THE APPLICATION OF MATRICES AND TENSORS TO THE ANALYSIS OF ELECTRIC NETWORKS

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PREFACE

The research which has resulted in this dissertation was started in the fall of 1949 when I first enrolled at the Oklahoma Agricultural and Mechanical College. Professor Charles F. Cameron of the School of Electrical Engineering originally suggested this subject, and we discussed the various ideas involved at great length. The very nature of the problem intrigued me, and I undertook the task of solving it. The thesis which I submitted in 1950 in partial fulfillment of the requirements for the Master of Science degree contained the findings of the first phase of my researches.

While the use of complex numbers in the conventional analysis of a-c circuits has many advantages over the trigonometric solution, there exist instances when the rules for manipulating complex numbers do not yield results that conform with the natural behavior of a-c quantities. It is quite customary among textbook writers to disregard these discrepancies and merely employ an artifice, where necessary, to obtain the correct results. Such an analysis leaves much to be desired.

In this dissertation, I have ignored all previous methods of network analysis that employ the complex notation. In conventional analysis currents, voltages, and impedances are all represented by exactly the same type of symbols, complex numbers. Since it is obvious that these three quantities are not alike, my first objective was to find a way to determine the basic nature of these quantities. Based upon the original work of Gabriel Kron, I have shown that current is a contravariant tensor of valence 1 (vector), voltage is a covariant tensor of valence 1 (vector), and impedance is a covariant tensor of valence 2. These findings are based upon the fundamental assumption that the nature of a quantity is determined by the manner in which it behaves when the coordinate system in which it is represented is subjected to a linear transformation.

Alternating currents and voltages of a given frequency have two degrees of freedom, amplitude and phase. It is convenient to represent a branch or mesh current as a single vector. In order to accommodate a two-dimensional current by a one-dimensional vector, I conceived the idea of representing alternating currents and voltages by complex vectors. The major portion of this thesis deals with the details of a-c network analysis where the complex vector voltages and currents have been expressed in equivalent matrix form.

An expression of appreciation is extended to Dr. Wayne Johnson and Dr. Alvin Pershing for the time which they so generously spent discussing details of this research with me. I am especially grateful to Professor Charles F. Cameron for his guidance and counsel during the past several years and for his willingness to serve as chairman of my advisory committee. Typing such a necessarily complicated notation as I have adopted in this thesis on a conventional typewriter is quite an accomplishment. I am deeply indebted to my wife for her patience and understanding in mastering the matrix equations on the typewriter.

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CHAPTER I

BASIC CONCEPTS OF A-C CIRCUIT QUANTITIES

Introduction

This research is a continuation of a study started by the author and Professor Charles F. Cameron in September, 1949. The earlier efforts on this study culminated in a thesis submitted by the author to fulfill the requirements for the Master of Science Degree in August, 1950. Frequently throughout this paper, references will be made to pertinent sections of the thesis submitted in 1950; this thesis will be referred to as the M. S. Thesis. Because of its pertinence, the Introduction to the M. S. Thesis is reproduced below.

Introduction to M. S. Thesis

Often in the early development of a particular branch of science, rules and regulations will be established which govern the existing knowledge of the field at that time. Later, however, invariably many new, and sometimes radically different, aspects of the science are discovered. These newly discovered truths may necessitate modifications in the statement of the rules and regulations formerly established, and in some instances may even render them void and useless. For example, the development of the theory of light propagation and the flow of light energy is yet to be completely explained. At present, the use of both the Wave and the Quantum Theories are required to explain all the phenomena of light. This is the way that knowledge has progressed from the very earliest beginning; first, the most elementary ideas and concepts, and then, as the known facts increase, the more complex aspects are explained, leading to the ultimate establishment of the science on a firm and sound theoretical basis, the theory being substantiated by observation and experiment.

The development of the field of electricity and magnetism has been no exception to this general rule. The first ideas were based almost entirely upon experimental data, but since that time more powerful mathematical tools have been discovered and developed that have tended to clarify and explain the wonders of electricity.

Many of the presently existing ideas in the field of alternating current electricity are even yet inconsistent.[1]

The student of elementary electrical technology learns Ohm's law in the symbolic form $I = E/R$, which is equivalent to the verbal statement that the current in an electric circuit is exactly proportional to the voltage applied to it. By assuming the truth of this statement of the law, the student is enabled to solve a large number of simple-circuit problems, and it is not until he progresses a little further in his studies that he finds its applications to be limited and true only under important qualifications.

In simple metallic circuits with steady voltages, for instance, this expression of the law holds good only when the temperature of the material of the circuit is maintained constant. Again, with alternating voltages and with inductive circuits, it is true only if the frequency is also constant. If the magnetic flux causing the inductance flows in iron this interpretation of the law is not correct; even with constant temperature and frequency, current is not exactly proportional to applied voltage. Further, Ohm's law is known not to hold in certain circuits of an electrolytic character.

The discrepancy in cases like these is sometimes explained by using the fiction of a back EMF, but this begs the whole question. It is better to recognize that there are two distinct kinds of electrical circuits, the one in which Ohm's law as defined above is obeyed and in which current is exactly proportional to voltage, and the other in which this proportionality does not hold. Circuits of the first class are generally said to have a linear impedance; in circuits of the other class the impedance is designated non-linear.[2]

The preceding discussion quoted from G. W. Stubbings points out several discrepancies which are present in the thinking of many electrical engineers. In this treatise, the discrepancies and shortcomings encountered in the usual application of complex scalar algebra to alternating-current circuit quantities will be discussed. The ultimate goal of this dissertation is to devise (or adapt) a branch of mathematics to solve a-c circuit problems that will avoid the discrepancies encountered in the use of the complex scalar algebra. Before proceeding further, it seems advisable to review the historical

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[1]Charles W. Jiles, A Comprehensive Study of Electrical Power Quantities (unpublished M. S. thesis, School of Electrical Engineering, Oklahoma A. and M. College, 1950), pp. 1-2.

[2]G. W. Stubbings, "Harmonics and P. F.," Electrical Review, (June 5, 1942).

manner in which a new scientific field is generally developed; the development of the field of electricity will be discussed.

The beginning of electrical engineering, as it is known today, was the discovery and collection of basic experimental information. Some of the most important of these experimental and fundamental data are (1) the concept of electric and magnetic lines of force pervading space by Faraday, (2) the connection between an electric current and a magnetic field, and the equivalence of permanent magnets and electromagnets by Ampere, (3) the electromagnetic wave equations by Maxwell, and (4) the beginning of the photoelectric and electron theories by Hertz. These basic researches and experiments have been given here solely for the purpose of describing the nature of a science, and, in particular, the nature of the different quantities that are studied in electrical engineering.

From this early beginning, as the field of electrical engineering grew in extent, more advanced mathematical techniques were adopted in an attempt to simplify the increasing complexity of solutions required for the types of problems that were encountered. When alternating-current electricity replaced direct-current electricity in many commercial uses, the solutions of even elementary problems became quite cumbersome and unwieldy. These solutions were accomplished with the voltages and currents expressed in trigonometric form. Then, immediately after Argand devised the complex algebra, Steinmetz applied the complex algebra to the solution of a-c circuits.

There is a tendency among electrical engineers to forget about the nature of the quantities with which they deal and to think of a-c quantities as actually being complex numbers. This kind of thinking (or lack of it) is erroneous and misleading. The fact is that a-c quantities, correctly represented by trigonometic functions and only partially represented by complex algebra, are real physically existing entities which obey certain natural laws. These natural laws are completely independent of any mathematical schemes or fictitious images dreamed up by the mind of man. If, however, it so happens that all the man-made laws of a particular branch of mathematics conform to all the natural laws of behavior of a set of physical objects, then the set of physical objects may be properly represented by symbols in that branch of mathematics. The mathematics may then be used to predict unknown characteristics of the physical objects. On the contrary, if it is known that at least one law of manipulation in the given branch of mathematics does not agree with the corresponding natural law of manipulating the physical objects, then the mathematics is at best only a partial representation of the physical objects; in this event, the mathematics should be used to predict unknown natural behavior of the physical objects only with great caution and insight.

The difference between a set of physical or abstract objects and the mathematics used to represent those objects should be clearly understood and appreciated. Once objects have been represented in a particular space or reference frame, there is a tendency to confuse the mathematical representation of the objects with the objects themselves. For example, the forces acting on a body may be represented by vectors, but the forces are not vectors and the vectors are not forces; the forces are physical objects and the vectors are mathematical entities. Physical or abstract objects may be represented by any mathematical system that is considered convenient. However, in order to have a useful representation, it is necessary to select a mathematical system for a given set of objects such that the algebraic laws of the mathematical system when applied to the set of objects yield results which agree with the known natural laws that govern the set of objects. When this has been accomplished, the mathematical system is a true representation of the set of objects. However, if an occasion should arise when a particular aspect of the set of objects disagrees with the corresponding result produced by the algebra of the selected mathematical system, then it should be recognized that the mathematical system simply does not completely represent the set of objects. In this event, either a more truly representative mathematical system should be devised, or, if such seems impossible, some artifice should be devised to correct this discrepancy. When such an artifice is used, the basic difficulty and the reason for using it should be made clear.

Trigonometric Representation of A-C Circuit Quantities

When a voltage of sine-wave form is impressed across a stationary electric circuit containing linear impedance elements, trigonometric functions may be used to correctly represent the various components of voltage, current, and power that exist in the circuit. Therefore, this problem will first be analyzed using trigonometry in order to determine the characteristics that any other mathematical representation of a-c quantities must have.

If the reference axis is so chosen that

$$v = V_m \sin wt \qquad (1.1)$$

then $$i = I_m \sin(wt - \theta) \qquad (1.2)$$

The total instantaneous power is defined as

$$p = vi \qquad (1.3)$$

Substituting (1.1) and (1.2) into (1.3) gives

$$p = V_m I_m \sin wt \sin(wt - \theta) \qquad (1.3a)$$

Since $\sin(wt - \theta) = \sin wt \cos \theta - \cos wt \sin \theta$

equation (1.3a) becomes

$$p = V_m I_m \sin wt\,(\sin wt \cos \theta - \cos wt \sin \theta) \qquad (1.3b)$$

or $$p = V_m I_m (\sin^2 wt \cos \theta - \sin wt \cos wt \sin \theta) \qquad (1.3c)$$

Using the identities that

$$\sin^2 wt = 1/2 - 1/2 \cos 2wt$$

and $$\sin wt \cos wt = 1/2 \sin 2wt$$

equation (1.3c) becomes

$$p = \frac{V_m I_m}{2} \cos \theta - \frac{V_m I_m}{2} \cos \theta \cos 2wt - \frac{V_m I_m}{2} \sin \theta \sin 2wt \qquad (1.4)$$

It should be noted that the total instantaneous power (1.4) is composed of three terms; the first term is a constant, and the other two terms are time functions which vary at twice the frequency of the impressed voltage. Equation (1.4) will be discussed in greater detail later in this chapter.

Complex Representation of A-C Circuit Quantities

From physics, it is well known that if a body travels in uniform circular motion the projection of its instantaneous velocity upon a diameter of its circle of rotation will describe simple harmonic motion. Equations (1.1) and (1.2) are obviously the equations of simple harmonic variations and hence could be represented as a linear projection of a line segment of a given length being rotated at a constant angular velocity w. It will be assumed that the current as given in (1.2) is lagging.

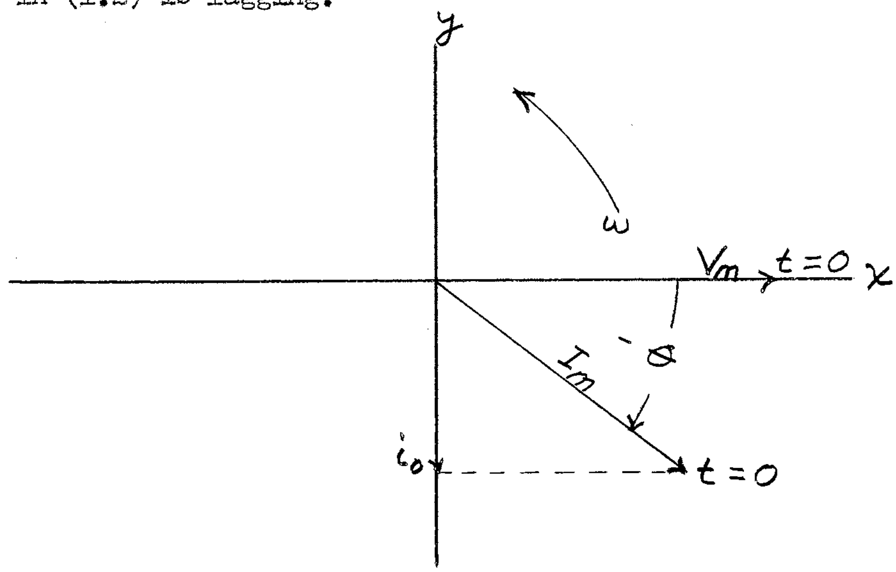

Figure 1
Sinusoidal Functions as Linear Projections of Rotating Vectors

Referring to figure 1, it is evident that equations (1.1) and (1.2) may be represented by the vertical (y) projections of the two vectors of constant length, V_m and I_m, respectively. These two vectors must start at t = 0 in the position shown, and then rotate in a counter-clockwise direction at a constant angular velocity w. This signifies that, if these rotating vectors were suddenly stopped for any given value of time, such a diagram could be used to indicate the correct phase relationship between two sine functions. This type of diagram is of great value to the electrical engineer. Since effective (r.m.s.) values rather than maximum values are usually used, the vectors in figure 1 must be multiplied by a constant (0.707). Capital letters without subscripts will be used to represent effective values.

If figure 1 were drawn for a value of time other than zero, the result would be that shown below in figure 2.

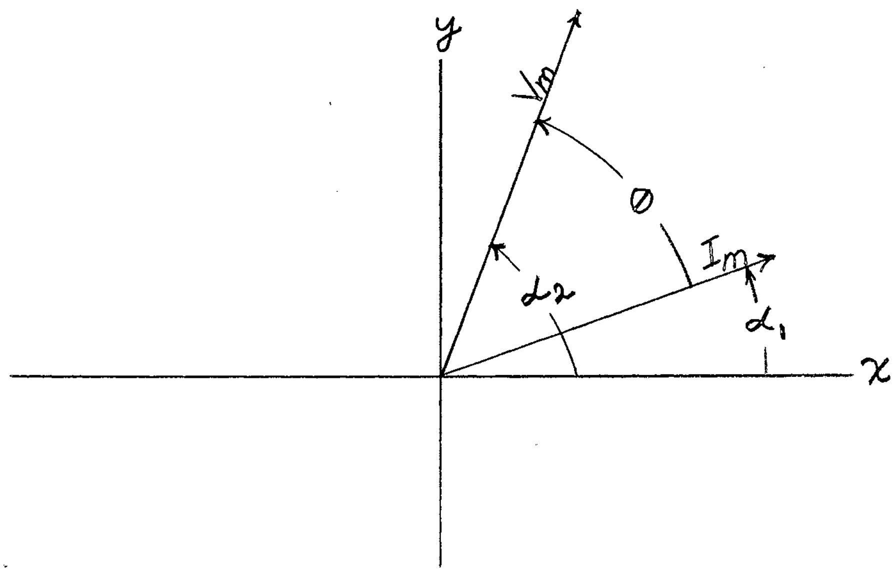

Figure 2
Rotating Vectors with General Reference Axis

For this condition, equations (1.1) and (1.2) become

$$v = V_m \sin(wt + \alpha_2) \qquad (1.1a)$$

and

$$i = I_m \sin(wt + \alpha_1) \qquad (1.2a)$$

If maximum values in figure 2 are replaced by effective values (by multiplying by 0.707), the X-axis is the real axis and the Y-axis is the imaginary axis, then the equations for voltage and current become

$$V = V_1 + jV_2 \qquad (1.5)$$

and

$$I = I_1 + jI_2 \qquad (1.6)$$

where

$$V_1 = V \cos \alpha_2 \quad : \quad V_2 = V \sin \alpha_2 \qquad (1.5a)$$

$$I_1 = I \cos \alpha_1 \quad : \quad I_2 = I \sin \alpha_1 \qquad (1.6a)$$

Figure 3 shows equations (1.5) and (1.6) in diagrammatical form. This is about as much of an introduction as most elementary texts usually give to the use of complex algebra in electrical engineering.

Since I and V as given in (1.5) and (1.6) differ in magnitude and phase only, the a-c impedance is written in the same form as V and I.

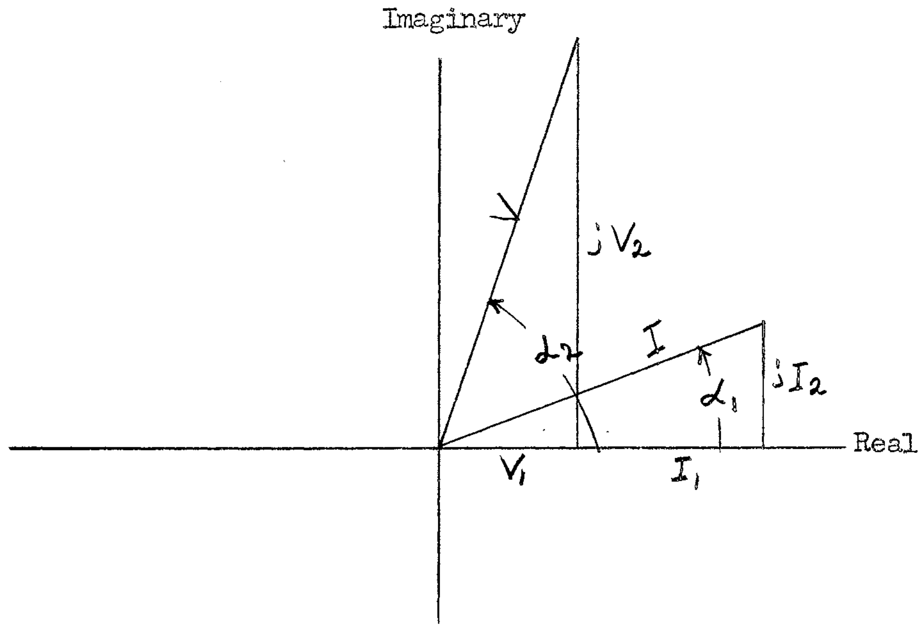

Figure 3
Effective Voltage and Current Expressed as Complex Numbers

$$Z = R + jX \tag{1.7}$$

The average real power in watts is defined as

$$P = VI\cos(\alpha_2 - \alpha_1) = VI\cos\theta \tag{1.8}$$

and the reactive power in vars is defined as

$$P_r = VI\sin(\alpha_2 - \alpha_1) = VI\sin\theta \tag{1.9}$$

Equations (1.8) and (1.9) indicate that P and P_r are the two legs of a right triangle with VI as the hypotenuse. The product of (1.5) and (1.6) should then give the sum of (1.8) and (1.9) at right angles. Taking the product of (1.5) and (1.6) gives

$$VI = (V_1I_1 - V_2I_2) + j(V_1I_2 + V_2I_1) \tag{1.10}$$

If this is correct, then

$$P = V_1I_1 - V_2I_2 \tag{1.11}$$

and $$P_r = V_1 I_2 + V_2 I_1 \quad (1.12)$$

Expanding (1.8) yields

$$P = VI(\cos \alpha_2 \cos \alpha_1 + \sin \alpha_2 \sin \alpha_1) \quad (1.8a)$$

Substituting from (1.5a) and (1.6a), equation (1.8a) becomes

$$P = V_1 I_1 + V_2 I_2 \quad (1.8b)$$

Similarly, expanding (1.9)

$$P_r = VI(\sin \alpha_2 \cos \alpha_1 - \sin \alpha_1 \cos \alpha_2) \quad (1.9a)$$

Again substituting from (1.5a) and (1.6a) gives

$$P_r = V_2 I_1 - V_1 I_2 \quad (1.9b)$$

Since (1.8b) and (1.9b) are known to be correct, equations (1.11) and (1.12) must be incorrect. The reason for this failure in the complex notation is not apparent; it is embodied within the theory of complex algebra. It should be recalled at this point (equation 1.4) that the instantaneous power is composed of a constant term and two double-frequency terms. The failure of the complex notation in (1.11) and (1.12) demonstrates that complex numbers can not be used to represent harmonic functions of different frequencies without more understanding of the theory of complex numbers than is included in most texts on a-c circuits. The definitions of average power in watts (1.8) and reactive power in vars (1.9) are quite misleading unless studied carefully. The implications of these two definitions will be analyzed in the following section.

Real and Reactive Power

The first two terms of (1.4) are defined as the instantaneous real power; the last term is defined as the instantaneous reactive power. The instantaneous real active power will be denoted by p_a, and the instantaneous reactive power will be denoted by p_r.

$$p_a = \frac{V_m I_m}{2} \cos \theta - \frac{V_m I_m}{2} \cos \theta \cos 2wt \qquad (1.13)$$

$$p_r = \frac{-V_m I_m}{2} \sin \theta \sin 2wt \qquad (1.14)$$

Using effective values, (1.13) and (1.14) become

$$p_a = VI \cos \theta - VI \cos \theta \cos 2wt \qquad (1.13a)$$

$$p_r = -VI \sin \theta \sin 2wt \qquad (1.14a)$$

The average value of a periodic function $y = f(x)$ is defined as

$$Y_{avg.} = \frac{1}{T} \int_0^T f(x)dx \qquad (1.15)$$

The average value of the instantaneous real power (1.13a) is

$$P = \frac{1}{T} \int_0^T (VI \cos \theta - VI \cos \theta \cos 2wt)dt \qquad (1.16)$$

where T is the period of the voltage or current. The second term in (1.16) obviously vanishes, leaving

$$P = VI \cos \theta \qquad (1.8)$$

Thus to read the average real power an electrodynamometer type movement calibrated to read (1.8) may be used. Equation (1.8) can therefore be experimentally determined in a straight-forward manner.

There seems to be no reason why the reasoning applied to p_a in (1.13a) through (1.16) should not also apply to p_r (1.14a). Applying definition (1.15) to (1.14a) gives

$$P_r = \frac{1}{T} \int_0^T -VI \sin \theta \sin 2wt \, dt \qquad (1.17)$$

The average value of p_r as given by (1.17) is obviously zero. Definition (1.9) must therefore be obtained from a different kind of reasoning than (1.8), and P_r obviously cannot be measured with the same type of instrument as P. It is expedient to investigate the type of reasoning required to obtain (1.9) from (1.14a) and to deterimine if (1.9) has any physical significance. One means of obtaining (1.9) from (1.14a) is to retard the voltage wave by 90°. This phase shift of -90° changes (1.1) to

$$v = V_m \sin(wt - 90^\circ) \qquad (1.18)$$

or

$$v = -V_m \cos wt \qquad (1.18a)$$

If (1.18a) is substituted into (1.3), then (1.3b) becomes

$$p = -V_m I_m \cos wt \sin wt \cos \theta + V_m I_m \cos^2 wt \sin \theta \qquad (1.19)$$

The average value of the first term, the instantaneous real power, in (1.19) is obviously zero. Using the identity

$$\cos^2 wt = 1/2 + 1/2 \cos 2wt$$

the instantaneous reactive power, the last term in (1.19), becomes

$$p_r = \frac{V_m I_m}{2} \sin \theta + \frac{V_m I_m}{2} \sin \theta \cos 2wt \qquad (1.20)$$

Changing to effective values, (1.20) becomes

$$p_r = VI \sin \theta + VI \sin \theta \cos 2wt \qquad (1.20a)$$

If definition (1.15) is now applied to (1.20a), the result is

$$P_r = VI \sin \theta$$

Experimentally, it is quite simple to accomplish the -90° phase shift in the voltage wave; an inductance, whose reactance is large compared to the internal resistance of the potential coil, is connected in series with the potential coil of the wattmeter. Since the p_a term vanishes, this modified wattmeter will now read (1.9).

It is now apparent that (1.8) and (1.9) embody two entirely different concepts. While (1.8) is a straight-forward definition, the definition of P_r implies a great deal more than the simple mathematical statement (1.9) would lead one to believe.

Addition of Displaced Sinusoidal Functions

It has been stated earlier that alternating currents and voltages are correctly represented by trigonometric functions. In order to investigate the manner in which two alternating currents or voltages of the same frequency combine by addition to form a single alternating current or voltage, the trigonometric representation of alternating currents or voltages may be used. For illustrative purposes, two voltages

$$v_1 = V_{m_1} \sin(wt + \alpha_1) \tag{1.21}$$

and

$$v_2 = V_{m_2} \sin(wt + \alpha_2) \tag{1.22}$$

will be chosen. Forming the sum of (1.21) and (1.22) yields

$$v = v_1 + v_2 = V_{m_1} \sin(wt + \alpha_1) + V_{m_2} \sin(wt + \alpha_2) \tag{1.23}$$

Expanding the sine functions by the identity

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$$

gives

$$v = V_{m_1}(\sin wt \cos \alpha_1 + \cos wt \sin \alpha_1) + V_{m_2}(\sin wt \cos \alpha_2 + \cos wt \sin \alpha_2) \tag{1.24}$$

$$v = (V_{m_1}\cos \alpha_1 + V_{m_2}\cos \alpha_2)\sin wt + (V_{m_1}\sin \alpha_1 + V_{m_2}\sin \alpha_2)\cos wt \tag{1.24a}$$

Equation (1.24a) may be written in the form

$$v = A \sin wt + B \cos wt \tag{1.24b}$$

where

$$A = V_{m_1}\cos \alpha_1 + V_{m_2}\cos \alpha_2 \tag{1.25}$$

$$B = V_{m_1}\sin \alpha_1 + V_{m_2}\sin \alpha_2 \tag{1.26}$$

Equation (1.24b) can be further simplified to

$$v = C \sin(wt + \theta) \tag{1.24c}$$

where

$$C^2 = A^2 + B^2 \tag{1.27}$$

and $$\Theta = \tan^{-1}\frac{B}{A} \qquad (1.28)$$

Substituting (1.25) and (1.26) into (1.27) gives

$$C^2 = V_{m_1}{}^2 + V_{m_2}{}^2 + 2\,V_{m_1}V_{m_2}\cos(\alpha_1 - \alpha_2) \qquad (1.29)$$

Equations (1.24c) and (1.29) indicate that C is the diagonal of a parallelogram having legs V_{m_1} and V_{m_2} separated by the angle $\alpha_1 - \alpha_2$.

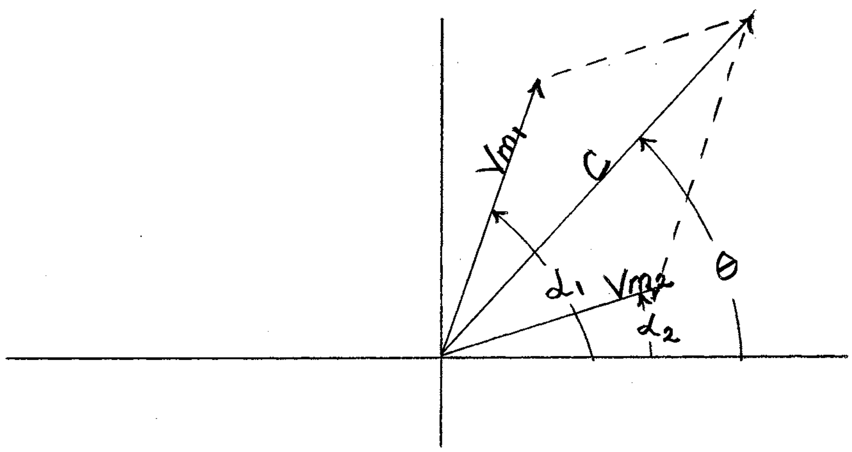

Figure 4
Addition of Sinusoidal Functions

The addition of two sinusoidal functions has been shown to follow the parallelogram law of addition. As far as the addition of alternating currents or voltages is concerned, these quantities may be represented by any form of mathematics for which addition follows the parallelogram law. Now it so happens that both vectors and complex numbers add according to the parallelogram law. Hence, as far as addition is concerned, either vectors or complex numbers could be used to represent alternating currents or voltages equally well. Since the complex represenitation of alternating currents and voltages has been demonstrated, it would be instructive to demonstrate the manner in which these quantities may be represented by vectors.

Real Vector Representation of Alternating Currents and Voltages

The following analysis follows the basic ideas contained in a paper entitled Double Frequency Quantities in Complex Notation written by Alexander S. Langsdorf. This paper was never published but was graciously loaned to the author because of mutual interests.

The Ohm's Law of a-c circuits is

$$V = IZ \qquad (1.30)$$

It must first be recognized that (1.30) is actually a vector equation; V and I have magnitude and phase (equivalent to direction). Since Z has no phase property, it is merely a complex number. Hereafter, symbols with bars under them will be used to represent vectors, and symbols with a dot over them will be used to represent complex numbers. In this notation, (1.30) becomes

$$\underline{V} = \underline{I}\dot{Z}$$

Since both $\underline{V}$ and $\underline{I}$ have two degrees of freedom, it will require two real vector components to describe each of them. These components will be taken along two orthogonal axes X_1 and X_2 along which two unit real vectors $\underline{e}_1$ and $\underline{e}_2$ are assumed to exist, respectively. The voltage will be assumed to be of the form

$$v = V_m \sin(wt + \alpha), \qquad (1.31)$$

the current of the form

$$i = I_m \sin(wt + \beta), \qquad (1.32)$$

and the impedance of the form

$$\dot{Z} = R + jX = Z\underline{/\theta} \qquad (1.33)$$

In vector notation, using effective values, these equation become

$$\underline{V} = V_1\underline{e}_1 + V_2\underline{e}_2 \qquad (1.34)$$

and $$\underline{I} = I_1\underline{e}_1 + I_2\underline{e}_2 \qquad (1.35)$$

If the current in (1.35) flows through a circuit having an impedance as given by (1.33), then (assuming that the ordinary laws of algebra hold without modification), the voltage drop across the circuit should be given by (1.30a).

$$\underline{V} = (I_1\underline{e}_1 + I_2\underline{e}_2)(R + jX) \qquad (1.36)$$

$$\underline{V} = I_1R\underline{e}_1 + I_2R\underline{e}_2 + I_1Xj\underline{e}_1 + I_2Xj\underline{e}_2 \qquad (1.36a)$$

Ascribing to the operator j its usual rotational property (see figure 5)

$$j\underline{e}_1 = \underline{e}_2 \qquad (1.37)$$

and

$$j\underline{e}_2 = -\underline{e}_1 \qquad (1.38)$$

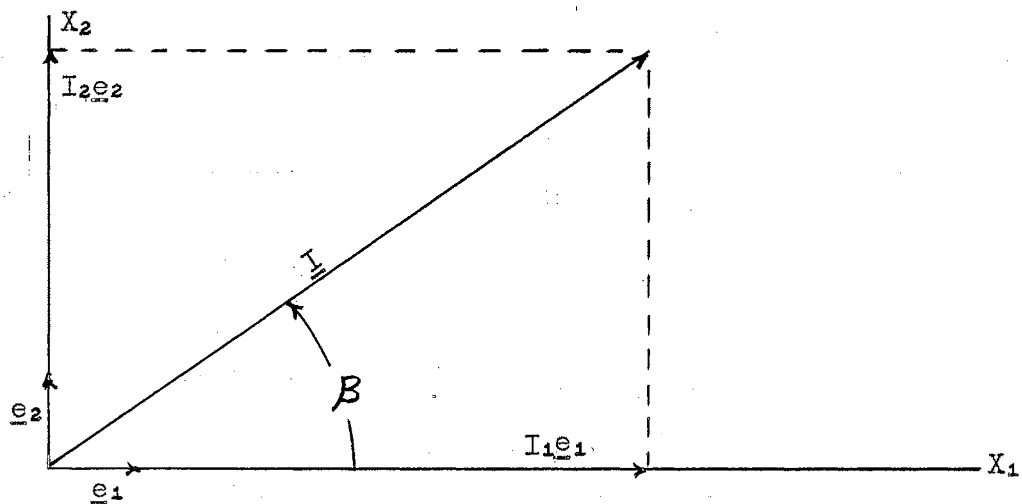

Figure 5
Two-Dimensional Real Vector Representation of A-C Current

Substituting (1.37) and (1.38) into (1.36a) gives

$$\underline{V} = (I_1R - I_2X)\underline{e}_1 + (I_1X + I_2R)\underline{e}_2 \qquad (1.39)$$

Equation (1.39) agrees with the results obtained by the trigonometric and complex number methods, both of which are known to be correct. Let it now be required to find the total power input to a circuit when the applied voltage is given by (1.34) and the resulting current by (1.35).

$$\underline{V}\underline{I} = (V_1\underline{e}_1 + V_2\underline{e}_2)(I_1\underline{e}_1 + I_2\underline{e}_2) \qquad (1.40)$$

$$\underline{V}\underline{I} = V_1I_1\underline{e}_1\underline{e}_1 + V_1I_2\underline{e}_1\underline{e}_2 + V_2I_1\underline{e}_2\underline{e}_1 + V_2I_2\underline{e}_2\underline{e}_2 \qquad (1.40a)$$

If, following Langsdorf, all the vector products in (1.40a) are interpreted to be the dot product of ordinary vector analysis, then

$$\begin{aligned} \underline{e}_1\underline{e}_1 &= 1 & \underline{e}_2\underline{e}_1 &= 0 \\ \underline{e}_1\underline{e}_2 &= 0 & \underline{e}_2\underline{e}_2 &= 1 \end{aligned} \qquad (1.41)$$

Substituting (1.41) into (1.40a) gives the average real power in watts

$$P = V_1I_1 + V_2I_2 \qquad (1.42)$$

It should be clearly understood that (1.42) was the result of straight-forward multiplication of $\underline{V}$ and $\underline{I}$ without recourse to the artifice of employing the conjugate as is required in the complex notation to get the correct answer. In other words, equations (1.34) and (1.35) appear to be a more natural representation of voltage and current than (1.5) and (1.6).

Actually, going beyond Langsdorf, (1.40a) can be made even more useful by defining the vector products as given below in the multiplication table.

	$\underline{e}_1$	$\underline{e}_2$
$\underline{e}_1$	1	$\underline{e}_3$
$\underline{e}_2$	$-\underline{e}_3$	1

Table 1
Multiplication Table for Three Orthogonal Real Unit Vectors

The cross product of $\underline{e}_1$ and $\underline{e}_2$ gives a third mutually perpendicular real unit vector, $\underline{e}_3$, the algebraic sign being determined by the order of multiplication. If this notation is used, (1.40a) becomes

$$\underline{VI} = (V_1I_1 + V_2I_2) + (V_1I_2 - V_2I_1)\underline{e}_3 \qquad (1.43)$$

The scalar component of (1.43) is the average real power in watts as defined by (1.8) and (1.42), and the vector part is the reactive power in vars as defined by (1.9). Using this notation, the product of vector voltage and vector current gives both the real and reactive power, which is more in keeping with the usual notions of volt-amperes. Equation (1.42) is actually a modified quaternion. The difference in the above procedure and true quaternion algebra is that the dot products of the unit vectors are minus one, i.e., $\underline{e}_1\underline{e}_1 = -1$, in quaternion algebra. Since this notation would make the average real power negative, such a definition of the dot product could not be tolerated in a-c circuit theory.

In addition to giving numerically correct results, equation (1.43) also indicates the physical properties of the real and reactive power components, a desirable feature that is lacking in the complex scalar treatment. In order to make this distinction clear, the various quantities have been listed below in trigonometric and vector notation.

Trigonometric Notation	Vector Notation
$v = V_m \sin(wt + \alpha)$	$\underline{V} = V_1\underline{e}_1 + V_2\underline{e}_2$
$i = I_m \sin(wt + \beta)$	$\underline{I} = I_1\underline{e}_1 + I_2\underline{e}_2$
$p_a = VI \cos\theta - VI \cos\theta \cos 2wt$	$P = V_1I_1 + V_2I_2$, or
	$P = VI \cos\theta$
$p_r = -VI \sin\theta \sin 2wt$	$\underline{P}_r = (V_1I_2 - V_2I_1)\underline{e}_3$, or
	$\underline{P}_r = (VI \sin\theta)\underline{e}_3$

Table 2
Comparison of Trigonometric and Real Vector Representations

The values of v and i as given in table 2 are both functions of their amplitudes (V_m and I_m) and their phase angles (α and β). Therefore, it is logical to represent both v and i as two-dimensional real vectors. The average values (over a complete period) of both v and i are zero. Since p_r has no phase angle, it is a function of its magnitude (-VI sin θ) only; the the average value of p_r (over a complete period) is also zero. The representation of p_r, similar to v and i, as a one-dimensional real vector is entirely appropriate. Futhermore, the double-frequency characteristic of p_r (with respect to v and i) is indicated by directing $\underline{P}_r$ normal to the plane of $\underline{V}$ and $\underline{I}$. Now the average value of p_a is not zero; it is VI cos θ. Since P is defined as the constant part of p_a, P is basically independent of time. If sinusoidal time functions are represented by vectors, then obviously terms that are independent of time should be described by some other notation. This is accomplished in (1.43) by representing the real number VI cos θ as a scalar. Thus, when properly interpreted, equations (1.34), (1.35), and (1.43) give a complete description of voltage, current, real power, and reactive power.

The term volt-amperes has been used in the preceding discussion to refer to the vector sum of the average real power and the reactive power as defined by (1.8) and (1.9); when harmonics are present, the term volt-amperes is often used in an entirely different sense. In the following section, the term volt-amperes will be analyzed in detail.

Volt-Amperes in A-C Circuits

Most electrical engineers tend to think of volt-amperes as either the product of effective voltage and effective current or the square root of the sum of the squares of the real and reactive power. If harmonics happen to be present in the voltage and current wave forms, these two concepts of volt-amperes are not equivalent. For the sake of brevity, only three different frequencies, designated by subscripts, will be assumed to be present; the general proof follows exactly analogous reasoning.

The effective values of voltage and current are

$$V = \sqrt{V_1^2 + V_2^2 + V_3^2} \qquad (1.44)$$

and

$$I = \sqrt{I_1^2 + I_2^2 + I_3^2} \qquad (1.45)$$

Volt-amperes defined as the product of (1.44) and (1.45) will be denoted by the symbol VA_1. Hence

$$VA_1 = \sqrt{V_1^2 + V_2^2 + V_3^2} \cdot \sqrt{I_1^2 + I_2^2 + I_3^2} \qquad (1.46)$$

$$(VA_1)^2 = V_1^2I_1^2 + V_2^2I_2^2 + V_3^2I_3^2 + V_1^2I_2^2 + V_1^2I_3^2 + V_2^2I_1^2 + V_2^2I_3^2 + V_3^2I_1^2 + V_3^2I_2^2 \qquad (1.47)$$

Volt-amperes defined as the square root of the sum of the squares of the real and reactive power will be denoted by the symbol VA_2. By definition

$$(VA_2)^2 = \left(\sum_{i=1}^{3} V_iI_i \cos \theta_i\right)^2 + \left(\sum_{i=1}^{3} V_iI_i \sin \theta_i\right)^2 \qquad (1.48)$$

If the indicated operations in (1.48) are performed, the result is

$$(VA_2)^2 = V_1^2I_1^2 + V_2^2I_2^2 + V_3^2I_3^2 + 2\,V_1I_1V_2I_2 \cos \theta_1 \cos \theta_2 + 2\,V_1I_1V_3I_3 \cos \theta_1 \cos \theta_3 + 2\,V_2I_2V_3I_3 \cos \theta_2 \cos \theta_3 + 2\,V_1I_1V_2I_2 \sin \theta_1 \sin \theta_2 + 2\,V_1I_1V_3I_3 \sin \theta_1 \sin \theta_3 + 2\,V_2I_2V_3I_3 \sin \theta_2 \sin \theta_3 \qquad (1.49)$$

By using common trigonometric identities, (1.49) reduces at once to

$$(VA_2)^2 = V_1^2I_1^2 + V_2^2I_2^2 + V_3^2I_3^2 + 2\,V_1I_1V_2I_2\cos(\theta_1 - \theta_2) + 2\,V_1I_1V_3I_3\cos(\theta_1 - \theta_3) + 2\,V_2I_2V_3I_3\cos(\theta_2 - \theta_3) \qquad (1.49a)$$

The problem is now to determine the conditions under which (1.47) is equal to (1.49a), that is

$$(VA_1)^2 = (VA_2)^2 \qquad (1.50)$$

In order for (1.50) to be true, two significant stipulations must be satisfied. The first condition is obvious; it is

$$\theta_1 = \theta_2 = \theta_3 \qquad (1.51)$$

If condition (1.51) is required, (1.49a) simplifies to

$$(VA_2)^2 = V_1^2I_1^2 + V_2^2I_2^2 + V_3^2I_3^2 + 2\,V_1I_1V_2I_2 + 2\,V_1I_1V_3I_3 + 2\,V_2I_2V_3I_3 \qquad (1.52)$$

Equation (1.52) is evidently in general not equal to (1.47). In addition to (1.51), if it is further stipulated that

$$\frac{V_1}{I_1} = \frac{V_2}{I_2} = \frac{V_3}{I_3} \qquad (1.53)$$

then

$$V_1^2I_2^2 = V_2^2I_1^2$$

$$V_1^2I_3^2 = V_3^2I_1^2 \qquad (1.53a)$$

$$V_2^2I_3^2 = V_3^2I_2^2$$

and

$$I_1V_2 = V_1I_2$$

$$I_1V_3 = V_1I_3 \qquad (1.53b)$$

$$I_2V_3 = V_2I_3$$

If (1.53a) is substituted into (1.47), (1.53b) substituted into (1.52), and each of these equations simplified by collecting like terms, the result is

$$2\,V_1^2I_2^2 + 2\,V_1^2I_3^2 + 2\,V_2^2I_3^2 = 2\,V_1^2I_2^2 + 2\,V_1^2I_3^2 + 2\,V_2^2I_3^2 \qquad (1.54)$$

The left side of (1.54) is (1.47), and the right side is (1.52).

From (1.51) and (1.53), the two conditions for the validity of (1.50) are (1) all harmonics must have equal displacement angles (referred to a base frequency) between current and voltage, and (2) the ratio of the amplitudes of voltage and current must be equal for all harmonics. It is easily shown that (1.53) and

$$\theta_1 = \theta_2 = \theta_3 = 0 \qquad (1.55)$$

are the necessary conditions for unity power factor.[3]

The preceding analysis has clearly established the conditions under which $(VA_1)^2$ and $(VA_2)^2$ are equal. In general, in actual circuits, (1.51) and (1.53) are not satisfied. Since in general (1.47) and (1.48) are not equal, it is desirable to obtain an equation for the difference in these two quantities. In order to avoid equations having excessive numbers of terms, the third harmonic components in (1.47) and (1.48) will be omitted. The definitions and terminology used in the M. S. Thesis will be incorporated.[4] Equation (1.47) becomes

$$P_{ap}^2 = V_1^2I_1^2 + V_2^2I_2^2 + V_1^2I_2^2 + V_2^2I_1^2 \qquad (1.56)$$

and (1.48) becomes

$$P_v^2 = P^2 + P_r^2 \qquad (1.57)$$

$$P^2 = V_1^2I_1^2 \cos^2\theta_1 + 2\, V_1I_1V_2I_2 \cos\theta_1 \cos\theta_2 + V_2^2I_2^2 \cos^2\theta_2 \qquad (1.58)$$

$$P_r^2 = V_1^2I_1^2 \sin^2\theta_1 + 2\, V_1I_1V_2I_2 \sin\theta_1 \sin\theta_2 + V_2^2I_2^2 \sin^2\theta_2 \qquad (1.59)$$

Equation (1.56) can be written as

$$\begin{aligned} P_{ap}^2 = {} & V_1^2I_1^2 \cos^2\theta_1 + 2\, V_1I_1V_2I_2 \cos\theta_1 \cos\theta_2 + V_2^2I_2^2 \cos^2\theta_2 + \\ & V_1^2I_1^2 \sin^2\theta_1 + 2\, V_1I_1V_2I_2 \sin\theta_1 \sin\theta_2 + V_2^2I_2^2 \sin^2\theta_2 + \\ & V_1^2I_2^2 + V_2^2I_1^2 - 2\, V_1I_1V_2I_2 \cos(\theta_1 - \theta_2) \end{aligned} \qquad (1.56a)$$

The sum of first three terms in (1.56a) is P^2 as given by (1.58); the sum of the next three terms is P_r^2 as given by (1.59). Since

$$P_{ap}^2 = P^2 + P_r^2 + P_d^2 \qquad (1.60)$$

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[3] R. M. Kerchner and G. F. Corcoran, Alternating-Current Circuits, 3rd Edition, (New York, 1951), pp. 192-193.

[4] Charles W. Jiles, A Comprehensive Study of Electrical Power Quantities, pp. 27-34.

the last three terms in (1.56a) constitute P_d^2, where P_d is the distortion power. Thus

$$P_d^2 = V_1^2 I_2^2 + V_2^2 I_1^2 - 2\, V_1 I_1 V_2 I_2 \cos(\theta_1 - \theta_2) \qquad (1.61)$$

The different terms in (1.56) may be analyzed as follows:

$V_1^2 I_1^2$ - part active and part reactive power

$V_2^2 I_2^2$ - part active and part reactive power

$V_1^2 I_2^2$ - all distortion power

and $V_2^2 I_1^2$ - all distortion power.

Moreover, (1.56) does not contain all the active, reactive, or distortion power terms. From (1.56a), it is evident that the sum of the $2\, V_1 I_1 V_2 I_2 \cos\theta_1 \cos\theta_2$ term in the active power and the $2\, V_1 I_1 V_2 I_2 \sin\theta_1 \sin\theta_2$ term in the reactive power cancels the $-2\, V_1 I_1 V_2 I_2 \cos(\theta_1 - \theta_2)$ term in the distortion power. Therefore, even though (1.56) is the correct expression for P_{ap}^2, the different voltage and current components do not combine algebraically to automatically give the three power components as given by (1.60).

Since none of the terms, or groups of terms, in (1.56) constitutes either the active, reactive, or distortion power components, it is indeed unfortunate that (1.56) was selected as the definition of total volt-amperes (P_{ap}). It would be difficult to devise a mathematical representation for voltage and current such that the product of voltage and current would give (1.56) in the form of (1.60) since, as discussed previously, part of the P, P_r, and P_d expressions are missing in (1.56). All of these difficulties would have been avoided had the total volt-amperes been defined to be the vector volt-amperes (P_v) as given by (1.57). If the total volt-amperes were defined by (1.57), a logical definition for power factor would be

$$P.F. = \frac{P}{\sqrt{P^2 + P_r^2}} \qquad (1.62)$$

rather than the existing definition of

$$P.F. = \frac{P}{P_{ap}} \qquad (1.63)$$

It was stated earlier that the voltages and currents of different frequencies would have to be in-phase and have equal ratios of amplitudes in order for the P.F. as defined by (1.63) to be unity. Such a specialized definition of unity power factor would obviously be avoided if (1.62) were adopted; the power factor would be unity whenever the reactive power (P_r) was zero. The association of unity power factor with zero reactive power is certainly more prevalent among engineers than the knowledge of the necessary conditons for unity power in (1.63). Therefore, in the remainder of this dissertation, the term volt-amperes will be used with reference to (1.57) rather than (1.56).

Summary of Chapter I

1. A group of physical entities may be represented by any mathematical system provided that there exists a one-to-one correspondence between the set of physical objects and the group of characters embraced by the mathematical system. In general, the representation will prove to be most useful if the laws for manipulating the mathematical characters conform with the natural laws which govern the behavior of the physical objects.
2. Sinusoidal functions are true representations of the instantaneous values of alternating currents and voltages.
3. Complex (scalar) algebra may be used to represent the effective values of a-c circuit quantities with the following shortcomings:

(a) May be directly applied only to functions of a single frequency.

(b) No distinction is made between functions of time and constants.

(c) When applied to functions of two different frequencies, the notation does not, itself, distinguish one from the other.

4. Two-dimensional real vectors may be used to represent alternating currents and voltages. This notation avoids the difficulties encountered in the use of complex scalars. All the advantages of this notation are summarized following table 2.

5. The definition of total volt-amperes (apparent power) as the product of effective voltage by effective current is unfortunate, since this definition does not lend itself easily to analytical methods. Therefore, in this dissertation, the term volt-amperes will be used to refer to the square root of the sum of the squares of the active and reactive power.

CHAPTER II

COMPLEX VECTORS

Introduction to Complex Vectors

Alternating currents and voltages of a given frequency have two degrees of freedom, namely amplitude and phase. In Chapter I, the manner in which sinusoidal functions could be represented by two-dimensional real vectors was demonstrated. The advantages of the use of vector notation were impressive when compared to the complex scalar notation. Even though the results obtained using real vector notation gave an excellent analytical description of the actual physical entities involved, this representation has the shortcoming that it does not lend itself easily to the simultaneous treatment of currents and voltages of multiple frequencies and about several loops. In this chapter, a new notation will be described that can be readily generalized to apply to any number of harmonics or any number of loops (or nodes).

It can not be emphasized too strongly that the term "vector" as used in this treatise is not the primitive notion of a quantity possessing magnitude, direction, and sense, generally entertained in elementary physics. Indeed, modern technology has progressed to the point where such an intuitive definition is no longer adequate.

It is now generally recognized that the mathematical equipment of the well-trained physicist or engineer of thirty years ago is no longer adequate for the physics and engineering of today. To understand wave mechanics it is not sufficient to master an old-fashioned treatment of vector analysis with its limitations to plane and space vectors with real coordinates and its emphasis on a visual realization of the basic concepts and relations. We must become familiar with multi-dimensional vectors with complex coordinates and with the matrices, or linear vector functions, which operate on these vectors.[1]

If a quantity is to be used in analytical expressions, it is desirable that it be defined in an analytical manner. The following definitions will be used throughout this dissertation. These definitions and concepts were not originally formulated by the author; they are commonly used in most recent advanced texts on the subject. The definitions as stated are the author's own concepts which were crystallized after careful study of several books, a complete list of which is included in the bibliography.

Definition 1 (space of n dimensions): A space of n dimensions is any set of objects, real or abstract, that can be placed in a one-to-one correspondence with the totality of ordered sets of numbers (real or complex) $x_1, x_2, x_3, \ldots, x_n$.

Definition 2 (coordinate system): The relation that expresses the one-to-one correspondence between the given set of objects and the ordered sets of numbers $x_1, x_2, \ldots, x_n$ is the coordinate system.

Definition 3 (points in n-space): The objects are themselves the points in the n-dimensional space, and the numbers $x_1, x_2, \ldots, x_n$ are the coordinates of points in the coordinate system.

Definition 4 (vector): The point $0_1, 0_2, \ldots, 0_n$ and every other point in the n-dimensional space determine an entity which is called a vector.

Definition 5 (Euclidean space): A space is called a Euclidean space

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[1]Francis D. Murnaghan, Introduction to Applied Mathematics (New York, 1948), p. v.

if it is possible to construct a coordinate system such that the distance, d, between two points, x_i and x_i', is given by the formula of Pythagoras. That is

$$d = [(x_1 - x_1')^2 + (x_2 - x_2')^2 + . . . + (x_n - x_n')^2]^{\frac{1}{2}} \qquad (2.1)$$

Definition 6 (scalar): A scalar is a number which is the same in every coordinate system.

The above definitions and concepts will undoubtedly seem somewhat vague and abstract at first to one unaccustomed to thinking in terms of generalized vectors and spaces. In particular, for one who has always thought of a vector as that which possesses, in addition to the quality of magnitude, the quality of direction, definition 4 will seem abstract.

Actually it is this "colloquial" definition that is vague. How can we tell when something "possesses the quality of direction"? The only answer is that it must have assigned to it, in each reference frame [coordinate system], a pair [set] of numbers, and the various pairs [sets], one in each reference frame, must be connected with each other in exactly the same way as are the projections of a line segment [linear transformations], i.e., by means of the table of direction cosines.[2]

Before proceeding further, a clear distinction must be made among the four kinds of vectors and numbers that will be considered in this treatise. First there are real numbers; real numbers, i.e., 0, 1, 2, -1, -2 are those numbers in common everyday usage that comprise the Field of Real Numbers. There are also complex numbers. Complex numbers are all numbers of the form a + jb, where "a" and "b" belong to the Field of Real Numbers and "j" is the complex operator, $\sqrt{-1}$. Analogous to real and complex numbers, there are also real and complex vectors. The totality of complex numbers is called the Field of Complex Numbers. A real or complex vector is a vector whose coordinates

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[2] Ibid., p. 5.

belong to the Field of Real or Complex Numbers, respectively. Of course, a scalar may be either a real or a complex number. A letter with no modification, such as "A" or "a", will be used to represent a real number. The symbol used to represent a complex number is a letter with a dot placed over it, i.e., $\dot{A}$ or $\dot{a}$. Real vectors will be represented by underscored letters, i.e., $\underline{A}$ or $\underline{a}$. An underscored letter with a dot placed over it, such as $\dot{\underline{A}}$ or $\dot{\underline{a}}$, will be used to represent a complex vector. The symbol for a complex number or vector followed by an asterisk will be used to represent the conjugate of the complex number or vector. In three-dimensional Cartesian coordinates, a complex vector would then be

$$\dot{\underline{A}} = \dot{A}_1\underline{i} + \dot{A}_2\underline{j} + \dot{A}_3\underline{k} \qquad (2.2)$$

where

$$\dot{A}_1 = A_1' + jA_1'' \qquad (2.3a)$$

$$\dot{A}_2 = A_2' + jA_2'' \qquad (2.3b)$$

and

$$\dot{A}_3 = A_3' + jA_3'' \qquad (2.3c)$$

are the complex coordinates of $\dot{\underline{A}}$. Substituting (2.3a-c) into (2.2) gives

$$\dot{\underline{A}} = (A_1' + jA_1'')\underline{i} + (A_2' + jA_2'')\underline{j} + (A_3' + jA_3'')\underline{k} \qquad (2.4)$$

If (2.4) is separated into real and imaginary parts, the result is

$$\dot{\underline{A}} = (A_1'\underline{i} + A_2'\underline{j} + A_3'\underline{k}) + j(A_1''\underline{i} + A_2''\underline{j} + A_3''\underline{k}) \qquad (2.5)$$

Equation (2.5) obviously expresses the complex vector $\dot{\underline{A}}$ as the sum of a real vector and an imaginary vector. Hence if

$$\underline{A}_1 = A_1'\underline{i} + A_2'\underline{j} + A_3'\underline{k} \qquad (2.6)$$

and

$$\underline{A}_2 = A_1''\underline{i} + A_2''\underline{j} + A_3''\underline{k} \qquad (2.7)$$

then (2.5) can be written as

$$\dot{\underline{A}} = \underline{A}_1 + j\underline{A}_2 \qquad (2.8)$$

The manipulations in (2.2) through (2.8) have been performed in order to emphasize the meaning of the notation being used and also to clarify the nature of a complex vector.

The Algebra of Complex Vectors

It will be assumed here that the algebras of real vectors and complex numbers as usually given in most elementary texts are known. The algebra of complex vectors is a combination of these two algebras with a few changes and additions. Only the algebraic laws that are different from the corresponding laws of real vectors will be discussed.

The rules for the addition and subtraction of complex vectors are the same as those for real vectors when applied to the real and imaginary components of the complex vectors separately. Given two complex vectors, $\dot{\underline{A}}$ and $\dot{\underline{B}}$, where

$$\dot{\underline{A}} = \underline{A}_1 + j\underline{A}_2 \tag{2.9}$$

and

$$\dot{\underline{B}} = \underline{B}_1 + j\underline{B}_2 \tag{2.10}$$

then

$$\dot{\underline{A}} + \dot{\underline{B}} = (\underline{A}_1 + \underline{B}_1) + j(\underline{A}_2 + \underline{B}_2) \tag{2.11}$$

Similarly

$$\dot{\underline{A}} - \dot{\underline{B}} = (\underline{A}_1 - \underline{B}_1) + j(\underline{A}_2 - \underline{B}_2) \tag{2.12}$$

Two complex vectors are equal if, and only if, their real and imaginary components are separately equal. That is, if

$$\dot{\underline{A}} = \dot{\underline{B}} \tag{2.13}$$

then

$$\underline{A}_1 = \underline{B}_1 \tag{2.14}$$

and

$$\underline{A}_2 = \underline{B}_2 \tag{2.15}$$

Analogous to the conjugate of a complex number, the conjugate of a complex vector $\dot{\underline{A}}$ (2.9) is defined as

$$\dot{\underline{A}}^* = \underline{A}_1 - j\underline{A}_2 \tag{2.16}$$

As an illustration, if $\dot{\underline{A}}$ is a complex vector in 3-space as in (2.2), using the results of (2.3), (2.4) and (2.5) the value of $\overset{\bullet}{\underline{A}}^{*}$ is

$$\overset{\bullet}{\underline{A}}^{*} = (A_1'\underline{i} + A_2'\underline{j} + A_3'\underline{k}) - j(A_1''\underline{i} + A_2''\underline{j} + A_3''\underline{k}) \qquad (2.17)$$

or

$$\overset{\bullet}{\underline{A}}^{*} = (A_1' - jA_1'')\underline{i} + (A_2' - jA_2'')\underline{j} + (A_3' - jA_3'')\underline{k} \qquad (2.18)$$

Equation (2.18) can also be written as

$$\overset{\bullet}{\underline{A}}^{*} = \overset{\bullet}{A}_1^{*}\underline{i} + \overset{\bullet}{A}_2^{*}\underline{j} + \overset{\bullet}{A}_3^{*}\underline{k} \qquad (2.19)$$

From (2.19), it is evident that changing the sign of the imaginary component of $\overset{\circ}{\underline{A}}$ (2.9) is equivalent to taking the conjugates of the complex scalar coefficients of $\overset{\bullet}{\underline{A}}$ in (2.4).

For vectors in generalized space, it is not possible, as is usually done with real plane and space vectors, to approach the idea of vector products by constructing coordinate systems and considering the relative orientation of the two vectors to be multiplied. Since in spaces of more than three dimensions direction means nothing, geometric intuition can no longer be used as a guide. The absence of geometric intuition actually aids in the determination of the most important feature of complex vector algebra. This characteristic, as in ordinary 2-space and 3-space, is the existence of a rule for calculating the distance between points. A space in which such a rule exists is called a metric space. The most useful metric space is the Euclidean space; in a Euclidean space, the distance between two points is given by (2.1). The magnitude of a vector in n-space is the distance from the point $0_1, 0_2, \ldots, 0_n$ to the point $x_1, x_2, \ldots, x_n$ (see Definition 4). In engineering, a vector is usually used to represent a physical object, and the magnitude of the vector has a physical significance. Obviously a physical entity is not itself altered merely by representing it in different coordinate systems. Therefore the magnitude of a vector is invariant with respect to changes in coordinate systems and hence is a scalar. Indeed, the characteristic of being invariant with respect to changes in coordinate systems is the cardinal feature of an entire branch of mathematics (Tensor Analysis) in which vectors appear as a special case (a vector is a tensor of valence one).

In order for (2.1) to be used in determining the magnitude of a vector, the coordinate system must be orthogonal. For the norm of a real vector

$$\underline{A} = A_1\underline{e}_1 + A_2\underline{e}_2 + \ldots + A_n\underline{e}_n \quad (2.20)$$

equation (2.1) becomes

$$A^2 = A_1^2 + A_2^2 + A_3^2 + \ldots + A_n^2 \quad (2.21)$$

The important aspect of (2.21) is that the zero vector

$$\underline{0} = 0\underline{e}_1 + 0\underline{e}_2 + \ldots + 0\underline{e}_n \quad (2.22)$$

is the only vector whose norm is zero. When the transition is made from real to complex vectors, this uniqueness would no longer exist if the magnitude of a complex vector

$$\dot{\underline{A}} = \dot{A}_1\underline{e}_1 + \dot{A}_2\underline{e}_2 + \ldots + \dot{A}_n\underline{e}_n \quad (2.23)$$

were defined as the square root of the sum of the squares of its complex coefficients. For example, if

$$\dot{\underline{B}} = \dot{B}_1\underline{e}_1 + \dot{B}_2\underline{e}_2 \quad (2.24)$$

and

$$\dot{B}_2 = j\dot{B}_1 \quad (2.25)$$

obviously

$$\dot{B}_1^2 + \dot{B}_2^2 = 0 \quad (2.26)$$

Thus there would be two distinct vectors, $\dot{\underline{B}}$ given by (2.24) and $\underline{0}$ given by (2.22), both having a zero magnitude. The norm of $\underline{A}$ in (2.21) is

$$A^2 = \underline{A} \cdot \underline{A} \quad (2.27)$$

It is desirable to extend the equation for the norm of real vectors (2.27) to apply to complex vectors. To avoid the occurrence of (2.26), for $\dot{B}_1$ and $\dot{B}_2$ not zero, the norm of a complex vector $\dot{\underline{A}}$ (2.23) is defined as $$A^2 = \overset{\circ *}{\underline{A}} \cdot \overset{\circ}{\underline{A}} \tag{2.28}$$

This definition also has the advantage that the dot product of a vector by itself is a real quantity. Thus the idea of length can be represented by the dot product for complex vectors (2.28) as is commonly done for real vectors (2.27). In most physical problems, the concept of length (or distance) is almost imperative, therefore one should suspect that a definition such as (2.28) would be required for the dot product of complex vectors if the results of using this product are to have useful physical significance. Using (2.19), equation (2.28) becomes $$A^2 = \overset{\circ *}{A}_1\overset{\circ}{A}_1 + \overset{\circ *}{A}_2\overset{\circ}{A}_2 + \ldots + \overset{\circ *}{A}_n\overset{\circ}{A}_n \tag{2.29}$$

The vectors $\underline{e}_i$ $(i = 1, 2, 3, \ldots, n)$ in equations (2.20) and (2.23) are unit, real, orthogonal vectors spanning the n-dimensional space. It is advantageous to introduce at this point the summation convention. The notation $x_i y_i$ $(i = 1, 2, 3, \ldots, n)$ will be used to represent the conventional summation $\sum_{i=1}^{n} (x_i y_i)$. Thus when two symbols are placed together having a repeated index, the symbols are to be summed for all permissible values of the index and, unless specified otherwise, the range of the index will be assumed to be 1, 2, 3, . . ., n. Using this notation, (2.23) can be written

$$\overset{\circ}{\underline{A}} = \overset{\circ}{A}_i \underline{e}_i \tag{2.30}$$

The norm of $\overset{\circ}{\underline{A}}$ is simply $$A^2 = \overset{\circ *}{A}_i\overset{\circ}{A}_i \tag{2.31}$$

The two conditions necessary for the simple expression of the norm of a vector as (2.28) are that the space be orthogonal and that the square of a complex coefficient be defined as the product of the coefficient and its conjugate.

Since it was necessary to define the dot (or inner) product of $\overset{\circ}{\underline{A}}$ by itself as $\overset{\circ *}{\underline{A}} \cdot \overset{\circ}{\underline{A}}$ to insure a unique value of the dot product and so that the magnitude of $\underline{\mathring{A}}$ would be given by the rule of Pythagoras, it is logical that the dot product of $\underline{\mathring{A}}$ and $\underline{\mathring{B}}$ should be defined as

$$\underline{\mathring{A}} \cdot \underline{\mathring{B}} = \mathring{A}^*_i \mathring{B}_i \qquad (2.32)$$

where

$$\underline{\mathring{B}} = \mathring{B}_1\underline{e}_1 + \mathring{B}_2\underline{e}_2 + \ldots + \mathring{B}_n\underline{e}_n \qquad (2.33)$$

The repeated subscripts in (2.32) indicate a summation of the indicated products for i = 1, 2, 3, . . ., n. Taking the conjugate of both sides of (2.32) gives

$$(\underline{\mathring{A}} \cdot \underline{\mathring{B}})^* = (\mathring{A}^*_i \mathring{B}_i)^* \qquad (2.34)$$

If

$$\mathring{A}_n = A'_n + jA''_n \qquad (2.35)$$

and

$$\mathring{B}_n = B'_n + jB''_n \qquad (2.36)$$

then

$$\mathring{A}^*_n \mathring{B}_n = (A'_n - jA''_n)(B'_n + jB''_n) \qquad (2.37)$$

or

$$\mathring{A}^*_n \mathring{B}_n = (A'_nB'_n + A''_nB''_n) - j(A''_nB'_n - A'_nB''_n) \qquad (2.37a)$$

Hence

$$(\mathring{A}^*_n \mathring{B}_n)^* = (A'_nB'_n + A''_nB''_n) + j(A''_nB'_n - A'_nB''_n) \qquad (2.38)$$

But (2.38) is $\mathring{B}^*_n \mathring{A}_n$, since

$$(B'_n - jB''_n)(A'_n + jA''_n) = (A'_nB'_n + A''_nB''_n) + j(A''_nB'_n - A'_nB''_n) \qquad (2.38)$$

therefore

$$(\mathring{A}^*_i \mathring{B}_i)^* = \mathring{B}^*_i \mathring{A}^{**}_i = \mathring{B}^*_i \mathring{A}_i = \underline{\mathring{B}} \cdot \underline{\mathring{A}} \qquad (2.39)$$

From (2.39)

$$\underline{\mathring{A}} \cdot \underline{\mathring{B}} = (\underline{\mathring{B}} \cdot \underline{\mathring{A}})^* \qquad (2.40)$$

In words, an interchange of two complex vectors in a scalar product changes the complex scalar product to its conjugate. Clearly, the scalar multiplication of two complex vectors is not in general commutative. Scalar multiplication is commutative only for the sinular case where the scalar product is real.

In equations (2.2) through (2.19), the algebraic laws for combining complex vectors were given for vectors of not more than three dimensions. Since these same laws may be extended directly to apply to n-dimensional complex vectors, they will not be repeated.

Representation of Sinusoidal Functions by Complex Vectors

Let $f(x)$ be a complex-valued function of the real variable x which is at least piecewise-continuous for values of x within a prescribed interval, $a \leq x \leq b$. The real and imaginary parts of $f(x)$ must each be piecewise-continuous over the interval ab (figure 6).

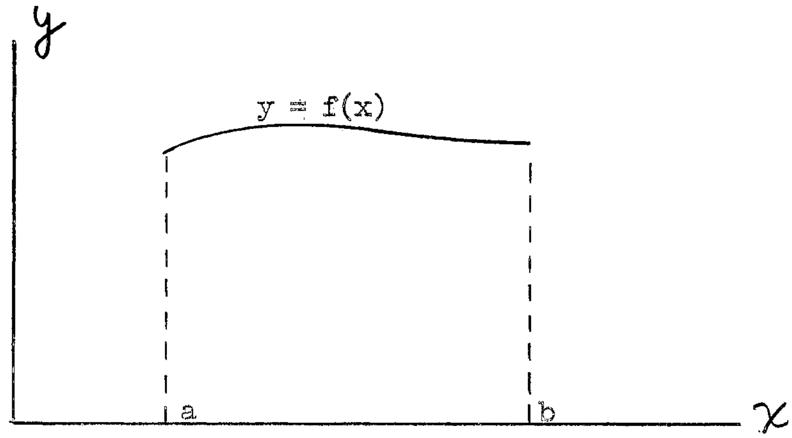

Figure 6
Complex-Valued Function to Be Represented by Complex Vector

It will now be shown that $f(x)$ can be represented by a k-dimensional complex vector, $\mathring{G}_i$ ($i = 1, 2, \ldots, k$), where k is the number of values which x may assume in the interval ab.

For ease of visualization, let $k = 3$ and $f(x)$ be real. If each value of x (x_1, x_2 and x_3) is substituted into $f(x)$, the resulting three values of y in

$$y = f(x) \tag{2.41}$$

will be y_1, y_2 and y_3. These three values of y as ordinates plotted against the three values of x as abscissas determine three points on the curve of figure 6. However, rather than thinking of figure 6, the function $f(x)$ may be considered as a 3-dimensional real vector G. The three values of $f(x)$ are then the components of G. The square of the magnitude of G is

$$G^2 = \sum_{i=1}^{3} f(x_i)f(x_i) = \sum_{i=1}^{3} [f(x_i)]^2 \qquad (2.42)$$

or

$$G^2 = [f(x_1)]^2 + [f(x_2)]^2 + [f(x_3)]^2 \qquad (2.43)$$

The components of $\underline{G}$ may then be interpreted in the 3-dimensional Cartesian coordinate system as shown in figure 7.

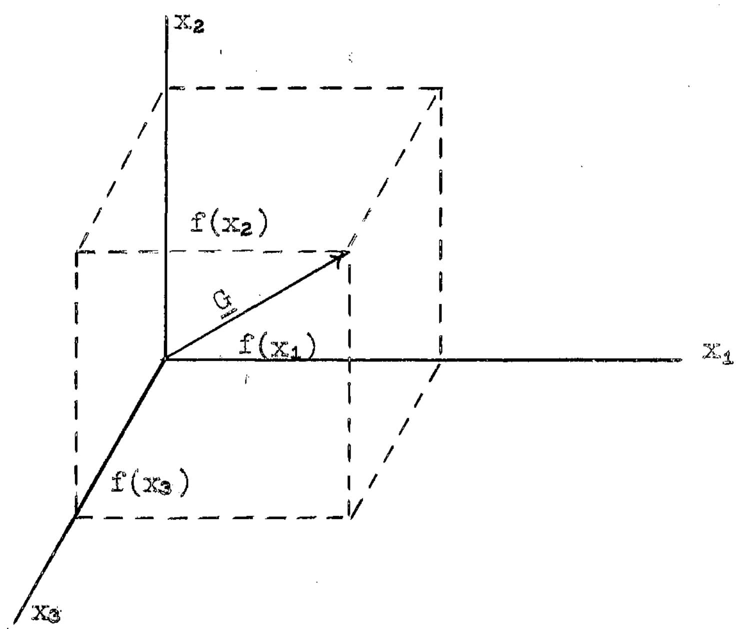

Figure 7
Graphical Representation of Function Vector

If f(x) is defined for all values of x in the interval ab (figure 6), then there is an infinite number of values of f(x) corresponding to each value of x in the interval. The summation of (2.42) necessarily changes to an integral and

$$G^2 = \int_a^b [f(x)]^2 dx \qquad \text{(for } f(x) \text{ real)} \qquad (2.44)$$

If f(x) is now considered as complex-valued, (2.44) becomes

$$G^2 = \int_a^b f(x)^* f(x) dx \qquad (2.44a)$$

By analogy with the scalar product of two complex vector (2.32), the scalar product of $\dot{\underline{G}}_i$ and $\dot{\underline{G}}_j$, representing the two complex-valued functions $f(x)_i$ and $f(x)_j$, is defined as

$$\dot{\underline{G}}_i \cdot \dot{\underline{G}}_j = \int_a^b f(x)_i^* \, f(x)_j \, dx \qquad (2.45)$$

The condition for the orthogonality of the functions $f(x)_i$ and $f(x)_j$ is that

$$\dot{\underline{G}}_i \cdot \dot{\underline{G}}_j = 0 \qquad (2.46)$$

A general sinusoidal function is represented by the equation

$$e_1 = \dot{E}_{m_1} \sin x \qquad (-\pi \leqq x \leqq \pi) \qquad (2.47)$$

where

$$\dot{E}_{m_1} = E'_{m_1} + jE''_{m_1} \qquad (2.48)$$

Employing the concept of the rotating vector (figure 1) and using effective values, equation (2.48) may then be represented by the complex vector $\dot{\underline{E}}_1$.

$$\dot{\underline{E}}_1 \sim \dot{E}_{m_1} \sin x \qquad (2.49)$$

In a similar manner, the function

$$e_2 = \dot{E}_{m_2} \sin 2x \qquad (2.50)$$

may be represented by the complex vector $\dot{\underline{E}}_2$.

$$\dot{\underline{E}}_2 \sim \dot{E}_{m_2} \sin 2x \qquad (2.51)$$

Applying (2.45)

$$\dot{\underline{E}}_1 \cdot \dot{\underline{E}}_2 = \int_{\pi}^{\pi} (\dot{E}_{m_1}^* \sin x)(\dot{E}_{m_2} \sin 2x) dx \qquad (2.52)$$

but

$$\dot{\underline{E}}_1 \cdot \dot{\underline{E}}_2 = (\dot{E}_{m_1}^* \dot{E}_{m_2}) \int_{-\pi}^{\pi} \sin x \sin 2x \, dx = 0 \qquad (2.53)$$

Thus the two functions $\dot{E}_{m_1} \sin x$ and $\dot{E}_{m_2} \sin 2x$ are orthogonal functions.

Geometrically, a complex vector determines a plane since it has two degrees of freedom. Therefore it is impossible to mentally picture $\dot{\underline{E}}_1$ and $\dot{\underline{E}}_2$ as two orthogonal vectors; to accomplish this would require the visual concept of a 4-dimensional space of which the limited human intellect cannot perceive. Since geometric intuition can no longer be relied upon, it is more satisfying to consider each harmonic as a function (similar to figure 6) defined over the interval $-\pi$ to $+\pi$, each function being represented by a complex vector. The totality of these complex vectors, one for each harmonic, forms an orthogonal set.

The idea of representing sinusoidal alternating currents and voltages as complex vectors was first conceived by the author without the knowledge that such a notation had been used elsewhere. Since that time, while making an intensive study of the available literature on this subject, two other texts have been found in which the authors made use of the complex vector concept.[3,4] However, in both instances the use actually made of the properties of complex vector algebra was so small that it could be considered trivial. It is felt that this slight use of the complex vector notation in no way detracts from the originality of this dissertation.

Summary of Chapter II

1. The metric property is one of the most important characteristics of a space. The most fundamentally important feature of a metric space is the nature of the rule which prescribes the manner in which the distance between points is to be measured.
2. In general, most of the rules for manipulating complex vectors are the same as the corresponding rules for real vector algebra and complex scalar algebra. The major exception to this statement is encountered in the scalar product. In order to insure that the norm of the zero vector only is zero and to make the norm of a non-zero vector be real, it was necessary to define the scalar product of complex vectors in a different manner than is commonly defined for real vectors.

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[3]Edith Clarke, Circuit Analysis of A-C Power Systems (New York, 1943), Volume 1, pp. 4-15.

[4]A. Pen-Tung Sah, Dyadic Circuit Analysis (Scranton, Pennsylvania, 1939), pp. 62-70.

3. When applied to alternating-current quantities, complex vectors fail to give results in a form which may be physically interpreted as well as the results which were obtained using two-dimensional real vectors in Chapter I. However, complex vectors lend themselves in a natural manner to the study of spaces of higher dimensions and to the methods of matrix analysis. The manipulation of two-dimensional real vectors generalized in this manner would be extremely awkward and clumsy, if not impossible.

CHAPTER III

LINEAR SINGLE-LOOP CIRCUIT ANALYSIS WITH SINUSOIDAL APPLIED VOLTAGE

In Chapter II, it was shown that sinusoidal functions may be represented by complex vectors. In this chapter, it will be demonstrated that by representing sinusoidal a-c quantities by complex vectors a consistent scheme of representation can be devised which, when applied by the rules of the algebra of complex vectors discussed earlier, yields results that agree with the known correct results.

In the following analysis, the circuits considered will be assumed to be linear and bilateral, and the voltage impressed across the terminals of these circuits will be assumed to be a pure sinusoid. Voltage and current, being sinusoidal functions of a single frequency, will be represented by one-dimensional vectors with complex coefficients. The a-c impedance has no sinusoidal property, being a constant independent of time, and hence will be represented by a complex number.

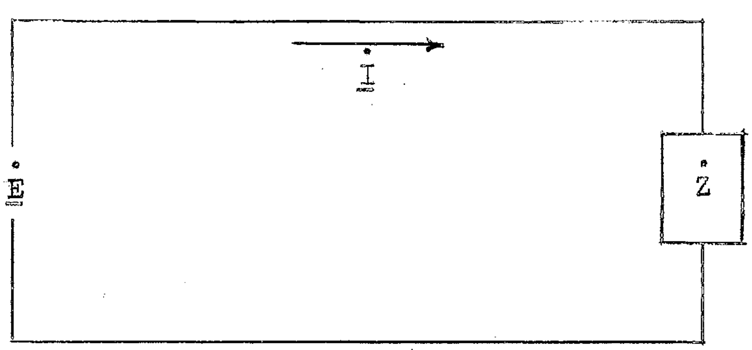

Figure 8
Linear Single-Loop Network

$$\mathring{Z} = R + jX \tag{3.1}$$

$$\dot{\underline{E}} = \dot{E}\underline{e}_1 = (E' + jE'')\underline{e}_1 \tag{3.2}$$

$$\dot{\underline{I}} = \dot{I}\underline{e}_1 = (I' + jI'')\underline{e}_1 \tag{3.3}$$

The complex number $\mathring{Z}$ is geometrically represented by a point in the complex plane as shown in figure 9.

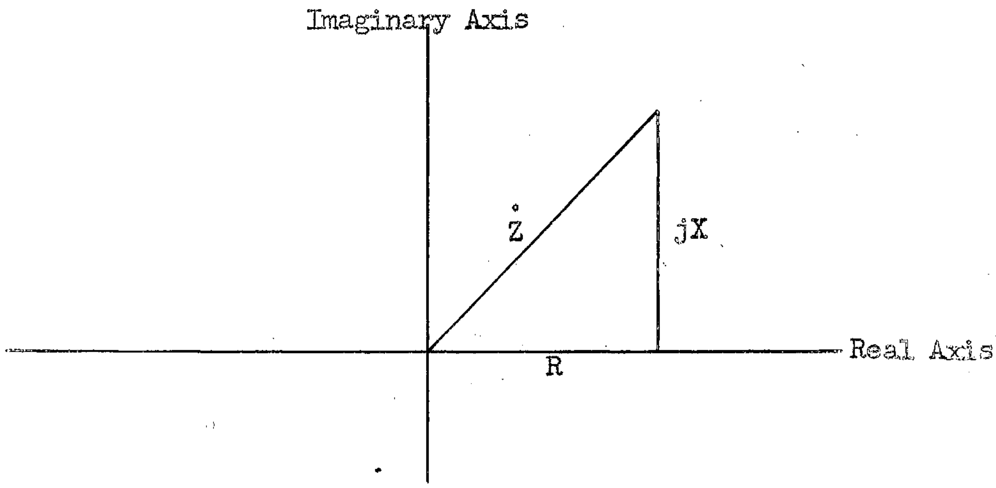

Figure 9
Impedance in Complex Plane

Recalling that complex vectors have two degrees of freedom, the complex vectors representing voltage (3.2) and current (3.3) determine a plane. In order to achieve a geometric concept of these two vectors, the plane of the paper may be arbitrarily taken as the plane of the complex vectors, and the complex coefficients plotted as shown in figure 10.

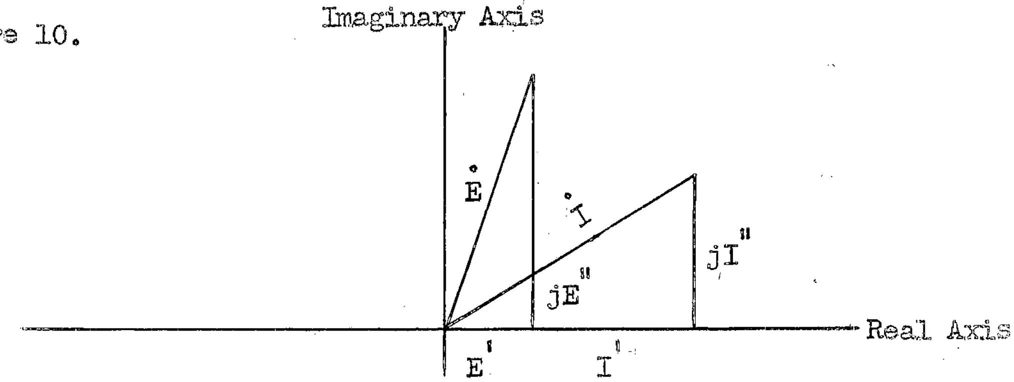

Figure 10
One-Dimensional Complex Vector Space (Plane)

Equations (3.2) and (3.3) may be written in the form

$$\overset{\circ}{\underline{E}} = E' \underline{e}_1 + jE'' \underline{e}_1 = \underline{E}_1 + j\underline{E}_2 \qquad (3.2a)$$

$$\overset{\circ}{\underline{I}} = I' \underline{e}_1 + jI'' \underline{e}_1 = \underline{I}_1 + j\underline{I}_2 \qquad (3.3a)$$

The complex vectors $\overset{\circ}{\underline{E}}$ and $\overset{\circ}{\underline{I}}$ may then be geometrically represented as shown in figure 11.

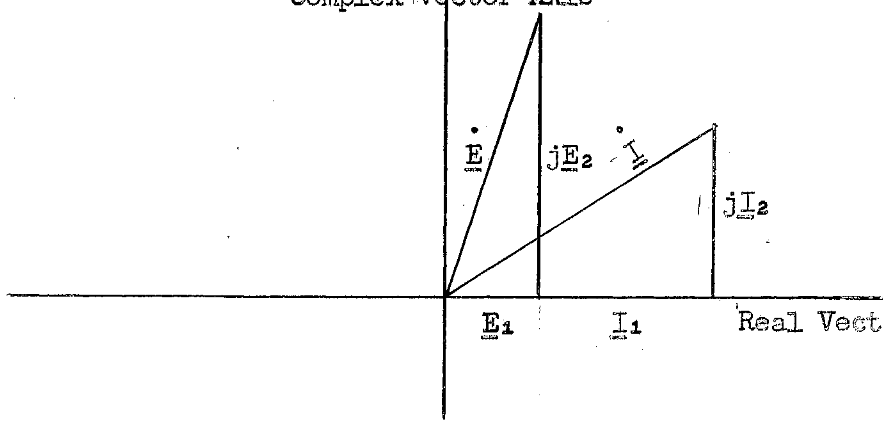

Figure 11
One-Dimensional Complex Vector Space (Plane)

There is another quantity, with its components, in addition to voltage, current, and impedance which is of importance in circuit analysis. This quantity is the a-c volt-amperes, which was discussed earlier in Chapter I. Before proceeding with the analysis of the circuit shown in figure 8, it would be instructive to examine a-c volt-amperes further.

In 1934, the Committee on Electrical and Magnetic Units which met in Paris decided that the reactive power in inductive circuits should be considered negative, and the reactive power in capacitive circuits should be considered positive. This decision merits careful thought; for it to be useful, it must conform with the basic equations of a-c circuits. It should be recalled from elementary trigonometry that

$$\cos(-\theta) = \cos\theta \qquad (3.4)$$

and

$$\sin(-\theta) = -\sin\theta \qquad (3.5)$$

From (1.8) and (1.9), the active and reactive powers are

$$P = VI \cos \theta \qquad (3.6)$$

and

$$P_r = VI \sin \theta \qquad (3.7)$$

The significance of defining P_r as being plus or minus for a given type of circuit is merely choosing either the voltage or current as the reference axis. For example, assuming an inductive circuit there would be two types of vector diagrams as shown in figure 12.

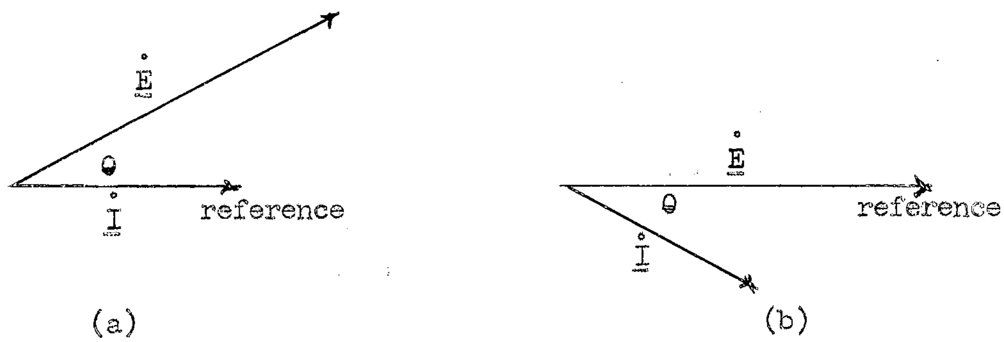

Figure 12
Two Different Reference Axes

Of course, actually figure 12(a) specifies that the phase angle shall be measured from the current vector to the voltage vector, and figure 12(b) specifies that the phase angle shall be measured from the voltage vector to the current vector. The sign of P as given by (3.6) is obviously positive for both figure 12(a) and figure 12(b). The sign of P_r (3.7) for figure 12(a) is positive, whereas the algebraic sign of P_r for figure 12(b) is negative. In accordance with the conclusions reached in Chapter I, the symbol $\dot{P}_v$ will be used to represent the so-called "vector volt-amperes". Using equations (3.2) and (3.3) for voltage and current, the value of $\dot{P}_v$ represented by figure 12(a) would be

$$\dot{P}_v = \dot{I}^* \cdot \dot{E} = (I' - jI'')(E' + jE'') \qquad (3.8)$$

or

$$\dot{P}_v = (I'E' + I''E'') + j(I'E'' - I''E') \qquad (3.8a)$$

Similarly, the value of $\mathring{P}_v$ represented by figure 12(b) would be

$$\mathring{P}_v = \mathring{\underline{E}}^* \cdot \mathring{\underline{I}} = (E' - jE'')(I' + jI'') \tag{3.9}$$

or

$$\mathring{P}_v = (E'I' + E''I'') - j(E''I' - E'I'') \tag{3.9a}$$

The equation that is selected for $\mathring{P}_v$, (3.8a) or (3.9a), must also satisfy the equation

$$I^2\mathring{Z} = \mathring{P}_v \tag{3.10}$$

Solving for $\mathring{Z}$ in (3.10) gives

$$\mathring{Z} = \frac{\mathring{P}_v}{I^2} \tag{3.10a}$$

In (3.10a) both $\mathring{Z}$ and $\mathring{P}_v$ are complex numbers, whereas I^2 (the norm of $\mathring{\underline{I}}$) is a real number. Therefore, if (3.10a) is an equality, the real and imaginary terms on both sides of (3.10a) must each be equal in magnitude and sign. Equation (3.8a) can be written as

$$\mathring{P}_v = P + jP_r \tag{3.11}$$

and (3.9a) can be written as

$$\mathring{P}_v = P - jP_r \tag{3.12}$$

Since the circuit was assumed to be inductive, $\mathring{Z}$ is correctly given by (3.1). Substituting (3.1) and (3.11) into (3.10a) gives

$$R + jX = \frac{P + jP_r}{I^2} \tag{3.13}$$

but substituting (3.1) and (3.12) into (3.10a) yields

$$R + jX = \frac{P - JP_r}{I^2} \tag{3.14}$$

Equation (3.14) is obviously false, since

$$+ jX \neq \frac{-jP_r}{I^2} \tag{3.15}$$

Therefore if basic equation (3.10) is to be used in circuit analysis, then (3.8) rather than (3.9) must be chosen as the defining equation for $\mathring{P}_v$. It is unfortunate that the Committee on Electrical and Magnetic Units chose the sign of P_r for inductive circuits to be negative. This choice would not allow the use of (3.10a) to calculate $\dot{Z}$, and $\dot{P}_v$ could not be calculated from the simple relation

$$\dot{P}_v = I^2\dot{Z} = I^2(R + jX) = I^2R + jI^2X \qquad (3.16)$$

As mentioned earlier, the use of (3.8) for $\dot{P}_v$ really means that positive phase angles will be measured from the current vector (as reference) to the voltage vector.

Example 1:

Let $\quad \dot{\underline{E}} = (50 + j86.6)\underline{e}_1$ volts

and $\quad \dot{\underline{I}} = (8.66 + j5)\underline{e}_1$ amperes

Using (3.8) $\quad \dot{P}_v = (8.66 - j5)(50 + j86.6)$ volt-amperes

$$\dot{P}_v = 866 + j500 \text{ volt-amperes}$$

$$I^2 = \dot{\underline{I}} \cdot \dot{\underline{I}} = (8.66 - j5)(8.66 + j5)$$

$$I^2 = 100 \text{ amperes squared}$$

Substituting into (3.10a)

$$\dot{Z} = \frac{866 + j500}{100} \text{ ohms}$$

$$\dot{Z} = 8.66 + j5 \text{ ohms}$$

where $\quad R = 8.66$ ohms resistance

and $\quad X_L = 5$ ohms inductive reactance

The nature of the calculated $\dot{Z}$ is inductive, which is as it should be since obviously the assumed $\dot{\underline{I}}$ lags the assumed $\dot{\underline{E}}$ by 30°. Had the recommendation of the Committee on Electrical and Magnetic Units been followed, the value of $\dot{P}_v$ would be,

$$\dot{P}_v = 866 - j500 \text{ volt-amperes}$$

where inductive vars are considered negative. The calculated value of $\dot{Z}$ would be, $\quad \dot{Z} = \frac{866 - j500}{100}$ ohms

or $\quad \dot{Z} = 8.66 - j5$ ohms

and thus $\dot{Z}$ is apparently capacitive, which is clearly incorrect. Therefore the recommendation that inductive vars be considered negative will be ignored; throughout the remainder of this dissertation inductive vars will be considered positive, i.e., equation (3.8) will be used for volt-amperes ($\dot{P}_v$).

There are essentially three basic types of problems encountered in the solution of single-loop networks. These problems are

Case I - Given $\dot{\underline{E}}$ and $\dot{\underline{I}}$, determine $\dot{Z}$,

Case II - Given $\dot{\underline{E}}$ and $\dot{Z}$, determine $\dot{\underline{I}}$, and

Case III - Given $\dot{\underline{I}}$ and $\dot{Z}$, determine $\dot{\underline{E}}$.

Of course, for any of these three cases additional information such as active power, power factor, etc., may be required. Cases I - III will be analyzed in chronological order, with a numerical example being included for each case.

<u>Case I</u> - Given $\dot{\underline{E}}$ and $\dot{\underline{I}}$, determine $\dot{Z}$:

Equations (3.2) and (3.3) may be written

$$\dot{\underline{E}} = \dot{E}\underline{e}_1 = (E' + jE'')\underline{e}_1 = Ee^{j\theta_1}\underline{e}_1 = E\underline{/\theta_1}\ \underline{e}_1 \qquad (3.17)$$

$$\dot{\underline{I}} = \dot{I}\underline{e}_1 = (I' + jI'')\underline{e}_1 = Ie^{j\theta_2}\underline{e}_1 = I\underline{/\theta_2}\ \underline{e}_1 \qquad (3.18)$$

The Ohm's law equation for $\dot{Z}$

$$\dot{Z} = \frac{\dot{\underline{E}}}{\dot{\underline{I}}} \qquad (3.19)$$

may be solved using either of the four notations for $\dot{\underline{E}}$ and $\dot{\underline{I}}$ in (3.17) and (3.18). The calculation of $\dot{Z}$ will be performed using the two most common forms of $\dot{\underline{E}}$ and $\dot{\underline{I}}$.

$$\dot{Z} = \frac{(E' + jE'')\underline{e}_1}{(I' + jI'')\underline{e}_1} \qquad (3.20)$$

In order to solve for $\dot{Z}$ in (3.20), it is necessary to take the scalar product of the numerator and denominator of the right-hand side of

(3.20) with $\dot{\underline{I}}$. Recalling that the scalar product of complex vectors is not commutative, the question is whether the left-hand or right-hand scalar product should be used. Equation (3.8) and the results of Example 1 indicate that the left-hand scalar product must be used.

$$\dot{Z} = \frac{(I' - jI'')(E' + jE'')}{(I' - jI'')(I' + jI'')} \tag{3.21}$$

or

$$\dot{Z} = \frac{(I'E' + I''E'') + j(I'E'' - I''E')}{(I')^2 + (I'')^2} \tag{3.22}$$

The value of $\dot{P}_v$ is actually the numerator of (3.22), but using (3.8)

$$\dot{P}_v = \dot{\underline{I}} \cdot \dot{\underline{E}} = (I' - jI'')(E' + jE'')$$

or

$$\dot{P}_v = (I'E' + I''E'') + j(I'E'' - I''E') \tag{3.23}$$

The power factor is

$$P.F. = \frac{P}{(P^2 + P_r^2)^{\frac{1}{2}}} \tag{3.24}$$

where

$$\dot{P}_v = P + jP_r$$

$$P.F. = \frac{I'E' + I''E''}{[(I'E' + I''E'')^2 + (I'E'' - I''E')^2]^{\frac{1}{2}}} \tag{3.25}$$

Employing the other most used notation, equations (3.20) through (3.25) become

$$\dot{Z} = \frac{E/\underline{\theta_1}\ \underline{e_1}}{I/\underline{\theta_2}\ \underline{e_1}} \tag{3.20a}$$

$$\dot{Z} = \frac{I/\underline{-\theta_2} \cdot E/\underline{\theta_1}}{I/\underline{-\theta_2} \cdot I/\underline{\theta_2}} \tag{3.21a}$$

or

$$\dot{Z} = \frac{EI}{I^2} /\underline{\theta_1 - \theta_2} \tag{3.22a}$$

$$\dot{P}_v = \dot{\underline{I}} \cdot \dot{\underline{E}} = I/\underline{-\theta_2} \cdot E/\underline{\theta_1}$$

or

$$\dot{P}_v = IE/\underline{\theta_1 - \theta_2} \tag{3.23a}$$

$$\dot{P}_v = P + jP_r$$

$$\dot{P}_v = IE\cos(\theta_1 - \theta_2) + jIE\sin(\theta_1 - \theta_2)$$

$$P.F. = \frac{IE\cos(\theta_1 - \theta_2)}{[(IE\cos(\theta_1 - \theta_2))^2 + (IE\sin(\theta_1 - \theta_2))^2]^{\frac{1}{2}}} \qquad (3.25a)$$

Example 2:

Let the applied voltage be

$$\dot{\underline{E}} = (50 + j50)\underline{e}_1 \quad \text{volts}$$

or

$$\dot{\underline{E}} = 70.7 \underline{/45^\circ}\ \underline{e}_1 \quad \text{volts}$$

and the resulting current be

$$\dot{\underline{I}} = (6 + j8)\underline{e}_1 \quad \text{amperes}$$

or

$$\dot{\underline{I}} = 10 \underline{/53.2^\circ}\ \underline{e}_1 \quad \text{amperes}$$

Equation (3.20) is then

$$\dot{Z} = \frac{(50 + j50)\underline{e}_1}{(6 + j8)\underline{e}_1} \quad \text{ohms}$$

and (3.20a) is

$$\dot{Z} = \frac{70.7 \underline{/45^\circ}\ \underline{e}_1}{10 \underline{/53.2^\circ}\ \underline{e}_1} \quad \text{ohms}$$

Equation (3.21) becomes

$$\dot{Z} = \frac{(6 - j8)(50 + j50)}{(6 - j8)(6 + j8)} \quad \text{ohms}$$

and (3.21a) becomes

$$\dot{Z} = \frac{10 \underline{/-53.2^\circ} \times 70.7 \underline{/45^\circ}}{10 \underline{/-53.2^\circ} \times 10 \underline{/53.2^\circ}} \quad \text{ohms}$$

The final expression for $\dot{Z}$ (3.22) is

$$\dot{Z} = \frac{(300 + 400) + j(300 - 400)}{36 + 64} = 7 - j1 \quad \text{ohms}$$

or using (3.22a)

$$\dot{Z} = \frac{10 \times 70.7}{100} \underline{/45^\circ - 53.2^\circ} = 7.07 \underline{/-8.2^\circ} \quad \text{ohms}$$

The value of $\dot{P}_v$ given by (3.23) is

$$\dot{P}_v = (6 - j8)(50 + j50) = (300 + 400) + j(300 - 400)$$

or $\dot{P}_v = 700 - j100$ volt-amperes

and the value of $\dot{P}_v$ given by (3.23a) is

$$\dot{P}_v = 10 \times 70.7 \,\underline{/45^\circ - 53.2^\circ} \text{ volt-amperes}$$

or $$\dot{P}_v = 707 \,\underline{/-8.2^\circ} \text{ volt-amperes}$$

The power factor given by (3.25) is

$$P.F. = \frac{700}{[(700)^2 + (-100)^2]^{\frac{1}{2}}} = \frac{700}{707} = 0.990$$

and the power factor given by (3.25a) is

$$P.F. = \frac{10 \times 70.7 \cos(-8.2^\circ)}{[(707 \cos(-8.2^\circ))^2 + (707 \sin(-8.2^\circ))^2]^{\frac{1}{2}}}$$

$$P.F. = \frac{700}{[(700)^2 + (-100)^2]^{\frac{1}{2}}} = \frac{700}{707} = 0.990$$

The calculated quantities are:

$Z = 7.07$ ohms, $R = 7.0$ ohms, $X_C = 1.0$ ohm

$P_v = 707$ volt-amperes, $P = 700$ watts, $P_r = -100$ vars (capacitive)

Power factor angle $= -8.2^\circ$ (leading) and P.F. $= 0.990$

Case II - Given $\dot{E}$ and $\dot{Z}$, determine $\dot{I}$:

From equations (3.1) and (3.2)

$$\dot{Z} = R + jX \qquad (3.26)$$

and $$\dot{E} = (E' + jE'')e_1 \qquad (3.27)$$

Ohm's Laws applied to a-c circuits states that

$$\dot{I} = \frac{\dot{E}}{\dot{Z}} \qquad (3.28)$$

Therefore $$\dot{I} = \frac{(E' + jE'')e_1}{R + jX} \qquad (3.29)$$

Rationalizing (3.29) gives

$$\dot{I} = \frac{[(RE' + XE'') + j(RE'' - XE')]e_1}{R^2 + X^2} \qquad (3.30)$$

If $\dot{\underline{I}}$ is expressed in the form

$$\dot{\underline{I}} = (I' + jI'')\underline{e}_1 \tag{3.31}$$

then from (3.30)

$$I' = \frac{RE' + XE''}{R^2 + X^2} \tag{3.32}$$

and

$$I'' = \frac{RE'' - XE'}{R^2 + X^2} \tag{3.33}$$

Using (3.29)

$$\dot{\underline{I}} \cdot \dot{\underline{I}} = I^2 = \frac{(E' - jE'')}{(R - jX)} \times \frac{(E' + jE'')}{(R + jX)} \tag{3.34}$$

Thus

$$I^2 = \frac{(E')^2 + (E'')^2}{R^2 + X^2} \tag{3.34a}$$

From (3.16)

$$\dot{P}_v = I^2\dot{Z} \tag{3.35}$$

Substituting (3.26) and (3.34a) into (3.35) gives

$$\dot{P}_v = \frac{[(E')^2 + (E'')^2](R + jX)}{R^2 + X^2} \tag{3.36}$$

or

$$\dot{P}_v = \frac{R[(E')^2 + (E'')^2]}{R^2 + X^2} + \frac{jX[(E')^2 + (E'')^2]}{R^2 + X^2} \tag{3.36a}$$

where

$$P = \frac{R[(E')^2 + (E'')^2]}{R^2 + X^2} \tag{3.37}$$

and

$$P_r = \frac{X[(E')^2 + (E'')^2]}{R^2 + X^2} \tag{3.38}$$

Substituting (3.37 and (3.38) into

$$P.F. = \frac{P}{(P^2 + P_r^2)^{\frac{1}{2}}} \tag{3.39}$$

gives the equation for power factor in terms of $\dot{\underline{E}}$ and $\dot{Z}$

$$P.F. = \frac{R}{(R^2 + X^2)^{\frac{1}{2}}} \tag{3.40}$$

Example 3:

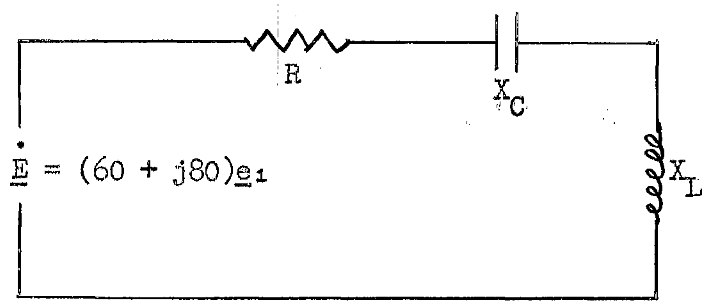

Figure 13
Circuit for Example 3

Given R = 30 ohms, X_C = 20 ohms and X_L = 60 ohms, let it be required to caculate $\dot{\underline{I}}$, P, P_r, P.F., P_v and the voltage drop across X_L, $\dot{\underline{V}}_L$.

$$\dot{Z} = R + j(X_L - X_C) \quad \text{ohms}$$

$$\dot{Z} = 30 + j(60 - 20) = 30 + j40 \quad \text{ohms}$$

Using (3.32)

$$I' = \frac{30 \times 60 + 40 \times 80}{30^2 + 40^2} = \frac{1800 + 3200}{900 + 1600}$$

$$I' = 2.0 \quad \text{amperes}$$

Solving for I'' (3.33) gives

$$I'' = \frac{30 \times 80 - 40 \times 60}{30^2 + 40^2}$$

$$I'' = 0 \quad \text{amperes}$$

Therefore

$$\dot{\underline{I}} = (I' + jI'')\underline{e}_1$$

$$\dot{\underline{I}} = 2\ \underline{e}_1 \quad \text{amperes}$$

Using (3.37)

$$P = \frac{30(60^2 + 80^2)}{30^2 + 40^2} = \frac{30 \times 10{,}000.}{2500}$$

$$P = 120 \text{ watts}$$

Using (3.38)

$$P_r = \frac{40(60^2 + 80^2)}{20^2 + 40^2} = \frac{40 \times 10{,}000.}{2500}$$

$$P_r = 160 \text{ vars (inductive)}$$

Hence $$\dot{P}_v = 120 + j160 \text{ volt-amperes}$$

and $$P_v = 200 \text{ volt-amperes}$$

The power factor as given by (3.39) is

$$P.F. = \frac{P}{P_v}$$

$$P.F. = \frac{120}{200} = 0.60$$

The impedance of the coil is

$$\dot{Z}_L = j60 \text{ ohms}$$

By Ohm's law, the voltage drop across the coil, $\dot{\underline{V}}_L$, is

$$\dot{\underline{V}}_L = \dot{Z}_L \times \dot{\underline{I}}$$

$$\dot{\underline{V}}_L = j60 \times 2\, \underline{e}_1$$

$$\dot{\underline{V}}_L = j120\, \underline{e}_1$$

Case III - Given $\dot{\underline{I}}$ and $\dot{Z}$, determine $\dot{\underline{E}}$:

Let $$\dot{\underline{I}} = (I' + jI'')\underline{e}_1 \qquad (3.41)$$

and $$\dot{Z} = R + jX \qquad (3.42)$$

The equation for $\dot{\underline{E}}$ is

$$\dot{\underline{E}} = \dot{Z} \times \dot{\underline{I}} \qquad (3.43)$$

Substituting (3.41) and (3.42) into (3.43)

gives $$\dot{\underline{E}} = (R + jX)(I' + jI'')\underline{e}_1$$

$$\dot{\underline{E}} = [(RI' - XI'') + j(RI'' + XI')]\underline{e}_1 \qquad (3.44)$$

If $$\dot{\underline{E}} = (E' + jE'')\underline{e}_1$$

then $$E' = RI' - XI'' \qquad (3.45)$$

and $$E'' = RI'' + XI' \qquad (3.46)$$

From (3.41) $$\dot{\underline{I}} \cdot \dot{\underline{I}} = \dot{I}^* \times \dot{I} = (I' - jI'')(I' + jI'')$$

or $$I^2 = (I')^2 + (I'')^2 \qquad (3.47)$$

The equation for volt-amperes is

$$\dot{P}_v = I^2\dot{Z}$$

$$\dot{P}_v = I^2(R + jX) = I^2R + jI^2X \qquad (3.48)$$

Substituting (3.47) into (3.48) yields

$$\dot{P}_v = R[(I')^2 + (I'')^2] + jX[(I')^2 + (I'')^2] \qquad (3.49)$$

Where

$$P = R[(I')^2 + (I'')^2] \qquad (3.50)$$

and

$$P_r = X[(I')^2 + (I'')^2] \qquad (3.51)$$

The equation for power factor is

$$P.F. = \frac{P}{(P^2 + P_r^2)^{\frac{1}{2}}} \qquad (3.52)$$

If (3.50) and (3.51) are substituted into (3.52), the result is the equation for power factor in terms of impedance

$$P.F. = \frac{R}{(R^2 + X^2)^{\frac{1}{2}}} \qquad (3.53)$$

Example 4:

Let it be required to solve the circuit of figure 13 when a current of $\dot{\underline{I}} = (8.66 + j5)\underline{e}_1$ amperes is flowing through the given impedances.

$R = 30$ ohms; $X = 60 - 20 = 40$ ohms; $\dot{Z} = R + jX = 30 + j40$ ohms. Using equation (3.45)

$$E' = 30 \times 8.66 - 40 \times 5$$

$$E' = 60$$

Using equation (3.46)

$$E'' = 30 \times 5 + 40 \times 8.66$$

$$E'' = 496.4$$

Hence

$$\dot{\underline{E}} = (60 + j496.4)\underline{e}_1 \text{ volts}$$

From equation (3.50)

$$P = 30[(8.66)^2 + (5)^2]$$

$$P = 3000 \text{ watts}$$

Using equation (3.51)

$$P_r = 40[(8.66)^2 + (5)^2]$$

$$P_r = 4000 \text{ vars (inductive)}$$

It follows that

$$\dot{P}_v = 3000 + j4000 \text{ volt-amperes}$$

and

$$\dot{P}_v = 5000 \text{ volt-amperes}$$

From (3.53)

$$P.F. = \frac{30}{(30^2 + 40^2)^{\frac{1}{2}}}$$

$$P.F. = 0.60$$

The voltage drop across the capacitor is

$$\dot{\underline{E}}_C = \dot{\underline{I}} \times \dot{Z}_C$$

$$\dot{\underline{E}}_C = (8.66 + j5)\underline{e}_1 \times (-j20)$$

$$\dot{\underline{E}}_C = (100 - j173.2)\underline{e}_1 \text{ volts}$$

Summary of Chapter III

1. The geometric representation of a one-dimensional complex vector requires a plane, since a complex vector has two degrees of freedom.
2. Using the complex vector notation, a-c impedance and volt-amperes are represented by complex scalars, whereas current and voltage are represented by one-dimensional complex vectors (for single-loop networks).
3. The recommendation of the Committee on Electrical and Magnetics Units, which met in Paris in 1934, that capacitive reactive power be considered positive is not consistent with the fundamental

Ohm's law equation for a-c circuits. Therefore in this dissertation inductive reactive power will be considered positive, and capacitive reactive power will be considered negative.

4. Complex vector algebra can be used to solve single-loop a-c circuits in a simple, straight-forward manner.

Conclusions

In this chapter, a new concept of complex vectors has been introduced to represent sinusoidal functions of time, and a method for solving single-loop networks has been presented using this new concept. It is apparent that the major difference in using the complex vector notation, rather than the commonly used complex scalar notation, lies in the definition of the scalar product of two complex vectors. This definition gives the correct expression for volt-amperes as discussed earlier. Since the reason for the failure of the complex scalar notation has been discussed, the question logically arises, why not explain the reason for the failure, use the conjugate of the current to arrive at the correct answer, and then continue using the complex scalar notation as before? Indeed, if this were to be the only use of the complex vector notation, this would certainly be the most satisfactory course to pursue! However, this is only a fragmentary portion of the reason for adopting the complex vector representation of sinusoidal functions of time. In the remainder of this dissertation, the concept of complex vectors will be generalized and applied to multiple-loop networks with both sinusoidal and nonsinusoidal wave forms of voltages and currents, where the nonsinusoidal wave forms are such that they can be analyzed into sinusoidal components by the methods of the Fourier Analysis. It will be seen later that the concept of the complex vector notation will readily lend itself to representation by matrices and subsequently to representation by tensors. The concepts and use of complex vectors developed up to this point are the author's own ideas; later this new concept will be adapted to a procedure formulated principally by Gabriel Kron.

CHAPTER IV

THE THEORY OF MATRICES

The Basic Concept of a Matrix

A large portion of the study of electrical engineering deals basically with transformations, most of which are assumed linear. Unfortunately, however, the mention of such a word as "transformation" is usually sufficient to scare away most electrical engineers. The matrix is a mathematical tool that is ideally suited for dealing with problems involving irrotational transformations. The rapidly increasing use of matrix methods by electrical engineers during the last several years is indeed encouraging, and it is the author's conviction that matrix algebra will take its place as a standard undergraduate course in the electrical engineering schools of the leading universities within the next decade.

The problems of the engineer are fundamentally the same as those of the physicist; both express physical phenomena in mathematical symbols. Generally speaking, the physicist endeavors to reduce natural phenomena to their simplest possible form, usually expressible by a few, mostly one, equations, introducing only as many mathematical symbols as there are corresponding physical concepts. That is, the physicist sets up an equation for, say, the conduction of electricity between "two" electrodes, or for an electro-magnetic wave traveling along a "single" conductor, or for the electromotive force generated in a "single" conductor moving in a magnetic field, or for the passage of light through a lens, etc. Once the equation for the phenomenon is set up, the physicist's role has ended.

This is where the engineer's role begins. The engineer takes a two-electrode tube and adds several additional electrodes; and for good measure he connects them to different types of networks; or he builds transmission networks covering whole continents; or he takes "several" moving conductors and constructs a large variety of complex rotating electrical machines; or he combines a series of lenses into an optical instrument, and so on.

That is, "the engineer generalizes the one-, two-, or three-dimensional problem of the physicist to k dimensions". And that is where his difficulty originates . . . . . .

In order to organize the large variety of engineering problems into the absolute minimum number of standardized types in which the physicist has expressed them, it is necessary to introduce new points of view, new symbols, new mental and physical concepts.[1]

A matrix is a highly condensed method of writing a system of linear equations. Let such a set of equations be

$$
\begin{aligned}
y_1 &= a_{11}x^1 + a_{12}x^2 + a_{13}x^3 \\
y_2 &= a_{21}x^1 + a_{22}x^2 + a_{23}x^3 \qquad (4.1) \\
y_3 &= a_{31}x^1 + a_{32}x^2 + a_{33}x^3
\end{aligned}
$$

Equations (4.1) represent a transformation of the variables x^1, x^2 and x^3 into the variables y_1, y_2 and y_3. The set of equations (4.1) may be interpreted in two different ways.[2]

(a) The quantities x^1, x^2 and x^3 may be regarded as components of a vector $\dot{\underline{X}}$, and the quantities y_1, y_2 and y_3 as components of another vector $\dot{\underline{Y}}$, where both $\dot{\underline{X}}$ and $\dot{\underline{Y}}$ are referred to the same coordinate system and set of base vectors $\underline{e}_1$, $\underline{e}_2$ and $\underline{e}_3$; in this case, equations (4.1) are to be thought of as representing a transformation of the vector $\dot{\underline{X}}$ into another vector $\dot{\underline{Y}}$.

(b) The two sets of quantities (x^1, x^2, x^3) and (y_1, y_2, y_3) may be regarded as components of the same vector $\dot{\underline{X}}$, when $\dot{\underline{X}}$ is referred to two different coordinate systems determined by the two sets of base vectors $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ and $(\underline{e}'_1, \underline{e}'_2, \underline{e}'_3)$; in this event, equations (4.1) are considered as transforming the coordinate axes.

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[1]Gabriel Kron, Tensor Analysis of Networks (New York, 1939), pp. 1-2.

[2]I. S. Sokolnikoff, Tensor Analysis (New York, 1951), p. 20.

Definition 1 (matrix): A table of mn numbers, called elements, arranged in a rectangular array of m rows and n columns is called a matrix with m rows and n columns.[3]

The matrix of (4.1) is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad (4.2)$$

A rectangular array of numbers enclosed in brackets, as in (4.2), will be used to represent a matrix in expanded form. A single capital-letter symbol enclosed in brackets will also be used to represent a matrix. For example, (4.2) will be written as

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad (4.3)$$

A matrix having m rows and n columns is called an $(m \times n)$ matrix; thus (4.3) is a (3×3) matrix. A matrix in which $m = n$, as in (4.3), is called a square matrix.

In order to illustrate the use of the matrix notation, equations (4.1) will be expressed in matrix form. If the quantities y_1, y_2 and y_3 are considered as the elements of a (3×1) matrix, then

$$[Y] = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad (4.4)$$

[3]Aristotle D. Michal, Matrix and Tensor Calculus (New York, 1947), p. 1.

Similarly $$[X] = \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} \qquad (4.5)$$

In matrix notation, (4.1) becomes

$$[Y] = [A] \times [X] \qquad (4.6)$$

or in expanded form

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \times \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} \qquad (4.7)$$

The space economy of matrix notation is amply illustrated by (4.6), when it is realized that (4.6) could refer to n equations in n unknowns just as easily as it refers to (4.1). The manner in which the two matrices on the right-hand side of (4.7) will be multiplied together to yield the right-hand side of (4.1) will be explained in the next section in this chapter.

Using the summation convention, (4.1) can be written

$$y_i = a_{ij}x^j \qquad (i, j = 1, 2, 3) \qquad (4.8)$$

It is apparent that the value assigned to i specifies the row and the value assigned to j specifies the column in which the element a_{ij} is located. Hereafter the symbol a_{ij} will be used to refer to the element in the ith row and jth column of [A]. The matrix (4.2) will also be written as $[a_{ij}]$, where the ranges of i and j are either specified or understood.

Equations (4.1) through (4.8) have been devised with the number of variables (x^j or y_i) purposely limited to three. This would permit matrices (4.4) and (4.5) to be interpreted as two vectors, $\overset{\circ}{\underline{Y}}$ and

and $\underline{\dot{X}}$, in ordinary three-dimensional Cartesian coordinates. This restriction causes no loss of generality, since all the discussion thus far would still apply if [A] were an (m x n) rather than a (3 x 3) matrix.

The Algebra of Matrices

Definition 2 (equality): Two matrices [A] and [B] are equal if, and only if, each element of [A] is equal to the corresponding element of [B]. That is

$$a_{ij} = b_{ij} \tag{4.9}$$

It should be noted that the equality of two matrices requires that the two matrices have the same number of rows and the same number of columns.

Definition 3 (sum and difference): The sum or difference of two matrices [A] and [B] is a matrix [C], each element of [C] being the sum or difference of the two corresponding elements of [A] and [B].

If

$$[A] + [B] = [C] \tag{4.10}$$

then

$$a_{ij} + b_{ij} = c_{ij} \tag{4.11}$$

Also, if

$$[A] - [B] = [C'] \tag{4.12}$$

then

$$a_{ij} - b_{ij} = c'_{ij} \tag{4.13}$$

In order to be added or subtracted, two matrices must have the same number of rows and the same number of columns. If [A] and [B] are (m x n) matrices, then in expanded form

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{4.14}$$

$$[B] = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} \qquad (4.15)$$

The sum of (4.14) and (4.15) is [C], where

$$[C] = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix} \qquad (4.16)$$

The matrix $[C']$ in (4.12) is obtained by replacing the plus signs in (4.16) by minus signs.

Definition 4 (conformable matrices): Two matrices [A] and [B] are said to be conformable if the number of columns in [A] equals the number of rows in [B].

If [A] is conformable to [B], then [A] has the same number of columns as [B] has rows; this does not necessarily imply that [B] is conformable to [A]. Two matrices $[a_{ij}]$ and $[b_{rs}]$ are conformable to each other only if $i = s$ and $j = r$.

Definition 5 (product of a matrix by a matrix): The product of two matrices $[a_{mn}]$ and $[b_{np}]$ is an (m x p) matrix $[C_{mp}]$, in which the elements are

$$c_{ij} = a_{ik}b_{kj} \qquad (4.17)$$

where ($i = 1, 2, \ldots, m$), ($j = 1, 2, \ldots, p$) and ($k = 1, 2, \ldots, n$). It should be remembered that in (4.17) the repeated index k denotes a summation. It is actually more informative to describe the process by which the product of two matrices is formed than to merely state the definition of the product. The steps to be followed in determining the product of two matrices are

(1) Multiply a_{11} by b_{11}, a_{12} by b_{21}, a_{13} by b_{31}, . . ., a_{1n} by b_{n1}, and form the sum of all these products; the result is the element in the first row and first column of [C], the product matrix.

(2) Using the elements of the second column of [B], b_{21}, b_{22}, . . ., b_{2n}, rather than the first column, repeat (1); the result is the element in the first row and second column of [C].

(3) Repeat the process of forming the sums of the products of the elements of the first row of [A] with the corresponding elements of successive columns of [B], until all p columns of [B] have been exhausted.

(4) Repeat steps (1) through (3) for successive rows of [A] to obtain successive rows of [C]. When all m rows of [A] have been used in this manner once, the matrix [C] will have been completed.

Example 1:

Let
$$[A] = \begin{bmatrix} 4 & -2 & 3 \\ -3 & 5 & 1 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 1 & -1 \\ 0 & 4 \\ 3 & 2 \end{bmatrix}$$

The product of [B] by [A] is

$$\begin{bmatrix} 4 & -2 & 3 \\ -3 & 5 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 0 & 4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 13 & -6 \\ 0 & 25 \end{bmatrix}$$

The multiplication of matrices is not commutative. That is

$$[A] \times [B] \neq [B] \times [A] \tag{4.18}$$

The validity of (4.18) is evident, since the product [B] x [A] may not even exist even though the product [A] x [B] is well defined. Even if the two products on both sides of (4.18) did exist, the two product matrices would not be of the same order unless [A] and [B] were both square matrices.

Example 2:

Using the two matrices [A] and [B] given in Example 1, let it be required to find the product [B] x [A].

$$[B] \times [A] = \begin{bmatrix} 1 & -4 \\ 0 & 4 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} 4 & -2 & 3 \\ -3 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -7 & 2 \\ -12 & 20 & 4 \\ 6 & 4 & 11 \end{bmatrix}$$

The numerical results of Examples 1 and 2 demonstrate the general validity of (4.18).

Definition 6 (product of a scalar and a matrix): The product of a matrix and a scalar is formed by multiplying each element of the matrix by the scalar.

Taking into consideration (4.1), the reason for Definition 6 is apparent. The matrix product [A] x [B] will be called the premultiplication of [B] by [A], and the matrix product [B] x [A] will be called the postmultiplication of [B] by [A].

There are several special types of matrices that are used extensively in applications. These matrices and their most important characteristics will now be described and defined.

Definition 7 (row or column matrix): A row or column matrix is a matrix containing a single row or column of elements.

The elements of a row or column matrix may be considered as the coordinates of a single row or column vector; the elements of the different rows and columns of any matrix may be considered as the coordinates of a set of row or column vectors. The result of premultiplying a general (m x n) matrix by a (1 x m) row matrix is a (1 x n) matrix; an (m x n) matrix postmultiplied by a (n x 1) column matrix yields an (m x 1) column matrix.

Definition 8 (null matrix): A matrix in which all the elements are zero is called the null matrix and is represented by the symbol [0].

Definition 9 (diagonal matrix): A square matrix in which all the elements are zero except those along the principal diagonal is called a diagonal matrix. The principal diagonal of a square matrix $[a_{nn}]$ contains the elements a_{ii} $(i = 1, 2, \ldots, n)$.

Definition 10 (scalar matrix): A diagonal matrix in which all the elements along the principal diagonal are equal is called a scalar matrix.

Definition 11 (unit matrix): A unit matrix [I] is a scalar matrix in which the elements of the principal diagonal are equal to one.

Definition 12 (transpose): If the rows and columns of a matrix [A] are interchanged, the result is called the transpose of [A] and is represented by the symbol $[A_t]$.

The following rules will be stated without proof:

$$([A] + [B])_t = [A_t] + [B_t]$$

$$([A] \times [B])_t = [B_t] \times [A_t]$$

Definition 13 (singular matrix): A square matrix [A] for which the determinant of the elements, written $|A|$, is zero is called a singular matrix; if the determinant of a square matrix is not zero, the matrix is said to be nonsingular.

Definition 14 (cofactor): The cofactor of the element a_{ij} of the square matrix [A], written A_{ij}, is $(-1)^{i+j}$ times the determinant formed by deleting the elements of the ith row and jth column. If this determinant, usually called the minor of a_{ij}, is represented by the symbol M_{ij}, then

$$A_{ij} = (-1)^{i+j} M_{ij} \qquad (4.19)$$

Definition 15 (adjoint): The matrix $[\underline{A}]$ which is the transpose of the matrix in which the elements are the cofactors of the elements of [A] is called the adjoint of [A].

$$[A] \times [\underline{A}] = |A|[I] \qquad (4.20)$$

Definition 16 (inverse): The inverse $[A^{-1}]$ of a matrix [A] satisfies the equation

$$[A] \times [A^{-1}] = [I] \qquad (4.21)$$

From (4.20), obviously

$$[A^{-1}] = \frac{[\underline{A}]}{|A|} \qquad (4.22)$$

Also

$$[A] \times [A^{-1}] = [A^{-1}] \times [A] = [I] \qquad (4.23)$$

By premultiplying both sides of (4.6) by $[A^{-1}]$, the inverse of the transformation (4.1) is

$$[X] = [A^{-1}] \times [Y] \qquad (4.24)$$

Definition 17 (symmetric matrix): If the transpose $[A_t]$ of a matrix [A] is the same as [A], the matrix [A] is called a symmetric matrix. That is, [A] is symmetric if

$$[A] = [A_t] \qquad (4.25)$$

Definition 18 (skew symmetric matrix):

If

$$[A] = -[A_t] \qquad (4.26)$$

the matrix [A] is called a skew symmetric matrix.

Definition 19 (orthogonal matrix): If the transpose $[A_t]$ of a matrix [A] is equal to the inverse $[A^{-1}]$ of [A], then the matrix [A] is called an orthogonal matrix.

$$[A_t] = [A^{-1}] \tag{4.27}$$

Premultiplying (4.27) by [A] gives

$$[A] \times [A_t] = [I] \tag{4.28}$$

From (4.28) or (4.27)

$$|A|^2 = 1 \tag{4.29}$$

$$|A| = \pm 1 \tag{4.30}$$

Summary of Chapter IV

1. The addition (or subtraction) of matrices is commutative and associative.
$$[A] + [B] = [B] + [A]$$
$$([A] + [B]) + [C] = [A] + ([B] + [C])$$

2. The multiplication of matrices is associative and distributive.
$$[A] \times ([B] \times [C]) = ([A] \times [B]) \times [C]$$
$$[A] \times ([B] + [C]) = [A] \times [B] + [A] \times [C]$$

3. The multiplication of matrices is not in general commutative
$$[A] \times [B] \neq [B] \times [A]$$
But
$$[A] \times [I] = [I] \times [A] = [A]$$

4. The matrix equation
$$[A] \times [B] = [0]$$
does not imply that either [A] or [B] is necessarily [0].

5. With the exception of special operations, such as the transpose, the differences in matrix algebra and scalar algebra are contained in items 3 and 4. That is, if a and b are scalars,
$$ab = ba$$
and
$$ab = 0 \text{ implies that either a or b}$$
must be zero.

CHAPTER V

THE TENSORIAL NATURE OF A-C NETWORK QUANTITIES

A major portion of the material presented in this chapter is based upon the works of Gabriel Kron and P. Le Corbeiller given in the general bibliography. While several new concepts have been introduced by the author, the tensor character of a-c circuit quantities was originally the work of Kron.

The Topology of Electric Networks

The terms branches, nodes, loops, and meshes are used extensively in electrical engineering. Since different writers use these terms to mean different things, to avoid confusion these terms will be defined below in the way that they will be used in this dissertation.

Definition 1 (branch): A branch of a network is a series combination of circuit elements between two terminals.

Definition 2 (junction): A junction is a point common to more than two branches.

Definition 3 (node): A node is a terminal; a node is usually, but not necessarily, a junction.

Definition 4 (mesh): A mesh (also called loop) is any closed contour drawn on a network diagram.

Definition 5 (subnetwork): The various parts of a given network that are coupled magnetically but not conductively are called subnetworks of the given network.

Definition 6 (node-pair): Any two nodes within a single network constitute a node-pair.

Let S = number of subnetworks

B = number of branches

N = number of nodes

and M = number of meshes

A network in which all coupled meshes are conductively coupled will be called a "completely connected" network. A network that is completely connected obviously has only one subnetwork.

In a completely connected network having B branches and N nodes, there are $\frac{1}{2}N(N - 1)$ node-pairs and hence, considering polarity, $N(N - 1)$ voltages. Let one of the N nodes be considered as the reference node (grounded); the voltages of all the other $N - 1$ nodes may be measured with respect to this reference node. The voltage between any two nodes is the difference between the voltages of these two nodes measured with respect to the reference node. Thus the $N(N - 1)$ voltages may be expressed linearly in terms of only $N - 1$ voltages.

If P = number of independent voltages

then
$$P = N - 1 \qquad (5.1)$$

For a network containing S subnetworks, equation (5.1) may be applied to each subnetwork
$$P_i = N_i - 1 \quad (i = 1, 2, \ldots, S) \qquad (5.2)$$

If the S equations represented by (5.2) are added, the left side becomes
$$P_1 + P_2 + \ldots + P_S = P \qquad (5.3)$$

and the right side becomes
$$(N_1 - 1) + (N_2 - 1) + \ldots + (N_S - 1) = N - S \qquad (5.4)$$

where P and N are the total number of independent voltages and nodes, respectively, in the network. Equating (5.3) and (5.4) gives

$$P = N - S \qquad (5.5)$$

independent voltage equations. For a network having B branches, a total of B independent equations are required to solve for the B unknown branch currents. Therefore the number of mesh equations, M, that must be formulated is

$$M = B - P = B - N + S \qquad (5.6)$$

For a completely connected network

$$M = B - N + 1 \qquad (5.7)$$

In solving networks, the number of independent equations required is of primary importance. Equations (5.5), (5.6), and (5.7) are fundamental topologic relations which form the basis of electrical network analysis.

A Particular Solution of a Given Network Using Matrix Algebra

The concepts that will be presented in the remainder of this chapter can be best introduced in the solution of a specific network.[1] It will be assumed that the voltages of the six generators, $\dot{\underline{E}}_a$, $\dot{\underline{E}}_b$, $\dot{\underline{E}}_c$, $\dot{\underline{E}}_d$, $\dot{\underline{E}}_f$ and $\dot{\underline{E}}_g$ are given. The generators, as usual, are assumed to be constant-voltage machines, all generating a single frequency. The six given branch impedances, $\dot{Z}_a$, $\dot{Z}_b$, $\dot{Z}_c$, $\dot{Z}_d$, $\dot{Z}_f$ and $\dot{Z}_g$ will be assumed to have no mutual magnetic coupling. The problem is to determine the six branch currents, $\dot{\underline{I}}^a$, $\dot{\underline{I}}^b$, $\dot{\underline{I}}^c$, $\dot{\underline{I}}^d$, $\dot{\underline{I}}^f$ and $\dot{\underline{I}}^g$ in

[1] P. Le Corbeiller, Matrix Analysis of Electric Networks (New York, 1950), pp. 27-34.

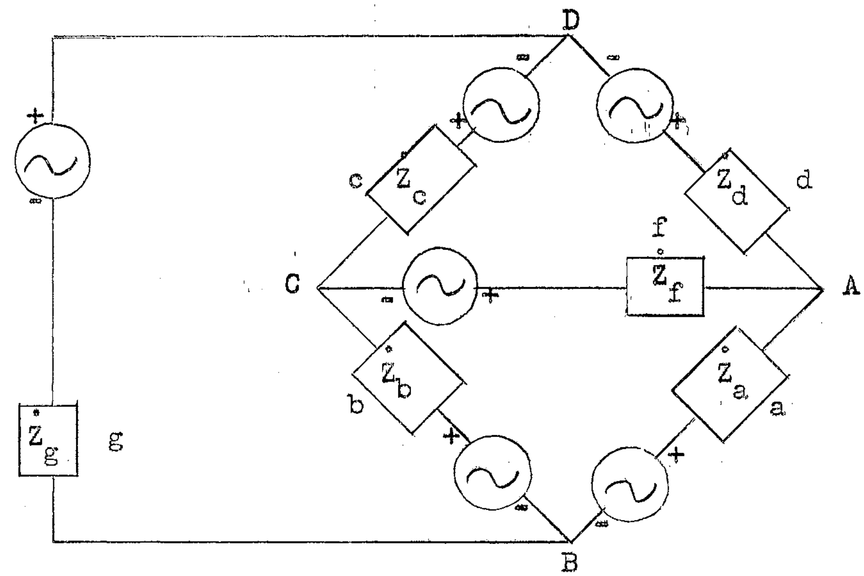

Figure 14
Given Network

figure 15. The reason for writing the current indices as superscripts, rather than subscripts as used for voltage and impedance, will be explained later in this chapter. The positive direction of each branch current will be assumed to be in the indicated positive direction of the generated voltage (figure 14) in that branch.

There are four nodes in figure 14 which are labeled as A, B, C, and D. Applying Kirchhoff's current law to each of these (N = 4) nodes gives the four equations

node A $\quad \dot{\underline{I}}^a + \dot{\underline{I}}^d + \dot{\underline{I}}^f = 0 \qquad (5.8)$

node B $\quad -\dot{\underline{I}}^a - \dot{\underline{I}}^b - \dot{\underline{I}}^g = 0 \qquad (5.9)$

node C $\quad \dot{\underline{I}}^b + \dot{\underline{I}}^c - \dot{\underline{I}}^f = 0 \qquad (5.10)$

node D $\quad -\dot{\underline{I}}^c - \dot{\underline{I}}^d + \dot{\underline{I}}^g = 0 \qquad (5.11)$

Since their sum is zero, these four equations are dependent. There are (see 5.1) only three (N - 1) independent node equations. There are six (B = 6) branch currents. Using (5.7), a total of

$$M = B - N + 1 = 6 - 4 + 1 = 3$$

more equations must be found in order to solve for the six branch currents. These three mesh equations may be obtained by applying Kirchhoff's second law to three independent meshes constructed on figure 14. A simple way to construct three independent meshes is to trace-out on figure 14 three closed contours that contain each circuit element of the network at least once. The three such meshes chosen for this example are shown in figure 15. The voltage equations around the three meshes are:

for mesh 1 $$\underline{\dot{E}}_a - \dot{Z}_a\underline{\dot{I}}^a + \dot{Z}_f\underline{\dot{I}}^f - \underline{\dot{E}}_f + \dot{Z}_b\underline{\dot{I}}^b - \underline{\dot{E}}_b = 0 \qquad (5.12)$$

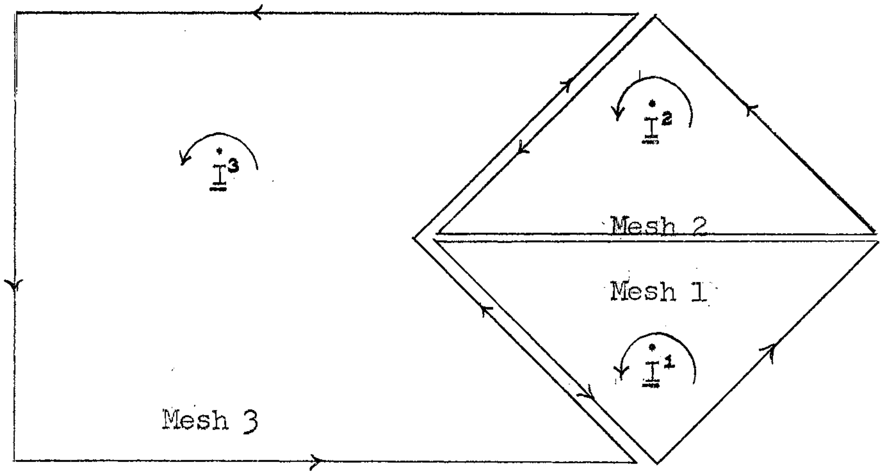

Figure 15
Mesh Contours

for mesh 2 $$\underline{\dot{E}}_f - \dot{Z}_f\underline{\dot{I}}_f + \dot{Z}_d\underline{\dot{I}}^d - \underline{\dot{E}}_d - \dot{Z}_c\underline{\dot{I}}^c + \underline{\dot{E}}_c = 0 \qquad (5.13)$$

and for mesh 3 $$\underline{\dot{E}}_b - \dot{Z}_b\underline{\dot{I}}^b + \dot{Z}_c\underline{\dot{I}}^c - \underline{\dot{E}}_c + \dot{Z}_g\underline{\dot{I}}^g - \underline{\dot{E}}_g = 0 \qquad (5.14)$$

The three independent equations out of the four equations (5.8) - (5.11) that will be used in this solution are equations (5.8), (5.9), and (5.10). A set of six equations (5.8, .9, .10, .12, .13, .14) has been obtained which may be solved for the six branch currents.

If the Maxwell cyclic current method had been used, a set of only three equations in three unknowns would have resulted. This method assumes that there are three mesh currents which circulate in the three meshes of figure 15. By correlating figure 15 with figure 14, the equations which express the branch currents in terms of the mesh currents can be written by inspection as

$$
\begin{array}{lllll}
\dot{I}^a & = & \dot{I}^1 & & \\
\dot{I}^b & = & -\dot{I}^1 & & +\dot{I}^3 \\
\dot{I}^c & = & & \dot{I}^2 & -\dot{I}^3 \\
\dot{I}^d & = & & -\dot{I}^2 & \\
\dot{I}^f & = & -\dot{I}^1 & +\dot{I}^2 & \\
\dot{I}^g & = & & & -\dot{I}^3
\end{array}
\qquad (5.15)
$$

Substituting (5.15) into equations (5.12) - (5.14) and simplifying gives

$$
\begin{array}{llll}
(\dot{Z}_a + \dot{Z}_f + \dot{Z}_b)\dot{I}^1 & -\dot{Z}_f\dot{I}^2 & -\dot{Z}_b\dot{I}^3 & = \dot{E}_a - \dot{E}_f - \dot{E}_b \\
-\dot{Z}_f\dot{I}^1 & +(\dot{Z}_f + \dot{Z}_d + \dot{Z}_c)\dot{I}^2 & -\dot{Z}_c\dot{I}^3 & = \dot{E}_b - \dot{E}_f - \dot{E}_d \\
-\dot{Z}_b\dot{I}^1 & -\dot{Z}_c\dot{I}^2 & +(\dot{Z}_b + \dot{Z}_c + \dot{Z}_g)\dot{I}^3 & = \dot{E}_b - \dot{E}_c - \dot{E}_g
\end{array}
\qquad (5.16)
$$

The system of three equations (5.17) with the three mesh currents as unknowns may now be readily solved. Once the mesh currents have been determined, the branch currents may be immediately found using (5.15).

The process of determing the unknown branch currents from the calculated mesh currents is so simple that nobody before Kron ever thought about the significance of such an obvious step. The branch currents, being one-dimensional complex vectors, can be expressed in matrix form as

$$[\dot{I}] = \begin{bmatrix} \dot{I}^a \\ \dot{I}^b \\ \dot{I}^c \\ \dot{I}^d \\ \dot{I}^f \\ \dot{I}^g \end{bmatrix} \quad (5.17)$$

Similarly, the three mesh currents may be expressed in matrix form as

$$[I'] = \begin{bmatrix} \dot{I}^1 \\ \dot{I}^2 \\ \dot{I}^3 \end{bmatrix} \quad (5.18)$$

The underscore notation for vectors has been dropped in (5.17) and (5.18) since no confusion can arise regarding the nature of the currents involved. In this treatise, the elements of a single column (or row) matrix will be considered as the projections of a vector in that particular reference frame. In (5.17), the vertical array of six complex quantities $\dot{I}^a \; . \; . \; . \; \dot{I}^g$ are the projections of a single complex vector $\dot{\underline{I}}$. Corresponding to (5.17) and (5.18) the branch voltage matrix and the mesh voltage matrix are

$$[\dot{E}] = \begin{bmatrix} \dot{E}_a \\ \dot{E}_b \\ \dot{E}_c \\ \dot{E}_d \\ \dot{E}_f \\ \dot{E}_g \end{bmatrix} \quad (5.19), \text{ and } \quad [\dot{E}'] = \begin{bmatrix} \dot{E}^1 \\ \dot{E}^2 \\ \dot{E}^3 \end{bmatrix} \quad (5.20)$$

Two equations derived in Chapter II

$$\dot{\underline{I}} \cdot \dot{\underline{I}} = \dot{I}^* \times \dot{I} = I^2 \quad (5.21)$$

and $$\dot{P}_v = \underline{\dot{I}} \cdot \underline{\dot{E}} = \dot{I}^* \times \dot{E} \qquad (5.22)$$

should be recalled at this point. From (5.21) and (5.22), it is seen that if $\underline{\dot{I}}$ and $\underline{\dot{E}}$ are to be expressed in matrix form then the multiplication of single-column matrices would be most useful if defined so that $[\dot{I}_t] \cdot [\dot{I}]$ would be a real magnitude. This product of 1-dimensional complex matrices (vectors) will be distinguished from the product of other matrices by placing a dot (rather than a cross) between the two column (or row) matrices.

Definition 7 (product of single-column matrices - vectors): The product of two single-column matrices is obtained by multiplying the conjugates of the elements of the first matrix by the corresponding elements of the second matrix and adding these products.

The coefficients of the mesh currents in (5.15) may be arranged in the matrix form

$$[C] = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad (5.23)$$

Using the rule for the multiplication of matrices given in Chapter IV, it can be readily verified that

$$[\dot{I}] = [C] \times [\dot{I}'] \qquad (5.24)$$

where $[\dot{I}]$ and $[\dot{I}']$ are given by (5.17) and (5.18). The transformation matrix (5.23) is of the greatest fundamental importance. As indicated by (5.24), the manner in which the closed contours are traced and the polarities assigned in the branches and meshes determine the reference frame (or coordinate system) in which these quantities are mathematically represented. A set of currents used in writing equations around one set of closed contours may be determined from the values of other currents calculated using another set of contours and a transformation matrix; this matrix will generally be different for each different set of meshes used in writing the equations.

The relations of the elements of the $[\dot{E}']$ matrix (5.20) to the branch voltages are contained in (5.16). Thus

$$\begin{aligned} \dot{E}_1 &= \dot{E}_a - \dot{E}_f - \dot{E}_b \\ \dot{E}_2 &= \dot{E}_c + \dot{E}_f - \dot{E}_d \\ \dot{E}_3 &= \dot{E}_b - \dot{E}_c - \dot{E}_g \end{aligned} \tag{5.25}$$

If the mesh impedances, the coefficients of the mesh currents in (5.16), are written in the matrix form

$$[\dot{Z}'] = \begin{bmatrix} \dot{Z}_a + \dot{Z}_f + \dot{Z}_b & -\dot{Z}_f & -\dot{Z}_b \\ -\dot{Z}_f & \dot{Z}_f + \dot{Z}_d + \dot{Z}_c & -\dot{Z}_c \\ -\dot{Z}_b & -\dot{Z}_c & \dot{Z}_b + \dot{Z}_c + \dot{Z}_g \end{bmatrix} \tag{5.26}$$

then the Ohm's law of the given network in matrix notation is

$$[\dot{E}'] = [\dot{Z}'] \times [\dot{I}'] \tag{5.27}$$

The elements of the principal diagonal of (5.26) are the self impedances of the three meshes, and the remaining elements are the mutual impedances among the three meshes. Since any two mesh currents always flow through a mutual impedance in opposite directions and the direction of flow of each mesh current around its own mesh is always considered positive, the self impedances will always be positive, and the mutual impedances will always be negative. A more compact notation for (5.26) is

$$[\dot{Z}'] = \begin{bmatrix} \dot{Z}_{11} & -\dot{Z}_{12} & -\dot{Z}_{13} \\ -\dot{Z}_{21} & \dot{Z}_{22} & -\dot{Z}_{23} \\ -\dot{Z}_{31} & -\dot{Z}_{32} & \dot{Z}_{33} \end{bmatrix} \qquad (5.26a)$$

where the repeated subscripts refer to the self impedances of the meshes, and two different subscripts indicate the two meshes to which the impedance is common.

Since no magnetic coupling was assumed to exist among the six branches, the branch impedance matrix is simply the diagonal matrix

$$[\dot{Z}] = \begin{bmatrix} \dot{Z}_a & 0 & 0 & 0 & 0 & 0 \\ 0 & \dot{Z}_b & 0 & 0 & 0 & 0 \\ 0 & 0 & \dot{Z}_c & 0 & 0 & 0 \\ 0 & 0 & 0 & \dot{Z}_d & 0 & 0 \\ 0 & 0 & 0 & 0 & \dot{Z}_f & 0 \\ 0 & 0 & 0 & 0 & 0 & \dot{Z}_g \end{bmatrix} \qquad (5.28)$$

If (5.23) is multiplied by (5.28), the result is

$$[\dot{Z}] \times [C] = \begin{bmatrix} \dot{Z}_a & 0 & 0 & 0 & 0 & 0 \\ 0 & \dot{Z}_b & 0 & 0 & 0 & 0 \\ 0 & 0 & \dot{Z}_c & 0 & 0 & 0 \\ 0 & 0 & 0 & \dot{Z}_d & 0 & 0 \\ 0 & 0 & 0 & 0 & \dot{Z}_f & 0 \\ 0 & 0 & 0 & 0 & 0 & \dot{Z}_g \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad (5.29)$$

$$[\dot{Z}] \times [C] = \begin{bmatrix} \dot{Z}_a & 0 & 0 \\ -\dot{Z}_b & 0 & \dot{Z}_b \\ 0 & \dot{Z}_c & -\dot{Z}_c \\ 0 & -\dot{Z}_d & 0 \\ -\dot{Z}_f & \dot{Z}_f & 0 \\ 0 & 0 & -\dot{Z}_g \end{bmatrix} \tag{5.29a}$$

The transpose of (5.23) is

$$[C_t] = \begin{bmatrix} 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 \end{bmatrix} \tag{5.30}$$

The matrix product of (5.29a) multiplied by (5.30) is

$$[C_t] \times [\dot{Z}] \times [C] = \begin{bmatrix} 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 \end{bmatrix} \times \begin{bmatrix} \dot{Z}_a & 0 & 0 \\ -\dot{Z}_b & 0 & \dot{Z}_b \\ 0 & \dot{Z}_c & -\dot{Z}_c \\ 0 & -\dot{Z}_d & 0 \\ -\dot{Z}_f & \dot{Z}_f & 0 \\ 0 & 0 & -\dot{Z}_g \end{bmatrix} \tag{5.31}$$

$$[C_t] \times [\dot{Z}] \times [C] = \begin{bmatrix} \dot{Z}_a + \dot{Z}_b + \dot{Z}_f & -\dot{Z}_f & -\dot{Z}_b \\ -\dot{Z}_f & \dot{Z}_c + \dot{Z}_d + \dot{Z}_f & -\dot{Z}_c \\ -\dot{Z}_b & -\dot{Z}_c & \dot{Z}_b + \dot{Z}_c + \dot{Z}_g \end{bmatrix} \tag{5.31a}$$

Equation (5.31a) is exactly the same as (5.26); that is

$$[\dot{Z}'] = [C_t] \times [\dot{Z}] \times [C] \tag{5.32}$$

Equation (5.32) is of equal importance to (5.24). The mesh impedance matrix is determined from the branch impedance matrix and the transformation matrix using this equation (5.32). Even with magnetic coupling of branches present, it is usually a simple matter to write down the branch impedance matrix $[\dot{Z}]$. In fairly simple circuits with magnetic coupling of branches, the task of writing down the mesh impedance matrix directly becomes highly complicated and strict attention must be paid to signs. However, even for more complicated networks, the transformation equation (5.32) always gives the correct magnitudes and signs of the various elements of the mesh impedance matrix $[\dot{Z}']$.

If the branch voltage matrix (5.19) is multiplied by the transpose of [C] (5.30), the result is

$$[C_t] \times [\dot{E}] = \begin{bmatrix} 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 \end{bmatrix} \times \begin{bmatrix} \dot{E}_a \\ \dot{E}_b \\ \dot{E}_c \\ \dot{E}_d \\ \dot{E}_f \\ \dot{E}_g \end{bmatrix} \qquad (5.33)$$

$$[C_t] \times [\dot{Z}] = \begin{bmatrix} \dot{E}_a - \dot{E}_b - \dot{E}_f \\ \dot{E}_c - \dot{E}_d + \dot{E}_f \\ \dot{E}_b - \dot{E}_c - \dot{E}_g \end{bmatrix} \qquad (5.33a)$$

The column matrix (5.33a) is exactly the same as the mesh voltage matrix (5.20), the elements of which are defined by (5.25). That is

$$[\dot{E}'] = [C_t] \times [\dot{E}] \qquad (5.34)$$

Equation (5.34) specifies the manner in which the mesh voltage matrix may be obtained from the branch voltage matrix and the transformation matrix.

Given the network in figure 14, the steps necessary to arrive at a solution for the branch currents may be summarized as follows:

1. From the given network, write down the two matrices $[\overset{\circ}{E}]$ (5.19) and $[\dot{Z}]$ (5.28).
2. Construct a suitable set of B-P (= 3) meshes, and construct the transformation matrix [C] (5.23).
3. By matrix multiplication, determine $[\dot{E}']$ (5.34) and $[\dot{Z}']$ (5.32).
4. Multiply both sides of (5.27) by the inverse of $[\dot{Z}']$ to give

$$[\dot{I}'] = [\dot{Z}'^{-1}] \times [\dot{E}']$$

5. Using (5.24), determine the branch currents from

$$[\dot{I}] = [C] \times [\dot{I}']$$

General Proof of the Transformation Equations for Voltage and Impedance

The network given in figure 14 will be used as a visual aid in formulating the general transformation equations of voltage and impedance. The proof will make use of two auxiliary networks; the first of these two networks Kron called the "primitive network". The primitive network consists of B meshes obtained by short-circuiting each branch of the given network. The primitive network of figure 14 is shown in figure 16.

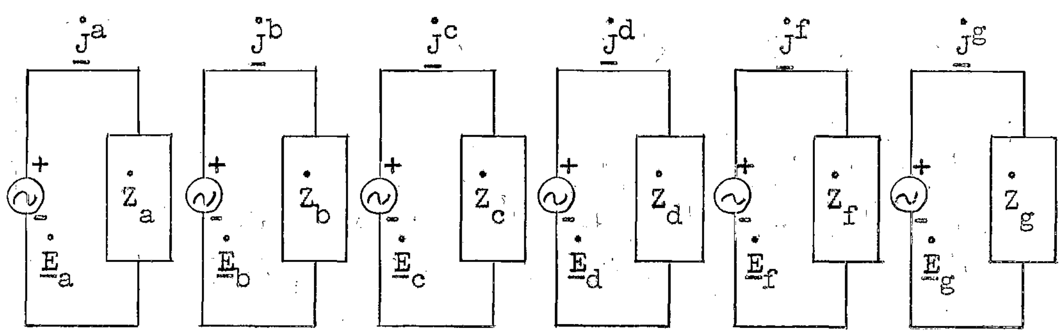

Figure 16
Primitive Network

It is evident that the branch voltage matrix and the branch impedance matrix are the same for the primitive network (figure 16) and for the given network (figure 14). Since the currents for the primitive network are obviously different from the branch currents of the given network, these currents will be represented by the matrix

$$[\overset{\circ}{J}] = \begin{bmatrix} \overset{\circ}{J}^a \\ \overset{\circ}{J}^b \\ \overset{\circ}{J}^c \\ \overset{\circ}{J}^d \\ \overset{\circ}{J}^f \\ \overset{\circ}{J}^g \end{bmatrix} \qquad (5.35)$$

The Ohm's law equation for the primitive network is

$$[\dot{E}] = [\overset{\circ}{Z}] \times [\dot{J}] \qquad (5.36)$$

where $[\overset{\circ}{E}]$ and $[\dot{Z}]$ are defined by (5.19) and (5.28), respectively.

The second auxiliary network needed in the proof Kron called the "intermediate network". The intermediate network is obtained from the given network by adding enough impedance-less connections so that each branch of the given network is short-circuited. The intermediate network of figure 14 is shown in figure 17.

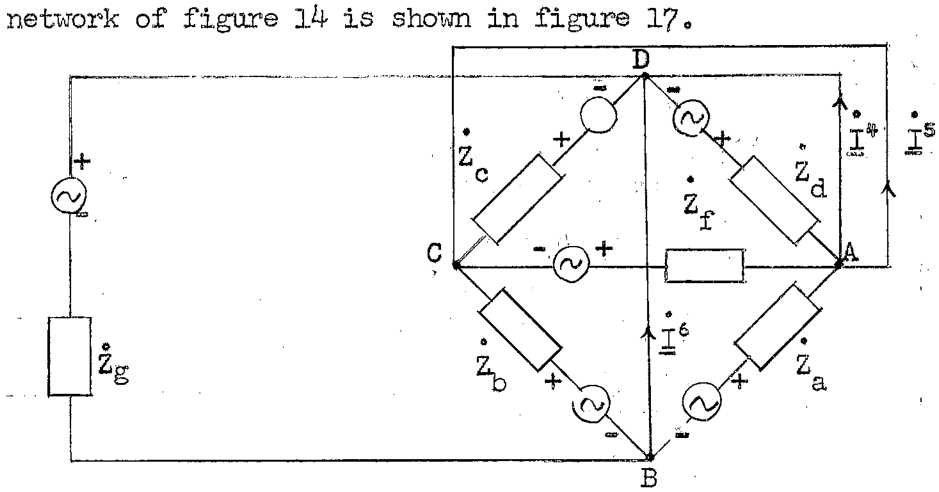

Figure 17
Intermediate Network

There are numerous ways of constructing the intermediate network pictured in figure 17. Six suitable meshes must now be selected on figure 17. The first three meshes selected will be the same three used in the previous solution of figure 14 as indicated in figure 15. Each of the last three meshes chosen must include at least one of the short-circuits, and all three of the short-circuits must be used. The last three meshes will be chosen as follows:

mesh 4 - branch d and short-circuit DA

mesh 5 - branch f and short-circuit CA

mesh 6 - branch g and short-circuit BD

Of course, many other choices were possible for meshes 4-6.

The branch current matrix $[(\dot{I})]$ and the mesh current matrix $[(\dot{I}')]$ of the intermediate network are

$$[(\dot{I})] = \begin{bmatrix} (\dot{I}^a) \\ (\dot{I}^b) \\ (\dot{I}^c) \\ (\dot{I}^d) \\ (\dot{I}^f) \\ (\dot{I}^g) \end{bmatrix} \quad (5.37), \text{ and} \qquad [(\dot{I}')] = \begin{bmatrix} (\dot{I}^1) \\ (\dot{I}^2) \\ (\dot{I}^3) \\ (\dot{I}^4) \\ (\dot{I}^5) \\ (\dot{I}^6) \end{bmatrix} \quad (5.38)$$

where the parentheses within the brackets refer to the currents of the intermediate network. From figure 17, the relation between the branch currents and mesh currents of the intermediate network is

$$\begin{bmatrix} (\dot{I}^a) \\ (\dot{I}^b) \\ (\dot{I}^c) \\ (\dot{I}^d) \\ (\dot{I}^f) \\ (\dot{I}^g) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \end{bmatrix} \times \begin{bmatrix} (\mathring{I}^1) \\ (\mathring{I}^2) \\ (\mathring{I}^3) \\ (\mathring{I}^4) \\ (\mathring{I}^5) \\ (\mathring{I}^6) \end{bmatrix} \tag{5.39}$$

or

$$[(\dot{I})] = [M] \times [(\mathring{I}')] \tag{5.39a}$$

The currents in the primitive network are numerically equal to the branch currents $[(\dot{I})]$ in the intermediate network. Equation (5.39a) is valid for all numerical values of the two sets of currents; this equation depends only on the topology of the network, i.e., the meshes chosen and the positive directions assigned in the branches and in the meshes. The values of the two sets of currents could be changed by changing the values of the branch impedances. In particular, if the impedance-less connections were open-circuited, i.e., the impedance made infinite, the two sets of currents would become the corresponding two sets of the given network (figure 14). Thus

$$[(\dot{I})] = \begin{bmatrix} (\dot{I}^a) \\ (\dot{I}^b) \\ (\dot{I}^c) \\ (\dot{I}^d) \\ (\dot{I}^f) \\ (\dot{I}^g) \end{bmatrix} \quad \text{becomes} \quad [\mathring{I}] = \begin{bmatrix} \mathring{I}^a \\ \mathring{I}^b \\ \mathring{I}^c \\ \mathring{I}^d \\ \mathring{I}^f \\ \mathring{I}^g \end{bmatrix} \tag{5.40}$$

$$[(\dot{I}')] = \begin{bmatrix} (\dot{I}^1) \\ (\dot{I}^2) \\ (\dot{I}^3) \\ (I^4) \\ (I^5) \\ (I^6) \end{bmatrix} \quad \text{becomes} \quad [\dot{I}'] = \begin{bmatrix} \dot{I}^1 \\ \dot{I}^2 \\ \dot{I}^3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (5.41)$$

since $\dot{I}^4$, $\dot{I}^5$, and $\dot{I}^6$ are now zero. However, regardless of the values of the two sets of currents, equation (5.39a) still holds, hence

$$[\dot{I}] = [M] \times [\dot{I}'] \qquad (5.42)$$

$$\begin{bmatrix} \dot{I}^a \\ \dot{I}^b \\ \dot{I}^c \\ \dot{I}^d \\ \dot{I}^f \\ \dot{I}^g \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \end{bmatrix} \times \begin{bmatrix} \dot{I}^1 \\ \dot{I}^2 \\ \dot{I}^3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (5.42a)$$

When multiplied together, the elements of the last three columns of the matrix [M] will be multiplied by the elements of the last three rows of $[\dot{I}']$, which are zero. Therefore the last three columns of the nonsingular matrix [M] may be set equal to zero giving the singular matrix $[M']$, since the values of the branch currents will not be affected.

$$\begin{bmatrix} \dot{I}^a \\ \dot{I}^b \\ \dot{I}^c \\ \dot{I}^d \\ \dot{I}^f \\ \dot{I}^g \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} \dot{I}^1 \\ \dot{I}^2 \\ \dot{I}^3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (5.43)$$

or $$[\dot{I}] = [M'] \times [\dot{I}']$$ (5.43a)

Equations (5.43) and (5.43a) are equivalent to

$$\begin{bmatrix} \dot{I}^a \\ \dot{I}^b \\ \dot{I}^c \\ \dot{I}^d \\ \dot{I}^f \\ \dot{I}^g \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \times \begin{bmatrix} \dot{I}^1 \\ \dot{I}^2 \\ \dot{I}^3 \end{bmatrix} \tag{5.44}$$

or $$[\dot{I}] = [C] \times [\dot{I}']$$ (5.44a)

Thus the rectangular (B x M) matrix [C] has been derived from the nonsingular square matrix [M].

The volt-amperes in the primitive network are given by

$$\dot{P}_v = [\dot{J}_t] \cdot [\dot{E}] \tag{5.45}$$

as given by definition 7 and (5.22). Each branch of the intermediate network (figure 17) is short-circuited the same as each element of the primitive network (figure 16). Therefore the a-c volt-amperes for the intermediate network are the same as for the primitive network. Hence

$$\dot{P}_v = [(\dot{I})_t] \cdot [\dot{E}] \tag{5.46}$$

Equating (5.45) and (5.46) and substituting for $[(\dot{I})]$ from (5.39a) gives

$$[\dot{J}_t] \cdot [\dot{E}] = ([M] \times [(\dot{I}')])_t \cdot [\dot{E}] \tag{5.47}$$

Denoting the mesh voltage matrix by $[(\dot{E}')]$, the volt-amperes for the intermediate network may be written in terms of the mesh voltage matrix $[(\dot{E}')]$ and the mesh current matrix $[(\dot{I}')]$ as

$$\dot{P}_v = [(\dot{I}')_t] \cdot [(\dot{E}')] \tag{5.48}$$

Equating (5.47) and (5.48) gives

$$[(\dot{I}')_t] \cdot [(\dot{E}')] = ([M] \times [(\dot{I}')])_t \cdot [\dot{E}] \tag{5.49}$$

or $$[(\dot{I}')_t] \cdot [(\dot{E}')] = [(\dot{I}')_t] \cdot [M_t] \times [\dot{E}] \tag{5.49a}$$

Since (5.49a) must hold for all values of $[(\dot{I}')]$, then

$$[(\dot{E}')] = [M_t] \times [\dot{E}] \tag{5.50}$$

The form of (5.50) is not affected by the topology of the network, hence removing the short-circuits (5.50) becomes

$$[\dot{E}'] = [M_t] \times [\dot{E}] \tag{5.51}$$

As shown in (5.43) and (5.44), when the impedance-less connections are removed the matrix [M] becomes [C]. Equation (5.51) can thus be written

$$[\dot{E}'] = [C_t] \times [\dot{E}] \tag{5.52}$$

Equation (5.52) is the same as (5.34); the voltage transformation equation has thus been proven.

For the primitive network, the equation

$$[\dot{E}] = [\dot{Z}] \times [\dot{I}] \tag{5.53}$$

becomes

$$[\dot{E}] = [\dot{Z}] \times [(\dot{I})] \tag{5.53}$$

when the branch currents of the intermediate network are used. Substituting $[(\dot{I})]$ from (5.39a) into (5.53) gives

$$[\dot{E}] = [\dot{Z}] \times [M] \times [(\dot{I}')] \tag{5.54}$$

Substituting (5.54) into (5.51) yields

$$[\dot{E}'] = [M_t] \times [\dot{Z}] \times [M] \times [(\dot{I}')] \tag{5.55}$$

The equations

$$[\dot{E}'] = [\dot{Z}'] \times [\dot{I}'] \tag{5.56}$$

and (5.55) must hold for all values of $[(\dot{I}')]$, hence

$$[\dot{Z}'] = [M_t] \times [\dot{Z}] \times [M] \tag{5.57}$$

Replacing [M] by [C], equation (5.57) becomes the same as (5.32), the second transformation relation that was to be proven.

Covariance and Contravariance of Voltage and Current Vectors

It was pointed out earlier that the transformation matrix [C] (5.23) is entirely determined by the topology of the given network (figure 14), *i.e.*, the location of the meshes and the positive directions assigned in the branches and meshes. Allowing the possibility of using the same branch more than once in a mesh, the number of different meshes that can be drawn is obviously infinite. Le Corbeiller has shown that for each of these (B x M) transformation matrices

$$[C_1],\ [C_2],\ [C_3],\ \ldots,\ [C_n],\ \ldots \tag{5.57}$$

there exists a corresponding (M x M) nonsingular matrix [K] satifying the equation[2]

$$[C_n] = [C_1] \times [K_n] \tag{5.58}$$

Thus there is a one-to-one correspondence between the two sets of matrices

$$[C_1],\ [C_2],\ [C_3],\ \ldots,\ [C_n],\ \ldots \tag{5.59}$$

and

$$[I],\ [K_2],\ [K_3],\ \ldots,\ [K_n],\ \ldots$$

The [C] matrices (5.57) are rectangular and consequently have no inverses. The nonsingular square [K] matrices (5.59) may be used in derivations requiring matrices which have inverses.

The concepts of covariance and contravariance of voltage and current are closely associated with similar concepts of the mechanical analogies of electrical quantities. Therefore an example from mechanics of a particle of unit mass moving in a plane under the influence of a force **F** will be used to introduce these concepts.

If a Cartesian coordinate system is constructed on the plane of motion of the particle, the projections of the force acting on the

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[2] *Ibid.*, pp. 52-56.

particle are

$$F_{x_1} = \frac{\partial \underline{V}(x_1, x_2)}{\partial x_1} \quad \text{and} \quad F_{x_2} = \frac{\partial \underline{V}(x_1, x_2)}{\partial x_2} \tag{5.60}$$

where $\underline{V}(x_1, x_2)$ is the velocity of the particle. A linear transformation from the old coordinates (x_1, x_2) to new ones (x_1', x_2')

$$\begin{aligned} x_1' &= ax_1 + bx_2 \\ x_2' &= cx_1 + dx_2 \end{aligned} \tag{5.61}$$

may be expressed in matrix form as

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} x \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{5.62}$$

or

$$[X'] = [T] \times [X] \tag{5.62a}$$

Expressed in terms of the new coordinates (x_1', x_2'), the projections of the velocity of the particle are

$$\begin{aligned} V_{x_1'} &= \frac{dx_1'}{dt} = a\frac{dx_1}{dt} + b\frac{dx_2}{dt} \\ V_{x_2'} &= \frac{dx_2'}{dt} = c\frac{dx_1}{dt} + d\frac{dx_2}{dt} \end{aligned} \tag{5.63}$$

Equation (5.63) expressed in matrix form is

$$\begin{bmatrix} V_{x_1'} \\ V_{x_2'} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} x \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} \tag{5.64}$$

or

$$[V'] = [T] \times [V] \tag{5.64a}$$

In terms of the new coordinates, the projections of the force are

$$\begin{aligned} F_{x_1'} &= \frac{\partial V(x_1', x_2')}{\partial x_1'} = \frac{\partial V}{\partial x_1}\frac{\partial x_1}{\partial x_1'} + \frac{\partial V}{\partial x_2}\frac{\partial x_2}{\partial x_1'} \\ F_{x_2'} &= \frac{\partial V(x_1', x_2')}{\partial x_2'} = \frac{\partial V}{\partial x_1}\frac{\partial x_1}{\partial x_2'} + \frac{\partial V}{\partial x_2}\frac{\partial x_2}{\partial x_2'} \end{aligned} \tag{5.65}$$

Solving for x_1 and x_2 in terms of x_1' and x_2' gives

$$x_1 = \frac{d}{ad - bc} x_1' - \frac{b}{ad - bc} x_2' \tag{5.66}$$

$$x_2 = \frac{-c}{ad - bc} x_1' + \frac{a}{ad - bc} x_2' \tag{5.66}$$

or $$[X] = [T^{-1}] \times [X'] \tag{5.66a}$$

Substituting the appropriate derivatives of (5.66) into (5.65) gives

$$F_{x_1'} = \frac{d}{ad - bc} F_{x_1} + \frac{-c}{ad - bc} F_{x_2}$$

$$F_{x_2'} = \frac{-b}{ad - bc} F_{x_1} + \frac{a}{ad - bc} F_{x_2} \tag{5.67}$$

or $$[F'] = [T_t^{-1}] \times [F] \tag{5.67}$$

Thus when the coordinates are transformed by the matrix [T], the velocity of the particle is transformed by [T] also, but the force acting on the particle is transformed by the transpose of the inverse of [T], $[T_t^{-1}]$. In general, quantities which are transformed by [T] when the coordinates are transformed by [T] are called "contra-variant" quantities; quantities that are transformed by $[T_t^{-1}]$ as a result of the coordinates being transformed by [T] are called "covariant" quantities.

In the solution of the network given in figure 14, it was shown (5.24) that the branch and mesh currents are related by the equation

$$[\mathring{I}] = [C_1] \times [\mathring{I}'] \tag{5.68}$$

If another set of meshes were chosen, the equation analogous to (5.68) would be

$$[\mathring{I}] = [C_2] \times [\mathring{I}''] \tag{5.69}$$

From (5.58) $$[C_2] = [C_1] \times [K_2] \tag{5.70}$$

Hence, equating (5.68) and (5.69)

$$[C_1] \times [\mathring{I}'] = [C_2] \times [\mathring{I}''] \tag{5.71}$$

or $$[\mathring{I}'] = [K_2] \times [\mathring{I}''] \tag{5.71a}$$

Considering voltages $$[\mathring{E}'] = [C_{1_t}] \times [\mathring{E}] \tag{5.72}$$

and $$[\mathring{E}''] = [C_{2_t}] \times [\mathring{E}] \tag{5.73}$$

but $$[C_{2_t}] = ([C_1] \times [K_2])_t = [K_{2_t}] \times [C_{1_t}] \qquad (5.70a)$$

Hence $$[\dot{E}''] = [K_{2_t}] \times [C_{1_t}] \times [\dot{E}] \qquad (5.74)$$

From (5.72) and (5.74), it follows that

$$[\dot{E}''] = [K_{2_t}] \times [\dot{E}'] \qquad (5.75)$$

Since $[K_2]$ is nonsingular, (5.75) may be solved for $[\dot{E}']$

$$[\dot{E}'] = [K_{2_t}^{-1}] \times [\dot{E}''] \qquad (5.76)$$

Equations (5.71a) and (5.76) show the basic difference in the transformation of currents and voltages. Equation (5.71a) and (5.76) are of the same type as (5.64a) and (5.67a). If one of the two sets of quantities, currents or voltages, are transformed by a certain nonsingular matrix, the other is transformed by the transpose of the inverse of that matrix. Actually, either set of quantities, currents or voltages, could be transformed first. The tendency of engineers to begin the analysis of a network by establishing the relations between the branch and mesh currents and the prevalent use of the force-voltage, mass-inductance and velocity-current system of mechanical analogies dictate the choice of currents as contravariant quantities and voltages as covariant quantities. The location of the indices will be used to indicate whether a vector is covariant or contravariant. A single superscript will be used to denote a contravariant vector, and a single subscript will be used to denote a covariant vector. This notation will be used throughout this dissertation. Later in this chapter, it will be shown that a covariant or contravariant vector is a special case of a more general entity, a tensor.

The Nature of A-C Network Quantities as Determined by Their Transformations

The manner in which sinusoidal voltages and currents could be represented by one-dimensional complex vectors was demonstrated in Chapter II. This demonstration was based upon the geometric concept of a rotating vector. Such an explanation of the representation of a-c voltages and currents by complex vectors was sufficient at that time and permitted the illustrations of the use of the notation to be presented immediately in Chapter III. However, to the truly inquiring mind, the establishment of an algebra upon a concept no more concrete than the intuitive idea of a rotating vector is far from satisfying. In this section, it will be shown that the fundamental reason for representing a-c voltages and currents as complex vectors is based upon the manner in which these quantities, currents and voltages, are transformed when the reference frame in which they are represented is subjected to a linear transformation.

Earlier in this chapter, it was shown that there existed a set of transformation matrices [C] (5.23), relating the branch and mesh currents, the elements of which were dependent upon the topology of the given network. For each choice of meshes, there exists a particular transformation matrix [C] and a corresponding mesh-current matrix $[\dot{I}']$.

Matrices were first introduced to reduce the large number of symbols or equations that had to be manipulated to a single symbol or equation. The first result of the adoption of the matrix notation was an obvious economy of space realized for a given number of manipulations. But now it is found that the entities, matrices, which were introduced to reduce the number of quantities to be manipulated have become great in number also. For example, there are many mesh-current matrices $[\dot{I}']$. This line of reasoning leads in a natural way to the following definition.

Definition 8 (tensor): A collection of n-way matrices forms a physical entity, a tensor of valence n, if with the aid of a group of transformation matrices [C] they can be transformed into one another.[3]

The process of changing from one set of meshes to another set of meshes is equivalent to changing coordinate systems. The collection of all the mesh-current matrices forms the mesh-current tensor; each of the mesh-current matrices is the projection of the mesh-current tensor in that particular coordinate system (reference frame). Since the mesh-current matrices are 1-way (column) matrices, the mesh-current tensor is a tensor of valence 1. A tensor of valence 1 is called a vector. Thus the mesh-current vector is the collection of all the mesh-current matrices $[\dot{I}']$, each particular mesh-current matrix being the projection of the mesh-current vector in that particular reference frame. It has been shown that the mesh-current matrices transform as contravariant quantities (5.71a). Hence the collection of the mesh-current matrices is a contravariant tensor (or vector) of valence 1.

For each of the mesh-current matrices resulting from choosing several different sets of meshes, there is a corresponding mesh-voltage matrix. The collection of all the 1-way mesh-voltage matrices is the mesh-voltage tensor (or vector) of valence 1. It has been

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[3]Gabriel Kron, <u>Tensor Analysis</u> (New York, 1942), p. 40.

shown that the mesh-voltage matrices transform as covariant quantities (5.76). Therefore the collection of all the mesh-voltage matrices constitutes a covariant tensor (or vector) of valence 1.

The transformation equation of the impedance $[\overset{\circ}{Z}]$ has been shown to be

$$[\overset{\circ}{Z}'] = [C_t'] \times [\overset{\circ}{Z}] \times [C] \qquad (5.32)$$

It should be noted that while the transformation equations of currents (5.24) and voltages (5.34) attract only one transformation matrix each, the transformation equation of impedance (5.32) attracts two transformation matrices. The term "valence" grew out of this "chemical" property of different kinds of tensors.

Definition 9 (valence): The valence of a given tensor is the number of transformation tensors required to transform the given tensor when the reference frame has been subjected to a linear transformation.

From the previous analysis and definitions 8 and 9, it is apparent that the collection of all the 2-way impedance matrices, corresponding to the various choices of meshes, constitutes a tensor of valence 2. In index notation, (5.32) is

$$\overset{\circ}{Z}_{m'n'} = \overset{\circ}{Z}_{mn} C^{m}_{m'} C^{n}_{n'} \qquad (5.77)$$

The equations of transformation for currents (5.24) and voltages (5.34) may be written in index notation as

$$\dot{\underline{I}}^{m} = C^{m}_{m'} \dot{\underline{I}}^{m'} \qquad (5.78)$$

and

$$\overset{\circ}{\underline{E}}_{m'} = C^{m}_{m'} \overset{\circ}{\underline{E}}_{m} \qquad (5.79)$$

The positions of the indices in (5.78) and (5.79) are used to indicate the contravariant nature of the currents and the covariant nature of the voltages. The double indices of $\overset{\circ}{Z}$ in (5.77) are both subscripts.

Equation (5.77) is a covariant transformation of $\dot{Z}_{mn}$ into $\dot{Z}_{m'n'}$. Thus the collection of all the impedance matrices $[\dot{Z}]$ is a tensor of covariant valence 2.

The law of transformation of the transformation matrix [C] (5.23) will now be derived. Given the equation

$$[\dot{I}] = [C] \times [\dot{I}'] \qquad (5.80)$$

let $[\dot{I}]$ and $[\dot{I}']$ be changed by

$$[\dot{I}] = [C_1] \times [\dot{I}''] \qquad (5.81)$$

and

$$[\dot{I}'] = [C_2] \times [\dot{I}'''] \qquad (5.82)$$

Substituting (5.81) and (5.82) in (5.80) gives

$$[C_1] \times [\dot{I}''] = [C] \times [C_2] \times [\dot{I}'''] \qquad (5.83)$$

$$[\dot{I}''] = [C_1^{-1}] \times [C] \times [C_2] \times [\dot{I}'''] \qquad (5.84)$$

If

$$[\dot{I}''] = [C'] \times [\dot{I}'''] \qquad (5.85)$$

then

$$[C'] = [C_1^{-1}] \times [C] \times [C_2] \qquad (5.86)$$

or in index notation

$$C^{n''}_{n'''} = C^{n}_{n'} \; C^{n''}_{n} \; C^{n'}_{n'''} \qquad (5.86a)$$

It is evident from (5.86) and (5.86a) that the collection of all the transformation matrices [C] is a tensor of valence two, but the positions of the indices indicate that the transformation tensor is different from the impedance tensor. The two indices on the transformation tensor $C^{n}_{n'}$ refer to two different reference frames, whereas the two indices on the impedance tensor Z_{mn} both refer to the same reference frame. Since the transformation tensor has one upper and one lower index, it is called a mixed tensor of covariant valence 1 and contravariant valence 1.

The equation for the total volt-amperes

$$[\dot{P}_v] = [\dot{I}_t] \cdot [\dot{E}] \qquad (5.87)$$

gives the same results regardless of what set of voltages and corresponding set of currents are used. The indicated matrix multiplication in (5.87) gives a single quantity, a scalar, hence $\overset{\circ}{P}_v$ is an invariant. The essence of (5.87) is that the mere representation of currents and voltages in different reference frames does not change the total energy input to the network in which these voltages and currents exist. This result is certainly logical, and its truth is self-evident. In the Summary of Chapter II, it was observed that the representation of a-c voltages and currents by complex vectors lost part of the geometric description gained by the use of two-dimensional real vectors. However, the invariant nature of volt-amperes gained by using tensor (complex vector) notation is of far more significance than the slight loss of the descriptiveness of two-dimensional real vectors.

The greatest single advantage of representing a-c network quantities as tensors has not yet been mentioned. A tensor equation is merely an expression of the natural behavior of several associated physical quantities. It can not be over-emphasized that the natural behavior of a physical quantity does not change regardless of the system (network) that it is placed in or the mathematical reference frame in which it might be represented. Thus a tensor equation that has been established for one system (network) is the same for all analogous systems (networks). The form of a tensor equation is invariant. Of course, the components of the tensors, the n-way matrices, are different for different systems (networks). The two major differences in n-way matrices and tensors are (1) a matrix equation is valid for only one reference frame, whereas a tensor equation is valid for all reference frames of all physically analogous systems, and (2) a tensor always has associated with it a definite law of transformation but an n-way matrix does not. For stationary networks, for example, the tensor equations

$$\dot{E} = \dot{Z} \times \dot{I}$$

$$\dot{P}_v = \dot{I}_t \cdot \dot{E}$$

are invariant. Since these are tensor equations, the brackets, used to identify matrices, have been dropped. Furthermore, the transformation equations of the tensors

$$\dot{I} = C \times \dot{I}'$$

$$\dot{E}' = C_t \times \dot{E}$$

$$\dot{Z}' = C_t \times \dot{Z} \times C$$

and

$$C' = C_1^{-1} \times C \times C_2$$

are invariant also. That is, each tensor has associated with it a permanent law of transformation.

Summary of Chapter V

1. (a) In a conductively connected network having N nodes, the number of independent current equations (P) that may be written is

$$P = N - 1.$$

(b) If the network in (a) has B branches, the number of mesh-voltage equations (M) required for a solution of the network is

$$M = B - P.$$

2. Stationary networks may be readily solved using matrices. The necessary steps may be summarized as follows:

(a) From the given network, write down the two matrices $[\dot{E}]$ (5.19) and $[\dot{Z}]$ (5.28).

(b) Construct a suitable (independent) set of B - P meshes, and determine the matrix [C] (5.23) which expresses the relations between the branch and mesh currents.

(c) Using the transformation equations (5.34) and (5.32), determine $[\dot{E}']$ and $[\dot{Z}']$, the mesh voltage and impedance matrices, from the branch voltage and impedance matrices $[\dot{E}]$ and $[\dot{Z}]$.

(d) Having determined $[\dot{Z}']$ and $[\dot{E}']$, solve for $[\dot{I}']$ in the equation $$[\dot{I}'] = [\dot{Z}'^{-1}] \times [\dot{E}']$$

(e) Using the transformation matrix [C], determine the desired branch currents from the calculated mesh currents using the equation $$[\dot{I}] = [C] \times [\dot{I}']$$

3. Because of the manner in which they transform, if either voltages or currents is covariant the other is necessarily contravariant. Following the force-voltage, velocity-current system of mechanical analogies, currents are contravariant and voltages are covariant. A contravariant vector is denoted by a single superscript, and a covariant vector is denoted by a single subscript.

4. In advanced analysis, the basic nature of a quantity is determined by the manner in which the quantity behaves when the reference frame in which it is represented is subjected to a linear transformation. Such intuitive and primitive concepts as "rotating vectors" and "a vector having magnitude, direction and sense" must be discarded.

5. As determined by their laws of transformation, the various electrical quantities have been found to be the following:

(a) current - a tensor of contravariant valence 1 - represented in each reference frame by a 1-way matrix - the elements of the 1-way matrices for different reference frames are different.

(b) voltage - a tensor of covariant valence 1 - represented in each reference frame by a 1-way matrix - the elements of the 1-way matrices for different reference frames are different.

(c) impedance - a tensor of covariant valence 2 - represented in each reference frame by a 2-way matrix - the elements of the 2-way matrices for different reference frames are different.

(d) volt-amperes - a tensor of valence zero, or a scalar - represented in each reference frame by a single complex number - the complex number is the same for all reference frames of a given network.

6. A tensor equation may be written in terms of its components (matrices) for any particular reference frame of a given system. A tensor equation established for a given system is also valid for all analogous systems.

7. This chapter is the work of Kron and Le Corbeiller adapted to the complex vector notation. Of course, the use of complex vectors necessitated the modification of several formulas, i.e., (5.87).

CHAPTER VI

AN APPLICATION OF THE TENSOR METHOD TO THREE-PHASE CIRCUITS

Basic Considerations

In Chapter V, the basic equations of stationary electric networks were derived and expressed as tensors. The elegance and simplicity of the forms of these basic formulas are indeed impressive. However, because of the necessary general character of these equations, the manipulations required for the numerical solution of a given problem may not be at all apparent to one having no previous training in tensor analysis. Consequently, as an illustration, the tensor equations of Chapter V will be used to solve a familiar three-phase network. The first two sections of this chapter are the author's own original work; the final two sections dealing with the combinations of series and parallel three-phase loads follow the original work of Sah. It is recognized, of course, that there are other, perhaps simpler, ways of solving the example which will be solved by the tensor method in this chapter, but it should be borne in mind that this is a very elementary illustration of a powerful tool which may be used to solve more complicated problems.

The stationary four-terminal network shown in figure 18 will now be considered. Terminals 1, 2, and 3 are assumed to be connected to the three lines of a three-phase grounded system, and terminal 4 is connected to the system ground. The assumed positive directions of the three line currents are indicated in figure 18. Denoting the

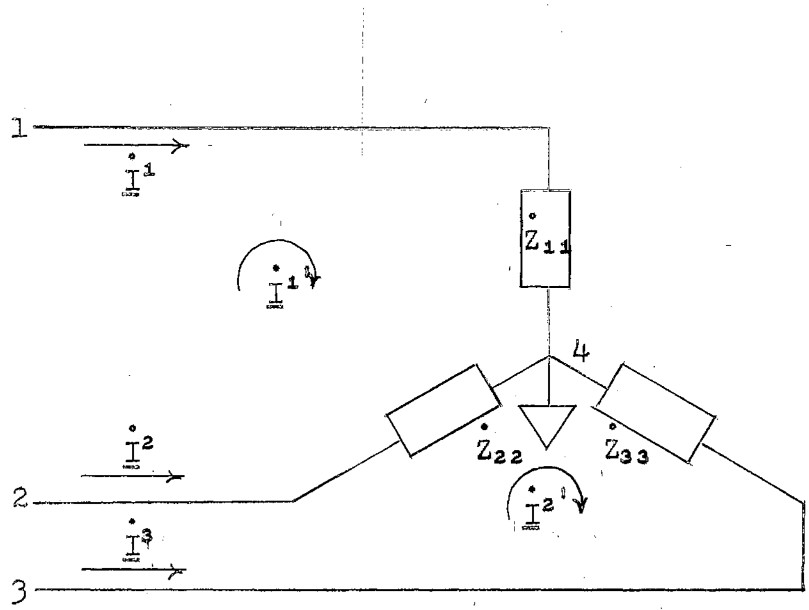

Figure 18
Three-Phase Grounded Load

three line-to-ground voltages by the symbols $\dot{E}_1$, $\dot{E}_2$, and $\dot{E}_3$, the Kirchhoff voltage equations for the three phases are

$$\begin{aligned} \dot{E}_1 &= \dot{Z}_{11}\dot{I}^1 + \dot{Z}_{12}\dot{I}^2 + \dot{Z}_{13}\dot{I}^3 \\ \dot{E}_2 &= \dot{Z}_{21}\dot{I}^1 + \dot{Z}_{22}\dot{I}^2 + \dot{Z}_{23}\dot{I}^3 \\ \dot{E}_3 &= \dot{Z}_{31}\dot{I}^1 + \dot{Z}_{32}\dot{I}^2 + \dot{Z}_{33}\dot{I}^3 \end{aligned} \qquad (6.1)$$

where: the $\dot{E}$'s and $\dot{I}$'s are one-dimensional complex vectors,

the $\dot{Z}$'s with repeated subscript are the series self impedances per phase, and

the $\dot{Z}$'s with two different subscripts are mutual impedances between the phases indicated by the two subscripts.

In index notation, (6.1) would be simply written as

$$\dot{E}_m = \dot{Z}_{mn}\dot{I}^n \quad (m, n = 1, 2, 3) \qquad (6.1a)$$

The significance of (6.1a) is that the second subscript of $\dot{Z}$ (n) acts as a dummy index with the superscript (n) on $\dot{I}$, thereby leaving the first subscript of $\dot{Z}$ (m) as the identifying index of $\dot{E}$. In terms of unit vectors, the vanishing of the n index on $\dot{Z}$ and $\dot{I}$ signifies that a unit vector associated with $\dot{Z}$ is being dot (scalar) multiplied by a similar unit vector associated with $\dot{\underline{I}}$, thus losing their vector identity. This leaves the unit vector associated with $\dot{Z}$ corresponding to the subscript m as the identifying unit vector for the corresponding component of $\dot{\underline{E}}$. Using the customary unit real vectors $\underline{i}$, $\underline{j}$, and $\underline{k}$, (6.1) can be written

$$\begin{aligned} \dot{E}_1\underline{i} &= \dot{Z}_{11}\underline{ii} \times \dot{I}^1\underline{i} + \dot{Z}_{12}\underline{ij} \times \dot{I}^2\underline{j} + \dot{Z}_{13}\underline{ik} \times \dot{I}^3\underline{k} \\ \dot{E}_2\underline{j} &= \dot{Z}_{21}\underline{ji} \times \dot{I}^1\underline{i} + \dot{Z}_{22}\underline{jj} \times \dot{I}^2\underline{j} + \dot{Z}_{23}\underline{jk} \times \dot{I}^3\underline{k} \\ \dot{E}_3\underline{k} &= \dot{Z}_{31}\underline{ki} \times \dot{I}^1\underline{i} + \dot{Z}_{32}\underline{kj} \times \dot{I}^2\underline{j} + \dot{Z}_{33}\underline{kk} \times \dot{I}^3\underline{k} \end{aligned} \qquad (6.2)$$

where

$$\begin{aligned} \dot{\underline{E}}_1 &= \dot{E}_1\underline{i} \qquad & \dot{\underline{I}}^1 &= \dot{I}^1\underline{i} \\ \dot{\underline{E}}_2 &= \dot{E}_2\underline{j} \qquad & \dot{\underline{I}}^2 &= \dot{I}^2\underline{j} \\ \dot{\underline{E}}_3 &= \dot{E}_3\underline{k} \qquad & \dot{\underline{I}}^3 &= \dot{I}^3\underline{k} \end{aligned} \qquad (6.3)$$

Expanding (6.2) gives

$$\begin{aligned} \dot{E}_1\underline{i} &= (\dot{Z}_{11}\dot{I}^1 + \dot{Z}_{12}\dot{I}^2 + \dot{Z}_{13}\dot{I}^3)\underline{i} \\ \dot{E}_2\underline{j} &= (\dot{Z}_{21}\dot{I}^1 + \dot{Z}_{22}\dot{I}^2 + \dot{Z}_{23}\dot{I}^3)\underline{j} \\ \dot{E}_3\underline{k} &= (\dot{Z}_{31}\dot{I}^1 + \dot{Z}_{32}\dot{I}^2 + \dot{Z}_{33}\dot{I}^3)\underline{k} \end{aligned} \qquad (6.4)$$

The three voltage vectors and the three current vectors of (6.3) may be thought of as the components along the three Cartesian coordinate axes of a space voltage vector and a space current vector. That is

$$\dot{\underline{E}} = \dot{E}_1\underline{i} + \dot{E}_2\underline{j} + \dot{E}_3\underline{k} \qquad (6.5)$$

and

$$\dot{\underline{I}} = \dot{I}^1\underline{i} + \dot{I}^2\underline{j} + \dot{I}^3\underline{k} \qquad (6.6)$$

The equations (6.2) are more neatly expressed in matrix notation as

$$\begin{bmatrix} \dot{E}_1 \\ \dot{E}_2 \\ \dot{E}_3 \end{bmatrix} = \begin{bmatrix} \dot{Z}_{11} & \dot{Z}_{12} & \dot{Z}_{13} \\ \dot{Z}_{21} & \dot{Z}_{22} & \dot{Z}_{23} \\ \dot{Z}_{31} & \dot{Z}_{32} & \dot{Z}_{33} \end{bmatrix} \times \begin{bmatrix} \dot{I}^1 \\ \dot{I}^2 \\ \dot{I}^3 \end{bmatrix} \qquad (6.7)$$

Each phase of the three-phase system is thus represented by a one-dimensional complex vector space. The collection of the three one-dimensional complex vector spaces forms a three-dimensional complex vector space.

Phase 1 is represented by the 1-dimensional space $\underline{i}$

Phase 2 is represented by the 1-dimensional space $\underline{j}$

Phase 3 is represented by the 1-dimensional space $\underline{k}$

Solution of Three-Phase Network

The three-phase network given in figure 18 will now be solved using the method presented in Chapter V. Reference will be made to each general equation of Chapter V as it is used in the solution. It will be assumed that a wye-connected generator with a grounded neutral supplies the three-phase voltages applied to terminals 1, 2, and 3 of figure 18.

The given network has two nodes, hence there is (5.1)

$$P = 2 - 1 = 1$$

one independent current equation. Since the given network has three branches, the number of additional mesh voltage equations required for a solution (5.6) is $M = 3 - 1 = 2$

The two meshes that will be chosen for this example are (a) the mesh composed of phases 1 and 2 of source and load, and (b) the mesh composed of phases 2 and 3 of source and load. The two mesh currents will be assumed to circulate in a clockwise direction around their respective meshes. The equations relating the two mesh currents, $\dot{I}^{1'}$ and $\dot{I}^{2'}$, and the three branch currents are

$$\dot{I}^1 = \dot{I}^{1'}$$

$$\dot{I}^2 = -\dot{I}^{1'} + \dot{I}^{2'}$$

$$\dot{I}^3 = \qquad - \dot{I}^{2'}$$

The transformation matrix [C] (5.23) is

$$[C] = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$$

and $[C_t]$ (5.30) is

$$[C_t] = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

The branch voltage matrix (5.19) is

$$[\dot{E}] = \begin{bmatrix} \dot{E}_1 \\ \dot{E}_2 \\ \dot{E}_3 \end{bmatrix}$$

and the branch impedance matrix (5.28) is

$$[\dot{Z}] = \begin{bmatrix} \dot{Z}_{11} & \dot{Z}_{12} & \dot{Z}_{13} \\ \dot{Z}_{21} & \dot{Z}_{22} & \dot{Z}_{23} \\ \dot{Z}_{31} & \dot{Z}_{32} & \dot{Z}_{33} \end{bmatrix}$$

Using the transformation equation (5.34), the mesh voltage matrix is

$$[\dot{E}'] = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \times \begin{bmatrix} \dot{E}_1 \\ \dot{E}_2 \\ \dot{E}_3 \end{bmatrix}$$

or

$$[\dot{E}'] = \begin{bmatrix} \dot{E}_1 - \dot{E}_2 \\ \dot{E}_2 - \dot{E}_3 \end{bmatrix}$$

Applying transformation equation (5.32), the mesh impedance matrix is

$$[\dot{Z}'] = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \times \begin{bmatrix} \dot{Z}_{11} & \dot{Z}_{12} & \dot{Z}_{13} \\ \dot{Z}_{21} & \dot{Z}_{22} & \dot{Z}_{23} \\ \dot{Z}_{31} & \dot{Z}_{32} & \dot{Z}_{33} \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$[\dot{Z}'] = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \times \begin{bmatrix} \dot{Z}_{11} - \dot{Z}_{12} & \dot{Z}_{12} - \dot{Z}_{13} \\ \dot{Z}_{21} - \dot{Z}_{22} & \dot{Z}_{22} - \dot{Z}_{23} \\ \dot{Z}_{31} - \dot{Z}_{32} & \dot{Z}_{32} - \dot{Z}_{33} \end{bmatrix}$$

$$[\dot{Z}'] = \begin{bmatrix} \dot{Z}_{11} - \dot{Z}_{12} - \dot{Z}_{21} + \dot{Z}_{22} & \dot{Z}_{12} - \dot{Z}_{13} - \dot{Z}_{22} + \dot{Z}_{23} \\ \dot{Z}_{21} - \dot{Z}_{22} - \dot{Z}_{31} + \dot{Z}_{32} & \dot{Z}_{22} - \dot{Z}_{23} - \dot{Z}_{32} + \dot{Z}_{33} \end{bmatrix}$$

The equation for the mesh currents is

$$[\dot{I}'] = [\dot{Z}'^{-1}] \times [\dot{E}']$$

The matrix the elements of which are cofactors of the elements of $[\dot{Z}']$ is

$$\text{Cof. matrix} = \begin{bmatrix} \dot{Z}_{22} - \dot{Z}_{23} - \dot{Z}_{32} + \dot{Z}_{33} & -(\dot{Z}_{21} - \dot{Z}_{22} - \dot{Z}_{31} + \dot{Z}_{32}) \\ -(\dot{Z}_{12} - \dot{Z}_{13} - \dot{Z}_{22} + \dot{Z}_{23}) & \dot{Z}_{11} - \dot{Z}_{12} - \dot{Z}_{21} + \dot{Z}_{22} \end{bmatrix}$$

The transpose of the cofactor matrix is

$$\begin{bmatrix} \dot{Z}_{22} - \dot{Z}_{23} - \dot{Z}_{32} + \dot{Z}_{33} & -(\dot{Z}_{12} - \dot{Z}_{13} - \dot{Z}_{22} + \dot{Z}_{23}) \\ -(\dot{Z}_{21} - \dot{Z}_{22} - \dot{Z}_{31} + \dot{Z}_{32}) & \dot{Z}_{11} - \dot{Z}_{12} - \dot{Z}_{21} + \dot{Z}_{22} \end{bmatrix}$$

Hence

$$[\dot{Z}'^{-1}] = \frac{1}{D}\begin{bmatrix} \dot{Z}_{22} - \dot{Z}_{23} - \dot{Z}_{32} + \dot{Z}_{33} & -(\dot{Z}_{12} - \dot{Z}_{13} - \dot{Z}_{22} + \dot{Z}_{23}) \\ -(\dot{Z}_{21} - \dot{Z}_{22} - \dot{Z}_{31} + \dot{Z}_{32}) & \dot{Z}_{11} - \dot{Z}_{12} - \dot{Z}_{21} + \dot{Z}_{22} \end{bmatrix}$$

where

$$D = (\dot{Z}_{11} - \dot{Z}_{12} - \dot{Z}_{21} + \dot{Z}_{22})(\dot{Z}_{22} - \dot{Z}_{23} - \dot{Z}_{32} + \dot{Z}_{33}) - (\dot{Z}_{12} - \dot{Z}_{13} - \dot{Z}_{22} + \dot{Z}_{23})(\dot{Z}_{21} - \dot{Z}_{22} - \dot{Z}_{31} + \dot{Z}_{32})$$

the determinant of the matrix $[\dot{Z}']$. If the given stationary network (figure 18) were also bilateral, then

$$\dot{Z}_{ij} = \dot{Z}_{ji}$$

and the expression for $[\dot{Z}'^{-1}]$ would be greatly simplified. Performing the indicated matrix multiplication in the equation

$$[\dot{I}'] = [\dot{Z}'^{-1}] \times [\dot{E}']$$

$$\begin{bmatrix}\dot{I}^{1'}\\ \dot{I}^{2'}\end{bmatrix} = \frac{1}{D}\begin{bmatrix}(\dot{Z}_{22}-\dot{Z}_{23}-\dot{Z}_{32}+\dot{Z}_{33})(\dot{E}_1-\dot{E}_2) - (\dot{Z}_{12}-\dot{Z}_{13}-\dot{Z}_{22}+\dot{Z}_{23})(\dot{E}_2-\dot{E}_3)\\ (\dot{Z}_{11}-\dot{Z}_{12}-\dot{Z}_{21}+\dot{Z}_{22})(\dot{E}_2-\dot{E}_3) - (\dot{Z}_{21}-\dot{Z}_{22}-\dot{Z}_{31}+\dot{Z}_{32})(\dot{E}_1-\dot{E}_2)\end{bmatrix}$$

The branch currents as determined by (5.24) are

$$\begin{bmatrix}\dot{I}^{1}\\ \dot{I}^{2}\\ \dot{I}^{3}\end{bmatrix} = \begin{bmatrix}1 & 0\\ -1 & 1\\ 0 & -1\end{bmatrix} \times \begin{bmatrix}\dot{I}^{1'}\\ \dot{I}^{2'}\end{bmatrix}$$

$$\begin{bmatrix}\dot{I}^{1}\\ \dot{I}^{2}\\ \dot{I}^{3}\end{bmatrix} = \frac{1}{D}\begin{bmatrix}(\dot{Z}_{22}-\dot{Z}_{23}-\dot{Z}_{32}+\dot{Z}_{33})(\dot{E}_1-\dot{E}_2) - (\dot{Z}_{12}-\dot{Z}_{13}-\dot{Z}_{22}+\dot{Z}_{23})(\dot{E}_2-\dot{E}_3)\\ \\ (\dot{Z}_{12}-\dot{Z}_{13}-\dot{Z}_{22}+\dot{Z}_{23})(\dot{E}_2-\dot{E}_3) - (\dot{Z}_{22}-\dot{Z}_{23}-\dot{Z}_{32}+\dot{Z}_{33})(\dot{E}_1-\dot{E}_2) -\\ (\dot{Z}_{21}-\dot{Z}_{22}-\dot{Z}_{31}+\dot{Z}_{32})(\dot{E}_2-\dot{E}_3) + (\dot{Z}_{11}-\dot{Z}_{12}-\dot{Z}_{21}+\dot{Z}_{22})(\dot{E}_1-\dot{E}_2)\\ \\ (\dot{Z}_{21}-\dot{Z}_{22}-\dot{Z}_{31}+\dot{Z}_{32})(\dot{E}_1-\dot{E}_2) - (\dot{Z}_{11}-\dot{Z}_{12}-\dot{Z}_{21}+\dot{Z}_{22})(\dot{E}_2-\dot{E}_3)\end{bmatrix}$$

From the definition of the equality of two matrices, the coefficients of the three branch currents are equal to the corresponding three rows of the matrix on the right of the above equation. That is

$$\dot{I}^{1} = (\dot{Z}_{22}-\dot{Z}_{23}-\dot{Z}_{32}+\dot{Z}_{33})(\dot{E}_1-\dot{E}_2) - (\dot{Z}_{12}-\dot{Z}_{13}-\dot{Z}_{22}+\dot{Z}_{23})(\dot{E}_2-\dot{E}_3)$$

$$\dot{I}^{2} = (\dot{Z}_{12}-\dot{Z}_{13}-\dot{Z}_{22}+\dot{Z}_{23})(\dot{E}_2-\dot{E}_3) - (\dot{Z}_{22}-\dot{Z}_{23}-\dot{Z}_{32}+\dot{Z}_{33})(\dot{E}_1-\dot{E}_2) - (\dot{Z}_{21}-\dot{Z}_{22}-\dot{Z}_{31}+\dot{Z}_{32})(\dot{E}_2-\dot{E}_3) + (\dot{Z}_{11}-\dot{Z}_{12}-\dot{Z}_{21}+\dot{Z}_{22})(\dot{E}_1-\dot{E}_2)$$

and

$$\dot{I}^{3} = (\dot{Z}_{21}-\dot{Z}_{22}-\dot{Z}_{31}+\dot{Z}_{32})(\dot{E}_1-\dot{E}_2) - (\dot{Z}_{11}-\dot{Z}_{12}-\dot{Z}_{21}+\dot{Z}_{22})(\dot{E}_2-\dot{E}_3)$$

Even though the three equations above for the three branch currents (line currents) are rather involved functions of the self impedances, the mutual impedances, and the phase voltages, the procedure used in deriving these equations is quite simple. Of course, the three-phase circuit of figure 18, having no neutral-to-ground impedance, is a simple two-mesh problem. Exactly the same procedure would have been followed regardless of the number of meshes involved, or whether the network were unilateral or bilateral, symmetric or antisymmetric.

The total volt-amperes input is

$$\dot{P}_v = [\dot{I}_t] \cdot [\dot{E}] = [\dot{I}'_t] \cdot [\dot{E}']$$

or in vector notation

$$\dot{P}_v = \dot{\underline{I}} \cdot \dot{\underline{E}}$$

Substituting $\dot{\underline{I}}$ and $\dot{\underline{E}}$ from (6.5) and (6.6)

$$\dot{P}_v = (\dot{I}^1 \underline{i} + \dot{I}^2 \underline{j} + \dot{I}^3 \underline{k}) \cdot (\dot{E}_1 \underline{i} + \dot{E}_2 \underline{j} + \dot{E}_3 \underline{k})$$

Thus

$$\dot{P}_v = \dot{I}^{1*}\dot{E}_1 + \dot{I}^{2*}\dot{E}_2 + \dot{I}^{3*}\dot{E}_3$$

The complex scalar $\dot{P}_v$ is the sum of three complex scalars, each being the volt-amperes for one of the three phases of the given three-phase network.

Methods for Combining Three-Phase Loads

In order to solve three-phase networks effectively, it will be necessary to show how to combine impedance tensors that are in series or parallel. Of course, for a given reference frame (choice of meshes and positive directions of currents) the impedance 2-tensor will be represented by a 2-way matrix.

Two sets of three impedances are shown connected in series in figure 19.

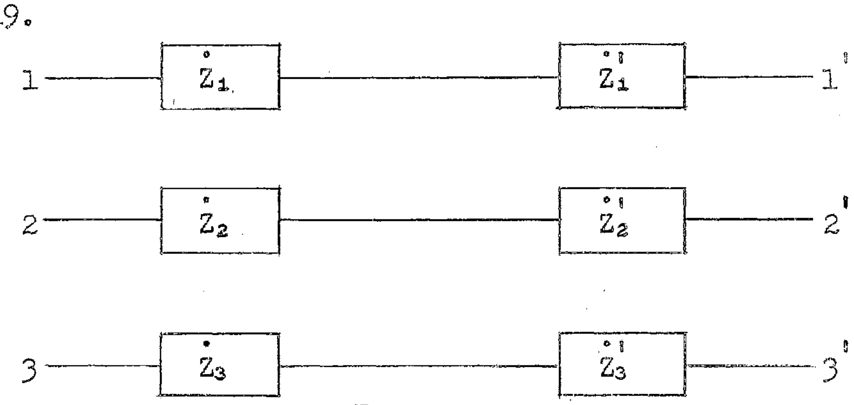

Figure 19
Three-Phase Series Loads

Assuming no coupling between the $\dot{Z}$ impedances and the $\dot{Z}'$ impedances, the two impedance matrices are

$$[\dot{Z}] = \begin{bmatrix} \dot{Z}_{11} & \dot{Z}_{12} & \dot{Z}_{13} \\ \dot{Z}_{21} & \dot{Z}_{22} & \dot{Z}_{23} \\ \dot{Z}_{31} & \dot{Z}_{32} & \dot{Z}_{33} \end{bmatrix} \quad (6.8), \text{ and } [\dot{Z}'] = \begin{bmatrix} \dot{Z}'_{11} & \dot{Z}'_{12} & \dot{Z}'_{13} \\ \dot{Z}'_{21} & \dot{Z}'_{22} & \dot{Z}'_{23} \\ \dot{Z}'_{31} & \dot{Z}'_{32} & \dot{Z}'_{33} \end{bmatrix} \quad (6.9)$$

The line currents are obviously common to both sets of impedances.

Thus

$$[\dot{E}] = [\dot{Z}] \times [\dot{I}] \tag{6.10}$$

where the elements of the matrix

$$[\dot{E}] = \begin{bmatrix} \dot{E}_1 \\ \dot{E}_2 \\ \dot{E}_3 \end{bmatrix} \tag{6.11}$$

are the voltage-drops across the corresponding elements of the impedance matrix $[\dot{Z}]$. Similarly

$$[\dot{E}'] = [\dot{Z}'] \times [\dot{I}] \tag{6.12}$$

The voltage-drops from terminals (1, 2, 3) to (1', 2', 3') are

$$[\dot{E}_s] = [\dot{E}] + [\dot{E}'] \tag{6.13}$$

Substituting (6.10) and (6.12) into (6.13)

gives

$$[\dot{E}_s] = [\dot{Z}] \times [\dot{I}] + [\dot{Z}'] \times [\dot{I}] \tag{6.14}$$

Using the distributive property of matrix multiplication, (6.14)

becomes

$$[\dot{E}_s] = ([\dot{Z}] + [\dot{Z}']) \times [\dot{I}] \tag{6.15}$$

Setting

$$[\dot{Z}_s] = [\dot{Z}] + [\dot{Z}'] \tag{6.16}$$

(6.15) can be written

$$[\dot{E}_s] = [\dot{Z}_s] \times [\dot{I}] \tag{6.17}$$

From (6.16), it is seen that two impedance matrices representing two sets of impedances which are connected as shown in figure 19 may be added to give the impedance matrix which represents the combination of both sets of impedances. In expanded form, (6.16) is

$$[\dot{Z}_s] = \begin{bmatrix} \dot{Z}_{11} + \dot{Z}'_{11} & \dot{Z}_{12} + \dot{Z}'_{12} & \dot{Z}_{13} + \dot{Z}'_{13} \\ \dot{Z}_{21} + \dot{Z}'_{21} & \dot{Z}_{22} + \dot{Z}'_{22} & \dot{Z}_{23} + \dot{Z}'_{23} \\ \dot{Z}_{31} + \dot{Z}'_{31} & \dot{Z}_{32} + \dot{Z}'_{32} & \dot{Z}_{33} + \dot{Z}'_{33} \end{bmatrix} \qquad (6.16a)$$

The three-phase circuit shown in figure 20 will now be considered.

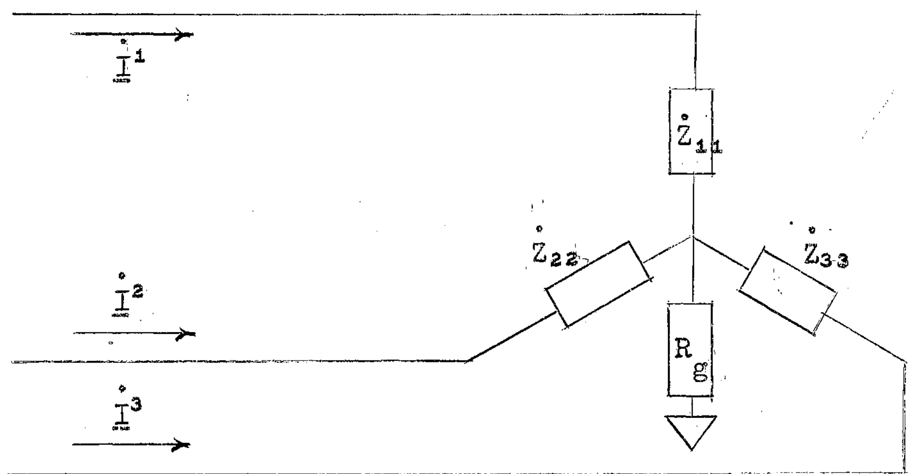

Figure 20
Three-Phase Load with Resistance Ground

Each of the three line-to-ground voltages is composed of two parts, the voltage drop from the line to point n and the voltage drop from point n to ground across the common grounding resistor R_g. The three line-to-ground Kirchhoff voltage equations are

$$\begin{aligned} \dot{E}_1 &= \dot{Z}_{11}\dot{I}^1 + \dot{Z}_{12}\dot{I}^2 + \dot{Z}_{13}\dot{I}^3 + R_g(\dot{I}^1 + \dot{I}^2 + \dot{I}^3) \\ \dot{E}_2 &= \dot{Z}_{21}\dot{I}^1 + \dot{Z}_{22}\dot{I}^2 + \dot{Z}_{23}\dot{I}^3 + R_g(\dot{I}^1 + \dot{I}^2 + \dot{I}^3) \qquad (6.18) \\ \dot{E}_3 &= \dot{Z}_{31}\dot{I}^1 + \dot{Z}_{32}\dot{I}^2 + \dot{Z}_{33}\dot{I}^3 + R_g(\dot{I}^1 + \dot{I}^2 + \dot{I}^3) \end{aligned}$$

The first three terms on the right of (6.18) are the same as (6.1) and hence can be expressed in matrix form as (6.7). The last term on the right of (6.18) may be written in matrix form as

$$[\dot{E}_g] = \begin{bmatrix} \dot{E}_{g1} \\ \dot{E}_{g2} \\ \dot{E}_{g3} \end{bmatrix} = \begin{bmatrix} R_g & R_g & R_g \\ R_g & R_g & R_g \\ R_g & R_g & R_g \end{bmatrix} \times \begin{bmatrix} \dot{I}^1 \\ \dot{I}^2 \\ \dot{I}^3 \end{bmatrix} \qquad (6.19)$$

or $$[\dot{E}_g] = R_g \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} \dot{I}^1 \\ \dot{I}^2 \\ \dot{I}^3 \end{bmatrix} \quad (6.20)$$

Equation (6.18) can then be written

$$[\dot{E}] = [\dot{Z}] \times [\dot{I}]$$

where $$[\dot{Z}] = \begin{bmatrix} Z_{11} + R_g & Z_{12} + R_g & Z_{13} + R_g \\ Z_{21} + R_g & Z_{22} + R_g & Z_{23} + R_g \\ Z_{31} + R_g & Z_{32} + R_g & Z_{33} + R_g \end{bmatrix} \quad (6.21)$$

As a final illustration, the circuit shown in figure 21 consisting of two three-phase loads in parallel will be analyzed.

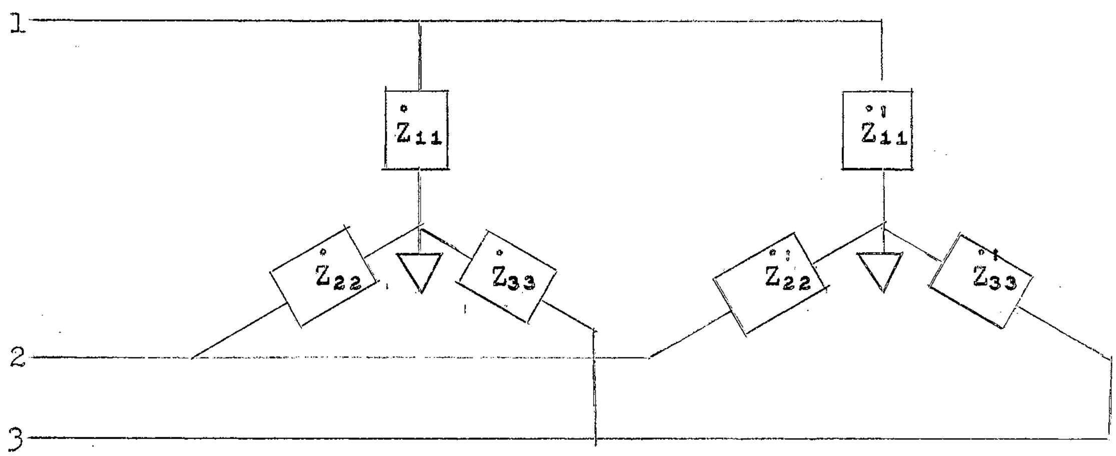

Figure 21
Parallel Three-Phase Loads Solidly Grounded

$$[\dot{E}] = [\dot{Z}] \times [\dot{I}] = [\dot{Z}'] \times [\dot{I}'] \quad (6.22)$$

Hence $$[\dot{I}] = [\dot{Z}^{-1}] \times [\dot{E}] \quad (6.23)$$

and $$[\dot{I}'] = [\dot{Z}'^{-1}] \times [\dot{E}] \quad (6.24)$$

The matrix of the total line currents $[\dot{I}_s]$ is

$$[\dot{I}_s] = [\dot{I}] + [\dot{I}'] \quad (6.25)$$

Substituting into (6.25) from (6.23) and (6.24), and simplifying gives

$$[\dot{I}_s] = ([\dot{Z}^{-1}] + [\dot{Z}'^{-1}]) \times [\dot{E}] \quad (6.26)$$

Solving for $[\dot{E}]$ gives

$$[\dot{E}] = ([\dot{Z}^{-1}] + [\dot{Z}'^{-1}])^{-1} \times [\dot{I}_s] \qquad (6.27)$$

Therefore

$$[\dot{E}] = [\dot{Z}_s] \times [\dot{I}_s] \qquad (6.28)$$

where

$$[\dot{Z}_s] = ([\dot{Z}^{-1}] + [\dot{Z}'^{-1}])^{-1} \qquad (6.29)$$

Thus the matrix which represents the combination of two parallel impedance matrices is the inverse of the sum of the inverses of the two impedance matrices. Equation (6.29) is entirely analogous to the familiar rule for combining parallel scalar impedances.

CHAPTER VII

APPLICATION OF TENSORS TO THE ANALYSIS OF ELECTRIC NETWORKS WITH NONSINUSOIDAL APPLIED VOLTAGES

In Chapter V, the basic tensor formulas of electric networks were derived, and in Chapter VI these equations were applied to a few simple three-phase circuits. Throughout these derivations it was always assumed, without being specifically pointed out, that the branch (or mesh) driving voltages were all true sinusoids of a single frequency. A method of solving electric networks with nonsinusoidal applied voltages using the tensor equations of Chapter V will be developed and illustrated in this chapter. Kron's original work which was adapted in Chapter V considered only sinusoidal functions of a single frequency. The extension of Kron's work developed in this chapter is the author's own original work. The impedance elements of the networks considered will be assumed to be linear.

Introductory Concepts

The solutions of a major portion of the problems encountered in electrical engineering are based upon the assumption that the driving voltage is a sine-wave variation. Such an assumption may or may not be justified. In many circuits, nonsinusoidal voltage variations are as common as sinusoidal variations, and in many supposedly "sine-wave circuits" nonsinusoidal variations must be occasionally considered.

Fortunately, most periodic nonsinusoidal variations that are encountered in electrical engineering vary in such a manner that they can be analyzed by the methods of the Fourier Analysis into the Fourier Sine Series. Any function f(x), that is periodic, single-valued, piece-wise continuous, and does not have an infinite number of maxima or minima in the neighborhood of any point, may be represented by the following series:

$$f(x) = A_0 + A_1 \sin x + A_2 \sin 2x + \ldots + A_n \sin nx$$
$$+ B_1 \cos x + B_2 \cos 2x + \ldots + B_n \cos nx \quad (7.1)$$

where the A's and the B's are real numbers. By simply combining corresponding sine and cosine terms, (7.1) can be written

$$f(x) = C_0 + \dot{C}_1 \sin x + \dot{C}_2 \sin 2x + \ldots + \dot{C}_n \sin nx \quad (7.2)$$

where $C_0 = A_0$ and the other C's are complex numbers defined as follows

$$\begin{aligned} \dot{C}_1 &= A_1 + jB_1 \\ \dot{C}_2 &= A_2 + jB_2 \\ &\vdots \\ \dot{C}_n &= A_n + jB_n \end{aligned} \quad (7.3)$$

As shown in Chapters II and V, the terms in (7.2) may be represented by complex vectors which form an orthogonal set. If f(x) is the voltage function e(t), then (6.2) can be written

$$\underline{\dot{E}} = {}_0\dot{E}\,{}_1\underline{e} + {}_1\dot{E}\,{}_2\underline{e} + {}_2\dot{E}\,{}_3\underline{e} + \ldots + {}_{n-1}\dot{E}\,{}_n\underline{e} \quad (7.4)$$

The presence of the real vector ${}_0E\,{}_1\underline{e}$ presents no difficulty in manipulating $\underline{\dot{E}}$, but it does cause ${}_{n-1}\dot{E}$ to be associated with ${}_n\underline{e}$. This would prevent the use of indices to indicate summation. Therefore (7.4) will be written in the symmetric form

$$\underline{\dot{E}} = {}_1\dot{E}\,{}_1\underline{e} + {}_2\dot{E}\,{}_2\underline{e} + {}_3\dot{E}\,{}_3\underline{e} + \ldots + {}_n\dot{E}\,{}_n\underline{e} \quad (7.5)$$

If the d-c term is present in (7.2) then it may be interpreted to be the ${}_1\dot{E}\ {}_1\underline{e}$ term in (7.5), the a-c components being represented by the remaining terms of (7.5).

The purpose of this chapter is to present a method of solving linear networks by the use of matrices (components of tensors in a given reference frame) when voltages of the form (7.5) are applied to the networks. Theoretically, there is an infinite number of terms in (7.5), but in practical problems only a finite number of terms, usually only a very few, is required to obtain the desired accuracy. Therefore this chapter will be concerned with the application of matrices having a finite number of elements.

Before proceeding further, it appears desirable to clarify the symbolism that will be used.

$[\dot{E}]$ and $[\dot{I}]$ refer to branch values.

$[\dot{E}']$ and $[\dot{I}']$ refer to mesh values.

$\dot{E}_1, \dot{E}_2, \ldots, \dot{E}_n$ are elements of $[\dot{E}]$.

$\dot{I}^1, \dot{I}^2, \ldots, \dot{I}^n$ are elements of $[\dot{I}]$.

$\dot{E}'_1, \dot{E}'_2, \ldots, \dot{E}'_n$ are elements of $[\dot{E}']$.

$\dot{I}^{1'}, \dot{I}^{2'}, \ldots, \dot{I}^{n'}$ are elements of $[\dot{I}']$.

Harmonics will be denoted by left-hand indices. The element representing the second harmonic of the voltage in branch three will be written symbolically as ${}_2\dot{E}_3$.

As demonstrated in Chapters V and VI, the branch (or mesh) voltages and currents of a given network may be represented as orthogonal components of the generalized space voltage and current vectors (tensors of valence 1). When harmonics are present, each of the branch (or mesh) voltages or currents is the sum of a set of orthogonal time (or frequency) vectors (7.5). Thus each of the branch (or mesh) one-dimensional space vectors spans an n-dimensional orthogonal time-vector-space. The n-dimensional orthogonal time-vector-space of (7.5) is called a Hilbert Space.

Single-Mesh Circuit Analysis

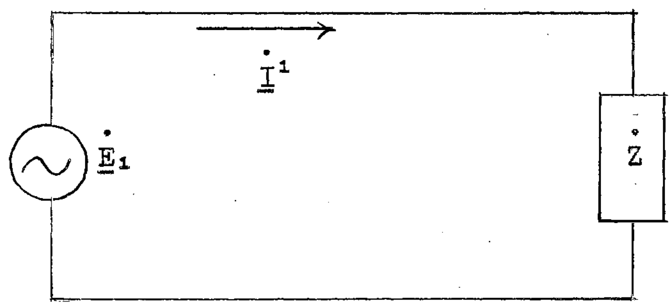

Figure 22
Single-Mesh Circuit with Nonsinusoidal Excitation

The voltage and current of figure 22 expressed as vectors may be written in terms of their components as

$$\dot{\underline{E}}_1 = {}_1\dot{E}_1\ {}_1\underline{e}_1 + {}_2\dot{E}_1\ {}_2\underline{e}_1 + {}_3\dot{E}_1\ {}_3\underline{e}_1 + \ldots + {}_n\dot{E}_1\ {}_n\underline{e}_1 \qquad (7.6)$$

and

$$\dot{\underline{I}}^1 = {}^1\dot{I}^1\ {}_1\underline{e}_1 + {}^2\dot{I}^1\ {}_2\underline{e}_1 + {}^3\dot{I}^1\ {}_3\underline{e}_1 + \ldots + {}^n\dot{I}^1\ {}_n\underline{e}_1 \qquad (7.7)$$

where the left indices refer to the order of the harmonics and the right indices (all 1) identify the mesh. Since there is only one mesh in figure 22, the right indices in (7.6) and (7.7) could be dropped with no loss of clarity, resulting in the form (7.5). However, in order to maintain the notation here that will be required in the following section, the right indices will not be dropped. The voltage and current vectors of (7.6) and (7.7) expressed as column matrices are

$$[\dot{E}_1] = \begin{bmatrix} {}_1\dot{E}_1 \\ {}_2\dot{E}_1 \\ {}_3\dot{E}_1 \\ \cdot \\ \cdot \\ \cdot \\ {}_n\dot{E}_1 \end{bmatrix} \quad (7.8), \text{ and } \quad [\dot{I}^1] = \begin{bmatrix} {}^1\dot{I}^1 \\ {}^2\dot{I}^1 \\ {}^3\dot{I}^1 \\ \cdot \\ \cdot \\ \cdot \\ {}^n\dot{I}^1 \end{bmatrix} \quad (7.9)$$

The a-c impedance of the circuit shown in figure 22 is generally different for each different harmonic (frequency). The scalar equations are

$$\begin{aligned} {}_1\dot{E}_1 &= {}_1\dot{Z}_{11} \times {}^1\dot{I}^1 \\ {}_2\dot{E}_1 &= {}_2\dot{Z}_{11} \times {}^2\dot{I}^1 \\ &\cdot \quad \cdot \quad \cdot \\ &\cdot \quad \cdot \quad \cdot \\ &\cdot \quad \cdot \quad \cdot \\ {}_n\dot{E}_1 &= {}_n\dot{Z}_{11} \times {}^n\dot{I}^1 \end{aligned} \qquad (7.10)$$

Expressed in tensor notation, (7.10) becomes

$$\begin{aligned} {}_1\dot{E}_1 \; {}_1\underline{e}_1 &= {}_1\dot{Z}_{11} \; {}_1\underline{e}_1 \; {}_1\underline{e}_1 \times {}^1\dot{I}^1 \; {}_1\underline{e}_1 \\ {}_2\dot{E}_1 \; {}_2\underline{e}_1 &= {}_2\dot{Z}_{11} \; {}_2\underline{e}_1 \; {}_2\underline{e}_1 \times {}^2\dot{I}^1 \; {}_2\underline{e}_1 \\ &\cdot \quad \cdot \quad \cdot \\ &\cdot \quad \cdot \quad \cdot \\ &\cdot \quad \cdot \quad \cdot \\ {}_n\dot{E}_1 \; {}_n\underline{e}_1 &= {}_n\dot{Z}_{11} \; {}_n\underline{e}_1 \; {}_n\underline{e}_1 \times {}^n\dot{I}^1 \; {}_n\underline{e}_1 \end{aligned} \qquad (7.11)$$

If the impedance tensor is represented by the matrix

$$[\dot{Z}_{11}] = \begin{bmatrix} {}_1\dot{Z}_{11} & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & {}_2\dot{Z}_{11} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & {}_3\dot{Z}_{11} & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & {}_n\dot{Z}_{11} \end{bmatrix} \tag{7.12}$$

then the matrix equation

$$\begin{bmatrix} {}_1\dot{E}_1 \\ {}_2\dot{E}_1 \\ {}_3\dot{E}_1 \\ \cdot \\ \cdot \\ \cdot \\ {}_n\dot{E}_1 \end{bmatrix} = \begin{bmatrix} {}_1\dot{Z}_{11} & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & {}_2\dot{Z}_{11} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & {}_3\dot{Z}_{11} & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & {}_n\dot{Z}_{11} \end{bmatrix} \times \begin{bmatrix} {}_1\dot{I}^1 \\ {}_2\dot{I}^1 \\ {}_3\dot{I}^1 \\ \cdot \\ \cdot \\ \cdot \\ {}_n\dot{I}^1 \end{bmatrix} \tag{7.13}$$

or

$$[\dot{E}_1] = [\dot{Z}_{11}] \times [\dot{I}^1] \tag{7.14}$$

is a correct representation of the set of equations (7.11). The matrix of the impedances to the various harmonics (7.12) is a diagonal matrix. The determinant of the matrix (7.12) is

$$D_1 = {}_1\dot{Z}_{11} \times {}_2\dot{Z}_{11} \times {}_3\dot{Z}_{11} \times \ldots \times {}_n\dot{Z}_{11} \tag{7.15}$$

The cofactor of any element of the principal diagonal of (7.12) is merely the product of all the other elements of the principal diagonal. The matrix whose elements are the cofactors of the elements of (7.12) is self-transposed. Therefore the inverse of (7.12) is

$$[Z_{11}^{-1}] = \begin{bmatrix} \frac{{}_1A}{D} & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \frac{{}_2A}{D} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \frac{{}_3A}{D} & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \frac{{}_nA}{D} \end{bmatrix} \tag{7.16}$$

where ${}_kA$ in (7.16) is the cofactor of ${}_k\dot{Z}_{11}$ in (7.12). Substituting for the A's in (7.16) in terms of the $\dot{Z}$'s of (7.12), equation (7.16) becomes

$$[\dot{Z}_{11}^{-1}] = \begin{bmatrix} \frac{1}{{}_1\dot{Z}_{11}} & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \frac{1}{{}_2\dot{Z}_{11}} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \frac{1}{{}_3\dot{Z}_{11}} & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \frac{1}{{}_n\dot{Z}_{11}} \end{bmatrix} \tag{7.17}$$

Multiplying both sides of (7.14) by $[\dot{Z}_{11}^{-1}]$, the current is

$$[\dot{I}^1] = [\dot{Z}_{11}^{-1}] \times [\dot{E}_1] \tag{7.18}$$

It is well known that currents and voltages of different integral frequencies do not combine to produce any average volt-amperes. The total complex volt-amperes ($\dot{P}_v$) is the algebraic sum of the complex scalar volt-amperes due to each harmonic. Hence

$$\dot{P}_v = [\dot{I}_t^1] \cdot [\dot{E}_1] \tag{7.19}$$

or $$\dot{P}_v = {}_1\overset{\circ}{I}{}^{1*} \times {}_1\dot{E}_1 + {}_2\overset{\circ}{I}{}^{1*} \times {}_2\dot{E}_1 + \ldots + {}_n\dot{I}{}^{1*} \times {}_n\dot{E}_1$$

Multi-Mesh Network Analysis with

Nonsinusoidal Applied Voltages

The analysis of general networks which may have nonsinusoidal voltages and currents present begins in a manner similar to the method used in analyzing the network of figure 14 in Chapter V. The essential difference between the method to be presented here and the method developed in Chapter V is that the elements of equations (5.24), (5.32), and (5.34) are themselves matrices; the matrices of Chapter V are actually "compound matrices" when harmonics are present. The rules for manipulating compound matrices are given and illustrated below.

A given matrix may be partitioned into several submatrices by drawing horizontal or vertical lines through the matrix. These submatrices may then be manipulated exactly as if they were elements of the matrix. The matrix

$$[Z] = \left[\begin{array}{cc|ccc} 1 & 3 & 2 & 4 & 0 \\ 0 & -1 & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 1 \\ \hline 2 & -1 & -3 & 1 & 0 \\ 0 & -2 & 0 & 1 & 1 \end{array}\right] \tag{7.20}$$

may be partitioned as indicated by the lines and written in the form

$$[Z] = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \tag{7.21}$$

where $$[Z_{11}] = \begin{bmatrix} 1 & 3 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}, \qquad [Z_{12}] = \begin{bmatrix} 2 & 4 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$[Z_{12}] = \begin{bmatrix} 2 & -1 \\ 0 & -2 \end{bmatrix} \text{ and } [Z_{22}] = \begin{bmatrix} -3 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Similarly the matrix

$$[I] = \begin{bmatrix} 1 \\ 3 \\ \hline 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} I^1 \\ I^2 \end{bmatrix} \qquad (7.22)$$

If

$$[E] = [Z] \times [I]$$

then

$$[E] = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \times \begin{bmatrix} I^1 \\ I^2 \end{bmatrix} \qquad (7.23)$$

or

$$[E] = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} Z_{11}I^1 + Z_{12}I^2 \\ Z_{21}I^1 + Z_{22}I^2 \end{bmatrix} \qquad (7.24)$$

Either of the voltages E_1 or E_2 may be determined from (7.24) without calculating the other. If E_1 and E_2 were given, then I^1 or I^2 could be determined. The calculation of I^1 or I^2 would involve at most the calculation of the inverse of a (3 x 3) matrix, rather than the given (5 x 5) matrix. If only the components of I^1 or I^2 were desired, the use of submatrices would achieve a tremendous economy of time and effort.

The matrices I^1 and I^2 of (7.24) may be evaluated by the following procedure. Of the four submatrices, only Z_{21} and Z_{12} have inverses. Equation (7.24) written as two equations is

$$E_1 = Z_{11}I^1 + Z_{12}I^2 \qquad (7.25)$$

$$E_2 = Z_{21}I^1 + Z_{22}I^2 \qquad (7.26)$$

To determine I^2, first solve (7.26) for $Z_{21}I^1$ giving

$$Z_{21}I^1 = E_2 - Z_{22}I^2 \tag{7.27}$$

Multiplying both sides of (7.27) by Z_{21}^{-1} gives

$$I^1 = Z_{21}^{-1}(E_2 - Z_{22}I^2) \tag{7.28}$$

By substitution of (7.28) into (7.25)

$$E_1 = Z_{11} Z_{21}^{-1}(E_2 - Z_{22}I^2) + Z_{12} I^2 \tag{7.29}$$

$$E_1 - Z_{11} Z_{21}^{-1} E_2 = (Z_{12} - Z_{11} Z_{21}^{-1} Z_{22})I^2$$

$$I^2 = (Z_{12} - Z_{11} Z_{21}^{-1} Z_{22})^{-1}(E_1 - Z_{11} Z_{21}^{-1} E_2) \tag{7.30}$$

A similar determination of I^1 gives

$$I^1 = (Z_{21} - Z_{22} Z_{12}^{-1} Z_{11})^{-1}(E_2 - Z_{22} Z_{12}^{-1} E_1) \tag{7.31}$$

It should be emphasized that in calculating I^2 and I^1 it is necessary to determine the inverses of only (2 x 2) and (3 x 3) matrices. In partitioning the two matrices (7.20) and (7.22), care must be taken that the submatrices so formed are conformable, else they cannot be multiplied. Two matrices in a matrix product are conformably partitioned if for every vertical partitioning line between columns of the matrix on the left there is a partitioning line between corresponding rows of the matrix on the right.

As a means of illustrating the procedure to be used in analyzing a general network, the network shown in figure 23 will be solved.

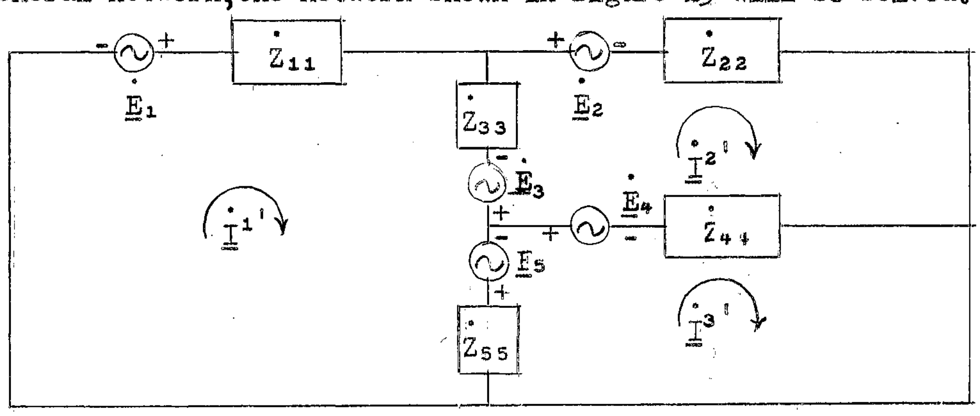

Figure 23
Illustrative Example

The positive directions of the five branch currents will be chosen so that the currents flow out of the positive terminals of the generators. It will be assumed that there is no magnetic coupling among branches, and the generators generate first, third, and fifth harmonics. The presence of magnetic coupling among branches would merely require additional non-zero elements in the branch impedance matrix; the procedure would be exactly the same.

From figure 23, the equations relating the branch and mesh currents are

$$\begin{aligned} \dot{I}^1 &= \dot{I}^{1'} + 0 + 0 \\ \dot{I}^2 &= 0 - \dot{I}^{2'} + 0 \\ \dot{I}^3 &= \dot{I}^{1'} - \dot{I}^{2'} + 0 \\ \dot{I}^4 &= 0 + \dot{I}^{2'} - \dot{I}^{3'} \\ \dot{I}^5 &= \dot{I}^{1'} + 0 - \dot{I}^{3'} \end{aligned} \qquad (7.32)$$

The transformation matrix is

$$[C] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \qquad (7.33)$$

The branch voltage matrix is

$$[\dot{E}] = \begin{bmatrix} \dot{E}_1 \\ \dot{E}_2 \\ \dot{E}_3 \\ \dot{E}_4 \\ \dot{E}_5 \end{bmatrix} \qquad (7.34)$$

From (7.34) and the voltage transformation equation

$$[\dot{E}'] = [C_t] \times [\dot{E}] \qquad (7.35)$$

the mesh voltage matrix is

$$[\dot{E}'] = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix} \times \begin{bmatrix} \dot{E}_1 \\ \dot{E}_2 \\ \dot{E}_3 \\ \dot{E}_4 \\ \dot{E}_5 \end{bmatrix} \tag{7.36}$$

$$[\dot{E}'] = \begin{bmatrix} \dot{E}_1 + \dot{E}_3 + \dot{E}_5 \\ -\dot{E}_2 - \dot{E}_3 + \dot{E}_4 \\ -\dot{E}_4 - \dot{E}_5 \end{bmatrix} \tag{7.37}$$

The product of the branch impedance matrix

$$[\dot{Z}] = \begin{bmatrix} \dot{Z}_{11} & 0 & 0 & 0 & 0 \\ 0 & \dot{Z}_{22} & 0 & 0 & 0 \\ 0 & 0 & \dot{Z}_{33} & 0 & 0 \\ 0 & 0 & 0 & \dot{Z}_{44} & 0 \\ 0 & 0 & 0 & 0 & \dot{Z}_{55} \end{bmatrix} \tag{7.38}$$

and [C] is

$$[\dot{Z}] \times [C] = \begin{bmatrix} \dot{Z}_{11} & 0 & 0 & 0 & 0 \\ 0 & \dot{Z}_{22} & 0 & 0 & 0 \\ 0 & 0 & \dot{Z}_{33} & 0 & 0 \\ 0 & 0 & 0 & \dot{Z}_{44} & 0 \\ 0 & 0 & 0 & 0 & \dot{Z}_{55} \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \tag{7.39}$$

$$[\dot{Z}] \times [C] = \begin{bmatrix} \dot{Z}_{11} & 0 & 0 \\ 0 & -\dot{Z}_{22} & 0 \\ \dot{Z}_{33} & -\dot{Z}_{33} & 0 \\ 0 & \dot{Z}_{44} & -\dot{Z}_{44} \\ \dot{Z}_{55} & 0 & -\dot{Z}_{55} \end{bmatrix} \tag{7.40}$$

Substituting (7.40) into the impedance transformation equation

$$[\dot{Z}'] = [C_t] \times [\dot{Z}] \times [C] \qquad (7.41)$$

gives

$$[\dot{Z}'] = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix} \times \begin{bmatrix} \dot{Z}_{11} & 0 & 0 \\ 0 & -\dot{Z}_{22} & 0 \\ \dot{Z}_{33} & -\dot{Z}_{33} & 0 \\ 0 & \dot{Z}_{44} & -\dot{Z}_{44} \\ \dot{Z}_{55} & 0 & -\dot{Z}_{55} \end{bmatrix}$$

$$[\dot{Z}'] = \begin{bmatrix} \dot{Z}_{11} + \dot{Z}_{33} + \dot{Z}_{55} & -\dot{Z}_{33} & -\dot{Z}_{55} \\ -\dot{Z}_{33} & \dot{Z}_{22} + \dot{Z}_{33} + \dot{Z}_{44} & -\dot{Z}_{44} \\ -\dot{Z}_{55} & -\dot{Z}_{44} & \dot{Z}_{44} + \dot{Z}_{55} \end{bmatrix} \qquad (7.42)$$

The mesh currents are

$$[\dot{I}'] = [\dot{Z}'^{-1}] \times [\dot{E}'] \qquad (7.44)$$

Using general impedance symbols, the inverse of (7.43) is too bulky to write down. For given numerical values of the parameters, of course, each element of $[\dot{Z}']$ and $[\dot{Z}'^{-1}]$ would be a single complex number. If the elements of the inverse of $[\dot{Z}']$ are denoted by

$$[\dot{Z}'^{-1}] = \begin{bmatrix} \dot{A}_{11} & \dot{A}_{12} & \dot{A}_{13} \\ \dot{A}_{21} & \dot{A}_{22} & \dot{A}_{23} \\ \dot{A}_{31} & \dot{A}_{32} & \dot{A}_{33} \end{bmatrix} \qquad (7.45)$$

then from (7.44)

$$[\dot{I}'] = \begin{bmatrix} \dot{A}_{11} & \dot{A}_{12} & \dot{A}_{13} \\ \dot{A}_{21} & \dot{A}_{22} & \dot{A}_{23} \\ \dot{A}_{31} & \dot{A}_{32} & \dot{A}_{33} \end{bmatrix} \times \begin{bmatrix} \dot{E}_1 + \dot{E}_3 + \dot{E}_5 \\ -\dot{E}_2 - \dot{E}_3 + \dot{E}_4 \\ -\dot{E}_4 - \dot{E}_5 \end{bmatrix} \qquad (7.46)$$

$$[\dot{I}'] = \begin{bmatrix} \dot{A}_{11}(\dot{E}_1+\dot{E}_3+\dot{E}_5) + \dot{A}_{12}(-\dot{E}_2-\dot{E}_3+\dot{E}_4) + \dot{A}_{13}(-\dot{E}_4-\dot{E}_5) \\ \dot{A}_{21}(\dot{E}_1+\dot{E}_3+\dot{E}_5) + \dot{A}_{22}(-\dot{E}_2-\dot{E}_3+\dot{E}_4) + \dot{A}_{23}(-\dot{E}_4-\dot{E}_5) \\ \dot{A}_{31}(\dot{E}_1+\dot{E}_3+\dot{E}_5) + \dot{A}_{32}(-\dot{E}_2-\dot{E}_3+\dot{E}_4) + \dot{A}_{33}(-\dot{E}_4-\dot{E}_5) \end{bmatrix} \qquad (7.47)$$

From the current transformation equation

$$[\dot{I}] = [C] \times [\dot{I}'] \qquad (7.48)$$

the branch currents may now be determined. If (7.47) is written in the form

$$[\dot{I}'] = \begin{bmatrix} \dot{I}^{1'} \\ \dot{I}^{2'} \\ \dot{I}^{3'} \end{bmatrix} \qquad (7.49)$$

then the branch current matrix is

$$[\dot{I}] = \begin{bmatrix} \dot{I}^{1'} & 0 & 0 \\ 0 & -\dot{I}^{2'} & 0 \\ \dot{I}^{1'} & -\dot{I}^{2'} & 0 \\ 0 & \dot{I}^{2'} & -\dot{I}^{3'} \\ \dot{I}^{1'} & 0 & -\dot{I}^{3'} \end{bmatrix} = \begin{bmatrix} \dot{I}^{1} \\ \dot{I}^{2} \\ \dot{I}^{3} \\ \dot{I}^{4} \\ \dot{I}^{5} \end{bmatrix} \qquad (7.50)$$

The branch impedance matrix (7.38), the branch voltage matrix (7.34), and the branch current matrix (7.50) have been determined.

From the results of the preceding section of this chapter, the elements of the branch matrices (7.34), (7.38), and (7.50) are themselves matrices. The elements of the mesh matrices (7.37), (7.43), and (7.49) are matrices also, but branch values are more useful in calculating voltage-drops, volt-amperes per branch, etc.

The voltage-drops across the branch impedance elements of figure 23 are

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \\ \dot{V}_4 \\ \dot{V}_5 \end{bmatrix} = \begin{bmatrix} \dot{Z}_{11} & 0 & 0 & 0 & 0 \\ 0 & \dot{Z}_{22} & 0 & 0 & 0 \\ 0 & 0 & \dot{Z}_{33} & 0 & 0 \\ 0 & 0 & 0 & \dot{Z}_{44} & 0 \\ 0 & 0 & 0 & 0 & \dot{Z}_{55} \end{bmatrix} \times \begin{bmatrix} \dot{I}^{1} \\ \dot{I}^{2} \\ \dot{I}^{3} \\ \dot{I}^{4} \\ \dot{I}^{5} \end{bmatrix} \qquad (7.51)$$

Equation (7.51) is actually the equation of the primitive network of the given network (see Chapter V). Recalling the assumption that only the first, third, and fifth harmonics were present in the generated voltages, the five equations given by (7.51) are actually matrix equations. Thus

$$[\dot{V}_1] = \begin{bmatrix} {}_1\dot{V}_1 \\ {}_3\dot{V}_1 \\ {}_5\dot{V}_1 \end{bmatrix} = \begin{bmatrix} {}_1\dot{Z}_{11} & 0 & 0 \\ 0 & {}_3\dot{Z}_{11} & 0 \\ 0 & 0 & {}_5\dot{Z}_{11} \end{bmatrix} \text{ x } \begin{bmatrix} {}^1\dot{I}^1 \\ {}^3\dot{I}^1 \\ {}^5\dot{I}^1 \end{bmatrix} \quad (7.52)$$

The other four equations for branches 2, 3, 4, and 5 are similar to (7.52). The voltage drop across any impedance due to any harmonic can be calculated using the five sets of equations of the type (7.52) as determined by (7.51).

The total volt-amperes of the given network are given by

$$\dot{P}_v = [\dot{I}_t] \cdot [\dot{E}] = [\dot{I}^1 \; \dot{I}^2 \; \dot{I}^3 \; \dot{I}^4 \; \dot{I}^5] \cdot \begin{bmatrix} \dot{E}_1 \\ \dot{E}_2 \\ \dot{E}_3 \\ \dot{E}_4 \\ \dot{E}_5 \end{bmatrix} \quad (7.53)$$

or $$\dot{P}_v = \dot{I}^{1*} \text{ x } \dot{E}_1 + \dot{I}^{2*} \text{ x } \dot{E}_2 + \dot{I}^{3*} \text{ x } \dot{E}_3 + \dot{I}^{4*} \text{ x } \dot{E}_4 + \dot{I}^{5*} \text{ x } \dot{E}_5 \quad (7.54)$$

Each of the elements in the matrices of (7.53) is a 3-element matrix. The volt-amperes delivered to branch 1 are

$$\dot{P}_{v_1} = [\dot{I}^1_t] \cdot [E_1] = [{}^1I^1 \; {}^3I^1 \; {}^5I^1] \cdot \begin{bmatrix} {}_1\dot{E}_1 \\ {}_3\dot{E}_1 \\ {}_5\dot{E}_1 \end{bmatrix} \quad (7.55)$$

or $$\dot{P}_{v_1} = {}^1\dot{I}^{1*} \text{ x } {}_1\dot{E}_1 + {}^3\dot{I}^{1*} \text{ x } {}_3\dot{E}_1 + {}^5\dot{I}^{1*} \text{ x } {}_5\dot{E}_1 \quad (7.56)$$

Equation (7.56) may be written

$$\dot{P}_{V1} = {}_{1}\dot{P}_{V1} + {}_{3}\dot{P}_{V1} + {}_{5}\dot{P}_{V1} \tag{7.57}$$

where ${}_{1}\dot{P}_{V1}$, ${}_{3}\dot{P}_{V1}$, and ${}_{5}\dot{P}_{V1}$ are the complex scalar volt-amperes in branch 1 due to the first, third, and fifth harmonics, respectively.

Summary of Chapter VII

1. In many network problems encountered in electrical engineering, the presence of harmonics must be considered in the solutions of the networks.
2. The harmonic components of a 1-dimensional voltage (or current) vector may be represented by an orthogonal set of complex vectors. The n harmonic components of a voltage vector span an n-dimensional time subspace of the 1-dimensional complex vector space in which the voltage vector is represented.
3. Single-mesh circuits in which the applied voltages contain harmonics may be solved using matrix algebra. The orthogonal components of the voltage and current vectors are represented by the elements of single-column matrices, and the a-c impedances to the different harmonics are represented by the elements on the principal diagonal of a 2-matrix, all elements not on the principal diagonal being zero (no magnetic coupling).
4. The addition of more than one branch or mesh to the network merely adds more space dimensions to the complex voltage and current vectors.
5. Matrices, the elements of which are themselves matrices, are called compound matrices, and the elements are called submatrices. Compound matrices are manipulated in the same manner as ordinary matrices.

6. The method of analysis of Chapter V may be applied to the solution of networks with nonsinusoidal applied voltages. This extension of the analysis of Kron was originally developed by the author. It is necessary to recognize that the matrices of Chapter V are then actually compound matrices; the submatrices (elements) of these compound matrices form the equations for each branch or mesh in terms of the harmonic components of the branch or mesh voltage, current, and impedance.

CHAPTER VIII

SUMMARY AND CONCLUSIONS

The analysis of a-c circuits was originally performed using unwieldly trigonometric equations. The use of complex numbers, first adapted to a-c circuit analysis by Steinmetz, was a tremendous stride in the advancement of electrical engineering technology. Rather than manipulating equations in terms of instantaneous values, the complex notation expressed a-c quantities in terms of effective (r.m.s.) or average values. Thus answers obtained using the complex notation were the same values that could be read directly from conventional electrical indicating instruments. The contrast between complex and trigonometric equations was equivalent to that between shorthand and longhand. The advantages of the compact complex notation over the cumbersome trigonometric notation were manifold. Within a few years after its inception, the complex notation had supplanted the trigonometric notation in most electrical engineering textbooks; the trigonometric notation was usually retained only for the purpose of introducing the student to the complex notation in a logical manner.

The relative advantages of the complex notation were so manifest that apparently few electrical engineers have ever questioned the validity of the complex representation of a-c circuit quantities. Yet there are glaring defects and inconsistencies in this representation of a-c quantities that cannot be compromised. Several of these deficiencies are as following:

(1) The complex notation can be simultaneously applied to quantities of different frequencies only by exercising great care and with an understanding of complex numbers not possessed by the average junior-level student in electrical engineering. For example, when the complex product of current and voltage is formed to give volt-amperes (a double-frequency quantity), the result is not the correct answer. It is customary in all elementary texts to employ the conjugate of either the current or the voltage simply because this artifice yields the known correct result. In engineering, correct answers are important, but to tacitly inject such an artifice with no explanation is indeed distressing to the true scholar.

(2) The same type of complex notation is used to represent constants and time-functions. No distinction is made between the sinusoidal current $\dot{I} = I' + jI''$ and the constant impedance $\dot{Z} = R + jX$.

(3) The complex notation does not distinguish between two sinusoidal functions of two different frequencies.

A method has been devised in Chapter II and illustrated in Chapter III for representing a-c voltages and currents by complex vectors. Although the representation of a-c voltages and currents by complex vectors was introduced in Chapter II using the intuitive (geometric) concept of a rotating vector, the analytical justification was purposely delayed until Chapter V. The use of the complex vector representation of a-c voltages and currents avoids most of the shortcomings of the complex scalar notation discussed above. At no place in the analysis is the use of artifices necessary in order to obtain the correct results. Rather the rules that have been given for manipulating complex vectors are based upon the metric properties of the space in which the complex vectors are represented. A space in which a rule exists for calculating the distance between two points (the length of a line segment or magnitude of a vector) is called a metric space. In engineering, the most useful expression for the magnitude of a vector is that the square of the real magnitude (norm) be equal to the sum of the squares of the components; a metric space in which such a rule exists is called a Euclidean space. The scalar product of two complex vectors was so formulated in Chapter II that the complex vector space would be Euclidean. The examples solved in Chapter II were included principally to demonstrate that complex vector algebra can be used to solve the simpler types of problems just as readily as complex scalar algebra.

The extension of complex vector analysis to multi-mesh networks adds additional dimensions to the voltage and current vectors. Matrix algebra was introduced in Chapter IV as a tool to be used in manipulating complex vectors of higher dimensions. In matrix notation, a complex current or voltage vector is represented by a one-column matrix the elements of which are the complex coefficients of the different components of the given vector. Matrix algebra is basically a condensed shorthand notation for representing a set (any number) of linear equations. A set of linear equations is a single matrix equation. Thus the use of matrices enables the engineer to focus his attention upon the basic physical principles of the problem which he is solving rather than a host of numbers, variables, and equations. Matrix algebra does not eliminate the necessity of making numerical calculations in the solution of a specific problem; it merely places the numerical calculations in an inconspicuous location at the end of the solution. However, in event that the values of only part of the variables involved are desired, then matrix algebra can greatly reduce the labor involved in making the numerical calculations.

The analytical foundation for the complex vector representation used throughout this dissertaion was presented in Chapter V. A major portion of the analysis presented in Chapter V is the author's adaptation of work previously published by Gabriel Kron and P. Le Corbeiller. The nature of a quantity is determined by the way it behaves when the coordinate system in which it is represented is subjected to a linear transformation. As determined by their laws of transformation, it was established in Chapter V that the various electrical quantities are as following:

(1) current - a tensor of contravariant valence 1 (a vector)

(2) voltage - a tensor of covariant valence 1 (a vector)

(3) impedance - a tensor of covariant valence 2

(4) volt-amperes - a tensor of valence 0 (a scalar)

These tensors are represented in a particular coordinate system by a matrix, the order of which is equal to the valence of the tensor. Therefore for a specific coordinate system (reference frame) currents and voltages (branch or mesh values) were expressed as the elements of 1-way (column) matrices, and impedances were represented by the elements of a 2-way matrix. In electric network analysis, the selection of the required meshes and the assignment of positive directions in the meshes and branches determine the coordinate system.

The components (matrices) of the voltage and current tensors will generally have different values (different elements) in different coordinate systems. In the electrical engineer's language, this means that the mesh currents will be different for each different set of meshes chosen. From a physical viewpoint, this result is almost self-evident. A similar argument is valid for the impedance tensor. The results of the analysis showed that the volt-amperes for a given network were a scalar. In tensor parlance, this means that the volt-amperes were invariant. That is, when the coördinate system was subjected to a linear transformation the volt-amperes did not change. Physically, this is exactly as it should be, since the mere choice of a different set of meshes in the analysis of a network certainly does not alter the volt-amperes input to the network.

The voltages or currents of independent branches or meshes may be considered as the orthogonal components of a generalized space voltage or current vector. Perhaps the most common multi-mesh networks are the variations of three-phase networks. In Chapter VI, several three-phase networks were solved illustrating the correlation between the orthogonal components of space vectors and the elements of 1-way matrices.

The method of analyzing networks using matrices that was developed in Chapter V was extended by the author in Chapter VII to apply to networks with nonsinusoidal applied voltages. The voltages were assumed to be such that they could be analyzed by the methods of Fourier Analysis into the Fourier Sine Series. It was shown that the harmonic components of a nonsinusoidal function, when expressed in complex vector notation, form an orthogonal set. For a single-mesh network the k harmonics of the voltage or current span a k-dimensional orthogonal time subspace of the 1-dimensional voltage or current space vectors. The addition of meshes (or branches) merely changes the space-dimensions of the voltage and current vectors.

The method to be followed in solving single-mesh circuits in which the applied voltage contains harmonics was devised in Chapter VII. The orthogonal harmonic components of the voltage and current vectors were represented by the elements of single-column matrices. The a-c impedances to the different harmonics were represented by the elements on the principal diagonal of a 2-way diagonal matrix. In extending the method of analysis of Chapter V to multi-mesh networks with nonsinusoidal applied voltages, it was necessary to recognize that the matrices of Chapter V were actually compound matrices. The submatrices (elements) of these compound matrices formed the equations for each branch (or mesh) in terms of the harmonic components of the branch (or mesh) voltage, current, and impedance.

A tensor equation is an expression of the natural behavior of several associated physical quantities. It follows then that the form of a tensor equation is the same for all analogous networks. The tensor equations in this dissertation were formulated for stationary networks. One of the most important aspects of tensor analysis is that once a tensor equation has been formulated for a given system, exactly the same equation applies to all analogous systems. Thus the tensor equations which have been established are valid for all stationary networks.

The analysis of electric networks which has been developed in this treatise using matrices and tensors is only a fragmentary portion of the possible applications of these powerful tools in the solution of electrical engineering problems. Engineers are currently using matrix algebra more and more in the analysis of electron tube and transistor circuits. There are myriad other applications of these tools to networks, rotating machinery, etc. With the ever increasing number of variations in circuits and equipments which he must understand, it is becoming increasingly imperative that the electrical engineer visualize physical laws in tensor form and think of the projections of tensors in different coordinate systems as matrices.

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APPENDIX

Key to Notation

Real number - A, D

Complex number - $\dot{Z}$, $\mathring{I}$

Conjugate of Complex number - $\dot{Z}^{*}$, $\dot{I}^{*}$

Real vector - $\underline{E}$, $\underline{A}$

Complex vector - $\underline{\dot{E}}$, $\underline{\mathring{I}}$

Conjugate of complex vector - $\underline{\dot{E}}^{*}$, $\underline{\dot{I}}^{*}$

Complex contravariant vector - $\underline{\dot{I}}^{k}$

Complex covariant vector - $\underline{\dot{E}}_{k}$

Matrix having real elements - $[C]$

Matrix having complex elements - $[\mathring{Z}]$

Matrix composed of branch elements - $[\dot{Z}]$, $[\mathring{I}]$

Matrix composed of mesh elements - $[\mathring{Z}']$, $[\mathring{I}']$

Total current in branch one - $\underline{\mathring{I}}^{1}$, $[\mathring{I}^{1}]$

Second harmonic current in branch one - ${}^{2}\underline{\dot{I}}^{1}$

Transpose of matrix - $[\mathring{Z}_{t}]$

Inverse of matrix - $[C^{-1}]$

$\sqrt{-1} = j$

VITA

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Thesis: THE APPLICATION OF MATRICES AND TENSORS TO THE ANALYSIS OF ELECTRIC NETWORKS

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