary in the whole bell-shaped area on the ($\beta1$-$\beta2$)-diagram, instead of just changing within a small sub-area. This means that different **types** of distributions should be studied instead of just the beta distributions.

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APPENDEX A: Computer Program for Generating Data Sets for Seven-fractile Models

```
//U14501AK JOB (*),'ZHANG',CLASS=3,
//         TIME=(10,0),MSGCLASS=X,NOTIFY=U14501A
// EXEC VSF2CLG,IMSL=DP
/*ROUTE PRINT BUS009
//SYSIN DD *

 C     PROGRAM: PERT TIME ESTIMATE, DATA GENERATION
 C     FOR SEVEN FRACTILES
   010 REAL PR(7), MU, SIGMA, S01, S10, S25, D01,
      &D25, P, Q, X(7), BETIN, RNUNF
       EXTERNAL BETIN, RNUNF
 C -------------------------
       PR(1)=.01
       PR(2)=.1
       PR(3)=.25
       PR(4)=.5
       PR(5)=.75
       PR(6)=.9
       PR(7)=.99
 C -------------------------
       I=2000
       DO 350 J = 1, I
       P = 1 + 99 * RNUNF()
       Q = 1 + 99 * RNUNF()
   310 MU = P/(P+Q)
       SIGMA = SQRT (P*Q/((P+Q)**2 * (P+Q+1)))
       DO 330 K = 1, 7
   330 X(K) = BETIN(PR(K),P,Q)
       S01 = X(7) + X(1)
       S10 = X(6) + X(2)
       S25 = X(5) + X(3)
       D01 = X(7) - X(1)
       D10 = X(6) - X(2)
       D25 = X(5) - X(3)
       WRITE(15,345)  MU,S01,S10,S25,X(4)
       WRITE(16,346) SIGMA,D01,D10,D25
   345 FORMAT(2X,F6.4,2X,F6.4,2X,F6.4,2X,F6.4,2X,F6.4)
   346 FORMAT(2X,F6.4,2X,F6.4,2X,F6.4,2X,F6.4)
   350 CONTINUE
       END

//GO.FT15F001 DD DSN=U14501A.OUT5.DATA,DISP=SHR
//GO.FT16F001 DD DSN=U14501A.OUT6.DATA,DISP=SHR
//GO.SYSIN DD *
```

APPENDEX B: Computer program for Determining Coefficients for Seven-fractile Models

```
//U14501AX JOB (*),
// 'ZHANG',TIME=(5,0),CLASS=3,MSGCLASS=X,
// NOTIFY=U14501A
/*ROUTE PRINT BUS009
// EXEC SAS
//OP1 DD DSN=U14501A.OUT5.DATA,DISP=SHR
//OP2 DD DSN=U14501A.OUT6.DATA,DISP=SHR
//SYSIN DD *


DATA OP1;
INFILE OP1;
INPUT MU S01 S10 S25 T50;
PROC REG DATA = OP1;
     MODEL MU = S01 S10 S25 T50;
DATA OP2;
INFILE OP2;
INPUT SIGMA D01 D10 D25;
PROC REG DATA = OP2;
     MODEL SIGMA = D01 D10 D25;
```

APPENDEX C: Computer Program for Generating Data Sets for Five-fractile Models

```
//U14501AK JOB (*),'ZHANG',CLASS=3,
//         TIME=(10,0),MSGCLASS=X,NOTIFY=U14501A
// EXEC VSF2CLG,IMSL=DP
/*ROUTE PRINT BUS009
//SYSIN DD *

 C     PROGRAM: PERT TIME ESTIMATE, DATA GENERATION
 C              FOR FIVE FRACTILES
   010 REAL PR(5), MU, SIGMA, S01, S10, S25, D01,
      &D25, P, Q, X(5), BETIN, RNUNF
       EXTERNAL BETIN, RNUNF
 C -------------------------
       PR(1)=.1
       PR(2)=.25
       PR(3)=.5
       PR(4)=.75
       PR(5)=.9
 C -------------------------
       I=2000
       DO 350 J = 1, I
       P = 1 + 99 * RNUNF()
       Q = 1 + 99 * RNUNF()
   310 MU = P/(P+Q)
       SIGMA = SQRT (P*Q/((P+Q)**2 * (P+Q+1)))
       DO 330 K = 1, 5
   330 X(K) = BETIN(PR(K),P,Q)
       S10 = X(5) + X(1)
       S25 = X(4) + X(2)
       D10 = X(5) - X(1)
       D25 = X(4) - X(2)
       WRITE(17,345)  MU,S10,S25,X(3)
       WRITE(18,346) SIGMA,D10,D25
   345 FORMAT(2X,F6.4,2X,F6.4,2X,F6.4,2X,F6.4)
   346 FORMAT(2X,F6.4,2X,F6.4,2X,F6.4)
   350 CONTINUE
       END

//GO.FT17F001 DD DSN=U14501A.OUT7.DATA,DISP=SHR
//GO.FT18F001 DD DSN=U14501A.OUT8.DATA,DISP=SHR
//GO.SYSIN DD *
```

APPENDEX D: Computer program for Determining Coefficients for Five-fractile Models

```
//U14501AX JOB (*),
// 'ZHANG',TIME=(5,0),CLASS=3,MSGCLASS=X,
// NOTIFY=U14501A
/*ROUTE PRINT BUS009
// EXEC SAS
//OP1 DD DSN=U14501A.OUT7.DATA,DISP=SHR
//OP2 DD DSN=U14501A.OUT8.DATA,DISP=SHR
//SYSIN DD *


DATA OP1;
INFILE OP1;
INPUT MU S10 S25 T50;
PROC REG DATA = OP1;
     MODEL MU = S10 S25 T50;
DATA OP2;
INFILE OP2;
INPUT SIGMA D10 D25;
PROC REG DATA = OP2;
     MODEL SIGMA = D10 D25;
```

APPENDIX E: Computer Program to Select "Clean" Coefficients

```
      REAL*8 FRM,TOL,BV,RV,DV,DFE,SCPE,XMIN,XMAX,FRC,BVA
      DIMENSION FRM(4000,30),INDIND(30),INDDEP(1)
      DIMENSION BV(30,30),RV(30,30),XMIN(30),XMAX(30)
      DIMENSION SCPE(30,30),BVA(30,30),DV(30),
      COMMON FRC(2,4000,30)
      DATA NRU,NCR,NCF,IPDV/4000,9,27,27/
      DATA IIND,INDIND/4,1,2,4,5,26*0/
      TOL=100.*AMACH(4)
      WRITE (6,971) TOL
  971 FORMAT (' TOL=',E15.5)
      DO 2 I=1,NRU
      FRM(I,IPDV)=1
      READ(11,*) (FRM(I,J),J=1,NCR)
    2 CONTINUE
  970 FORMAT (9F9.4)
      DO 12 I=1,NRU
      FRC(2,I,IPDV)=1
      READ (12,*) (FRC(2,I,J),J=1,9)
   12 CONTINUE
      DO 14 I=1,NRU
      FRC(1,I,IPDV)=1
      DO 16 J=1,9
   16 FRC(1,I,J)=FRM(I,J)
   14 CONTINUE
      ESMM=999.
      READ (5,*) (BV(I,1),I=1,IIND)
      WRITE (6,956) (BV(I,1),I=1,IIND)
  956 FORMAT (' BV=',4F12.6)
      CALL EVA(IIND,BV,INDIND,1,NRU,IPDV,2,ESM)
      IP=0
      DO 200 IB1=26,30
      BV(1,1)=IB1*1.D-2
      DO 200 IB2=1,5
      BV(2,1)=IB2*1.D-2
      DO 200 IB3=11,15
      IP=IP+1
      BV(3,1)=IB3*1.D-2
      BV(4,1)=(1-BV(1,1))/2 - BV(2,1)-BV(3,1)
      WRITE (6,952) IP,(BV(I,1),I=1,IIND)
  952 FORMAT (' IP=',I5,3X,'BV=',4F9.3)
   51 CALL EVA(IIND,BV,INDIND,1,NRU,IPDV,2,ESM)
      IF (ESM.LT.ESMM) THEN
        ESMM=ESM
        IMIN=IP
      END IF
```

```
  200 CONTINUE
      WRITE (6,955) ESMM,IMIN
  955 FORMAT (//' ESMM,IMIN=',E15.5,I5)
  800 STOP
      END
C
      SUBROUTINE EVA(IIND,BV,INDIND,IPB,IPE,IPDV,IDT,ESM)
      REAL*8 BV,FRC,ESTV,BVT
      COMMON FRC(2,4000,30)
      DIMENSION BV(30,30),INDIND(30),ESTV(4000)
  951 FORMAT (' BV=',6E14.5)
      DO 100 I=IPB,IPE
  100 ESTV(I)=0
      DO 110 J=1,IIND
      BVT=BV(J,1)
      IT=INDIND(J)
      DO 120 I=IPB,IPE
  120 ESTV(I)=ESTV(I)+BVT*FRC(IDT,I,IT)
  110 CONTINUE
      ESS=0.
      EMX=0.
      EAS=0.
      EV=0.
      DO 150 I=IPB,IPE
      ERR=DABS(FRC(IDT,I,IPDV)-ESTV(I))
      EAS=EAS+ERR
      ESS=ESS+ERR**2
      IF (ERR.GT.EMX) THEN
        EMX=ERR
        IPMAX=I
      END IF
  972 FORMAT (' ESTV,ERR,BVT,IT=',3F10.4,I4)
  150 CONTINUE
      NI=IPE-IPB+1
      EAM=EAS/NI
      ESM=ESS/NI
      WRITE (6,950) EAM,ESM,EMX,IPMAX
  950 FORMAT (' EAM,ESM,EMX=',3E14.4,3X,'IPMAX=',I5)
      RETURN
      END
//GO.FT11F001 DD DSN=U14501A.FRM,DISP=SHR
//GO.FT12F001 DD DSN=U14501A.FRMH,DISP=SHR
//GO.SYSIN DD *
```

VITA

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Doctor of Philosophy

Thesis: PERT TIME ESTIMATES: A LOGICAL ALTERNATIVE WITH IMPROVED ACCURACY

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