CONTRACTING RATIONAL CURVES ON SMOOTH COMPLEX THREEFOLDS

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TABLE OF CONTENTS

Chapter | Page

1. INTRODUCTION 1
   - 1.1 Statement of Problem and Results 4
   - 1.2 Formal Information 10
   - 1.3 Length 11
   - 1.4 Locally free sheaves on C 13
   - 1.5 Rational double points 15

2. THE LENGTH(1,1,$\cdots$,1) CASE 19

3. THE LENGTH(2,1) CASES 34
   - 3.1 The $\mathbf{D_4(2,1)}$ Case 34
   - 3.2 The $\mathbf{D_5(2,1)}$ Case 42

4. THE LENGTH(2,2) CASES 53
   - 4.1 The $\mathbf{D_5(2,2)}$ Case 54
   - 4.2 The $\mathbf{D_6(2,2)}$ Case 63

5. KAWAMATA TECHNIQUE 72
   - 5.1 The length(1,1, $\cdots$,1) case 74
   - 5.2 The length(2,1) case 75

6. CONTRACTION CRITERIA 83

BIBLIOGRAPHY 104

APPENDIX A: $\mathbf{D_4(2,1)}$ CALCULATIONS 106

APPENDIX B: $\mathbf{D_5(2,1)}$ CALCULATIONS 111

APPENDIX C: $\mathbf{D_5(2,2)}$ CALCULATIONS 120

APPENDIX D: $\mathbf{D_6(2,2)}$ CALCULATIONS 125

LIST OF TABLES

Table Page

1.1 Rational Double Points 16

LIST OF FIGURES

Figure		Page
3.1	$D_4(2,1)$ configuration	40
3.2	$D_5(2,1)$ configuration	50
4.1	$D_5(2,2)$ configuration	61
4.2	$D_6(2,2)$ configuration	69
5.1	$E_7(2,1)$	76
5.2	$E_6(2,1)$	79
5.3	$D_n(2,1)$	81

CHAPTER 1

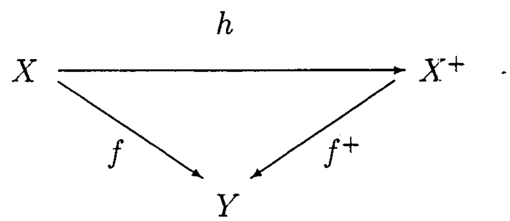
INTRODUCTION

The ideas for this work originated from the problem of classifying three dimensional algebraic varieties. An *algebraic variety* is the set of complex zeros in $\mathbf{P^n}$ of a collection of homogeneous polynomials. An algebraic variety can be regarded as an analytic subvariety of the compact complex manifold $\mathbf{P^n}$, so analytic methods can be used in the classification of algebraic varieties. The equivalence classes utilized to classify these varieties are called *birational* equivalence classes.

The classification of two dimensional smooth varieties, or *surfaces* was accomplished by studying special birational maps called *blow-ups.* If q is a point on a surface N, the *blow-up of* N *at* q, denoted $f : \tilde{N} \to N$, satisfies $f^{-1}(q) = E \cong \mathbf{P^1}$, $E^2 = -1$ and $\tilde{N} - E \cong N - \{q\}$. G. Castelnuovo, then, showed that if E is a curve on a smooth surface M with $E \cong \mathbf{P^1}$ and $E^2 = -1$, then there is a birational map $g : M \to N$ and a point $q \in N$ with $g(E) = q$ and $g : M - E \cong N - \{q\}$. In this situation, g is called a *blow-down* of E, or a *contraction* of E to the point q. A more general definition of contraction will be given later in the introduction. A surface M that contains no curves $E \cong \mathbf{P^1}$ with $E^2 = -1$, is said to be a *minimal model* in its birational equivalence class. This means that any birational map of M to a smooth surface is actually an isomorphism. The class ification of smooth algebraic surfaces was done by determining invariants and properties of a minimal model in each equivalence class. So, the importance of contracting smooth rational curves on surfaces played an important role in the classification of surfaces.

In the more recent study of the classification of three dimensional algebraic varieties, attempts have been made in determining minimal models in each birational equivalence class, and the concept of contracting rational curves was seen to be important in this process. The greatest advances in classifying threefolds were made by S. Mori in the development of the Minimal Model Program. In [Mo], Mori proved the existence of a birational map, called a *flip*, between varieties that was fundamental in this program. A closely related birational map of threefolds is the *flop*. In the context of this paper, the flop will be defined on smooth complex three dimensional manifolds. If X is a smooth complex 3-manifold and $E \subset X$ is a smooth rational curve with $\mathcal{K}_X \cdot E = 0$, then the *flop* is a birational map $h : X \to X^+$ with $h : X - E \cong X^+ - E^+$, where E^+ is a smooth rational curve in the 3-manifold X^+ also satisfying $\mathcal{K}_{X^+} \cdot E^+ = 0$. The flop can be described by two contraction maps.

That is, there exists an analytic 3-fold Y and a commutative diagram



where $f : X \to Y$ and $f^+ : X^+ \to Y$ contract E and E^+, respectively. This means there is a point $q \in Y$ with $f(E) = q$, $f^+(E^+) = q$, $f : X - E \cong Y - \{q\}$ and $f^+ : X^+ - E^+ \cong Y - \{q\}$. We also say that E (resp. E^+) is the *exceptional set* and f (resp. f^+) is a *resolution* of the singular point q.

The importance of the flop inspired work to understand contractions where X is a smooth complex 3-manifold and $E \cong \mathbf{P}^1$ in X with $\mathcal{K}_X \cdot E = 0$. The major works cited in this paper are those of M. Reid, [Re], and J. Jimenéz, [Ji]. It has long been known, [Wa], that if $f : X \to Y$ contracts a curve in X to a point q in Y, then q is a threefold singularity. So, unlike the surface situation where rational curves could be contracted to smooth points, contracting curves on threefolds presented new problems. One of the main considerations was to determine the singularity q. Reid in [Re] showed that if $f : X \to Y$ contracts a rational curve C to a point q, then q is a compound DuVal (cDV) singularity. This means that in a general hyperplane section, H, of q, q is a rational double point (RDP) and $f : f^*H \to H$ is a partial resolution of q. Also, since q is an RDP, it is singularity of type A_m, D_m, E_6, E_7 or E_8. (See section 1.5 for details and definitions concerning RDP's). For $C \cong \mathbf{P}^1$ an invariant of f called the *length* of C, introduced by J. Kollár in [CKM], proved to be the distinguishing factor by S. Katz and D. Morrison in [KM](Main Theorem) in determining the singularity type of q. That is, the length of C uniquely determines the singularity type of q. (See section 1.3 for details and definitions concerning length.) Prior to the work of Katz and Morrison, M. Reid in [Re], the "Pagoda" construction, showed that if $C \cong \mathbf{P}^1$ were of length 1, then q is an A_1 singularity in H. Y. Kawamata, in [Ka], proved the main theorem of Katz and Morrison using a more geometric technique. This method is discussed in chapter 5.

Another main consideration was to try to find an analogue to Castelnuovo's theorem for threefolds. In particular, given a smooth rational curve C in X with $\mathcal{K}_X \cdot C = 0$, work was done in trying to determine conditions to ensure that C could be contracted. H. B. Laufer, in [La], showed that if C were assumed just to be irreducible, then, if C were contracted, C must be a rational curve with its conormal sheaf in X isomorphic to $\mathcal{O}_C(1) \oplus \mathcal{O}_C(1)$, $\mathcal{O}_C \oplus \mathcal{O}_C(2)$ or $\mathcal{O}_C(-1) \oplus \mathcal{O}_C(3)$. (See section 1.4 for details concerning the conormal sheaf.) Also, if C were contractible, then Laufer in [La] showed that C has a rational formal neighborhood in X. This means that not only is C rational, but so is every curve supported on C. (See section 1.2 for a precise definition and section 1.4 to see how this implies the decomposition of the conormal sheaf.) The results that have been obtained to determine contractibility, then, depend on the conormal sheaf of C in X, denoted $\mathcal{I}_C/\mathcal{I}_C^2$, where $\mathcal{I}_C$ is the ideal sheaf of C in X.

H. Grauert, in [Gr], showed that if $\mathcal{I}_C/\mathcal{I}_C^2 = \mathcal{O}_C(1) \oplus \mathcal{O}_C(1)$ then C is contractible as the zero section of the normal sheaf of C in X. Reid, then, in [Re], investigated the case where $C \cong \mathbf{P}^1$ had a rational formal neighborhood in X and the conormal sheaf of C in X was $\mathcal{O}_C(1) \oplus \mathcal{O}_C(1)$ or $\mathcal{O}_C \oplus \mathcal{O}_C(2)$. In this situation, C has length 1 and the contractibility of C was determined from the higher order neighborhoods of C in X. In particular, a sequence of defining ideals, $\cdots \subset \mathcal{J}_{n+1} \subset \mathcal{J}_n \subset \cdots \subset \mathcal{J}_2 \subset \mathcal{J}_1 = \mathcal{I}$ was constructed such that $\mathcal{J}_k/\mathcal{I}\mathcal{J}_k \cong \mathcal{O}_C(1) \oplus \mathcal{O}_C(1)$ or $\mathcal{O}_C \oplus \mathcal{O}_C(2)$ for each $k \geq 1$. If $\mathcal{J}_k/\mathcal{I}\mathcal{J}_k \cong \mathcal{O}_C(1) \oplus \mathcal{O}_C(1)$ for some $k \geq 1$, then C could be contracted, otherwise C deformed in X. Jimenéz, [Ji], examined the remaining case where $C \cong \mathbf{P}^1$ has a rational formal neighborhood in X and $\mathcal{I}_C/\mathcal{I}_C^2 \cong \mathcal{O}_C(-1) \oplus \mathcal{O}_C(3)$. Here, where C has length greater than 1 and there are more possibilities for the higher order neighborhoods, a sequence of defining ideals, $\cdots \subset \mathcal{J}_{n+1} \subset \mathcal{J}_n \subset \cdots \subset \mathcal{J}_2 \subset \mathcal{J}_1 = \mathcal{I}$, was also constructed. It was shown that if $(\mathcal{J}_k/\mathcal{J}_k^2)/torsion$ decomposes with no negative summands for some k, then a formal map $\hat{f} : \hat{X} \to \hat{\mathbf{C}}^N$ could be constructed with $\hat{f}^{-1}(m_0) = \hat{\mathcal{J}}_k$, where m_0 is the maximal ideal at $0 \in \mathbf{C}^N$. It was concluded that from this formal map, there exists either an analytic contraction of C or an analytic deformation of C in X. Jimenéz's work, then, emphasized the importance of formal constructions and the infinitesimal neighborhoods of C in providing information about the analytic neighborhoods of C. Jimenéz attempted to bridge the gap between the formal construction and the analytic results by concluding that the formal map constructed gave a *formal modification* or *formal deformation* as defined by M. Artin in [Ar2](Definition 1.7). Artin's results show that such maps arise as the completions of analytic contractions or analytic deformations. So, in the situation of Jimenéz's work, the construction of the formal map would imply that the curve C either contracts or deforms in X, if Artin's definitions could be shown to apply.

1.1 Statement of Problem and Results

As discussed in the introduction, the work of M. Reid [Re] and J. Jimenéz [Ji] illustrate the benefit of acquiring formal results to answer questions about the contractibility of rational curves. In light of these and the other previous work in this area, the following situation is considered in this paper:
Let $C = \cup_{i=1}^{n} C_i$ with the C_i smooth rational curves satisfying

$$C_i \cap C_j = \begin{cases} \text{a point} & \text{if } |i-j| = 1 \\ \emptyset & \text{otherwise.} \end{cases}$$

Furthermore, assume $\mathcal{K}_X \cdot C_i = 0$ for each i, where X is a smooth complex threefold containing C and $\mathcal{K}_X$ is the canonical sheaf on X. Finally, assume that each C_i has a rational formal neighborhood in X. With these hypotheses, two main questions are considered:

1) If C contracts, what type of singularity results?

2) For which of these curves C can a contraction map be constructed?

To answer both of these questions, a formal construction is utilized. So, first, definitions of what is meant by formal in this paper needs to be made clear. In particular, the definitions of *formal cDV modification* and *formal cDV contraction* will be made precise (The basic definitions of formal schemes, maps, etc. can be found in section 1.2). To explain and motivate these definitions, some background on deformations of rational double points or DuVal singularities and the formal method employed in this paper will be given.

Definition 1.1 *An analytic map $f : X \to Y$, with $f : X - C \to Y - \{q\}$ an isomorphism for some point q in the analytic threefold Y and $f(C) = q$, is called an* **analytic contraction** *map.*

Assume an analytic contraction of C exists. Then since C is a closed subscheme of X, the formal completion, $\hat{X}$ of X is supported on C. Similarly, $\hat{Y}$, the formal completion of Y, is supported on q. Let $m_{q,Y}$ be the maximal ideal at the point $q \in Y$.

Definition 1.2 *Let $g : X \to Y$ contract C to the point $q \in Y$. The* **length** *of the component C_i is the length of the scheme with structure sheaf $\mathcal{O}_X / g^{-1}(m_{q,Y})$ at a generic point of C_i.*

Definition 1.3 *A* **formal deformation** *of C in $\hat{X}$ parameterized by a formal scheme $\hat{D}$, that contains a point q, is a family of abstract curves, $\hat{\mathcal{C}} \subset \hat{X} \times \hat{D}$, which is flat over D and restricts to the curve C over q.*

Using what was discussed in the introduction and definition 1.1, the two questions above can be restated as follows:

1)If $f : X \to Y$ is an analytic contraction of C, what type of cDV singularity, cA_m, cD_m, cE_6, cE_7 or cE_8, is $q \in Y$.

This question has been answered completely in the case of $C \cong \mathbf{P^1}$ by Katz and Morrison in [KM] and Y. Kawamata in [Ka]. In both of these works it is found that the singularity is determined completely by the length of the curve C. See section 1.3 for details of length. Therefore, the question of whether the same is true for curves with multiple components is an interesting one that will be addressed.

2)Given $C \subset X$ with the conormal bundle of each component C_i of C having conormal sheaf, $\mathcal{I}_{C_i}/\mathcal{I}^2_{C_i}$, isomorphic to $\mathcal{O}_{C_i}(1) \oplus \mathcal{O}_{C_i}(1)$, $\mathcal{O}_{C_i} \oplus \mathcal{O}_{C_i}(2)$ or $\mathcal{O}_{C_i}(-1) \oplus \mathcal{O}_{C_i}(3)$, when does there exist an analytic contraction $f : X \to Y$ of C.

This question has only partially been answered, even for the case of $C \cong \mathbf{P^1}$. The results that have been obtained for the one component case, mainly the works of M. Reid in [Re] and J. Jimenéz in [Ji], both use formal constructions. To get the analytic results that are desired, then, some way to toggle between the formal category and the analytic category had to be used. M. Artin in [Ar2] developed a way to do this. The main results of [Ar2] are dependent on the definition of *formal modification*, Definition 1.7. In particular, Artin shows that the existence of a formal modification implies the existence of an analytic contraction or a deformation, see Theorem 3.1 in [Ar2] and Theorem 6.2 in [Ar1]. So, in the formal constructions used in this thesis, a notion of a *formal cDV modification* and a *formal cDV contraction* is developed. To obtain analytic results from these definitions, then, it would suffice to show that these are equivalent to the definitions of Artin for these curves C. From the construction in this paper, and the definitions of formal cDV modification and formal cDV contraction that are formulated, it is determined that it is likely that a formal cDV modification is equivalent to the formal modification of Artin (See the conjecture at the end of this section). The significance of this conjecture being true is discussed at the end of this section also.

To motivate the definitions of *formal cDV modification* and *formal cDV contraction*, the formal methods used in answering each of these questions will be briefly described here.

Chapters 2 through 5 are devoted to answering question 1. As discussed in the introduction, it is known, see [Re], that if $f : X \to Y$ is an analytic contraction, then a general hyperplane section of the singular point q has q as a rational double point (RDP) or DuVal surface singularity. See section 1.5 for information concerning RDP's. As described in section 1.3, J. Jimenéz in [Ji] implemented an idea of J. Kollár to determine the length of the components in C by looking at sequences of defining ideals $\mathcal{I} \supset \mathcal{J}_2 \supset \cdots \supset \mathcal{J}_k$ of C. The construction of these ideals $\mathcal{J}_k$ result in a formal map $\hat{f} : \hat{X} \to \mathbf{C^4} \subset \hat{Y}$ for which $\hat{f}^{-1}(m_q) = \hat{\mathcal{J}}_k$ (see lemma 1.1) , where k depends on the length of the components. This map is defined by four global sections of $\hat{\mathcal{J}}_k$ that are lifted from the locally free sheaf $\mathcal{J}/\mathcal{J}^2$. A general section of m_q defines the formal DuVal singularity in a general hyperplane section of q in $\hat{Y}$, and $\hat{f}$ is a formal partial resolution of this singularity. This formal map, $\hat{f}$, is what will be called a *formal cDV modification.*

Definition 1.4 *A* **formal cDV modification** *consists of a map $\hat{f} : \hat{X} \to \hat{Y}$ of formal threefolds, with $\hat{X}$ supported on a curve C and $\hat{Y}$ supported on a point q, such that the general section $s \in m_q$ defines a formal Du Val surface singularity, while $\hat{f}^{-1}(s)$ defines a formal partial resolution.*

From [Ty], all DuVal surface singularities can be realized as the contraction of a curve in a threefold X, which can be viewed as a one parameter family of deformations of a hyperplane of C. See chapter 6 for a discussion of the versal deformations of RDP's and their simultaneous resolutions. The main point of this discussion is that, from a construction of Pinkham in [Pi], the versal deformation of DuVal singularities and their partial resolutions can be completely described by a map $\hat{\psi} : \mathrm{Spec}\mathbf{C}[[t]] \to \mathrm{Spec}\mathbf{C^n}[[t_1, \cdots, t_n]]$.

From the construction of the formal modification $\hat{f} : \hat{X} \to \hat{\mathbf{C}}^4 \subset \hat{Y}$ above, the one-parameter family of hypersurfaces $\{s = t\}$ gives a formal deformation of the singular surface, H, given by $s = 0$. Therefore, the inverse image of this family under $\hat{f}$ is a formal partial resolution of singularities. By the versal property of deformations of DuVal singularities and their partial resolutions, and the construction of Pinkham [Pi] described in chapter 6, the relationship of the versal family to this family given by $s \in m_p$ can be described. Let $\hat{\mathcal{X}}$ and $\hat{\mathcal{Z}}$ be the versal families of the formal deformations of H and the partial resolution $\hat{f}^{-1}(H)$, respectively. By the versal property, the formal threefolds, $\hat{X}$ and $\hat{Y}$, can be recovered from the spaces

$\hat{\mathcal{X}}$ and $\hat{\mathcal{Z}}$. In particular, there is a formal map $\omega : \text{Spec}\mathbf{C}[[t]] \to \text{Spec}\mathbf{C}^{\mathbf{n}}[[t_1, \cdots, t_n]]$ which defines a map of formal threefolds $\omega^*(\hat{\mathcal{Z}}) \to \omega^*(\hat{\mathcal{X}})$ that is isomorphic to the constructed map $\hat{f} : \hat{X} \to \hat{Y}$. So, as in the case of the versal deformations, determining if C contracts or components deform can be accomplished by looking at the discriminant locus in $\text{Spec}\mathbf{C}^{\mathbf{n}}[[t_1, \cdots, t_n]]$.

Definition 1.5 *$\hat{f} : \hat{X} \to \hat{Y}$ is a* **formal cDV contraction** *if $\hat{f} : \hat{X} \to \hat{Y}$ is a formal cDV modification such that the general section $s \in m_q$ defines a map $\omega : Spec\mathbf{C}[[t]] \to Spec\mathbf{C}^{\mathbf{n}}[[t_1, \cdots, t_n]]$ which does not factor through the inclusion of the discriminant locus in $Spec\mathbf{C}^{\mathbf{n}}[[t_1, \cdots, t_n]]$.*

As discussed in chapter 6, the curve C lies over the discriminant locus, and the components of C that deform, then, can be determined from the locally closed subsets of the discriminant locus. Over each subset is a flat family of deformations of some corresponding subset of C. So, if ω factors through the discriminant locus, then by looking at the locally closed subsets through which ω factors and pulling back the flat family of deformations to $\hat{\mathcal{X}}$ via ω^* gives the formal deformation.

If ω does not factor through the discriminant locus then, by definition, $\hat{f} : \hat{X} \to \hat{Y}$ is a formal cDV contraction. The following proposition has been proven:

Proposition 1.1 *If $\hat{f} : \hat{X} \to \hat{Y}$ is a formal cDV modification, then f is either a formal cDV contraction, or some component of C has a formal deformation.*

With these definitions, the results of this paper can now be stated. To answer both question the length of the components plays an essential role . Since the curve C is also studied in the formal threefold $\hat{X}$, the length of the components in $\hat{X}$ is also important.

Definition 1.6 *Let $\hat{f} : \hat{X} \to \hat{Y}$ be a formal cDV contraction. The* **formal length** *of the component $C_i \subset \hat{X}$ is the length of the formal scheme $\mathcal{O}_{\hat{X}}/\hat{f}^{-1}(m_{q,\hat{Y}})$ at a generic point of C_i.*

This means that the formal length of C_i is the multiplicity of the fiber of $\hat{f}$ over q. As mentioned above, the formal length is determined from the ideals $\mathcal{J}_k$ in the filtration of $\mathcal{I}$. Section 1.3 discusses the concept of length and formal length, and also compares the two.

To answer question 1, chapters 2-4 are divided up according to the lengths of the components of C. In each case a formal modification is constructed as defined above, and in this construction information is obtained about the formal neighborhood of C. From this, the singularity at q can be determined.

In chapter 2 the lengths of the components C_i are all assumed to be 1. The formal neighborhood of C can be determined from the sections of $\hat{\mathcal{I}}$. It is shown that a general section of m_q has a formal A_n singularity at q, where n is the number of components of C. Since it is assumed that an analytic contraction exists, this shows that an analytic hyperplane section of m_q has an A_n singularity at q. This proves, then, that if each component of C has length 1, then the length completely determines the singularity at q.

Similarly, analytic results are obtained in chapter 3 by first constructing a formal cDV modification.

We next consider the situation where one of the components of C has length 2. In particular, chapter 3 discusses the case where $C = C_1 \cup C_2$ with C_1 having length 2 and C_2 having length 1. In this situation a defining ideal $\mathcal{J}_2 \subset \mathcal{I}$ is used to determine the possible second order neighborhoods of C_1. There are two possible forms for $\mathcal{J}_2$, leading to two possible general hyperplane sections of q. A general section has either a D_4 singularity or a D_5 singularity at q.

Chapter 4, then, is the analysis of the case with $C = C_1 \cup C_2$ and C_1 and C_2 both have length 2. Here an ideal $\mathcal{J}_3 \subset \mathcal{J}_2 \subset \mathcal{I}$ is constructed giving 4 possible second order neighborhoods of C. Two of these four are discussed in detail, with results showing that a general hyperplane section of q has either a D_5 or a D_6 singularity at q.

In the work of Reid, [Re] and Katz and Morrison, [KM], with C a smooth rational curve, it was seen that the length of C completely determined the general hyperplane section of the singularity q. So, in the situation where C has more than one component, this work shows that the length of the components does not uniquely determine a general hyperplane section. The added local information provided by the defining ideals $\mathcal{J}_k$ is necessary to determine which rational double point results.

In 1993, as this work was in progress, Kawamata, in [Ka], also showed that the length of a smooth rational curve C completely determines the general hyperplane section. He accomplished this from a geometric approach to the blow-ups of rational double points. The work in this paper generalizes this approach to some curves $C = C_1 \cup .. \cup C_n$. If the C_i all have length 1, it is shown that a general hyperplane section (analytic) has q as an A_n singularity, confirming the formal results from Chapter 2. If $C = C_1 \cup C_2$ with C_1 of length 1 and C_2 of length 2, Kawamata's technique reduces the possibilities of the general hyperplane section, but it does not provide the added information necessary to determine the general hyperplane uniquely. In particular, Kawamata's method suffices in showing that the general hyperplane section has a singularity of type D_4 or D_5, but it does not provide a way of determining which singularity. Therefore, the formal construction of the defining ideals $\mathcal{J}_k$ appears to be necessary to determine the higher order neighborhoods of C and the resulting singularity from the contracting C.

Question 2 is discussed in chapter 6 in the case where each component of C has length 1. Knowing that if C contracts, it contracts to a compound A_n singularity, the deformation space of A_n singularities and their simultaneous resolutions is utilized. This chapter uses a method similar to that of Reid [Re] to construct a formal cDV modification. In particular, a sequence of defining ideals $\cdots \subset \mathcal{K}_m \subset \cdots \mathcal{K}_2 \subset \mathcal{I}$ is constructed where $\mathcal{K}_m/\mathcal{I}\mathcal{K}_m$ fits in an exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{K}_m/\mathcal{I}\mathcal{K}_m \longrightarrow \mathcal{O}_C(1,0,\cdots,0,1) \longrightarrow 0$$

for each m. See section 1.4 for the definition of $\mathcal{O}_C(1,0,\cdots,0,1)$. If this sequence splits, then the sequence $\mathcal{K}_m \subset \cdots \mathcal{K}_2 \subset \mathcal{I}$ can be extended to $\mathcal{K}_{m+1} \subset \mathcal{K}_m \subset \cdots \mathcal{K}_2 \subset \mathcal{I}$. By comparing the construction of this sequence with the versal deformation of an A_n singularity and the discriminant locus, as discussed above, it is shown that an infinite sequence of such ideals implies C deforms formally in $\hat{X}$. This argument can be applied to any component or union of components of C, so, since a cDV modification results, contraction criteria are also given. In particular, the following two theorems are proven:

Theorem 1.1 *C deforms formally in X if and only if there exists an infinite chain of subsheaves $\cdots \subset \mathcal{K}_{m+1} \subset \mathcal{K}_m \subset \cdots \subset \mathcal{K}_2 \subset \mathcal{I}$ such that $\mathcal{K}_m/\mathcal{K}_{m+1} \cong \mathcal{O}_C$ and $\mathcal{K}_{m+1}/\mathcal{I}\mathcal{K}_m \cong \omega_C^*$, where ω_C^* is the dual of the dualizing sheaf.*

Theorem 1.2 *A formal cDV contraction of C exists if and only if there is no infinite chain of subsheaves $\cdots \subset \mathcal{K}_{m+1} \subset \mathcal{K}_m \subset \cdots \subset \mathcal{K}_2 \subset \mathcal{I}_D$ such that $\mathcal{K}_m/\mathcal{K}_{m+1} \cong \mathcal{O}_D$ and $\mathcal{K}_{m+1}/\mathcal{I}\mathcal{K}_m \cong \omega_D^*$ for any $D = \cup_{j=i}^k C_j$ $(1 \leq i \leq k \leq n)$, where $\mathcal{I}_D$ is the ideal sheaf of D in X and ω_D^* is the dual of the dualizing sheaf.*

These theorems follow from the resolution of A_n singularities, but this work expresses these results in terms of the sequence of defining ideals $\mathcal{K}_m$ and it describes these ideals explicitly in local coordinates.

The main tool in answering both of the questions in this paper is the construction of a formal cDV modification, which is either a formal deformation or a formal cDV contraction, as defined above. The issue of whether this formal construction implies that C contracts or deforms in the analytic category is a very important one since it is in the analytic category where contraction criteria are most useful to the classification of threefolds. From Artin's results in [Ar2], this issue is equivalent to determining if the definition of cDV modification is the same as Artin's definition of formal modification. Proposition 1.1 proves that formal cDV modifications have the property of being either either formal cDV contractions or formal deformations. A formal modification has the property of being a formal contraction or formal deformation as well. This gives evidence that these definitions may be the same for the curves discussed in this paper. If they were, in fact, equivalent, then it could be concluded as in Jimenéz's corollary 3 of [Ji] that the existence of a formal cDV modification implies C contracts analytically or C moves in the analytic threefold X.
Conjecture If $\hat{f} : \hat{X} \to \hat{Y}$ is a formal cDV modification then $\hat{f}$ is a formal modification in the sense of Artin [Ar2] (Definition 1.7).

1.2 Formal Information

The definitions of formal contraction, deformation, and length were provided in the previous section. In this section some general definitions to help understand the formal techniques and results of this paper will be provided.

We have $C = \cup_{i=1}^{n} C_i$ a closed subscheme of the threefold X with $\mathcal{I}_C \subset \mathcal{O}_X$ the ideal sheaf of C in X defining the reduced structure of C. The following definitions can be found in [Ha2].

Definition 1.7 *The* **formal completion** *of X along C, denoted $\hat{X}$, is the curve C with the sheaf of rings $\mathcal{O}_{\hat{X}} = \varprojlim \mathcal{O}_X/\mathcal{I}^m$, where the inverse system arises from the natural maps $\cdots \mathcal{O}_X/\mathcal{I}^3 \to \mathcal{O}_X/\mathcal{I}^2 \to \mathcal{O}_X/\mathcal{I}$.*

A map of formal schemes, then, is a map that is compatible with the maps of the inverse system.

Definition 1.8 *A sheaf of ideals $\mathcal{L} \subset \mathcal{O}_{\hat{X}}$ is an* **ideal of definition** *for $\hat{X}$ if $Supp(\mathcal{O}_{\hat{X}}/\mathcal{L}) = \hat{X}$ and $(\hat{X}, \mathcal{O}_{\hat{X}}/\mathcal{L})$ is a Noetherian formal scheme.*

Definition 1.9 *If $\mathcal{L} \subset \mathcal{O}_X$ is a sheaf of ideals, then the* **completion of $\mathcal{L}$**, *denoted $\hat{\mathcal{L}}$, is $\varprojlim \mathcal{L}/\mathcal{L}^m$.*

In particular, $\hat{\mathcal{I}}_C$ is the unique largest ideal of definition for $\hat{X}$.

One of the hypotheses for the curve C is that each of its components, C_i, has a rational formal neighborhood in X. Let $\mathcal{I}_i$ be the ideal sheaf of C_i in X.

Definition 1.10 *C_i is said to have a* **rational formal neighborhood** *in X if* $H^1(\hat{X}, \mathcal{O}_{\hat{X}}) = 0$.

From the definitions above, this means that $H^1(C_i, \mathcal{O}_X/\mathcal{I}_i^m) = 0$ for all positive integers m. The significance of this assumption is discussed in section 1.4.

1.3 Length

As was discussed earlier, the length of the components, C_i, of C will be important in determining the type of singular point to which C will contract, assuming that C can be contracted. The length will also be important in determining the contractibility of the curve C. This section outlines an idea of Kollár that was used by Jiménez in [Ji] to construct a way of determining the length of a component from its defining ideal $\mathcal{I}_i$, and, furthermore, to construct a formal map $\hat{f} : \hat{X} \longrightarrow \hat{\mathbf{C}}^4$ whose central fiber is C. In the case of C having one component, Jimenéz claims this map construction was enough to show that C could either be contracted or it moved in the analytic threefold X.

Using the hypotheses as stated in section 1.1, we will first set up some notation for the length of curves with multiple components. If the component C_i has length a_i for each i, then it will be written that the curve C has length$(a_1, a_2, \cdots, a_i, \cdots a_n)$. For example, as seen in the table of contents, the length(2,1) case is where C has two components with C_1 having length 2 and C_2 length 1.

Assume there is a finite sequence of defining ideals, $\mathcal{I} = \mathcal{J}_1 \supset \mathcal{J}_2 \supset \cdots \supset \mathcal{J}_k$, such that $\mathcal{J}_l/\mathcal{J}_{l+1} \cong \mathcal{O}_{m(l)}(-1)$ for all $1 \leq l \leq k-1$ and some $m(l) \in \{1, 2, \cdots, n\}$. The significance of all quotients being $\mathcal{O}_{m(l)}(-1)$ for some $m(l)$ is that $H^0(\mathcal{J}_k) = H^0(\mathcal{I})$. Furthermore, assume $\mathcal{J}_k/\mathcal{J}_k^2$ is generated by global sections, call them $\{f_1, \cdots, f_N\}$, and $H^0(\hat{\mathcal{J}}_k) \longrightarrow H^0(\mathcal{J}_k/\mathcal{J}_k^2)$ is surjective. With these assumptions, the sections f_i can be lifted to sections, again denote them f_i, of $H^0(\hat{\mathcal{J}}_k)$. These sections define a formal map $\hat{f} : \hat{X} \longrightarrow \hat{\mathbf{C}}^N$. We also have the following lemma:

Lemma 1.1 $\hat{f}^{-1}(m_0) = \hat{\mathcal{J}}_k$.

Proof: $\hat{f}$ is defined by sections of $\hat{\mathcal{J}}_k$ coming from $\mathcal{J}_k/\mathcal{J}_k^2$, so $\hat{f}^{-1}(m_0) = \hat{\mathcal{L}}$ for some ideal $\hat{\mathcal{L}}$ that is congruent to $\mathcal{J}_k$ mod $\mathcal{J}_k^2$. That is, $\mathcal{L}/\mathcal{J}_k^2 \cong \mathcal{J}_k/\mathcal{J}_k^2$. $\hat{\mathcal{J}}_k$ is an ideal of definition, so for any point $p \in C$, $\mathcal{J}_k \subset m_p$, where m_p is the maximal ideal at p. And, since $\mathcal{J}_k/\mathcal{J}_k^2$ is generated by global sections, the map $H^0(\mathcal{J}_k/\mathcal{J}_k^2) \longrightarrow H^0(\mathcal{J}_k/m_p\mathcal{J}_k)$ is surjective for all $p \in C$. Therefore, $H^0(\mathcal{L}/\mathcal{J}_k^2) \longrightarrow H^0(\mathcal{J}_k/m_p\mathcal{J}_k)$ is surjective for all p also. This means that $\mathcal{J}_k/m_p\mathcal{J}_k \cong \mathcal{L}/m_p\mathcal{J}_k$ for all p. By Nakayama's Lemma, [Ma], pg. 11, $\hat{\mathcal{L}} = \hat{\mathcal{J}}_k$.

□

Therefore, if such a sequence of ideals exists with the above hypotheses holding, the central fiber of $\hat{f}$ is the curve C, and $\hat{f}$ is either a formal contraction or a formal deformation of C. In each of chapters 2, 3 and 4 it is shown that such a map can be constructed.

Assuming we have such a map, by definition, the formal length of C_i is the length of $\mathcal{O}_{\hat{X}}/\hat{\mathcal{J}}_k$ at a generic point of C_i. For $p \in C_i$ and $p \notin C_j$ for $j \neq i$ we have $\mathcal{J}_l = \mathcal{J}_{l+1}$ unless $\mathcal{J}_l/\mathcal{J}_{l+1} \cong \mathcal{O}_i(-1)$. So, there is a subsequence $\mathcal{I} = \mathcal{J}_{l_1} \supset \cdots \supset \mathcal{J}_{l_j}$ such that $\mathcal{J}_{l_t}/\mathcal{J}_{l_{t+1}} \cong \mathcal{O}_i(-1)$. At p, then, the sequence $\mathcal{I}_p = \mathcal{J}_{l_{1,p}} \supset \cdots \supset \mathcal{J}_{l_j,p}$ is such that $\mathcal{J}_{l_t,p}/\mathcal{J}_{l_{t+1},p} \cong \mathcal{O}_{i,p}$. The length of $\mathcal{O}_{\hat{X}}/\hat{\mathcal{J}}_k$, therefore, is the same as the length of $\mathcal{O}_{\hat{X}}/\hat{\mathcal{J}}_{l_j}$ and, by definition, the length of $\mathcal{O}_{\hat{X}}/\hat{\mathcal{J}}_{l_j}$ is l_j. In conclusion, this means that if we have a sequence $\mathcal{I} = \mathcal{J}_1 \supset \cdots \supset \mathcal{J}_l$ and there is an ideal $\mathcal{J}_{l+1} \subset \mathcal{J}_l$ such that $\mathcal{J}_l/\mathcal{J}_{l+1} \cong \mathcal{O}_i(-1)$, then the formal length of the component C_i increases by 1.

So, given the curve C, to determine the formal length of its components, C_i, it needs to be seen whether such a sequence with -1 quotients can be constructed. The method of determining the length of C_i in this paper will be, as in [Ji], by projecting to the $\mathcal{O}_i(-1)$ factors of $\mathcal{I}/\mathcal{I}^2|_{C_i} = \mathcal{O}_{C_i}(a_i) \oplus \mathcal{O}_{C_i}(b_i)$, if one exists. It will be shown in section 1.4 that $a_i, b_i \geq -1$ with the hypotheses stated in section 1.1. If a_i (or b_i) is -1, define $\mathcal{J} = Ker(\mathcal{I} \longrightarrow \mathcal{I}/\mathcal{I}_i\mathcal{I} \longrightarrow \mathcal{O}_i(-1))$, where this notation means $\mathcal{J}$ is the kernel of the composition of the two maps. From this definition, $\mathcal{J}$ is a sheaf satisfying $\mathcal{I} \supset \mathcal{J}$ and $\mathcal{I}/\mathcal{J} \cong \mathcal{O}_i(-1)$, so from the above discussion of length, the length of C_i is at least 2. This process of projecting to -1 factors will be utilized in chapters 3 and 4 to produce curves with length 2 components.

Assume now that C contracts. So, there is an analytic contraction map $g : X \to \mathbf{C}^4$ with $g^{-1}(m_q)$ an ideal of definition of C. We compare this to the formal map $\hat{f} : \hat{X} \to \hat{\mathbf{C}}^4$, described above, to establish the relationship between the length and the formal length of the components of C.

Lemma 1.2 $g^{-1}(m_q) \subset \mathcal{J}_k$.

Proof: By construction of the sequence of ideals, $\mathcal{J}_k \subset \mathcal{J}_{k-1} \subset \cdots \subset \mathcal{J}_2 \subset \mathcal{I}$, all quotients are $\mathcal{O}_i(-1)$ for some $i \in \{1, 2, \cdots, n\}$. Therefore, $H^0(\mathcal{I}) \subset H^0(\mathcal{J}_k)$. $g^{-1}(m_q)$ is an ideal of definition that is generated by global sections, so all global sections of $g^{-1}(m_q)$ are global sections of $\mathcal{I}$. But $H^0(\mathcal{I}) \subset H^0(\mathcal{J}_k)$, so $g^{-1}(m_q) \subset \mathcal{J}_k$.

□

Corollary 1.1 *The length of the component C_i of C is greater than or equal to the formal length of C_i for each $i \in \{1, 2, \cdots, n\}$.*

Proof: The formal length of C_i is the multiplicity of $\mathcal{J}_k$ at a generic point, and the length of C_i is the multiplicity of $g^{-1}(m_q)$ at a generic point of C_i. Since $g^{-1}(m_q) \subset \mathcal{J}_k$ by the previous lemma, the multiplicity of $g^{-1}(m_q)$ is greater than or equal to the multiplicity of $\mathcal{J}_k$.

1.4 Locally free sheaves on C

The main tools in answering both of the questions posed in section 1.1 are locally free sheaves on C and its components C_i. A locally free sheaf of rank 1 is called an *invertible* sheaf. An invertible sheaf on $C_i \cong \mathbf{P}^1$ is completely determined by its degree, a_i, and will be denoted $\mathcal{O}_i(a_i)$. A locally free rank 2 sheaf, $\mathcal{E}$, on C_i decomposes as the direct sum of invertible sheaves and, so, can be written $\mathcal{E} \cong \mathcal{O}_i(a_i) \oplus \mathcal{O}_i(b_i)$. If there is no confusion as to which component of C is being discussed, this will be expressed more briefly as $\mathcal{E} = (a_i, b_i)$. In general, a locally free sheaf of rank m on C_i can be decomposed as $\mathcal{O}_i(a_{i_1}) \oplus \cdots \oplus \mathcal{O}_i(a_{i_m})$. This will be expressed as $\sum_{j=1}^{m} \mathcal{O}_i(a_{i_j})$, and if $a_{i_j} = a_{i_k} = a$ for all $1 \leq j, k \leq m$ this sum will be denoted $\mathcal{O}_i(a)^{\oplus m}$. Similarly, the

tensor product of m copies of a sheaf $\mathcal{F}$ (not necessarily locally free) will be denoted $\mathcal{F}^{\otimes m}$. It is from the well known properties of locally free sheaves on the C_i that information will be obtained for locally free sheaves on C.

Invertible sheaves on C are classified by the cohomology group $H^1(C, \mathcal{O}_C^*)$, where $\mathcal{O}_C^*$ is the sheaf of invertible elements in $\mathcal{O}_C$. From the exponential exact sequence

$$0 \longrightarrow Z \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_C^* \longrightarrow 0,$$

we have the long exact cohomology sequence

$$\cdots H^1(\mathcal{O}_C) \to H^1(\mathcal{O}_C^*) \to H^2(Z) \to H^2(\mathcal{O}_C) \to \cdots.$$

C is a rational curve so $H^1(\mathcal{O}_C) = 0$, and since C has dimension 1, the group $H^2(\mathcal{O}_C)$ is also 0. Therefore, $H^1(\mathcal{O}_C^*) \cong H^2(Z)$. From [BPV], $H^2(Z) = \bigoplus H^2(C_i, Z) = \oplus Z$, which is the free group with one generator for each component of C. Invertible sheaves on C, then, are completely determined by invertible sheaves on its components. So, if $\mathcal{L}$ is an invertible sheaf on C, $\mathcal{L}|_{C_i} \cong \mathcal{O}_i(a_i)$ for each i, and two invertible sheaves $\mathcal{L}$ and $\mathcal{L}'$ are isomorphic if and only if $a_i = a_i'$ for all i. Invertible sheaves on C, then, will be denoted $\mathcal{O}(a_1, a_2, \cdots, a_n)$, where a_i is the degree of the sheaf on the component C_i. This argument holds for any such union of smooth rational curves, so an invertible sheaf on the curve $\bigcup_{i \neq j} C_i$ is uniquely determined by the degree of each of its components. Invertible sheaves on such a union will be denoted $\mathcal{O}(a_1, \cdots, a_{j-1}, \hat{a}_j, a_{j+1}, \cdots a_n)$.

The rank 2 sheaves called *conormal* sheaves, in particular, will be utilized extensively. Let $\mathcal{I}$ be the ideal sheaf of C in X and $\mathcal{I}_i$ the ideal sheaf of C_i in X. $\mathcal{I}$ and $\mathcal{I}_i$ will will define the reduced structure on C and C_i, respectively. Locally these ideals can be defined by their generators. If $\{x, y, z\}$ are the local coordinates at the point $p' = (0,0,0)$ on C, then coordinates can be chosen so that $\mathcal{I}_i$ is generated by $\{x, z\}$ at p' if p' lies on the component C_i. If p' is not a point of intersection of two components, then $\mathcal{I} = \mathcal{I}_i = (x, z)$. If p' is the point of intersection of C_i and C_j, then coordinates can be chosen so that $\mathcal{I}_i = (x, z)$, $\mathcal{I}_j = (y, z)$ and $\mathcal{I} = (xy, z)$. C and C_i for all i are local complete intersections in X with two generators at every point, so the respective conormal sheaves $\mathcal{I}/\mathcal{I}^2$ and $\mathcal{I}_i/\mathcal{I}_i{}^2$ are locally free sheaves of rank 2. The hypotheses stated in the first paragraph of this paper now will be used to give us a global description of the conormal sheaves $\mathcal{I}_i/\mathcal{I}_i^2$. In particular, $\mathcal{I}_i/\mathcal{I}_i^2 = (a_i, b_i)$ with degree $a_i + b_i$.

The hypotheses stated in section 1.1 put restrictions on the possible values of the integers a_i and b_i. The adjunction formula, [Ha2], pg. 361, shows that $-2 = C_i^2 + \mathcal{K}_X \cdot C_i$ for each i. But $C_i^2 = deg(\mathcal{N}_{C_i/X}) = deg((\mathcal{I}_i/\mathcal{I}_i^2)^*) = -(a_i + b_i)$, where $*$ denotes the dual sheaf. So, by the assumption that $\mathcal{K}_X \cdot C_i = 0$, we have $a_i + b_i = 2$.

The assumption that each C_i has a rational formal neighborhood in X gives further restrictions on a_i and b_i. The details of this restriction process, which is outlined here, can be found in [Pi], pgs. 363-367. By the definition of rational formal neighborhood, given in section 1.2, we have $H^1(C_i, \mathcal{O}_X/\mathcal{I}_i^m) = 0$ for all positive integers m. From the exact sequence

$$0 \longrightarrow \mathcal{I}_i/\mathcal{I}_i{}^2 \longrightarrow \mathcal{O}_X/\mathcal{I}_i^2 \longrightarrow \mathcal{O}_X/\mathcal{I}_i \longrightarrow 0,$$

we have the long exact cohomology sequence

$$\cdots H^0(\mathcal{O}_X/\mathcal{I}_i^2) \to H^0(\mathcal{O}_X/\mathcal{I}_i) \to H^1(\mathcal{I}_i/\mathcal{I}_i{}^2) \to H^1(\mathcal{O}_X/\mathcal{I}_i^2) \to \cdots.$$

The map on global sections $H^0(\mathcal{O}_X/\mathcal{I}_i^2) \to H^0(\mathcal{O}_X/\mathcal{I}_i)$ is surjective, as it takes the element 1 to the generator 1. $\mathcal{O}_X/\mathcal{I}_i = \mathcal{O}_i$, so the vector space $H^0(\mathcal{O}_X/\mathcal{I}_i)$ is one dimensional. This, together with the fact that $H^1(\mathcal{O}_X/\mathcal{I}_i^2) = 0$ proves that $H^1(\mathcal{I}_i/\mathcal{I}_i{}^2) = 0$. Therefore, the integers $a_i \geq -1$ and $b_i \geq -1$. The only two such integers satisfying $a_i + b_i = 2$, up to order, are the pairs $(a_i, b_i) = (1,1), (0,2)$ or $(-1,3)$. This shows that the conormal bundle of each component of C decomposes as $(1,1), (0,2)$ or $(-1,3)$.

Similarly, the assumption that C_i has a rational formal neighborhood in X implies that $H^1(C_i, \mathcal{O}_X/\mathcal{I}_i\mathcal{I}) = 0$, since $\mathcal{O}_X/\mathcal{I}_i\mathcal{I}$ is supported on C_i. Using the sequence

$$0 \longrightarrow \mathcal{I}/\mathcal{I}_i\mathcal{I} \longrightarrow \mathcal{O}_X/\mathcal{I}_i\mathcal{I} \longrightarrow \mathcal{O}_X/\mathcal{I} \longrightarrow 0$$

and arguing as above, $H^1(\mathcal{I}/\mathcal{I}_i\mathcal{I}) = 0$ and so $\mathcal{I}/\mathcal{I}_i\mathcal{I}$ has no factors $a_j < -1$.

1.5 Rational double points

The final preliminary results will concern rational double points, which will be necessary in answering questions 1 and 2. The following discussion of rational double points can be found in [BPV].

Definition 1.11 *Let S be a surface containing the singular point q. If $\mu : S' \to S$ is a resolution of q with exceptional set $C' = \cup_{j=1}^n C'_j$, with C'_j smooth, rational and satisfying $\mathcal{K}_{S'} \cdot C'_j = 0$ for all j, then q is called a* **rational double point (RDP)** *or a* **DuVal surface singularity**.

Definition 1.12 *A* **compound DuVal(cDV) singularity** *is a threefold singularity such that the general hyperplane section of this singularity has this point as a DuVal singularity.*

From Grauert's criterion, [Gr] pg. 367, the intersection matrix of the C'_j in S' must be negative definite if C' is exceptional. This implies that any two components of C' can intersect in at most one point, and if two components do intersect, they intersect transversally. Let Γ be the graph with the curves C'_j representing its vertices, and two vertices are joined by a line segment if the two corresponding curves intersect. Γ, then, must be one of the graphs A_n with $n \geq 0$, D_n with $n \geq 4$, E_6, E_7 or E_8 as shown

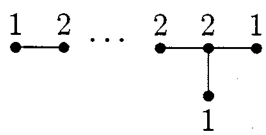
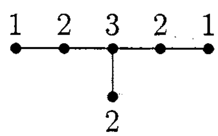
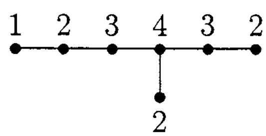
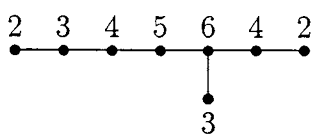
<table>
<tr><th>Γ</th><th>Dynkin Diagram</th><th>Labeling of Curves</th><th>Polynomial</th></tr>
<tr><td>A_n</td><td></td><td>$C'_1 - \cdots - C'_n$</td><td>$xy + z^{n+1}$</td></tr>
<tr><td>D_n</td><td></td><td>$C'_1 - \cdots - C'_{n-2} - C'_{n-1}$
$\vert$
C'_n</td><td>$x^2 + y^2 z + z^{n-1}$</td></tr>
<tr><td>E_6</td><td></td><td>$C'_1 - C'_2 - C'_3 - C'_4 - C'_5$
$\vert$
C'_6</td><td>$x^2 + y^3 + z^4$</td></tr>
<tr><td>E_7</td><td></td><td>$C'_1 - \cdots - C'_4 - C'_5 - C'_6$
$\vert$
C'_7</td><td>$x^2 + y^3 + yz^3$</td></tr>
<tr><td>E_8</td><td></td><td>$C'_1 - \cdots - C'_5 - C'_6 - C'_7$
$\vert$
C'_8</td><td>$x^2 + y^3 + z^5$</td></tr>
</table>

Table 1.1: Rational Double Points

in Table 1.1. Γ is called the *dual graph* of C'. It is also said that q is a *singularity of type* A_n, D_n, E_6, E_7 or E_8, [BPV] pg. 74. There is a "minimal" effective divisor $F = \sum_{j=1}^{n} m_j C'_j$ satisfying $F \cdot C'_j \leq 0$ for all j. F is called the *fundamental cycle of* q. The positive integer m_j is called the *multiplicity* of C'_j. The value of m_j in each case is next to the vertex representing C'_j in Table 1.1. See [BPV], section III.3.

Remark: The length of the component C_i of C coincides with its multiplicity in the fundamental cycle.

It is possible to identify rational double points from the following criterion [BPV], III.2.4.

Proposition 1.2 *Given a compact, reduced, connected curve D in a smooth surface S', if $\mu : S' \to S$ contracts D to the point q in the surface S and $\mathcal{K}_{S'} \cdot D_i = 0$ for all irreducible components D_i of D, then q is a RDP of type determined by the dual graph of D.*

The following theorem due to Reid in [Re] explains the relationship between the threefold singularity q resulting from a contraction f and rational double points.

Theorem 1.3 *Let $f : X \to Y$ be a resolution of the threefold singularity q with exceptional set C of pure dimension 1. If H is a general hyperplane section of q in Y, then H has a rational double point at q. Furthermore, the surface $f^*H = L$ is a normal surface in X containing the curve C, and the induced map $f_H : L \to H$ is a factor of the minimal resolution $g : M \to H$.*

This theorem states that the surface L is a partial resolution of the singularity $q \in H$, and the components, C_i, of C have length corresponding to their multiplicity in the fundamental cycle in M. The type of rational double point q is in H, then, is determined by resolving the singularities of L that lie on C and observing the dual graph of the resulting curve in M.

In each of chapters 2, 3 and 4 the pullback of a general hyperplane section of q is calculated locally. This allows the resolution of the singularities on C in the pullback to be resolved explicitly in coordinates and, therefore, the singularity can be determined. These calculations are made from the local description of the sequence of defining ideals $\mathcal{I} = \mathcal{J}_1 \supset \cdots \supset \mathcal{J}_k$ that is constructed as in the previous discussion of length. It was seen that the ideal $\mathcal{J}_k$ contains all the global sections of $\mathcal{I}$, so it is the general section of $\mathcal{J}_k$ that determines the pullback of a general hyperplane section of q. The singularities on C are calculated in these coordinates and can be determined by comparing with the polynomials in table 1.1. The resolution of these singularities gives the minimal resolution and, therefore, a specific Dynkin diagram, as in table 1.1, that distinguishes the type of rational point q is in H.

CHAPTER 2

THE LENGTH(1,1,···,1) CASE

This chapter will establish conditions on the curve C that will assure that if C contracts, then C contracts to a singularity q whose general hyperplane section has a RDP of type A_n at q, where n is the number of components of C. For C to contract to an A_n singularity in this hyperplane, the minimal resolution of q must have components all of multiplicity 1 in the fundamental cycle. The multiplicity coincides with the length of the component, which is invariant. Therefore, the length of each component of C must have length 1. It is necessarily assumed, then, that $\mathcal{I}_i/\mathcal{I}_i{}^2 = (1,1)$ or $(0,2)$ for all i, because if $\mathcal{I}_i/\mathcal{I}_i{}^2 = (-1,3)$ then the length of C_i is greater than or equal to two, [CKM], page 95.

Recall that the defining ideal sheaves for these curves can be defined in local coordinates $\{x,y,z\}$ at the point of intersection $p = (0,0,0)$ of C_i and C_j by $\mathcal{I}_i = (x,z)$, $\mathcal{I}_j = (y,z)$ and $\mathcal{I} = (xy,z)$. Being the restriction of the conormal sheaf on C, the sheaf $\mathcal{I}/\mathcal{I}^2|_{C_i} = \mathcal{I}/\mathcal{I}_i\mathcal{I}$, then, is locally free of rank 2 on C_i. These sheaves, therefore, have local generators at each point. From the generators of the ideal sheaves, it can be seen that $\mathcal{I}/\mathcal{I}^2$ and $\mathcal{I}/\mathcal{I}_i\mathcal{I}$ are generated by $\{xy,z\}$ at a point of intersection. $\mathcal{I}_i/\mathcal{I}_i{}^2$ and $\mathcal{I}_j/\mathcal{I}_j{}^2$ are generated by $\{x,z\}$ and $\{y,z\}$, respectively. Since $\mathcal{I}/\mathcal{I}_i\mathcal{I}$ is locally free of rank 2 on C_i, it is of the form (a,b) for integers a and b.

Lemma 2.1

$$\mathcal{I}/\mathcal{I}_i\mathcal{I} = \begin{cases} \textit{(0,0), (-1,1), or (-2,2)} & \textit{if } \ 2 \le i \le n-1 \\ \textit{(0,1) or (-1,2)} & \textit{if } \ i = \textit{1 or } n \end{cases}$$

Proof: The inclusion map $\mathcal{I}/\mathcal{I}_i\mathcal{I} \hookrightarrow \mathcal{I}_i/\mathcal{I}_i{}^2$ is well defined since $\mathcal{I} \subset \mathcal{I}_i$ for each i. The injection follows since $\mathcal{I} \bigcap \mathcal{I}_i{}^2 = \mathcal{I}_i\mathcal{I}$. This can be seen from a local calculation at each point. At a point $p \in C_i$ that is not a point of intersection we have $\mathcal{I} = \mathcal{I}_i$, so the equality is clear. If p is a point of intersection, from the local coordinates this intersection is $(xy,z) \cap (x,z)^2 = (xy,z) \cap (x^2,xz,z^2) = (x^2y,xyz,xyz^2,x^2z,xz,z^2) = (x^2y,xz,z^2)$, and this is the local product $(xy,z)(x,z)$ in $\mathcal{I}_i\mathcal{I}$.

If $p \in C_i$ is not a point of intersection, then $\mathcal{I} = \mathcal{I}_i$ near p. Therefore, the inclusion map is an isomorphism near any smooth point.

At a point of intersection, the map on generators is given by $xy \mapsto y \cdot x$, $z \mapsto z$, with y a local coordinate on C_i. The determinant of this map, $\mathcal{I}/\mathcal{I}_i\mathcal{I} \hookrightarrow \mathcal{I}_i/\mathcal{I}_i{}^2$, then, vanishes to order one at each point of intersection.

If C_i is not an end, i.e. $2 \leq i \leq n-1$,there are two points of intersection, so the degree of $\mathcal{I}/\mathcal{I}_i\mathcal{I}$ is two less than that of $\mathcal{I}_i/\mathcal{I}_i^2$. Since $\mathcal{I}_i/\mathcal{I}_i^2$ has degree two, $a+b=0$, and because it injects into $\mathcal{I}_i/\mathcal{I}_i^2$, $a, b \leq 2$. Therefore, $\mathcal{I}/\mathcal{I}_i\mathcal{I} = (0,0)$, $(-1,1)$, or $(-2,2)$. If C_i is an end, i.e $i = 1$ or n, there is just one point of intersection, so $a+b=1$. So, $\mathcal{I}/\mathcal{I}_i\mathcal{I} = (0,1)$ or $(-1,2)$.

□

As explained in section 1.4, $\mathcal{I}/\mathcal{I}_1\mathcal{I}$ cannot decompose as $(-2,2)$ since C_1 has a rational formal neighborhood. Furthermore, as discussed in section 1.3, if each component of C has length 1, then it is necessary that $\mathcal{I}/\mathcal{I}_i\mathcal{I} = (0,0)$ for $2 \leq i \leq n-1$ and is $(0,1)$ for $i = 1, n$. For the remainder of this chapter, then, this will be the case.

It is from the exact sequence of sheaves

$$0 \longrightarrow \mathcal{I}_i\mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{I}/\mathcal{I}_i\mathcal{I} \longrightarrow 0 \qquad (2.1)$$

and its long exact cohomology sequence that we will begin to establish properties of the conormal sheaf on C.

The sequence

$$0 \longrightarrow \mathcal{I}_i \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_i \longrightarrow 0$$

is exact, and tensoring with the local ring $\mathcal{O}_{X,p}$ for any $p \in C$, the resulting sequence

$$0 \longrightarrow \mathcal{I}_{i,p} \longrightarrow \mathcal{O}_{X,p} \longrightarrow \mathcal{O}_i \otimes \mathcal{O}_{X,p} \longrightarrow 0$$

remains exact. If p is not a point of the curve C_i, then $\mathcal{O}_i \otimes \mathcal{O}_{X,p} = 0$ since it is supported only at the point p. So, in this case, $\mathcal{I}_{i,p} \cong \mathcal{O}_{X,p}$. This shows that off of the component C_i, the ideal sheaf $\mathcal{I}_i$ is the trivial sheaf $\mathcal{O}_X$. This result will be used throughout the remainder of this paper.

Lemma 2.2 *$\mathcal{I}_i\mathcal{I}/\mathcal{I}^2$ is locally free of rank 2 on $\cup_{j\neq i}C_j$.*

Proof: Let $p \in \cup_{j\neq i}C_j$.

If p is not a point of C_i, then $\mathcal{I}_i = \mathcal{O}_X$. Therefore, this sheaf is isomorphic to $\mathcal{I}/\mathcal{I}^2$ off of C_i, and so is locally free here.

For $p \in C_i$, $p \in C_i \cap C_{i-1}$ or $p \in C_i \cap C_{i+1}$.

If $p \in C_i \cap C_{i-1}$, local coordinates can be chosen so that $\mathcal{I} = (xy, z)$, $\mathcal{I}_i = (x, z)$ and $\mathcal{I}_{i-1} = (y, z)$. In these coordinates $\mathcal{I}_i\mathcal{I} = (x^2y, xz, z^2)$ and $\mathcal{I}^2 = (x^2y^2, xyz, z^2)$, so the sheaf $\mathcal{I}_i\mathcal{I}/\mathcal{I}^2$ is generated by $\{x^2y, xz\}$ at p. Define a map $\mathcal{O}_{i-1} \oplus \mathcal{O}_{i-1} \to \mathcal{I}_i\mathcal{I}/\mathcal{I}^2$ by sending $(f, g) \mapsto f \cdot x^2y + g \cdot xz$. This takes the generators $(1, 0)$ and $(0, 1)$ of $\mathcal{O}_{i-1} \oplus \mathcal{O}_{i-1}$ to the generators x^2y and xz, respectively, of $\mathcal{I}_i\mathcal{I}/\mathcal{I}^2$, so this map is surjective. It is injective because $f \cdot x^2y + g \cdot xz \in \mathcal{I}^2$ implies that y and/or z must divide both f and g. That is, if $(f, g) \mapsto 0$ then $f, g \in \mathcal{I}_{i-1}$. Therefore, this map is an isomorphism, showing that $\mathcal{I}_i\mathcal{I}/\mathcal{I}^2$ is locally free at p.

For $p \in C_i \cap C_{i+1}$, local coordinates can be chosen such that $\mathcal{I} = (xy, z)$, $\mathcal{I}_i = (x, z)$ and $\mathcal{I}_{i+1} = (y, z)$, so the exact argument works by replacing $i - 1$ with $i + 1$.

□

The sheaf $\mathcal{I}_i\mathcal{I}/\mathcal{I}^2$ being locally free of rank 2 on $\bigcup_{j \neq i} C_j$ implies that its restriction to the components C_j, $j \neq i$, is locally free of rank 2 on C_j. Therefore, it decomposes as (a, b) on C_j, and the possible values of a and b can be calculated. This sheaf $\mathcal{I}_i\mathcal{I}/\mathcal{I}^2|_{C_j} = \mathcal{I}_i\mathcal{I}/(\mathcal{I}_i\mathcal{I}_j\mathcal{I} + \mathcal{I}^2)$ is generated locally at a point of intersection by $\{x^2y, xz\}$, as discussed in the proof of the previous lemma.

Lemma 2.3

$$\mathcal{I}_i\mathcal{I}/\mathcal{I}^2|_{C_j} = \begin{cases} (-1,-1) & \textit{if } |i-j| = 1 \textit{ and } 2 \leq j \leq n-1 \\ (-1,0) & \textit{if } |i-j| = 1 \textit{ and } j = 1 \textit{ or } n \\ (0,0) & \textit{if } |i-j| \neq 1 \textit{ and } 2 \leq j \leq n-1 \\ (0,1) & \textit{if } |i-j| \neq 1 \textit{ and } j = 1 \textit{ or } n \end{cases}$$

Proof: The global injection $\mathcal{I}_i\mathcal{I}/\mathcal{I}^2|_{C_j} \hookrightarrow \mathcal{I}/\mathcal{I}_j\mathcal{I}$ is well defined since $\mathcal{I}_i\mathcal{I} \bigcap \mathcal{I}_j\mathcal{I} = \mathcal{I}_i\mathcal{I}_j\mathcal{I} + \mathcal{I}^2$. The equality of these sheaves can be calculated in local coordinates at every point p of C_j. If p is not a point of intersection of C_i and C_j, then $\mathcal{I}_i = \mathcal{O}_X$, so $\mathcal{I}_i\mathcal{I} \bigcap \mathcal{I}_j\mathcal{I} = \mathcal{I}_j\mathcal{I}$ and $\mathcal{I}_i\mathcal{I}_j\mathcal{I} + \mathcal{I}^2 = \mathcal{I}_j\mathcal{I}$. If $p = C_i \bigcap C_j$, then local coordinates can be chosen so that $\mathcal{I}_i = (x, z)$, $\mathcal{I}_j = (y, z)$ and $\mathcal{I} = (xy, z)$. We have, then, that $\mathcal{I}_i\mathcal{I}_j \subset \mathcal{I}$, so $\mathcal{I}_i\mathcal{I}_j\mathcal{I} + \mathcal{I}^2 = \mathcal{I}^2$. Also, from these coordinates, $\mathcal{I}_i\mathcal{I} \bigcap \mathcal{I}_j\mathcal{I} = (x^2y, xz, z^2) \cap (xy^2, yz, z^2) = (x^2y^2, xyz, z^2) = \mathcal{I}^2$.

The decomposition of $\mathcal{I}_i\mathcal{I}/\mathcal{I}^2|_{C_j}$ will be determined from the vanishing of this injection at every point of C_j.

Case 1: If $C_i \cap C_j = \emptyset$, then $\mathcal{I}_i = \mathcal{O}_X$ near any point $p \in C_j$. So, $\mathcal{I}_i\mathcal{I}/\mathcal{I}^2|_{C_j} = \mathcal{I}/\mathcal{I}_j\mathcal{I}$ near p. This gives the last two results in the list above.

Case 2: If $C_i \bigcap C_j \neq \emptyset$: If $p \in C_j$ is not a point of intersection, then $\mathcal{I}_i = \mathcal{O}_X$ near p, so the inclusion map is an isomorphism. If $p = C_i \bigcap C_j$, then the map in terms of generators near p is given by $x^2y \mapsto x \cdot xy$ and $xz \mapsto x \cdot z$ with x a local coordinate on C_j. Since x is a local coordinate on C_j, this shows that $\mathcal{I}_i\mathcal{I}/\mathcal{I}^2|_{C_j} \cong \mathcal{I}_{j,p} \otimes \mathcal{I}/\mathcal{I}_j\mathcal{I}$ at p. But p is a divisor on C_j, so $\mathcal{I}_{j,p} \cong \mathcal{O}_j(-p) \cong \mathcal{O}_j(-1)$. From the assumption that $\mathcal{I}/\mathcal{I}_j\mathcal{I} = (0,1)$ at an end and $(0,0)$ otherwise, this gives the first two results in the list above.

□

From the exact sequence 2.1 above, we are able to begin the process of understanding the global sections of each of these sheaves. In fact, from the decomposition of the right term $\mathcal{I}/\mathcal{I}_i\mathcal{I}$, its cohomology groups $H^0(\mathcal{I}/\mathcal{I}_i\mathcal{I})$ and $H^1(\mathcal{I}/\mathcal{I}_i\mathcal{I})$ are known. So now, the sheaf $\mathcal{I}_i\mathcal{I}/\mathcal{I}^2$ is the target of interest.

We now have the exact sequence of sheaves

$$0 \longrightarrow (\mathcal{I}_i\mathcal{I}_j\mathcal{I} + \mathcal{I}^2)/\mathcal{I}^2 \longrightarrow \mathcal{I}_i\mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{I}_i\mathcal{I}/\mathcal{I}^2|_{C_j} \longrightarrow 0$$

Lemma 2.4 $(\mathcal{I}_i\mathcal{I}_j\mathcal{I} + \mathcal{I}^2)/\mathcal{I}^2$ *is locally free of rank 2 on* $\bigcup_{l \neq i,j} C_l$.

Proof:Let $p \in C_l$.

If p is not a point of intersection of C_i or C_j with C_l, then $\mathcal{I}_i = \mathcal{I}_j = \mathcal{O}_X$. $(\mathcal{I}_i\mathcal{I}_j\mathcal{I} + \mathcal{I}^2)/\mathcal{I}^2$, then, is $\mathcal{I}/\mathcal{I}^2$, which is locally free at p.

If $p = C_i \cap C_l$, then $\mathcal{I}_j = \mathcal{O}_X$ and so this sheaf is isomorphic to $\mathcal{I}_i\mathcal{I}/\mathcal{I}^2$, which is locally free at p from lemma 2.2.

If $p = C_j \cap C_l$, then $\mathcal{I}_i = \mathcal{O}_X$ and the argument goes through the same way.

□

Restricting the sheaf $(\mathcal{I}_i\mathcal{I}_j\mathcal{I} + \mathcal{I}^2)/\mathcal{I}^2$ to the component C_k of $\bigcup_{l \neq i,j} C_l$ yields a locally free sheaf of rank 2 on the smooth rational curve C_k; namely $(\mathcal{I}_i\mathcal{I}_j\mathcal{I} + \mathcal{I}^2)/\mathcal{I}^2|_{C_k} = (\mathcal{I}_i\mathcal{I}_j\mathcal{I} + \mathcal{I}^2)/(\mathcal{I}_i\mathcal{I}_j\mathcal{I}_k\mathcal{I} + \mathcal{I}^2)$.

Lemma 2.5

$$(\mathcal{I}_i\mathcal{I}_j\mathcal{I} + \mathcal{I}^2)/\mathcal{I}^2|_{C_k} = \begin{cases} \text{(-1,-1)} & \text{if } |k-i| = 1, |k-j| \neq 1 \text{ and } 2 \leq k \leq n-1 \\ \text{(-1,0)} & \text{if } |k-i| = 1, |k-j| \neq 1 \text{ and } k = 1 \text{ or } n \\ \text{(-2,-2)} & \text{if } |k-i| = 1, |k-j| = 1 \\ \text{(0,0)} & \text{if } |k-i| \neq 1, |k-j| \neq 1 \text{ and } 2 \leq k \leq n-1 \\ \text{(0,1)} & \text{if } |k-i| \neq 1, |k-j| \neq 1 \text{ and } k = 1 \text{ or } n \end{cases}$$

Proof: The injection $\mathcal{I}_i\mathcal{I}_j\mathcal{I}+\mathcal{I}^2/\mathcal{I}_i\mathcal{I}_j\mathcal{I}_k\mathcal{I}+\mathcal{I}^2 \hookrightarrow \mathcal{I}/\mathcal{I}_k\mathcal{I}$ is well defined since $\mathcal{I}_i\mathcal{I}_j\mathcal{I}+\mathcal{I}^2 \bigcap \mathcal{I}_k\mathcal{I} = \mathcal{I}_i\mathcal{I}_j\mathcal{I}_k\mathcal{I}+\mathcal{I}^2$. This equality can be calculated in local coordinates as in the proofs of lemma's 2.1 and 2.3.

Case 1: If $|k-i| \neq 1$ and $|k-j| \neq 1$ then, near any $p \in C_k$, $\mathcal{I}_i = \mathcal{I}_j = \mathcal{O}_X$, so the injection is an isomorphism. This determines the last two cases in the list above.

Case 2: If $|k-i| = 1$ and $|k-j| \neq 1$ then, near any $p \in C_k$, $\mathcal{I}_j = 0$. Therefore, $\mathcal{I}_i\mathcal{I}_j\mathcal{I}+\mathcal{I}^2/\mathcal{I}_i\mathcal{I}_j\mathcal{I}_k\mathcal{I}+\mathcal{I}^2 = \mathcal{I}_i\mathcal{I}/\mathcal{I}_i\mathcal{I}_k\mathcal{I}+\mathcal{I}^2 = \mathcal{I}_i\mathcal{I}/\mathcal{I}^2|_{C_k}$. So, this is the same as case 2 in the proof of the previous lemma where p is a point of intersection. This determines the first two results in the list above.

Case 3 : If $|k-i| = 1$ and $|k-j| = 1$ (so $k \neq 1$ or n), then there are two points of intersection. The injection is an isomorphism away from the intersection points, and at $p = C_k \bigcap C_i$, $\mathcal{I}_j = \mathcal{O}_X$, so as in case 2, this is the sheaf $\mathcal{I}_i\mathcal{I}/\mathcal{I}^2|_{C_k}$. Similarly, at $p = C_k \bigcap C_j$, this is the sheaf $\mathcal{I}_j\mathcal{I}/\mathcal{I}^2|_{C_k}$. Therefore, again we have reduced this to case 2 of lemma 2.3. In this case there are two points of intersection, so $(\mathcal{I}_i\mathcal{I}_j\mathcal{I}+\mathcal{I}^2)/\mathcal{I}^2|_{C_k} \cong \mathcal{O}_k(-1) \otimes \mathcal{O}_k(-1) \otimes \mathcal{I}/\mathcal{I}_k\mathcal{I}$. This gives the third result in the list above.

□

Since no more than two components can intersect a given component, it is not necessary to restrict the sheaf in lemma 2.5 to other components. In fact, it was seen in the proof of lemma 2.5 that it is always possible to reduce to the case of two components as in lemma 2.3. This is made more precise in the remark below.

To simplify the notation, let $\mathcal{F}_{i,k} = \mathcal{I}_i\mathcal{I}_{i+1}\cdots\mathcal{I}_{k-1}\mathcal{I}_k\mathcal{I}+\mathcal{I}^2$.

Remark: $\mathcal{F}_{i,k}/\mathcal{F}_{i,k+1} = \mathcal{F}_{i,k}/\mathcal{I}^2|_{C_{k+1}}$ is supported only on the curve C_{k+1}, so $\mathcal{I}_i = \cdots\mathcal{I}_{k-1} = \mathcal{O}_X$ along this support. This sheaf, then, is just $\mathcal{I}_k\mathcal{I}/\mathcal{F}_{k,k+1}$. But $\mathcal{F}_{k,k+1} = \mathcal{I}_k\mathcal{I}_{k+1}\mathcal{I}+\mathcal{I}^2$ and $\mathcal{I}_k\mathcal{I}_{k+1} \subset \mathcal{I}$, so $\mathcal{F}_{k,k+1} = \mathcal{I}^2$ and $\mathcal{F}_{i,k}/\mathcal{F}_{i,k+1} = \mathcal{I}_k\mathcal{I}/\mathcal{I}^2$ on C_{k+1}. From lemma 2.3 this sheaf decomposes as $(-1,-1)$ if $k+1 \neq n$ and as $(-1,0)$ if $k+1 = n$.

We are now ready to calculate the number of global sections of $\mathcal{I}_i\mathcal{I}/\mathcal{I}^2$.

Lemma 2.6

$$h^0(\mathcal{I}_i\mathcal{I}/\mathcal{I}^2) = \begin{cases} 1 & if \;\; i = 1 \; or \; n \\ 2 & if \;\; 2 \leq i \leq n-1 \end{cases}$$

Proof:*Step* 1: For $1 \leq i < n$, consider the exact sequence

$$0 \longrightarrow \mathcal{F}_{i,i+1}/\mathcal{I}^2 \longrightarrow \mathcal{I}\mathcal{I}_i/\mathcal{I}^2 \longrightarrow \mathcal{I}_i\mathcal{I}/\mathcal{I}^2|_{C_{i+1}} \longrightarrow 0. \quad (S_1)$$

The decomposition of the term on the right is known from lemma 2.3. The cohomology of the left term can be determined by restricting to the component C_{i+2}, giving the exact sequence:

$$0 \longrightarrow \mathcal{F}_{i,i+2}/\mathcal{I}^2 \longrightarrow \mathcal{F}_{i,i+1}/\mathcal{I}^2 \longrightarrow \mathcal{F}_{i,i+1}/\mathcal{I}^2|_{C_{i+2}} \longrightarrow 0 \quad (S_2)$$

Continuing to restrict the left term in each exact sequence to each successive component of C, we get in general:

$$0 \longrightarrow \mathcal{F}_{i,k}/\mathcal{I}^2 \longrightarrow \mathcal{F}_{i,k-1}/\mathcal{I}^2 \longrightarrow \mathcal{F}_{i,k-1}/\mathcal{I}^2|_{C_k} \longrightarrow 0 \quad (S_{k-i})$$

From the remark, the right term decomposes as $(-1,-1)$ if $k \neq n$ and $(-1,0)$ if $k = n$. The final sequence to consider will be

$$0 \longrightarrow \mathcal{F}_{i,n}/\mathcal{I}^2 \longrightarrow \mathcal{F}_{i,n-1}/\mathcal{I}^2 \longrightarrow \mathcal{F}_{i,n-1}/\mathcal{I}^2|_{C_n} \longrightarrow 0 \quad (S_{n-i})$$

Special Case $i = 1$: The first sequence in this process is

$$0 \longrightarrow \mathcal{F}_{1,2}/\mathcal{I}^2 \longrightarrow \mathcal{I}_1\mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{I}_1\mathcal{I}/\mathcal{I}^2|_{C_2} \longrightarrow 0$$

and the final sequence is

$$0 \longrightarrow \mathcal{F}_{1,n}/\mathcal{I}^2 \longrightarrow \mathcal{F}_{1,n-1}/\mathcal{I}^2 \longrightarrow \mathcal{F}_{1,n-1}/\mathcal{I}^2|_{C_n} \longrightarrow 0$$

The left term in this final sequence is 0 and the right term is locally free of rank 2 on the smooth rational curve C_n, and by the remark, then, decomposes as $(-1,0)$. The right term in each of the previous sequences decomposes as $(-1,-1)$ since $2 \leq k \leq n-1$. Working from the last sequence and backtracking to the first, we see that $H^1(\mathcal{F}_{1,n-1}/\mathcal{I}^2) = H^1(\mathcal{F}_{1,n-2}/\mathcal{I}^2) = \cdots H^1(\mathcal{F}_{1,k-1}/\mathcal{I}^2) = \cdots H^1(\mathcal{F}_{1,2}/\mathcal{I}^2) = H^1(\mathcal{I}_1\mathcal{I}/\mathcal{I}^2) = 0$. Therefore,

$$h^0(\mathcal{I}_1\mathcal{I}/\mathcal{I}^2) = \sum_{k=2}^{n} h^0(\mathcal{F}_{1,k-1}/\mathcal{I}^2|_{C_k})$$

However, all of these sheaves, $\mathcal{F}_{1,k-1}/\mathcal{I}^2|_{C_k}$, decompose as (-1,-1) except for when $k = n$, in which case it is $(-1, 0)$. So, $h^0(\mathcal{I}_1\mathcal{I}/\mathcal{I}^2) = 1$.

Step 2: For $1 < i \leq n$, continue from step 1. Begin with the sequence S_{n-1} and restrict the left term to C_{i-1}. The next sequence to consider, therefore, would be

$$0 \longrightarrow \mathcal{F}_{i-1,n}/\mathcal{I}^2 \longrightarrow \mathcal{F}_{i,n}/\mathcal{I}^2 \longrightarrow \mathcal{F}_{i,n}/\mathcal{I}^2|_{C_{i-1}} \longrightarrow 0 \quad (S_{n-i+1})$$

Proceed by restricting the left term of this sequence to C_{i-2}. A general sequence would be

$$0 \longrightarrow \mathcal{F}_{j-1,n}/\mathcal{I}^2 \longrightarrow \mathcal{F}_{j,n}/\mathcal{I}^2 \longrightarrow \mathcal{F}_{j,n}/\mathcal{I}^2|_{C_{j-1}} \longrightarrow 0 \quad (S_{n-j+1})$$

Continuing until $j = 2$, the final sequence to consider is

$$0 \longrightarrow \mathcal{F}_{1,n}/\mathcal{I}^2 \longrightarrow \mathcal{F}_{2,n}/\mathcal{I}^2 \longrightarrow \mathcal{F}_{2,n}/\mathcal{I}^2|_{C_1} \longrightarrow 0 \quad (S_{n-1})$$

The term on the left in this final sequence is 0 and the right term is locally free of rank 2 on C_1, and from lemma 2.3 it decomposes as $(-1, 0)$. The term on the right in all of the previous exact sequences in Step 2 is $(-1, -1)$, since they are restrictions to the curves $C_2, \cdots, C_{i-1}$. Working backwards from S_{n-1} to S_{n-i+1} it can be seen that $H^1(\mathcal{F}_{2,n}/\mathcal{I}^2) = H^1(\mathcal{F}_{i-1,n}/\mathcal{I}^2) = H^1(\mathcal{F}_{i,n}/\mathcal{I}^2) = 0$ Therefore,

$$h^0(\mathcal{F}_{i,n}/\mathcal{I}^2) = \sum_{j=2}^{i} h^0(\mathcal{F}_{j,n}/\mathcal{I}^2|_{C_{j-1}})$$

But the only term contributing to this sum is when $j = 2$, because by the remark, all terms in this sum are $(-1, -1)$ except for when $j = 2$. When $j = 2$, then

$$h^0(\mathcal{F}_{i,n}/\mathcal{I}^2) = h^0(\mathcal{F}_{2,n}/\mathcal{I}^2|_{C_1}) = 1$$

for all $2 \leq i \leq n$.

Special Case $i = n$: This equation shows that $h^0(\mathcal{I}_n\mathcal{I}/\mathcal{I}^2) = 1$.

For $1 < i < n$ combine this information with that from Step 1. Since $H^1(\mathcal{F}_{i,n}/\mathcal{I}^2) = 0$, by working backward from the sequences S_{n-i} to S_1, it can be seen from the corresponding cohomology sequences that

$$H^1(\mathcal{F}_{i,n-1}/\mathcal{I}^2) = \cdots H^1(\mathcal{F}_{i,i+1}/\mathcal{I}^2) = H^1(\mathcal{I}_i\mathcal{I}/\mathcal{I}^2) = 0.$$

Therefore, for $2 \leq i \leq n-1$,

$$h^0(\mathcal{I}_i\mathcal{I}/\mathcal{I}^2) = \sum_{k=i+1}^{n} h^0(\mathcal{F}_{i,k-1}/\mathcal{I}^2|_{C_k}) + h^0(\mathcal{F}_{i,n}/\mathcal{I}^2).$$

The first summation has all zero terms except for when $k = n$, because restricting to the curves $C_{i+1}, \cdots, C_{n-1}$ results in the decomposition $(-1,-1)$ by the remark. So,

$$h^0(\mathcal{I}_i\mathcal{I}/\mathcal{I}^2) = h^0(\mathcal{F}_{i,n-1}/\mathcal{I}^2|_{C_n}) + h^0(\mathcal{F}_{2,n}/\mathcal{I}^2).$$

Both terms have the value 1 for all $2 \leq i \leq n-1$. This proves the lemma.

□

Corollary 2.1 *We have*

1) $h^0(\mathcal{I}/\mathcal{I}^2) = 4$

2) $H^1(\mathcal{I}/\mathcal{I}^2) = 0$

3) The map $H^0(\mathcal{I}/\mathcal{I}^2) \to H^0(\mathcal{I}/\mathcal{I}_i\mathcal{I})$ *is surjective for all* i.

Proof: It was shown in the proof of lemma 2.6 that $H^1(\mathcal{I}_i\mathcal{I}/\mathcal{I}^2) = 0$. From the exact sequence 2.1, $h^0(\mathcal{I}/\mathcal{I}^2) = h^0(\mathcal{I}_i\mathcal{I}/\mathcal{I}^2) + h^0(\mathcal{I}/\mathcal{I}_i\mathcal{I})$. Now, using the assumption that $\mathcal{I}/\mathcal{I}_i\mathcal{I}$ decomposes as $(0,1)$ or $(0,0)$depending on if C_i is an end component or not, (1) follows immediately from lemma 2.6. Since $H^1(\mathcal{I}/\mathcal{I}_i\mathcal{I})$ is also 0, (2) follows from the cohomology exact sequence of the sequence 2.1. In proving lemma 2.6 it was shown that $H^1(\mathcal{I}_i\mathcal{I}/\mathcal{I}^2) = 0$ for all i. The statement of (3), then, also follows from sequence 2.1.

□

The remainder of this chapter focuses on the conormal sheaf and the lifting of its sections to the formal ideal sheaf $\hat{\mathcal{I}}$. This result is stated in proposition 2.1. The first preliminary result, lemma 2.7, concerns invertible sheaves on C. These are discussed in section 1.4.

Lemma 2.7 $\mathcal{I}_i/\mathcal{I}$ *is an invertible sheaf on* $\bigcup_{j\neq i} C_j$.

Proof:Let $p \in \bigcup_{j\neq i} C_j$.

If p does not lie on C_i, then $\mathcal{I}_i = \mathcal{O}_X$, so $\mathcal{I}_i/\mathcal{I} = \mathcal{O}_X/\mathcal{I} = \mathcal{O}_C$, which is locally free at p.

If $p \in C_i$, then $p = C_{i-1} \bigcap C_i$ or $p = C_i \bigcap C_{i+1}$. For p the point of intersection of C_{i-1} and C_i, local coordinates can be chosen at p such that $\mathcal{I}_i = (x,z)$, $\mathcal{I}_{i-1} = (y,z)$ and $\mathcal{I} = (xy, z)$. In these coordinates, the sheaf $\mathcal{I}_i/\mathcal{I}$ is generated by $\{x\}$. Define a map $\mathcal{O}_{i-1} \to \mathcal{I}_i/\mathcal{I}$ by $f \mapsto f \cdot x$. The generator (1) of $\mathcal{O}_{i-1}$ is mapped to the generator x, so this map is surjective. The injection follows from the fact that $f \cdot x \in \mathcal{I}$ implies that y and/or z divides f. That is $f \cdot x \in \mathcal{I}$ implies $f \in \mathcal{I}_{i-1}$, proving this is an isomorphism.

□

We know from this lemma and the discussion in section 1.4, then, that $\mathcal{I}_i/\mathcal{I}$ is of the form $\mathcal{O}(a_1, \cdot, \hat{a}_i, \cdots a_n)$, where a_j is the degree of this sheaf restricted to each component, C_j, of $\bigcup_{j\neq i} C_j$.

Lemma 2.8

$$\mathcal{I}_i/\mathcal{I} \cong \begin{cases} \mathcal{O}(\hat{a}_1, -1, 0, \cdots, 0) & \text{if } \ i = 1 \\ \mathcal{O}(0, \cdots, 0, -1, \hat{a}_n) & \text{if } \ i = n \\ \mathcal{O}(0, \cdots, 0, -1, \hat{a}_i, -1, 0, \cdots, 0) & \text{if } \ 2 \leq i \leq n-1 \end{cases}$$

Proof: The values of a_j uniquely determine this invertible sheaf. Therefore, it suffices to determine the degree of $\mathcal{I}_i/\mathcal{I}|_{C_j}$ for all $j \neq i$. $\mathcal{I}_i/\mathcal{I}|_{C_j} = \mathcal{I}_i/(\mathcal{I}_i\mathcal{I}_j + \mathcal{I})$ and if C_j does not intersect C_i, then $\mathcal{I}_i = \mathcal{O}_X$ and $\mathcal{I} = \mathcal{I}_j$, so $\mathcal{I}_i/\mathcal{I}|_{C_j} = \mathcal{O}_X/\mathcal{I}_j = \mathcal{O}_j$. Therefore, the degree of $\mathcal{I}_i/\mathcal{I}|_{C_j}$ is 0 if $j \neq i-1$ or $i+1$.

To determine the degree of this sheaf on the components C_{i-1} and C_{i+1} we will use the exact sequence

$$0 \longrightarrow \mathcal{I}_i/\mathcal{I} \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_i \longrightarrow 0. \tag{2.2}$$

For $i \geq 2$, tensor this sequence with the flat $\mathcal{O}_X$−module $\mathcal{O}_{i-1}$ to obtain the exact sequence

$$0 \longrightarrow \mathcal{I}_i/\mathcal{I}|_{C_{i-1}} \longrightarrow \mathcal{O}_{i-1} \longrightarrow \mathcal{O}_p \longrightarrow 0,$$

where p is the point of intersection of C_{i-1} and C_i. Notice that $\mathcal{O}_i \otimes \mathcal{O}_{i-1} \cong \mathcal{O}_p$ since it is supported only at the point p. Now p is an effective divisor on the curve C_{i-1}, so we also have the exact sequence

$$0 \longrightarrow \mathcal{O}_{i-1}(-p) \longrightarrow \mathcal{O}_{i-1} \longrightarrow \mathcal{O}_p \longrightarrow 0.$$

Comparing these two exact sequences, the isomorphism $\mathcal{I}_i/\mathcal{I}|_{C_{i-1}} \cong \mathcal{O}_{i-1}(-p)$ results. Since $C_{i-1} \cong \mathbf{P}^1$, $\mathcal{O}_{i-1}(-p) \cong \mathcal{O}_{i-1}(-1)$.

For $i \leq n-1$ restrict the sequence 2.2 to C_{i+1}. The exact argument as above, replacing $i-1$ with $i+1$, shows that $\mathcal{I}_i/\mathcal{I}|_{C_{i+1}} \cong \mathcal{O}_{i+1}(-1)$.

□

This lemma proves, then, that $\mathcal{I}_i/\mathcal{I}|_{C_{i+1}} \cong \mathcal{I}_i/(\mathcal{I}_i\mathcal{I}_{i+1} + \mathcal{I}) \cong \mathcal{O}_{i+1}(-1)$ for $1 \leq i \leq n-1$ and $\mathcal{I}_n/\mathcal{I}|_{C_{n-1}} \cong \mathcal{I}_n/(\mathcal{I}_{n-1}\mathcal{I}_n + \mathcal{I}) \cong \mathcal{O}_{n-1}(-1)$.

To simplify the notation in the remainder of this chapter, let

$$\mathcal{F}^m_{j,k} = \mathcal{I}_j\mathcal{I}_{j+1}\cdots\mathcal{I}_{k-1}\mathcal{I}_k + \mathcal{I}^{m+1}.$$

Notice that $\mathcal{F}^1_{j,k} = \mathcal{F}_{j,k}$, which was used earlier.

To prove the following two lemmas it will be necessary to do some local calculations at the point $p = C_{i-1} \bigcap C_i$, so let $\mathcal{I}_{i-1} = (x,z)$, $\mathcal{I}_i = (y,z)$ and $\mathcal{I} = (xy,z)$ at p.

$$\mathcal{I}^m = ((xy)^m, (xy)^{m-1}z, \cdots, (xy)^{m-j}z^j, \cdots, (xy)z^{m-1}, z^m)$$

and

$$\begin{aligned} \mathcal{I}_1\mathcal{I}^m &= x \cdot \mathcal{I}^m + z \cdot \mathcal{I}^m \\ &= (x(xy)^m, x(xy)^{m-1}z, \cdots, x(xy)^{m-j}z^j, \cdots, x(xy)z^{m-1}, xz^m) + \\ &\quad ((xy)^m z, (xy)^{m-1}z^2, \cdots, (xy)^{m-j}z^{j+1}, \cdots, (xy)z^m, z^{m+1}) \end{aligned}$$

if $i = 2$. But all of the elements in $z \cdot \mathcal{I}^m$ are contained in $x \cdot \mathcal{I}^m$ except for z^{m+1}, so

$$\mathcal{I}_1\mathcal{I}^m = (x(xy)^m, x(xy)^{m-1}z, \cdots, x(xy)^{m-j}z^j, \cdots, x(xy)z^{m-1}, xz^m, z^{m+1}).$$

The expressions for $\mathcal{I}^m$ and $\mathcal{I}_1\mathcal{I}^m$ show that an element of $\mathcal{I}_1\mathcal{I}^m$ is in $\mathcal{I}^m$ only if x or z divides this element. This means that the element must be in $\mathcal{I}_1$. Therefore, $\mathcal{I}^m/\mathcal{I}_1\mathcal{I}^m$ is a locally free sheaf of rank $m+1$ on C_1 and is generated by

$$\{(xy)^m, (xy)^{m-1}z, \cdots, (xy)^{m-j}z^j, \cdots, (xy)z^{m-1}, z^m\}$$

at the point p.

The sheaf $\mathcal{I}/\mathcal{I}_1\mathcal{I}$ is locally free of rank 2 on C_1 generated by $\{xy, z\}$ at $p = C_1 \bigcap C_2$, so the symmetric product $S^m(\mathcal{I}/\mathcal{I}_1\mathcal{I})$ is locally free of rank $m+1$ on C_1 and is generated by

$$\{xy, z\}^m = \{(xy)^m, (xy)^{m-1}z, \cdots, (xy)^{m-j}z^j, \cdots (xy)z^{m-1}, z^m\}.$$

Lemma 2.9 $\mathcal{I}^m/\mathcal{I}_1\mathcal{I}^m \cong S^m(\mathcal{I}/\mathcal{I}_1\mathcal{I})$

Proof: Define a global map $\mathcal{I}^{\otimes m} \longrightarrow \mathcal{I}^m/\mathcal{I}_1\mathcal{I}^m$ by multiplication of functions. This map kills $\mathcal{I}_1\mathcal{I} \otimes \mathcal{I}^{\otimes(m-1)}$, thus giving a well defined map $S^m(\mathcal{I}/\mathcal{I}_1\mathcal{I}) \longrightarrow \mathcal{I}^m/\mathcal{I}_1\mathcal{I}^m$. On $C_1 - \{p\}$, $\mathcal{I} = \mathcal{I}_1$, so $\mathcal{I}^m/\mathcal{I}_1\mathcal{I}^m = {\mathcal{I}_1}^m/{\mathcal{I}_1}^{m+1}$ and $S^m(\mathcal{I}/\mathcal{I}_1\mathcal{I}) = S^m(\mathcal{I}_1/{\mathcal{I}_1}^2)$. By [Ma], pg. 110, $S^m(\mathcal{I}_1/{\mathcal{I}_1}^2) \cong {\mathcal{I}_1}^m/{\mathcal{I}_1}^{m+1}$. So, it only remains to show that this isomorphism holds at the point p. From the calculations immediately preceding this lemma we see that $\mathcal{I}^m/\mathcal{I}_1\mathcal{I}^m$ and $S^m(\mathcal{I}/\mathcal{I}_1\mathcal{I})$ are both locally free sheaves of rank $m+1$ generated by the same elements at p. Therefore, the multiplication map is also an isomorphism at p.

□

In the next lemma we will look at the sheaf $\mathcal{F}^m_{1,i-1}/\mathcal{I}^{m+1}|_{C_i}$. By the remark immediately preceding lemma 2.6 this sheaf is $\mathcal{I}_{i-1}\mathcal{I}^m/\mathcal{I}^{m+1}$ locally near p. From the local expressions for $\mathcal{I}^m$ and $\mathcal{I}_{i-1}$ we have

$$\mathcal{I}_{i-1}\mathcal{I}^m = (x^{m+1}y^m, x^m y^{m-1}z, \cdots, x^{m-k+1}y^{m-k}z^k, \cdots, x^2yz^{m-1}, xz^m)$$

and

$$\mathcal{I}^{m+1} = ((xy)^{m+1}, (xy)^m z, \cdots, (xy)^{m-j}z^{j+1}, \cdots, (xy)z^m, z^{m+1}).$$

From these descriptions, an element of $\mathcal{I}^{m+1}$ can be in $\mathcal{I}_{i-1}\mathcal{I}^m$ only if it is divisible by y or z, or, in other words, only if the element is in $\mathcal{I}_i$. Therefore, $\mathcal{I}_{i-1}\mathcal{I}^m/\mathcal{I}^{m+1}$ is locally free of rank $m+1$ on C_i, generated by

$$\{x^{m+1}y^m, x^m y^{m-1}z, \cdots, x^{m-k+1}y^{m-k}z^k, \cdots, x^2yz^{m-1}, xz^m\}$$

at $p = C_{i-1} \bigcap C_i$.

The invertible sheaf $\mathcal{I}_{i-1}/(\mathcal{I}_{i-1}\mathcal{I}_i + \mathcal{I})$ is generated by $\{x\}$ and $\mathcal{I}/\mathcal{I}_i\mathcal{I}$ is generated by $\{xy, z\}$, so $S^m(\mathcal{I}/\mathcal{I}_i\mathcal{I}) \otimes \mathcal{I}_{i-1}/(\mathcal{I}_{i-1}\mathcal{I}_i + \mathcal{I})$ is generated by

$$\{xy, z\}^m \otimes \{x\} = \{x^{m+1}y^m, x^m y^{m-1}z, \cdots, x^{m-k+1}y^{m-k}z^k, \cdots, x^2yz^{m-1}, xz^m\}.$$

Lemma 2.10 *$\mathcal{F}^m_{1,i-1}/\mathcal{I}^{m+1}|_{C_i} \cong S^m(\mathcal{I}/\mathcal{I}_i\mathcal{I}) \otimes [\mathcal{I}_{i-1}/(\mathcal{I}_{i-1}\mathcal{I}_i + \mathcal{I})]$ for $2 \le i \le n$.*

Proof: Define a map $\mathcal{I}^{\otimes m} \otimes \mathcal{I}_{i-1} \longrightarrow \mathcal{I}_{i-1}\mathcal{I}^m/\mathcal{F}^m_{i-1,i}$ by multiplication of functions. The sheaves $\mathcal{I}^{\otimes(m)}\otimes(\mathcal{I}_{i-1}\mathcal{I}_i+\mathcal{I})$ and $\mathcal{I}_{i-1}\otimes\mathcal{I}_i\mathcal{I}\otimes\mathcal{I}^{\otimes(m-1)}$ are killed by this map because their images, $\mathcal{F}^m_{i-1,i}$ and $\mathcal{I}_{i-1}\mathcal{I}_i\mathcal{I}^m$, respectively, are contained in $\mathcal{F}^m_{i-1,i}$. Therefore, multiplication gives a well defined map on $S^m(\mathcal{I}/\mathcal{I}_i\mathcal{I}) \otimes [\mathcal{I}_{i-1}/(\mathcal{I}_{i-1}\mathcal{I}_i + \mathcal{I})]$. On $C_i - \{p\}$ we have $\mathcal{I}_{i-1} = \mathcal{O}_X$ and $\mathcal{I} = \mathcal{I}_i$, so $\mathcal{F}^m_{1,i-1} = \mathcal{I}^m_i$. $\mathcal{F}^m_{1,i-1}/\mathcal{I}^{m+1}|_{C_i}$, then, is the sheaf $\mathcal{I}^m_i/\mathcal{I}^{m+1}_i$ and the invertible sheaf $\mathcal{I}_{i-1}/(\mathcal{I}_{i-1}\mathcal{I}_i + \mathcal{I})$ is $\mathcal{O}_X/\mathcal{I}_i \cong \mathcal{O}_i$. This lemma, then, states that $\mathcal{I}^m_i/\mathcal{I}^{m+1}_i \cong S^m(\mathcal{I}_i/\mathcal{I}^2_i)$. This holds from [Ma] pg. 110. At the point p, $\mathcal{I}_{i-1}\mathcal{I}^m/\mathcal{F}^m_{i-1,i} \cong \mathcal{I}_{i-1}\mathcal{I}^m/\mathcal{I}^{m+1}$ has been shown to be generated by the same elements as $S^m(\mathcal{I}/\mathcal{I}_i\mathcal{I})\otimes\mathcal{I}_{i-1}/(\mathcal{I}_{i-1}\mathcal{I}_i+\mathcal{I})$ in the calculations above. Therefore, this map is an isomorphism everywhere.

□

Corollary 2.2

$$\mathcal{F}^m_{1,i-1}/\mathcal{I}^{m+1}|_{C_i} \cong \begin{cases} \bigoplus_{j=-1}^{m} \mathcal{O}_n(j) & \text{if } i = n \\ \mathcal{O}_i(-1)^{\oplus(m+1)} & \text{if } 2 \le i \le n-1 \end{cases}$$

Proof: From lemma 2.3, $\mathcal{I}/\mathcal{I}_n\mathcal{I} = (0,1)$, so $S^m(\mathcal{I}/\mathcal{I}_n\mathcal{I}) \cong \bigoplus_{j=0}^{m} \mathcal{O}_n(j)$. Lemma 2.8 gives $\mathcal{I}_{n-1}/(\mathcal{I}_{n-1}\mathcal{I}_n + \mathcal{I}) = \mathcal{I}_{n-1}/\mathcal{I}|_{C_n} \cong \mathcal{O}_n(-1)$, and combining this with the result of lemma 2.10,

$$\mathcal{F}^m_{1,n-1}/\mathcal{I}^{m+1}|_{C_n} \cong \left(\bigoplus_{j=0}^{m} \mathcal{O}_n(j)\right) \otimes \mathcal{O}_n(-1) \cong \bigoplus_{j=0}^{m} \mathcal{O}_n(j-1).$$

For $2 \le i \le n-1$ the only difference is $\mathcal{I}/\mathcal{I}_i\mathcal{I} = (0,0)$, so

$$\mathcal{F}^m_{1,i-1}/\mathcal{I}^{m+1}|_{C_i} \cong \left(\mathcal{O}_i^{\oplus(m+1)}\right) \otimes \mathcal{O}_i(-1) \cong \mathcal{O}_i(-1)^{\oplus(m+1)}.$$

□

Proposition 2.1 *The map on global sections, $H^0(\hat{\mathcal{I}}) \longrightarrow H^0(\mathcal{I}/\mathcal{I}^2)$, is surjective.*

Proof: The sequence of sheaves on C:

$$0 \to \mathcal{I}_1\mathcal{I}^m/\mathcal{I}^{m+1} \longrightarrow \mathcal{I}^m/\mathcal{I}^{m+1} \longrightarrow \mathcal{I}^m/\mathcal{I}_1\mathcal{I}^m \longrightarrow 0 \qquad (2.3)$$

is exact. Recall that $\mathcal{I}/\mathcal{I}_1\mathcal{I} \cong \mathcal{O}_1 \oplus \mathcal{O}_1(1)$, so from Lemma 2.9, the sheaf on the right in the above exact sequence has no factors, $\mathcal{O}(a)$, in its direct summand decomposition with $a \leq -1$. Therefore, $H^1(\mathcal{I}^m/\mathcal{I}_1\mathcal{I}^m) = 0$. To see that $H^1(\mathcal{I}_1\mathcal{I}^m/\mathcal{I}^{m+1}) = 0$, an argument similar to that used in *special case* 1 of lemma 2.6 is used. In fact, it is exactly the same argument for $m = 1$ The sequences to use are:

$$0 \longrightarrow \mathcal{F}^m_{1,2}/\mathcal{I}^{m+1} \longrightarrow \mathcal{I}_1\mathcal{I}^m/\mathcal{I}^{m+1} \longrightarrow \mathcal{I}_1\mathcal{I}^m/\mathcal{I}^{m+1}|_{C_2} \longrightarrow 0$$

$$\vdots$$

$$0 \longrightarrow \mathcal{F}^m_{1,n}/\mathcal{I}^{m+1} \longrightarrow \mathcal{F}^m_{1,n-1}/\mathcal{I}^{m+1} \longrightarrow \mathcal{F}^m_{1,n-1}/\mathcal{I}^{m+1}|_{C_n} \longrightarrow 0$$

The term on the left in this final sequence is 0. From Lemma 2.10, and the fact that $\mathcal{I}/\mathcal{I}_k\mathcal{I} = (0,0)$ or $(0,1)$ and $\mathcal{I}_{k-1}/\mathcal{I}_{k-1}\mathcal{I}_k + \mathcal{I} \cong \mathcal{O}_k(-1)$, H^1 of the term on the right in each of the sequences vanishes. Therefore, $H^1(\mathcal{F}^m_{1,n-1}/\mathcal{I}^{m+1}) = \cdots = H^1(\mathcal{F}^m_{1,2}/\mathcal{I}^{m+1}) = H^1(\mathcal{I}_1\mathcal{I}^m/\mathcal{I}^{m+1}) = 0$

From the cohomology exact sequence applied to sequence 2.3, $H^1(\mathcal{I}^m/\mathcal{I}^{m+1}) = 0$. An induction argument on $l - m$ will show that $H^1\left(\mathcal{I}^l/\mathcal{I}^m\right) = 0$ for all $l - m > 0$. The case for $l - m = 1$ has just been shown. Assuming this vanishing for $l - m \leq k$, the exact sequence

$$0 \longrightarrow \mathcal{I}^{m+k}/\mathcal{I}^{m+k+1} \longrightarrow \mathcal{I}^m/\mathcal{I}^{m+k+1} \longrightarrow \mathcal{I}^m/\mathcal{I}^{m+k} \longrightarrow 0$$

and its long exact cohomology sequence give the vanishing of the outer terms, which implies the vanishing of the middle term.

So, in particular, $H^1(\mathcal{I}^2/\mathcal{I}^m) = 0$ for all $m > 2$. The long exact cohomology sequence of

$$0 \longrightarrow \mathcal{I}^2/\mathcal{I}^m \longrightarrow \mathcal{I}/\mathcal{I}^m \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow 0$$

proves the surjection $H^0(\mathcal{I}/\mathcal{I}^m) \longrightarrow H^0(\mathcal{I}/\mathcal{I}^2)$ for all $m > 2$.

The sequence

$$0 \longrightarrow \mathcal{I}^m/\mathcal{I}^{m+1} \longrightarrow \mathcal{I}/\mathcal{I}^{m+1} \longrightarrow \mathcal{I}/\mathcal{I}^m \longrightarrow 0$$

is exact, and it has been shown that $H^1(\mathcal{I}^m/\mathcal{I}^{m+1}) = 0$ as well. Therefore, the map $H^0(\mathcal{I}/\mathcal{I}^{m+1}) \longrightarrow H^0(\mathcal{I}/\mathcal{I}^m)$ is surjective for all $m > 2$. By definition, then, $H^0(\hat{\mathcal{I}}) \longrightarrow H^0(\mathcal{I}/\mathcal{I}^2)$ is surjective.

$\square$

To show that q is a cA_n singularity, by the criterion stated in proposition 1.1, it is enough to show that a general section of $\hat{\mathcal{I}}$ is a smooth surface in which all the C_i have conormal bundle isomorphic to $\mathcal{O}_{C_i}(2)$.

The maps $H^0(\hat{\mathcal{I}}) \longrightarrow H^0(\mathcal{I}/\mathcal{I}^2) \longrightarrow H^0(\mathcal{I}/\mathcal{I}_i\mathcal{I})$ have both been shown to be surjective for all i.

Lemma 2.11 *$\mathcal{I}/\mathcal{I}^2$ is generated by global sections.*

Proof: It needs to be shown that at any point $p \in C$ there exists a basis of global sections generating $\mathcal{I}/\mathcal{I}^2$ at p. In particular, it will be shown that $H^0(\mathcal{I}/\mathcal{I}^2) \rightarrow H^0(\mathcal{I}/m_p\mathcal{I})$ is surjective for all $p \in C$. Recall that $\mathcal{I}_i/{\mathcal{I}_i}^2 = (1,1)$ or $(0,2)$, and $\mathcal{I}/\mathcal{I}_i\mathcal{I} = (0,1)$ or $(0,0)$ are both generated by global sections.

For $p \in C$, $p \in C_i$ for some i. If p is not a point of intersection, then $\mathcal{I}/\mathcal{I}^2 \cong \mathcal{I}_i/{\mathcal{I}_i}^2$ at p. Since $\mathcal{I}_i/{\mathcal{I}_i}^2$ is generated by global sections, there is a basis of global sections generating $\mathcal{I}/\mathcal{I}^2$ at p. If $p = C_i \bigcap C_j$, then, since $\mathcal{I}/\mathcal{I}_i\mathcal{I}$ is generated by global sections there exists a basis of global sections generating $\mathcal{I}/\mathcal{I}_i\mathcal{I}$ at p. From the surjection $H^0(\mathcal{I}/\mathcal{I}^2) \rightarrow H^0(\mathcal{I}/\mathcal{I}_i\mathcal{I})$, these sections can be lifted to global sections of $\mathcal{I}/\mathcal{I}^2$. But $\mathcal{I}/\mathcal{I}_1\mathcal{I}$ is generated by global sections, so $H^0(\mathcal{I}/\mathcal{I}_1\mathcal{I}) \rightarrow H^0(\mathcal{I}/m_p\mathcal{I})$ is surjective for all $p \in C_i$. Therefore, for any $p \in C$, the map $H^0(\mathcal{I}/\mathcal{I}^2) \rightarrow H^0(\mathcal{I}/m_p\mathcal{I})$ is surjective.

□

Lemma 2.12 *A general section of $\hat{I}$ defines a nonsingular surface along C.*

Proof: A general section of $\hat{\mathcal{I}}$ at any point of intersection p is of the form $g \cdot xy + h \cdot z$ with $g, h \in \mathcal{O}_{p,\hat{X}}$ with g or h a unit. Considering this as a local section of $\mathcal{I}/\mathcal{I}^2$, there exists a global section $s \in \mathcal{I}/\mathcal{I}^2$ that does not vanish at p, and, therefore, $h(p) \neq 0$. So, s is nonsingular at p. The condition $h(p) \neq 0$ defines an open dense subset of X on which h is non-vanishing. This being true at each point of intersection, and the intersection of these sets being open and dense in X, shows that a general section of $\mathcal{I}/\mathcal{I}^2$ is nonsingular at each point of intersection. Being surjective, lift this to a global section of $\hat{\mathcal{I}}$. At a smooth point of C, a general section of $\hat{\mathcal{I}}$ is of the form $g \cdot x + h \cdot z$ with g or h a unit. Therefore, a general section of $\hat{\mathcal{I}}$ is smooth away from the singular points of C as well. This shows that a general section of $\hat{\mathcal{I}}$ is smooth.

□

Take $s \in H^0(\mathcal{I}/\mathcal{I}^2)$ a nonzero section. It has been shown that for general s, the surface S defined by s is smooth. In particular, at any point $p \in C$, coordinates can be chosen such that (s) is defined by $(z = 0)$ and $\mathcal{I}_{C,S} = (xy)$ at a point of intersection and $\mathcal{I}_{C,S} \cong \mathcal{I}_{C_i,S} = (x)$ otherwise.

Because s is nonzero, it defines an injective map $0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{I}/\mathcal{I}^2$ (multiplication by s). The cokernel of this map is the line bundle $\mathcal{I}/\mathcal{I}^2|_S \cong \mathcal{I}_{C,S}/\mathcal{I}^2_{C,S}$. This gives the exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{I}_{C,S}/\mathcal{I}^2_{C,S} \longrightarrow 0$$

where S is the smooth surface defined by the section s. Restricting the sequence to C_i we get

$$0 \longrightarrow \mathcal{O}_{C_i} \longrightarrow \mathcal{I}/\mathcal{I}_1\mathcal{I} \longrightarrow \mathcal{I}_{C,S}/\mathcal{I}^2_{C,S}|_{C_i} \longrightarrow 0.$$

From the decomposition of $\mathcal{I}/\mathcal{I}_1\mathcal{I}$ on C_i we see that

$$\mathcal{I}_{C,S}/\mathcal{I}^2_{C,S}|_{C_i} \cong \begin{cases} \mathcal{O}_{C_i}(1) & \text{for } i = 1, n \\ \mathcal{O}_{C_i} & \text{for } 2 \le i \le n-1. \end{cases}$$

Therefore,

$$\mathcal{I}_{C,S}/\mathcal{I}^2_{C,S} \cong \mathcal{O}_C(1, 0, \cdots, 0, 1).$$

Theorem 2.1 *If $f : X \to Y$ is a contraction map with $f(C) = q$ and $C = \cup_{i=1}^n C_i$ with all components having length 1, then a general hyperplane section of q has an A_n type singularity at q.*

Proof: The injection $\mathcal{I}_{C,S}/\mathcal{I}^2_{C,S}|_{C_i} \hookrightarrow \mathcal{I}_{C_i,S}/\mathcal{I}^2_{C_i,S}$ is well defined since $\mathcal{I}_{C,S} \cap \mathcal{I}^2_{C_i,S} = \mathcal{I}_{C_i,S}\mathcal{I}_{C,S}$. This can be calculated in local coordinates at every point. This map is an isomorphism away from the singular points of C, and at a point p of intersection the map is defined by $xy \mapsto y \cdot x$. Since y is a local coordinate on C_i, this map vanishes to order 1 at each point of intersection. If $i = 1, n$ then there is just one singular point, and $\mathcal{I}_{C,S}/\mathcal{I}^2_{C,S}|_{C_i} \cong \mathcal{O}_{C_i}(1)$, so $\mathcal{I}_{C_i,S}/\mathcal{I}^2_{C_i,S} \cong \mathcal{O}_{C_i}(2)$. For $2 \le i \le n-1$ there are two points of intersection, and $\mathcal{I}_{C,S}/\mathcal{I}^2_{C,S}|_{C_i} \cong \mathcal{O}_{C_i}$, so again $\mathcal{I}_{C_i,S}/\mathcal{I}^2_{C_i,S} \cong \mathcal{O}_{C_i}(2)$. Therefore, if the curve C contracts, it will contract to a cA_n singularity where n is the number of components of C.

□

CHAPTER 3

THE LENGTH(2,1) CASES

This chapter deals with the case of $C = C_1 \bigcup C_2$ with $\mathcal{I}/\mathcal{I}_1\mathcal{I} = (-1, 2)$ and $\mathcal{I}/\mathcal{I}_2\mathcal{I} = (0, 1)$. From lemma 2.1 the only possibilities are $\mathcal{I}/\mathcal{I}_1\mathcal{I} = (0, 1)$ or $(-1, 2)$, and for the length of C_1 to be at least 2 there must be a -1 factor. Define

$$\mathcal{J} = Ker(\mathcal{I} \longrightarrow \mathcal{I}\mathcal{I}_1\mathcal{I} \longrightarrow \mathcal{O}_1(-1)).$$

By this definition, $\mathcal{I}/\mathcal{J} \cong \mathcal{O}_1(-1)$ and $\mathcal{J}/\mathcal{I}_1\mathcal{I} \cong \mathcal{O}_1(2)$. To calculate local coordinates of J at the point of intersection p notice that $\mathcal{J}/\mathcal{I}_1\mathcal{I}$ is a subsheaf of $\mathcal{I}/\mathcal{I}_1\mathcal{I}$, which is generated locally by $\{xy, z\}$ at p. Therefore, $\mathcal{J}/\mathcal{I}_1\mathcal{I}$ is generated by an element of the form $g \cdot z + h \cdot xy$ for some functions $g, h \in \mathcal{O}_{p,X}$ with g or h a unit.

If g is a unit, then this sheaf can be generated locally at p by an element of the form $z + h' \cdot xy$. So, $\mathcal{J} = (z + h' \cdot xy) + \mathcal{I}_1\mathcal{I}$, which in coordinates at p is $(z + h' \cdot xy, x^2y, xz, z^2) = (z + h' \cdot xy, x^2y)$. An analytic change of coordinates given by $z + h' \cdot xy \mapsto z$ shows that

$$\mathcal{J} = (x^2y, z).$$

If h is a unit, then $\mathcal{J}/\mathcal{I}_1\mathcal{I}$ is generated by an element of the form $g' \cdot z + xy$. Therefore, as above,

$$\mathcal{J} = (g' \cdot z + xy, x^2y, xz, z^2) = (g' \cdot z + xy, xz, z^2).$$

3.1 The $D_4(2, 1)$ Case

Case 1: $\mathcal{J} = (x^2y, z)$

From local calculations as in chapter 2, it can be shown that $\mathcal{J}/\mathcal{I}\mathcal{J}$ is locally free of rank 2 on C, and so the restriction to each component, $\mathcal{J}/\mathcal{I}_i\mathcal{J}$, is locally free of rank 2 on C_i. Consider the exact sequence

$$0 \longrightarrow \mathcal{I}_1\mathcal{I}/\mathcal{I}_1\mathcal{J} \longrightarrow \mathcal{J}/\mathcal{I}_1\mathcal{J} \longrightarrow \mathcal{J}/\mathcal{I}_1\mathcal{I} \longrightarrow 0 \tag{3.1}$$

Again, in coordinates, it can be seen that $\mathcal{I}_1\mathcal{I}/\mathcal{I}_1\mathcal{J}$ is locally free of rank 1 on C_1. To calculate the degree of this sheaf, we first calculate the degree of the sheaf $\mathcal{I}^2/\mathcal{I}\mathcal{J}$, which is also locally free of rank 1 on C_1.

Lemma 3.1 $\mathcal{I}_1\mathcal{I}/\mathcal{I}_1\mathcal{J} \cong \mathcal{O}_1(-1)$

Proof: The inclusion map $\mathcal{I}^2/\mathcal{I}\mathcal{J} \hookrightarrow \mathcal{I}_1\mathcal{I}/\mathcal{I}_1\mathcal{J}$ is well defined since $\mathcal{I}^2 \subset \mathcal{I}_1\mathcal{I}$ and $\mathcal{I}^2 \bigcap \mathcal{I}_1\mathcal{J} = \mathcal{I}\mathcal{J}$. Notice that on $C_1 - p$ the inclusion map is an isomorphism. At p it is defined in coordinates by $x^2y^2 \mapsto y \cdot x^2y$, and, therefore, vanishes to order 1 at p.

Now $\mathcal{I}^2/\mathcal{I}\mathcal{J} \cong \mathcal{I}/\mathcal{J} \otimes \mathcal{I}/\mathcal{J} \cong \mathcal{O}_1(-2)$, so $\mathcal{I}_1\mathcal{I}/\mathcal{I}_1\mathcal{J} \cong \mathcal{O}_1(-1)$.

□

Corollary 3.1 $\mathcal{J}/\mathcal{I}_1\mathcal{J} = (-1, 2)$ *or* $(0, 1)$.

Proof: This follows immediately from the exact sequence 3.1 above.

□

Lemma 3.2 $\mathcal{J}/\mathcal{I}_2\mathcal{J} = (-1, 1)$ *or* $(0, 0)$.

Proof: To calculate the decomposition of the sheaf $\mathcal{J}/\mathcal{I}_2\mathcal{J}$, consider the injection $\mathcal{J}/\mathcal{I}_2\mathcal{J} \hookrightarrow \mathcal{I}/\mathcal{I}_2\mathcal{I}$. This is a well defined injection since $\mathcal{J} \subset \mathcal{I}$ and $\mathcal{J} \bigcap \mathcal{I}_2\mathcal{I} = \mathcal{I}_2\mathcal{J}$, as can be determined by a local coordinate calculation. This map is defined on generators by $x^2y \mapsto x \cdot xy$ and $z \mapsto z$, so the map vanishes to order 1 at the point p. $\mathcal{J} \cong \mathcal{I}$ off of p, so the map is an isomorphism on $C_2 - \{p\}$. This shows that $\mathcal{J}/\mathcal{I}_2\mathcal{J} = (-1, 1)$ or $(0, 0)$.

□

Lemma 3.3 $\mathcal{J}^m/\mathcal{I}_1\mathcal{J}^m \cong S^m(\mathcal{J}/\mathcal{I}_1\mathcal{J})$

Proof: Define a map $\mathcal{J}^{\otimes m} \to \mathcal{J}^m/\mathcal{I}_1\mathcal{J}^m$ by multiplication of functions. This map kills $\mathcal{I}_1\mathcal{J} \otimes \mathcal{J}^{\otimes(m-1)}$, thus giving a well defined map $S^m(\mathcal{J}/\mathcal{I}_1\mathcal{J}) \to \mathcal{J}^m/\mathcal{I}_1\mathcal{J}^m$. Lemma A.2 from appendix A shows that these sheaves are generated by the same elements on all of C_1, and, therefore, this map is an isomorphism.

□

Lemma 3.4 $\mathcal{I}\mathcal{J}^m/\mathcal{J}^{m+1} \cong \mathcal{I}/\mathcal{J} \otimes S^m(\mathcal{J}/\mathcal{I}_1\mathcal{J})$

Proof: Define a map $\mathcal{J}^{\otimes m} \otimes \mathcal{I} \to \mathcal{I}\mathcal{J}^m/\mathcal{J}^{m+1}$ by multiplication of functions. The sheaves $\mathcal{J}^{\otimes(m+1)}$ and $\mathcal{I} \otimes \mathcal{I}_1\mathcal{J} \otimes \mathcal{J}^{\otimes(m-1)}$ are killed by this map since $\mathcal{J}^{m+1}$ and $\mathcal{I}_1\mathcal{I}\mathcal{J}^m$ are both contained in $\mathcal{J}^{m+1}$. Lemma A.3 shows that these sheaves are generated locally by the same elements on C_1. This map, then, is an isomorphism.

□

Lemma 3.5 $\mathcal{I}_1\mathcal{J}^m/\mathcal{I}\mathcal{J}^m \cong \mathcal{I}_1/\mathcal{I} \otimes S^m(\mathcal{J}/\mathcal{I}_2\mathcal{J})$

Proof: The multiplication map $\mathcal{I}_1 \otimes \mathcal{J}^{\otimes m} \to \mathcal{I}_1\mathcal{J}^m/\mathcal{I}\mathcal{J}^m$ kills $\mathcal{I} \otimes \mathcal{J}^m$ and $\mathcal{I}_1 \otimes \mathcal{I}_2\mathcal{J} \otimes \mathcal{J}^{\otimes(m-1)}$ since $\mathcal{I}\mathcal{J}^m$ and $\mathcal{I}_1\mathcal{I}_2\mathcal{J}^m$ are both subsheaves of $\mathcal{I}\mathcal{J}^m$ The isomorphism again follows from lemma A.4.

□

For the remainder of this case it will be assumed that $\mathcal{J}/\mathcal{I}_1\mathcal{J} = (0,1)$ and $\mathcal{J}/\mathcal{I}_2\mathcal{J} = (0,0)$. This means that C_1 has length 2 and C_2 has length 1 as defined and discussed in chapter 1. This assumption first allows us to prove the following proposition.

Proposition 3.1 *The map on global sections,* $H^0(\hat{\mathcal{J}}) \longrightarrow H^0(\mathcal{J}/\mathcal{J}^2)$ *, is surjective.*

Proof: The proof is by showing the vanishing of appropriate H^1's. Recall that $\mathcal{I}/\mathcal{J} \cong \mathcal{O}_1(-1)$ and $\mathcal{I}_1/\mathcal{I} \cong \mathcal{O}_2(-1)$. Therefore, the rank $m+1$ locally free sheaves $\mathcal{I}/\mathcal{J} \otimes S^m(\mathcal{J}/\mathcal{I}_1\mathcal{J})$ and $\mathcal{I}_1/\mathcal{I} \otimes S^m(\mathcal{J}/\mathcal{I}_2\mathcal{J})$ are isomorphic to $\bigoplus_{i=-1}^{m-1} \mathcal{O}_1(i)$ and $\mathcal{O}_2(-1)^{\oplus m}$, respectively. Also, since $\mathcal{J}/\mathcal{I}_1\mathcal{J} = (0,1)$, the rank $m+1$ sheaf $S^m(\mathcal{J}/\mathcal{I}_1\mathcal{J})$ is isomorphic to $\bigoplus_{i=0}^{m} \mathcal{O}_1(i)$. Therefore, from lemma 3.4, $H^1(\mathcal{I}\mathcal{J}^m/\mathcal{J}^{m+1}) = 0$, and from lemma 3.5, $H^1(\mathcal{I}_1\mathcal{J}^m/\mathcal{I}\mathcal{J}^m) = 0$.

The exact sequence

$$0 \longrightarrow \mathcal{I}_1\mathcal{J}^m/\mathcal{I}\mathcal{J}^m \longrightarrow \mathcal{J}^m/\mathcal{I}\mathcal{J}^m \longrightarrow \mathcal{J}^m/\mathcal{I}_1\mathcal{J}^m \longrightarrow 0 \tag{3.2}$$

shows, then, that $H^1(\mathcal{J}^m/\mathcal{I}\mathcal{J}^m) = 0$ and $H^0(\mathcal{J}^m/\mathcal{I}\mathcal{J}^m) \longrightarrow H^0(\mathcal{J}^m/\mathcal{I}_1\mathcal{J}^m)$ is surjective for all $m \geq 1$. Similarly, from

$$0 \longrightarrow \mathcal{I}\mathcal{J}^m/\mathcal{J}^{m+1} \longrightarrow \mathcal{J}^m/\mathcal{J}^{m+1} \longrightarrow \mathcal{J}^m/\mathcal{I}\mathcal{J}^m \longrightarrow 0 \tag{3.3}$$

we obtain $H^1(\mathcal{J}^m/\mathcal{J}^{m+1}) = 0$ and $H^0(\mathcal{J}^m/\mathcal{J}^{m+1}) \longrightarrow H^0(\mathcal{J}^m/\mathcal{I}\mathcal{J}^m)$ is surjective for all $m \geq 1$.

To complete the proof of the proposition, it needs to be shown that the maps $H^0(\mathcal{J}/\mathcal{J}^{m+1}) \longrightarrow H^0(\mathcal{J}/\mathcal{J}^m) \longrightarrow H^0(\mathcal{J}/\mathcal{J}^2)$ are surjective for all $m \geq 2$.

First it will be shown that $H^1(\mathcal{J}^l/\mathcal{J}^m) = 0$ for all $l < m$ by induction on $m-l > 0$. The case for $m-l = 1$ was shown above. Assuming the vanishing for all m, l with

$m - l \leq k$, we have $H^1(\mathcal{J}^{l+k}/\mathcal{J}^{l+k+1}) = 0$ and $H^1(\mathcal{J}^l/\mathcal{J}^{l+k}) = 0$, so the exact sequence

$$0 \longrightarrow \mathcal{J}^{l+k}/\mathcal{J}^{l+k+1} \longrightarrow \mathcal{J}^l/\mathcal{J}^{l+k+1} \longrightarrow \mathcal{J}^l/\mathcal{J}^{l+k} \longrightarrow 0$$

and its long exact cohomology sequence give the vanishing of $H^1(\mathcal{J}^l/\mathcal{J}^{l+k+1})$.

The induction argument shows that $H^1(\mathcal{J}^2/\mathcal{J}^m) = 0$ for all $m \geq 3$, so the exact sequence

$$0 \longrightarrow \mathcal{J}^2/\mathcal{J}^m \longrightarrow \mathcal{J}/\mathcal{J}^m \longrightarrow \mathcal{J}/\mathcal{J}^2 \longrightarrow 0$$

gives the surjective map on global sections $H^0(\mathcal{J}/\mathcal{J}^m) \longrightarrow H^0(\mathcal{J}/\mathcal{J}^2)$. Similarly, from the vanishing of $H^1(\mathcal{J}^m/\mathcal{J}^{m+1})$ and the exact sequence

$$0 \longrightarrow \mathcal{J}^m/\mathcal{J}^{m+1} \longrightarrow \mathcal{J}/\mathcal{J}^{m+1} \longrightarrow \mathcal{J}/\mathcal{J}^m \longrightarrow 0,$$

the map on global sections $H^0(\mathcal{J}/\mathcal{J}^{m+1}) \longrightarrow H^0(\mathcal{J}/\mathcal{J}^m)$ is also surjective.

□

Corollary 3.2 *We have*

1) $h^0(\mathcal{J}/\mathcal{J}^2) = 4$

2) $H^1(\mathcal{J}/\mathcal{J}^2) = 0$

3) The maps $H^0(\mathcal{J}/\mathcal{J}^2) \to H^0(\mathcal{J}/\mathcal{I}\mathcal{J}) \to H^0(\mathcal{J}/\mathcal{I}_1\mathcal{J}) \to H^0(\mathcal{J}/\mathcal{I}_1\mathcal{I})$ *are surjective.*

Proof: In lemma 3.1 it was shown that $\mathcal{I}_1\mathcal{I}/\mathcal{I}_1\mathcal{J} \cong \mathcal{O}_1(-1)$, so from the exact sequence 3.1 the map $H^0(\mathcal{J}/\mathcal{I}_1\mathcal{J}) \to H^0(\mathcal{J}/\mathcal{I}_1\mathcal{I})$ is surjective and $h^0(\mathcal{J}/\mathcal{I}_1\mathcal{J}) = h^0(\mathcal{J}/\mathcal{I}_1\mathcal{I}) = 3$.

$\mathcal{I}_1/\mathcal{I} \cong \mathcal{O}_2(-1)$ and $\mathcal{J}/\mathcal{I}_2\mathcal{J} = (0,0)$ by assumption, so letting $m = 1$ in lemma 3.5 shows that $\mathcal{I}_1\mathcal{J}/\mathcal{I}\mathcal{J} = (-1,-1)$. The map $H^0(\mathcal{J}/\mathcal{I}\mathcal{J}) \to H^0(\mathcal{J}/\mathcal{I}_1\mathcal{J})$ being surjective, then, is an immediate consequence of the exact sequence 3.2 with $m = 1$. Furthermore, this sequence shows that $h^0(\mathcal{J}/\mathcal{I}\mathcal{J}) = h^0(\mathcal{J}/\mathcal{I}_1\mathcal{J}) = 3$.

For $m = 1$ in lemma 3.4, $\mathcal{I}\mathcal{J}/\mathcal{J}^2 = (-1,0)$, so the exact sequence 3.3 gives the surjective map $H^0(\mathcal{J}/\mathcal{J}^2) \to H^0(\mathcal{J}/\mathcal{I}\mathcal{J})$ as well as the fact that $H^1(\mathcal{J}/\mathcal{J}^2) = 0$. $h^0(\mathcal{J}/\mathcal{J}^2) = 4$ also follows from the cohomology exact sequence associated to the exact sequence 3.3.

□

Lemma 3.6 *The map on global sections,* $H^0(\mathcal{J}/\mathcal{I}\mathcal{J}) \to H^0(\mathcal{J}/\mathcal{I}_2\mathcal{J})$*, is surjective.*

Proof: As in lemma 2.2 the sheaf $\mathcal{I}_2\mathcal{J}/\mathcal{I}\mathcal{J}$ is locally free of rank 2 on C_1. The global map $\mathcal{I}_2\mathcal{J}/\mathcal{I}\mathcal{J} \hookrightarrow \mathcal{J}/\mathcal{I}_1\mathcal{J}$ is injective since $\mathcal{I}_2\mathcal{J} \subset \mathcal{J}$ and, from a local coordinate calculation, $\mathcal{I}_2\mathcal{J} \bigcap \mathcal{I}_1\mathcal{J} = \mathcal{I}\mathcal{J}$. Furthermore, this map is an isomorphism away from the point of intersection p, as $\mathcal{I}_2 = \mathcal{O}_X$ and $\mathcal{I} = \mathcal{I}_1$. At p this injection is defined in coordinates by $x^2y^2 \mapsto y \cdot x^2y$, $yz \mapsto y \cdot z$. By assumption, $\mathcal{J}/\mathcal{I}_2\mathcal{J} = (0,1)$, so $\mathcal{I}_2\mathcal{J}/\mathcal{I}\mathcal{J} = (-1,0)$. In particular, this shows that $H^1(\mathcal{I}_2\mathcal{J}/\mathcal{I}\mathcal{J}) = 0$. From the exact sequence

$$0 \longrightarrow \mathcal{I}_2\mathcal{J}/\mathcal{I}\mathcal{J} \longrightarrow \mathcal{J}/\mathcal{I}\mathcal{J} \longrightarrow \mathcal{J}/\mathcal{I}_2\mathcal{J} \longrightarrow 0,$$

and its long exact cohomology sequence, then, $H^0(\mathcal{J}/\mathcal{I}\mathcal{J}) \to H^0(\mathcal{J}/\mathcal{I}_2\mathcal{J})$ is surjective.

□

Lemma 3.7 *The sheaves $\mathcal{J}/\mathcal{J}^2$ and $\mathcal{J}/\mathcal{I}\mathcal{J}$ are generated by global sections.*

Proof: This lemma will be proven first for the sheaf $\mathcal{J}/\mathcal{J}^2$. It will be shown that at any point $q \in C$, every local section, i.e. section of $\mathcal{J}/m_q\mathcal{J}$, is the restriction of a global section. That is, it will be shown that $H^0(\mathcal{J}/\mathcal{J}^2) \to H^0(\mathcal{J}/m_q\mathcal{J})$ is surjective for all $q \in C$. It was shown in corollary 3.2 and lemma 3.6 that the maps $H^0(\mathcal{J}/\mathcal{J}^2) \to H^0(\mathcal{J}/\mathcal{I}_i\mathcal{J})$ are surjective for $i = 1, 2$. By the assumption that $\mathcal{J}/\mathcal{I}_i\mathcal{J}$ decomposes with no negative factors, these two sheaves are generated by global sections. Let $q \in C$. q is on C_i for some i. But $\mathcal{J}/\mathcal{I}_i\mathcal{J}$ is generated by global sections, so $H^0(\mathcal{J}/\mathcal{I}_i\mathcal{J}) \to H^0(\mathcal{J}/m_q\mathcal{J})$ is surjective. Therefore the composition $H^0(\mathcal{J}/\mathcal{J}^2) \to H^0(\mathcal{J}/m_q\mathcal{J})$ is surjective.

The proof that $\mathcal{J}/\mathcal{I}\mathcal{J}$ is generated by global sections is the exact proof as that for $\mathcal{J}/\mathcal{J}^2$, but replacing $\mathcal{J}/\mathcal{J}^2$ with $\mathcal{J}/\mathcal{I}\mathcal{J}$.

□

By the discussion in chapter 1, there is a formal map $\hat{f} : \hat{X} \to \hat{\mathbf{C}}^4$ for which $\hat{f}^{-1}(0) = \hat{\mathcal{J}}$, so to determine the singularity from contracting C we will study the general section of the defining ideal $\hat{\mathcal{J}}$. In particular, a singularity can only occur on C if a section of $\hat{\mathcal{J}}$ vanishes at that point.

Lemma 3.8 *The zero scheme of a general section of $\hat{\mathcal{J}}$ is a smooth surface except for two distinct formal A_1 singularities on $C_1 - \{p\}$.*

Proof: A general section of $\hat{\mathcal{J}}$ is of the form $f \cdot x^2y + g \cdot z$ with $f, g \in \mathcal{O}_{p,\hat{X}}$. But this is also a general section of the sheaf $\mathcal{J}/\mathcal{I}\mathcal{J}$, and considering this as a section of this sheaf, a general one will satisfy $g(p) \neq 0$. Lifting this section to a general section of $\hat{\mathcal{J}}$ by the surjective map $H^0(\hat{\mathcal{J}}) \longrightarrow H^0(\mathcal{J}/\mathcal{I}\mathcal{J})$, we have that $g(p) \neq 0$, which shows that the zero scheme of a general section of $\hat{\mathcal{J}}$ is smooth at p.

On $C_2 - \{p\}$ a general section of $\hat{\mathcal{J}}$ is of the form $f \cdot y + g \cdot z$, and since one of f, g is a unit in $\mathcal{O}_{p,\hat{X}}$, a general section defines a smooth surface on $C_2 - \{p\}$ as well.

On $C_1 - \{p\}$ a general section of $\hat{\mathcal{J}}$ is $f \cdot x^2 + g \cdot z$. In this situation, the general section may be singular, and it will now be shown that this is the case. f or g is a unit in the ring $\mathcal{O}_{p,\hat{X}}$, and the only singularities can occur when f is a unit. Now that we are on C_1, the general section must come from the invertible sheaf $\mathcal{J}/\mathcal{I}_1\mathcal{I}$ because of the surjective map $H^0(\hat{\mathcal{J}}) \longrightarrow H^0(\mathcal{J}/\mathcal{I}_1\mathcal{I})$. Locally this map is defined by $f \cdot x^2 + g \cdot z \mapsto g \cdot z$. In these coordinates, $\mathcal{I}_1\mathcal{I} = (x^2, xz, z^2)$, so for a nonzero section of $\mathcal{J}/\mathcal{I}_1\mathcal{I}$ coordinates can be chosen so that g is a function of y only. Lifting this section to $\hat{\mathcal{J}}$, where f is a unit, a general section of $\hat{\mathcal{J}}$ is of the form $x^2 + g \cdot z$ with g a function of y only. The only way that such a section can define a rational double point is if g vanishes to order 1. This can be seen from the general equation of the rational double points. Since g vanishes to first order the map $(x, g, z) \mapsto (x, y, z)$ defines an analytic change of coordinates, and the general section of $\hat{\mathcal{J}}$ in these coordinates is $x^2 + yz$. This can be seen to be an A_1 singularity from table 1.1. In conclusion, on $C_1 - \{p\}$, a general section of $\mathcal{J}$ has at least one A_1 singularity.

We are now ready to determine the number of A_1 singularities that exist on the general section. In this case we show that there are two A_1 singularities on $C_1 - \{p\}$. From the exact sequence

$$0 \longrightarrow \mathcal{I}_1\mathcal{I}/\mathcal{I}_1\mathcal{J} \longrightarrow \mathcal{J}/\mathcal{I}_1\mathcal{J} \longrightarrow \mathcal{J}/\mathcal{I}_1\mathcal{I} \longrightarrow 0$$

and the calculations that showed that $\mathcal{I}_1\mathcal{I}/\mathcal{I}_1\mathcal{J} \cong \mathcal{O}_1(-1)$, $\mathcal{J}/\mathcal{I}_1\mathcal{J} = (0,1)$ and $\mathcal{J}/\mathcal{I}_1\mathcal{I} \cong \mathcal{O}_1(2)$, we can conclude from the long exact cohomology sequence that $H^0(\mathcal{J}/\mathcal{I}_1\mathcal{J}) \cong H^0(\mathcal{J}/\mathcal{I}_1\mathcal{I})$. Therefore, the global sections of $\mathcal{J}/\mathcal{I}_1\mathcal{J}$ can be considered as the three dimensional space of global sections of homogeneous quadratic polynomials on $C_1 \cong \mathbf{P}^1$. The subspace consisting of those quadratics with double roots is not the entire space, and, therefore, a general section of $H^0(\mathcal{J}/\mathcal{I}_1\mathcal{J})$ has two distinct roots. Since the singularities can only occur at the vanishing of a section, we have shown that there are exactly two A_1 singularities on $C_1 - \{p\}$.

□

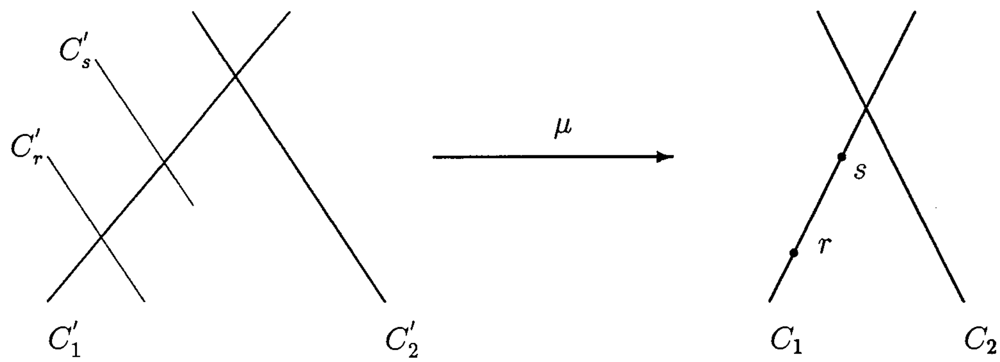


Figure 3.1: $D_4(2,1)$ configuration

The minimal resolution of the singularity resulting from contracting C is found by now resolving these two A_1 singularities. Let $\mu : S' \rightarrow S$ be the blow-up of S, the zero scheme of a general section of $\hat{\mathcal{J}}$, at the two A_1 singularities, which will be called r and s. The exceptional curves over r and s are smooth rational curves that do not intersect, since r and s are distinct. Away from these two points μ is an isomorphism, so the strict transforms of C_1 and C_2 are also smooth rational curves intersecting transversely in S', as C_1 and C_2 do in S. Therefore, S' contains a connected reducible rational curve, denoted C', with four components (see figure 3.1). These components are the exceptional curves over r and s, called C'_r and C'_s respectively, and the strict transforms C'_1 and C'_2 of C_1 and C_2. Let I', $I_r{}'$, $I_s{}'$, $I_1{}'$ and $I_2{}'$ be the ideal sheaves of C', C'_r,C'_s, C'_1 and C'_2 in S', respectively. By proposition 1.1, to complete the proof that C contracts to a cD_4 singularity it must be shown that $I_i'/I_i'^2 \cong \mathcal{O}_{C'_i}(2)$ for each $i \in \{1, 2, r, s\}$. To accomplish this we compare the conormal bundles on these components to related sheaves on the components of C.

If I_1, I_2 and I are the ideal sheaves of C_1, C_2 and C in S, then by definition $I/I^{(2)} = \mathcal{I}/Sat(\mathcal{I}^2, f)$, where f is a section of $\mathcal{J}$ defining the surface S. $Sat(\mathcal{I}^2, f) = (\mathcal{I}^2, f) + (torsion)$, where the torsion ideal consists locally of elements of $\mathcal{I}/(\mathcal{I}^2, f)$ that annihilate a power of the maximal ideal $m_{p',X} = (x, y, z)$ at p'. By modding out by torsion, $I/I^{(2)}$ is an invertible sheaf on C. The invertible sheaves $I_1/I_1{}^{(2)}$ and $I_2/I_2{}^{(2)}$ on C_1 and C_2 are defined similarly with $\mathcal{I}$ replaced with $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively. μ induces a map on sections defined by pullback $\mu^* : I/I^{(2)} \rightarrow I'/I'^2$.

μ^* is an isomorphism of sections of C_2 since neither r nor s lie on C_2. That is,

$I/I^{(2)} = I_2/I_2{}^{(2)} \overset{\mu^*}{\cong} I_2'/I_2'^2 = I'/I'^2$. S is smooth on C_2 so $I_2/I_2{}^{(2)} = I_2/I_2{}^2$, and this sheaf was seen to be isomorphic to $\mathcal{O}_2(2)$ in the previous section. $C_2 \cong C_2'$ under the blow-up, which proves that $I_2'/I_2'^2 \cong \mathcal{O}_{C_2'}(2)$.

The curves C_r' and C_s' are exceptional curves from the resolution of the rational double points r and s, so $I_r'/I_r'^2 \cong \mathcal{O}_{C_r'}(2)$ and $I_s'/I_s'^2 \cong \mathcal{O}_{C_s'}(2)$.

To show that $I_1'/I_1'^2 \cong \mathcal{O}_{C_1'}(2)$ the following two lemmas will be used.

Lemma 3.9 $I/I^{(2)}|_{C_1} \cong \mathcal{I}/\mathcal{J}$

Proof: From the definition of $I/I^{(2)}$, $I/I^{(2)}|_{C_1} \cong \mathcal{I}/\mathcal{I}_1\mathcal{I} + Sat(\mathcal{I}^2, f)$. $\mathcal{I}_1\mathcal{I}$ is also a subsheaf of $\mathcal{J}$, so the identity map $I/I^{(2)}|_{C_1} \to \mathcal{I}/\mathcal{J}$ is well defined. The lemma will be proven by showing that $\mathcal{J} = \mathcal{I}_1\mathcal{I} + Sat(\mathcal{I}^2, f)$ locally everywhere.

On $C_1 - \{p\}$ $I/I^{(2)}|_{C_1} = I_1/I_1{}^{(2)} = \mathcal{I}_1/Sat(\mathcal{I}_1{}^2, f)$ The defining polynomial f varies, so first calculate $Sat(\mathcal{I}_1{}^2, f)$ at the singular points r and s where $f = yz + x^2$ in appropriate coordinates. Recalling that $\mathcal{I}_1 = (x, z)$, $(\mathcal{I}_1{}^2, f) = (x^2, xz, z^2, yz + x^2)$, from which we see that $xz, yz, z^2 \in (\mathcal{I}_1{}^2, f)$. Therefore, the torsion element is z, and $Sat(\mathcal{I}_1{}^2, f) = (x^2, z)$. These are the generators of the ideal $\mathcal{J}$ as well.

On $C_1 - \{p, r, s\}$, S may be taken to be the smooth surface defined by $f = z$. Being smooth, $Sat(\mathcal{I}_1{}^2, f) = (\mathcal{I}_1{}^2, f) = (x^2, z)$, and again these are the local generators of $\mathcal{J}$.

At the point p, S again may be taken to be smooth with $f = z$ and $Sat(\mathcal{I}^2, f) = (\mathcal{I}^2, f) = (x^2y^2, z)$. $\mathcal{I}_1\mathcal{I} = (x^2y, z)$, so $\mathcal{I}_1\mathcal{I} + Sat(\mathcal{I}^2, f) = (x^2y, z) = \mathcal{J}$. Therefore, $\mathcal{I}_1\mathcal{I} + Sat(\mathcal{I}^2, f) = \mathcal{J}$ everywhere.

□

Lemma 3.10 $I_1/I_1{}^{(2)} \cong \mathcal{O}_1$

Proof: The inclusion map $I/I^{(2)}|_{C_1} \hookrightarrow I_1/I_1{}^{(2)}$ is an isomorphism on $C_1 - \{p\}$. To determine the degree of the invertible sheaf $I_1/I_1{}^{(2)}$, then, it suffices to find the order of vanishing at p. $I_1/I_1{}^{(2)} = \mathcal{I}_1/Sat(\mathcal{I}_1{}^2, f)$ with $\mathcal{I}_1 = (x, z)$ and $Sat(\mathcal{I}_1{}^2, f) = (\mathcal{I}_1{}^2, z) = (x^2, z)$. $I_1/I_1{}^{(2)}$, then, is generated by x at p. From the calculation in the previous lemma, $I/I^{(2)}|_{C_1}$is generated by xy at p. Therefore, in local coordinates, the inclusion map is defined by $xy \mapsto y \cdot x$, and y being a local coordinate on C_1 implies the order of vanishing is 1. By applying the previous lemma and recalling that $\mathcal{I}/\mathcal{J} \cong \mathcal{O}_1(-1)$, the proof is completed.

□

Now the map $\mu^* : I_1/I_1^{(2)} \to I_1'/I_1'^2$ is looked at more carefully to determine the degree of the sheaf $I_1'/I_1'^2$. μ is the blow-up of r and s, so μ^* is an isomorphism on $C_1 - \{r, s\}$. At the singularities S is defined by $f = yz + x^2$, and with (X, Y, Z) homogeneous coordinates on $\mathbf{P}^2$, S' is the surface in $\mathbf{C}^3 \times \mathbf{P}^2$ defined by

$$\begin{aligned} yz + x^2 &= 0 \\ xY &= yX \\ xZ &= zX \\ yZ &= zY \end{aligned}$$

On the affine piece $Y \neq 0$ we have $x^2 + yz = 0$, $x = yX$ and $z = yZ$. In these coordinates, (y, X, Z), $y^2(X^2 + Z) = 0$. The exceptional set is given by $y = 0$ and $I_1' = (X)$, so in this patch the map μ^* is defined by $x \mapsto y \cdot X$. This vanishes to order one on the exceptional set, so μ^* vanishes to order one at r and s. $I_1/I_1^{(2)}$has degree 0 from the previous lemma, which means $I_1'/I_1'^2 \cong \mathcal{O}_{C_1'}(2)$.

Since every component has been shown to have conormal sheaf isomorphic to $\mathcal{O}_i(2)$ in the smooth surface S', by proposition 1.1 the following theorem has been proven.

Theorem 3.1 *If $f : X \to Y$ is a contraction with $f(C) = q$ and $C = C_1 \cup C_2$ has length$(2,1)$ with defining ideal $\mathcal{J} = (x^2y, z)$ at $p = C_1 \bigcap C_2$, then a general hyperplane section of q has a D_4 type singularity at q*

3.2 The $\mathbf{D_5(2,1)}$ Case

Case 2: $\mathcal{J} = (xy + gz, xz, z^2)$

For this $\mathcal{J}$, a general section in coordinates at p is of the form $A \cdot (xy + gz) + B \cdot xz + D \cdot z^2$ with $A, B, D \in \mathcal{O}_{p,C}$. In the remainder of this section we will determine the rational double points on the surface defined by this section . In particular, it will be shown that the surface S defined by this section has an A_1 type singularity at p. As noted in [KM], S has a singularity of this type if and only if the quadratic part of the defining polynomial has rank 3. Expanding A, B, D and g in their power series expansions, we have:

$$\begin{aligned} A = &\ a_0 + &a_1x + a_2y + a_3z + H_A \\ B = &\ b_0 + &b_1x + b_2y + b_3z + H_B \\ D = &\ d_0 + &d_1x + d_2y + d_3z + H_D \\ g = & &g_1x + g_2y + g_3z + H_g \end{aligned}$$

where H_A, H_B, H_D and H_g represent the higher order terms. Separating the quadratic part of this section, we can write the general section as

$$a_0(xy + (g_1x + g_2y + g_3z) \cdot z) + b_0xz + d_0z^2 + H,$$

where H denotes the higher order terms in the expansion. To avoid repeating the xz and z^2 terms, we can assume that $g_1 = g_3 = 0$, resulting in the polynomial

$$a_0xy + g_2yz + b_0xz + d_0z^2 + H.$$

This being the defining polynomial of a general section, it can further be assumed that $A(p) \neq 0$, implying $a_0 \neq 0$. Eliminating this unit, then, leaves

$$\begin{aligned} &xy + g_2yz + b_0xz + d_0z^2 + H = \\ &\qquad (x + g_2z)(y + b_0z) + (d_0 - b_0g_2)z^2 + H \end{aligned}$$

Applying the analytic change of coordinates $(x + g_2z, y + b_0z, z) \mapsto (x, y, z)$, we can write the quadratic part as

$$xy + (d_0 - b_0g_2)z^2.$$

For general D, B, and g, we will have $d_0 - b_0g_2 \neq 0$, so the rank is three. This proves that a general section of $\mathcal{J}$ has an A_1 type singularity at p.

In the above argument it was shown that it was only necessary to consider $g = g_2y + H_g$. Under the analytic change of coordinates $(x + g_2z, y + b_0z, z) \mapsto (x, y, z)$, $g = g_2(y - b_0z) + H_g'$, with H_g' expressed in the new coordinates (x, y, z). The ideal $\mathcal{J}$,then, becomes $((x - g_2z)(y - b_0z) + (g_2(y - b_0z) + H_g')z, (x - g_2z)z, z^2)$, which simplifies to $(xy + H_g'z, xz, z^2)$. The ideal sheaves $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}$ are unchanged under this coordinate change. Since the elements xz and z^2 occur as generators of $\mathcal{J}$, in determining a section of $\mathcal{J}$ it is only necessary to consider the terms of H_g' containing powers of y only. That is, we can assume that $H_g' = \sum_{i=2}^{\infty} g_i' y^i$. Changing coordinates again by $(x + (z/y)H_g', y, z) \mapsto (x, y, z)$, we get $\mathcal{J} = (xy, (x - (z/y)H_g')z, z^2)$, which can be put in the more simple form

$$\mathcal{J} = (xy, xz, z^2).$$

These coordinate changes do not affect the description of $\mathcal{I} = (xy, z)$, $\mathcal{I}_1 = (x, z)$ or $\mathcal{I}_2 = (y, z)$, so in all the calculations that follow, this simpler description of $\mathcal{J}$ will be used.

From the definition of $\mathcal{J}$ as $Ker(\mathcal{I} \to \mathcal{I}/\mathcal{I}_1\mathcal{I} \to \mathcal{O}_1(-1))$, we have $\mathcal{J}/\mathcal{I}_1\mathcal{I} \cong \mathcal{O}_1(2)$, generated by $\{xy\}$ at p, and $\mathcal{I}/\mathcal{J} \cong \mathcal{O}_1(-1)$, generated by $\{z\}$ at p. In this case, $\mathcal{J}/\mathcal{I}\mathcal{J}$ is not locally free of rank 2 on C. However, the restricted sheaves $\mathcal{J}/Sat(\mathcal{I}_1\mathcal{J})$ and $\mathcal{J}/Sat(\mathcal{I}_2\mathcal{J})$ are locally free of rank 2 on C_1 and C_2, respectively. The torsion element of $\mathcal{J}/\mathcal{I}_1\mathcal{J}$ is xz, since $x^2z, xz^2, xyz \in \mathcal{I}_1\mathcal{J}$. So, $Sat(\mathcal{I}_1\mathcal{J}) = (x^2y, xz, z^3)$ and $\mathcal{J}/Sat(\mathcal{I}_1\mathcal{J})$ is generated by $\{xy, z^2\}$ at p. Similar calculations show that z^2 is the torsion element of $\mathcal{J}/\mathcal{I}_2\mathcal{J}$, proving that $Sat(\mathcal{I}_2\mathcal{J}) = (xy^2, xyz, z^2)$ and $\{xy, xz\}$ generate $\mathcal{J}/Sat(\mathcal{I}_2\mathcal{J})$ at p.

Lemma 3.11 *$\mathcal{J}/Sat(\mathcal{I}_1\mathcal{J}) = (-2, 2)$, $(-1, 1)$ or $(0, 0)$*

Proof: Consider the exact sequence

$$0 \to \mathcal{I}_1\mathcal{I}/Sat(\mathcal{I}_1\mathcal{J}) \to \mathcal{J}/Sat(\mathcal{I}_1\mathcal{J}) \to \mathcal{J}/\mathcal{I}_1\mathcal{I} \to 0. \tag{3.4}$$

The torsion element of $\mathcal{I}_1\mathcal{I}/\mathcal{I}_1\mathcal{J}$ can be calculated to be xz, and this invertible sheaf on C_1, then, is generated by z^2. The injection map

$$\mathcal{I}^2/\mathcal{I}\mathcal{J} \hookrightarrow \mathcal{I}_1\mathcal{I}/Sat(\mathcal{I}_1\mathcal{J})$$

is well defined since $\mathcal{I}^2 \subset \mathcal{I}_1\mathcal{I}$, and local calculations show that $\mathcal{I}^2 \bigcap Sat(\mathcal{I}_1\mathcal{J}) = \mathcal{I}\mathcal{J}$. The map on generators is defined by $z^2 \mapsto z^2$, and, therefore, these rank 1 sheaves are isomorphic. $\mathcal{I}^2/\mathcal{I}\mathcal{J} \cong \mathcal{I}/\mathcal{J} \otimes \mathcal{I}/\mathcal{J} \cong \mathcal{O}_1(-2)$, and so $\mathcal{I}_1\mathcal{I}/Sat(\mathcal{I}_1\mathcal{J}) \cong \mathcal{O}_1(-2)$. From the exact sequence 3.4, then, $\mathcal{J}/Sat(\mathcal{I}_1\mathcal{J}) = (-2, 2), (-1, 1)$ or $(0, 0)$.

□

Lemma 3.12 $\mathcal{J}/Sat(\mathcal{I}_2\mathcal{J}) = (-1,1)$ *or* $(0,0)$

Proof: The injection map

$$\mathcal{J}/Sat(\mathcal{I}_2\mathcal{J}) \hookrightarrow \mathcal{I}/\mathcal{I}_2\mathcal{I}$$

is well defined since $\mathcal{J} \subset \mathcal{I}$ and is injective since $\mathcal{J} \bigcap \mathcal{I}_2\mathcal{I} = Sat(\mathcal{I}_2\mathcal{J})$. It is an isomorphism on $C_2 - \{p\}$ since $\mathcal{J} = \mathcal{I} = \mathcal{I}_2$, and at p the map is defined on generators by $xy \mapsto xy$ and $xz \mapsto x \cdot z$. $\mathcal{I}/\mathcal{I}_2\mathcal{I} = (0,1)$ and this map vanishes to order 1 on C_2, which means that $\mathcal{J}/Sat(\mathcal{I}_2\mathcal{J}) = (-1,1)$ or $(0,0)$.

□

For C to have length$(2,1)$ it will be assumed that $\mathcal{J}/Sat(\mathcal{I}_1\mathcal{J}) = (0,0)$ and $\mathcal{J}/Sat(\mathcal{I}_2\mathcal{J}) = (0,0)$.

With these conditions, it will be shown that sections of $\mathcal{J}/Sat(\mathcal{I}_1\mathcal{J})$ and $\mathcal{J}/Sat(\mathcal{I}_2\mathcal{J})$ can be lifted to sections of $\hat{\mathcal{J}}$. As in Case 1, the following lemmas will be utilized.

Lemma 3.13 $H^1\left(Sat(\mathcal{J}^m)/Sat(\mathcal{I}_1\mathcal{J}^m)\right) = 0$ *for* $m \geq 1$.

Proof: From lemma B.4 in appendix B there is an injection map $S^m(\mathcal{J}/Sat(\mathcal{I}_1\mathcal{J})) \hookrightarrow Sat(\mathcal{J}^m)/Sat(\mathcal{I}_1\mathcal{J}^m)$ given locally at the point of intersection by

$$\begin{aligned} (xz^2)^k(xy)^{m-2k}y^k &\mapsto y^k \cdot (xz^2)^k(xy)^{m-2k} \\ (xz^2)^k(z^2)^{m-2k}y^k &\mapsto y^k \cdot (xz^2)^k(z^2)^{m-2k} \end{aligned} \tag{3.5}$$

for $0 \leq k \leq i$, where $i = \lfloor m/2 \rfloor$. Now $S^m(\mathcal{J}/Sat(\mathcal{I}_1\mathcal{J})) \cong \mathcal{O}_1^{\oplus(m+1)}$, and the injection is seen to vanish to order $2\sum_{k=0}^{i} k$ if m is odd or $i + 2\sum_{k=0}^{i-1} k$ if m is odd since y is a local coordinate on C_1. The degree of $Sat(\mathcal{J}^m)/Sat(\mathcal{I}_1\mathcal{J}^m)$, then, is $2\sum_{k=0}^{i} k$ i or $i + 2\sum_{k=0}^{i-1} k$, depending on m being odd or even. In either case, though, since $\mathcal{O}_1^{\oplus(m+1)}$ injects into this sheaf, $Sat(\mathcal{J}^m)/Sat(\mathcal{I}_1\mathcal{J}^m)$ can have no factors $\mathcal{O}_1(a)$ with $a < 0$. Therefore, $H^1\left(Sat(\mathcal{J}^m)/Sat(\mathcal{I}_1\mathcal{J}^m)\right) = 0$.

□

Lemma 3.14 *For $m \geq 1$,*

1) $H^1\left(Sat(\mathcal{I}\mathcal{J}^m)/Sat(\mathcal{J}^{m+1})\right) = 0$

2) $\mathcal{I}\mathcal{J}/Sat(\mathcal{J}^2) \cong \mathcal{O}_1(-1) \oplus \mathcal{O}_1(-1)$.

Proof: There is an injection $\mathcal{I}/\mathcal{J} \otimes S^m(\mathcal{J}/Sat(\mathcal{I}_1\mathcal{J})) \hookrightarrow Sat(\mathcal{I}\mathcal{J}^m)/Sat(\mathcal{J}^{m+1})$, from lemma B.6, given locally at p by the equations B.4. We have $\mathcal{I}/\mathcal{J} \otimes S^m(\mathcal{J}/Sat(\mathcal{I}_1\mathcal{J}))$ $\cong \mathcal{O}_1(-1)^{\oplus(m+1)}$ and, arguing as in lemma 3.13, $Sat(\mathcal{I}\mathcal{J}^m)/Sat(\mathcal{J}^{m+1})$ has degree $2\sum_{k=0}^{i} k$ i or $i + 2\sum_{k=0}^{i-1} k$, depending on m being odd or even. In particular, then, $Sat(\mathcal{I}\mathcal{J}^m)/Sat(\mathcal{J}^{m+1})$ can have no factors $\mathcal{O}_1(a)$ in its decomposition with $a < -1$. This shows (1) in the statement of the lemma.

If $m = 1$, then the injection is an isomorphism since $k = 0$ in the equations B.4. Now $\mathcal{I}\mathcal{J}$ has no torsion, so $Sat(\mathcal{I}\mathcal{J}) = \mathcal{I}\mathcal{J}$ and $\mathcal{I}\mathcal{J}/Sat(\mathcal{J}^2) \cong \mathcal{O}_1(-1) \oplus \mathcal{O}_1(-1)$.

□

Lemma 3.15 *For $m \geq 1$*

1) $H^1\left(Sat(\mathcal{I}_1\mathcal{J}^m)/Sat(\mathcal{I}\mathcal{J}^m)\right) = 0$

2) $Sat(\mathcal{I}_1\mathcal{J})/\mathcal{I}\mathcal{J} \cong \mathcal{O}_2 \oplus \mathcal{O}_2(-1)$.

Proof: The injection $\mathcal{I}_1/\mathcal{I} \otimes S^m(\mathcal{J}/Sat(\mathcal{I}_2\mathcal{J})) \hookrightarrow Sat(\mathcal{I}_1\mathcal{J}^m)/Sat(\mathcal{I}\mathcal{J}^m)$ of lemma B.8 in appendix B, and its local description at p given in equations B.5, shows that $Sat(\mathcal{I}_1\mathcal{J}^m)/Sat(\mathcal{I}\mathcal{J}^m)$ has degree $\sum j = 0^m j - \lfloor j/2 \rfloor$ more than the degree of $\mathcal{I}_1/\mathcal{I} \otimes S^m(\mathcal{J}/Sat(\mathcal{I}_2\mathcal{J}))$. Since $\mathcal{I}_1/\mathcal{I} \otimes S^m(\mathcal{J}/Sat(\mathcal{I}_2\mathcal{J})) \cong \mathcal{O}_2(-1)^{\oplus(m+1)}$ and it injects into $Sat(\mathcal{I}_1\mathcal{J}^m)/Sat(\mathcal{I}\mathcal{J}^m)$, $Sat(\mathcal{I}_1\mathcal{J}^m)/Sat(\mathcal{I}\mathcal{J}^m)$ can have no factors $\mathcal{O}_2(a)$ with $a < -1$. This shows (1) in the statement of the lemma.

If $m = 1$, then $Sat(\mathcal{I}_1\mathcal{J})/\mathcal{I}\mathcal{J}$ has degree 1 more than that of $\mathcal{I}_1/\mathcal{I} \otimes \mathcal{J}/Sat(\mathcal{I}_2\mathcal{J}) \cong \mathcal{O}_2(-1) \oplus \mathcal{O}_2(-1)$ and has no factors $\mathcal{O}_2(a)$ with $a < -1$. The only rank 2 locally free sheaf of degree -1 with factors $a \geq -1$ is $\mathcal{O}_2 \oplus \mathcal{O}_2(-1)$.

□

Lemma 3.16 *We have*

1) $h^0(\mathcal{J}/\mathcal{J}^2) = 4$

2) $H^1(\mathcal{J}/\mathcal{J}^2) = 0$

3) The maps $H^0(\mathcal{J}/\mathcal{J}^2) \to H^0(\mathcal{J}/\mathcal{I}\mathcal{J}) \to H^0(\mathcal{J}/Sat(\mathcal{I}_1\mathcal{J}))$ *are surjective.*

Proof: The sequence

$$0 \to Sat(\mathcal{I}_1\mathcal{J})/\mathcal{I}\mathcal{J} \to \mathcal{J}/\mathcal{I}\mathcal{J} \to \mathcal{J}/Sat(\mathcal{I}_1\mathcal{J}) \to 0$$

can be written

$$0 \longrightarrow \mathcal{O}_2(-1) \oplus \mathcal{O}_2 \longrightarrow \mathcal{J}/\mathcal{I}\mathcal{J} \longrightarrow \mathcal{O}_1 \oplus \mathcal{O}_1 \longrightarrow 0$$

by applying lemma 3.15 (2) to the term on the left. Therefore, the map on global sections $H^0(\mathcal{J}/\mathcal{I}\mathcal{J}) \to H^0(\mathcal{J}/Sat(\mathcal{I}_1\mathcal{J}))$ is surjective, $H^1(\mathcal{J}/\mathcal{I}\mathcal{J}) = 0$ and $h^0(\mathcal{J}/\mathcal{I}\mathcal{J}) = 3$.

Lemma 3.14 (2) shows that $\mathcal{I}\mathcal{J}/Sat(\mathcal{J}^2) = (-1,-1)$, so from the exact sequence

$$0 \longrightarrow \mathcal{I}\mathcal{J}/Sat(\mathcal{J}^2) \longrightarrow \mathcal{J}/Sat(\mathcal{J}^2) \longrightarrow \mathcal{J}/\mathcal{I}\mathcal{J} \longrightarrow 0$$

it can be seen that $H^0(\mathcal{J}/Sat(\mathcal{J}^2)) \to H^0(\mathcal{J}/\mathcal{I}\mathcal{J})$ is surjective, $H^1(\mathcal{J}/Sat(\mathcal{J}^2)) = 0$ and $h^0(\mathcal{J}/Sat(\mathcal{J}^2) = 3$. Then, from the torsion exact sequence

$$0 \longrightarrow (torsion) \longrightarrow \mathcal{J}/\mathcal{J}^2 \longrightarrow \mathcal{J}/Sat(\mathcal{J}^2) \longrightarrow 0$$

and the fact that $H^1(torsion) = 0$, the map $H^0(\mathcal{J}/\mathcal{J}^2) \to H^0(\mathcal{J}/Sat(\mathcal{J}^2))$ is surjective, $H^1(\mathcal{J}/\mathcal{J}^2) = 0$, and $h^0(\mathcal{J}/\mathcal{J}^2) = 4$.

□

Lemma 3.17 $H^0(\mathcal{J}/\mathcal{I}\mathcal{J}) \to H^0(\mathcal{J}/Sat(\mathcal{I}_2\mathcal{J}))$ *is surjective.*

Proof: It has been shown that $Sat(\mathcal{I}_2\mathcal{J}) = (xy^2, xyz, z^2)$ locally at p, and, since $\mathcal{I}\mathcal{J} = (x^2y^2, xyz, x^2z^2, z^3)$ at p, the sheaf $Sat(\mathcal{I}_2\mathcal{J})/\mathcal{I}\mathcal{J}$ can be shown to be locally free of rank 2 on C_1 generated by $\{xy^2, z^2\}$ at p. The inclusion map $Sat(\mathcal{I}_2\mathcal{J})/\mathcal{I}\mathcal{J} \hookrightarrow \mathcal{J}/Sat(\mathcal{I}_1\mathcal{J})$ is well defined since $Sat(\mathcal{I}_2\mathcal{J}) \subset \mathcal{J}$ and $Sat(\mathcal{I}_1\mathcal{J}) \bigcap Sat(\mathcal{I}_2\mathcal{J}) = \mathcal{I}\mathcal{J}$. These results follow from local calculations. On $C_1 - \{p\}$, $\mathcal{I}_2 = \mathcal{O}_X$ and $\mathcal{I} = \mathcal{I}_1$, so $Sat(\mathcal{I}_2\mathcal{J})/\mathcal{I}\mathcal{J} \cong \mathcal{J}/\mathcal{I}_1\mathcal{J} \cong \mathcal{J}/Sat(\mathcal{I}_1\mathcal{J})$, which shows that this map is an isomorphism away from p. At p, the inclusion is defined on generators by $xy^2 \mapsto y \cdot xy$ and $z^2 \mapsto z^2$. This map, then, vanishes to first order and, therefore, has degree one less than that of $\mathcal{J}/Sat(\mathcal{I}_1\mathcal{J})$. So, $Sat(\mathcal{I}_2\mathcal{J})/\mathcal{I}\mathcal{J}$ has degree -1 and, since it injects into $\mathcal{J}/Sat(\mathcal{I}_1\mathcal{J}) = (0,0)$, it must decompose as $(-1,0)$. Now that $H^1(Sat(\mathcal{I}_2\mathcal{J})/\mathcal{I}\mathcal{J}) = 0$ has been established, from the coho mology exact sequence of

$$0 \longrightarrow Sat(\mathcal{I}_2\mathcal{J})/\mathcal{I}\mathcal{J} \longrightarrow \mathcal{J}/\mathcal{I}\mathcal{J} \longrightarrow \mathcal{J}/Sat(\mathcal{I}_2\mathcal{J}) \longrightarrow 0,$$

we see that the map on global sections, $H^0(\mathcal{J}/\mathcal{I}\mathcal{J}) \to H^0(\mathcal{J}/Sat(\mathcal{I}_2\mathcal{J}))$, is surjective.

□

Proposition 3.2 *The map on global sections, $H^0(\hat{\mathcal{J}}) \to H^0(\mathcal{J}/\mathcal{J}^2)$, is surjective.*

Proof: From the exact sequence

$$0 \longrightarrow Sat(\mathcal{I}_1\mathcal{J}^m)/\mathcal{I}\mathcal{J}^m \longrightarrow \mathcal{J}^m/\mathcal{I}\mathcal{J}^m \longrightarrow \mathcal{J}^m/Sat(\mathcal{I}_1\mathcal{J}^m) \longrightarrow 0$$

and lemmas 3.15 and 3.13, we have $H^1(\mathcal{J}^m/\mathcal{I}\mathcal{J}^m) = 0$. Then, from

$$0 \longrightarrow \mathcal{I}\mathcal{J}^m/Sat(\mathcal{J}^{m+1}) \longrightarrow \mathcal{J}^m/Sat(\mathcal{J}^{m+1}) \longrightarrow \mathcal{J}^m/\mathcal{I}\mathcal{J}^m \longrightarrow 0$$

and lemma 3.14 , $H^1(\mathcal{J}^m/Sat(\mathcal{J}^{m+1})) = 0$. Therefore, the torsion sequence

$$0 \longrightarrow torsion \longrightarrow \mathcal{J}^m/\mathcal{J}^{m+1} \longrightarrow \mathcal{J}^m/Sat(\mathcal{J}^{m+1}) \longrightarrow 0$$

shows that $H^1(\mathcal{J}^m/\mathcal{J}^{m+1}) = 0$ for all $m \geq 1$. The induction argument used in the proof of proposition 3.1 proves the map $H^0(\hat{\mathcal{J}}) \to H^0(\mathcal{J}/\mathcal{J}^2)$ is surjective.

□

Lemma 3.18 *The sheaves $\mathcal{J}/\mathcal{J}^2$ and $\mathcal{J}/\mathcal{I}\mathcal{J}$ are generated by global sections.*

Proof: This lemma will be proven first for the sheaf $\mathcal{J}/\mathcal{J}^2$. It will be shown that at any point $q \in C$, every local section, i.e. section of $\mathcal{J}/m_q\mathcal{J}$, is the restriction of a global section. That is, it will be shown that $H^0(\mathcal{J}/\mathcal{J}^2) \to H^0(\mathcal{J}/m_q\mathcal{J})$ is surjective for all $q \in C$. It was shown in lemmas 3.16 and 3.17 that the maps $H^0(\mathcal{J}/\mathcal{J}^2) \to H^0(\mathcal{J}/\mathcal{I}_i\mathcal{J})$ are surjective for $i = 1, 2$. By the assumption that $\mathcal{J}/\mathcal{I}_i\mathcal{J}$ decomposes with no negative factors, these two sheaves are generated by global sections. Let $q \in C$. q is on C_i for some i. But $\mathcal{J}/\mathcal{I}_i\mathcal{J}$ is generated by global sections, so $H^0(\mathcal{J}/\mathcal{I}_i\mathcal{J}) \to H^0(\mathcal{J}/m_q\mathcal{J})$ is surjective. Therefore the composition $H^0(\mathcal{J}/\mathcal{J}^2) \to H^0(\mathcal{J}/m_q\mathcal{J})$ is surjective.

The proof that $\mathcal{J}/\mathcal{I}\mathcal{J}$ is generated by global sections is the exact proof as that for $\mathcal{J}/\mathcal{J}^2$, but replacing $\mathcal{J}/\mathcal{J}^2$ with $\mathcal{J}/\mathcal{I}\mathcal{J}$.

□

As in the previous case, the general section of $\hat{\mathcal{J}}$ determines the general hyperplane section of the singularity q.

Lemma 3.19 *A general section of $\hat{\mathcal{J}}$ defines a smooth surface except for two distinct A_1 singularities on $C_1 - \{p\}$ and an A_1 singularity at the point of intersection p.*

Proof: A general section has been shown to have an A_1 singularity at the point of intersection.

On $C_2 - \{p\}$ a general section of $\mathcal{J}$ is of the form $f \cdot y + g \cdot z$ with one of $f, g \in \mathcal{O}_{p,C}$ a unit. In either case, a general section is smooth on $C_2 - \{p\}$.

On $C_1 - \{p\}$ a general section is of the form $f \cdot x + g \cdot z^2$, so the only singularities can occur when g is a unit. Consider this as a section of $\mathcal{J}/Sat(\mathcal{I}_1\mathcal{J})$. From the results of the proof of lemma 3.11, the exact sequence 3.4 can be written

$$0 \longrightarrow \mathcal{O}_1(-2) \longrightarrow \mathcal{J}/Sat(\mathcal{I}_1\mathcal{J}) \longrightarrow \mathcal{O}_1(2) \longrightarrow 0,$$

where $\mathcal{J}/\mathcal{I}_1\mathcal{J} \longrightarrow \mathcal{O}_1(2)$ is given locally by $f \cdot x + g \cdot z^2 \mapsto f \cdot x$, since $\mathcal{J}/\mathcal{I}_1\mathcal{I}$ is generated by $\{x\}$. $\mathcal{I}_1\mathcal{I} = (x^2, xz, z^2)$, so a nonzero section of $\mathcal{J}/\mathcal{I}_1\mathcal{I}$ has f as a function of y only. Furthermore, the only way that $f \cdot x + z^2$ can define a rational double point is if f vanishes to first order. The analytic change of coordinates $(x, f, z) \mapsto (x, y, z)$ gives the general section of $\mathcal{J}$ as $xy + z^2$, which defines an A_1 singularity.

The long exact cohomology sequence associated to the above is

$$0 \longrightarrow H^0(\mathcal{J}/Sat(\mathcal{I}_1\mathcal{J})) \longrightarrow H^0(\mathcal{O}_1(2)) \longrightarrow H^1(\mathcal{O}_1(-2)) \longrightarrow 0,$$

where $H^1(\mathcal{O}_1(-2))$ is a one dimensional vector space. So, the image of the sheaf of global sections $H^0(\mathcal{J}/Sat(\mathcal{I}_1\mathcal{J}))$ is a two dimensional subspace of the three dimensional vector space, $H^0(\mathcal{O}_1(2))$, of homogeneous quadratic polynomials on $C_1 \cong \mathbf{P}^1$. This subspace cannot be contained in the subspace of homogeneous quadratics with double roots because the discriminant locus does not contain a two dimensional linear subspace. Therefore, a general section will have two distinct zeros.

□

To determine the minimal resolution, let $\mu : S' \to S$ be the blow up of the three A_1 singularities. These will be labeled p, r and s, recalling that p is the A_1 at the point of intersection. In resolving p, r and s, the exceptional curves over r and s are smooth rational curves that do not intersect since r and s are distinct. C_1 and C_2 intersect transversely at p, so resolving p results in a smooth exceptional curve that is intersected by the proper transforms of C_1 and C_2 in two distinct points. S', then, contains a curve C' which is the union of five irreducible components; the exceptional

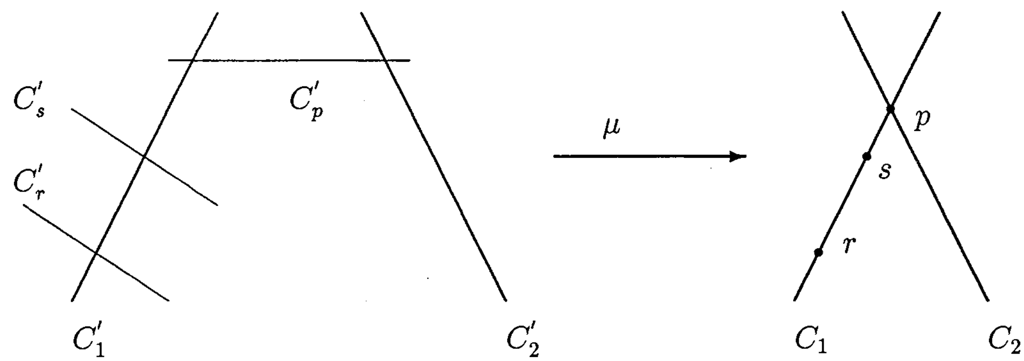


Figure 3.2: $D_5(2,1)$ configuration

curves C'_r, C'_s and C'_t of the three A_1 singularities and the strict transforms, C'_1 and C'_2, of C_1 and C_2 (see figure 3.2). Let I'_i be the ideal sheaf of C'_i in S' for $i \in \{1, 2, p, r, s\}$. From the configuration of these curves, by proposition 1.1, if $I'_i/I'^2_i \cong \mathcal{O}_{C'_i}(2)$ for each i, then C contracts to a cD_5 singularity.

The curves C'_i for $i = p, r, s$ are exceptional curves from the resolution of A_1 singularities, so $I'_i/I'^2_i \cong \mathcal{O}_{C'_i}(2)$ for these i.

As in the previous case, to prove this for $i = 1, 2$, we will study the map $\mu^* : I/I^{(2)} \to I'/I'^2$, where I is the ideal sheaf of C in S and I' is the ideal sheaf of C' in S'.

Lemma 3.20 $I/I^{(2)}|_{C_1} \cong \mathcal{I}/\mathcal{J}$

Proof: By definition $I/I^{(2)}|_{C_1} = \mathcal{I}/\mathcal{I}_1\mathcal{I} + Sat(\mathcal{I}^2, f)$, where F represents the local equation of S on C_1. Since $\mathcal{I}_1\mathcal{I} + Sat(\mathcal{I}^2, f) \subset \mathcal{J}$, the identity map $I/I^{(2)}|_{C_1} \to \mathcal{I}/\mathcal{J}$ is well defined. It will now be shown that $\mathcal{J} = \mathcal{I}_1\mathcal{I} + Sat(\mathcal{I}^2, f)$ locally everywhere on C_1.

On $C_1 - \{p\}$ $I/I^2|_{C_1} = I_1/{I_1}^{(2)} = \mathcal{I}_1/Sat({\mathcal{I}_1}^2, f)$. At the points r and s it was shown that coordinates can be chosen so that $f = xy + z^2$. $({\mathcal{I}_1}^2, f) = (x^2, xz, z^2, xy + z^2)$, so $Sat({\mathcal{I}_1}^2, f) = (x, z^2)$, since x is the torsion element. Therefore, $\mathcal{J} = Sat({\mathcal{I}_1}^2, f)$ at r and s.

On $C_1 - \{p, r, s\}$ coordinates can be chosen so that $f = x$, in which case $Sat({\mathcal{I}_1}^2, f)$ $= ({\mathcal{I}_1}^2, f) = (x, z^2)$. This again agrees locally with $\mathcal{J}$.

At the point p, $I/I^{(2)}|_{C_1} = \mathcal{I}/\mathcal{I}_1\mathcal{I} + Sat(\mathcal{I}^2, f)$, where again $f = xy + z^2$ in appropriately chosen coordinates. the saturation, $Sat(\mathcal{I}^2, f)$, is $(\mathcal{I}^2, f) = (xy, z^2)$, so

$\mathcal{I}_1\mathcal{I} + Sat(\mathcal{I}^2, f) = (x^2y, xz, z^2, xy, z^2) = (xy, xz, z^2) = \mathcal{J}$ at the point p.

□

Lemma 3.21 $I_1/I_1^{(2)} \cong \mathcal{O}_1(-1)$

Proof: The inclusion map $I/I^2|_{C_1} \hookrightarrow I_1/{I_1}^{(2)}$ is an isomorphism on $C_1 - \{p\}$, and at p, from the local calculation at p in the proof of the previous lemma, $I/I^{(2)}|_{C_1}$ is generated locally by z at p. The invertible sheaf $I_1/{I_1}^{(2)} = \mathcal{I}_1/Sat({\mathcal{I}_1}^2, f)$ is also generated by z at p, so the inclusion map is actually an isomorphism. Combining this information with the previous lemma, $I_1/{I_1}^{(2)} \cong I/I^{(2)}|_{C_1} \cong \mathcal{I}/\mathcal{J} \cong \mathcal{O}_1(-1)$.

□

Lemma 3.22 $I_1'/{I_1'}^2 \cong \mathcal{O}_1(2)$

Proof: The same argument as that following the proof of lemma 3.7 shows that $\mu^* : I_1/{I_1}^{(2)} \to I_1'/{I_1'}^2$ vanishes to order one at each A_1 singularity. Since there are three such singularities on C_1, namely p, r and s, the degree of $I_1'/{I_1'}^2$ is $-1 + 3 = 2$.

□

Lemma 3.23 $I_2/I_2^{(2)} \cong \mathcal{O}_2(1)$

Proof: The section defining the surface S defines an injective map $\mathcal{O}_C \to \mathcal{I}/\mathcal{I}^2$, giving us the exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{I}/\mathcal{I}^2 \to \mathcal{I}_{C,S}/\mathcal{I}_{C,S}^2 \to 0.$$

In the notation being used in this chapter, this is the sequence

$$0 \to \mathcal{O}_C \to \mathcal{I}/\mathcal{I}^2 \to I/I^2 \to 0.$$

Restricting to the curve C_2 we get the exact sequence

$$0 \to \mathcal{O}_2 \to \mathcal{I}/\mathcal{I}_2\mathcal{I} \to I/I^2|_{C_2} \to 0.$$

Using the fact that $\mathcal{I}/\mathcal{I}_2\mathcal{I} = (0,1)$, it must be that $I/I^2|_{C_2} \cong \mathcal{O}_2(1)$. There is a well defined injection $I/I^2|_{C_2} \hookrightarrow I_2/I_2^{(2)}$ since these are isomorphic away from p since C and S are both smooth. At p S is defined by $xy + z^2 = 0$, so $I = (z)$, $I_2 = (y, z)$ and $I_2^2 = (y, z^2)$. In coordinates, then, $I \bigcap I_2^{(2)} = (yz, z^2) = I_2I + I^2$, which shows that the injection is well defined everywhere. But $I/I^2|_{C_2}$ and $I_2/I_2^{(2)}$ are locally free sheaves on C_2 that are generated by the same element $\{z\}$, so this injection is in fact an isomorphism.

□

Lemma 3.24 $I_2'/I_2'^2 \cong \mathcal{O}_{C_2'}(2)$

Proof: $\mu^* : I_2/I_2{}^{(2)} \to \mathcal{I}_2'/I_2'^2$ is an isomorphism away from p since μ is an isomorphism away from p, r and s. It has been seen previously that μ^* vanishes to order 1 at p. Therefore, the degree of $I_2'/I_2'^2$ has degree 2.

□

Theorem 3.2 *If $f : X \to Y$ is a contraction with $f(C) = q$ and $C = C_1 \cup C_2$ has length $(2,1)$ with defining ideal $\mathcal{J} = (xy, xz, z^2)$ at $p = C_1 \bigcap C_2$, then a general hyperplane section of q has a D_5 type singularity.*

This completes the possible general hyperplane sections where one component has length 1 and one component has length 2. By the method that was used, the only way to determine if the RDP is a D_4 or a D_5 is to know the form of the defining ideal $\mathcal{J}$. In particular, knowing the length of each component is not enough information to determine which singularity results.

CHAPTER 4

THE LENGTH(2,2) CASES

We are now ready to discuss the possible singularities when both components have length 2. This will be accomplished by continuing from the length(2,1) cases in chapter 3. C_1 has length 2 for both of these cases, so to extend C_2 to a length 2 component, we must have $\mathcal{J}/\mathcal{I}_2\mathcal{J} = (-1,1)$ from case 1 and $\mathcal{J}/Sat(\mathcal{I}_2\mathcal{J}) = (-1,1)$ in case 2. It will be seen that with these conditions that C_2 has length greater than 1. Just as $\mathcal{J}$ was constructed by projecting to the -1 factor of $\mathcal{I}/\mathcal{I}_1\mathcal{I}$, a new ideal $\mathcal{K}$ will be created by projecting to the -1 factor of each of these sheaves. Case $1'$ will be the continuation from case 1 where $\mathcal{J} = (x^2y, z)$ and case $2'$ will continue from case 2 where $\mathcal{J} = (xy, xz, z^2)$. As there were two possible forms for the ideal $\mathcal{J}$, there will be two possible forms for the ideal $\mathcal{K}$ in case $1'$ and two forms for $\mathcal{K}$ in case $2'$. This gives us the new sub-cases $1'a$ and $1'b$ from case $1'$ and cases $2'a$ and $2'b$ from case $2'$. A complete analysis of one sub-case from each of these two cases will be be provided in this chapter.

Case 1′: $\mathcal{J} = (x^2y, z)$, $\mathcal{J}/\mathcal{I}_1\mathcal{J} = (0,1)$ and $\mathcal{J}/\mathcal{I}_2\mathcal{J} = (-1,1)$.

Define $\mathcal{K} = Ker(\mathcal{J} \to \mathcal{J}/\mathcal{I}_2\mathcal{J} \to \mathcal{O}_2(-1))$. By the definition of $\mathcal{K}$, $\mathcal{J}/\mathcal{K} \cong \mathcal{O}_2(-1)$ and $\mathcal{K}/\mathcal{I}_2\mathcal{J} \cong \mathcal{O}_2(1)$. This latter sheaf is a subsheaf of $\mathcal{J}/\mathcal{I}_2\mathcal{J}$, so it is generated at p by an element of the form $f \cdot x^2y + g \cdot z$, where one of $f, g \in \mathcal{O}_{p,C}$ is a unit.

If f is a unit, then the generator is of the form $x^2y + gz$ after eliminating f. So, $\mathcal{K} = (x^2y + gz) + \mathcal{I}_2\mathcal{J}$ and at p, $\mathcal{I}_2\mathcal{J} = (x^2y^2, yz, z^2)$, so

$$\mathcal{K} = (x^2y + gz, yz, z^2).$$

If g is a unit, then $\mathcal{K} = (fx^2y + z) + \mathcal{I}_2\mathcal{J} = (fx^2y + z, x^2y^2, yz)$. The analytic change of coordinates $(x, y, fx^2y + z) \mapsto (x, y, z)$ simplifies $\mathcal{K}$ to

$$\mathcal{K} = (x^2y^2, z)$$

and does not change $\mathcal{I}$, $\mathcal{I}_1$, $\mathcal{I}_2$ or $\mathcal{J}$.

Case 2′: $\mathcal{J} = (xy, xz, z^2)$, $\mathcal{J}/Sat(\mathcal{I}_1\mathcal{J}) = (0,0)$ and $\mathcal{J}/Sat(\mathcal{I}_2\mathcal{J}) = (-1,1)$.

Define $\mathcal{K} = Ker(\mathcal{J} \to \mathcal{J}/Sat(\mathcal{I}_2\mathcal{J}) \to \mathcal{O}_2(-1))$. By the definition of $\mathcal{K}$, $\mathcal{J}/\mathcal{K} \cong \mathcal{O}_2(-1)$ and $\mathcal{K}/Sat(\mathcal{I}_2\mathcal{J}) \cong \mathcal{O}_2(1)$. $\mathcal{K}/Sat(\mathcal{I}_2\mathcal{J})$ is a subsheaf of $\mathcal{J}/Sat(\mathcal{I}_2\mathcal{J})$, so it is generated at p by an element of the form $f \cdot xy + g \cdot xz$, where one of $f, g \in \mathcal{O}_{p,C}$ is a unit.

If f is a unit, then the generator is of the form $xy + gxz$ after eliminating f. So, $\mathcal{K} = (xy + gxz) + Sat(\mathcal{I}_2\mathcal{J})$ and at p, $Sat(\mathcal{I}_2\mathcal{J}) = (xy^2, xyz, z^2)$. This gives $\mathcal{K} = (xy + gxz, xy^2, xyz, z^2) = (xy + gxz, z^2$, and the analytic change of coordinates $(x, y + gz, z) \mapsto (x, y, z)$ simplifies $\mathcal{K}$ to

$$\mathcal{K} = (xy, z^2)$$

and does not change $\mathcal{I}$, $\mathcal{I}_1$, $\mathcal{I}_2$ or $\mathcal{J}$.

If g is a unit, then $\mathcal{K} = (fxy + xz) + Sat(\mathcal{I}_2\mathcal{J})$, which simplifies to

$$\mathcal{K} = (fxy + xz, xy^2, z^2).$$

4.1 The $\mathbf{D_5(2,2)}$ Case

Case 1′a: $\mathcal{K} = (x^2y^2, z)$

$\mathcal{K}$ is defined to be $Ker(\mathcal{J} \to \mathcal{J}/\mathcal{I}_2\mathcal{J} \to \mathcal{O}_2(-1))$ and $\mathcal{J} = (x^2y, z)$ at p, so $\mathcal{J}/\mathcal{K} \cong \mathcal{O}_2(-1)$, generated by $\{x^2y\}$ at p, and $\mathcal{K}/\mathcal{I}_2\mathcal{J} \cong \mathcal{O}_2(1)$, generated by $\{z\}$ at p. Since $\mathcal{K}/\mathcal{I}\mathcal{K}$ is a locally free sheaf of rank 2 on C, the sheaves $\mathcal{K}/\mathcal{I}_1\mathcal{K}$ and $\mathcal{K}/\mathcal{I}_2\mathcal{K}$ are locally free of rank 2 on C_1 and C_2, respectively, generated by $\{x^2y^2, z\}$ at the point p.

Lemma 4.1 $\mathcal{K}/\mathcal{I}_2\mathcal{K} = (0, 1)$.

Proof: The invertible sheaf $\mathcal{J}/\mathcal{K} \otimes \mathcal{J}/\mathcal{K} \cong \mathcal{J}^2/\mathcal{J}\mathcal{K}$ on C_2 is generated by $\{x^4y^2\}$ at p since $\mathcal{J}/\mathcal{K}$ is generated by $\{x^2y\}$. Also, $\mathcal{J}/\mathcal{K} \cong \mathcal{O}_2(-1)$ implies that $\mathcal{J}^2/\mathcal{J}\mathcal{K} \cong \mathcal{O}_2(-2)$.

In coordinates at p, $\mathcal{I}_2\mathcal{J} = (x^2y^2, yz, z^2)$ and $\mathcal{I}_2\mathcal{K} = (x^2y^3, yz, z^2)$, so local calculations show that $\mathcal{I}_2\mathcal{J}/\mathcal{I}_2\mathcal{K}$ is also an invertible sheaf on C_2 and it is generated by $\{x^2y^2\}$ at p. The degree of this sheaf can be determined from the injection

$$\mathcal{J}^2/\mathcal{J}\mathcal{K} \hookrightarrow \mathcal{I}_2\mathcal{J}/\mathcal{I}_2\mathcal{K}.$$

This is an injection since $\mathcal{J} \subset \mathcal{I}_2$ and $\mathcal{J}^2 \bigcap \mathcal{I}_2\mathcal{K} = \mathcal{J}\mathcal{K}$, as can be seen from local calculations. In particular, $\mathcal{J} = \mathcal{I}_2$ on $C_2 - \{p\}$ and $(x^2y, z) \subset (y, z)$ at p, which proves $\mathcal{J} \subset \mathcal{I}_2$. Now, $\mathcal{K} \subset \mathcal{J}$ by definition and $\mathcal{I}_2 = \mathcal{J}$ on $C_2 - \{p\}$, so $\mathcal{J}^2 \bigcap \mathcal{I}_2\mathcal{K} =$

${\mathcal{I}_2}^2 \bigcap \mathcal{I}_2\mathcal{K} = \mathcal{I}_2\mathcal{K}$ and $\mathcal{J}\mathcal{K} = \mathcal{I}_2\mathcal{K}$ away from p. Therefore, the intersection equality holds on $C_2 - \{p\}$. At the point p, $\mathcal{J}^2 = (x^4y^2, x^2yz, z^2)$, $\mathcal{I}_2\mathcal{K} = (x^2y^3, yz, z^2)$ and $\mathcal{J}\mathcal{K} = (x^4y^3, x^2yz, z^2)$. $\mathcal{J}^2 \bigcap \mathcal{I}_2\mathcal{K}$, then, is $(x^4y^3, x^2yz, z^2) = \mathcal{J}\mathcal{K}$ at p as well.

The injection is actually an isomorphism away from p since $\mathcal{J} = \mathcal{I}_2$ here, and so both sheaves are congruent to ${\mathcal{I}_2}^2/\mathcal{I}_2\mathcal{K}$. At the point p the map is defined on generators by $x^4y^2 \mapsto x^2 \cdot x^2y^2$. x is a local coordinate on C_2, so this map vanishes to second order and $\mathcal{I}_2\mathcal{J}/\mathcal{I}_2\mathcal{K}$ must have degree two greater than that of $\mathcal{J}^2/\mathcal{J}\mathcal{K}$. Invertible sheaves on C_2 being completely determined by their degree means that $\mathcal{I}_2\mathcal{J}/\mathcal{I}_2\mathcal{K} \cong \mathcal{O}_2$. The exact sequence

$$0 \longrightarrow \mathcal{I}_2\mathcal{J}/\mathcal{I}_2\mathcal{K} \longrightarrow \mathcal{K}/\mathcal{I}_2\mathcal{K} \longrightarrow \mathcal{K}/\mathcal{I}_2\mathcal{J} \longrightarrow 0, \tag{4.1}$$

then, can be expressed as

$$0 \longrightarrow \mathcal{O}_2 \longrightarrow \mathcal{K}/\mathcal{I}_2\mathcal{K} \longrightarrow \mathcal{O}_2(1) \longrightarrow 0. \tag{4.2}$$

Therefore, $\mathcal{K}/\mathcal{I}_2\mathcal{K} = (0, 1)$.

□

Lemma 4.2 $\mathcal{K}/\mathcal{I}_1\mathcal{K} = (0,0)$ *or* $(-1,1)$.

Proof: There is a well defined injection $\mathcal{K}/\mathcal{I}_1\mathcal{K} \hookrightarrow \mathcal{J}/\mathcal{I}_1\mathcal{J}$ since $\mathcal{K} \subset \mathcal{J}$ and $\mathcal{K} \bigcap \mathcal{I}_1\mathcal{J} = \mathcal{I}_1\mathcal{K}$. These results can be shown with local calculations as done in the previous lemma. $\mathcal{J} = \mathcal{K}$ on $C_1 - \{p\}$, so this map is an isomorphism away from p. At the point p this map is defined on generators by $x^2y^2 \mapsto y \cdot x^2y$ and $z \mapsto z$. The determinant map, then, vanishes to first order at p and so the degree of $\mathcal{K}/\mathcal{I}_1\mathcal{K}$ is one less than the degree of $\mathcal{J}/\mathcal{I}_1\mathcal{J} = (0,1)$. $\mathcal{K}/\mathcal{I}_1\mathcal{K}$ has degree 0 and injects into $\mathcal{O}_1 \oplus \mathcal{O}_1(1)$. The only possibilities are $(-1,1)$ and $(0,0)$.

□

Appendix C provides the local calculations used in proving lemmas 4.3, 4.4, 4.5 and 4.6.

Lemma 4.3 $\mathcal{K}^m/\mathcal{I}_2\mathcal{K}^m \cong S^m(\mathcal{K}/\mathcal{I}_2\mathcal{K})$.

Proof: : Define a map $\mathcal{K}^{\otimes m} \to \mathcal{K}^m/\mathcal{I}_2\mathcal{K}^m$ by multiplication of functions. Elements of the product sheaf $\mathcal{I}_2\mathcal{K}\otimes\mathcal{K}^{\otimes(m-1)}$ are killed by this map since their image is in the sheaf $\mathcal{I}_2\mathcal{K}^m$. Therefore, there is a well defined map $S^m(\mathcal{K}/\mathcal{I}_2\mathcal{K}) \to \mathcal{K}^m/\mathcal{I}_2\mathcal{K}^m$. Lemma C.2 of appendix C shows that both of these sheaves, $S^m(\mathcal{K}/\mathcal{I}_2\mathcal{K})$ and $\mathcal{K}^m/\mathcal{I}_2\mathcal{K}^m$, are generated locally everywhere on C_2 by the same elements. So, the map defined must be an isomorphism.

□

Lemma 4.4 $\mathcal{I}_2\mathcal{K}^m/\mathcal{I}\mathcal{K}^m \cong \mathcal{I}_2/\mathcal{I} \otimes S^m(\mathcal{K}/\mathcal{I}_1\mathcal{K})$.

Proof: Let $\mathcal{I}_2 \otimes \mathcal{K}^{\otimes m} \to \mathcal{I}_2\mathcal{K}^m/\mathcal{I}\mathcal{K}^m$ be defined by multiplication of functions. This map kills elements of $\mathcal{I}\otimes\mathcal{K}^{\otimes m}$ and $\mathcal{I}_2\otimes\mathcal{I}_1\mathcal{K}\otimes\mathcal{K}^{\otimes(m-1)}$ because $\mathcal{I}\mathcal{K}^m$ and $\mathcal{I}_1\mathcal{I}_2\mathcal{K}^m$ are contained in $\mathcal{I}\mathcal{K}^m$. This shows that there is a well defined map $\mathcal{I}_2/\mathcal{I}\otimes S^m(\mathcal{K}/\mathcal{I}_1\mathcal{K}) \to \mathcal{I}_2\mathcal{K}^m/\mathcal{I}\mathcal{K}^m$. The isomorphism follows from lemma C.3 in appendix C.

Lemma 4.5 $\mathcal{I}\mathcal{K}^m/\mathcal{J}\mathcal{K}^m \cong \mathcal{I}/\mathcal{J} \otimes S^m(\mathcal{K}/\mathcal{I}_1\mathcal{K})$.

Proof: The multiplication map $\mathcal{I} \otimes \mathcal{K}^{\otimes m} \to \mathcal{I}\mathcal{K}^m/\mathcal{J}\mathcal{K}^m$ kills both $\mathcal{J} \otimes \mathcal{K}^{\otimes m}$ and $\mathcal{I}\otimes\mathcal{I}_1\mathcal{K}\otimes\mathcal{K}^{\otimes(m-1)}$ because their images, $\mathcal{J}\mathcal{K}^m$ and $\mathcal{I}\mathcal{I}_1\mathcal{K}^m$, respectively, are contained in $\mathcal{J}\mathcal{K}^m$. This gives a well defined map $\mathcal{I}/\mathcal{J} \otimes S^m(\mathcal{K}/\mathcal{I}_1\mathcal{K}) \to \mathcal{I}\mathcal{K}^m/\mathcal{J}\mathcal{K}^m$. Lemma C.4 of appendix C proves that this map is an isomorphism.

□

Lemma 4.6 $\mathcal{J}\mathcal{K}^m/\mathcal{K}^{m+1} \cong \mathcal{J}/\mathcal{K} \otimes S^m(\mathcal{K}/\mathcal{I}_2\mathcal{K})$.

Proof: The multiplication map $\mathcal{J} \otimes \mathcal{K}^{\otimes m} \to \mathcal{J}\mathcal{K}^m/\mathcal{K}^{m+1}$ kills $\mathcal{K}^{\otimes(m+1)}$ and $\mathcal{J} \otimes \mathcal{I}_2\mathcal{K} \otimes \mathcal{K}^{\otimes(m-1)}$ because their images are contained in $\mathcal{K}^{m+1}$. The well defined map $\mathcal{J}/\mathcal{K} \otimes S^m(\mathcal{K}/\mathcal{I}_2\mathcal{K}) \to \mathcal{J}\mathcal{K}^m/\mathcal{K}^{m+1}$ can be seen to be an isomorphism from lemma C.5 in appendix C.

□

Assume that $\mathcal{K}/\mathcal{I}_1\mathcal{K} = (0,0)$, i.e. that C_1 has length 2. C_2 has length 2 as well since it has no negative factors in its decomposition, by lemma 4.1.

Lemma 4.7 *We have*

1) $h^0(\mathcal{K}/\mathcal{K}^2) = 4$

2) $H^1(\mathcal{K}/\mathcal{K}^2) = 0$

3) The maps $H^0(\mathcal{K}/\mathcal{K}^2) \to H^0(\mathcal{K}/\mathcal{I}\mathcal{K}) \to H^0(\mathcal{K}/\mathcal{I}_2\mathcal{K}) \to H^0(\mathcal{K}/\mathcal{I}_2\mathcal{J})$ *are surjective.*

Proof: From sequence 4.1 in the proof of lemma 4.1, $H^0(\mathcal{K}/\mathcal{I}_2\mathcal{K}) \to H^0(\mathcal{K}/\mathcal{I}_2\mathcal{J}$ is surjective. The statement of this lemma also shows that $h^0(\mathcal{K}/\mathcal{I}_2\mathcal{K}) = 3$. With the assumption that $\mathcal{K}/\mathcal{I}_1\mathcal{K} = (0,0)$, lemma 4.4, with $m = 1$, states that $\mathcal{I}_2\mathcal{K}/\mathcal{I}\mathcal{K} = (-1,-1)$. From the exact sequence

$$0 \longrightarrow \mathcal{I}_2\mathcal{K}/\mathcal{I}\mathcal{K} \longrightarrow \mathcal{K}/\mathcal{I}\mathcal{K} \longrightarrow \mathcal{K}/\mathcal{I}_2\mathcal{K} \longrightarrow 0,$$

then, $H^0(\mathcal{K}/\mathcal{I}\mathcal{K}) \to H^0(\mathcal{K}/\mathcal{I}_2\mathcal{K})$ is surjective, $H^1(\mathcal{K}/\mathcal{I}\mathcal{K}) = 0$ and $h^0(\mathcal{K}/\mathcal{I}\mathcal{K}) = 3$.

$\mathcal{J}\mathcal{K}/\mathcal{K}^2 \cong \mathcal{O}_2(-1) \oplus \mathcal{O}_2$ and $\mathcal{I}\mathcal{K}/\mathcal{J}\mathcal{K} \cong \mathcal{O}_1(-1) \oplus \mathcal{O}_1(-1)$ from lemmas 4.6 and 4.5 with $m = 1$, respectively. So, the exact sequence

$$0 \longrightarrow \mathcal{J}\mathcal{K}/\mathcal{K}^2 \longrightarrow \mathcal{I}\mathcal{K}/\mathcal{K}^2 \longrightarrow \mathcal{I}\mathcal{K}/\mathcal{J}\mathcal{K} \longrightarrow 0$$

shows $H^1(\mathcal{I}\mathcal{K}/\mathcal{K}^2) = 0$ and $h^0(\mathcal{I}\mathcal{K}/\mathcal{K}^2) = 1$. The cohomology exact sequence of

$$0 \longrightarrow \mathcal{I}\mathcal{K}/\mathcal{K}^2 \longrightarrow \mathcal{K}/\mathcal{K}^2 \longrightarrow \mathcal{K}/\mathcal{I}\mathcal{K} \longrightarrow 0,$$

then, implies that $H^0(\mathcal{K}/\mathcal{K}^2) \to H^0(\mathcal{K}/\mathcal{I}\mathcal{K})$ is surjective, $H^1(\mathcal{K}/\mathcal{K}^2) = 0$ and $h^0(\mathcal{K}/\mathcal{K}^2) = 4$.

□

Lemma 4.8 $H^0(\mathcal{K}/\mathcal{I}\mathcal{K}) \to H^0(\mathcal{K}/\mathcal{I}_1\mathcal{K})$ *is surjective.*

Proof: The sequence

$$0 \longrightarrow \mathcal{I}_1\mathcal{K}/\mathcal{I}\mathcal{K} \longrightarrow \mathcal{K}/\mathcal{I}\mathcal{K} \longrightarrow \mathcal{K}/\mathcal{I}_1\mathcal{K} \longrightarrow 0$$

is exact, so by showing that $H^1(\mathcal{I}_1\mathcal{K}/\mathcal{I}\mathcal{K}) = 0$ we will have shown the map in the statement of this lemma is surjective.

On $C_2 - \{p\}$, $\mathcal{I}_1 = \mathcal{O}_X$ and $\mathcal{I} = \mathcal{I}_2$, so the sheaf $\mathcal{I}_1\mathcal{K}/\mathcal{I}\mathcal{K} \cong \mathcal{K}/\mathcal{I}_2\mathcal{K}$ here. Therefore, $\mathcal{I}_1\mathcal{K}/\mathcal{I}\mathcal{K}$ is locally free of rank 2 on $C_2 - \{p\}$. At the point p, $\mathcal{I}_1\mathcal{K} = (x^3y^2, xz, z^2)$

and $\mathcal{I}\mathcal{K} = (x^3y^3, xyz, z^2)$, so $\mathcal{I}_1\mathcal{K}/\mathcal{I}\mathcal{K}$ is locally free of rank 2 on all of C_2 and is generated by $\{x^3y^2, xz\}$ at p.

Let $\mathcal{I}_1\mathcal{K}/\mathcal{I}\mathcal{K} \hookrightarrow \mathcal{K}/\mathcal{I}_2\mathcal{K}$ be the inclusion map. This is well defined since $\mathcal{I}_1\mathcal{K} \subset \mathcal{K}$ and $\mathcal{I}_1\mathcal{K} \bigcap \mathcal{I}_2\mathcal{K} = \mathcal{I}\mathcal{K}$. The equality of this intersection can be calculated locally everywhere. On $C_2 - \{p\}$, $\mathcal{I} = \mathcal{I}_2$ and $\mathcal{I}_1 = \mathcal{O}_X$, so $\mathcal{I}_1\mathcal{K} \bigcap \mathcal{I}_2\mathcal{K} = \mathcal{K} \bigcap \mathcal{I}_2\mathcal{K}$. But $\mathcal{I}_2\mathcal{K} \subset \mathcal{K}$ and, therefore, this intersection is $\mathcal{I}_2\mathcal{K}$. Since $\mathcal{I}\mathcal{K} = \mathcal{I}_2\mathcal{K}$ as well, the equality holds. At the point p, $\mathcal{I}_2\mathcal{K} = (x^2y^3, yz, z^2)$ and $\mathcal{I}_1\mathcal{K} \bigcap \mathcal{I}_2\mathcal{K} = (x^3y^2, xz, z^2) \bigcap (x^2y^3, yz, z^2) = (x^3y^3, xyz, z^2) = \mathcal{I}\mathcal{K}$.

It was seen above that these two sheaves are isomorphic on $C_2 - \{p\}$. At the point p, the inclusion is defined on generators by $x^3y^2 \mapsto x \cdot x^2y^2$ and $xz \mapsto x \cdot z$. So, $\mathcal{I}_1\mathcal{K}/\mathcal{I}\mathcal{K} \cong \mathcal{O}_2(-p) \otimes \mathcal{K}/\mathcal{I}_2\mathcal{K}$, and since $\mathcal{K}/\mathcal{I}_2\mathcal{K} = (0,0)$, we have shown $\mathcal{I}_1\mathcal{K}/\mathcal{I}\mathcal{K} = (-1,-1)$. In particular, this means that $H^1(\mathcal{I}_1\mathcal{K}/\mathcal{I}\mathcal{K}) = 0$.

□

Proposition 4.1 *The map on global sections, $H^0(\hat{\mathcal{K}}) \to H^0(\mathcal{K}/\mathcal{K}^2)$, is surjective.*

Proof: From lemma 4.1, $\mathcal{K}/\mathcal{I}_2\mathcal{K} \cong \mathcal{O}_2 \oplus \mathcal{O}_2(1)$. Therefore, $S^m(\mathcal{K}/\mathcal{I}_2\mathcal{K}) \cong \bigoplus_{i=0}^{m} \mathcal{O}_2(i)$, which implies $H^1(S^m(\mathcal{K}/\mathcal{I}_2\mathcal{K})) = 0$. The isomorphism of lemma 4.3, then, proves $H^1(\mathcal{K}^m/\mathcal{I}_2\mathcal{K}^m) = 0$.

With the assumption that $\mathcal{K}/\mathcal{I}_1\mathcal{K} = (0,0)$, we have $S^m(\mathcal{K}/\mathcal{I}_1\mathcal{K}) \cong \mathcal{O}_1^{\oplus(m+1)}$. Lemma 2.8 states that $\mathcal{I}_2/\mathcal{I} \cong \mathcal{O}_1(-1)$, so $S^m(\mathcal{K}/\mathcal{I}_1\mathcal{K}) \otimes \mathcal{I}_2/\mathcal{I} \cong \mathcal{O}_1(-1)^{\oplus(m+1)}$. From lemma 4.4, then, $H^1(\mathcal{I}_2\mathcal{K}^m/\mathcal{I}\mathcal{K}^m = 0$.

Now, $\mathcal{I}/\mathcal{J} \cong \mathcal{O}_1(-1)$ by definition of $\mathcal{J}$, so $S^m(\mathcal{K}/\mathcal{I}_1\mathcal{K}) \otimes \mathcal{I}/\mathcal{J} \cong \mathcal{O}_1(-1)^{\oplus(m+1)}$ as well. Therefore, from lemma 4.5, $H^1(\mathcal{I}\mathcal{K}^m/\mathcal{J}\mathcal{K}^m) = 0$.

Since $\mathcal{J}/\mathcal{K} \cong \mathcal{O}_2(-1)$, the sheaf $S^m(\mathcal{K}/\mathcal{I}_2\mathcal{K}) \otimes \mathcal{J}/\mathcal{K}$ is isomorphic to $\bigoplus_{i=-1}^{m-1} \mathcal{O}_2(i)$. Applying lemma 4.6, we have $H^1(\mathcal{J}\mathcal{K}^m/\mathcal{K}^{m+1}) = 0$.

Therefore, from

$$0 \longrightarrow \mathcal{I}_2\mathcal{K}^m/\mathcal{I}\mathcal{K}^m \longrightarrow \mathcal{K}^m/\mathcal{I}\mathcal{K}^m \longrightarrow \mathcal{K}^m/\mathcal{I}_2\mathcal{K}^m \longrightarrow 0$$

we have $H^1(\mathcal{K}^m/\mathcal{I}\mathcal{K}^m) = 0$, and from

$$0 \longrightarrow \mathcal{J}\mathcal{K}^m/\mathcal{K}^{m+1} \longrightarrow \mathcal{I}\mathcal{K}^m/\mathcal{K}^{m+1} \longrightarrow \mathcal{I}\mathcal{K}^m/\mathcal{J}\mathcal{K}^m \longrightarrow 0$$

we get $H^1(\mathcal{I}\mathcal{K}^m/\mathcal{K}^{m+1}) = 0$. The sequence

$$0 \longrightarrow \mathcal{I}\mathcal{K}^m/\mathcal{K}^{m+1} \longrightarrow \mathcal{K}^m/\mathcal{K}^{m+1} \longrightarrow \mathcal{K}^m/\mathcal{I}\mathcal{K}^m \longrightarrow 0,$$

then, shows that $H^1(\mathcal{K}^m/\mathcal{K}^{m+1}) = 0$ for all $m \geq 1$. The remainder of this proof is the induction argument used in the proof of proposition 3.1 with $\mathcal{K}$ replacing $\mathcal{J}$.

□

Lemma 4.9 *The sheaves $\mathcal{K}/\mathcal{K}^2$ and $\mathcal{K}/\mathcal{I}\mathcal{K}$ are generated by global sections.*

Proof: This lemma will be proven first for the sheaf $\mathcal{K}/\mathcal{K}^2$. It will be shown that at any point $q \in C$, every local section, i.e. section of $\mathcal{K}/m_q\mathcal{K}$, is the restriction of a global section. That is, it will be shown that $H^0(\mathcal{K}/\mathcal{K}^2) \to H^0(\mathcal{K}/m_q\mathcal{K})$ is surjective for all $q \in C$. It was shown in lemmas 4.7 and 4.8 that the maps $H^0(\mathcal{K}/\mathcal{K}^2) \to H^0(\mathcal{K}/\mathcal{I}_i\mathcal{K})$ are surjective for $i = 1, 2$. By the assumption that $\mathcal{K}/\mathcal{I}_i\mathcal{K}$ decomposes with no negative factors, these two sheaves are generated by global sections. Let $q \in C$. q is on C_i for some i. But $\mathcal{K}/\mathcal{I}_i\mathcal{K}$ is generated by global sections, so $H^0(\mathcal{K}/\mathcal{I}_i\mathcal{K}) \to H^0(\mathcal{K}/m_q\mathcal{K})$ is surjective. Therefore the composition $H^0(\mathcal{K}/\mathcal{K}^2) \to H^0(\mathcal{K}/m_q\mathcal{K})$ is surjective.

The proof that $\mathcal{K}/\mathcal{I}\mathcal{K}$ is generated by global sections is the exact proof as that for $\mathcal{K}/\mathcal{K}^2$, but replacing $\mathcal{K}/\mathcal{K}^2$ with $\mathcal{K}/\mathcal{I}\mathcal{K}$.

□

Again, as in the length(2,1) cases, to determine the general hyperplane section of the singularity q resulting from contracting C the general section of $\hat{\mathcal{K}}$ is looked at more closely. In particular, the surface singularities that occur on a general section of $\hat{\mathcal{K}}$ can now be determined.

Lemma 4.10 *A general section of $\hat{\mathcal{K}}$ defines a smooth surface, except for two distinct A_1 singularities on $C_1 - \{p\}$ and one A_1 singularity on $C_2 - \{p\}$.*

Proof: A general section of $\hat{\mathcal{K}}$, locally at p, is of the form $f \cdot x^2y^2 + g \cdot z$ where $f, g \in \mathcal{O}_{p,C}$. Considered as a lifting of a general section of $\mathcal{K}/\mathcal{K}^2$, which is generated by global sections, this local section is the restriction of a global section of $\mathcal{K}/\mathcal{K}^2$. And, being a general section, it will have g as a unit (i.e. $g(p) \neq 0$). But g being a unit implies that this local section is smooth at p. Lifting to a section of $\hat{\mathcal{K}}$ via the surjection $H^0(\hat{\mathcal{K}}) \to H^0(\mathcal{K}/\mathcal{K}^2)$, the section remains smooth at p as a section of $\hat{\mathcal{K}}$ as well.

Away from the point p, however, singularities do occur. On $C_1 - \{p\}$, $\mathcal{K} = (x^2, z) = \mathcal{J}$ in local coordinates. A general section of $\mathcal{K}$, then, in appropriately chosen coordinates is $fx^2 + gz$ where f and g are local functions on $C_1 - \{p\}$. If g is a unit then the section defines a smooth surface, so the only singularities can occur when f is a unit. Lemmas 4.7, 4.8, and proposition 4.1 show that $H^0(\hat{\mathcal{K}}) \longrightarrow H^0(\mathcal{K}/\mathcal{I}_1\mathcal{K})$ is surjective, and since $\mathcal{K} = \mathcal{J}$ on $C_1 - \{p\}$, we have $H^0(\hat{\mathcal{K}}) \longrightarrow H^0(\mathcal{J}/\mathcal{I}_1\mathcal{J})$ surjective. But from case 1 in chapter 3, corollary 3.2, $H^0(\mathcal{J}/\mathcal{I}_1\mathcal{J}) \longrightarrow H^0(\mathcal{J}/\mathcal{I}_1\mathcal{I})$ is surjective also. Therefore, a general section of $\mathcal{K}$ on $C_1 - \{p\}$ comes from a section of $\mathcal{J}/\mathcal{I}_1\mathcal{I} \cong \mathcal{O}_1(2)$. So, as in lemma 3.8, there are exactly two A_1 singularities on $C_1 - \{p\}$.

On $C_2 - \{p\}$ a general section of $\hat{\mathcal{K}}$ is defined locally by $f \cdot y^2 + g \cdot z$, so singularities can only appear when f is a unit in $\mathcal{O}_{p,C}$. From lemma 4.7 and proposition 4.1, we know that $H^0(\hat{\mathcal{K}}) \to H^0(\mathcal{K}/\mathcal{I}_2\mathcal{J})$ is surjective, so this section comes as the lifting of a section of $\mathcal{K}/\mathcal{I}_2\mathcal{J}$. This surjection is defined locally by $f \cdot y^2 + g \cdot z \mapsto g \cdot z$, and being general, its image will not be the zero section. Since $\mathcal{I}_2\mathcal{J} = (y^2, yz, z^2)$ away from p, this means that g can be a function of the variable x only. Now, lifting to $\hat{\mathcal{K}}$ and letting f be a unit, this section is $y^2 + g \cdot z$ with g a function of x only. The surface singularities can only be RDP's, and the only way $y^2 + g \cdot z$ can define a RDP is if g vanishes to first order. The analytic change of coordinates $(g, y, z) \mapsto (x, y, z)$, then, defines this singularity as $y^2 + xz$, which is of type A_1. Therefore, a general section of $\hat{\mathcal{K}}$ has at least one A_1 singularity on $C_2 - \{p\}$.

To find the number of A_1 singularities on $C_2 - \{p\}$, we will determine the number of zeros of a general section of $\mathcal{K}$. Considered as a section of $\mathcal{K}/\mathcal{I}_2\mathcal{K}$, it suffices to count the zeros of a section of this sheaf. From the exact sequence

$$0 \longrightarrow \mathcal{I}_2\mathcal{J}/\mathcal{I}_2\mathcal{K} \longrightarrow \mathcal{K}/\mathcal{I}_2\mathcal{K} \longrightarrow \mathcal{K}/\mathcal{I}_2\mathcal{J} \longrightarrow 0,$$

where $\mathcal{I}_2\mathcal{J}/\mathcal{I}_2\mathcal{K}$ has been seen to be isomorphic to $\mathcal{O}_2(-1)$ and $\mathcal{K}/\mathcal{I}_2\mathcal{J} \cong \mathcal{O}_2(1)$, the cohomology sequence shows that $H^0(\mathcal{K}/\mathcal{I}_2\mathcal{K}) \cong H^0(\mathcal{O}_2(1))$. Therefore, a section of $\mathcal{K}/\mathcal{I}_2\mathcal{K}$ can be viewed as a section of the sheaf of linear homogeneous polynomials on $C_2 \cong \mathbf{P}^1$, which vanish at one point. So, there is exactly one A_1 singularity on $C_2 - \{p\}$.

□

Let r and s be the A_1 singularities on $C_1 - \{p\}$ and t the A_1 on $C_2 - \{p\}$. Letting S be the zero scheme of the general section of $\hat{\mathcal{K}}$ and $\mu : S' \to S$ the blow up of r, s and t, the smooth surface S' will have five smooth rational curves, $C_i^{'}$ for $i \in \{1, 2, r, s, t\}$, with this notation consistent with that in the previous sections (see figure 4.1). To

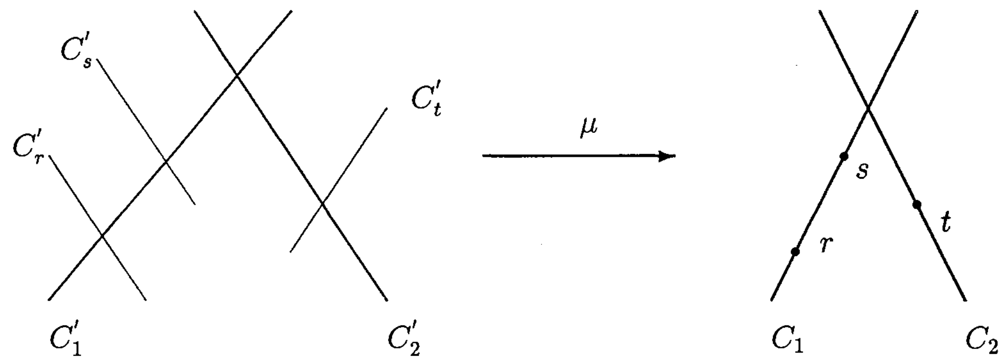


Figure 4.1: $D_5(2,2)$ configuration

show that the C contracts to a point q whose general hyperplane section has q as a D_5 singularity, it needs to be shown that $I_i'/I_i'^2 \cong \mathcal{O}_{C_i'}(2)$ for each i and that the components are in a D_5 configuration in S'. r, s and t are RDP's on S and μ is their resolution, so $I_i'/I_i'^2 \cong \mathcal{O}_{C_i'}(2)$ for $i = r, s, t$.

In this case the sheaf $\mathcal{I}/\mathcal{K}$ is an invertible sheaf on C, generated by $\{xy\}$ at the point p. As μ induces a map $\mu^* : I/I^{(2)} \to I'/I'^2$, we will study the sheaf $I/I^{(2)} = \mathcal{I}/Sat(\mathcal{I}^2, f)$, where f defines the surface S locally.

Lemma 4.11 $I/I^{(2)} \cong \mathcal{I}/\mathcal{K}$

Proof: It will be shown that locally everywhere $Sat(\mathcal{I}^2, f) = \mathcal{K}$. On $C_i - \{r, s, t\}$ for $i = 1, 2$, f can be chosen in suitable coordinates to be $f = z$. Therefore, $Sat(\mathcal{I}^2, f) = Sat(x^2y^2, xyz, z^2, z) = (x^2y^2, z) = \mathcal{K}$ at the point p. On $C_1 - \{p, r, s\}$, $Sat(\mathcal{I}^2, f) = Sat({\mathcal{I}_1}^2, f) = (x^2, xz, z^2, z) = (x^2, z) = \mathcal{K}$, and on $C_2 - \{p, t\}$, $Sat(\mathcal{I}^2, f) = (y^2, z) = \mathcal{K}$.

At the points r and s on C_1, $\mathcal{I} = \mathcal{I}_1$ and $f = yz + x^2$. So, $Sat(\mathcal{I}^2, f) = Sat(x^2, xz, z^2, yz + x^2)$ and the element z can be seen to be the torsion element giving $Sat(\mathcal{I}^2, f) = (x^2, xz, z^2, yz + x^2, z) = (x^2, z) = \mathcal{K}$. Similarly, at the point t where $\mathcal{I} = \mathcal{I}_2$ and $f = xz + y^2$, z is again the torsion element of $\mathcal{I}/(\mathcal{I}^2, f)$. Therefore, $Sat(\mathcal{I}^2, f) = (y^2, z) = \mathcal{K}$.

□

Lemma 4.12 $\mathcal{I}/\mathcal{K} \cong \mathcal{O}_C(-1, 0)$.

Proof: Since $\mathcal{I}/\mathcal{K}$ is an invertible sheaf on C we know that it is of the form $\mathcal{O}_C(a,b)$ for some integers a and b where $\mathcal{I}/\mathcal{K}|_{C_1} \cong \mathcal{O}_1(a)$ and $\mathcal{I}/\mathcal{K}|_{C_2} \cong \mathcal{O}_2(b)$.

$\mathcal{I}/\mathcal{K}|_{C_1} = \mathcal{I}/(\mathcal{I}_1\mathcal{I} + \mathcal{K})$, and $\mathcal{I}_1\mathcal{I} + \mathcal{K} = (x^2y, xz, z^2) + (x^2y^2, z) = (x^2y, z)$ in coordinates at p. These are the same local generators as $\mathcal{J}$. Away from the point p, $\mathcal{I} = \mathcal{I}_1 = (x, z)$ and $\mathcal{K} = (x^2, z)$ in coordinates, so $\mathcal{I}_1\mathcal{I} + \mathcal{K} = (x^2, z) = \mathcal{J}$. Locally everywhere, then, $\mathcal{I}_1\mathcal{I} + \mathcal{K} = \mathcal{J}$ and this proves that $\mathcal{I}/\mathcal{K}|_{C_1} \cong \mathcal{I}/\mathcal{J} \cong \mathcal{O}_1(-1)$.

Also, the exact sequence

$$0 \longrightarrow (\mathcal{I}_1\mathcal{I} + \mathcal{K})/\mathcal{K} \longrightarrow \mathcal{I}/\mathcal{K} \longrightarrow \mathcal{I}/\mathcal{K}|_{C_1} \longrightarrow 0$$

is the sequence

$$0 \longrightarrow \mathcal{J}/\mathcal{K} \longrightarrow \mathcal{I}/\mathcal{K} \longrightarrow \mathcal{I}/\mathcal{J} \longrightarrow 0$$

where $\mathcal{J}/\mathcal{K} \cong \mathcal{O}_2(-1)$. So, we have

$$0 \longrightarrow \mathcal{O}_2(-1) \longrightarrow \mathcal{O}_C(a,b) \longrightarrow \mathcal{O}_1(-1) \longrightarrow 0 \tag{4.3}$$

From Lemma 2.8, the sequence

$$0 \longrightarrow \mathcal{O}_2(-1) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_1 \longrightarrow 0$$

is exact and tensoring with the flat $\mathcal{O}_X$-module $\mathcal{O}_C(a,b)$, the resulting sequence is

$$0 \longrightarrow \mathcal{O}_2(b-1) \longrightarrow \mathcal{O}_C(a,b) \longrightarrow \mathcal{O}_1(a) \longrightarrow 0.$$

Letting $a = -1$ and comparing with sequence 4.3, we see that $b = 0$.

□

Corollary 4.1 *We have*

1) $I/I^{(2)}|_{C_1} \cong \mathcal{O}_1(-1)$

2) $I/I^{(2)}|_{C_2} \cong \mathcal{O}_2$

Proof: From lemmas 4.11 and 4.12, $I/I^{(2)} \cong \mathcal{I}/\mathcal{K} \cong \mathcal{O}_C(-1,0)$. Restricting to each component, the results follow immediately.

□

Lemma 4.13 $I_1/I_1^{(2)} \cong \mathcal{O}_1$.

Proof: $I_1/I_1^{(2)} = \mathcal{I}_1/Sat(\mathcal{I}_1{}^2, f)$ by definition, and at p, $Sat(\mathcal{I}_1{}^2, f) = (\mathcal{I}_1{}^2, f)$ since S is smooth at p. In coordinates this sheaf is $(x^2, xz, z^2, z) = (x^2, z)$. So, $I_1/I_1^{(2)}$ is generated by $\{x\}$ at p. The injection $I/I^{(2)}|_{C_1} \hookrightarrow I_1/I_1^{(2)}$ is an isomorphism away from p and at p is given by $xy \mapsto y \cdot x$. Vanishing to first order, this shows that the degree of $I_1/I_1^{(2)}$ has degree 0 by (1) in corollary 4.1.

□

Lemma 4.14 $I_2/I_2^{(2)} \cong \mathcal{O}_2(1)$.

Proof: $I_2/I_2^{(2)} = \mathcal{I}_2/Sat(\mathcal{I}_2{}^2, f)$ and at p, $f = z$, so $Sat(\mathcal{I}_2{}^2, f) = (y^2, z)$. $I_2/I_2^{(2)}$, then, is generated by y at p. The injection $I/I^{(2)}|_{C_2} \hookrightarrow I_2/I_2^{(2)}$ is an isomorphism on $C_2 - \{p\}$ and is defined by $xy \mapsto x \cdot y$ at p. This map vanishes to first order, so the degree of $I_2/I_2^{(2)}$ is 1 by applying (2) in corollary 4.1.

□

Lemma 4.15 $I'_i/I'^2_i \cong \mathcal{O}_i(2)$ *for* $i = 1, 2$.

Proof: The pullback $\mu^* : I_i/I_i^{(2)} \longrightarrow I'_i/I'^2_i$ is an isomorphism away from the singular points r, s, and t. As in the previous cases, μ^* vanishes to first order at each of these singular points. Therefore, the degree of I'_1/I'^2_1 is $0 + 2 = 2$ from lemma 4.13, and the degree of I'_2/I'^2_2 is $1 + 1 = 2$ from lemma 4.14.

□

It has been shown that $I'_i/I'^2_i \cong \mathcal{O}_{C'_i}(2)$ for $i = 1, 2$, so by proposition 1.1 the following theorem has been proven.

Theorem 4.1 *If* $f : X \longrightarrow Y$ *is a contraction map with* $f(C) = q$ *and* $C = C_1 \cup C_2$ *has length*$(2,2)$ *with defining ideals* $\mathcal{J} = (x^2y, z)$ *and* $\mathcal{K} = (x^2y^2, z)$ *at* $p = C_1 \bigcap C_2$, *then a general hyperplane section of* q *has a* D_5 *type singularity at* q.

4.2 The $\mathbf{D_6(2,2)}$ Case

Case 2′a: $\mathcal{K} = (xy, z^2)$.

$\mathcal{K}$ is defined to be $Ker(\mathcal{J} \to \mathcal{J}/Sat(\mathcal{I}_2\mathcal{J}) \to \mathcal{O}_2(-1))$ and $\mathcal{J} = (xy, xz, z^2)$ at p, so $\mathcal{J}/\mathcal{K} \cong \mathcal{O}_2(-1)$, generated by $\{xz\}$ at p, and $\mathcal{K}/Sat(\mathcal{I}_2\mathcal{J}) \cong \mathcal{O}_2(1)$, generated by

$\{xy\}$ at p. Since $\mathcal{K}/\mathcal{I}\mathcal{K}$ is a locally free sheaf of rank 2 on C, the sheaves $\mathcal{K}/\mathcal{I}_1\mathcal{K}$ and $\mathcal{K}/\mathcal{I}_2\mathcal{K}$ are locally free of rank 2 on C_1 and C_2, respectively, generated by $\{xy, z^2\}$ at the point p.

Lemma 4.16 $\mathcal{K}/\mathcal{I}_2\mathcal{K} = (0, 1)$.

Proof: Local calculations at p show that $Sat(\mathcal{I}_2\mathcal{J}) = (xy^2, xyz, z^2)$ and $\mathcal{I}_2\mathcal{K} = (xy^2, yz^2, xyz, z^2)$. $Sat(\mathcal{I}_2\mathcal{J})/\mathcal{I}_2\mathcal{K}$, then, is an invertible sheaf on C_2, generated by $\{z^2\}$ at p. The sheaf $\mathcal{J}/\mathcal{K} \otimes \mathcal{J}/\mathcal{K} \cong \mathcal{J}^2/\mathcal{J}\mathcal{K}$ is also an invertible sheaf on C_2. It is isomorphic to $\mathcal{O}_2(-2)$, because $\mathcal{J}/\mathcal{K} \cong \mathcal{O}_2(-1)$, and it is generated by $\{x^2z^2\}$ at p. There is an injection

$$\mathcal{J}^2/\mathcal{J}\mathcal{K} \hookrightarrow Sat(\mathcal{I}_2\mathcal{J})/\mathcal{I}_2\mathcal{K}$$

since $\mathcal{J}^2 \subset Sat(\mathcal{I}_2\mathcal{J})$ and $\mathcal{J}^2 \bigcap \mathcal{I}_2\mathcal{K} = \mathcal{J}\mathcal{K}$. On $C_2 - \{p\}$, $\mathcal{J} = \mathcal{I}_2$, so this map is an isomorphism. At p, it is defined on generators by $x^2z^2 \mapsto x^2 \cdot z^2$, so it vanishes to order 2 at p. The degree of $Sat(\mathcal{I}_2\mathcal{J})/\mathcal{I}_2\mathcal{K}$, then, is two more than that of $\mathcal{J}^2/\mathcal{J}\mathcal{K}$, so $Sat(\mathcal{I}_2\mathcal{J})/\mathcal{I}_2\mathcal{K} \cong \mathcal{O}_2$.

The exact sequence

$$0 \longrightarrow Sat(\mathcal{I}_2\mathcal{J})/\mathcal{I}_2\mathcal{K} \longrightarrow \mathcal{K}/\mathcal{I}_2\mathcal{K} \longrightarrow \mathcal{K}/Sat(\mathcal{I}_2\mathcal{J}) \longrightarrow 0 \tag{4.4}$$

can be expressed as

$$0 \longrightarrow \mathcal{O}_2 \longrightarrow \mathcal{K}/\mathcal{I}_2\mathcal{K} \longrightarrow \mathcal{O}_2(1) \longrightarrow 0. \tag{4.5}$$

Therefore, $\mathcal{K}/\mathcal{I}_2\mathcal{K} = (0, 1)$.

□

Lemma 4.17 $\mathcal{K}/\mathcal{I}_1\mathcal{K} = (0, 0)$.

Proof: There is a well defined injection $\mathcal{K}/\mathcal{I}_1\mathcal{K} \hookrightarrow \mathcal{J}/Sat(\mathcal{I}_1\mathcal{J})$ since $\mathcal{K} \subset \mathcal{J}$ and $\mathcal{K} \bigcap Sat(\mathcal{I}_1\mathcal{J}) = \mathcal{I}_1\mathcal{K}$. $\mathcal{J} = \mathcal{K}$ on $C_1 - \{p\}$, so this map is an isomorphism away from p. At the point p this map is defined on generators by $xy \mapsto xy$ and $z^2 \mapsto z^2$ since both sheaves are generated by the same elements. The degree of $\mathcal{K}/\mathcal{I}_1\mathcal{K}$, then, is the same as the degree of $\mathcal{J}/Sat(\mathcal{I}_1\mathcal{J}$, which is 0. But $\mathcal{K}/\mathcal{I}_1\mathcal{K}$ must also inject into this sheaf, so the only possibility is $\mathcal{K}/\mathcal{I}_1\mathcal{K} = (0, 0)$.

□

Remarks: It was shown in the proof of lemma 4.17 that $\mathcal{K}/\mathcal{I}_1\mathcal{K}$ is isomorphic to the sheaf $\mathcal{J}/Sat(\mathcal{I}_1\mathcal{J})$ from case 2. Notice also that from the last two lemmas, neither of the sheaves $\mathcal{K}/\mathcal{I}_1\mathcal{K}$ or $\mathcal{K}/\mathcal{I}_2\mathcal{K}$ has a negative factor, so neither C_1 nor C_2 can have length greater than two.

As in the previous cases, it will be shown that the generators of these two sheaves, $\mathcal{K}/\mathcal{I}_i\mathcal{K}$ for $i = 1, 2$, can be lifted to global sections of $\hat{\mathcal{K}}$. The following four lemmas will be used to show the appropriate maps on global sections are surjective. This procedure has been used throughout this paper and the proofs of the following lemmas are analogous to their counterparts, lemmas 4.3 - 4.6, in case$1'a$.

Lemma 4.18 $\mathcal{K}^m/\mathcal{I}_2\mathcal{K}^m \cong S^m(\mathcal{K}/\mathcal{I}_2\mathcal{K})$.

Proof: See lemma 4.3 for the appropriate multiplication map. Lemma D.2 of appendix D proves the isomorphism.

□

Lemma 4.19 $\mathcal{I}_2\mathcal{K}^m/\mathcal{I}\mathcal{K}^m \cong \mathcal{I}_2/\mathcal{I} \otimes S^m(\mathcal{K}/\mathcal{I}_1\mathcal{K})$.

Proof: See lemma 4.4 for the appropriate map. Lemma D.3 in appendix D shows this map is an isomorphism.

□

Lemma 4.20 $\mathcal{I}\mathcal{K}^m/\mathcal{J}\mathcal{K}^m \cong \mathcal{I}/\mathcal{J} \otimes S^m(\mathcal{K}/\mathcal{I}_1\mathcal{K})$.

Proof: See lemma 4.5 for the appropriate map. Lemma D.4 in appendix D proves this map is an isomorphism.

□

Lemma 4.21 $\mathcal{J}\mathcal{K}^m/\mathcal{K}^{m+1} \cong \mathcal{J}/\mathcal{K} \otimes S^m(\mathcal{K}/\mathcal{I}_2\mathcal{K})$.

Proof: See lemma 4.6 for the appropriate map and Lemma D.5 in appendix D to see that it is an isomorphism.

□

Lemma 4.22 *We have*

1) $h^0(\mathcal{K}/\mathcal{K}^2) = 4$

2) $H^1(\mathcal{K}/\mathcal{K}^2) = 0$

3) The maps $H^0(\mathcal{K}/\mathcal{K}^2) \to H^0(\mathcal{K}/\mathcal{I}\mathcal{K}) \to H^0(\mathcal{K}/\mathcal{I}_2\mathcal{K}) \to H^0(\mathcal{K}/Sat(\mathcal{I}_2\mathcal{J}))$ *are surjective.*

Proof: From sequence 4.4 in the proof of lemma 4.16, $H^0(\mathcal{K}/\mathcal{I}_2\mathcal{K}) \to H^0(\mathcal{K}/Sat(\mathcal{I}_2\mathcal{J})$ is surjective. The statement of this lemma also shows that $h^0(\mathcal{K}/\mathcal{I}_2\mathcal{K}) = 3$. Lemma 4.19, with $m = 1$, states that $\mathcal{I}_2\mathcal{K}/\mathcal{I}\mathcal{K} = (-1,-1)$, so, from the sequence

$$0 \longrightarrow \mathcal{I}_2\mathcal{K}/\mathcal{I}\mathcal{K} \longrightarrow \mathcal{K}/\mathcal{I}\mathcal{K} \longrightarrow \mathcal{K}/\mathcal{I}_2\mathcal{K} \longrightarrow 0,$$

$H^0(\mathcal{K}/\mathcal{I}\mathcal{K}) \to H^0(\mathcal{K}/\mathcal{I}_2\mathcal{K})$ is surjective, $H^1(\mathcal{K}/\mathcal{I}\mathcal{K}) = 0$ and $h^0(\mathcal{K}/\mathcal{I}\mathcal{K}) = 3$.

$\mathcal{J}\mathcal{K}/\mathcal{K}^2 \cong \mathcal{O}_2(-1) \oplus \mathcal{O}_2$ and $\mathcal{I}\mathcal{K}/\mathcal{J}\mathcal{K} \cong \mathcal{O}_1(-1) \oplus \mathcal{O}_1(-1)$ from lemmas 4.21 and 4.20 with $m = 1$, respectively. So, the exact sequence

$$0 \longrightarrow \mathcal{J}\mathcal{K}/\mathcal{K}^2 \longrightarrow \mathcal{I}\mathcal{K}/\mathcal{K}^2 \longrightarrow \mathcal{I}\mathcal{K}/\mathcal{J}\mathcal{K} \longrightarrow 0$$

shows $H^1(\mathcal{I}\mathcal{K}/\mathcal{K}^2) = 0$ and $h^0(\mathcal{I}\mathcal{K}/\mathcal{K}^2) = 1$. The sequence

$$0 \longrightarrow \mathcal{I}\mathcal{K}/\mathcal{K}^2 \longrightarrow \mathcal{K}/\mathcal{K}^2 \longrightarrow \mathcal{K}/\mathcal{I}\mathcal{K} \longrightarrow 0,$$

then, proves that $H^0(\mathcal{K}/\mathcal{K}^2) \to H^0(\mathcal{K}/\mathcal{I}\mathcal{K})$ is surjective, $H^1(\mathcal{K}/\mathcal{K}^2) = 0$ and $h^0(\mathcal{K}/\mathcal{K}^2) = 4$.

□

Lemma 4.23 *$H^0(\mathcal{K}/\mathcal{I}\mathcal{K}) \to H^0(\mathcal{K}/\mathcal{I}_1\mathcal{K})$ is surjective.*

Proof: At the point p, $\mathcal{I}_1\mathcal{K} = (x^2y, xz^2, xyz, z^3)$, $\mathcal{I}_2\mathcal{K} = (xy^2, yz^2, xyz, z^3)$ and $\mathcal{I}\mathcal{K} = (x^2y^2, xyz, z^3)$. This local information shows that $\mathcal{I}_1\mathcal{K}/\mathcal{I}\mathcal{K}$ is a locally free sheaf of rank 2 on C_2 and is generated by $\{x^2y, xz^2\}$ at p. The injection $\mathcal{I}_1\mathcal{K}/\mathcal{I}\mathcal{K} \hookrightarrow \mathcal{K}/\mathcal{I}_2\mathcal{K}$ is well defined since $\mathcal{I}_1\mathcal{K} \subset \mathcal{K}$ and $\mathcal{I}_1\mathcal{K} \bigcap \mathcal{I}_2\mathcal{K} = \mathcal{I}\mathcal{K}$. This can be checked with the local information given. On $C_2 - \{p\}$, $\mathcal{I}_1 = \mathcal{O}_X$ and $\mathcal{I} = \mathcal{I}_2$, so this map is an isomorphism away from p. At p, the injection is given by $x^2y \mapsto x \cdot xy$ and $xz^2 \mapsto x \cdot z^2$. That is, $\mathcal{I}_1\mathcal{K}/\mathcal{I}\mathcal{K} \cong \mathcal{O}_2(-p) \otimes \mathcal{K}/\mathcal{I}_2\mathcal{K}$ at p. Being an isomorphism away from p, we have $\mathcal{I}_1\mathcal{K}/\mathcal{I}\mathcal{K} \cong \mathcal{O}_2(-1) \otimes \mathcal{K}/\mathcal{I}_2\mathcal{K} = (-1,-1)$ and $H^1(\mathcal{I}_1\mathcal{K}/\mathcal{I}\mathcal{K}) = 0$. The statement of the lemma follows from the exact sequence

$$0 \longrightarrow \mathcal{I}_1\mathcal{K}/\mathcal{I}\mathcal{K} \longrightarrow \mathcal{K}/\mathcal{I}\mathcal{K} \longrightarrow \mathcal{K}/\mathcal{I}_1\mathcal{K} \longrightarrow 0.$$

□

Proposition 4.2 *The map on global sections, $H^0(\hat{\mathcal{K}}) \to H^0(\mathcal{K}/\mathcal{K}^2)$, is surjective.*

Proof: From lemmas 4.18, 4.19, 4.20 and 4.21, respectively, we have $H^1(\mathcal{K}^m/\mathcal{I}_2\mathcal{K}^m) = 0$, $H^1(\mathcal{I}_2\mathcal{K}^m/\mathcal{I}\mathcal{K}^m) = 0$, $H^1(\mathcal{I}\mathcal{K}^m/\mathcal{J}\mathcal{K}^m) = 0$ and $H^1(\mathcal{J}\mathcal{K}^m/\mathcal{K}^{m+1}) = 0$ for all $m \geq 1$. Therefore, from

$$0 \longrightarrow \mathcal{I}_2\mathcal{K}^m/\mathcal{I}\mathcal{K}^m \longrightarrow \mathcal{K}^m/\mathcal{I}\mathcal{K}^m \longrightarrow \mathcal{K}^m/\mathcal{I}_2\mathcal{K}^m \longrightarrow 0$$

we have $H^1(\mathcal{K}^m/\mathcal{I}\mathcal{K}^m) = 0$, and from

$$0 \longrightarrow \mathcal{J}\mathcal{K}^m/\mathcal{K}^{m+1} \longrightarrow \mathcal{I}\mathcal{K}^m/\mathcal{K}^{m+1} \longrightarrow \mathcal{I}\mathcal{K}^m/\mathcal{J}\mathcal{K}^m \longrightarrow 0$$

we get $H^1(\mathcal{I}\mathcal{K}^m/\mathcal{K}^{m+1}) = 0$. The sequence

$$0 \longrightarrow \mathcal{I}\mathcal{K}^m/\mathcal{K}^{m+1} \longrightarrow \mathcal{K}^m/\mathcal{K}^{m+1} \longrightarrow \mathcal{K}^m/\mathcal{I}\mathcal{K}^m \longrightarrow 0,$$

then, shows that $H^1(\mathcal{K}^m/\mathcal{K}^{m+1}) = 0$ for all $m \geq 1$. The remainder of this proof is the induction argument used in the proof of proposition 3.1 with $\mathcal{K}$ replacing $\mathcal{J}$.

□

Lemma 4.24 *The sheaves $\mathcal{K}/\mathcal{K}^2$ and $\mathcal{K}/\mathcal{I}\mathcal{K}$ are generated by global sections.*

Proof: This lemma will be proven first for the sheaf $\mathcal{K}/\mathcal{K}^2$. It will be shown that at any point $q \in C$, every local section, i.e. section of $\mathcal{K}/m_q\mathcal{K}$, is the restriction of a global section. That is, it will be shown that $H^0(\mathcal{K}/\mathcal{K}^2) \to H^0(\mathcal{K}/m_q\mathcal{K})$ is surjective for all $q \in C$. It was shown in lemmas 4.22 and 4.23 that the maps $H^0(\mathcal{K}/\mathcal{K}^2) \to H^0(\mathcal{K}/\mathcal{I}_i\mathcal{K})$ are surjective for $i = 1, 2$. By the assumption that $\mathcal{K}/\mathcal{I}_i\mathcal{K}$ decomposes with no negative factors, these two sheaves are generated by global sections. Let $q \in C$. q is on C_i for some i. But $\mathcal{K}/\mathcal{I}_i\mathcal{K}$ is generated by global sections, so $H^0(\mathcal{K}/\mathcal{I}_i\mathcal{K}) \to H^0(\mathcal{K}/m_q\mathcal{K})$ is surjective. Therefore the composition $H^0(\mathcal{K}/\mathcal{K}^2) \to H^0(\mathcal{K}/m_q\mathcal{K})$ is surjective.

The proof that $\mathcal{K}/\mathcal{I}\mathcal{K}$ is generated by global sections is the exact proof as that for $\mathcal{K}/\mathcal{K}^2$, but replacing $\mathcal{K}/\mathcal{K}^2$ with $\mathcal{K}/\mathcal{I}\mathcal{K}$.

□

The singularities of a general section of $\mathcal{K}$ can now be calculated. A general section of $\hat{\mathcal{K}}$ is of the form $Axy + Bz^2$. Expanding A and B as power series:

$$A = \quad a_0 + \quad a_1 x + a_2 y + a_3 z + H_A$$
$$B = \quad b_0 + \quad b_1 x + b_2 y + b_3 z + H_B$$

where H_A and H_B represent the higher order terms in the expansion. A general section of $\hat{\mathcal{K}}$ will have A and B non-vanishing at the point p. i.e. a_0 and b_0 can be assumed to be nonzero. So, the quadratic part of a local section is $a_0 xy + b_0 z^2$, and, having rank 3, the singularity at p is of type A_1.

Lemma 4.25 *A general section of $\hat{\mathcal{K}}$ defines a smooth surface except for two distinct A_1 singularities on $C_1 - \{p\}$, one A_1 singularity on $C_2 - \{p\}$, and an A_1 singularity at p.*

Proof: It has been shown that there is an A_1 singularity at the point of intersection, p.

On $C_1 - \{p\}$. In lemma 4.17 it was shown that $\mathcal{K}/\mathcal{I}_1\mathcal{K} \cong \mathcal{J}/Sat(\mathcal{I}_1\mathcal{J})$ and so, since $\mathcal{J}/Sat(\mathcal{I}_1\mathcal{J}) \rightarrow \mathcal{J}/\mathcal{I}_1\mathcal{I} \cong \mathcal{O}_1(1)$ is surjective, $\mathcal{K}/\mathcal{I}_1\mathcal{I} \rightarrow \mathcal{J}/\mathcal{I}_1\mathcal{I}$ is also surjective. Therefore, the map $H^0(\hat{\mathcal{K}}) \rightarrow H^0(\mathcal{J}/\mathcal{I}_1\mathcal{I})$ is surjective. A general section of $\hat{\mathcal{K}}$ is of the form $f \cdot x + g \cdot z^2$ on $C_1 - \{p\}$ with f or g a unit in the local ring $\mathcal{O}_{p,C}$. A singularity can occur, then, only when g is a unit. Assuming g is a unit, the map $H^0(\hat{\mathcal{K}}) \rightarrow H^0(\mathcal{J}/\mathcal{I}_1\mathcal{I})$ is defined by $f \cdot x + z^2 \mapsto f \cdot x$ in coordinates. $\mathcal{I}_1\mathcal{I} = (x^2, xz, z^2)$, so a nonzero section $f \cdot x$ of $\mathcal{J}/\mathcal{I}_1\mathcal{I}$ has f as a function of y only in appropriate coordinates. Lifting to the local section $f \cdot x + z^2$ we see that the only way this can be a RDP is if f vanishes to first order. The analytic change of coordinates $(x, f, z) \mapsto (x, y, z)$ gives this singularity as $xy + z^2$, which is an A_1. Therefore, there is at least one A_1 singularity on $C_1 - \{p\}$.

On $C_2 - \{p\}$ a general section of $\hat{\mathcal{K}}$ is of the form $f \cdot y + g \cdot z^2$ and is singular only when g is a unit. This section, because of the surjection $H^0(\hat{\mathcal{K}}) \rightarrow H^0(\mathcal{K}/Sat(\mathcal{I}_2\mathcal{J}))$, comes from the lifting of a section of $\mathcal{K}/Sat(\mathcal{I}_2\mathcal{J})$, which is generated by $\{y\}$ on $C_2 - \{p\}$. Locally, this surjection is defined by $f \cdot y + z^2 \mapsto f \cdot y$ with the assumption that g is a unit. $Sat(\mathcal{I}_2\mathcal{J}) = (y^2, yz, z^2)$ away from p, so a nonzero section of $\mathcal{K}/Sat(\mathcal{I}_2\mathcal{J})$ will have f as a function of x only. Lifting to $\hat{\mathcal{K}}$, $f \cdot y + z^2$ can define a RDP only if f vanishes to first order. The coordinate change $(f, y, z) \mapsto (x, y, z)$ gives this singularity as $xy + z^2$, which is an A_1 singularity. So, there is at least one A_1 singularity on $C_2 - \{p\}$.

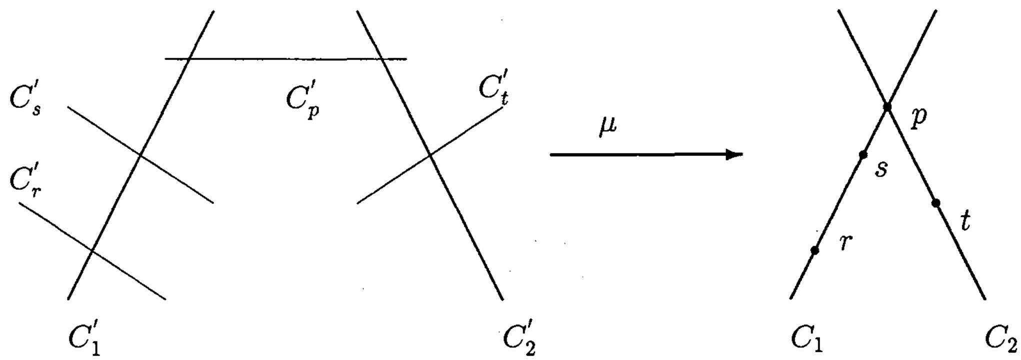


Figure 4.2: $D_6(2,2)$ configuration

We are now ready to count the number of A_1 singularities on $C_i - \{p\}$ for $i = 1, 2$. Recall that a general section has been shown to have an A_1 singularity at p. Lemma 4.17 showed that $\mathcal{K}/\mathcal{I}_1\mathcal{K} \cong \mathcal{J}/Sat(\mathcal{I}_1\mathcal{J})$, so counting the A_1 singularities on $C_1 - \{p\}$ is done exactly as in case 2. In lemma 3.19 it was shown that there were two such singularities.

The image of the map $H^0(\mathcal{K}/\mathcal{I}_2\mathcal{K}) \to H^0(\mathcal{K}/Sat(\mathcal{I}_2\mathcal{J})) \cong H^0(\mathcal{O}_2(1))$ is the sheaf of global sections $H^0(\mathcal{O}_2(1))$ since the sequence

$$0 \longrightarrow Sat(\mathcal{I}_2\mathcal{J})/\mathcal{I}_2\mathcal{K} \longrightarrow \mathcal{K}/\mathcal{I}_2\mathcal{K} \longrightarrow \mathcal{K}/Sat(\mathcal{I}_2\mathcal{J}) \longrightarrow 0$$

splits. Considering the general section as a section of $\mathcal{K}/\mathcal{I}_2\mathcal{K}$, then, its image under this map is a section of $\mathcal{O}_2(1)$, which has one root. This section can vanish at one point only, so there is exactly one A_1 on $C_2 - \{p\}$.

□

Let r and s be the A_1 singularities on $C_1 - \{p\}$ and t the A_1 on $C_2 - \{p\}$. Letting S be the zero scheme of the general section of $\hat{\mathcal{K}}$ and $\mu : S' \to S$ the blow up of r, s, t and p, the smooth surface S' will have six smooth rational curves, C'_i for $i \in \{1, 2, r, s, t, p\}$, with this notation consistent with that in the previous sections. To show that C contracts to a point q whose general hyperplane section has q as a D_6 singularity, it needs to be shown that $I'_i/I'^2_i \cong \mathcal{O}_{C'_i}(2)$ for each i and that the components are in a D_6 configuration in S'. r, s, t and p are RDP's and μ is their resolution, so $I'_i/I'^2_i \cong \mathcal{O}_{C'_i}(2)$ for $i = r, s, t, p$.

Lemma 4.26 $I/I^{(2)} \cong \mathcal{I}/\mathcal{K}$

Proof: As in the proof of lemma 4.10, it will be shown that locally everywhere $Sat(\mathcal{I}^2, f) = \mathcal{K}$. At r, s, t and p, f can be chosen in suitable coordinates to be $f = xy + z^2$. Furthermore, $f = x$ on $C_1 - \{p, r, s\}$ and $f = y$ on $C_2 - \{p, t\}$. Therefore, $Sat(\mathcal{I}^2, f) = Sat(x^2y^2, xyz, z^2, xy + z^2) = (xy, z^2) = \mathcal{K}$ at the point p. On $C_1 - \{p, r, s\}$, $Sat(\mathcal{I}^2, f) = Sat(\mathcal{I}_1{}^2, f) = (x^2, xz, z^2, x) = (x, z^2) = \mathcal{K}$, and on $C_2 - \{p, t\}$, $Sat(\mathcal{I}^2, f) = (y, z^2) = \mathcal{K}$.

At the points r and s on C_1, $\mathcal{I} = \mathcal{I}_1$, so $Sat(\mathcal{I}^2, f) = Sat(x^2, xz, z^2, xy+z^2)$ and the element x can be seen to be the torsion element giving $Sat(\mathcal{I}^2, f) = (x^2, xz, z^2, xy + z^2, x) = (x, z^2) = \mathcal{K}$. Similarly, at the point t where $\mathcal{I} = \mathcal{I}_2$, y is the torsion element of $\mathcal{I}/(\mathcal{I}^2, f)$. Therefore, $Sat(\mathcal{I}^2, f) = (y, z^2) = \mathcal{K}$.

□

Lemma 4.27 $\mathcal{I}/\mathcal{K} \cong \mathcal{O}_C(-1, 0)$.

Proof: As in the proof of lemma 4.12, it will be shown that $\mathcal{I}_1\mathcal{I} + \mathcal{K} = \mathcal{J}$ locally everywhere.

$\mathcal{I}/\mathcal{K}|_{C_1} = \mathcal{I}/(\mathcal{I}_1\mathcal{I} + \mathcal{K})$, and $\mathcal{I}_1\mathcal{I} + \mathcal{K} = (x^2y, xz, z^2) + (xy, z^2) = (xy, xz, z^2) = \mathcal{J}$ in coordinates at p. Away from the point p, $\mathcal{I} = \mathcal{I}_1 = (x, z)$ and $\mathcal{K} = (x, z^2)$, so $\mathcal{I}_1\mathcal{I} + \mathcal{K} = (x, z^2) = \mathcal{J}$. Locally everywhere, then, $\mathcal{I}_1\mathcal{I} + \mathcal{K} = \mathcal{J}$ and this proves that $\mathcal{I}/\mathcal{K}|_{C_1} \cong \mathcal{I}/\mathcal{J} \cong \mathcal{O}_1(-1)$.

The remainder of the proof is exactly the same as that in the proof of lemma 4.12.

□

Corollary 4.2 *We have*

1) $I/I^{(2)}|_{C_1} \cong \mathcal{O}_1(-1)$

2) $I/I^{(2)}|_{C_2} \cong \mathcal{O}_2$

Proof: $I/I^{(2)}|_{C_i} \cong \mathcal{I}/\mathcal{K}|_{C_i}$ from lemma 4.26. Lemma 4.27, then, gives $\mathcal{I}/\mathcal{K}|_{C_1} \cong \mathcal{O}_1(-1)$ and $\mathcal{I}/\mathcal{K}|_{C_2} \cong \mathcal{O}_2$.

□

Lemma 4.28 *We have*

1)$I_1/I_1^{(2)} \cong \mathcal{O}_1(-1)$.

2)$I_2/I_2^{(2)} \cong \mathcal{O}_2$.

Proof: $I_i/I_i^{(2)} = \mathcal{I}_i/Sat(\mathcal{I}_i^2, f)$ by definition, and at p, $Sat({\mathcal{I}_1}^2, f) = Sat(x^2, xz, z^2, xy + z^2)$. With x being the torsion element, this simplifies to (x, z^2). Similarly, y is the torsion element of $\mathcal{I}_2/Sat({\mathcal{I}_2}^2, xy + z^2)$, so $Sat({\mathcal{I}_2}^2, xy+z^2) = (z^2, y)$. $I_i/I_i^{(2)}$, then, is generated by $\{z\}$ at p for $i = 1, 2$. The injection $I/I^{(2)}|_{C_i} \hookrightarrow I_i/I_i^{(2)}$ is an isomorphism away from p, and at p is given by $z \mapsto z$. That is, this map is an isomorphism. The statement of the lemma follows from the preceding corollary.

□

Lemma 4.29 *$I_i'/I_i'^2 \cong \mathcal{O}_i(2)$ for $i = 1, 2$.*

$\mu^* : I_i/I_i^{(2)} \longrightarrow I_i'/I_i'^2$ vanishes to first order at each of the A_1 singularities. There are three such singular points on C_1, namely p, r and s. So, from lemma 4.28, the degree of $I_1'/I_1'^2$ is $-1 + 3 = 2$. p and t are A_1 singularities on C_2. From lemma 4.28 again, then, $I_2'/I_2'^2$ has degree $0 + 2 = 2$.

□

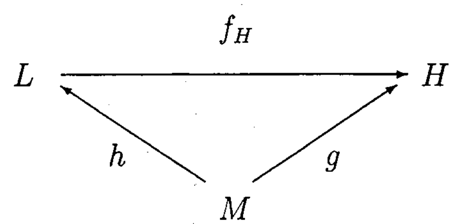
This completes the proof of the following theorem.

Theorem 4.2 *If $f : X \longrightarrow Y$ is a contraction map with $f(C) = q$ and $C = C_1 \cup C_2$ has length$(2,2)$ with defining ideals $\mathcal{J} = (xy, xz, z^2)$ and $\mathcal{K} = (xy, z^2)$ at $p = C_1 \bigcap C_2$, then a general hyperplane section of q has a D_6 type singularity at q.*

CHAPTER 5

KAWAMATA TECHNIQUE

Let $f : X \longrightarrow Y$ be the contraction map and H a general hyperplane section of the singular point q in Y. It has been noted that H has a rational double point (RDP) at q. The pullback $f^*H = L$ is a general section of the curve C, and it defines a map $f_H : L \longrightarrow H$. If $g : M \longrightarrow H$ is the minimal resolution of the RDP, then Reid in [Re] has proven that f_H factors g. That is, the diagram



commutes, where $h : M \longrightarrow L$ is the map contracting all of the exceptional curves of g except the strict transform C' of C.

It was proven in the paper [Ka] that the singularity of H can be completely determined by the multiplicity of the fundamental cycle at C' if C is an irreducible rational curve. However, if C is reducible, then the multiplicity of each component of C' in the fundamental cycle is not enough to determine the general hyperplane section. This chapter will utilize the geometric technique of Kawamata in [Ka] to try to determine the general hyperplane section H of q. It will be shown that if each component of C has length 1, then this technique confirms the result of theorem 2.1. However, for curves C with components of length 2, this technique does not appear to give the precise results that chapters 3 and 4 provide. That is, the additional information provided by the defining ideals $\mathcal{J}$ and $\mathcal{K}$ of these chapters to determine the type of rational double point is not apparently available using this technique.

The following notation and results will be used. The notation is as close to that as in [Ka] as possible, but a few changes were necessary to account for the added components of C.

- Γ : The dual graph of the exceptional curves of g.
- $F = \sum_{i=1}^{n} m_i C'_i$: The fundamental cycle of g on M.

- C'_{i_j}: The jth component of C' in the fundamental cycle.

- m_{i_j}: The multiplicity of C'_{i_j} in the fundamental cycle.

- $mC' = \sum_{j=1}^{k} m_{i_j} C'_{i_j}$

- H': Another general hyperplane section of Y through q. H' has the same type of singularities as H.

- $L' = f^*H'$. L' has the same type of singularities as L.

- $D = L \cap L'$. D is a general element of the linear system of effective Cartier divisors on L which contain C.

- D': A reduced nonsingular curve with no common irreducible components with F. D' comes from the first blow up of the point $q \in X$. From the calculations, if $\Gamma = A_n$ for $n \geq 2$, then D' has two irreducible components. Each component intersects transversely one of the end components of F. If $\Gamma = D_n$, then D' is irreducible and intersects C'_2 transversely. For $\Gamma = E_6, E_7$ and E_8, D' intersects C'_6, C'_6, and C'_1, respectively, transversely.

- $F + D'$: The total transform of D on M.

- p_i: The singular points of L.

- p'_i: The singular points of L'.

- Γ_i: The dual graph of the exceptional curves of h over p_i.

- F_i: The fundamental cycle corresponding to Γ_i.

- d_i: The multiplicity of D at the point p_i.

  This can be calculated from the blow up h of the singular points of L. h contracts everything except C', so the strict transform of D on M is $mC' + D'$. By definition, F_i is the exceptional set over p_i, so

- $d_i = (mC' + D') \cdot F_i$

To determine the general hyperplane H' of q, the hyperplane L' is studied. p_i is a singular point of L, (resp. L'), only if $\text{mult}_{p_i}(L) \geq 2$, ($\text{mult}_{p_i}(L') \geq 2$). Therefore, since $D = L \cap L'$, and L is known to be singular at p_i, L' can be singular at p_i only if $d_i \geq 4$. Kawamata also shows that away from the p_i, L' has only singularities of type $A_{m_{i_j}-1}$ on the component C'_{i_j}.

5.1 The length(1,1, $\cdots$,1) case

Theorem 5.1 *Let $C = C_1 \bigcup C_2 \bigcup \cdots \bigcup C_k$ with all curves having their strict transforms of multiplicity 1 in the fundamental cycle. If C contracts, then C contracts to a cA_k singularity.*

Proof: Since all of the components have multiplicity 1, the minimal resolution of p can only be one of the following:

Case 1: An A_n configuration with $k \leq n$.
Case 2: A D_n configuration with C having at most 2 components.
Case 3: An E_6 configuration with C having at most 2 components.

Case 1: Using the same notation as Kawamata we have $mC' = C'_{i_1} + C'_{i_2} + \cdots + C'_{i_k}$ with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. D' intersects only the components C'_1 and C'_n.Going from left to right: $F_1 = C'_1 + C'_2 + \cdots + C'_{i_1-1}$, $F_2 = C'_{i_1+1} + C'_{i_1+2} + \cdots C'_{i_2-1}$, $\cdots$, $F_j = C'_{i_{j-1}+1} + \cdots C'_{i_j-1}$, $\cdots$, $F_{k+1} = C'_{i_k+1} + \cdots + C'_n$. We calculate the multiplicity $d_i = (mC' + D') \cdot F_i$ of D at the points p_i for $i = 1, 2, \cdots k+1$. Since D' only intersects C'_1 and C'_n, $D' \cdot F_i = 1$ for $i = 1, k+1$ and $D' \cdot F_i = 0$ otherwise. From the configuration of the A_n Dynkin diagram it can be seen that

$$d_i = \begin{cases} 1 & \text{if} \quad i = 1 \text{ or } k+1 \\ 2 & \text{if} \quad 2 \leq i \leq k \end{cases}$$

Therefore, the maximum that d_i could be is 3. So, L' is smooth at the p_i and it must be smooth away from the p_i as well.

Case 2: Using symmetry of the Dynkin diagram, $mC' = C'_1 + C'_n$ is the only possibility. Then $F_1 = C'_2 + C'_3 + \cdots + C'_{n-1}$ since the singularity where C_1 and C_n intersect is an A_{n-2}. We have $mC' \cdot F_1 = 2$ and $D' \cdot F_1 = 1$ since D' intersects C_2 only. Therefore, $d_1 = 3$ and L' is smooth on all of C.

Case 3: $mC' = C_1' + C_5'$, D' intersects C_6' only, and $F_1 = C_2' + 2C_3' + C_4' + C_6'$ since it has a D_4 configuration. We have $mC' \cdot F_1 = 2$ and $D' \cdot F_1 = 1$, so $d_1 = 3$. Again L' is smooth on all of C.

□

5.2 The length(2,1) case

The next situation that will be discussed is when $C = C_1 \bigcup C_2$ with the strict transforms of C_1 and C_2 having multiplicities 2 and 1, respectively. From the Dynkin diagrams, Γ can be D_m for some $m \geq k$, E_6, or E_7. As in chapter 3, it will be shown that $\Gamma = D_4$ or D_5. Proceed by process of elimination.

Lemma 5.1 $\Gamma \neq E_7$.

Proof: From the E_7 configuration it can be seen that there are three cases to consider.

Case 1: $mC' = C_1' + 2C_2'$.

L has one singular point p_1, $\Gamma_1 = D_5$, $F_1 = C_6' + 2C_5' + 2C_4' + C_3' + C_7'$. Calculating from the definition above,

$$d_1 = 2C_2' \cdot C_3' + D' \cdot C_6' = 3.$$

Therefore, L' is smooth at p_1. On $C_1' - \{p_1\}$ L' must be smooth, and on $C_2' - \{p_1\}$ L' has A_1 singularities only. It follows from the possible Dynkin diagrams that $\Gamma = D_4$.

Case 2: $mC' = C_1' + 2C_6'$.

L has one singular point p_1, $\Gamma_1 = D_5$, $F_1 = C_2' + 2C_3' + 2C_4' + C_5' + C_7'$. Therefore,

$$d_1 = C_1' \cdot C_2' + 2C_6' \cdot C_5' = 3.$$

This means that L' is smooth at p_1. Away from p_1 on C_1', L' is known to be smooth, and on C_6', L' has singularities of type A_1 only. So, $\Gamma = D_4$.

Case 3: $mC' = C_1' + 2C_7'$.

L has one singular point p_1. $\Gamma_1 = A_5$, $F_1 = C_2' + C_3' + C_4' + C_5' + C_6'$. So,

$$d_1 = C_1' \cdot C_2' + 2C_7' \cdot C_3' + D' \cdot C_6' = 4.$$

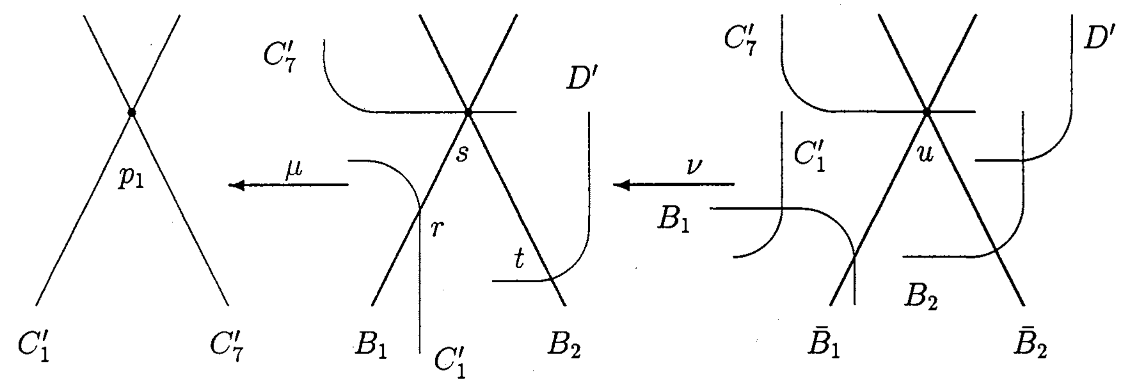


Figure 5.1: $E_7(2,1)$

In this case the singularity of L' needs to be determined because it cannot be concluded that L' is smooth at p_1 from the calculation of d_1. By symmetry with L, we can conclude that L' has at worst an A_5 singularity at p_1. It will be shown that, in fact, L' has at worst an A_2 singularity at p_1.

Let $\mu : X^{(1)} \to X$ be the blow up at p_1. In $X^{(1)}$, let E be the exceptional divisor, $L^{(1)}$, $L^{(1)'}$ the strict transforms of L and L', respectively. Assuming that p_1 is an A_5 type singularity, $L^{(1)} \cap E = B = B_1 + B_2$, where B_1 and B_2 are smooth rational curves corresponding to C_2' and C_6', respectively, in F. B_1 and B_2 meet transversely at a point s, which is a singularity of type A_3 on $L^{(1)}$. The strict transforms of D', C_1 and C_7 will be denoted the same on $X^{(1)}$ to simplify notation. They meet B_1 and B_2 transversely as shown in figure 5.1.

To find the singularity at p_1 on L', we look in the blow up at $B' = L^{(1)'} \cap E$. The pull back of L', $\mu^*(L') = L^{(1)'} + aE$ for some integer a, and B' is a curve of degree a. Furthermore, L' is smooth at p_1 if and only if $a = 1$. a is determined by calculating $\mu^*(L') \cap L^{(1)}$. From the multiplicities in the fundamental cycle,

$$\mu^*(L') \cap L^{(1)} = 2B_1 + 2B_2 + C_1' + 2C_7' + D'.$$

But we also know that

$$\mu^*(L') \cap L^{(1)} = (L^{(1)'} + aE) \cap L^{(1)} = L^{(1)} \cap L^{(1)'} + a(B_1 + B_2).$$

Equating the two equations for $\mu^*(L') \cap L^{(1)}$, we must have $a \leq 2$. If $a = 0$ or 1, then $B_1 + B_2 \subset L^{(1)} \cap L^{(1)'}$. But $B_1 + B_2 \subset E$ implies that $B_1 + B_2 \subset L^{(1)'} \cap E$.

This is impossible since either B' is a curve of degree 0, which makes no sense, or B' is a curve of degree 1, which makes it impossible to contain the degree 2 divisor B. Therefore, $a = 2$. It can now be concluded that

$$L^{(1)} \cap L^{(1)'} = C_1' + 2C_7' + D',$$

$$B' \cap B = L^{(1)} \cap L^{(1)'} \cap E = r + 2s + t,$$

as shown in figure 5.1, and B' is a conic in E.

There is no contradiction to the assumptions thus far, so continue by blowing up the point s in $X^{(1)}$. Let $\nu : X^{(2)} \to X^{(1)}$ denote this blow up with $E^{(1)}$ the exceptional set, and let $L^{(2)}$, $L^{(2)'}$and E' be the strict transforms of $L^{(1)}$, $L^{(1)'}$and E, respectively. Finally, let $\pi = \mu \circ \nu$. In this situation, since s is an A_3 singularity, $E^{(1)} \cap L^{(2)} = \bar{B}_1 + \bar{B}_2$, where $\bar{B}_1$ and $\bar{B}_2$ correspond to C_3' and C_5', respectively, in the fundamental cycle. Analogous computations are made in this blow up. $\nu^*(L^{(1)'}) = L^{(2)'} + bE^{(1)}$ with b an integer, and b is calculated from $\pi^*(L') \cap L^{(2)}$. From the fundamental cycle,

$$\pi^*(L') \cap L^{(2)} = 3\bar{B}_1 + 3\bar{B}_2 + 2B_1 + 2B_2 + C_1' + 2C_7' + D'.$$

Now, since $\pi^* = \nu^* \circ \mu^*$,

$$\begin{aligned} \pi^*(L') &= \nu^*(\mu^*(L') \\ &= \nu^*(L^{(1)'} + 2E) \\ &= L^{(2)'} + bE^{(1)} + 2\nu^*(E) \\ &= L^{(2)'} + bE^{(1)} + 2(E' + E^{(1)}). \end{aligned}$$

Substituting, we have

$$\pi^*(L') \cap L^{(2)} = L^{(2)} \cap L^{(2)'} + (b+2)(L^{(2)} \cap E^{(1)}) + 2(L^{(2)} \cap E').$$

In this case, we must have $b \leq 1$, but $b = 0$ makes no sense, so $b = 1$. Therefore, $L^{(1)'}$ is smooth at s, which means that the singularity at p_1 was resolved in one blow up, namely μ. The only RDP's that can be resolved in one blow up are A_1 and A_2. Using the fact that L' is smooth on C_1' and only A_1 singularities on C_7' away from p_1, if L' has an A_1 singularity at p_1, then $\Gamma = D_5$. If L' has an A_2 at then $\Gamma = D_6$. So, it has been shown that not only is $\Gamma \neq E_7$, but the singularity is at worst a D_6.

□

Lemma 5.2 $\Gamma \neq E_6$

Proof: From the Dynkin diagram of E_6, notice that again there are three different cases to consider.

Case 1: $mC' = C_1' + 2C_2'$. By symmetry this is the same as $mC' = C_5' + 2C_4'$.

In this case L has a singular point p_1 of type $\Gamma = A_4$ with $F_1 = C_6' + C'3 + C_4' + C_5'$. Recalling that D' intersects C_6' only, we have

$$d_1 = 2C_2' \cdot C_3' + D' \cdot C_6' = 3.$$

Therefore, L' is smooth at p_1, and again it has been shown that $\Gamma = D_4$.

Case 2: $mC' = C_1' + 2C_6'$. By symmetry this is the same as $mC' = C_5' + 2C_6'$.

L has a singularity, p_1, of type $\Gamma_1 = A_4$. $F_1 = C_2' + C_3' + C_4' + C_5'$, so

$$d_1 = C_1' \cdot C_2' + 2C_6' \cdot C_3' = 3.$$

L', then, is smooth at p_1 and $\Gamma = D_4$.

Case 3: $mC' = C_1' + 2C_4'$. By symmetry, this is the same as $mC' = C_5' + 2C_2'$.

L has two singularities p_1 of type $\Gamma_1 = A_1$ with $F_1 = C_5'$, and p_2 of type $\Gamma_2 = A_3$, making $F_2 = C_2' + C_3' + C_6'$. Calculating d_i for $i = 1, 2$ gives

$$d_1 = 2C_4' \cdot C_5' = 2,$$

and

$$d_2 = C_1' \cdot C_2' + 2C_4' \cdot C_3' + D' \cdot C_6' = 4.$$

Therefore, L' is nonsingular at p_1 and may be singular at p_2. We will continue as in the E_7 case with the point p_2.

Let $\mu : X^{(1)} \to X$ be the blow up at p_2, and let all the remaining notation be the same as that in Case 3 of the E_7 lemma. Here we have

$$\mu^*(L') \cap L^1 = 2B_1 + 2B_2 + C_1' + 2C_4' + D',$$

and, since $\mu^*(L') = L^{(1)'} + aE$,

$$\mu^*(L') \cap L^{(1)} = L^{(1)} \cap L^{(1)'} + a(B_1 + B_2)$$

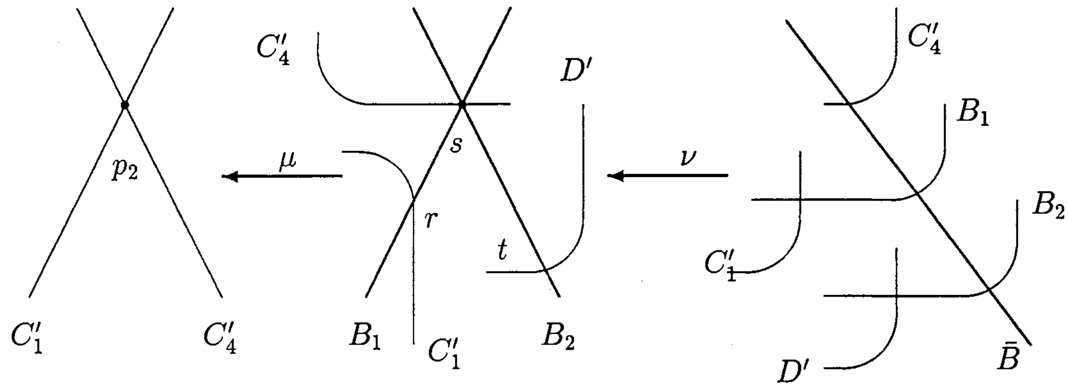


Figure 5.2: $E_6(2,1)$

as well. Again it can be concluded that $a = 2$, which means B' is a conic in E with $B \cap B' = r + 2s + t$ as shown in Figure 5.2.

$L^{(1)}$ has an A_1 singularity at s. Let $\nu : X^{(2)} \to X^{(1)}$ be the blow up at s, and let E', $E^{(1)}$, $L^{(2)}$, $L^{(2)'}$ and π^* the same as described earlier. In this case $L^{(2)} \cap E^{(1)} = \bar{B}$, where $\bar{B}$ corresponds to C_3' in F. So,

$$\pi^*(L') \cap L^{(2)} = 3\bar{B} + 2B_1 + 2B_2 + C_1' + 2C_4' + D'.$$

From the first blow up we showed that $\mu^*(L') = L^{(1)'} + 2E$, from which it follows that

$$\begin{aligned} \pi^*(L') &= \nu^*(L^{(1)'}) + 2\nu^*(E) \\ &= L^{(2)'} + bE^{(1)} + 2(E' + E^{(1)}). \end{aligned}$$

Now, then,

$$\begin{aligned} \pi^*(L') \cap L^{(2)} &= L^{(2)} \cap L^{(2)'} + (b+2)(L^{(2)} \cap E^{(1)}) + 2(L^{(2)} \cap E') \\ &= L^{(2)} \cap L^{(2)'} + (b+2)\bar{B} + 2(B_1 + B_2). \end{aligned}$$

Concluding as in the E_7 case, $b = 1$ and L' has at worst an A_2 singularity at p_2. So, $\Gamma = D_5$ or D_6, eliminating the possibility of E_6.

□

The final cases to consider are those where the minimal resolution has a D_n configuration as exceptional set. In chapter 3 it was shown that both D_4 and D_5 are possibilities, so the D_n cases with $n \geq 6$ will be considered.

Lemma 5.3 $\Gamma \neq D_n$ *for* $n \geq 6$.

Proof: There are two cases to consider.

Case 1: $mC' = C'_n + 2C'_j$ for $2 \leq j \leq n-2$. For fixed j, this is the same as $mC' = C'_{n-1} + 2C'_j$.

L has two singularities, p_1 and p_2. At p_1 we have $\Gamma_1 = A_{j-1}$ and $F_1 = C'_1 + C'_2 + \cdots C'_{j-2} + C'_{j-1}$. Knowing that D' intersects C'_2 only ,

$$d_1 = 2C'_j \cdot C'_{j-1} + D' \cdot C'_2 = \begin{cases} 2 & \text{if } j = 2 \\ 3 & \text{if } j \geq 3 \end{cases}$$

Therefore, L' is smooth at p_1.

At p_2 the dual graph $\Gamma_2 = A_{n-j-1}$ with $F_2 = C'_{j+1} + \cdots + C'_{n-2} + C'_{n-1}$.

$$d_2 = C'_n \cdot C'_{n-2} + 2C'_j \cdot C'_{j+1} = \begin{cases} 0+2=2 & \text{if } j = n-2 \\ 1+2=3 & \text{if } j \neq n-2 \end{cases}$$

So, L' is also smooth at p_2, showing that $\Gamma = D_4$.

Case 2: $mC' = C'_1 + 2C'_j$ for $2 \leq j \leq n-2$.

Sub-case 2a: $j = 2$

L has just one singular point p_1 at which $\Gamma_1 = D_{n-2}$ (A_{n-2} for $n \leq 5$) and $F_1 = C'_3 + 2C'_4 + \cdots + 2C'_{n-2} + C'_{n-1} + C'_n$. So $d_1 = 2C'_2 \cdot C'_3 = 2$, and L' is nonsingular at p_1. Therefore, $\Gamma = D_4$.

Sub-case 2b: $3 \leq j \leq n-3$

L has two singular points p_1 and p_2 with $\Gamma_1 = A_{j-2}$, $F_1 = C'_2 + \cdots C'_{j-1}$, $\Gamma_2 = D_{n-j}$, and $F_2 = C'_{j+1} + 2C'_{j+2} + \cdots + 2C'_{n-2} + C'_{n-1} + C'_n$.

At p_1, L' has at worst an A_{n-j} singularity at p_1. At p_2, $d_2 = 2C'_j \cdot C'_{j+1} = 2$, so L' is smooth at p_2. Therefore, L' has simpler singularities than L, which is a contradiction.

Sub-case 2c: $j = n-2$.

L has three singular points, p_1, p_2, and p_3. At p_2 and p_3 there are A_1 type singularities with $F_2 = C'_{n-1}$ and $F_3 = C'_n$, respectively. So, $d_2 = d_3 = 2$ and L' is smooth at these two points.

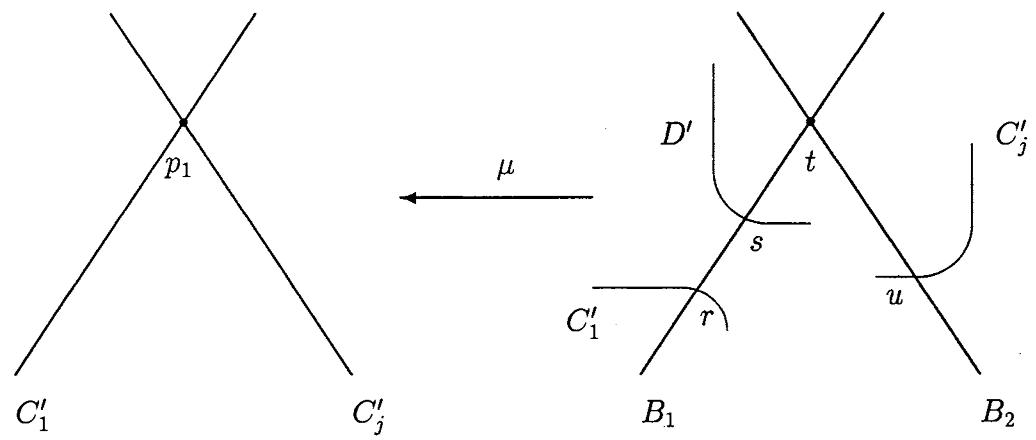


Figure 5.3: $D_n(2,1)$

The singularity at p_1 of type $\Gamma_1 = A_{j-2}$ with $F_1 = C'_2 + C'_3 + \cdots + C'_{j-1}$ has $d_1 = 4$. Therefore, L' may be singular at p_1. To show that $\Gamma = D_4$ or D_5, it must be shown that L' has at worst an A_1 singularity at p_1. That is, show that $j = 2$ or $j = 3$ are the only possible values of j if L' is a general hyperplane section. Assume $j \geq 4$, and let $\mu : X^{(1)} \to X$ be the blow up p_1 in X. Using the same notation as in the previous cases, then, $L^{(1)} \cap E = B = B_1 + B_2$, where B_1 and B_2 correspond to C'_2 and C'_{j-1}, respectively, in the fundamental cycle. That is, by assuming that $j \geq 4$, $L^{(1)} \cap E$ is a reducible curve in E.

Let $\{L_\lambda\}_{\lambda \in \mathbf{P1}}$ be a generic pencil of hyperplane sections of C. In particular, L and L' are generators of this pencil. Being a generic pencil, the intersection of any two elements of $\{L_\lambda\}$ is a curve. Therefore,

$$\mu^*(L_\lambda) \cap L^{(1)} = 2B_1 + 2B_2 + C'_1 + 2C'_j + D'. \tag{5.1}$$

Also, if $L^{(1)}_\lambda$ denotes the strict transform of L_λ, then $\mu^*(L_\lambda) = L^{(1)}_\lambda + a_\lambda E$, and we have that

$$\mu^*(L_\lambda) \cap L^{(1)} = L^{(1)} \cap L^{(1)}_\lambda + a_\lambda(B_1 + B_2). \tag{5.2}$$

But $L^{(1)} \cap L^{(1)}_\lambda$ is a curve, so $a_\lambda \leq 2$ for each L_λ in this pencil. If $a_\lambda = 0$ for some $\lambda \in \mathbf{P^1}$, then the degree 4 divisor $2B_1 + 2B_2 \subset L^{(1)}_\lambda \cap E$. But $L^{(1)}_\lambda \cap E$ is a degree 0 curve if $a_\lambda = 0$. This is a contradiction. Similarly, if $a_\lambda = 1$, then $B_1 + B_2$ would be contained in the degree 1 curve $L^{(1)}_\lambda \cap E$. Therefore, it has been shown that from the generic pencil of hyperplane sections $\{L_\lambda\}$, there is a pencil of degree 2 curves in $E \cong \mathbf{P^2}$ given by $\{L^{(1)}_\lambda \cap E\}$. That is, there is a pencil of conics generated by the curves $L^{(1)'} \cap E = B'$ and $L^{(1)} \cap E = B$. As B is a reducible conic, this is a pencil of reducible conics in E. Now $B \cap B' = r + s + 2u$ is the base locus of this pencil (see figure 5.3), so the general element of the pencil is smooth at t. B', then, is smooth on all of B, since t is the only singular point of B. In particular, B' is smooth at the base point u. But this means that B' is tangent to the component B_2 of B, and so B' could not be a reducible conic. Therefore, there could not be such a pencil of reducible conics in E and it must be that $\Gamma = D_4$ or D_5.

□

CHAPTER 6

CONTRACTION CRITERIA

The goal of this chapter is to give contraction criteria for those curves $C = \cup_{i=1}^{n} C_i$ for which all the components have length 1. From the explanation in chapter 2, this means that $\mathcal{I}/\mathcal{I}^2|_{C_i} = (0,1)$ if $i = 1, n$ and $(0,0)$ for $2 \leq i \leq n-1$. This will be accomplished from the theory of versal deformations of rational double points and their simultaneous resolutions, as well as a generalization of Reid's [Re] construction of a sequence of ideals in $\mathcal{I}$ that will determine contraction criteria for C. With the hypothesis that all components have length 1, it has been shown formally in chapter 2 and in the analytic category in chapter 5 that if C contracts it will contract to a point q whose general hyperplane section has an A_n singularity at q. So, in particular, the deformation space of A_n singularities will be utilized. Information concerning the deformations, and their simultaneous resolutions, of A_n singularities can be found in [Ty], [Kas] and [KM].

In the discussion following Lemma 2.12 of chapter 2, it was shown that a general section of $\mathcal{I}/\mathcal{I}^2$ defines a smooth surface S and the result is an exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{O}_C(1,0,\cdots,0,1) \longrightarrow 0. \tag{6.1}$$

This sequence splits if and only if there is a surjection $\mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_C$. The splitting of this exact sequence is also equivalent to $\mathcal{I}/\mathcal{I}^2 \cong \mathcal{O}_C \oplus \mathcal{O}_C(1,0,\cdots,0,1)$. The following lemma provides a constructive equivalency to the splitting of this sequence.

Lemma 6.1 *The sequence 6.1 splits if and only if there exists an ideal $\mathcal{K}_2$ satisfying $\mathcal{I}^2 \subset \mathcal{K}_2 \subset \mathcal{I}$, $\mathcal{I}/\mathcal{K}_2 \cong \mathcal{O}_C$ and $\mathcal{K}_2/\mathcal{I}^2 \cong \mathcal{O}_C(1,0,\cdots,0,1)$.*

Proof: If this sequence splits there is a surjection $\mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_C$, so define

$$\mathcal{K}_2 = Ker(\mathcal{I} \to \mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_C).$$

By definition, then, $\mathcal{I}^2 \subset \mathcal{K}_2 \subset \mathcal{I}$ and $\mathcal{I}/\mathcal{K}_2 \cong \mathcal{O}_C$. The exact sequence

$$0 \longrightarrow \mathcal{K}_2/\mathcal{I}^2 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{I}/\mathcal{K}_2 \longrightarrow 0 \tag{6.2}$$

restricted to the components C_i for $i = 1, n$ gives the exact sequence

$$0 \longrightarrow \mathcal{K}_2/\mathcal{I}^2|_{C_i} \longrightarrow \mathcal{O}_i \oplus \mathcal{O}_i(1) \longrightarrow \mathcal{O}_i \longrightarrow 0.$$

Therefore. $\mathcal{K}_2/\mathcal{I}^2|_{C_i} \cong \mathcal{O}_i(1)$ for $i = 1, n$. For $2 \le i \le n-1$, the restriction of this sequence becomes

$$0 \longrightarrow \mathcal{K}_2/\mathcal{I}^2|_{C_i} \longrightarrow \mathcal{O}_i \oplus \mathcal{O}_i \longrightarrow \mathcal{O}_i \longrightarrow 0.$$

This shows that $\mathcal{K}_2/\mathcal{I}^2|_{C_i} \cong \mathcal{O}_i$ for all $2 \le i \le n-1$. Since invertible sheaves on C are completely determined by their degree on each component, we have $\mathcal{K}_2/\mathcal{I}^2 \cong \mathcal{O}_C(1, 0, \cdots, 0, 1)$.

If there exists an ideal sheaf of C, $\mathcal{K}_2$, satisfying $\mathcal{I}^2 \subset \mathcal{K}_2 \subset \mathcal{I}$, $\mathcal{I}/\mathcal{K}_2 \cong \mathcal{O}_C$ and $\mathcal{K}_2/\mathcal{I}^2 \cong \mathcal{O}_C(1, 0, \cdots, 0, 1)$, then the exact sequence 6.2 results. So, $\mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{I}/\mathcal{K}_2$ is a surjection, implying that sequence 6.1 splits.

□

This new defining ideal for C can be calculated in local coordinates $\{x, y, z\}$ at a singular point of C. In fact, there are two possible forms for $\mathcal{K}_2$.

The invertible sheaf $\mathcal{K}_2/\mathcal{I}^2$ is a subsheaf of $\mathcal{I}/\mathcal{I}^2$, which is generated by $\{xy, z\}$. Therefore, $\mathcal{K}_2/\mathcal{I}^2$ is generated locally by an element of the form $f_0xy + f_1z$ with $f_0, f_1 \in \mathcal{O}_{p,X}$. Since this element is a generator, either f_0 or f_1 must be a unit in the ring $\mathcal{O}_{p,X}$. If f_0 is a unit, then, dividing by f_0, $\mathcal{K}_2/\mathcal{I}^2$ is generated by an element of the form $xy + g_1z$. So,

$$\mathcal{K}_2 = (xy + g_1z) + \mathcal{I}^2 = (xy + g_1z, z^2).$$

On the other hand, if f_1 is a unit, then, dividing by f_1, $\mathcal{K}_2/\mathcal{I}^2$ is generated by an element of the form $g_0xy + z$. In this case, the analytic change of coordinates inverse to $(x, y, z) \mapsto (x, y, g_0xy + z)$ gives $\mathcal{K}_2/\mathcal{I}^2$ being generated by z, and it does not affect the description of $\mathcal{I}$ as (xy, z). It can now be seen that

$$\mathcal{K}_2 = (z) + \mathcal{I}^2 = (x^2y^2, z).$$

Lemma 6.2 *$\mathcal{K}_2/\mathcal{I}\mathcal{K}_2$ is locally free of rank 2 on C.*

Proof: If $\mathcal{K}_2 = (xy + g_1z, z^2)$, define a map $\mathcal{O}_C \oplus \mathcal{O}_C \longrightarrow \mathcal{K}_2/\mathcal{I}\mathcal{K}_2$ by $(f, g) \mapsto f(xy + g_1z) + gz^2$. This map is surjective as it sends generators to generators. Now, $f(xy + g_1z) + gz^2 \in \mathcal{I}\mathcal{K}_2 = (x^2y^2 + g_1xyz, xyz + g_1z^2, z^3)$ implies xy or z must divide both f and g. That is, the kernel of this map is $\mathcal{I}_C \oplus \mathcal{I}_C$, proving it is injective. Therefore, this map is an isomorphism, showing that $\mathcal{K}_2/\mathcal{I}\mathcal{K}_2$ is locally free of rank 2 if $\mathcal{K}_2 = (xy + g_1z, z^2)$.

Similarly, for $\mathcal{K}_2 = (x^2y^2, z)$, the map $\mathcal{O}_C \oplus \mathcal{O}_C \longrightarrow \mathcal{K}_2/\mathcal{I}\mathcal{K}_2$ given by $(f,g) \mapsto fx^2y^2 + gz$ is an isomorphism.

□

This sheaf, $\mathcal{K}_2/\mathcal{I}\mathcal{K}_2$, fits in the exact sequence

$$0 \longrightarrow \mathcal{I}^2/\mathcal{I}\mathcal{K}_2 \longrightarrow \mathcal{K}_2/\mathcal{I}\mathcal{K}_2 \longrightarrow \mathcal{K}_2/\mathcal{I}^2 \longrightarrow 0. \qquad (6.3)$$

It has already been shown that $\mathcal{K}_2/\mathcal{I}^2 \cong \mathcal{O}_C(1,0,\cdots,0,1)$. The following lemma determines the invertible sheaf $\mathcal{I}^2/\mathcal{I}\mathcal{K}_2$.

Lemma 6.3 $\mathcal{I}^2/\mathcal{I}\mathcal{K}_2 \cong \mathcal{O}_C$

Proof: Define a map $\mathcal{I} \otimes \mathcal{I} \to \mathcal{I}^2/\mathcal{I}\mathcal{K}_2$ by multiplication of functions. This map clearly annihilates the sheaf $\mathcal{I} \otimes \mathcal{K}_2$, so this map induces a well defined map $\mathcal{I}/\mathcal{K}_2 \otimes \mathcal{I}/\mathcal{K}_2 \to \mathcal{I}^2/\mathcal{I}\mathcal{K}_2$. This map is an isomorphism due to local coordinate calculations, as performed in appendices A,B,C and D. Therefore, $\mathcal{I}^2/\mathcal{I}\mathcal{K}_2 \cong \mathcal{I}/\mathcal{K}_2 \otimes \mathcal{I}/\mathcal{K}_2 \cong \mathcal{O}_C$.

□

This lemma shows, then, that $\mathcal{I}^2/\mathcal{I}\mathcal{K}_2 \cong \mathcal{O}_C$, so we have the exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{K}_2/\mathcal{I}\mathcal{K}_2 \longrightarrow \mathcal{O}_C(1,0,\cdots,0,1) \longrightarrow 0,$$

which is the same as sequence 6.1 with $\mathcal{K}_2/\mathcal{I}\mathcal{K}_2$ replacing $\mathcal{I}/\mathcal{I}^2$. Just as in the proof of Lemma 6.1, then, sequence 6.3 splits if and only if there exists an ideal $\mathcal{K}_3$ satisfying $\mathcal{I}\mathcal{K}_2 \subset \mathcal{K}_3 \subset \mathcal{K}_2$, $\mathcal{K}_2/\mathcal{K}_3 \cong \mathcal{O}_C$ and $\mathcal{K}_3/\mathcal{I}\mathcal{K}_2 \cong \mathcal{O}_C(1,0,\cdots,0,1)$. To extend this process, local descriptions of the ideal sheaves $\mathcal{K}_i$ are needed.

Lemma 6.4 *Let $\mathcal{K}_m \subset \mathcal{K}_{m-1} \subset \cdots \subset \mathcal{K}_3 \subset \mathcal{K}_2 \subset \mathcal{I}$ be a sequence of ideals satisfying $\mathcal{I}\mathcal{K}_{i-1} \subset \mathcal{K}_i \subset \mathcal{K}_{i-1}$, $\mathcal{K}_{i-1}/\mathcal{K}_i \cong \mathcal{O}_C$ and $\mathcal{K}_i/\mathcal{I}\mathcal{K}_{i-1} \cong \mathcal{O}_C(1,0,\cdots,0,1)$. In local coordinates at p on C, $\mathcal{K}_m = (xy + g_1z + \cdots g_{m-1}z^{m-1}, z^m)$ for $m \geq 1$, or $\mathcal{K}_m = (x^my^m, z)$ for $m \geq 1$.*

Proof: For $m = 1$ we have $\mathcal{K}_1 = \mathcal{I} = (xy, z)$ for either case.

Assume that $\mathcal{K}_k = (xy + g_1z + \cdots + g_{k-1}z^{k-1}, z^k)$ for all $k \leq m$. It needs to be shown that $\mathcal{K}_{m+1} = (xy + g_1z + \cdots + g_mz^m, z^{m+1})$. The existence of $\mathcal{K}_{m+1}$ comes from the splitting of the exact sequence

$$0 \longrightarrow \mathcal{I}\mathcal{K}_{m-1}/\mathcal{I}\mathcal{K}_m \longrightarrow \mathcal{K}_m/\mathcal{I}\mathcal{K}_m \longrightarrow \mathcal{K}_m/\mathcal{I}\mathcal{K}_{m-1} \longrightarrow 0 \qquad (6.4)$$

Since $\mathcal{I}, \mathcal{K}_{m-1}$, and $\mathcal{K}_m$ have the desired form by the induction hypothesis, the generators for the invertible sheaves in the above sequence can be calculated in local coordinates: $\mathcal{IK}_{m-1}/\mathcal{IK}_m$ is generated by $\{z^m\}$ and $\mathcal{K}_m/\mathcal{IK}_{m-1}$ is generated by $\{xy + g_1z + \cdots + g_{m-1}z^{m-1}\}$. $\mathcal{K}_{m+1}$ is defined by $Ker(\mathcal{K}_m \to \mathcal{K}_m/\mathcal{IK}_m \to \mathcal{O}_C)$, where $\mathcal{O}_C = \mathcal{IK}_{m-1}/\mathcal{IK}_m$. So, the map $\mathcal{K}_m/\mathcal{IK}_m \longrightarrow \mathcal{O}_C$ is defined by $z^m \mapsto z^m$ and $xy + g_1z + \cdots + g_{m-1}z^{m-1} \mapsto f_mz^m$ for some function f_m. The kernel of this map is the subsheaf generated by $\{xy + g_1z + \cdots + g_{m-1}z^{m-1} - f_mz^m\}$. Therefore, $\mathcal{K}_{m+1} = (xy + g_1z + \cdots + g_{m-1}z^{m-1} - f_mz^m) + \mathcal{IK}_m$, which can be calculated in coordinates to be $(xy + g_1z + \cdots + g_{m-1}z^{m-1} - f_mz^m, z^{m+1})$. Let $g_m = -f_m$, then $\mathcal{K}_{m+1} = (xy + g_1z + \cdots + g_{m-1}z^{m-1} + g_mz^m, z^{m+1})$.

Now, assume that $\mathcal{K}_k = (x^ky^k, z)$ for all $k \leq m$. It needs to be shown that $\mathcal{K}_{m+1} = (x^{m+1}y^{m+1}, z)$. In this case, $\mathcal{IK}_{m-1}/\mathcal{IK}_m \cong \mathcal{O}_C$ is generated by $\{x^my^m\}$ and $\mathcal{K}_m/\mathcal{IK}_{m-1}$ is generated by $\{z\}$. The map $\mathcal{K}_m/\mathcal{IK}_m \longrightarrow \mathcal{O}_C$ is then defined by $x^my^m \mapsto x^my^m$ and $z \mapsto f_mx^my^m$ for some function f_m. The kernel is $(z - f_mx^my^m)$, and so $\mathcal{K}_{m+1} = (z - f_mx^my^m) + \mathcal{IK}_m = (z - f_mx^my^m, x^{m+1}y^{m+1})$. The analytic change of coordinates $(x, y, z) \mapsto (x, y, z - f_mx^my^m)$ has an inverse which gives $\mathcal{K}_{m+1} = (x^{m+1}y^{m+1}, z)$ and does not change the expressions for $\mathcal{K}_k$ for any $k \leq m$.

□

From these coordinate calculations it can be seen that $\mathcal{K}_m/\mathcal{IK}_m$ is locally free of rank 2 on C. Also, from the description of the $\mathcal{K}_m$, we have the exact sequence

$$0 \longrightarrow \mathcal{IK}_{m-1}/\mathcal{IK}_m \longrightarrow \mathcal{K}_m/\mathcal{IK}_m \longrightarrow \mathcal{K}_m/\mathcal{IK}_{m-1} \longrightarrow 0$$

with $\mathcal{K}_m/\mathcal{IK}_{m-1} \cong \mathcal{O}_C(1, 0, \cdots, 0, 1)$. The sheaf $\mathcal{IK}_{m-1}/\mathcal{IK}_m$ is now determined in the following lemma.

Lemma 6.5 $\mathcal{IK}_{m-1}/\mathcal{IK}_m \cong \mathcal{O}_C$

Proof: Define a map $\mathcal{I} \otimes \mathcal{K}_{m-1} \to \mathcal{IK}_{m-1}/\mathcal{IK}_m$ by multiplication of functions. From the local calculation of $\mathcal{K}_j$ in the previous lemma, this map kills $\mathcal{I} \otimes \mathcal{K}_m$ and $\mathcal{K}_2 \otimes \mathcal{K}_{m-1}$, since $\mathcal{K}_2\mathcal{K}_{m-1} \subset \mathcal{IK}_m$. Therefore, this induces a well defined map $\mathcal{I}/\mathcal{K}_2 \otimes \mathcal{K}_{m-1}/\mathcal{K}_m \longrightarrow \mathcal{IK}_{m-1}/\mathcal{IK}_m$, which is an isomorphism from local coordinate calculations. But $\mathcal{I}/\mathcal{K}_2$ and $\mathcal{K}_{m-1}/\mathcal{K}_m$ are both isomorphic to $\mathcal{O}_C$, so $\mathcal{IK}_{m-1}/\mathcal{IK}_m \cong \mathcal{O}_C$.

□

Sequence 6.4 can now be written

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{K}_m/\mathcal{I}\mathcal{K}_m \longrightarrow \mathcal{O}_C(1,0,\cdots,0,1) \longrightarrow 0.$$

Lemma 6.6 *The sequence 6.4 splits if and only if there is an ideal sheaf $\mathcal{K}_{m+1} \subset \mathcal{K}_m$ satisfying $\mathcal{K}_m/\mathcal{K}_{m+1} \cong \mathcal{O}_C$ and $\mathcal{K}_{m+1}/\mathcal{I}\mathcal{K}_m \cong \mathcal{O}_C(1,0,\cdots,0,1)$.*

Proof: If this sequence splits then there is a surjection $\mathcal{K}_m/\mathcal{I}\mathcal{K}_m \longrightarrow \mathcal{O}_C$, so define

$$\mathcal{K}_{m+1} = Ker(\mathcal{K}_m \longrightarrow \mathcal{K}_m/\mathcal{I}\mathcal{K}_m \longrightarrow \mathcal{O}_C).$$

By definition, $\mathcal{K}_{m+1} \subset \mathcal{K}_m$ and $\mathcal{K}_m/\mathcal{K}_{m+1} \cong \mathcal{O}_C$ and there is an exact sequence

$$0 \longrightarrow \mathcal{K}_{m+1}/\mathcal{I}\mathcal{K}_m \longrightarrow \mathcal{K}_m/\mathcal{I}\mathcal{K}_m \longrightarrow \mathcal{K}_m/\mathcal{K}_{m+1} \longrightarrow 0. \tag{6.5}$$

Restricting to each component as in the proof of lemma 6.2, we see that $\mathcal{K}_{m+1}/\mathcal{I}\mathcal{K}_m \cong \mathcal{O}_C(1,0,\cdots,0,1)$.

Conversely, the existence of $\mathcal{K}_{m+1}$ gives a surjection $\mathcal{K}_m/\mathcal{I}\mathcal{K}_m \longrightarrow \mathcal{K}_m/\mathcal{K}_{m+1} \cong \mathcal{O}_C$, which proves sequence 6.4 splits.

□

We will now establish the relationship between the existence of the ideals $\mathcal{K}_m$ and the deformation of the curve C.

Definition 6.1 *A* **first order (or infinitesimal) deformation** *of $C \subset X$ is a closed subscheme $\mathcal{C} \subset X \times Spec(\mathbf{C}[t]/t^2)$ which is flat over $Spec(\mathbf{C}[t]/t^2)$ and whose closed fiber is C.*

From [Ha2], pg. 267, the $\mathcal{C}$ satisfying this definition are classified by the sheaf of global sections $H^0(C, \mathcal{N}_C)$. In particular, this means that the dimension of $H^0(C, \mathcal{N}_C)$ gives the dimension of the the family of deformations of C in X. The sheaf $\mathcal{N}_C$ is defined to be the dual of the conormal bundle, and is called the *normal sheaf.* Being the dual of the conormal sheaf, the normal sheaf is also locally free of rank 2 on C.

For the remainder of this chapter the invertible sheaf $\mathcal{O}_C(1,0,\cdots,0,1)$ will be denoted $\omega_C{}^*$ since this is the dual of the dualizing sheaf, $\omega_C = \mathcal{O}_C(-1,0,\cdots,0,-1)$, of C.

Proposition 6.1 *C deforms to first order if and only if there exists an ideal sheaf $\mathcal{K}_2$ satisfying $\mathcal{I}^2 \subset \mathcal{K}_2 \subset \mathcal{I}$, $\mathcal{I}/\mathcal{K}_2 \cong \mathcal{O}_C$ and $\mathcal{K}_2/\mathcal{I}^2 \cong \omega_C{}^*$.*

Proof: By lemma 6.1, this second condition is equivalent to the splitting of the exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow {\omega_C}^* \longrightarrow 0.$$

This is equivalent to the splitting of the dual exact sequence

$$0 \longrightarrow \omega_C \longrightarrow \mathcal{N}_C \longrightarrow \mathcal{O}_C \longrightarrow 0. \tag{6.6}$$

By [Ha2], proposition III 6.3, $Ext^1_{\mathcal{O}_C}(\mathcal{O}_C, \omega_C) \cong H^1(C, \omega_C)$, and $H^1(C, \omega_C) \cong \mathbf{C}$ from [Ha3], corollary III.11.2. Now, $H^0(C, \omega_C) = 0$, since this is the geometric genus of C, so the long exact cohomology sequence

$$0 \longrightarrow H^0(\omega_C) \longrightarrow H^0(\mathcal{N}_C) \longrightarrow H^0(\mathcal{O}_C) \xrightarrow{\delta} H^1(\omega_C) \longrightarrow H^1(\mathcal{N}_C) \longrightarrow 0$$

can be written

$$0 \longrightarrow H^0(\mathcal{N}_C) \longrightarrow \mathbf{C} \xrightarrow{\delta} \mathbf{C} \longrightarrow H^1(\mathcal{N}_C) \longrightarrow 0,$$

where δ is the coboundary map given by $1 \mapsto$ extension class of $\mathcal{O}_C$ by ω_C. δ is a map between one-dimensional vector spaces, so δ is either an isomorphism or the zero map. But the extension class of $\mathcal{O}_C$ by ω_C is trivial, in which case $H^0(\mathcal{N}_C) = \mathbf{C}$, if and only if sequence 6.6 splits, and the extension class is non-trivial, meaning $H^0(\mathcal{N}_C) = 0$, if and only if sequence 6.6 does not split.

□

To see whether the curve C deforms, meaning to all orders, we need to first discuss the versal deformations of A_n singularities. It was shown that a general section of the ideal sheaf $\mathcal{I}$ is a smooth surface containing C in which the components, C_i, of C have conormal sheaves isomorphic to $\mathcal{O}_i(2)$ for each i. Therefore, the curve C comes from the deformation of an A_n singularity, and the threefold X is the pullback of the versal deformation of an A_n singularity. So, by understanding the versal deformation of an A_n singularity, the threefold X can be recovered. This will now be shown explicitly.

Definition 6.2 *A* **family of analytic spaces** *is a triple $(\pi, \mathcal{X}, \Sigma)$ consisting of two analytic spaces, $\mathcal{X}$ and Σ, and a flat analytic map $\pi : \mathcal{X} \rightarrow \Sigma$ with fibers $\mathcal{X}_\sigma = \pi^{-1}(\sigma)$. The space Σ is called the* **base** *of the family.*

Definition 6.3 *A* **deformation** *of an analytic space H over the base Σ consists of a base point $\sigma_0 \in \Sigma$, a family $(\pi, \mathcal{X}, \Sigma)$ and an isomorphism from H onto the fiber $\mathcal{X}_{\sigma_0}$.*

Definition 6.4 *A deformation* $(\pi, \mathcal{X}, \Sigma)$ *of* H *with base point* σ_0 *is* **locally complete** *if locally every deformation* $(\pi', \mathcal{X}', \Sigma')$ *of* H *with base point* σ_0' *is obtained as the pull-back from* $(\pi, \mathcal{X}, \Sigma)$ *by a suitable analytic map* $f : \Sigma' \to \Sigma$ *with* $f(\sigma_0') = \sigma_0$.

That is, the family $(\pi', \mathcal{X}', \Sigma')$ is the family $(\pi', \mathcal{X} \times_\Sigma \Sigma', \Sigma')$. The map f, however, is not uniquely determined by $\mathcal{X}'$ in general, as is the case of a universal deformation. The deformations of DuVal surface singularities do not have this universal property, but rather the following semi-universal or versal property:

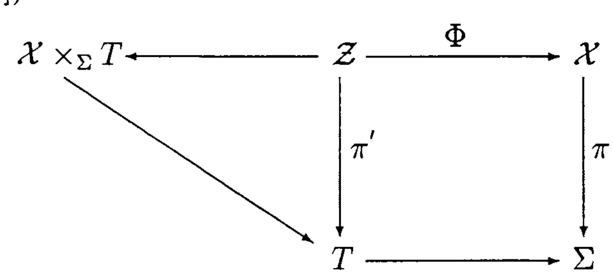
Definition 6.5 *A locally complete deformation* $(\pi, \mathcal{X}, \Sigma)$ *of* H *with base point* σ_0 *is called a* **versal deformation** *if for every deformation* $(\pi', \mathcal{X}', \Sigma')$ *of* H *with base point* σ_0' *there is an analytic map* $f : \Sigma' \to \Sigma$ *with* $f(\sigma_0' = \sigma_0$ *for which the derivative of* f *at* σ_0' *is uniquely determined.*

Recognizing that in our situation the family $\mathcal{X}$ contains singular fibers, in particular the central fiber $\mathcal{X}_{\sigma_0} \cong H$ has an A_n singularity. Therefore, we can talk about the resolution of the singularities of the fibers of π.

Definition 6.6 *A* **resolution** *of singularities of a family* $(\pi, \mathcal{X}, \Sigma)$ *is a a family* $(\pi', \mathcal{Z}, T)$ *with smooth fibers together with a morphism* $\Phi : (\pi', \mathcal{Z}, T) \to (\pi, \mathcal{X}, \Sigma)$ *having the property of* $\Phi|_{\mathcal{Z}_t} : \mathcal{Z}_t \to \mathcal{X}_{\Phi(t)}$ *being a resolution of singularities.*

From [Ty] it is known that if the fibers of π are surfaces which are smooth, or have finite sets of rational double points, then the family $(\pi, \mathcal{X}, \Sigma)$ has a local resolution of singularities.

It is also known from [Ty] that if X is a threefold and $C \subset X$ is a curve that would contract to a cDV singularity, then the space X can be viewed as the space of a one-parameter family of deformations of a general hyperplane section of C in X. It is from this viewpoint that Pinkham in [Pi] gives a construction for the singularities that come from analytic contractions of C. If $(\pi, \mathcal{X}, \Sigma)$ is a versal deformation of the general hyperplane section, H, of the cDV and $(\pi', \mathcal{Z}, T)$ is a versal deformation of the partial resolution $\tilde{H} \subset X$, then we have the following diagram (See [Ty], [Kas], [Pi] and [KM]):



Pinkham's construction shows that if $B \subset T$ is the germ of a smooth curve in T, then $\pi'^{-1}(B)$ is the threefold X and Y is the inverse image of B under the projection $\mathcal{X} \times_\Sigma T \to T$. All such cDV singularities arise as the contractions of such curves. Similarly, Pinkham's construction applies to the case where C is a curve in a formal threefold $\hat{X}$ which can be realized in the formal partial resolution, $\hat{\mathcal{Z}}$, of a formal cDV singularity. More precisely, the formal threefold $\hat{X}$ can be recovered by taking $\hat{B}$ to be the formal completion of the germ of a smooth curve in the completion, $\hat{T}$, of T at the origin. Also, by taking $\hat{\Sigma}$ to be the completion of the analytic space Σ, the versal deformation of the formal cDV singularity, $\hat{\mathcal{X}}$, can be constructed. So, $\hat{X} = \hat{\pi}'^{-1}(\hat{B})$, and $\hat{Y}$ is the inverse image of $\hat{B}$ under the projection $\hat{\mathcal{X}} \times_{\hat{\Sigma}} \hat{T} \to \hat{T}$. This will be made explicit in the case of the formal cA_m singularity in this section.

The versal deformation spaces, the parameter spaces, and the corresponding maps in this diagram have all been described explicitly in local coordinates in [Ty] and [KM]. These works are both in the analytic category, but, again, the formal spaces and maps can be obtained by taking appropriate completions of these analytic spaces. In general, for any of the formal DuVal surface singularities, the map between the base spaces $\hat{\phi} : \hat{T} \to \hat{\Sigma}$ is of the form $\hat{\phi} : \text{Spec}\mathbf{C}^{\mathbf{n}}[[t_1, \cdots, t_n]] \to \text{Spec}\mathbf{C}^{\mathbf{n}}[[s_1, \cdots, s_n]]$. It is from the discriminant locus of this map $\hat{\phi} : \hat{T} \to \hat{\Sigma}$ that the contractibility of C can be determined (See [KM]). More precisely, the curve C, the exceptional set in $\hat{\mathcal{Z}}$, lies over the discriminant in $\hat{T}$, so the components of C that deform can be determined from the local coordinates of the moduli space $\hat{T}$. From Pinkham's construction in [Pi], components of C that deform,then, can be found from the curve $\hat{B}$ in $\hat{T}$. The coordinates, t_i, of T are viewed as formal functions of the single local parameter t at $0 \in \hat{B}$. That is, there is a formal map $\hat{\psi} : \text{Spec}\mathbf{C}[[t]] \to \text{Spec}\mathbf{C}^{\mathbf{n}}[[t_1, \cdots, t_n]]$ which completely describes the deformation of the DuVal surface singularity and the partial resolution. Geometrically, this means that if $\hat{B}$ locally coincides with the discriminant locus, then the component over this part of the discriminant locus must deform. If no part of $\hat{B}$ coincides locally with the discriminant locus, then the curve C must contract. Again, all of this is made explicit in this section for the case of an A_n singularity. For these same results as well as those for the other RDP's, see [Ty] and [KM].

Recall, it is from these results of Pinkham and the versal deformations of RDP's that the definitions of *formal cDV modification* and *formal cDV contraction* are developed. These definitions are in section 1.2. The formal modification in this case is the formal map $\hat{f} : \hat{X} \to \hat{\mathbf{C}}^{\mathbf{4}}$ constructed from the four global sections of $\mathcal{I}$ as in chapters 2- 4. It is the ideals $\mathcal{K}_m$ that will determine if this formal modification is a formal cDV contraction or a formal deformation. It will be seen how the ideals $\mathcal{K}_m$ relate to the formal map $\text{Spec}\mathbf{C}[[t]] \to \text{Spec}\mathbf{C}^{\mathbf{n}}[[t_1, \cdots, t_n]]$ that results from the the formal modification. In particular, it will be seen when this map factors through the discriminant locus in the A_n case.

We are interested in the case where H is a singular surface having an A_n singularity. Near the singularity with coordinates $\{x, y, z\}$, H can be defined as a hypersurface in $\mathbf{C}^3(x, y, z)$ by the equation $-xy + z^{n+1}$. Notice this is adjusted slightly from the defining polynomial in Table 1.1. This is done to utilize the equations in [KM]. The analytic space $\mathcal{X}$ is defined as the hypersurface in $\mathbf{C}^3(x, y, z) \times \mathbf{C}^{\mathbf{n}}(\sigma_1, \cdots, \sigma_n)$ defined by

$$G = -xy + z^{n+1} + \sigma_1 z^{n-1} + \cdots \sigma_{n-1} z + \sigma_n = 0.$$

The base space is $\Sigma = \mathbf{C}^{\mathbf{n}}(\sigma_1, \cdots, \sigma_n)$ and the map $\pi : \mathcal{X} \to \Sigma$ is the map induced by projection.

The resolution corresponding to the versal family can also be explicitly described. Let T be the hyperplane in $\mathbf{C}^{\mathbf{n+1}}(t_1, \cdots, t_{n+1})$ defined by $\sum_{i=1}^{n+1} t_i = 0$. The map Φ on the base spaces, which we will denote $\phi : T \to \Sigma$, is defined by $\sigma_i =$ the $(i+1)$st symmetric polynomial in the t_i. Notice that by definition $\sigma_0 = \sum_{i=1}^{n+1} t_i = 0$. The smooth deformation $\pi' : V \to T$ induced by ϕ is defined in $\mathbf{C}^3(x, y, z) \times \mathbf{C}^{\mathbf{n+1}}(t_1, \cdots, t_{n+1})$ by the equations

$$\sum_{i=1}^{n+1} t_i = 0, \quad -xy + \prod_{i=1}^{n+1} (z + t_i) = 0.$$

Now, define a mapping

$$V \to (\mathbf{P}^1)^n$$

by

$$(x, y, z, t_1, \cdots, t_{n+1}) \to \{x, \prod_{j=1}^{i} (z + t_j)\}_i$$

for $i = 1, \cdots, n$. The analytic space $\mathcal{Z}$, then, is defined to be the closure of the graph of this map, and the mapping $\pi' : \mathcal{Z} \to T$ is defined by projection. If (u_k, v_k) are the homogeneous coordinates on the kth $\mathbf{P}^1$ from the resolution, then the equations defining $\mathcal{Z}$ are

$$-xy + \prod_{i=1}^{n+1} (z + t_i) = 0,$$

$$xv_j = u_j \prod_{i=1}^{j}(z+t_i) \quad (1 \le j \le n),$$

$$\prod_{i=k+1}^{j}(z+t_i)u_jv_k = u_kv_j \quad (1 \le k < j \le n).$$

From these equations it has been shown that C is the exceptional set of the fiber of $\mathcal{Z}$ over $(t_1, t_2, \cdots, t_{n+1}) = \vec{0}$ and the component C_i is defined by $x = y = z = 0$, $u_j = 0$ for $j < i$ and $v_k = 0$ for $k > i$. Furthermore, the curve $C_i + C_{i+1} + \cdots + C_j$ deforms when $t_i = t_{j+1}$.

Since X is being viewed as the space of a one parameter family of deformations of a resoluton of an A_n singularity, X is recovered, as described by Pinkham in [Pi], by introducing a smooth curve B in T with local parameter t near $0 \in B$. $\pi'^{-1}(B)$ is the smooth threefold X. The coordinates t_i of T, then, can be expressed as functions of t vanishing at $t = 0$. Let $f_i(t) = t_i$ under this parameterization, where the f_i are formal holomorphic functions on a neighborhood of $0 \in \mathbf{C}$. These functions have a power series expansion near $t = 0$. Let

$$f_i(t) = \sum_{j=1}^{\infty} a_{ij}t^j \quad (1 \le i \le n+1).$$

Pulling back B to $\mathcal{Z}$ by π', X can now be described from the equations defining the resolution with coordinates $\{x, y, z, (u_i, v_i), t\}$. In particular, we will be interested in defining X near an intersection point of two components, C_i and C_{i+1} with $1 \le i \le n-1$. $C_i \cong \mathbf{P}^1(u_i, v_i)$, $C_{i+1} \cong \mathbf{P}^1(u_{i+1}, v_{i+1})$ and X is defined near this point of intersection by the transition functions on the coordinate patches (u_{i-1}, v_i, t), (u_i, v_{i+1}, t) and (u_{i+1}, v_{i+2}, t), with the intersection point being in the coordinate patch (u_i, v_{i+1}, t). These transition functions are:

$$\begin{array}{lcl} u_{i-1} &=& {u_i}^2 v_{i+1} + u_i(f_i(t) - f_{i+1}(t)) \\ v_i &=& 1/u_i \\ t &=& t \end{array} \qquad \begin{array}{lcl} u_{i+1} &=& 1/v_{i+1} \\ v_{i+2} &=& {v_{i+1}}^2 u_i + v_{i+1}(f_{i+2}(t) - f_{i+1}(t)) \\ t &=& t \end{array}$$

with the convention that if $i = 1$ then $u_{i-1} = x$, and if $i = n-1$, $v_{i+2} = y$.

The functions $f_j(t)$ determine which components of C deform and which ones can be contracted. Since the deformation of $\cup_{i=j}^{k} C_i$ occurs when $t_j = t_{k+1}$, the deformation of this curve is determined by B coinciding with $f_j(t) = f_{k+1}(t)$. That is, B is contained in the discriminant locus of the curve. In particular, the curve C deforms when $f_1(t) = f_{n+1}(t)$. This explains the following two theorems.

Theorem 6.1 *$C = \bigcup_{i=1}^{n} C_i$ deforms formally in $\hat{X}$ if and only if $f_1(t) = f_{n+1}(t)$.*

Theorem 6.2 *$C = \bigcup_{i=1}^{n} C_i$ can be contracted via a formal cDV contraction if and only if $B \not\subset \{f_i(t) = f_j(t)\}$ for any $1 \leq i < j \leq n+1$.*

These known results mean that C deforms or contracts in this formal sense, and not necessarily in the analytic category.

With this explicit description of C in X and these results from the deformations and simultaneous resolutions of A_n singularities, the connection to the sequence of ideal sheaves

$$\cdots \subset \mathcal{K}_m \subset \cdots \subset \mathcal{K}_2 \subset \mathcal{I}$$

constructed earlier will now be established in the following theorem.

Theorem 6.3 *C deforms formally in $\hat{X}$ if and only if there exists an infinite chain of subsheaves $\cdots \subset \mathcal{K}_{m+1} \subset \mathcal{K}_m \subset \cdots \subset \mathcal{K}_2 \subset \mathcal{I}$ such that $\mathcal{K}_m/\mathcal{K}_{m+1} \cong \mathcal{O}_C$ and $\mathcal{K}_{m+1}/\mathcal{I}\mathcal{K}_m \cong {\omega_C}^*$.*

Proof: The method of proof will first be explained from the well known one component case. See [Re]. Then, the two component case will be worked out before induction is used to prove this result holds for any number of components.

From lemma 6.6, this is equivalent to showing C deforms in X if and only if there is a surjection $\mathcal{K}_m/\mathcal{I}\mathcal{K}_m \longrightarrow \mathcal{O}_C$ for all m. Theorem 6.1, then, says this is equivalent to proving $f_1(t) = f_{n+1}(t)$ if and only if there is a surjection $\mathcal{K}_m/\mathcal{I}\mathcal{K}_m \longrightarrow \mathcal{O}_C$. It is this last statement that will be proven in each case.

CASE 1: $\mathbf{C = C_1}$:

Notice that in this case $C = C_1$ is a smooth rational curve with conormal sheaf $\mathcal{I}/\mathcal{I}^2 = (1,1)$ or $(0,2)$.

The equations defining $\mathcal{Z}$ are

$$-xy + (z+t_1)(z+t_2) = 0$$

$$xv_1 = u_1(z+t_1).$$

On the affine set $u_1 \neq 0$, by combining these equations we have

$$-x(y - v_1(xv_1 - t_1 + t_2)) = 0.$$

The exceptional set is given by $x = 0$ and the strict transform is given by

$$y = xv_1{}^2 + v_1(t_2 - t_1).$$

In the coordinates $\{x, v_1, t_1, t_2\}$, C_1 is defined by $v_1 = 0$, and we have

$$\begin{aligned} x &= x \\ y &= xv_1{}^2 + v_1(t_2 - t_1) \\ z &= xv_1 - t_1 \end{aligned}$$

Similarly, on the affine piece $v_1 \neq 0$, in coordinates $\{y, u_1, t_1, t_2\}$, we have

$$\begin{aligned} x &= u_1(u_1 y + t_1 - t_2) \\ y &= y \\ z &= u_1 y - t_2 \end{aligned}$$

Expressing the t_i in terms of the single parameter, t, the transition functions defining the threefold X containing the curve $C = C_1$ is given by

$$\begin{aligned} x &= u_1{}^2 y + u_1(f_1(t) - f_2(t)) \\ t &= t \\ v_1 &= 1/u_1 \end{aligned}$$

The curve C is given by $y = t = 0$ in the $\{u_1, y, t\}$ coordinate patch, and by $x = t = 0$ in the $\{v_1, x, t\}$ patch. In other words, the ideal sheaf of C in X is $\mathcal{I} = (y, t) = (x, t)$. The t_i are analytic functions in t vanishing at $t = 0$, so they can be expressed in power series form.

$$t_1 = \sum_{j=1}^{\infty} a_j t^j$$

$$t_2 = \sum_{j=1}^{\infty} b_j t^j$$

Substituting these series for t_1 and t_2 in the transition functions we have

$$\begin{aligned} x &= {u_1}^2 y + u_1 \sum_{j=1}^{\infty} (a_j - b_j) t^j \\ t &= t \\ v_1 &= 1/u_1 \end{aligned} \tag{6.7}$$

It is now possible to explicitly determine the decomposition of the conormal sheaf and when there exist a surjection from the conormal sheaf to $\mathcal{O}_C$. Since $\mathcal{I}/\mathcal{I}^2$ decomposes as $(1,1)$ or $(0,2)$, there is a surjection $\mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_C$ if and only if $\mathcal{I}/\mathcal{I}^2 = (0,2)$.

Step 1: *There is a surjection $\mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_C$ if and only if $a_1 = b_1$.*

Proof: It suffices to show $\mathcal{I}/\mathcal{I}^2 = (0,2)$ if and only if $a_1 = b_1$. In calculating the decomposition of the conormal sheaf from the transition function, since $t^j \in \mathcal{I}^2$ for $j \geq 2$, it is only necessary to consider $x = {u_1}^2 y + u_1(a_1 - b_1)t$. $\mathcal{I}/\mathcal{I}^2$ is generated locally by $\{y, t\}$ and $\{x, t\}$ in these two coordinate patches and

$$(x,t) = \begin{pmatrix} {u_1}^2 & (a_1 - b_1)u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ t \end{pmatrix}$$

Therefore, $\mathcal{I}/\mathcal{I}^2 = (0,2)$ if and only if $a_1 = b_1$. See [Na], pgs. 519-520.

□

The map $\mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_C$, if it exists, can be calculated explicitly in coordinates to determine the ideal $\mathcal{K}_2$ of lemma 6.1. Note that in this case, though, $\mathcal{K}_2/\mathcal{I}^2 \cong \mathcal{O}_C(2)$, which is the dual of the dualizing sheaf of $C \cong \mathbf{P}^1$, and $\mathcal{K}_2 = (y, t^2) = (x, t^2)$ in local coordinates. So, from

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{K}_2/\mathcal{I}\mathcal{K}_2 \longrightarrow \mathcal{O}_C(2) \longrightarrow 0,$$

$\mathcal{K}_2/\mathcal{I}\mathcal{K}_2 = (1,1)$ or $(0,2)$.

Assume that we have $K_m \subset \mathcal{K}_{m-1} \subset \cdots \subset \mathcal{K}_2 \subset \mathcal{I}$ with $\mathcal{K}_{i-1}/\mathcal{K}_i \cong \mathcal{O}_C$, $\mathcal{K}_i/\mathcal{I}\mathcal{K}_{i-1} \cong \mathcal{O}_C(2)$ and $\mathcal{K}_i/\mathcal{I}\mathcal{K}_i = (0,2)$ if and only if $a_i = b_i$. Induction will now prove the following claim.

Step m+1: There is a surjection $\mathcal{K}_{m+1}/\mathcal{I}\mathcal{K}_{m+1} \longrightarrow \mathcal{O}_C$ if and only if $a_{m+1} = b_{m+1}$.

Proof: From [Re] $\mathcal{K}_j = (y, t^j) = (x, t^j)$ in the local coordinates and $\mathcal{K}_{m+1}$ exists satisfying the hypotheses $\mathcal{K}_m/\mathcal{K}_{m+1} \cong \mathcal{O}_C$, $\mathcal{K}_{m+1}/\mathcal{I}\mathcal{K}_m \cong \mathcal{O}_C(2)$ if and only if $\mathcal{K}_m/\mathcal{I}\mathcal{K}_m = (0,2)$. So, $\mathcal{K}_m/\mathcal{I}\mathcal{K}_m = (0,2)$ if and only if $a_m = b_m$. To establish that $\mathcal{K}_{m+1}/\mathcal{I}\mathcal{K}_{m+1} = (0,2)$ if and only if $a_{m+1} = b_{m+1}$, notice that $t^{m+2} \in \mathcal{I}\mathcal{K}_{m+1}$. In calculating the decomposition of this sheaf, then, it is only necessary to consider $f_i(t)$, $1 \leq i \leq 2$, up to the term t^{m+1}. Since we are assuming that $a_j = b_j$ for all $j \leq m$, $f_1(t) - f_2(t) = (a_{m+1} - b_{m+1})t^{m+1}$ is all that is needed. Substituting into the equations 6.7, we have

$$\begin{aligned} x &= {u_1}^2 y + u_1(a_{m+1} - b_{m+1})t^{m+1} \\ t &= t \\ v_1 &= 1/u_1 \end{aligned}$$

The generators of $\mathcal{K}_{m+1}/\mathcal{I}\mathcal{K}_{m+1}$ are known to be $(x, t^{m+1}) = (y, t^{m+1}$, and

$$(x, t^{m+1}) = \begin{pmatrix} {u_1}^2 & (a_{m+1} - b_{m+1})u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ t^{m+1} \end{pmatrix}$$

By [Na], pgs. 519-520, this shows that $\mathcal{K}_{m+1}/\mathcal{I}\mathcal{K}_{m+1} = (0,2)$ if and only if $a_{m+1} = b_{m+1}$.

□

From this inductive argument and the fact that C deforms in X if and only if $f_1(t) = f_2(t)$, the following results of Reid [Re] has been established.

Theorem: $C \cong \mathbf{P^1}$ *deforms in* X *if and only if there exists an infinite chain* $\cdots \subset \mathcal{K}_{m+1} \subset \mathcal{K}_m \subset \cdots \mathcal{K}_2 \subset \mathcal{I}$ *satisfying* $\mathcal{K}_m/\mathcal{K}_{m+1} \cong \mathcal{O}_C$ *and* $\mathcal{K}_{m+1}/\mathcal{I}\mathcal{K}_m \cong \mathcal{O}_C(2)$.

Furthermore, if for some m, $\mathcal{K}_m/\mathcal{I}\mathcal{K}_m = (1,1)$, then $f_1(t) \neq f_2(t)$. Therefore,

Theorem: $C \cong \mathbf{P^1}$ *contracts if and only if the chain* $\cdots \subset \mathcal{K}_m \subset \cdots \subset \mathcal{K}_2 \subset \mathcal{I}$ *terminates.*

Remark Reid, in [Re], showed not only that C contracts or deforms in this formal structure, but also that there is actually an analytic deformation or contraction of C.

This completes the discussion and proof of theorem 6.3 for the case of C being a smooth rational curve. It will now be established that similar results are true for C having several components.

CASE 2: $C = C_1 \bigcup C_2$.

From the earlier desription of X by transition functions, if $i = 1$ and $n = 2$, then X is defined by the transition functions

$$\begin{aligned} x &= {u_1}^2 v_2 + u_1(f_1(t) - f_2(t)) & u_2 &= 1/v_2 \\ v_1 &= 1/u_1 & y &= {v_2}^2 u_1 + v_2(f_3(t) - f_2(t)) \\ t &= t & t &= t \end{aligned}$$

where $\mathcal{I} = (u_1 v_2, t) = (x, t) = (y, t)$, $\mathcal{I}_1 = (v_2, t) = (x, t)$ and $\mathcal{I}_2 = (u_1, t) = (y, t)$ in the coordinate patches (u_1, v_2, t), (x, v_1, t) and (u_2, y, t).

Step 1: *There is a surjection $\mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{O}_C$ if and only if $a_{11} = a_{31}$.*

Proof: Assume that there is a surjection $\mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_C$. Defining this in local coordinates on the patch (u_1, v_2, t) containing the point of intersection, let

$$\begin{aligned} u_1 v_2 &\mapsto h_1(u_1, v_2) \\ t &\mapsto h_2(u_1, v_2) \end{aligned}$$

where the h_i are holomorphic functions in u_1 and v_2. In determining this map in coordinates, $t^2 \in \mathcal{I}^2$ in each patch, so it suffices to assume $f_i(t) = a_{i1}t$ for $1 \leq i \leq 3$. Then, in the remaining coordinate patches, $\mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_C$ is given by

$$\begin{aligned} x &\mapsto u_1 h_1 + u_1(a_{11} - a_{21})h_2 & y &\mapsto v_2 h_1 + v_2(a_{31} - a_{21})h_2 \\ t &\mapsto h_2 & t &\mapsto h_2 \end{aligned}$$

The images of the generators $\{x, t\}$ and $\{y, t\}$ of $\mathcal{I}/\mathcal{I}^2$ must be holomorphic in the coordinate patches (x, v_1, t) and (u_2, y, t) respectively. In particular, h_2 must be holomorphic in the coordinate $v_1 = 1/u_1$ and in $u_2 = 1/v_2$. This can only be possible if h_2 is a constant function. Let $h_2 = c$ where $c \in \mathbf{C}$ and $c \neq 0$ for a nontrivial map.

Expanding h_1 as a power series in u_1 and v_2, $h_1(u_1, v_2) = \sum_{i+j=0}^{\infty} \sum_{i,j=0}^{\infty} b_{ij} {u_1}^i {v_2}^j$. Since the image of x, $u_1 h_1 + u_1(a_{11} - a_{21})c$, must be holomorphic in $v_1 = 1/u_1$, replacing u_1 with $1/v_1$ in this function,

$$\begin{aligned} &1/v_1(h_1(1/v_1, v_2) + 1/v_1(a_{11} - a_{21})c = \\ &\quad 1/v_1(b_{00} + (a_{11} - a_{21})c) + 1/v_1 \left(\sum_{\substack{i,j=0 \\ i+j=1}}^{\infty} b_{ij}(1/v_1)^i (v_2)^j \right), \end{aligned}$$

we see that this can only be holomorphic in v_1 if it is the zero function. Therefore, $b_{00} = (a_{21} - a_{11})c$ and $b_{ij} = 0$ for all $(i, j) \neq (0, 0)$, so $h_1 = (a_{21} - a_{11})c$.

The surjection can exist, then, only if in the coordinates $\{y, t\}$,

$$\begin{aligned} y &\mapsto v_2(a_{21} - a_{11})c + v_2(a_{31} - a_{21})c \\ t &\mapsto c \end{aligned}$$

with the image of y holomorphic in $u_2 = 1/v_2$. That is, $1/u_2(a_{21} - a_{11})c + 1/u_2(a_{31} - a_{21})c$ must be holomorphic in u_2. Again, this is only possible if it is the zero function, which is equivalent to

$$(a_{21} - a_{11}) = (a_{21} - a_{31})$$

or

$$a_{11} = a_{31}.$$

It has been shown, then, that a surjection $\mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_C$ can occur only if $a_{11} = a_{31}$. Furthermore, assuming $c = 1$(since $c \neq 0$) and letting $g_1 = a_{11} - a_{21} = a_{31} - a_{21}$, the surjection is defined on the generators by the equations

$$\begin{array}{lll} u_1v_2 \mapsto -g_1 & x \mapsto 0 & y \mapsto 0 \\ t \mapsto 1 & t \mapsto 1 & t \mapsto 1. \end{array}$$

Conversely, if $a_{11} = a_{31}$ define $\mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{O}_C$ by the above equations.

□

The subsheaf $\mathcal{K}_2$ of $\mathcal{I}$ satisfying the conditions of lemma 5.1 can also be calculated explicitly. By definition, $\mathcal{K}_2 = Ker(\mathcal{I} \to \mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_C)$, so $t^2 \in \mathcal{I}^2$ and $u_1v_2 + g_1t$ generate $\mathcal{K}_2$ in the (u_1, v_2, t) patch. Similarly, from the equations of the map $\mathcal{I}/\mathcal{I}^2$ above $\{x, t^2\}$ and $\{y, t^2\}$ generate $\mathcal{K}_2$ in the (x, v_1, t) and (u_2, y, t) patches, respectively. The locally free rank 2 sheaf $\mathcal{K}_2/\mathcal{I}\mathcal{K}_2$ is generated by $\{u_1v_2 + g_1t, t^2\}$, $\{x, t^2\}$ and $\{y, t^2\}$ in the respective coordinate patches. Notice that this is the first form of lemma 6.4, so by the induction step shown in the proof of this lemma, successive surjections to $\mathcal{O}_C$ will result in subsheaves $\mathcal{K}_m$ of this form. This can also be calculated directly from the equations from the surjections as $\mathcal{K}_2$ was. Inductively, it will now be shown:

Step m+1 *There is a surjection $\mathcal{K}_{m+1}/\mathcal{I}\mathcal{K}_{m+1} \to \mathcal{O}_C$ if and only if $a_{1(m+1)} = a_{3(m+1)}$.*

Proof: Assume there is a surjection $\mathcal{K}_k/\mathcal{I}\mathcal{K}_k \to \mathcal{O}_C$ if and only if $a_{1k} = a_{3k}$ for all $k \leq m$.

To extend this to $m+1$, notice that, as mentioned, from lemma 6.4, $\mathcal{K}_{m+1}/\mathcal{I}\mathcal{K}_{m+1}$ is generated by $\{u_1v_2+g_1t+\cdots+g_mt^m, t^{m+1}\}$ at the point of intersection, where $g_i = a_{1i} - a_{2i} = a_{3i} - a_{2i}$ from the induction hypotheses. $\mathcal{K}_{m+1}/\mathcal{I}\mathcal{K}_{m+1}$ is generated, then, by $\{x, t^{m+1}\}$ and $\{y, t^{m+1}\}$ on the other coordinate patches. Since $t^{m+2} \in \mathcal{I}\mathcal{K}_{m+1}$ in each patch, to calculate the surjection $\mathcal{K}_{m+1}/\mathcal{I}\mathcal{K}_{m+1} \to \mathcal{O}_C$, it suffices to consider $f_i(t) = \sum_{j=1}^{m+1} a_{ij}t^j$ for $i = 1, 2, 3$. Defining this map on generators in the coordinates (u_1, v_2, t), let

$$\begin{aligned} u_1v_2 + g_1t\cdots + g_mt^m &\mapsto h_1(u_1, v_2) \\ t^{m+1} &\mapsto h_2(u_1, v_2) \end{aligned}$$

where h_1 and h_2 are holomorphic in u_1 and v_2. In the other patches we also have the generator t^{m+1} mapping to the function h_2. The exact reasoning from Step 1 shows that h_2 must be the constant function, and it can be assumed to be the constant 1. Notice that from the transition functions defining X,

$$x = u_1(u_1v_2 + \sum_{i=1}^{m} g_it^m + (a_{1(m+1)} - a_{2(m+1)})t^{m+1})$$

and

$$y = v_2(u_1v_2 + \sum_{i=1}^{m} g_it^m + (a_{3(m+1)} - a_{2(m+1)})t^{m+1}).$$

So, the surjection on the remaining generators is given by

$$\begin{aligned} x &\mapsto u_1(h_1 + (a_{1(m+1)} - a_{2(m+1)})) \qquad & y &\mapsto v_1(h_1 + (a_{3(m+1)} - a_{2(m+1)})) \\ t^{m+1} &\mapsto 1 & t^{m+1} &\mapsto 1 \end{aligned}$$

This map has the exact same form as that in Step 1, so we can conclude that $h_1 = a_{2(m+1)} - a_{1(m+1)} = a_{2(m+1)} - a_{3(m+1)}$, and this surjection can occur if and only if $a_{1(m+1)} = a_{3(m+1)}$.

□

Therefore, for the case where $C = C_1 \bigcup C_2$, C deforms if and only if there is an infinite chain $\cdots \subset \mathcal{K}_m \subset \cdots \mathcal{K}_2 \subset \mathcal{I}$ with $\mathcal{K}_m/\mathcal{K}_{m+1} \cong \mathcal{O}_C$ and $\mathcal{K}_{m+1}/\mathcal{I}\mathcal{K}_m \cong {\omega_C}^*$.

□

CASE n: $C = C_1 \bigcup C_2 \bigcup \cdots \bigcup C_n$

Proof: From the two component case we see that the surjection $\mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{O}_C$ was constructed from the patching data that describes X. This will be done in steps, as in case 2.

Step 1: *There is a surjection* $\mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{O}_C$ *if and only if* $a_{11} = a_{(n+1)1}$.

Proof: Assume there is a surjection $\mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{O}_C$. On the intersection of C_i and C_{i+1}, $\mathcal{I} = (u_i v_{i+1}, t)$ and in the patches (u_{i-1}, v_i, t) and (u_{i+1}, v_{i+2}, t), $\mathcal{I} = (u_{i-1}, t)$ and $\mathcal{I} = (v_{i+2}, t)$, respectively, away from the intersection points. This map, then, is defined by From the earlier desription of X by transition functions, X is defined by the transition functions

$$\begin{aligned} u_i v_{i+1} &\mapsto h_{i_1}(u_i, v_{i+1}) \\ t &\mapsto h_{i_2}(u_i, v_{i+1}) \end{aligned}$$

where the h_j are holomorphic functions in u_i and v_{i+1}. In determining this map in coordinates, $t^2 \in \mathcal{I}^2$ in each patch, so it suffices to assume $f_i(t) = a_{i1}t$ for $1 \leq i \leq n{+}1$. Then, in the remaining coordinate patches, $\mathcal{I}/\mathcal{I}^2 \rightarrow \mathcal{O}_C$ is given by

$$\begin{aligned} u_{i-1} &\mapsto u_i h_{i_1} + u_i(a_{i1} - a_{(i+1)1})t & \qquad v_{i+2} &\mapsto v_{i+1} h_{i_1} + v_{i+1}(a_{(i+2)1} - a_{(i+1)1})t \\ t &\mapsto h_{i_2} & t &\mapsto h_{i_2}. \end{aligned}$$

The image of t must be holomorphic in both of the coordinate patches (u_{i-1}, v_i, t) and (u_{i+1}, v_{i+2}, t) as well. In particular, h_{i_2} must be holomorphic in the coordinate $v_i = 1/u_i$ and in $u_{i+1} = 1/v_{i+1}$. This can only be possible if h_{i_2} is a constant function. Let $h_{i_2} = c$ where $c \in \mathbf{C}$ and $c \neq 0$ for a nontrivial map. For $c \neq 0$, though, we can assume the surjection has $c = 1$. Therefore,

$$\begin{aligned} u_{i-1} &\mapsto u_i(h_{i_1} + a_{i1} - a_{(i+1)1}) & \qquad v_{i+2} &\mapsto v_{i+1}(h_{i_1} + a_{(i+2)1} - a_{(i+1)1}) \\ t &\mapsto 1 & t &\mapsto 1. \end{aligned}$$

Arguing as in case 2, since the image of u_{i-1} must be holomorphic in $v_i = 1/u_i$, u_i and v_{i+2} must map to the zero function. This means that $h_{i_1} = a_{(i+1)1} - a_{i1} = a_{(i+1)1} - a_{(i+2)1}$. So, the surjection can only be defined on these patches if $a_{i1} = a_{(i+2)1}$. This has proven case 2 for the two components C_i and C_{i+1}.

Extending this argument to include the component C_{i+1}, we now work on the coordinate patch (u_{i+1}, v_{i+2}, t) where $\mathcal{I} = (u_{i+1} v_{i+2}, t)$. The surjection $\mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{O}_C$ is given by

$$\begin{aligned} u_{i+1}v_{i+2} &\mapsto h_{(i+1)_1}(u_{i+1}, v_{i+2}) \\ t &\mapsto h_{(i+1)_2}(u_{i+1}, v_{i+2}). \end{aligned}$$

As in the above argument, it must be that $h_{(i+1)_2} = c' \neq 0$ and $h_{(i+1)_1} = a_{(i+2)1} - a_{(i+1)1} = a_{(i+2)1} - a_{(i+3)1}$, which implies that $a_{(i+1)1} = a_{(i+3)1}$. Again, this is equivalent to $C_{i+1} \bigcup C_{i+2}$ deforming to first order.

Now to make sure that these maps patch together to give a well defined surjection. $t \mapsto 1$ and $t \mapsto c'$, so $c' = 1$. To see the relationship between h_{i_1} and $h_{(i+1)_1}$, the image of $u_i v_{i+1}$ in the coordinate patch (u_{i+1}, v_{i+2}, t) will be determined from the transition functions

$$\begin{aligned} u_i &\mapsto u_{i+1}(u_{i+1}v_{i+2} + a_{(i+1)1} - a_{(i+2)1}) \\ v_{i+1} &\mapsto 1/u_{i+1} \\ t &\mapsto t, \end{aligned}$$

where the constants a_j replace the holomorphic functions t_j. These show that

$$\begin{aligned} u_i v_{i+1} &\mapsto u_{i+1}v_{i+2} + a_{(i+1)1} - a_{(i+2)1} \\ t &\mapsto t. \end{aligned}$$

But, this means that

$$\begin{aligned} u_i v_{i+1} &\mapsto h_{(i+1)_1} + a_{(i+1)1} - a_{(i+2)1} \\ t &\mapsto t. \end{aligned}$$

So, $h_{(i+1)_1} + a_{(i+1)1} - a_{(i+2)1} = h_{i_1} = a_{(i+1)1} - a_{i1}$, as this generator maps to the same function. Therefore, $h_{(i+1)_1} = a_{(i+2)1} - a_{i1}$ and $h_{(i+1)_1} = a_{(i+2)1} - a_{(i+3)1}$, which shows $a_{i1} = a_{(i+3)1}$ if such a surjection exists. This shows that $C_i \bigcup C_{i+1} \bigcup C_{i+2}$ deforms to first order if and only if $a_{i1} = a_{(i+3)1}$.

Continuing to extend this argument component by component by increasing i by 1 each time and finding the image of $u_i v_{i+1}$ each time, we see that $C_i \bigcup \cdots \bigcup C_n$ deforms to first order if and only if $a_{i1} = a_{n+1}1$. Then decreasing i by 1 and arguing as above, this can be extended to $i = 1$, and, therefore, all of C.

$\square$

Step m+1: *There is a surjection* $\mathcal{K}_{m+1}/\mathcal{I}\mathcal{K}_{m+1} \longrightarrow \mathcal{O}_C$ *if and only if* $a_{1(m+1)} = a_{(n+1)(m+1)}$.

Proof: This is the same as the process for two components, case 2. Assume there is a surjection $\mathcal{K}_k/\mathcal{I}\mathcal{K}_k \to \mathcal{O}_C$ if and only if $a_{1k} = a_{(n+1)k}$ for all $k \leq m$. To extend this to $k = m+1$, notice that $\mathcal{K}_{m+1}/\mathcal{I}\mathcal{K}_{m+1}$ is generated by $\{u_i v_{i+1} + g_1 t + \cdots + g_m t^m, t^{m+1}\}$ with $g_j = a_{(i+1)j} - a_{ij} = a_{(i+1)j} - a_{(i+2)j}$, for $j = 1, \cdots, m$, at the point of intersection of C_i and C_{i+1}. Since $t^{m+2} \in \mathcal{I}\mathcal{K}_{m+1}$ in each patch, it suffices to consider formal functions $f_i(t) = \sum_{j=1}^{m+1} a_{ij} t^j$ for $i = 1, 2, \cdots, n+1$ in determining the surjection $\mathcal{K}_{m+1}/\mathcal{I}\mathcal{K}_{m+1} \to \mathcal{O}_C$ in the coordinates (u_i, v_{i+1}, t). This map is defined by

$$\begin{aligned} u_i v_{i+1} + g_1 t + \cdots + g_m t^m &\mapsto h_{(i)_1}(u_i, v_{i+1}) \\ t^{m+1} &\mapsto h_{i_2}(u_i, v_{i+1}) \end{aligned}$$

with h_{i_1} and h_{i_2} holomorphic in u_i and v_{i+1}. As the image of t^{m+1} must be holomorphic in $u_{i+1} = 1/v_{i+1}$ and $v_i = 1/u_i$ as well, h_{i_2} is constant, and it can be assumed to be 1. From the transition functions on the patches (u_{i-1}, v_i, t) and (u_{i+1}, v_{i+2}, t),

$$u_{i-1} = u_i(u_i v_{i+1} + \sum_{i=1}^{m} g_i t^m + (a_{i(m+1)} - a_{(i+1)(m+1)})t^{m+1})$$

and

$$v_{i+2} = v_{i+1}(u_i v_{i+1} + \sum_{i=1}^{m} g_i t^m + (a_{(i+2)(m+1)} - a_{(i+1)(m+1)})t^{m+1}),$$

the surjection $\mathcal{K}_{m+1}/\mathcal{I}\mathcal{K}_{m+1} \to \mathcal{O}_C$ is given by

$$\begin{aligned} u_{i-1} &\mapsto u_i(h_{i_1} + a_{i(m+1)} - a_{(i+1)(m+1)}) \\ t^{m+1} &\mapsto 1. \end{aligned}$$

and

$$\begin{aligned} v_{i+2} &\mapsto v_{i+1}(h_{i_1} + a_{(i+2)(m+1)} - a_{(i+1)(m+1)}) \\ t^{m+1} &\mapsto 1. \end{aligned}$$

The map, then, is determined exactly as in Step 1 except $m+1$ replaces 1 in the second subscript of a_{pq}. Therefore, it can now be concluded that on the patch containing $C_i \bigcap C_{i+1}$, there is a surjection if and only if $a_{i(m+1)} = a_{(i+2)(m+1)}$.

Continuing this process on the patches containing the remaining points of intersection, as in Step 1, we have $\mathcal{K}_{m+1}/\mathcal{I}\mathcal{K}_{m+1} \longrightarrow \mathcal{O}_C$ is surjective if and only if

$a_{1(m+1)} = a_{(n+1)(m+1)}$. This completes the proof of Step $m+1$.

□

We have now completed the proof of theorem 6.2.

□

The analogue to theorem 6.2, then, is the following theorem.

Theorem 6.4 *A formal cDV contraction of C exists if and only if there is no infinite chain of subsheaves $\cdots \subset \mathcal{K}_{m+1} \subset \mathcal{K}_m \subset \cdots \mathcal{K}_2 \subset \mathcal{I}_D$ satisfying $\mathcal{K}_m/\mathcal{K}_{m+1} \cong \mathcal{O}_D$ and $\mathcal{K}_{m+1}/\mathcal{I}\mathcal{K}_m \cong \omega_D{}^*$ for any $D = \bigcup_{j=i}^{k} C_j$ $(1 \leq i \leq k \leq n)$, where $\mathcal{I}_D$ is the ideal sheaf of D in X.*

Proof: For every i and k we can conclude from the proof of the previous theorem that $f_i \neq f_{k+1}$. Therefore, the curve B is not contained in the disriminant locus, which is equivalent to the induced formal map Spec$\mathbf{C}[[t]] \to$ Spec$\mathbf{C^n}[[t_1, \cdots, t_n]]$ does not factor through the discriminant locus.

□

Notice from this theorem, it can be concluded that even if every component of C can be contracted, this is not enough to ensure that C contracts.

Example: Using the description of X by transition functions, with $C = C_1 \bigcup C_2$, let $f_1(t) = 2t$, $f_2(t) = t$ and $f_3(t) = 2t$. Since $f_1(t) = f_3(t)$, the curve C deforms in X and so is not contractible. However, since $f_1(t) \neq f_2(t)$, C_1 can be contracted, and since $f_2(t) \neq f_3(t)$, C_2 can also be contracted. In fact, $\mathcal{I}_1/\mathcal{I}_1{}^2 = (1,1)$ and $\mathcal{I}_2/\mathcal{I}_2{}^2 = (1,1)$ (see [La]). The conormal sheaves of each component being ample implies that C_1 and C_2 can each be contracted separately.

BIBLIOGRAPHY

[Ar1] Artin, M. *Algebraization of formal moduli I.* Global Analysis. Papers in Honor of K. Kodaira, University of Tokyo Press. Princeton University Press (1969), 21 - 71.

[Ar2] Artin, M. *Algebraization of formal moduli: II. Existence of modifications.* Ann. of Math. **91**(1970), 88 - 135.

[BPV] Barth, W., Peters, C., Van de Ven, A. *Compact Complex Surfaces* Springer-Verlag, Berlin Heidelberg (1984).

[CKM] Clemens, H. , Kollár, J. and Mori, S. *Higher dimensional complex geometry,* Astérisque **166**(1988).

[GH] Griffiths, P. and Harris, J. *Principles of algebraic geometry,* John Wiley & Sons, New York (1978).

[Gr] Grauert, H. *Über Modifikationen und exzeptionnelle analytische Mengen.* Math.Ann. **146** (1962), 331-368.

[Ha1] Hartshorne, H. *Ample vector bundles* Publ. Math. IHES **29** (1966), 63 - 94.

[Ha2] Hartshorne, H. *Algebraic geometry.* Springer-Verlag, NY (1977).

[Ha3] Hartshorne, H. *Residues and duality* Springer, LNM **20**

[Ji] Jiménez, J. *Contraction of nonsingular curves*, Preprint, UC Riverside, (1991).

[Kas] Kas, A. *On the resolution of certain holomorphic mappings,* Global Analysis, Papers in honor of K. Kodaira (D.C. Spencer and S. iyanaga, eds.), University of Tokyo Press, Tokyo, and Princeton University Press, Princeton, 1969, 289-294.

[KM] Katz, S. and Morrison, D. *Gorenstein threefold singularities with small resolutions via invariant theory for Weyl groups,* J. of Alg. Geom. **1** (1992), 449 - 530.

[Ka] Kawamata, Y. *General hyperplane sections of nonsingular flops in dimension 3* , Preprint, University of Tokyo, (1993)

[La] Laufer, H.B. *On* $\mathbf{CP}^1$ *as exceptional set,* Ann. of Math. Stud., vol. 100, Princeton University Press, Princeton, 1981, 261-275.

[Ma] Matsumura, H. *Commutative algebra,* Benjamin, 1970.

[Mo] Mori, S. *Flip theorem and the existence of minimal models for 3-folds,* Journal AMS **1** (1988), 117-253.

[Na] Nakayama, N. *On smooth exceptional curves in threefolds,* J. Fac. Sci. Univ. Tokyo, Tokyo, **37** (1990), 511-525.

[Pi] Pinkham, H. *Factorization of birational maps in dimension 3*, Proceedings Symp. Pure Math. **40**, part 2, Ame. Math. Soc., Providence (1983), 343 - 371.

[Re] Reid, M. *Minimal models of canonical threefolds.* Algebraic varieties and analytic varieties. Adv. Studies in Pure Math. **1** (1983), 131 - 180.

[Ty] Tyurina, G.N. *Resolution of singularities of flat deformations of rational double points,* Functional Anal. Appl. **4** (1970), 324-356.

[Wa] van der Waerden, B. *Zur algebraischen geometrie VI: Algebraische korrespondenzen und rationale abbildungen,* Math. Ann. **110** (1934), 134-160.

APPENDIX A

$D_4(2,1)$ CALCULATIONS

This appendix provides the calculations used to prove lemmas 3.3, 3.4 and 3.5. In this case $\mathcal{I}_1 = (x,z)$, $\mathcal{I}_2 = (y,z)$, $\mathcal{I} = (xy,z)$ and $\mathcal{J} = (x^2y,z)$ in coordinates at $p = C_1 \bigcap C_2$. On $C_1 - \{p\}$, $\mathcal{I} = \mathcal{I}_1$, $\mathcal{I}_2 = \mathcal{O}_X$, and coordinates can be chosen so that $\mathcal{I} = (x,z)$, which implies $\mathcal{J} = (x^2,z)$. On $C_2 - \{p\}$, $\mathcal{I} = \mathcal{I}_2$, $\mathcal{I}_1 = \mathcal{O}_X$, and coordinates can be chosen so that $\mathcal{I} = (y,z)$, which means $\mathcal{J} = \mathcal{I} = (y,z)$.

Lemma A.1 *$S^m(\mathcal{J}/\mathcal{I}_1\mathcal{J})$ and $\mathcal{I}/\mathcal{J} \otimes S^m(\mathcal{J}/\mathcal{I}_1\mathcal{J})$ are locally free sheaves of rank $m+1$ on C_1. $\mathcal{I}_1/\mathcal{I} \otimes S^m(\mathcal{J}/\mathcal{I}_2\mathcal{J})$ is a locally free sheaf of rank $m+1$ on C_2.*

Proof: $\mathcal{J}/\mathcal{I}_1\mathcal{J}$ and $\mathcal{J}/\mathcal{I}_2\mathcal{J}$ are locally free sheaves of rank 2 on C_1 and C_2, respectively. Therefore, by [Ha2], pg. 127, $S^m(\mathcal{J}/\mathcal{I}_1\mathcal{J})$ is locally free of rank $m+1$ on C_1 and $S^m(\mathcal{J}/\mathcal{I}_2\mathcal{J})$ is locally free of rank $m+1$ on C_2. $\mathcal{I}/\mathcal{J}$ is an invertible sheaf on C_1, so $\mathcal{I}/\mathcal{J} \otimes S^m(\mathcal{J}/\mathcal{I}_1\mathcal{J})$ is locally free of rank $m+1$ on C_1. $\mathcal{I}_1/\mathcal{I}$ is an invertible sheaf on C_2, lemma 2.7, so $\mathcal{I}_1/\mathcal{I} \otimes S^m(\mathcal{J}/\mathcal{I}_2\mathcal{J})$ is locally free of rank $m+1$ on C_2.

□

Lemma A.2 *$\mathcal{J}^m/\mathcal{I}_1\mathcal{J}^m$ and $S^m(\mathcal{J}/\mathcal{I}_1\mathcal{J})$ are locally free sheaves of rank $m+1$ on C_1 and are generated by the same elements locally on all of C_1.* ∘

Proof: In coordinates,

$$\mathcal{J}^m = ((x^2y)^m, (x^2y)^{m-1}z, \cdots, (x^2y)^{m-j}z^j, \cdots, (x^2y)z^{m-1}, z^m) \qquad \text{(A.1)}$$

and

$$\begin{aligned}
\mathcal{I}_1\mathcal{J}^m &= x \cdot \mathcal{J}^m + z \cdot \mathcal{J}^m \\
&= (x(x^2y)^m, x(x^2y)^{m-1}z, \cdots, x(x^2y)^{m-j}z^j, \cdots, x(x^2y)z^{m-1}, xz^m) + \\
&\quad ((x^2y)^m z, (x^2y)^{m-1}z^2, \cdots, (x^2y)^{m-j}z^{j+1}, \cdots, (x^2y)z^m, z^{m+1}) \\
&= (x^{2m+1}y^m, \cdots, x^{2(m-j)+1}y^{m-j}z^j, \cdots, xz^m, z^{m+1}) \qquad \text{(A.2)}
\end{aligned}$$

since all of the elements in $z \cdot \mathcal{J}^m$ are in $x \cdot \mathcal{J}^m$ except for z^{m+1}.

Define a map $g : \mathcal{O}_1^{\oplus(m+1)} \to \mathcal{J}^m/\mathcal{I}_1\mathcal{J}^m$ by $(f_0, \cdots, f_m) \mapsto f_0(x^2y)^m + \cdots + f_j(x^2y)^{m-j}z^j + f_m z^m$. This map is surjective as it sends the generators of $\mathcal{O}_1^{\oplus(m+1)}$ to the generators of $\mathcal{J}^m/\mathcal{I}_1\mathcal{J}^m$. This map is also injective since an image element, $\sum_{j=0}^{m} f_j(x^2y)^{m-j}z^j$, is in $\mathcal{I}_1\mathcal{J}^m$ only if each f_i is divisible by x or z (compare equations 1 and 2). That is, $(f_0, \cdots, f_m) \mapsto 0$ implies $f_j \in \mathcal{I}_1$ for all $0 \le j \le m$. Therefore, g is an isomorphism and $\mathcal{J}^m/\mathcal{I}_1\mathcal{J}^m$ is locally free at p.

On $C_1 - \{p\}$, $\mathcal{I}_1 = (x, z)$ and $\mathcal{J} = (x^2, z)$ in local coordinates, so $\mathcal{J}^m$ in these coordinates is the same as in equation 1 with the variable y eliminated. Similarly, the expression for $\mathcal{I}_1\mathcal{J}^m$ is the same as equation 2 with y eliminated. The proof that $\mathcal{J}^m/\mathcal{I}_1\mathcal{J}^m$ is locally free at any point on $C_1 - \{p\}$ is the same as above except y is eliminated from all calculations. At p, $\mathcal{J}^m/\mathcal{I}_1\mathcal{J}^m$ is generated by

$$\{(x^2y)^m, (x^2y)^{m-1}z, \cdots, (x^2y)^{m-j}z^j, \cdots, (x^2y)z^{m-1}, z^m\} \tag{A.3}$$

and on $C_1 - \{p\}$ by

$$\{(x^2)^m, (x^2)^{m-1}z, \cdots, (x^2)^{m-j}z^j, \cdots, (x^2)z^{m-1}, z^m\} \tag{A.4}$$

The sheaf $\mathcal{J}/\mathcal{I}_1\mathcal{J}$ is generated locally by $\{x^2y, z\}$ at p, so $S^m(\mathcal{J}/\mathcal{I}_1\mathcal{J})$ is generated by

$$\{(x^2y)^m, \cdots, (x^2y)^{m-j}z^j, \cdots, (x^2y)z^{m-1}, z^m\}. \tag{A.5}$$

$\mathcal{J}/\mathcal{I}_1\mathcal{J}$ is generated locally by $\{x^2, z\}$ on $C_1 - \{p\}$, so $S^m(\mathcal{J}/\mathcal{I}_1\mathcal{J})$ is generated by

$$\{(x^2)^m, \cdots, (x^2)^{m-j}z^j, \cdots, (x^2)z^{m-1}, z^m\}. \tag{A.6}$$

Comparing the generators in (3) with those in (5), we see that these two sheaves are generated by the same elements at p. To see the same is true on $C_1 - \{p\}$ notice the generators in (4) and (6) are also the same.

□

Remark: Notice in the proof of the preceding lemma that the calculations on $C_1 - \{p\}$ are the same as those at p, but the local coordinate y on $C_1 - \{p\}$ is eliminated. This will happen for the remaining locally free sheaves on C_1 as well, so the calculations on $C_1 - \{p\}$ will be eliminated from the proof of lemma A.3, with the understanding that y can be eliminated from the calculations at p.

Also, when doing calculations on locally free sheaves on C_2, x is a local coordinate on $C_2 - \{p\}$, so the calculations on $C_2 - \{p\}$ are the same as those at p with x eliminated from each expression. Therefore, the calculations on $C_2 - \{p\}$ will be eliminated from the proof of lemma A.4.

Lemma A.3 *$\mathcal{I}\mathcal{J}^m/\mathcal{J}^{m+1}$ and $\mathcal{I}/\mathcal{J} \otimes S^m(\mathcal{J}/\mathcal{I}_1\mathcal{J})$ are locally free sheaves of rank $m+1$ on C_1 and are generated by the same elements locally on all of C_1.*

Proof: From (1)

$$\mathcal{J}^{m+1} = ((x^2y)^{m+1}, (x^2y)^m z, \cdots, (x^2y)^{m+1-j}z^j, \cdots, (x^2y)z^m, z^{m+1}) \tag{A.7}$$

and from the local coordinates at p,

$$\begin{aligned} \mathcal{I}\mathcal{J}^m &= xy \cdot \mathcal{J}^m + z \cdot \mathcal{J}^m \\ &= (xy(x^2y)^m, xy(x^2y)^{m-1}z, \cdots, xy(x^2y)^{m-j}z^j, \cdots, xy(x^2y)z^{m-1}, xz^m) + \\ &\quad ((x^2y)^m z, (x^2y)^{m-1}z^2, \cdots, (x^2y)^{m-j}z^{j+1}, \cdots, (x^2y)z^m, z^{m+1}) \\ &= (x^{2m+1}y^{m+1}, \cdots, x^{2(m-j)+1}y^{m-j+1}z^j, \cdots, xyz^m, z^{m+1}). \end{aligned} \tag{A.8}$$

Define a map $g : \mathcal{O}_1^{\oplus(m+1)} \to \mathcal{I}\mathcal{J}^m/\mathcal{J}^{m+1}$ by $(f_0, \cdots, f_m) \mapsto \sum_{j=0}^{m} f_j x^{2(m-j)+1}y^{m-j+1}z^j$. This map is surjective as it sends the generators of $\mathcal{O}_1^{\oplus(m+1)}$ to the generators of $\mathcal{I}\mathcal{J}^m/J^{m+1}$. This map is also injective since an image element $\sum_{j=0}^{m} f_j x^{2(m-j)+1}y^{m-j+1}z^j$ is in $\mathcal{J}^{m+1}$ only if each f_i is divisible by x or z (compare equations 7 and 8). That is, $(f_0, \cdots, f_m) \mapsto 0$ implies $f_j \in \mathcal{I}_1$ for all $0 \le j \le m$. Therefore, g is an isomorphism and $\mathcal{I}\mathcal{J}^m/\mathcal{J}^{m+1}$ is locally free at p. The remark preceding this lemma explains how this sheaf is also locally free on $C_1 - \{p\}$ as well. So $\mathcal{I}\mathcal{J}^m/\mathcal{J}^{m+1}$ is locally free of rank $m+1$ on C_1 generated by

$$\{(x^{2m+1}y^{m+1}, x^{2m-1}y^m z, \cdots, x^{2(m-j)+1}y^{m-j+1}z^j, \cdots, x^3y^2z^{m-1}, xyz^m\} \tag{A.9}$$

Now, $\mathcal{I}/\mathcal{J}$ is generated by $\{xy\}$, so $\mathcal{I}/\mathcal{J} \otimes S^m(\mathcal{J}/\mathcal{I}_1\mathcal{J})$ is generated by

$$\begin{aligned} &\{xy\} \otimes \{(x^2y)^m, \cdots, (x^2y)^{m-j}z^j, \cdots, x^2yz^{m-1}, z^m\} \\ &= \{x^{2m+1}y^{m+1}, \cdots, x^{2(m-j)+1}y^{m-j+1}z^j, \cdots, xyz^m\}. \end{aligned} \tag{A.10}$$

(9) and (10) show that these two sheaves are generated by the same elements at p. The same result holds on $C_1 - \{p\}$ by the remark above.

□

Lemma A.4 *$\mathcal{I}_1\mathcal{J}^m/\mathcal{I}\mathcal{J}^m$ and $\mathcal{I}_1/\mathcal{I} \otimes S^m(\mathcal{J}/\mathcal{I}_2\mathcal{J})$ are locally free sheaves of rank $m+1$ on C_2 and are generated by the same elements locally on all of C_2.*

Proof: To show $\mathcal{I}_1\mathcal{J}^m/\mathcal{I}\mathcal{J}^m$ is locally free, define a map $g : \mathcal{O}_2^{\oplus(m+1)} \to \mathcal{I}_1\mathcal{J}^m/\mathcal{I}\mathcal{J}^m$ by $(f_0, \cdots, f_m) \mapsto \sum_{j=0}^{m} f_j x^{2(m-j)+1}y^{m-j}z^j$. This map is surjective as it sends the generators of $\mathcal{O}_2^{\oplus(m+1)}$ to the generators of $\mathcal{I}_1\mathcal{J}^m/\mathcal{I}_j{}^m$. This map is also injective since an image element $\sum_{j=0}^{m} f_j x^{2(m-j)+1} y^{m-j} z^j$ is in $\mathcal{I}\mathcal{J}^m$ only if each f_i is divisible by y or z (compare equations 2 and 8). That is, $(f_0, \cdots, f_m) \mapsto 0$ implies $f_j \in \mathcal{I}_2$ for all $0 \leq j \leq m$. Therefore, g is an isomorphism and $\mathcal{I}_1\mathcal{J}^m/\mathcal{I}\mathcal{J}^m$ is locally free at p. The remark preceding this lemma explains how this sheaf is also locally free on $C_2 - \{p\}$ as well. So $\mathcal{I}_1\mathcal{J}^m/\mathcal{I}\mathcal{J}^m$ is locally free of rank $m+1$ on C_2 generated by

$$\{x^{2m+1}y^m, x^{2m-1}y^{m-1}z, \cdots, x^{2(m-j)+1}y^{m-j}z^j, \cdots, x^3yz^{m-1}, xz^m\}. \tag{A.11}$$

The invertible sheaf $\mathcal{I}_1/\mathcal{I}$ is generated by $\{x\}$ and $\mathcal{J}/\mathcal{I}_2\mathcal{J}$ is generated by $\{x^2y, z\}$ so $\mathcal{I}_1/\mathcal{I} \otimes S^m(\mathcal{J}/\mathcal{I}_2\mathcal{J})$ is generated by

$$\begin{aligned} &\{x\} \otimes \{(x^2y)^m, \cdots, (x^2y)^{m-j}z^j, \cdots, z^m\} \\ &\quad = \{x^{2m+1}y^m, x^{2m-1}y^{m-1}z, \cdots, x^{2(m-j)+1}y^{m-j}z^j, \cdots, x^3yz^{m-1}, xz^m\}. \end{aligned} \tag{A.12}$$

Comparing (12) and (13), this completes the proof.

□

APPENDIX B

$D_5(2,1)$ CALCULATIONS

This appendix provides the calculations required to prove lemmas 3.13, 3.14 and 3.15. All of the following calculations are done locally at the point, p, of intersection of C_1 and C_2. By the remark in appendix A, it can be seen that the following calculations can be extended to $C_1 - \{p\}$ and $C_2 - \{p\}$ as well.

Lemma B.1 *We have* $Sat(\mathcal{J}^m) = (xz^2)^i Sat(\mathcal{J}^{m-2i} + (xz^2)^{i-1}\mathcal{J}^{m-2(i-1)} + \cdots + xz^2\mathcal{J}^{m-2} + \mathcal{J}^m$ *for all integers* $m \geq 1$ *and* $i = \lfloor m/2 \rfloor$.

Proof: The proof is by induction on m.

If $m = 1$ then $Sat(\mathcal{J}^m) = \mathcal{J}$ since the sheaf $\mathcal{J}$ has no torsion.

Assume that the lemma holds for all $m \leq k$. The torsion element of $\mathcal{J}/\mathcal{J}^2$ is xz^2 at p since it annihilates the maximal ideal at p. Therefore, the torsion elements of $\mathcal{J}^k/\mathcal{J}^{k+1}$ are the elements of $xz^2 \cdot Sat(\mathcal{J}^{k-1})$ for all $k \geq 1$, where $\mathcal{J}^0 = (1)$. $Sat(\mathcal{J}^{k+1})$, then, is $xz^2 \cdot Sat(\mathcal{J}^{k-1}) + \mathcal{J}^{k+1}$ for $k \geq 0$. By the induction hypothesis, $Sat(\mathcal{J}^{k-1}) = (xz^2)^j Sat(\mathcal{J}^{k-1-2j}) + \cdots xz^2\mathcal{J}^{k-3} + \mathcal{J}^{k-1}$ for all $k \geq 2$ and $0 \leq j \leq \lfloor (k-1)/2 \rfloor$. So, $Sat(\mathcal{J}^{k+1}) = (xz^2)^{j+1} Sat(\mathcal{J}^{k-1-2j}) + \cdots (xz^2)^2\mathcal{J}^{k-3} + xz^2\mathcal{J}^{k-1} + \mathcal{J}^{k+1}$ for $0 \leq j \leq \lfloor (k+1)/2 \rfloor$.

□

Corollary B.1 *For* $i = \lfloor m/2 \rfloor$, $Sat(\mathcal{J}^m) = \sum_{k=0}^{i} (xz^2)^k \mathcal{J}^{m-2k}$

Proof: Apply the above lemma to $Sat(\mathcal{J}^{m-2i})$.

Lemma B.2 *For* $f \in (xz^2)^k\mathcal{J}^{m-2k}$, $f \notin (xz^2)^{k+1}\mathcal{J}^{m-2(k+1)}$ *if and only if*
$f \in \{(xz^2)^k((xy)^{m-2k}, (xy)^{m-2k-1}(xz), (z^2)^{m-2k})\}$.

Proof: Let $f \in (xz^2)^k\mathcal{J}^{m-2k}$. $f = (xz^2)^k g$ for some $g \in \mathcal{J}^{m-2k}$. But $f = (xz^2)^k g$ is in $(xz^2)^{k+1}\mathcal{J}^{m-2(k+1)}$ if and only if $g \in (xz^2)\mathcal{J}^{m-2(k+1)}$, so $f \in (xz^2)^{k+1}\mathcal{J}^{m-2(k+1)}$ if and only if g is divisible by xz^2. In other words, $f \notin (xz^2)^{k+1}\mathcal{J}^{m-2(k+1)}$ if and only if $g \in \mathcal{J}^{m-2k}$ is not divisible by xz^2. It suffices to show, then, that the the only elements of $\mathcal{J}^{m-2k}$ that are not divisible by xz^2 are elements generated by $\{(xy)^{m-2k}, (xy)^{m-2k-1}(xz), (z^2)^{m-2k}\}$. Since $\mathcal{J}^{m-2k}$ is generated by monomials, it suffices to show that this is true on the generators of $\mathcal{J}^{m-2k}$.

A general generator of $\mathcal{J}^{m-2k}$ is of the form $(xy)^a(xz)^b(z^2)^c$ with $a+b+c = m-2k$. Such terms are divisible by xz^2 if $b \geq 1$ and $c \geq 1$, so it can be assumed that $b = 0$ or $c = 0$.

If $b = 0$, the element $(xy)^a(z^2)^c$ is divisible by xz^2 if $a \geq 1$ and $c \geq 1$. So, it can be assumed that $a = 0$ or $c = 0$. If $a = 0$, then $(z^2)^c = (z^2)^{m-2k}$, which is not divisible by xz^2. If $c = 0$, then $(xy)^a = (xy)^{m-2k}$, which is not divisible by xz^2.

If $c = 0$, the element $(xy)^a(xz)^b$ is divisible by xz^2 if $b \geq 2$, so we must have $b = 0$ or $b = 1$. If $b = 0$, then $(xy)^a = (xy)^{m-2k}$, which is not divisible by xz^2. If $b = 1$, then $(xy)^a(xz) = (xy)^{m-2k-1}(xz)$, which is not divisible by xz^2.

So, the only possible generators of $\mathcal{J}^{m-2k}$ that are not divisible by xz^2 are $\{(xy)^{m-2k}, (xy)^{m-2k-1}(xz), (z^2)^{m-2k}\}$.

□

Corollary B.2 *For $i = \lfloor m/2 \rfloor$,*

$$Sat(\mathcal{J}^m) = \sum_{k=0}^{i}(xz^2)^k\left[(xy)^{m-2k}, (xy)^{m-2k-1}(xz), (z^2)^{m-2k}\right]$$

Proof: Applying lemma B2 to the expression of $Sat(\mathcal{J}^m)$ in corollary B1, we see that all elements can be eliminated from each term in the sum except for those in the set $\{(xy)^{m-2k}, (xy)^{m-2k-1}(xz), (z^2)^{m-2k}\}$.

□

Also, the expression for $Sat(\mathcal{J}^m)$ in corollary B2 gives a minimal set of generators for this sheaf at the point p. Local generators at p for the sheaf $Sat(\mathcal{I}_1\mathcal{J}^m)$ will now be calculated.

$\mathcal{I}_1\mathcal{J} = (x, z)(xy, xz, z^2) = (x^2y, x^2z, xz^2, xyz, z^2)$ and xz is the torsion element of $\mathcal{J}/\mathcal{I}_1\mathcal{J}$, so $Sat(\mathcal{I}_1\mathcal{J}) = (x^2y, xz, z^3) = xz + \mathcal{I}_1\mathcal{J}$. In general, then, $Sat(\mathcal{I}_1\mathcal{J}^m) = xzSat(\mathcal{J}^{m-1}) + \mathcal{I}_1\mathcal{J}^m$. But, from corollary B1, $Sat(\mathcal{J}^{m-1}) = \sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}(xz^2)^k\mathcal{J}^{m-2k-1}$, so

$$\begin{aligned} Sat(\mathcal{I}_1\mathcal{J}^m) &= \sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}(xz^2)^k xz\mathcal{J}^{m-2k-1} + \mathcal{I}_1\mathcal{J}^m \\ &= \sum_{k=1}^{\lfloor\frac{m-1}{2}\rfloor}(xz^2)^k xz\mathcal{J}^{m-2k-1} + xz\mathcal{J}^{m-1} + \mathcal{I}_1\mathcal{J}^m. \end{aligned}$$

Lemma B.3 *For $f \in \mathcal{I}_1\mathcal{J}^m$. $f \notin xz\mathcal{J}^{m-1}$ if and only if $f \in \{x(xy)^m, z(z^2)^m\}$.*

Proof: $f \in \mathcal{I}_1\mathcal{J}^m$ implies $f = xg$ for some $g \in \mathcal{J}^m$ or $f = zh$ for some $h \in \mathcal{J}^m$.

Now, $f = xg \in xz\mathcal{J}^{m-1}$ if and only if $g \in z\mathcal{J}^{m-1}$. That is, $f \in xz\mathcal{J}^{m-1}$ if and only if $g \in \mathcal{J}^m$ is divisible by z or equivalently, $f \notin xz\mathcal{J}^{m-1}$ if and only if $g \in \mathcal{J}^m$ is not divisible by z. But the only generator of $\mathcal{J}^m$ that is not divisible by z is $(xy)^m$. Therefore, $f = xg \notin xz\mathcal{J}^{m-1}$ if and only if $f = x(xy)^m$.

$f = zh \in xz\mathcal{J}^{m-1}$ with $h \in \mathcal{J}^m$ if and only if $h \in x\mathcal{J}^{m-1}$. So, $h \in \mathcal{J}^m$ is in $x\mathcal{J}^{m-1}$ if and only if h is divisible by x, which means $f \notin xz\mathcal{J}^{m-1}$ if and only if h is not divisible by x. All elements of $\mathcal{J}^m$ are divisible by x except for $(z^2)^m$, so $f = zh \notin xz\mathcal{J}^{m-1}$ if and only if $f = z(z^2)^m$.

□

From the local descriptions of the minimal set of generators for $Sat(\mathcal{J}^m)$ and $Sat(\mathcal{I}_1\mathcal{J}^m)$, the generators of the locally free sheaf $Sat(\mathcal{J}^m)/Sat(\mathcal{I}_1\mathcal{J}^m)$ can now be found. Since both sheaves in the quotient are generated by monomials, and the minimal set of generators of each has been determined, the generators of the quotient are the generators of $Sat(\mathcal{J}^m)$ that are not in $Sat(\mathcal{I}_1\mathcal{J}^m)$.

Corollary B.3 *$Sat(\mathcal{J}^m)/Sat(\mathcal{I}_1\mathcal{J}^m)$ is a locally free sheaf of rank $m+1$ on C_1 generated locally at p by $\sum_{k=0}^{i}(xz^2)^k[(xy)^{m-2k},(z^2)^{m-2k}]$, where $i = \lfloor m/2 \rfloor$.*

Proof: From corollary B2,

$$Sat(\mathcal{J}^{m-1}) = \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} (xz^2)^k[(xy)^{m-2k}, (xy)^{m-2k-1}xz, (z^2)^{m-2k}],$$

so

$$Sat(\mathcal{I}_1\mathcal{J}^m) = \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} (xz^2)^k[(xy)^{m-2k}(xz), (xy)^{m-2k-1}(xz)^2, (z^2)^{m-2k}(xz)] + \mathcal{I}_1\mathcal{J}^m.$$

From lemma B.3 only the elements of $\{x(xy)^m, z(z^2)^m\}$ are necessary from $\mathcal{I}_1\mathcal{J}^m$. Therefore,

$$\begin{aligned} Sat(\mathcal{I}_1\mathcal{J}^m) &= \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} (xz^2)^k[(xy)^{m-2k}(xz), (xy)^{m-2k-1}(xz)^2, (z^2)^{m-2k}(xz)] \\ &+ [x(xy)^m, z(z^2)^m]. \end{aligned}$$

Expanding this sum and regrouping terms gives

$$Sat(\mathcal{I}_1\mathcal{J}^m) = \sum_{k=0}^{i}(xz^2)^k[(xy)^{m-2k-1}(xz), x(xy)^{m-2k}, z(z^2)^{m-2k}] \qquad \text{(B.1)}$$

where $i = \lfloor m/2 \rfloor$. Now, comparing this with the expression for $Sat(\mathcal{J}^m)$ in corollary B2 we see that $Sat(\mathcal{J}^m)/Sat(\mathcal{I}_1\mathcal{J}^m)$ is generated by $\sum_{k=0}^{i}(xz^2)[(xy)^{m-2k},(z^2)^{m-2k}]$. This sheaf is locally free of rank $m+1$ on C_1 since each generator of $Sat(\mathcal{I}_1\mathcal{J}^m)$ that is not in $Sat(\mathcal{J}^m)$ is x or z times a generator of $Sat(\mathcal{J}^m)$ and $\mathcal{I}_1 = (x, z)$.

□

Lemma B.4 *$Sat(\mathcal{J}^m)/Sat(\mathcal{I}_1\mathcal{J}^m)$ and $S^m(\mathcal{J}/Sat(\mathcal{I}_1\mathcal{J}))$ are locally free sheaves of rank $m+1$ on C_1 and there is an injective map $S^m(\mathcal{J}/Sat(\mathcal{I}_1\mathcal{J})) \hookrightarrow Sat(\mathcal{J}^m)/Sat(\mathcal{I}_1\mathcal{J}^m)$.*

Proof: $\mathcal{J}/Sat(\mathcal{I}_1\mathcal{J})$ is a locally free sheaf of rank 2 on C_1 generated by $\{xy, z^2\}$ at p, so $S^m(\mathcal{J}/Sat(\mathcal{I}_1\mathcal{J}))$ is locally free of rank $m+1$ generated by the elements $\sum_{j=0}^{m}(xy)^{m-j}(z^2)^j$. These can also be expressed as $\sum_{k=0}^{i}[(xy)^{m-k}(z^2)^k,(xy)^k(z^2)^{m-k}]$ or $\sum_{k=0}^{i}(xz^2)^k[(xy)^{m-2k}y^k,(z^2)^{m-2k}y^k]$.

Define a map $\mathcal{J}^{\otimes m} \to Sat(\mathcal{J}^m)/Sat(\mathcal{I}_1\mathcal{J}^m)$ by multiplication of functions. This map kills $Sat(\mathcal{I}_1\mathcal{J}) \otimes \mathcal{J}^{\otimes(m-1)}$ since $Sat(\mathcal{I}_1\mathcal{J})\mathcal{J}^{m-1} \subset Sat(\mathcal{I}_1\mathcal{J}^m)$. So, there is a well defined map $S^m(\mathcal{J}/Sat(\mathcal{I}_1\mathcal{J})) \to Sat(\mathcal{J}^m)/Sat(\mathcal{I}_1\mathcal{J}^m)$, which is injective because $\mathcal{J}^m \cap Sat(\mathcal{I}_1\mathcal{J}^m) \subset Sat(\mathcal{I}_1\mathcal{J})\mathcal{J}^{m-1}$. From the expressions for the generators for each sheaf, the map on generators is given by

$$\begin{aligned}(xz^2)^k(xy)^{m-2k}y^k &\mapsto y^k \cdot (xz^2)^k(xy)^{m-2k}\\ (xz^2)^k(z^2)^{m-2k}y^k &\mapsto y^k \cdot (xz^2)^k(z^2)^{m-2k}\end{aligned} \tag{B.2}$$

for $0 \le k \le i$.

□

A minimal generating set for the sheaf $Sat(\mathcal{I}\mathcal{J}^m)$ will be found. The sheaf $\mathcal{I}\mathcal{J} = ((xy)^2, xyz, xz^2, z^3)$ has no torsion, so $Sat(\mathcal{I}\mathcal{J}) = \mathcal{I}\mathcal{J}$. However, $\mathcal{I}\mathcal{J}^2 = ((xy)^3, (xy)^2z, (xy)(xz)z, (xy)z^3, (xz)^2z, (xz)z^2z, z^5)$ has torsion element xz^3 since the elements xz^4, x^2z^3 and xyz^3 are all in $\mathcal{I}\mathcal{J}^2$. Therefore, $Sat(\mathcal{I}\mathcal{J}^2) = xz^3 + \mathcal{I}\mathcal{J}^2$ and, in general, $Sat(\mathcal{I}\mathcal{J}^m) = xz^3 Sat(\mathcal{J}^{m-2}) + \mathcal{I}\mathcal{J}^m$ for $m \ge 2$. Now, from corollary B1, $Sat(\mathcal{J}^{m-2}) = \sum_{k=0}^{i-1}(xz^2)^k\mathcal{J}^{m-2k-2} = \sum_{k=1}^{i-1}(xz^2)^k\mathcal{J}^{m-2k-2} + \mathcal{J}^{m-2}$, so

$$Sat(\mathcal{I}\mathcal{J}^m) = xz^3\sum_{k=1}^{i-1}(xz^2)^k\mathcal{J}^{m-2k-2} + xz^3\mathcal{J}^{m-2} + \mathcal{I}\mathcal{J}^m.$$

Lemma B.5 *For $f \in \mathcal{I}\mathcal{J}^m$. $f \notin xz^3\mathcal{J}^{m-2}$ if and only if $f \in \{(xy)^{m+1}, (xy)^m z, (xy)^{m-1}(xz)z, (z^2)^m z\}$.*

Proof: $f \in \mathcal{I}\mathcal{J}^m$ implies $f = xyg$ for some $g \in \mathcal{J}^m$ or $f = zh$ for some $h \in \mathcal{J}^m$.

Let $f = xyg$ with $g \in \mathcal{J}^m$. $f \in xz^3\mathcal{J}^{m-2}$ if and only if $yg \in z^3\mathcal{J}^{m-2}$. But this means that g must be divisible by z^3. The elements of $\mathcal{J}^m$ not divisible by z^3 are the elements of $\{(xy)^m, (xy)^{m-1}z^2, (xy)^{m-1}(xz)\}$. So, $f \notin xz^3\mathcal{J}^{m-2}$ if and only if $g \in \{(xy)^m, (xy)^{m-1}z^2, (xy)^{m-1}(xz)\}$ or, equivalently, $f \notin xz^3\mathcal{J}^{m-2}$ if and only if $f \in \{(xy)^{m+1}, (xy)^m z^2, (xy)^m(xz)\}$.

Let $f = zh$ with $h \in \mathcal{J}^m$. $f \in xz^3\mathcal{J}^{m-2}$ if and only if $h \in xz^2\mathcal{J}^{m-2}$, which can happen if and only if h is divisible by xz^2. The only elements of $\mathcal{J}^m$ that are not divisible by xz^2 are elements of $\{(xy)^m, (xy)^{m-1}(xz), (z^2)^m\}$. So, $f \notin xz^3\mathcal{J}^{m-2}$ if and only if $f \in \{z(xy)^m, z(xy)^{m-1}(xz), z(z^2)^m\}$.

Combining these two possible outcomes, we have $f \notin xz^3\mathcal{J}^{m-2}$ if and only if $f \in \{(xy)^{m+1}, (xy)^m z, (xy)^{m-1}(xz)z, (z^2)^m z\}$.

□

Corollary B.4 *$Sat(\mathcal{I}\mathcal{J}^m)/Sat(\mathcal{J}^{m+1})$ is a locally free of rank $m+1$ on C_1 generated by the elements $\sum_{k=0}^{i}(xz^2)^k[(xy)^{m-2k}z, (z^2)^{m-2k}z]$ where $i = \lfloor m/2 \rfloor$.*

Proof: From corollary B2,

$$Sat(\mathcal{J}^{m-2}) = \sum_{k=0}^{i-1}(xz^2)^k[(xy)^{m-2k-2}, (xy)^{m-2k-3}(xz), (z^2)^{m-2k-2}],$$

so, from the discussion preceding lemma B.5,

$$Sat(\mathcal{I}\mathcal{J}^m) = xz^3\sum_{k=0}^{i-1}(xz^2)^k[(xy)^{m-2k-2}, (xy)^{m-2k-3}(xz), (z^2)^{m-2k-2}] + \mathcal{I}\mathcal{J}^m.$$

From lemma B.5, though, the only generators of $\mathcal{I}\mathcal{J}^m$ that are necessary are those in the set $\{(xy)^{m+1}, (xy)^m z, (xy)^{m-1}(xz)z, (z^2)^m z\}$. This gives

$$\begin{aligned} Sat(\mathcal{I}\mathcal{J}^m) &= xz^3\sum_{k=0}^{i-1}(xz^2)^k[(xy)^{m-2k-2}, (xy)^{m-2k-3}(xz), (z^2)^{m-2k-2}] \\ &+ \{(xy)^{m+1}, (xy)^m z, (xy)^{m-1}(xz)z, (z^2)^m z\}. \end{aligned}$$

Expanding this sum and regrouping terms gives the expression

$$Sat(\mathcal{I}\mathcal{J}^m) = \sum_{k=0}^{j} (xz^2)^k [(xy)^{m-2k+1}, (xy)^{m-2k}z, (z^2)^{m-2k}z] \tag{B.3}$$

with $j = \lfloor (m+1)/2 \rfloor$.

Now, from corollary B.2,

$$Sat(\mathcal{J}^{m+1}) = \sum_{k=0}^{j} (xz^2)^k [(xy)^{m-2k+1}, (xy)^{m-2k}(xz), (z^2)^{m-2k+1}]$$

with $j = \lfloor (m+1)/2 \rfloor$. Comparing the expressions for $Sat(\mathcal{I}\mathcal{J}^m)$ and $Sat(\mathcal{J}^{m+1})$ we see that $Sat(\mathcal{I}\mathcal{J}^m)/Sat(\mathcal{J}^{m+1})$ is locally free of rank $m+1$, generated by the elements $\sum_{k=0}^{j} (xz^2)^k [(xy)^{m-2k}z, (z^2)^{m-2k}z]$. $m - 2k \geq 0$, and for $j = \lfloor (m+1)/2 \rfloor$, $m - 2k < 0$, so it suffices to sum to $i = \lfloor m/2 \rfloor$. This gives the generating set in the statement of this corollary. It is also seen that this sheaf is locally free on C_1 since $(xy)^{m-2k}(xz) = x \cdot (xy)^{m-2k}z$ and $(z^2)^{m-2k+1} = z \cdot (z^2)^{m-2k}z$ with $\mathcal{I}_1 = (x, z)$ in local coordinates.

□

Lemma B.6 *$Sat(\mathcal{I}\mathcal{J}^m)/Sat(\mathcal{J}^{m+1})$ and $\mathcal{I}/\mathcal{J} \otimes S^m(\mathcal{J}/Sat(\mathcal{I}_1\mathcal{J}))$ are locally free sheaves of rank $m+1$ on C_1 and there is an injective map $\mathcal{I}/\mathcal{J} \otimes S^m(\mathcal{J}/Sat(\mathcal{I}_1\mathcal{J})) \hookrightarrow Sat(\mathcal{I}\mathcal{J}^m)/Sat(\mathcal{J}^{m+1})$.*

Proof: $\mathcal{I}/\mathcal{J}$ is an invertible sheaf on C_1 generated by $\{z\}$ and $\mathcal{J}/Sat(\mathcal{I}_1\mathcal{J})$ is locally free of rank 2 on C_1 generated by $\{xy, z^2\}$. So, $\mathcal{I}/\mathcal{J} \otimes S^m(\mathcal{J}/Sat(\mathcal{I}_1\mathcal{J}))$ is locally free of rank $m+1$ generated by the elements $\sum_{j=0}^{m} (xy)^{m-j}(z^2)^j z$. This generating set can be expressed as $\sum_{k=0}^{i} [(xy)^{m-k}(z^2)^k z, (xy)^k (z^2)^{m-k} z]$ or, after factoring, as $\sum_{k=0}^{i} (xz^2)^k [(xy)^{m-2k} zy^k, (z^2)^{m-2k} zy^k]$.

Define a map $\mathcal{I} \otimes \mathcal{J}^{\otimes m} \to Sat(\mathcal{I}\mathcal{J}^m)/Sat(\mathcal{J}^{m+1})$ by multiplication of functions. This map kills $\mathcal{I} \otimes Sat(\mathcal{I}_1\mathcal{J}) \otimes \mathcal{J}^{\otimes(m-1)}$ since $\mathcal{I}\mathcal{J}^{m-1}Sat(\mathcal{I}_1\mathcal{J})$ is contained in $Sat(\mathcal{J}^{m+1})$. Therefore, there is a well defined map $\mathcal{I}/\mathcal{J} \otimes S^m(\mathcal{J}/Sat(\mathcal{I}_1\mathcal{J})) \to Sat(\mathcal{I}\mathcal{J}^m)/Sat(\mathcal{J}^{m+1})$. The inclusion $\mathcal{I}\mathcal{J}^m \cap Sat(\mathcal{J}^{m+1} \subset Sat(\mathcal{I}_1\mathcal{J})\mathcal{J}^m$, which follows from the local generators of each of these sheaves, proves that this map is injective.

From the local generator expressions for both sheaves, $\mathcal{I}/\mathcal{J} \otimes S^m(\mathcal{J}/Sat(\mathcal{I}_1\mathcal{J}))$ and $Sat(\mathcal{I}\mathcal{J}^m)/Sat(\mathcal{J}^{m+1})$, this injection is defined by

$$\begin{aligned} (xz^2)^k(xy)^{m-2k}zy^k &\mapsto y^k \cdot (xz^2)^k(xy)^{m-2k}z \\ (xz^2)^k(z^2)^{m-2k}zy^k &\mapsto y^k \cdot (xz^2)^k(z^2)^{m-2k}z \end{aligned} \tag{B.4}$$

for $0 \leq k \leq i$.

□

Lemma B.7 *$Sat(\mathcal{I}_1\mathcal{J}^m)/Sat(\mathcal{I}\mathcal{J}^m)$ is a locally free sheaf of rank $m+1$ on C_2 generated by the elements $\sum_{k=0}^{i}(xz^2)^k[x(xy)^{m-2k}, (xy)^{m-2k-1}(xz)]$ where $i = \lfloor m/2 \rfloor$.*

Proof: From equation B.1,

$$Sat(\mathcal{I}_1\mathcal{J}^m) = \sum_{k=0}^{i}(xz^2)^k[(xy)^{m-2k-1}(xz), x(xy)^{m-2k}, z(z^2)^{m-2k}]$$

with $i = \lfloor m/2 \rfloor$, and, from equation B.3,

$$Sat(\mathcal{I}\mathcal{J}^m) = \sum_{k=0}^{j}(xz^2)^k[(xy)^{m-2k+1}, (xy)^{m-2k}z, (z^2)^{m-2k}z]$$

with $j = \lfloor (m+1)/2 \rfloor$. The common generators to both of these sheaves are $(z^2)^{m-2k}z$ for $0 \leq k \leq i$, and since $m - 2k \geq 0$, these generators are exactly the same in each sheaf. So, the quotient sheaf $Sat(\mathcal{I}_1\mathcal{J}^m)/Sat(\mathcal{I}\mathcal{J}^m)$ is generated locally by $\sum_{k=0}^{i}(xz^2)^k[x(xy)^{m-2k}, (xy)^{m-2k-1}(xz)]$. The generators of $Sat(\mathcal{I}\mathcal{J}^m)$ can be expressed as $\sum_{k=0}^{j}(xz^2)^k[y \cdot x(xy)^{m-2k}, y \cdot (xy)^{m-2k-1}(xz), z(z^2)^{m-2k}]$, so, since $\mathcal{I}_2 = (y, z)$ in co-ordinates, the quotient has $m+1$ generators and is locally free on C_2.

□

Lemma B.8 *$Sat(\mathcal{I}_1\mathcal{J}^m)/Sat(\mathcal{I}\mathcal{J}^m)$ and $\mathcal{I}_1/\mathcal{I} \otimes S^m(\mathcal{J}/Sat(\mathcal{I}_2\mathcal{J}))$ are locally free sheaves of rank $m+1$ on C_2 and there is an injective map $\mathcal{I}_1/\mathcal{I} \otimes S^m(\mathcal{J}/Sat(\mathcal{I}_2\mathcal{J})) \hookrightarrow Sat(\mathcal{I}_1\mathcal{J}^m)/Sat(\mathcal{I}\mathcal{J}^m)$.*

Proof: $\mathcal{I}_1/\mathcal{I}$ is an invertible sheaf on C_2 generated by $\{x\}$ and $\mathcal{J}/Sat(\mathcal{I}_2\mathcal{J})$ is locally free of rank 2 on C_2 generated by $\{xy, xz\}$ locally at p. So, $\mathcal{I}_1/\mathcal{I} \otimes S^m(\mathcal{J}/Sat(\mathcal{I}_2\mathcal{J}))$ is locally free of rank $m+1$ generated by the elements $\sum_{j=0}^{m} x(xy)^{m-j}(xz)^j$.

The generating elements, $\sum_{k=0}^{i}(xz^2)^k[x(xy)^{m-2k}, (xy)^{m-2k-1}(xz)]$, of the quotient sheaf $Sat(\mathcal{I}_1\mathcal{J}^m)/Sat(\mathcal{I}\mathcal{J}^m)$ can also be expressed as $\sum_{j=0}^{m} x^{\lfloor j/2 \rfloor}x(xy)^{m-j}z^j$.

Define a map $\mathcal{I}_1 \otimes \mathcal{J}^{\otimes m} \to Sat(\mathcal{I}_1\mathcal{J}^m)/Sat(\mathcal{I}\mathcal{J}^m)$ by multiplication of functions. This map kills $\mathcal{I}_1 \otimes Sat(\mathcal{I}_2\mathcal{J}) \otimes J^{\otimes(m-1)}$, since $\mathcal{I}_1\mathcal{J}^{m-1}Sat(\mathcal{I}_2\mathcal{J}) \subset Sat(\mathcal{I}\mathcal{J}^m$, so there is a well defined map $\mathcal{I}_1/\mathcal{I} \otimes S^m(\mathcal{J}/Sat(\mathcal{I}_2\mathcal{J})) \to Sat(\mathcal{I}_1\mathcal{J}^m)/Sat(\mathcal{I}\mathcal{J}^m)$. This map is injective because $\mathcal{I}_1\mathcal{J}^m \cap Sat(\mathcal{I}\mathcal{J}^m) \subset \mathcal{I}\mathcal{J}^{m-1}Sat(\mathcal{I}_2\mathcal{J})$.

From the local generators for each sheaf, this injection is defined locally on generators by

$$x(xy)^{m-j}(xz)^j \mapsto x^{j-\lfloor j/2 \rfloor} \cdot x^{\lfloor j/2 \rfloor}x(xy)^{m-j}z^j \tag{B.5}$$

for $0 \leq j \leq m$.

□

APPENDIX C

$D_5(2, 2)$ CALCULATIONS

This appendix provides the calculations used to prove lemmas 4.3, 4.4, 4.5 and 4.6. In this case $\mathcal{I}_1 = (x,z)$, $\mathcal{I}_2 = (y,z)$, $\mathcal{I} = (xy,z)$, $\mathcal{J} = (x^2y,z)$ and $\mathcal{K} = ((xy)^2,z)$ in coordinates at $p = C_1 \bigcap C_2$. On $C_1 - \{p\}$, $\mathcal{I} = \mathcal{I}_1$, $\mathcal{I}_2 = \mathcal{O}_X$, and coordinates can be chosen so that $\mathcal{I} = (x,z)$, which implies $\mathcal{K} = \mathcal{J} = (x^2,z)$. On $C_2 - \{p\}$, $\mathcal{I} = \mathcal{I}_2$, $\mathcal{I}_1 = \mathcal{O}_X$, and coordinates can be chosen so that $\mathcal{I} = (y,z)$, which means $\mathcal{J} = \mathcal{I} = (y,z)$and $\mathcal{K} = (y^2,z)$.

As explained in the remark in appendix 1 it will only be necessary to prove each of the following lemmas in coordinates at p. On $C_1 - \{p\}$ the calculations go through the same, but eliminating the local coordinate y. On $C_2 - \{p\}$ the calculations are also the same, but eliminate the local coordinate x.

Lemma C.1 *The sheaves $S^m(\mathcal{K}/\mathcal{I}_2\mathcal{K})$ and $\mathcal{J}/\mathcal{K} \otimes S^m(\mathcal{K}/\mathcal{I}_2\mathcal{K})$ are locally free of rank $m+1$ on C_2, and the sheaves $\mathcal{I}_2/\mathcal{I} \otimes S^m(\mathcal{K}/\mathcal{I}_1\mathcal{K})$ and $\mathcal{I}/\mathcal{J} \otimes S^m(\mathcal{K}/\mathcal{I}_1\mathcal{K})$ are locally free of rank $m+1$ on C_1.*

Proof: The sheaves $\mathcal{K}/\mathcal{I}_2\mathcal{K}$ and $\mathcal{K}/\mathcal{I}_1\mathcal{K}$ are locally free of rank 2 on C_2 and C_1, respectively. By [Ha2], pg. 127, $S^m(\mathcal{K}/\mathcal{I}_2\mathcal{K})$ and $S^m(\mathcal{K}/\mathcal{I}_1\mathcal{K})$ are locally free of rank $m+1$ on C_2 and C_1, respectively. $\mathcal{I}_2/\mathcal{I}$ and $\mathcal{I}/\mathcal{J}$ are invertible sheaves on C_1, so tensoring these with $S^m(\mathcal{K}/\mathcal{I}_1\mathcal{K})$ results in locally free sheaves of rank $m+1$ on C_1. Similarly, since $\mathcal{J}/\mathcal{K}$ is invertible on C_2, $\mathcal{J}/\mathcal{K} \otimes S^m(\mathcal{K}/\mathcal{I}_2\mathcal{K})$ is locally free of rank $m+1$ on C_2.

□

Lemma C.2 *$\mathcal{K}^m/\mathcal{I}_2\mathcal{K}^m$ and $S^m(\mathcal{K}/\mathcal{I}_2\mathcal{K})$ are locally free of rank $m+1$ on C_2 generated by the same elements locally on all of C_2.*

Proof: At the point p,

$$\mathcal{K}^m = ((xy)^{2m}, (xy)^{2(m-1)}z, \cdots, (xy)^{2(m-j)}z^j, \cdots, z^m) \tag{C.1}$$

and

$$\begin{aligned}
\mathcal{I}_2\mathcal{K}^m &= y \cdot \mathcal{K}^m + z \cdot \mathcal{K}^m \\
&= (y(xy)^{2m}, y(xy)^{2(m-1)}z, \cdots, y(xy)^{2(m-j)}z^j, ḍots, yz^m) + \\
&\quad ((xy)^{2m}z, (xy)^{2(m-1)}z^2, \cdots, (xy)^{2(m-j)}z^{j+1}, \cdots, z^{m+1}) \\
&= (x^{2m}y^{2m+1},, \cdots, x^{2(m-j)}y^{2(m-j)+1}z^j, \cdots, z^{m+1}).
\end{aligned} \tag{C.2}$$

Define a map $g : \mathcal{O}_2^{\oplus(m+1)} \to \mathcal{K}^m/\mathcal{I}_2\mathcal{K}^m$ by $(f_0, \cdots, f_m) \mapsto \sum_{j=0}^{m} f_j(xy)^{2(m-j)}z^j$. This map is surjective as it sends the generators of $\mathcal{O}_2^{\oplus(m+1)}$ to the generators of $\mathcal{K}^m/\mathcal{I}_2\mathcal{K}^m$. This map is also injective since an image element $\sum_{j=0}^{m} f_j(xy)^{2(m-j)}z^j$ is in $\mathcal{I}_2\mathcal{K}^m$ only if each f_i is divisible by y or z (compare equations 1 and 2). That is, $(f_0, \cdots, f_m) \mapsto 0$ implies $f_j \in \mathcal{I}_2$ for all $0 \le j \le m$. Therefore, g is an isomorphism and $\mathcal{K}^m/\mathcal{I}_2\mathcal{K}^m$ is locally free at p. The rank of this sheaf is $m+1$, generated by the elements

$$\{(xy)^{2m}, (xy)^{2(m-1)}z, \cdots, (xy)^{2(m-j)}z^j, \cdots, z^m\}. \tag{C.3}$$

$\mathcal{K}/\mathcal{I}_2\mathcal{K}$ is generated by $\{(xy)^2, z\}$ at p, so $S^m(\mathcal{K}/\mathcal{I}_2\mathcal{K})$ is generated by

$$\{(xy)^{2m}, (xy)^{2(m-1)}z, \cdots, (xy)^{2(m-j)}z^j, \cdots, z^m\}. \tag{C.4}$$

Comparing (3) and (4), we see that these sheaves are generated by the same elements.

□

Remark: To show that the sheaf $\mathcal{K}^m/\mathcal{I}_2\mathcal{K}^m$ was locally free at p the map g was constructed and shown to be an isomorphism. g being surjective followed readily by definition. The injective property was concluded by observing that each of the monomial generators of the ideal $\mathcal{I}_2\mathcal{K}^m$ was the product of y or z times some monomial generator of $\mathcal{K}^m$.

In general, if $\mathcal{F}$ and $\mathcal{G}$ are ideal sheaves on C_i generated by a minimal set of monomials in coordinates at p, then the quotient sheaf $\mathcal{F}/\mathcal{G}$ is locally free at $p \in C_i$ if the monomials of $\mathcal{G}$ that differ from any of those in $\mathcal{F}$ are the product of a generator of $\mathcal{I}_i$ times some generator of $\mathcal{F}$. So, to show $\mathcal{F}/\mathcal{G}$ is locally free at $p \in C_1$, it is enough to observe that the generators of $\mathcal{G}$ are x or z times some monomial generator of $\mathcal{F}$. And, to show locally free at $p \in C_2$, it is enough to observe that each generator of $\mathcal{G}$ is y or z times some monomial generator of $\mathcal{F}$.

In the proofs of the following lemmas, then, such an observation will be pointed out, and nothing more will be said, to prove that the sheaves in question are locally free at p.

Lemma C.3 *$\mathcal{I}_2\mathcal{K}^m/\mathcal{I}\mathcal{K}^m$ and $\mathcal{I}_2/\mathcal{I} \otimes S^m(\mathcal{K}/\mathcal{I}_1\mathcal{K})$ are locally free sheaves of rank $m+1$ on C_1 generated by the same elements locally on all of C_1.*

Proof: In coordinates at p,

$$\begin{aligned}
\mathcal{I}\mathcal{K}^m &= xy \cdot \mathcal{K}^m + z \cdot \mathcal{K}^m \\
&= (xy(xy)^{2m}, xy(xy)^{2(m-1)}z, \cdots, xy(xy)^{2(m-j)}z^j, \cdots, xyz^m) + \\
&\quad ((xy)^{2m}z, (xy)^{2(m-1)}z^2, \cdots, (xy)^{2(m-j)}z^{j+1}, \cdots, z^{m+1}) \\
&= (x^{2m+1}y^{2m+1}, \cdots, x^{2(m-j)+1}y^{2(m-j)+1}z^j, \cdots, z^{m+1}). \qquad (C.5)
\end{aligned}$$

Comparing the expressions (2) and (5), we see that the first $m+1$ elements, in succession, of $\mathcal{I}\mathcal{K}^m$ are x times the corresponding element of $\mathcal{I}_2\mathcal{K}^m$. The final term z^{m+1} is common to both. Therefore, by the remark above, the quotient sheaf $\mathcal{I}_2\mathcal{K}^m/\mathcal{I}\mathcal{K}^m$ is locally free at $p \in C_1$. We can conclude, then, as mentioned at the beginning of this section, that $\mathcal{I}_2\mathcal{K}^m/\mathcal{I}\mathcal{K}^m$ is locally free of rank $m+1$ on C_1 generated by

$$\{x^{2m}y^{2m+1}, x^{2(m-1)}y^{2(m-1)+1}z, \cdots, x^{2(m-j)}y^{2(m-j)+1}z^j, \cdots, yz^m\} \qquad (C.6)$$

Since $\mathcal{K}/\mathcal{I}_1\mathcal{K}$ is generated by $\{(xy)^2, z\}$ and $\mathcal{I}_2/\mathcal{I}$ is generated by $\{y\}$ at p, $\mathcal{I}_2/\mathcal{I} \otimes S^m(\mathcal{K}/\mathcal{I}_1\mathcal{K})$ is generated by

$$\{x^{2m}y^{2m+1}, x^{2(m-1)}y^{2(m-1)+1}z, \cdots, x^{2(m-j)}y^{2(m-j)+1}z^j, \cdots, yz^m\} \qquad (C.7)$$

at p. These two expressions, (6) and (7) are the same, showing that these sheaves are generated by the same elements at p.

□

Lemma C.4 *$\mathcal{I}\mathcal{K}^m/\mathcal{J}\mathcal{K}^m$ and $\mathcal{I}/\mathcal{J} \otimes S^m(\mathcal{K}/\mathcal{I}_1\mathcal{K})$ are locally free of rank $m+1$ on C_1 generated by the same elements locally on all of C_1.*

Proof:

$$\begin{aligned}
\mathcal{J}\mathcal{K}^m &= x^2y \cdot \mathcal{K}^m + z \cdot \mathcal{K}^m \\
&= (x^2y(xy)^{2m}, x^2y(xy)^{2(m-1)}z, \cdots, x^2y(xy)^{2(m-j)}z^j, \cdots, x^2yz^m) + \\
&\quad ((xy)^{2m}z, (xy)^{2(m-1)}z^2, \cdots, (xy)^{2(m-j)}z^{j+1}, \cdots, z^{m+1}) \\
&= (x^{2(m+1)}y^{2m+1}, \cdots, x^{2(m-j+1)}y^{2(m-j)+1}z^j, \cdots, z^{m+1}). \qquad (C.8)
\end{aligned}$$

Comparing with (5), we see that each of the first $m+1$ terms of $\mathcal{J}\mathcal{K}^m$ is x times the corresponding element of $\mathcal{I}\mathcal{K}^m$, and the z^{m+1} element is common to both. Therefore, $\mathcal{I}\mathcal{K}^m/\mathcal{J}\mathcal{K}^m$ is locally free of rank $m+1$ on C_1, generated by

$$\{x^{2m+1}y^{2m+1}, x^{2(m-1)+1}y^{2(m-1)+1}z, \cdots, x^{2(m-j)+1}y^{2(m-j)+1}z^j, \cdots, xyz^m\}. \qquad (C.9)$$

$\mathcal{I}/\mathcal{J}$ is generated by $\{xy\}$ at p, so $\mathcal{I}/\mathcal{J} \otimes S^m(\mathcal{K}/\mathcal{I}_1\mathcal{K})$ is generated by

$$\begin{aligned}&\{xy\} \otimes \{(xy)^{2m}, (xy)^{2(m-1)}z, \cdots, (xy)^{2(m-j)}z^j, \cdots, (xy)^2 z^{m-1}, z^m\} \\ &\quad = \{(xy)^{2m+1}, (xy)^{2(m-1)+1}z, \cdots, (xy)^{2(m-j)+1}z^j, \cdots, xyz^m\}. \end{aligned} \tag{C.10}$$

These generators agree with those in (9).

□

Lemma C.5 *$\mathcal{J}\mathcal{K}^m/\mathcal{K}^{m+1}$ and $\mathcal{J}/\mathcal{K} \otimes S^m(\mathcal{K}/\mathcal{I}_2\mathcal{K})$ are locally free sheaves of rank $m+1$ on C_2 generated by the same elements locally on all of C_2.*

Proof: In coordinates at p,

$$\mathcal{K}^{m+1} = ((xy)^{2(m+1)}, (xy)^{2m}z, \cdots, (xy)^{2(m-j+1)}z^j, \cdots, z^{m+1}) \tag{C.11}$$

and comparing with the ideal sheaf $\mathcal{J}\mathcal{K}^m$ in (8) it can be seen that each generator of the ideal $\mathcal{K}^{m+1}$ is y times the corresponding generator of $\mathcal{J}\mathcal{K}^m$, except for the last element, which is common to both. So, $\mathcal{J}\mathcal{K}^m/\mathcal{K}^{m+1}$ is locally free of rank $m+1$ on C_2, generated by

$$\{(x^{2(m+1)}y^{2m+1}, x^{2m}y^{2(m-1)+1}z, \cdots, x^{2(m-j+1)}y^{2(m-j)+1}z^j, \cdots, x^2yz^m\}. \tag{C.12}$$

Now, $\mathcal{J}/\mathcal{K}$ is generated by $\{x^2y\}$, so $\mathcal{J}/\mathcal{K} \otimes S^m(\mathcal{K}/\mathcal{I}_2\mathcal{K})$ is generated by

$$\begin{aligned}&\{x^2y\} \otimes \{(xy)^{2m}, (xy)^{2(m-1)}z, \cdots, (xy)^{2(m-j)}z^j, \cdots, z^m\} \\ &\quad = \{x^{2(m+1)}y^{2m+1}, \cdots, x^{2(m-j+1)}y^{2(m-j)+1}z^j, \cdots, x^2yz^m\}. \end{aligned} \tag{C.13}$$

Equations (12) and (13) show that these two locally free sheaves on C_2 both have the same generators at p.

□

APPENDIX D

$D_6(2,2)$ CALCULATIONS

This appendix provides the calculations used to prove lemmas 4.18, 4.19, 4.20 and 4.21. In this case $\mathcal{I}_1 = (x,z)$, $\mathcal{I}_2 = (y,z)$, $\mathcal{I} = (xy,z)$, $\mathcal{J} = (xy,xz,z^2)$ and $\mathcal{K} = (xy,z^2)$ in coordinates at $p = C_1 \bigcap C_2$. On $C_1 - \{p\}$, $\mathcal{I} = \mathcal{I}_1$, $\mathcal{I}_2 = \mathcal{O}_X$, and coordinates can be chosen so that $\mathcal{I} = (x,z)$, which implies $\mathcal{K} = \mathcal{J} = (x,z^2)$. On $C_2 - \{p\}$, $\mathcal{I} = \mathcal{I}_2$, $\mathcal{I}_1 = \mathcal{O}_X$, and coordinates can be chosen so that $\mathcal{I} = (y,z)$, which means $\mathcal{J} = \mathcal{I} = (y,z)$and $\mathcal{K} = (y,z^2)$.

As explained in the remark in appendix A, it will only be necessary to prove each of the following lemmas in coordinates at p. On $C_1 - \{p\}$ the calculations go through the same, but eliminating the local coordinate y. On $C_2 - \{p\}$ the calculations are also the same, but eliminate the local coordinate x.

Lemma D.1 *The sheaves $S^m(\mathcal{K}/\mathcal{I}_2\mathcal{K})$ and $\mathcal{J}/\mathcal{K} \otimes S^m(\mathcal{K}/\mathcal{I}_2\mathcal{K})$ are locally free of rank $m+1$ on C_2, and the sheaves $\mathcal{I}_2/\mathcal{I} \otimes S^m(\mathcal{K}/\mathcal{I}_1\mathcal{K})$ and $\mathcal{I}/\mathcal{J} \otimes S^m(\mathcal{K}/\mathcal{I}_1\mathcal{K})$ are locally free of rank $m+1$ on C_1.*

Proof: The sheaves $\mathcal{K}/\mathcal{I}_2\mathcal{K}$ and $\mathcal{K}/\mathcal{I}_1\mathcal{K}$ are locally free of rank 2 on C_2 and C_1, respectively. By [Ha2], pg. 127, $S^m(\mathcal{K}/\mathcal{I}_2\mathcal{K})$ and $S^m(\mathcal{K}/\mathcal{I}_1\mathcal{K})$ are locally free of rank $m+1$ on C_2 and C_1, respectively. $\mathcal{I}_2/\mathcal{I}$ and $\mathcal{I}/\mathcal{J}$ are invertible sheaves on C_1, so tensoring these with $S^m(\mathcal{K}/\mathcal{I}_1\mathcal{K})$ results in locally free sheaves of rank $m+1$ on C_1. Similarly, since $\mathcal{J}/\mathcal{K}$ is invertible on C_2, $\mathcal{J}/\mathcal{K} \otimes S^m(\mathcal{K}/\mathcal{I}_2\mathcal{K})$ is locally free of rank $m+1$ on C_2.

□

Lemma D.2 *$\mathcal{K}^m/\mathcal{I}_2\mathcal{K}^m$ and $S^m(\mathcal{K}/\mathcal{I}_2\mathcal{K})$ are locally free of rank $m+1$ on C_2 generated by the same elements locally on all of C_2.*

Proof: At the point p,

$$\mathcal{K}^m = ((xy)^m, (xy)^{m-1}z^2, \cdots, (xy)^{m-j}(z^2)^j, \cdots, (z^2)^m) \tag{D.1}$$

and

$$\begin{aligned} \mathcal{I}_2\mathcal{K}^m &= y \cdot \mathcal{K}^m + z \cdot \mathcal{K}^m \\ &= (y(xy)^m, y(xy)^{m-1}z^2, \cdots, y(xy)^{m-j}(z^2)^j, \cdots, y(z^2)^m) + \\ &\quad ((xy)^m z, (xy)^{m-1}z^3, \cdots, (xy)^{m-j}z^{2j+1}, \cdots, z^{2m+1}) \end{aligned} \tag{D.2}$$

Define a map $g : \mathcal{O}_2^{\oplus(m+1)} \to \mathcal{K}^m/\mathcal{I}_2\mathcal{K}^m$ by $(f_0, \cdots, f_m) \mapsto \sum_{j=0}^{m} f_j(xy)^{m-j}(z^2)^j$. This map is surjective as it sends the generators of $\mathcal{O}_2^{\oplus(m+1)}$ to the generators of

$\mathcal{K}^m/\mathcal{I}_2\mathcal{K}^m$. This map is also injective since an image element $\sum_{j=0}^{m} f_j(xy)^{m-j}(z^2)^j$ is in $\mathcal{I}_2\mathcal{K}^m$ only if each f_i is divisible by y or z (compare equations 1 and 2). That is, $(f_0, \cdots, f_m) \mapsto 0$ implies $f_j \in \mathcal{I}_2$ for all $0 \leq j \leq m$. Therefore, g is an isomorphism and $\mathcal{K}^m/\mathcal{I}_2\mathcal{K}^m$ is locally free at p. The rank of this sheaf is $m+1$, generated by the elements

$$\{(xy)^m, (xy)^{m-1}z^2, \cdots, (xy)^{m-j}(z^2)^j, \cdots, (z^2)^m\}. \tag{D.3}$$

$\mathcal{K}/\mathcal{I}_2\mathcal{K}$ is generated by $\{xy, z^2\}$ at p, so $S^m(\mathcal{K}/\mathcal{I}_2\mathcal{K})$ is generated by

$$\{(xy)^m, (xy)^{m-1}z^2, \cdots, (xy)^{m-j}(z^2)^j, \cdots, (z^2)^m\}. \tag{D.4}$$

Comparing (3) and (4), we see that these sheaves are generated by the same elements.

□

From the remark in appendix C, the sheaves in question being locally free can be quickly determined from the monomial generators. This remark was illustrated in the proof of lemma D.2 as well. Such an observation will be pointed out to prove that these sheaves are locally free at p.

Lemma D.3 *$\mathcal{I}_2\mathcal{K}^m/\mathcal{I}\mathcal{K}^m$ and $\mathcal{I}_2/\mathcal{I} \otimes S^m(\mathcal{K}/\mathcal{I}_1\mathcal{K})$ are locally free sheaves of rank $m+1$ on C_1 generated by the same elements locally on all of C_1.*

Proof: In coordinates at p,

$$\begin{aligned} \mathcal{I}\mathcal{K}^m &= xy \cdot \mathcal{K}^m + z \cdot \mathcal{K}^m \\ &= (xy(xy)^m, xy(xy)^{m-1}z^2, \cdots, xy(xy)^{m-j}(z^2)^j, \cdots, xy(z^2)^m) + \\ &\quad ((xy)^m z, (xy)^{m-1)}z^3, \cdots, (xy)^{m-j}z^{2j+1}, \cdots, z^{2m+1}) \\ &= ((xy)^{m+1}, (xy)^m z, \cdots, (xy)^{m-j+1}z^{2j-1}, \cdots, xyz^{2m-1}, z^{2m+1}). \end{aligned} \tag{D.5}$$

Comparing the expressions (2) and (5), we see that the elements of $z\mathcal{K}^m$ are common to both $\mathcal{I}_2\mathcal{K}^m$ and $\mathcal{I}\mathcal{K}^m$, so it is only necessary to compare the elements of $y\mathcal{K}^m$ with those of $xy\mathcal{K}^m$. The elements of $xy\mathcal{K}^m$ are clearly x times the elements of $y\mathcal{K}^m$. Therefore, by the remark in appendix C, the quotient sheaf $\mathcal{I}_2\mathcal{K}^m/\mathcal{I}\mathcal{K}^m$ is locally free at $p \in C_1$. We can conclude, then, as mentioned at the beginning of this section, that $\mathcal{I}_2\mathcal{K}^m/\mathcal{I}\mathcal{K}^m$ is locally free of rank $m+1$ on C_1 generated by

$$\{y(xy)^m, y(xy)^{m-1}z^2, \cdots, y(xy)^{m-j}(z^2)^j, \cdots, y(z^2)^m\} \tag{D.6}$$

Since $\mathcal{K}/\mathcal{I}_1\mathcal{K}$ is generated by $\{xy, z^2\}$ and $\mathcal{I}_2/\mathcal{I}$ is generated by $\{y\}$ at p, $\mathcal{I}_2/\mathcal{I} \otimes S^m(\mathcal{K}/\mathcal{I}_1\mathcal{K})$ is generated by

$$\{y(xy)^m, y(xy)^{m-1}z^2, \cdots, y(xy)^{m-j}(z^2)^j, \cdots, y(z^2)^m\} \tag{D.7}$$

at p. These two expressions, (6) and (7), are the same, showing that these sheaves are generated by the same elements at p.

□

Lemma D.4 *$\mathcal{I}\mathcal{K}^m/\mathcal{J}\mathcal{K}^m$ and $\mathcal{I}/\mathcal{J} \otimes S^m(\mathcal{K}/\mathcal{I}_1\mathcal{K})$ are locally free of rank $m+1$ on C_1 generated by the same elements locally on all of C_1.*

Proof:

$$\begin{aligned}
\mathcal{J}\mathcal{K}^m &= xy\cdot\mathcal{K}^m + xz\cdot\mathcal{K}^m + z^2\mathcal{K}^m \\
&= (xy(xy)^m, xy(xy)^{m-1}z^2, \cdots, xy(xy)^{m-j}(z^2)^j, \cdots, xy(z^2)^m) + \\
&\quad (xz(xy)^m, xz(xy)^{m-1}z^2, \cdots, xz(xy)^{m-j}(z^2)^j, \cdots, xz(z^2)^m) + \\
&\quad ((xy)^m z^2, (xy)^{m-1}(z^2)^2, \cdots, (xy)^{m-j}(z^2)^{j+1}, \cdots, xy(z^2)^{m+1}).
\end{aligned} \tag{D.8}$$

Comparing with (5), we see that the elements of $xy\mathcal{K}^m$ are common to both $\mathcal{I}\mathcal{K}^m$ and $\mathcal{J}\mathcal{K}^m$. Furthermore, the elements of $xz\mathcal{K}^m$ are x times the elements of $z\mathcal{K}^m$ and the elements of $z^2\mathcal{K}^m$ are z times the elements of $z\mathcal{K}^m$. Therefore, $\mathcal{I}\mathcal{K}^m/\mathcal{J}\mathcal{K}^m$ is locally free of rank $m+1$ on C_1, generated by the elements of $z\mathcal{K}^m$, namely,

$$\{(xy)^m z, (xy)^{m-1)}z^3, \cdots, (xy)^{m-j}z^{2j+1}, \cdots, z^{2m+1}\} \tag{D.9}$$

$\mathcal{I}/\mathcal{J}$ is generated by $\{z\}$ at p, so $\mathcal{I}/\mathcal{J} \otimes S^m(\mathcal{K}/\mathcal{I}_1\mathcal{K})$ is generated by

$$\begin{aligned}
&\{z\} \otimes \{(xy)^m, (xy)^{m-1}z^2, \cdots, (xy)^{m-j}(z^2)^j, \cdots,, (z^2)^m\} \\
&\quad = \{(xy)^m z, (xy)^{m-1)}z^3, \cdots, (xy)^{m-j}z^{2j+1}, \cdots, z^{2m+1}\}
\end{aligned} \tag{D.10}$$

These generators agree with those in (9).

□

Lemma D.5 *$\mathcal{J}\mathcal{K}^m/\mathcal{K}^{m+1}$ and $\mathcal{J}/\mathcal{K} \otimes S^m(\mathcal{K}/\mathcal{I}_2\mathcal{K})$ are locally free sheaves of rank $m+1$ on C_2 generated by the same elements locally on all of C_2.*

Proof: In coordinates at p,

$$\mathcal{K}^{m+1} = ((xy)^{m+1}, (xy)^m z^2, \cdots, (xy)^{m-j+1}(z^2)^j, \cdots, (z^2)^{m+1}). \tag{D.11}$$

Noticing that $\mathcal{K}^{m+1} = xy\mathcal{K}^m + z^2\mathcal{K}^m$, from (8) we see that all these elements are common to those of $\mathcal{J}\mathcal{K}^m = xy\mathcal{K}^m + xz\mathcal{K}^m + z^2\mathcal{K}^m$. Therefore, $\mathcal{J}\mathcal{K}^m/\mathcal{K}^{m+1}$ is locally free of rank $m+1$ on C_2 generated by the elements of $xz\mathcal{K}^m$. In coordinates, these are the elements

$$\{xz(xy)^m, xz(xy)^{m-1}z^2, \cdots, xz(xy)^{m-j}(z^2)^j, \cdots, xz(z^2)^m\}. \tag{D.12}$$

Now, $\mathcal{J}/\mathcal{K}$ is generated by $\{xz\}$, so $\mathcal{J}/\mathcal{K} \otimes S^m(\mathcal{K}/\mathcal{I}_2\mathcal{K})$ is generated by

$$\begin{aligned}&\{xz\} \otimes \{(xy)^m, (xy)^{m-1}z^2, \cdots, (xy)^{m-j}(z^2)^j, \cdots, (z^2)^m\}\\ &\quad = \{xz(xy)^m, xz(xy)^{m-1}z^2, \cdots, xz(xy)^{m-j}(z^2)^j, \cdots, xz(z^2)^m\}\end{aligned} \tag{D.13}$$

as well.

□

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