THE DERIVATION OF AN ELASTIC RESTRAINT EQUATION FOR THE COMBINED TOP AND SEAT WITH WEB ANGLE SEMI-RIGID BEAM-COLUMN CONNECTION

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PREFACE

During the past twenty years, much interest has been shown[1 2 3 4 5] by the structural engineering profession, in a new type of beam-column connection. This connection has come to be known as the Semi-rigid connection and is now recognized by the AISC[6] specification as one of three types of connection. Its chief advantage is that of economy over the two other types.

There have been two disadvantages to the semi-rigid connection. One of these lies in the complicated nature of the resulting moment analysis and the other in the expensive and time consuming laboratory experiments necessary for each connection before analysis may be attempted. This thesis is a step toward the elimination of the second difficulty mentioned above.

Strictly speaking, all steel beam-column connections which are made through the medium of outstanding flanges are semi-rigid. This is true because of the tendency of these flanges to deflect. There are five main types as follows: (1) the bracket type[7], (2) the split-beam type[7], (3) the web-angle type[4], (4) the top angle with seat angle[8] and (5) a combination of the last two types.

For all practical purpose, types (1) and (2) may be considered as rigid. For the other three types a constant Z is necessary for analytical purposes. In the past, this constant has been found by measurement in the laboratory. During the past three or four years, equations for the computation of Z have been derived. The first by Professor J. E. Lothers was derived for type (3) in 1951[4] and the second was derived by Shan Yuan Yu for type (4) in 1953[8] under Professor Lothers' direction.

A derivation of an equation for Z for the type (5) Semi-rigid connection is the subject of this thesis.

This thesis was written under Professor J. E. Lothers of school of architectural engineering at Oklahoma A. and M. College. Grateful acknowledgement is due to Professor Lothers for his advice and encouragement as well as for the procedure laid down in his paper[4].

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ANALYSIS

INTRODUCTION

The analysis of the top, seat, and web connection angles is based on the bending strength of the angles and is solved by the slope deflection method. The procedure for the derivation of the necessary equations is as follows:

(1) Find the critical moment in the legs of the connection angles by the slope deflection method.

(2) Find the deflection Δ_B and Δ'_B.

(3) Find the horizontal pull P_1, P_2, and P'.

(4) Find the angle of strain, ϕ.

(5) Establish the relations among P_1, P_2 and P'.

(6) Find the resisting moment of the connection angles.

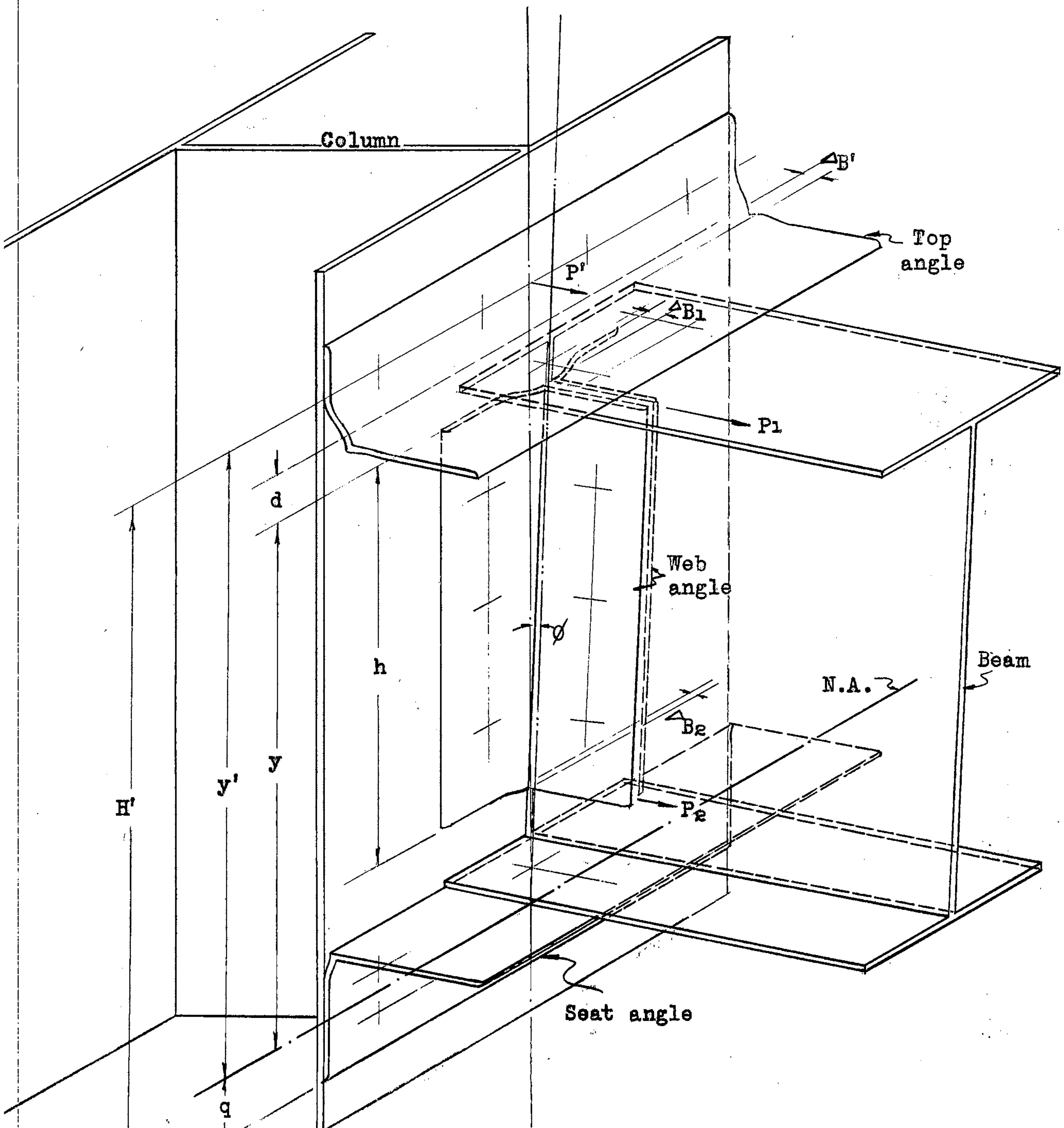
(7) Locate the neutral axis.

(8) Derive the equation of semi-rigid connection factor, Z.

(9) Make a comparison between the derived equation and the published laboratory results.

(1) The Critical Moment in the Legs of the Connection Angles.

For web-angles, at the top of the connection angles the bending moments of points A, B and C for load P_1 can be solved by the slope deflection method. A horizontal pull P_1 which is caused by the bending moment of the beam acts on the outstanding legs and pulls the heel away from the column, the maximum deflection is at the top of the connection angles and is expressed by Δ_{B1} (Fig.3). Similarly for top and seat angles, the horizontal force pulls the legs of the angles and causes a maximum deflection Δ'_B at the heel of the top angle. (Fig.4). The notation are shown in the figure.



Combined Top and Seat with Web angle
Semi-rigid Beam-column Connection

Fig. 1

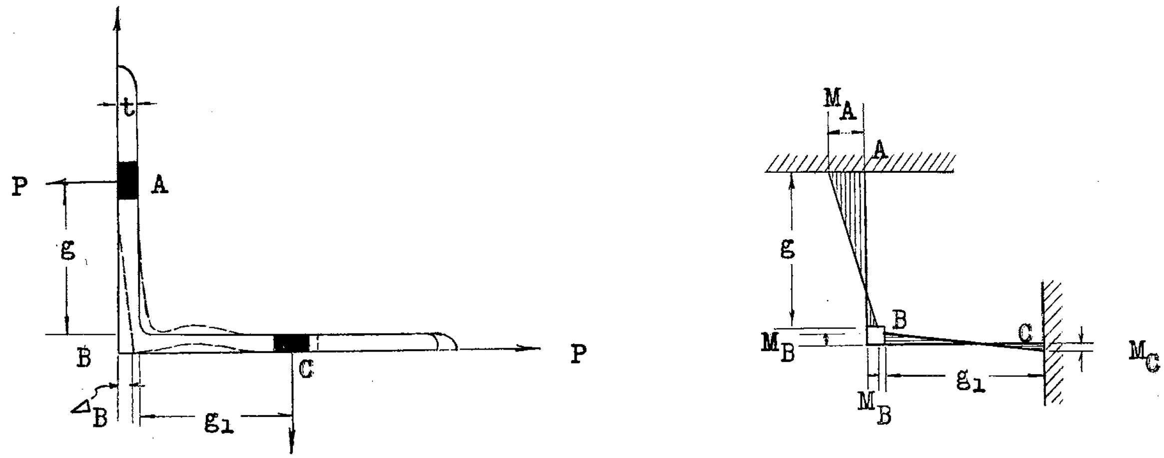
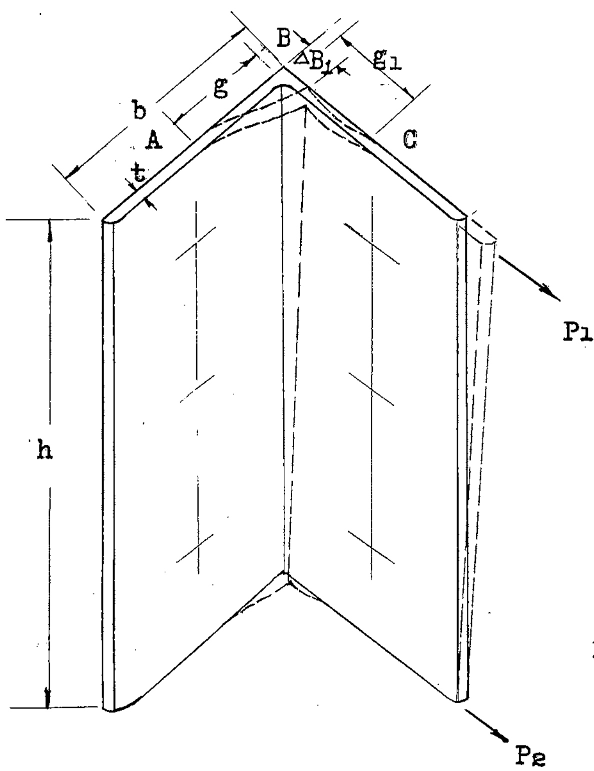
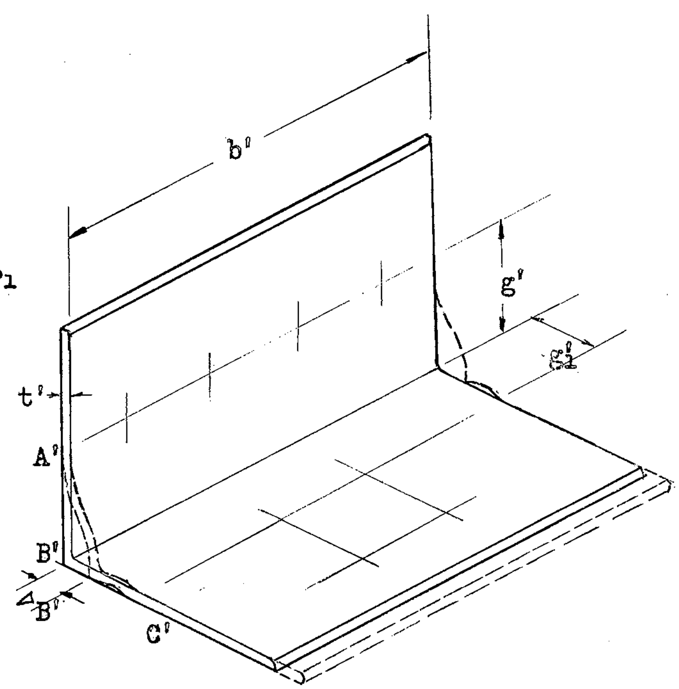


Fig. 2



Web angle

Fig. 3



Top angle

fig. 4

(A) Solve the End Moments for the Web Angles by the Slope Deflection Method.

From Figs. 2 and 3.

$L_{AB} = g, \quad L_{BC} = g_1, \quad \theta_A = 0, \quad \theta_C = 0, \quad R_{BC} = 0.$

(a) Elastic Equations.

$$M_{AB} = 2E\frac{I}{L_{AB}}(2\theta_A + \theta_B - 3R_{AB}) = 2E\frac{I}{g}(\theta_B - 3R_{AB}) \qquad (1)$$

$$M_{BA} = 2E\frac{I}{L_{AB}}(2\theta_B + \theta_A - 3R_{AB}) = 2E\frac{I}{g}(2\theta_B - 3R_{AB}) \qquad (2)$$

$$M_{BC} = 2E\frac{I}{L_{BC}}(2\theta_B + \theta_C - 3R_{BC}) = 2E\frac{I}{g_1}(2\theta_B) \qquad (3)$$

$$M_{CB} = 2E\frac{I}{L_{BC}}(2\theta_C + \theta_B - 3R_{BC}) = 2E\frac{I}{g_1}(\theta_B) \qquad (4)$$

(b) Joint Equation.

$\Sigma M_B = 0$, i.e. $M_{BA} + M_{BC} = 0.$

From Eq. (2) and (3),

$$2E\frac{I}{g_1}(2\theta_B - 3R_{AB}) + 2E\frac{I}{g_1}(2\theta_B) = 0 \qquad (5)$$

(c) Shear Equation.

$M_{AB} + M_{BA} = P_1 g$

From Eq. (1) and (2)

$$2E\frac{I}{g}(3\theta_B - 6R_{AB}) = P_1 g \qquad (6)$$

Solve Eqs. (5) and (6), get

$$\theta_B = -\frac{P_1 g^2}{2EI} \times \frac{g_1}{4g+g_1} \qquad (7)$$

$$R_{AB} = -\frac{P_1 g^2}{3EI} \times \frac{g+g_1}{4g+g_1} \qquad (8)$$

(d) End Moments.

Substitute values of θ_B and R_{AB} into Eqs.(1),(2),(3) and (4).

$$M_{AB} = 2E\frac{I}{g}\left(-\frac{P_1g^2g_1}{2EI(4g+g_1)} - 3\,\frac{-P_1g^2(g+g_1)}{3EI\;(4g+g_1)}\right)$$

$$= +\,P_1g\frac{2g+g_1}{4g+g_1} \qquad (9)$$

$$M_{BA} = 2E\frac{I}{g}\left(-\frac{2P_1g^2g_1}{2EI(4g+g_1)} - 3\,\frac{-P_1g^2(g+g_1)}{3EI\;(4g+g_1)}\right)$$

$$= +\,P_1g\,\frac{2g}{4g+g_1} \qquad (10)$$

$$M_{BC} = 2E\frac{I}{g_1}\left(-\,2\frac{P_1g^2g_1}{2EI(4g+g_1)}\right) = -\,P_1g\frac{2g}{4g+g_1} \qquad (11)$$

$$M_{CB} = 2E\frac{I}{g_1}\left(-\,\frac{P_1g^2g_1}{2EI(4g+g_1)}\right) = -\,P_1g\frac{g}{4g+g_1} \qquad (12)$$

(B) Solve the End Moments for Top And Seat angles.

From Fig. 4

$L_{A'B'} = g'$, $\quad L_{B'C'} = g_1'$, $\quad \theta_{A'} = 0$, $\quad \theta_{C'} = 0$, $\quad R_{B'C'} = 0$.

Similarly we get

$$M_{A'B'} = +\,P'g'\,\frac{2g'+g_1'}{4g'+g_1'} \qquad (13)$$

$$M_{B'A'} = +\,P'g'\,\frac{2g'}{4g'+g_1'} \qquad (14)$$

$$M_{B'C'} = -\,P'g'\,\frac{2g'}{4g'+g_1'} \qquad (15)$$

$$M_{C'B'} = -\,P'g'\,\frac{g'}{4g'+g_1'} \qquad (16)$$

(2) The Deflection Δ_{B1} (or $\Delta_{B'}$) of the Connection Angles.

(A) For web angles.

By definition $R_{AB} = \frac{\Delta_{B1}}{g}$, substitute value of R_{AB} and ignore the minus sign.

$$\triangle_{B1} = R_{AB} \times g = \frac{P_1 g^2}{3\,E\,I} \times \frac{g + g_1}{4g + g_1} \times g = \frac{P_1 g^3}{3\,E\,I} \times \frac{g+g_1}{4g+g_1} \tag{17}$$

(B) For Top and Seat Angles.

$$\triangle_{B'} = \frac{P' g'^3}{3\,E\,I} \times \frac{g' + g_1'}{4g' + g_1'} \tag{18}$$

(3) The horizontal Pull P_1, P_2, (or P') of the Connection Angles.

(A) For web angles.

At the top of the connection angles, the resisting moment of a 1-in. strip is governed by its bending strength due to M_{AB}, but $M_{AB} = \frac{Is}{c}$, in which I is the moment of inertia of a 1-in. strip, s is the bending stress, and c is half the thickness of the leg of the angles.

From Eq. (9)

$$M_A = P_1 g \frac{2g + g_1}{4g + g_1} = \frac{I\,s}{c},$$

Substituting $I = \frac{1}{12}(t^3)$, $c = \frac{t}{2}$.

$$P_1 = \frac{2\,I\,s}{g\,t} \times \frac{4g + g_1}{2g + g_1} = \frac{st^2}{6g} \times \frac{4g + g_1}{2g + g_1} \tag{19}$$

(B) For Top and Seat Angles.

$$M_{A'} = P' g' \frac{2g' + g_1'}{4g' + g_1'} = \frac{I' s}{c'}$$

in which $I' = \frac{b' t'^3}{12}$, $c' = t'/2$

$$P' = \frac{2\,I' s}{g' t'} \times \frac{4g' + g_1'}{2g' + g_1'} = \frac{b' t'^2 s}{6g'} \times \frac{4g' + g_1'}{2g' + g_1'} \tag{20}$$

(4) The Angle of Strain, ϕ.

(A) For web angles.

Referring to Fig.1 and Fig.5, the angle of strain ϕ is equal to the deflection at the top of connection angle $\triangle_{B1}$ divided by the distance from the top of the connection angles to the neutral axis.

$$\phi = \frac{\Delta B_1}{y} = \frac{P_1 g^3}{3EIy} \times \frac{g + g_1}{4g + g_1}$$

Substituting the value of P_1 from Eq.19 and putting $I = t^3/12$. Then

$$\phi = \frac{2g^2 s}{3Ety} \times \frac{g + g_1}{2g + g_1} \qquad (21)$$

(B) For top and seat angles.

Similarly as in the case of the web angles, referring to Fig. 1 and Fig. 5,

$$\phi = \frac{\Delta B'}{y+d} = \frac{P' g'^3}{3EI'(y+d)} \times \frac{g' + g_1'}{4g' + g_1'}$$

in which,

$$P' = \frac{b' t'^2 s}{6g'} \times \frac{4g' + g_1'}{2g' + g_1'}, \qquad I' = \frac{b' t'^3}{12}$$

simplify the above equation; then

$$\phi = \frac{2sg'^2}{3Et'(y+d)} \times \frac{g' + g_1'}{2g' + g_1'} \qquad (22)$$

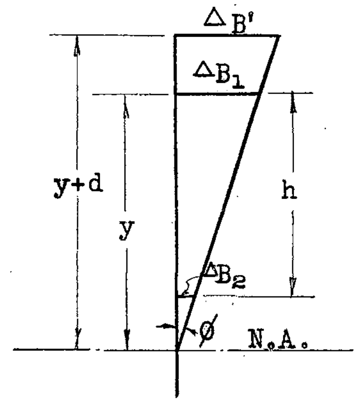


Fig. 5
Relation of Δ and P

(5) Relation Among P_1, P_2 and P'.

Referring to Fig. 5 the ratio of deflection is proportional to the ratio of the distance from neutral axis. Therefore

$$\frac{\Delta B_1}{\Delta B'} = \frac{y}{y + d}$$

from Eq. 17 and Eq. 18, substituting the values of Δ_{B_1} and $\Delta_{B'}$ into the above equation. Then

$$\frac{\Delta B_1}{\Delta B'} = \frac{y}{y + d} = \frac{\frac{P_1 g^3}{3EI} \times \frac{g+g_1}{4g+g_1}}{\frac{P' g'^3}{3EI'} \times \frac{g'+g_1'}{4g'+g_1'}}$$

simplify the above equation,

$$P_1 = \left(\frac{I}{I'}\right)\left(\frac{g'^3}{g^3}\right)\left(\frac{4g+g_1}{4g'+g_1'}\right)\left(\frac{g'+g_1'}{g+g_1}\right)\left(\frac{y}{y+d}\right) P'$$

$$= \alpha \frac{y}{y + d} P' \qquad (23)$$

where $\alpha = \frac{I}{I'} \times \frac{g'^3}{g^3} \times \frac{4g+g_1}{4g'+g_1'} \times \frac{g'+g_1'}{g+g_1}$

but $I = t^3/12$, $I' = b't'^3/12$. Then

$$\alpha = \frac{1}{b'} \times \frac{g'^3 t^3}{g^3 t'^3} \times \frac{4g + g_1}{4g'+g_1'} \times \frac{g' + g_1'}{g + g_1} \qquad (24)$$

similarly we get

$$P_2 = \alpha \frac{y-h}{y+d} P' \qquad (25)$$

$$P_1 - P_2 = \alpha \frac{P'}{y+d} (y-y+h) = \frac{\alpha h P'}{y+d} \qquad (26)$$

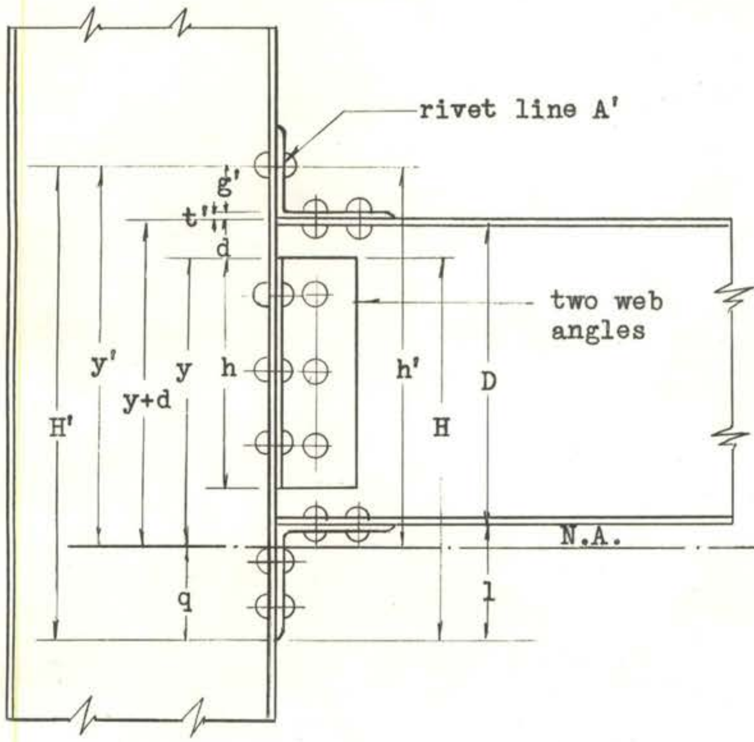
(6) The Resisting Moment of the Connection Angles.

Referring to Fig. 6 and Fig. 7, force P' for the top angle is the maximum pull. The resisting couple due to P' should be equal $M' = P'(y' + 2/3\, q)$,

where P' is maximum pull in pounds, y' is the distance between neutral axis and rivet line A', q is the distance between the neutral axis and the bottom of the seat angle.

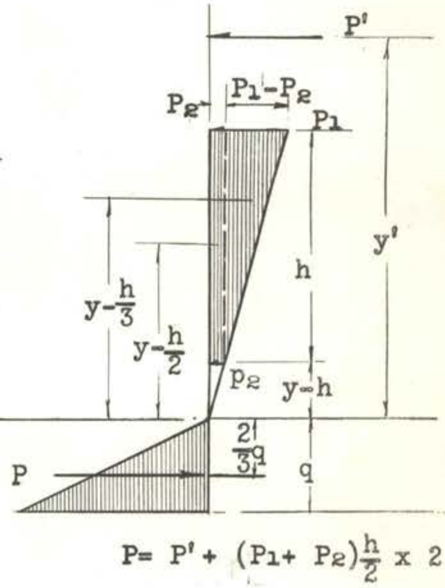
The resisting moment due to P_1 and P_2 for the web angles is equal to

$$M_{12} = 2\left[P_2 h \left(y - \frac{h}{2} + \frac{2}{3} q \right) + \frac{1}{2} (P_1 - P_2) \, h \times \left(y - \frac{h}{3} + \frac{2}{3} q \right)\right]$$



Detail of connection angles

Fig. 6



Resisting couple for all the angles

Fig. 7

where P_1 and P_2 are the pulling forces at the top and bottom of the connection angles, respectively, and are in pounds per inch of depth, per angle, h is the depth of the web angles.

The combined resisting moment is

$$M = M' + M_{12}$$
$$= P'\left(y' + \frac{2}{3}q\right) + 2\left[P_2 h\left(y - \frac{h}{2} + \frac{2}{3}q\right) + \frac{1}{2}(P_1 - P_2)h \times \left(y - \frac{h}{3} + \frac{2}{3}q\right)\right]$$

Substituting the value of P_1, P_2 and $P_1 - P_2$, from Eqs. 23, 25, and 26 into above Eq. and simplifying,

$$M = P'\left\{(y'+\frac{2}{3}q)+\frac{2\alpha h}{y+d}\left[(y-h)(y-\frac{h}{2}+\frac{2}{3}q)+\frac{1}{2}h(y-\frac{h}{3}+\frac{2}{3}q)\right]\right\}$$

$$= P'\left\{(y'+\frac{2}{3}q)+\frac{2\alpha h}{3(y+d)}\left[y^2+(H-h)(2y-h)\right]\right\}$$

$$= \frac{P'}{3}\left\{(y'+2H')+\frac{2\alpha h}{y+d}\left[y^2+(H-h)(2y-h)\right]\right\}$$

$$= \frac{b't'^2s}{18g'}\ \frac{4g'+g_1'}{2g'+g_1'}\left\{(y'+2H')+\frac{2\alpha h}{y+d}\left[y^2+(H-h)(2y-h)\right]\right\} \qquad (27)$$

(7) Location of the Neutral Axis.

For the location of the neutral axis, the transformed section in reinforced concrete beam analysis is adopted.[4, 8]

(a) The shear area for top angle along rivet line A' is

$A_s' = b't'$

but the shear area for two web angles along rivet line A is

$A_s = 2ht$

(b) The compression area A_c of the seat angle below the neutral axis referring to Fig. 6 and Fig. 7 is

$A_c = b'(H-y)$

in which b' is the width of seat angle, H is the distance between the top of web angles and the bottom of seat angle and y is the distance between the top of the web angle and the neutral axis.

(c) Transformed section area.

The stress-ratio factor of bending and shear stress for the top angle can be expressed as

$$m' = \frac{v'}{s}, \qquad v' = m's', \qquad \text{but}$$

$$v' = \frac{P'}{A_s'} = \frac{1}{b'\ t'}\times\frac{b'st'^2}{6g'}\times\frac{4g'+g_1'}{2g'+g_1'} = \frac{t'(4g'+g_1')}{6g'(2g'+g_1')}s$$

$$m' = \frac{t' \ (4g' + g_1')}{6g'(2g' + g_1')} \qquad (28)$$

where $m' = 1/n'$, refer to page 11, The derivation of an equation for the elastic restraint of the top and seat angle type semi-rigid connection — Yu, 1953. v' is the shearing stress in the top angle.

For web angles,

$m = \frac{v}{s}$, $v = m\ s$, but

$v = \frac{P_1}{A_s}$, where v is the shearing stress in the web angles, P_1 is the total shear force on the top inch of angle, and A_s is the area of the top inch of one web angle. From Eq. 19,

$$v = \frac{1}{t} \times \frac{st^2}{6g} \times \frac{4g + g_1}{2g + g_1} = \frac{t(\ 4g + g_1)}{6g(\ 2g + g_1)}\ s = m\ s$$

$$m = \frac{t(\ 4g + g_1)}{6g(\ 2g + g_1)} \qquad (29)$$

where $m = 1/n$, refer to page 485, ASCE Transaction Vol. 116, 1951.

Assume the ratio of bending stress to strain in the column-connected leg of the angle along the rivet line A' and A are the same as that in compression below the neutral axis, therefore, the transformed area (or shear area) is as follows:

for top angle = $m' A_s' = m'(b' t')$

for web angle = $m\ A_s = m\ (2ht\ )$

(d) Static moment of whole area about Neutral Axis,

Case I: Neutral axis is above the rivet line of the column connected leg of the seat angle.

$\Sigma M_{N.A.} = 0$, referring to Fig. 6,

$$b't'm'\left[y + (d+t'+g')\right] + 2htm\left(y-\frac{h}{2}\right) = b'(H-y)\frac{H-y}{2}$$

$$y = \frac{(b'H+2htm+b't'm' - \sqrt{(b'H+2htm+b't'm')^2 - b'\left[2h^2tm - 2b't'm'(d+t'+g') + b'H^2\right]}}{b'}$$

In the above equation such terms as 2htm, $b't'm'$, $4htmb't'm'$, $b'^2t'^2m'^2$ and $4h^2t^2m^2$ are relatively small. Therefore neglect these terms and simplify the above equation as

$$y = \frac{b'H - \sqrt{2b'htm(2H-h) + 2b'^2t'm'H'}}{b'} = H - \sqrt{\frac{2htm(2H-h)}{b'} + 2t'm'H'} \qquad (30)$$

The notation representing the dimensions are shown in Fig. 3 and Fig. 6.

Case II: When neutral axis is under the rivet line of the column connected leg of the seat angle, the effect of the seat angle is negligibly small and may be neglected.[8]

(8) The Semi-rigid Connection Factor, Z.

By definition, the semi-rigid connection factor Z is the angle change for unit moment, and the reciprocal, 1/Z is the slope of the moment-rotation curve.[2]

$Z = \frac{\phi}{M}$, From Eq. 22 and Eq. 27,

$$Z = \frac{\dfrac{2sg'^2}{3Et'(y+d)} \times \dfrac{g' + g_1'}{2g' + g_1'}}{\dfrac{b't'^2s}{18g'} \times \dfrac{4g'+g_1'}{2g'+g_1'}\left\{(y'+2H') + \dfrac{2\alpha h}{y+d}\left[y^2+(H-h)(2y-h)\right]\right\}}$$

$$= \frac{12g'^3}{Eb't'^3(y+d)\left\{(y'+2H') + \dfrac{2\alpha h}{y+d}\left[y^2+(H-h)(2y-h)\right]\right\}} \times \frac{g'+g_1'}{4g'+g_1'} \qquad (31)$$

(9) The Comparison of the Equation With the Published Laboratory Results.[2]

For a check of the above derived equation (31), with the laboratory results which is published by Professor Rathbun's paper in ASCE Transactions, Vol. 101, 1936. Use Eq. 31 to find Z and 1/Z. Draw the tangent line on Moment-Rotation Curve for Specimens 11 and 12 in Fig. 10 and Fig. 11.

(A) Specimen 11

$b' = 9$ in. $t' = 3/8$ in. $= 0.375$ in.

$g' = 2\ 1/2$ in. $- t' = 2.500 - 0.375 = 2.125$ in.

$g_1' = 2\ 1/4$ in. $- t' = 2.250 - 0.375 = 1.875$ in.

$t = 3/8$ in. $= 0.375$

$g_1 = 2\ 1/4$ in. $- t = 2.250 - 0.375 = 1.875$ in.

$g = \frac{1}{2}(5\frac{1}{2} - \frac{3}{8}) - 0.375 = 2.1875$ in.

$d = 1/2 (12 - 9) = 1.50$ in.

$h = 9$ in. $H = 12 + 6 - 1.5 = 16.50$ in.

$H' = 12 + 6 + 2.50 = 20.50$ in.

from Eq. (28)

$$m' = \frac{0.375 (4 \times 2.125 + 1.875)}{6 \times 2.125 (2 \times 2.125 + 1.875)} = 0.0498$$

from Eq. (29)

$$m = \frac{0.375 (4 \times 2.1875 + 1.875)}{6 \times 2.1875 (2 \times 2.1875 + 1.875)} = 0.0486.$$

from Eq. (30)

$$y = 16.5 - \sqrt{\frac{2 \times 9 \times 0.375 \times 0.0486 (2 \times 16.5 - 9)}{9} + 2 \times 0.375 \times 0.0498 \times 20.5}$$

$= 15.22$ in.

$y' = 15.22 + 1.50 + 0.375 + 2.125 = 19.22$ in.

from Eq. (24)

Specimen 11.

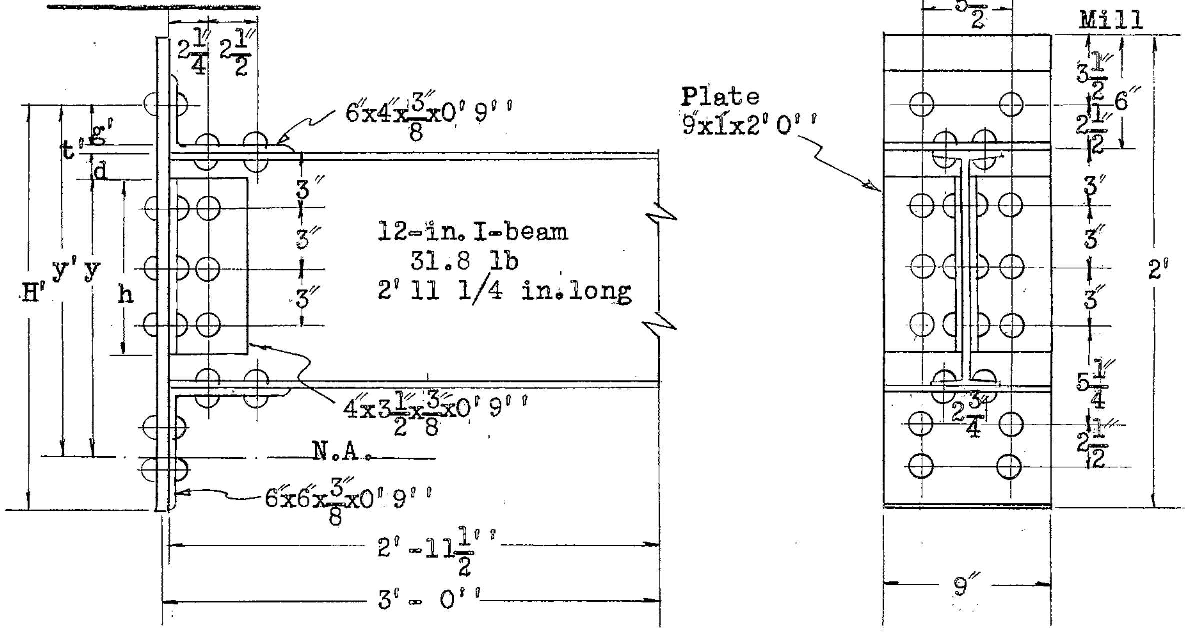


Fig. 8

Specimen 12.

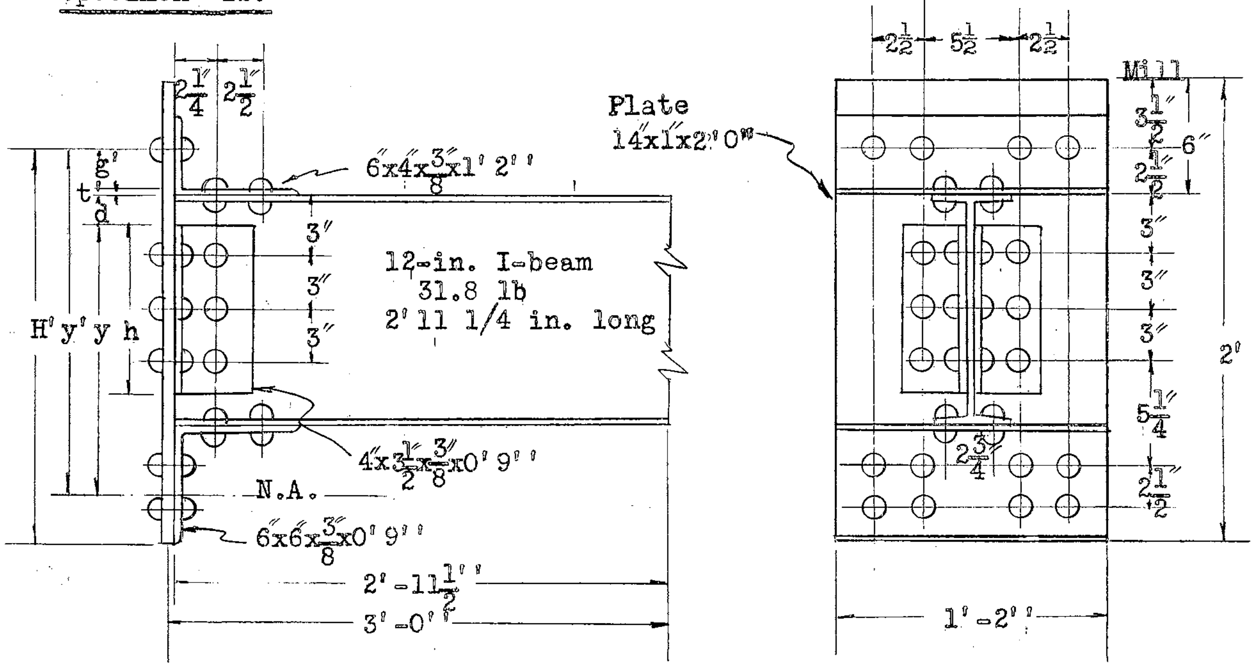


Fig. 9

$$\alpha = \frac{1}{9} \times \left(\frac{2.125 \times 0.375}{2.1875 \times 0.375}\right)^3 \times \frac{4 \times 2.1875 + 1.875}{4 \times 2.1250 + 1.875} \times \frac{2.125 + 1.875}{2.1875 + 1.875}$$

$$\doteq 0.0715$$

from Eq. (31)

$$Z = (12 \times 2.125^3)/29 \times 10^6 \times 9 \times 0.375^3 (15.22 + 1.5) \Big\{ (19.22 + 2 \times 20.5) + \frac{2 \times 0.0715 \times 9}{15.22 + 1.5} \big[15.22^2 + (16.5 - 9)(2 \times 15.22 - 9) \big] \Big\} \times \frac{2.125 + 1.875}{4 \times 2.125 + 1.875} = \frac{1}{469.0 \times 10^6}$$

$\frac{1}{Z} = 469.0 \times 10^6$ This defines the slope of the moment-rotation curve of specimen 11, Fig. 18, of Professor Rathbun's paper, ''Elastic properties of riveted connections'' Transactions ASCE, Vol. 101, 1936. From Fig. 10 the slope as computed by Eq. (31) is seen to fall tangent to Professor Rathbun's curve.

(B) Specimen 12.

$b' = 14$ in.

Other dimensions are the same as Specimen 11,

$m' \doteq 0.0498$ (Same as specimen 11)

$m = 0.0486$ (Same as Specimen 11)

from Eq. (30)

$$y = 16.5 - \sqrt{\frac{2 \times 9 \times 0.375 \times 0.0486 (2 \times 16.5 - 9)}{14} + 2 \times 0.375 \times 0.0498 \times 20.5}$$

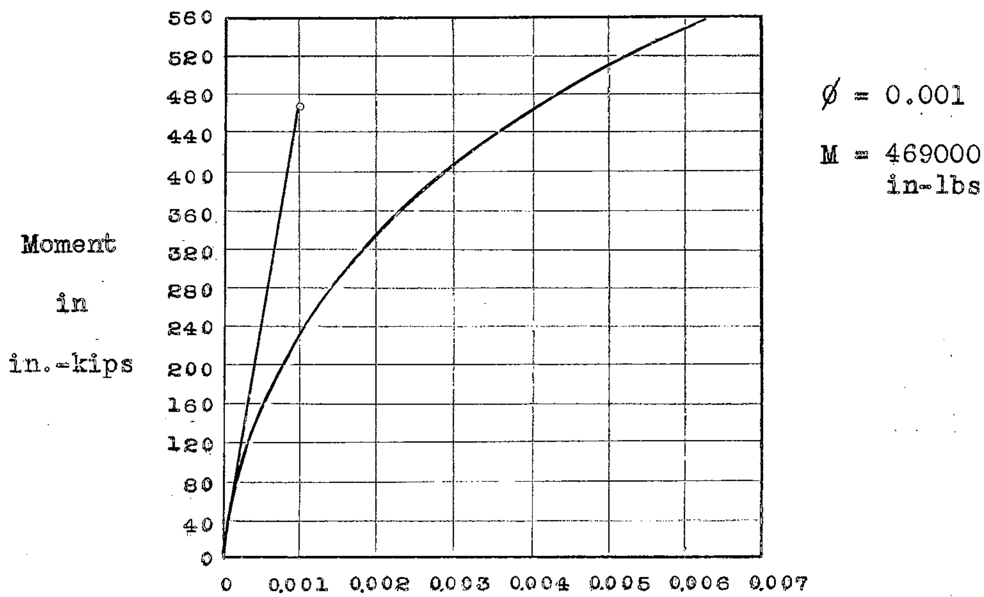
$= 15.35$ in. $y' = 15.35 + 1.5 + 0.375 + 2.215 = 19.35$ in.

from Eq. (24) $\alpha = 0.046$ from Eq. (31)

$$Z = (12 \times 2.125^3)/29 \times 10^6 \times 14 \times 0.375^3 \times (15.35 + 1.5) \Big\{ 19.35 + 2 \times 20.5) + \frac{2 \times 0.046 \times 9}{15.35 + 1.5} \big[15.35^2 + (16.5 - 9)(2 \times 15.35 - 9) \big] \Big\} \times \frac{2.125 + 1.875}{4 \times 2.125 + 1.875} = \frac{1}{649.3 \times 10^6}$$

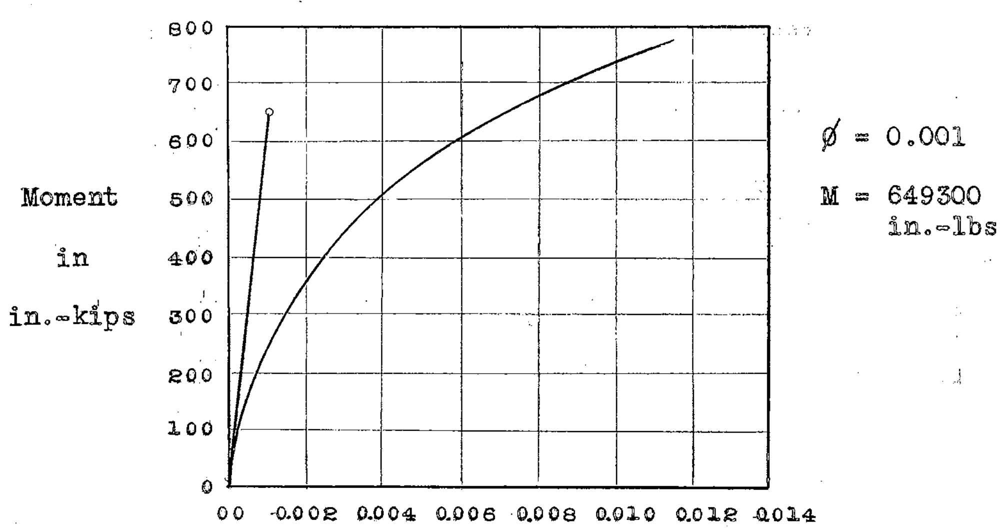
$$\frac{1}{Z} = 649.3 \times 10^6$$

See Fig. 11 for the slope, as computed above, plotted on Professor Rathbun's Specimen 12.[2]



Moment Rotation Curve for Specimen 11

Fig. 10



Moment Rotation Curve for Specimen 12

Fig.11

EXAMPLE

The economy in bent ABCD by using semi-rigid connections as compared to rigid connections can be demonstrated by the following example. All the dimensions and loads are as shown in Fig. 12.

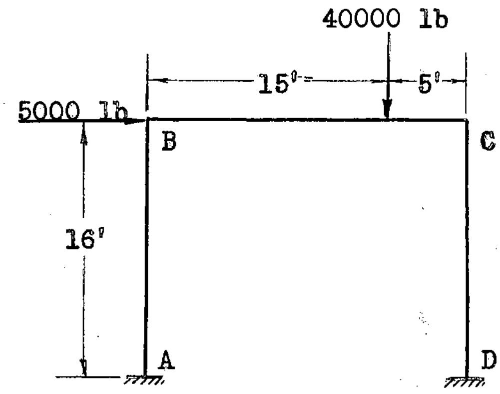


Fig. 12

I. Rigid Connections at Joint B and C.

A. Solved By Slope Deflection Method.

(1) Assumed sections.

For AB 8 WF 31

For BC 16 WF 40

For CD 10 WF 49

(2) Stiffness factors. (K's)

$$K_{AB} = \frac{I_{AB}}{L_{AB}} = \frac{109.7}{16} = 6.86 \qquad K_{BC} = \frac{I_{BC}}{L_{BC}} = \frac{515.5}{20} = 25.78$$

$$K_{CD} = \frac{I_{CD}}{L_{CD}} = \frac{272.9}{16} = 17.06$$

(3) Fixed end moments. (FM's)

$$FM_{BC} = -\frac{Pab^2}{L^2} - \frac{1}{12} w_{D.L.} L^2 = -\frac{40000 x 15 x 5^2}{20^2} - \frac{40 x 20^2}{12}$$

$$= -38833 \text{ ft-lb.}$$

$$FM_{CB} = +\frac{Pa^2b}{L^2} + \frac{1}{12} w_{D.L.} L^2 = +\frac{40000 x 15^2 x 5}{20^2} + \frac{40 x 20^2}{12}$$

$$= +113833 \text{ ft-lb.}$$

(4) Elastic Equations.

$\theta_A = 0, \quad \theta_D = 0, \quad R_{BC} = 0, \quad R_{AB} = R_{CD} = R$

$M_{AB} = 2EK_{AB}(2\theta_A + \theta_B - 3R) = 13.72E\theta_B - 41.16ER.$

$M_{BA} = 2EK_{BA}(2\theta_B + \theta_A - 3R) = 27.44E\theta_B - 41.16ER$

$M_{BC} = 2EK_{BC}(2\theta_B + \theta_C - 3R) - FM_{BC} = 103.12E\theta_B + 51.56E\theta_C - 38833$

$M_{CB} = 2EK_{CB}(2\theta_C + \theta_B - 3R) + FM_{CB} = 51.56E\theta_C + 103.12E\theta_B + 113833$

$M_{CD} = 2EK_{CD}(2\theta_C + \theta_D - 3R) = 68.24E\theta_C - 102.36ER$

$M_{DC} = 2EK_{DC}(2\theta_D + \theta_C - 3R) = 34.12E\theta_C - 102.36ER$

(5) Joint Equations.

$\Sigma M_B = 0 \qquad M_{BA} + M_{BC} = 0$

$$130.56E\theta_B + 51.56E\theta_C - 41.16ER - 38833 = 0 \tag{1}$$

$\Sigma M_C = 0 \qquad M_{CB} + M_{CD} = 0$

$$51.56E\,\theta_B + 171.36E\theta_C - 102.36ER + 113833 = 0 \tag{2}$$

(6) Shear Equations.

$R_{AX} + R_{DX} + 5000 = 0$

but $\quad R_{AX} = \dfrac{M_{AB} + M_{BA}}{16} \quad , \qquad R_{DX} = \dfrac{M_{CD} + M_{DC}}{16}$

$$M_{AB} + M_{BA} + M_{CD} + M_{DC} + 80000 = 0$$

i.e. $$41.16E\theta_B + 102.36E\theta_C - 287.04ER + 80000 = 0 \tag{3}$$

(7) Solution by Gauss Method and Iteration Method.[9 10]

There are three unknowns in above equations. The Gauss method is used for obtaining approximate values, the accurate values can be obtained by Iteration.

EQUA. NO.	E_B	E_C	ER	CONSTANT TERM	CHECK TERM	PROCEDURE
(1)	+130.560	+51.560	- 41.160	-38833.000	-38692.040	
(2)	+ 51.560	+171.360	-102.360	+113833.000	+113953.560	
(3)	+ 41.160	+102.360	-287.040	+80000.000	+79856.480	
(1)	+ 1.000	+ 0.395	- 0.315	- 297.434	-296 .354	(1)÷130.560
(2)	+ 1.000	+ 3.324	- 1.985	+ 2207.777	+ 2210.116	(2)÷51.560
(3)	+ 1.000	+ 2.487	- 6.974	+ 1943.635	+ 1940.148	(3)÷41.160
(4)		+ 2.929	- 1.670	+ 2505.211	+ 2506.470	(2)-(1)
(5)		+ 2.092	- 6.659	+ 2241.069	+ 2236.502	(3)-(1)
(4)		+ 1.000	- 0.570	+ 855.313	+ 855.743	(4)÷2.929
(5)		+ 1.000	- 3.183	+ 1071.257	+ 1069.074	(5)÷2.092
(6)			+ 2.613	- 215.944	- 213.331	(4)-(5)
(6)			+ 1.000	- 82.642	- 81.642	(6)÷2.613
Gauss Solution	+642.708	-808.207	+ 82.642			
Iteration	+642.660	-808.309	+ 82.657	1st.	Approximation	
	+642.705	-808.285	+ 82.613	2nd.	Approximation	
	+642.682	-808.325	+ 82.629	3rd.	Approximation	
	+642.702	-808.308	+ 82.611	4th.	Approximation	
	+642.690	-808.325	+ 82.620	5th.	Approximation	
	+642.700	-808.316	+ 82.612	6th.	Approximation	
	+642.694	-808.324	+ 82.616	7th.	Approximation	
	+642.698	-808.320	+ 82.613	8th.	Approximation	
	+642.696	-808.323	+ 82.615	9th.	Approximation	
	+642.697	-808.321	+ 82.614	10th.	Approximation	
	+642.697	-808.321	+ 82.614	11th.	Approximation	

(8) The End Moments.

M_{AB} = 13.72(+642.697)-41.16x(+82.614) = +5417.4 ft-lb

M_{BA} = 27.44(+642.697)-41.16x(+82.614) =+14235.2 ft-lb

M_{BC} =103.21(+642.697)+51.56x(-808.321)-38833 = -14235.1 ft-lb

M_{CB} = 51.56(+642.697)+103.12x(-808.321)+1138833 = +63616.4 ft-lb

M_{CD} = 68.24(-808.321)-102.36x(+82.614) = -63616.5 ft-lb

M_{DC} = 34.12(-808.321)-102.36x(+82.614) = -36036.4 ft-lb

B. Solve End Moments by Moment Distribution Method.[11]

(By Series)

(1) Stiffness Factors. (K's)

$K_{AB} = 6.86$, $K_{BC} = 25.78$, $K_{CD} = 17.06$

(2) Distribution Factors. (D's)

$D_{BA} = \frac{6.86}{6.86+25.78} = 0.21$ $D_{BC} = \frac{25.78}{6.86+25.78} = 0.79$

$D_{CB} = \frac{25.78}{25.78+17.06} = 0.60$ $D_{CD} = \frac{17.06}{25.78+17.06} = 0.40$

(3) Fixed End Moments. (FM's)

$FM_{BC} = -\ 38833$ ft-lb $FM_{CB} = +\ 113833$ ft-lb

(4) No Sidesway.

Joints	A	B		C		D
Moment	AB	BA	BC	CB	CD	DC
D'S	0	0.21 (a)	0.79 (b)	0.60 (c)	0.40 (d)	0
FM'S	0	0	-38833.000	+113833.000	0	0
1st. cycle	0	+ 8154.930	+30678.070	- 68299.800	-45533.200	0
Carry over	+ 4077.470	0	-34149.900	+ 15339.040	0	-22766.600
2nd. cycle	0	+ 7171.479	+26978.420	- 9203.424	- 6135.616	0
Carry over	+ 3585.739	0	- 4601.712	+ 13489.210	0	- 3067.808
1st. 2nd. C.O.	+7663 .209	+15326.409	+18904.879	- 48674.974	-51668.816	-25834.408

$x = 1 - \frac{bc}{4} = 1 - \frac{0.79x0.60}{4} = 0.8815$

$M^1_{AB} = \frac{7663.209}{0.8815} = +\ 8693.373$ $M^1_{BA} = +\frac{15326.409}{0.8815} = +17386.730$

$M^1_{BC} = -38833 + \frac{18904.879}{0.8815} = -\ 17386.740$

$M^1_{CB} = +113833 - \frac{48674.974}{0.8815} = +\ 58614.660$

$M^1_{CD} = -\frac{51668.816}{0.8815} = -\ 58614.65$, $M^1_{DC} = -\frac{25834.408}{0.8815} = -29307.32$

(5) With Sideway

From Fig. 13, fixed end moments due to deflection Δ as follows:

$$FM_{AB} = -\frac{6\ E\ I_{AB}\ \Delta}{L^2_{AB}}$$

$$FM_{BA} = -\frac{6\ E\ I_{BA}\ \Delta}{L^2_{BA}}$$

$$FM_{CD} = -\frac{6\ E\ I_{CD}\ \Delta}{L^2_{CD}}$$

$$FM_{DC} = -\frac{6\ E\ I_{DC}\ \Delta}{L^2_{DC}}$$

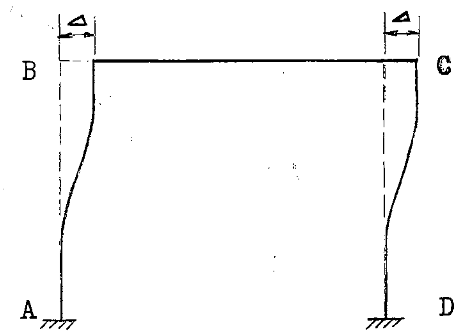


Fig.13

$$\frac{FM_{AB}}{FM_{DC}} = \frac{-\dfrac{6\ E\ I_{AB}\ \Delta}{L^2_{AB}}}{-\dfrac{6\ E\ I_{DC}\ \Delta}{L^2_{DC}}} \qquad \text{but} \quad L_{AB} = L_{DC}$$

$$\frac{FM_{AB}}{FM_{DC}} = \frac{I_{AB}}{I_{DC}} = \frac{109.7}{272.9} = \frac{100}{248.769}$$

Let $FM_{AB} = -100\ X$, then $FM_{DC} = -248.769\ X$

Joints	A	B		C		D
M	A B	B A	B C	C B	C D	D C
D'S	0	0.21	0.79	0.60	0.40	0
FM'S	-100.000	-100.000	0	0	-248.769	-248.769
1 st cycle	0	+ 21.000	+ 79.000	+149.261	+ 99.508	0
carry over	+ 10.500	0	+ 74.630	+ 39.500	0	+ 49.754
2 nd cycle	0	- 15.672	- 58.958	- 23.700	- 15.800	0
carry over	- 7.836	0	- 11.850	- 29.479	0	- 7.900
3 rd cycle	0	+ 2.489	- 9.361	+ 17.687	+ 11.792	0
Σ M	- 97.336	- 92.183	+ 92.183	+153.269	-153.269	-206.915

Moments due to sidesway.

$M^2_{AB} = -\ 97.336\ X$ $\qquad$ $M^2_{BA} = -\ 92.183\ X$

$M^2_{BC} = +\ 92.183\ X$ $\qquad$ $M^2_{CB} = +153.269\ X$

$M^2_{CD} = -153.269\ X$ $\qquad$ $M^2_{DC} = -206.915\ X$

Final moments:

$$M_{AB} = M^1_{AB} + M^2_{AB} = +\ 8693.373 - 97.336\ X$$

$$M_{BA} = M^1_{BA} + M^2_{BA} = +17386.730 - 92.183\ X$$

$$M_{BC} = M^1_{BC} + M^2_{BC} = -17386.740 + 92.183\ X$$

$$M_{CB} = M^1_{CB} + M^2_{CB} = +58614.660 + 153.269\ X$$

$$M_{CD} = M^1_{CD} + M^2_{CD} = -58614.650 - 153.269\ X$$

$$M_{DC} = M^1_{DC} + M^2_{DC} = -29307.320 - 206.915\ X$$

(6) Shear equation:

$$R_{Ax} + R_{Dx} + 5000 = 0$$

but $\quad R_{Ax} = (M_{AB} + M_{BA})/16, \quad R_{Dx} = (M_{CD} + M_{DC})/16$

$$M_{AB} + M_{BA} + M_{CD} + M_{DC} + 80000 = 0$$

Substitute these values of M_{AB}, M_{BA}, M_{CD} and M_{DC} into above equation, then

$$+\ 8693.373 - 97.336\ X + 17386.730 - 92.183\ X - 58614.650$$

$$-\ 153.269\ X - 29307.320 - 206.915\ X = 0$$

Solve above equation, we get

$$X = 33.03262$$

(7) End moments:

$M_{AB} = +\ 8693.373 - 97.336\ (33.03262) = +\ 5478.1$ ft-lbs

$M_{BA} = +17386.730 - 92.183\ (33.03262) = +14341.7$ ft-lbs

$M_{BC} = -17386.740 - 92.183\ (33.03262) = -14341.7$ ft-lbs

$M_{CB} = +58614.650 + 153.269\ (33.03262) = +63677.5$ ft-lbs

M_{CD} = - 58614.650 - 153.269 (33.03262) = - 63677.5 ft-lbs

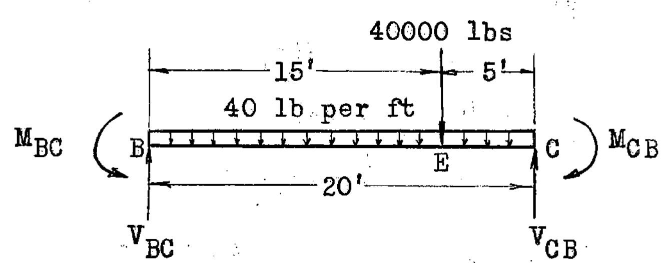
M_{DC} = - 29307.320 - 206.915 (33.03262) = - 36142.3 ft-lbs

(C) Comparison of end momentsas computed by the method of slope deflection and moment distribution.

Moment	Bending Moments (ft-lbs) Slope deflection M.	Moment distribution M.
M_{AB}	+ 5417.4	+ 5478.1
M_{BA}	+ 14235.2	+ 14341.7
M_{BC}	- 14235.1	- 14341.7
M_{CB}	+ 63616.4	+ 63677.5
M_{CD}	- 63616.5	- 63677.5
M_{DC}	- 36036.4	- 36142.3

(D) Design of members.

(1) Horizontal member BC



$\Sigma M_C = 0$

We can find the shear of end B, V_{BC}.

$$V_{BC} = \frac{14235.116 + 40000 \times 5 + \frac{1}{2} \times 40 \times 20^2 - 63616.395}{20} = 7931 \text{ lbs}$$

V_{CB} = 40000 + 40 x 20 - 7931 = 32869 lbs

M_E = 32869 x 5 - 63616.4 - $\frac{1}{2}$ x 40 x 5^2 = + 100228.6 ft-lbs

The critical moment for BC is 100228.6 ft-lbs.

$$\frac{I}{c} = \frac{M}{s} = \frac{100228.6 \times 12}{20000} = 60.14 \text{ in}^3$$

Use 16 WF 40 I/c = 64.4 60.14 in^3 (O.K.)

(2) Column AB.

s = allowable stress

$$= 17000 - 0.485 \left(\frac{16 \times 12}{3.47}\right)^2 = 15515 \text{ lb/in}^2$$

$N = 7931$ lbs, $A = 9.12$, $M = 14235.1$ ft-lbs, $I/c = 27.4$ in^3

$$\frac{N}{As_c} + \frac{M}{Ss_b} = \frac{7931}{9.12 \times 15515} + \frac{14235.1 \times 12}{27.4 \times 20000} = 0.056 + 0.312 = 0.368$$

$$< 1 \quad (\text{O. K.})$$

This is the minimum size for the width of the horizontal beam and results in waste but is easy to construct.

(3) Column CD.

$$s = 17000 - 0.485 \left(\frac{16 \times 12}{4.35} \right)^2 = 16055 \quad \text{lb/in}^2$$

$$\frac{N}{As_c} + \frac{M}{Ss_b} = \frac{32869}{14.4 \times 16055} + \frac{63616.4 \times 12}{54.6 \times 20000} = 0.842 < 1 \; (\text{O. K.})$$

(II) Design of the Semi-rigid Connections at B and C.

(1) Computation of moment area, a and b for horizontal member.

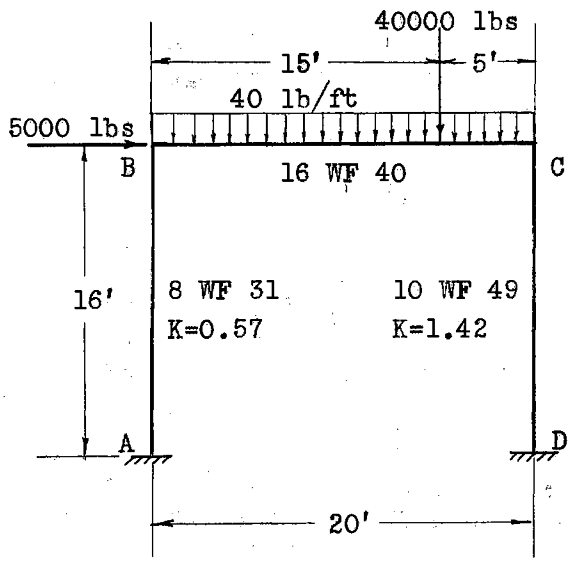


Fig. 14

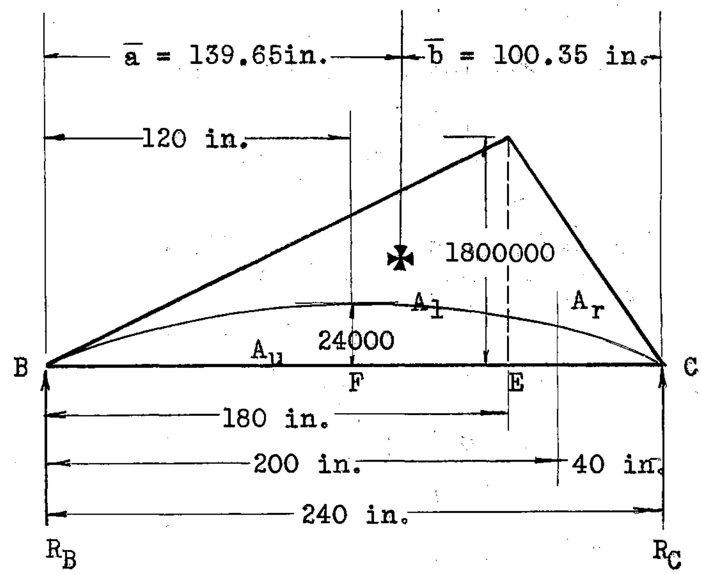


Fig. 15

As shown in Fig. 15

$R_B = \frac{1}{4} \times 40000 + 10 \times 40 = 10400$ lbs

$R_C = \frac{3}{4} \times 40000 + 10 \times 40 = 30400$ lbs

M_F = (due to uniform load) $= \frac{1}{8} wl^2 x12 = \frac{1}{8} x40x20^2x12 = 24000$ in-lb

M_E = (due to concentreted load) $= \frac{3}{4} \times 40000 \times 60 = 1800000$ in-lbs

A_u = Moment area due to uniform load

$= \frac{2}{3} \times 24000 \times 240 = 3840000$ lb-in^2

A_1 = Left part of moment area due to concentreted load

$= \frac{1}{2} \times 180 \times 1800000 = 162000000$ lb-in^2

A_r = Right part of moment area due to concentreted load

$= \frac{1}{2} \times 60 \times 1800000 = 54000000$ lb-in^2

A (total) = $A_u + A_l + A_r$ = 3840000 + 162000000 + 54000000

= 219840000 lb-in^2

$\bar{a}$ = Distance between the centroid of moment area and left support

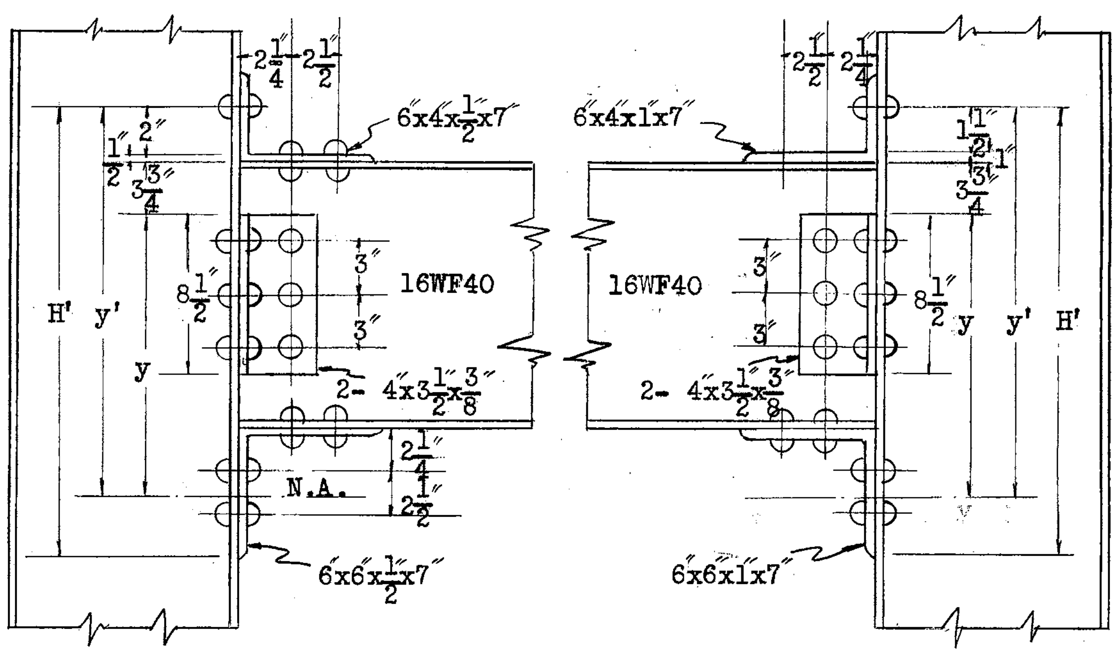
$$= \frac{120 \times 3840000 + 120 \times 162000000 + 200 \times 54000000}{219840000}$$

= 139.65 in.

$\bar{b}$ = Distance between the centroid of moment area and right support

= 240 - 139.65 = 100.35 in.

(2) The tentative connection at joint B.



Joint at B

Fig. 16

Joint at C

Fig. 17

(a) Fig. 16 is assumed connection.

$b' = 7$ in. $\quad t' = 1/2$ in. $= 0.5$ in.

$g' = 2\ 1/2$ in. $- 1/2$ in. $= 2.50 - 0.50 = 2.00$ in.

$g_1' = 2\ 1/4$ in. $- 1/2$ in. $= 2.25 - 0.50 = 1.75$ in.

$t = 3/8$ in. $= 0.375$ in.

$g_1 = 2\ 1/2$ in. $- 3/8$ in. $= 2.50 - 0.375 = 2.125$ in.

$g = 2$ in. $- 3/8$ in. $= 1\ 5/8$ in. $= 1.625$ in.

$d = 3\ 3/4$ in. $= 3.75$ in.

$h = 8\ 1/2$ in. $= 8.25$ in.

$H' = 6$ in. $+ 16$ in. $+ 2.5$ in $= 24.5$ in.

$h' = 16$ in. $+ 2\ 1/2$ in. $+ 2\ 1/4$ in. $= 20.75$ in.

$H = H' - g' - t' - d = 18.25$ in.

from Eq. (28)

$$m' = \frac{0.5\ (4 \times 2 + 1.75)}{6x2\ (2 \times 2 + 1.75)} = 0.0706$$

from Eq. (29)

$$m = \frac{0.375\ (4 \times 1.625 + 2.125)}{6x1.625(2 \times 1.625 + 2.125)} = 0.0617$$

from Eq. (30)

$$y = 18.25 - \sqrt{\frac{2x8.5x0.375x0.0617(2x18.25-8.5)}{7} + 2x0.5x0.0706x24.5}$$

$$= 16.435 \text{ in.}$$

$$y' = y + d + t' + g' = 16.435 + 3.75 + 0.5 + 2 = 22.685 \text{ in.}$$

from Eq. (24)

$$\alpha = \frac{1}{7}\left(\frac{0.375x2}{0.5x1.625}\right)^3\left(\frac{4x1.625+2.125}{4x2 + 1.75}\right)\left(\frac{2+1.75}{1.625 + 2.125}\right) = 0.0995$$

(b) Check the resisting moment for angles.

from Eq. (27)

$$M = \frac{b't'^2 s}{18g'}\ \frac{4g'+g_1'}{2g'+g_1'}\left\{(y'+2H') + \frac{2\alpha h}{y+d}\left[y^2+(H-h)(2y-h)\right]\right\}$$

$$M = \frac{7x0.5^2x20}{18x2}\ \frac{4x2+1.75}{2x2+1.75}\left\{(22.685+2x24.5)+\frac{2x0.0995x8.5}{16.435+3.75}\right.$$

$$\left. x\left[16.435^2+(18.25-8.5)(2x16.435-8.5)\right]\right\}$$

$$= 187 \text{ in-kips}$$

From the above equation the resisting moment is larger than the M_{BC} (170.8 in-kips) which we computed for the rigid connection.

(c) Check for rivets.

(1) For column rivets[7]

$$N = \frac{1}{ans}\sqrt{\frac{7Mans + 2pP^2}{2p}} \qquad (\ p = 26/8 = 3.25 \text{ in.}\ )$$

$$= \frac{1}{0.6013x2x20}\sqrt{\frac{7x170.8x0.6013x20+2x3.25x7.93^2}{2\ x\ 3.25}}$$

$$= 2.87$$

Use N = 6 (O. K.)

(2) For beam rivets

M = Resisting moment by top seat angle rivets

= 16 x 4 x 9.02 = 577.28 in-kips > 14.235 ft-kips

(48.10 ft-kips) (O. K.)

(3) The tentative connection at joint C.

(a) As Fig. 17 shown, is the assumed connection.

b' = 7 in. t' = 1 in.

g' = 2 1/2 in. - 1 in. = 1 1/2 = 1.500 in.

g_1' = 2 1/4 in. - 1 in. = 1 1/4 = 1.250 in.

t = 3/8 in. = 0.375 in.

g = 2 in. - 3/8 in. = 1.625 in.

g_1 = 2 1/2 in. - 3/8 in. = 2.125 in.

d = 3.75 in.

h = 8 1/2 in. = 8.25 in.

$H' = 6$ in. + 16 in. + 2.5 in. = 24.5 in.

$h' = 16$ in + 2 1/2 in. + 2 1/4 in. = 20.75 in.

$H = H' - g' - t' - d = 18.25$ in.

from Eq. (28)

$$m' = \frac{1 \times (4 \times 1.5 + 1.25)}{6 \times 1.5\,(2 \times 1.5 + 1.25)} = 0.1898$$

from Eq. (29)

$$m = \frac{0.375\,(4 \times 1.625 + 2.125)}{6 \times 1.625\,(2 \times 1.625 + 2.125)} = 0.0617$$

from Eq. (30)

$$y = 18.25 - \sqrt{\frac{2x8.5x0.375x0.0617(2x18.25-8.5)}{7} + 2x1x0.1898x24.5}$$

$$= 14.96 \text{ in.}$$

$$y' = 14.96 + 3.75 + 2.5 = 21.21 \text{ in.}$$

from Eq. (24)

$$\alpha = \frac{(1.5x0.375)^3(4x1.625+2.125)(1.5+1.25)}{7(1.625x1)^3(4x1.5x1.25)(1.625+2.125)} = 0.00516$$

(b) Check the resisting moment for angles

From Eq. (27)

$$M = \frac{(7x1^2x20)(4x1.5+1.25)}{(18\ x1.5)(2x1.5+1.25)}\left\{(21.21+2x24.5) + \frac{2x0.00516x8.5}{14.96+3.75}\right.$$

$$\left. x\left[14.96^2 + (18.25-8.5)(2x14.96-8.5)\right]\right\}$$

$$= 802 \text{ in-kips}$$

The resisting moment computed above is larger than the computed M_{CB} (763.4 in-kips) for rigid connection.

(c) Check for rivets.

(1) For column rivets.

$$N = \frac{1}{0.6013x2x20}\sqrt{\frac{7x763.4x0.6013x2x20+2x3.25x32.87^2}{2x3.25}}$$

$$= 6.01$$

Use $N = 6$ (O. K.)

(2) For beam rivets.

M = Resisting moment by top and seat angle rivets

= 16 x 4 x11.78 = 577.28 in-kips ≐ 63.616 ft-kips

(62.8 ft-kips) (O. K.)

(4) Semi-rigid connection factors. (Z)

(a) For connection C.

From Eq. (31)

$$Z_C = 12x1.5^3(1.5+1.25) \;/\; 29x10^6x7x1^3x(14.96+3.75)\Big\{(21.21+2x24.5) + \frac{2x0.00516x8.5}{14.96+3.75}\Big[14.96^2+(18.25-8.5)(2x14.96-8.5)(4x1.5+1.25)\Big]\Big\}$$

$$= \frac{1}{219.06 \times 10^8}$$

(b) For connection B.

$$Z_B = 12x2.0^3(2+1.75) \;/\; 29x10^6x7x0.5^3(16.435+3.75)\Big\{(22.685+2x24.5) + \frac{2x0.0995x8.5}{16.435+3.75}\Big[16.435^2+(18.25-8.5)(2x16.435-8.5)(4x2+1.75)\Big]\Big\}$$

$$= \frac{1}{15.85 \times 10^8}$$

(5) Fixed moments. (FM)

Equivalent length for B = L_B = L + $3EIZ_B$

$$= 240 + \frac{3x0.29x10^8x515.5}{15.85x10^8} = 268.3 \text{ in.}$$

Equivalent length for C = L_C = L + $3EIZ_C$

$$= 240 + \frac{3x0.29x10^8x515.5}{219.06^8} = 242.05 \text{ in.}$$

$$FM_{BC} = \frac{-6A}{L} x \frac{2\bar{b}L_C - \bar{a}L}{4L_BL_C - L^2}$$

$$= -\frac{6x219840000}{240} \quad \frac{2x100.35x242.05-139.65x240}{4x268.3x242.05-240^2}$$

$$= -409504 \text{ in-lbs}$$

$$FM_{CB} = +\frac{6A}{L} x \frac{2\bar{a}L_B - \bar{b}L}{4L_BL_C - L^2}$$

$$FM_{CB} = + \frac{6x219840000}{240} x \frac{2x139.65x268.3-100.35x240}{4x268.3x242.05-240^2}$$

$$= + 1381888.5 \text{ in-lbs}$$

(6) Carry over factors.

$$\text{B end of BC} = \frac{L}{2L_C} = \frac{240}{2 \times 242.05} = 0.4958$$

$$\text{C end of BC} = \frac{L}{2L_B} = \frac{240}{2 \times 268.30} = 0.4473$$

(7) Stiffness factors.

$$\text{B end of BC} = \frac{3I_{BC}L_C}{4L_BL_C-L^2} = \frac{3x515.5x242.05}{4x268.3x242.05-240^2}$$

$$= 1.85$$

$$\text{C end of BC} = \frac{3I_{BC}L_B}{4L_BL_C-L^2} = \frac{3x515.5x268.3}{4x268.3x242.05-240^2}$$

$$= 2.05$$

(8) Solve the moment by the slope deflection method.

(a) Elastic Equations.

$$M_{AB} = 2EK_{AB}(\theta_B-3R) = 2Ex0.57(\theta_B-3R) = 1.14E\theta_B - 3.42ER$$

$$M_{BA} = 2EK_{AB}(2\theta_B-3R) = 2Ex0.57(2\theta_B 3R) = 2.28E\theta_B - 3.42ER$$

$$M_{BC} = 6EI \frac{2L_C(\theta_B) + L(\theta_C)}{4L_BL_C - L^2} - FM_{BC}$$

$$= 6E(515.5) \frac{2x242.05\ \theta_B + 240\ \theta_C}{4x268.3x242.05-240^2} - 409504$$

$$= 7.41E\ \theta_B + 3.67E\ \theta_C - 409504$$

$$M_{CB} = 6EI \frac{2L_B(\theta_C) + L(\theta_B)}{4L_BL_C - L^2} + FM_{CB}$$

$$= 6E(515.5) \frac{2x268.30\ \theta_C + 240\ \theta_B}{4x268.3x242.05-240^2} + 1381888.5$$

$$= 3.67E\ \theta_B + 8.21E\ \theta_C + 1381888.5$$

$$M_{CD} = 2EK_{CD}\ (2\theta_C-3R) = 5.68E\ \theta_C - 8.52ER$$

$$M_{DC} = 2EK_{CD}\ (\theta_C - 3R) = 2.84E\ \theta_C - 8.52ER$$

(b) Joint Equations.

at joint B,

$$M_{BA} + M_{BC} = 0$$

at joint C,

$$M_{CB} + M_{CD} = 0$$

substitute these values of M_{BA}, M_{BC}, M_{CB} and M_{CD} into above two equations, then

$$9.69E\ \theta_B + 3.67E\ \theta_C - 3.42ER - 409504 = 0 \qquad (1)$$

$$3.67E\ \theta_B + 13.89E\ \theta_C - 8.52ER + 1381888.5 = 0 \qquad (2)$$

(c) Shear Equations.

$$R_{Ax} + R_{Dx} + 5000 = 0$$

but $R_{Ax} = (M_{AB} + M_{BA})/16 \times 12$, $R_{Dx} = (M_{CD} + M_{DC})/16x12$

$$M_{AB} + M_{BA} + M_{CD} + M_{DC} + 960000 = 0$$

$$3.42E\theta_B + 8.52E\theta_C - 23.88ER + 960000 = 0 \qquad (3)$$

(d) Solution by Gauss Method and Iteration.

EQUA. NO.	$E\theta_B$	$E\theta_C$	ER	CONSTANT TERM	CHECK TERM	PROCEDURE
(1)	+ 9.690	+ 3.670	- 3.420	- 409504.000	- 409494.060	
(2)	+ 3.670	+13.890	- 8.520	+1381888.500	+1381897.540	
(3)	+ 3.420	+ 8.520	-23.880	+ 960000.000	+ 959998.060	
(1)	+ 1.000	+ 0.379	- 0.353	- 42260.470	- 42259.444	(1)÷9.690
(2)	+ 1.000	+ 3.785	- 2.322	+ 376536.300	+ 376538.763	(2)÷3.670
(3)	+ 1.000	+ 2.491	- 6.982	+ 280701.750	+ 280698.259	(3)÷3.420
(4)		+ 3.406	- 1.969	+ 418796.770	+ 418798.205	(2)-(1)
(5)		+ 2.112	- 6.629	+ 322962.220	+ 322957.701	(3)-(1)
(4)		+ 1.000	- 0.578	+ 122958.530	+ 122958.952	(4)÷3.406
(5)		+ 1.000	- 3.139	+ 152917.710	+ 152915.571	(5)÷2.112
(6)			+ 2.561	- 29959.180	- 29956.619	(4)-(5)
(6)			+ 1.000	- 11698.235	- 11699.235	(6)÷2.561

Gauss Solution $E\theta_B = + 90660.985$ $E\theta_C = - 116196.950$

$ER = + 11698.235$

	$E\theta_B$	$E\theta_C$	ER	
Gauss Solution	+ 90660.985	-116196.950	+ 11698.235	
Itera-tion	+ 90397.800	-116266.760	+ 11727.912	1st. Approximation
	+ 90434.720	-116179.020	+ 11666.324	2nd. Approximation
	+ 90379.750	-116226.550	+ 11701.904	3rd. Approximation
	+ 90410.310	-116190.200	+ 11677.070	4th. Approximation
	+ 90387.780	-116213.520	+ 11694.420	5th. Approximation
	+ 90402.730	-116196.910	+ 11682.870	6th. Approximation
	+ 90392.370	-116207.940	+ 11690.940	7th. Approximation
	+ 90399.390	-116200.260	+ 11685.521	8th. Approximation
	+ 90394.570	-116205.440	+ 11689.270	9th. Approximation
	+ 90397.860	-116201.870	+ 11688.000	10th. Approximation
	+ 90396.050	-116203.520	+ 11688.890	11th. Approximation
	+ 90396.990	-116202.490	+ 11687.630	12th. Approximation
	+ 90396.160	-116203.510	+ 11688.120	13th. Approximation
	+ 90396.720	-116202.990	+ 11687.640	14th. Approximation
	+ 90396.350	-116203.440	+ 11687.910	15th. Approximation
	+ 90396.620	-116203.170	+ 11687.730	16th. Approximation
	+ 90396.450	-116203.350	+ 11687.740	17th. Approximation
	+ 90396.520	-116203.300	+ 11687.750	18th. Approximation
	+ 90396.510	-116203.310	+ 11687.770	19th. Approximation
	+ 90396.520	-116203.300	+ 11687.770	20th. Approximation
	+ 90396.520	-116203.300	+ 11687.770	21th. Approximation

(f) End Moments.

$M_{AB} = 1.14(+90396.52)-3.42(-11687.77)$

$= +\ 63079.80$ in-lbs. $= +\ 5256.65$ ft-lbs

$M_{BA} = 2.28(+90396.52)-3.42(-11687.77)$

$= +\ 166131.80$ in-lbs. $= +\ 13844.32$ ft-lbs

$M_{BC} = 7.41(+90396.52)+3.67(-11687.77)-409504$

$= -\ 166131.90$ in-lbs. $= -\ 13844.33$ ft-lbs

$M_{CB} = 3.67(+90396.52)+8.21(-11687.77)+1381888.50$

$= +\ 759614.64$ in-lbs $= +\ 63301.22$ ft-lbs

$M_{CD} = 5.68(-116203.30)-8.52(-11687.77)$

$= -\ 759614.54$ in-lbs $= -\ 63301.21$ ft-lbs

$M_{DC} = 2.84(-116203.30)-8.52(-11687.77)$

$= -\ 42957.17$ in-lbs $= -\ 35799.79$ ft-lbs

(9) Moment Distribution Method (by series) for a Semi-rigid Steel Structure.

Based on Analysis of Continuous Beams by Infinite Series[11], for semi-rigid steel structure, the moment distribution method (by series) can be derived as follows:

(1) Stiffness factors. (K's)

(2) Distribution factors. (D's)

assume $D_{BA} = a$, $D_{BC} = b$, $D_{CB} = c$, $D_{CD} = d$.

(3) Fixed moments. (FM's)

Assume the fixed moments for joints are FM_{AB}, FM_{BA}, FM_{BC}, FM_{CB}, FM_{CD}, FM_{DC}.

(4) Carry over factors. (C.O.'s)

For M_{BA}, and M_{CD}, we consider as rigid end. C.O.= 0.5

For M_{BC}, and M_{CB}, we consider as semi-rigid connections.

Assume $C.O._{BC} = m$, $C.O._{CB} = n$.

(5) Unbalance moment.

Assume for joint B = B.

for joint C = C.

(6) Moment distribution table.

Joints	A	B		C		D
Moment	A B	B A	B C	C B	C D	D C
D's	0	a	b	c	d	0
C.O.s	0	0.5	m	n	0.5	0
Fm's	FM_{AB}	FM_{BA}	FM_{BC}	FM_{CB}	FM_{CD}	FM_{DC}
1st cycle	0	$-aB$	$-bB$	$-cC$	$-dC$	0
Carry over	$-\frac{1}{2}aB$		$-nc$ C	$-mb$ B		$-\frac{1}{2}dC$
2nd cycle		$+nacC$	$+nbc$ C	$+mbc$ B	$+mbd$ B	
Carry over	$+\frac{1}{2}nacC$		$+mnbc$ B	$+mnbc$ C		$+\frac{1}{2}mbdB$
3rd cycle		$-mnabcB$	$-mnb^2c$ B	$-mnbc^2$ C	$-mnbcdC$	
Carry over	$-\frac{1}{2}mnabcB$		$-mn^2bc^2$ C	$-m^2nb^2c$ B		$-\frac{1}{2}mnbcd$ C
4th cycle		$+mn^2abc^2C$	$+mn^2b^2c^2$ C	$+m^2nb^2c^2$ B	$+m^2nb^2cdB$	
Carry over	$+\frac{1}{2}mn^2abc^2C$		$+m^2n^2b^2c^2B$	$+m^2n^2b^2c^2C$		$+\frac{1}{2}m^2nb^2cdB$

Final moments

$$M_{AB} = FM_{AB} + \left[- 1/2\ aB - 1/2\ mnabcB \cdots\cdots\right]$$
$$+ \left[1/2\ nacC + 1/2\ mn^2abc^2C + \cdots\cdots\right]$$
$$= FM_{AB} - 1/2\ aB\left[1 + (mnbc) + (mnbc)^2 + \cdots\cdots\right]$$
$$+ 1/2\ nacC\left[1 + (mnbc)^1 + (mnbc)^2 + \cdots\cdots\right]$$
$$= FM_{AB} + \left[- 1/2\ aB + 1/2\ nacC\right]\left[1 + (mnbc)^1 + (mnbc)^2 + \cdots\cdots\right]$$

Because the last term is a series and

$$1 + (mnbc)^1 + (mnbc)^2 + \cdots\cdots = \frac{1}{1 - mnbc}$$

Substitute into above equation, then

$$M_{AB} = FM_{AB} + \left[- 1/2\ aB + 1/2\ nacC\right] \frac{1}{1 - mnbc}$$
$$= FM_{AB} + \left[- 1/2\ aB + 1/2\ nacC\right] \frac{1}{x}$$

where $x = 1 - mnbc$

Similarly we get

$$M_{BA} = FM_{BA} + \left[- aB + nacC\right] \frac{1}{x}$$
$$M_{BC} = FM_{BC} + \left[- bB - ncC + nbcC + mnbcB\right] \frac{1}{x}$$

From above three moment equations, we can find the final moment is

$$M = FM + \left[\text{algebraic sum of 1st and 2nd cycle and carry over}\right] \frac{1}{x}$$

Where M is the final moment, FM is the fixed moment , and $x = 1 - mnbc$.

(10) Solve End Moments by the Moment Distribution Method.

(a) Stiffness factors. (K)

$K_{BA} = \frac{I_{AB}}{L_{AB}} = 0.57$ $K_{BC} = 1.85$

$K_{CB} = 2.05$ $K_{CD} = \frac{I_{CD}}{L_{CD}} = 1.42$

(b) Distribution factors. (D)

$D_{BA} = \frac{0.57}{0.57+1.85} = 0.236$

$D_{BC} = \frac{1.85}{0.57+1.85} = 0.764$

$D_{CB} = \frac{2.05}{2.05+1.42} = 0.591$

$D_{CD} = \frac{1.42}{2.05+1.42} = 0.409$

(c) Fixed moments. (FM)

$FM_{BC} = -409504$ in-lbs

$FM_{CB} = +1381888.5$ in-lbs

(d) Carry over factors. (C.O.)

$C.O._{BA} = 0.5000$ (Rigid connection)

$C.O._{BC} = 0.4958$ (Semi-rigid connection)

$C.O._{CB} = 0.4473$ (Semi-rigid connection)

$C.O._{CD} = 0.5000$ (Rigid connection)

(e) No sidesway.

Joints	A	B		C		D
Moment	A B	B A	B C	C B	C D	D C
D'S	0	(a) 0.236	(b) 0.764	(c) 0.591	(d) 0.409	0
C.O.S	0	0.500	(m) 0.4958	(n) 0.4473	0.500	0
FM' S	0	0	- 409504.00	+1381888.50	0	0
1st. cycle	0	+ 96642.94	+ 312861.06	- 816696.10	- 565192.40	0
Carry over	+ 48321.47	0	- 365308.17	+ 155116.51	0	- 282596.20
2nd cycle	0	+ 86212.73	+ 279095.44	- 91673.86	- 63442.65	0
Carry over	+ 43106.37	0	- 41005.72	+ 138375.52	0	- 31721.33
1st.2nd cycle carry o.	+ 91427.84	+ 182855.67	+ 185642.61	- 614877.93	- 628635.05	- 314317.53

$$x = 1 - m\,n\,b\,c = 1 - 0.4958 \times 0.4473 \times 0.764 \times 0.591 = 0.899865$$

$$M_{AB} = + \frac{91427.84}{0.899865} = + 101601.70 \quad \text{in-lbs}$$

$$M_{BA} = + \frac{182855.67}{0.899865} = + 203203.40 \quad \text{in-lbs}$$

$$M_{BC} = - 409504.0 + \frac{185642.61}{0.899865} = - 203203.5 \quad \text{in-lbs}$$

$$M_{CB} = +1381888.5 - \frac{614877.93}{0.899865} = + 698588.4 \quad \text{in-lbs}$$

$$M_{CD} = - \frac{628635.05}{0.899865} = - 698588.10 \quad \text{in-lbs}$$

$$M_{DC} = - \frac{314317.53}{0.899865} = - 349294.00 \quad \text{in-lbs}$$

(f) With Sidesway.

$FM_{AB} = FM_{BA} = -100\ X$ (See page 21)

$FM_{CD} = FM_{DC} = -248.769\ X$

Joint	A	B		C		D
M	A B	B A	B C	C B	C D	D C
D'S	0	0.236	0.764	0.591	0.409	0
C.O.S	0	0.500	0.4958	0.4473	0.500	0
FM'S	-100.000X	-100.000X	0	0	-248.769X	-248.769X
1st. cycle	0	+ 23.600	+ 76.400	+147.022	+107.747	0
Carry over	+ 11.800	0	+ 65.763	+ 37.879	0	+ 50.874
2nd cycle	0	- 15,520	- 50.243	- 22.386	- 15.493	0
Carry over	- 7.760	0	- 10.013	- 24.910	0	- 7.747
3rd cycle	0	+ 2.363	+ 7.650	+ 14.722	+ 10.188	0
Final M	- 95.960X	- 89.557X	+ 89.557X	+152.327X	-152.327X	-205.642X

(g) Combined Moments.

$M_{AB} = + 101601.70 - 95.960\ X$

$M_{BA} = + 203203.40 - 89.557\ X$

$M_{BC} = - 203203.50 + 89.557\ X$

$M_{CB} = + 698588.40 + 152.327\ X$

$M_{CD} = - 698588.10 - 152.327\ X$

$M_{DC} = - 349294.00 - 205.642\ X$

(h) Shear Equation.

$M_{AB} + M_{BA} + M_{CD} + M_{DC} + 960000 = 0$

Substituting the combined moments into above equation, then,

$+ 101601.70 + 203203.40 - 698588.1 - 349294.00$

$- 95.96X - 89.557X - 152.327X - 205.642X + 960000 = 0$

$X = 399.1326$

(i) End Moments.

$M_{AB} = + 101601.70 - 95.96(399.1326) = + 63301.0$ in-lbs

$= + 5275.1$ ft-lbs

$M_{BA} = + 203203.40 - 89.557(399.1326) = +167458.3$ in-lbs

$= +13954.8$ ft-lbs

$M_{BC} = - 203203.50 + 89.557(399.1326) = -167458.4$ in-lbs

$= -13954.8$ ft-lbs

$M_{CB} = + 698588.40 - 152.327(399.1326) = +759387.1$ in-lbs

$= +63282.3$ ft-lbs

$M_{CD} = - 698588.10 + 152.327(399.1326) = -759386.8$ in-lbs

$= -63282.2$ ft-lbs

$M_{DC} = - 349294.00 - 205.642(399.1326) = -431372.4$ in-lbs

$= -35947.7$ ft-lbs

(III) The Economy of Semi-Rigid Connections.

The end moments M_{BC} and M_{CB} which were computed for rigid and semi-rigid connections are as follows:

(a) For rigid connections.

$M_{BC} = - 14235.1$ ft-lbs

$M_{CB} = + 63616.4$ ft-lbs

(b) For semi-rigid connections

$M_{BC} = - 13844.5$ ft-lbs

$M_{CB} = + 63301.22$ ft-lbs

(c) The economy at joint B is

$$\frac{14235.1 - 13844.5}{14235.1} = 2.74\ \%$$

(d) The economy at joint C is

$$\frac{63616.4 - 63301.22}{63616.4} = 0.50\ \%$$

THE RELATION BETWEEN $\frac{L_E}{L}$ AND ECONOMY

Based on some comparisons between rigid and semi-rigid connections, we find that the economy is affected by

(1) the constant Z for the connection angles;

(2) the moment of inertia of the member;

(3) the modified stiffness factors for semi-rigid connections.

In other words, the economy which may be achieved is determined by the equivalent lengths of semi-rigid connection joints (L_E) or the ratio of equivalent length to the true length. ($\frac{L_E}{L}$)

From the example presented above and from the example in The Derivation of an Equation for the Elastic Restraint of the Top and Seat Angle Type of Semi-Rigid Connection, and the degsign problem of the writer's semi-rigid steel structure, we can find many data for comparing L_E/L and the corresponding economy as follows:

L_E/L	Economy(%)
1.005	1.08
1.01	0.50
1.11	2.74
1.14	4.43
1.21	7.03
1.21	7.06
1.26	9.25

Where L_E is the equivalent length of the semi-rigid connections and L_E = (L + 3 E I Z) , L is the true length of the member, and

$$\text{economy} = \frac{M_{Rigid} - M_{Semi\text{-}Rigid}}{M_{Rigid}} \quad (\%)$$

Using these data, we can plot a curve as shown in Fig. 18.

The complete design of a semi-rigid connection involves a considerable amount of work. Fig. 18 may be used to determine, in a given case, whether the economy to be achieved justifies so much work. To do this it is only necessary to compute Z and L_E/L.

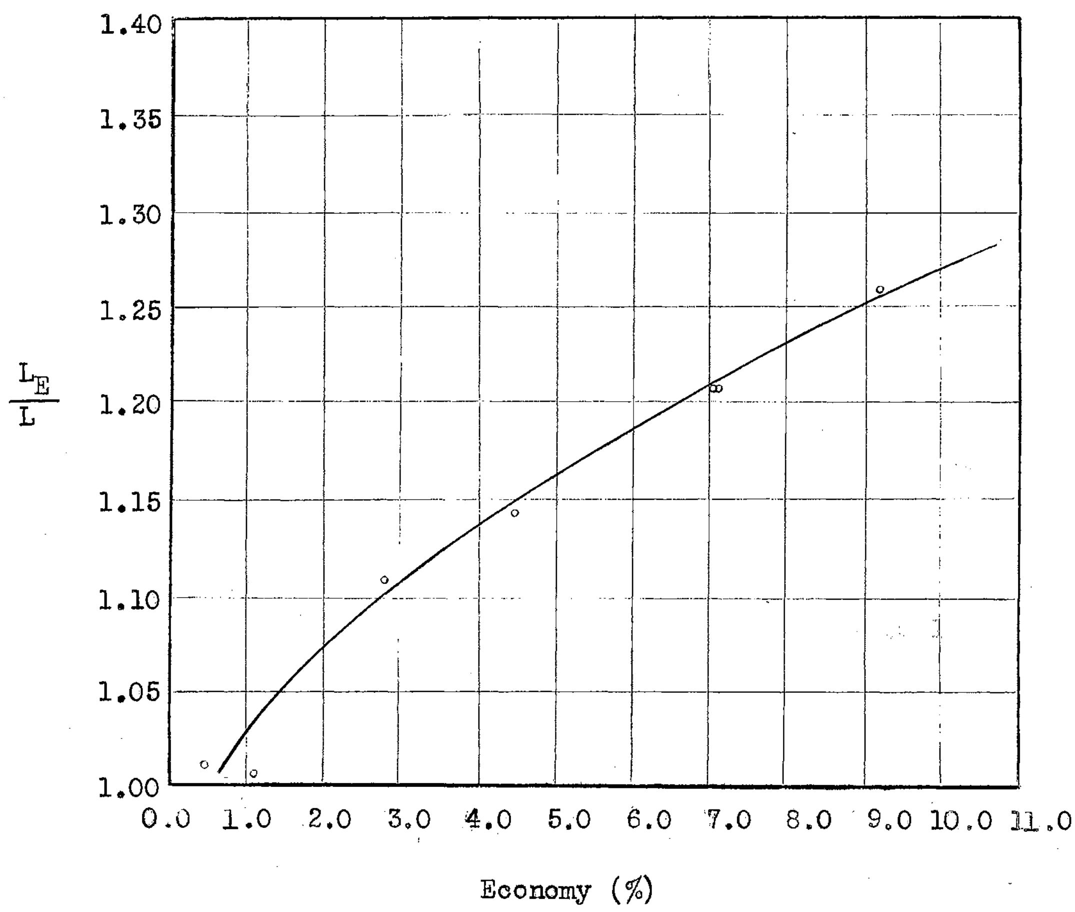


Fig. 18

CONCLUSION

In this thesis, the elastic restraint equation for the semi-rigid connection factor Z, or the angle change for unit moment, is derived for a combined top and seat, with web angle, semi-rigid connection. The evident good agreement between the slope of the moment-rotation curve($\frac{1}{Z}$) computed by this equation (Eq. 31) and the published laboratory results are shown in Figs. 10 and 11.

When the beam and column sections and connection angles are chosen, the connection factor Z is proportional to the cubic power of the distance (g') which is the length of the column-connected leg of the top angle, measured from the center of the rivet line to top face of the out standing leg, and is inversely proportional to the cubic power of the thickness of the top angle, (t'), the length of the top angle (b') and the relative factor α.(as between the top and web angles) (Eq.24). Other terms have no important influence.

Fig. 18 shows the relation between L_E/L and economy. The latter increases with the equivalent length of the beam. ($L_E = L + 3EIZ$). in other words, the economy is affected by the connection factor Z and the moment of inertia of the beam, I. The approximate economy is indicated by the curve in Fig. 18.

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