Approximation of conformal mapping via the Szegő kernel method

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Dedicated to the memory of Professor D. Gaier

Abstract

We study the uniform approximation of the canonical conformal mapping, for a Jordan domain onto the unit disk, by polynomials generated from the partial sums of the Szegő kernel expansion. These polynomials converge to the conformal mapping uniformly on the closure of any Smirnov domain. We prove estimates for the rate of such convergence on domains with piecewise analytic boundaries, expressed through the smallest exterior angle at the boundary. Furthermore, we show that the rate of approximation on compact subsets inside the domain is essentially the square of that on the closure. Two standard applications to the rate of decay for the contour orthogonal polynomials inside the domain, and to the rate of locally uniform convergence of Fourier series are also given.

Keywords. Conformal mapping, Szegő kernel, orthogonal polynomials, Fourier series.

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1 Convergence of the Szegő kernel expansion and approximation of conformal maps

Let G be a Jordan domain in the complex plane. There are two well known kernel methods used for approximation of the canonical conformal mappings of G onto a disk. The Bergman kernel method is associated with the L_2 spaces and orthogonal polynomials with respect to the area measure, while the Szegő kernel method is based on the inner product and orthogonal polynomials with respect to the arclength measure on the boundary of G (see Gaier [7], Smirnov and Lebedev

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[19]). The approximations related to the area orthogonality approach were first introduced by Bieberbach [5] through an extremal problem for polynomials. The first result about the uniform convergence of the Bieberbach polynomials was proved by Keldysh [14], for domains with sufficiently smooth boundaries. The uniform convergence of the Bieberbach polynomials has been extensively studied since then (see Gaier [8] and references therein). A selection of further results on this subject is in the papers of Andrievskii [1], Gaier [9]-[11], Andrievskii and Gaier [3], and Andrievskii and Pritsker [4]. In contrast, the method based on the Szegő kernel did not receive such a comprehensive attention. The goal of this paper is to show that the Szegő kernel method is also very useful for the uniform approximation of conformal mappings.

Suppose that G has rectifiable boundary L of length l. We consider the Smirnov spaces $E_p(G)$, $1 \le p < \infty$, of analytic functions in G, whose boundary values satisfy

$$\|f\|_p = \left(\frac{1}{l} \int_L |f(z)|^p |dz| \right)^{1/p} < \infty$$

(see Duren [6], Smirnov and Lebedev [19]). Our interest is focused on the Hilbert space $E_2(G)$, equipped with the inner product

$$(f,g) := \frac{1}{l} \int_L f(z) \overline{g(z)} |dz|, \quad f,g \in E_2(G).$$

Polynomials are dense in $E_p(G)$, $1 \le p < \infty$, if and only if G is a Smirnov domain [6]. Keldysh and Lavrentiev [15] characterized Smirnov domains by the property that, for a conformal mapping ψ of the unit disk D onto G, $\log|\psi|$ is represented by the Poisson integral of its boundary values. Although no complete geometric description of Smirnov domains is known, this class is sufficiently wide. In particular, it contains (in the decreasing order of generality) all Ahlfors-regular domains, Lavrentiev (cord-arc) domains, Lipschitz domains, domains with bounded boundary rotation (Radon domains), piecewise smooth and smooth domains (cf. Pommerenke [17, Chap. 7]). Applying Gram-Schmidt orthonormalization to monomials $\{z^n\}_{n=0}^{\infty}$ in the Smirnov domain G, we obtain a complete orthonormal system of polynomials $\{p_n(z)\}_{n=0}^{\infty}$ in $E_2(G)$ (see Szegő [21]). Next, we introduce the Szegő kernel

$$K(z,\zeta) = \sum_{k=0}^{\infty} \overline{p_k(\zeta)} p_k(z), \quad z,\zeta \in G, \tag{1.1}$$

where convergence of this bilinear series is uniform in z and ζ on compact subsets in G [21], [19], [7]. The importance of the Szegő kernel lies in its reproducing property

$$f(\zeta) = \frac{1}{l} \int_L f(z) \overline{K(z,\zeta)} |dz|, \quad \zeta \in G, \tag{1.2}$$

which holds for any $f \in E_2(G)$. Equivalently, every $f \in E_2(G)$ can be represented by its Fourier series

$$f(\zeta) = \sum_{k=0}^{\infty} a_k p_k(\zeta) = \sum_{k=0}^{\infty} \left(\frac{1}{l} \int_L f(z) \overline{p_k(z)} |dz| \right) p_k(\zeta), \tag{1.3}$$

convergent in $E_2(G)$ norm and, consequently, locally uniformly convergent in G (see Chapter 4 of [19]).

It is well known that the Szegő kernel is closely connected with the canonical conformal mapping φ of G onto the unit disk D:

$$K(z,\zeta) = \frac{l}{2\pi} \sqrt{\varphi'(z)\varphi'(\zeta)}, \tag{1.4}$$

where $\varphi(\zeta) = 0$ and $\varphi'(\zeta) > 0$ (cf. [19] and [21]). Hence we have that

$$\varphi'(\zeta) = \frac{2\pi}{l}\, K(\zeta,\zeta) = \frac{2\pi}{l} \sum_{k=0}^{\infty} |p_k(\zeta)|^2, \tag{1.5}$$

by (1.1) and (1.4). It follows that

$$\varphi'(z) = \frac{2\pi}{l}\, \frac{(K(z,\zeta))^2}{K(\zeta,\zeta)} = \frac{2\pi}{l}\, \frac{\left(\sum_{k=0}^{\infty} \overline{p_k(\zeta)} p_k(z) \right)^2}{\sum_{k=0}^{\infty} |p_k(\zeta)|^2}, \quad z \in G, \tag{1.6}$$

where $\zeta \in G$ is regarded as a fixed point. We now introduce the following sequence of approximating polynomials:

$$J_{2n+1}(z) = \frac{2\pi}{l}\, \frac{\int_\zeta^z \left(\sum_{k=0}^{n} \overline{p_k(\zeta)} p_k(t) \right)^2 dt}{\sum_{k=0}^{n} |p_k(\zeta)|^2}, \quad n \in \mathbb{N}. \tag{1.7}$$

Note that the degree of $J_{2n+1}(z)$ is $2n+1$. The sequence $\{J_{2n+1}\}_{n=0}^{\infty}$ converges to φ uniformly on compact subsets of G, which is inherited from the partial sums of the Szegő kernel. Similar approximating polynomials, but with a different normalization, were introduced via an extremal problem for any $E_p(\Gamma)$, $1 \le p < \infty$, by Keldysh and Lavrentiev (see [13] and [15]). They developed the ideas of Julia [12], who earlier considered the same extremal problem for the conformal mapping. Further study of the convergence properties in $E_2(G)$ is due to Warschawski [22] (also see Gaier [7] for a survey). Convergence questions for general Fourier expansions in contour orthogonal polynomials were considered in

Rosenbloom and Warschawski [18], Smirnov and Lebedev [19] and Suetin [20]. However, all of these studies impose quite strict smoothness assumptions on the boundary of G. We prove the first uniform convergence results for domains with corners, and give explicit rates of approximation in terms of the geometric properties of G.

We start with a general estimate for the uniform (sup) norm of the approximation error on $\overline{G}$. A similar result was proved by Warschawski [22], but in a somewhat different form (see also [7, pp. 130-131]).

Theorem 1.1 *Let G be a Smirnov domain, with $\zeta \in G$ fixed. If $\varphi : G \to D$ is a conformal mapping normalized by $\varphi(\zeta) = 0$ and $\varphi'(\zeta) > 0$, then*

$$\|\varphi - J_{2n+1}\|_\infty \le 8\pi \left\| K(\cdot,\zeta) - \sum_{k=0}^{n} \overline{p_k(\zeta)} p_k(\cdot) \right\|_2 \to 0 \quad \text{as } n \to \infty. \tag{1.8}$$

Recall that $\sum_{k=0}^{n} \overline{p_k(\zeta)} p_k(z)$ is the Fourier sum for $K(z,\zeta)$, i.e., it is the best $E_2(G)$ approximation from the subspace of polynomials of degree n. Hence we can give an upper estimate for the rate of convergence by appropriately choosing a sequence of approximating polynomials for $K(z,\zeta)$. The rate of convergence necessarily depends on the geometric properties of the domain G. We consider a class of domains with piecewise analytic boundaries, which is important in applications. An analytic arc is defined as the image of a segment under a mapping that is conformal in an open neighborhood of the segment. Thus a domain has piecewise analytic boundary if it is bounded by a Jordan curve consisting of a finite number of analytic arcs.

Theorem 1.2 *Let ∂G be piecewise analytic, with the smallest exterior angle $\lambda\pi$, $0 < \lambda < 2$, at the junction points of the analytic arcs. If $\zeta \in F$, where $F \subset G$ is compact, then*

$$\left\| K(\cdot,\zeta) - \sum_{k=0}^{n} \overline{p_k(\zeta)} p_k(\cdot) \right\|_2 \le C_1(G,F)\; n^{-\frac{\lambda}{4-2\lambda}}, \quad n \in \mathbb{N}. \tag{1.9}$$

Here, the constant $C_1(G,F) > 0$ depends only on G and F.

Combining Theorems 1.1 and 1.2, we obtain the main result on the uniform approximation of conformal mappings.

Theorem 1.3 *Let ∂G be piecewise analytic, with the smallest exterior angle $\lambda\pi$, $0 < \lambda < 2$, at the junction points of the analytic arcs. Suppose that $\zeta \in F$, where $F \subset G$ is compact. For the conformal mapping $\varphi : G \to D$, normalized by $\varphi(\zeta) = 0$ and $\varphi'(\zeta) > 0$, we have*

$$\|\varphi - J_{2n+1}\|_\infty \le C_2(G,F)\; n^{-\frac{\lambda}{4-2\lambda}}, \quad n \in \mathbb{N}, \tag{1.10}$$

where the constant $C_2(G,F) > 0$ depends only on G and F.

It is interesting to compare the convergence properties of our sequence $\{J_{2n+1}\}_{n=0}^{\infty}$ with those of the Bieberbach polynomials $\{B_n\}_{n=0}^{\infty}$. Gaier [10] proved for domains with piecewise analytic boundaries that

$$\|\varphi - B_n\|_{\infty} \leq C_3 \ \log n \ n^{-\frac{\lambda}{2-\lambda}}, \quad n \geq 2.$$

Later, Andrievskii and Gaier [3] replaced $\log n$ by $\sqrt{\log n}$, and relaxed the imposed geometric condition to piecewise quasianalytic boundary. Their estimate for the rate of uniform convergence of the Bieberbach polynomials remains the most precise known. Although (1.10) gives a slower rate of convergence, the polynomials J_{2n+1} have some advantages over the Bieberbach polynomials. They are free from the convergence anomalies exhibited by Keldysh's example [14], where only one singular point, at otherwise very smooth boundary, destroys the uniform convergence of the Bieberbach polynomials. (One can find further information about this example of Keldysh in [4].) In fact, Smirnov domains include any imaginable domain arising in numerical applications, guaranteeing the uniform convergence of $\{J_{2n+1}\}_{n=0}^{\infty}$ by Theorem 1.1. In addition, this sequence is easier to generate numerically, because the inner products defined by the boundary integrals are easier to compute than the area inner products in Gram-Schmidt orthonormalization.

We show that the rate of convergence for J_{2n+1} on compact subsets of G is better than on the whole domain, i.e., it is essentially squared comparing to (1.10).

Theorem 1.4 *If the conditions of Theorem 1.3 are satisfied, then*

$$\max_{z \in F} |\varphi(z) - J_{2n+1}(z)| \leq C_4(G,F) \ n^{-\frac{\lambda}{2-\lambda}}, \quad n \in \mathbb{N}, \tag{1.11}$$

where $C_4(G,F) > 0$ depends only on G and F.

Gaier [11] posed the question of possible improvement in locally uniform convergence rates for the Bieberbach polynomials and other approximations for conformal maps. The above theorem provides a partial answer for his question in the case of the Szegő kernel method. Furthermore, it is possible to give similar improvements for the Bieberbach polynomials too, by following the ideas of this paper.

Clearly, it was not our goal to achieve the highest possible level of generality here. Thus Theorem 1.2 is true for domains with piecewise quasianalytic boundaries [3] (therefore, all other results are valid for these domains too). One only needs to fill in a number of technical details on the behavior of $\sqrt{\varphi'}$ near corners, in our proof, to reach this conclusion. We are also able to prove Theorems 1.2-1.4 for Lavrentiev (cord-arc) domains, with the rates of convergence of the order $n^{-\gamma}$, $\gamma > 0$. Finally, we have analogues of these results in $E_p(G)$ for $p \neq 2$.

2 Orthogonal polynomials and Fourier series

We give two standard applications for approximation of the Szegő kernel here. Since polynomials $\{p_n\}_{n=0}^{\infty}$ form a complete orthonormal system, we can restate (1.9) in the following form:

$$\left(\sum_{k=n+1}^{\infty} |p_k(\zeta)|^2 \right)^{1/2} \le C_1(G,F)\ n^{-\frac{\lambda}{4-2\lambda}}, \quad n \in \mathbb{N}, \tag{2.1}$$

by (1.1). It is well known that p_n converge to zero on compact subsets of G as $n \to \infty$ (cf. Chapter XVI of [21]). Equation (2.1) immediately gives an estimate for the rate of decay of p_n inside G.

Theorem 2.1 *Let ∂G be piecewise analytic, with the smallest exterior angle $\lambda\pi,\ 0 < \lambda < 2$, at the junction points of the analytic arcs. If $\zeta \in F$, where $F \subset G$ is compact, then*

$$|p_n(\zeta)| \le C_1(G,F)\ n^{-\frac{\lambda}{4-2\lambda}}, \quad n \in \mathbb{N}. \tag{2.2}$$

The second application is related to the rates of convergence of the Fourier series (1.3) for $f \in E_2(G)$ on compact subsets of G. Observe that

$$\|f\|_2 = \left(\sum_{k=0}^{\infty} |a_k|^2 \right)^{1/2}.$$

Hence we have from (1.3) and Cauchy-Schwarz inequality that

$$\begin{aligned} \left| f(\zeta) - \sum_{k=0}^{n} a_k\, p_k(\zeta) \right| &= \left| \sum_{k=n+1}^{\infty} a_k\, p_k(\zeta) \right| \le \left(\sum_{k=n+1}^{\infty} |a_k|^2 \right)^{1/2} \left(\sum_{k=n+1}^{\infty} |p_k(\zeta)|^2 \right)^{1/2} \\ &= \left\| f - \sum_{k=0}^{n} a_k\, p_k \right\|_2 \left(\sum_{k=n+1}^{\infty} |p_k(\zeta)|^2 \right)^{1/2} \\ &\le \|f\|_2 \left(\sum_{k=n+1}^{\infty} |p_k(\zeta)|^2 \right)^{1/2}. \end{aligned}$$

Thus we obtain the following result from (2.1):

Theorem 2.2 *Let ∂G be piecewise analytic, with the smallest exterior angle $\lambda\pi,\ 0 < \lambda < 2$, at the junction points of the analytic arcs. Suppose that $f \in E_2(G)$ has the Fourier expansion (1.3). If $\zeta \in F$, where $F \subset G$ is compact, then*

$$\left| f(\zeta) - \sum_{k=0}^{n} a_k\, p_k(\zeta) \right| \le C_1(G,F)\ n^{-\frac{\lambda}{4-2\lambda}} \left\| f - \sum_{k=0}^{n} a_k\, p_k \right\|_2, \quad n \in \mathbb{N}. \tag{2.3}$$

Results of this kind for domains with smooth boundaries were previously proved by Szegő [21], Rosenbloom and Warschawski [18], Smirnov and Lebedev [19], Suetin [20], and others.

3 Proofs

Proof of Theorem 1.1. Let $\psi := \varphi^{-1}$. We have

$$\begin{aligned}|\varphi(z) - J_{2n+1}(z)| &= \left| \int_\zeta^z \left(\varphi'(t) - J'_{2n+1}(t)\right) dt \right| \\ &= \left| \int_0^{\varphi(z)} \left(\varphi'(\psi(u)) - J'_{2n+1}(\psi(u))\right) \psi'(u) du \right| \\ &\leq \int_0^{\varphi(z)} \left|\left(\varphi'(\psi(u)) - J'_{2n+1}(\psi(u))\right) \psi'(u)\right| |du|,\end{aligned}$$

where the integration is carried over the segment connecting 0 and $\varphi(z)$ in D. Note that the function $\left(\varphi'(\psi(u)) - J'_{2n+1}(\psi(u))\right)\psi'(u) = 1 - J'_{2n+1}(\psi(u))\psi'(u)$ belongs to the Hardy class $H^1(D)$, because L is rectifiable. Hence we obtain by Fejér-Riesz inequality (cf. Theorem 3.13 of [6]) that

$$\begin{aligned}|\varphi(z) - J_{2n+1}(z)| &\leq \frac{1}{2} \int_{|u|=1} \left|\left(\varphi'(\psi(u)) - J'_{2n+1}(\psi(u)\right) \psi'(u)\right| |du| \\ &= \frac{1}{2} \int_L \left|\varphi'(t) - J'_{2n+1}(t)\right| |dt|.\end{aligned}$$

Denote

$$Q_n(z) := \left(\frac{l}{2\pi} \sum_{k=0}^n |p_k(\zeta)|^2\right)^{-1/2} \sum_{k=0}^n \overline{p_k(\zeta)} p_k(z),$$

so that $J'_{2n+1}(z) = Q_n^2(z)$. We continue with this notation, by using Cauchy-Schwarz and Minkowski inequalities:

$$\begin{aligned}|\varphi(z) - J_{2n+1}(z)| &\leq \frac{1}{2} \int_L \left| \left(\sqrt{\varphi'(t)}\right)^2 - Q_n^2(t) \right| |dt| \\ &= \frac{1}{2} \int_L \left| \sqrt{\varphi'(t)} - Q_n(t) \right| \left| \sqrt{\varphi'(t)} + Q_n(t) \right| |dt| \\ &\leq \frac{l}{2} \, \|\sqrt{\varphi'} - Q_n\|_2 \, \|\sqrt{\varphi'} + Q_n\|_2 \\ &\leq \frac{l}{2} \, \|\sqrt{\varphi'} - Q_n\|_2 \left(\|\sqrt{\varphi'}\|_2 + \|Q_n\|_2\right).\end{aligned}$$

Observe that

$$\|\sqrt{\varphi'}\|_2 = \left(\frac{1}{l} \int_L |\varphi'(z)| \, |dz|\right)^{1/2} = \left(\frac{1}{l} \int_{|w|=1} |dw|\right)^{1/2} = \sqrt{\frac{2\pi}{l}},$$

and that

$$\|Q_n\|_2 = \left(\frac{l}{2\pi} \sum_{k=0}^n |p_k(\zeta)|^2\right)^{-1/2} \left(\sum_{k=0}^n |p_k(\zeta)|^2\right)^{1/2} = \sqrt{\frac{2\pi}{l}},$$

by orthonormality of the polynomials p_k. Thus

$$\|\varphi - J_{2n+1}\|_\infty \le \sqrt{2\pi l}\, \|\sqrt{\varphi'} - Q_n\|_2. \tag{3.1}$$

We now estimate the norm on the right of the above inequality. Recall that

$$\sqrt{\varphi'(z)} = \sqrt{\frac{2\pi}{l}}\, \frac{K(z,\zeta)}{\sqrt{K(\zeta,\zeta)}},$$

by (1.6). Therefore,

$$\begin{aligned}
\sqrt{\varphi'(z)} - Q_n(z) &= \sqrt{\frac{2\pi}{l}} \left(\frac{K(z,\zeta)}{\sqrt{K(\zeta,\zeta)}} - \left(\sum_{k=0}^n |p_k(\zeta)|^2 \right)^{-1/2} \sum_{k=0}^n \overline{p_k(\zeta)} p_k(z) \right) \\
&= \sqrt{\frac{2\pi}{lK(\zeta,\zeta)}} \left(K(z,\zeta) - \sum_{k=0}^n \overline{p_k(\zeta)} p_k(z) \right) \\
&\quad + \sqrt{\frac{2\pi}{lK(\zeta,\zeta)}} \left(1 - \left(\frac{K(\zeta,\zeta)}{\sum_{k=0}^n |p_k(\zeta)|^2} \right)^{1/2} \right) \sum_{k=0}^n \overline{p_k(\zeta)} p_k(z).
\end{aligned}$$

It follows that

$$\begin{aligned}
\|\sqrt{\varphi'} - Q_n\|_2 &\le \sqrt{\frac{2\pi}{lK(\zeta,\zeta)}} \left\| K(\cdot,\zeta) - \sum_{k=0}^n \overline{p_k(\zeta)} p_k(\cdot) \right\|_2 \\
&\quad + \sqrt{\frac{2\pi}{lK(\zeta,\zeta)}} \left(\left(\frac{K(\zeta,\zeta)}{\sum_{k=0}^n |p_k(\zeta)|^2} \right)^{1/2} - 1 \right) \left(\sum_{k=0}^n |p_k(\zeta)|^2 \right)^{1/2} \\
&= \sqrt{\frac{2\pi}{lK(\zeta,\zeta)}} \left\| K(\cdot,\zeta) - \sum_{k=0}^n \overline{p_k(\zeta)} p_k(\cdot) \right\|_2 \\
&\quad + \sqrt{\frac{2\pi}{lK(\zeta,\zeta)}} \left((K(\zeta,\zeta))^{1/2} - \left(\sum_{k=0}^n |p_k(\zeta)|^2 \right)^{1/2} \right).
\end{aligned}$$

Since

$$\begin{aligned}
(K(\zeta,\zeta))^{1/2} - \left(\sum_{k=0}^n |p_k(\zeta)|^2 \right)^{1/2} &= \|K(\cdot,\zeta)\|_2 - \left\| \sum_{k=0}^n \overline{p_k(\zeta)} p_k(\cdot) \right\|_2 \\
&\le \left\| K(\cdot,\zeta) - \sum_{k=0}^n \overline{p_k(\zeta)} p_k(\cdot) \right\|_2,
\end{aligned}$$

we obtain that

$$\|\sqrt{\varphi'} - Q_n\|_2 \le 2\sqrt{\frac{2\pi}{lK(\zeta,\zeta)}} \left\| K(\cdot,\zeta) - \sum_{k=0}^n \overline{p_k(\zeta)} p_k(\cdot) \right\|_2.$$

Equation (3.1) now gives that

$$\|\varphi - J_{2n+1}\|_\infty \leq \frac{4\pi}{\sqrt{K(\zeta,\zeta)}} \left\| K(\cdot,\zeta) - \sum_{k=0}^{n} \overline{p_k(\zeta)} p_k(\cdot) \right\|_2 . \tag{3.2}$$

Recall that

$$K(\zeta,\zeta) = \frac{l}{2\pi}\,\varphi'(\zeta),$$

by (1.5). Using Corollary 1.4 of [17, p. 9], we have

$$|\varphi'(\zeta)| \geq \frac{1}{4}\,\frac{1-|\varphi(\zeta)|^2}{\operatorname{dist}(\zeta,L)} = \frac{1}{4\operatorname{dist}(\zeta,L)},$$

where $\operatorname{dist}(\zeta,L)$ is the distance from ζ to L. Clearly, $l \geq 2\pi \operatorname{dist}(\zeta,L)$, so that

$$K(\zeta,\zeta) = \frac{l\,\varphi'(\zeta)}{2\pi} \geq \varphi'(\zeta)\operatorname{dist}(\zeta,L) \geq \frac{1}{4}.$$

Combining this with (3.2), we obtain (1.8).

■

Proof of Theorem 1.2. We start by recalling that the partial sum $\sum_{k=0}^{n} \overline{p_k(\zeta)} p_k(z)$ for $K(z,\zeta)$ is its best approximation in $E_2(G)$ (with $\zeta \in G$ fixed) among all polynomials of degree at most n. Thus we construct a sequence of polynomials with good approximative properties, which gives the desired upper bound (1.9).

It is clear from (1.4) that approximation of $K(z,\zeta)$ is equivalent to approximation of $\sqrt{\varphi'(z)}$. We use a method resembling that of Andrievskii and Gaier [3]. We first continue the mapping φ conformally beyond the boundary L, by using reflections across the analytic arcs L_i, $L = \cup_{i=1}^{m} L_i$. Suppose that τ_i is a mapping such that $L_i = \tau_i([0,1])$, which is conformal in an open neighborhood of $[0,1]$. Then we can find a symmetric lens shaped domain S_i, bounded by two circular arcs subtended by $[0,1]$, whose closure is contained in this open neighborhood of $[0,1]$. Defining

$$\tilde{G} := G \cup \left(\cup_{i=1}^{m} \tau_i(S_i)\right),$$

we extend φ into $\tilde{G}$ as follows:

$$\varphi(z) := \frac{1}{\overline{\varphi\left[\tau_i\left(\overline{\tau_i^{-1}(z)}\right)\right]}}, \qquad z \in \tau_i(S_i)\backslash\overline{G},$$

where $i = 1,\ldots,m$. The boundary $\partial\tilde{G}$ consists of m analytic arcs Γ_i that share endpoints with the arcs L_i of ∂G:

$$\partial\tilde{G} \cap \partial G = \{z_i\}_{i=1}^{m},$$

which are clearly the corner points of ∂G. Since each $\tau_i,\ i=1,\dots,m$, is conformal and has bounded derivative (together with its inverse) on S_i, we obtain the inequalities

$$\operatorname{dist}(z,\partial G)\ge c_1\min_{1\le i\le m}|z-z_i|,\qquad z\in\partial\tilde G, \tag{3.3}$$

where $\operatorname{dist}(z,\partial G)$ is the distance from z to ∂G, and

$$|\gamma|\le c_2|z-t|,\qquad z,t\in\partial\tilde G, \tag{3.4}$$

where $|\gamma|$ is the length of the shorter arc $\gamma\subset\partial\tilde G$, connecting z and t. We denote various positive constants by c_1,c_2, etc.

Let Γ_j be an arc of $\partial\tilde G$, with the endpoints z_j and z_{j+1}, and let $\zeta_j\in\Gamma_j$ be a fixed point, $j=1,\dots,m$. Note that ζ_j divides Γ_j into Γ_j^1 and Γ_j^2, so that $\partial\tilde G=\bigcup_{j=1}^m\bigcup_{i=1}^2\Gamma_j^i$. We obtain from Cauchy's integral formula for the continuation of $\sqrt{\varphi'}$ into $\tilde G$ that

$$\sqrt{\varphi'(z)}=\frac{1}{2\pi i}\int_{\partial\tilde G}\frac{\sqrt{\varphi'(t)}}{t-z}\,dt=\frac{1}{2\pi i}\sum_{j=1}^m\sum_{i=1}^2\int_{\Gamma_j^i}\frac{\sqrt{\varphi'(t)}}{t-z}\,dt,\quad z\in\tilde G. \tag{3.5}$$

Hence the problem is reduced to approximation of functions of the form

$$g(z):=\int_\gamma\frac{\sqrt{\varphi'(t)}}{t-z}\,dt \tag{3.6}$$

in $E_2(G)$ norm, where γ is any of the arcs Γ_j^i, with $i=1,2$ and $j=1,\dots,m$.

Let $\Omega:=\overline{\mathbb{C}}\setminus\overline{G}$. Consider the standard conformal mapping $\Phi:\Omega\to\Delta$, where $\Delta:=\{w:|w|>1\}$, normalized by $\Phi(\infty)=\infty$ and $\Phi'(\infty)>0$. We define the level curves of Φ by

$$L_n:=\{z:|\Phi(z)|=1+1/n\},\qquad n\in\mathbb{N}.$$

Denote by γ_2 the part of γ from its endpoint $\zeta_j\in\Gamma_j$ to the first point ξ of intersection with L_n, so that $\gamma_2\subset\{z:|\Phi(z)|>1+1/n\}$. Then $\gamma_1:=\gamma\setminus\gamma_2$ connects ξ with the corner point z_j of L. Write

$$g(z):=\int_{\gamma_1}\frac{\sqrt{\varphi'(t)}}{t-z}\,dt+\int_{\gamma_2}\frac{\sqrt{\varphi'(t)}}{t-z}\,dt=:g_1(z)+g_2(z). \tag{3.7}$$

We show that $\|g_1\|_2\to0$ sufficiently fast as $n\to\infty$, while g_2 is well approximated by polynomials of degree n. To estimate the norm of g_1, we need to know the behavior of $\sqrt{\varphi'}$ near the corner point $z_j\in L$. This is conveniently found from the asymptotic expansion of Lehman [16]. Assume that $z_j=0$ and that $\lambda_j\pi,\ 0<$

$\lambda_j < 2$, is the exterior angle formed by L at this point. Then we have in a neighborhood of $z_j = 0$ that

$$\varphi(z) - \varphi(0) = b\ z^{\frac{1}{2-\lambda_j}} + o\left(z^{\frac{1}{2-\lambda_j}}\right) \qquad \text{as } z \to 0,$$

where $b \neq 0$, and

$$\varphi'(z) = \frac{b}{2-\lambda_j}\ z^{\frac{1}{2-\lambda_j}-1} + o\left(z^{\frac{1}{2-\lambda_j}-1}\right) \qquad \text{as } z \to 0.$$

Hence there exists a constant $c_3 > 0$ such that

$$\left|\sqrt{\varphi'(z)}\right| \le c_3\ |z|^{\alpha}, \qquad z \in \tilde{G} \cup \partial\tilde{G}, \tag{3.8}$$

where we set

$$\alpha := \frac{1}{4-2\lambda_j} - \frac{1}{2}.$$

For the endpoints $\xi \in L_n$ and 0 of γ_1, we let

$$d_n := |\xi - 0| = |\xi|.$$

It follows from (3.4) that

$$|\gamma_1| \le c_2 d_n.$$

We now estimate that

$$\|g_1\|_2^2 = \frac{1}{l}\int_L \left|\int_{\gamma_1} \frac{\sqrt{\varphi'(t)}}{t-z}\,dt\right|^2 |dz| \le c_4 \int_L \left(\int_{\gamma_1} \frac{|t|^{\alpha}|dt|}{|t-z|}\right)^2 |dz|, \tag{3.9}$$

by (3.7) and (3.8). Note that if $z \in L$ satisfies $|z| \ge d_n$, then $|t-z| \sim |z|$ by (3.3). Consequently,

$$\int_{L\cap\{|z|\ge d_n\}} \left(\int_{\gamma_1} \frac{|t|^{\alpha}|dt|}{|t-z|}\right)^2 |dz| \le c_5 \int_{L\cap\{|z|\ge d_n\}} \left(\frac{d_n^{\alpha+1}}{|z|}\right)^2 |dz| \le c_6\, d_n^{2\alpha+1}. \tag{3.10}$$

On the other hand, if $z \in L$ satisfies $|z| \le d_n$, then $|t-z| \sim |t| + |z|$ by (3.3), and we obtain with help of (3.4) that

$$\begin{aligned}
\int_{L\cap\{|z|\le d_n\}} \left(\int_{\gamma_1} \frac{|t|^{\alpha}|dt|}{|t-z|}\right)^2 |dz| &\le c_7 \int_0^{c_8 d_n} \left(\int_0^{c_9 d_n} \frac{s^{\alpha}ds}{s+r}\right)^2 dr \\
&\le c_7 \int_0^{c_8 d_n} \left(\int_0^{r} \frac{s^{\alpha}}{r}\,ds + \int_r^{c_9 d_n} s^{\alpha-1}ds\right)^2 dr \\
&= c_7 \int_0^{c_8 d_n} \left(\frac{r^{\alpha}}{\alpha+1} + \frac{(c_9 d_n)^{\alpha} - r^{\alpha}}{\alpha}\right)^2 dr \\
&\le c_{10}\, d_n^{2\alpha+1},
\end{aligned} \tag{3.11}$$

for $\alpha \neq 0$. If $\alpha = 0$ then we estimate

$$\begin{aligned}\int_{L\cap\{|z|\le d_n\}} \left(\int_{\gamma_1} \frac{|dt|}{|t-z|}\right)^2 |dz| &= \int_{L\cap\{|z|\le d_n\}} \left(\int_{\gamma_1} \frac{|t|^{1/2}\,|t|^{-1/2}\,|dt|}{|t-z|}\right)^2 |dz| \\ &\le c_2\, d_n \int_{L\cap\{|z|\le d_n\}} \left(\int_{\gamma_1} \frac{|t|^{-1/2}\,|dt|}{|t-z|}\right)^2 |dz| \\ &\le c_2\, d_n\, c_{10}\, d_n^{2(-1/2)+1} = c_2\, c_{10}\, d_n,\end{aligned}$$

as above. Combining (3.9)-(3.11), we have that

$$\|g_1\|_2 \le c_{11}\, d_n^{\alpha+1/2} \le c_{11}\, d_n^{\frac{1}{4-2\lambda}}, \tag{3.12}$$

where $\lambda = \min_{1\le j\le m} \lambda_j$.

The next step is the construction of approximating polynomials P_n for g_2. This is accomplished by using Dzjadyk kernels (see, e.g., [2]) of the form

$$K_n(t,z) = \sum_{i=0}^{n} a_i(t) z^i, \qquad n \in \mathbb{N},$$

which approximate the Cauchy kernel. It was proved in Lemma 5 of [3] that a sequence of such kernels can be selected, so that for any fixed $m \in \mathbb{N}$, and for all $t \in \gamma$ with $|\Phi(t)| \ge 1 + 1/n$, we have

$$\left|\frac{1}{t-z} - K_n(t,z)\right| \le c_{12}\, \frac{d_n^m}{|t-z|^{m+1}}, \qquad z \in L, \tag{3.13}$$

for all sufficiently large $n \in \mathbb{N}$. In particular, (3.13) holds for $t \in \gamma_2$. Define the polynomials

$$P_n(z) := \int_{\gamma_2} \sqrt{\varphi'(t)}\, K_n(t,z)\, dt,$$

and estimate

$$\begin{aligned}\|g_2 - P_n\|_2^2 &= \frac{1}{l}\int_L \left|\int_{\gamma_2} \left(\frac{1}{t-z} - K_n(t,z)\right)\sqrt{\varphi'(t)}\, dt\right|^2 |dz| \\ &\le c_{13} d_n^{2m} \int_L \left(\int_{\gamma_2} \frac{|t|^{\alpha}|dt|}{|t-z|^{m+1}}\right)^2 |dz|,\end{aligned}$$

by (3.13) and (3.8). Observe that $|t-z| \sim |t| + |z|$ for $t \in \gamma_2$. Therefore, we have

for $m > \alpha$ that

$$\begin{aligned}\int_L \left(\int_{\gamma_2} \frac{|t|^\alpha |dt|}{|t-z|^{m+1}}\right)^2 |dz| &\le c_{14} \int_0^{c_{15}} \left(\int_{c_{16}d_n}^{c_{17}} \frac{s^\alpha ds}{(s+r)^{m+1}}\right)^2 dr \\ &\le c_{14} \int_0^{c_{16}d_n} \left(\int_{c_{16}d_n}^{c_{17}} s^{\alpha-m-1}\, ds\right)^2 dr \\ &+ c_{14} \int_{c_{16}d_n}^{c_{15}} \left(r^{-m-1} \int_{c_{16}d_n}^{r} s^\alpha\, ds + \int_r^{c_{17}} s^{\alpha-m-1}\, ds\right)^2 dr \\ &\le c_{18}\, d_n^{2(\alpha-m)+1} + c_{19} \int_{c_{16}d_n}^{c_{15}} r^{2(\alpha-m)}\, dr \\ &\le c_{20}\, d_n^{2(\alpha-m)+1}.\end{aligned}$$

It follows that

$$\|g_2 - P_n\|_2 \le c_{21}\, d_n^{\alpha+1/2} \le c_{21}\, d_n^{\frac{1}{4-2\lambda}}. \tag{3.14}$$

Collecting (3.12), (3.14) and (3.7) together, we obtain

$$\|g - P_n\|_2 \le \|g_1\|_2 + \|g_2 - P_n\|_2 \le c_{22}\, d_n^{\frac{1}{4-2\lambda}}. \tag{3.15}$$

Recall that $d_n = |\xi|$, where $\xi \in L_n \cap \gamma_1$. Applying the results of [16] to the conformal mapping $\Psi := \Phi^{-1}$, we obtain

$$z = \Psi(\Phi(z)) - \Psi(\Phi(0)) = a\,(\Phi(z) - \Phi(0))^{\lambda_j} + o\left((\Phi(z) - \Phi(0))^{\lambda_j}\right) \quad \text{as } z \to 0,$$

where $\lambda_j \pi$ is the exterior angle at $z_j = 0$, and $a \neq 0$. Thus

$$d_n = |\xi| \le c_{23} \min_{z \in L_n} |z| \le c_{24}\, n^{-\lambda_j} \le c_{24}\, n^{-\lambda}, \qquad n \in \mathbb{N},$$

and

$$\|g - P_n\|_2 \le c_{25}\, n^{-\frac{\lambda}{4-2\lambda}}, \qquad n \in \mathbb{N}, \tag{3.16}$$

by (3.15). Hence there exists a sequence of polynomials Q_n such that

$$\|\sqrt{\varphi'} - Q_n\|_2 \le c_{26}\, n^{-\frac{\lambda}{4-2\lambda}}, \qquad n \in \mathbb{N}. \tag{3.17}$$

Since

$$K(z,\zeta) = \frac{l}{2\pi} \sqrt{\varphi'(z)\varphi'(\zeta)},$$

we obtain (1.9) from the previous estimate:

$$\left\| K(\cdot,\zeta) - \frac{l}{2\pi} \sqrt{\varphi'(\zeta)}\; Q_n \right\|_2 \le \frac{c_{26} l}{2\pi} \sqrt{\varphi'(\zeta)}\; n^{-\frac{\lambda}{4-2\lambda}}, \qquad n \in \mathbb{N},$$

where $\zeta \in G$ was fixed throughout this proof. We now lift this restriction and allow ζ vary within a compact set $F \subset G$. Note that $|\varphi'(\zeta)|$ is uniformly bounded on F, and so are other constants in the above proof. Indeed, we only need to verify this for the constant c_3 of (3.8), arising from the first term in the expansion of $\varphi'(z)$. One can obtain a conformal mapping $\tilde{\varphi}$ of G onto D with $\tilde{\varphi}(\tilde{\zeta}) = 0$ and $\tilde{\varphi}'(\tilde{\zeta}) > 0$, for any $\tilde{\zeta} \in F$, by composing φ with a Möbius self-map of the unit disk, which is conformal in an open neighborhood of $\overline{D}$. It follows that (3.8) holds for all such mappings $\tilde{\varphi}$ with a constant c_3 uniformly bounded for $\tilde{\zeta} \in F$.

■

Proof of Theorem 1.4. We proceed as in the proof of Theorem 1.1, denoting

$$Q_n(z) := \left(\frac{l}{2\pi} \sum_{k=0}^{n} |p_k(\zeta)|^2 \right)^{-1/2} \sum_{k=0}^{n} \overline{p_k(\zeta)} p_k(z),$$

so that $J'_{2n+1}(z) = Q_n^2(z)$. It follows from (1.6) that

$$\begin{aligned} \sqrt{\varphi'(z)} - Q_n(z) &= \sqrt{\frac{2\pi}{l}} \left(\frac{K(z,\zeta)}{\sqrt{K(\zeta,\zeta)}} - \left(\sum_{k=0}^{n} |p_k(\zeta)|^2 \right)^{-1/2} \sum_{k=0}^{n} \overline{p_k(\zeta)} p_k(z) \right) \\ &= \sqrt{\frac{2\pi}{lK(\zeta,\zeta)}} \left(K(z,\zeta) - \sum_{k=0}^{n} \overline{p_k(\zeta)} p_k(z) \right) \\ &\quad + \sqrt{\frac{2\pi}{lK(\zeta,\zeta)}} \frac{\left(\sum_{k=0}^{n} |p_k(\zeta)|^2\right)^{1/2} - \left(K(\zeta,\zeta)\right)^{1/2}}{\left(\sum_{k=0}^{n} |p_k(\zeta)|^2\right)^{1/2}} \sum_{k=0}^{n} \overline{p_k(\zeta)} p_k(z). \end{aligned}$$

Recall that

$$\lim_{n\to\infty} \sum_{k=0}^{n} \overline{p_k(\zeta)} p_k(z) = K(z,\zeta),$$

where convergence is uniform for $z, \zeta \in F$, and that

$$0 < c_1 < |K(z,\zeta)| < c_2 < +\infty, \qquad z, \zeta \in F.$$

Hence

$$\begin{aligned} \max_{z\in F} |\sqrt{\varphi'(z)} - Q_n(z)| \le c_3 \max_{z\in F} \left| K(z,\zeta) - \sum_{k=0}^{n} \overline{p_k(\zeta)} p_k(z) \right| \\ + c_4 \left(\left(K(\zeta,\zeta)\right)^{1/2} - \left(\sum_{k=0}^{n} |p_k(\zeta)|^2 \right)^{1/2} \right). \end{aligned}$$

The first term is estimated by (1.9)-(2.1) and Cauchy-Schwarz inequality:

$$\begin{aligned}\max_{z,\zeta\in F}\left|K(z,\zeta)-\sum_{k=0}^{n}\overline{p_k(\zeta)}p_k(z)\right| &= \max_{z,\zeta\in F}\left|\sum_{k=n+1}^{\infty}\overline{p_k(\zeta)}p_k(z)\right| \\ &\le \left(\sum_{k=n+1}^{\infty}|p_k(z)|^2\right)^{1/2}\left(\sum_{k=n+1}^{\infty}|p_k(\zeta)|^2\right)^{1/2} \\ &\le (C_1(G,F))^2\, n^{-\frac{\lambda}{2-\lambda}}, \quad n\in\mathbb{N}.\end{aligned}$$

Thus for the second term we also have

$$\begin{aligned}(K(\zeta,\zeta))^{1/2}-\left(\sum_{k=0}^{n}|p_k(\zeta)|^2\right)^{1/2} &\le c_5\left(K(\zeta,\zeta)-\sum_{k=0}^{n}|p_k(\zeta)|^2\right) \\ &\le c_5\,(C_1(G,F))^2\, n^{-\frac{\lambda}{2-\lambda}}, \quad n\in\mathbb{N}.\end{aligned}$$

Combining these estimates, we obtain that

$$\max_{z\in F}|\sqrt{\varphi'(z)}-Q_n(z)|\le c_6\, n^{-\frac{\lambda}{2-\lambda}}, \qquad n\in\mathbb{N}.$$

It immediately follows that

$$\begin{aligned}\max_{z\in F}|\varphi'(z)-Q_n^2(z)| &\le \max_{z\in F}|\sqrt{\varphi'(z)}+Q_n(z)|\ \max_{z\in F}|\sqrt{\varphi'(z)}-Q_n(z)| \\ &\le c_7\, n^{-\frac{\lambda}{2-\lambda}}, \qquad n\in\mathbb{N},\end{aligned}$$

and that

$$\max_{z\in F}|\varphi(z)-J_{2n+1}(z)|\le \max_{z\in F}\int_{\zeta}^{z}|\varphi'(t)-Q_n^2(t)|\,|dt|\le c_8\, n^{-\frac{\lambda}{2-\lambda}}, \quad n\in\mathbb{N},$$

where all constants in this proof are independent of $z,\zeta\in F$ and $n\in\mathbb{N}$.
■

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