# NORMS OF PRODUCTS AND FACTORS POLYNOMIALS 

IGOR E. PRITSKER


#### Abstract

We study inequalities connecting a product of uniform norms of polynomials with the norm of their product. Generalizing Gel'fond-Mahler inequality for the unit disk and Kneser-Borwein inequality for the segment $[-1,1]$, we prove an asymptotically sharp inequality for norms of products of algebraic polynomials over an arbitrary compact set in plane. Applying similar techniques, we produce a related inequality for the norm of a single monic factor of a monic polynomial. The best constants in both inequalities are obtained by potential theoretic methods. We also consider applications of the general results to the cases of a disk and a segment.


## 1. Introduction

Let $E$ be a compact set in the complex plane $\mathbb{C}$. Define the uniform (sup) norm on $E$ as follows:

$$
\|f\|_{E}=\sup _{z \in E}|f(z)| .
$$

Consider algebraic polynomials $\left\{p_{k}(z)\right\}_{k=1}^{m}$ and their product

$$
p(z):=\prod_{k=1}^{m} p_{k}(z)
$$

We are interested here in polynomial inequalities of the form

$$
\begin{equation*}
\prod_{k=1}^{m}\left\|p_{k}\right\|_{E} \leq C\|p\|_{E} \tag{1.1}
\end{equation*}
$$

One of the first results in this direction is due to Kneser [17], for $E=[-1,1]$ and $m=2$ (see also Aumann [1), who proved that

$$
\begin{equation*}
\left\|p_{1}\right\|_{[-1,1]}\left\|p_{2}\right\|_{[-1,1]} \leq K_{\ell, n}\left\|p_{1} p_{2}\right\|_{[-1,1]} \tag{1.2}
\end{equation*}
$$

[^0]where
\[

$$
\begin{equation*}
K_{\ell, n}:=2^{n-1} \prod_{k=1}^{\ell}\left(1+\cos \frac{2 k-1}{2 n} \pi\right) \prod_{k=1}^{n-\ell}\left(1+\cos \frac{2 k-1}{2 n} \pi\right) \tag{1.3}
\end{equation*}
$$

\]

$\operatorname{deg} p_{1}=\ell$ and $\operatorname{deg}\left(p_{1} p_{2}\right)=n$. Note that (1.2) becomes an equality for the Chebyshev polynomial $t(z)=\cos n \arccos z=p_{1}(z) p_{2}(z)$, with a proper choice of the factors $p_{1}(z)$ and $p_{2}(z)$. P. B. Borwein (7] gave a new proof of (1.2)- (1.3) and generalized this to the multifactor inequality

$$
\begin{equation*}
\prod_{k=1}^{m}\left\|p_{k}\right\|_{[-1,1]} \leq 2^{n-1} \prod_{k=1}^{\left[\frac{n}{2}\right]}\left(1+\cos \frac{2 k-1}{2 n} \pi\right)^{2}\|p\|_{[-1,1]} \tag{1.4}
\end{equation*}
$$

He has also showed that

$$
\begin{equation*}
2^{n-1} \prod_{k=1}^{\left[\frac{n}{2}\right]}\left(1+\cos \frac{2 k-1}{2 n} \pi\right)^{2} \sim(3.20991 \ldots)^{n}, \text { as } n \rightarrow \infty \tag{1.5}
\end{equation*}
$$

Another case of the inequality (1.1) was considered by Gel'fond (14 p. 135] in connection with the theory of transcendental numbers, for $E=\bar{D}$, where $D:=\{w:|w|<1\}$ is the unit disk:

$$
\begin{equation*}
\prod_{k=1}^{m}\left\|p_{k}\right\|_{\bar{D}} \leq e^{n}\|p\|_{\bar{D}} \tag{1.6}
\end{equation*}
$$

The latter inequality was improved by Mahler [20, who replaced $e$ by 2:

$$
\begin{equation*}
\prod_{k=1}^{m}\left\|p_{k}\right\|_{\bar{D}} \leq 2^{n}\|p\|_{D} \tag{1.7}
\end{equation*}
$$

It is easy to see that the base 2 cannot be decreased, if $m=n$ and $n \rightarrow \infty$. However, (1.7) has recently been further improved in two directions. D. W. Boyd [9, 10] showed that, by taking in account the number of factors $m$ in (1.7), one has

$$
\begin{equation*}
\prod_{k=1}^{m}\left\|p_{k}\right\|_{\bar{D}} \leq\left(C_{m}\right)^{n}\|p\|_{\bar{D}} \tag{1.8}
\end{equation*}
$$

where

$$
\begin{equation*}
C_{m}:=\exp \left(\frac{m}{\pi} \int_{0}^{\pi / m} \log \left(2 \cos \frac{t}{2}\right) d t\right) \tag{1.9}
\end{equation*}
$$

is asymptotically best possible for each fixed $m$, as $n \rightarrow \infty$. Kroó and Pritsker [18] showed that, for any $m \leq n$,

$$
\begin{equation*}
\prod_{k=1}^{m}\left\|p_{k}\right\|_{\bar{D}} \leq 2^{n-1}\|p\|_{\bar{D}} \tag{1.10}
\end{equation*}
$$

where equality holds in (1.10) for each $n \in \mathbb{N}$, with $m=n$ and $p(z)=z^{n}-1$. We give an asymptotically sharp inequality for the norm of products of polynomials on arbitrary compact set in Section 2 which generalizes the results of Mahler, Kneser and Borwein. This inequality and other connected to it results were originally obtained in [23].

A closely related problem is to estimate the norm of a single factor via the norm of the whole polynomial. Clearly, we have to normalize the problem by assuming that $p(z)$ is a monic polynomial of degree $n$, with a monic factor $q(z)$, so that

$$
p(z)=q(z) r(z)
$$

In the case of the unit disk, Boyd 9 proved an asymptotically sharp inequality

$$
\begin{equation*}
\|q\|_{\bar{D}} \leq \beta^{n}\|p\|_{\bar{D}} \tag{1.11}
\end{equation*}
$$

with

$$
\begin{equation*}
\beta:=\exp \left(\frac{1}{\pi} \int_{0}^{2 \pi / 3} \log \left(2 \cos \frac{t}{2}\right) d t\right) \tag{1.12}
\end{equation*}
$$

This inequality improved upon a series of results by Mignotte [22], Granville [16] and Glesser [15].

Further progress was made by Borwein in [7], for the segment $[-a, a], a>$ 0 (see Theorems 2 and 5 there or see Section 5.3 in [8). In particular, Borwein proved that if $\operatorname{deg} q=m$ then

$$
\begin{equation*}
|q(-a)| \leq\|p\|_{[-a, a]} a^{m-n} 2^{n-1} \prod_{k=1}^{m}\left(1+\cos \frac{2 k-1}{2 n} \pi\right) \tag{1.13}
\end{equation*}
$$

where the bound is attained for a monic Chebyshev polynomial of degree $n$ on $[-a, a]$ and a factor $q$. He also showed that, for $E=[-2,2]$, the constant
in the above inequality satisfies

$$
\begin{aligned}
& \limsup _{n \rightarrow \infty}\left(2^{m-1} \prod_{k=1}^{m}\left(1+\cos \frac{2 k-1}{2 n} \pi\right)\right)^{1 / n} \\
\leq & \lim _{n \rightarrow \infty}\left(2^{[2 n / 3]-1} \prod_{k=1}^{[2 n / 3]}\left(1+\cos \frac{2 k-1}{2 n} \pi\right)\right)^{1 / n} \\
= & \exp \left(\int_{0}^{2 / 3} \log (2+2 \cos \pi x) d x\right)=1.9081 \ldots
\end{aligned}
$$

which hints that

$$
\begin{equation*}
C_{[-2,2]}=\exp \left(\int_{0}^{2 / 3} \log (2+2 \cos \pi x) d x\right)=1.9081 \ldots \tag{1.14}
\end{equation*}
$$

In Section 3 we find an asymptotically sharp inequality of this type for a rather arbitrary compact set $E$. The general result is then applied to the cases of a disk and a line segment, so that we recover (1.11)-(1.12) and confirm (1.14). Also see [24] for these results.

Considered problems have applications in transcendence theory (see [14]) and in designing algorithms for factoring polynomials (see [11 and [19]). We confine ourselves to studying the sup norms for polynomials of one variable only. A survey of the results involving other norms (e.g., Bombieri norms) can be found in 11. These inequalities are also of considerable interest for polynomials in several variables, where very little is known about sharp constants (cf. [2], 3], 5] and [21]).

## 2. Products of Polynomials in Uniform Norms

Inequalities (1.2)-1.10) clearly indicate that the constant $C$ in (1.1) grows exponentially fast with $n$, with the base for the exponential depending on the set $E$. A natural general problem arising here is to find the smallest constant $M_{E}>0$, such that

$$
\begin{equation*}
\prod_{k=1}^{m}\left\|p_{k}\right\|_{E} \leq M_{E}^{n}\|p\|_{E} \tag{2.1}
\end{equation*}
$$

for arbitrary algebraic polynomials $\left\{p_{k}(z)\right\}_{k=1}^{m}$ with complex coefficients, where $p(z)=\prod_{k=1}^{m} p_{k}(z)$ and $n=\operatorname{deg} p$. The solution of this problem is based on the logarithmic potential theory (cf. [27] and [26]). Let $\operatorname{cap}(E)$ be the logarithmic capacity of a compact set $E \subset \mathbb{C}$. For $E$ with $\operatorname{cap}(E)>0$, denote the equilibrium measure of $E$ by $\mu_{E}$. We remark that $\mu_{E}$ is a positive
unit Borel measure supported on $E$ (see [27] p. 55]). Define

$$
\begin{equation*}
d_{E}(z):=\max _{t \in E}|z-t|, \quad z \in \mathbb{C} \tag{2.2}
\end{equation*}
$$

which is clearly a positive and continuous function on $\mathbb{C}$.
Theorem 2.1. Let $E \subset \mathbb{C}$ be a compact set, $\operatorname{cap}(E)>0$. Then the best constant $M_{E}$ in (2.1) is given by

$$
\begin{equation*}
M_{E}=\frac{\exp \left(\int \log d_{E}(z) d \mu_{E}(z)\right)}{\operatorname{cap}(E)} \tag{2.3}
\end{equation*}
$$

One can see from (2.1) or (2.3) that $M_{E}$ is invariant with respect to the rigid motions and dilations of the set $E$ in the plane.

Note that the restriction $\operatorname{cap}(E)>0$ excludes only very thin sets from our consideration (see [27] pp. 63-66]), e.g., finite sets in the plane. On the other hand, Theorem[2.1] is applicable to any compact set with a connected component consisting of more than one point (cf. [27] p. 56]). In particular, if $E$ is a continuum, i.e., a connected set, then we obtain a simple universal bound for $M_{E}$.

Corollary 2.2. Let $E \subset \mathbb{C}$ be a bounded continuum (not a single point). Then we have

$$
\begin{equation*}
M_{E} \leq \frac{\operatorname{diam}(E)}{\operatorname{cap}(E)} \leq 4 \tag{2.4}
\end{equation*}
$$

where $\operatorname{diam}(E)$ is the Euclidean diameter of the set $E$.
For the unit disk $D=\{w:|w|<1\}$, we have that $\operatorname{cap}(\bar{D})=1$ [27] p. 84] and that

$$
\begin{equation*}
\mu_{\bar{D}}=\frac{1}{2 \pi} d \theta \tag{2.5}
\end{equation*}
$$

where $d \theta$ is the arclength on $\partial D$. Thus Theorem 2.1 yields

$$
\begin{equation*}
M_{\bar{D}}=\exp \left(\frac{1}{2 \pi} \int_{0}^{2 \pi} \log d_{\bar{D}}\left(e^{i \theta}\right) d \theta\right)=\exp \left(\frac{1}{2 \pi} \int_{0}^{2 \pi} \log 2 d \theta\right)=2 \tag{2.6}
\end{equation*}
$$

so that we immediately obtain Mahler's inequality (1.7).
If $E=[-1,1]$ then $\operatorname{cap}([-1,1])=1 / 2$ and

$$
\begin{equation*}
\mu_{[-1,1]}=\frac{d x}{\pi \sqrt{1-x^{2}}}, \quad x \in[-1,1] \tag{2.7}
\end{equation*}
$$

which is the Chebyshev (or arcsin) distribution (see [27, p. 84]). Using Theorem 2.1] we obtain

$$
\begin{align*}
M_{[-1,1]} & =2 \exp \left(\frac{1}{\pi} \int_{-1}^{1} \frac{\log d_{[-1,1]}(x)}{\sqrt{1-x^{2}}} d x\right)=2 \exp \left(\frac{2}{\pi} \int_{0}^{1} \frac{\log (1+x)}{\sqrt{1-x^{2}}} d x\right) \\
(2.8) & =2 \exp \left(\frac{2}{\pi} \int_{0}^{\pi / 2} \log (1+\sin t) d t\right) \approx 3.2099123 \tag{2.8}
\end{align*}
$$

This gives the asymptotic version of Borwein's inequality (1.4)- (1.5).
It appears that the upper bound 4 in Corollary 2.2 is not the best possible. One might conjecture that the sharp universal bounds are as follows

$$
\begin{equation*}
2=M_{\bar{D}} \leq M_{E} \leq M_{[-1,1]} \approx 3.2099123 \tag{2.9}
\end{equation*}
$$

for any bounded non-degenerate continuum $E$.
It is of interest to determine the nature of the extremal polynomials for (2.1). We characterized the asymptotically extremal polynomials for (2.1), i.e., those polynomials, for which (2.1) becomes an asymptotic equality as $n \rightarrow \infty$, by their asymptotic zero distributions. The precise statements of these results can be found in Theorems 2.3-2.5 and Corollaries 3.1-3.3 of [23].

## 3. Uniform Norm of a Single Factor

In the same way as in Section 2 we naturally arrive at the problem to find the best (the smallest) constant $C_{E}$, such that

$$
\begin{equation*}
\|q\|_{E} \leq C_{E}^{n}\|p\|_{E}, \quad \operatorname{deg} p=n \tag{3.1}
\end{equation*}
$$

is valid for any monic polynomial $p(z)$ and any monic factor $q(z)$. Our solution of this problem is based on similar ideas, involving the logarithmic capacity and the equilibrium measure of $E$.

Theorem 3.1. Let $E \subset \mathbb{C}$ be a compact set, $\operatorname{cap}(E)>0$. Then the best constant $C_{E}$ in (3.1) is given by

$$
\begin{equation*}
C_{E}=\frac{\max _{u \in \partial E} \exp \left(\int_{|z-u| \geq 1} \log |z-u| d \mu_{E}(z)\right)}{\operatorname{cap}(E)} \tag{3.2}
\end{equation*}
$$

Furthermore, if $E$ is regular then

$$
\begin{equation*}
C_{E}=\max _{u \in \partial E} \exp \left(-\int_{|z-u| \leq 1} \log |z-u| d \mu_{E}(z)\right) \tag{3.3}
\end{equation*}
$$

The above notion of regularity is to be understood in the sense of exterior Dirichlet problem (cf. [27] p. 7]).

One can readily see from (3.1) or (3.2) that the best constant $C_{E}$ is invariant under the rigid motions of the set $E$ in the plane. Therefore we consider applications of Theorem 3.1] to the family of disks $D_{r}:=\{z$ : $|z|<r\}$, which are centered at the origin, and to the family of segments $[-a, a], a>0$.

Corollary 3.2. Let $D_{r}$ be a disk of radius $r$. Then the best constant $C_{\bar{D}_{r}}$, for $E=\overline{D_{r}}$, is given by
(3.4) $C_{\bar{D}_{r}}=\left\{\begin{array}{l}\frac{1}{r}, \quad 0<r \leq 1 / 2, \\ \frac{1}{r} \exp \left(\frac{1}{\pi} \int_{0}^{\pi-2 \arcsin \frac{1}{2 r}} \log \left(2 r \cos \frac{x}{2}\right) d x\right), \quad r>1 / 2 .\end{array}\right.$

Note that (1.11)-(1.12) immediately follow from (3.4) for $r=1$. The graph of $C_{\bar{D}_{r}}$ is in Figure 1


Figure 1. $C_{\bar{D}_{r}}$ as a function of $r$.

Corollary 3.3. If $E=[-a, a]$, $a>0$, then

$$
C_{[-a, a]}=\left\{\begin{array}{l}
\frac{2}{a}, \quad 0<a \leq 1 / 2  \tag{3.5}\\
\frac{2}{a} \exp \left(\int_{1-a}^{a} \frac{\log (t+a)}{\pi \sqrt{a^{2}-t^{2}}} d t\right), \quad a>1 / 2
\end{array}\right.
$$

Observe that (3.5), with $a=2$, implies (1.14) by the change of variable $t=2 \cos \pi x$. We include the graph of $C_{[-a, a]}$ in Figure 2


Figure 2. $C_{[-a, a]}$ as a function of $a$.
We now state two general consequences of Theorem 3.1 They explain some interesting features of $C_{E}$, which the reader may have noticed in Corollaries 3.2 and 3.3 Recall that the Euclidean diameter of $E$ is defined by

$$
\operatorname{diam}(E):=\max _{z, \zeta \in E}|z-\zeta|
$$

Corollary 3.4. Suppose that $\operatorname{cap}(E)>0$. If $\operatorname{diam}(E) \leq 1$ then

$$
\begin{equation*}
C_{E}=\frac{1}{\operatorname{cap}(E)} \tag{3.6}
\end{equation*}
$$

It is well known that $\operatorname{cap}\left(D_{r}\right)=r$ and $\operatorname{cap}([-a, a])=a / 2$ (see [26] p. 135]), which clarifies the first lines of (3.4) and (3.5) by (3.6).

The next Corollary shows how the constant $C_{E}$ behaves under dilations of the set $E$. Let $\alpha E$ be the dilation of $E$ with a factor $\alpha>0$.

Corollary 3.5. If $E$ is regular then

$$
\begin{equation*}
\lim _{\alpha \rightarrow+\infty} C_{\alpha E}=1 . \tag{3.7}
\end{equation*}
$$

Thus Figures 1and $_{2}$ clearly illustrate (3.7).
We conclude this section with two remarks.
Remark 3.6. One can deduce inequalities of the type (3.1), for various $L_{p}$ norms, from Theorem 3.1 by using relations between $L_{p}$ and $L_{\infty}$ norms of polynomials on $E$ (see, e.g., [25]).

Remark 3.7. Note that the inequalities considered in this section hold for any monic factor $q(z)$ of a monic polynomial $p(z)$, i.e., they hold for the largest factor in the terminology of [12. However, if we are granted the existence of the factoring $p(z)=q(z) r(z)$, then the norm of the smallest factor (cf. [13) can be estimated from (2.1) as follows:

$$
\begin{equation*}
\|r\|_{E} \leq M_{E}^{n / 2}\|p\|_{E}^{1 / 2}, \quad \operatorname{deg} p=n \tag{3.8}
\end{equation*}
$$

which may be better than (3.1) in some cases.

## 4. Proofs

The following lemma is a generalization of Lemma 2 in [10. We refer to [23] for its proof.

Lemma 4.1. Let $E \subset \mathbb{C}$ be a compact set (not a single point) and let

$$
d_{E}(z):=\max _{t \in E}|z-t|, \quad z \in \mathbb{C} .
$$

Then $\log d_{E}(z)$ is a subharmonic function in $\mathbb{C}$ and

$$
\begin{equation*}
\log d_{E}(z)=\int \log |z-t| d \sigma_{E}(t), \quad z \in \mathbb{C} \tag{4.1}
\end{equation*}
$$

where $\sigma_{E}$ is a positive unit Borel measure in $\mathbb{C}$ with unbounded support, i.e.,

$$
\begin{equation*}
\sigma_{E}(\mathbb{C})=1 \quad \text { and } \quad \infty \in \operatorname{supp} \sigma_{E} . \tag{4.2}
\end{equation*}
$$

Lemma 4.2. (Bernstein-Walsh) Let $E \subset \mathbb{C}$ be a compact set, $\operatorname{cap}(E)>0$, with the unbounded component of $\overline{\mathbb{C}} \backslash E$ denoted by $\Omega$. Then, for any polynomial $p(z)$ of degree $n$, we have

$$
\begin{equation*}
|p(z)| \leq\|p\|_{E} e^{n g_{\Omega}(z, \infty)}, \quad z \in \mathbb{C} \tag{4.3}
\end{equation*}
$$

where $g_{\Omega}(z, \infty)$ is the Green function of $\Omega$, with pole at $\infty$, satisfying

$$
\begin{equation*}
g_{\Omega}(z, \infty)=\log \frac{1}{\operatorname{cap}(E)}+\int \log |z-t| d \mu_{E}(t), \quad z \in \mathbb{C} . \tag{4.4}
\end{equation*}
$$

This is a well known result about the upper bound (4.3) for the growth of $p(z)$ off the set $E$ (see [26, p. 156], for example). The representation (4.4) for $g_{\Omega}(z, \infty)$ is also classical (cf. Theorem III. 37 in [27] p. 82]).

Consider the $n$-th Fekete points $\left\{a_{k, n}\right\}_{k=1}^{n}$ for a compact set $E \subset \mathbb{C}$ (cf. [26, p. 152]). Let

$$
\begin{equation*}
F_{n}(z):=\prod_{k=1}^{n}\left(z-a_{k, n}\right) \tag{4.5}
\end{equation*}
$$

be the Fekete polynomial of degree $n$, and define the normalized counting measures in Fekete points by

$$
\begin{equation*}
\tau_{n}:=\frac{1}{n} \sum_{k=1}^{n} \delta_{a_{k, n}}, \quad n \in \mathbb{N} \tag{4.6}
\end{equation*}
$$

where $\delta_{a_{k, n}}$ is a unit point-mass at $a_{k, n}$.
Lemma 4.3. For a compact set $E \subset \mathbb{C}, \operatorname{cap}(E)>0$, we have that

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|F_{n}\right\|_{E}^{1 / n}=\operatorname{cap}(E) \tag{4.7}
\end{equation*}
$$

and

$$
\begin{equation*}
\tau_{n} \xrightarrow{*} \mu_{E}, \quad \text { as } n \rightarrow \infty . \tag{4.8}
\end{equation*}
$$

Equation (4.7) is standard (see Theorems 5.5.4 and 5.5 .2 in 26, pp. 153-155]), while (4.8) follows from 4.7) (see Ex. 5 on page 159 of [26]).
Proof of Theorem 2.1. First we show that the best constant $M_{E}$ in (2.1) is at most the right hand side of (2.3). Clearly, it is sufficient to prove an inequality of the type (2.1) for monic polynomials only. Thus, we assume that $p_{k}(z), 1 \leq k \leq m$, are all monic, so that $p(z)$ is monic too. Let $\left\{z_{k, n}\right\}_{k=1}^{n}$ be the zeros of $p(z)$ and let $\nu_{n}$ be the normalized zero counting measure for $p(z)$. Then, we use 4.1), Fubini's theorem and Lemma 4.2 in the following estimate:

$$
\begin{aligned}
& \frac{1}{n} \log \frac{\prod_{k=1}^{m}\left\|p_{k}\right\|_{E}}{\|p\|_{E}} \\
\leq & \frac{1}{n} \log \frac{\prod_{k=1}^{n}\left\|z-z_{k, n}\right\|_{E}}{\|p\|_{E}}=\log \frac{1}{\|p\|_{E}^{1 / n}}+\int \log d_{E}(z) d \nu_{n}(z) \\
= & \log \frac{1}{\|p\|_{E}^{1 / n}}+\iint \log |z-t| d \nu_{n}(z) d \sigma_{E}(t)=\int \log \frac{|p(t)|^{1 / n}}{\|p\|_{E}^{1 / n}} d \sigma_{E}(t) \\
\leq & \int g_{\Omega}(t, \infty) d \sigma_{E}(t)=\log \frac{1}{\operatorname{cap}(E)}+\iint \log |z-t| d \sigma_{E}(t) d \mu_{E}(z) \\
= & \log \frac{1}{\operatorname{cap}(E)}+\int \log d_{E}(z) d \mu_{E}(z)
\end{aligned}
$$

This gives that

$$
\begin{equation*}
M_{E} \leq \frac{\exp \left(\int \log d_{E}(z) d \mu_{E}(z)\right)}{\operatorname{cap}(E)} \tag{4.9}
\end{equation*}
$$

To show that equality holds in (4.9), we consider the $n$-th Fekete points $\left\{a_{k, n}\right\}_{k=1}^{n}$ for $E$ and the Fekete polynomials $F_{n}(z), n \in \mathbb{N}$. Observe that

$$
\left\|z-a_{k, n}\right\|_{E}=d_{E}\left(a_{k, n}\right), \quad 1 \leq k \leq n, \quad n \in \mathbb{N}
$$

Since $\operatorname{cap}(E) \neq 0$, the set $E$ consists of more than one point and, therefore, $d_{E}(z)$ is a strictly positive continuous function in $\mathbb{C}$. Consequently, $\log d_{E}(z)$ is also continuous in $\mathbb{C}$, and we obtain by (4.8) of Lemma 4.3 that

$$
\begin{align*}
& \lim _{n \rightarrow \infty}\left(\prod_{k=1}^{n}\left\|z-a_{k, n}\right\|_{E}\right)^{1 / n}=\lim _{n \rightarrow \infty} \exp \left(\frac{1}{n} \sum_{k=1}^{n} \log d_{E}\left(a_{k, n}\right)\right)  \tag{4.10}\\
= & \exp \left(\lim _{n \rightarrow \infty} \int \log d_{E}(z) d \tau_{n}(z)\right)=\exp \left(\int \log d_{E}(z) d \mu_{E}(z)\right) .
\end{align*}
$$

Finally, we have from the above and (4.7) that

$$
M_{E} \geq \lim _{n \rightarrow \infty} \frac{\left(\prod_{k=1}^{n}\left\|z-a_{k, n}\right\|_{E}\right)^{1 / n}}{\left\|F_{n}\right\|_{E}^{1 / n}}=\frac{\exp \left(\int \log d_{E}(z) d \mu_{E}(z)\right)}{\operatorname{cap}(E)}
$$

Proof of Corollary 2.2. Since $E$ is a bounded continuum, we obtain from Theorem 5.3.2(a) of [26, p. 138] that

$$
\operatorname{cap}(E) \geq \frac{\operatorname{diam}(E)}{4}
$$

Thus, the Corollary follows by combining this estimate with the obvious inequality

$$
d_{E}(z) \leq \operatorname{diam}(E), \quad z \in E
$$

and by using that $\mu_{E}(\mathbb{C})=1$, $\operatorname{supp} \mu_{E} \subset E$.
Proof of Theorem 3.1. The proof of this result is quite similar to that of Theorem 2.1 (also see [9]). For $u \in \mathbb{C}$, consider a function

$$
\rho_{u}(z):=\max (|z-u|, 1), \quad z \in \mathbb{C} .
$$

One can immediately see that $\log \rho_{u}(z)$ is a subharmonic function in $z \in \mathbb{C}$, which has the following integral representation (see [26] p. 29]):

$$
\begin{equation*}
\log \rho_{u}(z)=\int \log |z-t| d \lambda_{u}(t), \quad z \in \mathbb{C} \tag{4.11}
\end{equation*}
$$

where $d \lambda_{u}\left(u+e^{i \theta}\right)=d \theta /(2 \pi)$ is the normalized angular measure on $|t-u|=$ 1.

Let $u \in \partial E$ be such that

$$
\|q\|_{E}=|q(u)|
$$

If $z_{k}, k=1, \ldots, n$, are the zeros of $p(z)$, arranged so that the first $m$ zeros belong to $q(z)$, then

$$
\begin{align*}
\log \|q\|_{E} & =\sum_{k=1}^{m} \log \left|u-z_{k}\right| \leq \sum_{k=1}^{m} \log \rho_{u}\left(z_{k}\right) \leq \sum_{k=1}^{n} \log \rho_{u}\left(z_{k}\right) \\
& =\sum_{k=1}^{n} \int \log \left|z_{k}-t\right| d \lambda_{u}(t)=\int \log |p(t)| d \lambda_{u}(t) \tag{4.12}
\end{align*}
$$

by (4.11).
It follows from (4.11)-(4.12), Lemma 4.2 and Fubini's theorem that

$$
\begin{aligned}
\frac{1}{n} \log \frac{\|q\|_{E}}{\|p\|_{E}} & \leq \int \log \frac{|p(t)|^{1 / n}}{\|p\|_{E}^{1 / n}} d \lambda_{u}(t) \leq \int g_{\Omega}(t, \infty) d \lambda_{u}(t) \\
& =\log \frac{1}{\operatorname{cap}(E)}+\iint \log |z-t| d \lambda_{u}(t) d \mu_{E}(z) \\
& =\log \frac{1}{\operatorname{cap}(E)}+\int \log \rho_{u}(z) d \mu_{E}(z)
\end{aligned}
$$

Using the definition of $\rho_{u}(z)$, we obtain from the above estimate that

$$
\begin{aligned}
\|q\|_{E} & \leq\left(\frac{\max _{u \in \partial E} \exp \left(\int \log \rho_{u}(z) d \mu_{E}(z)\right)}{\operatorname{cap}(E)}\right)^{n}\|p\|_{E} \\
& =\left(\frac{\max _{u \in \partial E} \exp \left(\int_{|z-u| \geq 1} \log |z-u| d \mu_{E}(z)\right)}{\operatorname{cap}(E)}\right)^{n}\|p\|_{E}
\end{aligned}
$$

Hence

$$
\begin{equation*}
C_{E} \leq \frac{\max _{u \in \partial E} \exp \left(\int_{|z-u| \geq 1} \log |z-u| d \mu_{E}(z)\right)}{\operatorname{cap}(E)} \tag{4.13}
\end{equation*}
$$

In order to prove the inequality opposite to (4.13), we consider the $n$-th Fekete points $\left\{a_{k, n}\right\}_{k=1}^{n}$ for the set $E$ and the Fekete polynomials $F_{n}(z), n \in \mathbb{N}$. Let $u \in \partial E$ be a point, where the maximum of the right hand side of (4.13) is attained. Define the factor $q_{n}(z)$ for $F_{n}(z)$, with zeros
being the $n$-th Fekete points satisfying $\left|a_{k, n}-u\right| \geq 1$. Then we have by (4.8) that

$$
\begin{aligned}
\lim _{n \rightarrow \infty}\left\|q_{n}\right\|_{E}^{1 / n} & \geq \lim _{n \rightarrow \infty}\left|q_{n}(u)\right|^{1 / n}=\lim _{n \rightarrow \infty} \exp \left(\frac{1}{n} \sum_{\left|a_{k, n}-u\right| \geq 1} \log \left|u-a_{k, n}\right|\right) \\
& =\exp \left(\lim _{n \rightarrow \infty} \int_{|z-u| \geq 1} \log |u-z| d \tau_{n}(z)\right) \\
& =\exp \left(\int_{|z-u| \geq 1} \log |u-z| d \mu_{E}(z)\right) .
\end{aligned}
$$

Combining the above inequality with (4.7) and the definition of $C_{E}$, we obtain that

$$
C_{E} \geq \lim _{n \rightarrow \infty} \frac{\left\|q_{n}\right\|_{E}^{1 / n}}{\left\|F_{n}\right\|_{E}^{1 / n}} \geq \frac{\exp \left(\int_{|z-u| \geq 1} \log |z-u| d \mu_{E}(z)\right)}{\operatorname{cap}(E)}
$$

This shows that (3.2) holds true. Moreover, if $u \in \partial E$ is a regular point for $\Omega$, then we obtain by Theorem III. 36 of [27, p. 82]) and (4.4) that

$$
\log \frac{1}{\operatorname{cap}(E)}+\int \log |u-t| d \mu_{E}(t)=g_{\Omega}(u, \infty)=0
$$

Hence

$$
\log \frac{1}{\operatorname{cap}(E)}+\int_{|z-u| \geq 1} \log |u-t| d \mu_{E}(t)=-\int_{|z-u| \leq 1} \log |u-t| d \mu_{E}(t)
$$

which implies (3.3) by (3.2).
Proof of Corollary [3.2. It is well known [27] p. 84] that $\operatorname{cap}\left(\overline{D_{r}}\right)=r$ and $d \mu \overline{D_{r}}\left(r e^{i \theta}\right)=d \theta /(2 \pi)$, where $d \theta$ is the angular measure on $\partial D_{r}$. If $r \in$ $(0,1 / 2]$ then the numerator of (3.2) is equal to 1 , so that

$$
C_{\overline{D_{r}}}=\frac{1}{r}, \quad 0<r \leq 1 / 2
$$

Assume that $r>1 / 2$. We set $z=r e^{i \theta}$ and let $u_{0}=r e^{i \theta_{0}}$ be a point where the maximum in (3.2) is attained. On writing

$$
\left|z-u_{0}\right|=2 r\left|\sin \frac{\theta-\theta_{0}}{2}\right|,
$$

we obtain that

$$
\begin{aligned}
C_{\overline{D_{r}}} & =\frac{1}{r} \exp \left(\frac{1}{2 \pi} \int_{\theta_{0}+2 \arcsin \frac{1}{2 r}}^{2 \pi+\theta_{0}-2 \arcsin \frac{1}{2 r}} \log \left|2 r \sin \frac{\theta-\theta_{0}}{2}\right| d \theta\right) \\
& =\frac{1}{r} \exp \left(\frac{1}{2 \pi} \int_{2 \arcsin \frac{1}{2 r}-\pi}^{\pi-2 \arcsin \frac{1}{2 r}} \log \left(2 r \cos \frac{x}{2}\right) d x\right) \\
& =\frac{1}{r} \exp \left(\frac{1}{\pi} \int_{0}^{\pi-2 \arcsin \frac{1}{2 r}} \log \left(2 r \cos \frac{x}{2}\right) d x\right)
\end{aligned}
$$

by the change of variable $\theta-\theta_{0}=\pi-x$.
Proof of Corollary 3.3. Recall that $\operatorname{cap}([-a, a])=a / 2$ (see [27] p. 84]) and

$$
d \mu_{[-a, a]}(t)=\frac{d t}{\pi \sqrt{a^{2}-t^{2}}}, \quad t \in[-a, a]
$$

It follows from (3.2) that

$$
\begin{equation*}
C_{[-a, a]}=\frac{2}{a} \exp \left(\max _{u \in[-a, a]} \int_{[-a, a] \backslash(u-1, u+1)} \frac{\log |t-u|}{\pi \sqrt{a^{2}-t^{2}}} d t\right) \tag{4.14}
\end{equation*}
$$

If $a \in(0,1 / 2]$ then the integral in (4.14) obviously vanishes, so that $C_{[-a, a]}=2 / a$. For $a>1 / 2$, let

$$
\begin{equation*}
f(u):=\int_{[-a, a] \backslash(u-1, u+1)} \frac{\log |t-u|}{\pi \sqrt{a^{2}-t^{2}}} d t \tag{4.15}
\end{equation*}
$$

One can easily see from (4.15) that

$$
f^{\prime}(u)=\int_{u+1}^{a} \frac{d t}{\pi(u-t) \sqrt{a^{2}-t^{2}}}<0, \quad u \in[-a, 1-a]
$$

and

$$
f^{\prime}(u)=\int_{-a}^{u-1} \frac{d t}{\pi(u-t) \sqrt{a^{2}-t^{2}}}>0, \quad u \in[a-1, a]
$$

However, if $u \in(1-a, a-1)$ then

$$
f^{\prime}(u)=\int_{u+1}^{a} \frac{d t}{\pi(u-t) \sqrt{a^{2}-t^{2}}}+\int_{-a}^{u-1} \frac{d t}{\pi(u-t) \sqrt{a^{2}-t^{2}}}
$$

It is not difficult to verify directly that

$$
\int \frac{d t}{\pi(u-t) \sqrt{a^{2}-t^{2}}}=\frac{1}{\pi \sqrt{a^{2}-u^{2}}} \log \left|\frac{a^{2}-u t+\sqrt{a^{2}-t^{2}} \sqrt{a^{2}-u^{2}}}{t-u}\right|+C
$$

which implies that

$$
f^{\prime}(u)=\frac{1}{\pi \sqrt{a^{2}-u^{2}}} \log \left(\frac{a^{2}-u^{2}+u+\sqrt{a^{2}-(u-1)^{2}} \sqrt{a^{2}-u^{2}}}{a^{2}-u^{2}-u+\sqrt{a^{2}-(u+1)^{2}} \sqrt{a^{2}-u^{2}}}\right)
$$

for $u \in(1-a, a-1)$. Hence

$$
f^{\prime}(u)<0, u \in(1-a, 0), \quad \text { and } \quad f^{\prime}(u)>0, u \in(0, a-1)
$$

Collecting all facts, we obtain that the maximum for $f(u)$ on $[-a, a]$ is attained at the endpoints $u=a$ and $u=-a$, and it is equal to

$$
\max _{u \in[-a, a]} f(u)=\int_{1-a}^{a} \frac{\log (t+a)}{\pi \sqrt{a^{2}-t^{2}}} d t
$$

Thus (3.5) follows from (4.14) and the above equation.

Proof of Corollary 3.4. Note that the numerator of (3.2) is equal to 1 , because $|z-u| \leq 1, \forall z \in E, \forall u \in \partial E$. Thus (3.6) follows immediately.

Proof of Corollary 3.5. Observe that $C_{E} \geq 1$ for any $E \in \mathbb{C}$, so that $C_{\alpha E} \geq$ 1. Since $E$ is regular, we use the representation for $C_{E}$ in (3.3). Let $T: E \rightarrow \alpha E$ be the dilation mapping. Then $|T z-T u|=\alpha|z-u|, z, u \in E$, and $d \mu_{\alpha E}(T z)=d \mu_{E}(z)$. This gives that

$$
\begin{aligned}
C_{\alpha E} & =\max _{T u \in \partial(\alpha E)} \exp \left(-\int_{|T z-T u| \leq 1} \log |T z-T u| d \mu_{\alpha E}(T z)\right) \\
& =\max _{u \in \partial E} \exp \left(-\int_{|z-u| \leq 1 / \alpha} \log (\alpha|z-u|) d \mu_{E}(z)\right) \\
& =\max _{u \in \partial E} \exp \left(-\mu_{E}\left(\overline{D_{1 / \alpha}(u)}\right) \log \alpha-\int_{|z-u| \leq 1 / \alpha} \log |z-u| d \mu_{E}(z)\right) \\
& <\max _{u \in \partial E} \exp \left(-\int_{|z-u| \leq 1 / \alpha} \log |z-u| d \mu_{E}(z)\right)
\end{aligned}
$$

where $\alpha \geq 1$. Using the absolute continuity of the integral, we have that

$$
\lim _{\alpha \rightarrow+\infty} \int_{|z-u| \leq 1 / \alpha} \log |z-u| d \mu_{E}(z)=0
$$

which implies (3.7).

## References

[1] G. Aumann, Satz über das Verhalten von Polynomen auf Kontinuen, Sitz. Preuss. Akad. Wiss. Phys.-Math. Kl. (1933), 926-931.
[2] V. Avanissian and M. Mignotte, A variant of an inequality of Gel'fond and Mahler, Bull. London Math. Soc. 26 (1994), 64-68.
[3] B. Beauzamy, E. Bombieri, P. Enflo and H. L. Montgomery, Products of polynomials in many variables, J. Number Theory 36 (1990), 219-245.
[4] B. Beauzamy and P. Enflo, Estimations de produits de polynômes, J. Number Theory 21 (1985), 390-413.
[5] C. Benitez, Y. Sarantopoulos and A. Tonge, Lower bounds for norms of products of polynomials, Math. Proc. Cambridge Philos. Soc. 124 (1998), 395-408.
[6] H.-P. Blatt, E. B. Saff and M. Simkani, Jentzsch-Szegő type theorems for the zeros of best approximants, J. London Math. Soc. 38 (1988), 307-316.
[7] P. B. Borwein, Exact inequalities for the norms of factors of polynomials, Can. J. Math. 46 (1994), 687-698.
[8] P. Borwein and T. Erdélyi, Polynomials and Polynomial Inequalities, SpringerVerlag, New York, 1995.
[9] D. W. Boyd, Two sharp inequalities for the norm of a factor of a polynomial, Mathematika 39 (1992), 341-349.
[10] D. W. Boyd, Sharp inequalities for the product of polynomials, Bull. London Math. Soc. 26 (1994), 449-454.
[11] D. W. Boyd, Large factors of small polynomials, Contemp. Math. 166 (1994), 301308.
[12] D. W. Boyd, Bounds for the height of a factor of a polynomial in terms of Bombieri's norms: I. The largest factor, J. Symbolic Comp. 16 (1993), 115-130.
[13] D. W. Boyd, Bounds for the height of a factor of a polynomial in terms of Bombieri's norms: II. The smallest factor, J. Symbolic Comp. 16 (1993), 131145.
[14] A. O. Gel'fond, Transcendental and Algebraic Numbers, Dover, New York, 1960.
[15] P. Glesser, Nouvelle majoration de la norme des facteurs d'un polynôme, C. R. Math. Rep. Acad. Sci. Canada 12 (1990), 224-228.
[16] A. Granville, Bounding the coefficients of a divisor of a given polynomial, Monatsh. Math. 109 (1990), 271-277.
[17] H. Kneser, Das Maximum des Produkts zweies Polynome, Sitz. Preuss. Akad. Wiss. Phys.-Math. Kl. (1934), 429-431.
[18] A. Kroó and I. E. Pritsker, A sharp version of Mahler's inequality for products of polynomials, Bull. London Math. Soc. 31 (1999), 269-278.
[19] S. Landau, Factoring polynomials quickly, Notices Amer. Math. Soc. 34 (1987), 3-8.
[20] K. Mahler, An application of Jensen's formula to polynomials, Mathematika 7 (1960), 98-100.
[21] K. Mahler, On some inequalities for polynomials in several variables, J. London Math. Soc. 37 (1962), 341-344.
[22] M. Mignotte, Some useful bounds, In "Computer Algebra, Symbolic and Algebraic Computation" (B. Buchberger et al., eds.), pp. 259-263, Springer-Verlag, New York, 1982.
[23] I. E. Pritsker, Products of polynomials in uniform norms, to appear in Trans. Amer. Math. Soc.
[24] I. E. Pritsker, An inequality for the norm of a polynomial factor, to appear in Proc. Amer. Math. Soc.
[25] I. E. Pritsker, Comparing norms of polynomials in one and several variables, J. Math. Anal. Appl. 216 (1997), 685-695.
[26] T. Ransford, Potential Theory in the Complex Plane, Cambridge University Press, Cambridge, 1995.
[27] M. Tsuji, Potential Theory in Modern Function Theory, Chelsea Publ. Co., New York, 1975.

Department of Mathematics, 401 Mathematical Sciences, Oklahoma State University, Stillwater, OK 74078-1058, U.S.A.

E-mail address: igor@math.okstate.edu


[^0]:    1991 Mathematics Subject Classification. Primary 11C08, 30C10; Secondary 30C85, 31A15.

    Key words and phrases. Polynomials, products, factors, uniform norm, logarithmic capacity, equilibrium measure, subharmonic function, Fekete points.

    Research supported in part by the National Science Foundation grants DMS-9996410 and DMS-9707359.

