BEAMS ON ELASTIC FOUNDATIONS

By

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PREFACE

I wish to acknowledge my indebtedness to the following individuals:

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LIST OF SYMBOLS

SPECIAL GROUPINGS

V_A, V_B, M_A, M_B	End-conditioning elements
V_{SY}, M_{SY}	End-conditioning elements for symmetrical case
V_{AS}, M_{AS}	End-conditioning elements for anti-symmetrical case
y_{AP}, θ_{AP}, V_{AP}, M_{AP}	Beam functions of a finite beam being considered to be infinitely long
y', θ', V', M'	Beam functions for the symmetrical case of a finite beam being considered to be infinitely long
y'', θ'', V'', M''	Beam functions for the antisymmetrical case of a finite beam being considered to be infinitely long

GENERAL LISTING

a	Distance separating two concentrated forces
b	Width of the beam
c	Distance from the left end, point A, of a beam to the point of application of the load
du	Elemental length of the load diagram
E	Modulus of elasticity of the beam
I	Moment of inertia of the beam
k	Modulus of the elastic foundation
M	Bending moment or applied couple
q	Intensity of the load
R	Reaction force of the elastic foundation

U	Total strain energy
v	Variable intensity of load
v'	Non-uniformly varying intensity of load
V	Shear
w	Intensity of uniformly distributed load
y	Deflection
Z	Characteristic of the system $\sqrt[4]{\frac{k}{4EI}}$
Δu	Width of a strip
θ	Slope
$\doteq$	Is almost equal to

INTRODUCTION

The analysis of beams on elastic foundations is of great importance in many engineering problems and has been the subject of considerable study.

The first investigator of this problem, C. A. Coulomb in 1776, considering the beam to be perfectly rigid, assumed a linear relation of the external forces and moments to the reactive pressure in terms of the column formula.[3] Coulomb's column analogy has been widely used for nearly two hundred years and is still recognized as a satisfactory approximation in many practical problems.

Further progress in this subject was made in 1867 by E. Winkler[14] who developed a new and original solution based on the assumption that the intensity of the reactive force at any point of the beam must be a direct function of the deflection of the beam. A few years later, basing his analysis on this assumption, H. Zimmermann[15] published his classical work on the analysis of railroad track.

In 1921, the first completely general work[5] on the analysis of beams on elastic foundations was published. In this very complete work, K. Hayashi offers solutions to a great variety of problems. However, his methods are no longer being used because of their relative complexity.

For historical interest, a sample of Hayashi's method is presented in the appendix.

Twelve years after Hayashi's book was published, A. A. Umansky[10 and 11] presented a somewhat simpler method. This method, often called the method of initial end conditions, has likewise been supplanted by newer methods and only a brief discussion is given, also in the appendix.

In 1936, two new methods were published[7] by M. Hetenyi and have since been extended into a book[8] by the same author. These two new solutions offer considerable simplification over the earlier methods and make up the body of the material presented in this report. It should be noted that by the time M. Hetenyi's book was published the subject of beams on elastic foundations had an enlarged scope far beyond the usual application to foundation problems. These include, from page v of Hetenyi's[8] book:

> ......networks of beams and .... thin-walled tubes, shells and domes, where the elastic foundation for the beam is supplied by the resilience of the adjoining portions of a continuous elastic structure.

This report, however, will be limited to three principle problems:

1. Analysis of infinite beams on elastic foundations by the classical method,

2. Analysis of finite beams on elastic foundations by the method of superposition

and

3. Analysis of finite beams on elastic foundations by

the method of trigonometric series.

The application of various methods will be demonstrated by several examples and final formulas for specific cases will be derived.

I. THE CLASSICAL METHOD

A. DERIVATION

The solution of beams on elastic foundations is based on the equations of the elastic curve of the beam and on Winkler's assumption. The equations of the elastic curve are

$$y = \text{Deflection}, \qquad (1a)$$

$$\frac{dy}{dx} = \text{Slope}, \qquad (1b)$$

$$-EI\frac{d^2y}{dx^2} = \text{Bending Moment} = M, \qquad (1c)$$

$$-EI\frac{d^3y}{dx^3} = \text{Shear} = V = \frac{dM}{dx}, \qquad (1d)$$

$$-EI\frac{d^4y}{dx^4} = \text{Load} = q = \frac{dV}{dx} = \frac{d^2M}{dx^2}. \qquad (1e)$$

Winkler's assumption states that the intensity of the reactive force of the elastic foundation at any point is proportional to the deflection of the beam at that point. Thus

$$R = ky, \qquad (2)$$

where k is the modulus of the foundation ($\#/in^2$) and R is the reactive force of the elastic foundation (#/in).

Expressing Winkler's assumption algebraically in terms of equations (1e) and (2)

$$EI \frac{d^4y}{dx^4} = -ky. \tag{3}$$

Using the notation

$$\sqrt[4]{\frac{k}{4EI}} = Z, \tag{4}$$

equation (3) becomes

$$\frac{d^4y}{dx^4} + 4\,Z^4y = 0. \tag{5}$$

The general solution* of this differential equation is

$$y = e^{Zx}(C_1\cos Zx + C_2\sin Zx) + e^{-Zx}(C_3\cos Zx + C_4\sin Zx). \tag{6}$$

The evaluation of constants C_1, C_2, C_3 and C_4 will now be demonstrated on a typical example. An infinitely long beam supported by an elastic medium and loaded by a concentrated load P (Figure 1) will be considered. The origin

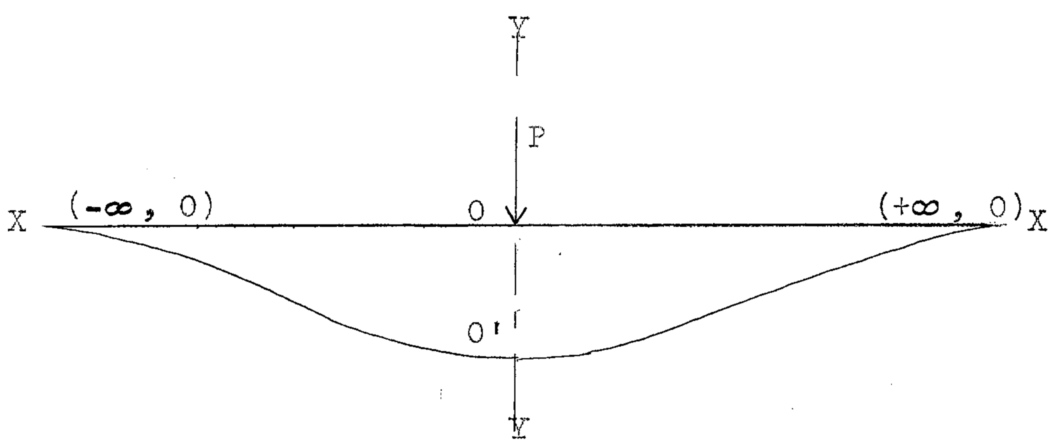

Figure 1. Infinite Beam

of the coordinate axis 0 is at the point of application of the load.

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* This solution is presented in the appendix.

Substituting for x and y in equation (6) the ordinates of the right end of the beam, the constants C_1 and C_2 become equal to zero and equation (6) can be reduced to

$$y = e^{-Zx}(C_3 \cos Zx + C_4 \sin Zx). \qquad (a)$$

Differentiating equation (a)

$$\frac{dy}{dx} = -Ze^{-Zx}(C_3 \cos Zx + C_4 \sin Zx + C_3 \sin Zx - C_4 \cos Zx)$$

and observing that at O' $\left[x = 0,\ \frac{dy}{dx} = 0\right]$, the equality of constants

$$C_3 = C_4 = C$$

becomes apparent.

Expressing equation (a) in terms of C, the consecutive derivatives become

$$y = Ce^{-Zx}(\cos Zx + \sin Zx), \qquad (b)$$

$$\frac{dy}{dx} = -2CZe^{-Zx} \sin Zx, \qquad (c)$$

$$\frac{d^2y}{dx^3} = 2CZ^2e^{-Zx}(\sin Zx - \cos Zx) \qquad (d)$$

and

$$\frac{d^3y}{dx^3} = 4CZ^3e^{-Zx} \cos Zx. \qquad (e)$$

Expressing the shear for the right portion of the beam as $-\frac{P}{2}$ in equation (1d)

$$-4\ EICZ^3 = -\frac{P}{2}$$

or

$$C = \frac{P}{8Z^3EI}. \qquad (f)$$

Substituting this value of C into equation (b), the equation of the deflection curve takes the form of

$$y = \frac{P}{8Z^3EI} e^{-Zx}(\cos Zx + \sin Zx), \tag{7}$$

or, in terms of equation (4),

$$y = \frac{PZ}{2k} e^{-Zx}(\cos Zx + \sin Zx). \tag{8}$$

To simplify the algebraic form of the final equations the following notations are introduced:

$$A_{Zx} = e^{-Zx}(\cos Zx + \sin Zx), \tag{9a}$$

$$B_{Zx} = e^{-Zx} \sin Zx, \tag{9b}$$

$$C_{Zx} = e^{-Zx}(\cos Zx - \sin Zx) \tag{9c}$$

and

$$D_{Zx} = e^{-Zx} \cos Zx. \tag{9d}$$

The final equations for deflection, slope, bending moment and shear for the right portion of an infinitely long beam supported by an elastic foundation and loaded by a single concentrated load can now be written as

$$y = \frac{PZ}{2k} e^{-Zx}(\cos Zx + \sin Zx) = \frac{PZ}{2k} A_{Zx}, \tag{10}$$

$$\theta = -\frac{PZ^2}{k} e^{-Zx} \sin Zx = -\frac{PZ^2}{k} B_{Zx}, \tag{11}$$

$$M = \frac{P}{4Z} e^{-Zx}(\cos Zx - \sin Zx) = \frac{P}{4Z} C_{Zx} \tag{12}$$

and

$$V = -\frac{P}{2} e^{-Zx} \cos Zx = -\frac{P}{2} D_{Zx}. \tag{13}$$

Graphical representation of the final equations (10-13) is shown in figure (2).

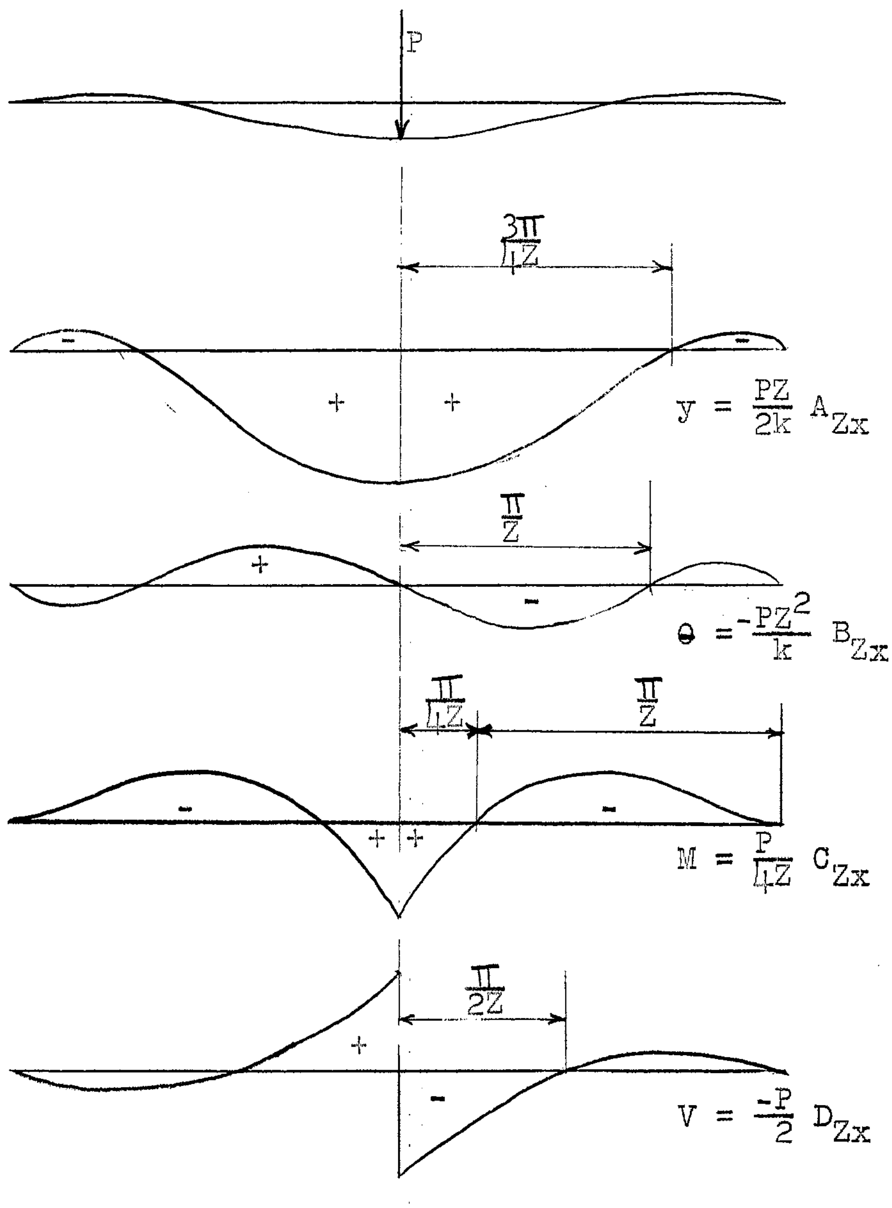


Figure 2.

Graphical Representation of Beam Function Equations.

Numerical values for A_{Zx}, B_{Zx}, C_{Zx} and D_{Zx} are given in the tables of numerical coefficients--Chapter IV.

B. PARTICULAR CASES

Any infinite beam on an elastic foundation, loaded by any system of loads, can be completely analized by using the general equations developed for a beam loaded by a concentrated load together with the principle of superposition.

1) System of Concentrated Loads

The analysis of a beam loaded by a system of concentrated loads can be accomplished by a simple superposition of particular solutions.

2) Applied Couple

In order to analize an infinite beam loaded by an applied couple M_o (Figure 3), it is necessary to replace the

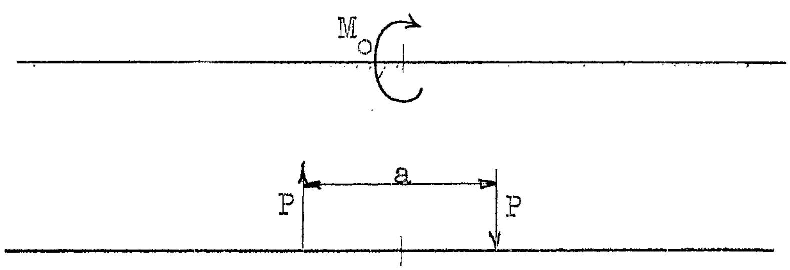


Figure 3. Applied Couple

couple by two equal forces acting in opposite directions, separated by a distance "$a \rightarrow 0$". Using equation (10), the equation for the deflection curve is

$$y = \frac{PZ}{2k} \quad A_{Zx} - A_{Z(x + a)}. \qquad (g)$$

The equation for the deflection curve for the applied couple M_o can now be found from equation (g) by substituting

$$P = \frac{M_o}{a}$$

and evaluating the entire expression for $a \longrightarrow 0$. Thus,

$$y = \frac{M_o Z}{2k}\left[\frac{A_{Zx} - A_{Z(x+a)}}{a}\right]_{a\to 0} = \frac{M_o Z}{2k}\frac{d}{dx}A_{Zx} = \frac{M_o Z^2}{k}B_{Zx}. \quad (14)$$

The slope, moment and shear can be found by differentiating as indicated by equations (1a-1d).

3) Uniformly Distributed Load

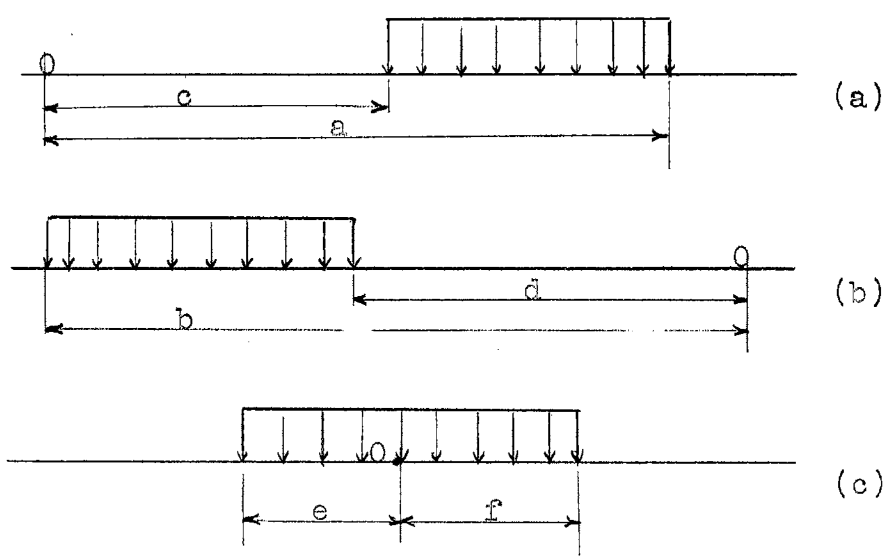


Figure 4.

Uniformly Distributed Load

The analysis of beams loaded by a partial uniformly distributed load must be considered as a three step procedure:

a.) Investigation of points located on the left side of the load (Figure 4a),

b.) Investigation of points located on the right side of the load (Figure 4b)

and

c.) Investigation of points located under the load (Figure 4 c).

Expressing the load P in the equation of the deflection curve (10) as

$$P = \int w \ du \qquad \text{(h)}$$

where w = intensity of the load,

du = elemental length of the load diagram

and

w du = elemental concentrated load dP,

and integrating between the limits c and a, the deflection equation for points located on the left side of the beam becomes

$$y = \frac{wZ}{2k}\int_c^a A_{Zx} \ du = -\frac{w}{2k}(D_{Za} - D_{Zc}). \qquad (15)$$

Since D_{Za} and D_{Zc} are finite values, the derivation of slope, bending moment and shear equations by a direct integration of equation (15) is not possible and equations (11, 12 and 13) must be applied. Expressing these equations in terms of equation (h) and integrating between given limits, the rest of the beam functions become

$$\theta = \frac{wZ}{k}(A_{Zc} - A_{Za}), \qquad (16)$$

$$M = -\frac{w}{2Z^2}(B_{Zc} - B_{Za}), \qquad (17)$$

and

$$V = \frac{w}{4Z}(C_{Zc} - C_{Za}). \tag{18}$$

The investigation of points located on the right side of the beam is a similar procedure.

For points located under the load, the two previous procedures have to be repeated and the integration of deflection equation (10) must be performed in two seperate operations between limits 0 to e and 0 to f. Thus,

$$y = \frac{wZ}{2k}\int_0^e e^{-Zx}(\cos Zx + \sin Zx)du + \int_0^f e^{-Zx}(\cos Zx + \sin Zx)du$$

or

$$y = \frac{w}{2k}(2 - D_{Ze} - D_{Zf}). \tag{19}$$

The derivation of slope, bending moment and shear equations is similar. All final equations were rearranged in tabular form and are presented in Chapter IV.

4) Uniformly Varying Load

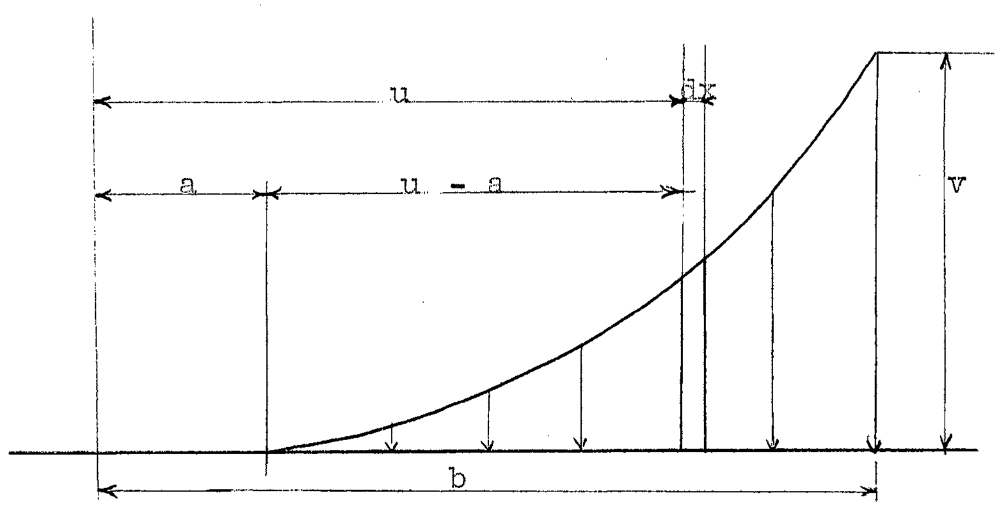


Figure 5. Uniformly Varying Load

The analysis of beams loaded by a partial uniformly varying load is considerably the same as the analysis of beams loaded by a partial uniformly distributed load. The load P in equation (10) of the deflection curve becomes

$$P = \int v \, du$$

where v = variable intensity of load,

du = elemental length of the load diagram,

$v\,du$ = elemental concentrated load dP

and

$v = F(u)$ as given by the generating line of the load diagram (Figure 5).

The equations of the beam functions are tabulated in Chapter IV.

5) Non-Uniformly Varying Loads

When the generating line of the load diagram does not offer a mathematical interpretation, no exact analysis is possible and the semi-graphical method by strips must be applied. The load diagram is divided into n-strips $(\Delta u)v'$ of equal or unequal width where

Δu = width of the strip

and v' = non-uniformly varying intensity of load,

and each strip is treated as a single concentrated load. It is apparent that the degree of accuracy is dependent on the number of strips and on the proper selection of Δu. In most practical cases, this procedure gives fair results.

C. REMARKS

In all derivations, the value of k is considered to have the dimensions lbs/in^2 and the equation $R = ky$ provides a dimensional check. However, this is the value only when the width of the beam is unity, since if the dimensions of the beam are considered

$$R = bky. \qquad (i)$$

This difference has no effect on derivations, but is quite an important consideration when determining the proper value of k experimentally.

In railroad work, the value of k is obtained by recording the deflection under a tie due to a concentrated force and dividing this value by the tie spacing. A typical calculation is found by applying a concentrated load of ten tons on rails supported by ties with 25 inch spacing. If the displacement is 0.4 inches, the resulting value of k is 2000 $\#/in^2$.

Z is often refered to as the characteristic of the system since it embodies the flexural rigidity of the beam as well as the elasticity of the supporting medium. Zx is an absolute number and must be taken in terms of radians.

D. CONCLUSIONS

1) The classical analysis of infinite beams on elastic foundations is based on the relationship between the differential equation of the elastic curve and Winkler's equation

$$EI \frac{d^4y}{dx^4} = -ky.$$

2) The general solution of this differential equation

is

$$y = e^{Zx}(C_1\cos Zx + C_2\sin Zx) + e^{-Zx}(C_3\cos Zx + C_4\sin Zx).$$

3) The numerical constants of the general solution are evaluated from special conditions.

4) The equations of particular beam functions,

a) deflection,

b) slope,

c) bending moment

and d) shear,

are found from their well known relations to the elastic curve. For a beam loaded by a concentrated load, these are

a) $y = \frac{PZ}{2k} A_{Zx}$,

b) $\theta = -\frac{PZ^2}{k} B_{Zx}$,

c) $M = \frac{P}{4Z} C_{Zx}$

and

d) $V = -\frac{P}{2} D_{Zx}$.

5) Any infinite beam on an elastic foundation, loaded by any system of loads, can be completely analized by means of:

a) the principle of superposition

and b) the equations of the beam functions produced by a concentrated load.

II. THE METHOD OF SUPERPOSITION

A. INTRODUCTORY STATEMENT

Finite beams on elastic foundations can be analized by combining:

1) the classical method of analysis of infinite beams

with 2) the principle of superposition.

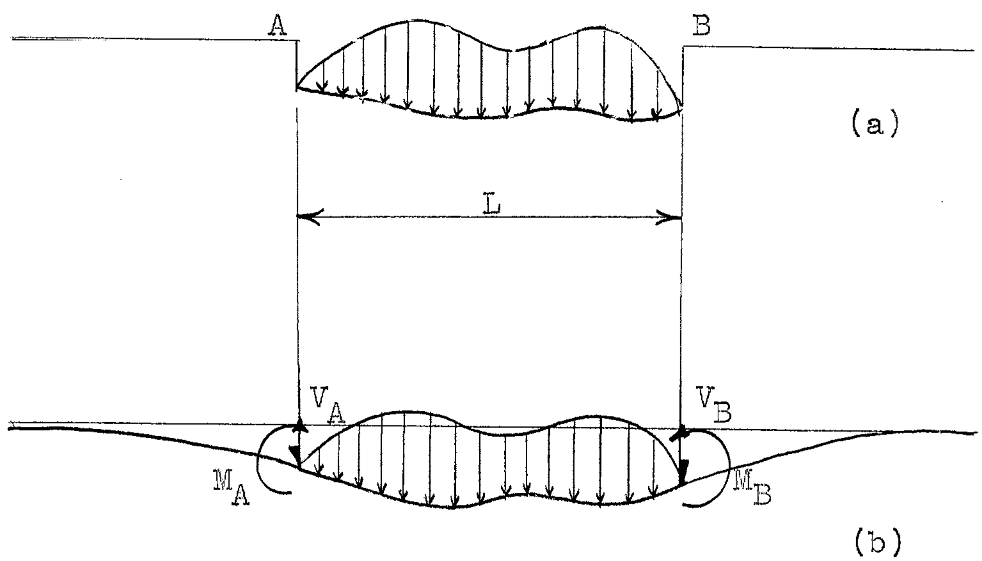


Figure 6. Free End Finite Beam

By this approach, the finite beam is replaced by an infinite beam as shown in figures (7a and 7b). The results of the analysis of the infinite beam are valid for the solution of the finite beam being considered.

The end-conditioning elements, V_A, V_B, M_A and M_B, may be obtained from equations containing the known beam functions (slope, deflection, bending moment or shear) at the ends of the finite beam and the unknown functions at the corresponding points of the infinite beam.

B. PARTICULAR CASES

1) Free Ends--General Solution

A finite beam of length L, loaded by a single concentrated load and supported by an elastic foundation, with ends free to displace, will be considered (Figure 7). The shear and the bending moment at both ends of the beam are equal to zero. Replacing this finite beam by five infinite beams loaded as shown in Figure (8), an equivalent beam system is developed. The values of bending moments and shears at points A and B are tabulated in Figure (8) and, by superposition of corresponding values, equations may be obtained:

The bending moment

at A

$$M_{AP} + \frac{V_A}{4Z} C_{ZO} + \frac{V_B}{4Z} C_{ZL} + \frac{M_A}{2} D_{ZO} + \frac{M_B}{2} D_{ZL} = 0, \qquad (20a)$$

at B

$$M_{BP} + \frac{V_A}{4Z} C_{ZL} + \frac{V_B}{4Z} C_{ZO} + \frac{M_A}{2} D_{ZL} + \frac{M_B}{2} D_{ZO} = 0, \qquad (20b)$$

The Shear

at A

$$V_{AP} - \frac{V_A}{2} D_{ZO} + \frac{V_B}{2} D_{ZL} - \frac{M_A Z}{2} A_{ZO} + \frac{M_B Z}{2} A_{ZL} = 0, \quad (20c)$$

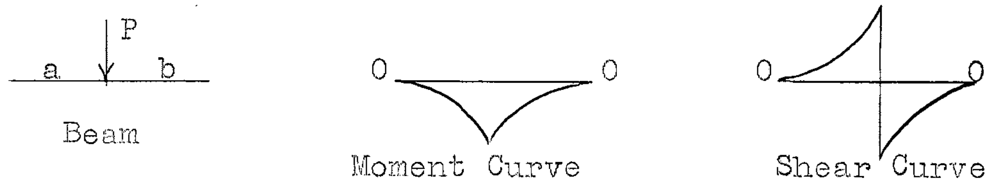


Figure 7. Free End Finite Beam with Concentrated Load.

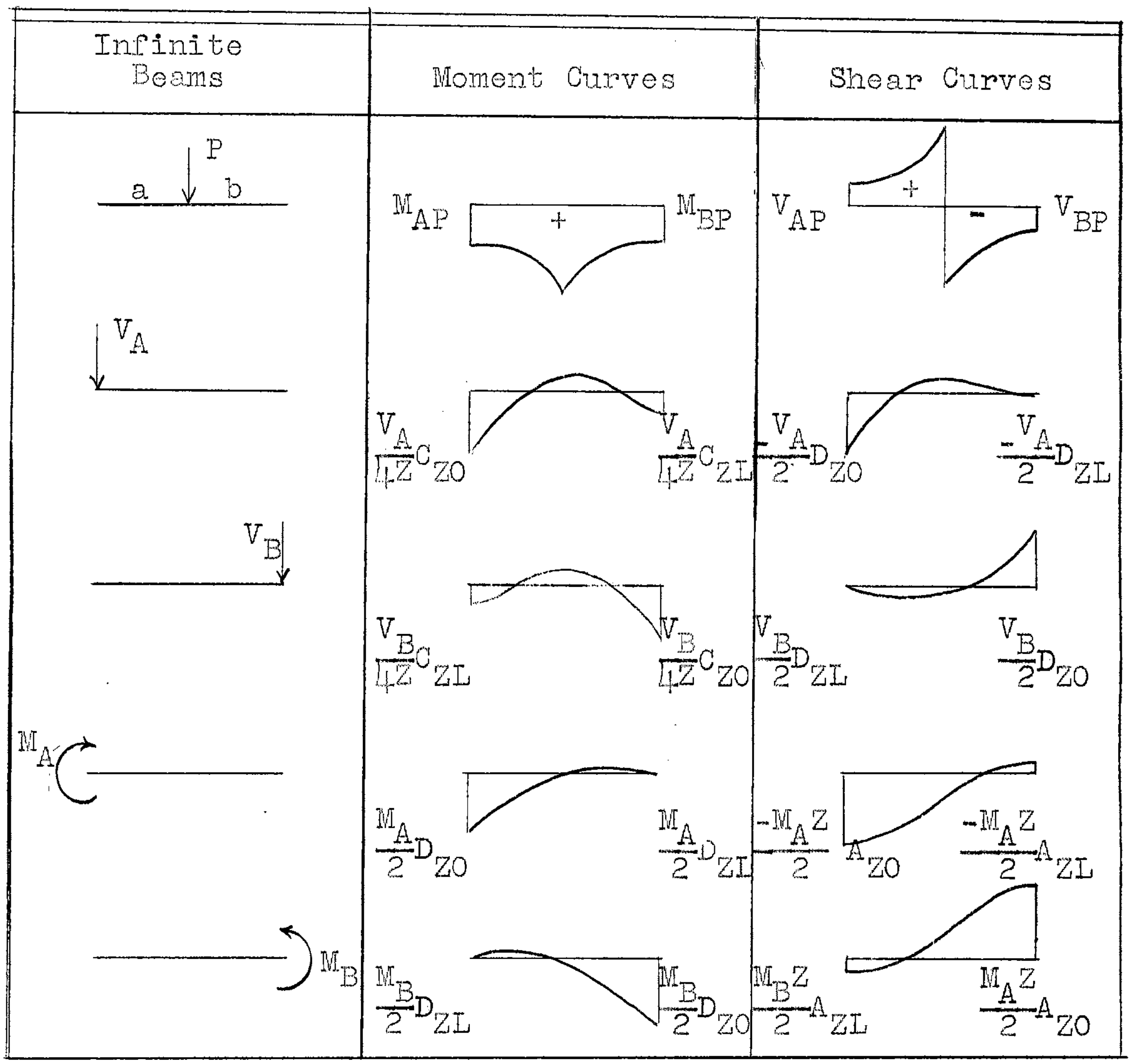


Figure 8. Equivalent Beam System

at B

$$-V_{BP} - \frac{V_A}{2} D_{ZL} + \frac{V_B}{2} D_{ZO} - \frac{M_A Z}{2} A_{ZL} + \frac{M_B Z}{2} A_{ZO} = 0. \quad (20d)$$

The end-conditioning elements are determined by evaluating the constants and solving the equations simultaneously. The solution may be completed by the classical method. For more complicated load diagrams, the method of superposition by integration may be readily applied.

2) Free Ends--Symmetrical and Antisymmetrical Resolution

A considerable simplification of the analysis may be obtained by resolving the load diagram into a symmetrical and an antisymmetrical one. The resolution of the system of loads, the shear and bending moment diagrams of the new systems and end-conditioning elements are shown in Figure (9). By superposition of corresponding values, two groups of indepent equations may be obtained:

The bending moment

at A

$$M_P' + M_P'' = M_{AP}, \quad (21a)$$

at B

$$M_P' - M_P'' = M_{BP}, \quad (21b)$$

The shear

at A

$$V_P' + V_P'' = V_{AP}, \quad (22a)$$

and at B

$$-V_P' + V_P'' = V_{BP}. \quad (22b)$$

From these equations

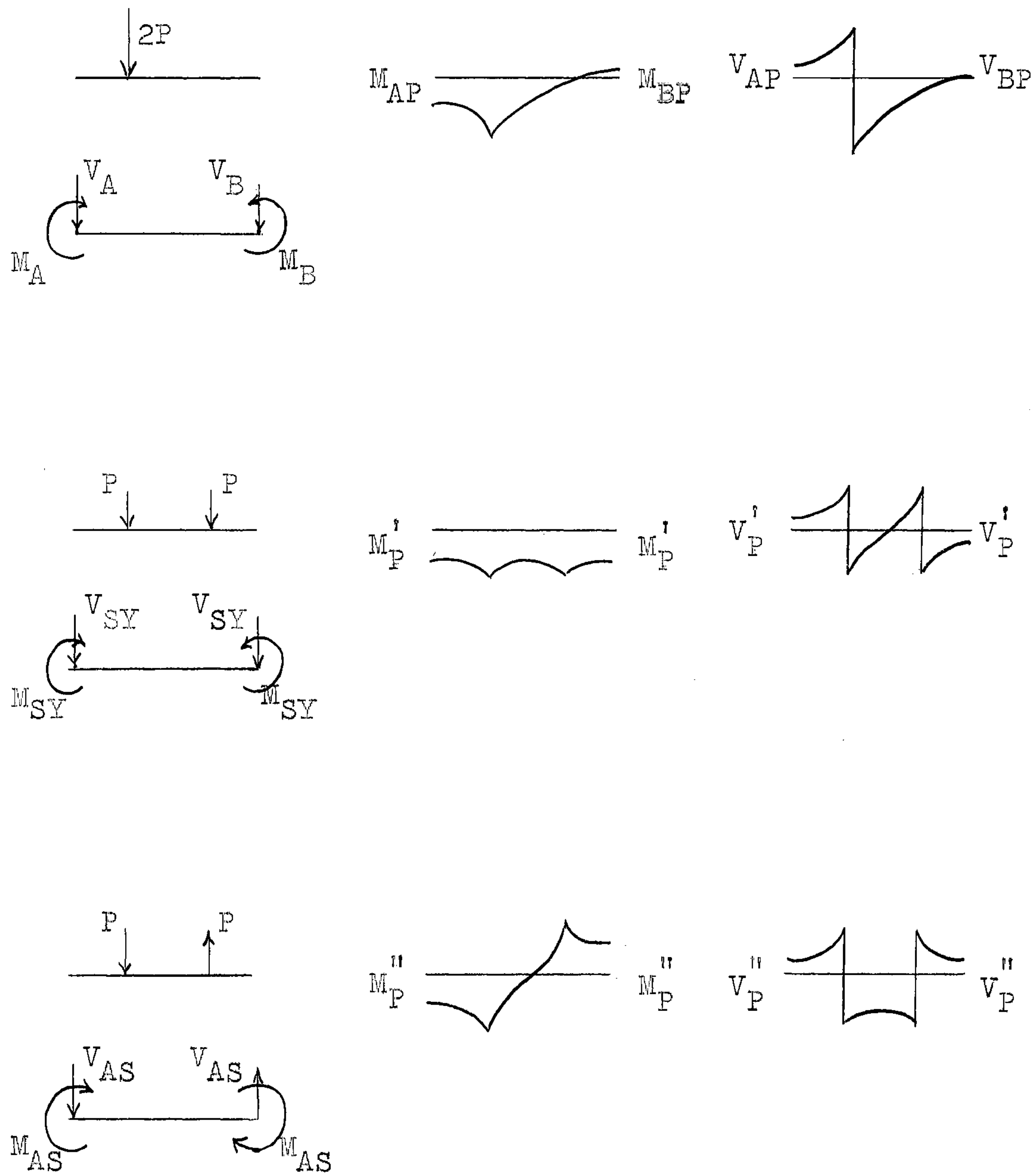


Figure 9.

Resolution of a System of Loads
with Bending Moment and Shear Diagrams.

$$M_P' = \tfrac{1}{2}(M_{AP} + M_{BP}), \tag{23a}$$

$$V_P' = \tfrac{1}{2}(V_{AP} - V_{BP}), \tag{23b}$$

$$M_P'' = \tfrac{1}{2}(M_{AP} - M_{BP}), \tag{24a}$$

and

$$V_P'' = \tfrac{1}{2}(V_{AP} + V_{BP}). \tag{24b}$$

Equations may now be written for the symmetrical case as

$$M_P' + \frac{V_{SY}}{4Z}(1 + C_{ZL}) + \frac{M_{SY}}{2}(1 + D_{ZL}) = 0, \tag{25a}$$

$$V_P' - \frac{V_{SY}}{2}(1 - D_{ZL}) - \frac{M_{SY}Z}{2}(1 - A_{ZL}) = 0, \tag{25b}$$

and for the antisymmetrical case as

$$M_P'' + \frac{V_{AS}}{4Z}(1 - C_{ZL}) + \frac{M_{AS}}{2}(1 - D_{ZL}) = 0 \tag{26a}$$

and

$$V_P'' - \frac{V_{AS}}{2}(1 + D_{ZL}) - \frac{M_{AS}Z}{2}(1 + A_{ZL}) = 0. \tag{26b}$$

Solving equations (25 a and b)

$$M_{SY} = \frac{2}{Z} E_I \left[V_P'(1 + C_{ZL}) + 2M_P'Z(1 - D_{ZL}) \right], \tag{27a}$$

$$V_{SY} = 4 E_I \left[V_P'(1 + D_{ZL}) + M_P' Z(1 - A_{ZL}) \right], \tag{27b}$$

and from equations (26 a and b)

$$M_{AS} = \frac{2}{Z} E_{II} \left[V_P''(1 - C_{ZL}) + 2M_P''Z(1 + D_{ZL}) \right], \tag{28a}$$

and

$$V_{AS} = 4 E_{II} \left[V_P''(1 - D_{ZL}) + M_P'' Z(1 + A_{ZL}) \right], \tag{28b}$$

where

$$E_I = \frac{1}{2(1 + D_{ZL})(1 - D_{ZL}) - (1 - A_{ZL})(1 + C_{ZL})} \tag{29a}$$

or

$$E_I = \frac{e^{ZL}}{2(\sinh ZL + \sin ZL)} \tag{29b}$$

and

$$E_{II} = \frac{1}{2(1 + D_{ZL})(1 - D_{ZL}) - (1 + A_{ZL})(1 - C_{ZL})} \tag{30a}$$

or

$$E_{II} = \frac{e^{ZL}}{2(\sinh ZL - \sin ZL)}. \tag{30b}$$

The final equations are found from superposition as

$$M_A = M_{SY} + M_{AS}, \tag{31a}$$

$$V_A = V_{SY} + V_{AS}, \tag{31b}$$

$$M_B = M_{SY} - M_{AS} \tag{31c}$$

and

$$V_B = V_{SY} - V_{AS}. \tag{31d}$$

The solutions of particular cases are tabulated in Chapter IV.

3) Hinged Ends

The end conditions of a finite hinged end beam may be determined by visual inspection and are

$$y = 0 \text{ and } M = 0.$$

The given system of loads is resolved into a symmetrical and an antisymmetrical one and the superposition of bending moments at points A and B and of shears at points A and B yields

$$M_{AP} = M_P' + M_P'', \qquad M_{BP} = M_P' - M_P'',$$

$$y_{AP} = y_P' + y_P'' \quad \text{and} \quad y_{BP} = y_P' - y_P'',$$

from which

$$M_P' = \tfrac{1}{2}(M_{AP} + M_{BP}), \qquad M_P'' = \tfrac{1}{2}(M_{AP} - M_{BP}),$$

$$y_P' = \tfrac{1}{2}(y_{AP} + y_{BP}) \quad \text{and} \quad y_P'' = \tfrac{1}{2}(y_{AP} - y_{BP}).$$

Equations can now be written for bending moment and deflection for the symmetrical case as

$$M_P' + \frac{M_{SY}}{2}(1 + D_{ZL}) + \frac{V_{SY}}{4Z}(1 + C_{ZL}) = 0 \tag{32a}$$

and

$$y_P' + \frac{M_{SY}Z^2}{k}(0 + B_{ZL}) + \frac{V_{SY}}{2k}(1 + A_{ZL}) = 0. \tag{32b}$$

Similarily for the antisymmetrical case

$$M_P'' + \frac{M_{AS}}{2}(1 - D_{ZL}) + \frac{V_{AS}}{4Z}(1 - C_{ZL}) = 0 \tag{33a}$$

and

$$y_P'' + \frac{M_{AS}Z^2}{k}(0 + B_{ZL}) + \frac{V_{AS}Z}{2k}(1 - A_{ZL}) = 0. \tag{33b}$$

The symmetrical and antisymmetrical end-conditioning elements are found from these equations to be

$$M_{SY} = 2\,F_I\left[-M_P'(1 + A_{ZL}) + 2Z^2EIy_P'(1 + C_{ZL})\right], \tag{34a}$$

$$V_{SY} = 4ZF_I\left[M_P'\,B_{ZL} - 2Z^2EI\,y_P'(1 + D_{ZL})\right], \tag{34b}$$

$$M_{AS} = -2F_{II}\left[M_P''(1 - A_{ZL}) - 2Z^2EIy_P''(1 - C_{ZL})\right]. \tag{34c}$$

and

$$V_{AS} = -4ZF_{II}\left[M_P''\,B_{ZL} + 2Z^2EI\,y_P''(1 - D_{ZL})\right], \tag{34d}$$

where

$$F_I = \frac{1}{B_{ZL}(1 - C_{ZL}) + (1 - D_{ZL})(1 - A_{ZL})} \tag{a}$$

or

$$F_I = \frac{e^{ZL}}{2(\cosh ZL + \cos ZL)} \tag{b}$$

and

$$F_{II} = \frac{1}{B_{ZL}(1 - C_{ZL}) + (1 - D_{ZL})(1 - A_{ZL})} \quad (c)$$

or

$$F_{II} = \frac{e^{ZL}}{2(\cosh ZL - \cos ZL)}. \quad (d)$$

From these symmetrical and antisymmetrical end-conditioning elements, the final end-conditioning elements are found by superposition as

$$M_A = M_{SY} + M_{AS}, \qquad M_B = M_{SY} - M_{AS}, \quad (35a)$$

$$V_A = V_{SY} + V_{AS} \quad \text{and} \quad V_B = V_{SY} - V_{AS}. \quad (35b)$$

4) Fixed Ends

The analysis of a fixed end finite beam originates with the known end conditions

$$\theta = 0 \quad \text{and} \quad M = 0.$$

The analysis proceeds in a manner similar to the previous cases and final results are shown in the tables of Chapter IV.

5) One End Hinged, One End Free

The simplification presented for beams with identical end conditions is not applicable in the case of an unsymmetrical beam (Figure 10a). However, the solution is still based on the principle of solving for M_A, M_B, V_A and V_B and of applying these values as loads to an infinite beam (Figure 10b) loaded as the finite beam under consideration.

Four equations satisfying the known end conditions are:

deflection at point A = 0,

$$y_{PA} + \frac{V_A Z}{2k} A_{ZO} - \frac{M_A Z^2}{k} B_{ZO} + \frac{V_B Z}{2k} A_{ZL} + \frac{M_B Z^2}{k} B_{ZL} = 0, \quad (e)$$

bending moment at point A = 0,

$$M_{PA} + \frac{V_A}{4Z} C_{ZO} + \frac{M_A}{2} D_{ZO} + \frac{V_B}{4Z} C_{ZL} + \frac{M_B}{2} D_{ZL} = 0, \quad (f)$$

shear at point B = 0,

$$V_{PB} + \frac{V_A}{2} D_{ZL} - \frac{M_A Z}{2} A_{ZL} + \frac{V_B}{2} D_{ZO} + \frac{M_B Z}{2} D_{ZO} = 0 \quad (g)$$

and

bending moment at point B = 0,

$$M_{PB} + \frac{V_A}{4Z} C_{ZL} + \frac{M_A}{2} D_{ZL} + \frac{V_B}{4Z} C_{ZO} + \frac{M_B}{2} D_{ZO} = 0. \quad (h)$$

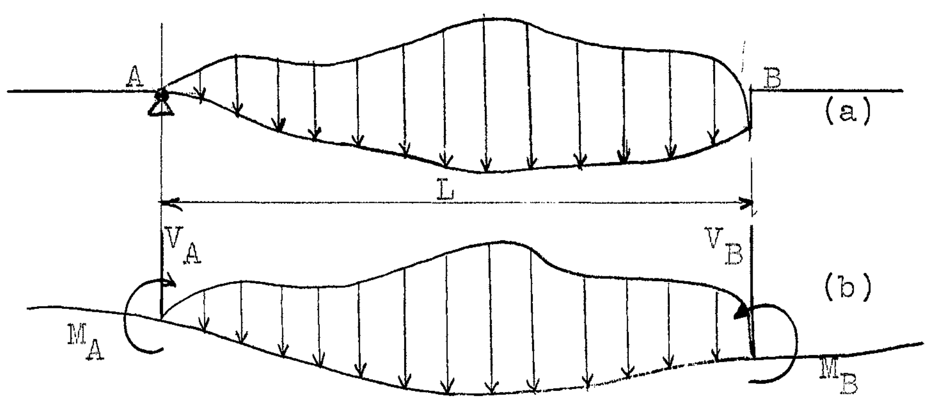


Figure 10. Finite Beam--One End Hinged, One End Free.

The analysis of beams with one end fixed and the other end free or with one end fixed and the other end hinged is a similar procedure. A set of tables covering these cases is enclosed in Chapter IV.

C. REMARKS

Finite beams on elastic foundations are classified according to stiffness. The relative stiffness is determined from the ZL quantity. This ZL quantity determines the magnitude of the curvature of the elastic curve and defines the rate at which the effect of the loading force vanishes in the form of a damped wave along the length of the beam.

Beams are classified according to the magnitude of this ZL value, thus,

Short beams	$ZL < 0.60$,
Medium beams	$0.60 > ZL < 5.00$

and

Long beams	$ZL > 5.00$.

Short beams have a relatively negligible deformation in the beam, and therefore are considered as completely rigid and may be analized by statics alone.

Medium beams require complete analysis based on the method of superposition.

Long beams have values of $A_{ZL} \doteq B_{ZL} \doteq C_{ZL} \doteq D_{ZL} \doteq 0$. Because of the negligible effect of forces applied at one end on the other end, these values can be taken equal to zero with little error. Long beams may then be analized at though they were infinitely long.

The effect of these approximations for short and long beams is illustrated in Chapter V.

D. CONCLUSIONS

1) The classical analysis of infinite beams on elastic foundations, in conjunction with the principle of superposition, is used to analize finite beams on elastic foundations.

2) The finite beam is replaced by an infinite beam loaded:

a) with the loading of the finite beam

and

b) with the end-conditioning elements.

3) The end-conditioning elements are obtained from equations which relate finite and infinite beam functions.

4) To simplify the procedure, the principle of symmetry and antisymmetry is utilized in cases of beams having identical end conditions.

5) The infinite beam is then analized by the classical method and the results are valid for the finite beam being considered.

6) Complete analysis is required only for medium beams where ZL is greater than 0.60 and less than 5.00.

7) For short beams (ZL less than 0.60), the equations of statics are sufficient; for long beams (ZL greater than 5.00) the functions A_{ZL}, B_{ZL}, C_{ZL} and D_{ZL} can be taken as equal to zero.

III. THE METHOD OF TRIGONOMETRIC SERIES

A. INTRODUCTORY STATEMENT

The second general method of analysis of finite beams on elastic foundations is based on two principles:

1) representation of the deflection curve in the form of a trigonometric series

and

2) the equality of the external and internal work of the system.

This method is a powerful tool in the analysis of certain types of beams on elastic foundations since E, I and k appear seperately. However, application is restricted to rather special cases and the complexity of the equations limits their application to the most elementary load conditions.

B. PARTICULAR CASES

1) Beams with Hinged Ends

The deflection curve of a finite hinged end beam on an elastic foundation can be expressed in the form of a sine series as

$$y = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}.$$

Consider the case of a finite beam loaded by a concentrated force P at a distance c from the left support.

The strain energy of bending of the beam is

$$U_1 = \frac{EI}{2}\int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx = \frac{\pi^4 EI}{4L^3}\sum_{n=1}^{\infty} n^4 a_n^2, \qquad (37a)$$

and the strain energy of the foundation is

$$U_2 = \frac{k}{2}\int_0^L y^2\ dx = \frac{kL}{4}\sum_{n=1}^{\infty} a^2 n. \qquad (37b)$$

If a small change is produced in the a_n th term, the changes in strain energy can be equated as

$$P\frac{\partial y_P}{\partial a_n}\,da_n = \frac{\partial U_1}{\partial a_n}\,da_n + \frac{\partial U_2}{\partial a_n}\,da_n, \qquad (38)$$

where

$$P\frac{\partial y_P}{\partial a_n} = P\ \sin\frac{n\pi c}{L},$$

$$\frac{\partial U_1}{\partial a_n} = \frac{\pi^4 EI}{2L^3}\,n^4\ a_n$$

and

$$\frac{\partial U_2}{\partial a_n} = \frac{kL}{2}\,a_n.$$

Expressing equation (38) in these terms

$$P\ \sin\frac{n\pi c}{L} = \frac{\pi^4 EI}{2L^3}\,n^4 a_n + \frac{kL}{2}\,a_n. \qquad (39)$$

From equation (39), the deflection as given in equation (36) becomes

$$y = \frac{2PL^3}{\pi^4 EI}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi c}{L}\ \sin\frac{n\pi x}{L}}{n^4 + \frac{kL^4}{\pi^4 EI}} \qquad (40)$$

Equation (40) may be obtained in a more rapidly convergent form by considering the deflection to be caused by the superposition of y_P and y_R, where

y_P = the deflection due to the load on a simple beam

and y_R = the deflection of the beam supported by distributed reactive forces rather than an elastic foundation.

The deflection y_R can now be evaluated from the differential equation of the elastic curve (equation 3) with y expressed by equation (40),

$$EI \frac{d^4 y_R}{dx^4} + k \left[\frac{2PL^3}{\pi^4 EI} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi c}{L} \sin \frac{n\pi x}{L}}{n^4 + \frac{kL^4}{\pi^4 EI}} \right] = 0, \qquad \text{(a)}$$

or

$$y_R = -\frac{2PL^3}{\pi^4 EI} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi c}{L} \sin \frac{n\pi x}{L}}{n^4 \left(1 + \frac{n^4 \pi^4 EI}{kL^4}\right)}. \qquad (41)$$

The form of the deflection equation of a finite hinged end beam on an elastic foundation now becomes

$$y = y_P - \frac{2PL^3}{\pi^4 EI} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi c}{L} \sin \frac{n\pi x}{L}}{n^4 \left(1 + n^4 \frac{\pi^4 EI}{kL^4}\right)}. \qquad (42)$$

The slope, moment and shear can be found by the successive differentiation of equation (42). The results of these differentiations are presented in the tables of Chapter IV, as well as equations for the beam functions of a beam loaded by either an applied moment or by a uniformly distributed load.

2) Fixed End Beam

The deflection curve of a fixed end beam takes the form

$$y = \sum_{n=1}^{\infty} \tfrac{1}{2}\, a_n \left(1 - \cos \frac{2n\pi x}{L}\right). \tag{43}$$

Every term of this series is symmetrical; therefore, only beams loaded symmetrically can be analized by this method.

Results for the deflection of a fixed end finite beam loaded by two symmetrical concentrated forces, by two symmetrical couples and by a symmetrical uniformly distributed load are presented in the tables of Chapter IV.

3) Beams with Free Ends

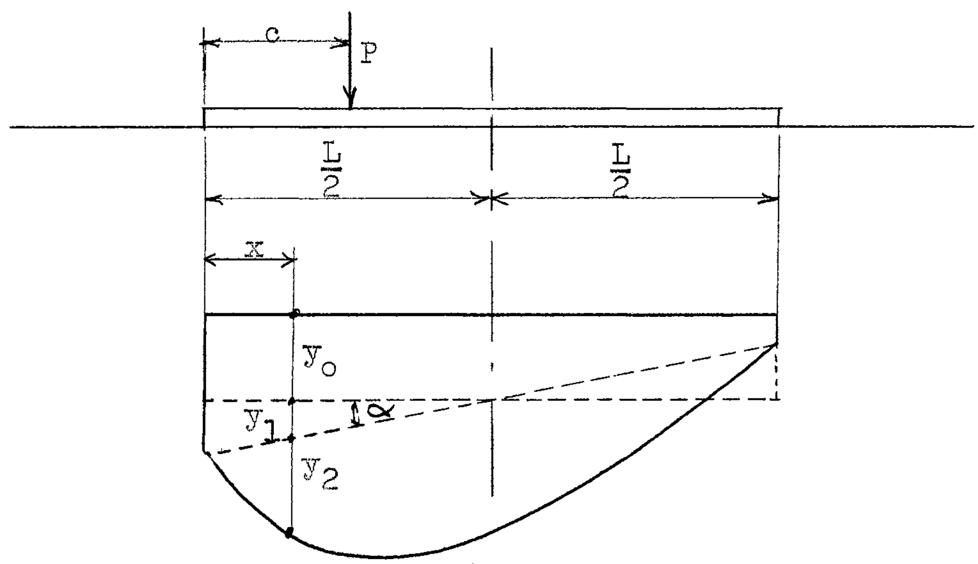


Figure 11. Free End Beam, Trigonometric Series

Modification of the basic sine series is necessary in the solution of free end beams on elastic foundations. Figure (11) represents the general case of such a beam loaded by a concentrated force P at a distance c from the left end of the beam.

It may be readily observed that the total deflection can be divided into three components:

a) y_o, the deflection parallel to the original axis of the beam,

b) y_1, the angular displacement related to the axis at the center of the beam

and

c) y_2, the portion of the deflection which can be represented by the basic sine series.

Therefore,

$$y = y_o + y_1 + y_2 \tag{44a}$$

or

$$y = y_o + \alpha\left(\frac{L}{2} - x\right) + \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}. \tag{44b}$$

However, the analysis can be simplified by resolving the load into symmetrical and antisymmetrical components as shown in Figure (12a and 12b).

The general deflection equations may now be written in the form

$$y = y_{SY} + y_{AS}, \tag{45a}$$

where

$$y_{SY} = y_o + y_2' = y_o + \sum_{n=1,3,5}^{\infty} a_n \sin\frac{n\pi x}{L} \tag{45b}$$

and

$$y_{AS} = y_1 + y_2'' = \alpha\left(\frac{L}{2} - x\right) + \sum_{n=2,4,6}^{\infty} a_n \sin\frac{n\pi x}{L}. \tag{45c}$$

The symmetrical and antisymmetrical cases can then be analized seperately and the results superimposed.

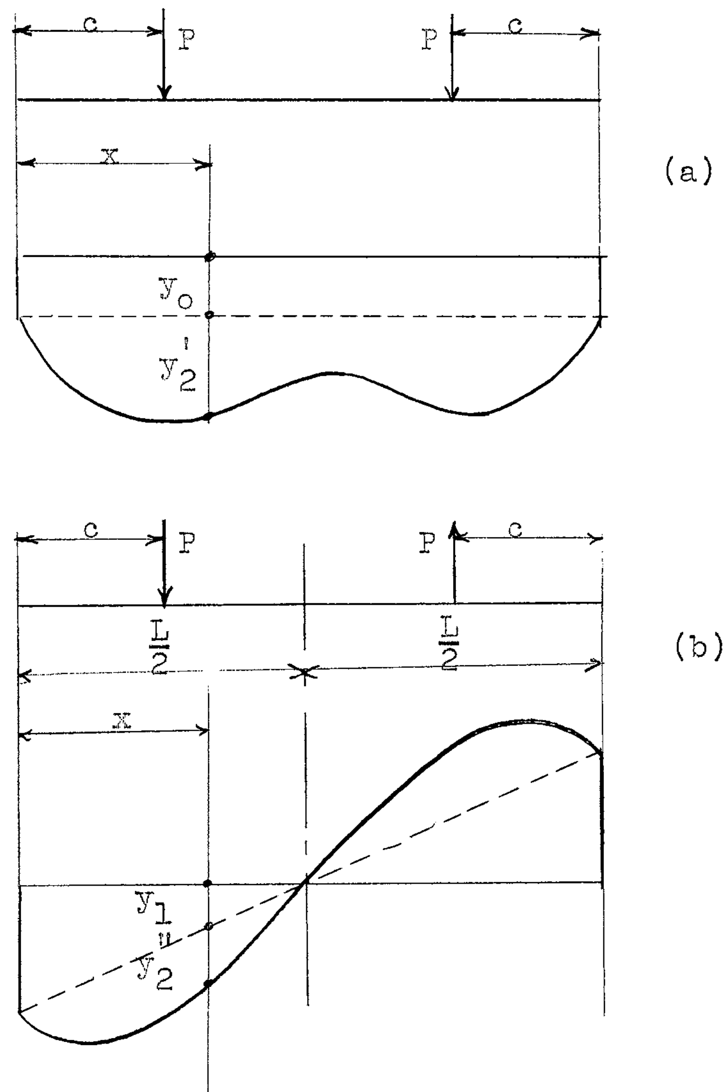


Figure 12.
Resolution of a System of Loads
for the Method of Trigonometric Series.

First, the symmetrical system is analized. From the basic assumption $R = ky$,

$$2P = k\int_0^L y\,dx = k\int_0^L \left(y_o + \sum_{n=1,3,5}^{\infty} a_n \sin \frac{n\pi x}{L}\right)dx. \qquad \text{(b)}$$

The displacement parallel to the original axis is

$$y_o = \frac{2P}{kL} - \frac{2}{\pi}\sum_{n=1,3,5}^{\infty}\frac{1}{n}a_n. \qquad (c)$$

Substituting these values into equation (45b)

$$y_{SY} = \frac{2P}{kL} - \sum_{n=1,3,5}^{\infty} a_n\left(\frac{2}{n\pi} - \sin\frac{n\pi x}{L}\right). \qquad (46)$$

The value a_n in equation (46) is unknown and can be calculated from the equality

$$2P\frac{\partial y_P}{\partial a_n}da_n = \frac{\partial U_1}{\partial a_n}da_n + \frac{\partial U_2}{\partial a_n}da_n, \qquad (47a)$$

or

$$2P\left(\sin\frac{n\pi x}{L} - \frac{2}{n\pi}\right) = a_n\left(n^4\frac{\pi^4 EI}{2L^3} + \frac{kL}{2}\right) - \frac{4kL}{2}\frac{1}{n}\sum_{n=1,3,5}^{\infty}\frac{1}{i}a_i. \qquad (47b)$$

The a terms can now be determined by evaluating n and i as

$$2P\left(\sin\frac{\pi x}{L} - \frac{2}{\pi}\right) = a_1\left(\frac{\pi^4 EI}{2L^3} + \frac{kL}{2}\right) - \frac{4kL}{\pi^2}\frac{1}{1}\left(\frac{a_1}{1} + \frac{a_3}{3} + \frac{a_5}{5} + \ldots\right)$$

$$2P\left(\sin\frac{3\pi x}{L} - \frac{2}{3\pi}\right) = a_3\left(3^4\frac{\pi^4 EI}{2L^3} + \frac{kL}{2}\right) - \frac{4kL}{2}\frac{1}{3}\left(\frac{a_1}{1} + \frac{a_3}{3} + \ldots\right), \text{etc.}$$

If the concentrated load is placed at the center or at the ends of the beam, only one term of the series is required and the deflection equation (46) may be written

$$y_{SY} = \frac{2P}{kL} + \frac{2P\left(\sin\frac{\pi x}{L} - \frac{2}{\pi}\right)^2}{\frac{\pi^4 EI}{2L^3} + \frac{kL}{2} - \frac{4kL}{\pi^2}}. \qquad (48)$$

For the analysis of the antisymmetrical system, the general deflection equation is

$$y_{AS} = \alpha\left(\frac{L}{2} - x\right) + \sum_{n=2,4,6}^{\infty} a_n \sin\frac{n\pi x}{L}, \qquad (49)$$

where α is evaluated from

$$M_A = 0 \quad \text{or} \quad P(L - 2c) + k\int_0^L y_{AS}\, x dx = 0 \qquad \text{(d)}$$

and

$$\alpha = \frac{12}{L^3}\left[\frac{P(L - 2c)}{k} - \frac{L^2}{\pi}\sum_{n=2,4,6}^{\infty} a_n \sin\frac{n\pi x}{L}\right]. \qquad \text{(e)}$$

The general deflection equation (49) in terms of equation (e) is

$$y_{AS} = \frac{12P(L-2c)}{kL^3}\left(\frac{L}{2} - x\right) - \left[\frac{12}{L\pi}\left(\frac{L}{2} - x\right) - 1\right]\sum_{n=2,4,6}^{\infty} a_n \sin\frac{n\pi x}{L}. \quad (50)$$

The a values can now be obtained by equating the strain energies as

$$2P\left[\sin\frac{n\pi c}{L} - \frac{6}{n\pi L}(L - 2c)\right] = a_n\left(n^4\frac{\pi^4 EI}{2L^3} + \frac{kL}{2}\right) - \frac{12kL}{n\pi^2}\sum_{n=2,4,6}^{\infty}\frac{1}{i}\, a_i. \quad (51)$$

If only the first term of the series is required,

$$a_2 = \frac{2P\left[\sin\frac{n\pi c}{L} - \frac{3}{L\pi}(L - 2c)\right]}{\frac{8\pi^4 EI}{L^3} + \frac{kL}{2} - \frac{3kL}{\pi^2}}. \qquad \text{(f)}$$

The final form of the deflection equation is obtained by substituting the result of equation (f) into equation (49). Finally, the total deflection is

$$y = y_{SY} + y_{AS}. \qquad (52)$$

From the performed investigation the complexity of the algebraic work is apparent and thus the limits of practical application of this method are established.

C. CONCLUSIONS

1) The method of trigonometric series is based on the representation of the elastic curve in the form of a trigonometric series.

2) Unknowns are evaluated from strain energy relations.

3) For hinged end beams the basic deflection equation is

$$y = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} ,$$

and the solutions are relatively simple and generally applicable.

4) For fixed end beams the basic deflection equation is

$$y = \sum_{n=1}^{\infty} \tfrac{1}{2}\, a_n(1 - \cos \frac{2n\pi x}{L}),$$

and the procedure is limited to the analysis of beams loaded by a symmetrical system of loads.

5) For free end beams the basic deflection equation must be modified to

$$y = y_o + \alpha(\frac{L}{2} - x) + \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L},$$

and the procedure is restricted to the solution of the most elementary problems.

6) In special cases the method of trigonometric series is superior since the modulus of the foundation, the modulus of elasticity of the beam and the moment of inertia of the beam appear seperately. In all other cases, the method of superposition should be used.

IV. TABLES

TABLE 1
Basic Relationships

1) Beam function equations for a point right of a concentrated load applied to an infinite beam on an elastic foundation:

deflection	$y = \frac{PZ}{2k} A_{Zx},$
slope	$\Theta = \frac{-PZ^2}{k} B_{Zx},$
bending moment	$M = \frac{P}{4Z} C_{Zx},$

and

shear	$V = \frac{-P}{2} D_{Zx}.$

2) Basic notations:

$$A_{Zx} = e^{-Zx}(\cos Zx + \sin Zx),$$

$$B_{Zx} = e^{-Zx} \sin Zx,$$

$$C_{Zx} = e^{-Zx}(\cos Zx - \sin Zx)$$

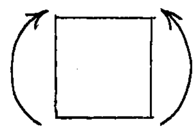
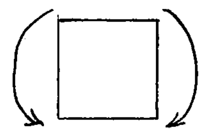
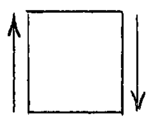
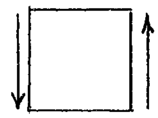
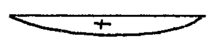
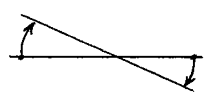
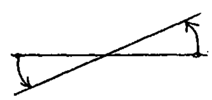
and

$$D_{Zx} = e^{-Zx} \cos Zx.$$

TABLE 2

Sign Conventions

1) Basic sign conventions:

	POSITIVE	NEGATIVE
Bending Moment		
Shear		
Deflection		
Load		
Slope		

2) Sign conventions for the method of superposition:

(for positive sense)

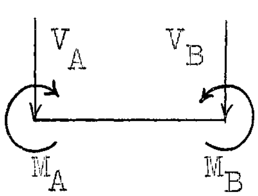
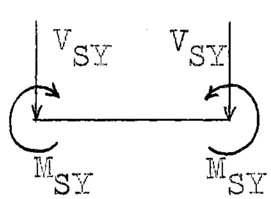
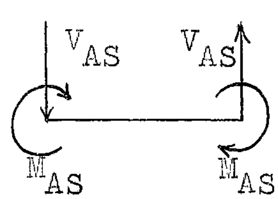
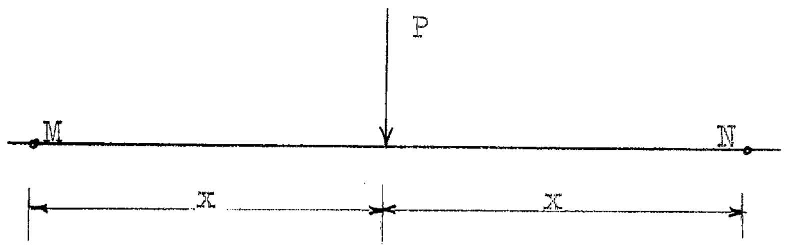
General Case	Symmetrical Case	Antisymmetrical Case
		

TABLE 3

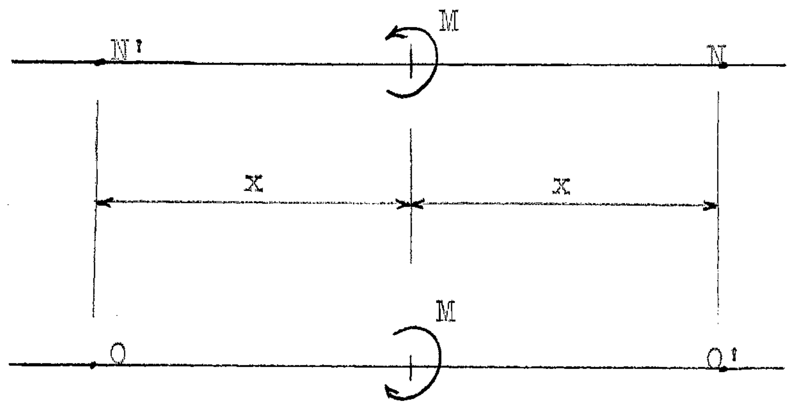
Infinite Beam - Concentrated Load



Deflection Equations	$y_M = \frac{PZ}{2k} A_{Zx}$ $y_N = \frac{PZ}{2k} A_{Zx}$
Slope Equations	$\theta_M = \frac{PZ^2}{k} B_{Zx}$ $\theta_N = \frac{-PZ^2}{k} B_{Zx}$
Bending Moment Equations	$M_M = \frac{P}{4Z} C_{Zx}$ $M_N = \frac{P}{4Z} C_{Zx}$
Shear Equations	$V_M = \frac{P}{2} D_{Zx}$ $V_N = \frac{-P}{2} D_{Zx}$

TABLE 4

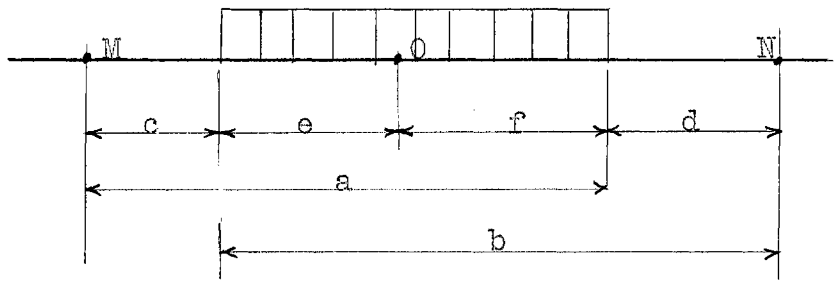
Infinite Beam - Couple



Deflection Equations	$-y_N = y_{N'} = \frac{MZ^2}{k} B_{Zx}$ $-y_O = y_{O'} = \frac{MZ^2}{k} B_{Zx}$
Slope Equations	$\theta_N = \theta_{N'} = \frac{-MZ^3}{k} C_{Zx}$ $\theta_O = \theta_{O'} = \frac{MZ^3}{k} C_{Zx}$
Bending Moment Equations	$-M_N = M_{N'} = \frac{M}{2} D_{Zx}$ $-M_O = M_{O'} = \frac{M}{2} D_{Zx}$
Shear Equations	$V_N = V_{N'} = \frac{MZ}{2} A_{Zx}$ $V_O = V_{O'} = \frac{-MZ}{2} A_{Zx}$

TABLE 5

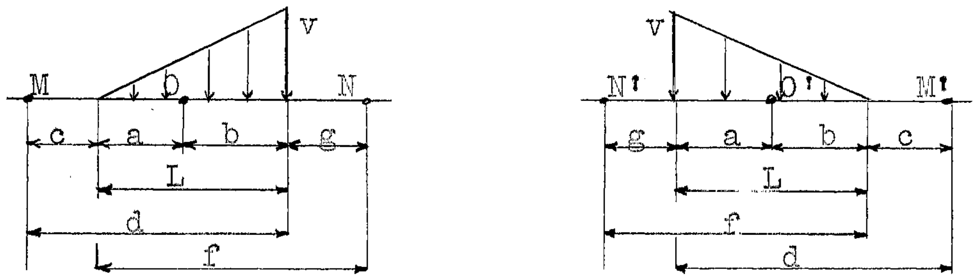
Infinite Beam - Uniformly Distributed Load



Deflection Equations	$y_M = \frac{w}{2k}(D_{Zc} - D_{Za})$ $y_N = \frac{-w}{2k}(D_{Zb} - D_{Zd})$ $y_O = \frac{w}{2k}(2 - D_{Ze} - D_{Zf})$
Slope Equations	$\theta_M = \frac{wZ}{k}(A_{Zc} - A_{Za})$ $\theta_N = \frac{wZ}{k}(A_{Zb} - A_{Zd})$ $\theta_O = \frac{wZ}{k}(A_{Zf} - A_{Ze})$
Bending Moment Equations	$M_M = \frac{-w}{2Z^2}(B_{Zc} - B_{Za})$ $M_N = \frac{w}{2Z^2}(B_{Zb} - B_{Zd})$ $M_O = \frac{w}{2Z^2}(B_{Zf} + B_{Ze})$
Shear Equations	$V_M = \frac{w}{4Z}(C_{Zc} - C_{Za})$ $V_N = \frac{w}{4Z}(C_{Zb} - C_{Zd})$ $V_O = \frac{w}{4Z}(C_{Zf} - C_{Ze})$

TABLE 6

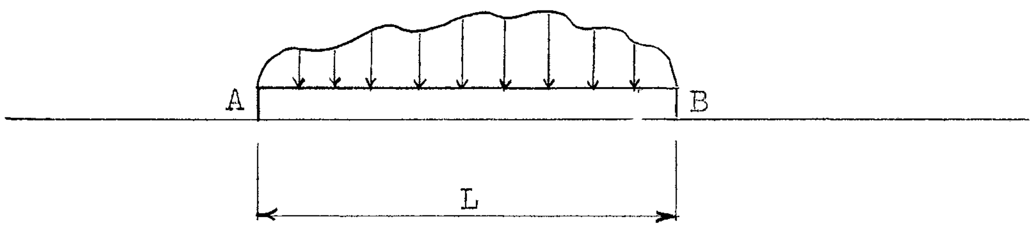
Infinite Beam - Triangular Load Distribution



Deflection Equations	$y_O = \frac{v}{4ZkL}(C_{Za} - C_{Zb} + 4Za - 2ZLD_{Zb})$
	$y_{O'} = \frac{v}{4ZkL}(C_{Zb} - C_{Za} + 4Zb - 2ZLD_{Za})$
	$y_M = y_{M'} = \frac{v}{4ZkL}(C_{Zc} - C_{Zd} - 2ZLD_{Zd})$
	$y_N = y_{N'} = \frac{v}{4ZkL}(C_{Zf} - C_{Zg} + 2ZLD_{Zg})$
Slope Equations	$\theta_O = \frac{-v}{2kL}(D_{Za} + D_{Zb} + ZL\ A_{Zb} - 2)$
	$\theta_{O'} = \frac{-v}{2kL}(D_{Zb} + D_{Za} + ZL\ A_{Za} - 2)$
	$\theta_M = -\theta_{M'} = \frac{v}{2kL}(D_{Zc} - D_{Zd} - ZL\ A_{Zd})$
	$-\theta_N = \theta_{N'} = \frac{v}{2kL}(D_{Zf} - D_{Zg} + ZL\ A_{Zg})$
Bending Moment Equations	$M_O = \frac{-v}{8Z^3L}(A_{Za} - A_{Zb} - 2ZL\ B_{Zb})$
	$M_{O'} = \frac{-v}{8Z^3L}(A_{Zb} - A_{Za} - 2ZL\ B_{Za})$
	$M_M = M_{M'} = \frac{-v}{8Z^3L}(A_{Zc} - A_{Zd} - 2ZL\ B_{Zd})$
	$-M_N = M_{N'} = \frac{v}{8Z^3L}(A_{Zf} - A_{Zg} + 2ZL\ B_{Zg})$
Shear Equations	$V_O = \frac{v}{4Z^2L}(B_{Za} + B_{Zb} - ZL\ C_{Zb})$
	$V_{O'} = \frac{v}{4Z^2L}(B_{Zb} + B_{Za} - ZL\ C_{Zb})$
	$-V_M = V_{M'} = \frac{v}{4Z^2L}(B_{Zc} - B_{Zd} + ZL\ C_{Zd})$
	$V_N = V_{N'} = \frac{v}{4Z^2L}(B_{Zf} - B_{Zg} - ZL\ C_{Zg})$

TABLE 7

Method of Superposition
Finite Beam - Free Ends



1) Consider beam AB as infinitely long and solve for M_{AP}, M_{BP}, V_{AP} and V_{BP}.

2) $M'_P = \frac{1}{2}(M_{AP} + M_{BP})$ $\qquad V'_P = \frac{1}{2}(V_{AP} - V_{BP})$

$M''_P = \frac{1}{2}(M_{AP} - M_{BP})$ $\qquad V''_P = \frac{1}{2}(V_{AP} + V_{BP})$

3)
$$M_{SY} = \frac{2}{Z} E_I \left[V'_P(1 + C_{ZL}) + 2M'_P\, Z(1 - D_{ZL}) \right]$$
$$V_{SY} = 4 E_I \left[V'_P(1 + D_{ZL}) + M'_P\, Z(1 - A_{ZL}) \right]$$
$$M_{AS} = \frac{2}{Z} E_{II} \left[V''_P(1 - C_{ZL}) + 2M''_P\, Z(1 + D_{ZL}) \right]$$
$$V_{AS} = 4 E_{II} \left[V''_P(1 - D_{ZL}) + M''_P\, Z(1 + A_{ZL}) \right]$$

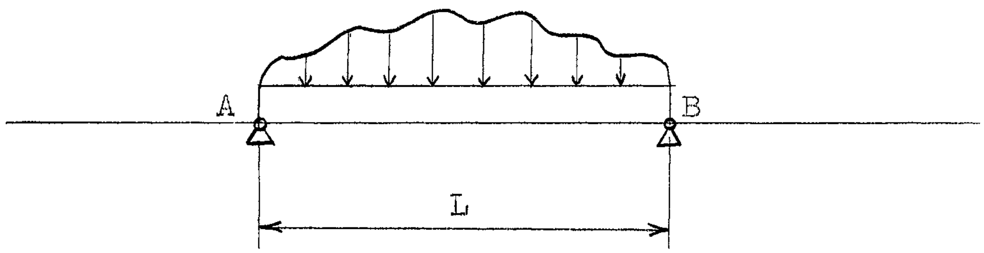
4) $M_A = M_{SY} + M_{AS}$ $\qquad V_A = V_{SY} + V_{AS}$

$M_B = M_{SY} - M_{AS}$ $\qquad V_A = V_{SY} - V_{AS}$

5) An infinite beam is now loaded with the original load and with the forces and moments V_A, V_B, M_A and M_B. The analysis of this infinite beam solves the finite beam.

TABLE 8

Method of Superposition
Finite Beam - Hinged Ends



1) Consider beam AB as infinitely long and solve for y_{AP}, y_{BP}, M_{AP} and M_{BP}.

2) $M'_P = \frac{1}{2}(M_{AP} + M_{BP})$ $\qquad$ $y'_P = \frac{1}{2}(y_{AP} + y_{BP})$

$M''_P = \frac{1}{2}(M_{AP} - M_{BP})$ $\qquad$ $y''_P = \frac{1}{2}(y_{AP} - y_{BP})$

3)
$$M_{SY} = 2 F_I \left[-M'_P(1 + A_{ZL}) + 2Z^2EI \; y'_P(1 + C_{ZL}) \right]$$
$$V_{SY} = 4Z F_I \left[M'_P B_{ZL} - 2Z^2EI \; y'_P(1 + D_{ZL}) \right]$$
$$M_{AS} = -2 F_{II} \left[M''_P(1 - A_{ZL}) - 2Z^2EI \; y''_P(1 - C_{ZL}) \right]$$
$$V_{AS} = -4Z F_{II} \left[M''_P B_{ZL} + 2Z^2EI \; y''_P(1 - D_{ZL}) \right]$$

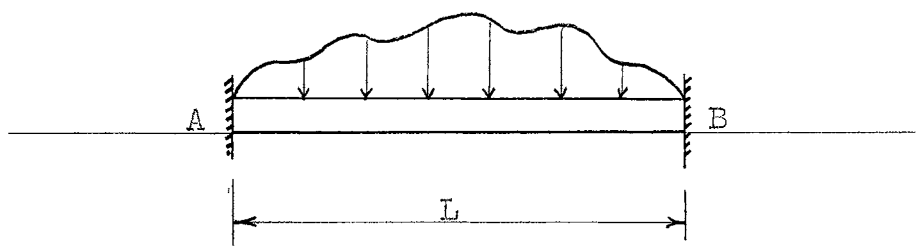
4) $M_{SY} + M_{AS} = M_A$ $\qquad$ $V_{SY} + V_{AS} = V_A$

$M_{SY} - M_{AS} = M_B$ $\qquad$ $V_{SY} - V_{AS} = V_B$

5) An infinite beam is now loaded with the original load and with the forces and moments V_A, V_B, M_A and M_B. The analysis of this infinite beam solves the finite beam.

TABLE 9

Method of Superposition
Finite Beam - Fixed Ends



1) Consider Beam AB as infinitely long and solve for y_{AP}, y_{BP}, θ_{AP} and θ_{BP}.

2) $\theta'_P = \frac{1}{2}(\theta_{AP} - \theta_{BP})$ $y'_P = \frac{1}{2}(y_{AP} + y_{BP})$

$\theta''_P = \frac{1}{2}(\theta_{AP} + \theta_{BP})$ $y''_P = \frac{1}{2}(y_{AP} - y_{BP})$

3)
$$M_{SY} = -4ZEI\ E_I\left[\theta'_P(1 + A_{ZL}) - 2y'_P Z\ B_{ZL}\right]$$
$$V_{SY} = 8Z^2EI\ E_I\left[\theta'_P\ B_{ZL} - y'_P Z(1 - C_{ZL})\right]$$
$$M_{AS} = -4ZEIE_{II}\left[\theta''_P(1 - A_{ZL}) + 2y''_P Z\ B_{ZL}\right]$$
$$V_{AS} = -8Z^2EIE_{II}\left[\theta''_P\ B_{ZL} + y''_P Z(1 + C_{ZL})\right]$$

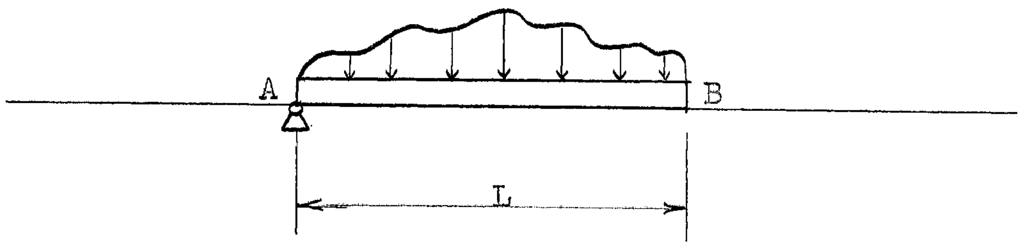
4) $M_{SY} + M_{AS} = M_A$ $V_{SY} + V_{AS} = V_A$

$M_{SY} - M_{AS} = M_B$ $V_{SY} - V_{AS} = V_B$

5) An infinite beam is now loaded with the original load and with the forces and moments V_A, V_B, M_A and M_B. The analysis of this infinite beam solves the finite beam.

TABLE 10

Method of Superposition
Finite Beam - One End Hinged, One End Free



1) Consider beam AB to be infinitely long and solve for y_{AP}, M_{AP}, V_{BP} and M_{BP}.

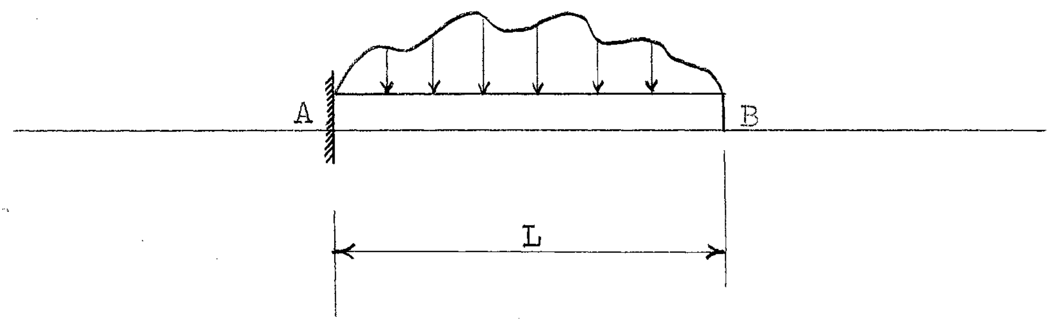
2) Solve the following determinant for V_A, V_B, M_A and M_B.

V_A	M_A	V_B	M_B	
Z	0	$Z\ A_{ZL}$	$2Z^2\ B_{ZL}$	$-2k\ y_{PA}$
1	$2Z$	C_{ZL}	$2Z\ D_{ZL}$	$-4Z\ M_{PA}$
D_{ZL}	$-Z\ A_{ZL}$	1	Z	$-2\ V_{PB}$
C_{ZL}	$2Z\ D_{ZL}$	1	$2Z$	$-4Z\ M_{PB}$

3) Load an infinite beam with the original loading on finite beam AB and with the forces and moments V_A, V_B, M_A and M_B. The analysis of this infinite beam by the classical method yields results which are valid for the finite beam being considered.

TABLE 11

Method of Superposition
Finite Beam - One End Fixed, One End Free



1) Consider beam AB to be infinitely long and solve for θ_{AP}, y_{AP}, M_{BP} and V_{BP}.

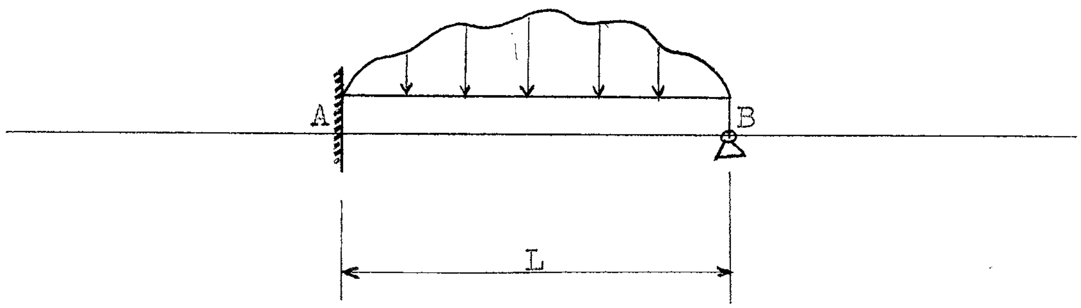
2) Solve the following determinant for V_A, V_B, M_A and M_B.

V_A	M_A	V_B	M_B	
0	Z^3	$Z^2\ B_{ZL}$	$-Z^3\ C_{ZL}$	$-k\ \theta_{AP}$
Z	0	$Z\ A_{ZL}$	$2Z^2\ B_{ZL}$	$-2k\ y_{AP}$
C_{ZL}	$2Z\ D_{ZL}$	1	2Z	$-4Z\ M_{BP}$
$-D_{ZL}$	$-Z\ A_{ZL}$	1	Z	$-2\ V_{BP}$

3) Load an infinite beam with the original loading on finite beam AB and with the forces and moments V_A, V_B, M_A and M_B. The analysis of this infinite beam by the classical method yields results which are valid for the finite beam being considered.

TABLE 12

Method of Superposition
Finite Beam - One End Fixed, One End Hinged



1) Consider beam AB to be infinitely long and solve for θ_{AP}, y_{AP}, y_{BP} and M_{BP}.

2) Solve the following determinant for V_A, V_B, M_A and M_B.

V_A	M_A	V_B	M_B	
0	Z^3	$Z^2\ B_{ZL}$	$-Z^3\ C_{ZL}$	$-k\ \theta_{AP}$
Z	0	$Z\ A_{ZL}$	$2Z^2\ B_{ZL}$	$-2k\ y_{AP}$
$Z\ A_{ZL}$	$2Z^2\ B_{ZL}$	Z	0	$-2k\ y_{BP}$
C_{ZL}	$2Z\ D_{ZL}$	1	2Z	$-4Z\ M_{BP}$

3) Load an infinite beam with the original loading on finite beam AB and with the forces and moments V_A, V_B, M_A and M_B. The analysis of this infinite beam by the classical method yields results which are valid for the finite beam being considered.

TABLE 13

Method of Trigonometric Series
Finite Hinged End Beam, Concentrated Load
(Figure on Page 52)

$$y = y^P - \frac{2PL^3}{\pi^4 EI}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi c}{L}\ \sin\frac{n\pi x}{L}}{n^4\left(1 + n^4\frac{\pi^4 EI}{kL^4}\right)}$$

$$\theta = \theta^P - \frac{2PL^2}{\pi^3 EI}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi c}{L}\ \cos\frac{n\pi x}{L}}{n^3\left(1 + n^4\frac{\pi^4 EI}{kL^4}\right)}$$

$$M = M^P - \frac{2PL}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi c}{L}\ \sin\frac{n\pi x}{L}}{n^2\left(1 + n^4\frac{\pi^4 EI}{kL^4}\right)}$$

$$V = V^P - \frac{2P}{\pi}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi c}{L}\ \cos\frac{n\pi x}{L}}{n\left(1 + n^4\frac{\pi^4 EI}{kL^4}\right)}$$

$$y^P_{(x<c)} = \frac{P}{6EI}\ \frac{L - c}{L}\left[(2L - c)cx - x^3\right]$$

$$y^P_{(x>c)} = \frac{P}{6EI}\frac{c}{L}\left[(L^2 - c^2)(L - x) - (L - x)^3\right]$$

TABLE 14

Method of Trigonometric Series
Finite Hinged End Beam, Loaded by a Couple
(Figure on Page 52)

$$y = y^M - \frac{2M_o L^2}{\pi^3 EI} \sum_{n=1}^{\infty} \frac{\cos\frac{n\pi c}{L} \sin\frac{n\pi x}{L}}{n^3\left(1 + n^4\frac{\pi^4 EI}{kL^4}\right)}$$

$$\theta = \theta^M - \frac{2M_o L}{\pi^2 EI} \sum_{n=1}^{\infty} \frac{\cos\frac{n\pi c}{L} \cos\frac{n\pi x}{L}}{n^2\left(1 + n^4\frac{\pi^4 EI}{kL^4}\right)}$$

$$M = M^M - \frac{2M_o}{\pi} \sum_{n=1}^{\infty} \frac{\cos\frac{n\pi c}{L} \sin\frac{n\pi x}{L}}{n\left(1 + n^4\frac{\pi^4 EI}{kL^4}\right)}$$

$$V = V^M - \frac{2M_o}{L} \sum_{n=1}^{\infty} \frac{\cos\frac{n\pi c}{L} \cos\frac{n\pi x}{L}}{1 + n^4\frac{\pi^4 EI}{kL^4}}$$

$$y^M_{(x<c)} = \frac{M_o}{6EI}\frac{1}{L}\left[(2L^2 - 6Lc + 3c^2)x + x^3\right]$$

$$y^M_{(x>c)} = \frac{M_o}{6EI}\frac{1}{L}\left[(L^2 - 3c^2)(L - x) - (L - x)^3\right]$$

TABLE 15

Method of Trigonometric Series
Finite Hinged End Beam, Uniformly Distributed Load
(Figure on Page 52)

$$y = y^{W} - \frac{2wL^4}{\pi^5 EI}\sum_{n=1}^{\infty}\frac{\left(\cos\frac{n\pi c_1}{L} - \cos\frac{n\pi c_2}{L}\right)\sin\frac{n\pi x}{L}}{n^5\left(1 + n^4\frac{{}^4EI}{kL^4}\right)}$$

$$\theta = \theta^{W} - \frac{2wL^3}{\pi^4 EI}\sum_{n=1}^{\infty}\frac{\left(\cos\frac{n\pi c_1}{L} - \cos\frac{n\pi c_2}{L}\right)\cos\frac{n\pi x}{L}}{n^4\left(1 + n^4\frac{\pi^4EI}{kL^4}\right)}$$

$$M = M^{W} - \frac{2wL^2}{\pi^3}\sum_{n=1}^{\infty}\frac{\left(\cos\frac{n\pi c_1}{L} - \cos\frac{n\pi c_2}{L}\right)\sin\frac{n\pi x}{L}}{n^3\left(1 + n^4\frac{\pi^4EI}{kL^4}\right)}$$

$$V = V^{W} - \frac{2wL}{\pi^2}\sum_{n=1}^{\infty}\frac{\left(\cos\frac{n\pi c_1}{L} - \cos\frac{n\pi c_2}{L}\right)\cos\frac{n\pi x}{L}}{n^2\left(1 + n^4\frac{\pi^4EI}{kL^4}\right)}$$

where for $x < c_1$

$$y^{W} = \frac{w}{24EI}\,\frac{bc}{L}\left\{\left[4(L^2 - b^2) - c^2\right]x - 4x^3\right\}$$

and for $c_1 < x < c_2$

$$y^{W} = \frac{w}{24EI}\,\frac{bc}{L}\left\{\left[4(L^2 - b^2) - c^2\right]x - 4x^3 + \frac{L(x - c_1)^4}{bc}\right\}$$

and for $x > c_2$

$$y^{W} = \frac{w}{24EI}\,\frac{ac}{L}\left\{\left[4(L^2 - a^2) - c^2\right](L - x) - 4(L - x)^3\right\}$$

Figure for Table 13

Finite Hinged End Beam - Concentrated Load

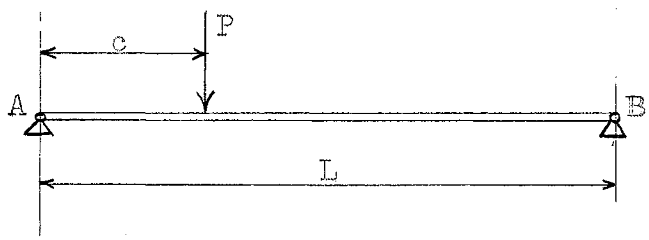


Figure for Table 14

Finite Hinged End Beam - Loaded by a Couple

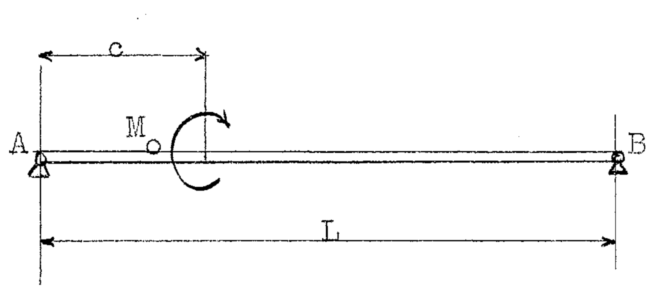


Figure for Table 15

Finite Hinged End Beam - Uniformly Distributed Load

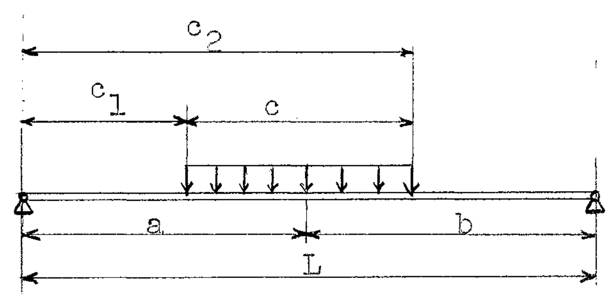
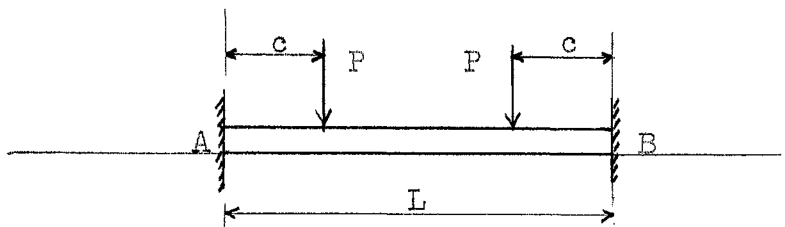


TABLE 16

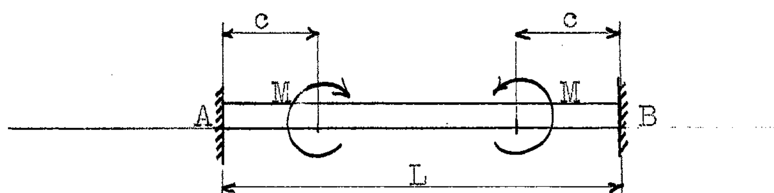
Method of Trigonometric Series
Deflection Equations for Finite Fixed End Beams

A. Two Symmetrical Concentrated Forces



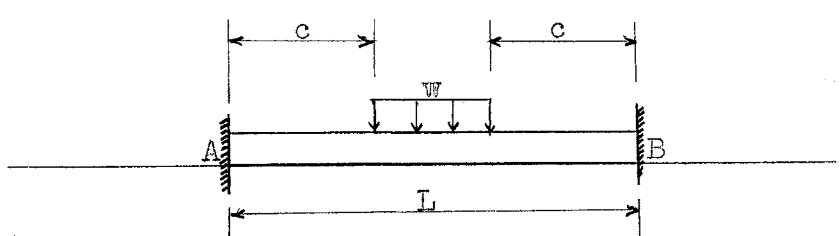
$$y = \frac{PL^3}{4\pi^4 EI}\sum_{n=1}^{\infty}\frac{\left(1 - \cos\frac{2n\pi c}{L}\right)\left(1 - \cos\frac{2n\pi x}{L}\right)}{n^4 + \frac{3}{16}\frac{kL^4}{\pi^4 EI}}$$

B. Two Symmetrical Couples



$$y = \frac{ML^2}{2\pi^3 EI}\sum_{n=1}^{\infty}\frac{n\sin\frac{2n\pi c}{L}\left(1 - \cos\frac{2n\pi x}{L}\right)}{n^4 + \frac{3}{16}\frac{kL^4}{\pi^4 EI}}$$

Uniformly Distributed Symmetrical Load



$$y = \frac{wL^3}{8\pi^4 EI}\sum_{n=1}^{\infty}\frac{\left(1 - 2c + \frac{1}{n\pi}\sin\frac{2n\pi c}{L}\right)\left(1 - \cos\frac{2n\pi x}{L}\right)}{n^4 + \frac{3}{16}\frac{kL^4}{\pi^4 EI}}$$

TABLE 17

Functions of A_{ZX}, B_{ZX}, C_{ZX} and D_{ZX}

Zx	A_{Zx}	B_{Zx}	C_{Zx}	D_{Zx}
0.0	1.0000	0.0000	1.0000	1.0000
0.1	0.9907	0.0903	0.8100	0.9003
0.2	0.9651	0.1627	0.6398	0.8024
0.3	0.9267	0.2189	0.4888	0.7077
0.4	0.8784	0.2610	0.3564	0.6174
0.5	0.8231	0.2908	0.2415	0.5323
0.6	0.7628	0.3099	0.1431	0.4530
0.7	0.6997	0.3199	0.0599	0.3798
0.8	0.6354	0.3223	-0.0093	0.3131
0.9	0.5712	0.3185	-0.0657	0.2527
1.0	0.5083	0.3096	-0.1108	0.1988
1.1	0.4476	0.2967	-0.1457	0.1510
1.2	0.3899	0.2807	-0.1716	0.1091
1.3	0.3355	0.2626	-0.1897	0.0729
1.4	0.2849	0.2430	-0.2011	0.0419
1.5	0.2384	0.2226	-0.2068	0.0158
1.6	0.1959	0.2018	-0.2077	-0.0059
1.7	0.1576	0.1812	-0.2047	-0.0235
1.8	0.1234	0.1610	-0.1985	-0.0376
1.9	0.0932	0.1415	-0.1899	-0.0484
2.0	0.0667	0.1230	-0.1794	-0.0563

Zx	A_{Zx}	B_{Zx}	C_{Zx}	D_{Zx}
2.2	0.0244	0.0895	-0.1548	-0.0652
2.4	-0.0056	0.0613	-0.1282	-0.0669
2.6	-0.0254	0.0383	-0.1019	-0.0636
2.8	-0.0369	0.0204	-0.0777	-0.0573
3.0	-0.0423	0.0070	-0.0563	-0.0493
3.2	-0.0431	-0.0024	-0.0383	-0.0407
3.4	-0.0408	-0.0085	-0.0237	-0.0323
3.6	-0.0366	-0.0121	-0.0124	-0.0245
3.8	-0.0314	-0.0137	-0.0040	-0.0177
4.0	-0.0258	-0.0139	0.0019	-0.0120
4.5	-0.0132	-0.0108	0.0085	-0.0023
5.0	-0.0046	-0.0065	0.0084	0.0019
5.5	0.0000	-0.0029	0.0058	0.0029
6.0	0.0017	-0.0007	0.0031	0.0024
6.5	0.0018	0.0004	0.0012	0.0015
7.0	0.0013	0.0006	0.0001	0.0007

TABLE 18

Functions of E_I, E_{II}, F_I and F_{II}

x	E_I	E_{II}	F_I	F_{II}
0.0	∞	∞	0.2500	∞
0.1	2.7634	1492.5	0.2763	55.157
0.2	1.5265	233.6	0.3053	15.293
0.3	1.1249	74.7	0.3373	7.5069
0.4	0.9322	34.9	0.3726	4.6614
0.5	0.8239	19.8	0.4111	3.2973
0.6	0.7584	12.7	0.4531	2.5292
0.7	0.7177	8.8028	0.4985	2.0534
0.8	0.6931	6.5147	0.5470	1.7370
0.9	0.6795	5.0582	0.5986	1.5154
1.0	0.6739	4.0740	0.6524	1.3551
1.1	0.6745	3.3807	0.7079	1.2361
1.2	0.6800	2.8746	0.7640	1.1461
1.3	0.6892	2.4967	0.8196	1.0770
1.4	0.7017	2.2066	0.8737	1.0235
1.5	0.7166	1.9802	0.9247	0.9821
1.6	0.7338	1.7997	0.9719	0.9501
1.7	0.7524	1.6550	1.0140	0.9254
1.8	0.7724	1.5369	1.0503	0.9070
1.9	0.7933	1.4396	1.0804	0.8935
2.0	0.8145	1.3593	1.1051	0.8842
2.1	0.8358	1.2927	1.1221	0.8781

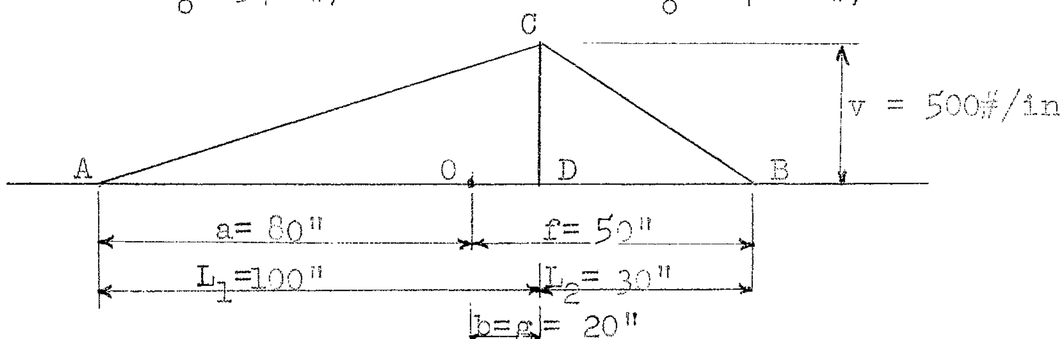
x	E_I	E_{II}	F_I	F_{II}
2.25	0.8674	1.2123	1.1383	0.8744
2.50	0.9160	1.1175	1.1427	0.8785
2.75	0.9572	1.0447	1.1288	0.8910
3.00	0.9983	1.0067	1.1063	0.9082
3.50	1.0226	0.9801	1.0590	0.9456
4.00	1.0290	0.9733	1.0242	0.9762
4.50	1.0222	0.9790	1.0045	0.9953
5.00	1.0132	0.9872	0.9962	1.0038

V. EXAMPLE PROBLEMS

A. INFINITE BEAMS - CLASSICAL METHOD

Example 1

Given: $b = 12''$
$d = 7\frac{1}{2}''$
$I = 422\ in^4$
$E = 3 \times 10^6$ (Concrete)
$k_o = 342\ \#/in^3$ $\qquad k = bk_o = 4100\ \#/in^2$



Find: Deflection, Moment and Shear at Point O

Procedure:

1) Determine Z

$$Z = \sqrt[4]{\frac{k}{4EI}} = \sqrt[4]{\frac{4100}{(4)(3)(10^6)(422)}} = 0.03\ in^{-1}$$

2) Resolve Load into Two Parts: ACD and BCD

3) Deflection at Point O

From Table 6,

$$y_o = \frac{v}{4ZkL_1}(C_{Za} - C_{Zb} + 4Za - 2ZLD_{Zb}) + \frac{v}{4ZkL_2}(C_{Zf} - C_{Zg} + 2ZLD_{Zg})$$

$Za = (.03)(80) = 2.40 \qquad Zf = (.03)(50) = 1.50$

$Zb = (.03)(20) = 0.60 \qquad Zg = (.03)(20) = 0.60$

From Table 17,

$C_{Za} = -0.1282$ $\quad$ $C_{Zf} = -0.2068$

$C_{Zb} = 0.1431$ $\quad$ $C_{Zg} = 0.1431$

$D_{Zb} = 0.4530$ $\quad$ $D_{Zg} = 0.4530$

$$y_o = \frac{500}{4(.03)(4100)(100)}\left[-.1282 - .1431 + 4(2.40) - 2(.03)(100).453\right]$$
$$+ \frac{500}{4(.03)(4100)(30)}\left[-.2068 - .1431 + 2(.03)(30)(.4530)\right]$$

$y_o = 0.0672 + 0.0158 = 0.0830$ inches

4) Moment at Point 0

From Table 6,

$$M_o = \frac{-V}{8Z^3L_1}(A_{Za} - A_{Zb} - 2ZL_1B_{Zb}) + \frac{V}{8Z^3L_2}(A_{Zf} - A_{Zg} + 2ZL_2B_{Zg})$$

From Table 17,

$A_{Za} = -0.0056$ $\quad$ $A_{Zf} = 0.2384$

$A_{Zb} = 0.7628$ $\quad$ $A_{Zg} = 0.7628$

$B_{Zb} = 0.3099$ $\quad$ $B_{Zg} = 0.3099$

$M_o = 60,900 + 2500 = 63,400$ inch pounds

5) Shear at Point 0

From Table 6,

$$V_o = \frac{V}{4Z^2L_1}(B_{Za} + B_{Zb} - ZL_1C_{Zb}) + \frac{V}{4Z^2L_2}(B_{Zf} - B_{Zg} - ZL_2C_{Zg})$$

From Table 6,

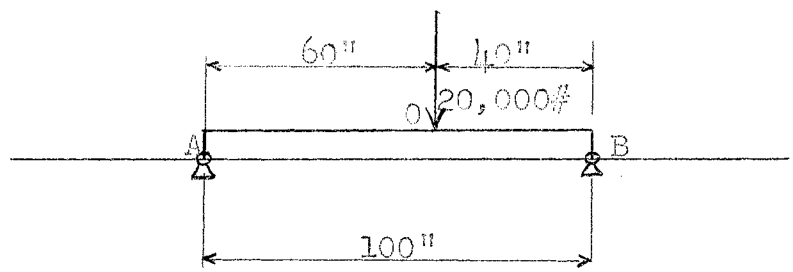
$B_{Za} = 0.0613$ $\quad$ $B_{Zf} = 0.2226$

$V_o = -81 - 1000 = -1081$ pounds

B. FINITE BEAMS - METHOD OF SUPERPOSITION

Example 2

Given: Hinged Ends
$E = 3 \times 10^6$
$k = 4100 \ \#/in^2$
$Z = 0.03 \ in^{-1}$
$I = 422 \ in^4$



Find: Deflection at Point 0

Procedure:

1) Determine y_{AP}, y_{BP}, M_{AP} and M_{BP} as if the beam were infinitely long.

$$y_{AP} = \frac{PZ}{2k} A_{Zx} = \frac{(20,000)(.03)}{2(4100)} A_{(.03)(60)} = .0090 \text{ in.}$$

$$y_{BP} = \frac{600}{8200} A_{(1.2)} = .0286 \text{ in.}$$

$$M_{AP} = \frac{P}{4Z} C_{Zx} = \frac{20,000}{4(.03)} C_{(1.8)} = -33,200 \text{ in}^{\#}$$

$$M_{BP} = \frac{20,000}{.12}(-.1716) = -28,600 \text{ in}^{\#}$$

2) Determine M_P', M_P'', V_P' and V_P''

$$M_P' = \tfrac{1}{2}(M_{AP} + M_{BP}) = \tfrac{1}{2}(-33,200 -28,600) = -30,900 \text{ in}^{\#}$$

$$M_P'' = \tfrac{1}{2}(M_{AP} - M_{BP}) = \tfrac{1}{2}(-33,200 +28,600) = - 2,300 \text{ in}^{\#}$$

$y_P' = .0188$ in. $\qquad$ $y_P'' = -.0098$ in.

3) Determine M_{SY}, V_{SY}, M_{AS} and V_{AS}

Constants required for ZL = 3.00:

$A_{ZL} = -0.0423$ | $C_{ZL} = -0.0563$

$B_{ZL} = 0.0070$ | $D_{ZL} = -0.0493$

$F_{I} = 1.1063$ | $F_{II} = 0.9082$

$$M_{SY} = 2\,F_{I}\left[-M_P'(1 + A_{ZL}) + 2Z^2EI\;y_P'(1 + C_{ZL})\right]$$

$M_{SY} = 154{,}000$ in# | $M_{AS} = -38{,}800$ in#

$V_{SY} = -5{,}550$ # | $V_{AS} = 2{,}540$ #

4) Determine End-Conditioning Elements

$M_A = M_{SY} + M_{AS} = 154{,}000 - 38{,}800 = 115{,}200$ in.#

$M_B = M_{SY} - M_{AS} = 154{,}000 + 38{,}800 = 192{,}800$ in#

$V_A = -3010$ # | $V_B = -8090$ #

5) Solve infinite beam loaded by P, M_A, M_B, V_A and V_B.

$$y = \frac{PZ}{2k} + \frac{V_AZ}{2k}\,A_{(Z)(60)} + \frac{V_BZ}{2k}\,A_{40Z} + \frac{M_AZ^2}{k}\,B_{60Z} + \frac{M_BZ^2}{k}B_{40Z}$$

$$y = 0.076 \text{ inches}$$

Example 3

Given: Beam AB with Free Ends

$E = 3 \times 10^6$ #/in^2

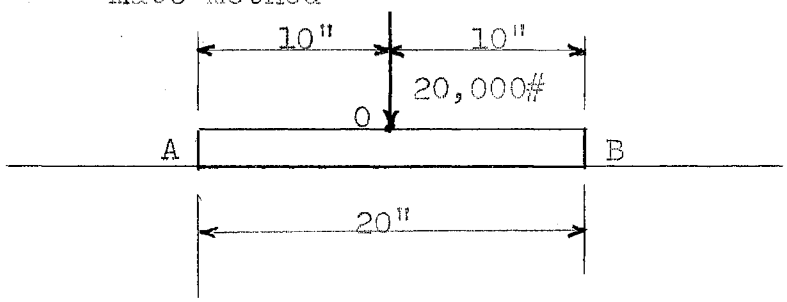
$k = 4100$ #/in^2

$Z = 0.03$ in^{-1}

$I = 422$ in^4

Find: (A) Deflection at Points A and O by the exact method

(B) Deflection at Points A and O by the approximate method



Procedure:

(A) 1) Solve for V_{AP}, V_{BP}, M_{AP} and M_{BP} as if beam AB were infinitely long

$$M_{AP} = M_{BP} = \frac{P}{4Z} C_{Zx} = \frac{20,000}{4(.03)} C_{(.30)} = 81,300 \text{ in}^{\#}$$

$$-V_{BP} = V_{AP} = \frac{P}{2} D_{Zx} = 7,077 \#$$

2) Determine M_P', M_P'', V_P' and V_P''

$$M_P' = \tfrac{1}{2}(M_{AP} + M_{BP}) = 81,300 \text{ in}^{\#} \qquad M_P'' = 0$$

$$V_P' = \tfrac{1}{2}(V_{AP} - V_{BP}) = 7,077 \# \qquad V_P'' = 0$$

3) Determine M_{SY}, V_{SY}, M_{AS} and V_{AS}

$$M_{SY} = 543,000 \text{ in}^{\#} \qquad M_{AS} = 0$$

$$V_{SY} = 33,000 \# \qquad V_{AS} = 0$$

4) Determine V_A, V_B, M_A and M_B

$$V_A = V_B = V_{SY} = 33,000 \#$$

$$M_A = M_B = M_{SY} = 543,000 \text{ in}^{\#}$$

5) Determine y_o

$$y_o = \frac{PZ}{2k} + \frac{2V_A Z}{2k} A_{(.30)} - \frac{2M_A Z^2}{k} B_{(.30)}$$

$$y_o = \frac{1}{4100} (300 + 918 - 215) = 0.2445 \text{ inches}$$

6) Determine y_A

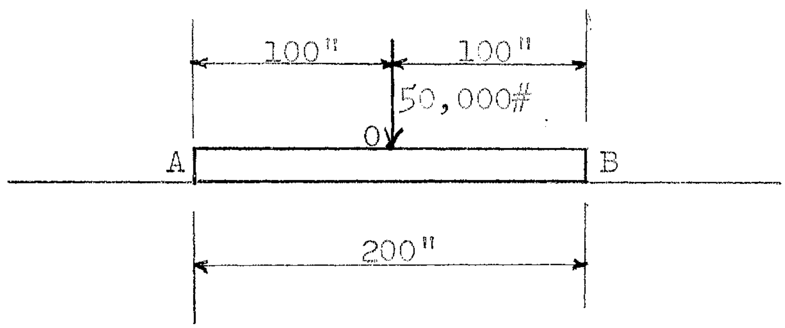
$$y_A = \frac{1}{8200} (1748 + 556 - 303) = 0.2450 \text{ inches}$$

(B) Determine approximate y by statics

$$y = \frac{P}{kL} = \frac{20,000}{20(4100)} = 0.2440 \text{ inches}$$

Example 4

Given: Beam AB with Free Ends
$E = 3 \times 10^6 \ \#/in^2$
$k = 4100 \ \#/in^2$
$Z = 0.03 \ in^{-1}$
$I = 422 \ in^4$



Find: (A) Deflection at Point O by considering the beam to be infinitely long

(B) Deflection at Point O by the Exact Method

(A) Determine y_o as if beam were infinitely long

$$y_o = \frac{PZ}{2k} = \frac{(50,000)(.03)}{2(4100)} = 0.183 \text{ inches}$$

(B) 1) Determine M_{AP}, M_{BP}, V_{AP} and V_{BP} as if the beam were infinitely long

$$M_{AP} = M_{BP} = -23400 \text{in}^{\#}$$

$$V_{AP} = -V_{BP} = -1232 \ \#$$

2) Determine E_I since 6 is beyond table limits

$$E_I = \frac{e^6}{2(\sinh 6 + \sin 6)} \doteq 1$$

3) Determine M_{SY} and V_{SY} ($M_{AS} = 0$ and $V_{AS} = 0$)

$$M_{SY} = -242,000 \ \text{in}^{\#}$$

$$V_{SY} = -7,740 \ \#$$

4) Determine V_A, V_B, M_A and M_B

$$V_A = V_B = V_{SY} \qquad M_A = M_B = M_{SY}$$

5) Determine the deflection at Point O

$$y_o = \frac{1}{8200} (1500 + 18.8 - 6.1) = 0.184 \text{ inches}$$

C. FINITE BEAMS - METHOD OF TRIGONOMETRIC SERIES

Example 5

Given: Beam AB of Example 2

1) Refer to Table 13

$$y = y^P + y^R$$

2) Determine y^P

$$y^P_{(x=c)} = \frac{P}{6EI}\ \frac{L - c}{L}\left[(2L - c)cx - x^3\right] = .304 \text{ in}$$

3) Determine y^R (one term of series only)

$$y^R = \frac{2PL^3}{\pi^4 EI}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi c}{L}\ \sin\frac{n\pi x}{L}}{n^4\left(1 + n^4\frac{\pi^4 EI}{kL^4}\right)} = -.226 \text{ in}$$

4) Determine y

$$y = .304 - .226 = 0.078 \text{ inches}$$

D. REMARKS

1) Comparison of Example 5 with Example 2 shows that for this special case, the error caused by evaluating only one term of the trigonometric series is only 2.5%.

2) Example 3 illustrates the small error (0.4%) resulting from using the approximate solution for short beams, where ZL is less than 0.60.

3) Example 4 illustrates the small error (0.6%) resulting from considering a long beam (ZL greater than 5.00) as an infinite beam.

CONCLUSIONS

1. Three methods of analysis of beams on elastic foundations were demonstrated in this report:

a. The classical method,

b. The method of superposition

and

c. The method of trigonometric series.

2. The classical method is based on the relationship between the differential equation of the elastic curve and Winkler's equation, and is directly applicable to the solution of infinite beams.

3. The method of superposition is an extension of the classical method, combined with the principle of superposition, and is applicable to the solution of finite beams.

4. The method of trigonometric series is an alternate method and is restricted to a limited number of cases.

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APPENDIX A

SOLUTION OF THE DIFFERENTIAL EQUATION

To solve $EI\,\frac{d^4y}{dx^4} = -ky$ or $EI\,y^{IV} = -ky$, (a)

express in terms of operators as $(EID^4 + k)y = 0$. (b)

From this, the reduced equation is $EIn^4 + k = 0$, making $n^4 = -\frac{k}{EI}$ the characteristic equation.

It is necessary to use De Moivre's Theorem to solve for the roots of n^4. Thus,

$$n = \left\{\frac{k}{EI}\left[\cos(180 + m360) + i\,\sin(180 + m360)\right]\right\}^{\frac{1}{4}},$$

$$n = \left(\frac{k}{EI}\right)^{\frac{1}{4}}\left[\cos\left(\frac{180 + m360}{4}\right) + i\,\sin\left(\frac{180 + m360}{4}\right)\right]$$

or

$$n = Z\left[\cos(45 + m90) + i\,\sin(45 + m90)\right].$$

Evaluating m as 0, 1, 2 and 3, the roots are found as $Z(1 + i)$, $Z(-1 + i)$, $-Z(1 + i)$ and $-Z(-1 + i)$.

Observing equation (b), the multiplicity is 4 and the general solution may be written as

$$y = C_1' e^{Zx+Zix} + C_2' e^{-Zx+Zix} + C_3' e^{-Zx-Zix} + C_4' e^{Zx-Zix}. \qquad (c)$$

Using the relationships

$$e^{Zix} = \cos Zx + i\,\sin Zx$$

and

$$e^{-Zix} = \cos Zx - i\,\sin Zx,$$

equation (c) can be rewritten in the form

$$y = C_1'e^{Zx}(\cos Zx + i \sin Zx) + C_2'e^{-Zx}(\cos Zx + i \sin Zx)$$

$$+ C_3'e^{-Zx}(\cos Zx - i \sin Zx) + C_4'e^{Zx}(\cos Zx - i \sin Zx).$$

Further simplification is possible from the notations

$$C_1' + C_4' = C_1, \qquad i(C_1' - C_4') = C_2,$$

$$C_2' + C_3' = C_3 \quad \text{and} \quad i(C_2' - C_3') = C_4,$$

and the general equation can be written

$$y = e^{Zx}(C_1 \cos Zx + C_2 \sin Zx) + e^{-Zx}(C_3 \cos Zx + C_4 \sin Zx).$$

APPENDIX B
HAYASHI'S METHOD

Notations: $T = \sqrt[4]{\frac{4EI}{bk}}$; $\frac{x}{T} = Z.$

Differential equation: $\frac{d^4y}{dZ^4} + 4y = \frac{4}{k} R.$

Solution:

$$y = \tfrac{1}{2} \left[(A_1e^Z + A_2e^{-Z})\cos Z + (A_3e^Z + A_4e^{-Z})\sin Z\right].$$

Consecutive derivatives:

$$\frac{dy}{dZ} = \tfrac{1}{2}\left[A_1e^Z(\cos Z - \sin Z) - A_2e^{-Z}(\cos Z + \sin Z) + A_3e^Z(\cos Z + \sin Z) + A_4e^{-Z}(\cos Z - \sin Z)\right],$$

$$\frac{d^2y}{dZ^2} = -(A_1e^Z - A_2e^{-Z}) \sin Z + (A_3e^Z - A_4e^{-Z}) \cos Z$$

and

$$\frac{d^3y}{dZ^3} = -A_1e^Z(\cos Z + \sin Z) + A_2e^{-Z}(\cos Z - \sin Z) + A_3e^Z(\cos Z - \sin Z) + A_4e^{-Z}(\cos Z + \sin Z).$$

Beam Functions:

$$\theta = \frac{1}{L}\frac{dy}{dZ}, \qquad M = \frac{-kL^2}{4}\frac{d^2y}{dZ^2} \qquad \text{and} \qquad V = \frac{-kL}{4}\frac{d^3y}{dZ^3}$$

Evaluation of constants:

1) At $x = 0$, $\frac{dy}{dx} = 0$, $Z = 0$ and $\frac{dy}{dZ} = 0$.

2) At $x = \frac{L}{2}$, $M = 0$. (Also use notation $Z = \frac{L}{T} = U$)

3) At $x = \frac{L}{2}$, $V = 0$. (Also use notation $A = \frac{L}{T} = U$)

4) At $x = 0$, $V = -\frac{P}{2}$.

Resulting equations expressed in determinant form:

A_1	A_2	A_3	A_4	
1	-1	1	1	0
$e^{\frac{U}{2}} \sin \frac{U}{2}$	$e^{\frac{-U}{2}} \sin \frac{U}{2}$	$e^{\frac{U}{2}} \cos \frac{U}{2}$	$e^{\frac{-U}{2}} \cos \frac{U}{2}$	0
$e^{\frac{U}{2}}(\cos\frac{U}{2}+\sin\frac{U}{2})$	$-e^{\frac{-U}{2}}(\cos\frac{U}{2}-\sin\frac{U}{2})$	$-e^{\frac{U}{2}}(\cos\frac{U}{2}-\sin\frac{U}{2})$	$-e^{\frac{-U}{2}}(\cos\frac{U}{2}+\sin\frac{U}{2})$	0
1	-1	-1	-1	$\frac{-2P}{kL}$

Using notations

$$a = \frac{2 + \cos U - \sin U + e^{-U}}{\sinh U + \sin U} \quad \text{and} \quad b = \frac{\cos U + \sin U - e^{-U}}{\sinh U + \sin U},$$

the determinant yields:

$$A_1 = \frac{P}{2kL}\, a, \qquad A_2 = \frac{P}{2kL}\,(2 + a),$$

$$A_3 = \frac{P}{2kL}\, b \quad \text{and} \quad A_4 = \frac{P}{2kL}\,(2 - b).$$

Additional notations:

$$[Z]_1 = e^{-Z} (\cos Z + \sin Z),$$

$$[Z]_2 = -e^{-Z} \sin Z,$$

$$[Z]_3 = e^{-Z} (\cos Z - \sin Z)$$

and

$$[Z]_4 = -e^{-Z} \cos Z.$$

Final Results:

$$y = \frac{P}{2kL}\left(\left[Z\right]_1 + a \cosh Z \cos Z + b \sinh Z \sin Z\right),$$

$$\theta = \frac{P}{2kL}\left(2\left[Z\right]_2 + (a+b)\sinh Z \cos Z - (a-b)\cosh Z \sin Z\right),$$

$$M = \frac{PL}{4}\left(\left[Z\right]_3 - b \cosh Z \cos Z + a \sinh Z \sin Z\right)$$

and

$$V = \frac{P}{4}\left(\left[Z\right]_4 + (a-b)\sinh Z \cos Z + (a+b)\cosh Z \sin Z\right).$$

Hayashi provides tables for $\left[Z\right]_1$, $\left[Z\right]_2$, $\left[Z\right]_3$, $\left[Z\right]_4$, a, b and Z.

APPENDIX C

UMANSKY'S METHOD

Basic equation:

$$y = e^{Zx}(C_1 \cos Zx + C_2 \sin Zx) + e^{-Zx}(C_3 \cos Zx + C_4 \sin Zx)$$

Evaluate constants for point $x = 0$:

$$y_{x=0} = y_o = C_1 + C_3,$$

$$\left(\frac{dy}{dx}\right)_{x=0} = \frac{\theta_o}{Z} = C_1 + C_2 - C_3 + C_4,$$

$$\left(-EI\frac{d^2y}{dx^2}\right)_{x=0} = \frac{M_o}{2Z^2EI} = -C_1 + C_4$$

and

$$\left(-EI\frac{d^3y}{dx^3}\right)_{x=0} = \frac{V_o}{2Z^3EI} = C_1 - C_2 - C_3 - C_4.$$

Constants:

$$C_1 = \frac{y_o}{2} + \frac{\theta_o}{4Z} + \frac{V_o}{8Z^3EI},$$

$$C_2 = \frac{\theta_o}{4Z} - \frac{M_o}{4Z^2EI} - \frac{V_o}{8Z^3EI},$$

$$C_3 = \frac{y_o}{Z} - \frac{\theta_o}{4Z} - \frac{V_o}{8Z^3EI}$$

and

$$C_4 = \frac{\theta_o}{4Z} + \frac{M_o}{4Z^2EI} - \frac{V_o}{8Z^3EI}.$$

Additional notations:

$$F_1(Zx) = \cosh Zx \cos Zx, \qquad \frac{dF_1}{dx} = -4ZF_4,$$

$$F_2(Zx) = \tfrac{1}{2}(\cosh Zx \sin Zx + \sinh Zx \cos Zx), \qquad \frac{dF_2}{dx} = ZF_1,$$

$$F_3(Zx) = \tfrac{1}{2}(\sinh Zx \sin Zx) \qquad \frac{dF_3}{dx} = ZF_2,$$

$$F_4(Zx) = \tfrac{1}{4}(\cosh Zx \sin Zx - \sinh Zx \cos Zx)$$

and

$$\frac{dF_4}{dx} = ZF_3.$$

Basic equation re-written:

$$y_x = y_o F_1(Zx) + \frac{\theta_o}{Z} F_2(Zx) - \frac{M_o}{Z^2 EI} F_3(Zx) - \frac{V_o}{Z^3 EI} F_4(Zx).$$

The particular equation for the deflection of a beam loaded by a concentrated load at $\frac{L}{2}$ must have the additional term:

$$+ \frac{P}{Z^3 EI} F_4\left[Z(x - \tfrac{L}{2})\right].$$

Rewrite equations (c and d) for the point $x = 0$, $M_o = 0$ and $V_o = 0$:

$$0 = \frac{y_o k}{Z^2} F_3(ZL) + \frac{\theta_o k}{Z^3} F_4(ZL) - \frac{P}{2} F_2\left[Z(L - \tfrac{L}{2})\right]$$

and

$$0 = \frac{y_o k}{Z} F_2(ZL) + \frac{\theta_o k}{Z^2} F_3(ZL) - P F_1\left[Z(L - \tfrac{L}{2})\right].$$

Solve for y_o and θ_o. Substitute these values into equation (a and b) to determine M_o and V_o. Now, the beam functions can be determined numerically for any point x.

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