LARGE SCALE DEFORMATIONS APPLIED TO MODEL METHOD OF STRESS ANALYSIS

By

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THESIS AND ABSTRACT APPROVED:

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PREFACE

This investigation stemmed from the writer's desire to add the complement of experiment to his study of mathematical methods of stress analysis. It was at the suggestion of Professor R. E. Means that the experimentation took the direction indicated by the title.

As the investigation proceeded, two objectives evolved. The first was to determine the accuracy of the model method when the deformations are large. The second was to develop a practical technique of model construction and manipulation.

Acknowledgement must be made to Professors J. E. Lothers and R. E. Means of the Department of Architecture at Oklahoma A. and M. College. It was from their inspired instruction in the field of structural engineering that the writer acquired the background necessary to proceed with this study.

The writer expresses appreciation to his wife, Jean Marie Cotner, for her excellent assistance as proof reader and editor of this paper; also for being, during the entire period of this study, the kind of encouraging critic that only a wife can be.

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A. GENERAL DISCUSSION OF THEORY

There appeared in the April 1864 issue of Philosophical Magazine, an English journal, a paper by that distinguished English mathematician and physicist, James Clerk Maxwell. Clerk Maxwell is rightly celebrated for his contribution to the expansion of knowledge in the field of electricity. This paper, entitled "The Calculation of the Equilibrium and Stiffness of Frames" was, however, an incursion into mechanics; and blessed that science with the wonderful principle known as Maxwell's Law of Reciprocal Deflections. A reprint of this paper is included in a recent article by Professor A. S. Niles of Stanford University, California.[1]

In his paper Maxwell presented a method of computing the axial loads in the members and the relative displacements of pairs of joints in a redundant pin-jointed truss. He derived his method by the principle of "Conservation of Energy"--more popularly known today as "Least Work." His paper attracted little notice, and much credit must be given to others (Professor Niles does so) for expanding the applications of the principle.[2]

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[1] A. S. Niles, "Clerk Maxwell and the Theory of Indeterminate Structures," Engineering (London), (September, 1950), p. 170.

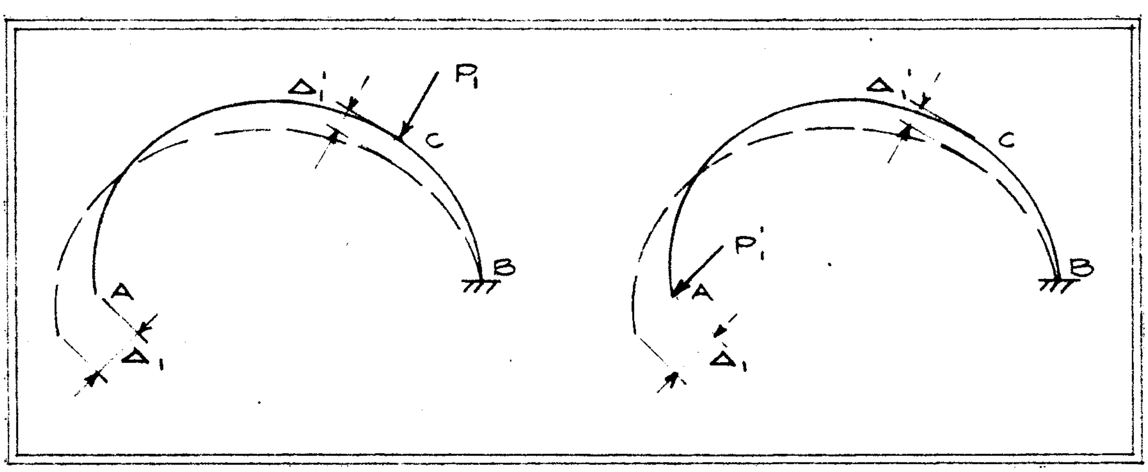
[2] Ibid., p. 172.

Maxwell's law can be presented in its general form in the following manner. Consider any elastic solid or framed structure to be in equilibrium under forces or moments represented by P_1, $P_2 \dots P_n$, M_1, $M_2, \dots M_n$, and let the displacements of these forces and moments be represented by $\Delta_1, \Delta_2, \dots \Delta_n, \theta_1, \theta_2 \dots \theta_n$. Let these forces and moments be replaced by a second system in equilibrium represented by P_1', $P_2' \dots P_n'$, M_1', $M_2' \dots M_n'$, acting in each case in the same direction as the corresponding forces and moments of the first system. Let the displacements of the second system be denoted by $\Delta_1', \Delta_2' \dots \Delta_n', \theta_1', \theta_2' \dots \theta_n'$. Maxwell's law then states that

$$P_1 \Delta'_1 + P_2 \Delta'_2 + \dots P_n \Delta'_n + M_1 \theta_1' + M_2 \theta_2' + \dots M_n \theta'_n =$$
$$P_1 \Delta_1 + P'_2 \Delta_2 + \dots P_n' \Delta_n + M_1' \theta_1 + M'_2 \theta_2 + \dots M_n' \theta_n$$

The law can perhaps be best illustrated by the simple particular case shown in Figure A-1.

FIGURE A-1

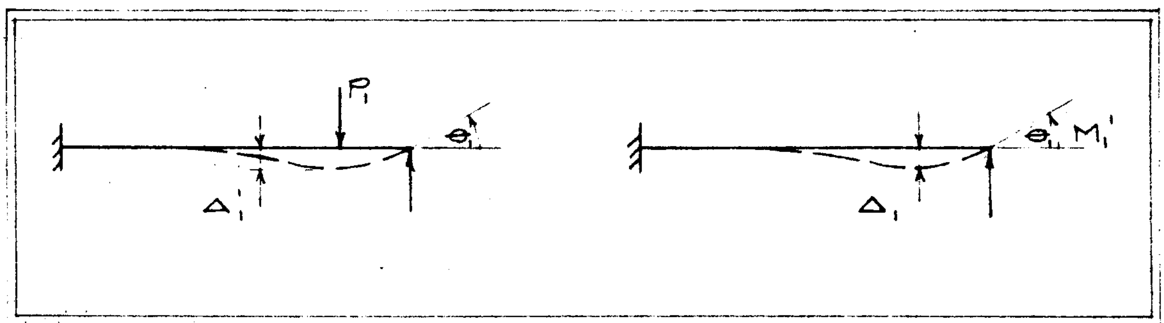


Let P_1 be any force acting in any direction at any point, C, on structure AB, and assume the structure will take the shape shown by the dashed line, the deflection at A and C being denoted by Δ_1 and Δ_1' respectively. Let P_1' acting through the line of movement of Δ_1 be the force required at A to cause Δ_1 at A and Δ_1' at C. Then

$$P_1\Delta_1' = P_1'\Delta_1 \text{ or } P_1'/P_1 = \Delta_1'/\Delta_1$$

A similar simple illustration can be made for moment.

FIGURE A-2



With reference to Figure A-2, Maxwell's law states that

$$P_1\Delta_1' = M_1'\theta_1 \text{ or } M_1'/P_1 = \Delta_1'/\theta_1.$$

Since mathematical proofs of this law are well known one will not be included at this point. Such a proof is included in the appendix. The writer feels that the best proof is to be found in the results of model testing.

Referring again to Figure A-1, it is apparent that if the value of either of the forces were known the other could be evaluated by causing a known deflection at either A or C and measuring the deflection at the other point.

It was this application of Maxwell's Law that the late George E. Beggs used when developing the model method of stress analysis that bears his name.[3]

In the Beggs method the models are made from paper, cardboard, acrylic base plastics, etc. They are scale models of the prototype structure. The deformations caused are microscopic, being produced by special gauges called deformeters. The method is now well established, the deformeters and special microscopes being produced commercially. Though the Beggs method is good, it has two disadvantages. The first is that the deformeters and microscopes are expensive. The second is that it is not possible to view the deformed structure as a whole due to the limited field of vision of the microscope. A. J. S. Pippard states that,

> Errors are introduced if the temperature of the model changes during an experiment as the consequent thermal movements are comparable with those produced by the imposed displacements.[4]

This writer has not had experience with the Beggs method and thus cannot accurately evaluate Mr. Pippard's statement. This writer feels, however, that thermal errors are probably not in most cases a real disadvantage of the method.

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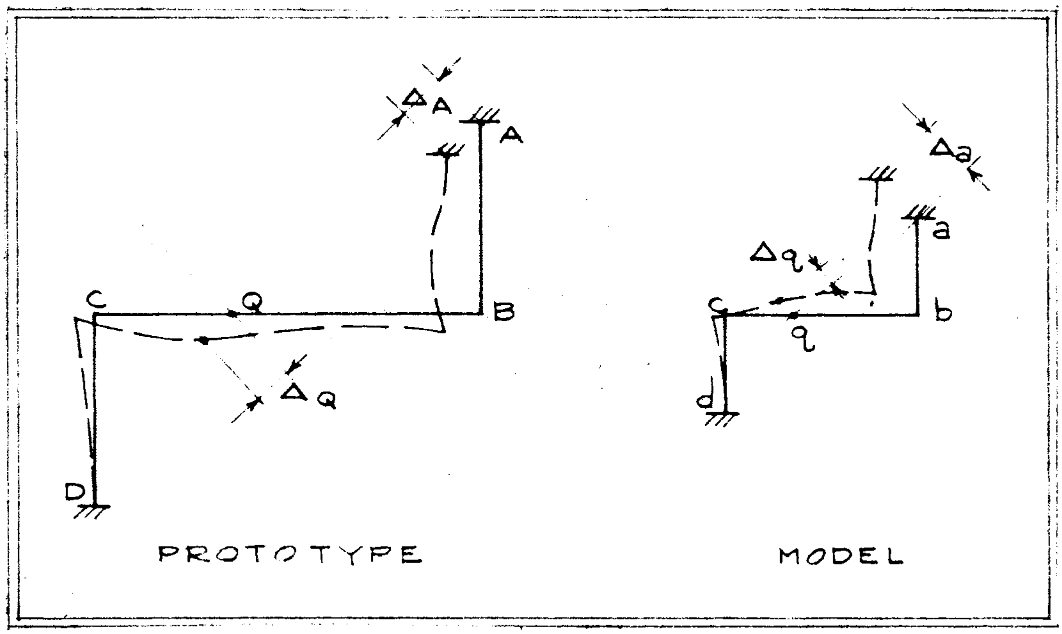
[3] George E. Beggs, Transactions, Am. Soc. C.E. Vol. 88 (1925), p. 1208.

[4] A. J. S. Pippard, The Experimental Study of Structures, p. 44.

It is apparent that a model method based upon the law of reciprocal deflections will yield accurate results only if the model offers a resistance to deformation similar to that of the prototype. (This is a loose expression.) The idea can perhaps be expressed better with reference to Figure A-3.

FIGURE A-3



Let ABCD represent any real structure of which abcd is a model to any scale. Let Q be any point on the structure and let q be the corresponding point on the model. Let Δ_Q be the deflection of point Q that results when Δ_A is produced at A; and let Δ_q be the deflection of point q when Δ_a is produced at a. Then Δ_q/Δ_a must equal $K\Delta_Q/\Delta_q$ wherein K is some constant (However, it is not necessary to evaluate K in order to use a model method.). If Δ_A were caused by a force, let that force be P, and let p be a force that could cause Δ_a. Let E_1 be the modulus of elasticity of the prototype and E_2 that of the model. For simplicity let all parts of the prototype have the same moment of inertia I_1; let I_2 be the moment of inertia of all parts of the model.

$$\Delta_Q = P\int\frac{M_A M_Q ds}{E_1 I_1} \text{ and } \Delta_q = P\int\frac{M_a M_q ds}{E_2 I_2}$$

wherein M_A and M_Q are the moments at A and Q due to a 1# load at A; and M_a and M_q are the moments at a and q due to a 1# load at a. These statements are proved in the appendix. They neglect the deformation caused by direct stress and shear. Such deformation is small and is practically always neglected in stress analysis methods based on the geometry of deformation--two prominent examples of which (in addition to the model method) are the methods of slope deflection and moment distribution. If the problem was the analysis of a truss, deformation due to direct stress could not be neglected.

In the statements for Δ_Q and Δ_q let $p = k_1P$, $E_2 = k_2E_1$, and $I_2 = k_3I_1$. Since M_AM_Q and M_aM_q are all functions of a 1# imaginary load, let $M_AM_Q = f(i)$ and $M_aM_q = k_4\ f(i)$. The statements may now be written in this form $\Delta_Q = P\int\frac{f(i)ds}{E_1I_1}$ and $\Delta_q = k_1P\int\frac{k_4f(i)ds}{k_2E_1k_3I_1}$ placing the constants in front of the integral sign the latter statement becomes

$\Delta_q = \frac{k_1 k_4}{K_2 K_3} \; P \int \frac{f(i)ds}{E_1 I_2}$. By letting $k = k_1 k_4 / k_2 k_3$ this becomes

$\Delta_q = kP \int \frac{f(i)ds}{E_1 I_1}$. Thus it can be seen that $\Delta_q = k' \Delta_Q$. It can be similarly proved that $\Delta_a = k'' \Delta_A$. Then letting $k'/k'' = k$ it can be stated that $\Delta_q / \Delta_a = k^{\Delta}Q / \Delta_q$.

It will be observed that the illustration is oversimplified because in an actual structure the values of I are usually different for each member; in fact, members often have a variable I. The writer does not apologize for the simplification, because the mathematical relationships are offered only as illustrations. The writer feels that the experiments that follow provide the real proof, and the experiments cover the cases of different I's in different members and varying I's.

As previously stated, the Beggs method uses models of the same shape as the prototype, thus the different constant relationships between the factors affecting deflection of the model and the prototype are taken care of automatically.

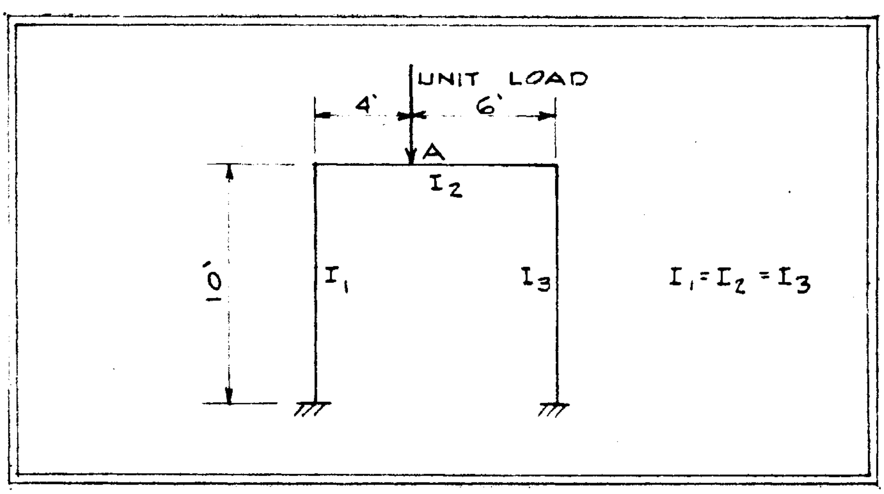
Since a model cut from a thin sheet would buckle laterally when subjected to large deformations, some other method of constructing the models is necessary if large deformations are to be applied. That other method must still satisfy the constant relationships between the factors affecting deflection. Such a method is presented later.

B. INVESTIGATION OF ACCURACY

As stated in The General Discussion of Theory the accuracy of the model method has been well established when the deformations are microscopic. One of the objectives of this study is to determine the accuracy of the method when the deformations are megascopic. Of course if the deformations are so great as to cause any portion of the model to be stressed beyond its elastic limit the results will not be accurate. Excluding such deformations it appears that Maxwell's Law will hold and that a large deformation method will give accurate results.

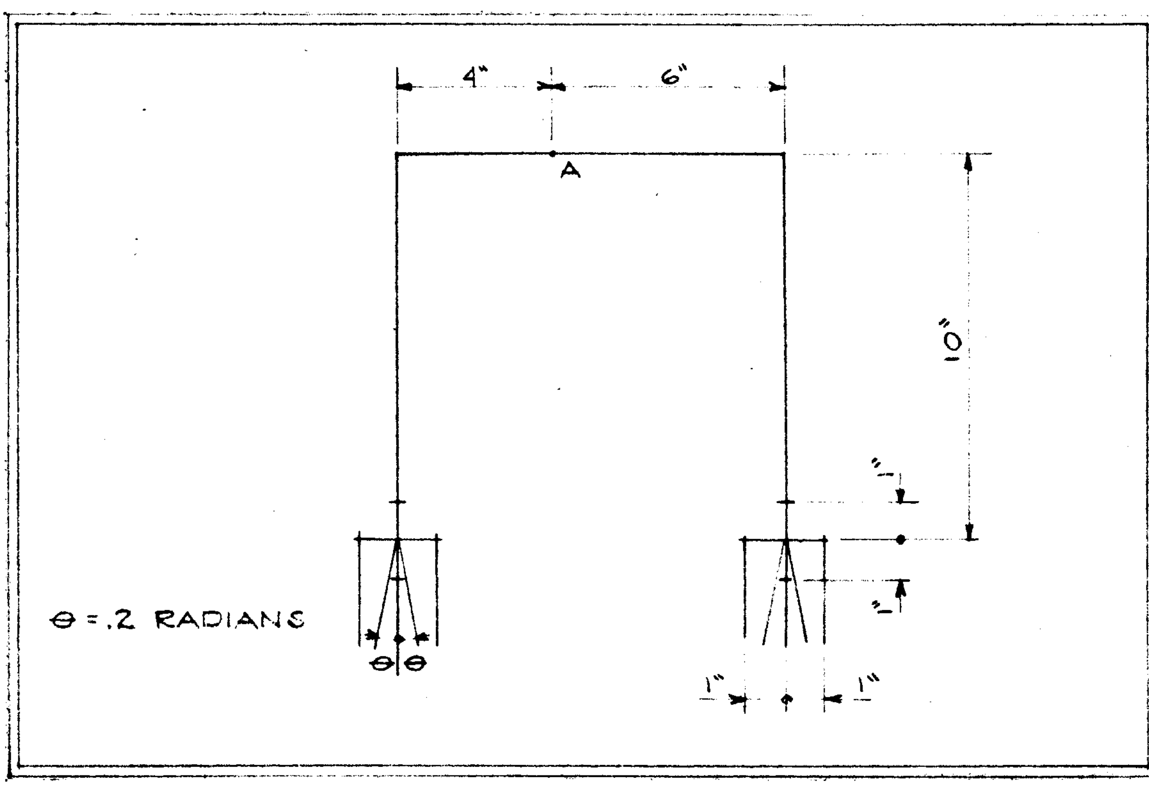
The first experiment was to determine the reactions of the statically indeterminate frame illustrated in Figure B-1.

FIGURE B-1



The model of this frame was constructed by bending a wire to the shape of the structure. The scale of the model was 1" = 10'. The wire was a piece of cold drawn steel wire (piano wire) of the type obtainable at model airplane supply shops. The diameter of the wire was 1/32". The vertical legs of the model were allowed to extend past the 10 inch length; however the points representing the end of the frame were marked on the wire. The point "A" on the horizontal member 4" from the left was also marked. A line drawing was made representing the axis of the model in a non-deformed state. At each reaction the lines shown on Figure B-2 were drawn.

FIGURE B-2



The wire was placed on the drawing and aligned on the axis. The end of the right member was fixed by sticking pins along side the wire into the drawing board underneath. The left member was then displaced to the left and aligned on the vertical line which was drawn one inch to the left of the axis. The deflection from the horizontal was measured at point "A" and found to be .14 inches. This measurement was made with a 12" triangular engineer's scale. This procedure is illustrated by Photograph I. Then according to Maxwell's Law $\frac{H}{P} = \frac{.14}{1}$, wherein H is the horizontal component of the left reaction and P is the unit load at the point "A". Since P is unity, H = .14; however, this is not the finally accepted value of H, as will be explained.

The left member was then freed; the right member remaining fixed all during the manipulation of the left. The left member was displaced 1" to the right, 1" up, and 1" down. It was rotated .2 radian to the left and .2 radian to the right. The deflection from horizontal of point "A" being read for each displacement of the left member. From these deflections values were obtained (in addition to the horizontal component of the left reaction) for the vertical component and the moment as illustrated in Table 1.

PHOTOGRAPH I

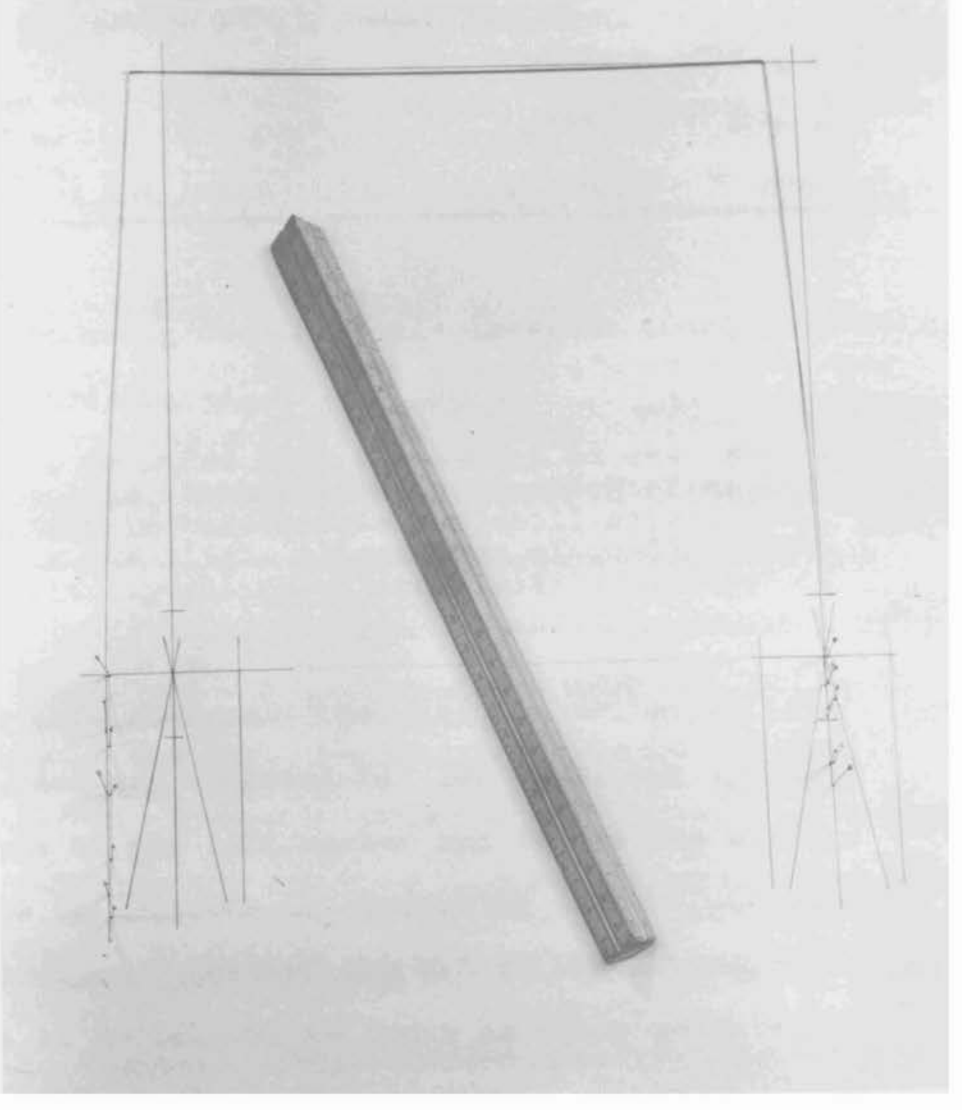


TABLE 1

Deflection of Model			Value of
At left reaction	At point A	Average	Reaction
1" to left	.14"	.12"	H = .12
1" to right	.10"		
1" up	.62"	.62"	V = .62
1" down	.62"		
.2 rad $\circlearrowright$	.04"	.07"	M = .35
.2 rad $\circlearrowleft$	.10"		

When discussing the large displacement model method A. J. S. Pippard states,

> It should be noted that a movement on each side of the normal is the usual technique adopted as to some extent it counteracts the small errors involved in the slight alterations of configuration due to the imposition of large displacements.[1]

It was this statement that led the writer to take deflection readings at point "A" for equal but opposite displacements of the left member and to use the average value of the deflections in evaluating the components of the left reaction. However the writer was not expecting differences in deflection as great as those measured for the two rotational displacements.

Being anxious to know if the average of two greatly differing deflections would give a correct answer for the reaction component, the writer next determined the reactions

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[1] A. J. S. Pippard, *The Experimental Study of Structures*, p.44.

of the frame by the method of moment distribution. The computations for which are shown in Figures B-3 and B-4. Comparative values for the left reaction are shown below:

	Model Method	Moment Distribution
H	.12	.12
V	.62	.60
M	.35	.36

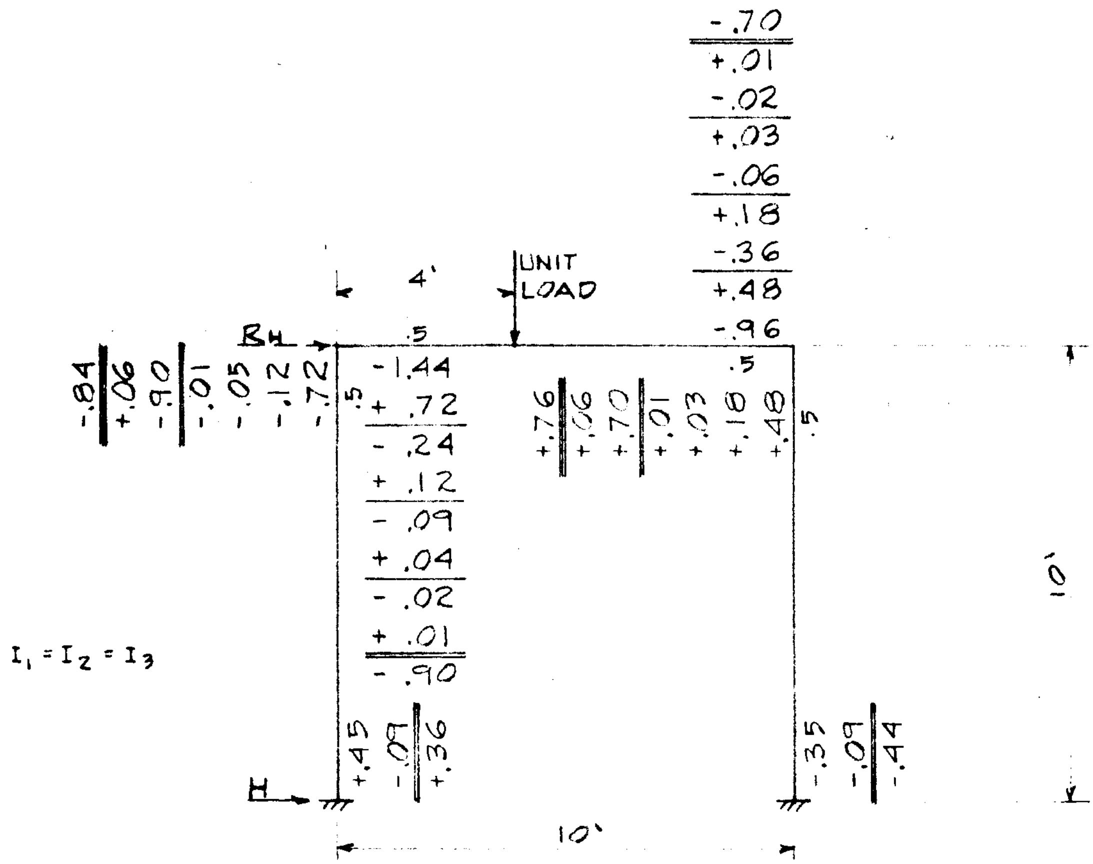
The correct value for V is of course .60, but the correct value for M may be just as near .35 as it is .36. In any event the values were close enough to the truth to satisfy the writer.

The next step was to complete the experiment by evaluating the components of the right reaction. The procedure was to fix the end of the left member; cause the various displacements of the right member; and measure the deflections of point "A" from the horizontal. The results are summarized in Table 2.

TABLE 2

Deflection of Model			Value of
At right reaction	At point A	Average	Reaction
1" to left	.12"	.125"	H = .125
1" to right	.13"		
1" up	.38"	.39"	V = .39
1" down	.40"		
.2 rad ↻	.10"	.085"	M = .425
.2 rad ↺	.07"		

FIGURE B-3



$$H_1 = \frac{.45 + .90}{10} = .135 \qquad H_2 = \frac{-.35 - .70}{10} = -.105$$

$$\Sigma H = 0 \;:\; .135 - .105 + R_H = 0 \;:\; R_H = -.030$$

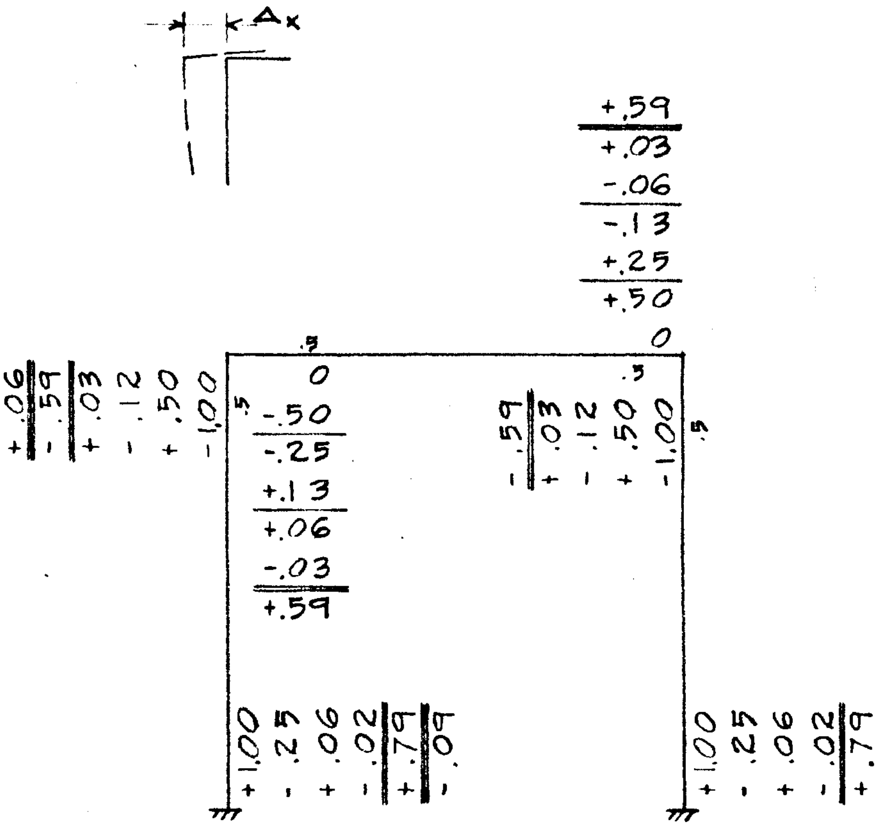
$$H = \frac{.36 + .84}{10} \; -.120$$

$$\text{CHECK} \;:\; \frac{-.44 - .76}{10} = -.120$$

$$V_{LEFT} = \frac{6 \times 1}{10} = .6$$

$$V_{RIGHT} = .4$$

FIGURE B-4



$$H_1^{\Delta x} = \frac{.79 + .59}{10} = .138$$

TOTAL SHEAR DUE TO $\Delta X = 2 \times .138 = .276$

$$C = \frac{R_H}{V^{\Delta x}} = \frac{-.030}{.276} = -.109$$

$-.109 \times -.59 = +.06^{+}$

$-.109 \times +.79 = -.09^{+}$

Comparative values for the left reaction are shown below:

	Model Method	Moment Distribution
H	.125	.12
V	.39	.40
M	.425	.44

The writer was satisfied with the accuracy of the results of this experiment, and decided to forego further experiments directed toward the sole aim of determining accuracy. It was decided to proceed toward the second objective of the study, which was to develop a practical technique of model construction and manipulation. It was felt that experiments along the latter line, if successful, would further substantiate the accuracy of the method.

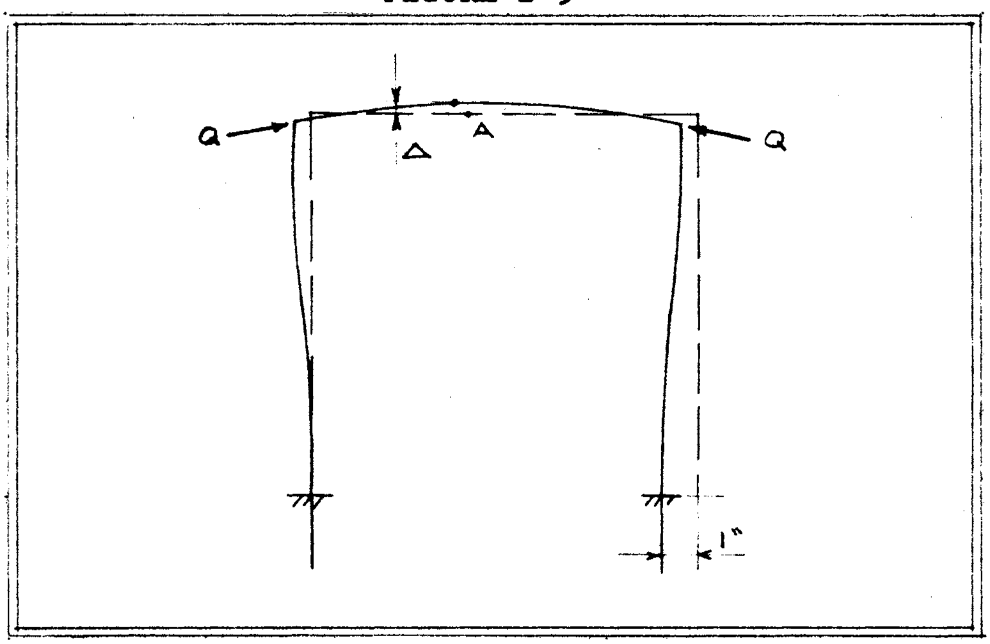
Before doing any further experimental work; however, the writer felt that it was necessary to know why displacements on opposite sides of the normal at the reaction did not give the same deflections at the point of load.

In the general discussion of theory it was stated that moment was the principal cause of deflection in a solid member structure, and the reader was referred to the appendix for a proof of Maxwell's Law. If the reader will again consider the proof (for any solid member structure) in the appendix, he will note that it is predicated on the proposition that the only "real" stress is due to moment, and that all of the deflection at "A" is due to moment.

Actually, if angle change in a structure is caused by a direct force acting on (or a linear displacement of) some section there will exist in the structure both moment and direct stress.

Consider Figure B-5 which shows the wire model in a deformed position. The axis of the model in the non-deformed state is shown by the dashed lines.

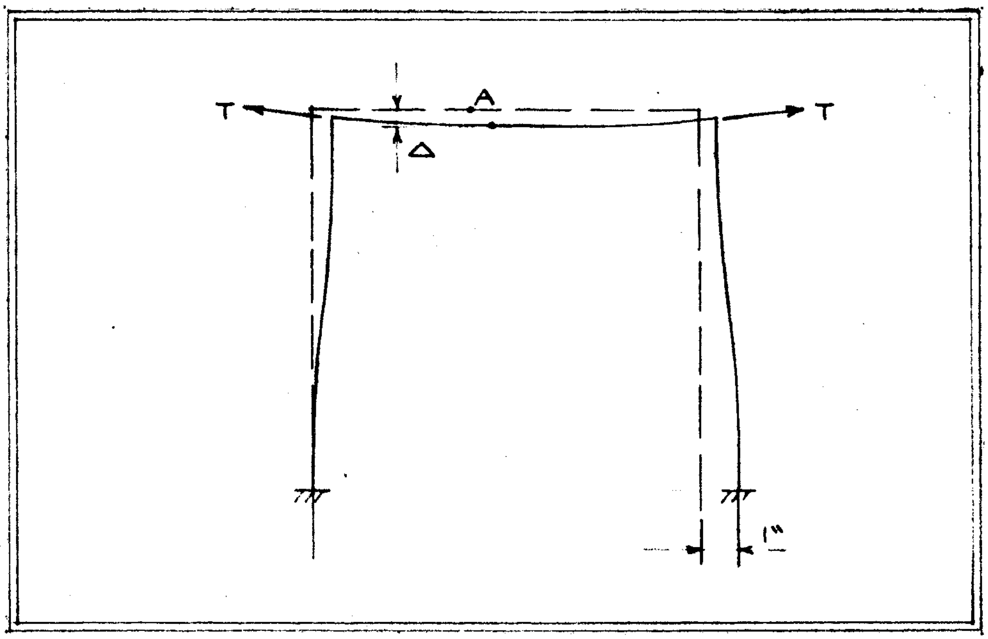
FIGURE B-5



Let $\triangle$ represent the total deflection from the horizontal of the point "A". Let "Q" represent the direct force applied to the ends of the horizontal member by the deformed vertical members. Angle change is also applied to the horizontal member, and the angle change, of course, causes an upward deflection, $\triangle_1$. Since a direct force applied to the end of a bent member will cause it to bend still more, the force "Q" will cause an additional deflection, Δ_2. It will be noted from the figure that the deformation of the vertical members has pulled the ends of the horizontal member below the axis. Let this downward deflection be Δ_3. Then $\Delta = \Delta_1 + \Delta_2 - \Delta_3$.

Consider now Figure B-6.

FIGURE B-6



In this case the vertical members apply a direct tensile force to the ends of the horizontal member. The deflection due to angle change, Δ_1, is downward. The deflection, Δ_2, due to "T" is upward; and there is again the downward deflection, Δ_3, at the ends of the horizontal member. Then in this case $\Delta = \Delta_1 - \Delta_2 + \Delta_3$.

Observation of the measured deflections recorded in Tables 1 and 2 indicate that the downward deflections at the ends of the beam have a larger relative effect on the total deflection than do the deflections due to direct stress.

The case just illustrated was one in which the deformed model assumed a shape that was symmetrical about a vertical axis. Similar illustrations could be made for the cases where the deformed model was not symmetrical. Angle change and vertical displacements would cause the unsymmetrical cases. If the model assumed an unsymmetrical shape, moment, direct stress, and translation of the horizontal member would all contribute to the deflection at the point of load as before; however, the ends of the horizontal member would not be translated an equal distance.

It is perhaps possible to proceed from the above generalized statements to mathematical relationships which would prove that the average of the two deflections on either side of the normal is the value that gives the correct answer when solving reciprocal deflection problems. However, the writer did not attempt to develop mathematical proofs, again preferring to let the experiments furnish the proofs.

Evidently in the Beggs method the deflection at a section is in general small enough that the small direct stresses do not appreciably affect the total deflection. Also the translation of entire members is generally of a negligible magnitude. It is to these very small deformations that the proof (for solid member structures) in the appendix applies.

C. THE EVOLUTION OF A TECHNIQUE

The type of wire used in the first experiment (or any other kind of tempered wire) has many qualities that would make it an excellent material for models. Wire has one serious drawback, however. The range of moment of inertia values of the different available diameters is limited. For example, the range of a few of the available diameters of piano wire is as follows: 1/64", 1/32", and 1/16". Since the moment of inertia of a circle varies directly as the fourth power of the diameter, it can readily be seen that it would be a practical impossibility to use wire in building a model of an actual structure.

As mentioned earlier the type of model used in the Beggs' method would buckle if subjected to large deformations and would thus be unsatisfactory. If a scale model of the same shape as the structure was constructed with a width sufficient to prevent buckling, it would be so stiff that the application of the large deformations would become a difficulty.

While following this line of reasoning the writer evolved the idea of making models that would have small depths and large widths of section. Such models would offer little resistance to the applied deformations, but would offer great resistance to buckling.

It was decided to try to solve the frame illustrated in Figure B-1 with such a model. The material chosen for the model was 1/32" sheet balsa wood. The reason for this choice was probably that this material had mildly stirred the writer's curiosity during the visit to the model airplane shop when purchasing the wire for the first experiment.

The balsa model was constructed with a constant width of section of 1/4". The members were made separately and the joints cemented with "model airplane" cement. The scale of lengths of members was 1" = 1' the same as was used for the wire model, and as before the vertical legs were allowed to extend past the 10" points.

A drawing was prepared very similar to that shown in Figure B-2. The same drawing was not used because the balsa model turned out to be about 1/32" wider than the wire model. The balsa model was then manipulated in the very same way the wire model had been. Pins alongside the extensions of the vertical legs again being used to hold the model in its various deformed shapes. Photograph II illustrates the model with the end of the left member rotated .25 radians counter-clockwise. The recorded deflections and computed values of the components of the reactions are shown in Table 3. They are also compared with the values obtained by moment distribution.

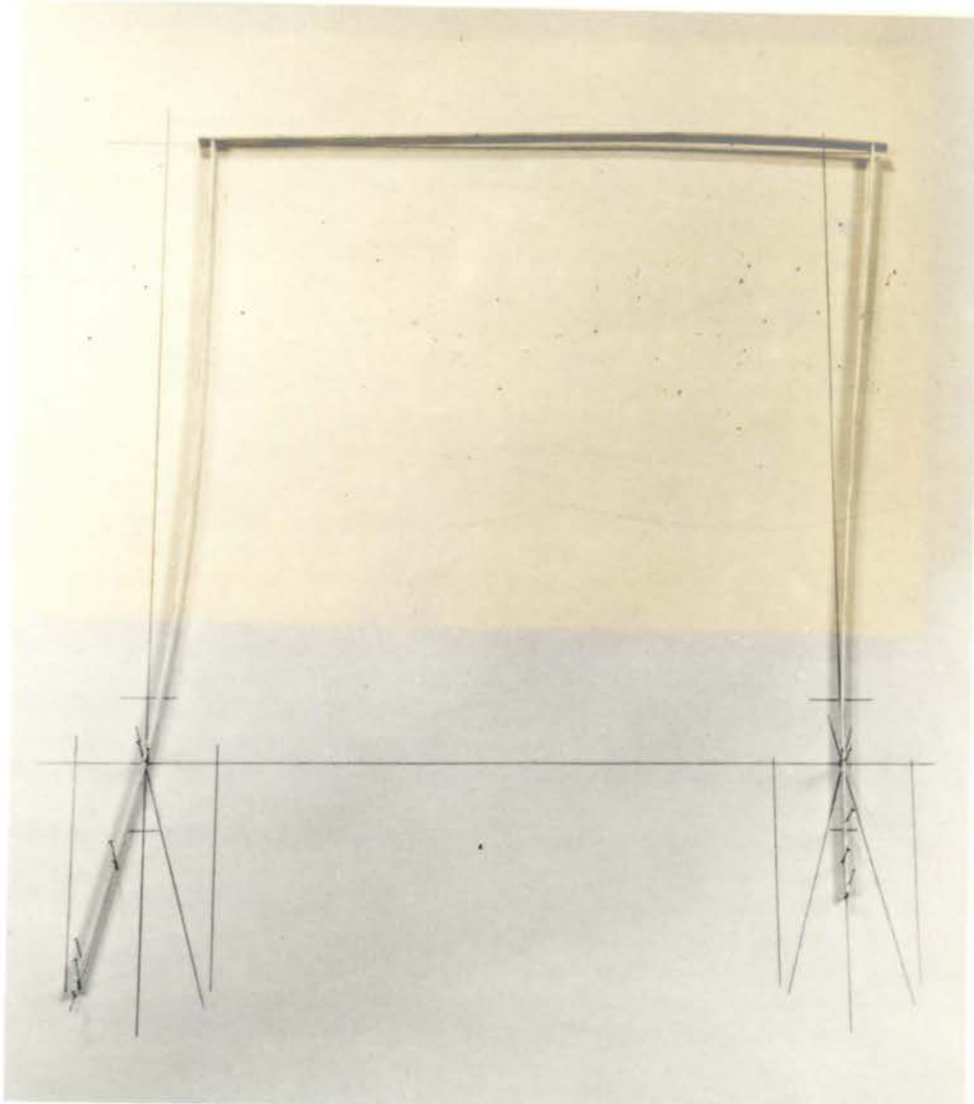
TABLE 3

Deflection of Model				Values by Moment
At left reaction	At point A	Av.	Value of Reaction	Distribution
1" left	.20"	.13"	H = .13	.12
1" right	.06"			
1" up	.56"	.63"	V = .63	.60
1" down	.70"			
.25 rad	.04"	.11"	M = .44	.35
.25 rad	.18"			
At right reaction				
1" left	.10"	.14"	H = .14	.12
1" right	.18"			
1" up	.30"	.35"	V = .35	.40
1" down	.40"			
.25 rad	.19"	.125"	M = .50	.44
.25 rad	.06"			

The values obtained with the balsa model would be accurate enough for design purposes but they are not as accurate as those obtained with the wire model. The writer discovered two possible reasons for this. The first is that pins (common straight pins as used by dressmakers) are tapered and are very unhandy for fixing a model of 1/4" breadth in position. Pins also become inconvenient when it is necessary to stick them in a spot that is very near to an existing pin hole. It is believed that part of the inaccuracy may have been due to inexactness of the caused deformations.

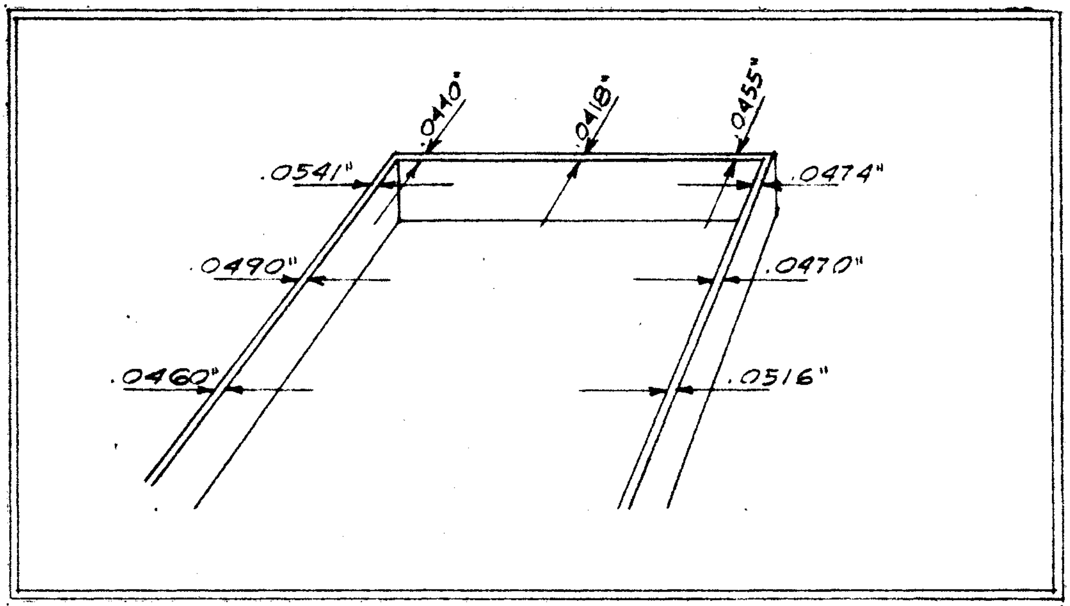
A second cause of inaccuracy may have been variations in depth of section of the model. Such variation was large

PHOTOGRAPH II



enough to be detected by eye, and after detection was measured with a micrometer. Figure C-1 illustrates these variations.

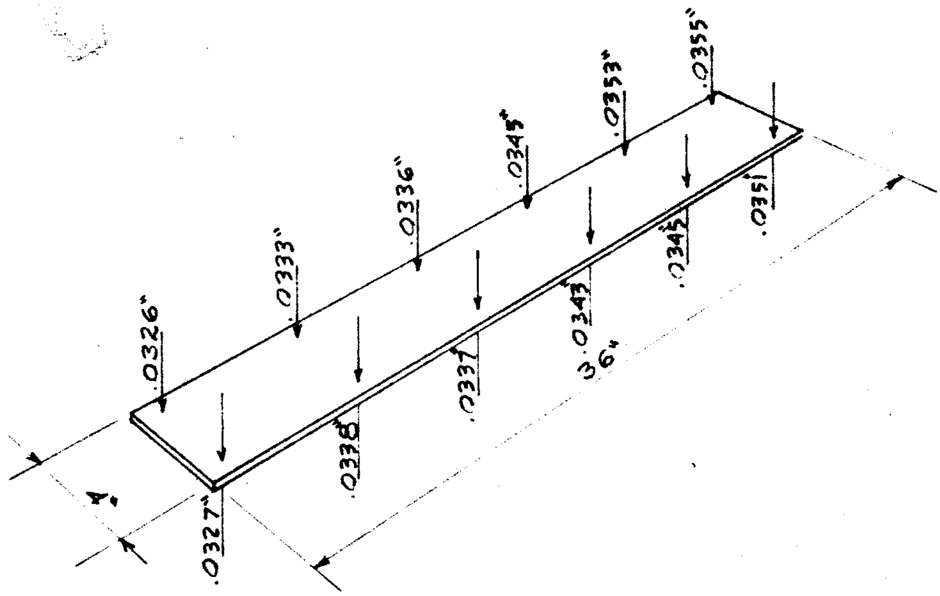
FIGURE C-1



The maximum difference in depth is .0541 - .0418 = .0123"; and .0123 is 22.7% of .0541. Since the moment of inertia varies directly as the depth to the third power it is felt that this variation in depth would certainly give rise to inaccuracies.

To see if all balsa wood sheet were subject to this variation, another 1/32" sheet was examined. In the second sheet no variation in thickness was discernible by visual inspection. The thickness of the sheet was then measured at several points with the micrometer. The measured thicknesses are illustrated in Figure C-2.

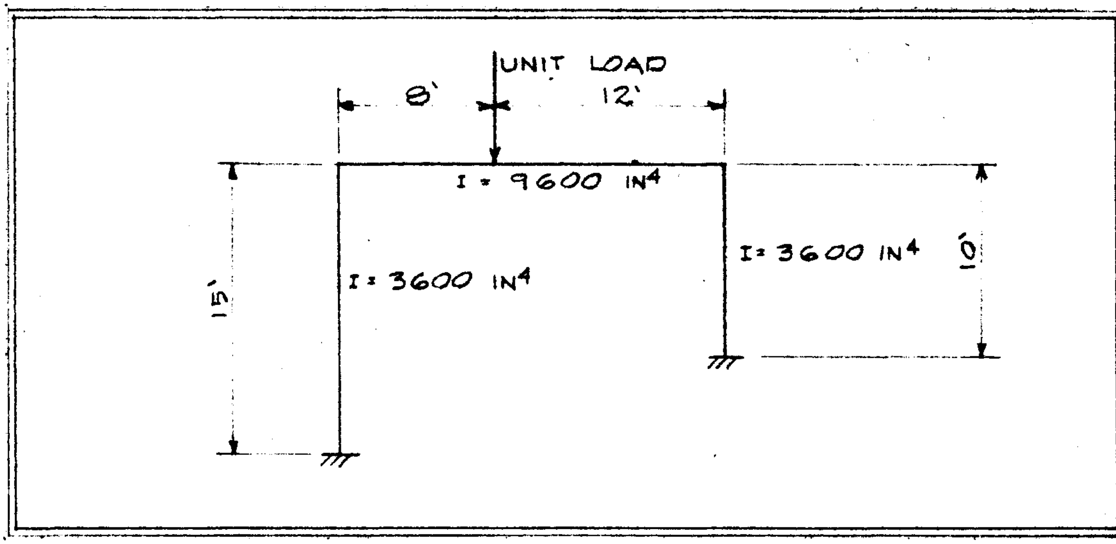
FIGURE C-2



For this sheet the maximum difference in thickness is .0355 - .0326 = .0029"; this difference is 8.17% of .0355. It was decided to try balsa wood again, building the model from this second sheet.

For this third experiment the frame illustrated in Figure C-3 was solved for the reactions.

FIGURE C-3



It will be noted that the moment of inertia of the horizontal member is different from that of the vertical members. From the illustration in the General Discussion of Theory it can be stated that the moment of inertia of each part of the model must have the same constant relationship to the corresponding part of the prototype. This statement is true if similar constant relationships exist for the modulus of elasticity and the scale of lengths of the parts. It is, of course, not convenient to vary the modulus of elasticity or the scale of lengths.

The moment of inertia of a rectangular section is expressed by the equation $I = \frac{bd^3}{12}$. If two rectangular sections had the same depth their moments of inertia would vary directly as their widths. It was arbitrarily decided to let the vertical members of the model have a width of

3/16". The width of the horizontal member of the model was then equal to $\frac{9600 \times .1875}{3600} = .50"$.

The scale of lengths chosen was 1" = 1'. The model was constructed by cutting the members separately and jointing them with model airplane cement. On this model the joints were reinforced with a little "haunch" of 1/16" x 1/16" strip balsa. This "haunch" was made to extend nearly to the edge of the wider member.

The axis and the guide lines for the caused deformations were drawn as before. This time, however, 2" linear displacements and 1/2 radian angular displacements were caused at the ends of the vertical members. These larger displacements were necessary because the model was so limber that inertia and frictional resistances could not be overcome by displacements of the magnitude previously used.

The model was placed on the drawing and manipulated in the previously described manner. Pins were discarded for this experiment, however; the extensions of the vertical members being fixed between two pieces of 2" x 2" x 4" steel bar stock. The forces exerted on the bars by the deformed model were not great enough to overcome the inertia of the bars, and thus they remained in whatever position they were placed. The bar method was very successful, and its great flexibility simplified and speeded up the manipulation of the model.

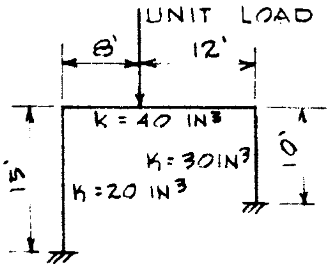
The results of the experiment are presented in Table 4. They are also compared with results obtained by moment distribution. The moment distribution solution is illustrated by Figure C-4.

TABLE 4

Deflection of Model: At left reaction	At point of load	av.	Value of Reaction	Value by Moment Distribution	% Difference
2" left	.30"	.26"	H = .13	.14	7.1%
2" right	.22"				
2" up	1.20"	1.20"	V = .60	.60	0 %
2" down	1.20"				
.5 rad ↻	.18"	.35"	M = .70	.76	7.9%
.5 rad ↺	.52"				
At right Reaction					
2" left	.22"	.26"	H = .13	.14	2.9%
2" right	.30"				
2" up	.74"	.74"	V = .37	.40	7.1%
2" down	.74"				
.5 rad ↻	.32"	.17"	M = .34	.35	7.5%
.5 rad ↺	.02"				

The writer feels that this experiment indicates that balsa wood is a good material for this type of model if the thickness of the sheet is uniform to the required degree. The "if" in the preceeding statement is not a disadvantage of the material because visual inspection is sufficient to establish whether or not the required degree of uniformity exists.

FIGURE C-4



$$\frac{K_1}{L_1^2} = \frac{20}{(15)^2} = .0889$$

$$\frac{K_2}{L_2^2} = \frac{30}{(10)^2} = .3000$$

$$\frac{K_1/L_1^2}{\Sigma K/L^2} = \frac{.0889}{.3889} = .228$$

$$\frac{K_2/L_2^2}{\Sigma K/L^2} = \frac{.3000}{.3889} = .772$$

$$\left.\begin{array}{l} \dfrac{3 \times -.48}{15} = -.0959 \\ \dfrac{3 \times .41}{10} = +.1230 \end{array}\right\} +.027$$

.228 × .027 = .00616   .00616 × 7.5 = .0452
.772 × .027 = .02080   .02080 × 5.0 = .1040

$$\left.\begin{array}{l} \dfrac{3 \times -.08}{15} = -.0160 \\ \dfrac{3 \times .23}{10} = +.0690 \end{array}\right\} +.053$$

.228 × .053 = .0121   .0121 × 7.5 = .0906
.772 × .053 = .0409   .0409 × 5.0 = .2045

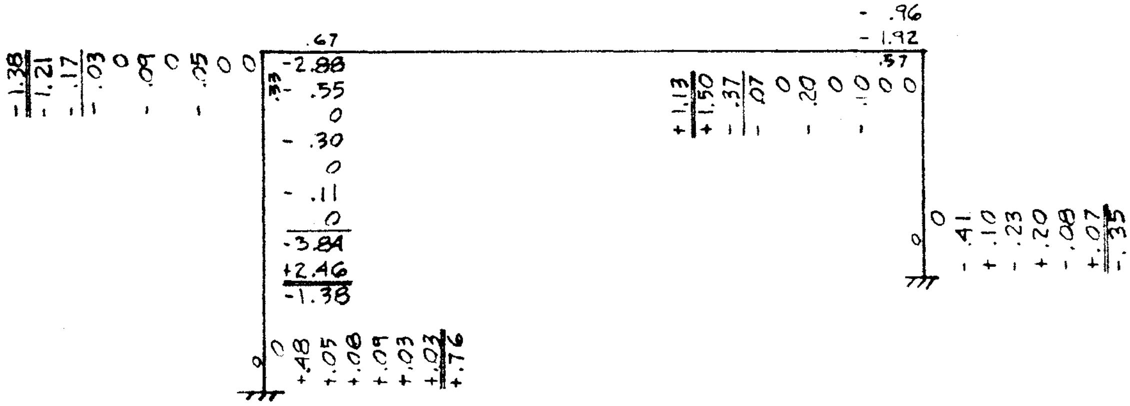
$$\left.\begin{array}{l} \dfrac{3 \times -.003}{15} = -.0060 \\ \dfrac{3 \times .08}{10} = +.0240 \end{array}\right\} +.018$$

.228 × .018 = .00411   .00411 × 7.5 = .0308
.772 × .018 = .01390   .01390 × 5.0 = .0695

CHECK STORY SHEAR :

$$V = \frac{-1.38 - .76}{15} + \frac{1.13 + .35}{10} = +.005$$

CLOSE ENOUGH



Not wishing to overlook the possibility of finding a better model material the writer decided to solve Frame B with a model made of sheet plastic (Plexiglass) and compare the results with those obtained with the balsa model.

A model was then constructed of 1/16" sheet plastic, the members being cut out separately and jointed with "model airplane" cement. A small "haunch" was used to reinforce the joint. The lengths of members and the widths of the sections were made the same as for the balsa model. Due to the human element, the plastic model was not exactly the same size as the balsa one; and so it was necessary to prepare a new drawing upon which to perform the manipulations. The various deformations were caused and the deflections at the point of load were measured. Photograph III illustrates the model with the left end rotated 1/2 radian counter-clockwise.

In Table 5 the results obtained with the plastic model are compared with those obtained from moment distribution.

PHOTOGRAPH III

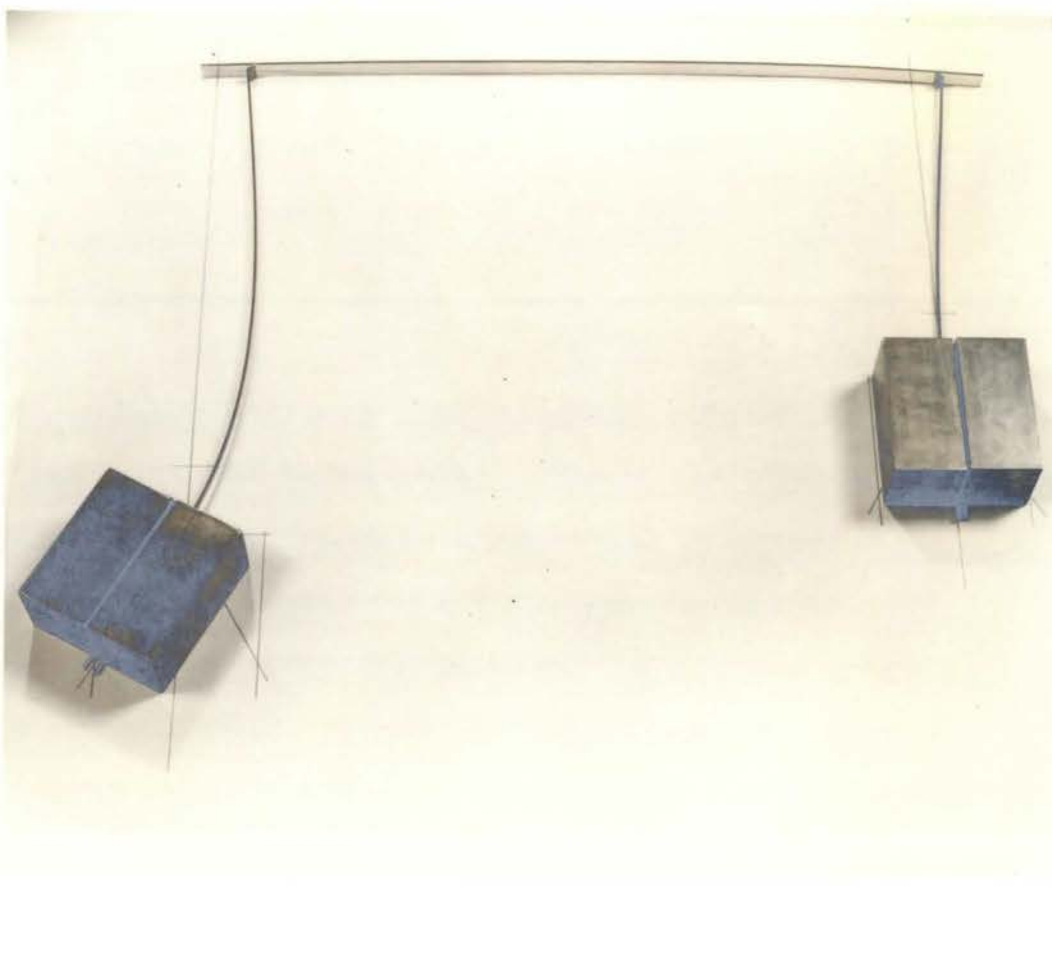
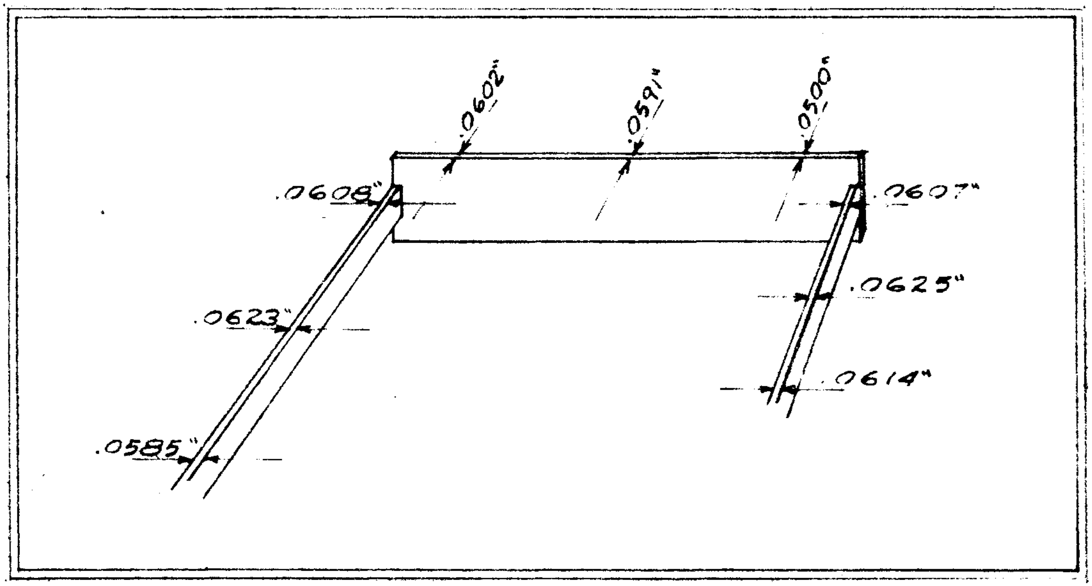


TABLE 5

Deflection of Model At left reaction	At point of load	av.	Value of Reaction	Value by Moment Distribution	% Difference
2" left	.36"	.27"	H = .135	.14	3.6
2" right	.18"				
2" up	1.16"	1.18"	V = .59	.60	1.7
2" down	1.20"				
.5 rad	.24"	.36"	M = .72	.76	5.3
.5 rad	.48"				
At right Reaction					
2" left	.26"	.30"	H = .15	.14	7.1
2" right	.34"				
2" up	.72"	.74"	V = .37	.40	7.5
2" down	.76"				
.5 rad	.38"	.20"	M = .40	.35	12.5
.5 rad	.02"				

These results indicate that slightly greater accuracy was obtained with the balsa model. The writer had expected the opposite to occur, because wood is not a homogenous material. Perhaps plastic is not homogenous either, and perhaps the homogenity of the material is not too great a factor in the results. It was observed that the plastic model was subject to variations in depth of section similar to that of the first balsa model. Possibly these variations affected the results. The model was checked with the micrometer. The depth variations are shown in Figure C-5.

FIGURE C-5



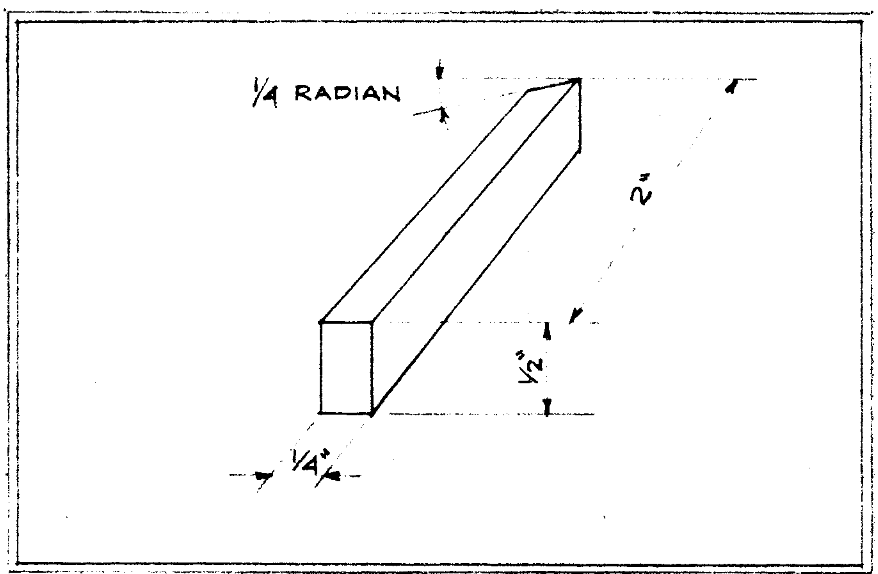
The maximum variation is .0625" - .0500" = .0125", and .0125" is 20.0% of .0625".

The writer feels that the results obtained with the plastic model would be good enough to use in the design of a structure, however plastic has two distinct disadvantages when compared to balsa wood. One is that it is more expensive; another is that it is much more difficult to cut. During the cutting of the members for the plastic model the writer resorted to a jig saw; a coping saw could have done the job if time had been of no importance. Thin sheet balsa, on the other hand, can be most readily cut with a razor blade or a model maker's knife.

So far all experiments had been confined to determining reactions and moments at the ends of framed structures. Maxwell's Law is not limited to such special cases; so, for the next experiment, the writer decided to try to determine the moment and shear at a point 8 inches from the right end on the horizontal member of the frame illustrated in Figure C-3. The previously constructed balsa model of this frame was used for this experiment.

In order to cause angle change or shear displacement at the selected point it was necessary to cut the model there. In order to apply the angle change "levers" were made from 1/4" sheet balsa and cemented to the horizontal member. These "levers" were made as indicated in Figure C-6.

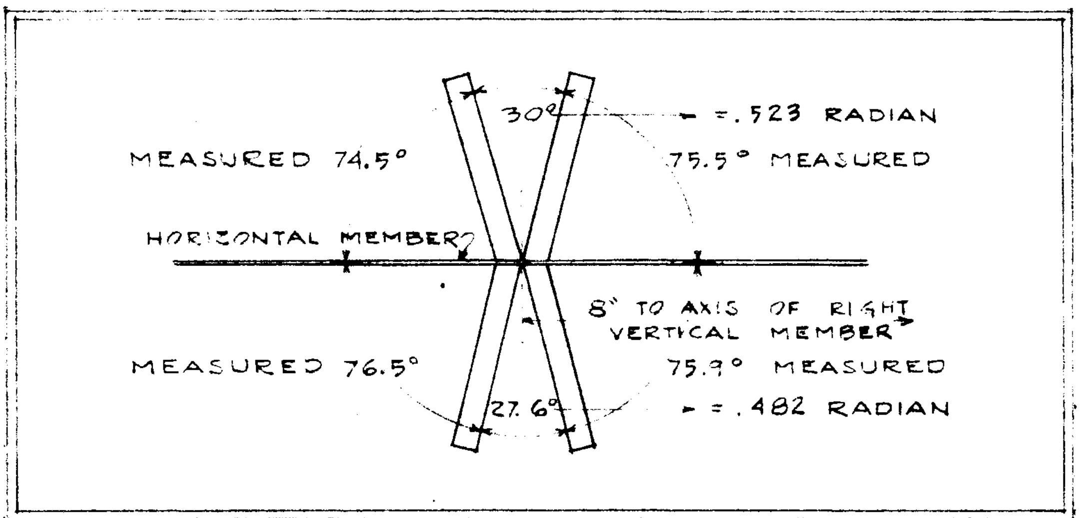
FIGURE C-6



An angle of 1/4 radian was laid off on the drawing board and used as a template for the angular end of the lever. The end of the lever was first sliced off at approximately the correct angle with a razor blade. Then by a trial and error process of making long sweeps across a large sheet of "00" sandpaper, flat on the drawing board, and checking the resulting angle with the template, surprisingly accurate results were obtained.

After the four levers were cemented to the model the actual angles were measured. Figure C-7 shows the measured angles.

FIGURE C-7



The angles were measured by placing the model on the drawing board and tracing with a pencil along the inside edge of the levers and along both sides of the horizontal member. The pencil lines were extended and right triangles were formed by drawing lines perpendicular to the horizontal member. The opposite and adjacent sides of the angle sought were measured and the tangent of the angle computed.

If the two bottom levers were rotated about the point of intersection of the four levers until they met, an angle change of .482 radians would be induced. The original thought had been to use a metal clamp (paper clamp) to hold the levers in such a position. However, it was decided to use a rubber band instead in order to keep the mass of the model at a minimum.

Since rubber has considerable frictional resistance it was necessary to suspend the model. Two cantilever frames were made by bending 1/16" welding rod and from these the model was suspended with threads.

The model was suspended above the drawing prepared for it when the reactions were being sought. This time the ends of the vertical members were fixed, with the steel bars, in an undisplaced position. A rubber band was then "doubled" several times and looped over two of the levers. The deflection of the point of load was then measured. Moment displacement is illustrated in Photograph IV.

It was discovered that it was difficult to measure this deflection accurately due to the vertical distance between the top of the model and the drawing of the axis. To overcome this difficulty a small flat mirror was obtained and marked with a glass cutter as shown in Figure C-8. Small pieces of drafting tape were stuck to the corners of the mirror. The mirror was then placed on the drawing board underneath the model and moved until the point of load on the model was directly over the point of intersection of the lines on the mirror. The mirror was then taped into

PHOTOGRAPH IV

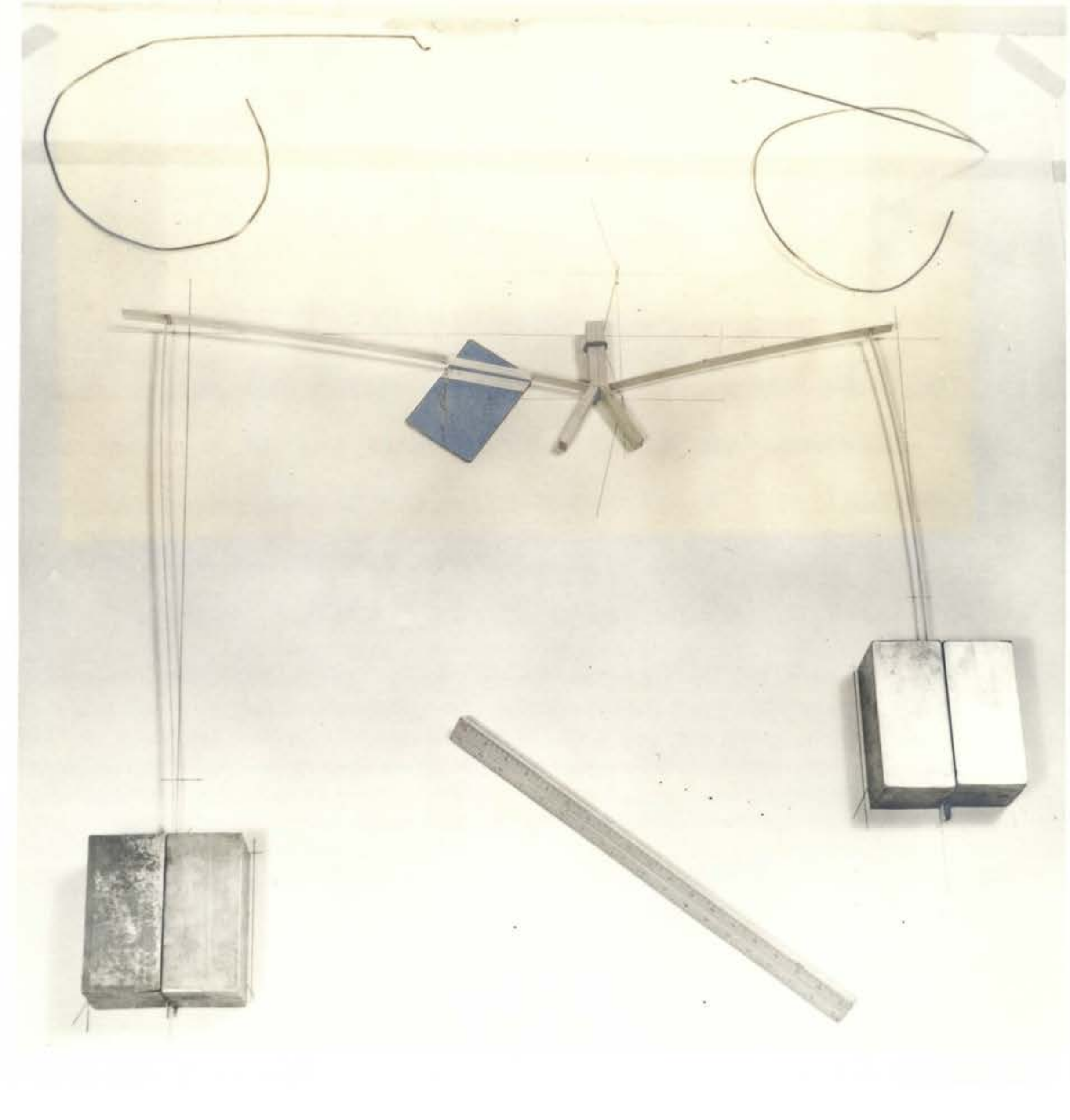
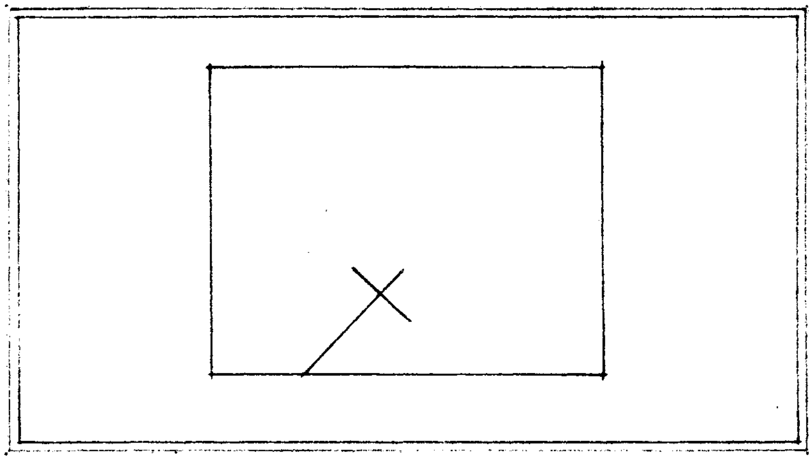


FIGURE C-8



position and the distance from the axis to the point of intersection of the lines on the mirror was measured. For very small deflections the intersection of the line and the edge of the mirror was used.

To solve for shear at the point where the model was cut the drawing was added to as indicated by Figure C-9.

PHOTOGRAPH V

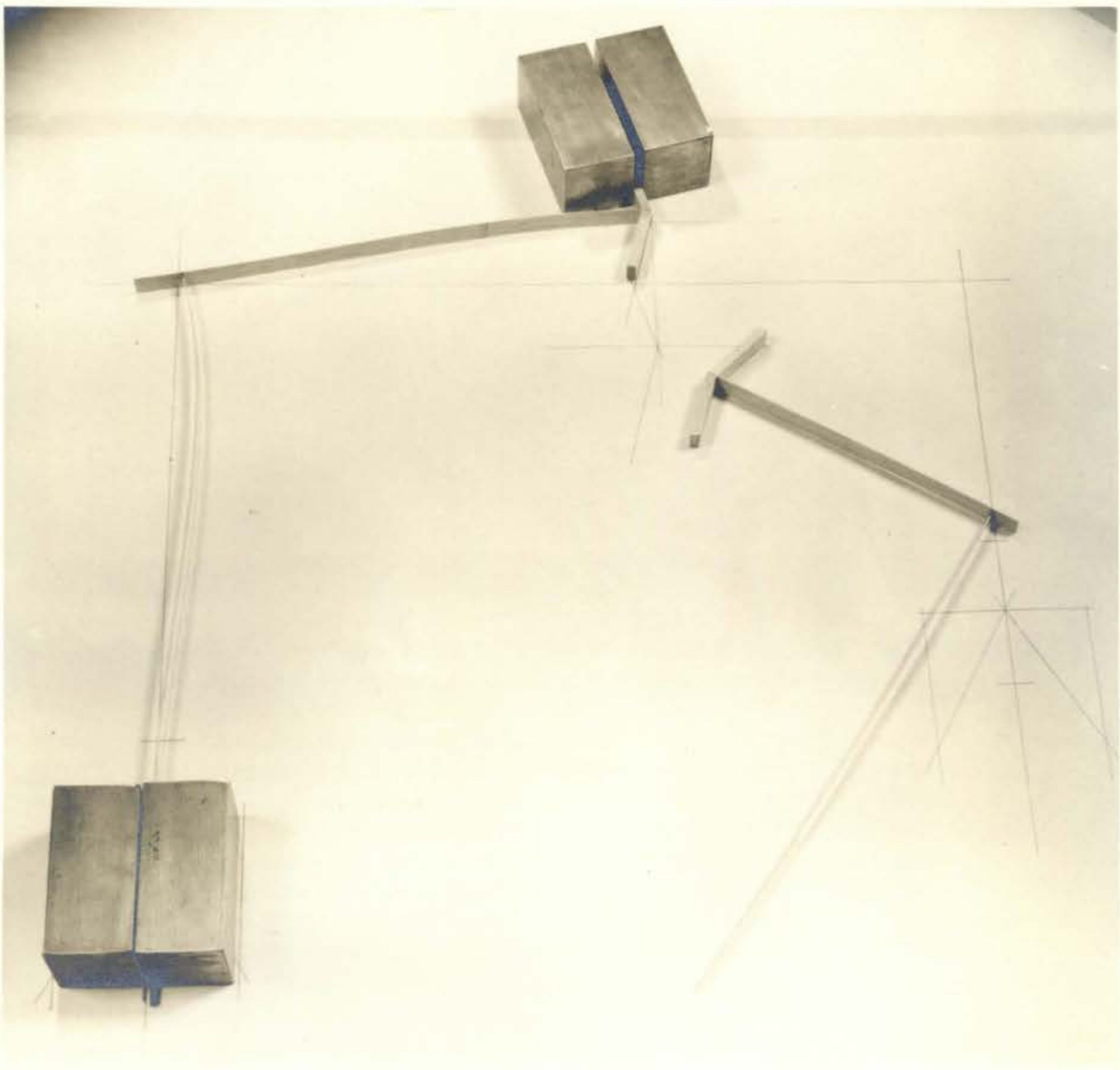
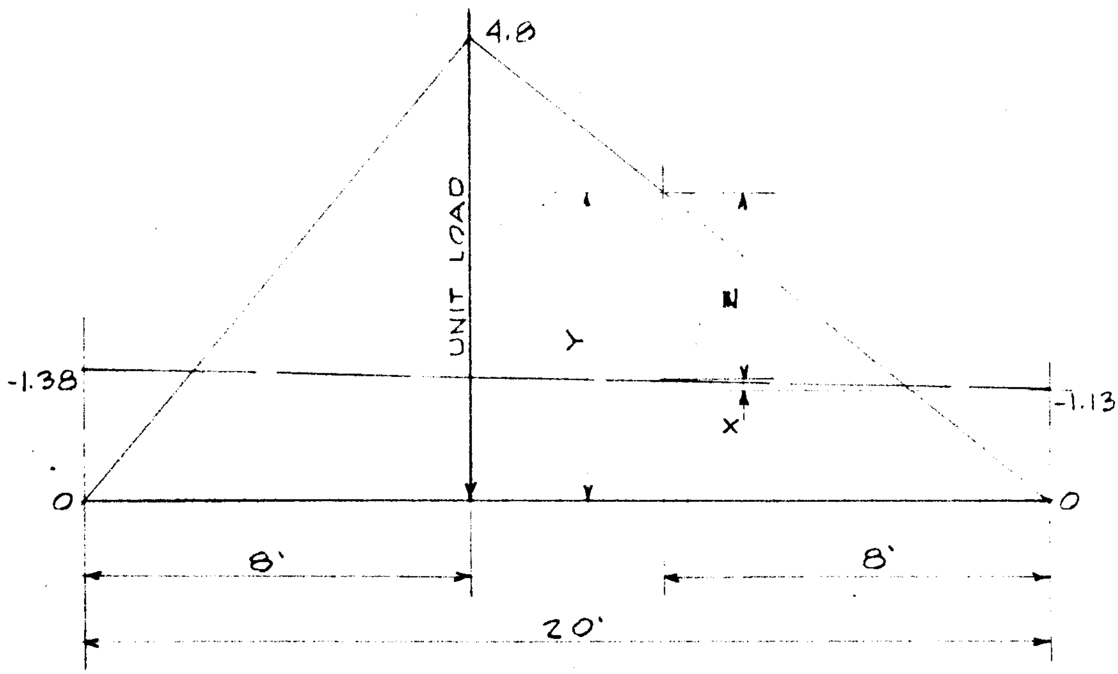


FIGURE C-10



SIMPLE MOMENT AT POINT OF LOAD

$$= \frac{Pab}{l} = \frac{1 \times 8 \times 12}{20} = 4.8$$

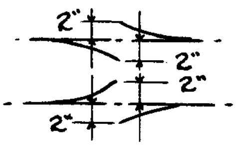
$$Y = \frac{4.8 \times 8}{12} = 3.2$$

$$X = \frac{8\,(1.38 - 1.13)}{20} = .10$$

M. AT POINT 8' FROM RIGHT = Z

$$Z = 3.2 - (1.13 + .10) = 1.97$$

TABLE 6

Deflection of Model: At point 8" from right on horizontal member	at point of bad	adjusted average	value of stress function	value by math.	% diff.
.482 rad	1.02"	.993	M = 1.90	1.97	3.6%
.523 rad	.88"				
	1.54"	1.60	V = .40	.40	0 %
	1.66"				

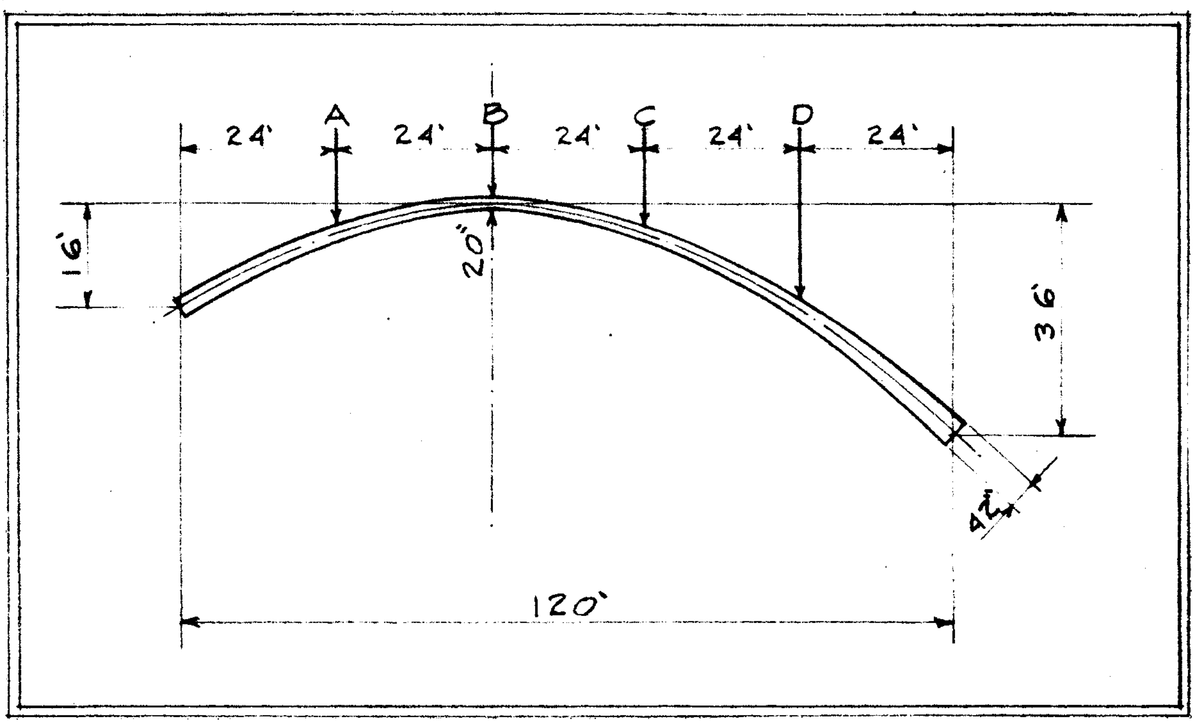
It was necessary to use an adjusted average in computing the moment, because the opposite angle changes caused were not equal. This adjusted average was arrived at in the following manner: The deflection at the point of load due to an angle change of .523 radian was assumed to equal $\frac{.523 \times 1.02}{.482} = 1.105"$; the average of this value and .88" was what the writer called the adjusted average. This adjusted average was then divided by .523 radian to obtain the moment.

It should be noted that the average value of the deflection of the point of load for the shear displacement was divided by 4" instead of 2". Even though the right part of the cut model was not used for the shear experiment, it had to be kept in mind that shear at a section tends to displace the parts adjacent to that section in opposite directions.

D. EXPANSION OF THE TECHNIQUE

Experiments thus far have indicated that the technique can be applied with success to the solution of structures in which the several members may have different, constant moments of inertia. The question next to arise was: Can the technique be applied to the solution of structures in which the member or members have varying moments of inertia? To obtain a manifestation of a possible answer to the question, it was decided to solve the parabolic, elastic arch illustrated by Figure C-1 for unit loads at "A, B, C, and D."

FIGURE D-1



Failing to conceive a method of causing sheet balsa to assume the parabolic shape in a non-stressed condition, the writer decided to build an approximate model by dividing the arch into parts. To do this it was necessary to know the depth of the arch section at the points of division. These depths could have been computed mathematically; they were, however, obtained by the more practical method of scaling them from a carefully prepared, large scale drawing of the arch. This drawing was made to the scale of 1" = 3'. For each of the curves of the extrados, intrados, and axis, two points were known (These points, at the crown and the right springing, were established graphically.). It was also known, of course, that the axes of the parabolas were vertical. The two known points of a curve thus were opposite corners of rectangles. Parabolas were inscribed in the three rectangles by the geometrical method of locating the intersection of the tangent and the normal at any point on the parabola.[1] Thirty-six points on each of the parabolas were thus plotted and connected with straight lines. Very close approximations of the true parabolas were thus obtained. In the interest of accuracy, all drawing was done with a light touch; and thus the drawing can not be reproduced in this paper.

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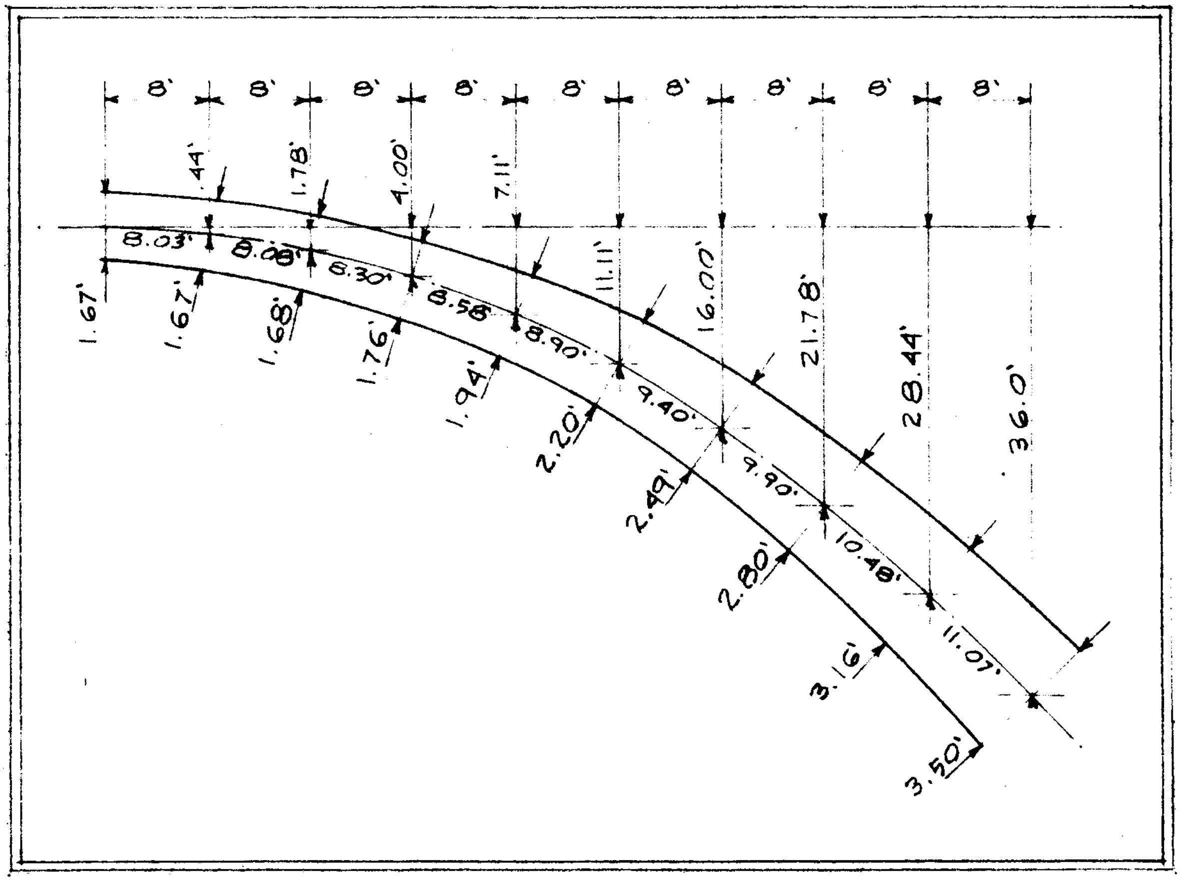
[1] Charles George Ramsey and Harold Reeve Sleeper, Architectural Graphic Standards, Third Edition, p. 297.

It was decided to divide the arch into fifteen parts, each having a horizontally projected length of eight feet. These eight foot lengths were laid off on the large scale drawing and projected vertically till they intersected the arch axis. At these points of intersection, perpendiculars to the tangent of the axis were constructed. The depth of the arch at each point was then scaled along these perpendiculars. The distance along the axis between division points was scaled; also the vertical distances from the horizontal line through the axis at the crown to the division points were scaled. These scaled distances are illustrated in Figure D-2.

The moment of inertia of a section of the arch is directly proportional to the third power of the depth at that section. The model, having a constant depth of section, had to be so constructed that the widths of the sections at the division points were all in the same constant ratio to the depths, to the third power, for the corresponding sections of the prototype. It was arbitrarily decided to let the model have a width of .25 inches at the crown. The width of the model at the right springing thus became $\frac{.25 \times (3.50)^3}{4.62} = 2.31$ inches. The required width of the model at each point of division was similarly computed.

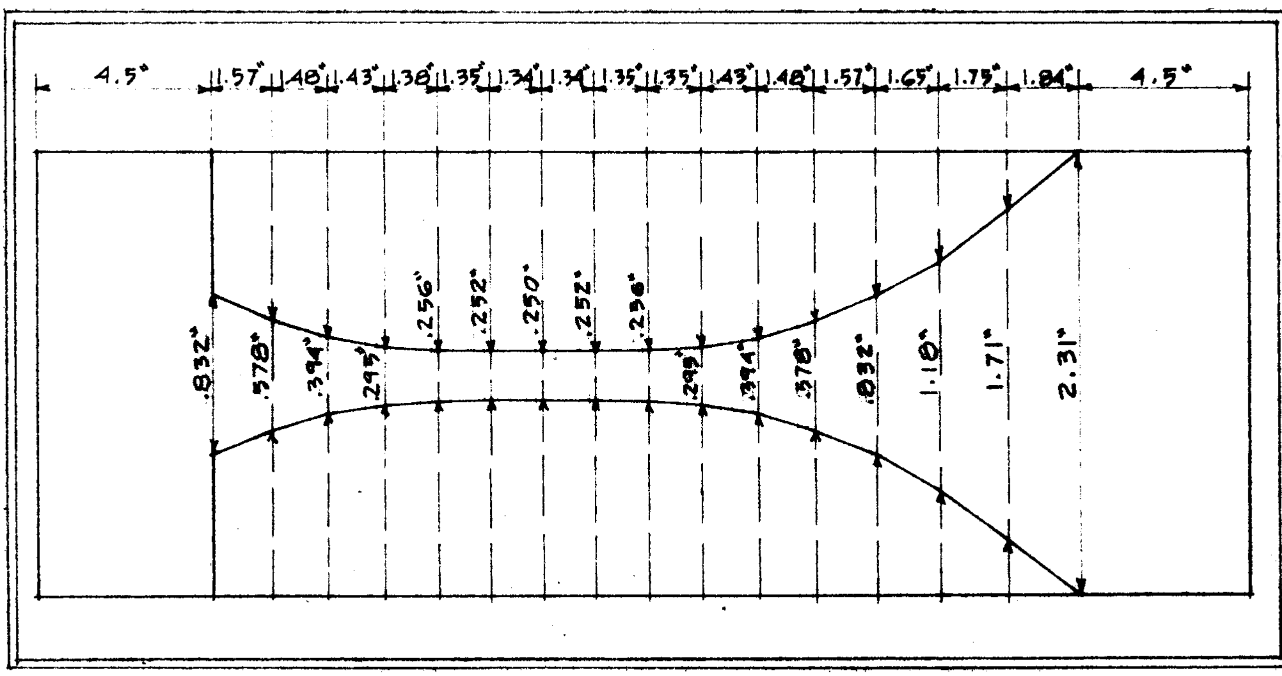
FIGURE D-2



The model was then laid out on a sheet of balsa selected for uniform thickness by visual inspection. The layout is illustrated by Figure D-3. The scale of lengths chosen was 1" = 6'.

A drawing of the "approximate" axis was made from the data illustrated in Figure D-2. The model layout was then cut nearly through along the lines (shown dashed in Figure D-3) separating the parts. The drawing of the axis was used as a template, and the sheet of balsa was bent at the cut lines so that it aligned with the axis. The steel bars were used to hold the sheet in place as "model airplane" cement was applied to both sides of the joint. It was thus necessary to cement one joint at a time. Four minutes was allowed for the cement to dry at each joint. After the model was formed to the shape of the axis it was cut to width, and the model was complete.

FIGURE D-3



Perhaps the procedure just outlined for constructing the model seems tedious. Certainly it is one of the disadvantages of the method being developed by the writer. However, it is not as great a disadvantage as the preceding description may have caused it to seem. After the first large scale drawing of the arch was finished, less than two hours were required to complete the model. The first drawing (or some other method of determining the dimensions of the arch at any section) would also be required for a mathematical solution of the problem.

The "feel" of the stiffness of the model indicated that the caused displacements at the springings should be of the magnitude of 1/2 inch for linear displacements and .2 radian for angular displacements. Guide lines for such displacements were then added to the drawing of the axis.

The model was placed on the drawing and manipulated as before. Using the mirror, as previously described, deflections from the horizontal were measured at points "A, B, C, and D". The horizontal and vertical components of the reaction, and the moment, were obtained at each springing for a unit load at each point. The results of this experiment are shown in Tables 7 and 8. Photograph VI illustrates the model in a deformed configuration due to angle change at the left springing.

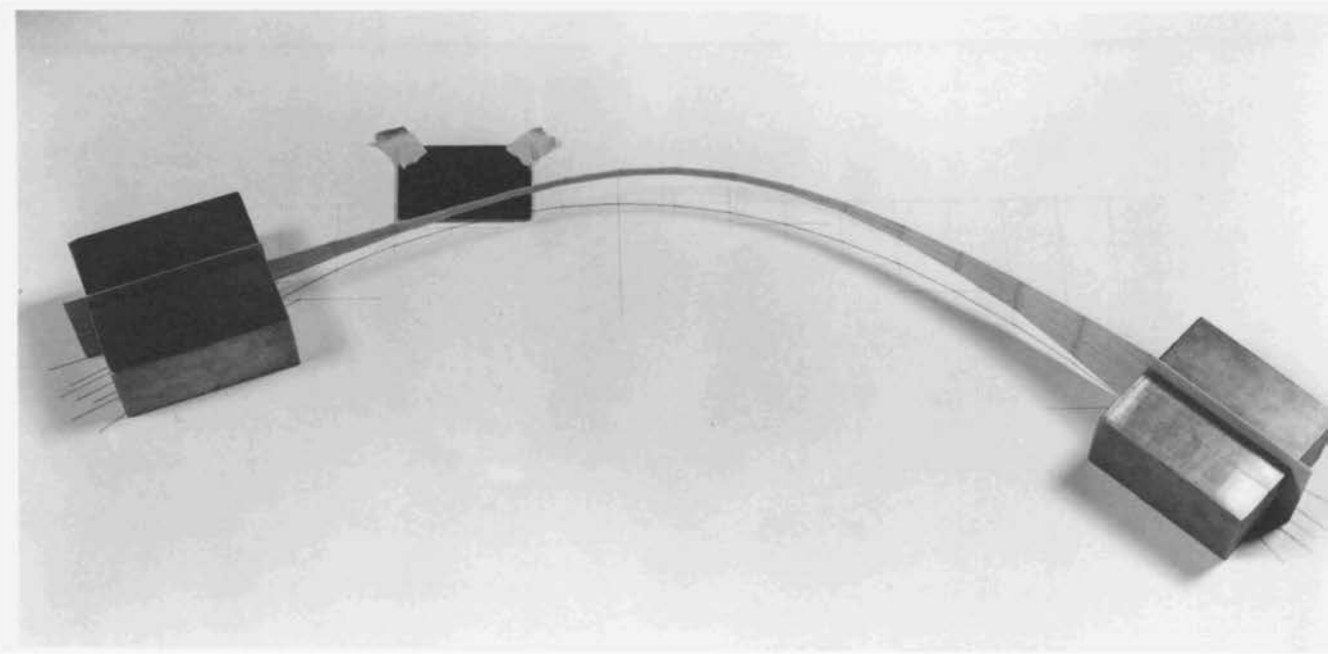
TABLE 7

Deflection of Model: At Left Springing	At Point of Load: A	B	C	D	Av.	Δ_1/Δ_2	Scale Factor	Value of Reaction
	.31" .27"				.29"	1.45	6	M_A = 8.70
.2 rad .2 rad		.09" .00"			.045"	.225	6	M_B = 1.35
			.28" .17"		.225"	1.125	6	M_C = 6.75
				.09" .11"	.10"	.50	6	M_D = 3.00
	.23" .24"				.235"			H_A = .47
½" left ½" right		.53" .62"			.575"			H_B = 1.15
			.44" .54"		.49"			H_C = .98
				.13" .17"	.15"			H_D = .30
	.45" .39"				.42"			V_A = .84
½" up ½" down		.37" .12"			.245"			V_B = .49
			.14" .00"		.07"			V_C = .14
				.00" .02"	.01"			V_D = .02

TABLE 8

Deflection of Model: At Right Springing	At Point of Load: A	B	C	D	Av.	Δ_1/Δ_2	Scale Factor	Value of Reaction
.2 rad ↻ .2 rad ↺	.22" .21"				.215"	1.075	6	M_A = 6.45
		.39" .37"			.38"	1.90	6	M_B = 11.40
			.07" .20"		.135"	.675	6	M_C = 4.05
				.34" .47"	.405"	2.025	6	M_D = 12.15
½" left ½" right	.23" .24"				.235"			H_A = .47
		.61" .52"			.565"			H_B = 1.13
			.50" .48"		.49"			H_C = .98
				.15" .17"	.16"			H_D = .32
½" up ½" down	.07" .07"				.07"			V_A = .14
		.33" .17"			.25"			V_B = .50
			.47" .36"		.415"			V_C = .83
				.48" .50"	.49"			V_D = .98

PHOTOGRAPH VI



It should be noted from Tables 7 and 8 that there is involved, in the computations for moment, what the writer calls a scale factor. Actually this scale factor was present in all the previous experiments; however, since its value was unity, a discussion of it was deferred. Moment has the dimensions of force-times-distance; while angle changes are dimensionless ratios. Consider the deflection at point "A" on the arch when an angle change of .2 radian was caused at the left springing (see Table 7). The average deflection was expressed in terms of inches, and was equal to .29 inches. Since one inch on the model was equal to 6 feet on the prototype, when .29 inches was divided by .2, the moment (1.45) at the left springing of the prototype due to a unit load at "A" was in terms of 1/6 foot pounds (if the unit load was one pound).

When single dimension quantities (shear, horizontal and vertical components of reactions, etc.) are being sought, there is no scale factor; because the applied displacements have the dimension of distance.

The arch was solved by the method developed by Professor Hardy Cross which combines the <u>Column Analogy</u> and graphic statics.[2] This solution is illustrated by Figures

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2 Hardy Cross and Newlin Dolbey Morgan, <u>Continuous Frames of Reinforced Concrete</u>, p. 279-288.

D-4, D-5, and D-6. The results are compared with the experimental results in Table 9.

TABLE 9

EXPERIMENTAL VALUES: From Displacements At Left Springing	EXPERIMENTAL VALUES: From Displacements At Right Springing	COLUMN ANALOGY VALUES
$M_A = 8.70$ $M_B = 1.35$ $M_C = 6.75$ $M_D = 3.00$		$M_A = 8.67$ $M_B = 1.55$ $M_C = 7.11$ $M_D = 2.95$
$H_A = .47$ $H_B = 1.15$ $H_C = .98$ $H_D = .30$		$H_A = .493$ $H_B = 1.186$ $H_C = 1.030$ $H_D = .317$
$V_A = .84$ $V_B = .49$ $V_C = .14$ $V_D = .02$		$V_C = .157$ $V_D = .023$
	$M_A = 6.45$ $M_B = 11.40$ $M_C = 4.05$ $M_D = 12.15$	$M_A = 6.96$ $M_B = 10.20$ $M_C = 1.65$ $M_D = 12.26$
	$H_A = .47$ $H_B = 1.13$ $H_C = .98$ $H_D = .32$	
	$V_A = .14$ $V_B = .50$ $V_C = .83$ $V_D = .98$	$V_A = .155$ $V_B = .524$

PROPERTIES OF THE SECTION											UNIT LOAD AT															
GIVEN					DERIVED						A				B				C				D			
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
No.	L	d	x	Y	a	ax	ay	ax^2	ay^2	axy	m_s	P	M_x	M_y	m_s	P	M_x	M_y	m_s	P	M_x	M_y	m_s	P	M_x	M_y
6	[illegible]43	2.33	-44	-13.43	9.66		-129.7	18702	1742.28		-20	-193.2	+8501	+2595	-44	-425.0	+18700	+5708								
5	6.95	2.06	-36	-9.00	11.93		-107.4	15461	966.33		-12	-143.2	+5155	+1289	-36	-429.5	+15462	+3866								
4	9.56	1.86	-25	-5.51	16.01		-88.2	12552	486.06		-4	-64.0	+1792	+353	-28	-448.3	+12552	+2470								
3	8.30	.72	-20	-2.78	19.58		-54.4	7832	151.35						-20	-391.6	+7832	+1089								
2	8.	.68	-12	-1.00	20.51		-20.5	2953	20.51						-12	-246.1	+2953	+246								
[illegible]	8.[illegible]	.67	-4	-.11	20.66		-2.3	330	.25						-4	-82[illegible]	+330	+9								
	8.[illegible]	.67	+4	-.11	20.66		-2.3	330	.25																	
2	8	.68	+12	-1.00	20.51		-20.5	2953	20.51																	
3	8.3[illegible]	.72	+20	-2.78	19.58		-54.4	7832	151.35																	
4	[illegible]	.86	+28	-5.51	16.01		-88.2	12552	486.06										-4	-64.0	-1792	+353				
5	[illegible]	2.06	+36	-9.00	11.93		-107.4	15461	966.33										-12	-143.2	-5155	+1289				
6	[illegible]	2.33	+44	-13.43	9.66		-129.7	18702	1742.28										-20	-193.2	-8501	+2595				
7	[illegible]	2.60	+52	-19.80	6.70	+348.4	-125.0	18117	2368.05	-6550									-28	-187.6	-9755	+3527	-4	-26.8	-1594	+504
8	[illegible]	2.73	+60	-25.00	6.11	+366.6	-152.8	21996	3818.75	-9165									-36	-220.0	-13200	+5500	-12	-73.3	-4398	+1833
9	10.5[illegible]	3.33	+68	-32.1[illegible]	3.43	+233.2	-110.1	15860	3534.31	-7488									-44	-151.0	-10268	+4847	-20	-68.6	-4665	+2202
Σ					[illegible].94	+948.2	-1192.9	171633	16455	-23203		-400.4	+15448	+4237		-2023.1	+57829	+13388		-959.0	-48671	+18111		-168.7	-10457	+4539
CORRECT FOR CENTROID $\bar{x} = +4.453$ $\bar{y} = -5.602$								+4222	+6682	-5312			-1783	+2253			-9009	+11333			-4270	+5372			-752	+945
CORRECT TO CENTROID								+167411	+9773	-17891			+17231	+1984			+66838	+2055			-44401	+12799			-9705	+3594
CORRECTION FOR DISSYMETRY								+32758	+2013				-3632	-1842			-3763	-7145			-23221	+4746			-6580	+1037
CORRECTED FOR DISSYMETRY								+134653	+7760				+20863	+3826			+70601	+9200			-21180	+7993			-3125	+2457
										P/A	-1.880				-9.502				-4.504				-.7922			
										vt			+.1550				+.5243				-.1574				-.02321	
										hi			+.4930				+1.186				+1.030				+.3166	
										x int.			+12.12				+18.12				-28.61				-34.13	
										y int.			+3.81				+8.01				+4.37				+.0232	

FIGURE D-4

FIGURE D-5

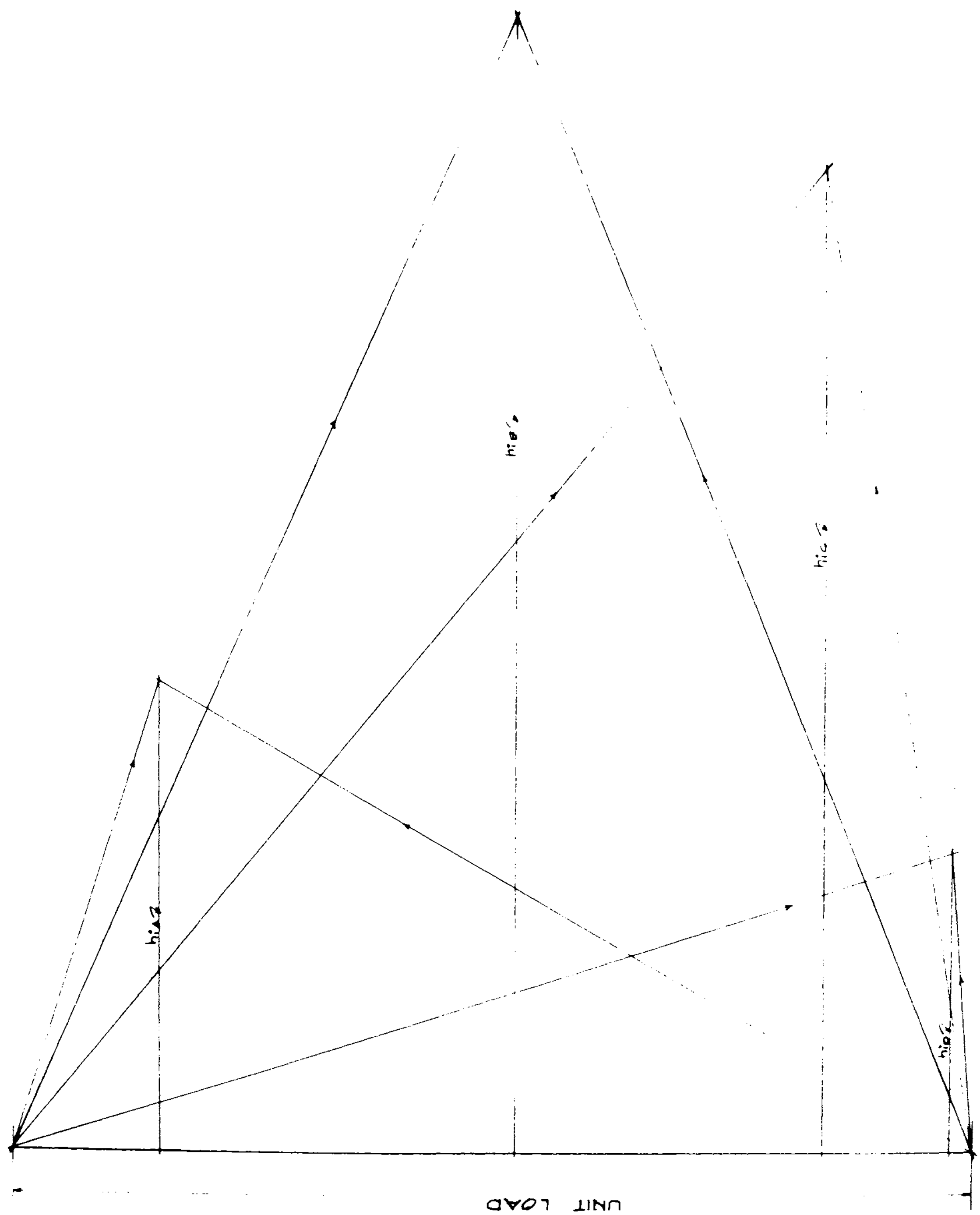
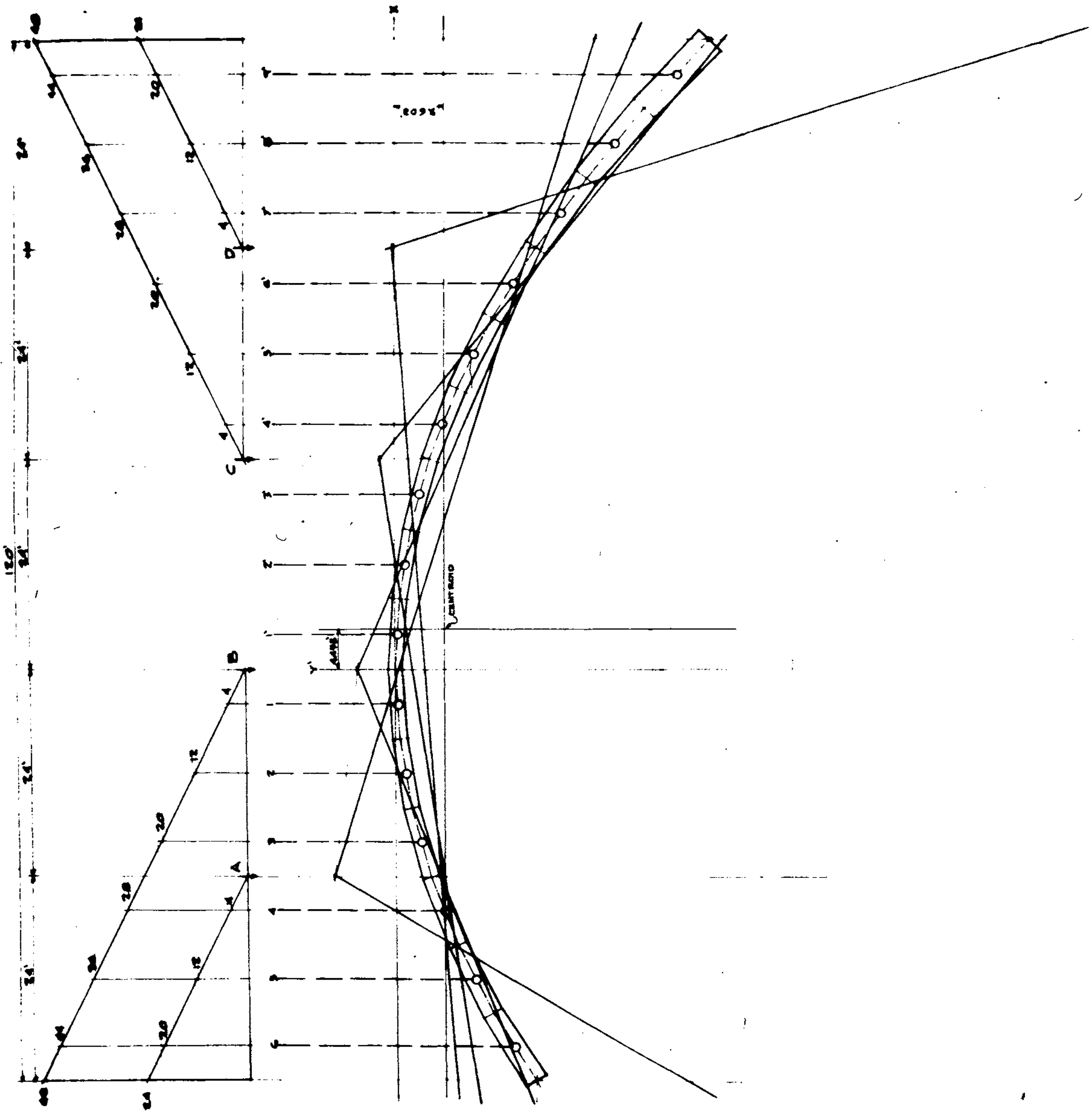


FIGURE D-6

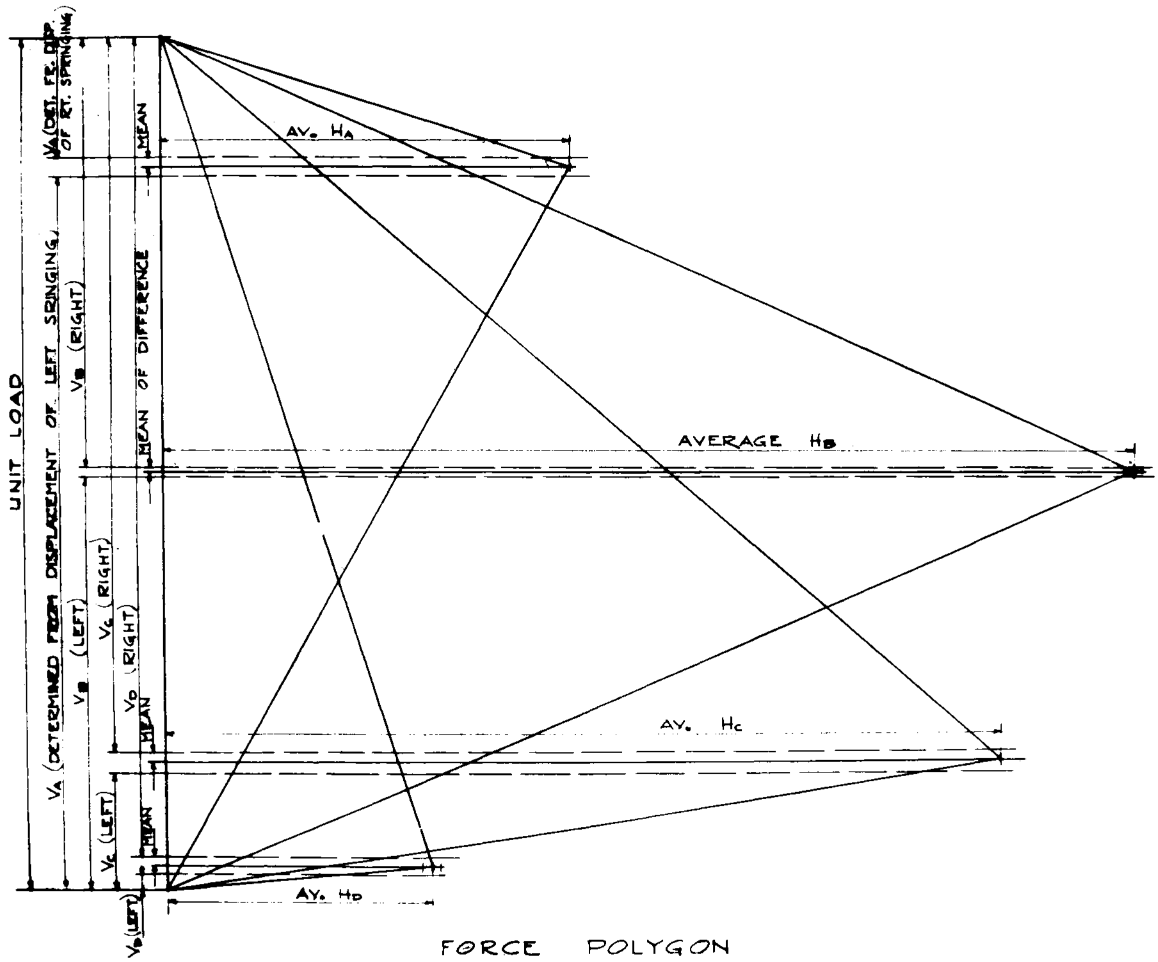


The values for the moments by the column analogy were obtained by multiplying the horizontal component of a pressure line by the scaled distance along a vertical line through the axis, at the springing, to the intersection of the vertical line and the pressure line.

Figure D-7 shows a force polygon constructed from the average values, experimentally obtained, of the horizontal and vertical components of the reactions. Figure D-8 compares the pressure lines obtained experimentally with those obtained by the column analogy (column analogy pressure lines are shown dashed). The experimentally obtained pressure lines were located on the arch by reversing the procedure described in the preceding paragraph.

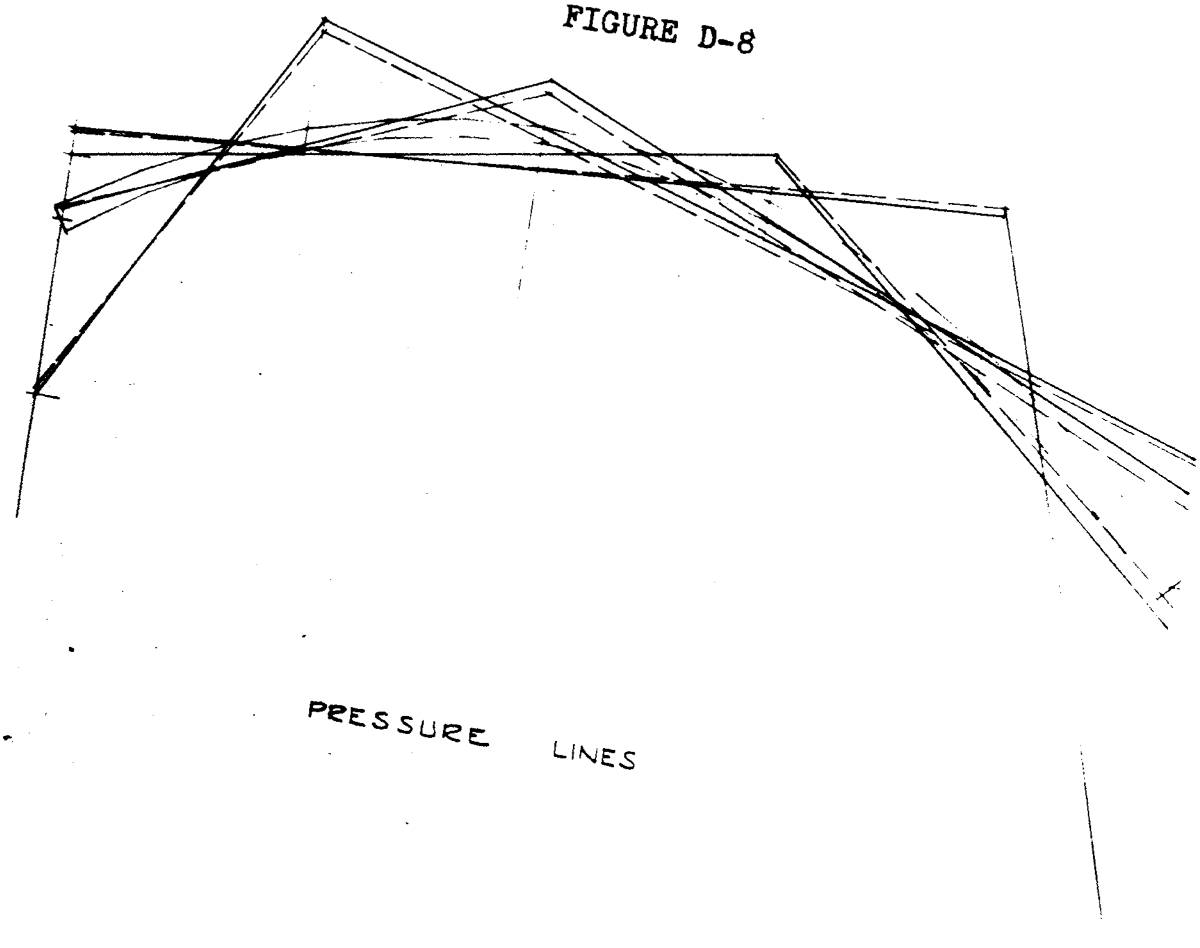
Table 8 and Figure D-8 indicate that the experimentally obtained moment at the right sprining due to a unit load at "C" is in error. This moment was rechecked; but the same result was obtained, so the error will have to remain. However, the writer feels that the experimental solution of the arch was quite successful. The values of stresses thus obtained would be well within the limits of accuracy necessary to design such a structure.

FIGURE D-7



FORCE POLYGON

FIGURE D-8



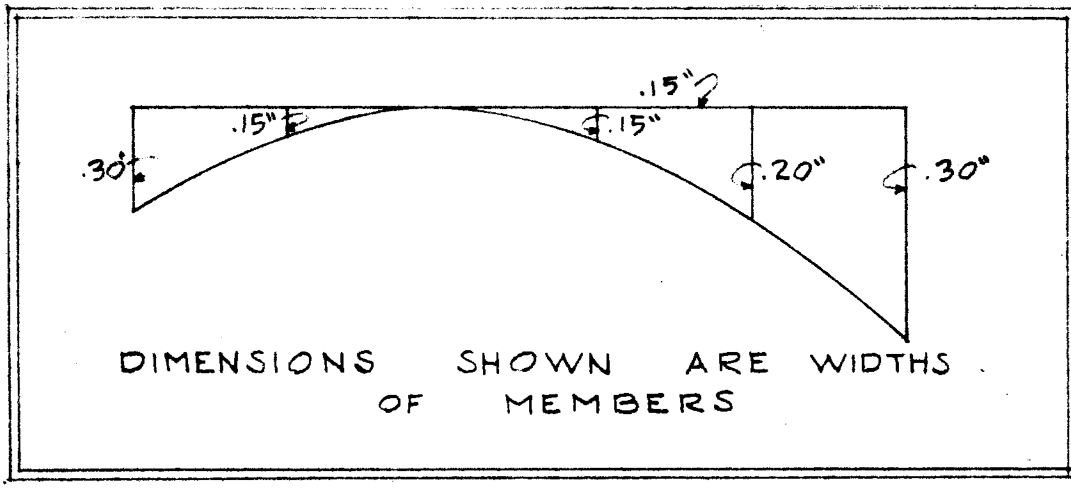
PRESSURE LINES

E. A PRACTICAL APPLICATION

The prototype structure of the preceding experiment could very well have been the arch of a bridge; if so, it would have had a superstructure. Undoubtedly such a structure would be of reinforced concrete which is an inherently continuous material. It was however analyzed, both experimentally and mathematically, as though the arch and superstructure acted independently. The assumption being made that the columns of the superstructure applied direct vertical loads to the arch at points "A, B, C, and D". This assumption is always made in any mathematical analysis of this type of problem, because an attempt to consider the arch and superstructure as monolithic in a mathematical analysis would prove to be too difficult to be practical. To be sure, a good designer would consider the effect of the superstructure on the arch; but his consideration would be based upon his judgement, and he would not have any "numbers" arrived at mathematically to aid such judgement.

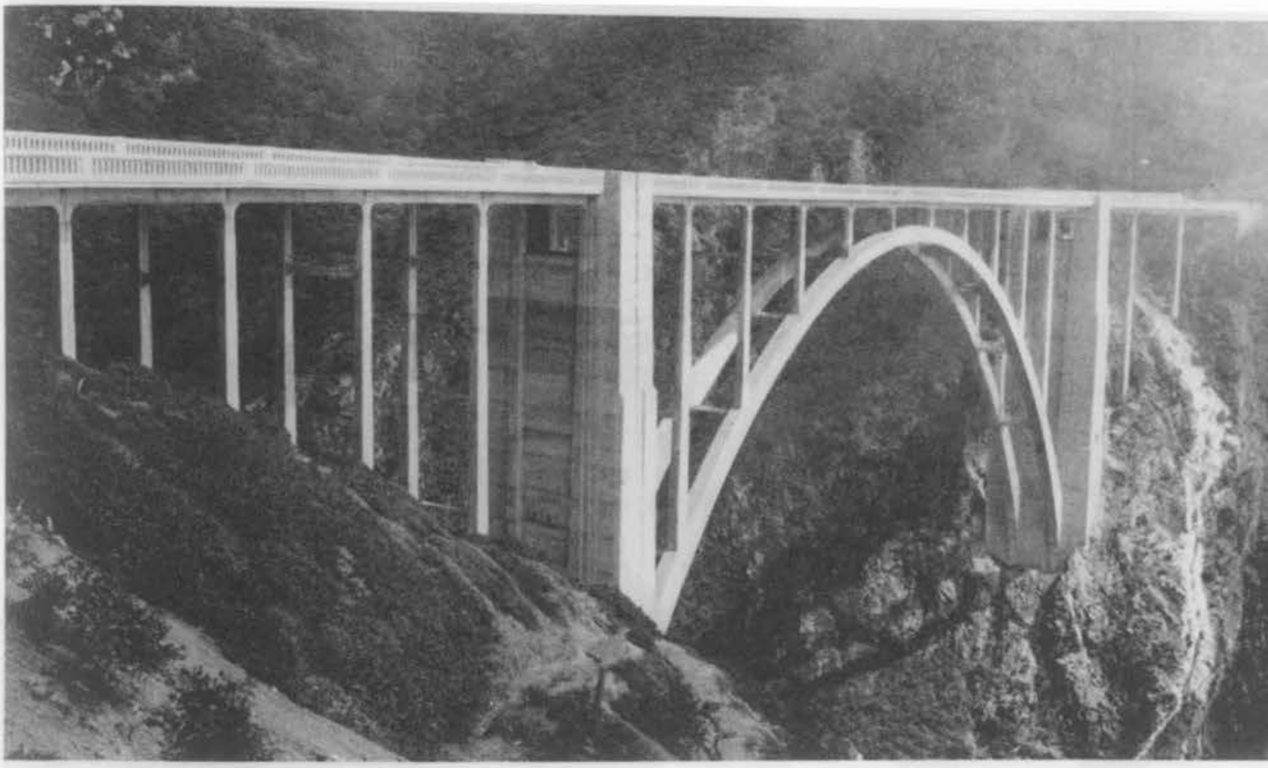
It appeared to the writer that it would not be difficult, by the experimental method presented in this paper, to make an analysis of the structure as an entity; so the superstructure illustrated by Figure E-1 was added to the model of the arch.

FIGURE E-1



The stiffnesses of the members of the superstructure, as determined by their widths of section, were made proportionate to what their actual stiffnesses might be, if the structure followed the form that American engineers customarily use (See Photograph VII). Of course those stiffnesses could take other relative values; the arch might become as thin as the superstructure members (See Photographs VIII and IX). The writer does not know how Mr. Maillart analyzed his bridge. There are rumors that he used a model method; other rumors have it that instinct was his primary tool of analysis. However that may be, the writer believes experimental methods of analysis to be instruments that could transport the designer of structures beyond the prosaic limits within which a sole dependence upon mathematics would confine him.

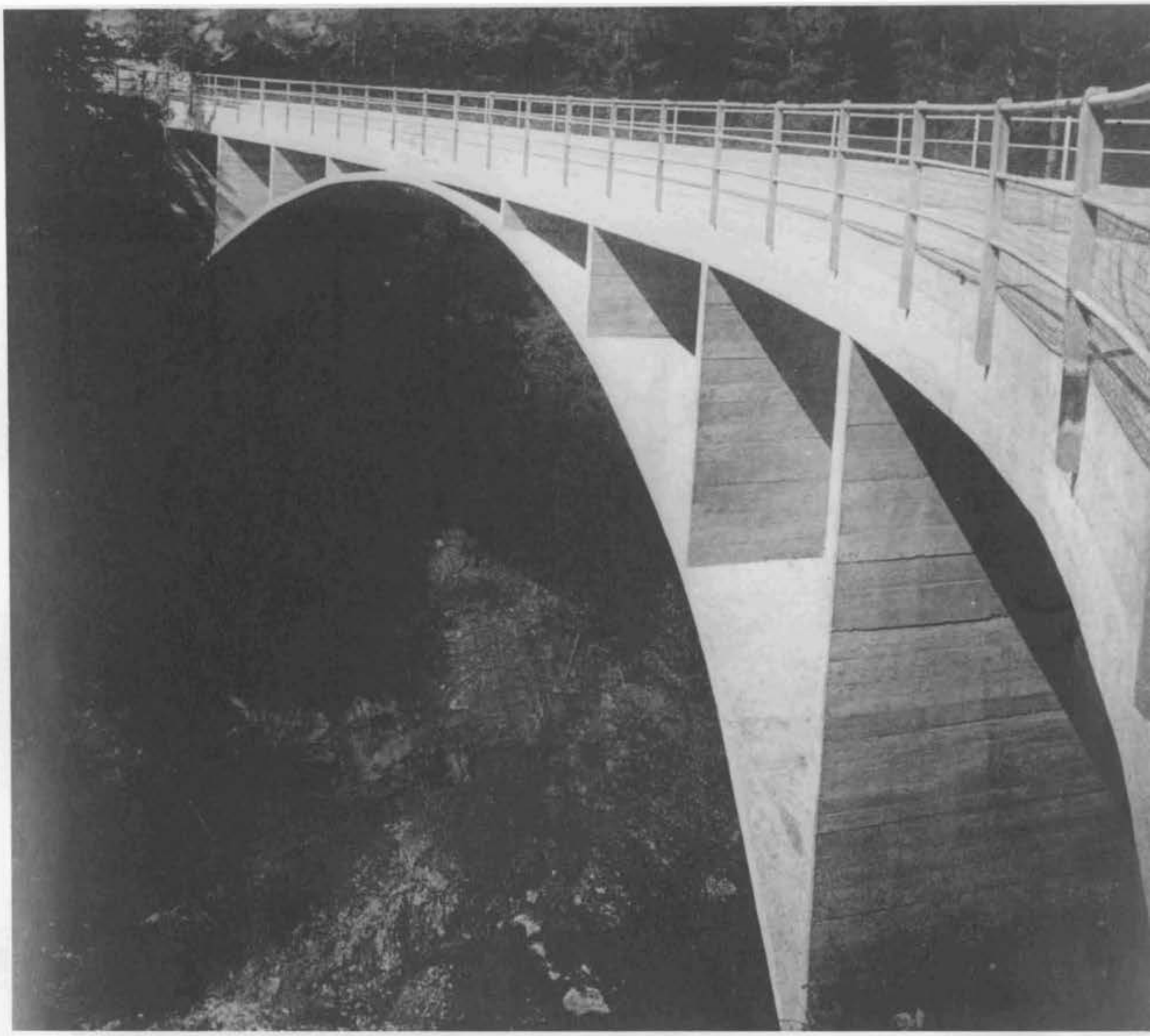
PHOTOGRAPH VII



216. Rainbow Bridge near Carmel, California. *The same problem — a narrow gorg as in the Swiss valley. Using the normal construction, the approach had to be built in gered sections; the alignment of the bridge could not be curved.*

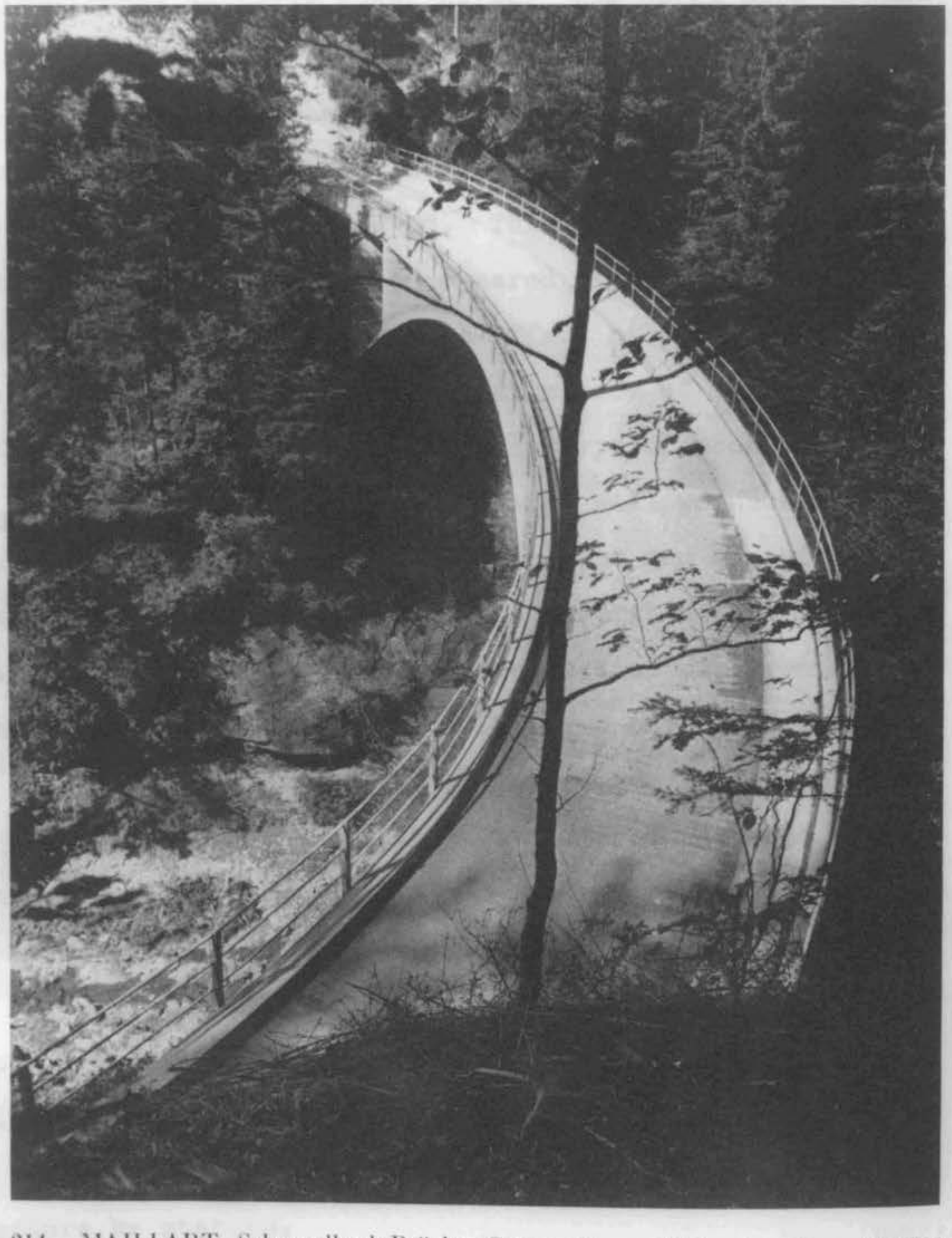
Photographs VII, VIII, and IX are reproduced from <u>Space</u>, <u>Time</u> <u>and</u> <u>Architecture</u> by Sigfried Gideon.

PHOTOGRAPH VIII



The bridge illustrated in Photographs VIII and IX was designed by the late Robert Maillart, renowned Swiss engineer.

PHOTOGRAPH IX



214. MAILLART, Schwandbach-Brücke, Canton Berne, 1933. *Air view. Maillart resolved bridge-building into a system of flat and curved slabs. In Maillart's hands the rigidity of the slab, hitherto an incalculable factor in construction, became an active bearing surface. The torsional strain that would have to be allowed for in a concrete bridge built on a curving alignment can be utilized only by this method of construction.*

To return to the experiment at hand, the superstructure was fabricated of 1/32" sheet balsa. The parts were joined and cemented to the arch with model airplane cement. The model was then placed on the drawing prepared for the preceding experiment, and the reactions were determined for unit loads at "A, B, C, and D". This was done in order to obtain results that could be compared with those of the previous experiment, so that the effect of the superstructure could be demonstrated. Actually, one of the advantages of the experimental solution of this type of problem is that the location of the load or loads that will produce the maximum values of the stress function sought can be quickly determined. Thus, if the arch with superstructure were being analyzed for moving loads, deflection measurements for points other than "A, B, C, and D" would have been taken. The results of the experiment are presented in Tables 10 and 11. In Table 12 these results are compared with the experimental results for the arch alone. Photograph X illustrates the model with the right springing displaced .25 radian .

The writer feels that the best procedure of analysis with the model would be to solve only for the reactions, including moment, experimentally; and then obtain values of the various stress functions at other sections of the structure by statics.

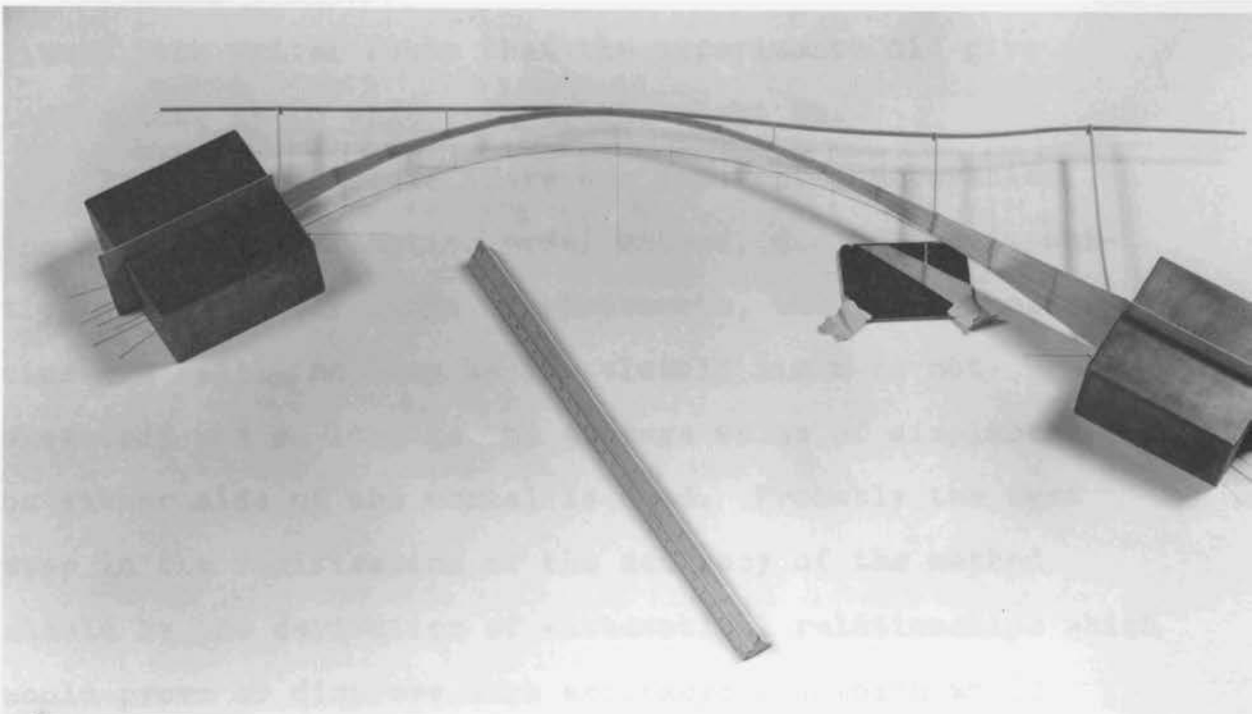
TABLE 11

Deflection of Model								
At Right Springing	At Point of Load A	B	C	D	Av.	Δ_1/Δ_2	Scale Factor	Value of Reaction
	.17"				.15	.75	6	M_A = 4.50
	.13"							
.5 rad		.11"			.175	.875	6	M_B = 5.25
.5 rad		.24"						
			.10"		.095	.475	6	M_C = 2.85
			.09"					
				.25"	.30	1.50	6	M_D = 9.00
				.35"				
	.26"				.27"			H_A = .54
	.28"							
½" left		.62"			.555"			H_B = 1.11
½" right		.49"						
			.49"		.455"			H_C = .91
			.42"					
				.16"	.185"			H_D = .37
				.21"				
	.11"				.10"			V_A = .20
	.09"							
½" up		.34"			.255"			V_B = .51
½" down		.17"						
			.46"		.395"			V_C = .79
			.33"					
				.44"	.46"			V_D = .92
				.48"				

TABLE 12

	REACTIONS AT LEFT SPRINGING		REACTIONS AT RIGHT SPRINGING	
	With Superstructure	Without Superstructure	With Superstructure	Without Superstructure
M_A	5.85	8.70	4.50	6.45
M_B	1.95	1.35	5.25	11.40
M_C	4.20	6.75	2.85	4.05
M_D	2.55	3.00	9.00	12.15
H_A	.56	.47	.54	.47
H_B	1.15	1.15	1.11	1.13
H_C	.93	.98	.91	.98
H_D	.36	.30	.37	.32
V_A	.81	.84	.20	.14
V_B	.49	.49	.51	.50
V_C	.19	.14	.79	.83
V_D	.10	.02	.92	.98

PHOTOGRAPH X



F. CONCLUSIONS

Though the number of experiments performed was not sufficiently exhaustive that absolute truths could be established, the writer feels that the experiments did give indications as to what such truths might be.

It is felt that if there are inherent inaccuracies, in the reciprocal deflection model method, due to the imposition of relatively large displacements, that such inaccuracies are small, so long as the elastic limit is not exceeded; and so long as the average value of displacements on either side of the normal is used. Probably the next step in the verification of the accuracy of the method should be the derivation of mathematical relationships which would prove or disprove such accuracy; and which would explain why it is necessary to use the average of the two deflections.

The experiments indicate that the method of model construction and manipulation evolved and presented in this paper can yield results sufficiently accurate for the design of structures; and that the method can be applied to structures too difficult to analyze by mathematics. This appears to be true in spite of the fact that a model constructed by the presented method does not take into account the cross sectional area of the prototype structure; even though direct stress in the members of the model would seem to be a factor that can not be neglected as a cause of deflection. Here again a mathematical study is in order.

The writer at this point would like to say that he has not intended to minimize the importance of mathematics. It is, of course, a most necessary adjunct to experimentation in the development of scientific theories; and it can be a powerful tool to the designer of structures. The writer does believe, however, that "calculations" should be servant and not master.

For a closing statement it would be impossible to do better than quote Leonardo Da Vinci who said,[1]

"Those sciences are vain and full of errors which do not end with one clear experiment."

---

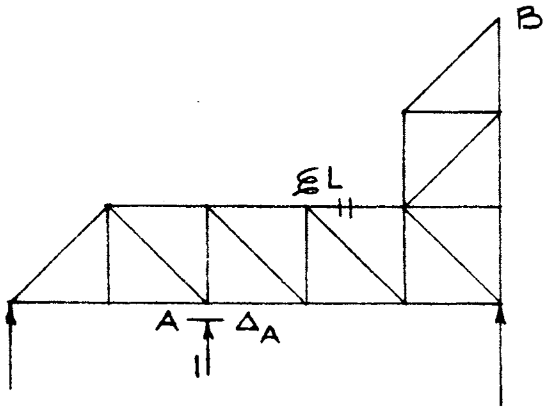
[1] Harvey F. Girvin, A Historical Appraisal of Mechanics, p. 57.

G. APPENDIX

The following proof of Maxwell's Law of Reciprocal Deflections was presented in $A_r$chitecture 434, a course in statically indeterminate structures at Oklahoma A. and M. College, by Professor R. E. Means.

In any structure made up of trusses or solid members, such as shown in the illustration on following page, assume an imaginary unit load at a point where the deflection is desired acting in the direction of the deflection. If the deflection is due to change in length of members in the truss or to change in length of fibers in the solid members (angle change), then the external virtual work done equals the deflection (distance through which the imaginary unit load is moved) times one; i.e., $1 \triangle_A$, and the internal virtual work done is equal to the stress in each member caused by the imaginary unit load times the change in length of the members (distance through which stress is moved).

ANY TRUSSED STRUCTURE



U_A = stress due to unit load at A

external work = internal work

$$1\,\Delta_A = U_A\,\delta L$$

if several members are deformed

$$\Delta_A = \sum U_A\,\delta L$$

if δL is due to stress

$$\delta L = \frac{f\,L}{E} = \frac{S\,L}{A\,E}$$

then

$$\Delta_A = \sum \frac{U_A\,S\,L}{A\,E}$$

for load P at B

$$\Delta_A^{P_B = P} = \sum \frac{U_A\,P\,U_B\,L}{AE} = P \sum \frac{U_A U_B L}{AE}$$

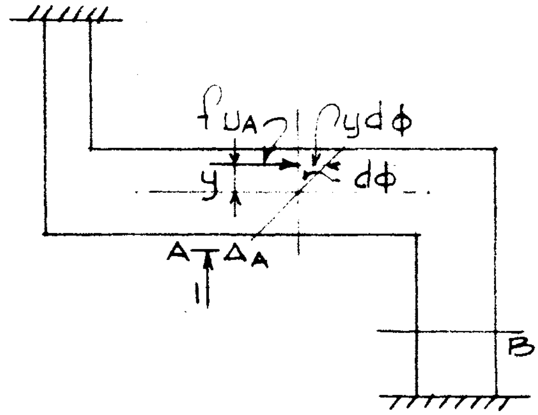
for load P at A

$$\Delta_B^{P_A = P} = \sum \frac{U_A\,P\,U_B\,L}{AE} = P \sum \frac{U_A U_B L}{AE}$$

therefore

$$\Delta_A^{P_B = P} = \Delta_B^{P_A = P}$$

ANY SOLID MEMBER STRUCTURE



f_{UA} = unit stress due to unit load at A

m_A = moment due to unit load at A

$$1\,\Delta_A = \int f_{UA}\,da\;y\,d\phi$$

if angle change occurs at more than one section

$$\Delta_A = \iint f_{UA}\,da\;y\,d\phi$$

if $d\phi$ is due to moment

$$d\phi = \frac{M\,ds}{EI} \quad \text{and} \quad f_{UA} = \frac{m_A\,y}{I}$$

then

$$\Delta_A = \iint \frac{m_A\,y}{I}\,\frac{M\,ds}{EI}\,y\,da$$

$$\Delta_A = \int \frac{m_A\,M\,ds}{EI}$$

for load P at B

$$\Delta_A^{P_B = P} = \int \frac{m_A\,P\,m_B\,ds}{EI} = P \int \frac{m_A m_B\,ds}{EI}$$

for load P at A

$$\Delta_B^{P_A = P} = \int \frac{m_B\,P\,m_A\,ds}{EI} = P \int \frac{m_B m_A\,ds}{EI}$$

therefore

$$\Delta_A^{P_B = P} = \Delta_B^{P_A = P}$$

Which is Maxwell's Law of Reciprocal Deflections

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THESIS TITLE: LARGE SCALE DEFORMATIONS APPLIED TO MODEL METHOD OF STRESS ANALYSIS

AUTHOR: DONALD LEE COTNER

THESIS ADVISER: PROFESSOR R. E. MEANS

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