PEAK SETS IN CONVEX DOMAINS WITH REAL-ANALYTIC BOUNDARIES

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CHAPTER 1

DEFINITIONS AND NOTATIONS

Throughout this paper D is a smoothly bounded domain in $\mathbb{C}^n$.

We denote by $A^\infty(D)$ the set of holomorphic functions in D which have a C^∞-extension to $\overline{D}$.

Definition 1.1 *A closed subset K of ∂D is a* peak set *for $A^\infty(D)$ if there exists a function $f \in A^\infty(D)$ so that $f = 1$ on K and $|f| < 1$ on $\overline{D} \setminus K$.*

K is locally *a* peak *set for $A^\infty(D)$ if for each $p \in K$, there exists a neighborhood V of p so that $K \cap \overline{V}$ is a peak set for $A^\infty(D)$.*

It is easy to see that a closed subset K of ∂D is a peak set for $A^\infty(D)$ if and only if there exists $g \in A^\infty(D)$ such that $g = 0$ on K and $\operatorname{Re} g > 0$ on $\overline{D} \setminus K$. Such a function g is called a strong suppo rt function for K.

Definition 1.2 *We denote by $T_p(M)$ the real tangent space to a smooth manifold M at the point $p \in M$.*

For a pont $p \in M$, the complex tangent space of M at p is the vector space

$$T_p^{\mathbb{C}}(M) = T_p(M) \cap J\{T_p(M)\}.$$

Here J is the almost complex structure corresponding to multiplication by i. $T_p^{\mathbb{C}}(M)$ is the maximal complex subspace of $T_p(M)$, of complex dimension $n-1$ if $M = \partial D$.

Definition 1.3 *A C^∞-submanifold $M \subseteq \partial D$ is integral at $p \in M$ if $T_p(M) \subseteq T_p^{\mathbb{C}}(\partial D)$. M is an integral manifold if it is integral at each point $p \in M$.*

Definition 1.4 *A C^∞-submanifold $M \subseteq \partial D$ is* totally real *if $T_p^{\mathbb{C}}(M) = \{0\}$ for every $p \in M$.*

Definition 1.5 *A defining function for a domain $D \subset \mathbb{C}^n$ is a C^∞ function*

$$r : \mathbb{C}^n \to \mathbb{R}$$

so that

(a) $D = \{z : r(z) < 0\}$

(b) $\nabla r \neq 0$ *on* ∂D.

If r is real-analytic, we say D has real-analytic boundary.

From now on, r denotes a defining function for D.

Definition 1.6 *We say that D is (Levi) pseudoconvex at p if*

$$L_r(p, w) = \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k}(p) w_j \overline{w}_k \geq 0, \tag{1.1}$$

for all $w \in T_p^{\mathbb{C}}(\partial D)$. *The expression on the left side of (1.1) is called the* Levi form *or the Complex Hessian of* r.

Definition 1.7 *Let D be pseudoconvex at p. The point p is said to be* strongly pseudoconvex *if the Levi form is positive whenever* $w \neq 0,\ w \in T_p^{\mathbb{C}}(\partial D)$.

The point p is said to be weakly pseudoconvex *if the Levi form is zero for some* $w \neq 0,\ w \in T_p^{\mathbb{C}}(\partial D)$.

We denote by $w(\partial D)$ the set of weakly pseudoconvex boundary points.

A domain is called pseudoconvex (resp. strongly pseudoconvex) if all its boundary points are pseudoconvex (resp. strongly pseudoconvex).

For $p \in \partial D$, we let $\mathcal{N}_p$ denote the null space in $T_p^{\mathbb{C}}(\partial D)$ of the Levi form at p. This, as well as notions in 1.6 and 1.7, are independent of the defining function.

Definition 1.8 *A domain $D \subset \mathbb{C}^n$ is* convex *if for all $p \in \partial D$ and $t \in \mathbb{R}^{2n}$,*

$$\sum_{j,k=1}^{2n} \frac{\partial^2 r}{\partial x_j \partial x_k}(p) t_j t_k \geq 0$$

whenever

$$\sum_{i=1}^{2n} \frac{\partial r}{\partial x_i}(p) t_i = 0.$$

Theorem 1.9 *Suppose $D \subset\subset \mathbb{C}^n$ is a convex domain with C^∞ boundary, then D is pseudoconvex.*

A proof of Theorem 1.9 can be found in [20].

The differential operator D^α is equal to

$$\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $|\alpha| = \alpha_1 + \ldots \alpha_n$.

We denote by $d(z, M)$ the Euclidean distance of z to a manifold M.

$\mathrm{Int}(A)_B$ will denote the interior of A in B.

CHAPTER 2

INTRODUCTION

The subject of peak sets has been studied in recent years by several authors [5], [6], [11], [12], [14], [18], [23]. If D is the unit disc in the complex plane, B. A. Taylor and D. L. Williams [23] proved that the only peak sets for $A^\infty(D)$ are the finite subsets of ∂D.

In $\mathbb{C}^n, n \geq 2$, the situation is somewhat different. In the strongly pseudoconvex case Hakim and Sibony [12], Chaumat and Chollet [5], [6], and Fornaess and Henriksen [11] gave the following characterization of locally peak sets for $A^\infty(D)$.

Theorem 2.1 *Let D be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary and K a closed subset of ∂D. The following conditions are equivalent:*

(i) K is locally a peak set for $A^\infty(D)$.

(ii) K is locally contained in an $(n-1)$-dimensional totally real submanifold of ∂D, which is integral at each point of K.

(iii) K is locally contained in an $(n-1)$-dimensional totally real integral submanifold of ∂D.

(iv) K is a peak set for $A^\infty(D)$.

For weakly pseudoconvex domains in $\mathbb{C}^n$, the aforementioned characterization of peak sets does not hold in general, including those which are convex with real-analytic boundary in $\mathbb{C}^2$.

In [18], Noell showed that there exists a convex domain D with real-analytic boundary in $\mathbb{C}^2$ and a peak set K for $A^\infty(D)$ which is not contained in any smooth curve. He also showed that there exists a convex domain D with real-analytic boundary in $\mathbb{C}^2$ and an integral curve $M \subset \partial D$ so that $M \cap \overline{U}$ is not a peak set for $A^\infty(D)$ for any neighborhood U of $0 \in M$. Hence the implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) do not hold for weakly pseudoconvex domains.

The implication (i) $\Rightarrow$ (iv) breaks down in general. Fornaess in [9] constructed a bounded pseudoconvex domain D with real-analytic boundary, and in [18], Noell showed that $K = w(\partial D)$ is locally a peak set for $A^\infty(D)$, but K is not globally a peak set for $A^\infty(D)$.

There is however, a positive result for the implication (i) $\Rightarrow$ (iv) in convex domains with real-analytic boundaries in $\mathbb{C}^2$.

Theorem 2.2 *[18]. Let $D \subset\subset \mathbb{C}^2$ be a convex domain with real-analytic boundary, then a compact set $K \subset \partial D$ is a peak set for $A^\infty(D)$ if and only if K is locally a peak set for $A^\infty(D)$.*

The main results of this thesis are:

Suppose $D \subset\subset \mathbb{C}^n$ is a convex domain with real-analytic boundary, K is a peak set for $A^\infty(D)$, and L a compact subset of K. Then L is a peak set for $A^\infty(D)$. (Theorem 5.6)

Suppose $D \subset\subset \mathbb{C}^3$ is a convex domain with real-analytic boundary. Then a compact subset K of ∂D is locally a peak set for $A^\infty(D)$ if and only if K is a peak set for $A^\infty(D)$. (Theorem 6.2)

Theorem 5.6 was proved by Chaumat and Chollet in [6] for strongly pseudoconvex domains with smooth boundaries in $\mathbb{C}^n$. Noell in [18] extended this result to pseudoconvex domains in $\mathbb{C}^2$ of finite type. He also showed that the finite type requirement can not be dropped, in fact, he gave in [18] an example of a pseudoconvex domain not of finite type, a compact set K which is a peak set for $A^\infty(D)$, and a compact subset L of K that is not a peak set for $A^\infty(D)$.

At this point, one asks the following question: What makes locally peak sets globally peak sets in convex domains with real-analytic boundaries in $\mathbb{C}^2$? The answer to this question depends on two criteria:

First, Noell in [18] imposed the (NP) condition on the domain which guarantees that we need only to patch peak functions at strongly pseudoconvex boundary points, and this is always achieved for such points by Theorem 2.1 ((i) $\Rightarrow$ (iv)). He also showed that every convex domain with real-analytic boundary satisfies the (NP) property. More precisely, Noell defined this "non-propagation" property as follows:

Definition 2.3 *Suppose $D \subset\subset \mathbb{C}^2$ is a pseudoconvex domain with real-analytic boundary. We say that D has property (NP) if there does not exist a real-analytic integral curve contained in $w(\partial D)$.*

Second, a decomposition of $w(\partial D)$ in $\mathbb{C}^2$ given by Fornaess and Øverlid in [10] is rather simple, and using this in conjunction with the (NP) property, we need to patch peak functions away from isolated sets. For more details, we refer the reader to [18].

In $\mathbb{C}^3$, bounded convex domains with real-analytic boundaries need not have the (NP) property.

Example *Let $D = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 + |z_3|^4 < 1\}$. It is clear that D is convex with real-analytic boundary, however*

$$w(\partial D) = \{(z_1, z_2, z_3) \in \partial D : z_3 = 0 \textit{ and } |z_1|^2 + |z_2|^2 = 1\}$$

contains a real-analytic integral curve, and therefore D does not have the (NP) property.

Instead, we will use the concept of linear regularity (Definition 3.1). This notion concerns the real tangent structure of $w(\partial D)$ in relation to the null space of the Levi form. Furthermore, the condition of linear regularity reduces to the (NP) property for bounded real-analytic domains in $\mathbb{C}^2$. Moreover, Noell showed in [19] that every bounded convex domain in $\mathbb{C}^n$ with real-analytic boundary is linearly regular. A proof of this fact is also included in this thesis (Theorem 3.2).

The idea of patching peak functions in convex domains with real-analytic boundaries in $\mathbb{C}^3$ came from the following example:

Let $D = \{z = (z_1, z_2, z_3) \in \mathbb{C}^3 : r(z) = Re\, z_3 + |z_1|^2 + |z_2|^4 < 0\}$.

It is obvious that D is a convex domain with real-analytic boundary and

$$w(\partial D) = \{z \in \partial D : z_2 = 0,\ Re\, z_3 = -|z_1|^2\}$$

Let $K = \{z \in \partial D : Re\, z_2 = Im\, z_1 = Im\, z_3 = 0\} \cup \{z \in \partial D : Im\, z_1 = Im\, z_2 = Im\, z_3 = 0\}$.

K is a peak set for $A^\infty(D)$ with strong support function $f(z) = z_3 + z_1^2 + z_2^4$.

$$K \cap w(\partial D) = \{z \in \partial D : z_2 = Im\, z_3 = Im\, z_1 = 0,\ Re\, z_3 = -(Re\, z_1)^2\}.$$

We note that $K \cap w(\partial D)$ is an integral curve which points in the strongly pseudoconvex direction. We patch peak functions near this curve.

This paper is divided into four parts:

In Chapter 3, we introduce the necessary definitions relevant to this section, and include a proof of the fact that every convex domain in $\mathbb{C}^n$ with real-analytic boundary is linearly regular. This theorem was proved by Noell in [19]. Our major goal in this chapter is to obtain a stratification of $w(\partial D)$ in $\mathbb{C}^n$ for convex domains with real-analytic boundaries (Theorem 3.9). In fact, the theorem holds for real-analytic domains which are linearly regular.

In Chapter 4, we apply Theorem 2.1 ((i)$\Rightarrow$(iii)), Theorem 3.9, and Rossi's theorem to show that $K \cap S$ is locally contained in an integral manifold of the boundary; here K is locally a peak set for $A^\infty(D)$, and S is any strata of $w(\partial D)$. Finally, we state Proposition 4.3 due to Chaumat and Chollet [5] which will be used in the proof of the Patching Lemma 6.1.

In Chapter 5, we turn our attention to the first main result, Theorem 5.6. The methods adopted there are based on those used by Chaumat and Chollet in [6], where they have proved the result for strongly pseudoconvex domains in $\mathbb{C}^n$.

In Chapter 6, we focus our attention on the proof of the second main result. We begin by proving the Patching Lemma 6.1, and use this in conjunction with Theorem 3.9 to show our second main result.

CHAPTER 3

LINEAR REGULARITY

In this chapter, we will find a local decomposition of $w(\partial D)$. This result (Theorem 3.9) is true for bounded convex domains in $\mathbb{C}^n$ with real-analytic boundaries. The main ingredient in the proof of Theorem 3.9 is to use the condition of "linear regularity" (Definition 3.1), which is the natural generalization of the "NP" property (introduced by Noell in [18]). Furthermore, Theorem 3.2 which appears in [19] shows that bounded convex domains in $\mathbb{C}^n$ satisfy such a condition. We apply Theorem 3.9 to prove the main results of this paper, namely, Theorem 5.6, and Theorem 6.2.

First, we give the basic definitions and concepts needed in this chapter.

Definition 3.1 *Suppose $D \subset\subset \mathbb{C}^n$ is a pseudoconvex domain with smooth boundary. We say D is "linearly regular" if there does not exist a smooth curve γ in ∂D so that $\gamma'(t) \in \mathcal{N}_{\gamma(t)}$ for all $t \in I$, where $I \subset \mathbb{R}$ is an interval.*

Theorem 3.2 *Suppose $D \subset\subset \mathbb{C}^n$ is a convex domain with real-analytic boundary. Then D is linearly regular.*

Proof. Assume to the contrary that there exists a smooth curve γ in ∂D so that $\gamma'(t) \in \mathcal{N}(\gamma(t))$ for all t. From this, we will show that ∂D contains a line segment, and this will be a contradiction.

Suppose γ is defined on an interval $I \subset \mathbb{R}$. Let $n(t) = \nabla r(\gamma(t))$, and $H(t)$ denote the $(2n \times 2n)$ matrix of real second-order partial derivatives of r, evaluated at $\gamma(t)$. Assume $||n|| = 1$.

Let $<,>$ denote the real inner product on $\mathbb{C}^n$ by identifying vectors in $\mathbb{C}^n$ with vectors in $\mathbb{R}^{2n}$.

Claim. $n(t)$ is constant for all $t \in I$.

Fix $t \in I$. Choose complex coordinates $z_j = x_j + iy_j$, $1 \leq j \leq n$ so that $\gamma'(t) = \frac{\partial}{\partial x_1}|_{\gamma(t)}$ and $n(t) = \frac{\partial}{\partial x_n}|_{\gamma(t)}$.

The Chain Rule gives

$$n'(s) = H(s)\gamma'(s), \tag{3.1}$$

for all $s \in I$.

Using the convexity of D, and since $\gamma'(t) \in \mathcal{N}_{\gamma(t)}$, we get

$$\frac{\partial^2 r}{\partial x_1^2}(\gamma(t)) = 0.$$

Also the convexity of D gives

$$\frac{\partial^2 r}{\partial x_1 \partial x_j}(\gamma(t)) = \frac{\partial^2 r}{\partial x_1 \partial y_k}(\gamma(t)) = 0,$$

when $1 \leq j \leq n-1,\ 1 \leq k \leq n$. So,

$$\langle H(t)\gamma'(t),\ \mu \rangle = 0$$

when $\mu \in T_{\gamma(t)}(\partial D)$.

Using this and (3.1), we conclude that $n'(t)$ is perpendicular to $T_{\gamma(t)}(\partial D)$.

Now, since $\langle n(s), n(s) \rangle \equiv 1$ for all $s \in I$, differentiation of both sides yields $\langle n'(s),\ n(s) \rangle \equiv 0$, which implies that $n'(s)$ is orthogonal to $n(s)$ for all $s \in I$. Because $n'(t)$ is orthogonal to $T_{\gamma(t)}(\partial D)$, we have $n'(t) \equiv 0$, and hence the claim follows.

Let $n(t) = \lambda$ for all $t \in I$, where λ is a constant. Define the function g as follows: for $s, t \in I$, put

$$g(s,t) = \langle \gamma(s) - \gamma(t), \lambda \rangle .$$

Then $\frac{\partial g}{\partial s} = \frac{\partial g}{\partial t} \equiv 0$, so g is a constant, since $g(s,s) = 0$ we get $g \equiv 0$. Thus $(\gamma(s) - \gamma(t)) \in T_{\gamma(t)}(\partial D)$, when $s, t \in I$. However, since D is convex, then the line segment through $\gamma(s)$ and $\gamma(t)$ must lie in ∂D. But this is a contradiction, since D is bounded, and D has real-analytic boundary. Thus D is linearly regular.

Definition 3.3 *Suppose S is a smooth submanifold of $\mathbb{C}^n$. We say S is a CR manifold if $dim_{\mathbb{C}} T_p^{\mathbb{C}}(S)$ is constant on S as a function of p.*

If $S \subset \partial D$ is a manifold, we say S has holomorphic dimension zero if for all $p \in S$ and all nonzero $(t_1, \ldots, t_n) \in T_p^{\mathbb{C}}(S)$ we have

$$\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k}(p) t_j \overline{t}_k > 0.$$

Definition 3.4 *Let S be a real-analytic manifold. We say $S' \subset S$ is a real-analytic subset of S if for every $p \in S$ there exists a neighborhood U of p and real-analytic map $F : U \rightarrow \mathbb{R}^m$, so that*

$$S' = \{q \in U : F(q) = 0\}.$$

The next theorem due to Frobenius is useful in the proof of 3.9. For a proof, consult [2].

Theorem 3.5 *A subbundle of the tangent bundle of a real manifold is the tangent bundle of a submanifold if and only if it is integrable. (Integrable means closed under the Lie bracket operation.)*

Now, we state Lojaciewicz's theorem which tells us that a real-analytic variety can be stratified into submanifolds of lower dimension. We will apply Theorem 3.6 below several times in the proof of Theorem 3.9 to stratify the real-analytic sets defined there.

Theorem 3.6 *Suppose that F is a nonconstant real-analytic function defined in a neighborhood $U \subset \mathbb{R}^n$ of the origin. Assume that the zero set Z of f in U is nonempty. Then Z has the following decomposition:*

$$Z = S_{(n-1)} \cup S_{(n-2)} \cup \ldots \cup S_0$$

where each $S_j(1 \leq j \leq n-1)$ *is a finite disjoint union of* j*-dimensional real-analytic submanifolds.*

Furthermore, S_j *is closed in* $Z \setminus ((\cup_{i=0}^{j-1} S_i)$, $j = 1, \ldots, n-1$.

The following theorem due to Diederich and Fornaess appears in [8], and will be used in the proof of Theorem 3.9.

Theorem 3.7 *Suppose* $D \subset\subset \mathbb{C}^n$ *is a pseudoconvex domain with real-analytic boundary. Then there exist real-analytic submanifolds* $S_1, \ldots, S_\tau$ *in* ∂D*, of holomorphic dimension zero, so that:*

(1) $w(\partial D) = \cup_{k=1}^{\tau} S_k$.

(2) S_k *is closed in* $\partial D \setminus \left(\cup_{i=1}^{k-1} S_i\right)$*, for* $k = 1, \ldots, \tau$.

The next proposition goes back to Bedford and Fornaess and can be found in [1].

Proposition 3.8 *Let* $D \subset\subset \mathbb{C}^n$ *be a pseudoconvex domain with* C^∞*-boundary. Suppose* $S \subset \partial D$ *is a smooth integral manifold. Then* $T_p^{\mathbb{C}}(S) \subset \mathcal{N}_p$*, for all* $p \in S$.

Now, we state and prove the main theorem of this chapter. The theorem below shows that the analysis on convex domains with real-analytic boundaries in $\mathbb{C}^n$ is similar to that of a strongly pseudoconvex domain.

Theorem 3.9 *Suppose* $D \subset\subset \mathbb{C}^n$ *is a convex domain with real-analytic boundary. Then for each* $p \in w(\partial D)$*, there exists a neighborhood* U *of* p *so that:*

(a)
$$w(\partial D) \cap U = \cup_{j=0}^{2n-3} S_j,$$

where each S_j *is a finite disjoint union of* j*-dimensional real-analytic CR submanifolds of* $\partial D \cap U$*. Furthermore, for all* $q \in S_j$,

$$T_q(S_j) \cap \mathcal{N}_q = \{0\}.$$

(b) If S is a component of some S_j and $T_q(S) \subset T_q^{\mathbb{C}}(\partial D)$ for some $q \in S$, then S is an integral submanifold of $\partial D \cap U$.

(c) S_j is closed in $\partial D \setminus \left(\cup_{i=0}^{j-1} S_i\right)$, $j = 1, \dots, 2n-3$.

Proof. By virtue of Theorem 3.7, we can find real-analytic submanifolds $N_1, \dots, N_\tau$ in ∂D of holomorphic dimension zero so that

$$w(\partial D) = \cup_{m=1}^{\tau} N_m,$$

with each N_k closed in $\partial D \setminus \left(\cup_{i=1}^{k-1} N_i\right)$, $k = 1, \dots, \tau$.

Let $V_k = \{z \in \partial D : \dim_{\mathbb{C}} \mathcal{N}_z \leq k\}$, $k = 1, \dots, n-1$. Then each V_k is a closed real-analytic subvariety of ∂D.

Put $W_k = V_k \setminus V_{k-1}$, and note that $w(\partial D) = \cup_{k=1}^{n-1} W_k$.

Fix m, and drop the subscript from N_m.

First, we note that N could not be of dimension $2n-1$, since N is of holomorphic dimension zero.

Assume for a contradiction that $\dim_{\mathbb{R}} N = 2n-2$.

We note that since $N = (W_1 \cup W_2 \cup \ldots \cup W_{n-1}) \cap N$, then

$$Int(W_i \cap N)_N \neq \emptyset \ for\ some\ i\ (1 \leq i \leq n-1).$$

Claim 1. $\mathrm{Int}(W_i \cap N)_N = \emptyset$ for all $i > 1$.

Proof. Suppose not, and let $q' \in Int\,(W_i \cap N)_N$ for some $i > 1$. Then there exists a neighborhood U' of q' in N so that $U' \subset W_i \cap N$. Since $T_{q'}^{\mathbb{C}}(U') \cap \mathcal{N}_{q'} = \{0\}$, and $\dim_{\mathbb{C}} T_{q'}^{\mathbb{C}}(U') \geq n-2$, this implies that $\dim_{\mathbb{C}} T_{q'}^{\mathbb{C}}(\partial D) \geq (n-2)+2 = n$. This is impossible, since $\dim_{\mathbb{C}} T_{q'}^{\mathbb{C}}(\partial D) = n-1$.

Thus the claim is true, and we conclude that $Int\,(W_1 \cap N)_N \neq \emptyset$.

Now, choose a nonzero real-analytic vector field X so that if $q \in W_1$ then $X(q)$ spans $\mathcal{N}_q$ over $\mathbb{C}$. For $q \in \mathit{Int}\,(W_1 \cap N)_N$, put

$$L_q = T_q(N) \cap \mathcal{N}_q.$$

We note that L_q is a real-subspace of $T_q(N)$, and L_q is of real dimension at most 1, since N is of holomorphic dimension zero.

Suppose $\dim_{\mathbb{R}} L_{q_0} = 0$ for some $q_0 \in \mathit{Int}\,(N \cap W_1)_N$. Then $T_{q_0}(N)$ contains no weakly pseudoconvex direction, but this is a contradiction, since $\dim_{\mathbb{R}} N = 2n-2$. Hence $\dim_{\mathbb{R}} L_q = 1$ for all $q \in \mathit{Int}\,(W_1 \cap N)_N$.

Because a one-dimensional subbundle of the real tangent of a manifold is integrable, then by Theorem 3.5, there exists a smooth curve $\gamma \subset \partial D$ so that $T_q(\gamma) = L_q$ for all $q \in \mathit{Int}\,(N \cap W_1)_N$. So $T_q(\gamma) \subset \mathcal{N}_q$. Therefore D is not linearly regular. But this is a contradiction, since by Theorem 3.2, every convex domain with real-analytic boundary is linearly regular.

Thus $\dim_{\mathbb{R}} N \leq 2n-3$.

Assume $\dim_{\mathbb{R}} N = 2n-3$.

Fix k, and let $p \in N \cap W_k$. For ease of notation, we drop the subscript from W_k. We will analyse the structure of $W \cap N$ in relation to the null space of the Levi form.

Choose nonzero real-analytic vector fields $\{X_j\}_{j=1}^k$ so that if $q \in W$ then $\{X_j(q)\}_{j=1}^k$ span $\mathcal{N}_q$.

Fix j, and let $M_q^{(j)} = \{\lambda X_j(q) : \lambda \in \mathbb{C}\}$, then $M_q^{(j)}$ is a complex subspace of $\mathcal{N}_q$ if $q \in W$.

For $q \in N \cap W$, put

$$L_q^{(j)} = T_q(N) \cap M_q^{(j)}.$$

We observe that $L_q^{(j)}$ is a real-subspace of $T_q(N)$, and $L_q^{(j)}$ is of real dimension at most 1, since N is of holomorphic dimension zero.

Let $S' = \{q \in N \cap W : \dim_{\mathbb{R}} L_q^{(j)} = 1\}$.

Claim 2. $(\,Int\,S')_N = \emptyset$.

Proof. Suppose not, and let $q_0 \in (\,Int\,S')_N$.

Choose a neighborhood V of q_0 in N so that $\dim_{\mathbb{R}} L_q^{(j)} = 1$ for all $q \in V$. By Theorem 3.5, there exists a smooth curve $\gamma \subset \partial D$ so that $T_q(\gamma) = L_q^{(j)} \subset \mathcal{N}_q$ for every $q \in \gamma \cap V$. This is impossible by linear regularity. So, $(\,Int\,S')_N = \emptyset$.

Claim 3. S' is a real-analytic subset of N.

Proof. Let $\{v_i\}_{i=1}^{2n-3}$ be nonzero real-analytic vector fields so that $\{v_i(q)\}_{i=1}^{2n-3}$ spans $T_q(N)$ for all $q \in N$.

Let $q \in S'$, and consider the $(2n-1) \times (2n)$ matrix

$$A_q = \begin{bmatrix} v_i(q) \\ X_j(q) \\ JX_j(q) \end{bmatrix} \quad (1 \leq i \leq 2n-3).$$

It follows from elementary linear algebra that if $q \in N \cap W$, then $q \in S'$ if and only if Rank $(A_q) \leq 2n-2$ if and only if $\det(B_q) = 0$ for every $(2n-1) \times (2n-1)$ submatrix B_q of A_q. Since $\det(B_q)$ is a real-analytic function, then S' is a real-analytic subset of S.

By Theorem 3.6, we may write:

$$S' \cap U = \Gamma_{(2n-4)} \cup \ldots \cup \Gamma_0,$$

where U is a small neighborhood of p, and each Γ_t, $(0 \leq t \leq 2n-4)$ is a finite disjoint union of t-dimensional real-analytic submanifolds of ∂D. One also notes that since $\Gamma_t \subset N$ and since N is of holomorphic dimension zero, so is Γ_t for all t, $(0 \leq t \leq 2n-4)$.

Let $S'' = (N \cap W) \setminus S'$.

Then S'' is an open subset of N, and S'' is of holomorphic dimension zero. Since $\dim_{\mathbb{R}} L_q^{(j)} = 0$ for all $q \in S''$, then $T_q(S'') \cap M_q^{(j)} = \{0\}$ for each $q \in S'' \cap U$.

Doing this for each $j = 1, \ldots, k$, we get a manifold still called S'' of dimension $2n-3$ such that for each q near p

$$T_q(S'') \cap \mathcal{N}_q = \{0\}.$$

Using this, and since $\dim_{\mathbb{R}} S'' = 2n-3$, we get for all q near p,

$$\dim_{\mathbb{C}} T_q^{\mathbb{C}}(S'') = n-2.$$

Hence S'' is a CR submanifold of $\partial D \cap U$. Since $\dim_{\mathbb{R}} S'' = 2n-3$, and $T_q(S'') \cap \mathcal{N}_q = \{0\}$, we must have that for all $q \in S''$ near p,

$$T_q(S'') \not\subseteq T_q^{\mathbb{C}}(\partial D).$$

Now,

$$(N \cap W) \cap U = S'' \cup \Gamma_{(2n-4)} \cup \ldots \cup \Gamma_0.$$

Since this is true for all m and k, we obtain:

$$w(\partial D) \cap U = \Lambda_{(2n-3)} \cup \Lambda_{(2n-4)} \cup \ldots \cup \Lambda_0,$$

where $\Lambda_{(2n-3)}$ is a finite disjoint union of $(2n-3)$-dimensional real-analytic CR submanifolds of $\partial D \cap U$, with

$$T_q(\Lambda_{(2n-3)}) \cap \mathcal{N}_q = \{0\}$$

for all q near p, and Λ_t, $(0 \leq t \leq 2n-4)$ are real-analytic submanifolds of $\partial D \cap U$ of holomorphic dimension zero.

The next thing we will do is to stratify the t-dimensional submanifolds of $\partial D \cap U$, $(0 \leq t \leq 2n-4)$ so that their tangent spaces do not contain any weakly pseudoconvex directions.

So, let Z be a real-analytic submanifold of $\partial D \cap U$, with $Z \subset w(\partial D)$, and Z is of holomorphic dimension zero, and assume $\dim_{\mathbb{R}} Z = 2n-4$.

Put $Z' = \{q \in Z : \dim_{\mathbb{R}} L_q'^{(j)} = 1\}$ where, for $q \in Z \cap W$,

$$L_q'^{(j)} = T_q(Z) \cap M_q^{(j)}.$$

We argue as in Claims 2 and 3, and deduce that $(Int\, Z')_Z = \emptyset$, and Z' is a real-analytic subset of Z.

Let $Z'' = \{q \in Z \cap W : T_q(Z) \subseteq T_q^{\mathbb{C}}(\partial D)\}$.

Claim 4. $(Int Z'')_Z = \emptyset$.

Proof. Suppose this is not the case, and let $q_0 \in (Int\, Z'')_Z$.

Then there exists a neighborhood U_1 of q_0 in Z so that $T_q(U_1) \subseteq T_q^{\mathbb{C}}(\partial D)$ for all $q \in U_1$. We consider two cases.

If $n > 3$, then because $T_{q_0}(U_1) \subseteq T_{q_0}^{\mathbb{C}}(\partial D)$, it follows that $\dim_{\mathbb{C}} T_{q_0}^{\mathbb{C}}(U_1) \geq n - 3 \geq 1$, and hence by Proposition 3.8 $T_{q_0}^{\mathbb{C}}(U_1) \subset \mathcal{N}_{q_0}$. But this is impossible, since U_1 is of holomorphic dimension zero.

If $n = 3$, then $\dim_{\mathbb{C}} T_q^{\mathbb{C}}(U_1) \leq 1$ for all $q \in U_1$.

Assume U_1 is totally real at $q \in U_1$, and hence in a neighborhood of q.

For $q_1 \in U_1 \cap W_1$, define

$$L''_{q_1} = T_{q_1}(U_1) \cap \mathcal{N}_q.$$

Then L''_{q_1} is a real-subspace of $T_{q_1}(U_1)$, and of dimension at most 1.

If $\dim_{\mathbb{R}} L''_{q_0} = 0$ for some $q_0 \in U_1 \cap W_1$, then since $T_{q_0}(U_1) \subset T_{q_0}^{\mathbb{C}}(\partial D)$, $T_{q_0}^{\mathbb{C}}(U_1) = \{0\}$, and $\dim_{\mathbb{R}} U_1 = 2$, this is impossible, so $\dim_{\mathbb{R}} L''_q = 1$ for all $q \in U_1$. Again, we argue as before, and obtain a contradiction to linear regularity. Thus U_1 is nowhere totally real. Hence $\dim_{\mathbb{C}} T_q^{\mathbb{C}}(U_1) = 1$ for all $q \in U_1$. Since $T_q(U_1) \subseteq T_q^{\mathbb{C}}(\partial D)$, this again gives a contradiction, by Proposition 3.8. Therefore the claim is true.

Claim 5. Z'' is a real-analytic subset of Z.

Proof. Let $F_1 = (f_1, \ldots, f_n)$ be a non-singular real-analytic parametrization of Z

defined near 0 in $\mathbb{R}^s$ $(s = 2n-4)$ so that $F_1'(0) = p$. Let

$$\varphi_j(u) = \sum_{k=1}^{n} \frac{\partial r}{\partial z_k}(F_1(u)) \cdot \frac{\partial f_k}{\partial u_j},\ (1 \leq j \leq s),$$

where r is a defining function for D.

Since r is a real-analytic function, so is $\{\varphi_j\}_{j=1}^{s}$, and hence,

$$Z'' = \{F_1(u) \in Z \cap W : \varphi_j(u) = 0 \ for\ all\ 1 \leq j \leq s\}$$

is a real-analytic subset of Z.

Because the union of two real-analytic sets is a real-analytic set, we conclude that $Z' \cup Z''$ is a real-analytic subset of Z.

By Theorem 3.6, and shrinking U if necessary, we write:

$$(Z' \cup Z'') \cap U = Q_{(2n-5)} \cup \ldots \cup Q_0,$$

where each Q_t, $(0 \leq t \leq 2n-5)$, is a finite disjoint union of t-dimensional real-analytic submanifolds of $\partial D \cap U$, and of holomorphic dimension zero.

Let $Z'' = (Z \cap W) \setminus (Z' \cup Z'')$.

Arguing as above, we conclude that Z'' is a submanifold of Z with

$$T_q(Z'') \cap \mathcal{N}_q = \{0\}$$

for all q near p.

Continuing this way, each time we stratify the sets:

$$\{q \in \Sigma \cap W : \dim_{\mathbb{R}} L_q^{(j)} = 1\},$$

and

$$\{q \in \Sigma \cap W : T_q(\Sigma) \subseteq T_q^{\mathbb{C}}(\partial D)\},$$

where $\Sigma \subset w(\partial D)$ is any submanifold of $\partial D \cap U$ of dimension less than or equal to $(2n-5)$, until its tangent space contains no weakly pseudoconvex tangent directions.

Finally, we may write $w(\partial D) \cap U$, for U sufficiently small neighborhood of p as:

$$w(\partial D) \cap U = T_{(2n-3)} \cup \ldots \cup T_0,$$

where each T_i, $(0 \leq i \leq 2n-3)$ is a finite disjoint union of real-analytic submanifolds of $\partial D \cap U$, and

$$T_q(T_i) \cap \mathcal{N}_q = \{0\}$$

for all $q \in T_i$.

All it remains to show in part (a) is that the T_i's,$(1 \leq i \leq 2n-4)$ can be modified to obtain CR manifolds. We will show this and part (b) simultaneously using the following procedure.

Let T be any real-analytic submanifold of $\partial D \cap U$, with $\dim_{\mathbb{R}} T = 2n-4$.

Assume T is connected.

Let

$$\lambda_1 = max\{\dim_{\mathbb{C}} T_q^{\mathbb{C}}(T) : q \in T\}.$$

Let

$$E = \{q \in T : \dim_{\mathbb{C}} T_q^{\mathbb{C}}(T) = \lambda_1\}.$$

We show in Claim 6 below that E is a real-analytic set. Suppose for a moment this is true.

Note that if E has interior in T, then since T is connected, $E = T$, and hence T is a CR manifold.

If E has empty interior in T, then by stratifying E using Theorem 3.6, we get lower dimensional manifolds. The complement E_1 of E in T is an open subset of T with

$$\dim_{\mathbb{C}} T_q^{\mathbb{C}}(E_1) \leq \lambda_1 - 1$$

for all $q \in E_1$. We repeat the procedure above, replacing T by E_1.

We continue this way, each time stratifying the sets of maximal complex tangent dimension until we obtain an open subset T' of T with constant CR dimension.

Claim 6. E is a real-analytic subset of T.

Proof. Suppose that T is given locally by

$$T = \{q : r_i(q) = 0 \ for \ all \ 1 \leq i \leq 4\},$$

where r_i are real-analytic functions on T. We know that

$$\dim_{\mathbb{C}} T_q^{\mathbb{C}}(T) = n - Rank_{\mathbb{C}} \left[\frac{\partial r_i}{\partial z_j}(p)\right]_{\substack{1 \leq i \leq 4 \\ 1 \leq j \leq n}} .$$

Now, if $q \in T$, then $q \in E$

$$\begin{aligned}
&\iff \dim_{\mathbb{C}} T_q^{\mathbb{C}}(T) = \lambda_1 \\
&\iff \dim_{\mathbb{C}} T_q^{\mathbb{C}}(T) > \lambda_1 - 1 \\
&\iff Rank_{\mathbb{C}} \left[\frac{\partial r_i}{\partial z_j}(q)\right] < n - (\lambda_1 - 1) \\
&\iff \det(B_q) = 0
\end{aligned}$$

for every $(n - \lambda_1 + 1) \times (n - \lambda_1 + 1)$ submatrix B_q of $\left[\frac{\partial r_i}{\partial z_j}(q)\right]_{\substack{1 \leq i \leq 4 \\ 1 \leq j \leq n}}$.

Thus E is a real-analytic subset of T.

Let T' be the set described before Claim 6, and let

$$F = \{q \in T' : T_q(T') \subset T_q^{\mathbb{C}}(\partial D)\}.$$

By the same argument as in Claim 5, F is a real-analytic subset of T'.

As above there are two cases to consider.

First, if F has interior in T', then $F = T'$, and so T' is integral.

If F has empty interior in T', then we apply Theorem 3.6 to stratify F, and we get lower dimensional manifolds. Then we consider the set $F' = T' \setminus F$, which is an open subset of T' with

$$T_q(F') \not\subseteq T_q^{\mathbb{C}}(\partial D)$$

for all $q \in F'$. Furthermore, we note that for all $q \in F'$,

$$dim_{\mathbb{C}} T_q^{\mathbb{C}}(F') = \dim_{\mathbb{C}} T_q^{\mathbb{C}}(T'),$$

and hence F' is a CR manifold.

This finishes the argument for the $(2n-4)$-dimensional submanifolds of $\partial D \cap U$.

Now, let M be any real-analytic submanifold of $\partial D \cap U$, with $\dim_{\mathbb{R}} M = 2n-5$.

We carry out the same reasoning as we did for T, stratifying first the set of points in M where $T^{\mathbb{C}}(M)$ has maximal dimension until we obtain an open subset M' of M which is a CR manifold, and then stratifying the set of points in M' where M' is integral.

We do this inductively until all the strata of $w(\partial D)$ satisfy the properties stated in the theorem.

This concludes the proof of the theorem.

Remark 3.10 Fornaess and Øverlid gave in [10] a global decomposition of $w(\partial D)$ for pseudoconvex domains with real-analytic boundaries. In fact, they have shown that $w(\partial D)$ for such domains can be written as:

$$w(\partial D) = S_0 \cup S_1 \cup S_2,$$

where each S_j is a finite disjoint union of j-dimensional totally real real-analytic submanifolds of ∂D. For convex domains with real-analytic boundaries in $\mathbb{C}^2$, Theorem 3.9 tells us that the maximal strata of $w(\partial D)$ is a curve, i.e., $S_2 = \emptyset$. That S_2 is empty in $\mathbb{C}^2$ is of course an immediate consequence of linear regularity.

CHAPTER 4

INTEGRAL MANIFOLDS

The purpose of this chapter is to build additional ingredients that will enable us to prove the main results of this thesis, that is, every set which is locally a peak set is a peak set, and compact subsets of peak sets are peak sets.

First of all, we state Proposition 4.1 due to Rossi [21], and we will apply this to show the main result of this chapter (Theorem 4.2), which in short says that the intersection of a peak set and any strata of $w(\partial D)$ is loc

ally contained in integral manifolds.

Second, we end this chapter with Proposition 4.3 due to Chaumat and Chollet which only holds for strongly pseudoconvex domains. The proposition shows the local behavior of peak functions. We will apply the proposition in Chapters 5 and 6.

The next proposition appears in [21]. This proposition enables us to put real-analytic CR submanifolds of $\mathbb{C}^n$ into lower dimensional $\mathbb{C}^k$.

Proposition 4.1 *Suppose $S \subset \mathbb{C}^n$ is a real-analytic CR manifold. Assume $\dim_{\mathbb{R}} S = 2t + \lambda$, with $\dim_{\mathbb{C}} T_q^{\mathbb{C}}(S) = t$ for all $q \in S$. Then for each $p \in S$, there exist a neighborhood U of p and a biholomorphic map $\Phi : U \longrightarrow \mathbb{C}^n$ so that:*

(a) $\Phi(p) = 0$

(b) $\Phi(S \cap U) \subset \mathbb{C}^{t+\lambda} \times \{0\}$.

Theorem 4.2 is fundamental to the proof of the main results. We will modify the peak set and patch peak functions along the integral manifold $\widehat{M}$ given there. For strongly pseudoconvex domains with smooth boundaries, Theorem 2.1 (i) $\Rightarrow$ (iii) says that if K is locally a peak set for $A^\infty(D)$, then K is locally contained in an $(n-1)$ dimensional integral manifold. Our result below is somewhat similar.

Theorem 4.2 *Suppose $D \subset\subset \mathbb{C}^n$ is a convex domain with real-analytic boundary. Assume K is a compact subset of ∂D which is locally a peak set for $A^\infty(D)$. Let S be any strata of $w(\partial D)$ as in Theorem 3.9, and suppose $\dim_{\mathbb{R}} S = 2t + \lambda$, where $\dim_{\mathbb{C}} T^{\mathbb{C}}_{q_0}(S) = t$ for all $q_0 \in S$. Suppose $p \in K \cap S$. Then there exist a neighborhood $U \subset \mathbb{C}^n$ of p, a holomorphic change of coordinates in U, in which $p = 0$ and $S \subset \mathbb{C}^{t+\lambda} \times \{0\}$, a neighborhood $U' \subset \mathbb{C}^{t+\lambda}$ of 0, a strongly pseudoconvex domain $\Omega \subset\subset U'$, a locally peak set $\widetilde{L} \subset \partial\Omega \cap V'$, where $V' \subset\subset U'$ is a neighborhood of 0, and a totally real smooth manifold $\widetilde{M}$ in $\partial\Omega \cap V'$ so that:*

(a) $K \cap S \cap V \subseteq \widehat{L} \subset \widehat{M} \subset (\partial\Omega \times \{0\}) \cap V'' \subset \partial D \cap V,$

where $\widehat{L} = \widetilde{L} \times \{0\}$, $\widehat{M} = \widetilde{M} \times \{0\}$, $V'' = V' \times \{0\}$, and $V \subset U$ is a neighborhood of 0.

(b) $T_q(\widehat{M}) \subseteq T^{\mathbb{C}}_q(\partial D)$ *for all* $q \in \widehat{M}$.

(c) $\dim_{\mathbb{R}} \widehat{M} \leq n - 2$.

Proof. We first do the case when $T_p(S) \not\subseteq T^{\mathbb{C}}_p(\partial D)$.

Since S is a real-analytic CR manifold, then by Proposition 4.1, there exists a neighborhood U of p in $\mathbb{C}^n$ and a biholomorphic map $\Phi : U \longrightarrow \mathbb{C}^n$ so that $\Phi(p) = 0$ and $\Phi(U \cap S) \subset \mathbb{C}^{t+\lambda} \times \{0\}$.

Let $z \in U$, and $\Phi(z) = (z', z'')$, with $z' = (z_1, \ldots, z_{t+\lambda})$, denotes the new holomorphic change of coordinates near 0, where $z_{t+\lambda} = u + iv$ is the complex normal direction to ∂D at 0. We assume that the new manifold obtained under Φ that sits in $\mathbb{C}^{t+\lambda} \times \{0\}$ is also denoted by S.

We define the function ρ by:

$$\rho(z') = r \circ h(z'),$$

where $h(z') = (z', 0, \ldots, 0)$. Let U' be a neighborhood of $0'$ in $\mathbb{C}^{t+\lambda}$, and put

$$\Omega = \{z' \in U' : \rho(z') < 0\}.$$

Note that Ω is a bounded domain in U', and S is locally contained in $(\partial\Omega \times \{0\}) \cap U'$, where

$$\partial\Omega \cap U' = \{z' \in U' : \rho(z') = 0\}.$$

We need to show that ρ is a defining function for Ω. So it suffices to show that if U' is small enough,

$$\nabla \rho \neq 0 \ on \ \partial\Omega \cap U'.$$

Assume $\dfrac{\partial r}{\partial z_{t+\lambda}}(0) = 1$.

By the Chain Rule,

$$\begin{aligned} \frac{\partial \rho}{\partial z_{t+\lambda}}(0') &= \sum_{k=1}^{n} \frac{\partial r}{\partial w_k}(0) \cdot \frac{\partial h_k}{\partial z_{t+\lambda}}(0') + \sum_{k=1}^{n} \frac{\partial r}{\partial \overline{w}_k}(0) \frac{\partial \overline{h}_k}{\partial z_{t+\lambda}}(0') \\ &= \sum_{k=1}^{n} \frac{\partial r}{\partial w_k}(0) \cdot \frac{\partial h_k}{\partial z_{t+\lambda}}(0') = 1 \end{aligned}$$

Thus, $\nabla\rho(0') \neq 0$, and hence $\nabla\rho \neq 0$ in a neighborhood of 0. We show that Ω is a strongly pseudoconvex domain near$0'$. So, it is enough to show that Ω is strongly pseudoconvex at $0'$. An easy computation of the Levi form yields,

$$L_\rho(0', \eta) = L_r(0, h'(0')\eta),$$

where,

$$\eta = (\eta_1, \ldots, \eta_{t+\lambda}) \in \mathbb{C}^{t+\lambda}, \ \eta \neq 0' \ and \ \eta \in T_{0'}^{\mathbb{C}}(\partial\Omega).$$

So,

$$L_\rho(0', \eta) = L_r(0, (\eta_1, \ldots, \eta_{t+\lambda}, 0, \ldots, 0)).$$

Since $T_0(S) \cap \mathcal{N}_0 = \{0\}$ by Theorem 3.9, we conclude that $L_\rho(0', \eta) > 0$.

Now, we show that there exists locally a peak set $\widetilde{L}$ for $A^{\infty}(\Omega)$ so that

$$K \cap S \cap V \subseteq \widetilde{L} \times \{0\},$$

where $V \subset U$ is a neighborhood of 0.

Let f be a strong support function for $K \cap \overline{V}$, where $V \subset U$ is an open neighborhood of 0 in $\mathbb{C}^n$.

Define the function g by

$$g(z') = f \circ h(z').$$

Put

$$\widetilde{L} = \{z' \in \overline{\Omega} \cap V' : g(z') = 0\},$$

with $V' \subset U'$ as a neighborhood of $0'$.

We claim that g is a strong support function for $\widetilde{L} \cap \overline{V}'$.

It is obvious that $g \in A^{\infty}(\Omega)$. We claim that $g \not\equiv 0$ on $\overline{\Omega} \cap V'$. To see this, assume to the contrary that $g \equiv 0$ on $\overline{\Omega} \cap V'$. Then by the Chain Rule:

$$\begin{aligned} 0 &= \frac{\partial g}{\partial z_{t+\lambda}}(0') \\ &= \sum_{k=1}^{n} \frac{\partial f}{\partial z_k}(0) \cdot \frac{\partial h_k}{\partial z_{t+\lambda}}(0') \\ &= \frac{\partial f}{\partial z_{t+\lambda}}(0), \end{aligned}$$

and hence $\dfrac{\partial f}{\partial u}(0) = 0$.

But this is absurd because $Re\, f$ is pluriharmonic in D, nonconstant, and has a local minimum at 0, so by the Hopf lemma,

$$\frac{\partial (\,Re\, f)}{\partial u}(0) < 0.$$

We note that if $Re\, g = 0$, then $g = 0$, and therefore $Re\, g > 0$ on $\overline{\Omega} \setminus \widetilde{L} \cap \overline{V}'$. In addition, $\widetilde{L} \subset \partial\Omega \cap V'$ which follows from the maximum modulus principle.

Let $z \in K \cap S \cap V$, with $z = (z_1, \dots z_{t+\lambda}, 0, \dots, 0) = (z', 0, \dots, 0)$. Then, $g(z') = f \circ h(z') = f(z', 0, \dots, 0) = 0$ since $z \in K$. So $z' \in \widetilde{L}$, and $(z', 0) \in \widetilde{L} \times \{0\}$. Thus $K \cap S \cap V \subset \widetilde{L} \times \{0\}$.

Now, we verify properties (a)–(c) of the theorem.

For this we use Theorem 2.1 ((i)$\Rightarrow$(iii)) due to Chaumat and Chollet, which is applicable only to strongly pseudoconvex domains.

Since Ω is a strongly pseudoconvex domain near $0'$, and $\widetilde{L}$ is locally a peak set for $A^\infty(\Omega)$, by Theorem 2.1 we obtain a totally real integral submanifold $\widetilde{M}$ of $\partial\Omega \cap V'$, if V' is small enough, so that:

$$\widetilde{L} \cap V' \subset \widetilde{M}.$$

Let $\widehat{M} = \widetilde{M} \times \{0\}$.

Then, for all $q \in \widehat{M}$,

$$T_q(\widehat{M}) \subseteq T_q^{\mathbb{C}}(\partial D),$$

and

$$\dim_{\mathbb{R}} \widehat{M} = t + \lambda - 1 \leq (n-1) - 1 = n - 2.$$

This finishes the proof in the case $T_p(S) \not\subseteq T_p^{\mathbb{C}}(\partial D)$.

If $T_p(S) \subset T_p^{\mathbb{C}}(\partial D)$, then S is an integral submanifold of ∂D by Theorem 3.9. Hence, S must be totally real by Proposition 3.8 and part (a) of Theorem 3.9. So the preceding proof is easily modified.

We note that the convexity of D was used only to get a real-analytic strata obtained by Theorem 3.9.

The proposition below appears in [6].

Proposition 4.3 *Suppose that $\Omega \subset\subset \mathbb{C}^k$ is a strongly pseudoconvex domain with smooth boundary. Let $\widetilde{L}$ be a peak set for $A^\infty(\Omega)$ with strong support function g. Then for each $p' \in \widetilde{L}$, there exists a neighborhood U' of p', a positive constant c', and* *a totally real submanifold* $\widetilde{M} \subset \partial\Omega \cap U'$ *of class* C^∞ *and of dimension* k*, containing* $\widetilde{L} \cap U'$*, and so that for all* $q' \in \overline{\Omega} \cap U'$

$$Re\, g(q') \geq c' d^2(q', \widetilde{M}).$$

CHAPTER 5

COMPACT SUBSETS OF PEAK SETS

The principal result of this chapter is Theorem 5.6, which is an extension of a result obtained by Noell in [18]. There he showed that compact subsets of peak sets are peak in convex domains with real-analytic boundaries in $\mathbb{C}^2$. In fact, Noell in [18] proved the aforementioned result for smooth pseudoconvex domains of finite type in $\mathbb{C}^2$.

The approach which we have carried out here to prove 5.6 is based on that used by Chaumat and Chollet in [6], where they proved the result for smooth strongly pseudoconvex domains in $\mathbb{C}^n$.

Our starting point is Proposition 5.1, which goes back to Chaumat and Chollet [5]. The proposition allows us to construct peak functions from the functions stated there.

Proposition 5.1 *Suppose $D \subset \mathbb{C}^n$ is a bounded pseudoconvex domain with smooth boundary. Let E be a compact subset of ∂D, W a neighborhood of E in $\mathbb{C}^n$, and ρ a non-negative continuous function on W which vanishes on E. Suppose that there exists a function $G \in C^\infty\ (W \cap \overline{D})$ such that:*

(a) $E = \{z \in W \cap \overline{D} : G(z) = 0\}$

(b) For each $\alpha \in \mathbb{N}^n$, $k \in \mathbb{N}$, there exists $C_{\alpha k} > 0$ such that for each $z \in W \cap \overline{D}$

$$|D^\alpha(\overline{\partial} G(z))| \leq C_{\alpha k}[\rho(z)]^k.$$

(c) There exists a constant $c > 0$ so that for all $z \in \overline{D} \cap W$,

$$Re\, G(z) \geq c\, \rho(z).$$

Then E is a peak set for $A^\infty(D)$.

Proof. The proof can be found in [5], so we will be brief.

Let X be a C^∞ real-valued function with compact support in W so that $0 \le X \le 1$ and $X \equiv 1$ on a neighborhood W_1 of E.

Let h be the $(0,1)$ form defined in $\overline{D} \setminus E$ by:

$$h = \begin{cases} \overline{\partial}\left(\frac{X}{G}\right) & \text{in } W \\ 0 & \text{elsewhere} \end{cases}$$

We note that h is C^∞ in $\overline{D} \setminus E$. We extend h, and all its derivatives to be C^∞ on $\overline{D}$.

By a theorem of J. J. Kohn [16], since $\overline{\partial}h$ vanishes on $\overline{D}$, there exists a C^∞-function u on $\overline{D}$ such that

$$\overline{\partial}u = h.$$

Let v be the function defined by

$$v = \frac{X}{G} - u.$$

Then v is holomorphic in D, and smooth on $\overline{D} \setminus E$. Furthermore,

$$Re\, v = X\frac{Re\, G}{|G|^2} - Re\, u.$$

Since u is of class C^∞ in $\overline{D}$, then $\mathrm{Re}\, u$ is bounded. Using (c), and adding a large constant, we may suppose that $\mathrm{Re}\, v > 0$ on $\overline{D} \setminus E$.

We deduce from this that

$$\frac{1}{v} = \frac{G}{1 - uG}$$

is holomorphic in D, of class C^∞ in $\overline{D} \setminus E$ on W_1. We note that $(1 - uG)$ does not vanish on $\overline{D}$ in a neighborhood of E, hence $\frac{1}{v}$ is of class C^∞ on $\overline{D}$. Thus,

$$\frac{1}{v} \in A^\infty(D),$$

$\frac{1}{v}$ extends to be 0 on E, and

$$Re\left(\frac{1}{v}\right) > 0$$

on $\overline{D} \setminus E$.

Therefore, E is a peak set for $A^\infty(D)$.

Now, we introduce the function $S_R(f)$. This function has already been used by Noell in [18].

For $R > 0$, let $S_R(f) = f - Rf^2$, where f is a strong support function for K in D.

We note that,

$$Re\,(S_R(f)) = (Re\,f)(1 - R(Re\,f)) + R(Im\,f)^2,$$

$S_R(f) = 0$ on K, and $\mathrm{Re}\,(S_R(f)) > 0$ on $(\overline{D} \cap U) \setminus K$ for a neighborhood U of K.

Theorem 5.2 *Suppose $D \subset\subset \mathbb{C}^n$ is a convex domain with real-analytic boundary. Let K be a compact subset of ∂D which is a peak set for $A^\infty(D)$ with strong support function f. Let $S \subset \partial D$ be any strata of $w(\partial D)$, and suppose $dim_{\mathbb{R}} S = 2t + r$, with $dim_{\mathbb{C}} T^{\mathbb{C}}_{q_0}(S) = t$ for all $q_0 \in S$. Assume $p \in K \cap S$. Let U, U', $\widetilde{L}$, and Ω be as in Theorem 4.2. Let R be a sufficiently large positive number. Then there exist neighborhoods $V \subset U$ of p, $V' \subset U'$ of p', a totally real manifold M' in $\partial\Omega \cap V'$, of dimension at most $(n-1)$ containing $\widetilde{L} \cap V'$ and a smooth manifold $N \subset V$ containing $M = M' \times \{0\}$ so that:*

(a) $K \cap S \cap V \subseteq \widehat{L} \subset M \subset (\partial\Omega \times \{0\}) \cap V'' \subset \partial D \cap V$,

where $V'' = V' \times \{0\}$.

(b) If $q \in \overline{D} \cap V$, *then*

$$Re\,S_R(f)(q) \geq c\,d^2(q, N),$$

where c is a positive constant.

Proof. Since $\Omega \subset\subset U'$ is a strongly pseudoconvex domain with smooth boundary, then by Proposition 4.3, we can find a neighborhood $V' \subset U'$ of $0'$, and a totally real submanifold $M' \subset \partial\Omega \cap V'$, of dimension $t + \lambda \leq n - 1$, containing $\widetilde{L} \cap V'$, and so that for all $q' \in \overline{\Omega} \cap V'$,

$$Re\, g(q') \geq c' d^2(q', M'),$$

where $c' > 0$ is a constant.

Now, part (a) of the theorem follows.

We define the manifold $N \subset U$ by

$$N = M' \times \mathbb{C}^{n-t-\lambda},$$

and we observe that N contains $M' \times \{0\}$.

It only remains to show part (b).

Let $q' \in M' \times \{0\}$. Since $M' \times \{0\}$ is totally real, we can make a holomorphic linear change of coordinates near q' that we denote by

$$z_j = x_j + iy_j,\ j = 1, \ldots, t + \lambda - 1,\ z_{t+\lambda} = u + iv$$

so that $q' = 0$, and

$$T_0(M') = \{z' \in \mathbb{C}^{t+\lambda} : y_1 = \ldots = y_{t+\lambda+1} = u = 0\}.$$

Assume also that

$$T_0(\partial\Omega) = \{z' : u = 0\} \text{ and } T_0(\partial D) = \{(z', z'') : u = 0\}.$$

Let $g(z') = f(z', 0, \ldots, 0)$ be a strong support function for $\widetilde{L} \cap \overline{V}'$, where $V' \subset U'$, obtained from the proof of Theorem 4.2.

Proposition 4.3 assures us that the real Hessian of $\mathrm{Re}\, g$ at 0 is positive definite when restricted to the orthogonal complement of $T_{0'}(M')$ in $T_{0'}^{\mathbb{C}}(\partial\Omega)$. In addition, by the Cauchy-Riemann equations and the Hopf lemma, we have:

$$\frac{\partial(Re\, f)}{\partial u}(0,0,0) < 0.$$

These facts give that for $q \in \overline{D}$ near q',

$$Re\, S_R(f)(q) \geq c\, d^2(q, N).$$

This finishes the proof of Theorem 5.2.

We will frequently use the next proposition in what follows. The proposition is due to Harvey and Wells, a proof is included in [13].

Proposition 5.3 *Suppose $\widehat{M} \subset \partial D \cap U$ is a totally real submanifold of ∂D, where U is an open subset of $\mathbb{C}^n$. Let χ be a C^∞-function in $\widehat{M}$. Then there exists a C^∞-function $\tilde{\chi}$ in U so that:*

(1) $\tilde{\chi} = \chi$ on $\widehat{M}$

(2) $\overline{\partial}\tilde{\chi}$ vanishes to infinite order along $\widehat{M}$, that is, $D^\alpha(\overline{\partial}\tilde{\chi}) \equiv 0$ along $\widehat{M} \cap U$, for each multi-index α.

(3) $\tilde{\chi}$ is locally constant near where χ is locally constant.

(4) If χ has compact support in $\widehat{M}$, then $\tilde{\chi}$ has compact support in U.

(5) First derivatives of $\tilde{\chi}$ vanish on $\widehat{M}$ in directions perpendicular to $T(\widehat{M}) + JT(\widehat{M})$.

Remark 5.4 Let $\{E_n\}_{n=1}^{\infty}$ be a collection of peak sets for $A^\infty(D)$, with strong support functions $\{f_n\}_{n=1}^{\infty}$. Then $E = \cap_{n=1}^{\infty} E_n$ is a peak set for $A^\infty(D)$. To see this, let

$$c_n = \max\{\|D^\alpha f_n\|_\infty : 0 \le |\alpha| \le n\},$$

and put

$$f = \sum_{n=1}^{\infty} \frac{1}{c_n 2^n} f_n.$$

We note that since $\|f_n\|_\infty \le c_n$, so $\left\|\frac{1}{c_n} f_n\right\|_\infty \le 1$, then f is well-defined. Also, $f \in A^\infty(D)$ and $\operatorname{Re} f_n(z) \ge 0$ for all n, and for each $z \in \overline{D}$. But $z \in E$ if and only if $f(z) = 0$.

Therefore, f is a strong support function for E.

The proposition below will be applied to prove the main result of this chapter, that is, Theorem 5.6. Noell in [18] obtained 5.5 for pseudoconvex domains with smooth boundary in $\mathbb{C}^2$, and in [15], A. Iordan generalized the result to such domains in $\mathbb{C}^n$.

Proposition 5.5 *Suppose $D \subset\subset \mathbb{C}^n$ is a pseudoconvex domain with smooth boundary. Let K be a peak set for $A^\infty(D)$ and L a compact subset of K. Then*

$$L_1 = [K \cap w(\partial D)] \cup L$$

is a peak set for $A^\infty(D)$.

The proposition above shows that we can take compact subsets of K away from $w(\partial D)$.

Theorem 5.6 *Suppose $D \subset\subset \mathbb{C}^n$ is a convex domain with real-analytic boundary. Let K be a compact subset of ∂D which is a peak set for $A^\infty(D)$, and L a compact subset of K. Then L is a peak set for $A^\infty(D)$.*

Proof. We apply Theorem 3.9 to get a finite covering of $K \cap w(\partial D)$ by open sets $\{U'_\alpha\}_{\alpha=1}^{\ell}$ so that on each U'_α, $(1 \le \alpha \le \ell)$, properties (a)$\longrightarrow$(c) of Theorem 3.9 are satisfied. We take compact subsets within each U'_α and then take the intersection.

Fix α, and drop the subscript from U'_α.

The idea of the proof is to take compact subsets in U' successively on $S_{2n-3}, S_{2n-4}, \ldots, S_1$, and S_0, where S_j $(0 \le j \le 2n-3)$ is a strata of $w(\partial D)$ obtained from Theorem 3.9, starting with the maximal dimensional strata S_{2n-3}. This technique was used by Noell in [18].

Let

$$L_2 = [L_1 \cap (S_0 \cup S_1 \cup \ldots \cup S_{2n-4})] \cup L.$$

and observe that $L \subseteq L_2 \subseteq L_1 \subseteq K$. We will show that L_2 is a peak set for $A^\infty(D)$. We will remove from L_1 points of $(K \setminus L)$ on S_{2n-3}.

Let $\{V_k\}_{k=1}^{\infty}$ be a family of open neighborhoods of $L_1 \cap (S_0 \cup \ldots \cup S_{2n-4})$ such that

$$V_{k+1} \subset\subset V_k$$

and

$$\cap_{k=1}^{\infty} V_k = L_1 \cap (S_0 \cup S_1 \cup \ldots S_{2n-4}).$$

Fix k. We first show that $(L_1 \cap \overline{V}_k) \cup L$ is a peak set for $A^\infty(D)$. Using this, and the above remark, we obtain that L_2 is a peak set for $A^\infty(D)$.

Let U be a neighborhood of $\overline{V}_k$.

Claim. There exists a peak set $L' \subset \partial D$ for $A^\infty(D)$ so that:

(1) $L' \subset L_1$

(2) $L' \setminus U = L \setminus U$

(3) $L \cup (L_1 \cap \overline{V}_k) \subset L'$.

Proof of the Claim. Let f be a strong support function for L_1. Apply 5.2 to get an open covering for $L_1 \setminus U$ by open sets U_j, $U_j', U_j \subset\subset U_j'$ $(1 \leq j \leq \ell)$, a smooth manifold $N_j \subset U_j'$, and a constant c_j such that

$$L_1 \cap S_{2n-3} \cap U_j' \subset N_j \quad (1 \leq j \leq \ell),$$

and for each $z \in \overline{D} \cap U_j'$.

$$Re\, S_R(f)(z) \geq c_j d^2(z, N_j). \tag{5.1}$$

Let $\chi_j' : \mathbb{C}^n \longrightarrow [0,1]$ be a C^∞-function so that

$$\chi_j' \equiv 1 \ on\ U_j \ (1 \leq j \leq \ell) \tag{5.2}$$

and

$$supp\, \chi_j' \subset U_j'. \tag{5.3}$$

For $z \in \mathbb{C}^n$, put

$$\rho(z) = \sum_{j=1}^{\ell} \chi_j'(z) d^2(z, N_j) \tag{5.4}$$

Then $\rho \geq 0$, and $\rho \equiv 0$ on $L_1 \cap S_{2n-3}$. We deduce from (5.1), for each $z \in \overline{D}$:

$$\begin{aligned} \textstyle\sum_{j=1}^{\ell} \chi_j'(z) Re\, S_R(f)(z) &\geq \textstyle\sum_{j=1}^{\ell} c_j \chi_j'(z) d^2(z, N_j) \\ &\geq c_0 \rho(z), \end{aligned}$$

where $c_0 = \min\{c_j,\ 1 \leq j \leq \ell\}$.

Thus

$$Re\, S_R(f)(z) \geq \frac{c_0}{\ell} \rho(z). \tag{5.5}$$

Let $\Omega_1 = \cup_{j=1}^{\ell} U_j$.

Let $\{\chi_j\}_{j=1}^{\ell}$ be a partition of unity on $L_1 \setminus U$ subordinate to the cover $\{U_j\}_{j=1}^{\ell}$.

Then we have for $1 \leq j \leq \ell$,

$$\chi_j \in C^\infty(U_j),$$

$$0 \leq \chi_j \leq 1,$$

$$supp\, \chi_j \subset U_j,$$

and

$$\sum_{j=1}^{\ell} \chi_j = 1 \ on \ L_1. \tag{5.6}$$

Put $\Omega_2 = \Omega_1 \cup U$, and let D' be a compact neighborhood of D containing Ω_2. Choose a C^∞-function s on $\mathbb{C}^n$ so that

$$s \geq 0,$$

$$supp\, s \subset D',$$

and

$$L = \{z \in D' : s(z) = 0\}. \tag{5.7}$$

Let

$$s_j = s\chi_j. \tag{5.8}$$

Let $M_j = M'_j \times \{0\}$ $(1 \leq j \leq \ell)$ be the totally real manifold obtained from Theorem 4.2 which is contained in N_j.

Let $z^{(j)} = (z'^{(j)}, z''^{(j)})$ be a holomorphic coordinate system on $U'_j (1 \leq j \leq \ell)$ as in the proof of Theorem 5.2, where $z'^{(j)} = (z_1^{(j)}, \ldots, z_{t+\lambda}^{(j)})$.

We apply 5.3 to s_j restricted to M'_j in a neighborhood V'_j in $\mathbb{C}^{t+\lambda}$ to get a function $\tilde{\tilde{s}}_j \in C^\infty(\mathbb{C}^{t+\lambda})$ so that:

$$\tilde{\tilde{s}}_j = s_j \ on \ M'_j,$$

$$\overline{\partial}\widetilde{\widetilde{s}}_j \textit{ vanishes to infinite order along } M_j',$$

and

$$supp\,\widetilde{\widetilde{s}}_j \subset V_j'',$$

where V_j'' is a compact neighborhood in V_j'. Furthermore, since $\overline{\partial}\widetilde{\widetilde{s}}_j \equiv 0$ on M_j', then as a consequence of the Canchy-Riemann equations, we get that the differential of $Re\,\widetilde{\widetilde{s}}_j$ is zero on $JT(M_j')$.

Extend $\widetilde{\widetilde{s}}_j$ trivially to get a function $\widetilde{s}_j$ so that $\widetilde{s}_j$ is defined on $\mathbb{C}^n$, and

$$\widetilde{s}_j(z^{(j)}) = \widetilde{\widetilde{s}}(z'^{(j)}).$$

Then

$$\widetilde{s}_j = s_j \textit{ on } N_j$$

(recall that $N_j = M_j' \times \mathbb{C}^{n-t-\lambda}$),

$$\overline{\partial}\widetilde{s}_j \textit{ vanishes to infinite order along } N_j \cap U_j'.$$

We modify $\widetilde{s}_j$ away from M_j to get

$$supp\,\widetilde{s}_j \subset U_j',$$

if U_j' is small enough. In addition, the differential of $Re\,\widetilde{s}_j$ vanishes on $JT(N_j)$, this is because $\overline{\partial}\widetilde{s}_j = \overline{\partial}\widetilde{\widetilde{s}}_j = 0$.

We conclude from this, by Taylor expanding $Re\,\widetilde{s}_j$, that there exists a constant $c_j' > 0$ so that for each $z \in \mathbb{C}^n$,

$$Re\,\widetilde{s}_j(z) \geq -c_j' d^2(z, N_j).$$

Let

$$\widetilde{s} = \sum_{j=1}^{\ell} \widetilde{s}_j.$$

We apply techniques used by Chaumat and Chollet in [5] to get a constant $C_{\alpha k} > 0$ and $d > 0$ such that for each $z \in \mathbb{C}^n$, $\alpha \in \mathbb{N}^n$, and $k \in \mathbb{N}$,

$$|D^\alpha \overline{\partial} \tilde{s}(z)| \leq C_{\alpha k} \rho^k(z)$$

and for all $z \in \mathbb{C}^n$,

$$Re\, \tilde{s}(z) \geq -d\rho(z). \tag{5.9}$$

Now, we define the function G by:

$$G = S_R(f) + \delta \tilde{s},$$

where $\delta > 0$ is sufficiently small.

Then, using (5.5) and (5.9), there exists a positive constant c so that:

$$Re\, G(z) \geq c\rho(z). \tag{5.10}$$

Let $L' = \{z \in \overline{D} \cap \Omega_2 : G(z) = 0\}$.

We deduce from (5.7) that

$$L' \subset \{z \in \Omega_2 \cap \overline{D} : \rho(z) = 0\}. \tag{5.11}$$

Thus, by using (5.9) and (5.11), we conclude by using Proposition 5.1, that L' is a peak set for $A^\infty(D)$.

Now, we verify properties (1)–(3) stated in the claim.

First we show that $L' \subset L_1$. We show this in two steps.

(a) If $z \in \Omega_1 \cap L'$, then by (5.6) we have $\tilde{s}(z) = \sum_{j=1}^{\ell} \chi_j(z) s(z) = s(z)$, and since $Re\, G = 0$ on L', we get $s(z) = 0$. Thus by (5.7), $z \in L$, and hence $z \in L_1$.

(b) If $z \in (U \setminus \Omega_1) \cap L'$, then $\chi'_j(z) = 0$ for all j $(1 \leq j \leq \ell)$, and hence $\tilde{s}(z) = 0$, so $\operatorname{Re} G(z) = Re\, S_R(f)(z) = 0$. Thus $z \in L_1$.

Combining (a) and (b), we obtain (1).

To show (2), we first show that:

(c) $\Omega_1 \cap L \subset L'$.

Let $z \in \Omega_1 \cap L$, then $\tilde{s}(z) = s(z) = 0$, and since $S_R(f)(z) = 0$ then $G(z) = 0$, so $z \in L'$.

Combining (a) and (c), we get (2).

Finally, (3) follows from (c) and the following:

(d) If $z \in L_1 \cap \overline{V}_k$, then $z \notin \Omega_1$, and so $\tilde{s}(z) = 0$. Since $z \in L_1$, then $G(z) = S_R(f)(z) = 0$, and hence $z \in L'$.

(e) If $z \in (U \setminus \Omega_1) \cap L$, then $\tilde{s}(z) = s(z) = 0$. Because $S_R(f)(z) = 0$, we get $G(z) = 0$, and so $z \in L_1$.

This ends the proof of the claim.

Now, we show that $(L_1 \cap \overline{V}_k) \cup L$ is a peak set for $A^\infty(D)$.

Let $\{U_i\}_{i=1}^\infty$ be a family of neighborhoods of $\overline{V}_k$ so that $U_{i+1} \subset\subset U_i$ and

$$\cap_{i=1}^\infty U_i = \overline{V}_k.$$

By virtue of the above claim, we get for each $i \geq 1$ a peak set $L'_i \in A^\infty(D)$ such that:

(1) $L'_i \subset L_1$

(2) $L'_i \setminus U_i = L \setminus U_1$

(3) $L \cup (L_1 \cap \overline{V}_k) \subset L_i'$

Thus by the remark above, $L' = \cap_{i=1}^{\infty} L_i$ is a peak set for $A^\infty(D)$, and by (3) we have $L \cup (L_1 \cap \overline{V}_k) \subset L'$.

We note further that since $L_i' \setminus U_i = L \setminus U_i$, then

$$L_i' = \left[L \cup (L_1 \cap \overline{V}_k)\right] \cup A_i,$$

with $A_i \subset (K \cap U_i) \setminus (K \cap \overline{V}_k)$. Taking intersections, we get $L' = L \cup (L_1 \cap \overline{V}_k)$.

Thus $L \cup (L_1 \cap \overline{V}_k)$ is a peak set for $A^\infty(D)$.

Hence by the above remark, it follows that L_2 is a peak set for $A^\infty(D)$.

Let $L_3 = L_2 \cap (S_0 \cup S_1 \cup \ldots \cup S_{2n-5})$.

We proceed along the same lines of the proof that L_2 is a peak set for $A^\infty(D)$.

Let $\{V_k\}_{k=1}^{\infty}$ be a family of neighborhoods of $L_2 \cap (S_0 \cup \ldots \cup S_{2n-5})$ so that

$$\cap_{k=1}^{\infty} V_k = L_2 \cap (S_2 \cup \ldots \cup S_{2n-5}).$$

Then one uses Theorem 5.2 with strong support function f for L_2, and $S = S_{2n-4}$.

Then we argue as above and obtain that L_3 is a peak set for $A^\infty(D)$.

Continuing inductively, and using the same process as above, we finally obtain that

$$L_m = (L_{m-1} \cap S_0) \cup L$$

is a peak set for $A^\infty(D)$, with $m = 2n-2$ and L_{m-1} a peak set for $A^\infty(D)$.

Now, we show that L is a peak set for $A^\infty(D)$. We use Proposition 5.1.

Choose a neighborhood W of L so that W does not contain the points of $(S_0 \cap L_m) \setminus L$, which are isolated in L_m.

Let h be a strong support function for L_m. Let

$$G = h \text{ on } W$$

Put

$$\rho = 0.$$

By Proposition 5.1, L is a peak set for $A^{\infty}(D)$.

CHAPTER 6

PROOF OF THE SECOND MAIN RESULT

Our major goal in this chapter is to apply all the machinery developed in Chapters 3, 4, and 5 to prove Theorem 6.2.

We first state and prove the Patching Lemma, which will be the main tool in proving the second main result. (Theorem 6.2)

Patching Lemma 6.1 *Suppose $D \subset\subset \mathbb{C}^3$ is a convex domain with real-analytic boundary. Let f_i $(i = 1, 2)$ be strong support functions for $K \cap \overline{U}_i$, where U_i are open subsets of $\mathbb{C}^3$ with $U_1 \cap U_2 \neq \emptyset$. Put $K_i = K \cap \overline{U}_i$ $(i = 1, 2)$. Let $p \in K_1 \cap K_2 \cap w(\partial D)$. Let $U \subset \mathbb{C}^3$, $U' \subset \mathbb{C}^2$, Ω, $\hat{L}$ and $\widehat{M}$ be as in Theorem 4.2. Assume $K_1 \cap K_2 \cap w(\partial D) \cap U = \widehat{M}$, and $\dim_{\mathbb{R}} \widehat{M} = 1$. Let R be a sufficiently large positive number. Then there exist a relatively compact neighborhood $V \subset U$ of p with $\partial V \cap \widehat{M} = \{p_1, p_2\}$, a smooth manifold $M \subset U$ containing $\widehat{M}$, and a function $g \in C^\infty(\overline{D} \cap V)$ with the following properties:*

(a) $g \equiv 0$ *on* $K_1 \cap K_2 \cap V$

(b) $Re\, g > 0$ *on* $(\overline{D} \cap V) \setminus (K_1 \cap K_2 \cap V)$

(c) $Re\, g(q') \geq c\, d^2(q', M)$ *, when* $q' \in \overline{D} \cap V$*, and where* c *is a positive constant*

(d) $D^\alpha(\overline{\partial} g(q')) = 0$ *for each* $q' \in \overline{D} \cap V \cap M$ *and for each multi-index* α

(e) $g = S_R(f_i)$ *near* p_i $(i = 1, 2)$

Proof. We choose a holomorphic coordinate system (z_1, z_2, z_3) in U as in the proof of Theorem 4.2, and assume $p = 0$.

Let $\widetilde{M}$ be as in Theorem 4.2, and assume without loss of generality that $\widetilde{M} \subset \mathbb{C}^2$.

Let $\gamma : (-3,3) \longrightarrow \partial\Omega$ be a non-singular parametrization of $\widetilde{M}$ so that $\gamma(0) = (0,0)$. Choose a C^∞-cut off function $\chi : \widetilde{M} \longrightarrow [0,1]$ so that:

$$\chi(\gamma(t)) = 1 \ for \ t \geq 1$$

and

$$\chi(\gamma(t)) = 0 \ for \ t \leq -1.$$

Proposition 5.3 guarantees the existence of an almost-holomorphic extension χ' of χ defined in U' and satisfying properties (1)–(5).

Extend χ' trivially to get a function $\widetilde{\chi}$ so that $\widetilde{\chi}$ is defined on $U' \times \mathbb{C}$, and

$$\widetilde{\chi}(z_1, z_2, z_3) = \chi'(z_1, z_2).$$

Put

$$G = \widetilde{\chi} S_R(f_2) + (1 - \widetilde{\chi}) S_R(f_1).$$

Let $V \subset\subset U$ be a small neighborhood of $L' = \{(\gamma(t), 0) : -2 < t < 2\}$ so that $\partial V \cap \widehat{M} = \{p_1, p_2\}$, where $p_1 = (\gamma(-2), 0)$, $p_2 = (\gamma(2), 0)$. Then $G \in C^\infty(\overline{D} \cap V)$. Put

$$M = \widetilde{M} \times \mathbb{C}.$$

Then M is a smooth submanifold of $\mathbb{C}^3$ containing $\widetilde{M}$.

Now, we show properties (a)–(e).

(a) Since $S_R(f_i) = 0$ $(i = 1, 2)$ on $K_1 \cap K_2 \cap V$, part (a) follows.

(e) is obvious because of part (c) of Proposition 5.3.

(d) Since $\overline{\partial}\chi'$ vanishes to infinite order along $\widetilde{M}$, then $\overline{\partial}\widetilde{\chi}$ vanishes to infinite order along M as well, and so part (d) of the lemma follows.

(b) To show part (b), we assume (c) holds. Then if V is small enough,

$$Re\, g > 0 \text{ on } (\overline{D} \cap V) \setminus (M \cap V).$$

Also, on $M \cap V$ we have

$$Re\, G = \chi S_R(f_2) + (1-\chi) S_R(f_1),$$

and so

$$Re\,(S_R(f_i)) > 0$$

on $(\overline{D} \cap V \cap M) \setminus (K_1 \cap K_2 \cap V)$; therefore (b) follows.

All it remains to show is part (c).

Pick $q \in \widehat{M} = \widetilde{M} \times \{0\}$.

Since $\widehat{M}$ is totally real, we can make a holomorphic change of coordinate in $\mathbb{C}^3$ near q, that we still denote by (z_1, z_2, z_3), so that $q = 0$ and

$$T_0(\widetilde{M}) = \{(z_1, z_2) : y_1 = z_2 = 0\},\ T_0(\partial\Omega) = \{(z_1, z_2) : u = 0\},$$

and

$$T_0(\partial D) = \{(z_1, z_2, z_3) : u = 0\}.$$

Here $z_j = x_j + iy_j$, $(j = 1, 3)$ and $z_2 = u + iv$.

Let $g(z_1, z_2) = f(z_1, z_2, 0)$.

By the proof of Theorem 4.2, we know that $g \in A^\infty(\Omega)$ is a strong support function for $\widetilde{L} \cap \overline{U}'$, if $U' \subset \mathbb{C}^2$ is a sufficiently small neighborhood of $(0,0)$.

Since by assumption $K_1 \cap K_2 \cap w(\partial D) \cap U'$ is an integral curve of $\partial\Omega$. Proposition 4.3 guarantees that $\dfrac{\partial^2 g}{\partial y_1^2}(0,0) > 0$, and hence $\dfrac{\partial^2 f}{\partial y_1^2}(0,0,0) = \dfrac{\partial^2 g}{\partial y_1^2}(0,0) > 0$. Let

$$B = \frac{\partial^2 f}{\partial y_1^2}(0,0,0).$$

As a consequence of the Canchy-Riemann equations and the Hopf lemma, we deduce that:

$$\frac{\partial(Re\, f)}{\partial u}(q) = \frac{\partial(Im\, f)}{\partial v}(q) = \alpha < 0.$$

So the Taylor expansion of $\mathrm{Re}(S_R(f))$ at q in directions perpendicular to M gives for $q' \in \overline{D}$ near q:

$$Re(S_R(f))(q') = \alpha u - R\alpha^2 u^2 + R\alpha^2 v^2 + Q(z_1, z_2),$$

where $Q(z_1, z_2)$ is of second order and is independent of R. This shows that we can control $\mathrm{Re}(S_R(f))$ in $J\vec{n}$ direction, where $\vec{n}$ is the outer normal vector to ∂D at q.

We observe that

$$Re\, g = (Re\, \tilde{\chi})(Re(S_R(f_2)) + Re(1-\tilde{\chi})\, Re(S_R(f_1)) + Im\, \tilde{\chi}(Im\, S_R(f_1) - Im\, S_R(f_2)).$$

Let G_1 denote the sum of the first two terms, and G_2 the last term.

By the above remarks, and since $\tilde{\chi}|_M = \chi$, the Taylor expansion of G_1, at $q' \in \overline{D}$ near q in directions perpendicular to M is greater than or equal to:

$$Au^2 + By_1^2 + C_R v^2,$$

for some positive constants A, B and C_R.

Using part (5) of Proposition 5.3, and Taylor expanding G_2 at q' in directions perpendicular to M, we get:

$$Ey_1 v + \mathit{error\ terms},$$

where E is a constant independent of R.

Now, for R large enough,

$$Au^2 + By_1^2 + C_R v^2 >> Ey_1 v.$$

Thus for all $q' \in \overline{D}$ near q,

$$Re\, g(q') \geq Au^2 + By_1^2 + C_R v^2.$$

Using compactness of L', we conclude that for all $q' \in \overline{D} \cap V$,

$$Re\, g(q') \geq cd^2(q', M),$$

where c is a positive constant.

This finishes the proof of the Patching Lemma.

Theorem 6.2 *Suppose $D \subset\subset \mathbb{C}^3$ is a convex domain with real-analytic boundary. Then a compact subset K of ∂D is locally a peak set for $A^\infty(D)$ if and only if K is a peak set for $A^\infty(D)$.*

Proof. We apply Theorem 4.2 and the decomposition of $w(\partial D)$ in $\mathbb{C}^3$ using Theorem 3.9 to get that $K \cap w(\partial D)$ is locally contained in the union of integral curves and a discrete set.

Since $K \cap w(\partial D)$ is compact, we deduce that $K \cap w(\partial D)$ is contained in the disjoint union of integral curves and a finite set. Say:

$$K \cap w(\partial D) \subseteq \left(\cup_{j=1}^{\ell} \widehat{M}_j\right) \cup M_0,$$

where $\widehat{M}_j$'s are integral curves, and M_0 is a finite set.

Choose open sets $\{U_i\}_{i=1}^m$ in $\mathbb{C}^3$ so that

$$(K \cap w(\partial D)) \subset \cup_{i=1}^m U_i,$$

and $K \cap \overline{U}_i$ is a peak set for $A^\infty(D)$ with strong support function f_i, $(1 \leq i \leq m)$.

We may assume that $M_0 \cap \partial U_i = \emptyset$ for all i.This is because compact subsets of peak sets are peak by Theorem 5.6, so we may shrink U_i to obtain a smaller neighborhood W_i so that $K \cap \overline{W}_i$ is a peak set for $A^\infty(D)$.

Fix i and j, and let $p \in K \cap w(\partial D) \cap \widehat{M}_j \cap \partial U_i$.

Choose k so that $p \in U_k$.

We will consider two cases.

Case 1: $K \cap w(\partial D)$ is a connected subset of $\widehat{M}_j$ near p.

Case 2: $K \cap w(\partial D)$ is not a connected subset of $\widehat{M}_j$ near p.

Suppose case 1 occurs.

We apply the Patching Lemma 6.1, with strong support functions f_i and f_k for $K \cap \overline{U}_i$ and $K \cap \overline{U}_k$ respectively, to obtain a C^∞-function g_{ij} defined in a small neighborhood V_{ij} of p so that $\partial V_{ij} \cap \widehat{M}_j = \{p_{ij}, p_{kj}\}$, and

$$g_{ij} = \begin{cases} S_R(f_i) & \text{near } p_{ij} \\ S_R(f_k) & \text{near } p_{kj} \end{cases}$$

Suppose case 2 occurs.

Choose disjoint neighborhoods V_1 and V_2 of $K \cap w(\partial D)$ in $U_i \cup U_k$ so that $K \cap \overline{V}_\ell$ ($\ell = 1, 2$) is a peak set for $A^\infty(D)$.

Let Q be neighborhood of $K \setminus (V_1 \cup V_2)$ so that Q does not intersect the weakly pseudoconvex boundary points in V_ℓ ($\ell = 1, 2$).

Let $K_1 = (K \cap Q) \setminus (\cup_{\ell=1}^2 V_\ell)$.

Fornaess and Henriksen in [11] showed that if a compact subset K_1 is contained in the boundary of a strongly pseudoconvex domain in $\mathbb{C}^3$, and K_1 is locally contained in integral manifolds, then there exists an integral manifold containing all of K_1.

Since $K_1 = (K \cap Q) \setminus (\cup_{\ell=1}^2 V_\ell)$ consists only of strongly pseudoconvex boundary points, applying the above fact, we obtain that K_1 is globally contained in a finite disjoint union of integral manifolds.

If $K \cap (Q \cap (V_1 \cup V_2))$ is nonempty, then we use the techniques of Theorem 2.1 ((i)$\Rightarrow$(iv)) to patch peak functions.

We do this for each i and j.

We are now prepared to construct a function ρ and a function G satisfying in a neighborhood of K the hypothesis of Proposition 5.1.

Put $E = K$.

Let

$$G = \begin{cases} g_{ij} & \text{in } V_{ij} \\ S_R(f_i) & \text{near } p_{ij} \\ S_R(f_k) & \text{near } p_{kj} \end{cases}$$

Pick a neighborhood W of E so that if $z \in \overline{D} \cap W$, then

$$G(z) = 0 \; if \; and \; only \; if \; z \in E. \tag{6.1}$$

Let M_j be the smooth manifold in $\mathbb{C}^3$ obtained in the Patching Lemma 6.1 so that $M_j \supset \widehat{M}_j$, and for all $q \in W \cap \overline{D}$ near p,

$$Re\, G(q) \geq c\, d^2(q, M_j). \tag{6.2}$$

Put $\rho(z) = d^2(z, M_j)$ in a neighborhood of $\widehat{M}_j$ containing p_{ij} and p_{ij}, and let $\rho \equiv 0$ outside a small neighborhood which includes p_{ij} and p_{kj}.

Now, we verify the hypothesis of Proposition 5.1.

Part (a) follows from (6.1).

Part (b) is true because of part (d) of the Patching Lemma 6.1, for points between p_{ij} and p_{kj}, and away from such points $G = S_R(f_i)$ or $G = S_R(f_k)$, and hence G is holomorphic in D. So, part (b) holds for all points in $W \cap \overline{D}$.

Part (c) follows because if $\rho(z) = d^2(z, M_j)$, and $G = g_{ij}$ then by (6.2), we have, for $z \in W \cap \overline{D}$ near p,

$$Re\, G(z) \geq c\, d^2(z, M_j)$$

and away from such points we have, $\mathrm{Re}\, G = S_R(f_\ell) \geq 0$ $(\ell = i, k)$.

Therefore K is a peak set for $A^\infty(D)$.

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