# CONSTRUCTIONS OF $\mathcal{L}_p$ SPACES

# FOR $p \in (1, \infty) \setminus \{2\}$

By

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## TABLE OF CONTENTS

# CHAPTER I

# INTRODUCTION

The $\mathcal{L}_p$ spaces are defined in terms of their finite-dimensional subspaces. However, in the category of separable infinite-dimensional Banach spaces, the $\mathcal{L}_p$ spaces for $1 < p < \infty$ with $p \neq 2$ are those spaces which are isomorphic to complemented subspaces of $L^p$, but not isomorphic to the Hilbert space $\ell^2$.

Rosenthal [**RI**], Schechtman [**S**], Alspach [**A**], and Bourgain [**B-R-S**] have developed methods of constructing $\mathcal{L}_p$ spaces for $1 < p < \infty$ with $p \neq 2$ which have a probabilistic aspect. These methods have enlarged the set of known $\mathcal{L}_p$ spaces from the classical examples [$\ell^p$, $\ell^2 \oplus \ell^p$, $\left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^p}$, and $L^p$] to a family indexed by the countable ordinals. We will examine these constructions, provide some details, clarify a few points, and to some extent interrelate the constructed spaces with respect to the relation $\overset{c}{\hookrightarrow}$.

## Preliminaries for $\mathcal{L}_p$ Spaces

### The $\mathcal{L}_p$ Spaces

The $\mathcal{L}_p$ spaces were introduced by Lindenstrauss and Pełczyński in [**L-P**], and were studied further by Lindenstrauss and Rosenthal in [**L-R**]. The definition and some basic results are presented below.

Definition. *Let $1 \leq p \leq \infty$ and $1 \leq \lambda < \infty$. A Banach space $X$ is an $\mathcal{L}_{p,\lambda}$ space if for each finite-dimensional subspace $Z$ of $X$, there is a finite-dimensional subspace $Y$*

*of $X$ containing $Z$ such that $d(Y, \ell^p_n) \leq \lambda$, where $n = \dim(Y)$ and $d(Y, \ell^p_n)$ is the Banach-Mazur distance between $Y$ and $\ell^p_n$. Finally, a Banach space is an $\mathcal{L}_p$ space if it is an $\mathcal{L}_{p,\gamma}$ space for some $1 \leq \gamma < \infty$.*

Let $1 < p < \infty$ where $p \neq 2$. In [**L-P**, Example 8.2], it is shown that $\ell^p$, $\ell^2 \oplus \ell^p$, $(\ell^2 \oplus \ell^2 \oplus \cdots)_{\ell^p}$, and $L^p$ are mutually nonisomorphic $\mathcal{L}_p$ spaces, although this is more easily seen in light of the subsequent results of [**L-R**]. These spaces are the classical $\mathcal{L}_p$ spaces.

Let $X$ be a Banach space. A bounded linear mapping $P : X \to X$ is called a projection if $P^2 = P$. Let $Y$ be a closed subspace of $X$. Then $Y$ is called a complemented subspace of $X$ if there is a (bounded linear) projection $P : X \to X$ mapping $X$ onto $Y$. If $Y$ is a complemented subspace of $X$, $P : X \to X$ is the (bounded linear) projection mapping $X$ onto $Y$, and $Z$ is the null space of $P$, then $X = Y \oplus Z$. Conversely, if $X = Y \oplus Z$ for some closed subspace $Z$ of $X$, then $Y$ is a complemented subspace of $X$ (as is $Z$).

We will restrict our attention to separable infinite-dimensional $\mathcal{L}_p$ spaces for $1 < p < \infty$ with $p \neq 2$. For these spaces, [**L-P**] and [**L-R**] each contribute one implication in the following characterization, but in greater generality.

**Theorem 1.1.** *Let $1 < p < \infty$ where $p \neq 2$, and let $X$ be a separable infinite-dimensional Banach space. Then $X$ is an $\mathcal{L}_p$ space if and only if $X$ is isomorphic to a complemented subspace of $L^p$ but $X$ is not isomorphic to $\ell^2$.*

The essence of the forward implication [**L-P**, Theorem 7.1] is the following.

**Proposition 1.2.** *Let $1 < p < \infty$ and let $X$ be an $\mathcal{L}_p$ space. Then $X$ is isomorphic to a complemented subspace of $L^p(\mu)$ for some measure $\mu$.*

Remark. In the above proposition, analogous statements for $p = 1$ and $p = \infty$

are false. For $p = 1$, [**L-P**, Example 8.1] provides a counterexample. For $p = \infty$, any separable infinite-dimensional $C(K)$ space provides a counterexample, as noted in [**L-P**]. However, by [**L-P**, Corollary 2 of Theorem 7.2], if $X$ is an $\mathcal{L}_1$ space, then $X$ is isomorphic to a subspace of $L^p(\mu)$ for some measure $\mu$.

The essence of the reverse implication [**L-R**, Theorem 2.1] is the following.

**Proposition 1.3.** *Let $1 < p < \infty$ and let $X$ be (isomorphic to) a complemented subspace of $L^p(\mu)$ for some measure $\mu$. Then either $X$ is an $\mathcal{L}_p$ space or $X$ is isomorphic to a Hilbert space.*

REMARK. In the above proposition, modified versions hold for $p = 1$ and $p = \infty$ [**L-R**, Theorem 3.2]. If $X$ is (isomorphic to) a complemented subspace of $L^1(\mu)$ for some measure $\mu$, then $X$ is an $\mathcal{L}_1$ space. If $X$ is (isomorphic to) a complemented subspace of a $C(K)$ space, then $X$ is an $\mathcal{L}_\infty$ space.

Let us assume the hypotheses of Theorem 1.1. The hypothesis that $X$ is infinite-dimensional excludes a class of spaces which are trivially $\mathcal{L}_p$. The hypothesis that $X$ is separable allows us to replace the $L^p(\mu)$ of Proposition 1.2 by $L^p = L^p(0,1)$. As noted in [**L-P**] and [**L-R**], the $\mathcal{L}_2$ spaces are precisely the spaces which are isomorphic to Hilbert spaces. However, the only separable infinite-dimensional Hilbert space (up to isometry) is $\ell^2$. Thus we may replace the Hilbert space of Proposition 1.3 by $\ell^2$. The conclusion of Theorem 1.1 now follows.

## The Relations $\hookrightarrow$ and $\overset{c}{\hookrightarrow}$

Let $X$ and $Y$ be Banach spaces. We write $X \hookrightarrow Y$ if $X$ is isomorphic to a closed subspace of $Y$. We write $X \overset{c}{\hookrightarrow} Y$ if $X$ is isomorphic to a complemented subspace of $Y$. Of course if $X \overset{c}{\hookrightarrow} Y$, then $X \hookrightarrow Y$. If $X \overset{c}{\hookrightarrow} Y$, then $X^* \overset{c}{\hookrightarrow} Y^*$. However if $X \hookrightarrow Y$, it does not follow that $X^* \hookrightarrow Y^*$. If $X$ is a closed subspace of $Y$ with $X \overset{c}{\hookrightarrow} Y$, it does not follow that $X$ itself is a complemented subspace of $Y$. The relations $\hookrightarrow$ and $\overset{c}{\hookrightarrow}$ are reflexive and transitive, but not antisymmetric.

We write $X \equiv Y$ if $X \hookrightarrow Y$ and $Y \hookrightarrow X$. We write $X \equiv_c Y$ if $X \overset{c}{\hookrightarrow} Y$ and $Y \overset{c}{\hookrightarrow} X$. We write $X \sim Y$ if $X$ is isomorphic to $Y$. The relations $\equiv$, $\equiv_c$, and $\sim$ are equivalence relations. Let $[\ ]_\sim$, $[\ ]_{\equiv_c}$, and $[\ ]_\equiv$ denote equivalence classes under $\sim$, $\equiv_c$, and $\equiv$, respectively. Then $[X]_\sim \subset [X]_{\equiv_c} \subset [X]_\equiv$.

If $X \equiv X'$ and $Y \equiv Y'$, then $X \hookrightarrow Y$ if and only if $X' \hookrightarrow Y'$. Similarly, if $X \equiv_c X'$ and $Y \equiv_c Y'$, then $X \overset{c}{\hookrightarrow} Y$ if and only if $X' \overset{c}{\hookrightarrow} Y'$. Thus $\hookrightarrow$ and $\overset{c}{\hookrightarrow}$ induce partial orderings on equivalence classes under $\equiv$ and $\equiv_c$, respectively.

### The Classical $\mathcal{L}_p$ Spaces

Let $2 < p < \infty$. Then $\ell^2$ and the classical separable infinite-dimensional $\mathcal{L}_p$ spaces are related by $\hookrightarrow$ as in diagram (1.1) below, where $X \to Y$ denotes $X \hookrightarrow Y$ but $Y \not\hookrightarrow X$, $X \equiv Y$ denotes $X \hookrightarrow Y$ and $Y \hookrightarrow X$, and the absence of a relation symbol between $X$ and $Y$ implies $X \not\hookrightarrow Y$ and $Y \not\hookrightarrow X$, unless some relation is implied by the transitivity of $\hookrightarrow$. The same conventions will apply in future diagrams relating spaces by $\hookrightarrow$.

$$\begin{array}{ccccccc} \ell^2 & & & & & & \\ & \searrow & & & & & \\ & & \ell^2 \oplus \ell^p & \rightarrow & (\ell^2 \oplus \ell^2 \oplus \cdots)_{\ell^p} & \rightarrow & L^p. \\ & \nearrow & & & & & \\ \ell^p & & & & & & \end{array} \tag{1.1}$$

Let $1 < p < \infty$ where $p \neq 2$. Then $\ell^2$ and the classical separable infinite-dimensional $\mathcal{L}_p$ spaces are related by $\overset{c}{\hookrightarrow}$ as in diagram (1.2) below. Conventions analogous to those described above will apply in this and in future diagrams relating spaces by $\overset{c}{\hookrightarrow}$ (with $\overset{c}{\hookrightarrow}$, $\overset{c}{\rightarrow}$, and $\equiv_c$ replacing $\hookrightarrow$, $\rightarrow$, and $\equiv$, respectively).

$$
\begin{array}{lllllll}
\ell^2 & & & & & & \\
& \overset{c}{\searrow} & & & & & \\
& & \ell^2 \oplus \ell^p & \overset{c}{\to} & \left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^p} & \overset{c}{\to} & L^p. \\
& \overset{c}{\nearrow} & & & & & \\
\ell^p & & & & & &
\end{array}
\tag{1.2}
$$

The positive relations asserted to exist above follow routinely from well-known results. Of course $\ell^2 \overset{c}{\hookrightarrow} \ell^2 \oplus \ell^p$ and $\ell^p \overset{c}{\hookrightarrow} \ell^2 \oplus \ell^p$. Letting $\mathbb{F}$ denote the scalar field,

$$\ell^2 \oplus \ell^p \sim \ell^2 \oplus (\mathbb{F} \oplus \mathbb{F} \oplus \cdots)_{\ell^p} \overset{c}{\hookrightarrow} \ell^2 \oplus \left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^p} \sim \left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^p}.$$

Khintchine's inequality [**W**, I.B.8] for the Rademacher functions $\{r_n\}$ shows that $[r_n]_{L^p} \sim \ell^2$. Moreover, for $2 < p < \infty$, the orthogonal projection of $L^p$ onto $[r_n]_{L^p}$ is bounded. Hence for $2 < p < \infty$, and for $1 < p < 2$ by duality, $\ell^2 \overset{c}{\hookrightarrow} L^p$. It follows that

$$\left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^p} \overset{c}{\hookrightarrow} (L^p \oplus L^p \oplus \cdots)_{\ell^p} \sim L^p.$$

Some of the the negative results are another matter, although $\ell^2 \not\hookrightarrow \ell^p$, $\ell^p \not\hookrightarrow \ell^2$, $\ell^2 \oplus \ell^p \not\hookrightarrow \ell^2$, and $\ell^2 \oplus \ell^p \not\hookrightarrow \ell^p$, all follow from the fact that $\ell^r \not\hookrightarrow \ell^s$ for $r, s \in [1, \infty)$ with $r \neq s$. The fact that $\left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^p} \not\hookrightarrow \ell^2 \oplus \ell^p$ for $2 < p < \infty$ is [**RI**, Lemma for Corollary 14], presented below as Lemma 2.23. The fact that $L^p \not\hookrightarrow \left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^p}$ for $2 < p < \infty$ is [**L-P 2**, Theorem 6.1].

**Elementary Constructions**

Fix $1 < p < \infty$ where $p \neq 2$.

Let $X$ and $Y$ be separable infinite-dimensional Banach spaces such that $X \overset{c}{\hookrightarrow} L^p$ and $Y \overset{c}{\hookrightarrow} L^p$. Then $X \oplus Y \overset{c}{\hookrightarrow} L^p \oplus L^p \sim L^p$. Note that since $\ell^2$ is prime, if $X \not\sim \ell^2$ and $Y \not\sim \ell^2$, then $X \oplus Y \not\sim \ell^2$. Hence if $X$ and $Y$ are $\mathcal{L}_p$ spaces, then $X \oplus Y$ is an $\mathcal{L}_p$ space.

A result of Pełczyński [**P**, Proposition $(*)$], presented below as Lemma 2.8, states that for Banach spaces $V$ and $W$ which are isomorphic to their squares in the sense that $V \oplus V \sim V$ and $W \oplus W \sim W$, if $V \overset{c}{\hookrightarrow} W$ and $W \overset{c}{\hookrightarrow} V$, then $V \sim W$.

Suppose $X$ and $Y$ are as above and are isomorphic to their squares. If $X \overset{c}{\hookrightarrow} Y$, then $X \oplus Y \sim Y$ [since $X \oplus Y$ and $Y$ are isomorphic to their squares, $X \oplus Y \overset{c}{\hookrightarrow} Y \oplus Y \sim Y$, and $Y \overset{c}{\hookrightarrow} X \oplus Y$]. If $X$ and $Y$ are incomparable in the sense that $X \overset{c}{\not\hookrightarrow} Y$ and $Y \overset{c}{\not\hookrightarrow} X$, then $X \oplus Y$ is isomorphically distinct from both $X$ and $Y$ [since $X \oplus Y \sim X$ would imply that $Y \overset{c}{\hookrightarrow} X$, and $X \oplus Y \sim Y$ would imply that $X \overset{c}{\hookrightarrow} Y$]. Hence if $X$ and $Y$ are $\mathcal{L}_p$ spaces which are isomorphic to their squares, then the $\mathcal{L}_p$ space $X \oplus Y$ is isomorphically distinct from both $X$ and $Y$ if and only if $X$ and $Y$ are incomparable in the sense mentioned above.

From the list $\ell^2$, $\ell^p$, $\ell^2 \oplus \ell^p$, $\left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^p}$, $L^p$ of five spaces, the only incomparable pair of spaces is $\left\{\ell^2, \ell^p\right\}$. However, $\ell^2 \oplus \ell^p$ has already been included in the list.

Let $Z$ be a separable infinite-dimensional Banach space such that $Z \overset{c}{\hookrightarrow} L^p$. Then $(Z \oplus Z \oplus \cdots)_{\ell^p} \overset{c}{\hookrightarrow} (L^p \oplus L^p \oplus \cdots)_{\ell^p} \sim L^p$. Note that $\ell^p \overset{c}{\hookrightarrow} (Z \oplus Z \oplus \cdots)_{\ell^p}$, whence $(Z \oplus Z \oplus \cdots)_{\ell^p} \not\sim \ell^2$ and $(Z \oplus Z \oplus \cdots)_{\ell^p}$ is an $\mathcal{L}_p$ space. The space $\left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^p}$ is an example. However, from the list $\ell^2$, $\ell^p$, $\ell^2 \oplus \ell^p$, $\left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^p}$, $L^p$ of five spaces, no space arises from this method of construction which has not already been included in the list.

## Preliminaries for Banach Spaces

We now introduce some terminology used in the study of Banach spaces. The presentation is unavoidably terse and a bit disjointed. General references for this material include [**L-T**] and [**W**]. Throughout the following discussion, $X$ and $Y$ will denote Banach spaces.

A Banach space is a complete normed vector space. Classical examples include the space $L^p(0,1)$ for $1 \leq p \leq \infty$, with $\|f\|_p = \left(\int_{(0,1)} |f|^p\right)^{\frac{1}{p}}$ for $1 \leq p < \infty$ and $\|f\|_\infty = \operatorname{ess\,sup} |f|$ for $p = \infty$, and the space $\ell^p$ for $1 \leq p \leq \infty$, with $\|\{a_i\}\|_{\ell^p} = (\sum |a_i|^p)^{\frac{1}{p}}$ for $1 \leq p < \infty$ and $\|\{a_i\}\|_{\ell^\infty} = \sup |a_i|$ for $p = \infty$. Here $\int$ denotes Lebesgue integration. Functions $f, g \in L^p(0,1)$ are identical as elements of $L^p(0,1)$ if they agree except on a set of measure zero, which is to say that strictly speaking, the elements of $L^p(0,1)$ are equivalence classes of functions.

Given Banach spaces $X_1, X_2, \ldots$ and $1 \leq p < \infty$, $(X_1 \oplus X_2 \oplus \cdots)_{\ell^p}$ is the set of all sequences $\{x_i\}$ with $x_i \in X_i$ such that $\|\{x_i\}\| = (\sum \|x_i\|_{X_i}^p)^{\frac{1}{p}} < \infty$. The sum $(X_1 \oplus X_2 \oplus \cdots)_{\ell^p}$ is a Banach space, and will also be denoted $\left(\sum^{\oplus} X_i\right)_{\ell^p}$.

Suppose $T : X \to Y$ is a linear operator. Then $T$ is said to be bounded if $\|T\| = \sup_{x \in X \setminus \{0\}} \frac{\|T(x)\|}{\|x\|} < \infty$. A linear operator is bounded if and only if it is continuous.

Suppose $T : X \to Y$ is a bounded linear operator. Then $T$ is said to be an isomorphism if $T$ has an inverse $T^{-1} : Y \to X$ which is a bounded linear operator. If $T$ is a bijection, then $T$ is an isomorphism by the open mapping theorem. If there is an isomorphism $S : X \to Y$, then $X$ and $Y$ are said to be isomorphic, and we write $X \sim Y$. If $X \sim Y$, the Banach-Mazur distance between $X$ and $Y$ is $d(X,Y) = \inf_S \left\{\|S\| \, \|S^{-1}\|\right\}$, where the infimum is taken over all isomorphisms $S : X \to Y$.

Suppose $T : X \to Y$ is a bounded linear operator. Then $T$ is called an isomorphic imbedding of $X$ into $Y$ if $T$ is an injection onto a closed subspace $Y'$ of $Y$. If there is an isomorphic imbedding $S : X \to Y$, we write $X \hookrightarrow Y$.

Suppose $P : X \to X$ is a bounded linear operator. Then $P$ is called a projection if $P^2 = P$. Suppose $P : X \to X$ is a projection. Then $P(X)$ is a closed subspace of $X$, and each $x \in X$ has a unique representation as $x = y + z$ where $y \in P(X)$ and $P(z) = 0$. Moreover, $I - P : X \to X$ is a projection as well, where $I : X \to X$ is the identity mapping. The range $R = P(X)$ and null space $N = (I - P)(X)$ of $P$ are said to be complemented subspaces of $X$, and $X = R \oplus N$. We write $R \overset{c}{\hookrightarrow} X$ and $N \overset{c}{\hookrightarrow} X$. More generally, we write $Y \overset{c}{\hookrightarrow} X$ if $Y$ is isomorphic to a complemented subspace of $X$.

The Rademacher functions $r_k : [0,1] \to \{-1,1\}$ for $k \in \mathbb{N}$ are defined by $r_k(t) = \operatorname{sgn} \sin(2^k \pi t)$.

For expressions $A$ and $B$ and constants $K_1$ and $K_2$, we write $A \underset{K_2}{\overset{K_1}{\approx}} B$ to signify that $A \leq K_1 B$ and $B \leq K_2 A$. We also write $A \approx B$ if $K_1$ and $K_2$ exist but are not specified. If so indicated, $A \approx B$ will refer to an approximation rather than to a pair of inequalities.

Khintchine's inequality states that for $1 \leq p < \infty$, there is a constant $K_p$ such that for all scalars $a_1, a_2, \ldots$, for the Rademacher functions $r_1, r_2, \ldots$, and for all $N \in \mathbb{N}$, $1/K_p \left(\sum_{i=1}^N |a_i|^2\right)^{\frac{1}{2}} \leq \left\|\sum_{i=1}^N a_i r_i\right\|_p \leq K_p \left(\sum_{i=1}^N |a_i|^2\right)^{\frac{1}{2}}$. This inequality could also be expressed as $\left\|\sum_{i=1}^N a_i r_i\right\|_p \underset{K_p}{\overset{K_p}{\approx}} \left(\sum_{i=1}^N |a_i|^2\right)^{\frac{1}{2}}$.

A sequence $\{x_i\}$ in $X$ is said to be a (Schauder) basis for $X$ if for each $x \in X$, there is a unique sequence $\{a_i\}$ of scalars such that $x = \sum a_i x_i$, with convergence in the norm of $X$.

Given a sequence $\{x_i\}$ in $X$, the closed linear span of $\{x_i\}$ in $X$ will be denoted $[x_i]_X$, or simply $[x_i]$ if the context is clear. Such a sequence is called a basic sequence if $\{x_i\}$ is a basis for $[x_i]_X$.

Given a sequence $\{x_i\}$ in $X$, the series $\sum x_i$ is said to converge unconditionally if any of the following equivalent conditions hold: (a) $\sum \epsilon_i x_i$ converges for all $\{-1,1\}$-valued sequences $\{\epsilon_i\}$, (b) $\sum x_{\sigma(i)}$ converges for all permutations $\sigma$ of $\mathbb{N}$, or (c) $\sum x_{n(i)}$

converges for all increasing sequences $\{n(i)\}$ in $\mathbb{N}$.

A basis $\{x_i\}$ for $X$ is said to be unconditional if for each sequence of scalars for which $\sum a_i x_i$ converges, the convergence is unconditional. If $\{x_i\}$ is an unconditional basis for $X$, then for $P_E : [x_i] \to [x_i]$ defined by $P_E\left(\sum_{i=1}^{\infty} a_i x_i\right) = \sum_{i \in E} a_i x_i$, we have $\sup_{E \subset \mathbb{N}} \|P_E\| < \infty$.

Suppose $\{x_i\}$ is a basic sequence in $X$. A sequence $\{y_j\}$ in $X$ is called a block basic sequence (with respect to $\{x_i\}$) if $y_j \neq 0$ for all $j \in \mathbb{N}$ and there are disjoint nonempty finite $E_1, E_2, \ldots \subset \mathbb{N}$ with $\max E_j < \min E_{j'}$ for $j < j'$ and scalars $a_1, a_2, \ldots$ such that $y_j = \sum_{i \in E_j} a_i x_i$ for all $j \in \mathbb{N}$. Suppose $\{y_j\}$ is a block basic sequence (with respect to $\{x_i\}$). Then $\{y_j\}$ is a basic sequence. If $\{x_i\}$ is unconditional, then $\{y_j\}$ is unconditional as well.

Suppose $\{x_i\}$ and $\{y_i\}$ are bases for $X$ and $Y$, respectively. Then $\{x_i\}$ and $\{y_i\}$ are said to be equivalent if for all sequences $\{a_i\}$ of scalars, $\sum a_i x_i$ converges if and only if $\sum a_i y_i$ converges. If $\{x_i\}$ and $\{y_i\}$ are equivalent, then there is a natural isomorphism between $X$ and $Y$ by the closed graph theorem.

Suppose $\{x_i\}$ and $\{y_i\}$ are normalized bases for $X$ and $Y$, respectively, which are equivalent. Let $K$ be a positive constant. Then $\{x_i\}$ and $\{y_i\}$ are said to be $K$-equivalent if for all sequences $\{a_i\}$ of scalars such that $\sum a_i x_i$ and $\sum a_i y_i$ converge, $\|\sum a_i x_i\| \underset{K}{\overset{K}{\approx}} \|\sum a_i y_i\|$.

A random variable is a measurable function on a probability space $(\Omega, \mu)$. For $N \in \mathbb{N}$, random variables $X_1, X_2, \ldots, X_N$ on $\Omega$ are said to be independent if for all Borel sets $B_1, B_2, \ldots, B_N$, $\mu\left(\bigcap_{i=1}^{N} \{t : X_i(t) \in B_i\}\right) = \prod_{i=1}^{N} \mu\left(\{t : X_i(t) \in B_i\}\right)$. Random variables $X_1, X_2, \ldots$ on $\Omega$ are said to be independent if $X_1, X_2, \ldots, X_N$ are independent for each $N \in \mathbb{N}$.

## Overview of Chapters

We briefly discuss the content of the succeeding chapters.

Chapter II reviews the construction of Rosenthal [**RI**]. Rosenthal's work is based on the study of the span in $L^p$ for $2 < p < \infty$ of sequences of independent mean zero random variables. A few nonclassical $\mathcal{L}_p$ spaces were found by Rosenthal, principal among them the space $X_p$. Chapter II includes a complete ordering of these spaces with respect to the (partial order) relation $\overset{c}{\hookrightarrow}$.

Chapter III reviews the construction of Schechtman [**S**]. Schechtman takes Rosenthal's space $X_p$ and iterates a tensor product operation to produce a sequence of $\mathcal{L}_p$ spaces. Chapter III includes a section on the sequence space realization of Schechtman's spaces, expanding on a remark found in [**S**].

Chapter IV reviews the construction of Alspach [**A**]. Alspach's work generalizes the construction of Rosenthal, and generates spaces by means of a notion of independent sum, but has only been available in manuscript form. A few nonclassical $\mathcal{L}_p$ spaces were found by Alspach, principal among them a space denoted $D_p$. Chapter IV includes a complete ordering of these and Rosenthal's spaces with respect to $\overset{c}{\hookrightarrow}$.

Chapter V reviews the construction of Bourgain, Rosenthal, and Schechtman [**B-R-S**]. These authors iterate and intertwine a notion of disjoint sum and a notion of independent sum to generate a family of $\mathcal{L}_p$ spaces indexed by the countable ordinals, and distinguish these spaces isomorphically by means of an isomorphic invariant, introduced in [**B-R-S**], which assigns an ordinal number to each separable Banach space.

Each chapter has a diagram relating the spaces under discussion with respect to $\overset{c}{\hookrightarrow}$. These diagrams are (1.2), (2.27), (3.2), (4.10), and (5.5).

# CHAPTER II

## THE NONCLASSICAL $\mathcal{L}_p$ SPACES OF ROSENTHAL

Let $1 < p < \infty$ where $p \neq 2$. Rosenthal [**RI**] was the first to extend the list of separable infinite-dimensional $\mathcal{L}_p$ spaces beyond the four previously known isomorphism types: $L^p$, $\ell^p$, $\ell^2 \oplus \ell^p$, and $\left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^p}$. The principal $\mathcal{L}_p$ spaces which Rosenthal constructed are $X_p$ and $B_p$, to be discussed presently. Using the newly revised list of six $\mathcal{L}_p$ spaces, Rosenthal constructed a few more such spaces by forming direct sums (pairwise and in the sense of $\ell^p$ for sequences) of these six.

### The Space $X_p$

In contrast to most classical Banach spaces, $X_p$ does not have a preferred standard realization. Let $2 < p < \infty$. One realization of $X_p$ is as the closed linear span in $L^p$ of a sequence $\{f_n\}$ of independent symmetric three-valued random variables such that the ratios $\|f_n\|_2/\|f_n\|_p$ approach zero slowly (in a sense to be made precise). On the other hand, given positive weights $w_n$ approaching zero slowly in the same sense, another realization of $X_p$ is as the set of all sequences $\{x_n\}$ in $\ell^p$ for which the weighted $\ell^2$ norm $\left(\sum |w_n x_n|^2\right)^{\frac{1}{2}}$ is finite. For the conjugate index $q$, $X_q$ is defined to be the dual of $X_p$.

#### The Space $X_{p,w}$

We first examine the sequence space realization of $X_p$.

Definition. *Let $2 < p < \infty$ and let $w = \{w_n\}$ be a sequence of positive scalars. Define $X_{p,w}$ to be the set of all sequences $x = \{x_n\}$ of scalars for which both $\sum |x_n|^p$ and $\sum |w_n x_n|^2$ are finite. For $x \in X_{p,w}$, define the norm $\|x\|_{X_{p,w}}$ to be the maximum of $\left(\sum |x_n|^p\right)^{\frac{1}{p}}$ and $\left(\sum |w_n x_n|^2\right)^{\frac{1}{2}}$.*

Thus $\|x\|_{X_{p,w}}$ is the maximum of the $\ell^p$ norm of $x$ and the weighted $\ell^2$ norm of $x$. Under this norm, it is a routine matter to show that $X_{p,w}$ is a Banach space with unconditional standard basis. The isomorphism type of $X_{p,w}$ depends on the sequence $w = \{w_n\}$ of weights, as partially outlined in the following proposition [**RI**].

**Proposition 2.1.** *Let $2 < p < \infty$ and let $w = \{w_n\}$ be a sequence of positive scalars.*

*(a) If $\inf w_n > 0$, then $X_{p,w}$ is isomorphic to $\ell^2$.*

*(b) If $\sum w_n^{\frac{2p}{p-2}} < \infty$, then $X_{p,w}$ is isomorphic to $\ell^p$.*

*(c) If there is some $\epsilon > 0$ for which $\{n\colon w_n \geq \epsilon\}$ and $\{n\colon w_n < \epsilon\}$ are both infinite and for which $\sum_{w_n<\epsilon} w_n^{\frac{2p}{p-2}} < \infty$, then $X_{p,w}$ is isomorphic to $\ell^2 \oplus \ell^p$.*

*(d) Otherwise, $w$ satisfies condition $(*)$:*

$$\text{for each } \epsilon > 0, \sum_{w_n<\epsilon} w_n^{\frac{2p}{p-2}} = \infty. \qquad (*)$$

**Proof.**

(a) Suppose $\inf w_n = C > 0$ and let $x = \{x_n\} \in X_{p,w}$. Then

$$\|x\|_{\ell^p} \leq \|x\|_{\ell^2} = \left(\sum |x_n|^2\right)^{\frac{1}{2}} \leq \tfrac{1}{C}\left(\sum |w_n x_n|^2\right)^{\frac{1}{2}}.$$

Hence

$$\left(\sum |w_n x_n|^2\right)^{\frac{1}{2}} \leq \|x\|_{X_{p,w}} \leq \max\left\{\tfrac{1}{C}, 1\right\}\left(\sum |w_n x_n|^2\right)^{\frac{1}{2}},$$

so $X_{p,w}$ is isomorphic to $\ell^2$ via the mapping $\{x_n\} \mapsto \{w_n x_n\}$.

(b) Suppose $\sum w_n^{\frac{2p}{p-2}} < \infty$ and let $x = \{x_n\} \in X_{p,w}$. Then by Hölder's inequality with conjugate indices $p' = \frac{p}{2} > 1$ and $q' = \frac{p}{p-2}$, we have

$$\sum |w_n x_n|^2 = \sum |w_n{}^2 x_n{}^2| \leq \left(\sum w_n{}^{2\frac{p}{p-2}}\right)^{\frac{p-2}{p}} \left(\sum |x_n|^{2\frac{p}{2}}\right)^{\frac{2}{p}}.$$

Let $K = \left(\sum w_n{}^{2\frac{p}{p-2}}\right)^{\frac{p-2}{2p}}$. Then $\left(\sum |w_n x_n|^2\right)^{\frac{1}{2}} \leq K \left(\sum |x_n|^p\right)^{\frac{1}{p}}$. Hence

$$\|x\|_{\ell^p} \leq \|x\|_{X_{p,w}} \leq \max\{1, K\} \|x\|_{\ell^p},$$

so $X_{p,w}$ is isomorphic to $\ell^p$ via the formal identity mapping.

(c) The hypothesis of part (c) is equivalent to the hypothesis that $\mathbb{N}$ is the disjoint union of two infinite sets $N_1$ and $N_2$ for which $inf_{n \in N_1} w_n > 0$ and $\sum_{n \in N_2} w_n^{\frac{2p}{p-2}} < \infty$. Thus part (c) follows from parts (a) and (b) and the unconditionality of the standard basis of $X_{p,w}$.

(d) Condition $(*)$ is equivalent to the conjunction of the negations of the hypotheses of parts (a), (b), and (c). □

Remark 1. We will show later that for fixed $2 < p < \infty$, all spaces $X_{p,w}$ for $w$ satisfying condition $(*)$ are mutually isomorphic, but isomorphically distinct from $\ell^2$, $\ell^p$, and $\ell^2 \oplus \ell^p$ (as well as $\left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^p}$ and $L^p$). Thus part (d) is indeed a different case, and part (d) does not split into subcases.

Remark 2. Let $2 < p < \infty$. If $inf\, w_n = 0$ (as occurs in parts (b), (c), and (d)), then $X_{p,w}$ contains a complemented subspace isomorphic to $\ell^p$, since some subsequence of $w$ satisfies the hypothesis of part (b). Hence in parts (b), (c), and (d), $X_{p,w}$ is not isomorphic to $\ell^2$. We will show later that the spaces $X_{p,w}$ are isomorphic to complemented subspaces of $L^p$. Thus only part (a) does not yield an $\mathcal{L}_p$ space, while parts (b) and (c) yield known $\mathcal{L}_p$ spaces, and part (d) yields a previously unknown $\mathcal{L}_p$ space. The spaces $X_{p,w}$ for $w$ satisfying condition $(*)$ will be our sequence space realizations of $X_p$.

**Rosenthal's Inequality**

Rosenthal proved the following fundamental probabilistic inequality [**RI**, Theorem 3], which (in its corollary) relates $X_{p,w}$ with the closed linear span of a sequence of independent mean zero random variables in $L^p$ $(2 < p < \infty)$.

**Theorem 2.2.** *Let $2 < p < \infty$. There is a constant $K_p$, depending only on $p$, such that if $f_1, \ldots, f_N$ are independent mean zero random variables in $L^p$, then*

*(a)* $\left\|\sum_{n=1}^N f_n\right\|_p \leq K_p \max\left\{\left(\sum_{n=1}^N \|f_n\|_p^p\right)^{\frac{1}{p}}, \left(\sum_{n=1}^N \|f_n\|_2^2\right)^{\frac{1}{2}}\right\}$, *and*

*(b)* $\left\|\sum_{n=1}^N f_n\right\|_p \geq \frac{1}{2} \max\left\{\left(\sum_{n=1}^N \|f_n\|_p^p\right)^{\frac{1}{p}}, \left(\sum_{n=1}^N \|f_n\|_2^2\right)^{\frac{1}{2}}\right\}$.

*If in addition $f_1, \ldots, f_N$ are assumed to be symmetric, then the constant $\frac{1}{2}$ can be replaced by* 1.

Remark. It is shown in [**J-S-Z**] that $K_p$ is of order $p/\log p$.

The proof of Rosenthal's inequality will not be presented, but we deduce its corollary [**RI**].

**Corollary 2.3.** *Let $2 < p < \infty$, let $\{f_n\}$ be a sequence of independent mean zero random variables in $L^p$, and let $w = \{w_n\} = \left\{\|f_n\|_2/\|f_n\|_p\right\}$. Then $[f_n]_{L^p}$ is isomorphic to $X_{p,w}$, and $\{f_n\}$ in $L^p$ is equivalent to the standard basis of $X_{p,w}$.*

**Proof.** Without loss of generality, suppose each $f_n$ is of norm one in $L^p$, so that $w_n = \|f_n\|_2$. Let $f \in \operatorname{span}\{f_n\}$ and express $f$ as $\sum_{n=1}^N c_n f_n$. Then by Theorem 2.2, we have

$$\left\|\sum_{n=1}^N c_n f_n\right\|_p \overset{K_p}{\underset{2}{\approx}} \max\left\{\left(\sum_{n=1}^N |c_n|^p\right)^{\frac{1}{p}}, \left(\sum_{n=1}^N |c_n w_n|^2\right)^{\frac{1}{2}}\right\}.$$

Hence $[f_n]_{L^p}$ is isomorphic to $X_{p,w}$ via the mapping $\sum c_n f_n \mapsto \{c_n\}$, and $\{f_n\}$ in $L^p$ is equivalent to the standard basis of $X_{p,w}$. $\square$

REMARK 1. Let $2 < p < \infty$. Given a sequence $w = \{w_n\}$ of positive scalars for which $sup\, w_n \leq 1$, $\{w_n\}$ can be realized as $\left\{\|f_n\|_2/\|f_n\|_p\right\}$ for $\{f_n\}$ satisfying the hypotheses of Corollary 2.3. If $sup\, w_n > 1$, then $X_{p,w} \sim X_{p,w'}$ for some sequence $w' = \{w'_n\}$ satisfying $sup\, w'_n \leq 1$. Thus there is a complete correspondence between the sequence spaces $X_{p,w}$ and the function spaces $[f_n]_{L^p}$ for $\{f_n\}$ satisfying the hypotheses of Corollary 2.3.

REMARK 2. For fixed $2 < p < \infty$, the spaces $[f_n]_{L^p}$ for $\{f_n\}$ satisfying the hypotheses of Corollary 2.3 and $w = \{w_n\} = \left\{\|f_n\|_2/\|f_n\|_p\right\}$ satisfying condition $(*)$ of Proposition 2.1 will be our function space realizations of $X_p$.

### The Complementation of $X_{p,w}$ in $L^p$

Let $2 < p < \infty$. In its sequence space realizations, it is not so clear that $X_p$ is an $\mathcal{L}_p$ space. However, we will soon show that in its function space realizations, the complementation of $[f_n]_{L^p}$ in $L^p$ follows if the sequence $\{f_n\}$ satisfies certain additional hypotheses. On the other hand, in its function space realizations, the isomorphic structure of $X_p$ is not so clear. We will go back and forth between realizations, depending on their relative advantages at the time.

Suppose $f_n$ is a symmetric three-valued random variable. Let $\alpha_n$ be the positive value attained by $|f_n|$ and let $\mu_n$ be the measure of the set on which $f_n$ is nonzero. Then for $1 \leq r < \infty$, we have

$$\|f_n\|_r = (\alpha_n{}^r \mu_n)^{\frac{1}{r}} = \alpha_n \mu_n{}^{\frac{1}{r}}.$$

Let $2 < p < \infty$. Then $w_n = \|f_n\|_2/\|f_n\|_p = \mu_n{}^{\frac{1}{2}-\frac{1}{p}} = \mu_n{}^{\frac{p-2}{2p}}$. Hence

$$w_n{}^{\frac{2p}{p-2}} = \mu_n.$$

This provides an interpretation for condition $(*)$ of Proposition 2.1 in terms of properties of a sequence $\{f_n\}$ of independent symmetric three-valued random variables, namely

$$\text{for each } \epsilon > 0, \sum_{\mu_n < \epsilon} \mu_n = \infty.$$

Let $q$ be the conjugate index of $p$. Then

$$\|f_n\|_p \|f_n\|_q = \alpha_n{}^2 \mu_n{}^{\frac{1}{p}+\frac{1}{q}} = \alpha_n{}^2 \mu_n = \left(\alpha_n \mu_n{}^{\frac{1}{2}}\right)^2 = \|f_n\|_2^2.$$

This provides a way to interrelate the $L^p$, $L^q$, and $L^2$ norms of a symmetric three-valued random variable. We will find this useful in the proof of the next theorem, where we show that a certain projection is bounded in both $L^2$ and $L^p$ norms. We will make explicit use of the fact that if $f_n$ is a symmetric three-valued random variable of norm one in $L^p$, then

$$\left\| \frac{f_n}{\|f_n\|_2^2} \right\|_q = \frac{\|f_n\|_q}{\|f_n\|_2^2} = \frac{1}{\|f_n\|_p} = 1. \tag{2.1}$$

Remark. If the scalars are complex, the hypothesis that $f_n$ is a symmetric three-valued random variable can be replaced by the hypothesis that $f_n$ is a mean zero random variable for which $|f_n|$ is $\{0, \alpha_n\}$-valued for $\alpha_n \neq 0$.

Rosenthal proved the following theorem [**RI**, Theorem 4], which (in its corollary) establishes that for $2 < p < \infty$, the spaces $X_{p,w}$ are isomorphic to complemented subspaces of $L^p$. To prove the theorem, we use the following probabilistic inequality [**RI**, Lemma 2b], which we state without proof.

**Lemma 2.4.** *Let $1 \leq q < 2$ and let $f_1, \ldots, f_N$ be independent mean zero random variables in $L^q$. Then*

$$\left\|\sum_{n=1}^N f_n\right\|_q \leq 2 \left(\sum_{n=1}^N \|f_n\|_q^q\right)^{\frac{1}{q}}.$$

*If in addition $f_1, \ldots, f_N$ are assumed to be symmetric, then the constant 2 can be replaced by 1.*

**Theorem 2.5.** *Let $1 < p < \infty$ and let $\{f_n\}$ be a sequence of independent symmetric three-valued random variables in $L^p$. Then there is a projection $P: L^p \to L^p$ onto $[f_n]_{L^p}$ with $\|P\| \leq C_p$, where $C_2 = 1$, $C_p = K_p$ (the constant in Theorem 2.2) for $2 < p < \infty$, and $C_p = C_q$ for conjugate indices $p$ and $q$.*

**Proof.** If $p = 2$, the orthogonal projection $\pi: L^2 \to L^2$ onto $[f_n]_{L^2}$ satisfies the requirements. We will presently show that for $2 < p < \infty$, the set-theoretic restriction of $\pi$ to $L^p$ yields a bounded projection $P: L^p \to L^p$ onto $[f_n]_{L^p}$ with $\|P\| \leq K_p$. This will suffice to prove the theorem in the general case, since the adjoint then induces a projection $Q: L^q \to L^q$ onto $[f_n]_{L^q}$ with $\|Q\| = \|P\|$.

Let $2 < p < \infty$, so that $L^p \subset L^2$. Let $w = \{w_n\} = \left\{\|f_n\|_2 / \|f_n\|_p\right\}$. Without loss of generality, suppose $f_n$ is real-valued with $\|f_n\|_p = 1$. Then $w_n = \|f_n\|_2$. Let $\pi: L^2 \to [f_n]_{L^2}$ be the orthogonal projection defined by

$$\pi(g) = \sum \left( \int_0^1 g(t) \frac{f_n}{\|f_n\|_2}(t)\, dt \right) \frac{f_n}{\|f_n\|_2}.$$

Then $\|\pi(g)\|_2 \leq \|g\|_2$. We will show that if $g \in L^p$, then $\pi(g) \in L^p$ and $\|\pi(g)\|_p \leq K_p \|g\|_p$. Thus

$$P(g) = \sum \left( \int_0^1 g(t) \frac{f_n}{\|f_n\|_2^2}(t)\, dt \right) f_n$$

defines a mapping $P: L^p \to [f_n]_{L^p}$. Set-theoretically, $P$ is the restriction of $\pi$ to $L^p$. It will follow that $P$ is a projection and $\|P\| \leq K_p$.

Fix $g \in L^p$ and let

$$x_n = \int_0^1 g(t) \frac{f_n}{\|f_n\|_2^2}(t)\, dt,$$

so that $\pi(g) = \sum x_n f_n$. We will show that $\{x_n\} \in X_{p,w}$ and $\|\{x_n\}\|_{X_{p,w}} \le \|g\|_p$. Corollary 2.3 will then yield $\|\pi(g)\|_p = \|\sum x_n f_n\|_p \le K_p \|\{x_n\}\|_{X_{p,w}} \le K_p \|g\|_p$.

First we examine the weighted $\ell^2$ norm of $\{x_n\}$. Let

$$y_n = \int_0^1 g(t) \frac{f_n}{\|f_n\|_2}(t)\, dt = x_n \|f_n\|_2 = x_n w_n.$$

Then

$$\left(\sum |w_n x_n|^2\right)^{\frac{1}{2}} = \|\{y_n\}\|_{\ell^2} = \left\|\sum y_n \tfrac{f_n}{\|f_n\|_2}\right\|_{L^2} = \|\pi(g)\|_2 \le \|g\|_2 \le \|g\|_p. \tag{2.2}$$

Next we examine the $\ell^p$ norm of $\{x_n\}$. We verify that $\{x_n\} \in \ell^p$ by testing against $\ell^q$. Let $\{c_n\} \in \ell^q$. Using Lemma 2.4 and equation (2.1), for each $N \in \mathbb{N}$

$$\begin{aligned}
\left\|\sum_{n=1}^N c_n \frac{f_n}{\|f_n\|_2^2}\right\|_q &\le \left(\sum_{n=1}^N \left\|c_n \frac{f_n}{\|f_n\|_2^2}\right\|_q^q\right)^{\frac{1}{q}} \\
&= \left(\sum_{n=1}^N |c_n|^q\right)^{\frac{1}{q}} \\
&\le \|\{c_n\}\|_{\ell^q}.
\end{aligned}$$

Now by Hölder's inequality and the observation above, for each $N \in \mathbb{N}$

$$\begin{aligned}
\left|\sum_{n=1}^N c_n x_n\right| &= \left|\sum_{n=1}^N c_n \int_0^1 g(t) \frac{f_n}{\|f_n\|_2^2}(t)\, dt\right| \\
&= \left|\int_0^1 g(t) \sum_{n=1}^N c_n \frac{f_n}{\|f_n\|_2^2}(t)\, dt\right| \\
&\le \|g\|_p \left\|\sum_{n=1}^N c_n \frac{f_n}{\|f_n\|_2^2}\right\|_q \\
&\le \|g\|_p \|\{c_n\}\|_{\ell^q}.
\end{aligned}$$

Hence $\{x_n\} \in \ell^p$ and

$$\|\{x_n\}\|_{\ell^p} \le \|g\|_p. \tag{2.3}$$

Combining (2.2) and (2.3), we see that $\{x_n\}$ is indeed in $X_{p,w}$ and $\|\{x_n\}\|_{X_{p,w}} \le \|g\|_p$.

Now by Corollary 2.3 (and the inequality appearing in its proof), we have $\|\sum x_n f_n\|_p \overset{K_p}{\underset{1}{\approx}} \|\{x_n\}\|_{X_{p,w}}$, so that

$$\|\pi(g)\|_p = \|\sum x_n f_n\|_p \leq K_p \|\{x_n\}\|_{X_{p,w}} \leq K_p \|g\|_p.$$

Hence $P(g) = \pi(g) \in [f_n]_{L^p}$ and $P$ is a projection from $L^p$ onto $[f_n]_{L^p}$ with $\|P\| \leq K_p$.

□

REMARK. If the scalars are complex, the hypothesis that each $f_n$ is symmetric and three-valued can be replaced by the hypothesis that each $f_n$ is mean zero and $|f_n|$ is $\{0, \alpha_n\}$-valued for $\alpha_n \neq 0$, but without the hypothesis of symmetry we have $\|P\| \leq 2C_p$.

We deduce the following corollary [**RI**].

**Corollary 2.6.** *Let $2 < p < \infty$ and let $w = \{w_n\}$ be a sequence of positive scalars. Then $X_{p,w}$ is isomorphic to a complemented subspace of $L^p$. If $\inf w_n = 0$, then $X_{p,w}$ is an $\mathcal{L}_p$ space. In particular, if $w$ satisfies condition $(*)$ of Proposition 2.1, then $X_{p,w}$ is an $\mathcal{L}_p$ space.*

**Proof.** First suppose that $sup\, w_n \leq 1$. Then $\{w_n\}$ can be realized as $\left\{\|f_n\|_2 / \|f_n\|_p\right\}$ for a sequence $\{f_n\}$ of independent symmetric (whence mean zero) three-valued random variables in $L^p$. Hence $X_{p,w}$ is isomorphic to $[f_n]_{L^p}$ by Corollary 2.3, and $[f_n]_{L^p}$ is complemented in $L^p$ by Theorem 2.5.

Now suppose that $sup\, w_n > 1$. Let $N_0 = \{n : w_n \leq 1\}$ and $N_1 = \{n : w_n > 1\}$. Let $w_{[0]} = \{w_n\}_{n \in N_0}$ and $w_{[1]} = \{w_n\}_{n \in N_1}$, and let $\{1\} = \{1\}_{n \in N_1}$ be the sequence with constant value one. Let $w' = \{w'_n\}_{n=1}^{\infty} = \{\min\{w_n, 1\}\}_{n=1}^{\infty}$, whence $sup\, w'_n \leq 1$ and $X_{p,w'} \overset{c}{\hookrightarrow} L^p$. Then

$$X_{p,w} \sim X_{p,w_{[0]}} \oplus X_{p,w_{[1]}} \sim X_{p,w_{[0]}} \oplus X_{p,\{1\}} \sim X_{p,w'} \overset{c}{\hookrightarrow} L^p,$$

where for an $N$-tuple $v = \{v_1, \ldots, v_N\}$ of positive scalars, $X_{p,v}$ is defined in the obvious way, and $X_{p,\emptyset} = \{0\}$.

If *inf* $w_n = 0$, then $X_{p,w}$ contains a complemented subspace isomorphic to $\ell^p$, whence $X_{p,w}$ is not isomorphic to $\ell^2$. Hence if *inf* $w_n = 0$, then $X_{p,w}$ is an $\mathcal{L}_p$ space by Theorem 1.1. Finally, note that if $w = \{w_n\}$ satisfies condition $(*)$ of Proposition 2.1, then *inf* $w_n = 0$. $\square$

**The Mutual Isomorphism of the Spaces $X_{p,w}$**

We will show that for fixed $2 < p < \infty$, all spaces $X_{p,w}$ for $w = \{w_n\}$ satisfying condition $(*)$ of Proposition 2.1 are mutually isomorphic, and isomorphically distinct from the previously known $\mathcal{L}_p$ spaces. These two results are our next major concerns. The following proposition [**RI**, Lemma 7] will be used in the proofs of both of these results.

**Proposition 2.7.** *Let $2 < p < \infty$ and let $w = \{w_n\}$ be a sequence of positive scalars. Suppose that $\{E_j\}$ is a sequence of disjoint nonempty finite subsets of $\mathbb{N}$. Let $b_j = \sum_{n \in E_j} w_n^{\frac{2}{p-2}} e_n$ and $\tilde{b}_j = b_j / \|b_j\|_{\ell^p}$, where $\{e_n\}$ is the standard basis of $X_{p,w}$. Let $v_j = \left(\sum_{n \in E_j} w_n^{\frac{2p}{p-2}}\right)^{\frac{p-2}{2p}}$ and $v = \{v_j\}$. Then*

*(a) $\left\{\tilde{b}_j\right\}$ is an unconditional basis for $\left[\tilde{b}_j\right]_{X_{p,w}}$ which is isometrically equivalent to the standard basis of $X_{p,v}$, and*

*(b) there is a projection $P\colon X_{p,w} \to \left[\tilde{b}_j\right]_{X_{p,w}}$ with $\|P\| = 1$.*

**Proof.** First we establish some notation. Let $\ell_{2,w}$ be the Hilbert space of all sequences $x = \{x_n\}$ of scalars for which $\|x\|_{\ell_{2,w}} = \left(\sum |w_n x_n|^2\right)^{\frac{1}{2}} < \infty$, where the inner product in $\ell_{2,w}$ is defined by $\langle x, y\rangle = \sum x_n \bar{y}_n {w_n}^2$ (where $x = \{x_n\}$, $y = \{y_n\}$, and bar is complex conjugation). Motivating the choice of the $b_j$ is the fact that

$$\|b_j\|_{\ell^p}^p = \sum\nolimits_{n\in E_j} w_n{}^{\frac{2p}{p-2}} = \sum\nolimits_{n\in E_j} w_n{}^{\frac{4}{p-2}} w_n{}^2 = \|b_j\|_{\ell_{2,w}}^2.$$

Let $\sigma_j$ denote the common value of $\|b_j\|_{\ell^p}^p$, $\|b_j\|_{\ell_{2,w}}^2$, and $\sum_{n\in E_j} w_n{}^{\frac{2p}{p-2}}$. Note that $v_j = \sigma_j{}^{\frac{p-2}{2p}}$ by our definitions.

(a) The unconditionality of $\{\tilde{b}_j\}$ follows from the unconditionality of $\{e_n\}$ in $X_{p,w}$.

We now examine the isometric equivalence of the bases. Let $J \in \mathbb{N}$ and let $\lambda_1, \ldots, \lambda_J$ be scalars. Then

$$\begin{aligned}\left\|\sum_{j=1}^{J} \lambda_j b_j\right\|_{\ell^p}^p &= \left\|\sum_{j=1}^{J} \lambda_j \sum_{n\in E_j} w_n{}^{\frac{2}{p-2}} e_n\right\|_{\ell^p}^p \\ &= \sum_{j=1}^{J} |\lambda_j|^p \sum_{n\in E_j} w_n{}^{\frac{2p}{p-2}} \\ &= \sum_{j=1}^{J} |\lambda_j|^p \sigma_j \end{aligned} \tag{2.4}$$

and

$$\begin{aligned}\left\|\sum_{j=1}^{J} \lambda_j b_j\right\|_{\ell_{2,w}}^2 &= \left\|\sum_{j=1}^{J} \lambda_j \sum_{n\in E_j} w_n{}^{\frac{2}{p-2}} e_n\right\|_{\ell_{2,w}}^2 \\ &= \sum_{j=1}^{J} |\lambda_j|^2 \sum_{n\in E_j} w_n{}^{\frac{4}{p-2}} w_n{}^2 \\ &= \sum_{j=1}^{J} |\lambda_j|^2 \sum_{n\in E_j} w_n{}^{\frac{2p}{p-2}} \\ &= \sum_{j=1}^{J} |\lambda_j|^2 \sigma_j. \end{aligned}$$

Normalizing each $b_j$ in $\ell^p$ and noting that $\|b_j\|_{\ell^p} = \sigma_j{}^{\frac{1}{p}}$, we have

$$\left\|\sum_{j=1}^{J} \lambda_j \tilde{b}_j\right\|_{\ell^p}^p = \sum_{j=1}^{J} |\lambda_j|^p \tag{2.5}$$

and

$$\left\|\sum_{j=1}^{J} \lambda_j \tilde{b}_j\right\|_{\ell_{2,w}}^2 = \sum_{j=1}^{J} |\lambda_j|^2 \frac{\sigma_j}{\sigma_j{}^{\frac{2}{p}}} = \sum_{j=1}^{J} |\lambda_j|^2 \sigma_j{}^{\frac{p-2}{p}} = \sum_{j=1}^{J} |\lambda_j|^2 v_j{}^2. \tag{2.6}$$

Thus

$$\left\|\sum_{j=1}^{J}\lambda_j\tilde{b}_j\right\|_{X_{p,w}} = \max\left\{\left\|\sum_{j=1}^{J}\lambda_j\tilde{b}_j\right\|_{\ell^p}, \left\|\sum_{j=1}^{J}\lambda_j\tilde{b}_j\right\|_{\ell_{2,w}}\right\}$$

$$= \max\left\{\left(\sum_{j=1}^{J}|\lambda_j|^p\right)^{\frac{1}{p}}, \left(\sum_{j=1}^{J}|v_j\lambda_j|^2\right)^{\frac{1}{2}}\right\}.$$

Hence $\left\{\tilde{b}_j\right\}$ in $X_{p,w}$ is isometrically equivalent to the standard basis of $X_{p,v}$.

(b) We wish to define a projection $P\colon X_{p,w} \longrightarrow [b_j]_{X_{p,w}}$ with $\|P\| = 1$. Recalling the inner product $\langle\ ,\ \rangle$ previously introduced on $\ell_{2,w}$, let $\pi\colon \ell_{2,w} \longrightarrow [b_j]_{\ell_{2,w}}$ be the orthogonal projection defined by

$$\pi(x) = \sum_{j=1}^{\infty}\left\langle x, \frac{b_j}{\|b_j\|_{\ell_{2,w}}}\right\rangle \frac{b_j}{\|b_j\|_{\ell_{2,w}}}.$$

Then $\|\pi(x)\|_{\ell_{2,w}} \le \|x\|_{\ell_{2,w}}$. We will show that if $x \in \ell^p \cap \ell_{2,w}$, then $\pi(x) \in \ell^p$ and $\|\pi(x)\|_{\ell^p} \le \|x\|_{\ell^p}$. Thus

$$P(x) = \sum_{j=1}^{\infty}\left\langle x, \frac{b_j}{\|b_j\|^2_{\ell_{2,w}}}\right\rangle b_j$$

defines a mapping $P\colon \ell^p \cap \ell_{2,w} \longrightarrow [b_j]_{\ell^p\cap\ell_{2,w}}$. Set-theoretically, $P$ is the restriction of $\pi$ to $\ell^p \cap \ell_{2,w}$. It will follow that if $x \in \ell^p \cap \ell_{2,w} = X_{p,w}$, then

$$\|P(x)\|_{X_{p,w}} = \max\left\{\|P(x)\|_{\ell_{2,w}}, \|P(x)\|_{\ell^p}\right\} \le \max\left\{\|x\|_{\ell_{2,w}}, \|x\|_{\ell^p}\right\} = \|x\|_{X_{p,w}}.$$

Fix $x = \{x_n\} \in \ell^p \cap \ell_{2,w}$ and let

$$\lambda_j = \left\langle x, \frac{b_j}{\|b_j\|^2_{\ell_{2,w}}}\right\rangle,$$

so that $\sum_{j=1}^{J}\lambda_j b_j$ is a partial sum of $\pi(x)$. We now show that $\pi(x) \in \ell^p$ and $\|\pi(x)\|_{\ell^p} \le \|x\|_{\ell^p}$. As in equation (2.4), we have

$$\left\|\sum_{j=1}^{J}\lambda_j b_j\right\|^p_{\ell^p} = \sum_{j=1}^{J}|\lambda_j|^p\sigma_j,$$

where

$$\lambda_j = \left\langle x, \frac{b_j}{\|b_j\|^2_{\ell_{2,w}}} \right\rangle = \frac{1}{\sigma_j} \langle x, b_j \rangle$$
$$= \frac{1}{\sigma_j} \sum\nolimits_{n \in E_j} x_n {w_n}^{\frac{2}{p-2}} {w_n}^2$$
$$= \frac{1}{\sigma_j} \sum\nolimits_{n \in E_j} x_n {w_n}^{\frac{2(p-1)}{p-2}}.$$

Now by Hölder's inequality, for $q = \frac{p}{p-1}$ we have

$$|\lambda_j| = \frac{1}{\sigma_j} \left| \sum\nolimits_{n \in E_j} x_n {w_n}^{\frac{2(p-1)}{p-2}} \right|$$
$$\leq \frac{1}{\sigma_j} \left( \sum\nolimits_{n \in E_j} |x_n|^p \right)^{\frac{1}{p}} \left( \sum\nolimits_{n \in E_j} {w_n}^{\frac{2(p-1)}{p-2} q} \right)^{\frac{1}{q}}$$
$$= \frac{1}{\sigma_j} \left( \sum\nolimits_{n \in E_j} |x_n|^p \right)^{\frac{1}{p}} \left( \sum\nolimits_{n \in E_j} {w_n}^{\frac{2p}{p-2}} \right)^{\frac{p-1}{p}}$$
$$= \frac{1}{\sigma_j} \left( \sum\nolimits_{n \in E_j} |x_n|^p \right)^{\frac{1}{p}} {\sigma_j}^{\frac{p-1}{p}}$$
$$= \frac{1}{{\sigma_j}^{\frac{1}{p}}} \left( \sum\nolimits_{n \in E_j} |x_n|^p \right)^{\frac{1}{p}}.$$

Hence $|\lambda_j|^p \sigma_j \leq \sum_{n \in E_j} |x_n|^p$. Referring again to equation (2.4), for each $J \in \mathbb{N}$

$$\left\| \sum_{j=1}^{J} \lambda_j b_j \right\|^p_{\ell^p} = \sum_{j=1}^{J} |\lambda_j|^p \sigma_j \leq \sum_{j=1}^{J} \sum_{n \in E_j} |x_n|^p \leq \|x\|^p_{\ell^p}. \tag{2.7}$$

Hence $\pi(x) = \sum_{j=1}^{\infty} \lambda_j b_j \in \ell^p$ and $\|\pi(x)\|_{\ell^p} \leq \|x\|_{\ell^p}$. □

We continue with results leading to the conclusion that for fixed $2 < p < \infty$, all spaces $X_{p,w}$ for $w = \{w_n\}$ satisfying condition $(*)$ of Proposition 2.1 are mutually isomorphic. The following result of Pełczyński [**P**, Proposition $(*)$] indicates the approach to be taken.

**Lemma 2.8.** *Let $X$ and $Y$ be Banach spaces. Suppose $X \overset{c}{\hookrightarrow} Y$ and $Y \overset{c}{\hookrightarrow} X$, where $X \sim X \oplus X$ and $Y \sim Y \oplus Y$. Then $X \sim Y$.*

**Proof.** Let $X'$ be a closed subspace of $X$ such that $X \sim Y \oplus X'$. Then

$X \sim Y \oplus X' \sim Y \oplus Y \oplus X' \sim Y \oplus X$. Similarly, $Y \sim X \oplus Y$. Hence

$X \sim Y \oplus X \sim X \oplus Y \sim Y$. □

First we examine the matter of mutual complementation [**RI**, Theorem 13].

**Proposition 2.9.** *Let $2 < p < \infty$ and let $w = \{w_n\}$ and $w' = \{w'_n\}$ be sequences of positive scalars satisfying condition $(*)$ of Proposition 2.1. Then $X_{p,w'} \overset{c}{\hookrightarrow} X_{p,w}$.*

**Proof.** By condition $(*)$, we may choose a sequence $\{E_j\}$ of disjoint nonempty finite subsets of $\mathbb{N}$ such that for each $j \in \mathbb{N}$,

$$(w'_j)^{\frac{2p}{p-2}} \leq \sum_{n \in E_j} w_n^{\frac{2p}{p-2}} \leq (2w'_j)^{\frac{2p}{p-2}}.$$

Then for $v_j = \left(\sum_{n \in E_j} w_n^{\frac{2p}{p-2}}\right)^{\frac{p-2}{2p}}$, $w'_j \leq v_j \leq 2w'_j$. Hence for $v = \{v_j\}$ and $x \in X_{p,w'}$, $\|x\|_{X_{p,w'}} \leq \|x\|_{X_{p,v}} \leq 2\|x\|_{X_{p,w'}}$. Thus $X_{p,w'} \sim X_{p,v}$ via the formal identity mapping. For $\tilde{b}_j$ as in Proposition 2.7, $X_{p,v} \sim \left[\tilde{b}_j\right]_{X_{p,w}} \overset{c}{\hookrightarrow} X_{p,w}$. Hence $X_{p,w'} \overset{c}{\hookrightarrow} X_{p,w}$. □

Next we examine the matter of $X_{p,w}$ being isomorphic to its square. As a preliminary, we show that a certain symmetric sum of $X_{p,w}$ is complemented in $X_{p,w}$ [**RI**, Proposition 12]. This symmetric sum is a special case of a more general sum which we now define.

Let $2 < p < \infty$. For each sequence $v = \{v_j\}$ of positive scalars, define a space $\ell_{2,v}$ as in the proof of Proposition 2.7. For each $k \in \mathbb{N}$, let $v^{(k)} = \left\{v_j^{(k)}\right\}_{j=1}^{\infty}$ be a sequence of positive scalars, and let $X_k$ be a closed subspace of $X_{p,v^{(k)}}$. Let $(X_1 \oplus X_2 \oplus \cdots)_{p,2,\{v^{(k)}\}}$ be the Banach space of all sequences $\{x_k\}$ with $x_k \in X_k$ such that $\|\{x_k\}\| = \max\left\{\left(\sum \|x_k\|_{\ell^p}^p\right)^{\frac{1}{p}}, \left(\sum \|x_k\|_{\ell_{2,v^{(k)}}}^2\right)^{\frac{1}{2}}\right\} < \infty$. If each $v^{(k)}$ is identical to a fixed sequence $v$, we will denote $(X_1 \oplus X_2 \oplus \cdots)_{p,2,\{v^{(k)}\}}$ by $(X_1 \oplus X_2 \oplus \cdots)_{p,2,v}$.

**Proposition 2.10.** *Let $2 < p < \infty$ and let $w = \{w_n\}$ be a sequence of positive scalars satisfying condition $(*)$ of Proposition 2.1. Let $\tilde{X}_{p,w} = (X_{p,w} \oplus X_{p,w} \oplus \cdots)_{p,2,w}$. Then $\tilde{X}_{p,w} \overset{c}{\hookrightarrow} X_{p,w}$.*

**Proof.** By condition $(*)$, we may choose a sequence $\{N_k\}$ of disjoint infinite subsets of $\mathbb{N}$ such that for each $\epsilon > 0$ and for each $k$,

$$\sum_{\substack{w_n<\epsilon\\ n\in N_k}} w_n^{\frac{2p}{p-2}} = \infty.$$

Hence for each $k$, we may choose a sequence $\left\{E_j^{(k)}\right\}_{j=1}^{\infty}$ of disjoint nonempty finite subsets of $N_k$ such that

$$w_j^{\frac{2p}{p-2}} \leq \sum_{n\in E_j^{(k)}} w_n^{\frac{2p}{p-2}} \leq (2w_j)^{\frac{2p}{p-2}}.$$

Then for $v_j^{(k)} = \left(\sum_{n\in E_j^{(k)}} w_n^{\frac{2p}{p-2}}\right)^{\frac{p-2}{2p}}$, $w_j \leq v_j^{(k)} \leq 2w_j$. Hence for $v^{(k)} = \left\{v_j^{(k)}\right\}_{j=1}^{\infty}$ and $x_k \in X_{p,w}$, $\|x_k\|_{\ell_{2,w}} \leq \|x_k\|_{\ell_{2,v^{(k)}}} \leq 2\|x_k\|_{\ell_{2,w}}$. Hence

$$(X_{p,w} \oplus X_{p,w} \oplus \cdots)_{p,2,w} \sim \left(X_{p,v^{(1)}} \oplus X_{p,v^{(2)}} \oplus \cdots\right)_{p,2,\{v^{(k)}\}} \tag{2.8}$$

via the formal identity mapping.

Let $b_j^{(k)} = \sum_{n\in E_j^{(k)}} w_n^{\frac{2}{p-2}} e_n$ (where $\{e_n\}$ is the standard basis of $X_{p,w}$). Let $\tilde{b}_j^{(k)} = b_j^{(k)} \Big/ \left\|b_j^{(k)}\right\|_{\ell^p}$. Then by part (a) of Proposition 2.7, and equations (2.5) and (2.6), for each $k$ there is an isometry $T_k: X_{p,v^{(k)}} \to \left[\tilde{b}_j^{(k)} : j \in \mathbb{N}\right]_{X_{p,w}}$ with $\|T_k(y_k)\|_{\ell^p} = \|y_k\|_{\ell^p}$ and $\|T_k(y_k)\|_{\ell_{2,w}} = \|y_k\|_{\ell_{2,v^{(k)}}}$ for $y_k \in X_{p,v^{(k)}}$. Hence

$$\left(X_{p,v^{(1)}} \oplus X_{p,v^{(2)}} \oplus \cdots\right)_{p,2,\{v^{(k)}\}} \sim \left(\left[\tilde{b}_j^{(1)}\right]_{X_{p,w}} \oplus \left[\tilde{b}_j^{(2)}\right]_{X_{p,w}} \oplus \cdots\right)_{p,2,w} \tag{2.9}$$

via the isometry $\{y_k\} \mapsto \{T_k(y_k)\}$.

The direct sum on the right side of (2.9) should be thought of as an internal direct sum of subspaces of $X_{p,w}$. We next show that

$$\left(\left[\tilde{b}_j^{(1)}\right]_{X_{p,w}} \oplus \left[\tilde{b}_j^{(2)}\right]_{X_{p,w}} \oplus \cdots\right)_{p,2,w} \sim \left[\tilde{b}_j^{(k)} : j,k \in \mathbb{N}\right]_{X_{p,w}} \tag{2.10}$$

via the mapping $\{z_k\} \mapsto \sum z_k$. For each $k$, let $z_k = \sum_{j=1}^{\infty} \lambda_j^{(k)} \tilde{b}_j^{(k)} \in \left[\tilde{b}_j^{(k)} : j \in \mathbb{N}\right]_{X_{p,w}}$.
Then by equations (2.5) and (2.6), and part (a) of Proposition 2.7, we have

$$\begin{aligned}
\|\{z_k\}\| &= \max\left\{\left(\sum_{k=1}^{\infty} \|z_k\|_{\ell^p}^p\right)^{\frac{1}{p}}, \left(\sum_{k=1}^{\infty} \|z_k\|_{\ell_{2,w}}^2\right)^{\frac{1}{2}}\right\} \\
&= \max\left\{\left(\sum_{k=1}^{\infty} \left\|\sum_{j=1}^{\infty} \lambda_j^{(k)} \tilde{b}_j^{(k)}\right\|_{\ell^p}^p\right)^{\frac{1}{p}}, \left(\sum_{k=1}^{\infty} \left\|\sum_{j=1}^{\infty} \lambda_j^{(k)} \tilde{b}_j^{(k)}\right\|_{\ell_{2,w}}^2\right)^{\frac{1}{2}}\right\} \\
&= \max\left\{\left(\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \left|\lambda_j^{(k)}\right|^p\right)^{\frac{1}{p}}, \left(\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \left|v_j^{(k)} \lambda_j^{(k)}\right|^2\right)^{\frac{1}{2}}\right\} \\
&= \left\|\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \lambda_j^{(k)} \tilde{b}_j^{(k)}\right\|_{X_{p,w}} \\
&= \left\|\sum_{k=1}^{\infty} z_k\right\|_{X_{p,w}}.
\end{aligned}$$

Hence the mapping $\{z_k\} \mapsto \sum z_k$ is an isometry.

By part (b) of Proposition 2.7, we have

$$\left[\tilde{b}_j^{(k)} : j, k \in \mathbb{N}\right]_{X_{p,w}} \overset{c}{\hookrightarrow} X_{p,w}. \tag{2.11}$$

Combining (2.8), (2.9), (2.10), and (2.11) yields

$$(X_{p,w} \oplus X_{p,w} \oplus \cdots)_{p,2,w} \overset{c}{\hookrightarrow} X_{p,w}.$$

□

The complementation of $\tilde{X}_{p,w}$ in $X_{p,w}$ is the key to showing that $X_{p,w}$ is isomorphic to its square [**RI**, Proposition 11].

**Proposition 2.11.** *Let $2 < p < \infty$ and let $w = \{w_n\}$ be a sequence of positive scalars satisfying condition $(*)$ of Proposition 2.1. Then $X_{p,w} \sim X_{p,w} \oplus X_{p,w}$.*

**Proof.** Let $\tilde{X}_{p,w}$ be as in Proposition 2.10. Then $\tilde{X}_{p,w} \overset{c}{\hookrightarrow} X_{p,w}$. Let $Y$ be a closed subspace of $X_{p,w}$ such that $X_{p,w} \sim \tilde{X}_{p,w} \oplus Y$. Note that $\tilde{X}_{p,w} \sim X_{p,w} \oplus \tilde{X}_{p,w}$. Hence

$$X_{p,w} \oplus X_{p,w} \sim X_{p,w} \oplus \tilde{X}_{p,w} \oplus Y \sim \tilde{X}_{p,w} \oplus Y \sim X_{p,w}.$$

□

REMARK. After noting that $\tilde{X}_{p,w} \sim \tilde{X}_{p,w} \oplus \tilde{X}_{p,w}$, we now see by Lemma 2.8 that $X_{p,w} \sim \tilde{X}_{p,w}$.

The above results immediately yield the following theorem [**RI**, Theorem 13].

**Theorem 2.12.** *Let $2 < p < \infty$ and let $w = \{w_n\}$ and $w' = \{w'_n\}$ be sequences of positive scalars satisfying condition $(*)$ of Proposition 2.1. Then $X_{p,w} \sim X_{p,w'}$.*

**Proof.** The spaces $X_{p,w}$ and $X_{p,w'}$ satisfy the hypotheses of Lemma 2.8. □

REMARK. For $p$, $w$, and $w'$ as above, there is a constant $C_p$, depending only on $p$, such that $d(X_{p,w}, X_{p,w'}) \leq C_p$, where $d(X_{p,w}, X_{p,w'})$ is the Banach-Mazur distance between $X_{p,w}$ and $X_{p,w'}$

DEFINITION. *Let $2 < p < \infty$. Define $X_p$ to be (the isomorphism type of) $X_{p,w}$ for any sequence $w = \{w_n\}$ of positive scalars satisfying condition $(*)$ of Proposition 2.1. For the conjugate index $q$, define $X_q$ to be the dual of $X_p$.*

By Theorem 2.12, $X_p$ is well-defined.

### The Isomorphism Type of $X_p$

We now present results leading to the conclusion that for $2 < p < \infty$ and for $w = \{w_n\}$ satisfying condition $(*)$ of Proposition 2.1, $X_{p,w}$ is isomorphically distinct from the previously known $\mathcal{L}_p$ spaces. The first result [**RI**, Corollary 8] establishes an unusual property of $X_{p,w}$.

**Proposition 2.13.** *Let $2 < p < \infty$ and let $w = \{w_n\}$ be a sequence of positive scalars satisfying condition $(*)$ of Proposition 2.1. Then for each $N \in \mathbb{N}$,*

*(a) there is a basic sequence $\left\{\tilde{b}_j\right\}$ in $X_{p,w}$, $2N$-equivalent to the standard basis of $\ell^2$, such that for all distinct $j_1, \ldots, j_N \in \mathbb{N}$, $\left\{\tilde{b}_{j_1}, \ldots, \tilde{b}_{j_N}\right\}$ is isometrically equivalent*

*to the standard basis of $\ell^p_N$, and*

(b) *there is a basic sequence $\{d_j\}$ in $X^*_{p,w}$, $2N$-equivalent to the standard basis of $\ell^2$, such that for all distinct $j_1, \ldots, j_N \in \mathbb{N}$, $\{d_{j_1}, \ldots, d_{j_N}\}$ is isometrically equivalent to the standard basis of $\ell^q_N$, where $q$ is the conjugate index of $p$.*

**Proof.** Fix $N \in \mathbb{N}$. By condition $(*)$, we may choose a sequence $\{E_j\}$ of disjoint nonempty finite subsets of $\mathbb{N}$ such that

$$\left(\frac{1}{2N}\right)^{\frac{2p}{p-2}} \leq \sum\nolimits_{n\in E_j} w_n{}^{\frac{2p}{p-2}} \leq \frac{1}{N}.$$

Define $b_j$, $\tilde{b}_j$, $v_j$, and $v$ as in Proposition 2.7. Recalling that $v_j = \left(\sum_{n\in E_j} w_n{}^{\frac{2p}{p-2}}\right)^{\frac{p-2}{2p}}$, we have

$$\frac{1}{2N} \leq v_j \leq \left(\frac{1}{N}\right)^{\frac{p-2}{2p}} \leq 1.$$

Hence $\mathit{inf}\, v_j \geq \frac{1}{2N} > 0$, $\mathit{sup}\, v_j \leq 1$, and $\mathit{sup}\, v_j{}^{\frac{2p}{p-2}} \leq \frac{1}{N}$.

(a) By part (a) of Proposition 2.7, $\left\{\tilde{b}_j\right\}$ is a basic sequence in $X_{p,w}$ which is isometrically equivalent to the standard basis of $X_{p,v}$. Since $\mathit{inf}\, v_j > 0$ and $\mathit{sup}\, v_j \leq 1$, the proof of part (a) of Proposition 2.1 shows that the standard basis of $X_{p,v}$ is equivalent to the standard basis of $\ell^2$, with $\|x\|_{X_{p,v}} \overset{1}{\underset{2N}{\approx}} \|x\|_{\ell^2}$ for every sequence $x = \{x_n\}$ of scalars. Hence $\left\{\tilde{b}_j\right\}$ in $X_{p,w}$ is $2N$-equivalent to the standard basis of $\ell^2$.

Let $j_1, \ldots, j_N \in \mathbb{N}$ be distinct and let $x_1, \ldots, x_N$ be scalars. Then by Hölder's inequality with conjugate indices $P = \frac{p}{2}$ and $Q = \frac{p}{p-2}$, and the fact that $\mathit{sup}\, v_j{}^{\frac{2p}{p-2}} \leq \frac{1}{N}$, we have

$$\begin{aligned}\sum\nolimits_{n=1}^N |v_{j_n} x_n|^2 = \sum\nolimits_{n=1}^N \left|x_n{}^2 v_{j_n}{}^2\right| &\leq \left(\sum\nolimits_{n=1}^N |x_n|^{2\frac{p}{2}}\right)^{\frac{2}{p}} \left(\sum\nolimits_{n=1}^N v_{j_n}{}^{2\frac{p}{p-2}}\right)^{\frac{p-2}{p}} \\ &\leq \left(\sum\nolimits_{n=1}^N |x_n|^p\right)^{\frac{2}{p}} \left(\sum\nolimits_{n=1}^N \frac{1}{N}\right)^{\frac{p-2}{p}} \\ &= \left(\sum\nolimits_{n=1}^N |x_n|^p\right)^{\frac{2}{p}}.\end{aligned}$$

Thus by part (a) of Proposition 2.7 and the above observation, we have

$$\left\|\sum_{n=1}^N x_n \tilde{b}_{j_n}\right\|_{X_{p,w}} = \max\left\{\left(\sum_{n=1}^N |x_n|^p\right)^{\frac{1}{p}}, \left(\sum_{n=1}^N |v_{j_n} x_n|^2\right)^{\frac{1}{2}}\right\}$$
$$= \left(\sum_{n=1}^N |x_n|^p\right)^{\frac{1}{p}}.$$

Hence $\left\{\tilde{b}_{j_1}, \ldots, \tilde{b}_{j_N}\right\}$ is isometrically equivalent to the standard basis of $\ell_N^p$.

(b) Define $\ell_{2,w}$ and its inner product $\langle\ ,\ \rangle$ as in Proposition 2.7. Let $d_j = b_j/\|b_j\|_{\ell^p}^{p-1}$ and consider $d_j$ as an element of $X_{p,w}^*$ with action $\langle\ , d_j\rangle$. Then $\left\langle \tilde{b}_j, d_{j'}\right\rangle = 0$ for $j \neq j'$, and

$$\left\langle \tilde{b}_j, d_j\right\rangle = \frac{1}{\|b_j\|_{\ell^p}^p} \langle b_j, b_j\rangle = \frac{\|b_j\|_{\ell_{2,w}}^2}{\|b_j\|_{\ell^p}^p} = 1.$$

Let $\{\alpha_n\}$ be a sequence of scalars and let $j_1, \ldots, j_N \in \mathbb{N}$ be distinct. We are trying to prove that

$$\left\|\sum_{n=1}^\infty \alpha_n d_n\right\|_{X_{p,w}^*} \overset{2N}{\underset{1}{\approx}} \left(\sum_{n=1}^\infty |\alpha_n|^2\right)^{\frac{1}{2}}$$

and

$$\left\|\sum_{n=1}^N \alpha_n d_{j_n}\right\|_{X_{p,w}^*} = \left(\sum_{n=1}^N |\alpha_n|^q\right)^{\frac{1}{q}}.$$

The proofs of these two relationships are quite similar. We introduce a shorthand to allow us to handle them simultaneously. Let $\sum'$ denote $\sum_{n=1}^\infty$ in the first setting and $\sum_{n=1}^N$ in the second setting. Let $\tau_n$ denote $n$ in the first setting and $j_n$ in the second setting. Then for sequences $\{\gamma_n\}$ of scalars, we have

$$\left\|\sum{}' \alpha_n d_{\tau_n}\right\|_{X_{p,w}^*} = \sup\left\{\left|\left\langle x, \sum{}' \alpha_n d_{\tau_n}\right\rangle\right| : \|x\|_{X_{p,w}} = 1\right\}$$
$$\geq \sup\left\{\left|\left\langle \sum{}' \gamma_n \tilde{b}_{\tau_n}, \sum{}' \alpha_n d_{\tau_n}\right\rangle\right| : \left\|\sum{}' \gamma_n \tilde{b}_{\tau_n}\right\|_{X_{p,w}} = 1\right\} \quad (2.12)$$
$$= \sup\left\{\left|\sum{}' \gamma_n \bar{\alpha}_n\right| : \left\|\sum{}' \gamma_n \tilde{b}_{\tau_n}\right\|_{X_{p,w}} = 1\right\}.$$

We will show that equality holds at (2.12). It will then follow by part (a) that

$$\left\|\sum_{n=1}^\infty \alpha_n d_n\right\|_{X_{p,w}^*} = \sup\left\{\left|\sum_{n=1}^\infty \gamma_n \bar{\alpha}_n\right| : \left\|\sum_{n=1}^\infty \gamma_n \tilde{b}_n\right\|_{X_{p,w}} = 1\right\}$$
$$\overset{2N}{\underset{1}{\approx}} \sup\left\{\left|\sum_{n=1}^\infty \gamma_n \bar{\alpha}_n\right| : \left(\sum_{n=1}^\infty |\gamma_n|^2\right)^{\frac{1}{2}} = 1\right\}$$
$$= \left(\sum_{n=1}^\infty |\alpha_n|^2\right)^{\frac{1}{2}}$$

and

$$\left\| \sum_{n=1}^{N} \alpha_n d_{j_n} \right\|_{X^*_{p,w}} = \sup\Big\{ \Big| \sum_{n=1}^{N} \gamma_n \bar{\alpha}_n \Big| : \Big\| \sum_{n=1}^{N} \gamma_n \tilde{b}_{j_n} \Big\|_{X_{p,w}} = 1 \Big\}$$

$$= \sup\Big\{ \Big| \sum_{n=1}^{N} \gamma_n \bar{\alpha}_n \Big| : \big( \sum_{n=1}^{N} |\gamma_n|^p \big)^{\frac{1}{p}} = 1 \Big\}$$

$$= \big( \sum_{n=1}^{N} |\alpha_n|^q \big)^{\frac{1}{q}},$$

which is what we are trying to prove.

We now show that equality holds at (2.12). It suffices to find a projection $P' : X_{p,w} \to X_{p,w}$ of norm one which is the set-theoretic restriction to $X_{p,w} = \ell^p \cap \ell_{2,w}$ of the orthogonal projection $\pi' : \ell_{2,w} \to \ell_{2,w}$ onto $\left[ \tilde{b}_n \right]_{\ell_{2,w}}$ in the first setting and onto *span* $\left\{ \tilde{b}_{j_n} \right\}_{n=1}^{N}$ in the second setting. For then we will have

$$\sup\Big\{ \big| \langle x, \textstyle\sum' \alpha_n d_{\tau_n} \rangle \big| : \|x\|_{X_{p,w}} = 1 \Big\}$$

$$= \sup\Big\{ \big| \langle x, (P')^* \left( \textstyle\sum' \alpha_n d_{\tau_n} \right) \rangle \big| : \|x\|_{X_{p,w}} = 1 \Big\}$$

$$= \sup\Big\{ \big| \langle P'(x), \textstyle\sum' \alpha_n d_{\tau_n} \rangle \big| : \|x\|_{X_{p,w}} = 1 \Big\}$$

$$\leq \sup\Big\{ \big| \langle P'(x), \textstyle\sum' \alpha_n d_{\tau_n} \rangle \big| : \|P'(x)\|_{X_{p,w}} = 1 \Big\}$$

$$= \sup\Big\{ \big| \langle \textstyle\sum' \gamma_n \tilde{b}_{\tau_n}, \sum' \alpha_n d_{\tau_n} \rangle \big| : \big\| \sum' \gamma_n \tilde{b}_{\tau_n} \big\|_{X_{p,w}} = 1 \Big\},$$

whence equality will hold at (2.12). Let $P' : X_{p,w} \to X_{p,w}$ be defined by

$$P'(x) = \sum{}' \left\langle x, \frac{b_{\tau_n}}{\|b_{\tau_n}\|^2_{\ell_{2,w}}} \right\rangle b_{\tau_n}.$$

In either setting, $P'$ is essentially the projection $P$ of part (b) of Proposition 2.7, the only difference between the settings being the choice of $\{E_j\}$ on which the projection is based. In either setting, $\|P'\| = 1$, as can be seen by (2.7). Thus equality indeed holds at (2.12). □

Following Rosenthal [**RI**], we say that a Banach space $X$ satisfies $\mathcal{P}_2$ if for each $\epsilon > 0$ and each sequence $\{f_n\}$ in $X$ equivalent to the standard basis $\{e_n\}$ of $\ell^2$, there is a subsequence $\{g_n\}$ of $\{f_n\}$ such that $\{g_n\}$ is $(1+\epsilon)$-equivalent to $\{e_n\}$.

The following result [**RI**] restates part (b) of Proposition 2.13 in terms of $\mathcal{P}_2$.

**Corollary 2.14.** *Let $2 < p < \infty$ and let $w = \{w_n\}$ be a sequence of positive scalars satisfying condition $(*)$ of Proposition 2.1. Then $X^*_{p,w}$ is not isomorphic to any Banach space satisfying $\mathcal{P}_2$.*

**Proof.** Suppose $X^*_{p,w}$ is isomorphic to a Banach space $Y$ satisfying $\mathcal{P}_2$. Let $K = d\left(X^*_{p,w}, Y\right)$, the Banach-Mazur distance between $X^*_{p,w}$ and $Y$. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $(1+\epsilon)(K+\epsilon) < d\left(\ell^2_N, \ell^q_N\right)$, the Banach-Mazur distance between $\ell^2_N$ and $\ell^q_N$, where $q$ is the conjugate index of $p$.

Choose a basic sequence $\{d_j\}$ in $X^*_{p,w}$ as in part (b) of Proposition 2.13. Then $\{d_j\}$ is equivalent to the standard basis of $\ell^2$, but for all distinct $j_1, \ldots, j_N \in \mathbb{N}$, $\{d_{j_1}, \ldots, d_{j_N}\}$ is isometrically equivalent to the standard basis of $\ell^q_N$.

Choose an isomorphism $T\colon X^*_{p,w} \to Y$ such that $\|T\|\,\|T^{-1}\| < K + \epsilon$. Let $\{y_j\} = \{T(d_j)\}$. Then $\{y_j\}$ is equivalent to the standard basis of $\ell^2$.

Suppose $\{y_{j_n}\}$ is a subsequence of $\{y_j\}$ such that $\{y_{j_n}\}$ is $(1+\epsilon)$-equivalent to the standard basis of $\ell^2$. Then the standard basis of $\ell^2_N$ is $(1+\epsilon)$-equivalent to $\{y_{j_1}, \ldots, y_{j_N}\}$, $\{y_{j_1}, \ldots, y_{j_N}\}$ is $(K+\epsilon)$-equivalent to $\{d_{j_1}, \ldots, d_{j_N}\}$, and $\{d_{j_1}, \ldots, d_{j_N}\}$ is isometrically equivalent to the standard basis of $\ell^q_N$. Hence the standard basis of $\ell^2_N$ is $(1+\epsilon)(K+\epsilon)$-equivalent to the standard basis of $\ell^q_N$, contrary to the choice of $N$.

□

It is a fairly routine matter to show that for $2 < p < \infty$, $\ell^*_2$, $\ell^*_p$, and $\left(\ell^2 \oplus \ell^p\right)^*$ satisfy $\mathcal{P}_2$. We will show that for $2 < p < \infty$, $\left(\ell^2 \oplus \ell^2 \oplus \cdots\right)^*_{\ell^p}$ satisfies $\mathcal{P}_2$ as well. Thus for $2 < p < \infty$, the duals of the classical sequence space $\mathcal{L}_p$ spaces satisfy $\mathcal{P}_2$. It follows that for $2 < p < \infty$ and $w$ satisfying condition $(*)$ of Proposition 2.1, $X_{p,w}$ is isomorphically distinct from the classical sequence space $\mathcal{L}_p$ spaces. Rather than take this approach, however, we will show that $\left(\ell^2 \oplus \ell^2 \oplus \cdots\right)^*_{\ell^p}$ satisfies $\mathcal{P}_2$ for $2 < p < \infty$

as a lemma for a somewhat stronger result.

The following example [**RI**, Sublemma 1] motivates the argument.

**Example 2.15.** *The space $\ell^2$ satisfies $\mathcal{P}_2$.*

**Proof.** Let $\{e_n\}$ be the standard basis of $\ell^2$. Suppose $\{f_n\}$ is a basic sequence in $\ell^2$ equivalent to $\{e_n\}$. Then $\{f_n\}$ is weakly null, $inf\,\|f_n\|_{\ell^2} > 0$, and $sup\,\|f_n\|_{\ell^2} < \infty$. Let $\epsilon > 0$ and choose $\delta > 0$ and $\gamma > 0$ such that $(1+\delta)^2 < 1+\epsilon$ and $(1+\gamma)^2 < 1+\delta$. By the method of Bessaga and Pełczyński [**B-P**, Theorem 3], choose a subsequence $\{g_n\}$ of $\{f_n\}$ such that $\{g_n\}$ is $(1+\delta)$-equivalent to a block basic sequence $\{b_n\}$ of $\{e_n\}$. It remains to show that $\{b_n\}$ has a subsequence which is $(1+\delta)$-equivalent to $\{e_n\}$.

Note that $\{b_n\}$ is equivalent to $\{e_n\}$, whence $inf\,\|b_n\|_{\ell^2} > 0$ and $sup\,\|b_n\|_{\ell^2} < \infty$. Choose a subsequence $\{b_{\alpha(n)}\}$ of $\{b_n\}$ such that $0 < L = \lim \|b_{\alpha(n)}\|_{\ell^2}$ exists, with

$$L\frac{1}{1+\gamma} < \left\|b_{\alpha(n)}\right\|_{\ell^2} < L(1+\gamma)$$

for all $n$. Then for scalars $\lambda_1, \lambda_2, \ldots$, we have

$$\left\|\sum_{n=1}^{\infty} \lambda_n b_{\alpha(n)}\right\|_{\ell^2} = \left(\sum_{n=1}^{\infty} |\lambda_n|^2 \left\|b_{\alpha(n)}\right\|_{\ell^2}^2\right)^{\frac{1}{2}} \underset{1+\gamma}{\overset{1+\gamma}{\approx}} L\left(\sum_{n=1}^{\infty} |\lambda_n|^2\right)^{\frac{1}{2}}.$$

Hence $\{b_{\alpha(n)}\}$ is $(1+\delta)$-equivalent to $\{e_n\}$, but $\{g_{\alpha(n)}\}$ is $(1+\delta)$-equivalent to $\{b_{\alpha(n)}\}$, so $\{g_{\alpha(n)}\}$ is $(1+\epsilon)$-equivalent to $\{e_n\}$. $\square$

The following result [**RI**, Sublemma 1] is similar, but is more technical than motivational. In our first application, $r = 2$.

**Lemma 2.16.** *Let $1 \le r < \infty$ and let $X$ be isomorphic to $\ell^r$. Suppose $\{f_n\}$ is a sequence in $X$ which is weakly null but not norm null. Then $\{f_n\}$ has a basic subsequence equivalent to the standard basis $\{e_n\}$ of $\ell^r$.*

**Proof.** Note that $M = \sup \|f_n\|_X < \infty$ since $\{f_n\}$ is weakly bounded. Let $\{g_n\}$

be a subsequence of $\{f_n\}$ such that $\mathit{inf}\,\|g_n\|_X > 0$. Choose $0 < \delta < 1$ such that $\delta \leq \inf\|g_n\|_X$. Fix an isomorphism $T: \ell^r \to X$ and its inverse $S: X \to \ell^r$.

By the method of Bessaga and Pełczyński [**B-P**, Theorem 3], choose a basic subsequence $\{h_n\}$ of $\{g_n\}$ such that $\{h_n\}$ is equivalent to a block basic sequence $\{b_n\}$ of $\{T(e_n)\}$, with $\|h_n - b_n\|_X < \frac{\delta}{2}$ for each $n$. Then for each $n$,

$$\|b_n\|_X \geq \|h_n\|_X - \|h_n - b_n\|_X > \delta - \frac{\delta}{2} = \frac{\delta}{2}$$

and

$$\|b_n\|_X \leq \|h_n\|_X + \|b_n - h_n\|_X < M + \frac{\delta}{2}.$$

Hence $\{S(b_n)\}$ is a block basic sequence of $\{e_n\}$, $\mathit{inf}\,\|S(b_n)\|_{\ell^r} > 0$, and $\mathit{sup}\,\|S(b_n)\|_{\ell^r} < \infty$. Hence $\{S(b_n)\}$ is equivalent to $\{e_n\}$, so $\{b_n\}$ is equivalent to $\{e_n\}$. Since $\{h_n\}$ is equivalent to $\{b_n\}$, $\{h_n\}$ is equivalent to $\{e_n\}$. $\square$

Let $1 \leq q < \infty$ and let $N \in \mathbb{N}$. Let $\Gamma$ be an index set, either $\{1, \ldots, N\}$ or $\mathbb{N}$. We now introduce some notation for $X = \left(\sum_{j\in\Gamma}^{\oplus} \ell^2\right)_{\ell^q(\Gamma)}$, that is, $X = \left(\ell^2 \oplus \cdots \oplus \ell^2\right)_{\ell^q_N}$ ($N$ summands) or $X = \left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^q}$. Denote a generic $x \in X$ by $\left\{x^{(j)}\right\}_{j\in\Gamma}$, with each $x^{(j)} \in \ell^2$. For each $J \in \Gamma$, define $\pi_J: X \to \ell^2$ by $\pi_J\left(\left\{x^{(j)}\right\}_{j\in\Gamma}\right) = x^{(J)}$. Let $\{e_k\}$ be the standard basis of $\ell^2$. Let $\{e_{i,j}\}$ be the standard basis of $X$, with $\pi_j(e_{i,j}) = e_i$ and $\pi_{j'}(e_{i,j}) = 0_{\ell^2}$ for $j, j' \in \Gamma$ such that $j \neq j'$.

The following somewhat idealized example provides a model to be approximated.

**Example 2.17.** *Let $1 \leq q < \infty$ and let $\Gamma$, $X = \left(\sum_{j\in\Gamma}^{\oplus} \ell^2\right)_{\ell^q(\Gamma)}$, $\pi_j: X \to \ell^2$, and $\{e_{i,j}\}$ be as above. Let $\{\alpha_j\}_{j\in\Gamma}$ be a sequence of nonnegative real numbers such that $\alpha = \left(\sum_{j\in\Gamma} {\alpha_j}^q\right)^{\frac{1}{q}} > 0$. Suppose $\left\{b_{[k]}\right\}$ is a basic sequence in $X$ which is disjointly supported with respect to $\{e_{i,j}\}$, such that for each $j \in \Gamma$, $\left\|\pi_j\left(b_{[k]}\right)\right\|_{\ell^2} = \alpha_j$ for all $k$. Then $\left\{b_{[k]}\right\}$ is 1-equivalent to the standard basis of $\ell^2$.*

**Proof.** For scalars $\lambda_1, \lambda_2, \ldots$, we have

$$\begin{aligned} \left\| \sum_{k=1}^{\infty} \lambda_k b_{[k]} \right\|_X &= \left[ \sum_{j \in \Gamma} \left\| \pi_j \left( \sum_{k=1}^{\infty} \lambda_k b_{[k]} \right) \right\|_{\ell^2}^q \right]^{\frac{1}{q}} \\ &= \left[ \sum_{j \in \Gamma} \left( \sum_{k=1}^{\infty} |\lambda_k|^2 \left\| \pi_j \left( b_{[k]} \right) \right\|_{\ell^2}^2 \right)^{\frac{1}{2} q} \right]^{\frac{1}{q}} \\ &= \left[ \sum_{j \in \Gamma} \left( \sum_{k=1}^{\infty} |\lambda_k|^2 {\alpha_j}^2 \right)^{\frac{1}{2} q} \right]^{\frac{1}{q}} \\ &= \left[ \sum_{j \in \Gamma} {\alpha_j}^q \right]^{\frac{1}{q}} \left[ \sum_{k=1}^{\infty} |\lambda_k|^2 \right]^{\frac{1}{2}} \\ &= \alpha \left[ \sum_{k=1}^{\infty} |\lambda_k|^2 \right]^{\frac{1}{2}}. \end{aligned} \tag{2.13}$$

Hence $\{b_{[k]}\}$ is 1-equivalent to the standard basis of $\ell^2$. □

The following lemma [**RI**, Sublemma 3] shows the relevance of Example 2.17 for $\Gamma = \{1, \ldots, N\}$ to the space $\left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^q}$ for $1 \leq q < 2$.

**Lemma 2.18.** *Let $1 \leq q < 2$ and let $X = \left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^q}$. Denote a generic $x \in X$ by $\{x^{(1)}, x^{(2)}, \ldots\}$, with $x^{(1)}, x^{(2)}, \ldots \in \ell^2$. For each $n \in \mathbb{N}$, define $P_n: X \to X$ by $P_n\left(\{x^{(1)}, x^{(2)}, \ldots\}\right) = \{x^{(1)}, \ldots, x^{(n)}, 0, 0, \ldots\}$ and define $Q_n: X \to X$ by $Q_n(x) = x - P_n(x)$. Suppose $Y$ is a subspace of $X$ isomorphic to $\ell^2$. Then $\lim_{n\to\infty} \|Q_n|_Y\| = 0$ and $\lim_{n\to\infty} \|P_n|_Y\| = 1$.*

**Proof.** For each $n \in \mathbb{N}$, $1 - \|Q_n|_Y\| \leq \|P_n|_Y\| \leq 1$. Hence it suffices to show that $\lim_{n\to\infty} \|Q_n|_Y\| = 0$. Fix an ordering of the standard basis $\{e_{i,j}\}$ of $X$.

Suppose the conclusion is false. Then we may choose $0 < \delta < 1$ and $y_1, y_2, \ldots \in Y$ of norm one such that $\|Q_n(y_n)\|_X \geq \delta$ for each $n$, and (by the reflexivity of $Y$) such that $\{y_n\}$ is weakly convergent. Choose positive integers $n_1 < n_2 < \ldots$ such that for $k < k'$, $\left\|Q_{n_{k'}}(y_{n_k})\right\|_X < \frac{\delta}{8}$.

Let $d_k = y_{n_{2k}} - y_{n_{2k-1}}$ and let $T_k = Q_{n_{2k}}$. Then $\{d_k\}$ is weakly null,

$$\|T_k(d_k)\|_X \geq \|Q_{n_{2k}}(y_{n_{2k}})\|_X - \left\|Q_{n_{2k}}\left(y_{n_{2k-1}}\right)\right\|_X > \delta - \frac{\delta}{8} = \frac{7}{8}\delta,$$

and for $k < k'$,

$$\|T_{k'}(d_k)\|_X \leq \left\|Q_{n_{2k'}}(y_{n_{2k}})\right\|_X + \left\|Q_{n_{2k'}}(y_{n_{2k-1}})\right\|_X < \frac{\delta}{8} + \frac{\delta}{8} = \frac{\delta}{4}.$$

Note that $\|d_k\|_X \geq \|T_k(d_k)\|_X > \frac{7}{8}\delta$, whence $\{d_k\}$ is not norm null. Hence by the method of Bessaga and Pełczyński [**B-P**, Theorem 3] and Lemma 2.16, we may choose a subsequence $\{d_{\alpha(k)}\}$ of $\{d_k\}$ such that $\{d_{\alpha(k)}\}$ is equivalent to a block basic sequence $\left\{\tilde{d}_{\alpha(k)}\right\}$ of the standard basis $\{e_{i,j}\}$ of $X$, and such that $\{d_{\alpha(k)}\}$ and $\left\{\tilde{d}_{\alpha(k)}\right\}$ are equivalent to the standard basis $\{e_k\}$ of $\ell^2$, where $\tilde{d}_{\alpha(k)} = d_{\alpha(k)} \cdot 1_{\operatorname{supp} \tilde{d}_{\alpha(k)}}$, $\left\|d_{\alpha(k)} - \tilde{d}_{\alpha(k)}\right\|_X < \frac{\delta}{8}$, and there is a $C > 0$ such that for each $K \in \mathbb{N}$,

$$\left\|\sum_{k=1}^{K} \tilde{d}_{\alpha(k)}\right\|_X \leq C\left\|\sum_{k=1}^{K} e_k\right\|_{\ell^2} = CK^{\frac{1}{2}}.$$

Hence

$$\left\|T_{\alpha(k)}\left(\tilde{d}_{\alpha(k)}\right)\right\|_X \geq \left\|T_{\alpha(k)}\left(d_{\alpha(k)}\right)\right\|_X - \left\|T_{\alpha(k)}\left(d_{\alpha(k)} - \tilde{d}_{\alpha(k)}\right)\right\|_X > \frac{7}{8}\delta - \frac{\delta}{8} = \frac{3}{4}\delta,$$

and for $k < k'$,

$$\left\|T_{\alpha(k')}\left(\tilde{d}_{\alpha(k)}\right)\right\|_X \leq \left\|T_{\alpha(k')}\left(d_{\alpha(k)}\right)\right\|_X < \frac{\delta}{4}.$$

Let $b_{\alpha(k)} = \left(T_{\alpha(k)} - T_{\alpha(k+1)}\right)\left(\tilde{d}_{\alpha(k)}\right)$. Then

$$\left\|b_{\alpha(k)}\right\|_X \geq \left\|T_{\alpha(k)}\left(\tilde{d}_{\alpha(k)}\right)\right\|_X - \left\|T_{\alpha(k+1)}\left(\tilde{d}_{\alpha(k)}\right)\right\|_X > \frac{3}{4}\delta - \frac{\delta}{4} = \frac{\delta}{2}.$$

Hence for each $K \in \mathbb{N}$,

$$\left\|\sum_{k=1}^{K} \tilde{d}_{\alpha(k)}\right\|_X \geq \left\|\sum_{k=1}^{K} b_{\alpha(k)}\right\|_X = \left(\sum_{k=1}^{K} \|b_{\alpha(k)}\|_X^q\right)^{\frac{1}{q}} > \frac{\delta}{2}K^{\frac{1}{q}}.$$

Thus for each $K \in \mathbb{N}$, $\frac{\delta}{2}K^{\frac{1}{q}} < CK^{\frac{1}{2}}$, which is impossible for sufficiently large $K$.

□

We have laid the groundwork for the following result [**RI**, Lemma 10].

**Proposition 2.19.** *Let $1 \leq q < 2$. Then $X = \left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^q}$ satisfies $\mathcal{P}_2$.*

**Proof.** Define $\pi_j : X \to \ell^2$ and the standard basis $\{e_{i,j}\}$ of $X$ as in the discussion preceding Example 2.17. Let $\{e_k\}$ be the standard basis of $\ell^2$. Fix an ordering of $\{e_{i,j}\}$.

Suppose $\{f_{[k]}\}$ is a basic sequence in $X$ equivalent to $\{e_k\}$. Then $\{f_{[k]}\}$ is weakly null, *inf* $\|f_{[k]}\|_X > 0$, and *sup* $\|f_{[k]}\|_X < \infty$. Let $\epsilon > 0$. Choose $\delta > 0$, $\gamma > 0$, and $\eta > 0$ such that $(1+\delta)^2 < 1+\epsilon$, $(1+\gamma)^2 < 1+\delta$, and $\eta = \frac{\gamma}{2}$, so that $1 + 2\eta = 1 + \gamma$ and $1 + \eta < 1 + \gamma$.

By the method of Bessaga and Pełczyński [**B-P**, Theorem 3], choose a subsequence $\{g_{[k]}\}$ of $\{f_{[k]}\}$ such that $\{g_{[k]}\}$ is $(1+\delta)$-equivalent to a block basic sequence $\{b_{[k]}\}$ of $\{e_{i,j}\}$. It remains to show that $\{b_{[k]}\}$ has a subsequence which is $(1+\delta)$-equivalent to $\{e_k\}$.

We will choose a subsequence of $\{b_{[k]}\}$ in such a way as to approximate the situation of Example 2.17 for $\Gamma = \{1, \ldots, N\}$, after the application of the projection $P_N$ of Lemma 2.18 for sufficiently large $N$.

Note that $\{b_{[k]}\}$ is equivalent to $\{e_k\}$, whence *inf* $\|b_{[k]}\|_X > 0$, *sup* $\|b_{[k]}\|_X < \infty$, and $[b_{[k]}]_X \sim \ell^2$. By Lemma 2.18, we may choose $N \in \mathbb{N}$ such that for all $x \in [b_{[k]}]_X$,

$$\frac{1}{1+\gamma}\|x\|_X \leq \|P_N(x)\|_X \leq \|x\|_X,$$

where $P_N$ is as in Lemma 2.18. Choose a subsequence $\{b_{[\alpha(k)]}\}$ of $\{b_{[k]}\}$ such that for each $j \in \{1, \ldots, N\}$, $L_j = \lim_{k\to\infty} \left\|\pi_j\left(b_{[\alpha(k)]}\right)\right\|_{\ell^2}$ exists. Let

$$L = \lim_{k\to\infty} \left\|P_N\left(b_{[\alpha(k)]}\right)\right\|_X = \lim_{k\to\infty} \left(\sum_{j=1}^{N} \left\|\pi_j\left(b_{[\alpha(k)]}\right)\right\|_{\ell^2}^q\right)^{\frac{1}{q}} = \left(\sum_{j=1}^{N} {L_j}^q\right)^{\frac{1}{q}}.$$

Then $L \geq \frac{1}{1+\gamma} \inf \left\|b_{[\alpha(k)]}\right\|_X > 0$ and some $L_j$ is nonzero. Let $J_1 = \{1 \leq j \leq N : L_j > 0\}$ and $J_0 = \{1 \leq j \leq N : L_j = 0\}$. Choose a subsequence $\{b_{[\beta(k)]}\}$ of $\{b_{[\alpha(k)]}\}$ such that for each $j \in J_1$,

$$L_j \frac{1}{1+\eta} < \left\| \pi_j \left( b_{[\beta(k)]} \right) \right\|_{\ell^2} < L_j (1+\eta)$$

for all $k$, and for each $j \in J_0$,

$$L_j \frac{1}{1+\eta} = 0 \leq \left\| \pi_j \left( b_{[\beta(k)]} \right) \right\|_{\ell^2} < \frac{L\eta}{N}$$

for all $k$. Then for scalars $\lambda_1, \lambda_2, \ldots$, we have

$$\begin{aligned} \left[ \sum_{j=1}^{N} \left( \sum_{k=1}^{\infty} |\lambda_k|^2 \left\| \pi_j \left( b_{[\beta(k)]} \right) \right\|_{\ell^2}^2 \right)^{\frac{1}{2}q} \right]^{\frac{1}{q}} &\geq \left[ \sum_{j=1}^{N} \left( \sum_{k=1}^{\infty} |\lambda_k|^2 \left( L_j \tfrac{1}{1+\eta} \right)^2 \right)^{\frac{1}{2}q} \right]^{\frac{1}{q}} \\ &= \frac{1}{1+\eta} \left( \sum_{j=1}^{N} L_j{}^q \right)^{\frac{1}{q}} \left( \sum_{k=1}^{\infty} |\lambda_k|^2 \right)^{\frac{1}{2}} \\ &= \frac{1}{1+\eta} L \left( \sum_{k=1}^{\infty} |\lambda_k|^2 \right)^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} \left[ \sum_{j=1}^{N} \left( \sum_{k=1}^{\infty} |\lambda_k|^2 \left\| \pi_j \left( b_{[\beta(k)]} \right) \right\|_{\ell^2}^2 \right)^{\frac{1}{2}q} \right]^{\frac{1}{q}} &\leq \left[ \sum_{j \in J_1} \left( \sum_{k=1}^{\infty} |\lambda_k|^2 \left\| \pi_j \left( b_{[\beta(k)]} \right) \right\|_{\ell^2}^2 \right)^{\frac{1}{2}q} \right]^{\frac{1}{q}} \\ &\quad + \left[ \sum_{j \in J_0} \left( \sum_{k=1}^{\infty} |\lambda_k|^2 \left\| \pi_j \left( b_{[\beta(k)]} \right) \right\|_{\ell^2}^2 \right)^{\frac{1}{2}q} \right]^{\frac{1}{q}} \\ &\leq \left[ \sum_{j \in J_1} \left( \sum_{k=1}^{\infty} |\lambda_k|^2 \left( L_j (1+\eta) \right)^2 \right)^{\frac{1}{2}q} \right]^{\frac{1}{q}} \\ &\quad + \left[ \sum_{j \in J_0} \left( \sum_{k=1}^{\infty} |\lambda_k|^2 \left( \tfrac{L\eta}{N} \right)^2 \right)^{\frac{1}{2}q} \right]^{\frac{1}{q}} \\ &\leq (1+\eta) \left( \sum_{j \in J_1} L_j{}^q \right)^{\frac{1}{q}} \left( \sum_{k=1}^{\infty} |\lambda_k|^2 \right)^{\frac{1}{2}} \\ &\quad + L\eta \left( \sum_{k=1}^{\infty} |\lambda_k|^2 \right)^{\frac{1}{2}} \\ &\leq (1+2\eta) L \left( \sum_{k=1}^{\infty} |\lambda_k|^2 \right)^{\frac{1}{2}} \end{aligned}$$

(compare with equation (2.13) and its consequents). Noting that

$$\frac{1}{1+\gamma} \left\| \sum_{k=1}^{\infty} \lambda_k b_{[\beta(k)]} \right\|_X \leq \left\| P_N \left( \sum_{k=1}^{\infty} \lambda_k b_{[\beta(k)]} \right) \right\|_X \leq \left\| \sum_{k=1}^{\infty} \lambda_k b_{[\beta(k)]} \right\|_X$$

by the choice of $N$, and

$$\left\| P_N \left( \sum_{k=1}^{\infty} \lambda_k b_{[\beta(k)]} \right) \right\|_X = \left[ \sum_{j=1}^{N} \left\| \pi_j \left( \sum_{k=1}^{\infty} \lambda_k b_{[\beta(k)]} \right) \right\|_{\ell^2}^q \right]^{\frac{1}{q}}$$

$$= \left[ \sum_{j=1}^{N} \left( \sum_{k=1}^{\infty} |\lambda_k|^2 \left\| \pi_j \left( b_{[\beta(k)]} \right) \right\|_{\ell^2}^2 \right)^{\frac{1}{2}q} \right]^{\frac{1}{q}}$$

(compare with equation (2.13) and its antecedents), we have

$$\frac{1}{1+\gamma} \left\| \sum_{k=1}^{\infty} \lambda_k b_{[\beta(k)]} \right\|_X \leq (1+2\eta) L \left( \sum_{k=1}^{\infty} |\lambda_k|^2 \right)^{\frac{1}{2}}$$

and

$$\frac{1}{1+\eta} L \left( \sum_{k=1}^{\infty} |\lambda_k|^2 \right)^{\frac{1}{2}} \leq \left\| \sum_{k=1}^{\infty} \lambda_k b_{[\beta(k)]} \right\|_X .$$

Hence

$$\frac{1}{(1+\gamma)^2} \left\| \sum_{k=1}^{\infty} \lambda_k b_{[\beta(k)]} \right\|_X \leq L \left( \sum_{k=1}^{\infty} |\lambda_k|^2 \right)^{\frac{1}{2}} \leq (1+\gamma)^2 \left\| \sum_{k=1}^{\infty} \lambda_k b_{[\beta(k)]} \right\|_X .$$

Thus $\{b_{[\beta(k)]}\}$ is $(1+\delta)$-equivalent to $\{e_k\}$, but $\{g_{[\beta(k)]}\}$ is $(1+\delta)$-equivalent to $\{b_{[\beta(k)]}\}$, so $\{g_{[\beta(k)]}\}$ is $(1+\epsilon)$-equivalent to $\{e_k\}$. □

The preceding proposition, together with the following lemma [**RI**], will lead to the main result concerning the isomorphic distinctness of $X_{p,w}$.

**Lemma 2.20.** *Let $1 < q < 2$. Suppose $X$ is a Banach space satisfying $\mathcal{P}_2$. Suppose $Y$ is isomorphic to a quotient space of $\ell^q$. Then $Z = X \oplus Y$ satisfies $\mathcal{P}_2$.*

**Proof.** Let $\{e_n\}$ be the standard basis of $\ell^2$. Suppose $\{z_n\}$ is a basic sequence in $Z$ equivalent to $\{e_n\}$. Let $\epsilon > 0$ and choose $\delta > 0$ such that $(1+\delta)^2 < 1+\epsilon$.

Express each $z_n$ as $x_n \oplus y_n$ with $x_n \in X$ and $y_n \in Y$. Then there is a bounded linear operator $T: \ell^2 \to Y$ such that $T(e_n) = y_n$ for all $n$ [$e_n \mapsto z_n = x_n \oplus y_n \mapsto y_n$]. The adjoint $T^*$ induces a bounded linear operator from a closed subspace of $\ell^p$ to $\ell^2$, where $p$ is the conjugate index of $q$. Hence $T^*$ is compact since $2 < p < \infty$

[**R**, Theorem A2]. Thus $T$ is compact as well. Moreover, $\{e_n\}$ is weakly null. Hence $\lim_{n\to\infty} \|y_n\|_Y = \lim_{n\to\infty} \|T(e_n)\|_Y = 0$.

Choose a subsequence $\{y_{\alpha(n)}\}$ of $\{y_n\}$ such that $\{z_{\alpha(n)}\} = \{x_{\alpha(n)} \oplus y_{\alpha(n)}\}$ is $(1+\delta)$-equivalent to $\{x_{\alpha(n)}\}$. Choose a subsequence $\{x_{\beta(n)}\}$ of $\{x_{\alpha(n)}\}$ such that $\{x_{\beta(n)}\}$ is $(1+\delta)$-equivalent to $\{e_n\}$, as we may since $X$ satisfies $\mathcal{P}_2$. Then $\{z_{\beta(n)}\} = \{x_{\beta(n)} \oplus y_{\beta(n)}\}$ is $(1+\epsilon)$-equivalent to $\{e_n\}$. □

Finally we present the theorem [**RI**, Theorem 9] which (in its corollary) establishes that for $2 < p < \infty$ and $w$ satisfying condition $(*)$ of Proposition 2.1, $X_{p,w}$ is isomorphically distinct from the classical sequence space $\mathcal{L}_p$ spaces.

**Theorem 2.21.** *Let $2 < p < \infty$ and let $w = \{w_n\}$ be a sequence of positive scalars satisfying condition $(*)$ of Proposition 2.1. Let $V$ be a closed subspace of $\ell^p$. Then $X_{p,w}$ is not a continuous linear image of $(\ell^2 \oplus \ell^2 \oplus \cdots)_{\ell^p} \oplus V$.*

**Proof.** Equivalently, we show that for $Y$ isometric to a quotient space of $\ell^q$, where $q$ is the conjugate index of $p$, $X^*_{p,w}$ is not isomorphic to a closed subspace of $(\ell^2 \oplus \ell^2 \oplus \cdots)_{\ell^q} \oplus Y$.

Let $Y$ be isometric to a quotient space of $\ell^q$. By Corollary 2.14, $X^*_{p,w}$ is not isomorphic to any Banach space satisfying $\mathcal{P}_2$. However, $(\ell^2 \oplus \ell^2 \oplus \cdots)_{\ell^q} \oplus Y$ satisfies $\mathcal{P}_2$ (as do all of its closed subspaces) by Proposition 2.19 and Lemma 2.20. □

The following corollary [**RI**, Corollary 14] extracts only part of the information available from the preceding theorem.

**Corollary 2.22.** *Let $2 < p < \infty$ and let $w = \{w_n\}$ be a sequence of positive scalars satisfying condition $(*)$ of Proposition 2.1. Then $X_{p,w}$ is isomorphically distinct from $\ell^2$, $\ell^p$, $\ell^2 \oplus \ell^p$, and $(\ell^2 \oplus \ell^2 \oplus \cdots)_{\ell^p}$.*

**Proof.** Each of the spaces $\ell^2$, $\ell^p$, $\ell^2 \oplus \ell^p$, and $(\ell^2 \oplus \ell^2 \oplus \cdots)_{\ell^p}$ is a continuous linear image of $\left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^p} \oplus \ell^p$. However, $X_{p,w}$ is not such an image, as established by Theorem 2.21. □

## Complementation and Imbedding Relations for $X_p$

The following lemma [**RI**, Corollary 14] distinguishes the isomorphism types of two classical sequence space $\mathcal{L}_p$ spaces, and is used in the proof that $\left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^p} \not\hookrightarrow X_p$ for $2 < p < \infty$.

**Lemma 2.23.** *Let* $2 < p < \infty$. *Then* $\left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^p} \not\hookrightarrow \ell^2 \oplus \ell^p$.

**Proof.** Suppose $T\colon \left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^p} \to \ell^2 \oplus \ell^p$ is an isomorphic imbedding. Let $P\colon \ell^2 \oplus \ell^p \to \ell^2 \oplus \left\{0_{\ell^p}\right\}$ and $Q\colon \ell^2 \oplus \ell^p \to \left\{0_{\ell^2}\right\} \oplus \ell^p$ be the obvious projections, with $P + Q = I$, the identity operator on $\ell^2 \oplus \ell^p$.

For each $N \in \mathbb{N}$, let $X_N$ be the set of all $s^{(1)} \oplus s^{(2)} \oplus \cdots \in \left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^p}$ with $s^{(n)} = 0_{\ell^2}$ if $n \leq N$. Then each $X_N$ is a subspace of $\left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^p}$ isometric to $\left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^p}$.

We will show that $lim_{N\to\infty} \|PT|_{X_N}\| = 0$. Assuming this for now, $lim_{N\to\infty} \left\|P|_{T(X_N)}\right\| = 0$ as well, so we may choose $N \in \mathbb{N}$ such that $\left\|I|_{T(X_N)} - Q|_{T(X_N)}\right\| = \left\|P|_{T(X_N)}\right\| < 1$. Hence $Q|_{T(X_N)}\colon T\left(X_N\right) \to \left\{0_{\ell^2}\right\} \oplus \ell^p$ is an isomorphic imbedding, and for an isomorphic imbedding $R\colon \ell^2 \to \left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^p}$, the operator $QTR\colon \ell^2 \to \left\{0_{\ell^2}\right\} \oplus \ell^p$ is an isomorphic imbedding as well. However, no such imbedding exists, and the lemma will follow.

It remains to show that $lim_{N\to\infty} \|PT|_{X_N}\|$ is indeed zero. Suppose $lim_{N\to\infty} \|PT|_{X_N}\| \neq 0$. Then we may choose $\epsilon > 0$ and a normalized sequence $\{x_N\}$ with $x_N \in X_N$ such that $\|PT\left(x_N\right)\|_{\ell^2\oplus\{0\}} \geq \epsilon$ for each $N$. Let $\tau_N\colon \left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^p} \to \left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^p}$ be the truncation operator defined by

$\tau_N\left(s^{(1)} \oplus s^{(2)} \oplus \cdots\right) = s^{(1)} \oplus \cdots \oplus s^{(N)} \oplus 0_{\ell^2} \oplus 0_{\ell^2} \oplus \cdots$. Choose positive integers $N_1 < N_2 < \cdots$ such that for $\tilde{x}_{N_k} = \tau_{N_{k+1}}(x_{N_k})$, $\frac{1}{2} \leq \|\tilde{x}_{N_k}\|_{(\ell^2 \oplus \ell^2 \oplus \cdots)_{\ell^p}} \leq 1$ and $\|PT(\tilde{x}_{N_k})\|_{\ell^2 \oplus \{0\}} \geq \frac{\epsilon}{2}$. Then $\{\tilde{x}_{N_k}\}$ is equivalent to the standard basis of $\ell^p$. Hence $PT|_{[\tilde{x}_{N_k}]_{(\ell^2 \oplus \ell^2 \oplus \cdots)_{\ell^p}}}$ induces a bounded linear operator from $\ell^p$ into $\ell^2$, so $PT|_{[\tilde{x}_{N_k}]_{(\ell^2 \oplus \ell^2 \oplus \cdots)_{\ell^p}}}$ must be compact. Hence some subsequence $\left\{PT\left(\tilde{x}_{N_{k(\alpha)}}\right)\right\}$ of $\{PT(\tilde{x}_{N_k})\}$ converges in norm. Since $\{\tilde{x}_{N_k}\}$ is weakly null, $\{PT(\tilde{x}_{N_k})\}$ is weakly null as well. Hence $\left\{PT\left(\tilde{x}_{N_{k(\alpha)}}\right)\right\}$ must converge to $0_{\ell^2 \oplus \ell^p}$ in norm, contrary to $\|PT(\tilde{x}_{N_k})\|_{\ell^2 \oplus \{0\}} \geq \frac{\epsilon}{2}$ for all $k$. □

We are now ready to see how $X_p$ is related to the classical $\mathcal{L}_p$ spaces under the relations $\hookrightarrow$ and $\overset{c}{\hookrightarrow}$. Recall that $X \equiv Y$ means $X \hookrightarrow Y$ and $Y \hookrightarrow X$.

**Proposition 2.24.** *Let $2 < p < \infty$. Then*

*(a)* $X_p \hookrightarrow \ell^2 \oplus \ell^p$,

*(b)* $\ell^2 \oplus \ell^p \overset{c}{\hookrightarrow} X_p$,

*(c)* $X_p \equiv \ell^2 \oplus \ell^p$,

*(d)* $X_p \overset{c}{\not\hookrightarrow} \ell^2 \oplus \ell^p$,

*(e)* $\left(\sum^{\oplus} \ell^2\right)_{\ell^p} \not\hookrightarrow X_p$,

*(f)* $X_p \overset{c}{\not\hookrightarrow} \left(\sum^{\oplus} \ell^2\right)_{\ell^p}$,

*(g)* $L^p \not\hookrightarrow X_p$, *and*

*(h) parts (b), (d), and (f) hold for $1 < p < 2$ by duality.*

**Proof.**

(a) We norm $\ell^2 \oplus \ell^p$ by $\|a \oplus b\|_{\ell^2 \oplus \ell^p} = \max\left\{\|a\|_{\ell^2}, \|b\|_{\ell^p}\right\}$. Let $w = \{w_n\}$ be a sequence of positive scalars satisfying condition $(*)$ of Proposition 2.1. Then $X_{p,w} \sim X_p$. Define $T\colon X_{p,w} \to \ell^2 \oplus \ell^p$ by $T(\{x_n\}) = \{w_n x_n\} \oplus \{x_n\}$. Then $T$ is an isometry. It follows that $X_p \hookrightarrow \ell^2 \oplus \ell^p$.

(b) Let $w = \{w_n\}$ be a sequence of positive scalars such that $w_{[1]} = \{w_{3n-2}\}$ satisfies $\inf w_{3n-2} > 0$, $w_{[2]} = \{w_{3n-1}\}$ satisfies $\sum (w_{3n-1})^{\frac{2p}{p-2}} < \infty$, and $w_{[3]} = \{w_{3n}\}$ satisfies condition $(*)$ of Proposition 2.1. Then $w$ satisfies condition $(*)$ as well. Hence

$$X_p \sim X_{p,w} \sim X_{p,w_{[1]}} \oplus X_{p,w_{[2]}} \oplus X_{p,w_{[3]}} \sim \ell^2 \oplus \ell^p \oplus X_p.$$

It follows that $\ell^2 \oplus \ell^p \overset{c}{\hookrightarrow} X_p$.

(c) The fact that $X_p \equiv \ell^2 \oplus \ell^p$ is an immediate consequence of parts (a) and (b).

(d) Suppose $X_p \overset{c}{\hookrightarrow} \ell^2 \oplus \ell^p$. Then $X_p$ is a continuous linear image of $\ell^2 \oplus \ell^p$, contrary to Theorem 2.21. It follows that $X_p \overset{c}{\not\hookrightarrow} \ell^2 \oplus \ell^p$.

(e) Suppose $\left(\sum^{\oplus} \ell^2\right)_{\ell^p} \hookrightarrow X_p$. Then $\left(\sum^{\oplus} \ell^2\right)_{\ell^p} \hookrightarrow X_p \hookrightarrow \ell^2 \oplus \ell^p$ by part (a), so $\left(\sum^{\oplus} \ell^2\right)_{\ell^p} \hookrightarrow \ell^2 \oplus \ell^p$, contrary to Lemma 2.23. It follows that $\left(\sum^{\oplus} \ell^2\right)_{\ell^p} \not\hookrightarrow X_p$.

(f) Suppose $X_p \overset{c}{\hookrightarrow} \left(\sum^{\oplus} \ell^2\right)_{\ell^p}$. Then $X_p$ is a continuous linear image of $\left(\sum^{\oplus} \ell^2\right)_{\ell^p}$, contrary to Theorem 2.21. It follows that $X_p \overset{c}{\not\hookrightarrow} \left(\sum^{\oplus} \ell^2\right)_{\ell^p}$.

(g) Suppose $L^p \hookrightarrow X_p$. Then $\left(\sum^{\oplus} \ell^2\right)_{\ell^p} \hookrightarrow L^p \hookrightarrow X_p$, so $\left(\sum^{\oplus} \ell^2\right)_{\ell^p} \hookrightarrow X_p$, contrary to part (e). It follows that $L^p \not\hookrightarrow X_p$.

(h) Parts (b), (d), and (f) are the parts involving $\overset{c}{\hookrightarrow}$. $\square$

Building on diagrams (1.1) and (1.2), for $2 < p < \infty$ we have

$$\begin{array}{ccccccc} \ell^2 & & & & & & \\ & \searrow & & & & & \\ & & \ell^2 \oplus \ell^p & \to & \left(\sum^{\oplus} \ell^2\right)_{\ell^p} & \to & L^p, \\ & \nearrow & \equiv & & & & \\ \ell^p & & X_p & & & & \end{array} \tag{2.14}$$

and for $1 < p < \infty$ where $p \neq 2$, we have

$$\begin{array}{ccccccc} \ell^2 & & & & \left(\sum^{\oplus} \ell^2\right)_{\ell^p} & & \\ & \overset{c}{\searrow} & & \overset{c}{\nearrow} & & \overset{c}{\searrow} & \\ & & \ell^2 \oplus \ell^p & & & & L^p. \\ & \overset{c}{\nearrow} & & \overset{c}{\searrow} & & \overset{c}{\nearrow} & \\ \ell^p & & & & X_p & & \end{array} \tag{2.15}$$

## The Space $B_p$

Let $2 < p < \infty$. The Banach space $B_p$ is of the form $(X_1 \oplus X_2 \oplus \cdots)_{\ell^p}$, where each space $X_N$ is isomorphic to $\ell^2$, but $\{X_N\}_{N=1}^{\infty}$ is chosen so that $sup_{N\in\mathbb{N}}\, d\left(X_N, \ell^2\right) = \infty$, where $d\left(X_N, \ell^2\right)$ is the Banach-Mazur distance between $X_N$ and $\ell^2$. Each space $X_N$ is of the form $X_{p,v^{(N)}}$ where $v^{(N)}$ is an appropriately chosen constant sequence. The specifics are outlined below. For the conjugate index $q$, $B_q$ is defined to be the dual of $B_p$.

### The Space $X_{p,v^{(N)}}$

Let $2 < p < \infty$ and fix $N \in \mathbb{N}$. Let $v_j^{(N)} = \left(\frac{1}{N}\right)^{\frac{p-2}{2p}}$ for each $j \in \mathbb{N}$, and let $v^{(N)}$ be the constant sequence $\left\{v_j^{(N)}\right\}_{j=1}^{\infty}$. Then $X_{p,v^{(N)}}$ is isomorphic to $\ell^2$ by part (a) of Proposition 2.1.

The following observation [**RI**] concerning $X_{p,v^{(N)}}$ is analogous to Propositions 2.7 and 2.13, but starts with $v^{(N)}$ and produces $w^{(N)}$ rather that the reverse. The lemma eventually leads to information about $B_p$.

**Lemma 2.25.** *Let $2 < p < \infty$ and fix $N \in \mathbb{N}$. Let $v^{(N)} = \left\{v_j^{(N)}\right\}_{j=1}^{\infty}$ where $v_j^{(N)} = \left(\frac{1}{N}\right)^{\frac{p-2}{2p}}$ as above. Then there is a sequence $w^{(N)} = \left\{w_n^{(N)}\right\}_{n=1}^{\infty}$ of positive scalars satisfying condition $(*)$ of Proposition 2.1, a basic sequence $\left\{\tilde{b}_j^{(N)}\right\}_{j=1}^{\infty}$ in $X_{p,w^{(N)}}$, and a basic sequence $\left\{d_j^{(N)}\right\}_{j=1}^{\infty}$ in $X^*_{p,w^{(N)}}$ such that*

*(a) $\left\{\tilde{b}_j^{(N)}\right\}_{j=1}^{\infty}$ is isometrically equivalent to the standard basis of $X_{p,v^{(N)}}$,*

*(b) there is a projection $P_N\colon X_{p,w^{(N)}} \to \left[\tilde{b}_j^{(N)} : j \in \mathbb{N}\right]_{X_{p,w^{(N)}}}$ of norm one,*

*(c) $\left\{\tilde{b}_j^{(N)}\right\}_{j=1}^{\infty}$ is $2N$-equivalent to the standard basis of $\ell^2$, but for all distinct $j_1, \ldots, j_N \in \mathbb{N}$, $\left\{\tilde{b}_{j_1}^{(N)}, \ldots, \tilde{b}_{j_N}^{(N)}\right\}$ is isometrically equivalent to the standard basis of $\ell_N^p$, and*

*(d)* $\left\{d_j^{(N)}\right\}_{j=1}^{\infty}$ *is* $2N$*-equivalent to the standard basis of* $\ell^2$*, but for all distinct* $j_1, \ldots, j_N \in \mathbb{N}$, $\left\{d_{j_1}^{(N)}, \ldots, d_{j_N}^{(N)}\right\}$ *is isometrically equivalent to the standard basis of* $\ell_N^q$*, where* $q$ *is the conjugate index of* $p$*.*

**Proof.** Choose a sequence $\left\{E_j^{(N)}\right\}_{j=1}^{\infty}$ of disjoint nonempty finite subsets of $\mathbb{N}$ and a sequence $w^{(N)} = \left\{w_n^{(N)}\right\}$ of positive scalars satisfying condition $(*)$ of Proposition 2.1 such that for each $j \in \mathbb{N}$, $\sum_{n \in E_j^{(N)}} \left(w_n^{(N)}\right)^{\frac{2p}{p-2}} = \frac{1}{N}$. [We may take $E_j^{(N)}$ of cardinality $j$ and $\left(w_n^{(N)}\right)^{\frac{2p}{p-2}} = \frac{1}{jN}$ for $n \in E_j^{(N)}$.] Then for each $j \in \mathbb{N}$, $v_j^{(N)} = \left(\sum_{n \in E_j^{(N)}} \left(w_n^{(N)}\right)^{\frac{2p}{p-2}}\right)^{\frac{p-2}{2p}}$. Let $b_j^{(N)} = \sum_{n \in E_j^{(N)}} \left(w_n^{(N)}\right)^{\frac{2}{p-2}} e_n$ and $\tilde{b}_j^{(N)} = b_j^{(N)} \Big/ \left\|b_j^{(N)}\right\|_{\ell^p}$ (analogous to $b_j$ and $\tilde{b}_j$ in Proposition 2.7), where $\{e_n\}$ is the standard basis of $X_{p,w^{(N)}}$. Then parts (a) and (b) follow from Proposition 2.7.

Note that $\left\{E_j^{(N)}\right\}_{j=1}^{\infty}$ satisfies the condition in the proof of Proposition 2.13. Let $b_j^{(N)}$ and $\tilde{b}_j^{(N)}$ be as above (analogous to $b_j$ and $\tilde{b}_j$ in Proposition 2.13), and let $d_j^{(N)} = b_j^{(N)} \Big/ \left\|b_j^{(N)}\right\|_{\ell^p}^{p-1}$ (analogous to $d_j$ in Proposition 2.13, and considered as an element of $X^*_{p,w^{(N)}}$). Then parts (c) and (d) follow from Proposition 2.13. $\square$

**The Space $B_p$**

The following definition was suggested above, but we now present it formally.

DEFINITION. *Let* $2 < p < \infty$*. For each* $N \in \mathbb{N}$*, let* $v^{(N)} = \left\{v_j^{(N)}\right\}_{j=1}^{\infty}$ *where* $v_j^{(N)} = \left(\frac{1}{N}\right)^{\frac{p-2}{2p}}$ *as above. Define* $B_p$ *to be* $\left(X_{p,v^{(1)}} \oplus X_{p,v^{(2)}} \oplus \cdots\right)_{\ell^p}$*. For the conjugate index* $q$*, define* $B_q$ *to be the dual of* $B_p$*.*

The following proposition [**RI**] is the first step in showing that $B_p$ is an $\mathcal{L}_p$ space. The subsequent proposition [**RI**] is somewhat stronger.

**Proposition 2.26.** *Let* $1 < p < \infty$ *where* $p \neq 2$*. Then* $B_p \overset{c}{\hookrightarrow} L^p$*.*

**Proof.** First suppose $2 < p < \infty$. For each $N \in \mathbb{N}$, let $v^{(N)}$ be as above. Then as in the first part of the proof of Corollary 2.6, for each $N \in \mathbb{N}$ there is a sequence $\left\{f_j^{(N)}\right\}_{j=1}^{\infty}$ of independent symmetric three-valued random variables in $L^p$ such that $X_{p,v^{(N)}} \sim \left[f_j^{(N)} : j \in \mathbb{N}\right]_{L^p} \overset{c}{\hookrightarrow} L^p$, where the isomorphism is uniform in $N$ by the proof of Corollary 2.3, and the complementation is uniform in $N$ by Theorem 2.5. Hence

$$B_p = \left(X_{p,v^{(1)}} \oplus X_{p,v^{(2)}} \oplus \cdots\right)_{\ell^p} \overset{c}{\hookrightarrow} (L^p \oplus L^p \oplus \cdots)_{\ell^p} \sim L^p,$$

and $B_p \overset{c}{\hookrightarrow} L^p$. The result now holds for $1 < p < 2$ by duality. □

**Proposition 2.27.** *Let $1 < p < \infty$ where $p \neq 2$. Then $B_p \overset{c}{\hookrightarrow} \left(\sum^{\oplus} X_p\right)_{\ell^p}$.*

**Proof.** First suppose $2 < p < \infty$. For each $N \in \mathbb{N}$ let $v^{(N)}$, $w^{(N)}$, and $\left\{\tilde{b}_j^{(N)}\right\}_{j=1}^{\infty}$ be as in Lemma 2.25. Then by parts (a) and (b) of Lemma 2.25, there is a projection $P_N \colon X_{p,w^{(N)}} \to \left[\tilde{b}_j^{(N)} : j \in \mathbb{N}\right]_{X_{p,w^{(N)}}}$ of norm one, and there is an isometry $T_N \colon \left[\tilde{b}_j^{(N)} : j \in \mathbb{N}\right]_{X_{p,w^{(N)}}} \to X_{p,v^{(N)}}$. Thus by the remark following Theorem 2.12, for any sequence $w$ satisfying condition $(*)$ of Proposition 2.1,

$$B_p = \left(X_{p,v^{(1)}} \oplus X_{p,v^{(2)}} \oplus \cdots\right)_{\ell^p} \overset{c}{\hookrightarrow} \left(X_{p,w^{(1)}} \oplus X_{p,w^{(2)}} \oplus \cdots\right)_{\ell^p}$$
$$\sim (X_{p,w} \oplus X_{p,w} \oplus \cdots)_{\ell^p}.$$

Hence $B_p \overset{c}{\hookrightarrow} (X_{p,w} \oplus X_{p,w} \oplus \cdots)_{\ell^p}$. The result now holds for $1 < p < 2$ by duality. □

REMARK. Alternatively, the proof of parts (a) and (b) of Lemma 2.25 could be slightly modified to produce a sequence $w = \{w_n\}$ of positive scalars satisfying condition $(*)$ of Proposition 2.1 such that $B_p \overset{c}{\hookrightarrow} (X_{p,w} \oplus X_{p,w} \oplus \cdots)_{\ell^p}$, without the passage through $\left(X_{p,w^{(1)}} \oplus X_{p,w^{(2)}} \oplus \cdots\right)_{\ell^p}$.

Let $2 < p < \infty$. We will show that $B_p^*$ is not isomorphic to any Banach space satisfying $\mathcal{P}_2$. This will distinguish $B_p$ isomorphically from $\ell^2$, $\ell^p$, $\ell^2 \oplus \ell^p$, and

$\left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^p}$. The proof follows the same pattern as the proof that $X_p^*$ is not isomorphic to any Banach space satisfying $\mathcal{P}_2$. The following proposition **[RI]** is analogous to Proposition 2.13.

**Proposition 2.28.** *Let $2 < p < \infty$. Then for each $N \in \mathbb{N}$,*

*(a) there is a basic sequence $\left\{\dot{b}_j^{(N)}\right\}_{j=1}^{\infty}$ in $B_p$, $2N$-equivalent to the standard basis of $\ell^2$, such that for all distinct $j_1, \ldots, j_N \in \mathbb{N}$, $\left\{\dot{b}_{j_1}^{(N)}, \ldots, \dot{b}_{j_N}^{(N)}\right\}$ is isometrically equivalent to the standard basis of $\ell_N^p$, and*

*(b) there is a basic sequence $\left\{\dot{d}_j^{(N)}\right\}_{j=1}^{\infty}$ in $B_p^*$, $2N$-equivalent to the standard basis of $\ell^2$, such that for all distinct $j_1, \ldots, j_N \in \mathbb{N}$, $\left\{\dot{d}_{j_1}^{(N)}, \ldots, \dot{d}_{j_N}^{(N)}\right\}$ is isometrically equivalent to the standard basis of $\ell_N^q$, where $q$ is the conjugate index of $p$.*

**Proof.** Fix $N \in \mathbb{N}$. Let $v^{(N)}$, $w^{(N)}$, $\tilde{b}_j^{(N)}$, and $d_j^{(N)}$ be as in Lemma 2.25. Let $T_N$: $\left[\tilde{b}_j^{(N)} : j \in \mathbb{N}\right]_{X_{p,w^{(N)}}} \to X_{p,v^{(N)}}$ be the isometry cited in the proof of Proposition 2.27, and let $S_N$: $\left[\tilde{b}_j^{(N)} : j \in \mathbb{N}\right]^*_{X_{p,w^{(N)}}} \to X^*_{p,v^{(N)}}$ be the isometry $S_N = \left(T_N^{-1}\right)^*$. Let $\iota_N : X_{p,v^{(N)}} \to B_p$ and $\kappa_N : X^*_{p,v^{(N)}} \to B_p^*$ be the obvious isometric injections.

Now $\left\{\tilde{b}_j^{(N)}\right\}_{j=1}^{\infty}$ and $\left\{d_j^{(N)}\right\}_{j=1}^{\infty}$ have the properties asserted in parts (c) and (d) of Lemma 2.25. Let $\dot{b}_j^{(N)} = \iota_N\left(T_N\left(\tilde{b}_j^{(N)}\right)\right)$. Then the sequence $\left\{\dot{b}_j^{(N)}\right\}_{j=1}^{\infty}$ in $B_p$ is isometrically equivalent to $\left\{\tilde{b}_j^{(N)}\right\}_{j=1}^{\infty}$, and part (a) follows.

Let $\tilde{d}_j^{(N)}$ be the restriction of $d_j^{(N)}$ to $\left[\tilde{b}_j^{(N)} : j \in \mathbb{N}\right]_{X_{p,w^{(N)}}}$. Then $\left\{\tilde{d}_j^{(N)}\right\}_{j=1}^{\infty}$ is isometrically equivalent to $\left\{d_j^{(N)}\right\}_{j=1}^{\infty}$ by the argument in the proof of part (b) of Proposition 2.13, where it is shown that equality holds at (2.12). Let $\dot{d}_j^{(N)} = \kappa_N\left(S_N\left(\tilde{d}_j^{(N)}\right)\right)$. Then the sequence $\left\{\dot{d}_j^{(N)}\right\}_{j=1}^{\infty}$ in $B_p^*$ is isometrically equivalent to $\left\{\tilde{d}_j^{(N)}\right\}_{j=1}^{\infty}$ and $\left\{d_j^{(N)}\right\}_{j=1}^{\infty}$, and part (b) follows. □

The proof of the following corollary **[RI]** is virtually identical to the proof of Corollary 2.14, with $B_p^*$ replacing $X_{p,w}^*$, $\dot{d}_j^{(N)}$ replacing $d_j$, and Proposition 2.28

replacing Proposition 2.13.

**Corollary 2.29.** *Let $2 < p < \infty$. Then $B_p^*$ is not isomorphic to any Banach space satisfying $\mathcal{P}_2$.*

The following theorem [**RI**] now follows as in the proof of Theorem 2.21, with $B_p^*$ replacing $X_{p,w}^*$ and Corollary 2.29 replacing Corollary 2.14.

**Theorem 2.30.** *Let $2 < p < \infty$ and let $V$ be a closed subspace of $\ell^p$. Then $B_p$ is not a continuous linear image of $\left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^p} \oplus V$.*

The following corollary [**RI**, Corollary 14] is analogous to Corollary 2.22.

**Corollary 2.31.** *Let $1 < p < \infty$ where $p \neq 2$. Then $B_p$ is isomorphically distinct from $\ell^2$, $\ell^p$, $\ell^2 \oplus \ell^p$, and $\left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^p}$. In particular, $B_p$ is an $\mathcal{L}_p$ space.*

**Proof.** First suppose $2 < p < \infty$. Then each of the spaces $\ell^2$, $\ell^p$, $\ell^2 \oplus \ell^p$, and $\left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^p}$ is a continuous linear image of $\left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^p} \oplus \ell^p$, but by Theorem 2.30, $B_p$ is not such an image. Finally, $B_p \overset{c}{\hookrightarrow} L^p$ by Proposition 2.26, but the fact that $B_p \not\sim \ell^2$ has just been established. Hence $B_p$ is an $\mathcal{L}_p$ space. The result now holds for $1 < p < 2$ by duality. $\square$

We now know that $B_p$ is isomorphically distinct from the classical sequence space $\mathcal{L}_p$ spaces. We present next some results to distinguish $B_p$ isomorphically from $X_p$ and $L^p$. The first result [**RI**] will distinguish $B_p$ from $X_p$, and the three subsequent results will refine the distinction.

**Proposition 2.32.** *Let $1 < p < \infty$ where $p \neq 2$. Then $\left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^p} \overset{c}{\hookrightarrow} B_p$.*

**Proof.** First suppose $2 < p < \infty$. Let $v^{(N)} = \left\{v_j^{(N)}\right\}_{j=1}^{\infty}$ where $v_j^{(N)} = \left(\frac{1}{N}\right)^{\frac{p-2}{2p}}$ as above. Choose a doubly indexed sequence $\left\{E_J^{(N)}\right\}_{J,N\in\mathbb{N}}$ of disjoint nonempty finite subsets of $\mathbb{N}$ such that for each $J, N \in \mathbb{N}$,

$$\sum_{j\in E_J^{(N)}} \left(v_j^{(N)}\right)^{\frac{2p}{p-2}} = \sum_{j\in E_J^{(N)}} \frac{1}{N} \geq 1.$$

[We may take $E_J^{(N)}$ of cardinality $N$.] Let $u_J^{(N)} = \left(\sum_{j\in E_J^{(N)}} \left(v_j^{(N)}\right)^{\frac{2p}{p-2}}\right)^{\frac{p-2}{2p}}$ and let $u^{(N)} = \left\{u_J^{(N)}\right\}_{J=1}^{\infty}$. Then $\inf u_J^{(N)} \geq 1$. Hence by part (a) of Proposition 2.1, and the inequality appearing in its proof, $X_{p,u^{(N)}}$ is isometric to $\ell^2$. Moreover, by Proposition 2.7, $X_{p,u^{(N)}} \overset{c}{\hookrightarrow} X_{p,v^{(N)}}$, and the implied projection is of norm one. Hence

$$\left(\ell^2 \oplus \ell^2 \oplus \cdots\right)_{\ell^p} \sim \left(X_{p,u^{(1)}} \oplus X_{p,u^{(2)}} \oplus \cdots\right)_{\ell^p} \overset{c}{\hookrightarrow} \left(X_{p,v^{(1)}} \oplus X_{p,v^{(2)}} \oplus \cdots\right)_{\ell^p} = B_p.$$

The result now holds for $1 < p < 2$ by duality. □

The following lemma [**RI**] is a modification of Lemma 2.18. The proof is virtually identical, with $\ell^r$ replacing $\ell^2$ and $K^{\frac{1}{r}}$ replacing $K^{\frac{1}{2}}$.

**Lemma 2.33.** *Let $1 < q < r \leq 2$ and let $X = \left(X^*_{p,v^{(1)}} \oplus X^*_{p,v^{(2)}} \oplus \cdots\right)_{\ell^q}$, where $p$ is the conjugate index of $q$. Denote a generic $x \in X$ by $\{x^{(1)}, x^{(2)}, \ldots\}$, with each $x^{(k)} \in X^*_{p,v^{(k)}}$. For each $n \in \mathbb{N}$, define $P_n: X \to X$ by $P_n\left(\{x^{(1)}, x^{(2)}, \ldots\}\right) = \{x^{(1)}, \ldots, x^{(n)}, 0, 0, \ldots\}$ and define $Q_n: X \to X$ by $Q_n(x) = x - P_n(x)$. Suppose $Y$ is a subspace of $X$ isomorphic to $\ell^r$. Then $\lim_{n\to\infty} \|Q_n|_Y\| = 0$ and $\lim_{n\to\infty} \|P_n|_Y\| = 1$.*

As a corollary, we have the following [**RI**].

**Lemma 2.34.** *Let $1 < q < r < 2$. Then $\ell^r \not\hookrightarrow B_q$.*

**Proof.** Suppose $\ell^r \hookrightarrow B_q$. Then $\ell^r \hookrightarrow X = \left(X^*_{p,v^{(1)}} \oplus X^*_{p,v^{(2)}} \oplus \cdots\right)_{\ell^q}$, where $p$ is the conjugate index of $q$, since $B_q = B_p^* \sim \left(X^*_{p,v^{(1)}} \oplus X^*_{p,v^{(2)}} \oplus \cdots\right)_{\ell^q}$. Let $T: \ell^r \to X$ be an isomorphic imbedding and let $Y = T(\ell^r)$. For each $n \in \mathbb{N}$, let $P_n: X \to X$ and $Q_n: X \to X$ be as in Lemma 2.33, with $P_n + Q_n = I$, the identity operator on

$X$. By Lemma 2.33, we may choose $N \in \mathbb{N}$ such that $\|I|_Y - P_N|_Y\| = \|Q_N|_Y\| < 1$. Hence $P_N|_Y : Y \to P_N(Y)$ is an isomorphism. Now $Y \sim \ell^r$ and $P_N(Y) \sim \ell^2$, so $P_N|_Y$ induces an isomorphism between $\ell^r$ and $\ell^2$. However, no such isomorphism exists, and the lemma follows. □

We state without proof [**RII**, Corollary 4.2].

**Lemma 2.35.** *Let $1 < q \leq r \leq 2$. Then $\ell^r \hookrightarrow X_q$.*

The following observation [**RI**] will distinguish $B_p$ from $L^p$.

**Lemma 2.36.** *Let $2 < p < \infty$. Then $\left(\sum^{\oplus} X_p\right)_{\ell^p} \hookrightarrow \left(\sum^{\oplus} \ell^2\right)_{\ell^p}$.*

**Proof.** By part (a) of Proposition 2.24, $X_p \hookrightarrow \ell^2 \oplus \ell^p$. Hence, letting $\mathbb{F}$ denote the scalar field,

$$\begin{aligned}\left(\sum^{\oplus} X_p\right)_{\ell^p} &\hookrightarrow \left(\sum^{\oplus} (\ell^2 \oplus \ell^p)\right)_{\ell^p} \\ &\sim \left(\sum^{\oplus} \ell^2\right)_{\ell^p} \oplus \left(\sum^{\oplus} \ell^p\right)_{\ell^p} \\ &\sim \left(\sum^{\oplus} \ell^2\right)_{\ell^p} \oplus \ell^p \\ &\sim \left(\sum^{\oplus} \ell^2\right)_{\ell^p} \oplus \left(\sum^{\oplus} \mathbb{F}\right)_{\ell^p} \\ &\sim \left(\sum^{\oplus} (\ell^2 \oplus \mathbb{F})\right)_{\ell^p} \\ &\sim \left(\sum^{\oplus} \ell^2\right)_{\ell^p}.\end{aligned}$$

□

Collecting our results and deducing simple consequences yields the following.

**Proposition 2.37.** *Let $2 < p < \infty$. Then*

*(a)* $B_p \hookrightarrow \left(\sum^{\oplus} \ell^2\right)_{\ell^p}$,

*(b)* $\left(\sum^{\oplus} \ell^2\right)_{\ell^p} \overset{c}{\hookrightarrow} B_p$,

*(c)* $B_p \equiv \left(\sum^{\oplus} \ell^2\right)_{\ell^p}$,

*(d)* $B_p \overset{c}{\not\hookrightarrow} \left(\sum^{\oplus} \ell^2\right)_{\ell^p}$,

*(e)* $X_p \hookrightarrow B_p$,

*(f)* $B_p \not\hookrightarrow X_p$,

*(g)* $X_p \overset{c}{\not\hookrightarrow} B_p$,

*(h)* $L^p \not\hookrightarrow B_p$, *and*

*(i) parts (b), (d), and (g) hold for* $1 < p < 2$ *by duality.*

**Proof.**

(a) We know $B_p \overset{c}{\hookrightarrow} \left(\sum^{\oplus} X_p\right)_{\ell^p} \hookrightarrow \left(\sum^{\oplus} \ell^2\right)_{\ell^p}$ by Proposition 2.27 and Lemma 2.36. It follows that $B_p \hookrightarrow \left(\sum^{\oplus} \ell^2\right)_{\ell^p}$.

(b) Part (b) is a restatement of Proposition 2.32.

(c) The fact that $B_p \equiv \left(\sum^{\oplus} \ell^2\right)_{\ell^p}$ is an immediate consequence of parts (a) and (b).

(d) Suppose $B_p \overset{c}{\hookrightarrow} \left(\sum^{\oplus} \ell^2\right)_{\ell^p}$. Then $B_p$ is a continuous linear image of $\left(\sum^{\oplus} \ell^2\right)_{\ell^p}$, contrary to Theorem 2.30. It follows that $B_p \overset{c}{\not\hookrightarrow} \left(\sum^{\oplus} \ell^2\right)_{\ell^p}$.

(e) We know $X_p \hookrightarrow \ell^2 \oplus \ell^p \overset{c}{\hookrightarrow} \left(\sum^{\oplus} \ell^2\right)_{\ell^p} \overset{c}{\hookrightarrow} B_p$ by part (a) of Proposition 2.24 and part (b) above. It follows that $X_p \hookrightarrow B_p$.

(f) Suppose $B_p \hookrightarrow X_p$. Then $\left(\sum^{\oplus} \ell^2\right)_{\ell^p} \overset{c}{\hookrightarrow} B_p \hookrightarrow X_p \hookrightarrow \ell^2 \oplus \ell^p$ by part (b) above and part (a) of Proposition 2.24, so $\left(\sum^{\oplus} \ell^2\right)_{\ell^p} \hookrightarrow \ell^2 \oplus \ell^p$, contrary to Lemma 2.23. It follows that $B_p \not\hookrightarrow X_p$.

(g) Suppose $X_p \overset{c}{\hookrightarrow} B_p$. Then $X_q \overset{c}{\hookrightarrow} B_q$, where $q$ is the conjugate index of $p$. Hence for $1 < q < r < 2$, $\ell^r \hookrightarrow X_q \overset{c}{\hookrightarrow} B_q$ by Lemma 2.35, so $\ell^r \hookrightarrow B_q$, contrary to Lemma 2.34. It follows that $X_p \overset{c}{\not\hookrightarrow} B_p$.

(h) Suppose $L^p \hookrightarrow B_p$. Then $L^p \hookrightarrow B_p \hookrightarrow \left(\sum^{\oplus} \ell^2\right)_{\ell^p}$ by part (a) above, so $L^p \hookrightarrow \left(\sum^{\oplus} \ell^2\right)_{\ell^p}$, contrary to [**L-P 2**, Theorem 6.1]. It follows that $L^p \not\hookrightarrow B_p$.

(i) Parts (b), (d), and (g) are the parts involving $\overset{c}{\hookrightarrow}$. $\square$

Building on diagrams (2.14) and (2.15), for $2 < p < \infty$ we have

$$\begin{array}{ccccccc} \ell^2 & & & & B_p & & \\ & \searrow & & & \parallel\!\! \mid & & \\ & & \ell^2 \oplus \ell^p & \rightarrow & \left(\sum^{\oplus} \ell^2\right)_{\ell^p} & \rightarrow & L^p, \\ & \nearrow & \parallel\!\! \mid & & & & \\ \ell^p & & X_p & & & & \end{array} \tag{2.16}$$

and for $1 < p < \infty$ where $p \neq 2$, we have

$$\begin{array}{ccccccc} \ell^2 & & & & \left(\sum^{\oplus} \ell^2\right)_{\ell^p} \overset{c}{\rightarrow} B_p & & \\ & \overset{c}{\searrow} & & \overset{c}{\nearrow} & & \overset{c}{\searrow} & \\ & & \ell^2 \oplus \ell^p & & & & L^p. \\ & \overset{c}{\nearrow} & & \overset{c}{\searrow} & & \overset{c}{\nearrow} & \\ \ell^p & & & & X_p & & \end{array} \tag{2.17}$$

**Sums of $B_p$**

We now present results leading to the conclusion that $B_p \sim B_p \oplus B_p$ and $\left(\sum^{\oplus} B_p\right)_{\ell^p} \sim B_p$. Along the way, we will show that the sequence used in the definition of $B_p$ can be modified to some extent without changing the isomorphism type of the space.

**Lemma 2.38.** *Let $2 < p < \infty$. Let $r = \{r_n\}$ and $s = \{s_n\}$ be sequences of positive scalars, and suppose that $\inf_{n \in \mathbb{N}} s_n = 0$. For each $n \in \mathbb{N}$, let $r^{(n)}$ be the constant sequence $\{r_n, r_n, \ldots\}$ and let $s^{(n)}$ be the constant sequence $\{s_n, s_n, \ldots\}$. Let $B_{p,r} = \left(X_{p,r^{(1)}} \oplus X_{p,r^{(2)}} \oplus \cdots\right)_{\ell^p}$ and $B_{p,s} = \left(X_{p,s^{(1)}} \oplus X_{p,s^{(2)}} \oplus \cdots\right)_{\ell^p}$. Then $B_{p,r} \overset{c}{\hookrightarrow} B_{p,s}$.*

**Proof.** Fix a subsequence $\{s_{\alpha(n)}\}$ of $\{s_n\}$ such that for each $n \in \mathbb{N}$, $s_{\alpha(n)} \leq r_n$. Let $S_{\alpha(n)} = s_{\alpha(n)}^{\frac{2p}{p-2}}$ and $R_n = r_n^{\frac{2p}{p-2}}$. Then $S_{\alpha(n)} \leq R_n$ for each $n$. Let $\{K_n\}$ be the sequence of positive integers such that for each $n \in \mathbb{N}$,

$$K_n S_{\alpha(n)} \leq R_n < (K_n + 1) S_{\alpha(n)} \leq 2 K_n S_{\alpha(n)} \leq 2^{\frac{2p}{p-2}} K_n S_{\alpha(n)}.$$

Fix $n \in \mathbb{N}$. Let $\left\{E_j^{(n)}\right\}_{j=1}^{\infty}$ be a sequence of disjoint subsets of $\mathbb{N}$ such that each $E_j^{(n)}$ has cardinality $K_n$. Then for each $j \in \mathbb{N}$,

$$\sum_{k \in E_j^{(n)}} S_{\alpha(n)} \leq R_n < 2^{\frac{2p}{p-2}} \sum_{k \in E_j^{(n)}} S_{\alpha(n)}.$$

Let $t_n = \left(\sum_{k \in E_j^{(n)}} S_{\alpha(n)}\right)^{\frac{p-2}{2p}}$ [which does not depend on $j$]. Then $t_n \leq r_n < 2t_n$. Hence for $t^{(n)} = \{t_n, t_n, \dots\}$ and $x \in X_{p,t^{(n)}}$, $\|x\|_{X_{p,t^{(n)}}} \leq \|x\|_{X_{p,r^{(n)}}} \leq 2\,\|x\|_{X_{p,t^{(n)}}}$. Thus $X_{p,r^{(n)}} \sim X_{p,t^{(n)}}$ via the formal identity mapping. Moreover, $X_{p,t^{(n)}} \overset{c}{\hookrightarrow} X_{p,s(\alpha(n))}$ by Proposition 2.7, where the implied projection is of norm one.

Release $n$ as a free variable. Then for each $n \in \mathbb{N}$, $X_{p,r^{(n)}} \sim X_{p,t^{(n)}} \overset{c}{\hookrightarrow} X_{p,s(\alpha(n))}$, where the isomorphism $X_{p,r^{(n)}} \sim X_{p,t^{(n)}}$ is uniform in $n$. It follows that

$$\begin{aligned} B_{p,r} = \left(X_{p,r^{(1)}} \oplus X_{p,r^{(2)}} \oplus \cdots\right)_{\ell^p} &\sim \left(X_{p,t^{(1)}} \oplus X_{p,t^{(2)}} \oplus \cdots\right)_{\ell^p} \\ &\overset{c}{\hookrightarrow} \left(X_{p,s(\alpha(1))} \oplus X_{p,s(\alpha(2))} \oplus \cdots\right)_{\ell^p} \\ &\overset{c}{\hookrightarrow} \left(X_{p,s(1)} \oplus X_{p,s(2)} \oplus \cdots\right)_{\ell^p} \\ &= B_{p,s}. \end{aligned}$$

□

REMARK. For $2 < p < \infty$, the space $B_p$ is of the form $B_{p,s}$ where $s = \{s_n\}$ and $B_{p,s}$ are as above, with $inf_{n \in \mathbb{N}}\, s_n = 0$.

**Lemma 2.39.** *Let $2 < p < \infty$. Let $r = \{r_n\}$, $r^{(n)}$, and $B_{p,r}$ be as in Lemma 2.38. Then $B_{p,r} \sim B_{p,r} \oplus B_{p,r}$.*

**Proof.** Recall that $B_{p,r} = \left(X_{p,r^{(1)}} \oplus X_{p,r^{(2)}} \oplus \cdots\right)_{\ell^p}$. For each $n \in \mathbb{N}$, let $\left\{z_k^{(n)}\right\}_{k=1}^{\infty}$ represent an element of $X_{p,r^{(n)}}$. Define a projection $P\colon B_{p,r} \to B_{p,r}$ by $P\left(\left\{z_k^{(1)}\right\} \oplus \left\{z_k^{(2)}\right\} \oplus \cdots\right) = \left(\left\{x_k^{(1)}\right\} \oplus \left\{x_k^{(2)}\right\} \oplus \cdots\right)$, where for $k, n \in \mathbb{N}$, $x_k^{(n)} = z_k^{(n)}$ if $k$ is even and $x_k^{(n)} = 0$ if $k$ is odd. Then the image of $B_{p,r}$ under $P$ is isomorphic to $B_{p,r}$, as is the kernel of $P$. Hence $B_{p,r} \sim B_{p,r} \oplus B_{p,r}$. □

By the remark above, we have the following corollary (true for $1 < p < 2$ by duality) of Lemma 2.39.

**Corollary 2.40.** *Let $1 < p < \infty$ where $p \neq 2$. Then $B_p \sim B_p \oplus B_p$.*

We also have the following corollary of Lemmas 2.38 and 2.39.

**Corollary 2.41.** *Let $2 < p < \infty$. Let $r = \{r_n\}$ and $s = \{s_n\}$ be sequences of positive scalars such that $\inf_{n\in\mathbb{N}} r_n = 0$ and $\inf_{n\in\mathbb{N}} s_n = 0$. Let $r^{(n)}$, $s^{(n)}$, $B_{p,r}$, and $B_{p,s}$ be as in Lemma 2.38. Then $B_{p,r} \sim B_{p,s}$.*

**Proof.** The spaces $B_{p,r}$ and $B_{p,s}$ satisfy the hypotheses of Lemma 2.8. □

REMARK 1. Recalling the remark above, one consequence of Corollary 2.41 is that for $2 < p < \infty$, and for $1 < p < 2$ by duality, the isomorphism type of $B_p$ does not depend on the specific sequence $\left\{\left(\frac{1}{N}\right)^{\frac{p-2}{2p}}\right\}_{N=1}^{\infty}$ used in its definition, but only on the fact that the infimum of the sequence is zero.

REMARK 2. Let $2 < p < \infty$. Then $B_p$ is of the form $\left(X_{p,w^{(1)}} \oplus X_{p,w^{(2)}} \oplus \cdots\right)_{\ell^p}$ where for each $N \in \mathbb{N}$, $w^{(N)}$ is a sequence $\left\{w_k^{(N)}\right\}_{k=1}^{\infty}$ of positive scalars. The above remark gives a sufficient condition for $B_p \sim \left(X_{p,w^{(1)}} \oplus X_{p,w^{(2)}} \oplus \cdots\right)_{\ell^p}$ in the case where each $w^{(N)}$ is a constant sequence. Although the details will not be given, $B_p \sim \left(X_{p,w^{(1)}} \oplus X_{p,w^{(2)}} \oplus \cdots\right)_{\ell^p}$ if and only if the following two conditions hold: (a) for each $N \in \mathbb{N}$, $w^{(N)}$ fails condition $(*)$ of Proposition 2.1, and (b) there is an increasing sequence $\{\alpha(N)\}_{N=1}^{\infty}$ of positive integers and a sequence $\{S_N\}_{N=1}^{\infty}$ of infinite subsets of $\mathbb{N}$ such that for each $N \in \mathbb{N}$, $c_N = \liminf_{k\in S_N} w_k^{(\alpha(N))} > 0$, but $lim_{N\to\infty} c_N = 0$.

Just as $B_p \oplus B_p \sim B_p$, $(B_p \oplus B_p \oplus \cdots)_{\ell^p} \sim B_p$, as shown below.

**Corollary 2.42.** *Let $1 < p < \infty$ where $p \neq 2$. Then $(B_p \oplus B_p \oplus \cdots)_{\ell^p} \sim B_p$.*

**Proof.** First suppose that $2 < p < \infty$. Then $B_p$ is of the form $B_{p,s}$ where $s = \{s_n\}$ satisfies $inf_{n\in\mathbb{N}}\, s_n = 0$, and $s^{(n)}$ and $B_{p,s}$ are as in Lemma 2.38. Let $S$ be the sequence $\{s_1; s_1, s_2; s_1, s_2, s_3; \ldots\} = \left\{\{s_n\}_{n=1}^{T}\right\}_{T=1}^{\infty}$. Then $S$ has infimum zero as well. Hence $B_{p,S} \sim B_{p,s}$ by Corollary 2.41. It follows that

$$
\begin{aligned}
(B_{p,s} \oplus B_{p,s} \oplus \cdots)_{\ell^p} &= \left((X_{p,s^{(1)}} \oplus X_{p,s^{(2)}} \oplus \cdots)_{\ell^p} \oplus (X_{p,s^{(1)}} \oplus X_{p,s^{(2)}} \oplus \cdots)_{\ell^p} \oplus \cdots\right)_{\ell^p} \\
&\sim \left(\textstyle\sum^{\oplus}_{T\in\mathbb{N}} \sum^{\oplus}_{1\le n\le T} X_{p,s^{(n)}}\right)_{\ell^p} \\
&\sim B_{p,S} \\
&\sim B_{p,s}.
\end{aligned}
$$

The result now holds for $1 < p < 2$ by duality. $\square$

## Sums Involving $X_p$ or $B_p$

As observed by Rosenthal [**RI**], a few more $\mathcal{L}_p$ spaces can be constructed by forming sums involving $X_p$ or $B_p$. The resulting spaces are $\left(\sum^{\oplus} \ell^2\right)_{\ell^p} \oplus X_p$, $B_p \oplus X_p$, and $\left(\sum^{\oplus} X_p\right)_{\ell^p}$. The following proposition [**RI**] shows that these spaces cannot be distinguished by the relation $\hookrightarrow$.

**Proposition 2.43.** *Let $2 < p < \infty$. Then*

*(a)* $B_p \oplus X_p \overset{c}{\hookrightarrow} \left(\sum^{\oplus} X_p\right)_{\ell^p}$ *(whence the same is true for $1 < p < 2$ by duality),*

*(b)* $\left(\sum^{\oplus} \ell^2\right)_{\ell^p}$, $B_p$, $\left(\sum^{\oplus} \ell^2\right)_{\ell^p} \oplus X_p$, $B_p \oplus X_p$, *and* $\left(\sum^{\oplus} X_p\right)_{\ell^p}$ *are equivalent under* $\equiv$, *and*

*(c) letting $Y$ denote any of the five spaces of part (b) and letting $X$ denote either $\ell^2 \oplus \ell^p$ or $X_p$, we have $X \hookrightarrow Y \hookrightarrow L^p$ but $L^p \not\hookrightarrow Y \not\hookrightarrow X$.*

**Proof.**

(a) By Proposition 2.27, we have $B_p \oplus X_p \overset{c}{\hookrightarrow} \left(\sum^{\oplus} X_p\right)_{\ell^p} \oplus X_p \sim \left(\sum^{\oplus} X_p\right)_{\ell^p}$.

(b) Consider the chains

$$\left(\sum^{\oplus} \ell^2\right)_{\ell^p} \overset{c}{\hookrightarrow} B_p \overset{c}{\hookrightarrow} B_p \oplus X_p \overset{c}{\hookrightarrow} \left(\sum^{\oplus} X_p\right)_{\ell^p}$$

and

$$\left(\sum^{\oplus} \ell^2\right)_{\ell^p} \overset{c}{\hookrightarrow} \left(\sum^{\oplus} \ell^2\right)_{\ell^p} \oplus X_p \overset{c}{\hookrightarrow} B_p \oplus X_p \overset{c}{\hookrightarrow} \left(\sum^{\oplus} X_p\right)_{\ell^p}$$

established by part (b) of Proposition 2.37 and part (a) above. Now $\left(\sum^{\oplus} X_p\right)_{\ell^p} \hookrightarrow \left(\sum^{\oplus} \ell^2\right)_{\ell^p}$ by Lemma 2.36, which completes each of the two cycles. It follows that the listed spaces are equivalent under $\equiv$.

(c) We know $\ell^2 \oplus \ell^p \overset{c}{\hookrightarrow} \left(\sum^{\oplus} \ell^2\right)_{\ell^p} \overset{c}{\hookrightarrow} L^p$ but $L^p \not\hookrightarrow \left(\sum^{\oplus} \ell^2\right)_{\ell^p} \not\hookrightarrow \ell^2 \oplus \ell^p$ as in the discussion of diagrams (1.1) and (1.2). The result now follows from the fact that $X \equiv \ell^2 \oplus \ell^p$ by part (c) of Proposition 2.24 and $Y \equiv \left(\sum^{\oplus} \ell^2\right)_{\ell^p}$ by part (b) above. $\square$

Building on diagram (2.16), for $2 < p < \infty$ we have

$$\begin{array}{ccccccccc}
\ell^2 & & & & B_p & & & & \\
& \searrow & & & \parallel\!| & & & & \\
& & \ell^2 \oplus \ell^p & \rightarrow & \left(\sum^{\oplus} \ell^2\right)_{\ell^p} & \equiv \; B_p \oplus X_p & \equiv & \left(\sum^{\oplus} X_p\right)_{\ell^p} & \rightarrow \; L^p. \\
& \nearrow & \parallel\!| & & \parallel\!| & & & & \\
\ell^p & & X_p & & \left(\sum^{\oplus} \ell^2\right)_{\ell^p} \oplus X_p & & & &
\end{array} \tag{2.18}$$

As we have seen, the relation $\hookrightarrow$ is inadequate to distinguish $\left(\sum^{\oplus} \ell^2\right)_{\ell^p} \oplus X_p$, $B_p \oplus X_p$, and $\left(\sum^{\oplus} X_p\right)_{\ell^p}$ isomorphically. We will distinguish these three spaces via the relation $\overset{c}{\hookrightarrow}$. The next three results will distinguish $B_p \oplus X_p$ and $\left(\sum^{\oplus} X_p\right)_{\ell^p}$. The first result is a corollary of Lemma 2.34.

**Lemma 2.44.** *Let $1 < q < r < 2$. Suppose $S\colon \ell^r \rightarrow B_q$ is a bounded linear operator. Then given a sequence $\{\epsilon_n\}$ of positive scalars, there is a normalized block basic sequence $\{x_n\}$ of the standard basis $\{e_k\}$ of $\ell^r$ such that $\|S(x_n)\|_{B_q} < \epsilon_n$ for each $n \in \mathbb{N}$.*

**Proof.** It suffices to show that there is a normalized block basic sequence $\{x_n\}$ of the standard basis $\{e_k\}$ of $\ell^r$ such that $\|S(x_n)\|_{B_q} \leq \frac{\|S\|}{n}$ for each $n \in \mathbb{N}$, for the result will then follow upon passing to an appropriately chosen subsequence of $\{x_n\}$.

We define $\{x_n\}$ by induction, where each $x_n$ is of the form $\sum_{k\in E_n} \lambda_k e_k$, each $E_n$ is a finite subset of $\mathbb{N}$, each $\{\lambda_k : k \in E_n\}$ is a set of nonzero scalars, and $\max E_i < \min E_j$ for $1 \leq i < j$.

Let $x_1 = \sum_{k\in E_1} \lambda_k e_k$ be a normalized block of $\{e_k\}$. Then $\|S(x_1)\|_{B_q} \leq \frac{\|S\|}{1}$. Suppose normalized disjointly supported blocks $x_1, \ldots, x_N$ have been chosen, where $x_n = \sum_{k\in E_n} \lambda_k e_k$ and $\|S(x_n)\|_{B_q} \leq \frac{\|S\|}{n}$ for each $1 \leq n \leq N$, and $\max E_i < \min E_j$ for $1 \leq i < j \leq N$. Let $M = \max E_N$. Then as we verify below, we may choose $x_{N+1} \in \operatorname{span}\{e_k : k \geq M+1\}$ of norm one such that $\|S(x_{N+1})\|_{B_q} \leq \frac{\|S\|}{N+1}$.

Suppose for a moment that no such $x_{N+1}$ exists. Let $X_{M+1} = [e_k : k \geq M+1]_{\ell^r}$, which is isometric to $\ell^r$. Then for each normalized $x \in X_{M+1}$, $\|S(x)\|_{B_q} > \frac{1}{2}\frac{\|S\|}{N+1}$. Hence $S|_{X_{M+1}}$ induces an isomorphic imbedding of $\ell^r$ into $B_q$. However, by Lemma 2.34, no such imbedding exist. Thus $x_{N+1}$ can be chosen as claimed, and the result follows. $\square$

**Lemma 2.45.** *Let $1 < q < r < 2$. Then* $\left(\sum^{\oplus} \ell^r\right)_{\ell^q} \not\hookrightarrow B_q \oplus X_q$.

**Proof.** Suppose $\left(\sum^{\oplus} \ell^r\right)_{\ell^q} \hookrightarrow B_q \oplus X_q$. Let $T: \left(\sum^{\oplus} \ell^r\right)_{\ell^q} \to B_q \oplus X_q$ be an isomorphic imbedding. Let $Q: B_q \oplus X_q \to B_q \oplus \{0_{X_q}\}$ be the obvious projection. Then $QT: \left(\sum^{\oplus} \ell^r\right)_{\ell^q} \to B_q \oplus \{0_{X_q}\}$ is a bounded linear operator.

We will show that there is a subspace $X$ of $\left(\sum^{\oplus} \ell^r\right)_{\ell^q}$, isometric to $\left(\sum^{\oplus} \ell^r\right)_{\ell^q}$, such that $\left\|Q|_{T(X)}\right\| < 1$, whence $(I - Q)|_{T(X)}$ induces an isomorphic imbedding of $\left(\sum^{\oplus} \ell^r\right)_{\ell^q}$ into $X_q$. However by [**S**, Proposition 2], presented below as Lemma 3.7, no such imbedding exists, and the lemma will follow.

Let $\{e_{m,n}\}$ be the standard basis of $\left(\sum^{\oplus} \ell^r\right)_{\ell^q}$, where for each $n \in \mathbb{N}$, $\{e_{m,n}\}_{m=1}^{\infty}$ is isometrically equivalent to the standard basis of $\ell^r$. By Lemma 2.44, for each $n \in \mathbb{N}$ we may choose a normalized block basic sequence $\left\{x_k^{(n)}\right\}_{k=1}^{\infty}$ of $\{e_{m,n}\}_{m=1}^{\infty}$ such that $\left\|QT\left(x_k^{(n)}\right)\right\|_{B_q} < \frac{1}{\|T^{-1}\|2^{k+n}}$. Let $X = \left[x_k^{(n)} : k, n \in \mathbb{N}\right]$. Then $X$ is isometric to $\left(\sum^{\oplus} \ell^r\right)_{\ell^q}$. Let $\left\{\lambda_k^{(1)}\right\} \oplus \left\{\lambda_k^{(2)}\right\} \oplus \cdots \in (\ell^r \oplus \ell^r \oplus \cdots)_{\ell^q}$ be of norm one. Then

$$\begin{aligned}
\left\|QT\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\lambda_k^{(n)}x_k^{(n)}\right)\right\|_{B_q} &= \left\|\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\lambda_k^{(n)}QT\left(x_k^{(n)}\right)\right\|_{B_q} \\
&\leq \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left\|QT\left(x_k^{(n)}\right)\right\|_{B_q} \\
&< \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{\|T^{-1}\|2^{k+n}} \\
&= \frac{1}{\|T^{-1}\|}.
\end{aligned}$$

Hence $\|QT|_X\| < \frac{1}{\|T^{-1}\|}$, so $\|Q|_{T(X)}\| \leq \|T^{-1}\| \, \|QT|_X\| < 1$. Thus $(I - Q)|_{T(X)}$ induces an isomorphic imbedding of $\left(\sum^{\oplus} \ell^r\right)_{\ell^q}$ into $X_q$, where $I$ is the formal identity mapping, but no such imbedding exists. $\square$

**Proposition 2.46.** *Let $1 < p < \infty$ where $p \neq 2$. Then $\left(\sum^{\oplus} X_p\right)_{\ell^p} \overset{c}{\not\hookrightarrow} B_p \oplus X_p$.*

**Proof.** First let $1 < q < 2$ and suppose $\left(\sum^{\oplus} X_q\right)_{\ell^q} \overset{c}{\hookrightarrow} B_q \oplus X_q$. For $1 < q < r < 2$, $\ell^r \hookrightarrow X_q$ by Lemma 2.35, so $\left(\sum^{\oplus} \ell^r\right)_{\ell^q} \hookrightarrow \left(\sum^{\oplus} X_q\right)_{\ell^q} \overset{c}{\hookrightarrow} B_q \oplus X_q$. Hence $\left(\sum^{\oplus} \ell^r\right)_{\ell^q} \hookrightarrow B_q \oplus X_q$, contrary to Lemma 2.45. It follows that $\left(\sum^{\oplus} X_q\right)_{\ell^q} \overset{c}{\not\hookrightarrow} B_q \oplus X_q$. The result now holds for $2 < p < \infty$ by duality. $\square$

The next two results will distinguish $\left(\sum^{\oplus} \ell^2\right)_{\ell^p} \oplus X_p$ and $B_p \oplus X_p$ isomorphically. The lemma isolates some preliminary calculations.

**Lemma 2.47.** *Let $2 < p < \infty$ with conjugate index $q$, and let $n \in \mathbb{N}$. Let $X_{p,v^{(n)}}$ be as in the definition of $B_p$, and let $v_n$ denote $\left(\frac{1}{n}\right)^{\frac{p-2}{2p}}$, the value taken by the constant sequence $v^{(n)}$. Let $\mathcal{B}_n$ be the closed unit ball of $X_{p,v^{(n)}}$. Then for $M_n \in \mathbb{N}$*

*such that* $M_n \le v_n^{-\frac{2p}{p-2}} = n$,

$$\sup_{\{d_m\}\in\mathcal{B}_n} \left|\sum_{m=1}^{M_n} d_m\right| = M_n^{\frac{1}{q}}.$$

*Moreover, for* $K \in \mathbb{N}$ *and* $\{\lambda_k\} \in \ell^2$,

$$\sup_{\{d_{k,\ell}\}\in\mathcal{B}_n} \left|\sum_{k=1}^{K}\sum_{\ell=1}^{n} \lambda_k d_{k,\ell}\right| = n^{\frac{1}{q}} \left(\sum_{k=1}^{K} |\lambda_k|^2\right)^{\frac{1}{2}}.$$

**Proof.** Let $M \in \mathbb{N}$ and let $\{d_m\}$ be a sequence of scalars. Then by Hölder's inequality,

$$\left|\sum_{m=1}^{M} d_m\right| = \left|\sum_{m=1}^{M} 1 d_m\right| \le \left(\sum_{m=1}^{M} 1^q\right)^{\frac{1}{q}} \left(\sum_{m=1}^{M} |d_m|^p\right)^{\frac{1}{p}}$$
$$= M^{\frac{1}{q}} \left(\sum_{m=1}^{M} |d_m|^p\right)^{\frac{1}{p}}$$

and

$$\left|\sum_{m=1}^{M} d_m\right| = \frac{1}{v_n}\left|\sum_{m=1}^{M} 1 d_m v_n\right| \le \frac{1}{v_n}\left(\sum_{m=1}^{M} 1^2\right)^{\frac{1}{2}} \left(\sum_{m=1}^{M} |d_m v_n|^2\right)^{\frac{1}{2}}$$
$$= \frac{1}{v_n} M^{\frac{1}{2}} \left(\sum_{m=1}^{M} |d_m v_n|^2\right)^{\frac{1}{2}}.$$

Suppose $\{d_m\} \in \mathcal{B}_n$. Then $\left(\sum_{m=1}^{M} |d_m|^p\right)^{\frac{1}{p}} \le 1$ and $\left(\sum_{m=1}^{M} |d_m v_n|^2\right)^{\frac{1}{2}} \le 1$. Hence $\left|\sum_{m=1}^{M} d_m\right| \le M^{\frac{1}{q}}$ and $\left|\sum_{m=1}^{M} d_m\right| \le \frac{1}{v_n} M^{\frac{1}{2}}$. It follows that

$$\sup_{\{d_m\}\in\mathcal{B}_n} \left|\sum_{m=1}^{M} d_m\right| \le \min\left\{M^{\frac{1}{q}}, \frac{1}{v_n} M^{\frac{1}{2}}\right\}.$$

Let $M_n \in \mathbb{N}$ such that $M_n \le v_n^{-\frac{2p}{p-2}}$. Then $M_n^{\frac{1}{q}-\frac{1}{2}} = M_n^{\frac{p-1}{p}-\frac{1}{2}} = M_n^{\frac{p-2}{2p}} \le \frac{1}{v_n}$, so $M_n^{\frac{1}{q}} \le \frac{1}{v_n} M_n^{\frac{1}{2}}$. Hence with no loss of sharpness,

$$\sup_{\{d_m\}\in\mathcal{B}_n} \left|\sum_{m=1}^{M_n} d_m\right| \le M_n^{\frac{1}{q}}.$$

Let $\tilde{d}_m = \frac{1}{M_n} M_n^{\frac{1}{q}} = M_n^{\frac{1}{q}-1} = M_n^{-\frac{1}{p}}$ for $1 \le m \le M_n$, and $\tilde{d}_m = 0$ otherwise. Then $\sum_{m=1}^{M_n} \left|\tilde{d}_m\right|^p = 1$ and $\sum_{m=1}^{M_n} \left|\tilde{d}_m v_n\right|^2 = v_n^2 M_n^{\frac{2}{q}-1} = \left(v_n M_n^{\frac{1}{q}-\frac{1}{2}}\right)^2 \le 1$, whence $\left\{\tilde{d}_m\right\} \in \mathcal{B}_n$. Moreover, $\left|\sum_{m=1}^{M_n} \tilde{d}_m\right| = M_n^{\frac{1}{q}}$. Hence

$$\sup_{\{d_m\}\in\mathcal{B}_n} \left|\sum_{m=1}^{M_n} d_m\right| \ge M_n^{\frac{1}{q}}.$$

It follows that

$$\sup_{\{d_m\}\in\mathcal{B}_n}\left|\sum_{m=1}^{M_n} d_m\right| = M_n^{\frac{1}{q}}. \tag{2.19}$$

Let $K \in \mathbb{N}$, let$\{\lambda_k\} \in \ell^2$, and let $\{d_{k,\ell}\}$ be a sequence of scalars. Note that $\frac{1}{v_n}n^{\frac{1}{2}} = n^{\frac{p-2}{2p}}n^{\frac{1}{2}} = n^{\frac{p-1}{p}} = n^{\frac{1}{q}}$. Then by Hölder's inequality,

$$\begin{aligned}\left|\sum_{k=1}^{K}\sum_{\ell=1}^{n}\lambda_k d_{k,\ell}\right| &\le \left(\sum_{k=1}^{K}\sum_{\ell=1}^{n}|\lambda_k|^q\right)^{\frac{1}{q}}\left(\sum_{k=1}^{K}\sum_{\ell=1}^{n}|d_{k,\ell}|^p\right)^{\frac{1}{p}}\\ &= n^{\frac{1}{q}}\left(\sum_{k=1}^{K}|\lambda_k|^q\right)^{\frac{1}{q}}\left(\sum_{k=1}^{K}\sum_{\ell=1}^{n}|d_{k,\ell}|^p\right)^{\frac{1}{p}}\end{aligned}$$

and

$$\begin{aligned}\left|\sum_{k=1}^{K}\sum_{\ell=1}^{n}\lambda_k d_{k,\ell}\right| &= \frac{1}{v_n}\left|\sum_{k=1}^{K}\sum_{\ell=1}^{n}\lambda_k d_{k,\ell}v_n\right|\\ &\le \frac{1}{v_n}\left(\sum_{k=1}^{K}\sum_{\ell=1}^{n}|\lambda_k|^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{K}\sum_{\ell=1}^{n}|d_{k,\ell}v_n|^2\right)^{\frac{1}{2}}\\ &= \frac{1}{v_n}n^{\frac{1}{2}}\left(\sum_{k=1}^{K}|\lambda_k|^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{K}\sum_{\ell=1}^{n}|d_{k,\ell}v_n|^2\right)^{\frac{1}{2}}\\ &= n^{\frac{1}{q}}\left(\sum_{k=1}^{K}|\lambda_k|^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{K}\sum_{\ell=1}^{n}|d_{k,\ell}v_n|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Suppose $\{d_{k,\ell}\} \in \mathcal{B}_n$. Then $\left(\sum_{k=1}^{K}\sum_{\ell=1}^{n}|d_{k,\ell}|^p\right)^{\frac{1}{p}} \le 1$ and $\left(\sum_{k=1}^{K}\sum_{\ell=1}^{n}|d_{k,\ell}v_n|^2\right)^{\frac{1}{2}} \le 1$. Hence $\left|\sum_{k=1}^{K}\sum_{\ell=1}^{n}\lambda_k d_{k,\ell}\right| \le n^{\frac{1}{q}}\left(\sum_{k=1}^{K}|\lambda_k|^q\right)^{\frac{1}{q}}$ and $\left|\sum_{k=1}^{K}\sum_{\ell=1}^{n}\lambda_k d_{k,\ell}\right| \le n^{\frac{1}{q}}\left(\sum_{k=1}^{K}|\lambda_k|^2\right)^{\frac{1}{2}}$. It follows that

$$\begin{aligned}\sup_{\{d_{k,\ell}\}\in\mathcal{B}_n}\left|\sum_{k=1}^{K}\sum_{\ell=1}^{n}\lambda_k d_{k,\ell}\right| &\le n^{\frac{1}{q}}\min\left\{\left(\sum_{k=1}^{K}|\lambda_k|^q\right)^{\frac{1}{q}},\left(\sum_{k=1}^{K}|\lambda_k|^2\right)^{\frac{1}{2}}\right\}\\ &= n^{\frac{1}{q}}\left(\sum_{k=1}^{K}|\lambda_k|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Let $\tilde{d}_{k,\ell} = \frac{1}{v_n}n^{-\frac{1}{2}}\bar{\lambda}_k$ for $1 \le \ell \le n$ and $1 \le k \le K$, and $\tilde{d}_{k,\ell} = 0$ otherwise, where $\bar{\lambda}_k$ is the complex conjugate of $\lambda_k$. Note that $\frac{1}{v_n}n^{-\frac{1}{2}} = n^{\frac{p-2}{2p}}n^{-\frac{1}{2}} = n^{-\frac{1}{p}}$. Hence

$$\begin{aligned}\left(\sum_{k=1}^{K}\sum_{\ell=1}^{n}\left|\tilde{d}_{k,\ell}\right|^p\right)^{\frac{1}{p}} &= n^{-\frac{1}{p}}\left(\sum_{k=1}^{K}\sum_{\ell=1}^{n}|\lambda_k|^p\right)^{\frac{1}{p}}\\ &= \left(\sum_{k=1}^{K}|\lambda_k|^p\right)^{\frac{1}{p}}\\ &\le \left(\sum_{k=1}^{K}|\lambda_k|^2\right)^{\frac{1}{2}}\end{aligned}$$

and

$$\left(\sum_{k=1}^{K}\sum_{\ell=1}^{n}\left|\tilde{d}_{k,\ell}v_n\right|^2\right)^{\frac{1}{2}} = \left(\sum_{k=1}^{K}\sum_{\ell=1}^{n} n^{-1}\left|\lambda_k\right|^2\right)^{\frac{1}{2}} = \left(\sum_{k=1}^{K}\left|\lambda_k\right|^2\right)^{\frac{1}{2}}.$$

Thus for $\left(\sum_{k=1}^{K}|\lambda_k|^2\right)^{\frac{1}{2}} \leq 1$, $\left\{\tilde{d}_{k,\ell}\right\} \in \mathcal{B}_n$. Moreover, for $\left(\sum_{k=1}^{K}|\lambda_k|^2\right)^{\frac{1}{2}} = 1$,

$$\left|\sum_{k=1}^{K}\sum_{\ell=1}^{n}\lambda_k\tilde{d}_{k,\ell}\right| = \sum_{k=1}^{K}\sum_{\ell=1}^{n} n^{-\frac{1}{p}}\left|\lambda_k\right|^2 = n^{\frac{1}{q}}\sum_{k=1}^{K}\left|\lambda_k\right|^2 = n^{\frac{1}{q}}\left(\sum_{k=1}^{K}\left|\lambda_k\right|^2\right)^{\frac{1}{2}}.$$

Hence

$$\sup_{\{d_{k,\ell}\}\in\mathcal{B}_n}\left|\sum_{k=1}^{K}\sum_{\ell=1}^{n}\lambda_k d_{k,\ell}\right| \geq n^{\frac{1}{q}}\left(\sum_{k=1}^{K}\left|\lambda_k\right|^2\right)^{\frac{1}{2}}.$$

It follows that

$$\sup_{\{d_{k,\ell}\}\in\mathcal{B}_n}\left|\sum_{k=1}^{K}\sum_{\ell=1}^{n}\lambda_k d_{k,\ell}\right| = n^{\frac{1}{q}}\left(\sum_{k=1}^{K}\left|\lambda_k\right|^2\right)^{\frac{1}{2}}. \tag{2.20}$$

□

**Proposition 2.48.** *Let $1 < p < \infty$ where $p \neq 2$. Then $B_p \overset{c}{\not\hookrightarrow} \left(\sum^{\oplus}\ell^2\right)_{\ell^p} \oplus X_p$.*

**Proof.** By duality, it suffices to show that $B_q \overset{c}{\not\hookrightarrow} \left(\sum^{\oplus}\ell^2\right)_{\ell^q} \oplus X_q$ for $1 < q < 2$. Let $1 < q < 2$ and suppose $B_q \overset{c}{\hookrightarrow} \left(\sum^{\oplus}\ell^2\right)_{\ell^q} \oplus X_q$. Let $p$ be the conjugate index of $q$. For each $n \in \mathbb{N}$, let $v_n$ and $\mathcal{B}_n$ be as in Lemma 2.47. Now $B_q \sim B_p^* \sim \left(\sum^{\oplus} X^*_{p,v(n)}\right)_{\ell^q}$, so $\left(\sum^{\oplus} X^*_{p,v(n)}\right)_{\ell^q} \overset{c}{\hookrightarrow} \left(\sum^{\oplus}\ell^2\right)_{\ell^q} \oplus X_q$. Let $T\colon \left(\sum^{\oplus} X^*_{p,v(n)}\right)_{\ell^q} \to \left(\sum^{\oplus}\ell^2\right)_{\ell^q} \oplus X_q$ be an isomorphic imbedding with complemented range. Let
$Q\colon \left(\sum^{\oplus}\ell^2\right)_{\ell^q} \oplus X_q \to \left(\sum^{\oplus}\ell^2\right)_{\ell^q} \oplus \{0_{X_q}\}$ be the obvious projection. Then
$QT\colon \left(\sum^{\oplus} X^*_{p,v(n)}\right)_{\ell^q} \to \left(\sum^{\oplus}\ell^2\right)_{\ell^q} \oplus \{0_{X_q}\}$ is a bounded linear operator.

We will show that there is a subspace $Y$ of $\left(\sum^{\oplus} X^*_{p,v(n)}\right)_{\ell^q}$ isometric to $\left(\sum^{\oplus}\ell^2\right)_{\ell^q}$ such that $\left\|Q|_{T(Y)}\right\| < 1$, whence $(I - Q)|_{T(Y)}$ induces an isomorphic imbedding of $\left(\sum^{\oplus}\ell^2\right)_{\ell^q}$ into $X_q$, where $I$ is the formal identity mapping. However by [**S**, Proposition 2], presented below as Lemma 3.7, no such imbedding exists, and the proposition will follow.

Let $\{e_{m,n}\}$ be the standard basis of $\left(\sum^{\oplus} X^*_{p,v(n)}\right)_{\ell^q}$, where for each $n \in \mathbb{N}$, $\{e_{m,n}\}_{m=1}^{\infty}$ is isometrically equivalent to the standard basis of $X^*_{p,v(n)}$ and equivalent to the standard basis of $\ell^2$. Let $\{\tilde{e}_{m,n}\}$ be the standard basis of $\left(\sum^{\oplus} X_{p,v(n)}\right)_{\ell^p}$, where for each $n \in \mathbb{N}$, $\{\tilde{e}_{m,n}\}_{m=1}^{\infty}$ is isometrically equivalent to the standard basis of $X_{p,v(n)}$.

For $K \in \mathbb{N}$, let $\Gamma(K)$ denote a subset of $\mathbb{N}$ having cardinality $K$. Let $M \in \mathbb{N}$. Then for fixed $n \in \mathbb{N}$, letting $\langle\ ,\ \rangle$ denote the action of $X^*_{p,v(n)}$ on $X_{p,v(n)}$,

$$\begin{aligned}\left\|\sum_{m\in\Gamma(M)} e_{m,n}\right\| &= \sup_{\{d_k\}\in\mathcal{B}_n}\left|\left\langle \sum_{k=1}^{\infty} d_k\tilde{e}_{k,n}, \sum_{m\in\Gamma(M)} e_{m,n}\right\rangle\right| \\ &= \sup_{\{d_k\}\in\mathcal{B}_n}\left|\sum_{k\in\Gamma(M)} d_k\right| \\ &= \sup_{\{d_k\}\in\mathcal{B}_n}\left|\sum_{k=1}^{M} d_k\right|. \end{aligned} \tag{2.21}$$

Now for fixed $n \in \mathbb{N}$, letting $M_n \le v_n^{-\frac{2p}{p-2}} = n$ as in Lemma 2.47, equations (2.21) and (2.19) yield

$$\left\|\sum_{m\in\Gamma(M_n)} e_{m,n}\right\| = M_n^{\frac{1}{q}},$$

or upon normalization,

$$\left\|M_n^{-\frac{1}{q}} \sum_{m\in\Gamma(M_n)} e_{m,n}\right\| = 1. \tag{2.22}$$

We now introduce a construction which will be used in two different settings. Fix $n \in \mathbb{N}$ and let $\tilde{M}_n = v_n^{-\frac{2p}{p-2}} = n$. Let $\left\{E_k^{(n)}\right\}_{k=1}^{\infty}$ be a sequence of disjoint subsets of $\mathbb{N}$, each of cardinality $\tilde{M}_n$. Let $\{\tau(m)\}$ be an increasing sequence of positive integers. For each $k \in \mathbb{N}$, let $x_k^{(n)} = \tilde{M}_n^{-\frac{1}{q}} \sum_{m\in E_k^{(n)}} e_{\tau(m),n}$. Then each $x_k^{(n)}$ is of norm one by equation (2.22), and $\left\{x_k^{(n)}\right\}_{k=1}^{\infty}$ is equivalent to the standard basis of $\ell^2$. Recalling equation (2.20) for the last step, for $K \in \mathbb{N}$ and $\{\lambda_k\} \in \ell^2$,

$$\left\| \sum_{k=1}^{K} \lambda_k x_k^{(n)} \right\| = \tilde{M}_n^{-\frac{1}{q}} \left\| \sum_{k=1}^{K} \lambda_k \sum_{m \in E_k^{(n)}} e_{\tau(m),n} \right\|$$

$$= n^{-\frac{1}{q}} \sup_{\{d_\ell\} \in \mathcal{B}_n} \left| \left\langle \sum_{\ell=1}^{\infty} d_\ell \tilde{e}_{\ell,n}, \sum_{k=1}^{K} \lambda_k \sum_{m \in E_k^{(n)}} e_{\tau(m),n} \right\rangle \right|$$

$$= n^{-\frac{1}{q}} \sup_{\{d_\ell\} \in \mathcal{B}_n} \left| \sum_{k=1}^{K} \lambda_k \sum_{\ell \in E_k^{(n)}} d_\ell \right|$$

$$= \left( \sum_{k=1}^{K} |\lambda_k|^2 \right)^{\frac{1}{2}}. \tag{2.23}$$

Hence $\left\{ x_k^{(n)} \right\}_{k=1}^{\infty}$ is in fact isometrically equivalent to the standard basis of $\ell^2$.

We now distinguish two exhaustive but not mutually exclusive cases. In the first case, there are infinitely many $n \in \mathbb{N}$ such that $lim_{m \to \infty} \|QT(e_{m,n})\| = 0$. In the second case, there are infinitely many $n \in \mathbb{N}$ such that $lim\, sup_{m \in \mathbb{N}} \|QT(e_{m,n})\| > 0$.

We will show that in either case, there is an increasing sequence $\{n(i)\}_{i=1}^{\infty}$ of positive integers and a sequence $\{X_{n(i)}\}_{i=1}^{\infty}$ of subspaces of $\left(\sum^{\oplus} X_{p,v(n)}^*\right)_{\ell^q}$ such that for each $i \in \mathbb{N}$, $X_{n(i)}$ is a subspace of $\left[e_{m,n(i)}: m \in \mathbb{N}\right]$ isometric to $\ell^2$ with $\left\|Q|_{T(X_{n(i)})}\right\| \leq \|T^{-1}\| \left\|QT|_{X_{n(i)}}\right\| < \frac{1}{2^i}$. It will follow that there is a subspace $Y = \left(\sum^{\oplus} Y_n\right)_{\ell^q}$ of $\left(\sum^{\oplus} X_{p,v(n)}^*\right)_{\ell^q}$ isometric to $\left(\sum^{\oplus} \ell^2\right)_{\ell^q}$ such that $\left\|Q|_{T(Y)}\right\| \leq \|T^{-1}\| \left\|QT|_Y\right\| < 1$. [$Y_{n(i)} = X_{n(i)}$ and $Y_k = \{0\}$ if $k \notin \{n(i)\}$.] As noted before, the proposition will then follow.

*The first case.*

Fix $n \in \mathbb{N}$ such that $lim_{m \to \infty} \|QT(e_{m,n})\| = 0$. Choose a subsequence $\{e_{\alpha(m),n}\}_{m=1}^{\infty}$ of $\{e_{m,n}\}_{m=1}^{\infty}$ such that for each $m \in \mathbb{N}$,
$$\left\|QT\left(e_{\alpha(m),n}\right)\right\| < \frac{1}{2^{m+n} n^{\frac{1}{p}} \|T^{-1}\|}.$$

Let $\tilde{M}_n = v_n^{-\frac{2p}{p-2}} = n$. Let $\left\{E_k^{(n)}\right\}_{k=1}^{\infty}$ be a sequence of disjoint subsets of $\mathbb{N}$, each of cardinality $\tilde{M}_n$, such that for each $k \in \mathbb{N}$, $\inf E_k^{(n)} \geq k$. Then for each $m \in E_k^{(n)}$,

$\left\|QT\left(e_{\alpha(m),n}\right)\right\| < \dfrac{1}{2^{k+n}n^{\frac{1}{p}}\left\|T^{-1}\right\|}$. For each $k \in \mathbb{N}$, let $x_k^{(n)} = \tilde{M_n}^{-\frac{1}{q}} \sum_{m \in E_k^{(n)}} e_{\alpha(m),n}$. Then each $x_k^{(n)}$ is of norm one by equation (2.22), $\left\{x_k^{(n)}\right\}_{k=1}^{\infty}$ is isometrically equivalent to the standard basis of $\ell^2$ as in equation (2.23), and for each $x_k^{(n)}$,

$$\begin{aligned}\left\|QT\left(x_k^{(n)}\right)\right\| = \tilde{M_n}^{-\frac{1}{q}}\left\|\sum_{m \in E_k^{(n)}} QT\left(e_{\alpha(m),n}\right)\right\| &\leq n^{-\frac{1}{q}} \sum_{m \in E_k^{(n)}} \left\|QT\left(e_{\alpha(m),n}\right)\right\| \\ &< n^{-\frac{1}{q}} n \frac{1}{2^{k+n}n^{\frac{1}{p}}\left\|T^{-1}\right\|} \\ &= \frac{1}{2^{k+n}\left\|T^{-1}\right\|}.\end{aligned}$$

Let $\{\lambda_k\} \in \ell^2$ be of norm one. Then

$$\begin{aligned}\left\|QT\left(\sum_{k=1}^{\infty} \lambda_k x_k^{(n)}\right)\right\| = \left\|\sum_{k=1}^{\infty} \lambda_k QT\left(x_k^{(n)}\right)\right\| &\leq \sum_{k=1}^{\infty} \left\|QT\left(x_k^{(n)}\right)\right\| \\ &< \sum_{k=1}^{\infty} \frac{1}{2^{k+n}\|T^{-1}\|} \\ &= \frac{1}{2^n\left\|T^{-1}\right\|}.\end{aligned}$$

Letting $X_n = \left[x_k^{(n)} : k \in \mathbb{N}\right]$, it follows that $\left\|Q|_{T(X_n)}\right\| \leq \left\|T^{-1}\right\| \left\|QT|_{X_n}\right\| < \frac{1}{2^n}$.

Release $n \in \mathbb{N}$ as a free variable. Let $\{n(i)\}_{i=1}^{\infty}$ be an increasing sequence of positive integers such that for each $i \in \mathbb{N}$, $\lim_{m\to\infty} \left\|QT(e_{m,n(i)})\right\| = 0$. Then for each $i \in \mathbb{N}$, there is a subspace $X_{n(i)}$ of $\left(\sum^{\oplus} X^*_{p,v(n)}\right)_{\ell^q}$ isometric to $\ell^2$ such that $X_{n(i)}$ is a subspace of $\left[e_{m,n(i)} : m \in \mathbb{N}\right]$ with $\left\|Q|_{T(X_{n(i)})}\right\| \leq \left\|T^{-1}\right\| \left\|QT|_{X_{n(i)}}\right\| < \frac{1}{2^{n(i)}} \leq \frac{1}{2^i}$. Thus the proposition follows in the first case.

*The second case.*

Fix $n \in \mathbb{N}$ such that $c_n = \limsup_{m \in \mathbb{N}} \left\|QT\left(e_{m,n}\right)\right\| > 0$. Then $c_n \leq \|QT\|$. Given $0 < \epsilon < 1$, we may choose a subsequence $\left\{e_{\alpha(m),n}\right\}_{m=1}^{\infty}$ of $\{e_{m,n}\}_{m=1}^{\infty}$ such that $lim_{m\to\infty} \left\|QT\left(e_{\alpha(m),n}\right)\right\| = c_n$, with $sup_{m \in \mathbb{N}} \left|\left\|QT\left(e_{\alpha(m),n}\right)\right\| - c_n\right| < \epsilon c_n$, and such that $\left\{QT\left(e_{\alpha(m),n}\right)\right\}_{m=1}^{\infty}$ is a basic sequence [**B-P**, Theorem 3], whence $QT|_{[e_{\alpha(m),n} : m \in \mathbb{N}]}$ is an isomorphic imbedding and $\left\{QT\left(e_{\alpha(m),n}\right)\right\}_{m=1}^{\infty}$ is equivalent to the standard basis of $\ell^2$. Now by Proposition 2.19, given $0 < \epsilon < 1$ and such a sequence $\{e_{\alpha(m),n}\}_{m=1}^{\infty}$, we may choose a subsequence $\{e_{\beta(m),n}\}_{m=1}^{\infty}$ such that $\{QT(e_{\beta(m),n})\}_{m=1}^{\infty}$ is $(1+\epsilon)$-equivalent to the standard basis of $\ell^2$.

Let $\tilde{M}_n = v_n^{-\frac{2p}{p-2}} = n$. Let $\left\{E_k^{(n)}\right\}_{k=1}^{\infty}$ be a sequence of disjoint subsets of $\mathbb{N}$, each of cardinality $\tilde{M}_n$. Given $0 < \epsilon < 1$ and $\{e_{\beta(m),n}\}_{m=1}^{\infty}$ as above, for each $k \in \mathbb{N}$ let $x_k^{(n)} = \tilde{M}_n^{-\frac{1}{q}} \sum_{m \in E_k^{(n)}} e_{\beta(m),n}$. Then each $x_k^{(n)}$ is of norm one by equation (2.22), $\left\{x_k^{(n)}\right\}_{k=1}^{\infty}$ is

isometrically equivalent to the standard basis of $\ell^2$ as in equation (2.23), and for each $x_k^{(n)}$,

$$\left\|QT\left(x_k^{(n)}\right)\right\| = \tilde{M}_n^{-\frac{1}{q}} \left\| \sum_{m \in E_k^{(n)}} QT\left(e_{\beta(m),n}\right) \right\| \approx \tilde{M}_n^{-\frac{1}{q}} \tilde{M}_n^{\frac{1}{2}} c_n = \tilde{M}_n^{\frac{1}{2}-\frac{1}{q}} c_n, \tag{2.24}$$

where the approximation can be improved to any degree by the choice of ($\epsilon$ and) $\left\{x_k^{(n)}\right\}_{k=1}^{\infty}$.

Given $0 < \epsilon < 1$, we may choose a sequence $\left\{x_k^{(n)}\right\}_{k=1}^{\infty}$ as above such that $\left|\left\|QT\left(x_k^{(n)}\right)\right\| - \tilde{M}_n^{\frac{1}{2}-\frac{1}{q}} c_n\right| < \epsilon \tilde{M}_n^{\frac{1}{2}-\frac{1}{q}} c_n$, where $QT|_{\left[x_k^{(n)}:k\in\mathbb{N}\right]}$ is an isomorphic imbedding and $\left\{QT\left(x_k^{(n)}\right)\right\}_{k=1}^{\infty}$ is equivalent to the standard basis of $\ell^2$. Thus by Proposition 2.19, given $0 < \epsilon < 1$ and such a sequence $\left\{x_k^{(n)}\right\}_{k=1}^{\infty}$, there is a subsequence $\left\{x_{\gamma(k)}^{(n)}\right\}_{k=1}^{\infty}$ such that $\left\{QT\left(x_{\gamma(k)}^{(n)}\right)\right\}_{k=1}^{\infty}$ is $(1+\epsilon)$-equivalent to the standard basis of $\ell^2$. Recalling (2.24), it follows that for $\{\lambda_k\} \in \ell^2$,

$$\left\|QT\left(\sum_{k=1}^{\infty} \lambda_k x_{\gamma(k)}^{(n)}\right)\right\| = \left\|\sum_{k=1}^{\infty} \lambda_k QT\left(x_{\gamma(k)}^{(n)}\right)\right\| \approx \left(\sum_{k=1}^{\infty} |\lambda_k|^2\right)^{\frac{1}{2}} \tilde{M}_n^{\frac{1}{2}-\frac{1}{q}} c_n, \tag{2.25}$$

where the approximation can be improved to any degree by the choice of ($\epsilon$ and) $\left\{x_{\lambda(k)}^{(n)}\right\}_{k=1}^{\infty}$.

Now $\left\{x_k^{(n)}\right\}_{k=1}^{\infty}$ is isometrically equivalent to the standard basis of $\ell^2$ as noted above, and the same is true of $\left\{x_{\gamma(k)}^{(n)}\right\}_{k=1}^{\infty}$. Let $X_n = \left[x_{\gamma(k)}^{(n)} : k \in \mathbb{N}\right]$. Then by (2.25), it follows that

$$\|QT|_{X_n}\| \approx \tilde{M}_n^{\frac{1}{2}-\frac{1}{q}} c_n \leq n^{\frac{1}{2}-\frac{1}{q}} \|QT\|, \tag{2.26}$$

where the approximation can be improved to any degree as in (2.25).

Release $n$ as a free variable and note that $lim_{n\to\infty} n^{\frac{1}{2}-\frac{1}{q}} \|QT\| = 0$. Hence by the hypothesis of the second case and by (2.26), we may choose an increasing sequence $\{n(i)\}_{i=1}^{\infty}$ of positive integers such that for each $i \in \mathbb{N}$,
$c_{n(i)} = \limsup_{m\in\mathbb{N}} \left\|QT(e_{m,n(i)})\right\| > 0$ and there is a subspace $X_{n(i)}$ of $\left(\sum^{\oplus} X^*_{p,v(n)}\right)_{\ell^q}$ isometric to $\ell^2$ such that $X_{n(i)}$ is a subspace of $\left[e_{m,n(i)}: m \in \mathbb{N}\right]$ with
$\left\|Q|_{T(X_{n(i)})}\right\| \leq \left\|T^{-1}\right\| \left\|QT|_{X_{n(i)}}\right\| < 2^i$. Thus the proposition follows in the second case, and in the general case. $\square$

Collecting our results and deducing simple consequences yields the following.

**Proposition 2.49.** *Let $1 < p < \infty$ where $p \neq 2$. Then*

*(a)* $B_p \overset{c}{\not\hookrightarrow} \left(\sum^{\oplus} \ell^2\right)_{\ell^p}$,

*(b)* $\left(\sum^{\oplus} \ell^2\right)_{\ell^p} \oplus X_p \overset{c}{\not\hookrightarrow} \left(\sum^{\oplus} \ell^2\right)_{\ell^p}$,

*(c)* $B_p \overset{c}{\not\hookrightarrow} \left(\sum^{\oplus} \ell^2\right)_{\ell^p} \oplus X_p$,

*(d)* $\left(\sum^{\oplus} \ell^2\right)_{\ell^p} \oplus X_p \overset{c}{\not\hookrightarrow} B_p$,

*(e)* $B_p \oplus X_p \overset{c}{\not\hookrightarrow} B_p$,

*(f)* $B_p \oplus X_p \overset{c}{\not\hookrightarrow} \left(\sum^{\oplus} \ell^2\right)_{\ell^p} \oplus X_p$, *and*

*(g)* $\left(\sum^{\oplus} X_p\right)_{\ell^p} \overset{c}{\not\hookrightarrow} B_p \oplus X_p$.

**Proof.**

(a) Part (a) is a restatement of part (d) of Proposition 2.37.

(b) Part (b) follows from part (f) of Proposition 2.24: $X_p \overset{c}{\not\hookrightarrow} \left(\sum^{\oplus} \ell^2\right)_{\ell^p}$.

(c) Part (c) is a restatement of Proposition 2.48.

(d) Part (d) follows from part (g) of Proposition 2.37: $X_p \overset{c}{\not\hookrightarrow} B_p$.

(e) Part (e) follows from part (g) of Proposition 2.37: $X_p \overset{c}{\not\hookrightarrow} B_p$.

(f) Part (f) follows from part (c) above.

(g) Part (g) is a restatement of Proposition 2.46. $\square$

Building on diagram (2.17), for $1 < p < \infty$ where $p \neq 2$, we have

$$
\begin{array}{ccccccccc}
 & & & & B_p & & & & L^p \\
 & & & \overset{c}{\nearrow} & & \overset{c}{\searrow} & & & \uparrow c \\
\ell^2 & & \left(\sum^{\oplus} \ell^2\right)_{\ell^p} & & & & B_p \oplus X_p & \overset{c}{\hookrightarrow} & \left(\sum^{\oplus} X_p\right)_{\ell^p}. \\
\downarrow c & \overset{c}{\nearrow} & & \overset{c}{\searrow} & & \overset{c}{\nearrow} & & & \\
\ell^2 \oplus \ell^p & & & & \left(\sum^{\oplus} \ell^2\right)_{\ell^p} \oplus X_p & & & & \\
\uparrow c & \overset{c}{\searrow} & & \overset{c}{\nearrow} & & & & & \\
\ell^p & & X_p & & & & & &
\end{array}
\tag{2.27}
$$

## Concluding Remarks

Fix $1 < p < \infty$ where $p \neq 2$.

If $X$ and $Y$ are separable infinite-dimensional $\mathcal{L}_p$ spaces, then $X \oplus Y$ is a separable infinite-dimensional $\mathcal{L}_p$ space as well. Suppose $X$ and $Y$ are as above and are isomorphic to their squares. If $X$ and $Y$ are incomparable in the sense that $X \overset{c}{\not\hookrightarrow} Y$ and $Y \overset{c}{\not\hookrightarrow} X$, then $X \oplus Y$ is isomorphically distinct from both $X$ and $Y$, while if $X \overset{c}{\hookrightarrow} Y$, then $X \oplus Y \sim Y$.

From the list $\ell^p$, $\ell^2 \oplus \ell^p$, $\left(\sum^{\oplus} \ell^2\right)_{\ell^p}$, $X_p$, $B_p$, $\left(\sum^{\oplus} X_p\right)_{\ell^p}$, $L^p$ of seven spaces, the only incomparable pairs of spaces are $\left\{\left(\sum^{\oplus} \ell^2\right)_{\ell^p}, X_p\right\}$ and $\{B_p, X_p\}$. As has been shown, $\left(\sum^{\oplus} \ell^2\right)_{\ell^p} \oplus X_p$ and $B_p \oplus X_p$ are isomorphically distinct from each of the seven listed spaces and from each other. Augmenting the list of seven spaces with the two new ones, the only new incomparable pair of spaces is $\left\{B_p, \left(\sum^{\oplus} \ell^2\right)_{\ell^p} \oplus X_p\right\}$. However $\left(\sum^{\oplus} \ell^2\right)_{\ell^p} \overset{c}{\hookrightarrow} B_p$, so $B_p \oplus \left(\sum^{\oplus} \ell^2\right)_{\ell^p} \sim B_p$, whence
$B_p \oplus \left(\left(\sum^{\oplus} \ell^2\right)_{\ell^p} \oplus X_p\right) \sim \left(B_p \oplus \left(\sum^{\oplus} \ell^2\right)_{\ell^p}\right) \oplus X_p \sim B_p \oplus X_p$, which has already been included in the augmented list.

If $Z$ is a separable infinite-dimensional Banach space such that $Z \overset{c}{\hookrightarrow} L^p$, then

$\left(\sum^{\oplus} Z\right)_{\ell^p}$ is a separable infinite-dimensional $\mathcal{L}_p$ space. However, from the augmented list of nine spaces above, no space arises from this method of construction which has not already been included in the list.

# CHAPTER III

## THE TENSOR PRODUCT CONSTRUCTION OF SCHECHTMAN

Let $1 < p < \infty$ where $p \neq 2$. Schechtman [S] constructed a sequence of isomorphically distinct separable infinite-dimensional $\mathcal{L}_p$ spaces by iterating a certain tensor product of Rosenthal's space $X_p$ with itself. Using $X_p^{\otimes n}$ to denote $X_p \otimes \cdots \otimes X_p$ ($n$ factors), the resulting sequence is $\{X_p^{\otimes n}\}_{n=1}^{\infty}$.

For closed subspaces $X$ and $Y$ of $L^p$, $X \otimes Y$ is defined to be the closed linear span in $L^p([0,1] \times [0,1])$ of products of the form $x(s)y(t)$ where $x \in X$ and $y \in Y$. It is a :airly routine matter to show that if $X$ and $Y$ are separable infinite-dimensional $\mathcal{L}_p$ spaces, then $X \otimes Y$ is a separable infinite-dimensional $\mathcal{L}_p$ space. More work is required ;o show that for $m \neq n$, $X_p^{\otimes m} \not\sim X_p^{\otimes n}$.

### The Tensor Product Construction

We begin with some preliminary definitions and lemmas. For each $k \in \mathbb{N}$, let $^{k} = [0,1]^k$. Let $m, n \in \mathbb{N}$.

DEFINITION. *Let $1 \leq p < \infty$ and let $X$ and $Y$ be closed subspaces of $L^p(I^m)$ and $^p(I^n)$, respectively. Define the tensor product $X \otimes Y$ of $X$ and $Y$ by*

$$X \otimes Y = [x(s)y(t) : x \in X, y \in Y, s \in I^m, t \in I^n]_{L^p(I^{m+n})}.$$

*)enote the element $x(s)y(t)$ by $x \otimes y$.*

Let $X$ and $Y$ be as above, and let $Z$ be a closed subspace of $L^p(I^k)$ for some $k \in \mathbb{N}$. Then $X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$. Thus the expressions $X \otimes Y \otimes Z$ and $\bigotimes_{i=1}^N X$ are unambiguous. The tensor power $\bigotimes_{i=1}^N X$ will also be denoted $X^{\otimes N}$.

The following lemma will be used in the proof of the fact that the tensor product of complemented subspaces of $L^p$ is a complemented subspace of $L^p\left(I^2\right)$.

**Lemma 3.1.** *Let $1 \leq p < \infty$. Then $L^p(I^m) \otimes L^p(I^n) = L^p(I^{m+n})$.*

**Proof.** Note that $L^p(I^m) \otimes L^p(I^n)$ is a closed subspace of $L^p(I^{m+n})$. Thus it will suffice to show that $L^p(I^m) \otimes L^p(I^n)$ is dense in $L^p(I^{m+n})$. Let $f \in L^p(I^{m+n})$ and let $\epsilon > 0$. Choose $g \in C(I^{m+n})$ such that $\|f-g\|_{L^p(I^{m+n})} < \frac{\epsilon}{2}$. By the Stone-Weierstrass theorem, choose $h \in \operatorname{span}_{C(I^{m+n})}\{h_1(s)h_2(t) : h_1 \in C(I^m), h_2 \in C(I^n)\}$ such that $\|g-h\|_{L^p(I^{m+n})} \leq \|g-h\|_{L^\infty(I^{m+n})} < \frac{\epsilon}{2}$. Then $\|f-h\|_{L^p(I^{m+n})} < \epsilon$. □

The tensor product preserves the property of having an unconditional basis, as shown in the following lemma [**S**, Lemma 3].

**Lemma 3.2.** *Let $1 \leq p < \infty$ and let $X$ and $Y$ be as above. Suppose $\{x_i\}$ and $\{y_j\}$ are unconditional bases for $X$ and $Y$, respectively. Then $\{x_i \otimes y_j\}_{i,j\in\mathbb{N}}$ is an unconditional basis for $X \otimes Y$.*

**Proof.** Note that $[x_i \otimes y_j : i,j \in \mathbb{N}] = X \otimes Y$. Let $\{r_k\}$ be the sequence of Rademacher functions. Then by the unconditionality of $\{x_i(s)\}$ for each $t$, Fubini's theorem, and a generalization of Khintchine's inequality, for scalars $a_{i,j}$

$$\left\| \sum_i \sum_j a_{i,j}(x_i \otimes y_j) \right\|^p_{L^p(I^{m+n})} = \int\int \left| \sum_i \sum_j a_{i,j} x_i(s) y_j(t) \right|^p ds\, dt$$
$$\approx \int\int\int\int \left| \sum_i \sum_j a_{i,j} r_i(u) r_j(v) y_j(t) x_i(s) \right|^p ds\, du\, dv\, dt$$
$$= \int\int\int\int \left| \sum_i \sum_j a_{i,j} x_i(s) y_j(t) r_i(u) r_j(v) \right|^p du\, dv\, ds\, dt$$
$$\approx \int\int \left( \sum_i \sum_j |a_{i,j} x_i(s) y_j(t)|^2 \right)^{\frac{p}{2}} ds\, dt.$$

If $\sum_i \sum_j a_{i,j}(x_i \otimes y_j) = 0$, then $\int\int \left(\sum_i \sum_j |a_{i,j} x_i(s) y_j(t)|^2\right)^{\frac{p}{2}} ds\, dt = 0$ by the inequalities above, and $a_{i,j} = 0$ for all $i, j \in \mathbb{N}$. Hence $\{x_i \otimes y_j\}_{i,j \in \mathbb{N}}$ is a basis for $X \otimes Y$. The unconditionality of $\{x_i \otimes y_j\}_{i,j \in \mathbb{N}}$ is similarly clear from the inequalities above. □

Definition. *Let $1 \leq p < \infty$. Let $X$ and $X'$ be closed subspaces of $L^p(I^m)$, and let $Y$ and $Y'$ be closed subspaces of $L^p(I^n)$. Suppose $S: X \to X'$ and $T: Y \to Y'$ are bounded linear operators. Define the tensor product $S \otimes T : X \otimes Y \to X' \otimes Y'$ of $S$ and $T$ by*

$$(S \otimes T)\left(\sum_i x_i(s) y_i(t)\right) = \sum_i S(x_i)(s) T(y_i)(t)$$

*for sequences $\{x_i\}$ in $X$ and $\{y_i\}$ in $Y$ such that $\sum_i x_i(s) y_i(t) \in L^p\left(I^{m+n}\right)$.*

The tensor product of bounded linear operators is bounded and linear, as shown in the following lemma [**S**]. Moreover, the tensor product of projections is a projection, and the tensor product of isomorphisms is an isomorphism, as shown in the subsequent lemma [**S**, Lemmas 1 and 2].

**Lemma 3.3.** *Let $1 \leq p < \infty$ and let $X$, $X'$, $Y$, $Y'$, $S$, and $T$ be as above. Then $S \otimes T$ is well-defined and linear, with $\|S \otimes T\| \leq \|S\|\,\|T\|$.*

**Proof.** For $i \in \mathbb{N}$, let $x_i \in X$ and $y_i \in Y$. Then $S \otimes T$ is formally linear by an easy computation. Suppose only finitely many elements of $\{x_i\}$ and $\{y_i\}$ are nonzero. Then by Fubini's theorem,

$$\begin{aligned}
\left\|(S \otimes T)\left(\sum_i x_i(s)y_i(t)\right)\right\|^p_{L^p(I^{m+n})} &= \int\int\left|\sum_i S(x_i)(s)T(y_i)(t)\right|^p ds\,dt \\
&= \int\left\|S\left(\sum_i T(y_i)(t)x_i\right)\right\|^p_{L^p(I^m)} dt \\
&\le \|S\|^p \int\left\|\sum_i T(y_i)(t)x_i\right\|^p_{L^p(I^m)} dt \\
&= \|S\|^p \int\int\left|\sum_i T(y_i)(t)x_i(s)\right|^p ds\,dt \\
&= \|S\|^p \int\int\left|\sum_i T(y_i)(t)x_i(s)\right|^p dt\,ds \\
&= \|S\|^p \int\left\|T\left(\sum_i x_i(s)y_i\right)\right\|^p_{L^p(I^n)} ds \\
&\le \|S\|^p \|T\|^p \int\left\|\sum_i x_i(s)y_i\right\|^p_{L^p(I^n)} ds \\
&= \|S\|^p \|T\|^p \int\int\left|\sum_i x_i(s)y_i(t)\right|^p dt\,ds \\
&= \|S\|^p \|T\|^p \left\|\sum_i x_i(s)y_i(t)\right\|^p_{L^p(I^{m+n})}.
\end{aligned}$$

If $z = \sum_i x_i(s)y_i(t) = 0$, then $(S \otimes T)(z) = 0$ by the inequality above, whence $(S \otimes T)(0) = 0$ independently of the representation of 0, and $S \otimes T$ is well-defined. Moreover, $\|S \otimes T\| \le \|S\|\,\|T\|$ by the inequality above. □

**Lemma 3.4.** *Let $1 \le p < \infty$ and let $X$, $X'$, $Y$, $Y'$, $S$, and $T$ be as above.*

*(a) If $S$ and $T$ are projections, then $S \otimes T$ is a projection.*

*(b) If $S$ and $T$ are isomorphisms, then $S \otimes T$ is an isomorphism.*

**Proof.**

(a) Suppose $S$ and $T$ are projections. Then

$(S \otimes T)^2 = (S \otimes T)(S \otimes T) = S^2 \otimes T^2 = S \otimes T$. Hence $S \otimes T$ is a projection.

(b) Suppose $S$ and $T$ are isomorphisms. Then $S \otimes T$ and $S^{-1} \otimes T^{-1}$ are formal inverses, and $\|S^{-1} \otimes T^{-1}\| \leq \|S^{-1}\| \, \|T^{-1}\|$ by Lemma 3.3. Hence $S^{-1} \otimes T^{-1}$ is bounded and $S \otimes T$ is an isomorphism. $\square$

REMARK. Let $1 \leq p < \infty$. Suppose $X \hookrightarrow L^p(I^m)$ and $Y \hookrightarrow L^p(I^n)$. By part (b) above, $X \otimes Y$ is well-defined up to isomorphism if we identify $X \otimes Y$ with $X' \otimes Y'$ for closed subspaces $X'$ and $Y'$ of $L^p(I^m)$ and $L^p(I^n)$ isomorphic to $X$ and $Y$, respectively.

The tensor product of complemented subspaces of $L^p$ is complemented, and the tensor product of $\mathcal{L}_p$ spaces is an $\mathcal{L}_p$ space, as shown in the following proposition [**S**, Lemma 1].

**Proposition 3.5.** *Let $1 < p < \infty$ where $p \neq 2$. Suppose $X$ and $Y$ are separable infinite-dimensional $\mathcal{L}_p$ spaces. Then $X \otimes Y$ is a separable infinite-dimensional $\mathcal{L}_p$ space.*

**Proof.** It is clear that $X \otimes Y$ is separable and infinite-dimensional. Let $X'$ and $Y'$ be complemented subspaces of $L^p$ isomorphic to $X$ and $Y$, respectively. Then there are projections $P_{X'}\colon L^p \to X'$ and $P_{Y'}\colon L^p \to Y'$. By part (a) of Lemma 3.4, $P_{X'} \otimes P_{Y'}\colon L^p \otimes L^p \to X' \otimes Y'$ is a projection as well, so $X' \otimes Y'$ is a complemented subspace of $L^p \otimes L^p$, which by Lemma 3.1 is equal to $L^p(I^2)$. Hence $X \otimes Y \sim X' \otimes Y' \overset{c}{\hookrightarrow} L^p \otimes L^p = L^p(I^2) \sim L^p$.

It remains to show that $X \otimes Y \not\sim \ell^2$. By [**L-P**, Proposition 7.3], $\ell^p \overset{c}{\hookrightarrow} Z$ for every infinite-dimensional $\mathcal{L}_p$ space $Z$. Now $\ell^p \overset{c}{\hookrightarrow} X$ and $[y_0] \overset{c}{\hookrightarrow} Y$ for $y_0 \in Y \setminus \{0\}$, whence $\ell^p \sim \ell^p \otimes [y_0] \overset{c}{\hookrightarrow} X \otimes Y$. It follows that $X \otimes Y \not\sim \ell^2$. $\square$

Of course it follows that $X_p^{\otimes n}$ is an $\mathcal{L}_p$ space for $1 < p < \infty$ with $p \neq 2$.

### The Isomorphic Distinctness of $X_p^{\otimes m}$ and $X_p^{\otimes n}$

We now present results leading to the conclusion that the various tensor powers of $X_p$ are isomorphically distinct. The main result is Theorem 3.10 below.

First we state some facts about stable random variables.

Let $1 \leq T \leq 2$. Then there is a distribution $\mu$ such that $\int_{\mathbb{R}} e^{i\alpha x}\, d\mu(\alpha) = e^{-|x|^T}$ and a random variable $f\colon [0,1] \to \mathbb{R}$ having distribution $\mu$. Such a random variable $f$ is said to be $T$-stable [**W**, III.A. 13 and 14].

If $f$ is a $T$-stable random variable, then $f \in L^t$ for each $1 \leq t < T \leq 2$. Let $\{f_n\}$ be a sequence of independent $T$-stable random variables. Then for each $1 \leq t < T \leq 2$, $[f_n]_{L^t}$ is isometric to $\ell^T$ [**W**, III.A. 15 and 16].

Let $1 \leq t < T \leq 2$, and let $\{f_n\}$ be a sequence of independent identically distributed $T$-stable random variables normalized in $L^t$. Then the sequence $\{f_n\}$ in $L^t$ is isometrically equivalent to the standard basis of $\ell^t$, and equivalent to the standard basis of $\ell^{t'}$ for all $1 \leq t' < T \leq 2$ [**RII**, Corollary 4.2].

The following lemma is [**S**, Proposition 1].

**Lemma 3.6.** *Let $1 \leq q < r < s \leq 2$. Let $X$ and $Y$ be closed subspaces of $L^q$ isomorphic to $\ell^r$ and $\ell^s$, respectively. Then $\ell^r \otimes \ell^s \sim X \otimes Y \sim \left(\sum^{\oplus} \ell^s\right)_{\ell^r}$ via equivalence of their standard bases.*

**Proof.** Choose a sequence $\{x_i\}$ in $X$ of independent identically distributed $r$-stable random variables normalized in $L^q$, and a sequence $\{y_j\}$ in $Y$ of independent identically distributed $s$-stable random variables normalized in $L^r$. Then $X \sim \ell^r \sim [x_i]_{L^q}$ and $Y \sim \ell^s \sim [y_j]_{L^q}$.

For scalars $a_{i,j}$, by the $r$-stability and $q$-normalization of $\{x_i\}$ with $q < r$, we have

$$\left\|\sum_i\sum_j a_{i,j}\left(x_i\otimes y_j\right)\right\|_{L^q(I^2)}^q = \int\int\left|\sum_i\sum_j a_{i,j}x_i(u)y_j(v)\right|^q du\,dv$$

$$\approx \int\int\left|\sum_i\left(\sum_j a_{i,j}y_j(v)\right)x_i(u)\right|^q du\,dv$$

$$= \int\left(\sum_i\left|\sum_j a_{i,j}y_j(v)\right|^r\right)^{\frac{q}{r}} dv.$$

Hence by the concavity of $(\ )^{\frac{q}{r}}$, and the $s$-stability and $r$-normalization of $\{y_j\}$ with $r < s$, we have

$$\left\|\sum_i\sum_j a_{i,j}\left(x_i\otimes y_j\right)\right\|_{L^q(I^2)}^q \approx \int\left(\sum_i\left|\sum_j a_{i,j}y_j(v)\right|^r\right)^{\frac{q}{r}} dv$$

$$\leq \left(\int\sum_i\left|\sum_j a_{i,j}y_j(v)\right|^r dv\right)^{\frac{q}{r}}$$

$$= \left(\sum_i\int\left|\sum_j a_{i,j}y_j(v)\right|^r dv\right)^{\frac{q}{r}}$$

$$= \left(\sum_i\left(\sum_j |a_{i,j}|^s\right)^{\frac{r}{s}}\right)^{\frac{q}{r}}.$$

Moreover, by the triangle inequality and the $s$-stability of $\{y_j\}$ with $q < s$, we have

$$\left\|\sum_i\sum_j a_{i,j}\left(x_i\otimes y_j\right)\right\|_{L^q(I^2)}^q \approx \int\left(\sum_i\left|\sum_j a_{i,j}y_j(v)\right|^r\right)^{\frac{q}{r}} dv$$

$$= \int\left\|\left\{\left|\sum_j a_{i,j}y_j(v)\right|^q\right\}_{i=1}^{\infty}\right\|_{\ell^{\frac{r}{q}}} dv$$

$$\geq \left\|\left\{\int\left|\sum_j a_{i,j}y_j(v)\right|^q dv\right\}_{i=1}^{\infty}\right\|_{\ell^{\frac{r}{q}}}$$

$$\approx \left\|\left\{\left(\sum_j |a_{i,j}|^s\right)^{\frac{q}{s}}\right\}_{i=1}^{\infty}\right\|_{\ell^{\frac{r}{q}}}$$

$$= \left(\sum_i\left(\sum_j |a_{i,j}|^s\right)^{\frac{r}{s}}\right)^{\frac{q}{r}}.$$

Hence $\{x_i\otimes y_j\}$ is equivalent to the standard basis of $\left(\sum^{\oplus}\ell^s\right)_{\ell^r}$, and

$\ell^r\otimes\ell^s \sim X\otimes Y \sim \left(\sum^{\oplus}\ell^s\right)_{\ell^r}$. □

Let $1 \leq p < \infty$ and let $\{x_i\}$ be a sequence in $L^p$. Then $\{x_i\}$ is said to be uniformly $p$-integrable if for each $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $\int_{\{t:|x_i(t)|>N\}} |x_i(t)|^p \, dt < \epsilon^p$ for each $i \in \mathbb{N}$.

A basis $\{x_i\}$ for a space $X$ is said to be symmetric if for all permutations $\tau$ of scalars $a_i$, $\sum_i \tau(a_i) x_i$ converges if and only if $\sum_i a_i x_i$ converges.

The following lemma is [**S**, Proposition 2].

**Lemma 3.7.** *Let $1 < q < r < s \leq 2$. Then there is no sequence $\{x_{i,j}\}_{i,j\in\mathbb{N}}$ of independent random variables in $L^q$ equivalent to the standard basis of $\left(\sum^{\oplus} \ell^s\right)_{\ell^r}$.*

**Proof.** Suppose $\{x_{i,j}\}_{i,j\in\mathbb{N}}$ is a sequence of independent random variables in $L^q$ equivalent to the standard basis of $\left(\sum^{\oplus} \ell^s\right)_{\ell^r}$, where for each $j \in \mathbb{N}$, $\{x_{i,j}\}_{i\in\mathbb{N}}$ is equivalent to the standard basis of $\ell^s$. Now $\ell^q \not\hookrightarrow \left(\sum^{\oplus} \ell^s\right)_{\ell^r}$. Hence $\{x_{i,j}\}_{i,j\in\mathbb{N}}$ is uniformly $q$-integrable [**J-O**, third lemma].

Let $\epsilon > 0$, and choose $N \in \mathbb{N}$ such that $\int_{\{|x_{i,j}|>N\}} |x_{i,j}|^q \, d\mu < \epsilon^q$ for all $i, j \in \mathbb{N}$. Let $\delta = \frac{1}{D}$ for some $D \in \mathbb{N}$, and let $\{I_k\}_{k=1}^K$ be a partition of the interval $[-N, N]$ into $K = D(2N+1)$ intervals of equal length $|I_k| = \frac{2N}{K} = \frac{2N}{D(2N+1)} < \delta$.

Let $\rho = \delta^{2q}$. For each $j \in \mathbb{N}$, choose a subsequence $\{x_{i,j}\}_{i\in M_j}$ of $\{x_{i,j}\}_{i\in\mathbb{N}}$ such that for each $i, i' \in M_j$ and $k \in \{1, \ldots, K\}$,

$$|\mu(\{x_{i,j} \in I_k\}) - \mu(\{x_{i',j} \in I_k\})| < \frac{\rho}{3}.$$

Then $\{x_{i,j}\}_{i\in M_j, j\in\mathbb{N}}$ is still equivalent to the standard basis of $\left(\sum^{\oplus} \ell^s\right)_{\ell^r}$. Without loss of generality, suppose $1 \in M_j$ for each $j \in \mathbb{N}$.

Choose a subsequence $\{x_{1,j}\}_{j\in L}$ of $\{x_{1,j}\}_{j\in\mathbb{N}}$ such that for each $j, j' \in L$ and $k \in \{1, \ldots, K\}$,

$$|\mu(\{x_{1,j} \in I_k\}) - \mu(\{x_{1,j'} \in I_k\})| < \frac{\rho}{3}.$$

Then $\{x_{i,j}\}_{i\in M_j, j\in L}$ is still equivalent to the standard basis of $\left(\sum^{\oplus} \ell^s\right)_{\ell^r}$. Without loss of generality, suppose $1 \in L$. Note that for each $j, j' \in L$, $i \in M_j$, $i' \in M_{j'}$, and $k \in \{1, \ldots, K\}$,

$$|\mu(\{x_{i,j} \in I_k\}) - \mu(\{x_{i',j'} \in I_k\})| < \rho.$$

For each $k \in \{1, \ldots, K\}$, let $c_k$ be the center of $I_k$. Let $\{z_{i,j}\}_{i\in M_j, j\in L}$ be a sequence of $\{c_1, \ldots, c_K\}$-valued independent random variables in $L^q$ such that for each $j \in L$, $i \in M_j$, and $k \in \{1, \ldots, K\}$,

$$\mu(\{z_{i,j} = c_k\}) = \mu(\{x_{1,1} \in I_k\}),$$

and such that $\{z_{i,j} = c_k\}$ is chosen either as a subset of $\{x_{i,j} \in I_k\}$ or as a superset of $\{x_{i,j} \in I_k\}$. Then $\{z_{i,j}\}_{i\in M_j, j\in L}$ is identically distributed, whence $\{z_{i,j}\}_{i\in M_j, j\in L}$ is a symmetric basis, and for each $j \in L$, $i \in M_j$, and $k \in \{1, \ldots, K\}$,

$$|\mu(\{x_{i,j} \in I_k\}) - \mu(\{z_{i,j} = c_k\})| < \rho.$$

Hence for each $j \in L$, $i \in M_j$, and $k \in \{1, \ldots, K\}$,

$$\mu(\{x_{i,j} \in I_k\} \setminus \{z_{i,j} = c_k\}) < \rho.$$

Now for each $j \in L$ and $i \in M_j$,

$$\begin{aligned}
\|z_{i,j} - x_{i,j}\|_q \le & \left(\int_{\{|x_{i,j}|>N\}} |z_{i,j} - x_{i,j}|^q\right)^{\frac{1}{q}} \\
& + \left(\int_{\bigcup_{k=1}^{K}(\{x_{i,j}\in I_k\}\cap\{z_{i,j}=c_k\})} |z_{i,j} - x_{i,j}|^q\right)^{\frac{1}{q}} \\
& + \sum_{k=1}^{K} \left(\int_{\{x_{i,j}\in I_k\}\setminus\{z_{i,j}=c_k\}} |z_{i,j} - x_{i,j}|^q\right)^{\frac{1}{q}} \\
& < 2\epsilon + \frac{\delta}{2} + K\rho^{\frac{1}{q}}(2N+1),
\end{aligned}$$

where $K\rho^{\frac{1}{q}}(2N+1) = D(2N+1)\delta^2(2N+1) = \delta(2N+1)^2$.

Fix $J \in \mathbb{N}$ and assume $\{1, \ldots, J\}$ is a subset of $L$ and each $M_j$. Then

$$\begin{aligned}
&\left\| \sum_{i=1}^{J} \sum_{j=1}^{J} a_{i,j} x_{i,j} - \sum_{i=1}^{J} \sum_{j=1}^{J} a_{i,j} z_{i,j} \right\|_q \\
&\leq \sum_{i=1}^{J} \sum_{j=1}^{J} |a_{i,j}| \left\| x_{i,j} - z_{i,j} \right\|_q \\
&\leq \sum_{i=1}^{J} \sum_{j=1}^{J} |a_{i,j}| \max_{i,j \in \{1,\ldots,J\}} \left\| x_{i,j} - z_{i,j} \right\|_q \\
&\leq \left( \sum_{i=1}^{J} \left( \sum_{j=1}^{J} |a_{i,j}|^s \right)^{\frac{r}{s}} \right)^{\frac{1}{r}} |J|^{\left(1-\frac{1}{r}\right)+\left(1-\frac{1}{s}\right)} \left( 2\epsilon + \frac{\delta}{2} + \delta(2N+1)^2 \right).
\end{aligned}$$

For any $J \in \mathbb{N}$ and $\gamma > 0$, we can choose $\epsilon > 0$ and $\delta > 0$ such that $|J|^{\left(1-\frac{1}{r}\right)+\left(1-\frac{1}{s}\right)} \left( 2\epsilon + \frac{\delta}{2} + \delta(2N+1)^2 \right) < \gamma$. Hence we can find a symmetric sequence equivalent to the standard basis of $\left( \sum^{\oplus} \ell^s \right)_{\ell^r}$, contrary to fact. $\square$

A basis $\{e_i\}$ for a Banach space $E$ is said to be reproducible if for each Banach space $X$ with basis $\{x_i\}$ such that $E \hookrightarrow X$, there is a block basic sequence $\{z_i\}$ with respect to $\{x_i\}$ equivalent to $\{e_i\}$. For $r, s \in [1, \infty)$, the standard basis of $\left( \sum^{\oplus} \ell^s \right)_{\ell^r}$ is reproducible [**L-P 2**, Section 4].

The following proposition has been extracted from the proof of [**S**, Theorem]. The subsequent corollary is essentially [**S**, Remark 1].

**Proposition 3.8.** *Let $1 < q < 2$ and let $n \in \mathbb{N}$. Then $\bigotimes_{i=1}^{2n} \ell^{r_i} \not\hookrightarrow X_q^{\otimes n}$ for $q < r_1 < r_2 < \cdots < r_{2n} \leq 2$.*

**Proof.** Suppose $n = 1$. Let $q < r < s \leq 2$ and suppose $\ell^r \otimes \ell^s \hookrightarrow X_q$. Then by Lemma 3.6, $\left( \sum^{\oplus} \ell^s \right)_{\ell^r} \hookrightarrow X_q$. Now $X_q \sim [x_{i,j}]_{L^q}$ for some sequence $\{x_{i,j}\}$ of independent random variables in $L^q$. By the reproducibility of the standard basis $\{e_{i,j}\}$ of $\left( \sum^{\oplus} \ell^s \right)_{\ell^r}$, there is a block basic sequence $\{z_{i,j}\}$ with respect to $\{x_{i,j}\}$ equivalent to $\{e_{i,j}\}$. However, $\{z_{i,j}\}$ is a sequence of independent random variables in $L^q$ equivalent to $\{e_{i,j}\}$, contrary to Lemma 3.7. Hence the result holds for $n = 1$.

Suppose the result is true for $n = k-1$, but there are $q < r_1 < r_2 < \cdots < r_{2k} \leq 2$ such that $\bigotimes_{i=1}^{2k} \ell^{r_i} \hookrightarrow X_q^{\otimes k}$ via a mapping $\tau$.

Let $\{e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_{2k}}\}_{j_1, j_2, \ldots, j_{2k} \in \mathbb{N}}$ be the standard basis of $\bigotimes_{i=1}^{2k} \ell^{r_i}$, and let $y_{j_1, j_2, \ldots, j_{2k}} = \tau(e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_{2k}})$ for $j_1, j_2, \ldots, j_{2k} \in \mathbb{N}$.

Let $\{x_j\}$ be a basis for $X_q$. For each $m \in \mathbb{N}$, let $P_m$ be the obvious projection of $X_q^{\otimes k}$ onto $[x_{j_1} \otimes x_{j_2} \otimes \cdots \otimes x_{j_k} : \max\{j_1, j_2, \ldots, j_k\} \leq m]$, and let $Q_m$ be the obvious projection of $X_q^{\otimes k}$ onto $[x_{j_1} \otimes x_{j_2} \otimes \cdots \otimes x_{j_k} : \min\{j_1, j_2, \ldots, j_k\} > m]$.

Recalling that $X_q \sim X_q \oplus X_q$, for each $s \in \mathbb{N}$

$$X_q^{\otimes s} \sim (X_q \oplus X_q) \otimes X_q^{\otimes(s-1)} \sim X_q^{\otimes s} \oplus X_q^{\otimes s}.$$

Hence for each $s, t \in \mathbb{N}$,

$$\sum_{i=1}^{t}{}^{\oplus} X_q^{\otimes s} \sim X_q^{\otimes s}.$$

Note that for each $m \in \mathbb{N}$, $(I - Q_m)(X_q^{\otimes k}) \sim \sum_{i=1}^{t}{}^{\oplus} X_q^{\otimes(k-1)}$ for some $t \in \mathbb{N}$, whence $(I - Q_m)(X_q^{\otimes k}) \sim X_q^{\otimes(k-1)}$.

Let $\{e_{j_1} \otimes e_{j_2}\}_{j_1, j_2 \in \mathbb{N}}$ be the standard basis of $\ell^{r_1} \otimes \ell^{r_2}$ with order determined by a bijection $\phi: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$.

For each $j \in \mathbb{N}$, let $Y_j = \left[y_{\phi(j), j_3, j_4, \ldots, j_{2k}} : j_3, j_4, \ldots, j_{2k} \in \mathbb{N}\right]$, which is isomorphic to $\bigotimes_{i=1}^{2(k-1)} \ell^{r_{i+2}}$. Then by the inductive hypothesis, for each $j, m \in \mathbb{N}$

$$Y_j \sim \bigotimes_{i=1}^{2(k-1)} \ell^{r_{i+2}} \not\hookrightarrow X_q^{\otimes(k-1)} \sim (I - Q_m)(X_q^{\otimes k}),$$

whence $(I - Q_m)|_{Y_j}$ is not an isomorphism.

Let $\{\epsilon_j\}$ be a sequence of positive scalars. Let $m_0 = 0$ and $Q_{m_0} = I$. Choose $z_1 \in Y_1$ with $\|z_1\| = 1$ and $m_1 \in \mathbb{N}$ such that $\|(I - Q_{m_0})(z_1)\| < \frac{\epsilon_1}{2}$ and $\|(I - P_{m_1})(z_1)\| < \frac{\epsilon_1}{2}$. Choose $z_2 \in Y_2$ with $\|z_2\| = 1$ and a positive integer $m_2 > m_1$ such that $\|(I - Q_{m_1})(z_2)\| < \frac{\epsilon_2}{2}$ and $\|(I - P_{m_2})(z_2)\| < \frac{\epsilon_2}{2}$. Continuing as above, we may inductively define a sequence $\{z_j\}$ and an increasing sequence $\{m_j\}$ of positive integers such that for each $j \in \mathbb{N}$, $z_j \in Y_j$ with $\|z_j\| = 1$, $\left\|(I - Q_{m_{j-1}})(z_j)\right\| < \frac{\epsilon_j}{2}$, and $\left\|(I - P_{m_j})(z_j)\right\| < \frac{\epsilon_j}{2}$. Hence for each $j \in \mathbb{N}$, $\left\|(I - Q_{m_{j-1}} \circ P_{m_j})(z_j)\right\| < \epsilon_j \left\|P_{m_j}\right\|$. Thus for an appropriate choice of $\{\epsilon_j\}$, $\{z_j\}$ is equivalent to $\left\{(Q_{m_{j-1}} \circ P_{m_j})(z_j)\right\}$. However, $\{z_j\}$ is equivalent to the standard basis $\{e_{j_1} \otimes e_{j_2}\}_{j_1, j_2 \in \mathbb{N}}$ of $\ell^{r_1} \otimes \ell^{r_2}$, and $\left\{(Q_{m_{j-1}} \circ P_{m_j})(z_j)\right\}$ is a sequence of independent random variables. Hence there is a sequence of independent random variables equivalent to the standard basis of $\ell^{r_1} \otimes \ell^{r_2}$, contrary to Lemma 3.7. □

**Corollary 3.9.** *Let $1 < q < 2$. Then for each $n \in \mathbb{N}$, $X_q^{\otimes(n+1)} \not\hookrightarrow X_q^{\otimes n}$.*

**Proof.** Let $n \in \mathbb{N}$ and let $q < r_1 < r_2 < \cdots < r_{2n} \leq 2$. Then for each $1 \leq i \leq 2n$, $\ell^{r_i} \hookrightarrow X_q$ by Lemma 2.35. Hence $\bigotimes_{i=1}^{2n} \ell^{r_i} \hookrightarrow X_q^{\otimes 2n}$. However, $\bigotimes_{i=1}^{2n} \ell^{r_i} \not\hookrightarrow X_q^{\otimes n}$ by Proposition 3.8. It follows that $X_q^{\otimes 2n} \not\hookrightarrow X_q^{\otimes n}$.

Now suppose that $X_q^{\otimes(n+1)} \hookrightarrow X_q^{\otimes n}$. Then there is a chain

$$\cdots \hookrightarrow X_q^{\otimes(n+2)} \hookrightarrow X_q^{\otimes(n+1)} \hookrightarrow X_q^{\otimes n}.$$

In particular, $X_q^{\otimes 2n} \hookrightarrow X_q^{\otimes n}$, contrary to fact. It follows that $X_q^{\otimes(n+1)} \not\hookrightarrow X_q^{\otimes n}$. □

Note that $X \overset{c}{\hookrightarrow} X \otimes Y$ [where $1 \leq p < \infty$, $X$ and $Y$ are isomorphic to closed subspaces of $L^p$, and $\dim Y > 0$], since $X \sim X \otimes [y_0] \overset{c}{\hookrightarrow} X \otimes Y$ for $y_0 \in Y \setminus \{0\}$. Hence for $n \in \mathbb{N}$ and $1 < p < \infty$ with $p \neq 2$, $X_p^{\otimes n} \overset{c}{\hookrightarrow} X_p^{\otimes(n+1)}$.

For $1 < q < 2$, we have

$$X_q \to X_q^{\otimes 2} \to X_q^{\otimes 3} \to \cdots \to L^q. \tag{3.1}$$

Note that $(X \otimes Y)^* \sim X^* \otimes Y^*$ [where $1 < p < \infty$, and $X$ and $Y$ are isomorphic to closed subspaces of $L^p$]. Let $2 < p < \infty$ with conjugate index $q$. Then for each $k \in \mathbb{N}$,

$\left(X_p^{\otimes k}\right)^* \sim \left(X_p^*\right)^{\otimes k} \sim X_q^{\otimes k}$. Let $n \in \mathbb{N}$. Then the fact that $X_p^{\otimes(n+1)} \stackrel{c}{\not\hookrightarrow} X_p^{\otimes n}$ follows from $\left(X_p^{\otimes(n+1)}\right)^* \sim X_q^{\otimes(n+1)} \stackrel{c}{\not\hookrightarrow} X_q^{\otimes n} \sim \left(X_p^{\otimes n}\right)^*$.

For $1 < p < \infty$ with $p \neq 2$, we have

$$X_p \stackrel{c}{\to} X_p^{\otimes 2} \stackrel{c}{\to} X_p^{\otimes 3} \stackrel{c}{\to} \cdots \stackrel{c}{\to} L^p. \tag{3.2}$$

Finally we have the main result [**S**, Theorem].

**Theorem 3.10.** *Let $1 < p < \infty$ where $p \neq 2$. Then $\{X_p^{\otimes n}\}_{n=1}^{\infty}$ is a sequence of mutually nonisomorphic $\mathcal{L}_p$ spaces.*

**Proof.** Each $X_p^{\otimes n}$ is an $\mathcal{L}_p$ space by Proposition 3.5. For $m \neq n$, the fact that $X_p^{\otimes m} \not\sim X_p^{\otimes n}$ follows from Corollary 3.9 and the discussion leading to diagrams (3.1) and (3.2). In particular, if $X_p^{\otimes m} \sim X_p^{\otimes n}$ for $m < n$, then $X_p^{\otimes(m+1)} \stackrel{c}{\hookrightarrow} X_p^{\otimes n} \stackrel{c}{\hookrightarrow} X_p^{\otimes m}$, contrary to fact. $\square$

### The Sequence Space Realization of $X_p^{\otimes n}$

For $n \in \mathbb{N}$, $X_p^{\otimes n}$ has a realization as a sequence space, as follows from Proposition 3.13 below. This proposition is essentially contained in [**S**, Section 4], although the presentation via Lemmas 3.11 and 3.12 owes more to Dale Alspach.

**Lemma 3.11.** *Let $2 < p < \infty$ and $k \in \mathbb{N}$. Let $\{x_i\}$ be a sequence of normalized independent mean zero random variables in $L^p$. Let $\{y_j\}$ be an unconditional basic sequence in $L^p(I^k)$ with closed linear span $Y = [y_j]_{L^p(I^k)}$. Let $\{r_i\}$ be the sequence of Rademacher functions. Then for scalars $a_{i,j}$*

$$\left\|\sum_i \sum_j a_{i,j}(x_i \otimes y_j)\right\|_{L^p(I^{k+1})}$$
$$\approx \max\left\{\left(\sum_i \left\|\sum_j a_{i,j} y_j\right\|_Y^p\right)^{\frac{1}{p}}, \left(\int \left\|\sum_j \left(\sum_i a_{i,j} \|x_i\|_2 r_i(u)\right) y_j\right\|_Y^p du\right)^{\frac{1}{p}}\right\}.$$

**Proof.** For each $i \in \mathbb{N}$, let $f_i(t) = \sum_j a_{i,j} y_j(t)$. Then for each $t \in [0,1]$, $\{x_i(s)f_i(t)\}_{i=1}^{\infty}$ is a sequence of independent mean zero random variables in $L^p$. Thus by Theorem 2.2 [Rosenthal's inequality], for each $t \in [0,1]$

$$\left(\int \left|\sum_i x_i(s) f_i(t)\right|^p ds\right)^{\frac{1}{p}}$$
$$\approx \max\left\{\left(\sum_i \int |x_i(s)f_i(t)|^p\, ds\right)^{\frac{1}{p}}, \left(\sum_i \int |x_i(s)f_i(t)|^2\, ds\right)^{\frac{1}{2}}\right\}.$$

Hence

$$\left(\int\int \left|\sum_i x_i(s) f_i(t)\right|^p ds\, dt\right)^{\frac{1}{p}}$$
$$\approx \max\left\{\left(\sum_i \int\int |x_i(s)f_i(t)|^p\, ds\, dt\right)^{\frac{1}{p}}, \left(\int\left(\sum_i \int |x_i(s)f_i(t)|^2\, ds\right)^{\frac{p}{2}} dt\right)^{\frac{1}{p}}\right\}.$$

Now

$$\int\int |x_i(s)f_i(t)|^p\, ds\, dt = \|x_i\|_p^p \, \|f_i\|_{L^p(I^k)}^p = \|f_i\|_{L^p(I^k)}^p = \left\|\sum_j a_{i,j} y_j\right\|_Y^p$$

and

$$\int\left(\sum_i \int |x_i(s)f_i(t)|^2\, ds\right)^{\frac{p}{2}} dt = \int\left(\sum_i \|x_i\|_2^2\, |f_i(t)|^2\right)^{\frac{p}{2}} dt$$
$$\approx \int\int \left|\sum_i \|x_i\|_2\, f_i(t) r_i(u)\right|^p du\, dt$$
$$= \int\int \left|\sum_i \|x_i\|_2 \sum_j a_{i,j} y_j(t) r_i(u)\right|^p dt\, du$$
$$\approx \int \left\|\sum_j \left(\sum_i a_{i,j}\, \|x_i\|_2\, r_i(u)\right) y_j\right\|_Y^p du.$$

Hence

$$
\begin{aligned}
&\left\|\sum_i \sum_j a_{i,j}(x_i \otimes y_j)\right\|_{L^p(I^{k+1})} \\
&= \left(\int\int \left|\sum_i \sum_j a_{i,j} x_i(s) y_j(t)\right|^p ds\, dt\right)^{\frac{1}{p}} \\
&\approx \left(\int\int \left|\sum_i x_i(s) \sum_j a_{i,j} y_j(t)\right|^p ds\, dt\right)^{\frac{1}{p}} \\
&= \left(\int\int \left|\sum_i x_i(s) f_i(t)\right|^p ds\, dt\right)^{\frac{1}{p}} \\
&\approx \max\left\{\left(\sum_i \int\int |x_i(s) f_i(t)|^p\, ds\, dt\right)^{\frac{1}{p}}, \left(\int \left(\sum_i \int |x_i(s) f_i(t)|^2\, ds\right)^{\frac{p}{2}} dt\right)^{\frac{1}{p}}\right\} \\
&\approx \max\left\{\left(\sum_i \left\|\sum_j a_{i,j} y_j\right\|_Y^p\right)^{\frac{1}{p}}, \left(\int \left\|\sum_j \left(\sum_i a_{i,j} \|x_i\|_2\, r_i(u)\right) y_j\right\|_Y^p du\right)^{\frac{1}{p}}\right\}.
\end{aligned}
$$

□

Let $\{r_j\}$ be the sequence of Rademacher functions. Kahane's inequality [**W**, Theorem III.A.18] states that for each $1 \leq p < \infty$, there is a constant $C_p$ such that for each Banach space $X$ and for each finite sequence $\{x_j\}$ in $X$, $\left(\int \left\|\sum_j r_j(u) x_j\right\|_X^p du\right)^{\frac{1}{p}} \overset{C_p}{\underset{1}{\approx}} \int \left\|\sum_j r_j(u) x_j\right\|_X du$.

**Lemma 3.12.** *Let $1 \leq p < \infty$ and let $\{r_j\}$ be the sequence of Rademacher functions. Then for scalars $a_{i,j}$*

$$
\int \left(\sum_i \left|\sum_j a_{i,j} r_j(u)\right|^2\right)^{\frac{p}{2}} du \approx \left(\sum_i \sum_j |a_{i,j}|^2\right)^{\frac{p}{2}}.
$$

**Proof.** Let $\{e_i\}$ be the standard basis of $\ell^2$. Then by Kahane's inequality,

$$\begin{aligned}
\int \left( \sum_i \left| \sum_j a_{i,j} r_j(u) \right|^2 \right)^{\frac{p}{2}} du &= \int \left\| \sum_i \left( \sum_j a_{i,j} r_j(u) \right) e_i \right\|_{\ell^2}^p du \\
&= \int \left\| \sum_j r_j(u) \left( \sum_i a_{i,j} e_i \right) \right\|_{\ell^2}^p du \\
&\overset{C_p^p}{\underset{1}{\approx}} \left( \int \left\| \sum_j r_j(u) \left( \sum_i a_{i,j} e_i \right) \right\|_{\ell^2} du \right)^p \\
&\overset{1}{\underset{C_2^p}{\approx}} \left( \int \left\| \sum_j r_j(u) \left( \sum_i a_{i,j} e_i \right) \right\|_{\ell^2}^2 du \right)^{\frac{p}{2}} \\
&= \left( \int \left\| \sum_i \left( \sum_j a_{i,j} r_j(u) \right) e_i \right\|_{\ell^2}^2 du \right)^{\frac{p}{2}} \\
&= \left( \int \sum_i \left| \sum_j a_{i,j} r_j(u) \right|^2 du \right)^{\frac{p}{2}} \\
&= \left( \sum_i \int \left| \sum_j a_{i,j} r_j(u) \right|^2 du \right)^{\frac{p}{2}} \\
&= \left( \sum_i \sum_j |a_{i,j}|^2 \right)^{\frac{p}{2}}.
\end{aligned}$$

□

**Proposition 3.13.** *Let $2 < p < \infty$ and $n \in \mathbb{N}$. Let $\{x_i\}$ be a sequence of normalized independent mean zero random variables in $L^p$. For each $i \in \mathbb{N}$, let $w_i = \|x_i\|_2$. Then for scalars $a_{i_1,\dots,i_n}$*

$$\begin{aligned}
&\left\| \sum_{i_1,\dots,i_n} a_{i_1,\dots,i_n} (x_{i_1} \otimes \cdots \otimes x_{i_n}) \right\|_{L^p(I^n)} \\
&\approx \max_{S_n} \left\{ \left( \sum_{i_k:\, k \in S_n} \left( \sum_{i_\ell:\, \ell \in S_n^c} |a_{i_1,\dots,i_n}|^2 \prod_{\ell \in S_n^c} w_{i_\ell}^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \right\}
\end{aligned}$$

*where the* max *is taken over all subsets $S_n$ of $\{1, \dots, n\}$, and $S_n^c = \{1, \dots, n\} \setminus S_n$.*

**Proof.** For $n = 1$ [with $i_1 = i$], the statement is

$$\left\|\sum_i a_i x_i\right\|_p \approx \max\left\{\left(\left(\sum_i |a_i|^2 w_i^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}, \left(\sum_i \left(|a_i|^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}\right\}$$

$$= \max\left\{\left(\sum_i |a_i|^2 w_i^2\right)^{\frac{1}{2}}, \left(\sum_i |a_i|^p\right)^{\frac{1}{p}}\right\},$$

which is immediate from Corollary 2.3 [Rosenthal's inequality].

Assume the statement is true for $n = N$. We wish to prove the statement for $n = N + 1$.

Let $\{r_i\}$ be the sequence of Rademacher functions. By Lemma 3.11,

$$\left\|\sum_{i_1,\ldots,i_N} \sum_{i_{N+1}} a_{i_1,\ldots,i_{N+1}} (x_{i_1} \otimes \cdots \otimes x_{i_N}) \otimes x_{i_{N+1}}\right\|_{L^p(I^{N+1})} \approx \max\{E_1, E_2\}$$

where

$$E_1 = \left(\sum_{i_{N+1}} \left\|\sum_{i_1,\ldots,i_N} a_{i_1,\ldots,i_{N+1}} (x_{i_1} \otimes \cdots \otimes x_{i_N})\right\|_{L^p(I^N)}^p\right)^{\frac{1}{p}}$$

and

$$E_2 = \left(\int \left\|\sum_{i_1,\ldots,i_N} \left(\sum_{i_{N+1}} a_{i_1,\ldots,i_{N+1}} \left\|x_{i_{N+1}}\right\|_2 r_{i_{N+1}}(u)\right) (x_{i_1} \otimes \cdots \otimes x_{i_N})\right\|_{L^p(I^N)}^p du\right)^{\frac{1}{p}}.$$

Let

$$A_{i_1,\ldots,i_N}(u) = \sum_{i_{N+1}} a_{i_1,\ldots,i_{N+1}} \left\|x_{i_{N+1}}\right\|_2 r_{i_{N+1}}(u)$$

and

$$B^{(S_N^c)}_{i_1,\ldots,i_{N+1}} = a_{i_1,\ldots,i_{N+1}} \left\|x_{i_{N+1}}\right\|_2 \prod_{\ell \in S_N^c} w_{i_\ell}.$$

By the inductive hypothesis, and then a rearrangement, we have

$$E_1 \approx \left(\sum_{i_{N+1}} \max_{S_N}\left\{\sum_{i_k:\, k\in S_N} \left(\sum_{i_\ell:\, \ell\in S_N^c} \left|a_{i_1,\ldots,i_{N+1}}\right|^2 \prod_{\ell\in S_N^c} w_{i_\ell}^2\right)^{\frac{p}{2}}\right\}\right)^{\frac{1}{p}}$$

$$\approx \max_{S_N}\left\{\left(\sum_{i_k:\, k\in S_N\cup\{N+1\}} \left(\sum_{i_\ell:\, \ell\in S_N^c} \left|a_{i_1,\ldots,i_{N+1}}\right|^2 \prod_{\ell\in S_N^c} w_{i_\ell}^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}\right\}$$

$$= \max_{\substack{S_{N+1}:\\ N+1\in S_{N+1}}}\left\{\left(\sum_{i_k:\, k\in S_{N+1}} \left(\sum_{i_\ell:\, \ell\in S_{N+1}^c} \left|a_{i_1,\ldots,i_{N+1}}\right|^2 \prod_{\ell\in S_{N+1}^c} w_{i_\ell}^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}\right\}.$$

By the inductive hypothesis, a rearrangement, and Lemma 3.12, we have

$$
\begin{aligned}
E_2 &= \left( \int \left\| \sum_{i_1,\ldots,i_N} A_{i_1,\ldots,i_N}(u)\,(x_{i_1} \otimes \cdots \otimes x_{i_N}) \right\|_{L^p(I^N)}^p du \right)^{\frac{1}{p}} \\
&\approx \left( \int \max_{S_N} \left\{ \sum_{i_k:\, k\in S_N} \left( \sum_{i_\ell:\, \ell\in S_N^c} |A_{i_1,\ldots,i_N}(u)|^2 \prod_{\ell\in S_N^c} w_{i_\ell}^2 \right)^{\frac{p}{2}} \right\} du \right)^{\frac{1}{p}} \\
&\approx \max_{S_N} \left\{ \left( \sum_{i_k:\, k\in S_N} \int \left( \sum_{i_\ell:\, \ell\in S_N^c} |A_{i_1,\ldots,i_N}(u)|^2 \prod_{\ell\in S_N^c} w_{i_\ell}^2 \right)^{\frac{p}{2}} du \right)^{\frac{1}{p}} \right\} \\
&= \max_{S_N} \left\{ \left( \sum_{i_k:\, k\in S_N} \int \left( \sum_{i_\ell:\, \ell\in S_N^c} \left| \sum_{i_{N+1}} B_{i_1,\ldots,i_{N+1}}^{(S_N^c)} r_{i_{N+1}}(u) \right|^2 \right)^{\frac{p}{2}} du \right)^{\frac{1}{p}} \right\} \\
&\approx \max_{S_N} \left\{ \left( \sum_{i_k:\, k\in S_N} \left( \sum_{i_\ell:\, \ell\in S_N^c\cup\{N+1\}} \left| B_{i_1,\ldots,i_{N+1}}^{(S_N^c)} \right|^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \right\} \\
&= \max_{S_N} \left\{ \left( \sum_{i_k:\, k\in S_N} \left( \sum_{i_\ell:\, \ell\in S_N^c\cup\{N+1\}} |a_{i_1,\ldots,i_{N+1}}|^2 \prod_{\ell\in S_N^c\cup\{N+1\}} w_{i_\ell}^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \right\} \\
&= \max_{\substack{S_{N+1}: \\ N+1\notin S_{N+1}}} \left\{ \left( \sum_{i_k:\, k\in S_{N+1}} \left( \sum_{i_\ell:\, \ell\in S_{N+1}^c} |a_{i_1,\ldots,i_{N+1}}|^2 \prod_{\ell\in S_{N+1}^c} w_{i_\ell}^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \right\}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\left\| \sum_{i_1,\ldots,i_{N+1}} a_{i_1,\ldots,i_{N+1}}\,(x_{i_1} \otimes \cdots \otimes x_{i_{N+1}}) \right\|_{L^p(I^{N+1})} \\
&\approx \max\{E_1, E_2\} \\
&\approx \max_{S_{N+1}} \left\{ \left( \sum_{i_k:\, k\in S_{N+1}} \left( \sum_{i_\ell:\, \ell\in S_{N+1}^c} |a_{i_1,\ldots,i_{N+1}}|^2 \prod_{\ell\in S_{N+1}^c} w_{i_\ell}^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \right\}.
\end{aligned}
$$

□

For $2 < p < \infty$ and $n \in \mathbb{N}$, Proposition 3.13 yields a representation of $X_p^{\otimes n}$ as a sequence space, taking $\{x_i\}$ to be a sequence of normalized independent mean zero random variables in $L^p$ with $w = \{w_i\} = \{\|x_i\|_2\}$ satisfying condition $(*)$ of Proposition 2.1.

In particular, for $n = 2$ and $S_2 \subset \{i, j\}$, for scalars $a_{i,j}$

$$\left\| \sum_{i,j} a_{i,j} \left( x_i \otimes y_j \right) \right\|_{L^p(I^2)} \approx \max \left\{ \mathcal{N}_{[S_2=\emptyset]}, \mathcal{N}_{[S_2=\{i\}]}, \mathcal{N}_{[S_2=\{j\}]}, \mathcal{N}_{[S_2=\{i,j\}]} \right\},$$

where

$$\begin{aligned}
\mathcal{N}_{[S_2=\emptyset]} &= \left( \left( \sum_{i,j} |a_{i,j}|^2 w_i^2 w_j^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} = \left( \sum_{i,j} |a_{i,j}|^2 w_i^2 w_j^2 \right)^{\frac{1}{2}}, \\
\mathcal{N}_{[S_2=\{i\}]} &= \left( \sum_i \left( \sum_j |a_{i,j}|^2 w_j^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}, \\
\mathcal{N}_{[S_2=\{j\}]} &= \left( \sum_j \left( \sum_i |a_{i,j}|^2 w_i^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}, \\
\mathcal{N}_{[S_2=\{i,j\}]} &= \left( \sum_{i,j} \left( |a_{i,j}|^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} = \left( \sum_{i,j} |a_{i,j}|^p \right)^{\frac{1}{p}}.
\end{aligned}$$

# CHAPTER IV

## THE INDEPENDENT SUM CONSTRUCTION OF ALSPACH

Let $2 < p < \infty$ and let $\Omega = \prod_{i=1}^{\infty}[0,1]$. Alspach [**A**] developed a general method for constructing complemented subspaces of $L^p(\Omega)$, given spaces $X_i$ of mean zero functions which are complemented in $L^p[0,1]$ in a special way. The construction produces spaces $Z_i$ of mean zero functions which are similarly complemented in $L^p(\Omega)$, such that $Z_i$ is isometric to $X_i$, each function in $Z_i$ depends only on component $i$ of $\Omega$, there is a common supporting set $S_i$ for all functions in $Z_i$, and the measure of $S_i$ approaches zero slowly as $i$ increases. The independent sum of $\{X_i\}_{i=1}^{\infty}$ is then $[Z_i : i \in \mathbb{N}]_{L^p(\Omega)}$.

The rate at which the measure of $S_i$ approaches zero is controlled by a sequence $w$, which plays a role similar to the role of $w$ in Rosenthal's space $X_{p,w}$. Indeed, Alspach's construction generalizes the construction of Rosenthal's space $X_{p,w}$.

All of the $\mathcal{L}_p$ spaces of Chapter II can be constructed as independent sums in the above sense. The principal new separable infinite-dimensional $\mathcal{L}_p$ space constructed by Alspach as an independent sum is $D_p$, which is the independent sum of copies of $\ell^2$, with $\ell^2$ realized as the span of the Rademachers in $L^p$. Also new is $B_p \oplus D_p$. The method of taking independent sums has the potential to generate a sequence of $\mathcal{L}_p$ spaces by iteration. However, no general method has been developed for distinguishing the isomorphism types of the resulting spaces.

**The Independent Sum $\left(\sum^{\oplus} X_i\right)_{I,w}$**

Fix $2 < p < \infty$. Let $\Omega = \prod_{i=1}^{\infty}[0,1]$. For $t = (t_1, t_2, \ldots) \in \Omega$ and $i \in \mathbb{N}$, let $\pi_i : \Omega \to [0,1]$ be the projection $\pi_i(t) = t_i$. Let $L_0^p[0,1]$ be the space of mean zero functions in $L^p[0,1]$. For $0 < k \leq 1$, identify $L^p[0,k]$ with the space of functions in $L^p[0,1]$ supported on $[0,k]$. Let $\{X_i\}$ be a sequence of closed subspaces of $L_0^p[0,1]$. Let $w = \{w_i\}$ and $\{k_i\}$ be sequences of scalars from $(0,1]$ such that $k_i = w_i^{\frac{2p}{p-2}}$. Let $T_i : L^p[0,1] \to L^p[0,k_i] \subset L^p[0,1]$ be defined by

$$T_i(f)(s) = \begin{cases} k_i^{-\frac{1}{p}} f\left(\frac{s}{k_i}\right) & \text{if } 0 \leq s \leq k_i \\ 0 & \text{if } k_i < s \leq 1 \end{cases}.$$

Let $Y_i = T_i(X_i)$ and let $\tilde{Y}_i = \{\tilde{y}_i = y_i \circ \pi_i : y_i \in Y_i\} \subset L^p(\Omega)$.

Definition. *Let $p$, $\Omega$, $\pi_i$, $\{X_i\}$, $w = \{w_i\}$, $\{k_i\}$, $T_i$, $Y_i$, and $\tilde{Y}_i$ be as above. Suppose*

*(a) for each $i \in \mathbb{N}$, the orthogonal projection of $L^2[0,1]$ onto $\overline{X_i} \subset L^2[0,1]$, when restricted to $L^p[0,1]$, yields a bounded projection $P_i : L^p[0,1] \to X_i \subset L^p[0,1]$ onto $X_i$, and*

*(b) the sequence $\{P_i\}_{i=1}^{\infty}$ satisfies $\sup_{i\in\mathbb{N}} \|P_i\| < \infty$.*

*Define $\left(\sum^{\oplus} X_i\right)_{I,w}$, the independent sum of $\{X_i\}$ with respect to $w$, by*

$$\left(\sum^{\oplus} X_i\right)_{I,w} = \left[\tilde{Y}_i : i \in \mathbb{N}\right]_{L^p(\Omega)}.$$

Remark. The mapping $T_i$ is an isometry, and the spaces $X_i$, $Y_i$, and $\tilde{Y}_i$ are isometric. If $\tilde{y}_i \in \tilde{Y}_i$ for each $i \in \mathbb{N}$, then $\{\tilde{y}_i\}_{i=1}^{\infty}$ is a sequence of independent mean zero random variables. The sequence $w$ plays a role similar to the role of $w$ in Rosenthal's space $X_{p,w}$. In particular, $w_i^{\frac{2p}{p-2}}$ is related to the measure of the support of $\tilde{y}_i \in \tilde{Y}_i$.

**Example 4.1.** *Let $2 < p < \infty$, let $r_1$ be the first Rademacher function $1_{[0,\frac{1}{2})} - 1_{[\frac{1}{2},1]}$, let $X = [r_1]_{L^p[0,1]}$, and let $w = \{w_i\}$ be a sequence from $(0,1]$. Then*

$\left(\sum^{\oplus} X\right)_{I,w}$ *is isomorphic to* $\ell^2$, $\ell^p$, $\ell^2 \oplus \ell^p$, *or* $X_p$, *where each can be realized by an appropriate choice of* $w$ *as in Proposition 2.1.*

**Proof.** Let $\{k_i\}$ and $\{T_i\}$ correspond with $w = \{w_i\}$ as above. Let $y_i = T_i(r_1)$ and $\tilde{y}_i = y_i \circ \pi_i$. Then $\left(\sum^{\oplus} X\right)_{I,w} = [\tilde{y}_i : i \in \mathbb{N}]_{L^p(\Omega)}$. Now $\{\tilde{y}_i\}_{i=1}^{\infty}$ is a sequence of independent symmetric three-valued random variables in $L^p(\Omega)$, with $\tilde{y}_i$ supported on a set of measure $k_i = w_i^{\frac{2p}{p-2}}$. Moreover, $w_i = k_i^{\frac{p-2}{2p}} = k_i^{\frac{1}{2}-\frac{1}{p}} = \|\tilde{y}_i\|_{L^2(\Omega)} / \|\tilde{y}_i\|_{L^p(\Omega)}$. Hence $\left(\sum^{\oplus} X\right)_{I,w} \sim X_{p,w}$ (essentially) by Corollary 2.3, so $\left(\sum^{\oplus} X\right)_{I,w}$ is isomorphic to $\ell^2$, $\ell^p$, $\ell^2 \oplus \ell^p$, or $X_p$, depending on $w$ as in Proposition 2.1 and the definition of $X_p$. $\square$

### The Complementation of $\left(\sum^{\oplus} X_i\right)_{I,w}$ in $L^p(\Omega)$

Fix $2 < p < \infty$ and $0 < k \leq 1$. For $1 \leq r < \infty$, identify $L^r[0,k]$ with the space of functions in $L^r[0,1]$ supported on $[0,k]$, and for a measure space $E$, let $L_0^r(E)$ be the space of mean zero functions in $L^r(E)$.

Let $T : L^1[0,1] \to L^1[0,k] \subset L^1[0,1]$ be defined by

$$T(f)(s) = \begin{cases} k^{-\frac{1}{p}} f\left(\frac{s}{k}\right) & \text{if } 0 \leq s \leq k \\ 0 & \text{if } k < s \leq 1 \end{cases}.$$

For $1 \leq r < \infty$, let $T_r = T|_{L^r[0,1]}$.

**Lemma 4.2.** *Let* $p$, $k$, *and* $T$ *be as above. For* $1 \leq r < \infty$, *let* $f, g \in L^r[0,1]$. *Then*

(a) $\|T(f)\|_r = k^{\frac{p-r}{rp}} \|f\|_r$,

(b) $T_r : L^r[0,1] \to L^r[0,k] \subset L^r[0,1]$,

(c) $T_r$ *maps* $L^r[0,1]$ *onto* $L^r[0,k]$,

(d) $T_p$ *is an isometry,*

(e) $T_p = T_2|_{L^p[0,1]}$,

*(f) $f$ has mean zero if and only if $T(f)$ has mean zero, and*

*(g) $f$ and $g$ are orthogonal if and only if $T(f)$ and $T(g)$ are orthogonal.*

**Proof.** Part (a) follows from the computation

$$\|T(f)\|_r^r = \int_0^k |T(f)(s)|^r \, ds = \int_0^k \left| k^{-\frac{1}{p}} f\left(\tfrac{s}{k}\right)\right|^r \, ds = k^{1-\frac{r}{p}} \int_0^1 |f(t)|^r \, dt = k^{\frac{p-r}{p}} \|f\|_r^r.$$

Part (b) follows from (a) and the definition of $T$. Considering $T_r$ as a mapping from $L^r[0,1]$ to $L^r[0,k]$, $T_r$ has inverse $T_r^{-1} : L^r[0,k] \to L^r[0,1]$ with $T_r^{-1}(h)(t) = k^{\frac{1}{p}} h(kt)$, and (c) follows. Taking $r = p$, (d) follows from (a). Part (e) is clear. As in the computation for (a), but taking $r = 1$ and deleting the absolute values, $\int_0^k T(f)(s)\, ds = k^{1-\frac{1}{p}} \int_0^1 f(t)\, dt$, and (f) follows. Finally, $\int_0^k T(f)(s) \cdot T(g)(s)\, ds = k^{-\frac{2}{p}} \int_0^k f\left(\frac{s}{k}\right) \cdot g\left(\frac{s}{k}\right)\, ds = k^{1-\frac{2}{p}} \int_0^1 f(t) \cdot g(t)\, dt$, and (g) follows. □

Let $R : L^1[0,1] \to L^1[0,k]$ be defined by $R(f) = 1_{[0,k]} \cdot f$. For $1 \le r < \infty$, let $R_r = R|_{L^r[0,1]}$.

Let $X$ be a closed subspace of $L_0^p[0,1]$ such that the orthogonal projection $P_2$ of $L^2[0,1]$ onto $\overline{X} \subset L^2[0,1]$, when restricted to $L^p[0,1]$, yields a bounded projection $P_p : L^p[0,1] \to X \subset L^p[0,1]$ onto $X$. Let $Y = T(X)$.

**Lemma 4.3.** *Let $p$, $k$, $T$, $R$, $X$, $P_2$, $P_p$, and $Y$ be as above. Let $1 \le r < \infty$. Then*

*(a) $R_r : L^r[0,1] \to L^r[0,k]$ is a projection of $L^r[0,1]$ onto $L^r[0,k]$ with $\|R_r\| = 1$,*

*(b) $R_2$ is the orthogonal projection of $L^2[0,1]$ onto $L^2[0,k]$,*

*(c) $R_p = R_2|_{L^p[0,1]}$,*

*(d) $Y$ is a subspace of $L_0^p[0,k]$ isometric to $X$,*

*(e) the closure of $X$ in $L^2[0,1]$ is contained in $L_0^2[0,1]$,*

*(f) the closure of $Y$ in $L^2[0,k]$ is contained in $L_0^2[0,k]$,*

*(g) $T_2\left(\overline{X}\right) = \overline{Y}$, where $\overline{X}$ and $\overline{Y}$ are the closures of $X$ and $Y$ in $L^2[0,1]$,*

(h) $T_2P_2T_2^{-1}$ *is the orthogonal projection of* $L^2[0,k]$ *onto* $\overline{Y} \subset L^2[0,k]$,

(i) $T_pP_pT_p^{-1} = \left(T_2P_2T_2^{-1}\right)|_{L^p[0,k]}$, *and*

(j) $T_pP_pT_p^{-1}$ *maps* $L^p[0,k]$ *onto* $Y$.

**Proof.** Part (a) is clear. For $f, g \in L^2[0,1]$, $(f - R_2(f)) \perp R_2(g)$, so $(f - R_2(f)) \in \left(R_2\left(L^2[0,1]\right)\right)^{\perp}$, and (b) follows. Part (c) is clear. Part (d) follows from the fact that $T_p : L^p[0,1] \to L^p[0,k]$ is an isometry which preserves mean zero functions. First noting that $X \subset L_0^2[0,1]$ and $Y \subset L_0^2[0,k]$, parts (e) and (f) are clear. Part (g) is clear. For $f, g \in L^2[0,k]$, $\left(T_2^{-1}(f) - P_2\left(T_2^{-1}(f)\right)\right) \perp P_2\left(T_2^{-1}(g)\right)$, so $\left(f - \left(T_2P_2T_2^{-1}\right)(f)\right) \perp \left(T_2P_2T_2^{-1}\right)(g)$, and (h) follows after noting (g). Parts (i) and (j) are clear. $\square$

For $r \in \{2, p\}$, let $Q_r = T_rP_rT_r^{-1}R_r$.

**Lemma 4.4.** *Let* $p$, $r$, $k$, $T$, $R$, $X$, $P_r$, $Y$, *and* $Q_r$ *be as above. Then*

(a) $Q_p : L^p[0,1] \to Y \subset L^p[0,1]$ *maps* $L^p[0,1]$ *onto* $Y$,

(b) $\|Q_p\| = \|P_p\|$,

(c) $Q_2$ *is the orthogonal projection of* $L^2[0,1]$ *onto* $\overline{Y} \subset L^2[0,1]$,

(d) $Q_p = Q_2|_{L^p[0,1]}$, *and*

(e) $Q(1) = 0$.

**Proof.** Note that $T_p^{-1}R_p : L^p[0,1] \to L^p[0,1]$ is surjective, with right inverse $T_p$. Thus (a) follows, and $Q_pT_p = \left(T_pP_pT_p^{-1}R_p\right)T_p = T_pP_p\left(T_p^{-1}R_pT_p\right) = T_pP_p$. Since $T_p$ is an isometry, (b) follows. Part (c) follows from the fact that $R_2$ and $T_2P_2T_2^{-1}$ are orthogonal projections mapping $L^2[0,1]$ onto $L^2[0,k]$ and $L^2[0,k]$ onto $\overline{Y} \subset L^2[0,k]$, respectively. Part (d) follows from the fact that $R_p = R_2|_{L^p[0,1]}$ and $T_pP_pT_p^{-1} = \left(T_2P_2T_2^{-1}\right)|_{L^p[0,k]}$. Noting that $1 \overset{R_p}{\mapsto} 1_{[0,k]} \overset{T_p^{-1}}{\mapsto} k^{\frac{1}{p}} \cdot 1_{[0,1]} \overset{P_p}{\mapsto} 0 \overset{T_p}{\mapsto} 0$, (e) follows. $\square$

The relevant subspaces of $L^p[0,1]$ are related as in the diagram

$$\begin{array}{ccccc} L^p[0,1] & \overset{P_p}{\rightarrow} & X & \subset L_0^p[0,1] \subset L^p[0,1] \\ R_p \downarrow\uparrow T_p^{-1} & \overset{Q_p}{\searrow} & \downarrow T_p & \\ L^p[0,k] & \overset{T_pP_pT_p^{-1}}{\rightarrow} & Y & \subset L_0^p[0,k] \subset L^p[0,1]. \end{array} \tag{4.1}$$

We now perform a similar construction for each $i \in \mathbb{N}$.

Let $\{k_i\}$ be a sequence of scalars from $(0,1]$. Then for $r \in \{1,2,p\}$, $\{k_i\}$ determines sequences $\{T_{i,r}\}$ and $\{R_{i,r}\}$ of mappings, where $T_{i,r}$ and $R_{i,r}$ are simply $T_r$ and $R_r$, respectively, with $k_i$ replacing $k$. Let $\{X_i\}$ be a sequence of closed subspaces of $L_0^p[0,1]$ such that the orthogonal projection $P_{i,2}$ of $L^2[0,1]$ onto $\overline{X_i} \subset L^2[0,1]$, when restricted to $L^p[0,1]$, yields a bounded projection $P_{i,p} : L^p[0,1] \rightarrow X_i \subset L^p[0,1]$ onto $X_i$. Let $Y_i = T_{i,p}(X_i)$, and for $r \in \{2,p\}$, let $Q_{i,r} = T_{i,r}P_{i,r}T_{i,r}^{-1}R_{i,r}$. Then $X_i$, $Y_i$, $P_{i,r}$, and $Q_{i,r}$ are simply $X$, $Y$, $P_r$, and $Q_r$, respectively, with $k_i$ replacing $k$. Thus as in diagram (4.1), we have the diagram

$$\begin{array}{ccccc} L^p[0,1] & \overset{P_{i,p}}{\rightarrow} & X_i & \subset L_0^p[0,1] \subset L^p[0,1] \\ R_{i,p} \downarrow\uparrow T_{i,p}^{-1} & \overset{Q_{i,p}}{\searrow} & \downarrow T_{i,p} & \\ L^p[0,k_i] & & Y_i & \subset L_0^p[0,k_i] \subset L^p[0,1], \end{array} \tag{4.2}$$

and Lemmas 4.2, 4.3, and 4.4 hold, with the obvious notational changes.

Let $1 \leq r < \infty$ and let $i \in \mathbb{N}$. Let $\Pi_{i,r} : L^r[0,1] \rightarrow L^r[\Omega]$ be the isometry $\Pi_{i,r}(f) = f \circ \pi_i$. Then for $f, g \in L^r[0,1]$, $f$ has mean zero if and only if $\Pi_{i,r}(f)$ has mean zero, and $f$ and $g$ are orthogonal if and only if $\Pi_{i,r}(f)$ and $\Pi_{i,r}(g)$ are orthogonal.

Given a closed subspace $Z_{i,r}$ of $L^r[0,1]$, let $\tilde{Z}_{i,r} = \Pi_{i,r}(Z_{i,r}) \subset L^r(\Omega)$. Let $\tilde{L}_i^r[0,1] = \Pi_{i,r}(L^r[0,1])$ and $\tilde{L}_{0,i}^r[0,1] = \Pi_{i,r}(L_0^r[0,1])$.

Given closed subspaces $Z_{i,r}$ and $Z'_{i,r}$ of $L^r[0,1]$ and a mapping $L_{i,r} : Z_{i,r} \rightarrow Z'_{i,r}$, let $\tilde{L}_{i,r} : \tilde{Z}_{i,r} \rightarrow \tilde{Z}'_{i,r}$ be the mapping $\tilde{L}_{i,r} = \Pi_{i,r}L_{i,r}\Pi_{i,r}^{-1}$. Then diagram (4.2) induces the diagram

$$\begin{array}{ccccc} \tilde{L}_i^p[0,1] & \overset{\tilde{P}_{i,p}}{\rightarrow} & \tilde{X}_{i,p} & \subset \tilde{L}_{0,i}^p[0,1] \subset \tilde{L}_i^p[0,1] \\ \tilde{R}_{i,p} \downarrow\uparrow \tilde{T}_{i,p}^{-1} & \overset{\tilde{Q}_{i,p}}{\searrow} & \downarrow \tilde{T}_{i,p} & \\ \tilde{L}^p[0,k_i] & & \tilde{Y}_{i,p} & \subset \tilde{L}_0^p[0,k_i] \subset \tilde{L}_i^p[0,1], \end{array} \tag{4.3}$$

and results analogous to Lemmas 4.2, 4.3, and 4.4 hold.

Let $E_i : L^1(\Omega) \to \tilde{L}_i^1[0,1] \subset L^1(\Omega)$ be the projection onto $\tilde{L}_i^1[0,1] = \Pi_{i,1}\left(L^1[0,1]\right)$ of norm one defined by $E_i(f) = \mathcal{E}_{\mathcal{B}_i} f$, where $\mathcal{E}_{\mathcal{B}_i}$ is conditional expectation with respect to the $\sigma$-algebra $\mathcal{B}_i = \left\{\prod_{j=1}^{\infty} B_j : B_i \subset [0,1] \text{ is measurable}, B_j = [0,1] \text{ for } j \neq i\right\}$. For $1 < r < \infty$, let $E_{i,r} = E_i|_{L^r(\Omega)}$. [See Chapter V, The Complementation of $R_\alpha^p$ in $L^p$, Preliminaries, for properties of conditional expectation.]

**Lemma 4.5.** *Let $p$, $\Pi_{i,r}$, $\tilde{L}_i^r[0,1]$, $\mathcal{B}_i$, and $E_i$ be as above for $1 < r < \infty$ with conjugate index $s$, and let $f \in L^r(\Omega)$. Then*

*(a) $E_{i,r} : L^r(\Omega) \to L^r(\Omega)$ with $\|E_{i,r}\| = 1$,*

*(b) $E_{i,r}$ maps $L^r(\Omega)$ onto $\tilde{L}_i^r[0,1] = \Pi_{i,r}(L^r[0,1])$,*

*(c) $f$ has mean zero if and only if $E_{i,r}(f)$ has mean zero,*

*(d) if $\{f_i\}_{i=1}^{\infty}$ is a sequence in $L^r(\Omega)$, then $\{E_{i,r}(f_i)\}_{i=1}^{\infty}$ is independent,*

*(e) $E_{i,r}^* = E_{i,s}$,*

*(f) $E_{i,2}$ is the orthogonal projection of $L^2(\Omega)$ onto $\tilde{L}_i^2[0,1]$, and*

*(g) $E_{i,p} = E_{i,2}|_{L^p(\Omega)}$.*

**Proof.** By the convexity of $|\ |^r$, $\int_\Omega |E_i(f)|^r \leq \int_\Omega E_i(|f|^r) = \int_\Omega |f|^r$, and (a) follows. The fact that $E_{i,r}$ maps $L^r(\Omega)$ into $\tilde{L}_i^r[0,1] = \Pi_{i,r}(L^r[0,1])$ follows from the choice of the $\sigma$-algebra $\mathcal{B}_i$. For $f \in \tilde{L}_i^r[0,1] = \Pi_{i,r}(L^r[0,1])$, $E_{i,r}(f) = f$, and (b) follows. Since $\int_\Omega E_i(f) = \int_\Omega f$, (c) follows. Part (d) follows from the choice of the $\sigma$-algebra $\mathcal{B}_i$. Noting that $\int_\Omega f \cdot E_{i,r}^*(g) = \int_\Omega E_{i,r}(f) \cdot g = \int_\Omega E_i(f) \cdot g = \int_\Omega E_i(E_i(f) \cdot g) =$

$\int_\Omega E_i(f) \cdot E_i(g) = \int_\Omega E_i\left(E_i(g) \cdot f\right) = \int_\Omega E_i(g) \cdot f = \int_\Omega E_{i,s}(g) \cdot f$ for $g \in L^s(\Omega)$, (e) follows. Part (g) is clear.

Now $E_{i,2} : L^2(\Omega) \to \tilde{L}_i^2[0,1] \subset L^2(\Omega)$ maps $L^2(\Omega)$ onto $\tilde{L}_i^2[0,1] = \Pi_{i,2}\left(L^2[0,1]\right)$ by parts (a) and (b). Let $f \in L^2(\Omega)$. Then $\int_B (f - E_i(f)) = 0$ for all $B \in \mathcal{B}_i$, and $\int_\Omega (f - E_i(f)) \cdot g = 0$ for all $g \in \tilde{L}_i^2[0,1]$. Hence $f - E_i(f) \in \left(\tilde{L}_i^2[0,1]\right)^\perp$, and (f) follows. □

For $r \in \{2, p\}$, let $S_{i,r} = \tilde{Q}_{i,r} E_{i,r}$, where $\tilde{Q}_{i,r}$ and $E_{i,r}$ are as above.

**Lemma 4.6.** *Let $p$, $r$, $P_i$, $\tilde{Y}_{i,p}$, and $S_{i,r}$ be as above. Let $f \in L^r(\Omega)$ and $g \in L^q(\Omega)$, where $q$ is the conjugate index of $p$. Then*

*(a) $S_{i,p} : L^p(\Omega) \to \tilde{Y}_{i,p} \subset L^p(\Omega)$ maps $L^p(\Omega)$ onto $\tilde{Y}_{i,p}$,*

*(b) $S_{i,2}$ is the orthogonal projection of $L^2(\Omega)$ onto $\overline{\tilde{Y}_{i,p}} \subset L^2(\Omega)$,*

*(c) $S_{i,p} = S_{i,2}|_{L^p(\Omega)}$,*

*(d) $\|S_{i,p}\| \le \|P_i\|$,*

*(e) $S_{i,p}(1) = 0$,*

*(f) $\int S_{i,r}(f) = 0$,*

*(g) $\int S_{i,p}^*(g) = 0$,*

*(h) $\{S_{i,r}(f)\}_{i=1}^\infty$ is independent, and*

*(i) if $\{g_i\}_{i=1}^\infty$ is a sequence in $L^q(\Omega)$, then $\left\{S_{i,p}^*(g_i)\right\}_{i=1}^\infty$ is independent.*

**Proof.** Part (a) is clear. Since $S_{i,2} = \tilde{Q}_{i,2} E_{i,2}$ is the composition of orthogonal projections, where $L^2(\Omega) \overset{E_{i,2}}{\to} \tilde{L}_i^2[0,1]$ surjectively and $\tilde{L}_i^2[0,1] \overset{\tilde{Q}_{i,2}}{\to} \overline{\tilde{Y}_{i,p}}$ surjectively, (b) follows. Part (c) is clear. Noting that $\|S_{i,p}\| \le \left\|\tilde{Q}_{i,p}\right\| \|E_{i,p}\| = \left\|\tilde{Q}_{i,p}\right\| = \|Q_i\| = \|P_i\|$, (d) follows. Since $E_{i,p}(1) = 1$ and $\tilde{Q}_{i,p}(1) = 0$, (e) follows. Since $\tilde{Y}_{i,p} \subset \tilde{L}_0^p[0, k_i]$ and $\overline{\tilde{Y}_{i,p}} \subset \tilde{L}_0^2[0, k_i]$, (f) follows. Noting that $\int S_{i,p}^*(g) = \int g \cdot S_{i,p}(1) = \int g \cdot 0 = 0$, (g) follows. For reference, $S_{i,r} = \tilde{Q}_{i,r} E_{i,r}$ and $S_{i,p}^* = E_{i,p}^* \tilde{Q}_{i,p}^*$. Part (h) follows from an analogous property of $E_{i,r}$ which $\tilde{Q}_{i,r}$ preserves. Recalling that $E_{i,q}$ has an analogous property and $E^*_{i,p} = E_{i,q}$, (i) follows. □

For $r \in \{2,p\}$, let $S_r = \sum_{i=1}^{\infty} S_{i,r}$. We show below that the formal series defines a bounded linear operator on $L^r(\Omega)$.

**Lemma 4.7.** *Let $p$, $\tilde{Y}_{i,p}$, and $S_2$ be as above. Then $S_2$ is the orthogonal projection of $L^2(\Omega)$ onto $\left[\overline{\tilde{Y}_{i,p}} \subset L^2(\Omega) : i \in \mathbb{N}\right]_{L^2(\Omega)}$.*

**Proof.** For $f \in L^2(\Omega)$, $S_2(f) = \sum_{i=1}^{\infty} S_{i,2}(f)$, where $S_{i,2}(f) \in \overline{\tilde{Y}_{i,p}} \subset \tilde{L}_0^2[0,k_i] \subset L^2(\Omega)$, $S_{i,2}(f)$ is the orthogonal projection of $f$ onto the span of $S_{i,2}(f)$ in $L^2(\Omega)$, and $\{S_{i,2}(f)\}_{i=1}^{\infty}$ is an orthogonal sequence of random variables. Hence $S_2 : L^2(\Omega) \to \left[\overline{\tilde{Y}_{i,p}} \subset L^2(\Omega) : i \in \mathbb{N}\right]_{L^2(\Omega)}$ is the orthogonal projection of $L^2(\Omega)$ onto $\left[\overline{\tilde{Y}_{i,p}} \subset L^2(\Omega) : i \in \mathbb{N}\right]_{L^2(\Omega)}$. □

**Theorem 4.8.** *Let $2 < p < \infty$ and let $w = \{w_i\}$ be a sequence of scalars from $(0,1]$. Let $\{X_i\}$ be a sequence of closed subspaces of $L_0^p[0,1]$ satisfying the hypotheses (a) and (b) in the definition of $\left(\sum^{\oplus} X_i\right)_{I,w}$. Then $\left(\sum^{\oplus} X_i\right)_{I,w}$ is a complemented subspace of $L^p(\Omega)$ via the projection $S_p$.*

**Proof.** Let $f \in L^p(\Omega)$. Then $\{S_{i,p}(f)\}_{i=1}^{\infty}$ is a sequence of independent mean zero random variables in $L^p(\Omega)$. Hence (essentially) by Theorem 2.2 [Rosenthal's inequality],

$$\begin{aligned} \|S_p(f)\|_{L^p(\Omega)} &= \left\| \sum_{i=1}^{\infty} S_{i,p}(f) \right\|_{L^p(\Omega)} \\ &\overset{K_p}{\underset{2}{\approx}} \max\left\{ \left( \sum_{i=1}^{\infty} \|S_{i,p}(f)\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}, \left( \sum_{i=1}^{\infty} \|S_{i,p}(f)\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} \right\}. \end{aligned}$$

By the orthogonality of $\{S_{i,p}(f)\}_{i=1}^{\infty}$ and the fact that $S_p = S_2|_{L^p(\Omega)}$ where $S_2$ is orthogonal projection,

$$\left( \sum_{i=1}^{\infty} \|S_{i,p}(f)\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} = \left\| \sum_{i=1}^{\infty} S_{i,p}(f) \right\|_{L^2(\Omega)} = \|S_p(f)\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \leq \|f\|_{L^p(\Omega)}.$$

Let $G = \left\{ \{g_i\}_{i=1}^{\infty} : g_i \in L^q(\Omega), \left( \sum_{i=1}^{\infty} \|g_i\|_{L^q(\Omega)}^q \right)^{\frac{1}{q}} \leq 1 \right\}$, where $q$ is the conjugate index of $p$. Then for $g_i \in L^q(\Omega)$, $\left\{ S_{i,p}^* (g_i) \right\}_{i=1}^{\infty}$ is a sequence of independent mean zero random variables in $L^q(\Omega)$. Hence by Hölder's inequality and (essentially) Lemma 2.4,

$$\begin{aligned}
\left( \sum_{i=1}^{\infty} \|S_{i,p}(f)\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}} &= \sup_{\{g_i\} \in G} \left| \sum_{i=1}^{\infty} \langle S_{i,p}(f), g_i \rangle \right| \\
&= \sup_{\{g_i\} \in G} \left| \left\langle f, \sum_{i=1}^{\infty} S_{i,p}^*(g_i) \right\rangle \right| \\
&\leq \sup_{\{g_i\} \in G} \left\| \sum_{i=1}^{\infty} S_{i,p}^*(g_i) \right\|_{L^q(\Omega)} \|f\|_{L^p(\Omega)} \\
&\leq 2 \sup_{\{g_i\} \in G} \left( \sum_{i=1}^{\infty} \left\| S_{i,p}^*(g_i) \right\|_{L^q(\Omega)}^q \right)^{\frac{1}{q}} \|f\|_{L^p(\Omega)} \\
&\leq 2 \sup_{i \in \mathbb{N}} \left\| S_{i,p}^* \right\| \sup_{\{g_i\} \in G} \left( \sum_{i=1}^{\infty} \|g_i\|_{L^q(\Omega)}^q \right)^{\frac{1}{q}} \|f\|_{L^p(\Omega)} \\
&\leq 2 \sup_{i \in \mathbb{N}} \|P_i\| \, \|f\|_{L^p(\Omega)} .
\end{aligned}$$

It now follows that $\|S_p(f)\|_{L^p(\Omega)} \leq K_p \max \{2 \sup_{i \in \mathbb{N}} \|P_i\|, 1\} \|f\|_{L^p(\Omega)}$. Hence $S_p : L^p(\Omega) \to \left[ \tilde{Y}_{i,p} : i \in \mathbb{N} \right]_{L^p(\Omega)}$ maps $L^p(\Omega)$ onto $\left[ \tilde{Y}_{i,p} : i \in \mathbb{N} \right]_{L^p(\Omega)}$ with $\|S_p\| \leq K_p \max \{2 \sup_{i \in \mathbb{N}} \|P_i\|, 1\}$, and $\left( \sum^{\oplus} X_i \right)_{I,w} = \left[ \tilde{Y}_{i,p} : i \in \mathbb{N} \right]_{L^p(\Omega)}$ is complemented in $L^p(\Omega)$. □

### Independent Sums with Basis

Now suppose in addition to the hypotheses (a) and (b) in the definition of $\left( \sum^{\oplus} X_i \right)_{I,w}$, the sequence $\{X_i\}$ of closed subspaces of $L_0^p[0,1]$ satisfies

(c) for each $i \in \mathbb{N}$, $X_i$ has an unconditional orthogonal basis $\{x_{i,n}\}_{n=1}^{\infty}$.

Then of course $X_i = [x_{i,n} : n \in \mathbb{N}]_{L^p[0,1]}$.

Letting $Y_i = T_i(X_i)$ as before, and letting $y_{i,n} = T_i(x_{i,n})$, we have $Y_i = [y_{i,n} : n \in \mathbb{N}]_{L^p[0,1]}$, and $\{y_{i,n}\}_{n=1}^{\infty}$ is an unconditional orthogonal basis for $Y_i$ isometrically equivalent to $\{x_{i,n}\}_{n=1}^{\infty}$.

Letting $\tilde{Y}_i = \{\tilde{y}_i = y_i \circ \pi_i : y_i \in Y_i\}$ as before, and letting $\tilde{y}_{i,n} = y_{i,n} \circ \pi_i$, we have $\tilde{Y}_i = [\tilde{y}_{i,n} : n \in \mathbb{N}]_{L^p(\Omega)}$, and $\{\tilde{y}_{i,n}\}_{n=1}^{\infty}$ is an unconditional orthogonal basis for $\tilde{Y}_i$ isometrically equivalent to $\{y_{i,n}\}_{n=1}^{\infty}$ and $\{x_{i,n}\}_{n=1}^{\infty}$.

In this context, $\left(\sum^{\oplus} X_i\right)_{I,w} = [\tilde{y}_{i,n} : i, n \in \mathbb{N}]_{L^p(\Omega)}$, and $\{\tilde{y}_{i,n}\}_{i,n\in\mathbb{N}}$ is an unconditional orthogonal basis for $\left(\sum^{\oplus} X_i\right)_{I,w}$.

REMARK. Noting that $y_{i,n} = T_i(x_{i,n})$ and $k_i^{\frac{p-2}{2p}} = w_i$, by part (a) of Lemma 4.2 we have $\|\tilde{y}_{i,n}\|_{L^2(\Omega)} = \|y_{i,n}\|_2 = w_i \|x_{i,n}\|_2$.

**Proposition 4.9.** *Let $2 < p < \infty$ and let $w = \{w_i\}$ be a sequence of scalars from $(0, 1]$. Let $\{X_i\}$ be a sequence of closed subspaces of $L_0^p[0,1]$ such that each $X_i$ has an unconditional orthogonal basis $\{x_{i,n}\}_{n=1}^{\infty}$. Let $\tilde{y}_{i,n} = (T_i(x_{i,n})) \circ \pi_i \in L^p(\Omega)$, where $T_i$ and $\pi_i$ are as in the definition of $\left(\sum^{\oplus} X_i\right)_{I,w}$. Then for $K_p$ as in Theorem 2.2 and for scalars $a_{i,n}$,*

$$\left\|\sum_i \sum_n a_{i,n}\tilde{y}_{i,n}\right\|_{L^p(\Omega)} \overset{K_p}{\underset{2}{\approx}} \max\left\{\left(\sum_i \left\|\sum_n a_{i,n}x_{i,n}\right\|_p^p\right)^{\frac{1}{p}}, \left(\sum_i w_i^2 \sum_n |a_{i,n}|^2 \|x_{i,n}\|_2^2\right)^{\frac{1}{2}}\right\}.$$

**Proof.** Let $z_i = \sum_n a_{i,n}\tilde{y}_{i,n}$. Then $\{z_i\}$ is a sequence of independent mean zero random variables in $L^p(\Omega)$. Hence (essentially) by Corollary 2.3 [Rosenthal's inequality],

$$\left\|\sum_i z_i\right\|_{L^p(\Omega)} \overset{K_p}{\underset{2}{\approx}} \max\left\{\left(\sum_i \|z_i\|_{L^p(\Omega)}^p\right)^{\frac{1}{p}}, \left(\sum_i \|z_i\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}\right\}.$$

Note that $\|z_i\|_{L^p(\Omega)}^p = \|\sum_n a_{i,n}\tilde{y}_{i,n}\|_{L^p(\Omega)}^p = \|\sum_n a_{i,n}x_{i,n}\|_p^p$. Moreover, by the orthogonality of $\{\tilde{y}_{i,n}\}_{n=1}^{\infty}$ and by the remark above,

$\|z_i\|_{L^2(\Omega)}^2 = \|\sum_n a_{i,n}\tilde{y}_{i,n}\|_{L^2(\Omega)}^2 = \sum_n |a_{i,n}|^2 \|\tilde{y}_{i,n}\|_{L^2(\Omega)}^2 = w_i^2 \sum_n |a_{i,n}|^2 \|x_{i,n}\|_2^2$.

The result now follows from the displayed inequality. □

**Corollary 4.10.** *Let $2 < p < \infty$ and let $w = \{w_i\}$ be a sequence of scalars from $(0, 1]$. Let $\{X_i\}$ be a sequence of closed subspaces of $L_0^p[0,1]$ satisfying the hypotheses* *(a) and (b) in the definition of* $\left(\sum^{\oplus} X_i\right)_{I,w}$ *such that each* $X_i$ *has an unconditional orthogonal basis* $\{x_{i,n}\}_{n=1}^{\infty}$. *Suppose* $\sum w_i^{\frac{2p}{p-2}} < \infty$. *Then* $\left(\sum^{\oplus} X_i\right)_{I,w} \sim \left(\sum^{\oplus} X_i\right)_{\ell^p}$.

**Proof.** Let $\tilde{y}_{i,n}$ be as in Proposition 4.9. Let $K = \left(\sum w_i^{\frac{2p}{p-2}}\right)^{\frac{p-2}{2p}}$. By Hölder's inequality with conjugate indices $p' = \frac{p}{2}$ and $q' = \frac{p}{p-2}$, and the orthogonality of $\{x_{i,n}\}_{n=1}^{\infty}$, for scalars $a_{i,n}$ we have

$$\begin{aligned}
\left(\sum_i w_i^2 \left(\sum_n |a_{i,n}|^2 \|x_{i,n}\|_2^2\right)\right)^{\frac{1}{2}} &\leq \left(\left(\sum_i w_i^{2\frac{p}{p-2}}\right)^{\frac{p-2}{p}} \left(\sum_i \left(\sum_n |a_{i,n}|^2 \|x_{i,n}\|_2^2\right)^{\frac{p}{2}}\right)^{\frac{2}{p}}\right)^{\frac{1}{2}} \\
&= \left(\sum_i w_i^{\frac{2p}{p-2}}\right)^{\frac{p-2}{2p}} \left(\sum_i \left(\sum_n \|a_{i,n}x_{i,n}\|_2^2\right)^{\frac{1}{2}p}\right)^{\frac{1}{p}} \\
&= K\left(\sum_i \left\|\sum_n a_{i,n}x_{i,n}\right\|_2^p\right)^{\frac{1}{p}} \\
&\leq K\left(\sum_i \left\|\sum_n a_{i,n}x_{i,n}\right\|_p^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Hence by Proposition 4.9 and the above bound, for $\tilde{K} = \max\{1, K\}$ we have

$$\begin{aligned}
\left\|\sum_i \sum_n a_{i,n}\tilde{y}_{i,n}\right\|_{L^p(\Omega)} &\overset{K_p}{\underset{2}{\approx}} \max\left\{\left(\sum_i \left\|\sum_n a_{i,n}x_{i,n}\right\|_p^p\right)^{\frac{1}{p}}, \left(\sum_i w_i^2 \sum_n |a_{i,n}|^2 \|x_{i,n}\|_2^2\right)^{\frac{1}{2}}\right\} \\
&\overset{\tilde{K}}{\underset{1}{\approx}} \left(\sum_i \left\|\sum_n a_{i,n}x_{i,n}\right\|_p^p\right)^{\frac{1}{p}}.
\end{aligned}$$

It follows that $\left(\sum^{\oplus} X_i\right)_{I,w} \sim \left(\sum^{\oplus} X_i\right)_{\ell^p}$. □

**Example 4.11.** *Let* $2 < p < \infty$ *and let* $w = \{w_i\}$ *be a sequence of scalars from* $(0,1]$ *such that* $\sum w_i^{\frac{2p}{p-2}} < \infty$. *Then* $\left(\sum^{\oplus} \ell^2\right)_{\ell^p}$, $\left(\sum^{\oplus} X_p\right)_{\ell^p}$, $B_p$, $X_p \oplus \left(\sum^{\oplus} \ell^2\right)_{\ell^p}$, *and* $X_p \oplus B_p$ *can be realized as* $\left(\sum^{\oplus} X_i\right)_{I,w}$ *for appropriately chosen* $X_i$.

**Proof.** Let $\{x_n\}$ be the sequence of Rademacher functions and let $X = [x_n]_{L^p} \sim \ell^2$. Then $\left(\sum^{\oplus} X\right)_{I,w} \sim \left(\sum^{\oplus} \ell^2\right)_{\ell^p}$.

Let $\{x_n\}$ be a sequence of independent mean zero random variables in $L^p$ such that $v = \{v_n\} = \left\{\|x_n\|_2 / \|x_n\|_p\right\}$ satisfies condition $(*)$ of Proposition 2.1, and let $X = [x_n]_{L^p} \sim X_p$. Then $\left(\sum^{\oplus} X\right)_{I,w} \sim \left(\sum^{\oplus} X_p\right)_{\ell^p}$.

For each $i \in \mathbb{N}$, let $\{x_{i,n}\}_{n=1}^{\infty}$ be a sequence of independent mean zero random variables in $L^p$ such that $v^{(i)} = \{v_{i,n}\}_{n=1}^{\infty} = \left\{\|x_{i,n}\|_2 / \|x_{i,n}\|_p\right\}_{n=1}^{\infty}$ satisfies $v_{i,n}^{\frac{2p}{p-2}} = \frac{1}{i}$ for each $n \in \mathbb{N}$. Let $X_i = [x_{i,n} : n \in \mathbb{N}]_{L^p} \sim X_{p,v^{(i)}}$. Then $\left(\sum^{\oplus} X_i\right)_{I,w} \sim \left(\sum^{\oplus} X_{p,v^{(i)}}\right)_{\ell^p} \sim B_p$.

Let $\{x_{1,n}\}_{n=1}^{\infty}$ be a sequence of independent mean zero random variables in $L^p$ such that $v^{(1)} = \{v_{1,n}\}_{n=1}^{\infty} = \left\{\|x_{1,n}\|_2 / \|x_{1,n}\|_p\right\}_{n=1}^{\infty}$ satisfies condition (*) of Proposition 2.1, and let $X_1 = [x_{1,n} : n \in \mathbb{N}]_{L^p} \sim X_p$. For each $i \in \mathbb{N} \setminus \{1\}$, let $\{x_{i,n}\}_{n=1}^{\infty}$ be the sequence of Rademacher functions and let $X_i = [x_{i,n} : n \in \mathbb{N}]_{L^p} \sim \ell^2$. Then $\left(\sum^{\oplus} X_i\right)_{I,w} \sim \left(\sum^{\oplus} X_i\right)_{\ell^p} \sim \left(X_p \oplus \sum^{\oplus} \ell^2\right)_{\ell^p} \sim X_p \oplus \left(\sum^{\oplus} \ell^2\right)_{\ell^p}$.

Let $\{x_{1,n}\}_{n=1}^{\infty}$ be a sequence of independent mean zero random variables in $L^p$ such that $v^{(1)} = \{v_{1,n}\}_{n=1}^{\infty} = \left\{\|x_{1,n}\|_2 / \|x_{1,n}\|_p\right\}_{n=1}^{\infty}$ satisfies condition (*) of Proposition 2.1, and let $X_1 = [x_{1,n} : n \in \mathbb{N}]_{L^p} \sim X_p$. For each $i \in \mathbb{N} \setminus \{1\}$, let $\{x_{i,n}\}_{n=1}^{\infty}$ be a sequence of independent mean zero random variables in $L^p$ such that $v^{(i)} = \{v_{i,n}\}_{n=1}^{\infty} = \left\{\|x_{i,n}\|_2 / \|x_{i,n}\|_p\right\}_{n=1}^{\infty}$ satisfies $v_{i,n}^{\frac{2p}{p-2}} = \frac{1}{i}$ for each $n \in \mathbb{N}$, and let $X_i = [x_{i,n} : n \in \mathbb{N}]_{L^p} \sim X_{p,v^{(i)}}$. Then $\left(\sum^{\oplus} X_i\right)_{I,w} \sim \left(\sum^{\oplus} X_i\right)_{\ell^p} \sim \left(X_p \oplus \sum_{i\geq 2}^{\oplus} X_{p,v^{(i)}}\right)_{\ell^p} \sim X_p \oplus \left(\sum_{i\geq 2}^{\oplus} X_{p,v^{(i)}}\right)_{\ell^p} \sim X_p \oplus B_p$. □

## The Independent Sum $\left(\sum^{\oplus} X\right)_I$

Let $2 < p < \infty$. Suppose $X$ is a closed subspace of $L_0^p[0,1]$ satisfying

(a′) the orthogonal projection of $L^2[0,1]$ onto $\overline{X} \subset L^2[0,1]$, when restricted to $L^p[0,1]$, yields a bounded projection $P : L^p[0,1] \to X \subset L^p[0,1]$ onto $X$, and

(c′) $X$ has an unconditional orthogonal normalized basis $\{x_n\}$.

We adopt notation as before, with $X$ replacing $X_i$ and $x_n$ replacing $x_{i,n}$. In particular, $\tilde{y}_{i,n} = (T_i(x_n)) \circ \pi_i \in L^p(\Omega)$, where $T_i$ and $\pi_i$ are as in the definition of $\left(\sum^{\oplus} X_i\right)_{I,w}$.

For $2 < p < \infty$, we will show that for a fixed closed subspace $X$ of $L_0^p[0,1]$

satisfying the hypotheses (a′) and (c′) above, all spaces $\left(\sum^{\oplus} X\right)_{I,w}$ for sequences $w = \{w_i\}$ from $(0,1]$ satisfying condition $(*)$ of Proposition 2.1 are mutually isomorphic. The following results follow the pattern of Propositions 2.7, 2.9, 2.10, 2.11, and Theorem 2.12, where it is shown that the isomorphism type of $X_{p,w}$ does not depend on $w$ as long as $w$ satisfies condition $(*)$.

**Proposition 4.12.** *Let $2 < p < \infty$ and let $w = \{w_i\}$ be a sequence of scalars from $(0,1]$. Let $X$ be a closed subspace of $L_0^p[0,1]$ satisfying the hypotheses (a′) and (c′) above. Suppose $\{E_j\}$ is a sequence of disjoint nonempty finite subsets of $\mathbb{N}$ such that $\sum_{i\in E_j} w_i^{\frac{2p}{p-2}} \leq 1$ for each $j \in \mathbb{N}$. Let $z_{j,n} = \sum_{i\in E_j} w_i^{\frac{2}{p-2}} \tilde{y}_{i,n}$ and let $\tilde{z}_{j,n}$ be the normalization of $z_{j,n}$ in $L^p(\Omega)$. Let $v_j = \left(\sum_{i\in E_j} w_i^{\frac{2p}{p-2}}\right)^{\frac{p-2}{2p}}$ and $v = \{v_j\}$. Then*

*(a) $\{\tilde{z}_{j,n}\}$ is an unconditional basis for $[\tilde{z}_{j,n} : j,n \in \mathbb{N}]_{\left(\sum^{\oplus} X\right)_{I,w}}$ which is equivalent to the standard basis of $\left(\sum^{\oplus} X\right)_{I,v}$, and*

*(b) there is a projection $P : \left(\sum^{\oplus} X\right)_{I,w} \to [\tilde{z}_{j,n} : j,n \in \mathbb{N}]_{\left(\sum^{\oplus} X\right)_{I,w}}$.*

**Proof.** First we establish some notation. Let $Y_{p,\{x_n\}}$ be the Banach space of all sums of the form $y = \sum_i \sum_n a_{i,n}\tilde{y}_{i,n}$ (for scalars $a_{i,n}$) such that

$\|y\|_{Y_{p,\{x_n\}}} = \left(\sum_i \|\sum_n a_{i,n}\tilde{y}_{i,n}\|_{L^p(\Omega)}^p\right)^{\frac{1}{p}} = \left(\sum_i \|\sum_n a_{i,n}x_n\|_p^p\right)^{\frac{1}{p}} < \infty$. Let $Y_{2,w,\{x_n\}}$ be the Hilbert space of all sums of the form $y = \sum_i \sum_n a_{i,n}\tilde{y}_{i,n}$ (for scalars $a_{i,n}$) such that $\|y\|_{Y_{2,w,\{x_n\}}} = \left(\sum_i \|\sum_n a_{i,n}\tilde{y}_{i,n}\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} = \left(\sum_i w_i^2 \sum_n |a_{i,n}|^2 \|x_n\|_2^2\right)^{\frac{1}{2}} < \infty$, where the inner product in $Y_{2,w,\{x_n\}}$ is defined by

$\langle y_a, y_b\rangle = \sum_i \int \left(\sum_n a_{i,n}\tilde{y}_{i,n}\right) \overline{\left(\sum_n b_{i,n}\tilde{y}_{i,n}\right)} = \sum_i w_i^2 \sum_n a_{i,n}\bar{b}_{i,n} \|x_n\|_2^2$

(where $y_a = \sum_i \sum_n a_{i,n}\tilde{y}_{i,n}$, $y_b = \sum_i \sum_n b_{i,n}\tilde{y}_{i,n}$, and bar is complex conjugation).

Let $|||\ |||$ be the norm on $\left(\sum^{\oplus} X\right)_{I,w}$ defined by

$|||y||| = \max\left\{\|y\|_{Y_{p,\{x_n\}}}, \|y\|_{Y_{2,w,\{x_n\}}}\right\}$. By Proposition 4.9, $|||\ |||$ is equivalent to the standard norm on $\left(\sum^{\oplus} X\right)_{I,w}$. Without loss of generality, we will proceed in the context of $\left(\sum^{\oplus} X\right)_{I,w}$ endowed with the norm $|||\ |||$.

We now find the normalizing factor for $z_{j,n}$. Let $\sigma_j = \sum_{i\in E_j} w_i^{\frac{2p}{p-2}}$. Noting that $2+\frac{4}{p-2}=\frac{2p}{p-2}$, $1=\|x_n\|_p \geq \|x_n\|_2$, and $\sigma_j^{\frac{1}{p}} \geq \sigma_j^{\frac{1}{2}}$, we have

$$\begin{aligned}
|||z_{j,n}||| = \left|\left|\left| \sum_{i\in E_j} w_i^{\frac{2}{p-2}} \tilde{y}_{i,n} \right|\right|\right| &= \max\left\{ \left( \sum_{i\in E_j} \left\| w_i^{\frac{2}{p-2}} x_n \right\|_p^p \right)^{\frac{1}{p}}, \left( \sum_{i\in E_j} w_i^2 w_i^{\frac{4}{p-2}} \|x_n\|_2^2 \right)^{\frac{1}{2}} \right\} \\
&= \max\left\{ \left( \sum_{i\in E_j} w_i^{\frac{2p}{p-2}} \|x_n\|_p^p \right)^{\frac{1}{p}}, \left( \sum_{i\in E_j} w_i^{\frac{2p}{p-2}} \|x_n\|_2^2 \right)^{\frac{1}{2}} \right\} \\
&= \max\left\{ \sigma_j^{\frac{1}{p}} \|x_n\|_p, \sigma_j^{\frac{1}{2}} \|x_n\|_2 \right\} \\
&= \sigma_j^{\frac{1}{p}}.
\end{aligned}$$

Hence $\tilde{z}_{j,n} = \sigma_j^{-\frac{1}{p}} z_{j,n} = \sigma_j^{-\frac{1}{p}} \sum_{i\in E_j} w_i^{\frac{2}{p-2}} \tilde{y}_{i,n}$.

(a) The unconditionality of $\{\tilde{z}_{j,n}\}$ follows from the unconditionality of $\{\tilde{y}_{i,n}\}$ in $\left(\sum^{\oplus} X\right)_{I,w}$. We now examine the equivalence of the bases. For scalars $a_{j,n}$, we have

$$\begin{aligned}
\left\| \sum_j \sum_n a_{j,n} \tilde{z}_{j,n} \right\|_{Y_{p,\{x_n\}}}^p &= \left\| \sum_j \sum_n a_{j,n} \sigma_j^{-\frac{1}{p}} \sum_{i\in E_j} w_i^{\frac{2}{p-2}} \tilde{y}_{i,n} \right\|_{Y_{p,\{x_n\}}}^p \\
&= \left\| \sum_j \sum_{i\in E_j} \sum_n \sigma_j^{-\frac{1}{p}} w_i^{\frac{2}{p-2}} a_{j,n} \tilde{y}_{i,n} \right\|_{Y_{p,\{x_n\}}}^p \\
&= \sum_j \sum_{i\in E_j} \left\| \sum_n \sigma_j^{-\frac{1}{p}} w_i^{\frac{2}{p-2}} a_{j,n} x_n \right\|_p^p \\
&= \sum_j \sigma_j^{-1} \sum_{i\in E_j} w_i^{\frac{2p}{p-2}} \left\| \sum_n a_{j,n} x_n \right\|_p^p \\
&= \sum_j \left\| \sum_n a_{j,n} x_n \right\|_p^p \qquad (4.4) \\
&= \left\| \sum_j \sum_n a_{j,n} \tilde{y}_{j,n}^{(v)} \right\|_{Y_{p,\{x_n\}}}^p,
\end{aligned}$$

and noting that $2+\frac{4}{p-2}=\frac{2p}{p-2}$ and $1-\frac{2}{p}=\frac{p-2}{2p}2$,

$$\left\| \sum_j \sum_n a_{j,n} \tilde{z}_{j,n} \right\|_{Y_{2,w,\{x_n\}}}^2 = \left\| \sum_j \sum_n a_{j,n} \sigma_j^{-\frac{1}{p}} \sum_{i\in E_j} w_i^{\frac{2}{p-2}} \tilde{y}_{i,n} \right\|_{Y_{2,w,\{x_n\}}}^2$$

$$= \left\| \sum_j \sum_{i \in E_j} \sum_n \sigma_j^{-\frac{1}{p}} w_i^{\frac{2}{p-2}} a_{j,n} \tilde{y}_{i,n} \right\|^2_{Y_{2,w,\{x_n\}}}$$

$$= \sum_j \sum_{i \in E_j} w_i^2 \sum_n \left| \sigma_j^{-\frac{1}{p}} w_i^{\frac{2}{p-2}} a_{j,n} \right|^2 \|x_n\|_2^2$$

$$= \sum_j \sigma_j^{-\frac{2}{p}} \sum_{i \in E_j} w_i^{\frac{2p}{p-2}} \sum_n |a_{j,n}|^2 \|x_n\|_2^2$$

$$= \sum_j \left( \sigma_j^{\frac{p-2}{2p}} \right)^2 \sum_n |a_{j,n}|^2 \|x_n\|_2^2$$

$$= \sum_j v_j^2 \sum_n |a_{j,n}|^2 \|x_n\|_2^2 \tag{4.5}$$

$$= \left\| \sum_j \sum_n a_{j,n} \tilde{y}_{j,n}^{(v)} \right\|^2_{Y_{2,v,\{x_n\}}},$$

where $\tilde{y}_{j,n}^{(v)}$ is analogous to $\tilde{y}_{j,n}$ with $v$ replacing $w$. Hence

$$\left\|\left\| \sum_j \sum_n a_{j,n} \tilde{z}_{i,n} \right\|\right\| = \max \left\{ \left\| \sum_j \sum_n a_{j,n} \tilde{z}_{j,n} \right\|_{Y_{p,\{x_n\}}}, \left\| \sum_j \sum_n a_{j,n} \tilde{z}_{j,n} \right\|_{Y_{2,w,\{x_n\}}} \right\}$$

$$= \max \left\{ \left\| \sum_j \sum_n a_{j,n} \tilde{y}_{j,n}^{(v)} \right\|_{Y_{p,\{x_n\}}}, \left\| \sum_j \sum_n a_{j,n} \tilde{y}_{j,n}^{(v)} \right\|_{Y_{2,v,\{x_n\}}} \right\}$$

$$= \left\|\left\| \sum_j \sum_n a_{j,n} \tilde{y}_{j,n}^{(v)} \right\|\right\|_v,$$

where $|||\ |||_v$ is analogous to $|||\ |||$ with $v$ replacing $w$. Hence $\{\tilde{z}_{j,n}\}$ is equivalent to the standard basis $\left\{\tilde{y}_{j,n}^{(v)}\right\}$ of $\left(\sum^{\oplus} X\right)_{I,v}$.

(b) Let $\pi : Y_{2,w,\{x_n\}} \to [z_{j,n} : j, n \in \mathbb{N}]_{Y_{2,w,\{x_n\}}}$ be the orthogonal projection onto $[z_{j,n} : j, n \in \mathbb{N}]_{Y_{2,w,\{x_n\}}}$ defined by

$$\pi(y) = \sum_j \sum_n \frac{\langle y, z_{j,n} \rangle}{\langle z_{j,n}, z_{j,n} \rangle} z_{j,n}.$$

Let $y \in \left(\sum^{\oplus} X\right)_{I,w} = Y_{p,\{x_n\}} \cap Y_{2,w,\{x_n\}}$. Then $\|\pi(y)\|_{Y_{2,w,\{x_n\}}} \leq \|y\|_{Y_{2,w,\{x_n\}}}$. We will show that $\|\pi(y)\|_{Y_{p,\{x_n\}}} \leq \|y\|_{Y_{p,\{x_n\}}}$ as well, whence

$$|||\pi(y)||| = \max \left\{ \|\pi(y)\|_{Y_{p,\{x_n\}}}, \|\pi(y)\|_{Y_{2,w,\{x_n\}}} \right\}$$

$$\leq \max \left\{ \|y\|_{Y_{p,\{x_n\}}}, \|y\|_{Y_{2,w,\{x_n\}}} \right\} = |||y|||.$$

Thus letting $P : \left(\sum^{\oplus} X\right)_{I,w} \to [z_{j,n} : j, n \in \mathbb{N}]_{\left(\sum^{\oplus} X\right)_{I,w}}$ be the restriction of $\pi$ to $\left(\sum^{\oplus} X\right)_{I,w}$, $P$ will satisfy our requirements.

Fix $y = \sum_i \sum_n a_{i,n}\tilde{y}_{i,n} \in \left(\sum^{\oplus} X\right)_{I,w}$. Let $\lambda_{j,n} = \langle y, z_{j,n}\rangle / \langle z_{j,n}, z_{j,n}\rangle$, so that $\pi(y) = \sum_j \sum_n \lambda_{j,n} z_{j,n}$. Noting that $2 + \frac{2}{p-2} = \frac{2(p-1)}{p-2}$ and $2 + \frac{4}{p-2} = \frac{2p}{p-2}$, we have

$$\begin{aligned}
\lambda_{j,n} &= \langle y, z_{j,n}\rangle / \langle z_{j,n}, z_{j,n}\rangle \\
&= \left\langle \sum_i \sum_n a_{i,n}\tilde{y}_{i,n}, \sum_{i\in E_j} w_i^{\frac{2}{p-2}} \tilde{y}_{i,n} \right\rangle \Big/ \left\langle \sum_{i\in E_j} w_i^{\frac{2}{p-2}} \tilde{y}_{i,n}, \sum_{i\in E_j} w_i^{\frac{2}{p-2}} \tilde{y}_{i,n} \right\rangle \\
&= \left( \sum_{i\in E_j} w_i^2 a_{i,n} w_i^{\frac{2}{p-2}} \|x_n\|_2^2 \right) \Big/ \left( \sum_{i\in E_j} w_i^2 w_i^{\frac{4}{p-2}} \|x_n\|_2^2 \right) \\
&= \left( \sum_{i\in E_j} w_i^{\frac{2(p-1)}{p-2}} a_{i,n} \right) \Big/ \left( \sum_{i\in E_j} w_i^{\frac{2p}{p-2}} \right) \\
&= \sigma_j^{-1} \sum_{i\in E_j} w_i^{\frac{2(p-1)}{p-2}} a_{i,n}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\|\pi(y)\|_{Y_{p,\{x_n\}}} &= \left\| \sum_j \sum_n \lambda_{j,n} z_{j,n} \right\|_{Y_{p,\{x_n\}}} \\
&= \left\| \sum_j \sum_n \lambda_{j,n} \sum_{i\in E_j} w_i^{\frac{2}{p-2}} \tilde{y}_{i,n} \right\|_{Y_{p,\{x_n\}}} \\
&= \left\| \sum_j \sum_{i\in E_j} \sum_n \lambda_{j,n} w_i^{\frac{2}{p-2}} \tilde{y}_{i,n} \right\|_{Y_{p,\{x_n\}}} \\
&= \left( \sum_j \sum_{i\in E_j} \left\| \sum_n \lambda_{j,n} w_i^{\frac{2}{p-2}} x_n \right\|_p^p \right)^{\frac{1}{p}} \\
&= \left( \sum_j \sum_{i\in E_j} w_i^{\frac{2p}{p-2}} \left\| \sum_n \lambda_{j,n} x_n \right\|_p^p \right)^{\frac{1}{p}} \\
&= \left( \sum_j \sigma_j \left\| \sum_n \sigma_j^{-1} \sum_{i\in E_j} w_i^{\frac{2(p-1)}{p-2}} a_{i,n} x_n \right\|_p^p \right)^{\frac{1}{p}} \\
&= \left( \sum_j \sigma_j^{1-p} \left\| \sum_n \sum_{i\in E_j} w_i^{\frac{2(p-1)}{p-2}} a_{i,n} x_n \right\|_p^p \right)^{\frac{1}{p}},
\end{aligned}$$

where by Hölder's inequality, letting $q$ be the conjugate index of $p$ and noting that $(p-1)q = p$ and $\frac{p}{q} = p-1$,

$$\begin{aligned}\left\|\sum_n \sum_{i\in E_j} w_i^{\frac{2(p-1)}{p-2}} a_{i,n}x_n\right\|_p^p &= \int \left|\sum_{i\in E_j} \left(w_i^{\frac{2(p-1)}{p-2}}\right)\left(\sum_n a_{i,n}x_n\right)\right|^p \\ &\le \int \left|\left(\sum_{i\in E_j}\left(w_i^{\frac{2(p-1)}{p-2}}\right)^q\right)^{\frac{1}{q}}\left(\sum_{i\in E_j}\left|\sum_n a_{i,n}x_n\right|^p\right)^{\frac{1}{p}}\right|^p \\ &= \left(\sum_{i\in E_j} w_i^{\frac{2p}{p-2}}\right)^{\frac{p}{q}} \sum_{i\in E_j}\int\left|\sum_n a_{i,n}x_n\right|^p \\ &= \sigma_j^{p-1}\sum_{i\in E_j}\left\|\sum_n a_{i,n}x_n\right\|_p^p,\end{aligned}$$

whence

$$\|\pi(y)\|_{Y_{p,\{x_n\}}} \le \left(\sum_j\sum_{i\in E_j}\left\|\sum_n a_{i,n}x_n\right\|_p^p\right)^{\frac{1}{p}} \le \left(\sum_i\left\|\sum_n a_{i,n}x_n\right\|_p^p\right)^{\frac{1}{p}} = \|y\|_{Y_{p,\{x_n\}}}.$$

□

REMARK. We have actually shown that for $\left(\sum^{\oplus} X\right)_{I,w}$ and $\left(\sum^{\oplus} X\right)_{I,v}$ endowed with the norms $|||\ \ |||$ and $|||\ \ |||_v$, respectively, $\{\tilde{z}_{j,n}\}$ is isometrically equivalent to the standard basis of $\left(\sum^{\oplus} X\right)_{I,v}$, and there is a projection $P : \left(\sum^{\oplus} X\right)_{I,w} \to [\tilde{z}_{j,n} : j,n \in \mathbb{N}]_{\left(\sum^{\oplus} X\right)_{I,w}}$ with $\|P\| = 1$.

**Proposition 4.13.** *Let $2 < p < \infty$ and let $X$ be a closed subspace of $L_0^p[0,1]$ satisfying the hypotheses (a′) and (c′) above. Let $w = \{w_i\}$ and $w' = \{w'_i\}$ be sequences of scalars from $(0,1]$ satisfying condition $(*)$ of Proposition 2.1. Then* $\left(\sum^{\oplus} X\right)_{I,w'} \overset{c}{\hookrightarrow} \left(\sum^{\oplus} X\right)_{I,w}$.

**Proof.** By condition $(*)$, we may choose a sequence $\{E_j\}$ of disjoint nonempty finite subsets of $\mathbb{N}$ such that for each $j \in \mathbb{N}$, $\left(\frac{w'_j}{2}\right)^{\frac{2p}{p-2}} \le \sum_{i\in E_j} w_i^{\frac{2p}{p-2}} \le (w'_j)^{\frac{2p}{p-2}}$. Then for $v_j = \left(\sum_{i\in E_j} w_i^{\frac{2p}{p-2}}\right)^{\frac{p-2}{2p}}$, $\frac{w'_j}{2} \le v_j \le w'_j$. Let $v = \{v_j\}$ and let $y \in \left(\sum^{\oplus} X\right)_{I,w'}$. Then $\frac{1}{2}\|y\|_{\left(\sum^{\oplus} X\right)_{I,w'}} \le \|y\|_{\left(\sum^{\oplus} X\right)_{I,v}} \le \|y\|_{\left(\sum^{\oplus} X\right)_{I,w'}}$. Hence

$\left(\sum^{\oplus} X\right)_{I,w'} \sim \left(\sum^{\oplus} X\right)_{I,v}$. However, $\left(\sum^{\oplus} X\right)_{I,v} \overset{c}{\hookrightarrow} \left(\sum^{\oplus} X\right)_{I,w}$ by Proposition 4.12. It follows that $\left(\sum^{\oplus} X\right)_{I,w'} \overset{c}{\hookrightarrow} \left(\sum^{\oplus} X\right)_{I,w}$. □

Let $2 < p < \infty$ and let $X$ be a closed subspace of $L_0^p[0,1]$ satisfying the hypotheses (a′) and (c′) above. For each sequence $v = \{v_i\}$ from $(0,1]$, define spaces $Y_{p,\{x_n\}}$ and $Y_{2,v,\{x_n\}}$ as in the proof of Proposition 4.12. For each $k \in \mathbb{N}$, let $v^{(k)} = \left\{v_i^{(k)}\right\}_{i=1}^{\infty}$ be a sequence from $(0,1]$, and let $Y_k$ be a closed subspace of $\left(\sum^{\oplus} X\right)_{I,v^{(k)}}$. Let $(Y_1 \oplus Y_2 \oplus \cdots)_{p,2,\{v^{(k)}\}}$ be the Banach space of all sequences $\{y_k\}$ with $y_k \in Y_k$ such that $\|\{y_k\}\| = \max\left\{\left(\sum \|y_k\|^p_{Y_{p,\{x_n\}}}\right)^{\frac{1}{p}}, \left(\sum \|y_k\|^2_{Y_{2,v^{(k)},\{x_n\}}}\right)^{\frac{1}{2}}\right\} < \infty$.

For each sequence $v = \{v_i\}$ from $(0,1]$, let $S(X,v)$ denote $\left(\sum^{\oplus} X\right)_{I,v}$, and let $\tilde{S}(X,v)$ denote $(S(X,v) \oplus S(X,v) \oplus \cdots)_{p,2,\{v\}}$, where $\{v\}$ is the sequence $\{v, v, \ldots\}$.

**Proposition 4.14.** *Let $2 < p < \infty$ and let $X$ be a closed subspace of $L_0^p[0,1]$ satisfying the hypotheses (a′) and (c′) above. Let $w = \{w_i\}$ be a sequence of scalars from $(0,1]$ satisfying condition $(*)$ of Proposition 2.1. Let $S(X,w)$ and $\tilde{S}(X,w)$ be as above. Then $\tilde{S}(X,w) \overset{c}{\hookrightarrow} S(X,w)$.*

**Proof.** By condition $(*)$, we may choose a sequence $\{N_k\}$ of disjoint infinite subsets of $\mathbb{N}$ such that for each $\epsilon > 0$ and for each $k$,

$$\sum_{\substack{w_i < \epsilon \\ i \in N_k}} w_i^{\frac{2p}{p-2}} = \infty.$$

Hence for each $k$, we may choose a sequence $\left\{E_j^{(k)}\right\}_{j=1}^{\infty}$ of disjoint nonempty finite subsets of $N_k$ such that for each $j$,

$$\left(\frac{w_j}{2}\right)^{\frac{2p}{p-2}} \leq \sum_{i \in E_j^{(k)}} w_i^{\frac{2p}{p-2}} \leq w_j^{\frac{2p}{p-2}}.$$

Then for $v_j^{(k)} = \left(\sum_{i \in E_j^{(k)}} w_i^{\frac{2p}{p-2}}\right)^{\frac{p-2}{2p}}$, $\frac{w_j}{2} \leq v_j^{(k)} \leq w_j$. Hence for $v^{(k)} = \left\{v_j^{(k)}\right\}_{j=1}^{\infty}$ and

$y_k \in S(X,w)$, $\frac{1}{2}\|y_k\|_{Y_{2,w,\{x_n\}}} \le \|y_k\|_{Y_{2,v^{(k)},\{x_n\}}} \le \|y_k\|_{Y_{2,w,\{x_n\}}}$. Hence

$$\tilde{S}(X,w) = (S(X,w) \oplus S(X,w) \oplus \cdots)_{p,2,\{w\}} \sim \left(S\left(X, v^{(1)}\right) \oplus S\left(X, v^{(2)}\right) \oplus \cdots\right)_{p,2,\left\{v^{(k)}\right\}} \tag{4.6}$$

via the formal identity mapping.

Let $z_{j,n}^{(k)} = \sum_{i \in E_j^{(k)}} w_i^{\frac{2}{p-2}} \tilde{y}_{i,n}$ and let $\tilde{z}_{j,n}^{(k)}$ be the normalization of $z_{j,n}^{(k)}$ in $L^p(\Omega)$. Then by part (a) of Proposition 4.12, for each $k$ there is an isomorphism $J_k \colon S\left(X, v^{(k)}\right) \to \left[\tilde{z}_{j,n}^{(k)} : j,n \in \mathbb{N}\right]_{S(X,w)}$. Moreover, for $y_k \in S\left(X, v^{(k)}\right)$, $\|J_k(y_k)\|_{Y_{p,\{x_n\}}} = \|y_k\|_{Y_{p,\{x_n\}}}$ and $\|J_k(y_k)\|_{Y_{2,w,\{x_n\}}} = \|y_k\|_{Y_{2,v^{(k)},\{x_n\}}}$ by equations (4.4) and (4.5), respectively. Hence

$$\left(S\left(X, v^{(1)}\right) \oplus S\left(X, v^{(2)}\right) \oplus \cdots\right)_{p,2,\left\{v^{(k)}\right\}} \sim \left(\left[\tilde{z}_{j,n}^{(1)}\right]_{S(X,w)} \oplus \left[\tilde{z}_{j,n}^{(2)}\right]_{S(X,w)} \oplus \cdots\right)_{p,2,\{w\}} \tag{4.7}$$

via the isometry $\{y_k\} \mapsto \{J_k(y_k)\}$.

The direct sum on the right side of (4.7) should be thought of as an internal direct sum of subspaces of $S(X,w)$. We next show that

$$\left(\left[\tilde{z}_{j,n}^{(1)}\right]_{S(X,w)} \oplus \left[\tilde{z}_{j,n}^{(2)}\right]_{S(X,w)} \oplus \cdots\right)_{p,2,\{w\}} \sim \left[\tilde{z}_{j,n}^{(k)} : j,n,k \in \mathbb{N}\right]_{S(X,w)} \tag{4.8}$$

via the mapping $\{s_k\} \mapsto \sum s_k$. For each $k$ and for scalars $a_{j,n}^{(k)}$, let $s_k = \sum_j \sum_n a_{j,n}^{(k)} \tilde{z}_{j,n}^{(k)} \in \left[\tilde{z}_{j,n}^{(k)} : j,n \in \mathbb{N}\right]_{S(X,w)}$. Then by equations (4.4) and (4.5),

$$\begin{aligned}
\|\{s_k\}\| &= \max\left\{\left(\sum \|s_k\|^p_{Y_{p,\{x_n\}}}\right)^{\frac{1}{p}}, \left(\sum \|s_k\|^2_{Y_{2,w,\{x_n\}}}\right)^{\frac{1}{2}}\right\} \\
&= \max\left\{\left(\sum_k \left\|\sum_j \sum_n a^{(k)}_{j,n}\tilde{z}^{(k)}_{j,n}\right\|^p_{Y_{p,\{x_n\}}}\right)^{\frac{1}{p}}, \left(\sum_k \left\|\sum_j \sum_n a^{(k)}_{j,n}\tilde{z}^{(k)}_{j,n}\right\|^2_{Y_{2,w,\{x_n\}}}\right)^{\frac{1}{2}}\right\} \\
&= \max\left\{\left(\sum_k \sum_j \left\|\sum_n a^{(k)}_{j,n} x_n\right\|^p_p\right)^{\frac{1}{p}}, \left(\sum_k \sum_j \left(v^{(k)}_j\right)^2 \sum_n \left|a^{(k)}_{j,n}\right|^2 \|x_n\|^2_2\right)^{\frac{1}{2}}\right\} \\
&= \max\left\{\left(\left\|\sum_k \sum_j \sum_n a^{(k)}_{j,n}\tilde{z}^{(k)}_{j,n}\right\|^p_{Y_{p,\{x_n\}}}\right)^{\frac{1}{p}}, \left(\left\|\sum_k \sum_j \sum_n a^{(k)}_{j,n}\tilde{z}^{(k)}_{j,n}\right\|^2_{Y_{2,w,\{x_n\}}}\right)^{\frac{1}{2}}\right\} \\
&= \left|\left|\left|\sum_k \sum_j \sum_n a^{(k)}_{j,n}\tilde{z}^{(k)}_{j,n}\right|\right|\right| \approx \left\|\sum_k \sum_j \sum_n a^{(k)}_{j,n}\tilde{z}^{(k)}_{j,n}\right\|_{S(X,w)} = \left\|\sum s_k\right\|_{S(X,w)},
\end{aligned}$$

where $|||\ \ |||$ is as in the proof of Proposition 4.12. Hence the mapping $\{s_k\} \mapsto \sum s_k$ is an isomorphism.

By part (b) of Proposition 4.12, we have

$$\left[\tilde{z}^{(k)}_{j,n} : j,n,k \in \mathbb{N}\right]_{S(X,w)} \overset{c}{\hookrightarrow} \left(\sum\nolimits^{\oplus} X\right)_{I,w} = S(X,w). \tag{4.9}$$

Combining (4.6), (4.7), (4.8), and (4.9) yields $\tilde{S}(X,w) \overset{c}{\hookrightarrow} S(X,w)$. □

**Proposition 4.15.** *Let $2 < p < \infty$ and let $X$ be a closed subspace of $L^p_0[0,1]$ satisfying the hypotheses (a′) and (c′) above. Let $w = \{w_i\}$ be a sequence of scalars from $(0,1]$ satisfying condition $(*)$ of Proposition 2.1. Then*

$$\left(\sum\nolimits^{\oplus} X\right)_{I,w} \sim \left(\sum\nolimits^{\oplus} X\right)_{I,w} \oplus \left(\sum\nolimits^{\oplus} X\right)_{I,w}.$$

**Proof.** Let $S(X,w)$ and $\tilde{S}(X,w)$ be as in Proposition 4.14. Then $\tilde{S}(X,w) \overset{c}{\hookrightarrow} S(X,w)$. Let $Y$ be a closed subspace of $S(X,w)$ such that $S(X,w) \sim \tilde{S}(X,w) \oplus Y$. Note that $\tilde{S}(X,w) \sim S(X,w) \oplus \tilde{S}(X,w)$. Hence $S(X,w) \oplus S(X,w) \sim S(X,w) \oplus \tilde{S}(X,w) \oplus Y \sim \tilde{S}(X,w) \oplus Y \sim S(X,w)$. □

**Theorem 4.16.** *Let $2 < p < \infty$ and let $X$ be a closed subspace of $L^p_0[0,1]$ satisfying the hypotheses (a′) and (c′) above. Let $w = \{w_i\}$ and $w' = \{w'_i\}$ be* *sequences of scalars from* $(0,1]$ *satisfying condition* $(*)$ *of Proposition 2.1. Then* $\left(\sum^{\oplus} X\right)_{I,w} \sim \left(\sum^{\oplus} X\right)_{I,w'}$.

**Proof.** The spaces $\left(\sum^{\oplus} X\right)_{I,w}$ and $\left(\sum^{\oplus} X\right)_{I,w'}$ satisfy the hypotheses of Lemma 2.8. □

DEFINITION. *Let* $2 < p < \infty$. *Let* $X$ *be a closed subspace of* $L_0^p[0,1]$ *satisfying*

*(a′) the orthogonal projection of* $L^2[0,1]$ *onto* $\overline{X} \subset L^2[0,1]$, *when restricted to* $L^p[0,1]$, *yields a bounded projection* $P : L^p[0,1] \to X \subset L^p[0,1]$ *onto* $X$, *and*

*(c′)* $X$ *has an unconditional orthogonal normalized basis* $\{x_n\}$.

*Define* $\left(\sum^{\oplus} X\right)_I$, *the independent sum of* $X$, *to be (the isomorphism type of)* $\left(\sum^{\oplus} X\right)_{I,w}$ *for any sequence* $w = \{w_i\}$ *of scalars from* $(0,1]$ *satisfying condition* $(*)$ *of Proposition 2.1.*

By Theorem 4.16, $\left(\sum^{\oplus} X\right)_I$ is well-defined.

## The Space $D_p$

DEFINITION. *Let* $2 < p < \infty$, *let* $\{x_n\}$ *be the sequence of Rademacher functions, and let* $X = [x_n]_{L^p} \sim \ell^2$. *Define* $D_p$ *to be* $\left(\sum^{\oplus} X\right)_I$. *For the conjugate index* $q$, *define* $D_q$ *to be* $D_p^*$.

**Proposition 4.17.** *Let* $1 < p < \infty$ *where* $p \neq 2$. *Then*

*(a)* $X_p \overset{c}{\hookrightarrow} D_p$,

*(b)* $\left(\sum^{\oplus} \ell^2\right)_{\ell^p} \overset{c}{\hookrightarrow} D_p$, *and*

*(c)* $\left(\sum^{\oplus} \ell^2\right)_{\ell^p} \oplus X_p \overset{c}{\hookrightarrow} D_p$.

**Proof.** It suffices to prove the result for $2 < p < \infty$, since the result for $1 < p < 2$ will then follow by duality.

Suppose $2 < p < \infty$. Realize $D_p$ as $\left(\sum^{\oplus} X\right)_{I,w}$, where $X$ and $\{x_n\}$ are as in the definition of $D_p$, and $w = \{w_i\}$ is a sequence of scalars from $(0,1]$ satisfying condition $(*)$ of Proposition 2.1. Then $D_p = [\tilde{y}_{i,n} : i, n \in \mathbb{N}]_{L^p(\Omega)}$, where $\tilde{y}_{i,n} = (T_i(x_n)) \circ \pi_i \in L^p(\Omega)$, and $T_i$ and $\pi_i$ are as in the definition of $\left(\sum^{\oplus} X_i\right)_{I,w}$.

(a) Let $D_p^{(1)} = [\tilde{y}_{i,1} : i \in \mathbb{N}]_{L^p(\Omega)}$. Then $D_p^{(1)}$ is a complemented subspace of $D_p$ by the unconditionality of $\{\tilde{y}_{i,n}\}$, and $D_p^{(1)} = \left(\sum^{\oplus} X^{(1)}\right)_{I,w}$ where $X^{(1)} = [x_1]_{L^p}$ and $x_1 = 1_{[0,\frac{1}{2})} - 1_{[\frac{1}{2},1]}$. As noted in Example 4.1, $\left(\sum^{\oplus} X^{(1)}\right)_{I,w} \sim X_p$. Hence $X_p \sim D_p^{(1)} \overset{c}{\hookrightarrow} D_p$.

(b) Choose an increasing sequence $\{i_k\}$ of positive integers such that $\sum w_{i_k}^{\frac{2p}{p-2}} < \infty$, and let $w' = \{w_{i_k}\}$. Let $D'_p = [\tilde{y}_{i_k,n} : k, n \in \mathbb{N}]_{L^p(\Omega)}$. Then $D'_p$ is a complemented subspace of $D_p$ by the unconditionality of $\{\tilde{y}_{i,n}\}$, and
$D'_p = \left(\sum^{\oplus} X\right)_{I,w'} \sim \left(\sum^{\oplus} X\right)_{\ell^p} \sim \left(\sum^{\oplus} \ell^2\right)_{\ell^p}$ by Corollary 4.10. Hence
$\left(\sum^{\oplus} \ell^2\right)_{\ell^p} \sim D'_p \overset{c}{\hookrightarrow} D_p$.

(c) By Proposition 4.15 and parts (a) and (b) above,
$\left(\sum^{\oplus} \ell^2\right)_{\ell^p} \oplus X_p \overset{c}{\hookrightarrow} D_p \oplus D_p \sim D_p$.

□

For $2 < p < \infty$, it is clear that $D_p \overset{c}{\not\hookrightarrow} B_p$, since otherwise $X_p \overset{c}{\hookrightarrow} D_p \overset{c}{\hookrightarrow} B_p$ by part (a) of Proposition 4.17, so $X_p \overset{c}{\hookrightarrow} B_p$, contrary to part (g) of Proposition 2.37.

We now present results leading to the conclusion that $B_p \overset{c}{\not\hookrightarrow} D_p$ [**A**]. We begin with a definition and some preliminary observations used in the proof of the subsequent lemma.

Let $2 < p < \infty$ and let $\{r_n\}$ be the sequence of Rademacher functions. Given a sequence $w = \{w_i\}$ of positive scalars, let $\tilde{y}_{i,n} = T_i(r_n) \circ \pi_i$, where $T_i$ and $\pi_i$ are as in the definition of $\left(\sum^{\oplus} X_i\right)_{I,w}$. Let $P_0 : D_p \to D_p$ be the zero mapping. For each

$m \in \mathbb{N}$, let $P_m : D_p \to D_p$ be the natural projection of $D_p$ onto $[\tilde{y}_{i,n} : i \in \{1, \ldots, m\}, n \in \mathbb{N}]_{D_p}$. A sequence $\{z_k\}$ in $D_p$ will be said to be strip disjoint if there is an increasing sequence $\{m_k\}$ in $\mathbb{N}$ such that $\left\| \left(P_{m_k} - P_{m_{k-1}}\right)(z_k) \right\|_{D_p} \geq \left(1 - \frac{1}{2^k}\right) \|z_k\|_{D_p}$ for all $k \in \mathbb{N}$.

Let $2 < p < \infty$, let $w$ be a positive scalar, and let $\{w\} = \{w, w, \ldots\}$. Let $\{e_n\}$ be the standard basis for $X_{p,\{w\}}$. Let $T : X_{p,\{w\}} \to D_p$ be an isomorphic imbedding. Suppose $\epsilon > 0$ is such that for each $m \in \mathbb{N}$, $\|P_m(T(e_n))\|_{D_p} < \epsilon$ for infinitely many $n \in \mathbb{N}$.

Then we may choose increasing sequences $\{\gamma(n)\}$ and $\{m(n)\}$ in $\mathbb{N}$ such that $T\left(e_{\gamma(n)}\right) = x_n + y_n$, where $x_n = P_{m(n)}\left(T\left(e_{\gamma(n)}\right)\right)$, $\|x_n\|_{D_p} < \epsilon$, $\{y_n\}$ is strip disjoint, and $\{x_n\}$ and $\{y_n\}$ are block basic sequences with respect to the standard basis of $D_p$.

There are constants $K$ and $C$ such that for each finite $F \subset \mathbb{N}$,

$$\left\|T^{-1}\right\|^{-1} \left\| \sum_{n \in F} e_{\gamma(n)} \right\|_{X_{p,\{w\}}} \leq \left\| \sum_{n \in F} T\left(e_{\gamma(n)}\right) \right\|_{D_p} \leq \left\| \sum_{n \in F} x_n \right\|_{D_p} + \left\| \sum_{n \in F} y_n \right\|_{D_p},$$

where [letting $|F|$ denote the cardinality of $F$]

$$\left\| \sum_{n \in F} e_{\gamma(n)} \right\|_{X_{p,\{w\}}} = \max\left\{ |F|^{\frac{1}{p}}, |F|^{\frac{1}{2}} w \right\},$$

$$\left\| \sum_{n \in F} x_n \right\|_{D_p} \leq K \left( \sum_{n \in F} \|x_n\|_{D_p}^2 \right)^{\frac{1}{2}} \leq K \left( \sum_{n \in F} \epsilon^2 \right)^{\frac{1}{2}} = K |F|^{\frac{1}{2}} \epsilon,$$

and

$$\begin{aligned} \left\| \sum_{n \in F} y_n \right\|_{D_p} &\leq C \max\left\{ \left( \sum_{n \in F} \|y_n\|_p^p \right)^{\frac{1}{p}}, \left( \sum_{n \in F} \|y_n\|_2^2 \right)^{\frac{1}{2}} \right\} \\ &\leq C \max\left\{ |F|^{\frac{1}{p}} \|T\|, |F|^{\frac{1}{2}} \max_{n \in F} \|y_n\|_2 \right\}. \end{aligned}$$

Thus for $F$ such that $|F|^{\frac{1}{2}} w > |F|^{\frac{1}{p}}$ and $|F|^{\frac{1}{2}} \max_{n \in F} \|y_n\|_2 > |F|^{\frac{1}{p}} \|T\|$,

$$\left\|T^{-1}\right\|^{-1} |F|^{\frac{1}{2}} w \leq K |F|^{\frac{1}{2}} \epsilon + C |F|^{\frac{1}{2}} \max_{n \in F} \|y_n\|_2,$$

so

$$\max_{n\in F}\|y_n\|_2 \geq \frac{\left\|T^{-1}\right\|^{-1}w - K\epsilon}{C}.$$

Hence we may choose an increasing sequence $\{\beta(n)\}$ in $\mathbb{N}$ such that for all $n \in \mathbb{N}$

$$\left\|y_{\beta(n)}\right\|_2 \geq \frac{\left\|T^{-1}\right\|^{-1}w - K\epsilon}{C}.$$

**Lemma 4.18.** *Let $2 < p < \infty$. Let $\{e_{i,n}\}$ be the standard basis for $B_p$ and let $w_i = \left(\frac{1}{i}\right)^{\frac{p-2}{2p}}$. Suppose $T : B_p \to D_p$ is an isomorphic imbedding. Then there is an $\epsilon > 0$ such that for all but a finite number of $i \in \mathbb{N}$, there is an $m_i \in \mathbb{N}$ and an infinite $K_i \subset \mathbb{N}$ such that $\|P_{m_i}(T(e_{i,n}))\|_{D_p} \geq w_i\epsilon$ for all $n \in K_i$.*

**Proof.** Suppose the conclusion is false. Then for each $\epsilon > 0$, there is an infinite $\mathbb{N}_\epsilon \subset \mathbb{N}$ such that for all $i \in \mathbb{N}_\epsilon$, all $m \in \mathbb{N}$, and all infinite $K \subset \mathbb{N}$, there is an $n \in K$ such that $\|P_m(T(e_{i,n}))\|_{D_p} < w_i\epsilon$.

Fix $\epsilon > 0$ and let $\epsilon_i = \frac{\epsilon}{2^i}$. For $i \in \mathbb{N}$, choose $\alpha(i) \in \mathbb{N}_{\epsilon_i}$ such that $\{\alpha(i)\}$ is an increasing sequence in $\mathbb{N}$. Let $i \in \mathbb{N}$. Then for each $m \in \mathbb{N}$, $\left\|P_m\left(T\left(e_{\alpha(i),n}\right)\right)\right\|_{D_p} < w_{\alpha(i)}\epsilon_i = \frac{w_{\alpha(i)}}{2^i}\epsilon$ for infinitely many $n \in \mathbb{N}$.

We may choose increasing sequences $\{\gamma_i(n)\}$ and $\{m_i(n)\}$ in $\mathbb{N}$ such that $T\left(e_{\alpha(i),\gamma_i(n)}\right) = x_{i,n} + y_{i,n}$, where $x_{i,n} = P_{m_i(n)}\left(T\left(e_{\alpha(i),\gamma_i(n)}\right)\right)$, $\|x_{i,n}\|_{D_p} < \frac{w_{\alpha(i)}}{2^i}\epsilon$, $\{y_{i,n}\}_{i,n\in\mathbb{N}}$ is strip disjoint, and $\{x_{i,n}\}_{i,n\in\mathbb{N}}$ and $\{y_{i,n}\}_{i,n\in\mathbb{N}}$ are block basic sequences with respect to the standard basis of $D_p$.

There are constants $K$ and $C$, and there is an increasing sequence $\{\beta_i(n)\}$ in $\mathbb{N}$, such that for all $n \in \mathbb{N}$

$$\left\|y_{i,\beta_i(n)}\right\|_2 \geq \frac{\left\|T^{-1}\right\|^{-1}w_{\alpha(i)} - K\frac{w_{\alpha(i)}}{2^i}\epsilon}{C}.$$

By the fact that $L^p$ is of type 2 [**W**, III.A.17,23], and by Hölder's inequality for conjugate indices $p' = \frac{p}{2}$ and $q' = \frac{p}{p-2}$, there is a constant $K$ such that for scalars $a_{i,n}$

$$
\begin{aligned}
\left\|\sum_i \sum_n a_{i,n} x_{i,n}\right\|_{D_p} &\le K \left(\sum_i \sum_n |a_{i,n}|^2 \|x_{i,n}\|_{D_p}^2\right)^{\frac{1}{2}} \\
&\le K \left(\sum_i \sum_n |a_{i,n}|^2 \left(\tfrac{w_{\alpha(i)}}{2^i}\epsilon\right)^2\right)^{\frac{1}{2}} \\
&= K\epsilon \left(\sum_i \left(\sum_n |a_{i,n}|^2 w_{\alpha(i)}^2\right) \left(\tfrac{1}{2^i}\right)^2\right)^{\frac{1}{2}} \\
&\le K\epsilon \left(\left(\sum_i \left(\sum_n |a_{i,n}|^2 w_{\alpha(i)}^2\right)^{\frac{p}{2}}\right)^{\frac{2}{p}} \left(\sum_i \left(\tfrac{1}{2^i}\right)^{2\frac{p}{p-2}}\right)^{\frac{p-2}{p}}\right)^{\frac{1}{2}} \\
&= K\epsilon \left(\sum_i \left(\sum_n |a_{i,n}|^2 w_{\alpha(i)}^2\right)^{\frac{1}{2}p}\right)^{\frac{1}{p}} \left(\sum_i \left(\tfrac{1}{2^i}\right)^{\frac{2p}{p-2}}\right)^{\frac{p-2}{2p}} \\
&\le K\epsilon \left\|\sum_i \sum_n a_{i,n} e_{\alpha(i),n}\right\|_{B_p} \left(\sum_i \left(\tfrac{1}{2^i}\right)^{\frac{2p}{p-2}}\right)^{\frac{p-2}{2p}} \\
&= K\epsilon \left\|\sum_i \sum_n a_{i,n} e_{\alpha(i),\gamma_i(n)}\right\|_{B_p} \left(\sum_i \left(\tfrac{1}{2^i}\right)^{\frac{2p}{p-2}}\right)^{\frac{p-2}{2p}} .
\end{aligned}
$$

Thus given $\delta > 0$, $\|\sum_i \sum_n a_{i,n} x_{i,n}\|_{D_p} \le \delta \left\|\sum_i \sum_n a_{i,n} e_{\alpha(i),\gamma_i(n)}\right\|_{B_p}$ for $\epsilon$ sufficiently small. Define $S : \left[e_{\alpha(i),\gamma_i(n)} : i, n \in \mathbb{N}\right]_{B_p} \to D_p$ by $S\left(\sum_i \sum_n a_{i,n} e_{\alpha(i),\gamma_i(n)}\right) = \sum_i \sum_n a_{i,n} y_{i,n}$. Then for $\epsilon$ sufficiently small, $S$ is an isomorphic imbedding. Since $\{y_{i,n}\}_{i,n\in\mathbb{N}}$ is strip disjoint, $[y_{i,n} : i, n \in \mathbb{N}]_{D_p} \sim X_{p,v}$ for some $v$. However, $\left[e_{\alpha(i),\gamma_i(n)} : i, n \in \mathbb{N}\right]_{B_p} \sim B_p$. Since $X_{p,v} \hookrightarrow \ell^2 \oplus \ell^p$ by Proposition 2.1, Theorem 2.12, and part (a) of Proposition 2.24, $B_p \sim \left[e_{\alpha(i),\gamma_i(n)} : i, n \in \mathbb{N}\right]_{B_p} \hookrightarrow [y_{i,n} : i, n \in \mathbb{N}]_{D_p} \sim X_{p,v} \hookrightarrow \ell^2 \oplus \ell^p$, so $B_p \hookrightarrow \ell^2 \oplus \ell^p$, contrary to Lemma 2.23 and part (a) of Proposition 2.37. □

**Lemma 4.19.** *Let $2 < p < \infty$. Let $w = \{w_i\}$ where $w_i = \left(\frac{1}{i}\right)^{\frac{p-2}{2p}}$, and let $\tilde{y}_{i,n}$ be as above. Let $\{E_\ell\}$ be a sequence of disjoint nonempty finite subsets of $\mathbb{N}$. Let $\{z_{k,\ell}\}$ be a sequence in $D_p$ which is normalized with respect to $|||\ |||_{D_p}$ such that for each $\ell \in \mathbb{N}$, $z_{k,\ell} \in [\tilde{y}_{i,n} : i \in E_\ell, n \in \mathbb{N}]_{D_p}$ for all $k \in \mathbb{N}$ and $\{z_{k,\ell}\}_{k\in\mathbb{N}}$ is equivalent to the standard basis of $\ell^2$. Then there is an infinite $L \subset \mathbb{N}$, and for each $\ell \in L$ there is* *an infinite $K_\ell \subset \mathbb{N}$, such that $\{z_{k,\ell}\}_{k\in K_\ell,\, \ell\in L}$ is equivalent to either the standard basis of $\ell^2$ or the standard basis of $\left(\sum^{\oplus} \ell^2\right)_{\ell^p}$.*

**Proof.** By passing to a subsequence, we may assume that $\{z_{k,\ell}\}$ is a block basic sequence with respect to the standard basis of $D_p$.

Let $z_{k,\ell} = \sum_{i\in E_\ell} v_{i,k,\ell}$ where $v_{i,k,\ell} = \sum_{n\in N_{i,k,\ell}} b_{i,n}\tilde{y}_{i,n}$ for $N_{i,k,\ell} \subset \mathbb{N}$ and scalars $b_{i,n}$. Let $\lambda_{i,k,\ell} = \left(\sum_{n\in N_{i,k,\ell}} |b_{i,n}|^2\right)^{\frac{1}{2}}$. Then for scalars $a_{k,\ell}$

$$\left|\left|\left|\sum_\ell \sum_k a_{k,\ell} z_{k,\ell}\right|\right|\right|_{D_p} = \left|\left|\left|\sum_\ell \sum_k a_{k,\ell} \sum_{i\in E_\ell} v_{i,k,\ell}\right|\right|\right|_{D_p} = \left|\left|\left|\sum_\ell \sum_k a_{k,\ell} \sum_{i\in E_\ell} \sum_{n\in N_{i,k,\ell}} b_{i,n}\tilde{y}_{i,n}\right|\right|\right|_{D_p}$$

$$= \left|\left|\left|\sum_\ell \sum_{i\in E_\ell} \sum_k \sum_{n\in N_{i,k,\ell}} a_{k,\ell} b_{i,n}\tilde{y}_{i,n}\right|\right|\right|_{D_p}$$

$$= \max\left\{\left(\sum_\ell \sum_{i\in E_\ell}\left(\sum_k \sum_{n\in N_{i,k,\ell}} |a_{k,\ell}b_{i,n}|^2\right)^{\frac{1}{2}p}\right)^{\frac{1}{p}}, \left(\sum_\ell \sum_{i\in E_\ell} w_i^2 \sum_k \sum_{n\in N_{i,k,\ell}} |a_{k,\ell}b_{i,n}|^2\right)^{\frac{1}{2}}\right\}$$

$$= \max\left\{\left(\sum_\ell \sum_{i\in E_\ell}\left(\sum_k |a_{k,\ell}|^2 \lambda_{i,k,\ell}^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}, \left(\sum_\ell \sum_{i\in E_\ell} w_i^2 \sum_k |a_{k,\ell}|^2 \lambda_{i,k,\ell}^2\right)^{\frac{1}{2}}\right\}.$$

As a special case of the above,

$$1 = |||z_{k,\ell}|||_{D_p} = \max\left\{\left(\sum_{i\in E_\ell} \lambda_{i,k,\ell}^p\right)^{\frac{1}{p}}, \left(\sum_{i\in E_\ell} w_i^2\lambda_{i,k,\ell}^2\right)^{\frac{1}{2}}\right\} \geq \left(\sum_{i\in E_\ell} \lambda_{i,k,\ell}^p\right)^{\frac{1}{p}},$$

whence $\lambda_{i,k,\ell} \leq 1$ for $k, \ell \in \mathbb{N}$ and $i \in E_\ell$. Let $\{\epsilon_k\}$ be a sequence of positive scalars with limit zero. For each $\ell \in \mathbb{N}$, choose an increasing sequence $\{\alpha_\ell(k)\}$ in $\mathbb{N}$ and scalars $\Lambda_i$ for $i \in E_\ell$ such that $\left|\lambda_{i,\alpha_\ell(k),\ell} - \Lambda_i\right| < \epsilon_k$ for $k \in \mathbb{N}$ and $i \in E_\ell$. Then

$$\left|\left|\left|\sum_\ell \sum_k a_{k,\ell} z_{\alpha_\ell(k),\ell}\right|\right|\right|_{D_p}$$

$$= \max\left\{\left(\sum_\ell \sum_{i\in E_\ell}\left(\sum_k |a_{k,\ell}|^2 \lambda_{i,\alpha_\ell(k),\ell}^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}, \left(\sum_\ell \sum_{i\in E_\ell} w_i^2 \sum_k |a_{k,\ell}|^2 \lambda_{i,\alpha_\ell(k),\ell}^2\right)^{\frac{1}{2}}\right\}$$

$$\approx \max\left\{\left(\sum_\ell \sum_{i\in E_\ell} \Lambda_i^p\left(\sum_k |a_{k,\ell}|^2\right)^{\frac{1}{2}p}\right)^{\frac{1}{p}}, \left(\sum_\ell \sum_{i\in E_\ell} w_i^2\Lambda_i^2\left(\sum_k |a_{k,\ell}|^2\right)^{\frac{1}{2}2}\right)^{\frac{1}{2}}\right\}$$

$$= \max\left\{\left(\sum_\ell \sum_{i\in E_\ell} \Lambda_i^p \left\|\{a_{k,\ell}\}_k\right\|_{\ell^2}^p\right)^{\frac{1}{p}}, \left(\sum_\ell \sum_{i\in E_\ell} w_i^2\Lambda_i^2 \left\|\{a_{k,\ell}\}_k\right\|_{\ell^2}^2\right)^{\frac{1}{2}}\right\},$$

where the approximation can be improved to any degree by the choice of $\{\epsilon_k\}$ and $\{\alpha_\ell(k)\}$. As a special case of the above,

$1 = \left|\left|\left| z_{\alpha_\ell(k),\ell} \right|\right|\right|_{D_p} \approx \max\left\{ \left(\sum_{i\in E_\ell} \Lambda_i^p\right)^{\frac{1}{p}}, \left(\sum_{i\in E_\ell} w_i^2 \Lambda_i^2\right)^{\frac{1}{2}} \right\}$, where the approximation can be improved to any degree by the choice of $\{\epsilon_k\}$ and $\{\alpha_\ell(k)\}$. Hence $\left\{z_{\alpha_\ell(k),\ell}\right\}$ can be chosen to be equivalent to the standard basis of $\left(\sum^\oplus \ell^2\right)_{I,W}$ where $W = \{W_\ell\}$ and

$$W_\ell = \frac{\left(\sum_{i\in E_\ell} w_i^2 \Lambda_i^2\right)^{\frac{1}{2}}}{\left(\sum_{i\in E_\ell} \Lambda_i^p\right)^{\frac{1}{p}}}.$$

If $inf_{\ell\in\mathbb{N}} W_\ell > 0$, then $\left\{z_{\alpha_\ell(k),\ell}\right\}$ is equivalent to the standard basis of $\ell^2$.

If $inf_{\ell\in\mathbb{N}} W_\ell = 0$, then $\left\{z_{\alpha_\ell(k),\ell}\right\}$ is equivalent to the standard basis of $\left(\sum^\oplus \ell^2\right)_{\ell^p}$.

□

REMARK. As a special case of the first display in the above proof, $|||v_{i,k,\ell}|||_{D_p} = \max\{\lambda_{i,k,\ell}, w_i\lambda_{i,k,\ell}\} = \lambda_{i,k,\ell}$.

**Lemma 4.20.** *Let $2 < p < \infty$. Suppose $T : B_p \to D_p$ is an isomorphic imbedding. Then $B_p$ has a complemented subspace $X$ isomorphic to $B_p$, and $D_p$ has a closed subspace $Y$ isomorphic to $\ell^2 \oplus X_{p,v}$ or $\left(\sum^\oplus \ell^2\right)_{\ell^p} \oplus X_{p,v}$ for some $v$, such that $T(X) \subset Y$.*

**Proof.** Choose (as we may by Lemma 4.18) $\epsilon > 0$ and $\mathbb{N}' \subset \mathbb{N}$ with finite complement such that for each $i \in \mathbb{N}'$, there is an $m_i \in \mathbb{N}$ and an infinite $K_i \subset \mathbb{N}$ such that $\|P_{m_i}(T(e_{i,n}))\|_{D_p} \geq w_i\epsilon$ for all $n \in K_i$.

For each $i \in \mathbb{N}'$ and $n \in K_i$, let $T(e_{i,n}) = x_{i,n} + y_{i,n}$, where $x_{i,n} = P_{m_i}(T(e_{i,n}))$. For each $i \in \mathbb{N}'$, choose an infinite $H_i \subset K_i$ such that $y_{i,n} = r_{i,n} + s_{i,n}$ for $n \in H_i$, where $\|r_{i,n}\|_{D_p} < \frac{w_i}{2^i}\epsilon$ for $n \in H_i$, and $\{s_{i,n}\}_{n\in H_i}$ is strip disjoint. Choose infinite $G_i \subset H_i$ for $i \in \mathbb{N}'$ such that $\{s_{i,n}\}_{i\in\mathbb{N}',n\in G_i}$ is strip disjoint.

Now for $i \in \mathbb{N}'$ and $n \in G_i$, $T(e_{i,n}) = x_{i,n} + r_{i,n} + s_{i,n}$, where

$x_{i,n} = P_{m_i}(T(e_{i,n}))$, $\|r_{i,n}\|_{D_p} < \frac{w_i}{2^i}\varepsilon$, and $\{s_{i,n}\}_{i\in\mathbb{N}', n\in G_i}$ is strip disjoint.

For each $i \in \mathbb{N}'$, choose an infinite $F_i \subset G_i$ such that $\left\{x_{i,n}/\|x_{i,n}\|_{D_p}\right\}_{n\in F_i}$ is $(1+\frac{1}{2^i})$-equivalent to the standard basis of $\ell^2$. Choose (as we may by Lemma 4.19) an infinite $\mathbb{N}'' \subset \mathbb{N}'$, and for each $i \in \mathbb{N}''$ choose an infinite $E_i \subset F_i$, such that $[x_{i,n} : i \in \mathbb{N}'', n \in E_i]_{D_p}$ is isomorphic to $\ell^2$ or $\left(\sum^{\oplus} \ell^2\right)_{\ell^p}$. Now $[x_{i,n} : i \in \mathbb{N}'', n \in E_i]_{D_p} \sim [x_{i,n} + r_{i,n} : i \in \mathbb{N}'', n \in E_i]_{D_p}$, since $\|r_{i,n}\|_{D_p} < \frac{w_i}{2^i}\epsilon \leq \frac{\|x_{i,n}\|_{D_p}}{2^i}$ for $i \in \mathbb{N}''$ and $n \in E_i$, and $\{r_{i,n}\}_n$ has an upper $\ell^2$ estimate.

Let $X = [e_{i,n} : i \in \mathbb{N}'', n \in E_i]_{B_p} \sim B_p$, and let $Y = [x_{i,n} + r_{i,n} : i \in \mathbb{N}'', n \in E_i]_{D_p} \oplus [s_{i,n} : i \in \mathbb{N}'', n \in E_i]_{D_p}$. Then $T(X) = [x_{i,n} + r_{i,n} + s_{i,n} : i \in \mathbb{N}'', n \in E_i]_{D_p} \subset Y$, and $Y \sim [x_{i,n} : i \in \mathbb{N}'', n \in E_i]_{D_p} \oplus [s_{i,n} : i \in \mathbb{N}'', n \in E_i]_{D_p}$ is isomorphic to $\ell^2 \oplus X_{p,v}$ or $\left(\sum^{\oplus} \ell^2\right)_{\ell^p} \oplus X_{p,v}$ for some $v$. □

**Proposition 4.21.** *Let $1 < p < \infty$ where $p \neq 2$. Then $B_p \overset{c}{\not\hookrightarrow} D_p$.*

**Proof.** Suppose $2 < p < \infty$ and $B_p \overset{c}{\hookrightarrow} D_p$. Then $B_p \overset{c}{\hookrightarrow} \left(\sum^{\oplus} \ell^2\right)_{\ell^p} \oplus X_{p,v}$ for some $v$ by Lemma 4.20, but $X_{p,v} \overset{c}{\hookrightarrow} X_p$ for all $v$ by Proposition 2.1, Theorem 2.12, and part (b) of Proposition 2.24. Hence $B_p \overset{c}{\hookrightarrow} \left(\sum^{\oplus} \ell^2\right)_{\ell^p} \oplus X_p$, contrary to Proposition 2.48. The result for $1 < p < 2$ now follows by duality. □

## Sums Involving $D_p$

A few more $\mathcal{L}_p$ spaces can be constructed by forming sums involving $D_p$. The resulting spaces are $B_p \oplus D_p$ and $\left(\sum^{\oplus} X_p\right)_{\ell^p} \oplus D_p$.

We first present results leading to the conclusion that $D_p \overset{c}{\not\hookrightarrow} \left(\sum^{\oplus} X_p\right)_{\ell^p}$ **[A]**.

Given $E \subset \mathbb{N}$, let $P_E : \left(\sum^{\oplus} \ell^2\right)_{\ell^r} \to \left(\sum^{\oplus} \ell^2\right)_{\ell^r}$ be the natural projection onto the subspace $\left(\sum^{\oplus} X_i\right)_{\ell^r}$ with $X_i = \ell^2$ if $i \in E$ and $X_i = \{0\}$ otherwise. Given $M \in \mathbb{N}$, let $P_M = P_{\{1,\ldots,M\}}$.

Given $F \subset \mathbb{N}$, let $P'_F : \left(\sum^{\oplus} X_q\right)_{\ell^q} \to \left(\sum^{\oplus} X_q\right)_{\ell^q}$ be the natural projection onto the subspace $\left(\sum^{\oplus} Y_i\right)_{\ell^q}$ with $Y_i = X_q$ if $i \in F$ and $Y_i = \{0\}$ otherwise. Given $N \in \mathbb{N}$, let $P'_N = P'_{\{1,\ldots,N\}}$.

**Lemma 4.22.** *Let $1 < q < r < 2$. Then $\left(\sum^{\oplus} \ell^2\right)_{\ell^r} \not\hookrightarrow \left(\sum^{\oplus} X_q\right)_{\ell^q}$.*

**Proof.** Suppose $\left(\sum^{\oplus} \ell^2\right)_{\ell^r} \hookrightarrow \left(\sum^{\oplus} X_q\right)_{\ell^q}$. Let $T : \left(\sum^{\oplus} \ell^2\right)_{\ell^r} \to \left(\sum^{\oplus} X_q\right)_{\ell^q}$ be an isomorphic imbedding. Then given $n \in \mathbb{N}$, $P'_n \circ T : \left(\sum^{\oplus} \ell^2\right)_{\ell^r} \to \left(\sum^{\oplus} X_q\right)_{\ell^q}$ is not an isomorphic imbedding, essentially by Lemma 3.7. Thus given $\epsilon > 0$ and $m \in \mathbb{N}$, there is an $x \in \left(\sum^{\oplus} \ell^2\right)_{\ell^r}$ with $P_m(x) = 0$ such that $\|P'_n(T(x))\| < \frac{\epsilon}{2\|T^{-1}\|}\|x\|$. Hence there is an $M \in \mathbb{N}$ with $m < M$ such that $\|P'_n(T(P_M(x)))\| < \frac{\epsilon}{2\|T^{-1}\|}\|P_M(x)\| \leq \frac{\epsilon}{2}\|T(P_M(x))\|$. Letting $y = P_M(x)$ and $E = \{m+1,\ldots,M\}$, $P_E(y) = y$ and $\|P'_n(T(y))\| < \frac{\epsilon}{2}\|T(y)\|$. Now there is an $N \in \mathbb{N}$ with $n < N$ such that $\|P'_N(T(y))\| > \left(1 - \frac{\epsilon}{2}\right)\|T(y)\|$. Letting $F = \{n+1,\ldots,N\}$, $(1-\epsilon)\|T(y)\| < \|P'_F(T(y))\| \leq \|T(y)\|$.

Given $\epsilon_1, \epsilon_2, \ldots > 0$, we will inductively find disjoint nonempty finite sets $E_1, E_2, \ldots \subset \mathbb{N}$ with $\max E_i < \min E_{i'}$ for $i < i'$, $y_1, y_2, \ldots \in \left(\sum^{\oplus} \ell^2\right)_{\ell^r}$ with $P_{E_i}(y_i) = y_i$, and disjoint nonempty finite sets $F_1, F_2, \ldots \subset \mathbb{N}$ with $\max F_i < \min F_{i'}$ for $i < i'$, such that $(1-\epsilon_i)\|T(y_i)\| < \left\|P'_{F_i}(T(y_i))\right\| \leq \|T(y_i)\|$ for each $i \in \mathbb{N}$.

Given $\epsilon_1 > 0$, the argument above with $n = 1$ and $m = 1$ shows how to find a finite $E_1 \subset \mathbb{N}$ and $y_1 \in \left(\sum^{\oplus} \ell^2\right)_{\ell^r}$ with $P_{E_1}(y_1) = y_1$, and a finite $F_1 \subset \mathbb{N}$, such that $(1-\epsilon_1)\|T(y_1)\| < \left\|P'_{F_1}(T(y_1))\right\| \leq \|T(y_1)\|$.

Let $\{\epsilon_i\}$ be a sequence of positive scalars and let $k \in \mathbb{N}$. Suppose $E_1, \ldots, E_k$,

$y_1, \ldots, y_k$, and $F_1, \ldots, F_k$ satisfying our requirements for all $i \in \{1, \ldots, k\}$ have been found. The argument above with $n > \max F_k$ and $m > \max E_k$ shows how to find a finite $E_{k+1} \subset \mathbb{N}$ and $y_{k+1} \in \left(\sum^{\oplus} \ell^2\right)_{\ell^r}$ with $\max E_k < \min E_{k+1}$ and $P_{E_{k+1}}(y_{k+1}) = y_{k+1}$, and a finite $F_{k+1} \subset \mathbb{N}$ with $\max F_k < \min F_{k+1}$, such that $(1 - \epsilon_{k+1}) \|T(y_{k+1})\| < \left\|P'_{F_{k+1}}(T(y_{k+1}))\right\| \leq \|T(y_{k+1})\|$. Thus $\{E_i\}$, $\{y_i\}$, and $\{F_i\}$ can be found as claimed.

For $\{\epsilon_i\}$ approaching zero rapidly and $\{y_i\}$ normalized, $[y_i] \sim \ell^r$, but $[T(y_i)] \sim [P'_{F_i}(T(y_i))] \sim \ell^q$. Hence $\ell^r \hookrightarrow \ell^q$, contrary to fact. It follows that no such isomorphic imbedding $T$ exists. □

**Lemma 4.23.** *Let $1 \leq q < \infty$ and let $\{x_i\}$ be unconditional in $L^q$. Let $C$ be the sign-unconditional constant for $\{x_i\}$ and let $K_q$ be Khintchine's constant for $L^q$. Then for scalars $d_i$,*

$$\left\|\sum_i d_i x_i\right\|_q^q \overset{C^q K_q^q}{\underset{C^q K_q^q}{\approx}} \int \left(\sum_i |d_i x_i(s)|^2\right)^{\frac{1}{2}q} ds = \left\|\sum_i |d_i x_i|^2\right\|_{\frac{q}{2}}^{\frac{q}{2}}.$$

**Proof.** Let $\{r_i\}$ be the sequence of Rademacher functions. Then by the unconditionality of $\{x_i\}$, Fubini's theorem, and Khintchine's inequality, we have

$$\begin{aligned}
\left\|\sum_i d_i x_i\right\|_q^q &\overset{C^q}{\underset{C^q}{\approx}} \int \left(\int \left|\sum_i d_i r_i(t) x_i(s)\right|^q ds\right) dt \\
&= \int \left(\int \left|\sum_i d_i x_i(s) r_i(t)\right|^q dt\right) ds \\
&\overset{K_q^q}{\underset{K_q^q}{\approx}} \int \left(\sum_i |d_i x_i(s)|^2\right)^{\frac{1}{2}q} ds \\
&= \left\|\sum_i |d_i x_i|^2\right\|_{\frac{q}{2}}^{\frac{q}{2}}.
\end{aligned}$$

□

**Lemma 4.24.** *Let $1 < q < r < 2$. Then $\left(\sum^{\oplus} \ell^2\right)_{\ell^r} \hookrightarrow D_q$.*

**Proof.** Let $p$ be the conjugate index of $q$, let $\{r_n\}$ be the sequence of Rademacher functions, let $\Omega = \prod_{i=1}^{\infty}[0,1]$, and let $\{N_i\}$ be a sequence of disjoint infinite subsets of $\mathbb{N}$ with $\mathbb{N} = \bigcup_{i\in\mathbb{N}} N_i$. For each $i \in \mathbb{N}$, let $\{r_{i,n}\}_{n\in\mathbb{N}} = \{r_n\}_{n\in N_i}$, and let $z_i : [0,1] \to \mathbb{R}$ be the normalization in $L^q$ of $1_{[0,k_i]}$, where $k_i = w_i^{\frac{2p}{p-2}}$ and $\{w_i\}$ is a sequence of positive scalars satisfying condition $(*)$ of Proposition 2.1.

Let $u = (u_1, u_2, \ldots)$ and $v = (v_1, v_2, \ldots)$. Now $\{z_i(u_i)r_i(v_i)\}_{i\in\mathbb{N}}$, being a sequence of independent symmetric three-valued random variables, and is equivalent to the standard basis of $X_{q,\{w_i\}}$. Thus by [RII, Corollary 4.2], we may choose a sequence $\{a_i\}$ of scalars and a sequence $\{F_j\}$ of nonempty finite intervals in $\mathbb{N}$ with $\mathbb{N} = \bigcup_{j\in\mathbb{N}} F_j$ and $1 + \max F_j = \min F_{j+1}$, such that for $y_j(u,v) = \sum_{i\in F_j} a_i z_i(u_i) r_i(v_i)$, $\{y_j(u,v)\}$ is a (perturbation of) a sequence of independent $r$-stable normalized random variables in $L^q(\Omega^2)$. Then for scalars $b_{j,n}$, letting $c_j = \left(\sum_n |b_{j,n}|^2\right)^{\frac{1}{2}}$, by Khintchine's inequality, Lemma 4.23, and the $r$-stability of $\{y_j(u,v)\}$, for $t = (t_{i,n})_{i\in\mathbb{N},\, n\in N_i}$ we have

$$\begin{aligned}
&\left\| \sum_j \sum_n b_{j,n} \sum_{i\in F_j} a_i z_i(u_i) r_i(v_i) r_{i,n}(t_{i,n}) \right\|_{L^q(\Omega^3)}^q \\
&= \int_\Omega \int_\Omega \left( \int_\Omega \left| \sum_j \sum_n b_{j,n} \sum_{i\in F_j} a_i z_i(u_i) r_i(v_i) r_{i,n}(t_{i,n}) \right|^q dt \right) du\, dv \\
&\approx \int_\Omega \int_\Omega \left( \sum_j \sum_n |b_{j,n}|^2 \sum_{i\in F_j} |a_i z_i(u_i) r_i(v_i)|^2 \right)^{\frac{1}{2}q} du\, dv \\
&= \int_\Omega \int_\Omega \left( \sum_j \sum_{i\in F_j} |c_j a_i z_i(u_i) r_i(v_i)|^2 \right)^{\frac{1}{2}q} du\, dv \\
&\approx \left\| \sum_j \sum_{i\in F_j} c_j a_i z_i(u_i) r_i(v_i) \right\|_{L^q(\Omega^2)}^q \\
&= \left\| \sum_j c_j y_j(u,v) \right\|_{L^q(\Omega^2)}^q \approx \left( \sum_j |c_j|^r \right)^{\frac{1}{r}q} = \left( \sum_j \left( \sum_n |b_{j,n}|^2 \right)^{\frac{1}{2}r} \right)^{\frac{1}{r}q}.
\end{aligned}$$

Hence

$$\left[ \sum_{i\in F_j} a_i z_i(u_i) r_i(v_i) r_{i,n}(t_{i,n}) : j, n \in \mathbb{N} \right]_{L^q(\Omega^3)} \sim \left( \sum\nolimits^{\oplus} \ell^2 \right)_{\ell^r}.$$

Moreover, by the choice of $\{z_i\}$,

$$\left[\sum_{i\in F_j} a_i z_i(u_i) r_i(v_i) r_{i,n}(t_{i,n}) : j,n \in \mathbb{N}\right]_{L^q(\Omega^3)} \hookrightarrow D_q.$$

It follows that $\left(\sum^{\oplus} \ell^2\right)_{\ell^r} \hookrightarrow D_q$. □

**Proposition 4.25.** *Let $1 < p < \infty$ where $p \neq 2$. Then $D_p \overset{c}{\not\hookrightarrow} \left(\sum^{\oplus} X_p\right)_{\ell^p}$.*

**Proof.** Suppose $1 < q < 2$ and $D_q \overset{c}{\hookrightarrow} \left(\sum^{\oplus} X_q\right)_{\ell^q}$. Then for $1 < q < r < 2$, $\left(\sum^{\oplus} \ell^2\right)_{\ell^r} \hookrightarrow D_q \overset{c}{\hookrightarrow} \left(\sum^{\oplus} X_q\right)_{\ell^q}$ by Lemma 4.24, so $\left(\sum^{\oplus} \ell^2\right)_{\ell^r} \hookrightarrow \left(\sum^{\oplus} X_q\right)_{\ell^q}$, contrary to Lemma 4.22. Hence $D_q \overset{c}{\not\hookrightarrow} \left(\sum^{\oplus} X_q\right)_{\ell^q}$, and the result for $2 < p < \infty$ holds by duality. □

Next we present results leading to the conclusion that $\left(\sum^{\oplus} X_p\right)_{\ell^p} \overset{c}{\not\hookrightarrow} B_p \oplus D_p$ [**A**].

Let $1 < q < r < 2$, and let $p$ be the conjugate index of $q$. Let $\{e_i\}$ be the standard basis of $\ell^r$. Let $\{z_{i,j}\}$ be the standard basis of $D_p$, and let $\{z^*_{i,j}\}$ be the corresponding dual basis of $D_q$, where for each $j \in \mathbb{N}$, $[z_{i,j} : i \in \mathbb{N}]_{D_p} \sim \ell^2$.

Given $E \subset \mathbb{N}$, let $P_E : \ell^r \to \ell^r$ be the natural projection onto the subspace $\ell^r_{(E)} = [e_i : i \in E]_{\ell^r}$. Given $M \in \mathbb{N}$, let $P_M = P_{\{1,\ldots,M\}}$.

Given $F \subset \mathbb{N}$, let $P'_F : D_q \to D_q$ be the natural projection onto the subspace $D_q^{(F)} = \left[z^*_{i,j} : i \in \mathbb{N},\ j \in F\right]_{D_q}$. Given $N \in \mathbb{N}$, let $P'_N = P'_{\{1,\ldots,N\}}$ and let $D_q^{(N)} = D_q^{\{1,\ldots,N\}}$.

**Lemma 4.26.** *Let $1 < q < r < 2$. Suppose $T : \ell^r \to D_q$ is an isomorphic imbedding. Then for each sequence $\{\epsilon_i\}$ of positive scalars, there is a normalized block basic sequence $\{y_i\}$ in $\ell^r$ and a sequence $\{F_i\}$ of disjoint nonempty finite subsets of $\mathbb{N}$ with $\max F_i < \min F_{i'}$ for $i < i'$, such that $\ell^r \sim [y_i]_{\ell^r} \sim [T(y_i)]_{D_q} \sim \left[P'_{F_i}(T(y_i))\right]_{D_q}$ via equivalence of natural bases, with $(1-\epsilon_i)\|T(y_i)\| < \left\|P'_{F_i}(T(y_i))\right\| \le \|T(y_i)\|$ for each $i \in \mathbb{N}$.*

**Proof.** Given $n \in \mathbb{N}$, $D_q^{(n)} \sim \ell^2$, so $P_n' \circ T : \ell^r \to D_q^{(n)}$ is not an isomorphic imbedding. Thus given $\epsilon > 0$ and $m \in \mathbb{N}$, there is an $x \in \ell^r$ with $P_m(x) = 0$ such that $\|P_n'(T(x))\| < \frac{\epsilon}{2\|T^{-1}\|}\|x\|$. Hence there is an $M \in \mathbb{N}$ with $m < M$ such that $\|P_n'(T(P_M(x)))\| < \frac{\epsilon}{2\|T^{-1}\|}\|P_M(x)\| \leq \frac{\epsilon}{2}\|T(P_M(x))\|$. Letting $y = P_M(x)$ and $E = \{m+1, \ldots, M\}$, $P_E(y) = y$ and $\|P_n'(T(y))\| < \frac{\epsilon}{2}\|T(y)\|$. Now there is an $N \in \mathbb{N}$ with $n < N$ such that $\|P_N'(T(y))\| > \left(1 - \frac{\epsilon}{2}\right)\|T(y)\|$. Letting $F = \{n+1, \ldots, N\}$, $(1-\epsilon)\|T(y)\| < \|P_F'(T(y))\| \leq \|T(y)\|$.

Given $\epsilon_1, \epsilon_2, \ldots > 0$, we will inductively find disjoint nonempty finite sets $E_1, E_2, \ldots \subset \mathbb{N}$ with $\max E_i < \min E_{i'}$ for $i < i'$, $y_1, y_2, \ldots \in \ell^r$ with $P_{E_i}(y_i) = y_i$, and disjoint nonempty finite sets $F_1, F_2, \ldots \subset \mathbb{N}$ with $\max F_i < \min F_{i'}$ for $i < i'$, such that $(1-\epsilon_i)\|T(y_i)\| < \left\|P_{F_i}'(T(y_i))\right\| \leq \|T(y_i)\|$ for each $i \in \mathbb{N}$.

Given $\epsilon_1 > 0$, the argument above with $n = 1$ and $m = 1$ shows how to find a finite $E_1 \subset \mathbb{N}$ and $y_1 \in \ell^r$ with $P_{E_1}(y_1) = y_1$, and a finite $F_1 \subset \mathbb{N}$, such that $(1-\epsilon_1)\|T(y_1)\| < \left\|P_{F_1}'(T(y_1))\right\| \leq \|T(y_1)\|$.

Let $\{\epsilon_i\}$ be a sequence of positive scalars and let $k \in \mathbb{N}$. Suppose $E_1, \ldots, E_k$, $y_1, \ldots, y_k$, and $F_1, \ldots, F_k$ satisfying our requirements for all $i \in \{1, \ldots, k\}$ have been found. The argument above with $n > \max F_k$ and $m > \max E_k$ shows how to find a finite $E_{k+1} \subset \mathbb{N}$ and $y_{k+1} \in \ell^r$ with $\max E_k < \min E_{k+1}$ and $P_{E_{k+1}}(y_{k+1}) = y_{k+1}$, and a finite $F_{k+1} \subset \mathbb{N}$ with $\max F_k < \min F_{k+1}$, such that $(1-\epsilon_{k+1})\|T(y_{k+1})\| < \left\|P_{F_{k+1}}'(T(y_{k+1}))\right\| \leq \|T(y_{k+1})\|$. Thus $\{E_i\}$, $\{y_i\}$, and $\{F_i\}$ can be found as claimed.

For $\{\epsilon_i\}$ approaching zero rapidly and $\{y_i\}$ normalized, $\ell^r \sim [y_i]_{\ell^r} \sim [T(y_i)]_{D_q} \sim \left[P_{F_i}'(T(y_i))\right]_{D_q}$ via equivalence of natural bases. $\square$

**Lemma 4.27.** *Let $1 < q < r < 2$. Then* $\left(\sum^{\oplus} \ell^r\right)_{\ell^q} \not\hookrightarrow D_q$.

**Proof.** Suppose $\left(\sum^{\oplus} \ell^r\right)_{\ell^q} \hookrightarrow D_q$. Let $T : \left(\sum^{\oplus} \ell^r\right)_{\ell^q} \to D_q$ be an isomorphic imbedding. Let $\{e_{i,j}\}$ be the standard basis of $\left(\sum^{\oplus} \ell^r\right)_{\ell^q}$, where for each $j \in \mathbb{N}$, $\{e_{i,j}\}_{i\in\mathbb{N}}$ is isometrically equivalent to the standard basis of $\ell^r$. For each $j \in \mathbb{N}$, let $\ell^r_{(j)} = [e_{i,j} : i \in \mathbb{N}]$, and for a sequence $\left\{\epsilon_i^{(j)}\right\}_{i\in\mathbb{N}}$ of positive scalars, choose (as we may by Lemma 4.26) a normalized block basic sequence $\left\{y_i^{(j)}\right\}_{i\in\mathbb{N}}$ in $\ell^r_{(j)}$ and disjoint nonempty finite subsets $F_1^{(j)}, F_2^{(j)}, \ldots$ of $\mathbb{N}$ with $\max F_i^{(j)} < \min F_{i'}^{(j)}$ for $i < i'$, such that

$\ell^r \sim \ell^r_{(j)} \sim \left[y_i^{(j)} : i \in \mathbb{N}\right]_{\ell^r_{(j)}} \sim \left[T\left(y_i^{(j)}\right) : i \in \mathbb{N}\right]_{D_q} \sim \left[P'_{F_i^{(j)}}\left(T\left(y_i^{(j)}\right)\right) : i \in \mathbb{N}\right]_{D_q}$ via equivalence of natural bases, with

$\left(1 - \epsilon_i^{(j)}\right)\left\|T\left(y_i^{(j)}\right)\right\| < \left\|P'_{F_i^{(j)}}\left(T\left(y_i^{(j)}\right)\right)\right\| \le \left\|T\left(y_i^{(j)}\right)\right\|$ for each $i \in \mathbb{N}$.

For $\epsilon_i^{(j)}$ approaching zero rapidly and for infinite subsets $M_1, M_2, \ldots$ of $\mathbb{N}$ chosen so that $\left\{F_i^{(j)}\right\}_{i\in M_j,\, j\in\mathbb{N}}$ is disjoint,

$\left(\sum^{\oplus} \ell^r\right)_{\ell^q} \sim \left[T\left(y_i^{(j)}\right) : i \in M_j,\, j \in \mathbb{N}\right]_{D_q} \sim \left[P'_{F_i^{(j)}}\left(T\left(y_i^{(j)}\right)\right) : i \in M_j,\, j \in \mathbb{N}\right]_{D_q}$ via equivalence of natural bases. Hence the standard basis of $\left(\sum^{\oplus} \ell^r\right)_{\ell^q}$ is equivalent to the span in $L^q$ of a sequence of independent random variables, contrary to Lemma 3.7. It follows that $\left(\sum^{\oplus} \ell^r\right)_{\ell^q} \not\hookrightarrow D_q$. □

**Lemma 4.28.** *Let $1 < q < r < 2$. Then $\left(\sum^{\oplus} \ell^r\right)_{\ell^q} \not\hookrightarrow B_q \oplus D_q$.*

**Proof.** Suppose $\left(\sum^{\oplus} \ell^r\right)_{\ell^q} \hookrightarrow B_q \oplus D_q$. Let $T : \left(\sum^{\oplus} \ell^r\right)_{\ell^q} \to B_q \oplus D_q$ be an isomorphic imbedding. Let $Q : B_q \oplus D_q \to B_q \oplus \{0_{D_q}\}$ be the obvious projection. Then $QT : \left(\sum^{\oplus} \ell^r\right)_{\ell^q} \to B_q \oplus \{0_{D_q}\}$ is a bounded linear operator. As in the proof of Lemma 2.45, there is a subspace $X$ of $\left(\sum^{\oplus} \ell^r\right)_{\ell^q}$, isometric to $\left(\sum^{\oplus} \ell^r\right)_{\ell^q}$, such that $\left\|Q|_{T(X)}\right\| < 1$, whence $(I - Q)|_{T(X)}$ induces an isomorphic imbedding of $\left(\sum^{\oplus} \ell^r\right)_{\ell^q}$ into $D_q$. However by Lemma 4.27, no such imbedding exists. It follows that $\left(\sum^{\oplus} \ell^r\right)_{\ell^q} \not\hookrightarrow B_q \oplus D_q$. □

**Proposition 4.29.** *Let $1 < p < \infty$ where $p \neq 2$. Then* $\left(\sum^{\oplus} X_p\right)_{\ell^p} \overset{c}{\not\hookrightarrow} B_p \oplus D_p$.

**Proof.** First let $1 < q < 2$ and suppose $\left(\sum^{\oplus} X_q\right)_{\ell^q} \overset{c}{\hookrightarrow} B_q \oplus D_q$. For $1 < q < r < 2$, $\ell^r \hookrightarrow X_q$ by Lemma 2.35, so $\left(\sum^{\oplus} \ell^r\right)_{\ell^q} \hookrightarrow \left(\sum^{\oplus} X_q\right)_{\ell^q} \overset{c}{\hookrightarrow} B_q \oplus D_q$. Hence $\left(\sum^{\oplus} \ell^r\right)_{\ell^q} \hookrightarrow B_q \oplus D_q$, contrary to Lemma 4.28. It follows that $\left(\sum^{\oplus} X_q\right)_{\ell^q} \overset{c}{\not\hookrightarrow} B_q \oplus D_q$. The result now holds for $2 < p < \infty$ by duality. □

Finally, we distinguish $D_p$, $B_p \oplus D_p$, and $\left(\sum^{\oplus} X_p\right)_{\ell^p} \oplus D_p$ from each other and from the $\mathcal{L}_p$ spaces of Rosenthal.

**Proposition 4.30.** *Let $1 < p < \infty$ where $p \neq 2$. Then*

*(a)* $D_p \overset{c}{\not\hookrightarrow} B_p$,

*(b)* $B_p \overset{c}{\not\hookrightarrow} D_p$,

*(c)* $B_p \oplus X_p \overset{c}{\not\hookrightarrow} D_p$,

*(d)* $B_p \oplus D_p \overset{c}{\not\hookrightarrow} D_p$,

*(e)* $\left(\sum^{\oplus} X_p\right)_{\ell^p} \overset{c}{\not\hookrightarrow} D_p$,

*(f)* $D_p \overset{c}{\not\hookrightarrow} \left(\sum^{\oplus} X_p\right)_{\ell^p}$,

*(g)* $B_p \oplus D_p \overset{c}{\not\hookrightarrow} \left(\sum^{\oplus} X_p\right)_{\ell^p}$,

*(h)* $\left(\sum^{\oplus} X_p\right)_{\ell^p} \oplus D_p \overset{c}{\not\hookrightarrow} \left(\sum^{\oplus} X_p\right)_{\ell^p}$,

*(i)* $D_p \overset{c}{\not\hookrightarrow} B_p \oplus X_p$,

*(j)* $B_p \oplus D_p \overset{c}{\not\hookrightarrow} B_p \oplus X_p$,

*(k)* $D_p \overset{c}{\not\hookrightarrow} \left(\sum^{\oplus} \ell^2\right)_{\ell^p} \oplus X_p$,

*(l)* $\left(\sum^{\oplus} X_p\right)_{\ell^p} \overset{c}{\not\hookrightarrow} B_p \oplus D_p$, *and*

*(m)* $\left(\sum^{\oplus} X_p\right)_{\ell^p} \oplus D_p \overset{c}{\not\hookrightarrow} B_p \oplus D_p$.

**Proof.** Suppose $2 < p < \infty$.

(a) Suppose $D_p \overset{c}{\hookrightarrow} B_p$. Then $X_p \overset{c}{\hookrightarrow} D_p \overset{c}{\hookrightarrow} B_p$ by part (a) of Proposition 4.17, so $X_p \overset{c}{\hookrightarrow} B_p$, contrary to part (g) of Proposition 2.37.

(b) Part (b) is a restatement of Proposition 4.21.

(c) Part (c) is immediate from part (b).

(d) Part (d) is immediate from part (b).

(e) Suppose $\left(\sum^{\oplus} X_p\right)_{\ell^p} \stackrel{c}{\hookrightarrow} D_p$. Then $B_p \stackrel{c}{\hookrightarrow} \left(\sum^{\oplus} X_p\right)_{\ell^p} \stackrel{c}{\hookrightarrow} D_p$ by Proposition 2.27, so $B_p \stackrel{c}{\hookrightarrow} D_p$, contrary to part (b) above.

(f) Part (f) is a restatement of Proposition 4.25.

(g) Part (g) is immediate from part (f).

(h) Part (h) is immediate from part (f).

(i) Suppose $D_p \stackrel{c}{\hookrightarrow} B_p \oplus X_p$. Then $D_p \stackrel{c}{\hookrightarrow} B_p \oplus X_p \stackrel{c}{\hookrightarrow} \left(\sum^{\oplus} X_p\right)_{\ell^p}$ by part (a) of Proposition 2.43, so $D_p \stackrel{c}{\hookrightarrow} \left(\sum^{\oplus} X_p\right)_{\ell^p}$, contrary to part (f) above.

(j) Part (j) is immediate from part (i).

(k) Suppose $D_p \stackrel{c}{\hookrightarrow} \left(\sum^{\oplus} \ell^2\right)_{\ell^p} \oplus X_p$. Then $D_p \stackrel{c}{\hookrightarrow} \left(\sum^{\oplus} \ell^2\right)_{\ell^p} \oplus X_p \stackrel{c}{\hookrightarrow} B_p \oplus X_p$ by Proposition 2.32, so $D_p \stackrel{c}{\hookrightarrow} B_p \oplus X_p$, contrary to part (i) above.

(l) Part (l) is a restatement of Proposition 4.29.

(m) Part (m) is immediate from part (l).

The result for $1 < p < 2$ follows by duality. □

Building on diagram (2.27), for $1 < p < \infty$ where $p \neq 2$, we have

$$
\begin{array}{ccccccccc}
B_p & & & & \left(\sum^{\oplus} X_p\right)_{\ell^p} & & & & \\
 & \stackrel{c}{\searrow} & & \stackrel{c}{\nearrow} & & \stackrel{c}{\searrow} & & & \\
 & & B_p \oplus X_p & & & & \left(\sum^{\oplus} X_p\right)_{\ell^p} \oplus D_p & \stackrel{c}{\to} & L^p. \\
 & \stackrel{c}{\nearrow} & & \stackrel{c}{\searrow} & & \stackrel{c}{\nearrow} & & & \\
\left(\sum^{\oplus} \ell^2\right)_{\ell^p} \oplus X_p & & & & B_p \oplus D_p & & & & \\
 & \stackrel{c}{\searrow} & & \stackrel{c}{\nearrow} & & & & & \\
 & & D_p & & & & & &
\end{array}
\tag{4.10}
$$

# CHAPTER V

# THE CONSTRUCTION AND ORDINAL INDEX OF BOURGAIN, ROSENTHAL, AND SCHECHTMAN

Let $1 < p < \infty$ and let $B$ and $B_1, B_2, \ldots$ be separable Banach spaces with $B \hookrightarrow L^p$ and $B_i \hookrightarrow L^p$. Bourgain, Rosenthal, and Schechtman [**B-R-S**] iterate and intertwine two constructions, a disjoint sum of $B$ with itself and an independent sum of $B_1, B_2, \ldots$, to produce a chain $\{R^p_\alpha\}_{\alpha<\omega_1}$ of separable $\mathcal{L}_p$ spaces. An ordinal index is introduced which assigns to each separable Banach space $B$ an ordinal number $h_p(B)$. The index $h_p(\ )$ proves to be an isomorphic invariant, and is used to select a subchain $\left\{R^p_{\tau(\alpha)}\right\}_{\alpha<\omega_1}$ of [infinite-dimensional] isomorphically distinct spaces. Thus Bourgain, Rosenthal, and Schechtman show that there are uncountably many separable infinite-dimensional $\mathcal{L}_p$ spaces [up to isomorphism].

## Preliminaries

We let $\omega_1$ denote the first uncountable ordinal, and we let $\omega$ denote the first infinite ordinal [except in some contexts where $\omega$ will denote an element of a space $\Omega$].

A strict partial order on a nonempty set $X$ is a relation $\prec$ on $X$ which is transitive and anti-reflexive.

A tree is a nonempty set $T$ with a strict partial order $\prec$ such that for each $x \in T$, $\{y \in T : y \prec x\}$ is well-ordered by $\prec$. We say that a tree $(T, \prec)$ is a CFRE (countable finite-ranked elements) tree if $T$ is finite or countable, and for each $x \in T$,

$\{y \in T : y \prec x\}$ is finite.

Let $(T, \prec)$ be a tree. A subtree of $T$ is a nonempty subset $S$ of $T$ with partial order $\prec$ [suitably restricted] such that for each $x \in S$, the set $\{y \in T : y \prec x\}$ is contained in $S$.

Let $(T, \prec)$ be a tree. A branch of $T$ is a maximal totally ordered subset of $T$. Suppose $(T, \prec)$ is a CFRE tree. We say that $B$ is a finite branch of $T$ if $B$ is of the form $\{y \in T : y \preceq x\}$ for some $x \in T$. We call $\{y \in T : y \preceq x\}$ the finite branch of $T$ generated by $x$. Note that a finite branch of $T$ need not be a branch of $T$, although a finite branch of $T$ is a branch of some subtree of $T$.

Let $\triangleleft$ be a relation on a nonempty set $X$.

An infinite $\triangleleft$-chain $x_1 \triangleleft x_2 \triangleleft \cdots$ in $X$ is a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $X$ such that $x_n \triangleleft x_{n+1}$ for all $n \in \mathbb{N}$. A finite $\triangleleft$-chain $x_1 \triangleleft \cdots \triangleleft x_N$ in $X$ is a sequence $\{x_n\}_{n=1}^{N}$ in $X$ such that $x_n \triangleleft x_{n+1}$ for all $1 \leq n < N$. An $x \in X$ is $\triangleleft$-terminal in $X$ if there is no $y \in X$ such that $x \triangleleft y$.

The relation $\triangleleft$ is well-founded in $X$ if there is no infinite $\triangleleft$-chain $x_1 \triangleleft x_2 \triangleleft \cdots$ in $X$. Note that if $\triangleleft$ is well-founded, then $\triangleleft$ must be anti-reflexive and there can be no finite $\triangleleft$-chain $x_1 \triangleleft \cdots \triangleleft x_N$ with $x_1 = x_N$.

For $n \in \mathbb{N}$, an $n$-string is an $n$-tuple which is not delimited by punctuation. We will identify a 0-string with the empty set. For $n \in \mathbb{N} \cup \{0\}$, let $D_n$ be the set of all $n$-strings of 0's and 1's. Then $D_n = \{t_1 \cdots t_n : t_i \in \{0, 1\}$ for all $1 \leq i \leq n\}$ for $n \in \mathbb{N}$, and $D_0 = \{\emptyset\}$. Fix $n \in \mathbb{N} \cup \{0\}$. Then $D_n$ has cardinality $2^n$. There is a natural identification of $D_n$ with $S_n = \{0, \ldots, 2^n - 1\}$, namely $t_1 \cdots t_n \mapsto \sum_{i=1}^{n} t_i 2^{n-i}$ for $n \in \mathbb{N}$, and $\{\emptyset\} \mapsto 0$. Thus for $n \in \mathbb{N}$, $t_1 \cdots t_n \in D_n$ is the $n$-place binary expansion [possibly with leading 0's] of some $r \in S_n$.

Let $n, m \in \mathbb{N} \cup \{0\}$. Given $t \in D_n$ and $s \in D_m$, let $t{\cdot}s$ be the element of $D_{n+m}$ formed by the concatenation of $t$ and $s$.

Let $(\Omega, \mathcal{M}, \mu)$ and $(\Omega', \mathcal{M}', \mu')$ be probability spaces, and let $X$ and $X'$ be spaces of measurable functions on $\Omega$ and $\Omega'$, respectively. We say that $X$ and $X'$ are distributionally isomorphic, denoted $X \overset{\text{dist}}{\sim} X'$, if there is a linear bijection $T : X \to X'$ such that $\operatorname{dist}(Tx) = \operatorname{dist}(x)$ for all $x \in X$.

## The Ordinal Index

Before introducing the ordinal index $h_p$, we introduce a general ordinal index $h$ based on essentially the same concept, but applicable to a simpler class of spaces.

### A General Ordinal Index $h$

Let $\triangleleft$ be a relation on a nonempty set $X$.

For each ordinal $\alpha$, we define a subset $H_\alpha(\triangleleft)$ of $X$. Let $H_0(\triangleleft) = X$. If $\alpha = \beta + 1$ and $H_\beta(\triangleleft)$ has been defined, let $H_\alpha(\triangleleft) = \{x \in H_\beta(\triangleleft) : x \triangleleft y \text{ for some } y \in H_\beta(\triangleleft)\}$. If $\alpha$ is a limit ordinal and $H_\beta(\triangleleft)$ has been defined for all $\beta < \alpha$, let $H_\alpha(\triangleleft) = \bigcap_{\beta<\alpha} H_\beta(\triangleleft)$.

If $\beta < \alpha$, then $H_\beta(\triangleleft) \supset H_\alpha(\triangleleft)$. The members of the nonincreasing family $(H_\alpha(\triangleleft))$ cannot all be distinct. For suppose the members are distinct. Then there is a family $(x_\alpha)$ of distinct elements of $X$, with $x_\alpha \in H_\alpha(\triangleleft) \setminus H_{\alpha+1}(\triangleleft)$. Thus for a sufficiently large ordinal $\Gamma$, $\{x_\alpha : \alpha < \Gamma\}$ has cardinality larger than the cardinality of $X$, contrary to $\{x_\alpha : \alpha < \Gamma\} \subset X$. Hence there is a least ordinal $\gamma$ such that $H_\gamma(\triangleleft) = H_{\gamma+1}(\triangleleft)$. Let $h(\triangleleft)$ denote this least ordinal $\gamma$, and let $S(\triangleleft)$ denote the stable set $H_\gamma(\triangleleft)$. Then the cardinality of $h(\triangleleft)$ is bounded by the cardinality of $X$. Note that if $H_\gamma(\triangleleft) = H_{\gamma+1}(\triangleleft)$, then $H_\gamma(\triangleleft) = H_{\gamma'}(\triangleleft)$ for all $\gamma' > \gamma$.

Suppose $\lhd$ is not well-founded. Then there is an infinite $\lhd$-chain $x_1 \lhd x_2 \lhd \cdots$ in $X$. For such a chain, $\{x_1, x_2, \ldots\} \subset H_\alpha(\lhd)$ for all $\alpha$. Thus $\{x_1, x_2, \ldots\} \subset S(\lhd)$ and $S(\lhd) \neq \emptyset$. For the converse, suppose $S(\lhd) \neq \emptyset$. Let $x \in S(\lhd)$. Then $x$ is not $\lhd$-terminal in $S(\lhd)$, so there is some $y \in S(\lhd)$ with $x \lhd y$. By induction, there is an infinite $\lhd$-chain $x_1 \lhd x_2 \lhd \cdots$ in $S(\lhd) \subset X$. Thus $\lhd$ is not well-founded. It follows that $\lhd$ is well-founded if and only if $S(\lhd) = \emptyset$.

Let $\lhd$ and $\lhd'$ be relations on nonempty sets $X$ and $X'$, respectively. A function $\tau : (X, \lhd) \to (X', \lhd')$ preserves relations if $\tau x \lhd' \tau y$ whenever $x \lhd y$.

The following lemma [**B-R-S**, Lemma 2.4] establishes a property of the ordinal index $h$ with respect to relation-preserving maps.

**Lemma 5.1.** *Let $\lhd$ and $\lhd'$ be relations on nonempty sets $X$ and $X'$, respectively. Suppose $\tau : (X, \lhd) \to (X', \lhd')$ preserves relations. Then $\tau(H_\alpha(\lhd)) \subset H_\alpha(\lhd')$ for all ordinals $\alpha$. If in addition $\lhd'$ is well-founded, then $h(\lhd) \leq h(\lhd')$.*

**Proof.** Clearly $\tau(H_0(\lhd)) = \tau(X) \subset X' = H_0(\lhd')$. Suppose $\alpha = \beta + 1$ and $\tau(H_\beta(\lhd)) \subset H_\beta(\lhd')$. Then $\tau : H_\beta(\lhd) \to H_\beta(\lhd')$ [suitably restricted]. Since $\tau$ preserves relations, if $x$ is not $\lhd$-terminal in $H_\beta(\lhd)$, then $\tau(x)$ is not $\lhd'$-terminal in $H_\beta(\lhd')$. Hence $\tau(H_\alpha(\lhd)) \subset H_\alpha(\lhd')$. Suppose $\alpha$ is a limit ordinal and $\tau(H_\beta(\lhd)) \subset H_\beta(\lhd')$ for all $\beta < \alpha$. Then $\tau(H_\alpha(\lhd)) = \tau\left(\bigcap_{\beta<\alpha} H_\beta(\lhd)\right) \subset \bigcap_{\beta<\alpha} \tau(H_\beta(\lhd)) \subset \bigcap_{\beta<\alpha} H_\beta(\lhd') = H_\alpha(\lhd')$.

Suppose $\lhd'$ is well-founded. Let $\gamma = h(\lhd)$ and $\gamma' = h(\lhd')$. Then $\tau(H_{\gamma'}(\lhd)) \subset H_{\gamma'}(\lhd') = \emptyset$. Thus $H_{\gamma'}(\lhd) = \emptyset$ as well. Hence $\gamma \leq \gamma'$ and $h(\lhd) \leq h(\lhd')$. $\square$

### Motivation from $L^p$

Let $1 \leq p < \infty$. Let $\{g_n\}_{n\in\mathbb{N}}$ be the sequence of normalized functions in $L^p$ given by $g_1 = 1_{[0,1]}$, $g_2 = 2^{\frac{1}{p}} 1_{[0,1/2]}$, $g_3 = 2^{\frac{1}{p}} 1_{[1/2,1]}$, $\ldots$ , $g_n = 2^{\frac{k}{p}} 1_{[r/2^k,(r+1)/2^k]}$,

..., where $n = 2^k + r$ such that $k \in \mathbb{N} \cup \{0\}$ and $0 \le r < 2^k$. For $n$, $k$, and $r$ as above, $2n = 2^{k+1} + 2r$ where $0 \le 2r < 2^{k+1}$, and $2n+1 = 2^{k+1} + (2r+1)$ where $0 < 2r+1 < 2^{k+1}$. Thus $g_{2n} = 2^{\frac{(k+1)}{p}} 1_{[2r/2^{k+1},(2r+1)/2^{k+1}]} = 2^{\frac{(k+1)}{p}} 1_{[r/2^k,(r+1/2)/2^k]}$, $g_{2n+1} = 2^{\frac{(k+1)}{p}} 1_{[(2r+1)/2^{k+1},(2r+2)/2^{k+1}]} = 2^{\frac{(k+1)}{p}} 1_{[(r+1/2)/2^k,(r+1)/2^k]}$, and $g_n = 2^{-\frac{1}{p}} (g_{2n} + g_{2n+1})$. This reflects the fact that $supp\, g_n = supp\, g_{2n} \cup supp\, g_{2n+1}$ [with the union being essentially disjoint]. The coefficient $2^{-\frac{1}{p}}$ is simply a normalization factor. Thus the functions $g_1, g_2, \ldots$ can be arranged in a binary tree

$$\begin{array}{lccccccc}
\text{[level 0:]} & & & & g_1 & & & \\
\text{[level 1:]} & & g_2 & & & & g_3 & \\
\text{[level 2:]} & g_4 & & g_5 & & g_6 & & g_7 \\
\vdots & & & & \vdots & & &
\end{array}$$

according to their supports, where the functions at level $k$ are of the form $g_{2^k+r}$ with $0 \le r < 2^k$.

Indexing by binary expansions, $g_t = 2^{-\frac{1}{p}} (g_{t\cdot 0} + g_{t\cdot 1})$, where $t$ is the binary expansion of $n \in \mathbb{N}$, and $t\cdot 0$ and $t\cdot 1$ are the binary expansions of $2n$ and $2n+1$, respectively. The corresponding tree is

$$\begin{array}{lccccccc}
\text{[level 0:]} & & & & g_1 & & & \\
\text{[level 1:]} & & g_{10} & & & & g_{11} & \\
\text{[level 2:]} & g_{100} & & g_{101} & & g_{110} & & g_{111} \\
\vdots & & & & \vdots & & & ,
\end{array}$$

where the functions at level $k$ are of the form $g_{1\cdot s}$ where $s$ is the $k$-place binary expansion of $r$ for $0 \le r < 2^k$.

Dropping the superfluous leading 1's and indexing by strings of 0's and 1's, $g_s = 2^{-\frac{1}{p}} (g_{s\cdot 0} + g_{s\cdot 1})$, where $s$ is a string of 0's and 1's. The corresponding tree is

$$\begin{array}{lccccccc}
\text{[level 0:]} & & & & g_\emptyset & & & \\
\text{[level 1:]} & & g_0 & & & & g_1 & \\
\text{[level 2:]} & g_{00} & & g_{01} & & g_{10} & & g_{11} \\
\vdots & & & & \vdots & & & ,
\end{array}$$

where the functions at level $k$ are indexed by $k$-strings of 0's and 1's.

Level $k$ itself can be thought of as a $2^k$-tuple of elements of $L^p$. Recalling that $D_k$ is the set of all $k$-strings of 0's and 1's, the cardinality of $D_k$ is $2^k$. Thus level $k$ can be thought of as a function from $D_k$ to $L^p$, or an element of $(L^p)^{D_k}$. Letting $u_k$ denote level $k$,

$$\begin{array}{lcl} u_0 & = & ( \qquad\qquad\qquad g_\emptyset \qquad\qquad\qquad ) \\ u_1 & = & ( \qquad g_0 \qquad , \qquad g_1 \qquad ) \\ u_2 & = & ( \; g_{00} \; , \; g_{01} \; , \; g_{10} \; , \; g_{11} \; ) \\ \vdots & & \qquad\qquad\qquad \vdots \qquad\qquad\qquad , \end{array} \tag{5.1}$$

where for each $s \in D_k$, $u_k(s) = 2^{-\frac{1}{p}}(u_{k+1}(s\cdot 0) + u_{k+1}(s\cdot 1))$. Moreover, for each $s \in D_k$ and each $d \in \mathbb{N}$,

$$u_k(s) = 2^{-\frac{d}{p}} \sum_{r \in D_d} u_{k+d}(s\cdot r). \tag{5.2}$$

Furthermore, for each $k \in \mathbb{N} \cup \{0\}$ and each $c \in \mathbb{R}^{D_k}$,

$$\int \left| \sum_{s \in D_k} c(s) u_k(s) \right|^p = \sum_{s \in D_k} |c(s)|^p \int |u_k(s)|^p = \sum_{s \in D_k} |c(s)|^p . \tag{5.3}$$

**The Space $\left(\overline{B}^\delta, \prec\right)$**

For $n \in \mathbb{N} \cup \{0\}$, recall that $D_n$ is the set of all $n$-strings of 0's and 1's, and there is a natural identification of $D_n$ with $\{0, \ldots, 2^n - 1\}$, namely $t_1 \cdots t_n \mapsto \sum_{i=1}^n t_i 2^{n-i}$ for $n \in \mathbb{N}$, and $\{\emptyset\} \mapsto 0$. For a vector space $B$, $B^{D_n}$ is the set of all functions from $D_n$ to $B$, which can be identified with the set of all $2^n$-tuples $(b_0, \ldots, b_{2^n-1})$ of elements of $B$. We identify $B^{D_0}$ with $B$.

We do not assign an independent meaning to $\mathcal{D}$, but given a vector space $B$, we let $B^{\mathcal{D}}$ denote $\bigcup_{n=0}^\infty B^{D_n}$.

Let $B$ be a vector space. If $u \in B^{\mathcal{D}}$, then $u \in B^{D_n}$ for a unique $n \in \mathbb{N} \cup \{0\}$, denoted $|u|$. Define $\prec$ on $B^{\mathcal{D}}$ by $u \prec v$ if $|u| < |v|$ and for $k = |v| - |u|$, $u(t) = 2^{-\frac{k}{p}} \sum_{s \in D_k} v(t\cdot s)$ for all $t \in D^{|u|}$. Then $\prec$ is a strict partial order.

Definition. *Suppose $B$ is a separable Banach space, $1 \le p < \infty$, and $0 < \delta \le 1$. Let $\overline{B}^{\delta}$ be the set of all $u \in B^{\mathcal{D}}$ such that*

$$\delta\left(\sum_{t\in D_{|u|}} |c(t)|^p\right)^{\frac{1}{p}} \le \left\|\sum_{t\in D_{|u|}} c(t)u(t)\right\|_B \le \left(\sum_{t\in D_{|u|}} |c(t)|^p\right)^{\frac{1}{p}}$$

*for all $c \in \mathbb{R}^{D_{|u|}}$. Let $\prec$ on $\overline{B}^{\delta}$ be the strict partial order $\prec$ on $B^{\mathcal{D}}$ [suitably restricted].*

Remark. For $B = L^p$, equation (5.2) implies that $u_0 \prec u_1 \prec \cdots$, and equation (5.3) implies that $u_k \in \overline{L^p}^1$ for all $k \in \mathbb{N} \cup \{0\}$, whence $\left(\overline{L^p}^1, \prec\right)$ is not well-founded.

## A Characterization of $L^p \hookrightarrow B$

The following proposition [**B-R-S**, Proposition 2.2] characterizes those spaces $B$ for which $L^p \hookrightarrow B$. Essentially, the issue is whether or not $B$ contains a sequence which simulates the behavior of the sequence $\{u_k(t)\}_{k\ge 0, t\in D_k}$ in $L^p$.

**Proposition 5.2.** *Let $B$ be a separable Banach space and let $1 \le p < \infty$. Then $L^p \hookrightarrow B$ if and only if there is a $0 < \delta \le 1$ such that $\left(\overline{B}^{\delta}, \prec\right)$ is not well-founded.*

**Proof.** Suppose $L^p \hookrightarrow B$. Let $T : L^p \to B$ be an isomorphic imbedding with $\|T\| \le 1$, and let $0 < \delta \le 1$ be such that $\delta \|x\|_p \le \|T(x)\|_B \le \|x\|_p$ for all $x \in L^p$. Let $\tau : (L^p)^{\mathcal{D}} \to B^{\mathcal{D}}$ be defined by $(\tau u)(t) = T(u(t))$ for $u \in (L^p)^{\mathcal{D}}$ and $t \in D_{|u|}$. Then $\tau$ preserves order by the linearity of $T$.

Let $u \in \overline{L^p}^1$. Then for all $c \in \mathbb{R}^{D_{|u|}}$,

$$\begin{aligned}
&\delta\left(\sum_{t\in D_{|u|}} |c(t)|^p\right)^{\frac{1}{p}} \\
&= \delta\left\|\sum_{t\in D_{|u|}} c(t)u(t)\right\|_p \le \left\|T\left(\sum_{t\in D_{|u|}} c(t)u(t)\right)\right\|_B \le \left\|\sum_{t\in D_{|u|}} c(t)u(t)\right\|_p \\
&= \left(\sum_{t\in D_{|u|}} |c(t)|^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Since $\left\|\sum_{t\in D_{|u|}} c(t)(\tau u)(t)\right\|_B = \left\|T\left(\sum_{t\in D_{|u|}} c(t)u(t)\right)\right\|_B$, it follows that $\tau u \in \overline{B}^\delta$. Hence $\tau : \overline{L^p}^1 \to \overline{B}^\delta$ [suitably restricted].

As noted in the remark above, there is a sequence $\{u_k\}$ in $\overline{L^p}^1$ with $u_0 \prec u_1 \prec \cdots$. Since $\tau : \overline{L^p}^1 \to \overline{B}^\delta$ preserves order, $\tau u_0 \prec \tau u_1 \prec \cdots$ in $\overline{B}^\delta$. Hence $\left(\overline{B}^\delta, \prec\right)$ is not well-founded.

For the converse, suppose there is a $0 < \delta \leq 1$ such that $\left(\overline{B}^\delta, \prec\right)$ is not well-founded. Then there is a sequence $\{v_k\}$ in $\overline{B}^\delta$ with $v_0 \prec v_1 \prec \cdots$. Let $\{r(k)\}$ be the increasing sequence in $\mathbb{N} \cup \{0\}$ with $r(k) = |v_k|$ for all $k$. For $\{u_k\}$ as in (5.1), let $\{\tilde{u}_k\}$ be the subsequence of $\{u_k\}$ such that $|\tilde{u}_k| = r(k) = |v_k|$ for all $k$. For $k \in \mathbb{N} \cup \{0\}$, let $X_k = \left[\tilde{u}_k(t) : t \in D_{r(k)}\right]_{L^p}$, let $B_k = \left[v_k(t) : t \in D_{r(k)}\right]_B$, and let $T_k : X_k \to B_k$ be defined by

$$T_k\left(\sum_{t\in D_{r(k)}} c(t)\tilde{u}_k(t)\right) = \sum_{t\in D_{r(k)}} c(t)v_k(t)$$

for $c \in \mathbb{R}^{D_{r(k)}}$. Then $T_k$ is well-defined and linear, and $T_i = T_j|_{X_i}$ for $i < j$. Since $\left\|\sum_{t\in D_{r(k)}} c(t)\tilde{u}_k(t)\right\|_p = \left(\sum_{t\in D_{r(k)}} |c(t)|^p\right)^{\frac{1}{p}}$ by equation (5.3), and $\delta\left(\sum_{t\in D_{r(k)}} |c(t)|^p\right)^{\frac{1}{p}} \leq \left\|\sum_{t\in D_{r(k)}} c(t)v_k(t)\right\|_B \leq \left(\sum_{t\in D_{r(k)}} |c(t)|^p\right)^{\frac{1}{p}}$, we have

$$\delta\left\|\sum_{t\in D_{r(k)}} c(t)\tilde{u}_k(t)\right\|_p \leq \left\|T_k\left(\sum_{t\in D_{r(k)}} c(t)\tilde{u}_k(t)\right)\right\|_B \leq \left\|\sum_{t\in D_{r(k)}} c(t)\tilde{u}_k(t)\right\|_p,$$

whence $\delta\|x\|_p \leq \|T_k(x)\|_B \leq \|x\|_p$ for $k \in \mathbb{N} \cup \{0\}$ and $x \in X_k$.

Given $x \in \bigcup_{k=0}^\infty X_k$, $x \in X_k$ for some $k \in \mathbb{N} \cup \{0\}$. Let $\tilde{T} : \bigcup_{k=0}^\infty X_k \to \bigcup_{k=0}^\infty B_k$ be defined by $\tilde{T}(x) = T_k(x)$ for $x \in X_k$. Then $\delta\|x\|_p \leq \left\|\tilde{T}(x)\right\|_B \leq \|x\|_p$ for all $x \in \bigcup_{k=0}^\infty X_k$. Since $\bigcup_{k=0}^\infty X_k$ is dense in $L^p$, $\tilde{T}$ extends to an isomorphic imbedding of $L^p$ into $B$. $\square$

**The Ordinal Index $h_p(\delta, \;)$**

The ordinal index $h(\triangleleft)$ serves as a model for the ordinal index $h_p(\delta, B)$, for which the underlying set is $\overline{B}^{\delta}$. The ordinal index $h_p(B)$ is then derived from the indices $h_p(\delta, B)$.

DEFINITION. *Suppose $B$ is a separable Banach space, $1 \leq p < \infty$, and $0 < \delta \leq 1$. Let $H_0^{\delta}(B) = \overline{B}^{\delta}$. If $\alpha = \beta + 1$ and $H_{\beta}^{\delta}(B)$ has been defined, let $H_{\alpha}^{\delta}(B) = \left\{ u \in H_{\beta}^{\delta}(B) : u \prec v \text{ for some } v \in H_{\beta}^{\delta}(B) \right\}$. If $\alpha$ is a limit ordinal and $H_{\beta}^{\delta}(B)$ has been defined for all $\beta < \alpha$, let $H_{\alpha}^{\delta}(B) = \bigcap_{\beta<\alpha} H_{\beta}^{\delta}(B)$.*

DEFINITION. *Suppose $B$ is a separable Banach space, $1 \leq p < \infty$, and $0 < \delta \leq 1$. Let $h_p(\delta, B)$ be the least ordinal $\alpha$ such that $H_{\alpha}^{\delta}(B) = H_{\alpha+1}^{\delta}(B)$.*

The following proposition [**B-R-S**, Proposition 2.3] leads to one half of the characterization contained in Theorem 5.5.

**Proposition 5.3.** *Let $B$ be a separable Banach space. Let $1 \leq p < \infty$ and $0 < \delta \leq 1$. If $L^p \not\hookrightarrow B$, then $h_p(\delta, B) < \omega_1$.*

**Proof.** Suppose $L^p \not\hookrightarrow B$. Let $B_\omega$ be a countable dense subset of $B$. Let $\overline{B_\omega}^{\delta,2}$ be the countable set of all $u \in B_\omega^{\mathcal{D}}$ such that

$$\frac{\delta}{2}\left(\sum_{t\in D_{|u|}} |c(t)|^p\right)^{\frac{1}{p}} \leq \left\|\sum_{t\in D_{|u|}} c(t)u(t)\right\|_B \leq 2\left(\sum_{t\in D_{|u|}} |c(t)|^p\right)^{\frac{1}{p}}$$

for all $c \in \mathbb{R}^{D_{|u|}}$. Let $\lhd$ be the relation on $\overline{B_\omega}^{\delta,2}$ defined by $u \lhd v$ if (a) $|u| < |v|$ and (b) for $k = |v| - |u|$ and for $\delta_\ell = \delta 4^{-(\ell+1)}$, $\left\|u(t) - 2^{-\frac{k}{p}} \sum_{s\in D_k} v(t^\frown s)\right\|_B \leq \delta_{|u|}$ for all $t \in D_{|u|}$.

We will show that $\lhd$ is well-founded and there is a relation-preserving map $\tau : \left(\overline{B}^{\delta}, \prec\right) \to \left(\overline{B_\omega}^{\delta,2}, \lhd\right)$. It will follow by Lemma 5.1 that $h_p(\delta, B) \leq h(\lhd) < \omega_1$.

First we show that $\lhd$ is well-founded. Suppose $\lhd$ is not well-founded. Let $u_1 \lhd u_2 \lhd \cdots$ be an infinite $\lhd$-chain in $\overline{B_\omega}^{\delta,2}$. We will show that there is a corresponding infinite $\prec$-chain $\bar{u}_1 \prec \bar{u}_2 \prec \cdots$ in $\overline{B}^{\delta}$, whence $L^p \hookrightarrow B$ by Proposition 5.2, contrary to hypothesis. It will follow that $\lhd$ is well-founded.

Given $i, j \in \mathbb{N}$ with $i < j$, let $\Delta(i,j) = |u_j| - |u_i|$. Fix $i \in \mathbb{N}$. For $i < j \in \mathbb{N}$ and $t \in D_{|u_i|}$, let $\tilde{u}_j^{(i)}(t) = 2^{-\frac{\Delta(i,j)}{p}} \sum_{s \in D_{\Delta(i,j)}} u_j(t^\frown s)$. Then $\tilde{u}_j^{(i)} \prec u_j$. For $t \in D_{|u_i|}$,

$$\begin{aligned}
&\left\|\tilde{u}_j^{(i)}(t) - \tilde{u}_{j+1}^{(i)}(t)\right\|_B \\
&= \left\| 2^{-\frac{\Delta(i,j)}{p}} \sum_{s \in D_{\Delta(i,j)}} u_j(t^\frown s) - 2^{-\frac{\Delta(i,j+1)}{p}} \sum_{x \in D_{\Delta(i,j+1)}} u_{j+1}(t^\frown x) \right\|_B \\
&= \left\| 2^{-\frac{\Delta(i,j)}{p}} \sum_{s \in D_{\Delta(i,j)}} u_j(t^\frown s) - 2^{-\frac{\Delta(i,j)+\Delta(j,j+1)}{p}} \sum_{s \in D_{\Delta(i,j)}} \sum_{r \in D_{\Delta(j,j+1)}} u_{j+1}(t^\frown s^\frown r) \right\|_B \\
&\le 2^{-\frac{\Delta(i,j)}{p}} \sum_{s \in D_{\Delta(i,j)}} \left\| u_j(t^\frown s) - 2^{-\frac{\Delta(j,j+1)}{p}} \sum_{r \in D_{\Delta(j,j+1)}} u_{j+1}(t^\frown s^\frown r) \right\|_B \\
&\le 2^{-\frac{\Delta(i,j)}{p}} \cdot 2^{\Delta(i,j)} \cdot \delta_{|u_j|} \\
&= 2^{\Delta(i,j)\frac{p-1}{p}} \cdot \delta_{|u_j|} \\
&< 2^{|u_j|} \cdot \delta_{|u_j|}.
\end{aligned}$$

Hence for $i < j < k \in \mathbb{N}$ and $t \in D_{|u_i|}$,

$$\begin{aligned}
\left\|\tilde{u}_j^{(i)}(t) - \tilde{u}_{j+k}^{(i)}(t)\right\|_B &\le \sum_{n=j}^{j+k-1} \left\|\tilde{u}_n^{(i)}(t) - \tilde{u}_{n+1}^{(i)}(t)\right\|_B \\
&< \sum_{n=j}^{j+k-1} 2^{|u_n|} \cdot \delta_{|u_n|} \\
&< \sum_{n=j}^{\infty} 2^{|u_n|+1} \cdot \delta 4^{-(|u_n|+1)} \\
&= \delta \sum_{n=j}^{\infty} 2^{-(|u_n|+1)} \\
&\le \delta \sum_{n=j}^{\infty} 2^{-n} \\
&= \delta 2^{1-j}.
\end{aligned}$$

Now $lim_{j\to\infty} \delta 2^{1-j} = 0$, so $\left\{\tilde{u}_j^{(i)}(t)\right\}_{j=i+1}^{\infty}$ is Cauchy. Let $\overline{u}_i(t) = \lim_{j\to\infty} \tilde{u}_j^{(i)}(t)$. Releasing $i$ as a free variable, $\overline{u}_i(t)$ is defined for all $i \in \mathbb{N}$ and all $t \in D_{|u_i|}$.

Fix $i, j \in \mathbb{N}$ with $i < j$. Then for $t \in D_{|u_i|}$,

$$\begin{aligned}
\overline{u}_i(t) = \lim_{k\to\infty} \tilde{u}_k^{(i)}(t) &= \lim_{k\to\infty} 2^{-\frac{\Delta(i,k)}{p}} \sum_{x\in D_{\Delta(i,k)}} u_k(t^\cdot x) \\
&= \lim_{k\to\infty} 2^{-\frac{\Delta(i,j)+\Delta(j,k)}{p}} \sum_{s\in D_{\Delta(i,j)}} \sum_{r\in D_{\Delta(j,k)}} u_k(t^\cdot s^\cdot r) \\
&= 2^{-\frac{\Delta(i,j)}{p}} \sum_{s\in D_{\Delta(i,j)}} \lim_{k\to\infty} 2^{-\frac{\Delta(j,k)}{p}} \sum_{r\in D_{\Delta(j,k)}} u_k(t^\cdot s^\cdot r) \\
&= 2^{-\frac{\Delta(i,j)}{p}} \sum_{s\in D_{\Delta(i,j)}} \lim_{k\to\infty} \tilde{u}_k^{(j)}(t^\cdot s) \\
&= 2^{-\frac{\Delta(i,j)}{p}} \sum_{s\in D_{\Delta(i,j)}} \overline{u}_j(t^\cdot s).
\end{aligned}$$

Hence $\overline{u}_i \prec \overline{u}_j$. More generally, $\overline{u}_1 \prec \overline{u}_2 \prec \cdots$. As noted previously, it follows that $L^p \hookrightarrow B$, contrary to hypothesis, so $\triangleleft$ is well-founded.

We next show that there is a relation-preserving map $\tau : \left(\overline{B}^\delta, \prec\right) \to \left(\overline{B_\omega}^{\delta,2}, \triangleleft\right)$. Let $u \in \overline{B}^\delta$. For each $t \in D_{|u|}$, choose $v(t) \in B_\omega$ such that $\|u(t) - v(t)\|_B \leq \epsilon_{|u|}$, where $\epsilon_\ell = \delta 8^{-(\ell+1)}$ for $\ell \in \mathbb{N}$. Let $\tau u = v$.

First we show that $\tau u \in \overline{B_\omega}^{\delta,2}$. Note that $2^\ell \cdot \epsilon_\ell = 2^\ell 8^{-(\ell+1)}\delta < \frac{\delta}{2} < 1$. Thus for $t \in D_{|u|}$ and $c \in \mathbb{R}^{D_{|u|}}$,

$$\begin{aligned}
\left\|\sum_{t\in D_{|u|}} c(t)v(t)\right\|_B &= \left\|\sum_{t\in D_{|u|}} c(t)u(t) + \sum_{t\in D_{|u|}} c(t)\left(v(t) - u(t)\right)\right\|_B \\
&\leq \left\|\sum_{t\in D_{|u|}} c(t)u(t)\right\|_B + \sum_{t\in D_{|u|}} |c(t)| \cdot \epsilon_{|u|} \\
&\leq \left(\sum_{t\in D_{|u|}} |c(t)|^p\right)^{\frac{1}{p}} + 2^{|u|} \cdot \epsilon_{|u|} \cdot \left(\sum_{t\in D_{|u|}} |c(t)|^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{t\in D_{|u|}} |c(t)|^p\right)^{\frac{1}{p}} \left(1 + 2^{|u|} \cdot \epsilon_{|u|}\right) \\
&\leq 2\left(\sum_{t\in D_{|u|}} |c(t)|^p\right)^{\frac{1}{p}}
\end{aligned}$$

and

$$\begin{aligned}\left\|\sum_{t\in D_{|u|}} c(t)v(t)\right\|_B &= \left\|\sum_{t\in D_{|u|}} c(t)u(t) - \sum_{t\in D_{|u|}} c(t)\left(u(t)-v(t)\right)\right\|_B \\ &\geq \left\|\sum_{t\in D_{|u|}} c(t)u(t)\right\|_B - \sum_{t\in D_{|u|}} |c(t)|\cdot\epsilon_{|u|} \\ &\geq \delta\left(\sum_{t\in D_{|u|}} |c(t)|^p\right)^{\frac{1}{p}} - 2^{|u|}\cdot\epsilon_{|u|}\cdot\left(\sum_{t\in D_{|u|}} |c(t)|^p\right)^{\frac{1}{p}} \\ &= \left(\sum_{t\in D_{|u|}} |c(t)|^p\right)^{\frac{1}{p}}\left(\delta - 2^{|u|}\cdot\epsilon_{|u|}\right) \\ &\geq \frac{\delta}{2}\left(\sum_{t\in D_{|u|}} |c(t)|^p\right)^{\frac{1}{p}}.\end{aligned}$$

Hence $\tau u = v \in \overline{B_\omega}^{\delta,2}$.

We next show that $\tau$ preserves relations. Suppose $u, v \in \overline{B}^{\delta}$ with $u \prec v$. Let $k = |v| - |u|$. Then for all $t \in D_{|u|}$,

$$\begin{aligned}&\left\|\tau u(t) - 2^{-\frac{k}{p}}\sum_{s\in D_k}\tau v(t^\frown s)\right\|_B \\ &\leq \|\tau u(t) - u(t)\|_B + \left\|u(t) - 2^{-\frac{k}{p}}\sum_{s\in D_k} v(t^\frown s)\right\|_B + 2^{-\frac{k}{p}}\sum_{s\in D_k}\|v(t^\frown s) - \tau v(t^\frown s)\|_B \\ &\leq \epsilon_{|u|} + 0 + 2^{-\frac{k}{p}}\cdot 2^k\cdot\epsilon_{|v|} \\ &< \epsilon_{|u|} + 2^k\cdot\epsilon_{|u|+k} \\ &= \delta\left(\frac{1}{8^{|u|+1}} + \frac{2^k}{8^{|u|+k+1}}\right) < \delta\frac{2}{8^{|u|+1}} < \frac{\delta}{4^{|u|+1}} = \delta_{|u|} = \delta_{|\tau u|}.\end{aligned}$$

Hence $\tau u \lhd \tau v$ and $\tau$ preserves relations. As noted previously, since $\lhd$ is well-founded, it follows that $h_p(\delta, B) \leq h(\lhd) < \omega_1$. □

The following lemma **[B-R-S]** provides useful information about the behavior of $h_p(\delta, B)$ as a function of $\delta$.

**Lemma 5.4.** *Let $B$ be a separable Banach space and let $1 \leq p < \infty$. Suppose $0 < \delta_1 < \delta_2 \leq 1$. Then $H_\alpha^{\delta_1}(B) \supset H_\alpha^{\delta_2}(B)$ for each ordinal $\alpha$. If in addition $L^p \not\hookrightarrow B$, then $h_p(\delta_1, B) \geq h_p(\delta_2, B)$, whence $h_p(\delta, B)$ is a nonincreasing function of $\delta$.*

**Proof.** Let $0 < \delta_1 < \delta_2 \le 1$. Then $H_0^{\delta_1}(B) = \overline{B}^{\delta_1} \supset \overline{B}^{\delta_2} = H_0^{\delta_2}(B)$. Suppose $\alpha = \beta + 1$ and $H_\beta^{\delta_1}(B) \supset H_\beta^{\delta_2}(B)$. If $x \in H_\alpha^{\delta_2}(B)$, then $x$ is nonmaximal in $H_\beta^{\delta_2}(B)$, so $x$ is nonmaximal in $H_\beta^{\delta_1}(B)$, whence $x \in H_\alpha^{\delta_1}(B)$. Hence $H_\alpha^{\delta_1}(B) \supset H_\alpha^{\delta_2}(B)$.
Suppose $\alpha$ is a limit ordinal and $H_\beta^{\delta_1}(B) \supset H_\beta^{\delta_2}(B)$ for all $\beta < \alpha$. Then $H_\alpha^{\delta_1}(B) = \bigcap_{\beta<\alpha} H_\beta^{\delta_1}(B) \supset \bigcap_{\beta<\alpha} H_\beta^{\delta_2}(B) = H_\alpha^{\delta_2}(B)$. It follows that for each ordinal $\alpha$, $H_\alpha^{\delta_1}(B) \supset H_\alpha^{\delta_2}(B)$.

Suppose $L^p \not\hookrightarrow B$. Then by Proposition 5.2, $\left(\overline{B}^\delta, \prec\right)$ is well-founded for all $0 < \delta \le 1$, so $H_{\gamma_i}^{\delta_i}(B) = \emptyset$ for $\gamma_i = h_p(\delta_i, B)$. Thus $H_{\gamma_2}^{\delta_1}(B) \supset H_{\gamma_2}^{\delta_2}(B) = \emptyset$, so $\gamma_1 \ge \gamma_2$ and $h_p(\delta_1, B) \ge h_p(\delta_2, B)$. Hence $h_p(\delta, B)$ is a nonincreasing function of $\delta$. □

## The Ordinal Index $h_p$

Finally we define the ordinal index $h_p$.

Definition. *Suppose $B$ is a separable Banach space and $1 \le p < \infty$. If $L^p \not\hookrightarrow B$, let $h_p(B) = \sup_{0<\delta\le 1} h_p(\delta, B)$. If $L^p \hookrightarrow B$, let $h_p(B) = \omega_1$.*

We presently show that if $L^p \not\hookrightarrow B$, then $\{h_p(\delta, B) : 0 < \delta \le 1\}$ is bounded, whence $h_p(B)$ is well-defined. Note that the hypothesis $L^p \not\hookrightarrow B$ is equivalent to asserting that for each $0 < \delta \le 1$, there is an ordinal $\alpha$ such that $H_\alpha^\delta(B) = \emptyset$.

The following two results [**B-R-S**, Theorem 2.1] establish a countability criterion for $h_p$ and the monotonicity of $h_p$.

**Theorem 5.5.** *Let $B$ be a separable Banach space and let $1 \le p < \infty$. Then $h_p(B) \le \omega_1$, with $h_p(B) < \omega_1$ if and only if $L^p \not\hookrightarrow B$.*

**Proof.** If $L^p \hookrightarrow B$, then $h_p(B) = \omega_1$. Henceforth suppose $L^p \not\hookrightarrow B$. Now $h_p(\delta, B)$ is a nonincreasing function of $\delta$ by Lemma 5.4, and $h_p(\delta, B) < \omega_1$ for all $0 < \delta \le 1$ by Proposition 5.3. Hence $h_p(B) = \sup_{0<\delta\le 1} h_p(\delta, B) = \sup_{n\in\mathbb{N}} h_p\left(\frac{1}{n}, B\right) < \omega_1$. □

**Theorem 5.6.** *Let $X$ and $Y$ be separable Banach spaces and let $1 \leq p < \infty$. If $X \hookrightarrow Y$, then $h_p(X) \leq h_p(Y)$.*

**Proof.** Suppose $X \hookrightarrow Y$. If $L^p \hookrightarrow Y$, then $h_p(X) \leq \omega_1 = h_p(Y)$ by Theorem 5.5. Henceforth suppose $L^p \not\hookrightarrow Y$, whence $L^p \not\hookrightarrow X$. Then by Proposition 5.2, $\left(\overline{Y}^{\gamma}, \prec\right)$ is well-founded for each $0 < \gamma \leq 1$.

Let $T : X \to Y$ be an isomorphic imbedding with $\|T\| \leq 1$, and let $0 < \eta \leq 1$ be such that for each $x \in X$, $\eta \|x\|_X \leq \|T(x)\|_Y \leq \|x\|_X$. Let $\tau : X^{\mathcal{D}} \to Y^{\mathcal{D}}$ be defined by $(\tau u)(t) = T(u(t))$ for $u \in X^{\mathcal{D}}$ and $t \in D_{|u|}$. Then $\tau$ preserves order by the linearity of $T$.

Fix $0 < \delta \leq 1$ and let $u \in \overline{X}^{\delta}$. Then for all $c \in \mathbb{R}^{D_{|u|}}$,

$$\begin{aligned}
&\eta\delta \left( \sum_{t \in D_{|u|}} |c(t)|^p \right)^{\frac{1}{p}} \\
&\leq \eta \left\| \sum_{t \in D_{|u|}} c(t)u(t) \right\|_X \leq \left\| T\left( \sum_{t \in D_{|u|}} c(t)u(t) \right) \right\|_Y \leq \left\| \sum_{t \in D_{|u|}} c(t)u(t) \right\|_X \\
&\leq \left( \sum_{t \in D_{|u|}} |c(t)|^p \right)^{\frac{1}{p}}.
\end{aligned}$$

Since $\left\|\sum_{t\in D_{|u|}} c(t)(\tau u)(t)\right\|_Y = \left\|T\left(\sum_{t\in D_{|u|}} c(t)u(t)\right)\right\|_Y$, it follows that $\tau u \in \overline{Y}^{\eta\delta}$. Hence $\tau : \overline{X}^{\delta} \to \overline{Y}^{\eta\delta}$ [suitably restricted]. Since $\tau$ preserves order and $\left(\overline{Y}^{\eta\delta}, \prec\right)$ is well-founded, $h_p(\delta, X) \leq h_p(\eta\delta, Y)$ by Lemma 5.1. Releasing $\delta$ as a free variable, $h_p(X) = \sup_{0<\delta\leq 1} h_p(\delta, X) \leq \sup_{0<\delta\leq 1} h_p(\eta\delta, Y) = \sup_{0<\gamma\leq\eta} h_p(\gamma, Y) = h_p(Y)$, since $h_p(\gamma, Y)$ is a nonincreasing function of $\gamma$ by Lemma 5.4. □

Remark. It follows that $h_p(\ )$ is an isomorphic invariant.

### The Disjoint and Independent Sum Constructions

Let $(\Omega, \mu)$ be a probability space, let $\left(\Omega^{\mathbb{N}}, \mu^{\mathbb{N}}\right)$ be the corresponding product space, and let $(\{0,1\}, m)$ be the probability space with $m(0) = \frac{1}{2} = m(1)$. Suppose

$1 \le p < \infty$, and let $B$ and $B_1, B_2, \ldots$ be closed subspaces of $L^p(\Omega)$.

Given $b_0, b_1 \in B$, let $b(\omega, \epsilon)$ be the element of $L^p(\Omega \times \{0,1\})$ such that $b(\omega, 0) = 2^{\frac{1}{p}} b_0(\omega)$ and $b(\omega, 1) = 2^{\frac{1}{p}} b_1(\omega)$ for all $\omega \in \Omega$. Let $b_0 \oplus b_1$ denote the element $b(\omega, \epsilon)$ of $L^p(\Omega \times \{0,1\})$ corresponding to $b_0, b_1 \in B$.

Definition. *Let $1 \le p < \infty$ and let $B$ be a closed subspace of $L^p(\Omega)$. Define the $L^p$-disjoint sum $(B \oplus B)_p$ to be any space of random variables distributionally isomorphic to the subspace $\tilde{B}$ of $L^p(\Omega \times \{0,1\})$ defined by*

$$\tilde{B} = \{b(\omega, \epsilon) \in L^p(\Omega \times \{0,1\}) : b(\omega, \epsilon) = b_0 \oplus b_1 \text{ for some } b_0, b_1 \in B\}.$$

Note that $1_\Omega \oplus 1_\Omega = 2^{\frac{1}{p}} \cdot 1_{\Omega \times \{0,1\}}$, and if $b(\omega, \epsilon) = b_0 \oplus b_1$, then

$$\begin{aligned} \|b_0 \oplus b_1\|_\oplus^p = \|b(\omega, \epsilon)\|_{\tilde{B}}^p &= \int_{\Omega \times \{0,1\}} |b(\omega, \epsilon)|^p \\ &= \int_{\Omega \times \{0\}} |b(\omega, \epsilon)|^p + \int_{\Omega \times \{1\}} |b(\omega, \epsilon)|^p \\ &= \frac{1}{2} \int_\Omega 2\,|b_0(\omega)|^p + \frac{1}{2} \int_\Omega 2\,|b_1(\omega)|^p \\ &= \|b_0\|_B^p + \|b_1\|_B^p . \end{aligned}$$

Hence for $b \in B$, $\|b \oplus 0\|_\oplus = \|b\|_B = \|0 \oplus b\|_\oplus$.

Given $i \in \mathbb{N}$ and $b_i \in B_i$, let $\tilde{b}_i$ be the element of $L^p\left(\Omega^{\mathbb{N}}\right)$ such that $\tilde{b}_i(\omega_1, \omega_2, \ldots) = b_i(\omega_i)$ for all $\omega_1, \omega_2, \ldots \in \Omega$.

Definition. *Let $1 \le p < \infty$ and let $B_1, B_2, \ldots$ be closed subspaces of $L^p(\Omega)$. For each $i \in \mathbb{N}$, let*

$$\tilde{B}_i = \left\{ b \in L^p\left(\Omega^{\mathbb{N}}\right) : b = \tilde{b}_i \text{ for some } b_i \in B_i \right\}.$$

*Define the $L^p$-independent sum $\left(\sum^\oplus B_i\right)_{\mathrm{Ind},p}$ to be any space of random variables distributionally isomorphic to $\left[\tilde{B}_i : i \in \mathbb{N}\right]_{L^p(\Omega^{\mathbb{N}})}$.*

Finally, the spaces $R_\alpha^p$ for $0 < \alpha < \omega_1$ are defined as disjoint or independent sums, depending on whether $\alpha$ is a successor or limit ordinal, respectively.

Definition. *Let $1 \le p < \infty$. Let $R_0^p = [1]_{L^p}$. Suppose $0 < \alpha < \omega_1$. If $\alpha = \beta + 1$ and $R_\beta^p$ has been defined, let $R_\alpha^p = \left(R_\beta^p \oplus R_\beta^p\right)_p$. If $\alpha$ is a limit ordinal and $R_\beta^p$ has been defined for all $\beta < \alpha$, let $R_\alpha^p = \left(\sum_{\beta<\alpha}^{\oplus} R_\beta^p\right)_{\mathbf{Ind},p}$.*

Remark 1. It is shown in [**B-R-S**, Proposition 2.8] that for $1 < p < \infty$ and $\alpha < \omega_1$, $R_\alpha^p$ has an unconditional basis.

Remark 2. Technically, $R_\alpha^p = \left(\sum_{\beta_i<\alpha}^{\oplus} R_{\beta_i}^p\right)_{\mathbf{Ind},p}$ for an enumeration $\{\beta_i\}$ of the ordinals less than $\alpha$, but it is clear that the definition of $R_\alpha^p$ does not depend on the order.

The following two results serve as lemmas for the subsequent theorem [**B-R-S**, Proposition 2.7], which distinguishes $R_\alpha^p$ from $L^p$ isomorphically. Proposition 5.7 is a corollary of [**J-M-S-T**, Theorem 9.1]. Proposition 5.8 is [**B-R-S**, Theorem 1.1].

**Proposition 5.7.** *Let $1 < p < \infty$. Suppose $X$ is a closed subspace of $L^p$ such that $L^p \hookrightarrow X$. Then $L^p \overset{c}{\hookrightarrow} X$.*

**Proof.** Let $Y$ be a closed subspace of $X$ such that $L^p \sim Y \subset L^p$. By [**J-M-S-T**, Theorem 9.1], choose a closed subspace $Z$ of $Y$ such that $L^p \sim Z$ where $Z$ is complemented in $L^p$. Let $P$ be a projection from $L^p$ onto $Z$. Since $P(Z) = Z$ and $Z \subset X \subset L^p$, the restriction of $P$ to $X$ is a projection from $X$ onto $Z$. Hence $L^p \sim Z \overset{c}{\hookrightarrow} X$. $\square$

**Proposition 5.8.** *Let $1 < p < \infty$. Let $X$ be a Banach space with an unconditional Schauder decomposition $\{X_i\}$ such that $L^p \overset{c}{\hookrightarrow} X$. Then either $L^p \overset{c}{\hookrightarrow} X_i$ for some $i$, or there is a block basic sequence with respect to $\{X_i\}$ equivalent to the Haar basis of $L^p$, with closed linear span complemented in $X$.*

The proof of Proposition 5.8 consumes [**B-R-S**, Section 1], and will not be presented here.

**Theorem 5.9.** *Let $1 < p < \infty$ where $p \neq 2$, and let $\alpha < \omega_1$. Then $L^p \not\hookrightarrow R^p_\alpha$.*

**Proof.** Clearly $L^p \not\hookrightarrow [1]_{L^p} = R^p_0$.

Suppose $\alpha = \beta + 1$ and $L^p \not\hookrightarrow R^p_\beta$. Suppose for the moment that $L^p \hookrightarrow R^p_\alpha$. Then $L^p \hookrightarrow \tilde{R}^p_\alpha \subset L^p$ for some $\tilde{R}^p_\alpha$ distributionally isomorphic to $R^p_\alpha$. Hence $L^p \overset{c}{\hookrightarrow} R^p_\alpha$ by Proposition 5.7. Now $R^p_\alpha = \left(R^p_\beta \oplus R^p_\beta\right)_p$, so $L^p \overset{c}{\hookrightarrow} \left(R^p_\beta \oplus R^p_\beta\right)_p$, whence $L^p \overset{c}{\hookrightarrow} R^p_\beta$ by Proposition 5.8, contrary to the inductive hypothesis. Hence $L^p \not\hookrightarrow R^p_\alpha$.

Suppose $\alpha$ is a limit ordinal and $L^p \not\hookrightarrow R^p_\beta$ for all $\beta < \alpha$. Suppose for the moment that $L^p \hookrightarrow R^p_\alpha$. Then $L^p \overset{c}{\hookrightarrow} R^p_\alpha$ as above. Let $\{\beta_i\}_{i=0}^\infty$ be an enumeration of the ordinals less than $\alpha$, with $\beta_0 = 0$. Let $X_0 = R^p_{\beta_0} = R^p_0 = [1]_{L^p}$, and for $i \geq 1$, let $X_i = \left(R^p_{\beta_i}\right)_0$, the space of mean zero functions in $R^p_{\beta_i}$. Now $L^p \overset{c}{\hookrightarrow} \left(\sum^\oplus_{i\geq 0} X_i\right)_{\text{Ind},p}$, since $R^p_\alpha = \left(\sum^\oplus_{\beta<\alpha} R^p_\beta\right)_{\text{Ind},p} = \left(\sum^\oplus_{i\geq 0} X_i\right)_{\text{Ind},p}$, but $L^p \not\hookrightarrow X_i$ for $i \geq 0$. Let $\tilde{X}_i = \left\{x \in L^p\left([0,1]^{\mathbb{N}}\right) : x = \tilde{x}_i \text{ for some } x_i \in X_i\right\}$, with notation as in the definition of $\left(\sum^\oplus B_i\right)_{\text{Ind},p}$. Then by Proposition 5.8, there is a block basic sequence $\{z_i\}_{i\geq 0}$ with respect to $\left\{\tilde{X}_i\right\}_{i\geq 0}$ [with at most $z_0$ not mean zero] equivalent to the Haar basis of $L^p$. Hence $L^p \sim [z_i : i \geq 0]_{L^p([0,1]^{\mathbb{N}})} \sim [z_i : i \geq 1]_{L^p([0,1]^{\mathbb{N}})}$. Since $\{z_i\}_{i\geq 1}$ is a sequence of independent mean zero random variables in $L^p\left([0,1]^{\mathbb{N}}\right)$, $[z_i : i \geq 1]_{L^p([0,1]^{\mathbb{N}})} \hookrightarrow X_p$ [by Corollary 2.3, Proposition 2.1, Theorem 2.12, and part (b) of Proposition 2.24 for $2 < p < \infty$, and by [**RII**, Corollary 4.3] for $1 < p < 2$].

Hence $L^p \hookrightarrow X_p$, directly contrary to part (g) of Proposition 2.24 for $2 < p < \infty$, and indirectly contrary to the same result for $1 < p < 2$ as we presently show. Thus it will follow that $L^p \not\hookrightarrow R^p_\alpha$.

Suppose $L^s \hookrightarrow X_s$ for $1 < s < 2$, and let $r$ be the conjugate index of $s$. Then $L^s \hookrightarrow X_s \subset L^s$, whence $L^s \overset{c}{\hookrightarrow} X_s$ by Proposition 5.7. Hence $L^r \overset{c}{\hookrightarrow} X_r$, contrary to part (g) of Proposition 2.24. $\square$

REMARK. As shown in [**B-R-S**], Theorem 5.9 is true for $p = 1$ as well, but the proof is not identical.

**The Interaction of the Constructions and the Ordinal Index**

The disjoint and independent sum constructions are designed to force the ordinal index $h_p(R^p_\alpha)$ to increase [not necessarily strictly, but in the sense that the set $\{h_p(R^p_\alpha) : \alpha < \omega_1\}$ has no maximum]. The first results in this direction are the following proposition [**B-R-S**, Lemma 2.5] and corollary [**B-R-S**].

**Proposition 5.10.** *Let $1 \leq p < \infty$, $0 < \delta \leq 1$, and $\alpha < \omega_1$. Suppose $B$ is a closed subspace of $L^p$. Then for each $e \in H^\delta_\alpha(B)$, there is some $\bar{e} \in H^\delta_{\alpha+1}(B \oplus B)_p$.*

**Proof.** Suppose $e = x_0 \in B^{D_0}$. Let $\tau e = (x_0 \oplus 0, 0 \oplus x_0) \in (B \oplus B)^{D_1}_p$. Then $\tau e(0) = x_0 \oplus 0 \in (B \oplus B)_p$ and $\tau e(1) = 0 \oplus x_0 \in (B \oplus B)_p$. Let

$$\bar{e} = \frac{x_0 \oplus x_0}{2^{\frac{1}{p}}}. \tag{5.4}$$

Then $\bar{e} \in (B \oplus B)^{D_0}_p$ and $\bar{e} = 2^{-\frac{1}{p}}(\tau e(0) + \tau e(1))$. Hence $\bar{e} \prec \tau e$.

Let $k \in \mathbb{N}$ and suppose $e = (x_0, \ldots, x_{2^k-1}) \in B^{D_k}$. Then $e(t) \in B$ for $t \in D_k$. Let $\tau e = (x_0 \oplus 0, \ldots, x_{2^k-1} \oplus 0, 0 \oplus x_0, \ldots, 0 \oplus x_{2^k-1}) \in (B \oplus B)^{D_{k+1}}_p$. Then for $t \in D_k$, $\tau e(0{\cdot}t) = e(t) \oplus 0 \in (B \oplus B)_p$ and $\tau e(1{\cdot}t) = 0 \oplus e(t) \in (B \oplus B)_p$. Let

$$\bar{e} = \left(\frac{x_0 + x_1}{2^{\frac{1}{p}}} \oplus 0, \ldots, \frac{x_{2^k-2} + x_{2^k-1}}{2^{\frac{1}{p}}} \oplus 0, 0 \oplus \frac{x_0 + x_1}{2^{\frac{1}{p}}}, \ldots, 0 \oplus \frac{x_{2^k-2} + x_{2^k-1}}{2^{\frac{1}{p}}}\right).$$

Then $\bar{e} \in (B \oplus B)^{D_k}_p$ and $\bar{e}(t) = 2^{-\frac{1}{p}}(\tau e(t{\cdot}0) + \tau e(t{\cdot}1))$ for $t \in D_k$. Hence $\bar{e} \prec \tau e$.

We will show that if $e \in H^\delta_\alpha(B)$, then $\tau e \in H^\delta_\alpha(B \oplus B)_p$. Since $\bar{e} \prec \tau e$, it will follow that $\bar{e}$ is a nonmaximal element of $H^\delta_\alpha(B \oplus B)_p$, so $\bar{e} \in H^\delta_{\alpha+1}(B \oplus B)_p$.

First we show that $\tau$ preserves order. Suppose $e \prec d$. Without loss of generality suppose $|d| - |e| = 1$. Then for $t \in D_{|e|}$, $e(t) = 2^{-\frac{1}{p}}(d(t{\cdot}0) + d(t{\cdot}1))$. Thus for $t \in D_{|e|}$

$$\tau e(0{\cdot}t) = e(t) \oplus 0 = \frac{(d(t{\cdot}0) \oplus 0) + (d(t{\cdot}1) \oplus 0)}{2^{\frac{1}{p}}} = \frac{\tau d(0{\cdot}t{\cdot}0) + \tau d(0{\cdot}t{\cdot}1)}{2^{\frac{1}{p}}}$$

and

$$\tau e(1{\cdot}t) = 0 \oplus e(t) = \frac{(0 \oplus d(t{\cdot}0)) + (0 \oplus d(t{\cdot}1))}{2^{\frac{1}{p}}} = \frac{\tau d(1{\cdot}t{\cdot}0) + \tau d(1{\cdot}t{\cdot}1)}{2^{\frac{1}{p}}}.$$

Hence for $s = (0{\cdot}t)$ or $s = (1{\cdot}t)$, $\tau e(s) = 2^{-\frac{1}{p}}\left(\tau d(s{\cdot}0) + \tau d(s{\cdot}1)\right)$, so $\tau e \prec \tau d$ and $\tau$ preserves order.

We now show by induction on $\alpha$ that if $e \in H^{\delta}_{\alpha}(B)$, then $\tau e \in H^{\delta}_{\alpha}\left(B \oplus B\right)_p$.

Suppose $\alpha = 0$ and let $e \in H^{\delta}_{0}(B) = \overline{B}^{\delta}$. Then for $k = |e|$ and $c \in \mathbb{R}^{D_{k+1}}$,

$$\begin{aligned}
\left\| \sum_{\substack{t \in D_k \\ b \in \{0,1\}}} c(b{\cdot}t)\tau e(b{\cdot}t) \right\|_{\oplus}^{p} &= \left\| \left( \sum_{t \in D_k} c(0{\cdot}t)\tau e(0{\cdot}t) \right) + \left( \sum_{t \in D_k} c(1{\cdot}t)\tau e(1{\cdot}t) \right) \right\|_{\oplus}^{p} \\
&= \left\| \left( \sum_{t \in D_k} c(0{\cdot}t)(e(t) \oplus 0) \right) + \left( \sum_{t \in D_k} c(1{\cdot}t)(0 \oplus e(t)) \right) \right\|_{\oplus}^{p} \\
&= \left\| \left( \sum_{t \in D_k} c(0{\cdot}t)e(t) \right) \oplus \left( \sum_{t \in D_k} c(1{\cdot}t)e(t) \right) \right\|_{\oplus}^{p} \\
&= \left\| \sum_{t \in D_k} c(0{\cdot}t)e(t) \right\|_{B}^{p} + \left\| \sum_{t \in D_k} c(1{\cdot}t)e(t) \right\|_{B}^{p} \\
&\underset{\delta^{-p}}{\overset{1}{\approx}} \sum_{t \in D_k} |c(0{\cdot}t)|^p + \sum_{t \in D_k} |c(1{\cdot}t)|^p \\
&= \sum_{\substack{t \in D_k \\ b \in \{0,1\}}} |c(b{\cdot}t)|^p .
\end{aligned}$$

Hence $\tau e \in \overline{\left(B \oplus B\right)_p}^{\delta} = H^{\delta}_{0}\left(B \oplus B\right)_p$.

Suppose $\alpha = \beta + 1$, where if $d \in H^{\delta}_{\beta}(B)$, then $\tau d \in H^{\delta}_{\beta}\left(B \oplus B\right)_p$. Let $e \in H^{\delta}_{\alpha}(B)$. Then $e \in H^{\delta}_{\beta}(B)$, there is some $d \in H^{\delta}_{\beta}(B)$ such that $e \prec d$, and $\tau d \in H^{\delta}_{\beta}\left(B \oplus B\right)_p$. Since $\tau$ preserves order, $\tau e \prec \tau d$. Thus $\tau e$ is a nonmaximal element of $H^{\delta}_{\beta}\left(B \oplus B\right)_p$, whence $\tau e \in H^{\delta}_{\alpha}\left(B \oplus B\right)_p$.

Suppose $\alpha$ is a limit ordinal, where for each $\beta < \alpha$, if $d \in H^{\delta}_{\beta}(B)$, then $\tau d \in H^{\delta}_{\beta}\left(B \oplus B\right)_p$. Let $e \in H^{\delta}_{\alpha}(B)$. Then $e \in H^{\delta}_{\beta}(B)$ for all $\beta < \alpha$, and $\tau e \in H^{\delta}_{\beta}\left(B \oplus B\right)_p$ for all $\beta < \alpha$, whence $\tau e \in H^{\delta}_{\alpha}\left(B \oplus B\right)_p$.

Hence if $e \in H^\delta_\alpha(B)$, then $\tau e \in H^\delta_\alpha (B \oplus B)_p$. Now as previously noted, if $e \in H^\delta_\alpha(B)$, then $\bar{e} \prec \tau e \in H^\delta_\alpha (B \oplus B)_p$, so $\bar{e} \in H^\delta_{\alpha+1} (B \oplus B)_p$. $\square$

**Corollary 5.11.** *Let $1 \leq p < \infty$ and $\alpha < \omega_1$. Suppose $B$ is a closed subspace of $L^p$ such that $L^p \not\hookrightarrow B$. If $h_p(B) > \alpha$, then $h_p (B \oplus B)_p > \alpha + 1$.*

**Proof.** Suppose $h_p(B) > \alpha$. Then $h_p(\delta, B) > \alpha$ for some $0 < \delta \leq 1$. Thus $H^\delta_\alpha(B) \neq \emptyset$, so $H^\delta_{\alpha+1} (B \oplus B)_p \neq \emptyset$ by Proposition 5.10. Hence $h_p \left(\delta, (B \oplus B)_p\right) > \alpha + 1$, so $h_p (B \oplus B)_p > \alpha + 1$. $\square$

Remark. It follows that if $h_p(B)$ is a successor ordinal, then $h_p(B) < h_p (B \oplus B)_p$, while if $h_p(B)$ is a limit ordinal, then $h_p(B) \leq h_p (B \oplus B)_p$. Thus this result is not sufficient to force $h_p (R^p_\alpha)$ to increase.

For each ordinal $\alpha < \omega_1$, we define a probability space $\Omega_\alpha$. Let $\Omega_0 = [0,1]$. If $\alpha = \beta + 1$ and $\Omega_\beta$ has been defined, let $\Omega_\alpha = \Omega_\beta \times \{0,1\}$. If $\alpha$ is a limit ordinal and $\Omega_\beta$ has been defined for all $\beta < \alpha$, let $\Omega_\alpha = \prod_{\beta<\alpha} \Omega_\beta$.

The following theorem [**B-R-S**, Theorem 2.6] leads almost immediately to the subsequent corollary [**B-R-S**, Theorem B(2)], which is the key to forcing $h_p (R^p_\alpha)$ to increase in the sense mentioned previously.

**Theorem 5.12.** *Let $1 \leq p < \infty$ and $\alpha < \omega_1$. Then $1_{\Omega_\alpha} \in H^1_\alpha (R^p_\alpha)$.*

**Proof.** First we show that $1_{\Omega_\alpha} \in R^p_\alpha$. Clearly $1_{\Omega_0} \in [1]_{L^p} = R^p_0$. Suppose $\alpha = \beta + 1$ and $1_{\Omega_\beta} \in R^p_\beta$. Then $1_{\Omega_\alpha} = 2^{-\frac{1}{p}}(1_{\Omega_\beta} \oplus 1_{\Omega_\beta}) \in \left(R^p_\beta \oplus R^p_\beta\right)_p = R^p_\alpha$. Suppose $\alpha$ is a limit ordinal and $1_{\Omega_\beta} \in R^p_\beta$ for all $\beta < \alpha$. Fix $\beta < \alpha$, so $1_{\Omega_\beta} \in R^p_\beta$. Now $R^p_\beta$ is distributionally isomorphic to some closed subspace $\tilde{R}^p_\beta$ of $R^p_\alpha$. Let $T : R^p_\beta \to \tilde{R}^p_\beta \subset R^p_\alpha$ be the distributional isomorphism. Then $T(1_{\Omega_\beta}) = 1_{\Omega_\alpha} \in \tilde{R}^p_\beta \subset R^p_\alpha$. Hence $1_{\Omega_\alpha} \in R^p_\alpha$.

We now show that $1_{\Omega_\alpha} \in H^1_\alpha (R^p_\alpha)$. Clearly $1_{\Omega_0} \in \overline{[1]}^1_{L^p} = H^1_0 \left([1]_{L^p}\right) = H^1_0 (R^p_0)$. Suppose $\alpha = \beta + 1$ and $1_{\Omega_\beta} \in H^1_\beta \left(R^p_\beta\right)$. Then $1_{\Omega_\beta} \in R^p_\beta$, so $\bar{1}_{\Omega_\beta} = 2^{-\frac{1}{p}}(1_{\Omega_\beta} \oplus 1_{\Omega_\beta})$ for

$\bar{1}_{\Omega_\beta}$ as in equation (5.4). Hence by Proposition 5.10, $1_{\Omega_\alpha} = 2^{-\frac{1}{p}}\left(1_{\Omega_\beta} \oplus 1_{\Omega_\beta}\right) = \bar{1}_{\Omega_\beta} \in H^1_\alpha\left(R^p_\beta \oplus R^p_\beta\right)_p = H^1_\alpha\left(R^p_\alpha\right)$. Suppose $\alpha$ is a limit ordinal and $1_{\Omega_\beta} \in H^1_\beta\left(R^p_\beta\right)$ for all $\beta < \alpha$. Fix $\beta < \alpha$, so $1_{\Omega_\beta} \in H^1_\beta\left(R^p_\beta\right)$. Let $T : R^p_\beta \to \tilde{R}^p_\beta \subset R^p_\alpha$ be as above. Let $\tau : \left(R^p_\beta\right)^{\mathcal{D}} \to \left(R^p_\alpha\right)^{\mathcal{D}}$ be defined by $(\tau u)(t) = T(u(t))$ for $u \in \left(R^p_\beta\right)^{\mathcal{D}}$ and $t \in D_{|u|}$. Since $T$ is an isometry, $\tau$ maps $\overline{R^p_\beta}^1$ into $\overline{R^p_\alpha}^1$. Hence $\tau : \overline{R^p_\beta}^1 \to \overline{R^p_\alpha}^1$ [suitably restricted]. Since $1_{\Omega_\beta} \in \left(R^p_\beta\right)^{D_0}$, $\tau 1_{\Omega_\beta} = T(1_{\Omega_\beta}) = 1_{\Omega_\alpha}$. Since $T$ is linear, $\tau$ preserves order. Thus by Lemma 5.1, $\tau\left(H^1_\beta\left(R^p_\beta\right)\right) \subset H^1_\beta\left(R^p_\alpha\right)$. Hence $1_{\Omega_\alpha} = \tau 1_{\Omega_\beta} \in H^1_\beta\left(R^p_\alpha\right)$. Now $1_{\Omega_\alpha} \in H^1_\beta\left(R^p_\alpha\right)$ for all $\beta < \alpha$. Hence $1_{\Omega_\alpha} \in \bigcap_{\beta<\alpha} H^1_\beta\left(R^p_\alpha\right) = H^1_\alpha\left(R^p_\alpha\right)$. □

**Corollary 5.13.** *Let $1 < p < \infty$ where $p \neq 2$, and let $\alpha < \omega_1$. Then $h_p\left(R^p_\alpha\right) \geq \alpha + 1$.*

**Proof.** By Theorem 5.9, $L^p \not\hookrightarrow R^p_\alpha$, and $H^1_\alpha\left(R^p_\alpha\right) \neq \emptyset$ by Theorem 5.12. Thus $h_p\left(1, R^p_\alpha\right) > \alpha$, whence $h_p\left(R^p_\alpha\right) \geq h_p\left(1, R^p_\alpha\right) \geq \alpha + 1$. □

We collect our main results concerning the ordinal index $h_p$, the spaces $R^p_\alpha$, and their interaction. The proof of the subsequent theorem [**B-R-S**, Theorem A] will make implicit use of these results.

**Proposition 5.14.** *Let $1 < p < \infty$ where $p \neq 2$. Let $B$, $X$, and $Y$ be separable Banach spaces. Let $\alpha, \beta < \omega_1$. Then*

*(a) $L^p \not\hookrightarrow B$ if and only if $h_p(B) < \omega_1$,*

*(b) if $X \hookrightarrow Y$, then $h_p(X) \leq h_p(Y)$,*

*(c) $L^p \not\hookrightarrow R^p_\alpha$,*

*(d) if $\alpha < \beta$, then $R^p_\alpha \overset{c}{\hookrightarrow} R^p_\beta$,*

*(e) $h_p\left(R^p_\alpha\right) < \omega_1$, and*

*(f) $h_p\left(R^p_\alpha\right) \geq \alpha + 1$.*

**Proof.** Parts (a), (b), (c), and (f) are restatements of Theorem 5.5, Theorem 5.6, Theorem 5.9, and Corollary 5.13, respectively. Part (d) is clear from definitions. Part (e) is clear from parts (c) and (a). $\square$

**Theorem 5.15.** *Let $1 < p < \infty$ where $p \neq 2$. There is a strictly increasing function $\tau : \omega_1 \to \omega_1$ such that for $\gamma, \delta < \omega_1$,*

*(a) if $\gamma < \delta$, then $R^p_{\tau(\gamma)} \overset{c}{\hookrightarrow} R^p_{\tau(\delta)}$ but $R^p_{\tau(\delta)} \not\hookrightarrow R^p_{\tau(\gamma)}$, and*

*(b) if $Y$ is a separable Banach space such that $R^p_{\tau(\alpha)} \hookrightarrow Y$ for all $\alpha < \omega_1$, then $L^p \hookrightarrow Y$.*

**Proof.** Let $\tau(0) = \omega < \omega_1$ [so $R^p_{\tau(0)}$ is infinite-dimensional]. If $\tau(\beta)$ has been defined with $\tau(\beta) < \omega_1$, let $\tau(\beta+1) = h_p\left(R^p_{\tau(\beta)}\right) < \omega_1$. Then
$h_p\left(R^p_{\tau(\beta+1)}\right) \geq \tau(\beta+1)+1 > \tau(\beta+1) = h_p\left(R^p_{\tau(\beta)}\right)$. More generally, if $0 < \alpha < \omega_1$ and $\tau(\beta)$ has been defined with $\tau(\beta) < \omega_1$ for all $\beta < \alpha$, let $\tau(\alpha) = \sup_{\beta<\alpha} h_p\left(R^p_{\tau(\beta)}\right) < \omega_1$ [each $h_p\left(R^p_{\tau(\beta)}\right) < \omega_1$ and $\{\beta : \beta < \alpha\}$ is countable]. Then
$h_p\left(R^p_{\tau(\alpha)}\right) \geq \tau(\alpha) + 1 > \tau(\alpha) = \sup_{\beta<\alpha} h_p\left(R^p_{\tau(\beta)}\right)$, so $h_p\left(R^p_{\tau(\alpha)}\right) > h_p\left(R^p_{\tau(\beta)}\right)$ for all $\beta < \alpha$. Thus $R^p_{\tau(\alpha)} \not\hookrightarrow R^p_{\tau(\beta)}$ for all $\beta < \alpha$, so $\tau(\alpha) > \tau(\beta)$ for all $\beta < \alpha$, and $\tau$ is strictly increasing.

(a) Suppose $\gamma < \delta < \omega_1$. Then $\tau(\gamma) < \tau(\delta)$ and $R^p_{\tau(\gamma)} \overset{c}{\hookrightarrow} R^p_{\tau(\delta)}$, but $R^p_{\tau(\delta)} \not\hookrightarrow R^p_{\tau(\gamma)}$ as shown above.

(b) Let $Y$ be a separable Banach space such that $R^p_{\tau(\alpha)} \hookrightarrow Y$ for all $\alpha < \omega_1$. Then $\alpha < \tau(\alpha) + 1 \leq h_p\left(R^p_{\tau(\alpha)}\right) \leq h_p(Y) \leq \omega_1$ for all $\alpha < \omega_1$. Thus $h_p(Y) = \omega_1$, whence $L^p \hookrightarrow Y$.

$\square$

Remark. Let $1 < p < \infty$ where $p \neq 2$. We will show that $R^p_\alpha \overset{c}{\hookrightarrow} L^p$ for all $\alpha < \omega_1$. Thus part (a) will yield uncountably many isomorphically distinct $\mathcal{L}_p$ spaces [at most one $R^p_\alpha \sim \ell^2$]. By [**J-M-S-T**, Corollary 9.2], if $L^p \hookrightarrow Y \overset{c}{\hookrightarrow} L^p$, then $Y \sim L^p$.

Thus part (b) will imply that there is no separable $\mathcal{L}_p$ space $Y$, other than $L^p$ itself, such that $R^p_{\tau(\alpha)} \hookrightarrow Y$ for all $\alpha < \omega_1$.

### The Complementation of $R^p_\alpha$ in $L^p$

This section is devoted to the proof that $R^p_\alpha \overset{c}{\hookrightarrow} L^p$ for $1 < p < \infty$ and $\alpha < \omega_1$. We proceed by showing that $R^p_\alpha \sim Z^p_{T_\alpha} \overset{c}{\hookrightarrow} Z^p_{\mathbb{N}} \sim L^p$ for spaces $Z^p_{T_\alpha}$ and $Z^p_{\mathbb{N}}$ to be defined. The major components of the proof are Theorem 5.22, Proposition 5.25, and Proposition 5.26.

#### Preliminaries

Let $\mathbb{T}$ be a countable set, and let $\{0,1\}^{\mathbb{T}}$ be the standard product space.

We say that a measurable function $f$ on $\{0,1\}^{\mathbb{T}}$ depends on $E \subset \mathbb{T}$ if $f(x) = f(y)$ for all $x, y \in \{0,1\}^{\mathbb{T}}$ such that $x|_E = y|_E$. We say that a measurable set $S \subset \{0,1\}^{\mathbb{T}}$ depends on $E \subset \mathbb{T}$ if the indicator function $1_S$ depends on $E$. Thus $S \subset \{0,1\}^{\mathbb{T}}$ depends on $E \subset \mathbb{T}$ if $1_S(x) = 1_S(y)$ for all $x, y \in \{0,1\}^{\mathbb{T}}$ such that $x|_E = y|_E$.

It is easy to check that given $E \subset \mathbb{T}$, the set $\mathcal{A}$ of all measurable $S \subset \{0,1\}^{\mathbb{T}}$ which depend on $E$ is a $\sigma$-algebra, which we call the $\sigma$-algebra corresponding to $E$. Given $E \subset \mathbb{T}$, let $\mathcal{A}_E$ be the $\sigma$-algebra corresponding to $E$. It is easy to check that

(a) if $A \subset B \subset \mathbb{T}$, then $\mathcal{A}_A \subset \mathcal{A}_B$, and

(b) if $A, B \subset \mathbb{T}$, then $\mathcal{A}_{A\cap B} = \mathcal{A}_A \cap \mathcal{A}_B$.

Let $f$ be a measurable function on $\{0,1\}^{\mathbb{T}}$ and let $E \subset \mathbb{T}$. It is easy to check that

(c) $f$ is $\mathcal{A}_E$-measurable if and only if $f$ depends on $E$.

Let $(\Omega, \mathcal{M}, \mu)$ be a probability space. Given a sub $\sigma$-algebra $\mathcal{A}$ of $\mathcal{M}$, let $\mathcal{E}_{\mathcal{A}}$ be the conditional expectation operator with respect to $\mathcal{A}$.

Let $\mathcal{A}$ be a sub $\sigma$-algebra of $\mathcal{M}$. Then for each integrable function $f$ on $\Omega$,

(a) $\mathcal{E}_{\mathcal{A}} f$ is $\mathcal{A}$-measurable, and

(b) $\int_S \mathcal{E}_{\mathcal{A}} f = \int_S f$ for all $S \in \mathcal{A}$.

Moreover, $\mathcal{E}_{\mathcal{A}} f$ is essentially defined by these two conditions.

Let $\mathcal{A}$ and $\mathcal{B}$ be sub $\sigma$-algebras of $\mathcal{M}$, let $f$ and $g$ be integrable functions on $\Omega$, and let $1 \leq p < \infty$. Conditional expectation has the following properties ([**Ch**], [**Db**], and [**Stn**]):

(c) if $f$ is $\mathcal{A}$-measurable, then $\mathcal{E}_{\mathcal{A}} f = f$,

(d) $\mathcal{E}_{\mathcal{A}} \mathcal{E}_{\mathcal{A}} f = \mathcal{E}_{\mathcal{A}} f$,

(e) if $f \in L^p(\Omega)$, then $\mathcal{E}_{\mathcal{A}} f \in L^p(\Omega)$, with $\|\mathcal{E}_{\mathcal{A}} f\|_p \leq \|f\|_p$,

(f) if $f, g \in L^2(\Omega)$, then $\int g \mathcal{E}_{\mathcal{A}} f = \int f \mathcal{E}_{\mathcal{A}} g$,

(g) if $f \in L^2(\Omega)$, then $f = \mathcal{E}_{\mathcal{A}} f + f'$, where $f' \in L^2(\Omega)$ such that $\int f'h = 0$ for all $\mathcal{A}$-measurable $h \in L^2(\Omega)$,

(h) if $\mathcal{A} \subset \mathcal{B}$, then $\mathcal{E}_{\mathcal{A}} f = \mathcal{E}_{\mathcal{B}} f$ if and only if $\mathcal{E}_{\mathcal{B}} f$ is $\mathcal{A}$-measurable, and

(i) if $\mathcal{A} \subset \mathcal{B}$, then $\mathcal{E}_{\mathcal{A}} \mathcal{E}_{\mathcal{B}} f = \mathcal{E}_{\mathcal{A}} f = \mathcal{E}_{\mathcal{B}} \mathcal{E}_{\mathcal{A}} f$.

Suppose $\mathcal{E}_{\mathcal{A}}$ and $\mathcal{E}_{\mathcal{B}}$ commute. Then $\mathcal{E}_{\mathcal{A}} \mathcal{E}_{\mathcal{B}} f$, which is equal to $\mathcal{E}_{\mathcal{B}} \mathcal{E}_{\mathcal{A}} f$, is in turn $\mathcal{A}$-measurable and $\mathcal{B}$-measurable, whence $\mathcal{A} \cap \mathcal{B}$-measurable. Now $F = \mathcal{E}_{\mathcal{A}} f$ is integrable on $\Omega$, $\mathcal{A} \cap \mathcal{B} \subset \mathcal{B}$, and $\mathcal{E}_{\mathcal{B}} F = \mathcal{E}_{\mathcal{B}} \mathcal{E}_{\mathcal{A}} f$ is $\mathcal{A} \cap \mathcal{B}$-measurable. Thus
$\mathcal{E}_{\mathcal{A} \cap \mathcal{B}} f = \mathcal{E}_{\mathcal{A} \cap \mathcal{B}} \mathcal{E}_{\mathcal{A}} f = \mathcal{E}_{\mathcal{A} \cap \mathcal{B}} F = \mathcal{E}_{\mathcal{B}} F = \mathcal{E}_{\mathcal{B}} \mathcal{E}_{\mathcal{A}} f$. Hence

(j) if $\mathcal{E}_{\mathcal{A}} \mathcal{E}_{\mathcal{B}} = \mathcal{E}_{\mathcal{B}} \mathcal{E}_{\mathcal{A}}$, then $\mathcal{E}_{\mathcal{A}} \mathcal{E}_{\mathcal{B}} = \mathcal{E}_{\mathcal{A} \cap \mathcal{B}} = \mathcal{E}_{\mathcal{B}} \mathcal{E}_{\mathcal{A}}$.

Let $\left(\{0,1\}^{\mathbb{N}}, \mathcal{M}, \mu\right)$ be the standard product space. Let $A$ and $B$ be subsets of $\mathbb{N}$, with corresponding $\sigma$-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $f$ be an integrable function on $\{0,1\}^{\mathbb{N}}$. Consider $f$ as a function of $t = (t_1, t_2, \ldots)$ where $t_i \in \{0,1\}$. Then $\mathcal{E}_{\mathcal{A}} f$ is given by integration with respect to those $t_i$ such that $i \in \mathbb{N} \setminus A$. Hence

(a) $\mathcal{E}_{\mathcal{A}} \mathcal{E}_{\mathcal{B}} f = \mathcal{E}_{\mathcal{B}} \mathcal{E}_{\mathcal{A}} f$, and

(b) $\mathcal{E}_{\mathcal{A}}\mathcal{E}_{\mathcal{B}}f = \mathcal{E}_{\mathcal{A}\cap\mathcal{B}}f = \mathcal{E}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}f$.

## The Isomorphism of $Z^p_{\mathbb{N}}$ and $L^p$

Let $\{A_n\}$ be a sequence of sets. We say that $\{A_n\}$ is monotonic if it is either nondecreasing or nonincreasing, and $\{A_n\}$ is compatible if there is a permutation $\tau$ such that $\{A_{\tau(n)}\}$ is monotonic.

The following result [**Stn**, Theorem 8] substitutes for [**B-R-S**, Lemma 3.2]. We do not present the proof, but apply the result in the proof of the subsequent corollary, which substitutes for [**B-R-S**, Lemma 3.3]. This alternative approach was suggested in a remark of [**B-R-S**].

**Proposition 5.16.** *Let $1 < p < \infty$, let $(\Omega, \mathcal{M}, \mu)$ be a probability space, and let $\{f_n\}$ be a sequence of integrable functions on $\Omega$. Suppose $\{\mathcal{A}_n\}$ is a compatible sequence of sub $\sigma$-algebras of $\mathcal{M}$. Then there is a constant $A_p$, depending only on p, such that*

$$\left\| \left( \sum_n |\mathcal{E}_{\mathcal{A}_n} f_n|^2 \right)^{\frac{1}{2}} \right\|_p \le A_p \left\| \left( \sum_n |f_n|^2 \right)^{\frac{1}{2}} \right\|_p .$$

**Corollary 5.17.** *Let $1 < p < \infty$, let $(\Omega, \mathcal{M}, \mu)$ be a probability space, let $\{f_n\}$ be a sequence of integrable functions on $\Omega$, and let $\{\mathcal{B}_n\}$ be a sequence of sub $\sigma$-algebras of $\mathcal{M}$. Suppose $\{\mathcal{L}_n\}$, $\{\mathcal{R}_n\}$, and $\{\mathcal{T}_n\}$ are sequences of sub $\sigma$-algebras of $\mathcal{M}$ such that*

*(a) each of $\{\mathcal{L}_n\}$, $\{\mathcal{R}_n\}$, and $\{\mathcal{T}_n\}$ is compatible,*

*(b) for each $n$, $\mathcal{E}_{\mathcal{L}_n}$, $\mathcal{E}_{\mathcal{R}_n}$, and $\mathcal{E}_{\mathcal{T}_n}$ commute, and*

*(c) for each $n$, $\mathcal{B}_n = \mathcal{L}_n \cap \mathcal{R}_n \cap \mathcal{T}_n$.*

*Then for $A_p$ as above,*

$$\left\| \left( \sum_n |\mathcal{E}_{\mathcal{B}_n} f_n|^2 \right)^{\frac{1}{2}} \right\|_p \le A_p^3 \left\| \left( \sum_n |f_n|^2 \right)^{\frac{1}{2}} \right\|_p .$$

**Proof.** By part (c), $\mathcal{E}_{\mathcal{B}_n} = \mathcal{E}_{\mathcal{L}_n \cap \mathcal{R}_n \cap \mathcal{T}_n}$. By part (b), $\mathcal{E}_{\mathcal{L}_n \cap \mathcal{R}_n \cap \mathcal{T}_n} = \mathcal{E}_{\mathcal{L}_n} \mathcal{E}_{\mathcal{R}_n} \mathcal{E}_{\mathcal{T}_n}$. Thus $\mathcal{E}_{\mathcal{B}_n} = \mathcal{E}_{\mathcal{L}_n} \mathcal{E}_{\mathcal{R}_n} \mathcal{E}_{\mathcal{T}_n}$. Hence by Proposition 5.16 (applied three times), we have

$$\left\| \left( \sum_n |\mathcal{E}_{\mathcal{B}_n} f_n|^2 \right)^{\frac{1}{2}} \right\|_p = \left\| \left( \sum_n |\mathcal{E}_{\mathcal{L}_n} (\mathcal{E}_{\mathcal{R}_n} (\mathcal{E}_{\mathcal{T}_n} f_n))|^2 \right)^{\frac{1}{2}} \right\|_p \leq A_p^3 \left\| \left( \sum_n |f_n|^2 \right)^{\frac{1}{2}} \right\|_p .$$

□

Let $n \in \mathbb{N}$. Then $n$ has a unique expression as $n = 2^k + r$ for $k \in \mathbb{N} \cup \{0\}$ and $0 \leq r < 2^k$. For $n = 2^k + r$ as above, let $\lambda(n) = k$.

Let $D'_0 = \{1\}$. For $k \in \mathbb{N}$, let $D'_k = \{t_0 \cdots t_k : t_0 = 1 \text{ and } t_i \in \{0,1\} \text{ for } 1 \leq i \leq k\}$. Let $\mathcal{D}' = \bigcup_{k=0}^{\infty} D'_k$.

Now $\mathcal{D}'$ has a natural strict partial order $\prec$ defined by $s_0 \cdots s_{k_1} \prec t_0 \cdots t_{k_2}$ if $k_1 < k_2$ and $s_i = t_i$ for all $0 \leq i \leq k_1$.

Let $\gamma : (\mathbb{N}, <) \to (\mathcal{D}', \prec)$ be defined by $\gamma(n) = t_0 \cdots t_k \in D'_k$ for $k = \lambda(n)$, where $t_0 \cdots t_k$ is the binary expansion of $n$. Then $\gamma$ is a bijection, and $\gamma^{-1}$ preserves order. Let $\dot{\prec}$ be the strict partial order on $\mathbb{N}$ induced by $\prec$ via $\gamma$ $[m \mathrel{\dot{\prec}} n \iff \gamma(m) \prec \gamma(n)]$. Then $<$ extends $\dot{\prec}$.

The following application of Corollary 5.17 substitutes for [**B-R-S**, Scholium 3.4]. The result serves as a lemma for Theorem 5.22.

**Proposition 5.18.** *Let $1 < p < \infty$, let $\left(\{0,1\}^{\mathbb{N}}, \mathcal{M}, \mu\right)$ be the standard product space, and let $\{f_n\}$ be a sequence of integrable functions on $\{0,1\}^{\mathbb{N}}$. Given $n \in \mathbb{N}$, let $B_n = \{m \in \mathbb{N} : m \mathrel{\dot{\preceq}} n\}$, and let $\mathcal{B}_n$ be the corresponding sub $\sigma$-algebra of $\mathcal{M}$. Then for $A_p$ as above and $N \in \mathbb{N}$,*

$$\left\| \left( \sum_{n=1}^{N} |\mathcal{E}_{\mathcal{B}_n} f_n|^2 \right)^{\frac{1}{2}} \right\|_p \leq A_p^3 \left\| \left( \sum_{n=1}^{N} |f_n|^2 \right)^{\frac{1}{2}} \right\|_p .$$

**Proof.** Given $k \in \mathbb{N} \cup \{0\}$, let $\Lambda_k = \{m \in \mathbb{N} : \lambda(m) = k\}$, and let

$T_{[k]} = \{m \in \mathbb{N} : \lambda(m) \leq k\}$. Given $n \in \mathbb{N}$, let $\Lambda(n) = \{m \in \mathbb{N} : \lambda(m) = \lambda(n)\}$, and let

$$T_n = \{m \in \mathbb{N} : m \leq n\},$$

$$B_n = \{m \in \mathbb{N} : m \dot{\preceq} n\}$$

as above, which is the branch of $(T_n, \dot{\prec})$ generated by $n$,

$$L_n = \{m \in \mathbb{N} : m \dot{\preceq} n' \text{ for some } n' \in \Lambda(n) \text{ with } n' \leq n\},$$

the union of the branches $B_{n'}$ for $n' \in \Lambda(n)$ with $n' \leq n$, and

$$R_n = \{m \in \mathbb{N} : m \dot{\preceq} n' \text{ for some } n' \in \Lambda(n) \text{ with } n' \geq n\},$$

the union of the branches $B_{n'}$ for $n' \in \Lambda(n)$ with $n' \geq n$.

Fix $K \in \mathbb{N} \cup \{0\}$. For each $n \in T_{[K]}$, choose $N(n) \in \Lambda_K$ such that $n \dot{\preceq} N(n)$. Then given $n \in T_{[K]}$, $B_{N(n)}$ is an extension of $B_n$ to a branch of $T_{[K]}$, and

$$B_n = B_{N(n)} \cap T_n = L_{N(n)} \cap R_{N(n)} \cap T_n.$$

Note that $\{L_N\}_{N \in \Lambda_K}$, $\{R_N\}_{N \in \Lambda_K}$, and $\{T_n\}_{n \in T_{[K]}}$ are each monotonic. Hence $\{L_{N(n)}\}_{n \in T_{[K]}}$, $\{R_{N(n)}\}_{n \in T_{[K]}}$, and $\{T_n\}_{n \in T_{[K]}}$ are each compatible.

For $n \in T_{[K]}$, let $\mathcal{B}_n$, $\mathcal{L}_n$, $\mathcal{R}_n$, and $\mathcal{T}_n$ be the $\sigma$-algebras corresponding to $B_n$, $L_n$, $R_n$, and $T_n$, respectively. Then $\{\mathcal{L}_{N(n)}\}_{n \in T_{[K]}}$, $\{\mathcal{R}_{N(n)}\}_{n \in T_{[K]}}$, and $\{\mathcal{T}_n\}_{n \in T_{[K]}}$ are each compatible. Moreover, for $n \in T_{[K]}$, $\mathcal{B}_n = \mathcal{L}_{N(n)} \cap \mathcal{R}_{N(n)} \cap \mathcal{T}_n$, and $\mathcal{E}_{\mathcal{L}_{N(n)}}$, $\mathcal{E}_{\mathcal{R}_{N(n)}}$, and $\mathcal{E}_{\mathcal{T}_n}$ commute.

Hence for $N = 2^{K+1} - 1 \in T_{[K]}$ and $f_1, \ldots, f_N$ integrable on $\{0,1\}^{\mathbb{N}}$, $\left\| \left( \sum_{n=1}^N |\mathcal{E}_{\mathcal{B}_n} f_n|^2 \right)^{\frac{1}{2}} \right\|_p \leq A_p^3 \left\| \left( \sum_{n=1}^N |f_n|^2 \right)^{\frac{1}{2}} \right\|_p$ by Corollary 5.17. Releasing $K \in \mathbb{N} \cup \{0\}$ as a free variable, we have the same result for arbitrary $N \in \mathbb{N}$. □

The following square function inequality [**Burk**, Theorem 9] is quoted in [**B-R-S**, Scholium 3.5]. We do not present the proof, but apply the result in the proof of Theorem 5.22.

**Proposition 5.19.** *Let $1 < p < \infty$, let $(\Omega, \mathcal{M}, \mu)$ be a probability space, and let $\{\mathcal{T}_n\}_{n=0}^{\infty}$ be a nondecreasing sequence of sub $\sigma$-algebras of $\mathcal{M}$. Suppose $\{g_n\}_{n=0}^{\infty}$ is a sequence in $L^p(\Omega)$ such that $g_n$ is $\mathcal{T}_n$-measurable for all $n \in \mathbb{N} \cup \{0\}$ and $\mathcal{E}_{\mathcal{T}_{n-1}} g_n = 0$ for all $n \in \mathbb{N}$. Then there is a constant $K_p$, depending only on $p$, such that*

$$\frac{1}{K_p} \left\| \left( \sum_n |g_n|^2 \right)^{\frac{1}{2}} \right\|_p \leq \left\| \sum_n g_n \right\|_p \leq K_p \left\| \left( \sum_n |g_n|^2 \right)^{\frac{1}{2}} \right\|_p .$$

For $n \in \mathbb{N}$, let $B_n$, $T_n$, $\mathcal{B}_n$, and $\mathcal{T}_n$ be as above. Then for $n \in \mathbb{N}$, $T_n$ is the subtree $\{1, \ldots, n\}$ of $(\mathbb{N}, \dot{\prec})$, $B_n$ is the branch of $T_n$ generated by $n$, and $\mathcal{T}_n$ and $\mathcal{B}_n$ are the $\sigma$-algebras corresponding to $T_n$ and $B_n$, respectively. Let $T_0 = B_0 = \emptyset$ and let $\mathcal{T}_0$ and $\mathcal{B}_0$ be the trivial algebras. Let

$$\begin{aligned} Z_{\mathbb{N}}^p &= [f : f \text{ is } \mathcal{B}_n\text{-measurable for some } n \in \mathbb{N}]_{L^p(\{0,1\}^{\mathbb{N}})} \\ &= [f : f \text{ is measurable and depends on } B_n \text{ for some } n \in \mathbb{N}]_{L^p(\{0,1\}^{\mathbb{N}})}. \end{aligned}$$

Let $\Delta_0 = \Gamma_0 = \left\{\text{constant functions on } \{0,1\}^{\mathbb{N}}\right\}$. For $n \in \mathbb{N}$, let

$$\Delta_n = \left\{ f \text{ on } \{0,1\}^{\mathbb{N}} : f \text{ is } \mathcal{T}_n\text{-measurable and } \mathcal{E}_{\mathcal{T}_{n-1}} f = 0 \right\}$$

and

$$\Gamma_n = \left\{ f \in \Delta_n : f \text{ is } \mathcal{B}_n\text{-measurable} \right\}.$$

Suppose $f$ is measurable and $n \in \mathbb{N}$. Then $\left(\mathcal{E}_{\mathcal{T}_n} - \mathcal{E}_{\mathcal{T}_{n-1}}\right) f$ is $\mathcal{T}_n$-measurable, and $\mathcal{E}_{\mathcal{T}_{n-1}} \left(\mathcal{E}_{\mathcal{T}_n} - \mathcal{E}_{\mathcal{T}_{n-1}}\right) f = \mathcal{E}_{\mathcal{T}_{n-1}} f - \mathcal{E}_{\mathcal{T}_{n-1}} f = 0$, whence $\left(\mathcal{E}_{\mathcal{T}_n} - \mathcal{E}_{\mathcal{T}_{n-1}}\right) f \in \Delta_n$. Note that if $f \in \Delta_n$, then $f = \left(\mathcal{E}_{\mathcal{T}_n} - \mathcal{E}_{\mathcal{T}_{n-1}}\right) f$. Hence for $n \in \mathbb{N}$,

$$\Delta_n = \left\{ \left(\mathcal{E}_{\mathcal{T}_n} - \mathcal{E}_{\mathcal{T}_{n-1}}\right) f : f \text{ on } \{0,1\}^{\mathbb{N}} \text{ is measurable} \right\}.$$

The following lemmas for Theorem 5.22 have been extracted from the proof of [**B-R-S**, Theorem 3.1].

**Lemma 5.20.** *Let $1 \leq p < \infty$, and let $Z^p_{\mathbb{N}}$ and $\Gamma_n$ be as above. Then* $Z^p_{\mathbb{N}} = [\Gamma_n : n \geq 0]_{L^p(\{0,1\}^{\mathbb{N}})}$.

**Proof.** Note that $\Gamma_n \subset Z^p_{\mathbb{N}}$ for $n \in \mathbb{N} \cup \{0\}$, whence $[\Gamma_n : n \geq 0]_{L^p(\{0,1\}^{\mathbb{N}})} \subset Z^p_{\mathbb{N}}$. We now show that $Z^p_{\mathbb{N}} \subset [\Gamma_n : n \geq 0]_{L^p(\{0,1\}^{\mathbb{N}})}$, whence $Z^p_{\mathbb{N}} = [\Gamma_n : n \geq 0]_{L^p(\{0,1\}^{\mathbb{N}})}$.

Let $n \in \mathbb{N}$ and let $f$ be $\mathcal{B}_n$-measurable. Now $B_n \subset T_n$, so $\mathcal{B}_n \subset \mathcal{T}_n$, whence $f$ is $\mathcal{T}_n$-measurable and $\mathcal{E}_{\mathcal{T}_n} f = f$. Moreover, $\mathcal{E}_{\mathcal{T}_0} f$ is $\mathcal{T}_0$-measurable, whence $\mathcal{E}_{\mathcal{T}_0} f$ is constant, and $\int \mathcal{E}_{\mathcal{T}_0} f = \int f$, whence $\mathcal{E}_{\mathcal{T}_0} f = \int \mathcal{E}_{\mathcal{T}_0} f = \int f$. Thus

$$f = \int f - \mathcal{E}_{\mathcal{T}_0} f + \mathcal{E}_{\mathcal{T}_n} f = \int f + \sum_{i=1}^{n} \left(\mathcal{E}_{\mathcal{T}_i} - \mathcal{E}_{\mathcal{T}_{i-1}}\right) f.$$

Let $1 \leq i \leq n$. Then $\left(\mathcal{E}_{\mathcal{T}_i} - \mathcal{E}_{\mathcal{T}_{i-1}}\right) f \in \Delta_i$. We now show that $\left(\mathcal{E}_{\mathcal{T}_i} - \mathcal{E}_{\mathcal{T}_{i-1}}\right) f$ is $\mathcal{B}_i$-measurable, whence it will follow that $\left(\mathcal{E}_{\mathcal{T}_i} - \mathcal{E}_{\mathcal{T}_{i-1}}\right) f \in \Gamma_i$.

Note that $f = \mathcal{E}_{\mathcal{B}_n} f$, whence

$$\left(\mathcal{E}_{\mathcal{T}_i} - \mathcal{E}_{\mathcal{T}_{i-1}}\right) f = \left(\mathcal{E}_{\mathcal{T}_i} - \mathcal{E}_{\mathcal{T}_{i-1}}\right) \mathcal{E}_{\mathcal{B}_n} f = \mathcal{E}_{\mathcal{T}_i} \mathcal{E}_{\mathcal{B}_n} f - \mathcal{E}_{\mathcal{T}_{i-1}} \mathcal{E}_{\mathcal{B}_n} f = \mathcal{E}_{\mathcal{T}_i \cap \mathcal{B}_n} f - \mathcal{E}_{\mathcal{T}_{i-1} \cap \mathcal{B}_n} f.$$

Suppose first that $i \notin B_n$. Then $T_i \cap B_n = T_{i-1} \cap B_n$, so $\mathcal{T}_i \cap \mathcal{B}_n = \mathcal{T}_{i-1} \cap \mathcal{B}_n$, whence

$$\left(\mathcal{E}_{\mathcal{T}_i} - \mathcal{E}_{\mathcal{T}_{i-1}}\right) f = \mathcal{E}_{\mathcal{T}_i \cap \mathcal{B}_n} f - \mathcal{E}_{\mathcal{T}_{i-1} \cap \mathcal{B}_n} f = 0,$$

which is $\mathcal{B}_i$-measurable. Next suppose that $i \in B_n$. Then $T_i \cap B_n = B_i$, so $\mathcal{T}_i \cap \mathcal{B}_n = \mathcal{B}_i$, and $T_{i-1} \cap B_n \subset B_i$, so $\mathcal{T}_{i-1} \cap \mathcal{B}_n \subset \mathcal{B}_i$, whence

$$\left(\mathcal{E}_{\mathcal{T}_i} - \mathcal{E}_{\mathcal{T}_{i-1}}\right) f = \mathcal{E}_{\mathcal{T}_i \cap \mathcal{B}_n} f - \mathcal{E}_{\mathcal{T}_{i-1} \cap \mathcal{B}_n} f = \mathcal{E}_{\mathcal{B}_i} f - \mathcal{E}_{\mathcal{B}'_i} f$$

for some $\mathcal{B}'_i \subset \mathcal{B}_i$. Now $\mathcal{E}_{\mathcal{B}_i} f$ is $\mathcal{B}_i$-measurable, and $\mathcal{E}_{\mathcal{B}'_i} f$ is $\mathcal{B}'_i$-measurable, whence $\mathcal{B}_i$-measurable. Thus $\left(\mathcal{E}_{\mathcal{T}_i} - \mathcal{E}_{\mathcal{T}_{i-1}}\right) f$ is $\mathcal{B}_i$-measurable [now in both cases]. As noted above, it follows that $\left(\mathcal{E}_{\mathcal{T}_i} - \mathcal{E}_{\mathcal{T}_{i-1}}\right) f \in \Gamma_i$.

We now have

$$f = \int f + \sum_{i=1}^{n} \left(\mathcal{E}_{\mathcal{T}_i} - \mathcal{E}_{\mathcal{T}_{i-1}}\right) f \in [\Gamma_i : 0 \leq i \leq n]_{L^p(\{0,1\}^{\mathbb{N}})}.$$

Thus $f \in [\Gamma_n : n \geq 0]_{L^p(\{0,1\}^{\mathbb{N}})}$. It follows that $Z^p_{\mathbb{N}} \subset [\Gamma_n : n \geq 0]_{L^p(\{0,1\}^{\mathbb{N}})}$, whence $Z^p_{\mathbb{N}} = [\Gamma_n : n \geq 0]_{L^p(\{0,1\}^{\mathbb{N}})}$. □

**Lemma 5.21.** *Let* $2 \leq p < \infty$, *and let* $\Delta_i$ *be as above. Then* $\{\Delta_i\}_{i \geq 0}$ *is an unconditional Schauder decomposition of* $L^p\left(\{0,1\}^{\mathbb{N}}\right)$.

**Proof.** Suppose $f, g \in L^2\left(\{0,1\}^{\mathbb{N}}\right)$, and let $i \in \mathbb{N}$. If $f \in \Delta_i$ and $g \in \Delta_j$ for $i < j \in \mathbb{N}$, then $\mathcal{E}_{\mathcal{T}_{j-1}} g = 0$ and $f$ is $\mathcal{T}_{j-1}$-measurable, so

$$\int fg = \int f\left(g - \mathcal{E}_{\mathcal{T}_{j-1}} g\right) = \int fg - \int f \mathcal{E}_{\mathcal{T}_{j-1}} g = \int fg - \int g \mathcal{E}_{\mathcal{T}_{j-1}} f = \int fg - \int gf = 0,$$

whence $f$ and $g$ are orthogonal. If $f \in \Delta_i$ and $g \in \Delta_0$, then $g$ is constant, and $\int f = \int \mathcal{E}_{\mathcal{T}_{i-1}} f$, but $\mathcal{E}_{\mathcal{T}_{i-1}} f = 0$, so

$$\int fg = g \int f = g \int \mathcal{E}_{\mathcal{T}_{i-1}} f = 0,$$

whence $f$ and $g$ are orthogonal. Hence $\{\Delta_i\}_{i \geq 0}$ is orthogonal.

Suppose $f \in L^2\left(\{0,1\}^{\mathbb{N}}\right)$. Let $f_0 = \mathcal{E}_{\mathcal{T}_0} f \in \Delta_0$, and for $i \in \mathbb{N}$, let $f_i = \left(\mathcal{E}_{\mathcal{T}_i} - \mathcal{E}_{\mathcal{T}_{i-1}}\right) f \in \Delta_i$. Then for $n \in \mathbb{N}$,

$$\sum_{i=0}^{n} f_i = \mathcal{E}_{\mathcal{T}_0} f + \sum_{i=1}^{n} \left(\mathcal{E}_{\mathcal{T}_i} - \mathcal{E}_{\mathcal{T}_{i-1}}\right) f = \mathcal{E}_{\mathcal{T}_n} f.$$

Note that $L^p\left(\{0,1\}^{\mathbb{N}}\right) \subset L^2\left(\{0,1\}^{\mathbb{N}}\right)$. If $f \in L^p\left(\{0,1\}^{\mathbb{N}}\right)$, then $\lim_{n\to\infty} \|f - \mathcal{E}_{\mathcal{T}_n} f\|_p = 0$, whence $f = \sum_{i=0}^{\infty} f_i$ in $L^p\left(\{0,1\}^{\mathbb{N}}\right)$. By the orthogonality of $\{\Delta_i\}_{i \geq 0}$, the representation $f = \sum_{i=0}^{\infty} f_i'$ with $f_i' \in \Delta_i$ is unique. By Proposition 5.19, the convergence is unconditional. Hence $\{\Delta_i\}_{i \geq 0}$ is an unconditional Schauder decomposition of $L^p\left(\{0,1\}^{\mathbb{N}}\right)$. □

Remark. The above result can be viewed as a consequence of the unconditionality of the Haar system.

We are now prepared to prove the following theorem [**B-R-S**, Theorem 3.1], which is a major component of the proof that $R^p_\alpha \overset{c}{\hookrightarrow} L^p$.

**Theorem 5.22.** *Let $1 < p < \infty$, and let $Z_{\mathbb{N}}^p$ be as above. Then* $Z_{\mathbb{N}}^p \overset{c}{\hookrightarrow} L^p\left(\{0,1\}^{\mathbb{N}}\right)$.

**Proof.** First suppose $2 \leq p < \infty$, whence $L^p\left(\{0,1\}^{\mathbb{N}}\right) \subset L^2\left(\{0,1\}^{\mathbb{N}}\right)$. Fix $i \in \mathbb{N} \cup \{0\}$. Let $f \in \Delta_i$ and let $g = \mathcal{E}_{\mathcal{B}_i} f$. If $i = 0$, then $\Gamma_i = \Delta_i$, $\mathcal{E}_{\mathcal{B}_i} f = f$, and $\mathcal{E}_{\mathcal{B}_i}|_{\Delta_i}$ is the identity mapping. Suppose $i \in \mathbb{N}$. Then $g$ is $\mathcal{B}_i$-measurable. Now $B_i \subset T_i$, so $\mathcal{B}_i \subset \mathcal{T}_i$, whence $g$ is $\mathcal{T}_i$-measurable. Moreover, $\mathcal{E}_{\mathcal{T}_{i-1}} g = \mathcal{E}_{\mathcal{T}_{i-1}} \mathcal{E}_{\mathcal{B}_i} f = \mathcal{E}_{\mathcal{B}_i} \mathcal{E}_{\mathcal{T}_{i-1}} f = 0$. Thus $g$ is a $\mathcal{B}_i$-measurable element of $\Delta_i$, whence $g \in \Gamma_i$. If $f \in \Gamma_i$, then $\mathcal{E}_{\mathcal{B}_i} f = f$. Hence for $i \in \mathbb{N} \cup \{0\}$, $\mathcal{E}_{\mathcal{B}_i}|_{\Delta_i}$ is the orthogonal projection of $\Delta_i$ onto $\Gamma_i$.

By Lemma 5.21, $\{\Delta_i\}_{i \geq 0}$ is an unconditional Schauder decomposition of $L^2\left(\{0,1\}^{\mathbb{N}}\right)$. For $f \in L^2\left(\{0,1\}^{\mathbb{N}}\right)$, let $\{f_i\}$ be the unique sequence with $f_i \in \Delta_i$ such that $f = \sum_{i=0}^{\infty} f_i$. Let $\pi : L^2\left(\{0,1\}^{\mathbb{N}}\right) \to L^2\left(\{0,1\}^{\mathbb{N}}\right)$ be defined by

$$\pi f = \sum_{i=0}^{\infty} \mathcal{E}_{\mathcal{B}_i} f_i.$$

Then $\pi$ is the orthogonal projection of $L^2\left(\{0,1\}^{\mathbb{N}}\right)$ onto $[\Gamma_i : i \geq 0]_{L^2(\{0,1\}^{\mathbb{N}})}$, where $[\Gamma_i : i \geq 0]_{L^2(\{0,1\}^{\mathbb{N}})} = Z_{\mathbb{N}}^2$ by Lemma 5.20.

Let $P$ be the restriction of $\pi$ to $L^p\left(\{0,1\}^{\mathbb{N}}\right)$, let $f \in L^p\left(\{0,1\}^{\mathbb{N}}\right)$, and let $\{f_i\}$ be as above. Then by Proposition 5.19, Proposition 5.18, and Proposition 5.19 again, for $n \in \mathbb{N}$ we have

$$\left\| \sum_{i=0}^{n} \mathcal{E}_{\mathcal{B}_i} f_i \right\|_p \leq K_p \left\| \left( \sum_{i=0}^{n} |\mathcal{E}_{\mathcal{B}_i} f_i|^2 \right)^{\frac{1}{2}} \right\|_p \leq K_p A_p^3 \left\| \left( \sum_{i=0}^{n} |f_i|^2 \right)^{\frac{1}{2}} \right\|_p \leq K_p^2 A_p^3 \left\| \sum_{i=0}^{n} f_i \right\|_p,$$

where the constants $K_p$ and $A_p$ are as in the cited propositions. Hence $\|Pf\|_p \leq K_p^2 A_p^3 \|f\|_p$, and $P : L^p\left(\{0,1\}^{\mathbb{N}}\right) \to L^p\left(\{0,1\}^{\mathbb{N}}\right)$ is bounded. Of course $P$ is a projection, and $P$ maps $L^p\left(\{0,1\}^{\mathbb{N}}\right)$ onto $[\Gamma_i : i \geq 0]_{L^p(\{0,1\}^{\mathbb{N}})}$, where $[\Gamma_i : i \geq 0]_{L^p(\{0,1\}^{\mathbb{N}})} = Z_{\mathbb{N}}^p$ by Lemma 5.20.

For $2 < p < \infty$ with conjugate index $q$, the adjoint of $P$ induces a bounded projection of $L^q\left(\{0,1\}^{\mathbb{N}}\right)$ onto $Z_{\mathbb{N}}^q$. □

REMARK. While $Z^p_{\mathbb{N}} \overset{c}{\hookrightarrow} L^p$ is our major concern, in fact $Z^p_{\mathbb{N}} \sim L^p$.

## The Complementation of $R^p_\alpha$ in $Z^p_{\mathbb{N}}$

Recall that a tree $(T, \prec)$ is a CFRE tree if $T$ is finite or countable, and for each $x \in T$, $\{y \in T : y \prec x\}$ is finite. Let $(T, \prec)$ be a CFRE tree. For $t \in T$, let $B_t$ be the finite branch of $T$ generated by $t$. For $1 \leq p < \infty$, let

$$Z^p_T = [f : f \text{ is measurable and depends on } B_t \text{ for some } t \in T]_{L^p(\{0,1\}^T)}.$$

The space $Z^p_T$ is similar to the previously defined space $Z^p_{\mathbb{N}}$.

Let $S$ be a nonempty subset of $\mathbb{N}$. Then $(S, \dot{\prec})$ is a CFRE tree, where $\dot{\prec}$ is the previously introduced partial order on $\mathbb{N}$ [suitably restricted]. The finite branches of $S$ are precisely those sets of the form $B_n \cap S$ for $n \in S$, where $B_n$ is the finite branch of $(\mathbb{N}, \dot{\prec})$ generated by $n$. For $1 \leq p < \infty$, $L^p\left(\{0,1\}^S\right)$ is isomorphic to the subspace of $L^p\left(\{0,1\}^{\mathbb{N}}\right)$ consisting of those functions which depend on $S$, and $Z^p_S$ is isomorphic to the space

$$\tilde{Z}^p_S = [f : f \text{ is measurable and depends on } B_n \cap S \text{ for some } n \in S]_{L^p(\{0,1\}^{\mathbb{N}})}.$$

The following lemmas [**B-R-S**, Lemmas 3.6 and 3.7] lead to the subsequent proposition [**B-R-S**, Theorem 3.8], which is a component of the proof that $R^p_\alpha \overset{c}{\hookrightarrow} L^p$.

**Lemma 5.23.** *Let $1 \leq p < \infty$ and let $\emptyset \neq S \subset \mathbb{N}$. Then $Z^p_S \overset{c}{\hookrightarrow} Z^p_{\mathbb{N}}$.*

**Proof.** Let $\mathcal{S}$ be the $\sigma$-algebra corresponding to $S$, and let $P : Z^p_{\mathbb{N}} \to Z^p_{\mathbb{N}}$ be defined by $Pf = \mathcal{E}_{\mathcal{S}} f$. Note that $\tilde{Z}^p_S \subset Z^p_{\mathbb{N}}$. If $f \in Z^p_{\mathbb{N}}$ depends on $B_n$, then $Pf$ depends on $B_n \cap S$, which is either the empty set or a finite branch of $S$ of the form $B_m \cap S$ for some $m \in S$, whence $P$ maps $Z^p_{\mathbb{N}}$ into $\tilde{Z}^p_S$. Now $Pf = f$ for $f \in \tilde{Z}^p_S$. Hence

$P$ maps $Z^p_{\mathbb{N}}$ onto $\tilde{Z}^p_S$, and $P^2 = P$. Finally, $\|Pf\|_p = \|\mathcal{E}_S f\|_p \leq \|f\|_p$, whence $\|P\| = 1$. Hence $Z^p_S \sim \tilde{Z}^p_S \overset{c}{\hookrightarrow} Z^p_{\mathbb{N}}$. □

For $n \in \mathbb{N}$, let $N_n = \{t_1 \cdots t_n : t_i \in \mathbb{N} \text{ for all } 1 \leq i \leq n\}$. Let $\mathcal{N} = \bigcup_{n=1}^{\infty} N_n$, and define a strict partial order $\prec$ on $\mathcal{N}$ by $s_1 \cdots s_n \prec t_1 \cdots t_m$ if $n < m$ and $s_i = t_i$ for all $1 \leq i \leq n$.

**Lemma 5.24.** *Let* $(T, \prec)$ *be a CFRE tree. Then* $(T, \prec)$ *is order-isomorphic to a subset of* $(\mathbb{N}, \dot{\prec})$.

**Proof.** Clearly $T$ is order-isomorphic to a subset of $\mathcal{N}$. We will show that $\mathcal{N}$ is order-isomorphic to a subset of $\mathcal{D}'$. The result will then follow upon noting that $\mathcal{D}'$ is order-isomorphic to $\mathbb{N}$ endowed with $\dot{\prec}$.

We describe a subset $\mathcal{S}$ of $\mathcal{D}'$ such that $\mathcal{N}$ is order-isomorphic to $\mathcal{S}$. Given $t \in \mathcal{D}'$, let $S(t) = \{t{\cdot}1, t{\cdot}01, t{\cdot}001, \ldots\}$. Then $S(t)$ is a countable set of distinct and mutually incomparable successors of $t$. Moreover, if $s$ and $t$ are distinct and incomparable elements of $\mathcal{D}'$, then $S(s)$ and $S(t)$ are disjoint, and the elements of $S(s) \cup S(t)$ are mutually incomparable elements of $\mathcal{D}'$. For $A \subset \mathcal{D}'$, let $S(A) = \bigcup_{a \in A} S(a)$. Finally, let $\mathcal{S} = S(1) \cup S(S(1)) \cup \cdots$. Then $\mathcal{N}$ is order-isomorphic to $\mathcal{S} \subset \mathcal{D}'$, and the result follows as noted above. □

**Proposition 5.25.** *Let* $1 \leq p < \infty$ *and let* $T$ *be a CFRE tree. Then* $Z^p_T \overset{c}{\hookrightarrow} Z^p_{\mathbb{N}}$.

**Proof.** If trees $T$ and $T'$ are order-isomorphic, then $Z^p_T \sim Z^p_{T'}$. Thus by Lemma 5.24, we may choose $T' \subset \mathbb{N}$ such that $Z^p_T \sim Z^p_{T'}$. Now $Z^p_{T'} \overset{c}{\hookrightarrow} Z^p_{\mathbb{N}}$ by Lemma 5.23. Hence $Z^p_T \overset{c}{\hookrightarrow} Z^p_{\mathbb{N}}$. □

Remark. By Proposition 5.25 and Theorem 5.22, for $1 < p < \infty$ and $T$ a CFRE tree, $Z^p_T \overset{c}{\hookrightarrow} L^p\left(\{0,1\}^{\mathbb{N}}\right)$, whence $Z^p_T \overset{c}{\hookrightarrow} L^p\left(\{0,1\}^T\right)$.

The following proposition [**B-R-S**, Lemma 3.9] is the final component of the proof that $R^p_\alpha \overset{c}{\hookrightarrow} L^p$.

**Proposition 5.26.** *Let $1 \le p < \infty$ and $\alpha < \omega_1$. Then there is a well-founded CFRE tree $T_\alpha$ such that $R^p_\alpha$ is distributionally isomorphic to $Z^p_{T_\alpha}$.*

**Proof.** Clearly $R^p_0 = [1]_{L^p}$ is distributionally isomorphic to $Z^p_{T_0}$ where $T_0 = \emptyset$. Moreover, $R^p_1 = (R^p_0 \oplus R^p_0)_p$ is distributionally isomorphic to $Z^p_{T_1}$ where $T_1 = \{1\}$.

Suppose $\alpha = \beta + 1 > 1$, where $R^p_\beta$ is distributionally isomorphic to $Z^p_{T_\beta}$ for some well-founded CFRE tree $(T_\beta, \prec_\beta)$. Without loss of generality, suppose $R^p_\beta = Z^p_{T_\beta}$. Choose $\theta \notin T_\beta$. Let $T_\alpha = T_\beta \cup \{\theta\}$, and let $\prec_\alpha$ extend $\prec_\beta$ by declaring $\theta \prec_\alpha \tau$ for all $\tau \in T_\beta$. Then $(T_\alpha, \prec_\alpha)$ is a well-founded CFRE tree. For the case $\alpha = \beta + 1 > 1$, it remains to show that $R^p_\alpha$ is distributionally isomorphic to $Z^p_{T_\alpha}$.

Let $\bar{0}, \bar{1} \in \{0,1\}^{\{\theta\}}$ be defined by $\bar{0}(\theta) = 0$ and $\bar{1}(\theta) = 1$, so that $\bar{\jmath}(\theta) = j$. Note that $\{0,1\}^{\{\theta\}} = \{\bar{0}, \bar{1}\}$. Let $e_0, e_1 : \{0,1\}^{\{\theta\}} \to \{0,1\}$ be defined by $e_0(t) = 1 - t(\theta)$ and $e_1(t) = t(\theta)$. Then $e_i(\bar{\jmath}) = 1$ if $i = j$ and $e_i(\bar{\jmath}) = 0$ if $i \ne j$.

Given $s \in \{0,1\}^{T_\beta}$ and $t \in \{0,1\}^{\{\theta\}}$, we associate $(s,t) \in \{0,1\}^{T_\beta} \times \{0,1\}^{\{\theta\}} = \{0,1\}^{T_\beta} \times \{\bar{0}, \bar{1}\}$ with the element $[s,t] \in \{0,1\}^{T_\alpha}$ which extends both $s$ and $t$. Thus there is an association $J : L^p\left(\{0,1\}^{T_\beta} \times \{\bar{0},\bar{1}\}\right) \to L^p\left(\{0,1\}^{T_\alpha}\right)$. Let $\left(Z^p_{T_\beta} \oplus Z^p_{T_\beta}\right)_p$ be identified with the subspace of $L^p\left(\{0,1\}^{T_\beta} \times \{\bar{0},\bar{1}\}\right)$ which is related to $Z^p_{T_\beta}$ as in the definition of $(B \oplus B)_p$. Let $\left[Z^p_{T_\beta} \oplus Z^p_{T_\beta}\right]_p = J\left(Z^p_{T_\beta} \oplus Z^p_{T_\beta}\right)_p$. Then $\left[Z^p_{T_\beta} \oplus Z^p_{T_\beta}\right]_p \overset{\text{dist}}{\sim} \left(Z^p_{T_\beta} \oplus Z^p_{T_\beta}\right)_p$.

Let $b_0, b_1 \in Z^p_{T_\beta}$. Then $b_i \otimes e_i \in Z^p_{T_\alpha}$, where $(b_i \otimes e_i)[s,t] = 2^{\frac{1}{p}} b_i(s) e_i(t)$ for $s \in \{0,1\}^{T_\beta}$ and $t \in \{0,1\}^{\{\theta\}} = \{\bar{0},\bar{1}\}$. If $b = b_0 \otimes e_0 + b_1 \otimes e_1$, then $b[s,\bar{\jmath}] = 2^{\frac{1}{p}} b_0(s) e_0(\bar{\jmath}) + 2^{\frac{1}{p}} b_1(s) e_1(\bar{\jmath})$, so $b[s,\bar{0}] = 2^{\frac{1}{p}} b_0(s)$ and $b[s,\bar{1}] = 2^{\frac{1}{p}} b_1(s)$, whence $b \in \left[Z^p_{T_\beta} \oplus Z^p_{T_\beta}\right]_p$. Conversely, if $b \in \left[Z^p_{T_\beta} \oplus Z^p_{T_\beta}\right]_p$, then $b = b_0 \otimes e_0 + b_1 \otimes e_1$ for $b_0(s) = 2^{-\frac{1}{p}} b[s,\bar{0}]$ and $b_1(s) = 2^{-\frac{1}{p}} b[s,\bar{1}]$. Hence

$\left[Z^p_{T_\beta} \oplus Z^p_{T_\beta}\right]_p = \left\{b_0 \otimes e_0 + b_1 \otimes e_1 : b_0, b_1 \in Z^p_{T_\beta}\right\} \subset Z^p_{T_\alpha}$.

Let $f \in Z^p_{T_\alpha}$. For $s \in \{0,1\}^{T_\beta}$, let $b_0(s) = 2^{-\frac{1}{p}} f[s,\bar{0}]$ and $b_1(s) = 2^{-\frac{1}{p}} f[s,\bar{1}]$. Then $b_i \in Z^p_{T_\beta}$, and $f = b_0 \otimes e_0 + b_1 \otimes e_1$, so $f \in \left[Z^p_{T_\beta} \oplus Z^p_{T_\beta}\right]_p$. Thus $Z^p_{T_\alpha} \subset \left[Z^p_{T_\beta} \oplus Z^p_{T_\beta}\right]_p$, whence $Z^p_{T_\alpha} = \left[Z^p_{T_\beta} \oplus Z^p_{T_\beta}\right]_p$. For the case $\alpha = \beta + 1 > 1$, it now follows that $R^p_\alpha = \left(R^p_\beta \oplus R^p_\beta\right)_p = \left(Z^p_{T_\beta} \oplus Z^p_{T_\beta}\right)_p \overset{\text{dist}}{\sim} \left[Z^p_{T_\beta} \oplus Z^p_{T_\beta}\right]_p = Z^p_{T_\alpha}$.

Suppose $\alpha$ is a limit ordinal, where for each $\beta < \alpha$, $R^p_\beta$ is distributionally isomorphic to $Z^p_{T_\beta}$ for some well-founded CFRE tree $(T_\beta, \prec_\beta)$. Without loss of generality, suppose $R^p_\beta = Z^p_{T_\beta}$ for all $\beta < \alpha$, and suppose $T_\gamma \cap T_\beta = \emptyset$ for all $\gamma \neq \beta$ with $\gamma, \beta < \alpha$. Let $T_\alpha = \bigcup_{\beta<\alpha} T_\beta$, and let $\sigma \prec_\alpha \tau$ if there is some $\beta < \alpha$ such that $\sigma, \tau \in T_\beta$ with $\sigma \prec_\beta \tau$. Then $(T_\alpha, \prec_\alpha)$ is a well-founded CFRE tree.

Note that $B$ is a finite branch of $T_\alpha$ if and only if $B$ is a finite branch of $T_\beta$ for some $\beta < \alpha$. Thus $f$ depends on a finite branch $B$ of $T_\alpha$ if and only if $f$ depends on a finite branch $B$ of $T_\beta$ for some $\beta < \alpha$, so $Z^p_{T_\alpha} = \left[Z^p_{T_\beta} : \beta < \alpha\right]_{L^p(\{0,1\}^{T_\alpha})}$. Since $\{T_\beta\}_{\beta<\alpha}$ is disjoint, $\left[Z^p_{T_\beta} : \beta < \alpha\right]_{L^p(\{0,1\}^{T_\alpha})} \overset{\text{dist}}{\sim} \left(\sum^{\oplus}_{\beta<\alpha} Z^p_{T_\beta}\right)_{\text{Ind},p}$. Hence $Z^p_{T_\alpha} = \left[Z^p_{T_\beta} : \beta < \alpha\right]_{L^p(\{0,1\}^{T_\alpha})} \overset{\text{dist}}{\sim} \left(\sum^{\oplus}_{\beta<\alpha} Z^p_{T_\beta}\right)_{\text{Ind},p} = \left(\sum^{\oplus}_{\beta<\alpha} R^p_\beta\right)_{\text{Ind},p} = R^p_\alpha$. □

The following theorem [**B-R-S**, Theorem B(3)] is now almost immediate.

**Theorem 5.27.** *Let* $1 < p < \infty$ *and* $\alpha < \omega_1$. *Then* $R^p_\alpha \overset{c}{\hookrightarrow} L^p$.

**Proof.** By Proposition 5.26, we may choose a well-founded CFRE tree $T_\alpha$ such that $R^p_\alpha \sim Z^p_{T_\alpha}$. Then $Z^p_{T_\alpha} \overset{c}{\hookrightarrow} Z^p_{\mathbb{N}}$ by Proposition 5.25, and $Z^p_{\mathbb{N}} \overset{c}{\hookrightarrow} L^p\left(\{0,1\}^{\mathbb{N}}\right)$ by Theorem 5.22. Hence $R^p_\alpha \overset{c}{\hookrightarrow} L^p\left(\{0,1\}^{\mathbb{N}}\right) \sim L^p$. □

## Concluding Remarks

Let $1 < p < \infty$ where $p \neq 2$.

Conceivably $R^p_{\tau(\alpha)} \sim \ell^2$ for some $\alpha < \omega_1$, but in light of part (a) of Theorem

5.15, this is possible only for $\alpha = 0$. Thus as in the remark following Theorem 5.15, $\left\{R^p_{\tau(\alpha)}\right\}_{0<\alpha<\omega_1}$ is an uncountable chain of isomorphically distinct $\mathcal{L}_p$ spaces, and there is no separable $\mathcal{L}_p$ space $Y$, other than $L^p$ itself, such that $R^p_{\tau(\alpha)} \hookrightarrow Y$ for all $\alpha < \omega_1$. By Theorem 5.27 and part (a) of Theorem 5.15, for $\gamma < \delta < \omega_1$ we have

$$R^p_{\tau(\gamma)} \overset{c}{\to} R^p_{\tau(\delta)} \overset{c}{\to} L^p. \tag{5.5}$$

The isomorphism type of $R^p_\alpha$ for $\omega < \alpha < \omega_1$ is not well understood. Recent work by Dale Alspach indicates that $R^p_\omega \sim X_p$.

We know that $\{h_p(R^p_\alpha)\}_{\alpha<\omega_1}$ is a nondecreasing chain of ordinals such that $\{h_p(R^p_\alpha) : \alpha < \omega_1\}$ has no maximum, but little is known about the specific values of $h_p(R^p_\alpha)$ for $\omega \leq \alpha < \omega_1$, or precisely where the increases occur.

Part (b) of Theorem 5.15 reflects one way in which $\{R^p_\alpha\}_{\alpha<\omega_1}$ reaches toward $L^p$. However, it is not known whether for each separable $\mathcal{L}_p$ space $Y \not\sim L^p$, there is an $\alpha < \omega_1$ such that $Y \hookrightarrow R^p_\alpha$, nor whether there is an $\alpha < \omega_1$ such that $Y \hookrightarrow R^p_\alpha$ for uncountably many $\mathcal{L}_p$ spaces $Y$.

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2

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