

HOUSE OF ALGEBRAIC INTEGERS SYMMETRIC ABOUT THE UNIT CIRCLE

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ABSTRACT. We give a Schinzel-Zassenhaus-type lower bound for the maximum modulus of roots of a monic integer polynomial with all roots symmetric with respect to the unit circle. Our results extend a recent work of Dimitrov, who proved the general Schinzel-Zassenhaus conjecture by using the Pólya rationality theorem for a power series with integer coefficients, and some estimates for logarithmic capacity (transfinite diameter) of sets. We use an enhancement of Pólya's result obtained by Robinson, which involves Laurent-type rational functions with small supremum norms, thereby replacing the logarithmic capacity with a smaller quantity. This smaller quantity is expressed via a weighted Chebyshev constant for the set associated with Dimitrov's function used in Robinson's rationality theorem. Our lower bound for the house confirms a conjecture of Boyd.

1. INTRODUCTION AND MAIN RESULT

The subject of algebraic integers located near (or on) the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ is classical. Kronecker [8] proved that if an algebraic integer and all of its conjugates are located in the closed unit disk $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$, then it is either a root of unity or zero. For an algebraic integer $\alpha = \alpha_1$, with the complete set of conjugates $\{\alpha_k\}_{k=1}^n$, the house of this algebraic integer is defined by

$$\overline{|\alpha|} := \max_{1 \leq k \leq n} |\alpha_k|.$$

The result of Kronecker may also be written in the form: If α is a non-zero algebraic integer that is not a root of unity, then $\overline{|\alpha|} > 1$. Another form of the same result can be recorded by using the Mahler measure of α defined by

$$M(\alpha) := \prod_{k=1}^n \max(1, |\alpha_k|).$$

Thus if α is a non-zero algebraic integer that is not a root of unity, then $M(\alpha) > 1$. Both versions indicate that any algebraic integer that is not a root of unity must either be off the unit circle itself, or have conjugates off the unit circle. A natural question is how far away those algebraic integers should be from the unit circle. This brings us to celebrated Lehmer's conjecture [9], see also [1], [7], [17], [18] for more details and references. Lehmer observed from computations that the smallest Mahler measure of a non-zero and non-cyclotomic algebraic integer seems to be coming from the largest (in absolute value sense) root α_L of the polynomial $L(z) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$. It turns out that all but two roots of Lehmer's polynomial are on the unit circle, with the remaining roots being $\alpha_L > 1$

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and $1/\alpha_L$. This prompted Lehmer to conjecture that any non-zero algebraic integer α that is not a root of unity must satisfy

$$M(\alpha) \geq M(\alpha_L) = \alpha_L \approx 1.176280818.$$

Note that $L(z)$ is a reciprocal polynomial, i.e., it satisfies $L(z) = z^{10}L(1/z)$. An algebraic integer of degree n is called reciprocal if its minimal polynomial P is reciprocal, meaning $P(z) = z^n P(1/z)$. It is not difficult to see that all conjugates of a reciprocal algebraic integer are symmetric with respect to the unit circle. If α is a root of a non-reciprocal irreducible polynomial with integer coefficients, Smyth [16] proved that

$$M(\alpha) \geq \theta_0 \approx 1.3247,$$

where θ_0 is the positive root of $z^3 - z - 1$. Lehmer's conjecture was proved for many other classes of algebraic integers, but the case of general reciprocal α remains open. A related conjecture for the house of algebraic integer was made by Schinzel and Zassenhaus [15]: If α is a non-zero algebraic integer of degree n that is not a root of unity, then

$$(1.1) \quad \overline{|\alpha|} \geq 1 + c/n$$

for an absolute constant $c > 0$. Note that Lehmer's conjecture implies that of Schinzel and Zassenhaus by the inequality

$$\overline{|\alpha|} \geq M(\alpha)^{1/n} > 1 + \frac{\log M(\alpha)}{n}.$$

Dimitrov recently proved the conjecture of Schinzel and Zassenhaus by showing that

$$(1.2) \quad \overline{|\alpha|} \geq 2^{\frac{1}{4n}} > 1 + \frac{\log 2}{4n},$$

see Theorem 1 of [3]. However, questions on the optimal values of c in (1.1) for specific classes of algebraic integers remain open. The latter questions were raised by Boyd [2] on the bases of computations, see Conjectures (A)-(D) in his paper. In particular, Boyd conjectured that the optimal (largest possible) lower bounds for the house are all coming from non-reciprocal algebraic integers, in contrast to Lehmer's conjecture. Moreover, if n is divisible by 3, then Boyd conjectured that (1.1) holds with

$$(1.3) \quad c = \frac{3}{2} \log \theta_0 \approx 0.4217.$$

The best bounds for non-reciprocal algebraic integers are due to Dubickas [4] and [5], who showed that (1.1) holds with

$$c \approx 0.30965.$$

Dimitrov proved that for reciprocal algebraic integers with all conjugates off the unit circle (1.1) holds with

$$(1.4) \quad c = \frac{\log 2}{2} \approx 0.34657,$$

see Theorem 6 in [3]. We improve this result to $c \approx 0.44068$, and thereby confirm Boyd's conjectures in the special case when α is a non-zero reciprocal algebraic integer with all conjugates outside the unit circle.

Theorem 1. *If α is a reciprocal algebraic integer of degree n , with complete set of conjugates $\{\alpha_k\}_{k=1}^n \cap \mathbb{T} = \emptyset$, then*

$$(1.5) \quad \overline{|\alpha|} \geq (1 + \sqrt{2})^{\frac{1}{2n}} > 1 + \frac{\log(1 + \sqrt{2})}{2n}.$$

It is clear that the degree n must be even in the settings of the above theorem as one half of conjugates are located inside \mathbb{T} and the other half is outside \mathbb{T} due to symmetry.

We give an outline of proof of Theorem 1 in the next section. Some technical results from potential theory that are necessary for the proof are established in Section 3. A complete proof of the main result is contained in Section 4.

2. ESSENTIAL IDEAS OF THE PROOF

We first discuss a sketch of Dimitrov's proof for (1.2). As the non-reciprocal case is known (cf. [4]), we let α be a reciprocal algebraic integer with the complete set of conjugates $\{\alpha_k\}_{k=1}^n$, where $\alpha = \alpha_1$, and with the minimal polynomial

$$P(z) = \prod_{k=1}^n (z - \alpha_k).$$

Since P is reciprocal, we have that $P(0) = 1$. Define the auxiliary polynomials

$$P_2(z) = \prod_{k=1}^n (z - \alpha_k^2) \quad \text{and} \quad P_4(z) = \prod_{k=1}^n (z - \alpha_k^4),$$

and note that $P_2, P_4 \in \mathbb{Z}[z]$. The arithmetic information was captured by Dimitrov in the function

$$(2.1) \quad D(z) := \sqrt{P_2(z)P_4(z)}.$$

We state the following result, condensed from Proposition 2.2 and Lemma 2.3 of [3].

Proposition 2. *The Maclaurin series of $D(z)$ has integer coefficients. Assuming that P_2 is not a perfect square, we have that P is cyclotomic if and only if $D(z)$ is rational.*

Rationality of $D(z)$ is deduced from the well known results due to Pólya, see the original papers [10] and [11], and also [13] for further history and discussion. Consider the function $f(z) := D(1/z)$. It is clear that the Laurent series expansion of f near ∞ consists of powers of $1/z$, and has integer coefficients by Proposition 2. This function can be defined as analytic in $\overline{\mathbb{C}} \setminus K$, where $K := \bigcup_{1 \leq k \leq n} ([0, \alpha_k^2] \cup [0, \alpha_k^4])$, by introducing cuts from the origin to the zeros of P_2 and P_4 in order to define appropriate branches of complex square root. Pólya's theorem states that if the transfinite diameter of K is less than one, meaning that this set is sufficiently small, then f must be a rational function, hence D is so too, and hence P is cyclotomic. Complete discussions of transfinite diameter (identical to logarithmic capacity and Chebyshev constant) may be found in [12] and [20]. Using the simple fact that the transfinite diameter is increasing with the set (cf. Theorem 5.1.2 of [12, p. 128]), we enlarge K to \tilde{K} by extending each segment of K to the length $\overline{|\alpha|}^4$. A much deeper result of Dubinin, see Corollary 4.7 of [6, p. 118], states that the transfinite diameter of \tilde{K} is largest when all segments of \tilde{K} are equally spaced in the angular sense. The transfinite diameter of this equally spaced configuration of $2n$ segments is known to be $(\overline{|\alpha|}^{8n}/4)^{1/(2n)} = \overline{|\alpha|}^4/2^{1/n}$ by

Theorem 5.2.5 and Corollary 5.2.4 of [12, p. 134], where we applied the mapping z^{2n} that transforms \tilde{K} into a segment emanating from the origin of length $\sqrt[n]{\alpha}^{8n}$, whose transfinite diameter is $\sqrt[n]{\alpha}^{8n}/4$. It follows from the above argument that if $\sqrt[n]{\alpha}^{4}/2^{1/n} < 1$ then P is cyclotomic. Thus if P is not cyclotomic, then the opposite inequality holds, and we arrive at (1.2).

The proof of Theorem 1 follows a similar scheme. We also use Proposition 2 in the same fashion, but slightly modify the domain of $D(z)$. Due to symmetry of the set $\{\alpha_k\}_{k=1}^n$ with respect to the unit circle, we can assume that all conjugates come in symmetric pairs α_k and α_{n-k+1} , with $\alpha_k = 1/\bar{\alpha}_{n-k+1}$, for $k = 1, \dots, n/2$. We set

$$(2.2) \quad F(z) := \prod_{k=1}^{n/2} \sqrt{(z - \alpha_k^2)(z - \alpha_{n-k+1}^2)} \prod_{k=1}^{n/2} \sqrt{(z - \alpha_k^4)(z - \alpha_{n-k+1}^4)},$$

where $\sqrt{(z - \alpha_k^2)(z - \alpha_{n-k+1}^2)}$ is defined as holomorphic in $\mathbb{C} \setminus [\alpha_k^2, \alpha_{n-k+1}^2]$ by selecting a single valued branch of the root that is asymptotic to z at ∞ , and the same approach is applied to the second product of roots as well. Thus $F(z)$ is analytic in $\mathbb{C} \setminus E$, where

$$(2.3) \quad E := \bigcup_{1 \leq k \leq n/2} ([\alpha_k^2, \alpha_{n-k+1}^2] \cup [\alpha_k^4, \alpha_{n-k+1}^4]).$$

Observe that E is a proper subset of K , so that E has smaller transfinite diameter than that of K . Another advantage of this construction is that we can now use two (Laurent) series expansions of F : one about ∞ , as in Dimitrov's proof, and another one about the origin. In a neighborhood of the origin, we have $F(z) = D(z)$ so that the Maclaurin series of F has integer coefficients by Proposition 2. For the Laurent series at ∞ , we use the identity $F(z) = z^n F(1/z)$, $z \in \mathbb{C} \setminus E$, which is verified in Lemma 4, and conclude that this expansion also has integer coefficients. This enables us to use the following result of Robinson that enhances the rationality theorem of Pólya, see [13, p. 533].

Theorem 3. *Suppose that $F(z)$ is analytic in a domain G that contains both 0 and ∞ , and that $F(z)$ has Laurent expansions with integer coefficients of the form:*

$$F(z) = \sum_{k=0}^{\infty} a_k z^{-k} \text{ near } \infty \quad \text{and} \quad F(z) = \sum_{k=0}^{\infty} b_k z^k \text{ near } 0.$$

If there is a Laurent-type rational function with complex coefficients of the form

$$h(z) = \sum_{k=-l}^m A_k z^k, \quad \text{with } |A_m| \geq 1 \text{ and } |A_{-l}| \geq 1,$$

such that $|h(z)| < 1$ for $z \in E := \mathbb{C} \setminus G$, then $F(z)$ is rational.

It is clear from (2.2) that rationality of F implies rationality of D , hence implies in turn that P is cyclotomic by Proposition 2 as before. Robinson's theorem essentially replaces the transfinite diameter of E used in Pólya's theorem, which is equal to the Chebyshev constant of E defined via monic polynomials with the least supremum norms on E (cf. [12] and [20]), by a smaller quantity defined via the supremum norms of Laurent-type rational functions

$h(z)$. We may assume that $h(z)$ has equal number of positive and negative powers due to symmetry of our set E defined in (2.3) with respect to \mathbb{T} , and write

$$|h(z)| = \left| \sum_{k=-m}^m A_k z^k \right| = |z|^{-m} \left| \sum_{k=0}^{2m} A_{k-m} z^k \right| = w(z)^{2m} |Q_{2m}(z)|.$$

Here, we introduce the weight function $w(z) := |z|^{-1/2}$ and consider weighted polynomials of the form $w^{2m}Q_{2m}$, where $\deg(Q_{2m}) = 2m$. Such weighted polynomials were studied in detail by the methods of potential theory in [14], see Chapter III in particular. The relevant quantity we need in this paper is the weighted Chebyshev constant of E [14, p. 163] defined by

$$(2.4) \quad t_w := \lim_{m \rightarrow \infty} (\inf\{\|w^m Q_m\|_E : Q_m \in \mathbb{C}[z] \text{ is monic, } \deg Q_m = m\})^{1/m},$$

where $\|\cdot\|_E$ denotes the standard supremum norm on E , see Chapter III of [14] for a complete exposition. Note that if $w \equiv 1$ on E , then t_w reduces to the regular Chebyshev constant of E that is equal to the transfinite diameter or capacity of E . In the context of our weight $w(z) := |z|^{-1/2}$, $z \in E$, the main application to our problems is that $t_w < 1$ implies existence of weighted polynomials $w^m Q_m$ with geometrically small supremum norms on E , hence existence of rational functions $h(z)$ of the form required in Theorem 3. However, completing the proof of our main result in Theorem 1 via this approach needs a detailed and somewhat technical study by using weighted potential theory (or potential theory with external fields), which is carried out in the next section.

3. TECHNICAL INGREDIENTS

This section contains various auxiliary statements necessary to justify all steps of our argument, and complete a proof of Theorem 1. The first lemma provides more details on F and its Laurent series used in Theorem 3.

Lemma 4. *Let F be as defined in (2.2). Then F is analytic in $G := \mathbb{C} \setminus E$, and satisfies $F(z) = z^n F(1/z)$, $z \in G$. Moreover, $F(z)$ has Laurent expansions with integer coefficients of the form:*

$$F(z) = \sum_{k=0}^{\infty} b_k z^k \text{ near } 0 \quad \text{and} \quad F(z) = \sum_{k=0}^{\infty} b_k z^{n-k} \text{ near } \infty.$$

Proof. The definition of F and its analyticity in $\mathbb{C} \setminus E$ was discussed in the previous section. Recall that $P(z) = \prod_{k=1}^n (z - \alpha_k)$ is reciprocal, which implies that $P(0) = (-1)^n \prod_{k=1}^n \alpha_k = 1$. Hence $\prod_{k=1}^n \alpha_k^2 = \prod_{k=1}^n \alpha_k^4 = 1$. It also follows that the complete set of conjugates $\{\alpha_k\}_{k=1}^n$ is invariant under complex conjugation and inversion in \mathbb{T} given by the map $z \rightarrow 1/\bar{z}$, so that this set is invariant under the map $z \rightarrow 1/z$. This further entails similar symmetry properties of E as defined in (2.3), under our convention of symmetric pairing for α_k and α_{n-k+1} by setting $\alpha_k = 1/\bar{\alpha}_{n-k+1}$, for $k = 1, \dots, n/2$. Thus if $z \in G$ then $1/z \in G$. For z in

a neighborhood of the origin, we obtain from the definition of F that

$$\begin{aligned} z^n F(1/z) &= z^n \sqrt{\prod_{k=1}^n (1/z - \alpha_k^2)(1/z - \alpha_k^4)} = z^n \sqrt{z^{-2n} \prod_{k=1}^n \alpha_k^2 \prod_{k=1}^n \alpha_k^4 \prod_{k=1}^n (\alpha_k^{-2} - z)(\alpha_k^{-4} - z)} \\ &= z^n z^{-n} \sqrt{\prod_{k=1}^n (z - \alpha_k^2)(z - \alpha_k^4)} = F(z), \end{aligned}$$

where we also used our choice of branch for the square root. Since both $z^n F(1/z)$ and $F(z)$ are analytic in G and coincide on an open set, they are identical for all $z \in G$. The fact that F has Maclaurin expansion with integer coefficients near the origin is immediate from Proposition 2. Then the stated Laurent expansion near infinity follows from the latter Maclaurin expansion by the formula $F(z) = z^n F(1/z)$ for z near infinity. \square

As explained in the end of Section 2, we need to develop a detailed study of the weighted Chebyshev constant t_w for $w(z) := |z|^{-1/2}$, $z \in E$. In the remaining part of this section, we obtain an explicit form of t_w in terms of standard logarithmic potential theory, which allows to find a sharp estimate expressed through the house of α . We use many facts and ideas from potential theory in the complex plane below, and refer to [12] and [20] for the classical version, as well as to [14] for the weighted version. For a positive unit Borel measure μ with compact support, define its logarithmic potential by

$$U^\mu(z) := - \int \log |z - t| d\mu(t).$$

The weighted equilibrium measure μ_w is a unique probability measure supported on E that expresses a steady state distribution of the unit charge in presence of the external field $Q(z) = -\log w(z)$, where we assume here that w is a positive continuous function defined on E . The equilibrium is described by the following equations for the combined potential of μ_w and the external field Q :

$$(3.1) \quad U^{\mu_w}(z) + Q(z) \geq F_w, \quad z \in E,$$

and

$$(3.2) \quad U^{\mu_w}(z) + Q(z) = F_w, \quad z \in \text{supp } \mu_w,$$

where F_w is a constant (see Theorems I.1.3 and I.5.1 in [14]). These equations are of importance for us because the weighted Chebyshev constant is given by

$$(3.3) \quad t_w = e^{-F_w}.$$

Note that for $w \equiv 1$ on E we have $Q \equiv 0$, so that μ_w reduces to the classical (not weighted) equilibrium measure μ_E , and F_w reduces to the classical Robin's constant V_E for E . Since logarithmic capacity (and the transfinite diameter) of E is given by $\text{cap}(E) = e^{-V_E}$, the connection of t_w with these classical notions is apparent.

Lemma 5. *Let $E \subset \mathbb{C}$ be a compact set with no interior such that $G = \overline{\mathbb{C}} \setminus E$ is connected and $0 \in G$. If $w(z) := |z|^{-1/2}$, $z \in E$, then*

$$(3.4) \quad t_w = e^{-g_G(0, \infty)/2} \sqrt{\text{cap}(E)}.$$

where $g_G(t, \infty)$ is the Green function of G with logarithmic pole at ∞ .

Proof. Equations (3.1) and (3.2) characterize μ_w in the sense that if for a positive unit Borel measure μ supported on E one has

$$(3.5) \quad U^\mu(z) + \frac{1}{2} \log |z| \geq C, \quad z \in E,$$

and

$$(3.6) \quad U^\mu(z) + \frac{1}{2} \log |z| = C, \quad z \in S \subset E,$$

where C is a constant, then Theorem 3.3 of [14, Ch. I]) implies that $\mu_w = \mu$ and $F_w = C$. We show that the equilibrium measure is given by

$$(3.7) \quad \mu := \frac{\omega_G(\infty, \cdot) + \omega_G(0, \cdot)}{2},$$

where $\omega_G(\xi, \cdot)$ is the harmonic measure at $\xi \in G$, relative to G . In fact, we verify that (3.6) holds for all $z \in E$. The harmonic measure $\omega_G(0, \cdot)$ is the balayage of the point mass δ_0 from the domain G onto its boundary $\partial G = E$, see Section II.4 of [14]. It follows from Theorem 4.4 of [14, p. 115] that the potential of $\omega_G(0, \cdot)$ satisfies

$$(3.8) \quad U^{\omega_G(0, \cdot)}(z) + \log |z| = U^{\omega_G(0, \cdot)}(z) - U^{\delta_0}(z) = \int g_G(t, \infty) d\delta_0(t) = g_G(0, \infty), \quad z \in E.$$

We recall from Frostman's Theorem that the potential of $\omega_G(\infty, \cdot) = \mu_E$, which is the standard equilibrium measure of E , is equal to Robin's constant on E :

$$(3.9) \quad U^{\omega_G(\infty, \cdot)}(z) = V_E, \quad z \in E,$$

see, e.g., [12, p. 59]. Combining (3.8) with (3.9), we obtain that

$$\begin{aligned} U^\mu(z) + \frac{1}{2} \log |z| &= \frac{1}{2} (U^{\omega_G(0, \cdot)}(z) + \log |z|) + \frac{1}{2} U^{\omega_G(\infty, \cdot)}(z) \\ &= \frac{1}{2} g_G(0, \infty) + \frac{1}{2} V_E, \quad z \in E, \end{aligned}$$

and so conclude that (3.6) is satisfied with the constant given by

$$(3.10) \quad F_w = \frac{V_E + g_G(0, \infty)}{2}.$$

Hence (3.4) follows from (3.10) and (3.3). □

In the subsequent analysis, we use the notation $t_w(S)$ for the weighted Chebyshev constant of a compact set S with respect to the weight $w(z) := |z|^{-1/2}$, $z \in S$. Our next goal is to find a convenient upper bound for t_w via symmetrization.

Lemma 6. *In the settings of Theorem 1, let E be as defined in (2.3), and let*

$$(3.11) \quad E^* := \bigcup_{1 \leq k \leq n} (e^{2\pi k i/n} [\overline{|\alpha|}^{-4}, \overline{|\alpha|}^4]).$$

For the weight $w(z) := |z|^{-1/2}$, we have

$$(3.12) \quad t_w(E) \leq t_w(E^*).$$

Proof. It is clear from the definition (2.3) that E is contained in the closed annulus $\{z \in \mathbb{C} : \lceil \alpha \rceil^{-4} \leq |z| \leq \lceil \alpha \rceil^4\}$. The first step is to construct the set \tilde{E} by extending all radial segments of E so that they connect the circles $|z| = \lceil \alpha \rceil^{-4}$ and $|z| = \lceil \alpha \rceil^4$. Theorem 5.1.2 of [12, p. 128] gives that $\text{cap}(E) \leq \text{cap}(\tilde{E})$ because $E \subset \tilde{E}$. Also, Corollary 4.4.5 of [12, p. 108] gives that $g_G(0, \infty) \geq g_{\tilde{G}}(0, \infty)$ because $\tilde{G} := \overline{\mathbb{C}} \setminus \tilde{E} \subset G := \overline{\mathbb{C}} \setminus E$. It follows from (3.4) that

$$(3.13) \quad t_w(E) \leq t_w(\tilde{E}).$$

Alternatively, one can observe the above inequality directly from the definition of t_w in (2.4), as the supremum norms in that definition increase with the set.

On the second step, we show that t_w increases if the radial segments of the set \tilde{E} become equally spaced in angular sense, matching the property of capacity (transfinite diameter) proved by Dubinin, see Corollary 4.7 of [6, p. 118] and the sketch of proof for Dimitrov's result in Section 2. In fact, Dubinin's result states that

$$(3.14) \quad \text{cap}(\tilde{E}) \leq \text{cap}(E^*).$$

In view of (3.4), it remains to prove that $g_{\tilde{G}}(0, \infty) \geq g_{G^*}(0, \infty)$, where $G^* := \overline{\mathbb{C}} \setminus E^*$. We deduce this fact from Theorem A of Solynin [19, p. 1702] that states a corresponding result for harmonic measures:

$$(3.15) \quad \omega_{\tilde{D}_R}(0, \tilde{E}) \leq \omega_{D_R^*}(0, E^*),$$

where $\omega_{\tilde{D}_R}(0, \tilde{E})$ is the harmonic measure of \tilde{E} at 0, relative to $\tilde{D}_R := \{z \in \mathbb{C} : |z| < R\} \setminus \tilde{E}$, and $\omega_{D_R^*}(0, E^*)$ is the harmonic measure of E^* at 0, relative to $D_R^* := \{z \in \mathbb{C} : |z| < R\} \setminus E^*$, for sufficiently large R . We note that the result of Solynin in [19] is stated for the unit disk, i.e., for $R = 1$, but (3.15) immediately follows by using the scaling map $z \rightarrow Rz$. Recall that $\omega_{\tilde{D}_R}(z, \tilde{E})$ is a harmonic function of z in \tilde{D}_R , which is continuous on the closure of this domain, and has boundary values $\omega_{\tilde{D}_R}(z, \tilde{E}) = 1$, $z \in \tilde{E}$, and $\omega_{\tilde{D}_R}(z, \tilde{E}) = 0$, $|z| = R$. Hence the function

$$(3.16) \quad \tilde{u}_R(z) := (1 - \omega_{\tilde{D}_R}(z, \tilde{E}))(\log R + V_{\tilde{E}})$$

is harmonic in \tilde{D}_R , continuous on the closure of this domain, and has boundary values $\tilde{u}_R(z) = 0$, $z \in \tilde{E}$, and $\tilde{u}_R(z) = \log R + V_{\tilde{E}}$, $|z| = R$, where $V_{\tilde{E}}$ denotes the Robin's constant of \tilde{E} . Since the Green function $g_{\tilde{G}}(z, \infty)$ has similar properties of being harmonic in $\tilde{G} \setminus \{\infty\}$, with boundary value 0 on \tilde{E} and the asymptotic $g_{\tilde{G}}(z, \infty) = \log R + V_{\tilde{E}} + o(1)$ as $|z| = R \rightarrow \infty$, we obtain from the Maximum-Minimum Principle that

$$|\tilde{u}_R(z) - g_{\tilde{G}}(z, \infty)| \leq o(1), \quad z \in \tilde{D}_R, \quad R \rightarrow \infty.$$

Setting $R = N \in \mathbb{N}$, we produce a sequence of harmonic functions $\tilde{u}_N(z)$ that converges to $g_{\tilde{G}}(z, \infty)$ uniformly on compact subsets of \mathbb{C} as $N \rightarrow \infty$. The same construction yields the sequence of harmonic functions

$$u_N^*(z) := (1 - \omega_{D_R^*}(z, E^*))(\log N + V_{E^*})$$

that converges to $g_{G^*}(z, \infty)$ uniformly on compact subsets of \mathbb{C} . Applying (3.15) and the equivalent of (3.14) written as $V_{\tilde{E}} \geq V_{E^*}$, we conclude that

$$\tilde{u}_N(0) \geq u_N^*(0) \quad \text{for large } N \in \mathbb{N}.$$

Passing to the limit as $N \rightarrow \infty$, we arrive at $g_{\tilde{G}}(0, \infty) \geq g_{G^*}(0, \infty)$, which implies that

$$t_w(\tilde{E}) \leq t_w(E^*)$$

by (3.14) and (3.4). Thus (3.12) follows from the latter inequality and (3.13). \square

We are now ready to find an explicit bound for t_w expressed via the house of α .

Lemma 7. *In the settings of Theorem 1, let E^* be as defined in (3.11) and let $I := [\overline{|\alpha|}^{-4n}, \overline{|\alpha|}^{4n}] \subset \mathbb{R}$. For the weight $w(z) := |z|^{-1/2}$, we have*

$$(3.17) \quad t_w(E^*) \leq (t_w(I))^{1/n},$$

where

$$(3.18) \quad t_w(I) = \frac{|\overline{|\alpha|}^{2n} - \overline{|\alpha|}^{-2n}}{2}.$$

Proof. We use the definition of $t_w(I)$ given in (2.4), first replacing E with I in that equation. Let Q_m be a sequence of monic polynomials that realize the inf in (2.4) for $t_w(I)$. For $z \in E^*$, we have that $t = z^n \in I$ and

$$(t_w(I))^{1/n} = \lim_{m \rightarrow \infty} \left\| |t|^{-m/2} Q_m(t) \right\|_I^{\frac{1}{mn}} \geq \liminf_{m \rightarrow \infty} \left\| |z|^{-mn/2} Q_m(z^n) \right\|_{E^*}^{\frac{1}{mn}} \geq t_w(E^*).$$

In fact, equality holds in (3.17), but the stated inequality is sufficient for our purpose. We now compute $t_w(I)$ from (3.4). Corollary 5.2.4 of [12, p. 134] immediately gives that

$$(3.19) \quad \text{cap}(I) = (|\overline{|\alpha|}^{4n} - \overline{|\alpha|}^{-4n}) / 4.$$

We now need to find the value of $g_\Omega(0, \infty)$, where $\Omega := \mathbb{C} \setminus I$. This is conveniently available from the well known connection with the conformal mapping $\Phi : \Omega \rightarrow \{w : |w| > 1\}$ normalized by $\Phi(\infty) = \infty$:

$$g_\Omega(z, \infty) = \log |\Phi(z)|, \quad z \in \Omega,$$

see, e.g., the proof of Theorem 4.4.1 in [12, p. 113]. If $I = [a, b] \subset \mathbb{R}$ then Φ is given explicitly by the (shifted and scaled) inverse of the Joukowski conformal mapping:

$$\Phi_\infty(z) := \frac{2z - a - b + 2\sqrt{(z-a)(z-b)}}{b-a}, \quad z \in \Omega.$$

Applying this with $a = \overline{|\alpha|}^{-4n}$ and $b = \overline{|\alpha|}^{4n}$, we obtain that

$$e^{-g_\Omega(0, \infty)} = 1/|\Phi(0)| = \frac{|\overline{|\alpha|}^{4n} - \overline{|\alpha|}^{-4n}}{|\overline{|\alpha|}^{4n} + \overline{|\alpha|}^{-4n} + 2}.$$

The latter formula together with (3.19) and (3.4) yield

$$t_w(I) = \frac{|\overline{|\alpha|}^{4n} - \overline{|\alpha|}^{-4n}}{2(|\overline{|\alpha|}^{2n} + \overline{|\alpha|}^{-2n})} = \frac{|\overline{|\alpha|}^{2n} - \overline{|\alpha|}^{-2n}}{2}.$$

\square

Our last lemma gives information about constant terms of polynomials with asymptotically minimal weighted norms, which is necessary for the application of Theorem 3.

Lemma 8. *Let E be as defined in (2.3), and let $w(z) = |z|^{-1/2}$, $z \in E$. Then there is a sequence of monic polynomials Q_m , $\deg(Q_m) = m$, that satisfies*

$$(3.20) \quad t_w(E) = \lim_{m \rightarrow \infty} \| |z|^{-m/2} Q_m(z) \|_E^{1/m}$$

and

$$(3.21) \quad \lim_{m \rightarrow \infty} |Q_m(0)|^{1/m} = 1.$$

Proof. The required sequence of polynomials is selected as the weighted Fekete polynomials introduced and studied in Section III.1 of [14]. In particular, Theorem 1.9 of [14, p. 150] states that the weighted Fekete polynomials Q_m associated with the weight $w(z) = |z|^{-1/2}$, $z \in E$, satisfy (3.20), and Theorem 1.8 of [14, p. 150] states that

$$\lim_{m \rightarrow \infty} |Q_m(z)|^{1/m} = \exp(-U^{\mu_w}(z)), \quad z \in G,$$

where μ_w is the weighted equilibrium measure found in the proof of Lemma 5, and explicitly given in (3.7). Hence (3.21) is equivalent to $U^{\mu_w}(0) = 0$, which further translates into

$$(3.22) \quad U^{\omega_G(\infty, \cdot)}(0) + U^{\omega_G(0, \cdot)}(0) = 0.$$

For the classical equilibrium (conductor) potential $U^{\omega_G(\infty, \cdot)}$, we have

$$(3.23) \quad U^{\omega_G(\infty, \cdot)}(z) = V_E - g_G(z, \infty), \quad z \in G,$$

by Theorem III.37 in [20, p. 82]. We also use that $\omega_G(0, \cdot)$ is the balayage of δ_0 from G , as in the proof of Lemma 5. It follows from Theorem 5.1 of [14, p. 124] that

$$U^{\delta_0}(z) - \int g_G(z, t) d\delta_0(t) = U^{\omega_G(0, \cdot)}(z) - \int g_G(t, \infty) d\delta_0(t), \quad z \in G,$$

consequently,

$$(3.24) \quad U^{\omega_G(0, \cdot)}(z) = g_G(0, \infty) - g_G(z, 0) - \log |z|, \quad z \in G.$$

Since G is invariant under the transformation $w = 1/z$, we also have the connection [12, p. 108]

$$g_G(z, 0) = g_G(1/z, \infty), \quad z \in G.$$

Combining (3.23) and (3.24) with the latter fact, we verify (3.22) as follows:

$$\begin{aligned} U^{\omega_G(\infty, \cdot)}(0) + U^{\omega_G(0, \cdot)}(0) &= V_E - g_G(0, \infty) + g_G(0, \infty) - \lim_{z \rightarrow 0} (g_G(z, 0) + \log |z|) \\ &= V_E - \lim_{w \rightarrow \infty} (g_G(w, \infty) - \log |w|) = V_E - V_E = 0, \end{aligned}$$

where we used (3.23) and Theorem 3.1.2 of [12, p. 53] to compute the last limit. \square

4. PROOF OF THE MAIN RESULT

Proof of Theorem 1. The lower bound for the house of α in (1.5) follows from the inequality

$$(4.1) \quad \frac{|\alpha|^{2n} - |\alpha|^{-2n}}{2} \geq 1,$$

which turns into the quadratic inequality $x^2 - 2x - 1 \geq 0$ after the substitution $x = \sqrt{\alpha}^{2n}$. We show that (4.1) holds by contradiction, following the argument sketched in Section 2. Indeed, if we assume that the opposite of (4.1) holds, then Lemmas 6 and 7 give that

$$t_w(E) \leq t_w(E^*) \leq (t_w(I))^{1/n} = \left(\frac{|\alpha|^{2n} - |\alpha|^{-2n}}{2} \right)^{1/n} < 1.$$

Applying Lemma 8, we find a sequence of monic polynomials Q_m , $\deg(Q_m) = m$, that satisfies

$$\lim_{m \rightarrow \infty} \| |z|^{-m/2} Q_m(z) \|_E^{1/m} < 1$$

and (3.21). For $m = 2k$, $k \in \mathbb{N}$, the weighted polynomials $z^{-m/2} Q_m(z)$ take the following form:

$$h_k(z) := z^{-k} Q_{2k}(z) = A_{-k,k} z^{-k} + \sum_{j=-k+1}^{k-1} A_{j,k} z^j + z^k,$$

with $|h_k(z)| < 1$, $z \in E$, for all large $k \in \mathbb{N}$. If $|A_{-k,k}| \geq 1$ for some sufficiently large k , then the corresponding h_k can be used directly as $h(z)$ in Theorem 3. However, it may happen that $|A_{-k,k}| < 1$ for all large $k \in \mathbb{N}$, in which case we consider

$$g_k(z) := h_k(z)/A_{-k,k} = z^{-k} + \sum_{j=-k+1}^{k-1} A_{j,k} z^j / A_{-k,k} + z^k / A_{-k,k}, \quad k \in \mathbb{N} \text{ is large,}$$

that satisfies the requirements of Theorem 3 for $h(z)$ because $A_{-k,k} = Q_{2k}(0)$, and (3.21) gives

$$\lim_{k \rightarrow \infty} \|g_k(z)\|_E^{1/k} = \lim_{k \rightarrow \infty} \|z^{-k} Q_{2k}(z)/A_{-k,k}\|_E^{1/k} = \lim_{k \rightarrow \infty} \|z^{-k} Q_{2k}(z)\|_E^{1/k} < 1.$$

We would like to apply Theorem 3 to our function $F(z)$ defined in (2.2), which has Laurent expansions with integer coefficients at ∞ and at 0 according to Lemma 4. Note however that near ∞

$$F(z) = \sum_{k=0}^{\infty} b_k z^{n-k},$$

which formally does not fit the assumptions on this expansion in Theorem 3, as it contains n positive powers of z . The latter problem is easily remedied by subtracting the polynomial part of this expansion from F , and instead applying Theorem 3 to

$$F(z) - \sum_{k=0}^n b_k z^{n-k}.$$

The conclusion of Theorem 3 is that the above function is rational, hence F is rational too. But Proposition 2 implies now that α is a root of unity, which is an obvious contradiction to our original assumptions of Theorem 1, provided that P_2 is not a perfect square.

We now use induction to handle the remaining case when P_2 is a perfect square. The basis is easily established by considering quadratic reciprocal polynomials $P(z) = z^2 + bz + 1$, $b \in \mathbb{Z}$, where $|b| \geq 3$ because P cannot have roots on \mathbb{T} . For a root α of such P , we clearly have

$$|\alpha| = \frac{|b| + \sqrt{b^2 - 4}}{2} \geq \frac{3 + \sqrt{5}}{2} > (1 + \sqrt{2})^{1/4},$$

confirming (1.5) for $n = 2$. Assume now that (1.5) holds for all α of degree less than n , satisfying the assumptions of this theorem. Let $P_2(z) = (R(z))^2$, where $R \in \mathbb{Z}[z]$ is a monic polynomial with roots from the set $\{\alpha_k^2\}_{k=1}^n$, which is symmetric with respect to \mathbb{T} . As P_2 has double roots according to our assumption, the set $\{\alpha_k^2\}_{k=1}^n$ is composed of pairs $\alpha_j^2 = \alpha_k^2$, $j \neq k$. Hence we can assume that $\alpha_{n/2+k} = -\alpha_k$, $k = 1, \dots, n/2$, after a proper rearrangement. Thus $R(z) = \prod_{k=1}^{n/2} (z - \alpha_k^2)$, and it inherits the reciprocal property from P_2 . Since $P(z) = \prod_{k=1}^n (z - \alpha_k) = \prod_{k=1}^{n/2} (z^2 - \alpha_k^2) = R(z^2)$, where P is irreducible, we conclude that R is also irreducible. Letting $\overline{|\alpha|} = |\alpha_M|$, $1 \leq M \leq n/2$, we obtain that R is the minimal polynomial of α_M^2 of degree $n/2$. The induction hypothesis implies that

$$|\alpha_M|^2 = \overline{|\alpha_M^2|} \geq (1 + \sqrt{2})^{\frac{1}{n}},$$

which gives

$$\overline{|\alpha|} = |\alpha_M| \geq (1 + \sqrt{2})^{\frac{1}{2n}}$$

as required. □

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