# ATTACHING BOUNDARY PLANES TO IRREDUCIBLE OPEN 3-MANIFOLDS 

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#### Abstract

Given any connected, open 3-manifold $U$ having finitely many ends, a non-compact 3-manifold $M$ is constructed having the following properties: the interior of $M$ is homeomorphic to $U$; the boundary of $M$ is the disjoint union of finitely many planes; $M$ is not almost compact; $M$ is eventually end-irreducible; there are no proper, incompressible embeddings of $S^{1} \times \mathbf{R}$ in $M$; every compact subset of $M$ is contained in a larger compact subset whose complement is anannular; there is a compact subset of $M$ whose complement is $\mathbf{P}^{2}$-irreducible.

If $U$ is irreducible it also has the following two properties: every proper, non-trivial plane in $M$ is boundary-parallel; every proper surface in $M$ each component of which has non-empty boundary and is non-compact and simply connected lies in a collar on $\partial M$.

This construction can be chosen so that $M$ admits no homeomorphisms which take one boundary plane to another or reverse orientation. For the given $U$ there are uncountably many non-homeomorphic such $M$.

Two auxiliary results may be of independent interest. First, general conditions are given under which infinitely many "trivial" compact components of the intersection of two proper, non-compact surfaces in an irreducible 3-manifold can be removed by an ambient isotopy. Second, $n$ component tangles in a 3 -ball are constructed such that every non-empty union of components of the tangle has hyperbolic exterior.


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## Introduction

Suppose $U$ is an open 3 -manifold and $M$ is a non-compact 3-manifold each of whose boundary components is homeomorphic to $\mathbf{R}^{2}$. If $U$ is homeomorphic to the interior of $M$, then one may say that $M$ is obtained by attaching boundary planes to $U$. The fact that this can be done in different ways is most dramatically illustrated by the existence of 3-manifolds with interior homeomorphic to $\mathbf{R}^{3}$ and boundary homeomorphic to $\mathbf{R}^{2}$ which are not homeomorphic to $\mathbf{R}^{2} \times[0, \infty)$. The first such example was constructed by Fox and Artin [9]. Tucker [18] later gave a different method for constructing such examples and showed that planes were the only surfaces which could be, in a sense explained below, "bad" boundary components of 3-manifolds.

A non-compact 3 -manifold $M$ is almost compact if there is a compact 3manifold $Q$ and a closed subset $K$ of $\partial Q$ such that $M$ is homeomorphic to $Q-K$. The examples mentioned above fail to be almost compact. Tucker [18] showed that if one attaches a connected surface with finitely generated fundamental group to a $\mathbf{P}^{2}$-irreducible, almost compact 3-manifold in such a way that it becomes an incompressible boundary component, then either the new 3-manifold is almost compact or the surface is a plane. Scott and Tucker [17] gave other examples of 3-manifolds which are not almost compact but have almost compact interiors, including an example with boundary consisting of two disjoint planes such that the complement of each plane is homeomorphic to $\mathbf{R}^{2} \times[0,1)$ and an example with boundary a single plane whose complement is homeomorphic to $S^{1} \times \mathbf{R}^{2}$. It can be shown that the latter example contains no proper non-separating planes.

This paper gives a general procedure for attaching boundary planes to $U$ to obtain an $M$ having special properties which may be very different from those of $U$. The open 3-manifold $U$ is required to be connected and irreducible and to have finitely many ends. It is not assumed to be orientable or $\mathbf{P}^{2}$-irreducible. One attaches any finite number of boundary planes to $U$ subject to the restriction that one attaches at least one plane to each end. The resulting 3-manifold $M$ is not almost compact. However, it has an important property which is shared by those almost compact 3 -manifolds whose boundaries are finite disjoint unions of planes, namely eventual end-irreducibility. This property, introduced by Brown [8], ensures that the ends of the 3 -manifold, while perhaps not tame, are at least not excessively wild, in the sense that they can be analyzed using incompressible surface theory. This has proven to be a fruitful concept in the study of non-compact 3-manifolds. See $[2,3,4,6,7,8]$. The next section includes discussions of ends and of eventual end-irreducibility.

The most important special properties considered in this paper concern certain embeddings of surfaces. A surface $S$ embedded in a 3 -manifold $M$ is proper if $S \cap \partial M=\partial S$ and $S \cap C$ is compact for every compact subset $C$ of $M$. A proper surface $S$ with $\partial S=\emptyset$ is $\partial$-parallel if some component of $M-S$ has closure homeomorphic to $S \times[0,1]$ with $S=S \times\{0\}$ and $S \times\{1\}$ a component of $\partial M$; it is end-parallel or trivial if some component of $M-S$ has closure homeomorphic to $S \times[0, \infty)$ with $S=S \times\{0\}$. A 3-manifold $M$ is aplanar if every proper
plane in $M$ is either $\partial$-parallel or end-parallel; it is acylindrical if the same is true for every proper incompressible cylinder $S^{1} \times \mathbf{R}$. It is totally acylindrical if it contains no proper incompressible cylinders. The boundary planes will be attached so that $M$ is aplanar and totally acylindrical. In particular one can take an irreducible one ended open 3-manifold which contains non-trivial planes or nontrivial incompressible cylinders, such as the interior of a cube with handles or $S^{3}-K$ for $K$ a torus knot, cable knot, or composite knot, and create an aplanar, totally acylindrical 3 -manifold by attaching a single boundary plane.

It will be shown that $M$ has two further embedding properties. It is strongly aplanar, meaning that in addition to being aplanar it has the property that given any proper surface $\mathcal{P}$ such that each component of $\mathcal{P}$ is non-compact, is simply connected, and has non-empty boundary, there exists a collar on $\partial M$ containing $\mathcal{P}$. It is also anannular at infinity in the sense that for every compact subset $K$ of $M$ there is a compact subset $L$ of $M$ containing $K$ such that $M-L$ is anannular. Note for comparison that $\mathbf{R}^{2} \times[0,1)$ is aplanar but not strongly aplanar, while $\mathbf{R}^{2} \times[0,1]$ is aplanar but neither strongly aplanar nor anannular at infinity. (See [15].) These two properties are involved in the study of "plane sums" of noncompact 3 -manifolds, i.e. 3 -manifolds obtained by gluing together a collection of 3 -manifolds whose boundary components are planes along these planes. In [15] it will be shown that (subject to some mild additional hypotheses) if the summands are irreducible, strongly aplanar, and anannular at infinity, then the image of each gluing plane is non-trivial in the sum, and every non-trivial plane in the sum is ambient isotopic to one of these planes. This result is then used to investigate a non-compact analogue of the connected sum called the "end sum." The present paper provides the mechanism for generating the relevant examples.

By a modification of the basic construction $M$ can be built so that, in addition to the previous properties, it admits no orientation reversing self-homeomorphisms, there are no self-homeomorphisms taking one boundary plane to another, and there are uncountably many pairwise non-homeomorphic such $M$ having the same number of boundary planes per end of $U$. Moreover all these properties hold for the 3manifolds obtained by deleting any collection of boundary planes from $M$ subject to the restriction that there remains at least one boundary plane attached to each end. These properties are also relevant to the study of plane sums and end sums.

Although irreducible 3-manifolds are the main objects of interest in this paper it is worth noting that some of our results generalize to the reducible case. In fact the only properties of $M$ which require the irreducibility of $U$ are aplanarity and strong aplanarity. Moreover $M$ can be constructed so as to be eventually $\mathbf{P}^{2}$ irreducible in the sense that there is a compact subset of $M$ whose complement is $\mathbf{P}^{2}$-irreducible. One can, for example, create a totally acylindrical, eventually $\mathbf{P}^{2}$-irreducible 3-manifold by attaching two boundary planes to the product of a closed, connected surface with $\mathbf{R}$, including the cases when the surface is $S^{2}$ or $\mathbf{P}^{2}$.

Two auxiliary results in this paper may be of independent interest. First, we give general conditions under which infinite collections of "trivial" intersection curves of two non-compact proper surfaces in an irreducible 3-manifold can be removed by an ambient isotopy. By a "trivial" intersection curve we mean a simple closed curve
which bounds disks on both surfaces or a proper arc which is $\partial$-parallel on both surfaces. These results are used both in the present paper and in [15]. Second, we prove the existence of "poly-excellent tangles." A poly-excellent $n$-tangle is the union of $n$ disjoint proper arcs in a 3 -ball such that the union of any non-empty collection of its components has hyperbolic exterior. This result is required in our modification of the basic construction.

The paper is organized as follows. Section 1 contains background material and discusses exhaustions of non-compact 3-manifolds. In particular it introduces the concept of a "nice" exhaustion. It is readily seen that a 3 -manifold $M$ with a nice exhaustion is not almost compact but is eventually end-irreducible, eventually $\mathbf{P}^{2}$-irreducible, and anannular at infinity. Section 2 gives conditions under which "trivial" intersections of non-compact surfaces can be removed by an ambient isotopy. Section 3 reformulates some work of Winters [19, 20] to show that a 3-manifold with a nice exhaustion is totally acylindrical and, if it is irreducible, is aplanar. Sections 4 and 5 show that an irreducible 3 -manifold with a nice exhaustion is strongly aplanar. Section 6 shows how to construct $M$ from $U$ so that $M$ has a nice exhaustion; it also describes the modification of this basic construction and proves the additional properties listed above. The proof of the existence of poly-excellent tangles has a different flavor from the rest of the paper and is given in an appendix so as not to disrupt the main line of the argument.

## 1. Preliminaries

We shall work throughout in the PL category. An m-manifold $M$ may or may not have boundary but is assumed to be second countable. $\partial M$ and int $M$ denote the manifold theoretic boundary and interior of $M$, respectively. Let $A$ be a subset of $M$. The topological boundary, interior, and closure of $A$ in $M$ are denoted by $F r_{M} A, I n t_{M} A$, and $C l_{M} A$, respectively, with the subscript deleted when $M$ is clear from the context. All isotopies of $A$ in $M$ will be ambient. $A$ is bounded if $C l A$ is compact. $M$ is open if $\partial M=\emptyset$ and no component of $M$ is compact.

A surface is a 2-manifold; no assumptions are made about its being connected or compact or having a boundary.

A map $f: M \rightarrow N$ of manifolds is $\partial$-proper if $f^{-1}(\partial N)=\partial M$. It is endproper if preimages of compact sets are compact. It is proper if it has both these properties. These terms are applied to a submanifold if its inclusion map has the corresponding property.

Let $S$ be a proper codimension one submanifold of the $m$-manifold $M$. Suppose $S^{\prime}$ is either another such submanifold such that int $S \cap$ int $S^{\prime}=\emptyset$ or is an end-proper submanifold of $\partial M$. Assume that $\partial S=\partial S^{\prime}$. Then $S$ and $S^{\prime}$ are parallel if some component of $M-\left(S \cup S^{\prime}\right)$ has closure homeomorphic to $S \times[0,1]$ with $S \times\{0\}=S$ and $S \times\{1\}=S^{\prime}$ when $\partial S=\emptyset$, while $((\partial S) \times[0,1]) \cup(S \times\{1\})=S^{\prime}$ when $\partial S \neq \emptyset$. The product $S \times[0,1]$ is a parallelism between $S$ and $S^{\prime}$. When $S^{\prime} \subseteq \partial M$ one says that $S$ is $\partial$-parallel. We say that $S$ is end-parallel or trivial if some component of $M-S$ has closure homeomorphic to $S \times[0, \infty)$ with $S=S \times\{0\}$.

Infinite sequences, unless indicated otherwise, will be indexed by the set of nonnegative integers.

An exhausting sequence $C=\left\{C_{n}\right\}$ for a non-compact $m$-manifold $M$ is a sequence $C_{0} \subseteq C_{1} \subseteq C_{2} \subseteq \cdots$ of compact subsets of $M$ whose union is $M$. A sequence $V_{0} \supseteq V_{1} \supseteq V_{2} \supseteq \cdots$ of open subsets of $M$ is an end sequence associated to $C$ if each $V_{n}$ is a component of $M-C_{n}$. Two end sequences $\left\{V_{n}\right\}$ and $\left\{W_{p}\right\}$ associated to exhausting sequences $C$ and $K$ for $M$ are cofinal if for every $n$ there is a $p$ such that $V_{n} \supseteq W_{p}$ and for every $p$ there is a $q$ such that $W_{p} \supseteq V_{q}$. Cofinality is an equivalence relation on end sequences of $M$. The equivalence classes are called the ends of $M$. The set of all ends of $M$ is denoted by $\varepsilon(M)$. An endproper map $M \rightarrow N$ induces a well defined function $\varepsilon(M) \rightarrow \varepsilon(N)$. If $\partial M$ has no compact components, then the inclusion map induces a well defined bijection $\varepsilon($ int $M) \rightarrow \varepsilon(M)$.

An exhausting sequence $C$ for a connected, non-compact $m$-manifold $M$ is an exhaustion for $M$ if each $C_{n}$ is a compact, connected $m$-manifold, $C_{n} \cap \partial M$ is either empty or an ( $m-1$ )-manifold, $C_{n} \subseteq \operatorname{Int} C_{n+1}$, and $M-C_{n}$ has no bounded components. Connected non-compact $m$-manifolds always have exhaustions. Given an exhaustion $C$ for $M$ and a subsequence $\left\{n_{k}\right\}$ of the non-negative integers, let $C_{k}^{\prime}=C_{n_{k}}$. Then $C^{\prime}$ is also an exhaustion for $M$ and will be called a subexhaustion of $C$.

The reader is referred to [10] or [11] for basic 3-manifold topology, including the definition of incompressible surface. We adopt the conventions of [11] that every proper disk in a 3 -manifold $M$ is incompressible and that a proper 2 -sphere is compressible if and only if it bounds a 3 -ball. $M$ is irreducible if every 2 -sphere in $M$ is compressible; it is $\mathbf{P}^{2}$-irreducible if it contains no 2-sided projective planes. It is $\partial$-irreducible if $\partial M$ is incompressible in $M$. It is anannular if every proper incompressible annulus in $M$ is $\partial$-parallel. It is atoroidal if every proper incompressible torus in $M$ is $\partial$-parallel.

A partial disk is a pair $\left(D, \partial_{0} D\right)$, where $D$ is a disk and $\partial_{0} D$ is a non-empty finite union of disjoint arcs in $\partial D$; the order of $\left(D, \partial_{0} D\right)$ is the number of these arcs. A halfdisk is a partial disk of order one; a band is a partial disk of order two. $\partial D-\operatorname{int} \partial_{0} D$ is denoted by $\partial_{1} D$. A partial disk may be denoted by $D$ when $\partial_{0} D$ is clear from the context. Suppose $D$ is a partial disk contained in a surface $S$. Then $D$ is proper in $S$ if $D \cap \partial S=\partial_{0} D$. If $D$ is a proper partial disk in $S$ such that no component of $S-I n t_{S} D$ is a proper halfdisk $D^{\prime}$ in $S$ with $\partial_{1} D^{\prime}=D \cap D^{\prime}$, then $D$ is well embedded in $S$.

A proper surface $S$ in $M$ is $\partial$-incompressible if it is not a $\partial$-parallel disk and whenever $D$ is a halfdisk in $M$ such that $D \cap \partial M=\partial_{0} D$ and $D \cap S=\partial_{1} D$, one has that $\partial_{1} D$ is $\partial$-parallel in $S$.

We now give some terminology for some standard isotopies which will be used later. Suppose $S$ and $T$ are end-proper surfaces in an irreducible 3-manifold $M$. Suppose $S$ and $T$ are in general position and $J$ is a simple closed curve component of int $S \cap \operatorname{int} T$ which bounds a disk $D$ on $S$ and a disk $G$ on $T$. Then $J$ is innermost on $S$ if $D \cap T=J$. In this case there is a 3-ball $B$ in $M$ with $\partial B=D \cup G$. Let $B^{+}$be a regular neighborhood of $B$ in $M$. There is an ambient isotopy of $S$ in $M$ supported in $B^{+}$which carries $D$ to $G$ and then off $G$ into $B^{+}-B$. This is called a disk push of $D$ across $B$ past $G$. Let $S^{*}$ and $D^{*}$ be the images of $S$ and $D$,
respectively, under this isotopy. Then $S^{*}$ is in general position with respect to $T$ and $\left(\left(S^{*}-D^{*}\right) \cap T\right) \subseteq((S-D) \cap T)$ and $D^{*} \cap T=\emptyset$.

Now suppose that $S$ and $T$ are also $\partial$-proper in $M$ and that there is an endproper surface $R$ in $\partial M$ such that $\partial S \cup \partial T$ lies in int $R$ and $R$ is incompressible in $M$. Suppose $\alpha$ is a component of $S \cap T$ which is an arc such that $\alpha=\partial_{1} D=\partial_{1} G$, where $D$ and $G$ are proper halfdisks in $S$ and $T$, respectively. Then $\alpha$ is innermost on $S$ if $D \cap T=\alpha$. In this case $\partial_{0} D \cup \partial_{0} G=\partial D^{\prime}$ for a disk $D^{\prime}$ in int $R$, and there is a 3 -ball $B$ in $M$ such that $\partial B=D \cup G \cup D^{\prime}$. Let $B^{+}$be a regular neighborhood of $B$ in $M$. There is an ambient isotopy of $S$ in $M$ supported in $B^{+}$which carries $D$ to $G$ and then off $G$ into $B^{+}-B$. This is called a halfdisk push of $D$ across $B$ past $G$. The images $S^{*}$ and $D^{*}$ then satisfy the same conditions as for a disk push.

A partial plane $P$ is a non-compact simply connected 2 -manifold with $\partial P \neq \emptyset$. When $\partial P$ has exactly one component $P$ is called a halfplane. We next give criteria for a proper plane or halfplane to be trivial. Note that a proper halfplane is trivial if and only if it is $\partial$-parallel.

Lemma 1.1. Let $M$ be a connected, irreducible, non-compact 3-manifold.
(1) A proper plane $P$ in $M$ is trivial if and only if there exist sequences $\left\{D_{n}\right\}$ and $\left\{D_{n}^{\prime}\right\}$ of disks in $M$ such that $\left\{D_{n}\right\}$ is an exhaustion for $P, D_{n}^{\prime} \cap P=\partial D_{n}$, and $\cup D_{n}^{\prime}$ is end-proper in $M$.
(2) A proper halfplane $P$ in $M$ is trivial if and only if there exist sequences $\left\{D_{n}\right\}$ and $\left\{D_{n}^{\prime}\right\}$ of halfdisks in $M$ such that $D_{n}$ is proper in $P,\left\{D_{n}\right\}$ is an exhaustion for $P, D_{n}^{\prime} \cap P=\partial_{1} D_{n}=\partial_{1} D_{n}^{\prime}, D_{n}^{\prime} \cap \partial M=\partial_{0} D_{n}^{\prime}$, $\cup D_{n}^{\prime}$ is end-proper in $M$, and $\partial_{0} D_{n} \cup \partial_{0} D_{n}^{\prime}$ bounds a disk in $\partial M$.

Proof. Necessity is obvious in both cases.
(1) Given any compact subset $K$ of $M$ there is an $n$ such that $D_{n}$ contains $K \cap P$ and $D_{m}^{\prime} \cap K=\emptyset$ for all $m \geq n$. If $K$ is a simple closed curve which meets $P$ transversely in a single point, then $K$ meets the 2 -sphere $D_{n} \cup D_{n}^{\prime}$ transversely in a single point, contradicting the fact that $D_{n} \cup D_{n}^{\prime}$ bounds a 3 -ball in $M$. Thus $P$ must separate $M$ into two components with closures $X$ and $Y$. By passing to a subsequence we may assume that all $D_{n}^{\prime}$ lie in $X$. Then for each compact subset $K$ of $X$ there is an $n$ such that $K$ lies in the 3 -ball in $X$ bounded by $D_{n} \cup D_{n}^{\prime}$. It follows that $X$ is homeomorphic to $\mathbf{R}^{2} \times[0,1)$.
(2) The proof is similar to that of (1) and is left to the reader.

Let $M$ be a connected, non-compact 3-manifold. It is said to be eventually end-irreducible if it contains a compact subset $J$ such that for every compact subset $K$ containing $J$ there is a compact subset $L$ containing $K$ such that every loop in $M-L$ which is null-homotopic in $M-J$ must be null-homotopic in $M-K$.

Let $C$ be an exhaustion for a connected, non-compact 3-manifold $M$. Denote $\operatorname{Fr} C_{n}$ by $F_{n}, C_{n+1}-\operatorname{Int} C_{n}$ by $X_{n+1}$, and $\cup_{n \geq k} F_{n}$ by $\mathcal{F}_{k}$. The exhaustion is good if for each $n \geq 0$ one has that $F_{n} \cup F_{n+1}$ is incompressible in $X_{n+1}, X_{n+1}$ is $\mathbf{P}^{2}$ irreducible, no component of $\mathcal{F}_{0}$ is a disk, and no component of $X_{n+1}$ has the form $F \times[0,1]$, where $F \times\{0\}$ and $F \times\{1\}$ are components of $F_{n}$ and $F_{n+1}$ respectively. Standard arguments show that in this case $\mathcal{F}_{0}$ is incompressible in $M-\operatorname{Int} C_{0}$
and has no 2 -sphere or projective plane components and that $M-\operatorname{Int} C_{0}$ is $\mathbf{P}^{2}$ irreducible. It is easily seen that every subexhaustion of a good exhaustion is good. Note that $M$ itself need not be $\mathbf{P}^{2}$-irreducible or even irreducible.

Lemma 1.2. If the connected, non-compact 3-manifold $M$ admits a good exhaustion, then $M$ is eventually end-irreducible and eventually $\mathbf{P}^{2}$-irreducible and is not almost compact.

Proof. The first property is well known; its proof will be sketched for completeness. Let $J=C_{0}$. Suppose $K$ is compact and contains $J$. There is an $n>0$ such that $K \subseteq C_{n}$. Let $L=C_{n+1}$. Then the incompressibility of $F_{n}$ in $M-I n t C_{0}$ implies that any null-homotopy in $M-J$ of a loop in $M-L$ can be cut off on $F_{n}$ so as to obtain a null-homotopy in $M-K$.

The $\mathbf{P}^{2}$-irreducibility of $M-\operatorname{Int} C_{0}$ implies that $M$ is eventually $\mathbf{P}^{2}$-irreducible.
To show that $M$ is not almost compact it suffices to show that the fundamental group of some component of $M-C_{0}$ is not finitely generated. In fact, this is the case for every component $V$ of $M-C_{0}$. We may assume that $H_{1}(V)$ is finitely generated. Then for all sufficiently large $n$ one has that the intersection of each component of $V \cap X_{n+1}$ with each component of $V \cap X_{n+2}$ is connected. Since no component of $X_{n+1}$ has the form $F \times[0,1]$ described above it follows from Theorem 10.2 of [10] that for every component $Y$ of $X_{n+1}$ and component $S$ of $\operatorname{Fr} Y$ we have that $\pi_{1}(S) \rightarrow \pi_{1}(Y)$ is a non-surjective monomorphism. It then follows that $\pi_{1}(V)$ is an infinite non-trivial free product with amalgamation, hence is not finitely generated.

Now suppose $M$ is a connected 3 -manifold with a finite number $\mu>0$ of ends whose boundary consists of a finite number $\nu \geq 0$ of disjoint planes $E^{i}$. An exhaustion $C$ for $M$ is nice if $C_{n} \cap \partial M$ consists of a single disk in each $E^{i}, X_{n+1}$ is $\mathbf{P}^{2}$-irreducible, $\partial$-irreducible, and anannular, each component of $F_{n}$ has negative Euler characteristic, each orientable component of $F_{n}$ has positive genus, and $M-\operatorname{Int} C_{n}$ has $\mu$ components for all $n \geq 0$. Note again that $M$ need not be $\mathbf{P}^{2}$-irreducible or irreducible.

Lemma 1.3. Let $C$ be a nice exhaustion for the connected, non-compact 3-manifold $M$. Then the following conditions hold.
(1) $C$ is a good exhaustion for $M$.
(2) $X_{n+1} \cap \partial M$ consists of one annulus in each component of $\partial M$ and is incompressible in $X_{n+1}$.
(3) $M-C_{0}, M-\operatorname{Int} C_{0}$, and, for $n \geq 1, C_{n}-\operatorname{Int} C_{0}$ are $\mathbf{P}^{2}$-irreducible, $\partial$-irreducible, and anannular.
(4) In $M-$ Int $C_{0}$ one has that $\partial M-\operatorname{Int}\left(C_{0} \cap \partial M\right)$ is incompressible and $F_{n}$ is $\partial$-incompressible.
(5) Every subexhaustion of $C$ is nice.
(6) $M$ is anannular at infinity.

Proof. Clearly $\left\{C_{n} \cap E^{i}\right\}$ is an exhaustion for $E^{i}$ by concentric disks, and so $X_{n+1} \cap$ $E^{i}$ is an annulus with one boundary component in $\partial F_{n}$ and the other in $\partial F_{n+1}$.

Suppose $D$ is a compressing disk for $F_{n} \cup F_{n+1}$ in $X_{n+1}$. Since $X_{n+1}$ is $\partial$-irreducible there is a disk $D^{\prime}$ in $\partial X_{n+1}$ with $\partial D^{\prime}=\partial D$ such that $D^{\prime}$ must contain some annulus $X_{n+1} \cap E_{i}$ and hence some component of $F_{n} \cup F_{n+1}$, contradicting the fact that no component of $\mathcal{F}_{0}$ is a planar surface. Thus $F_{n} \cup F_{n+1}$ is incompressible in $X_{n+1}$. If some component of $X_{n+1}$ is a product $F \times[0,1]$ with $F \times\{0\}$ in $F_{n}$ and $F \times\{1\}$ in $F_{n+1}$, then since $F$ has negative Euler characteristic there is an incompressible product annulus in $F \times[0,1]$ which is not $\partial$-parallel, contradicting the fact that $X_{n+1}$ is anannular. Thus $C$ satisfies (1) and (2).

Now suppose $D$ is a compressing disk for the boundary of $Y_{n}=C_{n}-\operatorname{Int} C_{0}$, where $n>1$. Let $F=F_{1} \cup \cdots \cup F_{n-1}$. If $D \cap F=\emptyset$, then $D$ lies in $X_{1}$ or $X_{n}$, say $X_{1}$. Isotop $D$ so that $\partial D$ lies in $F_{0}$. Then $\partial D=\partial D^{\prime}$ for a disk $D^{\prime}$ in $\partial X_{1}$. If $D^{\prime}$ does not lie in $\partial Y_{n}$, then it must contain a component of $F_{1}$, contradicting the positive genus condition. Thus we may assume that $D \cap F \neq \emptyset$. Suppose $D$ does not meet both $F_{0}$ and $F_{n}$; say it misses $F_{n}$. Then $D$ can be isotoped so that $\partial D$ lies in $F_{0}$. By the incompressibility of the $F_{i}$ and the irreducibility of the $X_{j}$ we may, if necessary, apply a finite sequence of disk pushes to $D$ so that $D \cap F=\emptyset$. Thus we may assume that $D$ meets both $F_{0}$ and $F_{n}$. By a disk push argument similar to that just given one may assume that $D \cap F$ contains no simple closed curves. Assume further that $D$ has been isotoped so that among all such $D$ the number of components of $D \cap F$ is minimal. Now $D \cap F$ splits $D$ into partial disks. Among these is a halfdisk $H$ which meets $\partial D$ in $\partial_{0} H$ and $F$ in $\partial_{1} H$. We may assume $H \subseteq X_{1}$, and so $\partial_{1} H \subseteq F_{1}$. Then $\partial H=\partial H^{\prime}$ for a disk $H^{\prime}$ in $\partial X_{1}$. This implies that $\partial\left(\partial_{1} H\right)$ lies in a single component $J$ of $\partial F_{1}$, and so $J \cap H^{\prime}$ is an arc which splits $H^{\prime}$ into a disk $H_{1}^{\prime}$ in $F_{1}$ and a disk $H_{0}^{\prime}$ in $\partial X_{1}-i n t F_{1}$. Since $X_{1}$ is irreducible $H \cup H^{\prime}$ bounds a ball $B$ in $X_{1}$, and so there is a halfdisk push of $H$ across $B$ past $H_{1}^{\prime}$ which removes at least $\partial_{1} H$ from the intersection, thereby contradicting minimality. Thus $Y_{n}$ is $\partial$-irreducible. The $\mathbf{P}^{2}$-irreducibility of $Y_{n}$ and $M-I n t C_{0}$ follows from that of the $X_{j}$ together with the incompressibility of the $F_{i}$. This implies that $M-C_{0}$ is $\mathbf{P}^{2}$-irreducible as well.

Suppose $D$ is a compressing disk for $\partial\left(M-\operatorname{Int} C_{0}\right)$. Then for some $n$ one has $D \subseteq\left(Y_{n}-F_{n}\right)$, and so $\partial D=\partial D^{\prime}$ for a disk $D^{\prime}$ in $\partial Y_{n}$. If $D^{\prime}$ does not lie in $\partial\left(M-\operatorname{Int} C_{0}\right)$, then it must contain a component of $F_{n}$. But this contradicts the positive genus condition, and so $M-\operatorname{Int} C_{0}$ is $\partial$-irreducible. The positive genus condition applied to $F_{0}$ now implies that $\partial M-\operatorname{Int}\left(C_{0} \cap \partial M\right)$ is incompressible in $M-\operatorname{Int} C_{0}$ and hence in $M-C_{0}$.

Suppose $D$ is a $\partial$-compressing halfdisk in $M-\operatorname{Int} C_{0}$ for some $F_{n}$. So $D \cap F_{n}=$ $\partial_{1} D$ and $D \cap \partial\left(M-\operatorname{Int} C_{0}\right)=\partial_{0} D$. Now either $D \subseteq Y_{n}$ or $D \cap Y_{n}=\partial_{1} D$. In the first case the $\partial$-irreducibility of $Y_{n}$ implies that $\partial D=\partial D^{\prime}$ for a disk $D^{\prime}$ in $\partial Y_{n}$. It follows that $\partial\left(\partial_{1} D\right)$ lies in a component $J$ of $\partial F_{n}$ and that $J \cap D^{\prime}$ is an arc which splits $D^{\prime}$ into a disk in $F_{n}$ and a disk in $\partial Y_{n}-i n t F_{n}$; thus $\partial_{1} D$ is $\partial$-parallel in $F_{n}$. In the second case $D$ lies in some $Y^{\prime}=C_{n+k}-\operatorname{Int} C_{n}$, and one applies the same argument to this manifold.

We have now established (4) and the portions of (3) not dealing with annuli. So suppose $A$ is an incompressible proper annulus in $Y_{n}$. Assume that the number of components of $A \cap F$ is minimal. Then a disk push argument shows that it
contains no simple closed curve components which bound disks on $A$. Suppose $\alpha$ is a component of $A \cap F$ which is an arc. If $\alpha$ is $\partial$-parallel in $A$, then we may assume that $\alpha$ is innermost on $A$, hence $\alpha=\partial_{1} D$ for a proper halfdisk $D$ on $A$ such that $D \cap F=\alpha$. Since $F$ is $\partial$-incompressible in $M-\operatorname{Int} C_{0}$ one has that $\alpha$ is $\partial$-parallel in $F$, and so $\alpha=\partial_{1} D^{\prime}$ for a proper halfdisk $D^{\prime}$ in $F$. Then the disk $D \cup D^{\prime}$ is $\partial$-parallel in $Y_{n}$ via a 3 -ball $B$. Thus there is a halfdisk push of $D$ across $B$ past $D^{\prime}$ which removes at least $\alpha$ from $A \cap F$, contradicting minimality. Thus $\alpha$ is a spanning arc in $A$. Since each $F_{j}$ separates $Y_{n}$ there must be two such arcs $\alpha$ and $\alpha^{\prime}$ such that $\alpha \cup \alpha^{\prime}=\partial_{1} D$ for a proper band $D$ in $A$ such that $D \cap F=\alpha \cup \alpha^{\prime}$. Now $D$ lies in some $X_{j}$, and so $\partial D=\partial D^{\prime}$ for a disk $D^{\prime}$ in $\partial X_{j}$. Now $D \cup D^{\prime}$ bounds a 3-ball $B$ in $X_{j}$. Then $D^{\prime} \cap F$ consists of one or two disks. If it is a single disk $D^{\prime \prime}$, then $D^{\prime}$ is the union of $D^{\prime \prime}$ and two disks in $\partial Y_{n}$. There is an isotopy (a band push) which moves the disk $D$ across $B$ past $D^{\prime \prime}$ which removes at least $\alpha \cup \alpha^{\prime}$ from $A \cap F$, contradicting minimality. If $D^{\prime} \cap F$ consists of two disks then there are $\partial$-compressing halfdisks for $A$ in $Y_{n}$, and hence $A$ is $\partial$-parallel in $Y_{n}$. (See Lemma 2.2 of [14].) Thus we may assume $A \cap F$ contains no spanning arcs of $A$. Suppose $J$ is a simple closed curve component of $A \cap F$ which is non-contractible in $A$. Then $A$ is $\partial$-parallel in $A$ via an annulus $A_{0}$ in $A$; we may assume that $A_{0} \cap F=J$. Then $A_{0}$ is a proper incompressible annulus in some $X_{j}$. Since $X_{j}$ is anannular $A_{0}$ is parallel in $X_{j}$ to an annulus $A_{1}$ in $\partial X_{j}$ with $\partial A_{1}=\partial A_{0}$. The parallelism $T$ is a solid torus with $\partial T=A_{0} \cup A_{1}$. There is a component $A_{2}$ of $A_{1} \cap F$ which is an annulus one of whose boundary components is $J$. Then there is an isotopy supported in a regular neighborhood of $T$ which moves $A_{0}$ across $T$ and past $A_{2}$, thereby removing at least $J$ from $A \cap F$ and thereby contradicting minimality. Thus we assume that $A \cap F=\emptyset$. Therefore $A$ lies in some $X_{j}$ and is parallel in $X_{j}$ to an annulus $A^{\prime}$ in $\partial X_{j}$ whose boundary misses that of $F$. By the positive genus condition $A^{\prime} \cap F=\emptyset$, and so $A^{\prime} \subseteq \partial Y_{n}$. Thus $Y_{n}$ is anannular.

Now suppose $A$ is an incompressible proper annulus in $M-\operatorname{Int} C_{0}$. Then for some $n$ one has $A \subseteq\left(Y_{n}-F_{n}\right)$, and so $A$ is parallel to an annulus $A^{\prime}$ in $\partial Y_{n}$. If $A^{\prime}$ does not lie in $\partial\left(M-\operatorname{Int} C_{0}\right)$, then it must contain a component of $F_{n}$. But this contradicts the positive genus condition, and so $M-\operatorname{Int} C_{0}$ is anannular. The positive genus condition applied to $F_{0}$ implies that $M-C_{0}$ is anannular. This completes the proof of (3).

Clearly the properties we have proven for $Y_{n}$ also hold for $C_{n}-\operatorname{Int} C_{m}$ whenever $n>m$. It follows from this that every subexhaustion of $C$ is nice, thus establishing (5).

Finally we note that given any compact subset $J$ of $M$ we can choose an $m>0$ such that $J \subseteq C_{m}$. Then by (5) and (3) we have that $M-I n t C_{m}$ is anannular. Suppose $A$ is an incompressible proper annulus in $M-C_{m}$. Then $A$ is parallel in $M-\operatorname{Int} C_{m}$ to an annulus $A^{\prime}$ in $\partial\left(M-\operatorname{Int} C_{m}\right)$. Since $F_{m}$ is incompressible in $M-\operatorname{Int} C_{m}$ and has no disk or annulus components $A^{\prime}$ must lie in $\partial\left(M-C_{m}\right)$. Thus $M$ is anannular at infinity.

## 2. Removing Trivial Intersections

Given two compact, proper, incompressible surfaces in an irreducible 3-manifold
it is a standard result that one can ambiently isotop one of the surfaces so that the two surfaces are in general position and no simple closed curve component of their intersection is trivial, i.e. bounds a disk in one (and hence both) of the surfaces. This isotopy consists of a finite series of disk pushes. In the case of non-compact surfaces one is faced with the possibility of an infinite series of disk pushes, which might not converge to an ambient isotopy. This section gives, in a slightly more general setting, conditions under which the isotopy can be accomplished, treating as well the removal of $\partial$-parallel arcs in $\partial$-incompressible surfaces.

Proposition 2.1. Let $M$ be a connected, irreducible, non-compact 3-manifold which is not homeomorphic to $\mathbf{R}^{3}$. Let $\mathcal{P}$ and $\mathcal{Q}$ be proper surfaces in $M$ which are in general position. Let $\mathcal{J}$ be a union of simple closed curve components of $\mathcal{P} \cap \mathcal{Q}$. Assume that the following conditions are satisfied.
(1) No component of $\mathcal{P}$ or of $\mathcal{Q}$ is a 2-sphere.
(2) Each component $J$ of $\mathcal{J}$ bounds a disk $D(J)$ on $\mathcal{P}$ and a disk $G(J)$ on $\mathcal{Q}$.
(3) There is no infinite sequence $\left\{J_{m}\right\}$ of distinct components of $\mathcal{J}$ such that either $D\left(J_{m}\right) \subseteq \operatorname{int} D\left(J_{m+1}\right)$ for all $m$ or $G\left(J_{m}\right) \subseteq \operatorname{int} G\left(J_{m+1}\right)$ for all $m$, i.e. there is no infinite nesting on $\mathcal{P}$ or on $\mathcal{Q}$ among the components of $\mathcal{J}$.
Then there is an ambient isotopy of $\mathcal{P}$ in $M$, fixed on $\partial M$, which takes $\mathcal{P}$ to a surface $\mathcal{P}^{\prime}$ such that $\mathcal{P}^{\prime}$ and $\mathcal{Q}$ are in general position and $\left(\mathcal{P}^{\prime} \cap \mathcal{Q}\right) \subseteq(\mathcal{P} \cap \mathcal{Q})-\mathcal{J}$. Moreover, the isotopy is fixed on $\mathcal{P}^{\prime} \cap \mathcal{Q}$.

Proof. Since neither $\mathcal{P}$ nor $\mathcal{Q}$ contains 2-spheres the disks $D(J)$ and $G(J)$ are unique. Call $D(J)$ maximal if there is no $D\left(J^{\prime}\right)$ such that $D(J) \subseteq \operatorname{int} D\left(J^{\prime}\right)$. These maximal disks are disjoint and their union contains all the $D(J)$. Let $\left\{D\left(J_{i}\right)\right\}$ be the set of all maximal disks. Let $D_{i}$ be a regular neighborhood of $D\left(J_{i}\right)$ in $\mathcal{P}$ chosen so that $D_{i} \cap \mathcal{Q}=D\left(J_{i}\right) \cap \mathcal{Q}$ and distinct $D_{i}$ are disjoint. Let $\mathcal{D}=\cup D_{i}$. Similar remarks apply to the maximal disks on $\mathcal{Q}$, yielding $\mathcal{G}=\cup G_{j}$. It suffices to ambiently isotop $\mathcal{P}$ to $\mathcal{P}^{\prime}$ such that, denoting the image of $\mathcal{D}$ under the isotopy by $\mathcal{D}^{\prime}$, we have that $\mathcal{P}^{\prime}$ and $\mathcal{Q}$ are in general position, $\mathcal{D}^{\prime} \cap \mathcal{Q}=\emptyset,\left(\left(\mathcal{P}^{\prime}-\mathcal{D}^{\prime}\right) \cap \mathcal{Q}\right)$ is a union of components of $((\mathcal{P}-\mathcal{D}) \cap \mathcal{Q})$, and the isotopy is fixed on $\left(\left(\mathcal{P}^{\prime}-\mathcal{D}^{\prime}\right) \cap \mathcal{Q}\right)$.

Choose an exhaustion $C$ for $M$ such that for each $n \geq 0$ and each $m>n$ one has that $C_{n}$ and $C_{m}-\operatorname{Int} C_{n}$ are irreducible. This is possible because $M$ is irreducible and is not homeomorphic to $\mathbf{R}^{3}$. We may assume that $\mathcal{G} \cap \mathcal{F}_{0}=\emptyset$ because the $G_{j}$ are disjoint disks in the interior of $M$. By passing to a subexhaustion of $C$ we may also assume that $\operatorname{Int} C_{n+1}$ contains all those $D_{i}$ which meet $F_{n}$. Let $Y_{0}=\operatorname{Int} C_{1}$, and let $Y_{n}=\left(\operatorname{Int} C_{n+1}\right)-C_{n-1}$ for $n \geq 1$. Thus if $D_{i}$ meets $F_{n}$, then $D_{i} \subseteq Y_{n}$.

Suppose $n$ is even, $D_{i}$ meets $F_{n}$, and $D_{i}$ meets $\mathcal{G}$. There is a disk $D$ in $D_{i}$ such that $D \cap \mathcal{G}=\partial D$. Note that $\partial D$ need not be a component of $\mathcal{J}$ and that int $D$ need not be disjoint from $\mathcal{Q}$. Let $D_{+}$be a regular neighborhood of $D$ in $\mathcal{P}$ chosen so that $D_{+} \cap \mathcal{Q}=D \cap \mathcal{Q}$. Let $G$ be the disk on $\mathcal{G}$ bounded by $\partial D$. For a subset $A$ of $M$ let $A^{*}$ denote the image of $A$ under the disk push of $D$ past $G$ across the 3-ball $B$ in $Y_{n}$ bounded by $D \cup G$. We may assume that $\partial D_{+}$lies on $\partial B^{+}$, where $B^{+}$is the regular neighborhood of $B$ supporting the isotopy. Then $\mathcal{P}^{*}$ and $\mathcal{Q}$ are
in general position, $\left(\mathcal{D}^{*}-D_{+}^{*}\right) \cap \mathcal{Q}$ and $\left(\left(\mathcal{P}^{*}-\mathcal{D}^{*}\right) \cap \mathcal{Q}\right)$ are unions of components of, respectively, $\left(\mathcal{D}-D_{+}\right) \cap \mathcal{Q}$ and $((\mathcal{P}-\mathcal{D}) \cap \mathcal{Q})$, and $D_{+}^{*} \cap \mathcal{Q}=\emptyset$.

Now suppose $K$ is a component of $\mathcal{P} \cap \mathcal{Q}$. If $K$ lies outside $B$, then we may assume that the isotopy is fixed on $K$, and so $K$ is a component of $\mathcal{P}^{*} \cap \mathcal{Q}$. If $K$ meets $B$, then it is a component of $D \cap \mathcal{Q}$ or of $\mathcal{P} \cap G$, and so it is removed by the isotopy. Since all the components of $\mathcal{P}^{*} \cap \mathcal{Q}$ are components of $\mathcal{P} \cap \mathcal{Q}$, we have that the isotopy is fixed on $\mathcal{P}^{*} \cap \mathcal{Q}$.

We next consider the effect of the isotopy on other disks $D_{k}$. Suppose $D_{k} \cap F_{n}=\emptyset$. If $D_{k} \cap B=\emptyset$, then we may assume that $D_{k}^{*}=D_{k}$. If $D_{k} \cap B \neq \emptyset$, then $D_{k} \cap G \neq \emptyset$. The isotopy might move $D_{k} \cap B$ across some portion of $F_{n}$, but since $\left(D_{k} \cap B\right)^{*}$ lies within a regular neighborhood of $G$ and $G \cap F_{n}=\emptyset$ we may assume that $D_{k}^{*} \cap F_{n}=\emptyset$. Thus the number of disks $D_{k}^{*}$ in $\mathcal{P}^{*}$ which meet $F_{n}$ is no greater than the number of disks $D_{k}$ in $\mathcal{P}$ which meet $F_{n}$. Therefore after performing a finite sequence of these isotopies we may assume that if $D_{i} \cap F_{n} \neq \emptyset$, then $D_{i} \cap \mathcal{G}=\emptyset$. Since these isotopies are supported in $Y_{n}$ we may do this simultaneously for all even $n$.

Now for $n$ odd we still have that if $D_{i} \cap F_{n} \neq \emptyset$, then $D_{i} \subseteq Y_{n}$. Thus if $D_{i}$ is a component of $\mathcal{D}$ such that $D_{i} \cap \mathcal{G} \neq \emptyset$, then $D_{i} \subseteq Y_{n}$ for some odd $n$. By performing a finite sequence of disk pushes in each such $Y_{n}$ we obtain the desired surface $\mathcal{P}^{\prime}$.

Corollary 2.2. Let $M$ be a connected, irreducible, non-compact 3-manifold which is not homeomorphic to $\mathbf{R}^{3}$. Let $\mathcal{P}$ and $\mathcal{Q}$ be proper, incompressible surfaces in $M$ such that no component of $\mathcal{P}$ or of $\mathcal{Q}$ is a 2-sphere or a trivial plane. Assume that there do not exist plane components $P$ of $\mathcal{P}$ and $Q$ of $\mathcal{Q}$ on both of which there is infinite nesting among the components of $P \cap Q$. Then there is an ambient isotopy of $\mathcal{P}$ in $M$, fixed on $\partial M$, which takes $\mathcal{P}$ to a surface $\mathcal{P}^{\prime}$ such that $\mathcal{P}^{\prime}$ and $\mathcal{Q}$ are in general position and no simple closed curve component of $\mathcal{P}^{\prime} \cap \mathcal{Q}$ bounds a disk on $\mathcal{P}^{\prime}$ or on $\mathcal{Q}$. This isotopy is fixed on $\mathcal{P}^{\prime} \cap \mathcal{Q}$.

Proof. Let $\mathcal{J}$ be the set of all simple closed curve components of $\mathcal{P} \cap \mathcal{Q}$ which bound disks on $\mathcal{P}$ (or equivalently on $\mathcal{Q}$.) Infinite nesting on $\mathcal{P}$ or on $\mathcal{Q}$ among the elements of $\mathcal{J}$ implies by Lemma 1.1 (1) that either one of these surfaces has a component which is a trivial plane or there are two plane components $P$ and $Q$ as above.

We now consider the removal of trivial arcs of intersection.
Proposition 2.3. Let $M$ be a connected, irreducible, non-compact 3-manifold which has non-empty boundary and is not homeomorphic to $\mathbf{R}^{2} \times[0,1)$. Let $\mathcal{P}$ and $\mathcal{Q}$ be proper surfaces in $M$ which are in general position. Let $\mathcal{A}$ be a union of components of $\mathcal{P} \cap \mathcal{Q}$, each of which is an arc. Let $R$ be an end-proper surface in $\partial M$. Assume that the following conditions are satisfied.
(1) No component of $\mathcal{P}$ or of $\mathcal{Q}$ is a 2-sphere or a disk.
(2) Each component $\alpha$ of $\mathcal{A}$ is $\partial$-parallel in $\mathcal{P}$ across a halfdisk $D(\alpha)$ and is $\partial$-parallel in $\mathcal{Q}$ across a halfdisk $G(\alpha)$.
(3) There is no infinite sequence $\left\{\alpha_{m}\right\}$ of distinct components of $\mathcal{A}$ such that either $D\left(\alpha_{m}\right) \subseteq \operatorname{Int}_{\mathcal{P}} D\left(\alpha_{m+1}\right)$ for all $m$ or $G\left(\alpha_{m}\right) \subseteq \operatorname{Int}_{\mathcal{Q}} G\left(\alpha_{m+1}\right)$ for all
$m$, i.e. there is no infinite nesting on $\mathcal{P}$ or on $\mathcal{Q}$ among the components of $\mathcal{A}$.
(4) $\partial \mathcal{P} \cup \partial \mathcal{Q}$ lies in int $R$.
(5) $R$ is incompressible in $M$.
(6) Each component $J$ of $\mathcal{J}$ bounds a disk $G(J)$ in $\mathcal{Q}$, where $\mathcal{J}$ is the union of all those simple closed curve components of $\mathcal{P} \cap \mathcal{Q}$ which lie in some $D(\alpha)$.
(7) There is no infinite nesting on $\mathcal{Q}$ among the components of $\mathcal{J}$.

Then there is an ambient isotopy of $\mathcal{P}$ in $M$, fixed on $(\partial M)-$ int $R$ which takes $\mathcal{P}$ to a surface $\mathcal{P}^{\prime}$ such that $\mathcal{P}^{\prime}$ and $\mathcal{Q}$ are in general position and $\left(\mathcal{P}^{\prime} \cap \mathcal{Q}\right) \subseteq$ $(\mathcal{P} \cap \mathcal{Q})-(\mathcal{A} \cup \mathcal{J})$. Moreover, the isotopy is fixed on $\mathcal{P}^{\prime} \cap \mathcal{Q}$.

Proof. Since neither $\mathcal{P}$ nor $\mathcal{Q}$ has a disk component the halfdisks $D(\alpha)$ and $G(\alpha)$ are unique. As before call a halfdisk $D(\alpha)$ maximal if there is no $D\left(\alpha^{\prime}\right)$ such that $D(\alpha) \subseteq \operatorname{Int}_{\mathcal{P}} D\left(\alpha^{\prime}\right)$. By hypothesis each $D(\alpha)$ lies in some maximal halfdisk and these maximal halfdisks are disjoint. Let $\left\{D\left(\alpha_{i}\right)\right\}$ be the set of maximal halfdisks. Let $D_{i}$ be a regular neighborhood of $D\left(\alpha_{i}\right)$ in $\mathcal{P}$ chosen so that $D_{i} \cap \mathcal{Q}=D\left(\alpha_{i}\right) \cap \mathcal{Q}$ and distinct $D_{i}$ are disjoint. Let $\mathcal{D}=\cup D_{i}$. The maximal halfdisks in $\mathcal{Q}$ yield $\mathcal{G}=\cup G_{j}$. It suffices to ambiently isotop $\mathcal{P}$ to $\mathcal{P}^{\prime}$ such that, denoting the image of $\mathcal{D}$ under the isotopy by $\mathcal{D}^{\prime}$, we have that $\mathcal{P}^{\prime}$ and $\mathcal{Q}$ are in general position, $\mathcal{D}^{\prime} \cap \mathcal{G}=\emptyset$, $\left(\left(\mathcal{P}^{\prime}-\mathcal{D}^{\prime}\right) \cap \mathcal{Q}\right)$ is a union of components of $((\mathcal{P}-\mathcal{D}) \cap \mathcal{Q})$, and the isotopy is fixed on $\left(\left(\mathcal{P}^{\prime}-\mathcal{D}^{\prime}\right) \cap \mathcal{Q}\right)$.

Since $\mathcal{J} \subseteq \mathcal{D}$ there is no infinite nesting on $\mathcal{P}$ among the components of $\mathcal{J}$. Thus by Proposition 2.1 we may assume $\mathcal{J}=\emptyset$.

As before we can choose an exhaustion $C$ for $M$ such that for each $n \geq 0$ and each $m>n$ one has that $C_{n}$ and $C_{m}-\operatorname{Int} C_{n}$ are irreducible. Since the $G_{j}$ are disjoint disks each meeting $\partial M$ in a single arc we may assume that $\mathcal{G} \cap \mathcal{F}_{0}=\emptyset$. We may also assume that $\partial \mathcal{F}_{0}$ and $\partial R$ are in general position, and that if $D_{i}$ meets $F_{n}$, then $D_{i} \subseteq \operatorname{Int} C_{n+1}$. Note that since $M$ is not homeomorphic to $\mathbf{R}^{2} \times[0,1)$ we may assume that no $C_{n}$ lies in a 3 -ball in $M$ which meets $\partial M$ in a single disk. We claim that $C_{n} \cap R$ and $\left(C_{m}-\operatorname{Int} C_{n}\right) \cap R$ are incompressible in $C_{n}$ and $\left(C_{m}-\operatorname{Int} C_{n}\right)$ respectively. If $D$ is a compressing disk for $C_{n} \cap R$ in $C_{n}$, then $\partial D=\partial D^{\prime}$ for a disk $D^{\prime}$ in $R$. If $D^{\prime}$ does not lie in $C_{n} \cap R$, then it must meet $M-C_{n}$. Since $M$ is irreducible $D \cup D^{\prime}$ bounds a 3-ball containing a component of $M-C_{n}$, contradicting the fact that $M-C_{n}$ has no bounded components. If $D$ is a compressing disk for $\left(C_{m}-\operatorname{Int} C_{n}\right) \cap R$ in $C_{m}-\operatorname{Int} C_{n}$, then $\partial D=\partial D^{\prime}$ for a disk $D^{\prime}$ in $R$ which must meet $C_{n}$ or $M-C_{m}$. Let $B$ be the 3 -ball in $M$ bounded by $D \cup D^{\prime}$. Then $D^{\prime}$ does not meet $C_{n}$, for otherwise $C_{n}$ would lie in $B$. So $D^{\prime}$ meets $M-C_{m}$ and hence a component of $M-C_{m}$ is contained in $B$, contradicting the fact that $M-C_{m}$ has no bounded components.

We now proceed as in the proof of Proposition 2.1. We let $Y_{0}=\operatorname{Int} C_{1}$ and $Y_{n}=\left(\operatorname{Int} C_{n+1}\right)-C_{n-1}$ for $n \geq 1$. Let $R_{n}=Y_{n} \cap R$. Suppose $D_{i}$ meets both $F_{n}$ and $\mathcal{G}$. The role of an innermost disk is now given to an innermost halfdisk, i.e. a proper halfdisk $D$ in $\mathcal{P}$ which lies in $D_{i}$ such that $\partial_{1} D=D_{i} \cap \mathcal{G}$, although $\partial_{1} D$ need not be a component of $\mathcal{A}$ and $\operatorname{Int}_{\mathcal{P}} D$ need not be disjoint from $\mathcal{Q}$. There is a unique proper halfdisk $G$ in $\mathcal{Q}$ such that $G$ lies in $\mathcal{G}$ and $\partial_{1} G=\partial_{1} D$. Then $D \cup G$
is a proper disk in $Y_{n}$ with $\partial(D \cup G)$ in $R_{n}$. Since $R_{n}$ is incompressible in $Y_{n}$ there is a disk $D^{\prime}$ in $R_{n}$ with $\partial D^{\prime}=\partial(D \cup G)$. Since $Y_{n}$ is irreducible $D \cup G \cup D^{\prime}$ bounds a 3-ball $B$ in $Y_{n}$. A halfdisk push of $D$ across $B$ past $G$ removes at least $\partial_{1} D$ from the intersection, adds no new components of intersection, either fixes or removes each component of $\mathcal{P} \cap \mathcal{Q}$, and does not increase the number of $D_{k}$ meeting $F_{n}$. As before we remove all intersections of $D_{i}$ with $\mathcal{G}$ for those $D_{i}$ which meet $F_{n}$ for $n$ even and then remove all remaining intersections of $\mathcal{D}$ with $\mathcal{G}$.

Corollary 2.4. Let $M$ be a connected, irreducible, non-compact 3-manifold which has non-empty boundary and is not homeomorphic to $\mathbf{R}^{2} \times[0,1)$. Let $\mathcal{P}$ and $\mathcal{Q}$ be proper, incompressible, $\partial$-incompressible surfaces in $M$ such that no component of $\mathcal{P}$ or of $\mathcal{Q}$ is a 2-sphere, a disk, a trivial plane, or a trivial halfplane. Assume that there do not exist components $P$ of $\mathcal{P}$ and $Q$ of $\mathcal{Q}$ which are either both planes or both halfplanes on both of which there is infinite nesting among the components of $P \cap Q$. Suppose $R$ is an end-proper surface in $\partial M$ which is incompressible in $M$ and whose interior contains $\partial \mathcal{P} \cup \partial \mathcal{Q}$. Then there is an ambient isotopy of $\mathcal{P}$ in $M$, fixed on $(\partial M)-i n t R$, which takes $\mathcal{P}$ to a surface $\mathcal{P}^{\prime}$ such that $\mathcal{P}^{\prime}$ and $\mathcal{Q}$ are in general position, no simple closed curve component of $\mathcal{P}^{\prime} \cap \mathcal{Q}$ bounds a disk on $\mathcal{P}^{\prime}$ or on $\mathcal{Q}$, and no component of $\mathcal{P}^{\prime} \cap \mathcal{Q}$ is an arc which is $\partial$-parallel in $\mathcal{P}^{\prime}$ or in $\mathcal{Q}$. This isotopy is fixed on $\mathcal{P}^{\prime} \cap \mathcal{Q}$.

Proof. First apply Corollary 2.2 to remove all trivial simple closed curve components of $\mathcal{P} \cap \mathcal{Q}$. Then let $\mathcal{A}$ be the set of all those components of $\mathcal{P} \cap \mathcal{Q}$ which are $\partial$ parallel arcs in $\mathcal{P}$ (or equivalently in $\mathcal{Q}$.) Infinite nesting among the components of $\mathcal{A}$ implies by Lemma 1.1 (2) that either one of these surfaces has a component which is a trivial halfplane or there are halfplane components $P$ and $Q$ as above.

## 3. Aplanarity and total acylindricality

The goal of this section is to show that a connected, non-compact 3-manifold which possesses a nice exhaustion must be totally acylindrical and, if it is irreducible, must be aplanar. These results are basically due to Winters and are contained, either explicitly or implicitly, in his thesis [19], where they sometimes appear in a slightly different form and context. We include proofs of them here for several reasons. First, the paper [20] containing the relevant portions of [19] has not yet been published, and so giving proofs here will make the argument of the present paper more complete. Second, the proof given here of Lemmas 3.1 and 3.2 is somewhat different from that of the corresponding Lemma X. 1 of [19] in that it applies the general machinery for removing trivial intersections developed in the previous section of this paper rather than the direct arguments of [19]; this remark also applies to the proof of Theorem 3.5. Third, Lemma 3.3 and Theorems 3.4 and 3.5 are not stated explicitly in [19] in the forms we shall need, although their proofs either are contained in or can be easily deduced from the proofs of Lemmas II.2, II.3, and XII. 3 of [19]. Finally, the terminology and the organization of the proof of aplanarity given in this section establish the background and conceptual framework for the more difficult analysis of partial planes in the next section.

In the first three lemmas $M$ is a connected non-compact 3-manifold and $C$ is an exhaustion for $M$. Whenever $J$ is a simple closed curve on a plane $P$, let $D(J)$ be the disk on $P$ bounded by $J$. Let $n_{0}>0$. A proper plane $P$ is in $n_{0}$-standard position with respect to $C$ if $P$ is in general position with respect to $\mathcal{F}_{0}, P \cap \mathcal{F}_{n_{0}}$ is a sequence $\left\{J_{m}\right\}$ of simple closed curves such that $\left\{D\left(J_{m}\right)\right\}$ is an exhaustion for $P, D\left(J_{0}\right)$ is a proper disk in $C_{n_{0}}$, and $\left(P \cap C_{0}\right) \subseteq \operatorname{int} D\left(J_{0}\right)$. Note that if this is the case and $n_{1}>n_{0}$, then $P$ is also in $n_{1}$-standard position with respect to $C$.

Lemma 3.1. Suppose $C$ is a good exhaustion for $M$ and $P$ is a proper plane in $M$. Then for some $n_{0}>0$ one has that $P$ is ambient isotopic to a plane which is in $n_{0}$-standard position with respect to $C$.

Proof. First ambiently isotop $P$ so that it is in general position with respect to $\mathcal{F}_{0}$. Since $P$ is proper there is a disk $D$ in $P$ which contains $P \cap C_{0}$ in its interior. (If $P \cap C_{0}=\emptyset$, let $D$ be any disk in $P$.) Since $C$ is an exhaustion and $D$ is compact there is an $n_{0}>0$ such that $D \subseteq \operatorname{Int} C_{n_{0}}$. There is then a component $J$ of $P \cap F_{n_{0}}$ such that $D \subseteq \operatorname{int} D(J)$. Let $J_{0}$ be the innermost such component of $P \cap F_{n_{0}}$, i.e. there is no component $J$ of $P \cap F_{n_{0}}$ such that $D \subseteq \operatorname{int} D(J) \subseteq \operatorname{int} D\left(J_{0}\right)$.

Now let $J$ be a component of $D\left(J_{0}\right) \cap F_{n_{0}}$ other than $J_{0}$. Assume that $J$ is innermost on $P$ among such curves. Then $D(J) \cap F_{n_{0}}=J$. Moreover $D(J) \cap D=\emptyset$, and so $D(J)$ lies in $M$ - Int $C_{0}$. Since $F_{n_{0}}$ is incompressible in $M$ - Int $C_{0}$ there is a disk $D^{\prime}$ in $F_{n_{0}}$ bounded by $J$. Since $M$ - Int $C_{0}$ is irreducible the 2-sphere $D(J) \cup D^{\prime}$ bounds a 3-ball $B$ in $M-\operatorname{Int} C_{0}$, and so one can perform a disk push of $D(J)$ across $B$ past $D^{\prime}$. This ambient isotopy of $P$ is supported in $M-\operatorname{Int} C_{0}$, removes $J$ from the intersection, and adds no new components. We continue performing such isotopies until $D\left(J_{0}\right)$ becomes a proper disk in $C_{n_{0}}$.

Suppose $J$ is a component of $P \cap F_{n_{0}}$ such that $D(J)$ does not contain $D\left(J_{0}\right)$. Then $D(J)$ lies in $M-\operatorname{Int} C_{0}$ and so a sequence of disk pushes similar to that described above removes all such $J$ from the intersection.

Let $Y_{n_{0}}=M-\operatorname{Int} C_{n_{0}}$ and $\mathcal{P}=P \cap Y_{n_{0}}$. Then $Y_{n_{0}}$ is irreducible and $\partial$ irreducible and the set of components of $\mathcal{P}$ consists of a half-cylinder (homeomorphic to $S^{1} \times[0, \infty)$ ) and possibly a finite collection of annuli, all of which are proper in $Y_{n_{0}}$. Let $\mathcal{Q}=\mathcal{F}_{n_{0}+1}$. Let $\mathcal{J}$ be the union of all those components $J$ of $\mathcal{P} \cap \mathcal{Q}$ such that $D(J)$ does not contain $D\left(J_{0}\right)$. Then $Y_{n_{0}}, \mathcal{P}, \mathcal{Q}$, and $\mathcal{J}$ satisfy the hypotheses of Proposition 2.1, and so there is an ambient isotopy in $Y_{n_{0}}$, fixed on $F_{n_{0}}$, which removes $\mathcal{J}$ from the intersection and adds no new components to it. Since the isotopy is fixed on those intersection curves which are not removed they all bound disks which contain $D\left(J_{0}\right)$. The isotopy thus extends by the identity isotopy on $C_{n_{0}}$ to an ambient isotopy in $M$ which carries $P$ to a plane $P^{\prime}$ which is in $n_{0}$-standard position with respect to $C$.

A proper plane $P$ is in non-trivial $n_{0}$-standard position with respect to an exhaustion $C$ if it is in $n_{0}$-standard position with respect to $C$ and no component of $P \cap \mathcal{F}_{n_{0}}$ bounds a disk in $\mathcal{F}_{n_{0}}$.

Lemma 3.2. Let $C$ be a good exhaustion for $M$, and let $P$ be a non-trivial proper plane in $M$. Suppose $M$ is irreducible. Then for some $n_{0}>0$ one has that $P$ is
ambient isotopic to a plane which is in non-trivial $n_{0}$-standard position with respect to $C$.

Proof. By Lemma 3.1 we may assume that $P$ is in $k$-standard position with respect to $C$ for some $k>0$. Then $P \cap \mathcal{F}_{k}$ is a nested sequence $\left\{J_{m}\right\}$ of simple closed curves on $P$. If there is an increasing sequence $\left\{m_{i}\right\}$ such that $J_{m_{i}}$ bounds a disk $D^{\prime}\left(J_{m}\right)$ in $\mathcal{F}_{k}$, then by Lemma 1.1 (1) we have that $P$ is trivial in $M$.

Thus no such sequence exists, and so there is an $m_{0} \geq 0$ such that for all $m \geq m_{0}$ $J_{m}$ does not bound a disk in $\mathcal{F}_{k}$. There is then an $m_{1} \geq m_{0}$ and an $n_{0} \geq k$ such that $J_{m_{1}} \subseteq F_{n_{0}}$ and $D\left(J_{m_{1}}\right) \subseteq C_{n_{0}}$. Then $P$ is in non-trivial $n_{0}$-standard position with respect to $C$.

A proper plane $P$ in $M$ is in $n_{0}$-monotone position with respect to an exhaustion $C$ if it is in $n_{0}$-standard position with respect to $C$ and for $n \geq n_{0}$ one has that $P \cap X_{n+1}$ is an annulus with one boundary component in $F_{n}$ and the other in $F_{n+1}$. We apply the adjective "non-trivial" in the same sense as above. Again $P$ is in $n_{1}$-monotone position for all $n_{1}>n_{0}$. We now assume that $M$ is a connected non-compact 3 -manifold having finitely many ends and finitely many boundary components and that each boundary component is a plane.

Lemma 3.3. Suppose $C$ is a nice exhaustion for $M$ and $P$ is a non-trivial proper plane in $M$. Assume that $M$ is irreducible. Then for some $n_{0}>0$ one has that $P$ is ambient isotopic to a plane which is in non-trivial $n_{0}$-monotone position with respect to $C$.

Proof. By Lemma 3.2 we may assume that $P$ is in non-trivial $n_{0}$-standard position with respect to $C$ for some $n_{0}>0$. Then there is an exhaustion $\left\{D_{m}\right\}$ of $P$ with each $D_{m}$ a disk such that $\left(P \cap C_{0}\right) \subseteq$ int $D_{0}$ and $D_{0}$ is a proper disk in $C_{n_{0}}$. Moreover, since $\partial D_{m}$ does not bound a disk in $\mathcal{F}_{n_{0}}$ and $\mathcal{F}_{n_{0}}$ is incompressible in $M-\operatorname{Int} C_{0}$, one has that each annulus $D_{m+1}-i n t D_{m}$ is incompressible in the $X_{n+1}$ containing it. Denote $\partial D_{m}$ by $J_{m}$.

Let $m_{0}=0$. Let $m_{1}$ be the smallest index for which $J_{m_{1}} \subseteq F_{n_{0}+1}$ and $\left(P \cap C_{n_{0}}\right) \subseteq$ $D_{m_{1}}$. Assume $m_{1}, \ldots, m_{r}$ have been defined. Let $m_{r+1}$ be the smallest index for which $J_{m_{r+1}} \subseteq F_{n_{0}+r+1}$ and $\left(P \cap C_{n_{0}+r}\right) \subseteq D_{m_{r+1}}$. Then $\left\{D_{m_{r}}\right\}$ is an exhaustion for $P$. Let $A_{r+1}=D_{m_{r+1}}-\operatorname{int} D_{m_{r}}$. Suppose $\left(\right.$ int $\left.A_{r+1}\right) \cap \mathcal{F}_{n_{0}} \neq \emptyset$. Call any component of this set a redundant intersection. Since each $F_{n}$ separates $M$ the redundant intersections occur in pairs, with each component of a pair lying in the same component of $\mathcal{F}_{n_{0}}$ and the pair forming the boundary of an annulus contained in $A_{r+1}$. Call such an annulus a redundant annulus. There is then a redundant annulus $A$ which is innermost in the sense that its interior misses $\mathcal{F}_{n_{0}}$. Then $A$ is a proper incompressible annulus in some $X_{n+1}$, and $\partial A \subseteq F_{n}$ or $\partial A \subseteq F_{n+1}$.

Assume $\partial A \subseteq F_{n}$. Then $A$ is parallel in $X_{n+1}$ to an annulus $A^{\prime}$ in $\partial X_{n+1}$. Suppose $A^{\prime}$ does not lie in $F_{n}$. By the incompressibility of $F_{n}, F_{n+1}$, and $X_{n+1} \cap \partial M$ in $X_{n+1}$ the components of $A^{\prime} \cap F_{n}, A^{\prime} \cap F_{n+1}$ and $A^{\prime} \cap \partial M$ must be annuli. $A^{\prime} \cap F_{n+1}=\emptyset$ since its components would be components of $F_{n+1}$, which, since $C$ is nice, has no annulus components. Since $F_{n}$ has no annulus components $A^{\prime} \cap F_{n}$ has exactly two components, each of which is a collar on a boundary component
of $F_{n}$. Thus $A^{\prime} \cap \partial M$ is an annulus whose boundary lies entirely in $F_{n}$, again contradicting the fact that $C$ is nice. Therefore $A^{\prime}$ lies in $F_{n}$ and so there is an ambient isotopy supported in a regular neighborhood of $X_{n+1}$ which reduces the number of redundant intersections. A similar argument holds for $\partial A \subseteq F_{n+1}$. Note that this process may move some redundant annuli and remove others. But it introduces no new redundant annuli and leaves any remaining redundant annuli lying in the union of the same set of $X_{k+1}$ 's.

Thus any one redundant annulus can be removed by an ambient isotopy of $P$ supported in $M-\operatorname{Int} C_{0}$. However, since there may be infinitely many redundant annuli one must avoid performing an infinite sequence of these isotopies which fails to converge to an ambient isotopy of $M$. We proceed as follows.

Choose $n_{1}>n_{0}$ so that if $A$ is any redundant annulus with $\partial A \subseteq F_{n_{0}}$, then $A$ lies in $\left(\operatorname{Int} C_{n_{1}}\right)-C_{0}$, and if $H$ is any redundant annulus with $\partial H \subseteq F_{n_{1}}$, then $H$ lies in $M-C_{n_{0}}$. Now suppose $n_{0}, n_{1}, \ldots, n_{i}, i>0$, have been chosen. Choose $n_{i+1}>n_{i}$ so that if $A$ is any redundant annulus with $A \subseteq F_{n_{i}}$, then $A$ lies in $\left(\operatorname{Int} C_{n_{i+1}}\right)-C_{n_{i-1}}$, and if $H$ is any redundant annulus with $\partial H \subseteq F_{n_{i+1}}$, then $H$ lies in $M-C_{n_{i}}$.

By an ambient isotopy supported in $\left(\operatorname{Int} C_{n_{1}}\right)-C_{0}$ remove all redundant intersections with $F_{n_{0}}$. For each even $i>0$ perform an ambient isotopy supported in ( $\operatorname{Int} C_{n_{i+1}}$ ) - $C_{n_{i-1}}$ which removes all redundant intersections with $F_{n_{i}}$. Since they have disjoint compact supports these isotopies give a single ambient isotopy supported in $M-C_{0}$. Now for each odd $i \geq 1$ perform an ambient isotopy which removes all redundant intersections with $F_{n_{i}}$. Again this gives a single ambient isotopy supported in $M-C_{0}$.

One now has no redundant intersections with the $F_{n_{i}}$. For each $i>0$ perform an ambient isotopy supported in $\left(\operatorname{Int} C_{n_{i}}\right)-C_{n_{i-1}}$ which removes all redundant intersections with those $F_{j}$ with $n_{i-1}<j<n_{i}$. This completes the removal of all redundant intersections and puts $P$ in non-trivial $n_{0}$-monotone position with respect to $C$.

Theorem 3.4. Let $M$ be a connected, irreducible, non-compact 3-manifold which has a nice exhaustion. Then $M$ is aplanar.

Proof. Suppose $P$ is a nontrivial proper plane in $M$, and $C$ is a nice exhaustion for $M$. By Lemma 3.3 we may assume that $P$ is in non-trivial $n_{0}$-monotone position with respect to $C$. For simplicity of notation we shall reindex so that $n_{0}=0$. Then $P \cap C_{n}$ is a disk $D_{n},\left\{D_{n}\right\}$ is an exhaustion of $P$, and $P \cap X_{n+1}=D_{n+1}-\operatorname{int} D_{n}$ is an annulus $A_{n+1}$ with one boundary component in $F_{n}$ and the other in $F_{n+1}$. Moreover $J_{n}=\partial D_{n}$ does not bound a disk in $F_{n}$.

It follows that each $A_{n+1}$ is parallel in $X_{n+1}$ to an annulus $A_{n+1}^{\prime}$ in $\partial X_{n+1}$. Since $\mathcal{F}_{0}$ has no disk components and is incompressible in $M-\operatorname{Int} C_{0}$ one has that each component of $A_{n+1}^{\prime} \cap\left(F_{n} \cup F_{n+1}\right)$ is an annulus. Since $\mathcal{F}_{0}$ has no annulus components this intersection consists of collars $G_{n+1}$ and $H_{n+1}$, respectively, on $J_{n}$ and $J_{n+1}$ in $A_{n+1}^{\prime}$. Let $R_{n+1}=A_{n+1}^{\prime} \cap \partial M$. Then $A_{n+1}^{\prime}=G_{n+1} \cup R_{n+1} \cup H_{n+1}$. The parallelism between $A_{n+1}$ and $A_{n+1}^{\prime}$ defines an embedding of $S^{1} \times[0,1] \times[0,1]$ in $X_{n+1}$, with $S^{1} \times[0,1] \times\{0\}=A_{n+1}, S^{1} \times[0,1] \times\{1\}=R_{n+1}, S^{1} \times\{0\} \times I=$
$G_{n+1}$, and $S^{1} \times\{1\} \times I=H_{n+1}$. Consider the analogous situation in $X_{n+2}$. If $G_{n+2} \neq H_{n+1}$, then $G_{n+2} \cup H_{n+1}$ is a component of $F_{n+1}$, contradicting the fact that $\mathcal{F}_{0}$ has no annulus components. Therefore $G_{n+2}=H_{n+1}$, all the $R_{n+1}$ lie in the same component $E_{i}$ of $\partial M$, and so one can fit together the embeddings of $S^{1} \times[0,1] \times[0,1]$ to get an end proper embedding of $S^{1} \times[1, \infty) \times[0,1]$ in $M-\operatorname{Int} C_{0}$ with $S^{1} \times[1, \infty) \times\{0\}=P-$ int $D_{0}, S^{1} \times[1, \infty) \times\{1\}=E_{i}-\left(E_{i} \cap C_{0}\right)$, and $S^{1} \times\{1\} \times[0,1]=G_{1}$. Now $D_{0} \cup G_{1}$ is a proper disk in $M$ whose boundary is that of $E_{i} \cap C_{0}$. Since $M$ is irreducible the union of these disks bounds a 3-ball in $M$ which can be used to extend the product structure to obtain a parallelism between $P$ and $E_{i}$.

Now recall that $M$ is totally acylindrical if it admits no proper incompressible embeddings of a cylinder $S^{1} \times \mathbf{R}$.

Theorem 3.5. Let $M$ be a connected, non-compact 3-manifold which has a nice exhaustion. Then $M$ is totally acylindrical.
Proof. Suppose $S=S^{1} \times \mathbf{R}$ is a proper, incompressible cylinder in $M$. Let $C$ be a nice exhaustion for $M$. Put $S$ in general position with respect to $\mathcal{F}_{0}$. Since $S$ is proper there exist $a, b \in \mathbf{R}, a<b$, such that the annulus $A=S^{1} \times[a, b]$ contains $S \cap C_{0}$. (If $S \cap C_{0}=\emptyset$, then choose $a<b$ arbitrarily.) There is then an $n_{0}>0$ such that $A \subseteq \operatorname{Int} C_{n_{0}}$. There are components $J_{0}^{+}$and $J_{0}^{-}$of $S \cap F_{n_{0}}$, neither of which bounds a disk on $S$, such that $A$ is contained in the interior of the annulus $A_{0}$ on $S$ bounded by $J_{0}^{+} \cup J_{0}^{-}$. We may assume that $A_{0}$ is an innermost such annulus, i.e. there are no components $J^{+}, J^{-}$of $S \cap F_{n_{0}}$ which are non-contractible on $S$ such that the annulus bounded by $J^{+} \cup J^{-}$contains $A$ and is contained in, but does not equal, the annulus $A_{0}$.

We now proceed as in the proof of Lemma 3.1. All the components of $A_{0} \cap F_{n_{0}}$ other than $J^{+}$and $J^{-}$bound disks in $A_{0}$ which miss $A$ and lie in $M-\operatorname{Int} C_{0}$. We perform a finite sequence of disk pushes which remove these curves. We then isotop the two half-cylinders composing $S-I n t A_{0}$ to remove all the other components of $S \cap F_{n_{0}}$ which bound disks on $S$.

Let $Y_{n_{0}}=M-\operatorname{Int} C_{n_{0}}$ and $\mathcal{P}=S \cap Y_{n_{0}}$. Then $\mathcal{P}$ consists of two half-cylinders and possibly a finite collection of annuli, all of which are proper in $Y_{n_{0}}$. Let $\mathcal{Q}=$ $\mathcal{F}_{n_{0}+1}$. Let $\mathcal{J}$ be the union of all those components of $\mathcal{P} \cap \mathcal{Q}$ which bound disks on $S$ and hence on $\mathcal{P}$. Since $Y_{n_{0}}$ is irreducible Proposition 2.1 gives an ambient isotopy of $S$ in $M$ which removes $\mathcal{J}$ from the intersection, adds no new components, and is fixed on those which are not removed.

Now each $X \cap X_{n+1}$ is a union of disjoint annuli. Since $S$ is proper this intersection is non-empty for all sufficiently large $n$ and in fact must contain an annulus $A^{\prime}$ running from $F_{n}$ to $F_{n+1}$. Since $X_{n+1}$ is anannular $A^{\prime}$ is parallel to an annulus $A^{\prime \prime}$ in $\partial X_{n+1}$. Now $A^{\prime \prime}$ must contain the annulus $E \cap X_{n+1}$ for some component $E$ of $\partial M$. Since $E$ is a plane it follows that $S$ is compressible in $M$, a contradiction.

## 4. Strong Aplanarity: The Special Case

Recall that a partial plane $P$ is a non-compact, simply-connected 2-manifold with non-empty boundary. In this section we show that given any proper partial plane
$P$ in an irreducible 3-manifold $M$ which has a nice exhaustion there exists a collar on $\partial M$ which contains $P$. This is a strong condition on $M$ since such a $P$ cannot meet distinct components of $\partial M$. One should note, however, that $P$ need not be $\partial$-parallel. In the next section this result will be extended to proper surfaces each of whose components is a partial plane.

The general line of argument will be similar to that of the previous section. Instead of working with simple closed curves and the disks they bound we shall work with finite collections of proper arcs which cut off disks on $P$. Specifically, let $\alpha$ be the union of finitely many disjoint proper $\operatorname{arcs} \alpha^{1}, \ldots, \alpha^{k}$ in a partial plane $P$. If for some proper partial disk $D$ in $P$ one has $\alpha=\partial_{1} D$, then $\alpha$ is called a bounding arc system in $P$, and $D$ is denoted by $D(\alpha)$.

A proper partial plane $P$ is in $n_{0}$-standard position with respect to an exhaustion $C$ if $P$ is in general position with respect to $\mathcal{F}_{0}$, each component of $P \cap \mathcal{F}_{n_{0}}$ is an arc, there is a sequence $\left\{D_{m}\right\}$ of well embedded partial disks in $P$ which is an exhaustion for $P,\left(P \cap C_{0}\right) \subseteq \operatorname{Int}_{P} D_{0}, D_{0}$ is a proper disk in $C_{n_{0}}$, and $P \cap \mathcal{F}_{n_{0}}=\cup_{m \geq 0} F r_{P} D_{m}$. In this case it is in $n_{1}$-standard position for all $n_{1}>n_{0}$.

Lemma 4.1. Let $C$ be a nice exhaustion for $M$, and let $P$ be a proper partial plane in $M$. Then for some $n_{0}>0$ one has that $P$ is ambient isotopic to a partial plane which is in $n_{0}$-standard position with respect to $C$.

Proof. First ambiently isotop $P$ so that it is in general position with respect to $\mathcal{F}_{0}$. Since $\partial P \neq \emptyset$ and $P$ is proper $\partial M \neq \emptyset$. Since $P$ is proper there is a disk $D$ in $P$ such that $\left(P \cap C_{0}\right) \subseteq \operatorname{Int}_{P} D$ and $D \cap \partial P \neq \emptyset$. (If $P \cap C_{0}=\emptyset$, let $D$ be any disk in $P$ with $D \cap \partial P \neq \emptyset$.) Since $C$ is an exhaustion and $D$ is compact there is an $n_{0}>0$ such that $D \subseteq \operatorname{Int} C_{n_{0}}$. There is a unique component $L$ of $P \cap C_{n_{0}}$ such that $D \subseteq \operatorname{Int}_{P} L$. Then $L$ is a planar surface whose boundary consists of one simple closed curve $K$ which meets $\partial P$ and possibly some other simple closed curves $K_{j}$ which do not meet $\partial P$. Let $\beta=K \cap F_{n_{0}}$. Then $\beta$ is a bounding arc system in $P$ such that $D(\beta)$ is the union of $L$ with the disks in $P$ bounded by the $K_{j}$.

If $D(\beta)$ is not well embedded in $P$, then for some components $\beta^{i}$ of $\beta$ there exist proper halfdisk components $D^{i}$ of $P-\operatorname{Int} t_{P} D(\beta)$ such that $\partial_{1} D^{i}=\beta^{i}$. Delete these $\beta^{i}$ from $\beta$ to obtain a new arc system $\alpha$. Then $\alpha$ is a bounding arc system such that $D(\alpha)$ is well embedded in $P, \alpha \subseteq\left(P \cap F_{n_{0}}\right)$, and $D \subseteq \operatorname{Int}_{P} D(\alpha)$. Note that $D(\alpha)$ need not lie in $C_{n_{0}}$. We shall next isotop $P$ so that afterwards $D(\alpha)$ does lie in $C_{n_{0}}$. To simplify the notation denote $D(\alpha)$ by $D_{0}$.

Suppose $J$ is a simple closed curve component of $D_{0} \cap F_{n_{0}}$. We may assume that $J$ is innermost on $P$ among such curves. Since $D \cap \partial P \neq \emptyset$ one has that $D(J) \cap D=\emptyset$, and so $D(J)$ lies in $M-\operatorname{Int} C_{0}$. Since $M-I n t C_{0}$ is irreducible and $F_{n_{0}}$ is incompressible in $M-\operatorname{Int} C_{0}$, there is a disk push which removes $J$ from the intersection and adds no new components. Continue in this fashion until all such $J$ are removed.

Now suppose $\gamma$ is a component of $\left(\right.$ Int $\left.D_{0}\right) \cap F_{n_{0}}$ which is a proper arc in $D_{0}$. We may assume that $\gamma$ is innermost on $P$ among such arcs. Hence there is a proper halfdisk $H$ in $P$ with $\partial_{1} H=\gamma=H \cap F_{n_{0}}$. We may further assume that $H$ does not contain $D$, and so $H$ lies in $M$-Int $C_{0}$. Since $F_{n_{0}}$ is $\partial$-incompressible in $M$-Int $C_{0}$
there is a proper halfdisk $H^{\prime}$ in $F_{n_{0}}$ with $\partial_{1} H^{\prime}=\gamma$. Since $\partial M-\operatorname{Int}\left(C_{0} \cap \partial M\right)$ is incompressible in $M-\operatorname{Int} C_{0}$, the simple closed curve $\partial_{0} H \cup \partial_{0} H^{\prime}$ bounds a disk $H^{\prime \prime}$ in $\partial M-\operatorname{Int}\left(C_{0} \cap \partial M\right)$. Hence by the irreducibility of $M-\operatorname{Int} C_{0}$ there is a halfdisk push of $H$ past $H^{\prime}$ across the ball bounded by $H \cup H^{\prime} \cup H^{\prime \prime}$ which removes $\gamma$ from the intersection and adds no new components. We continue in this fashion until $D_{0}$ becomes a proper disk in $C_{n_{0}}$.

We now consider components of $P \cap F_{n_{0}}$ which do not lie in $D_{0}$. As above we first remove all such components which are simple closed curves and then remove all those which are $\partial$-parallel arcs in $P$. Note that afterwards $P \cap F_{n_{0}}$ splits $P$ into finitely many partial disks and partial planes.

Let $Y_{n_{0}}=M-\operatorname{Int} C_{n_{0}}$. We now have that $P \cap C_{n_{0}}$ consists of $D_{0}$ and possibly a finite number of other disks. $P \cap Y_{n_{0}}$ consists of a finite number of partial planes and perhaps finitely many disks $D_{j}$, each of which meets $F_{n_{0}}$. Let $\mathcal{P}$ be the union of the partial planes, and let $\mathcal{Q}=\mathcal{F}_{n_{0}+1}$. By an isotopy fixed on $\partial Y_{n_{0}}$ we remove all simple closed curve intersections of the $D_{j}$ with $\mathcal{Q}$. Then by an isotopy fixed on $F_{n_{0}}$ we remove all those components of $\left(\cup D_{j}\right) \cap \mathcal{Q}$ which are arcs which are $\partial$-parallel on $P$ across halfdisks which do not contain $D_{0}$.

Now $\partial D_{j}=\partial D_{j}^{\prime}$ for a disk $D_{j}^{\prime}$ in $\partial Y_{n_{0}}$ such that $D_{j} \cup D_{j}^{\prime}=\partial B_{j}$ for a 3-ball $B_{j}$ in $Y_{n_{0}}$. Let $R^{\prime}$ be the union of $F_{n_{0}}$ and the $D_{j}^{\prime}$. Let $R=\left(\partial Y_{n_{0}}\right)-i n t R^{\prime}$. Since each component of $\mathcal{P}$ is non-compact we have that $\mathcal{P}$ lies outside the union of the $B_{j}$ and thus $\mathcal{P} \cap \mathcal{Q} \cap \partial Y_{n_{0}}$ lies in $R$. We claim that $R$ is incompressible in $Y_{n_{0}}$. For suppose $G$ is a compressing disk. Then $\partial G=\partial G^{\prime}$ for a disk $G^{\prime}$ in $\partial Y_{n_{0}}$. Since no component of $F_{n_{0}}$ is a planar surface $G^{\prime}$ must lie in $\left(\partial Y_{n_{0}}\right)-F_{n_{0}}$. But then int $G^{\prime}$ must contain some $D_{j}^{\prime}$, and so $D_{j}$ cannot meet $F_{n_{0}}$, a contradiction.

Now let $\mathcal{J}$ be the union of all the simple closed curve components of $\mathcal{P} \cap \mathcal{Q}$. Then $Y_{n_{0}}, \mathcal{P}, \mathcal{Q}$, and $\mathcal{J}$ satisfy the hypotheses of Proposition 2.1, and so $\mathcal{J}$ can be removed from the intersection by an isotopy which is fixed on those components of $\mathcal{P} \cap \mathcal{Q}$ which are not removed. Next let $\mathcal{A}$ be the union of all the components of $\mathcal{P} \cap \mathcal{Q}$ which are arcs that are $\partial$-parallel in $P$ across a halfdisk which does not contain $D_{0}$. Each such component is $\partial$-parallel in $\mathcal{P}$ to an arc which does not meet $F_{n_{0}}$. Then $Y_{n_{0}}, \mathcal{P}, \mathcal{Q}, R$, and $\mathcal{A}$ satisfy the hypotheses of Proposition 2.3, and so $\mathcal{A}$ can be removed from the intersection by an isotopy fixed on the components which are not removed. Note that since these isotopies are fixed on $R^{\prime}$ we may assume that they are fixed on the union of $F_{n_{0}}$ and the $B_{j}$. We thus get an ambient isotopy of $P$ in $M$ which puts it into $n_{0}$-standard position with respect to $C$.

Let $P$ be in $n_{0}$-standard position with respect to $C$; let $\left\{D_{m}\right\}$ be the corresponding exhaustion for $P$. For each $m \geq 0$ each component $Z$ of $D_{m+1}-\operatorname{Int}_{P} D_{m}$ is a partial disk and will be called a patch. $Z$ lies in $X_{n+1}$ for some $n \geq n_{0}$ or in $C_{n_{0}}-$ Int $C_{0}$. In the first case let $\partial_{+} Z=\partial_{1} Z \cap F_{n+1}$ and $\partial_{-} Z=\partial_{1} Z \cap F_{n}$; in the second case let $\partial_{+} Z=\partial_{1} Z$ and $\partial_{-} Z=\emptyset$. The set of all patches together with $D_{0}$ forms the vertex set of a locally finite graph $\Gamma$ in which the edges are all the components of all the $\operatorname{Fr}_{P} D_{m}$; two vertices are joined by an edge whenever the two disks meet along the corresponding arc. Since $P$ is simply connected $\Gamma$ is a tree. A subgraph of $\Gamma$ will often be identified with the submanifold of $P$ consisting of the
union of its vertices. For $n \geq n_{0}$ let $\Gamma_{n}=\Gamma \cap C_{n}$. If $T$ is a component of $\Gamma_{n}$, then $T$ is a finite tree and $F r_{P} T \subseteq F_{n}$. If $T$ does not contain $D_{0}$, then $T$ is called a falling tree with frontier in $F_{n}$. We also say that the falling tree $T$ descends from $F_{n}$. A falling tree which is the star of some vertex $Z$ is called a falling star with falling vertex or center $Z$. Every falling tree contains a falling star.
$P$ is in $n_{0}$-monotone position with respect to $C$ if it is in $n_{0}$-standard position with respect to $C$ and for each patch $Z$ one has that $\partial_{-} Z$ is a single arc. This is equivalent to each $\Gamma_{n}$ being connected and hence to there being no falling trees.

Lemma 4.2. Let $C$ be a nice exhaustion for $M$, and let $P$ be a proper partial plane in $M$. Then for some $n_{0}>0$ one has that $P$ is ambient isotopic to a partial plane which is in $n_{0}$-monotone position with respect to $C$.

Proof. Use Lemma 4.1 to put $P$ in $n_{0}$-standard position. By choosing $n_{0}$ sufficiently large we may assume that $D_{0} \cap C_{0} \neq \emptyset$. We first describe isotopies supported in 3 -balls which reduce the number of edges of a falling star and which eliminate falling stars with two edges. We then examine the effect of these isotopies on other portions of $P$ and describe further isotopies which may be needed to keep $P$ in $n_{0}$-standard position. The concatenation of these isotopies gives an isotopy with compact support which eliminates a falling star but may create new ones. We then show how to organize these isotopies so as to eliminate all falling trees.

Suppose $Z$ is the center of a falling star. Then $\partial_{1} Z \subseteq F_{n+1}$ for some $n \geq n_{0}-1$, and $Z$ lies in $X$, where $X$ is $X_{n+1}$ if $n \geq n_{0}$ and $C_{n_{0}}-\operatorname{Int} C_{0}$ if $n=n_{0}-1$. Let $X^{+}$be the union of $X$ and a collar on $F_{n+1}$ in $X_{n+2}$. Since $X$ is irreducible and $\partial$-irreducible there is a disk $Z^{\prime}$ in $\partial X$ to which $Z$ is $\partial$-parallel across a 3 -ball $B$ in $X$. Let $A$ be a component of $X \cap \partial M$ which meets $Z$. Then $A$ is an annulus which meets $F_{n+1}$ in a simple closed curve $K$. Since $Z \cap F_{n}=\emptyset$ each component $\beta$ of $A \cap \partial Z$ is an arc which is $\partial$-parallel in $A$ to an $\operatorname{arc} \beta^{\prime}$ in $K$. Let $\beta$ be an innermost such component, i.e. $\beta \cup \beta^{\prime}$ bounds a disk $G$ in $A$ such that $G \cap Z=\beta$.

Suppose $Z$ has order at least three. Let $B_{1}$ be a regular neighborhood of $G$ in $X^{+}$. There is an ambient isotopy of $P$ in $M$ supported in $B_{1}$ which moves $\beta$ to $\beta^{\prime}$ and then past $\beta^{\prime}$ into $\operatorname{int}\left(X_{n+2} \cap \partial M\right)$. Such an isotopy (regardless of the order of $Z)$ is called a boundary slide of $\beta$ past $\beta^{\prime}$. The endpoints of $\beta$ must lie on distinct $\operatorname{arcs} \gamma_{1}, \gamma_{2}$ of $\partial_{1} Z$; otherwise they would be joined by a single arc $\gamma$ in $\partial_{1} Z$ and so one would have $\partial Z=\beta \cup \gamma$, contradicting the fact that $P$ is in $n_{0}$-standard position. Now $\gamma_{1}$ and $\gamma_{2}$ are components of $\partial_{-} Z_{1}$ and $\partial_{-} Z_{2}$, respectively, for distinct patches $Z_{1}$ and $Z_{2}$ in $X_{n+2}$. The boundary slide replaces $Z_{1}$ and $Z_{2}$ by a new patch $W$ obtained by joining them by a band. $W$ has order one less than the sum of the orders of $Z_{1}$ and $Z_{2}$; since these orders are at least two $W$ has order at least three. The boundary slide replaces $Z$ by a partial disk $V$ with order one less than that of $Z$; hence $V$ has order at least two.

Suppose $Z$ has order two. Then the endpoints of the arc $\gamma_{1} \cup \beta \cup \gamma_{2}$ are joined by an $\operatorname{arc} \delta$ in some component $A^{*}$ of $X \cap \partial M$, and the union of these two arcs is $\partial Z^{\prime}$. Moreover $\delta$ is $\partial$-parallel in $A^{*}$ to an arc $\delta^{\prime}$ in $A^{*} \cap F_{n+1}$ across a disk $G^{*}$.

Suppose $G \cap G^{*}=\emptyset$. Then $\beta^{\prime} \cup \gamma_{1} \cup \delta^{\prime} \cup \gamma_{2}$ bounds a disk $H$ in $F_{n+1}$, and $Z^{\prime}=G \cup H \cup G^{*}$. Let $B_{2}$ be a regular neighborhood in $X^{+}$of $B$. There is an
ambient isotopy of $P$ in $M$ supported in $B_{2}$ which carries $Z$ to $H$ and then past $H$ into Int $X_{n+2}$. This band push replaces $Z, Z_{1}$, and $Z_{2}$ by a patch $Y$ in $X_{n+2}$ whose order is two less than the sum of the orders of $Z_{1}$ and $Z_{2}$ and hence is at least two.

Suppose $G \cap G^{*} \neq \emptyset$. Then $A^{*}=A, G \subseteq I n t_{A} G^{*}$, and $\beta^{\prime} \subseteq \operatorname{int} \delta^{\prime}$. Moreover each $\gamma_{i}$ is $\partial$-parallel in $F_{n+1}$ across a disk $H_{i}$ to an arc $\delta_{i}^{\prime}$ in $K, H_{1} \cap H_{2}=\emptyset$, $Z^{\prime}=H_{1} \cup\left(G^{*}-\operatorname{Int}_{G^{*}} G\right) \cup H_{2}$, and $\delta^{\prime}=\delta_{1}^{\prime} \cup \beta^{\prime} \cup \delta_{2}^{\prime}$. Let $B_{3}$ be a regular neighborhood in $X^{+}$of $B \cup G$. There is an ambient isotopy of $P$ in $M$ supported in $B_{3}$ which consists of a boundary slide of $\beta$ past $\beta^{\prime}$, followed by a boundary slide of $\delta$ past $\delta^{\prime}$, followed by a disk push which replaces $Z, Z_{1}$, and $Z_{2}$ by a patch $Y$ in $X_{n+2}$ of order at least two. This is a band unfolding.

Now suppose $Z_{0}$ is some other vertex of $\Gamma \cap X$ which meets the ball $B_{i}(i=1,2$, or 3) supporting one of these isotopies. For a band push we have $Z_{0} \subseteq B_{2}$, and so $Z_{0}$ is the center of a falling star. The isotopy replaces this star by a new vertex in $X_{n+2}$ of order at least two, and so $P$ is still in $n_{0}$-standard position. Now consider a boundary slide. Let $D=\beta^{\prime} \times[0,1]$ be a disk in $F_{n+1}$ with $\beta^{\prime} \times\{0\}=\beta^{\prime}$, and $\beta^{\prime} \times\{0,1\} \subseteq\left(\gamma_{1} \cup \gamma_{2}\right)$. We may asssume that $D \subseteq B_{1}$ and that $Z_{0} \cap D$ consists of product arcs. This isotopy deletes each of the arcs of $Z_{0} \cap(G \cup D)$ and joins its endpoints by an arc in $D$. It joins together the vertices in $X_{n+2}$ adjacent to $Z_{0}$ by bands corresponding to the new arcs in $D$ to get a 2 -manifold $W_{0}$; it also replaces $Z_{0}$ by a disk $V_{0}$. Let $J=W_{0} \cap V_{0}$. Then $J \subseteq \partial V_{0}$. If $J$ has at least two components, then each of these components is an arc, and $V_{0}$ and each component of $W_{0}$ is a partial disk of order at least two. If $J$ has exactly one component, then it is an arc or a simple closed curve. Suppose $J$ is an arc. Then $W_{0}$ is a partial disk of order at least three. If $V_{0}$ has order one, then $\partial V_{0}=\partial V_{0}^{\prime}$ for a disk $V_{0}^{\prime}$ in $\partial X$ such that $V_{0}^{\prime}$ is the union of a disk $V_{1}^{\prime}$ in $X \cap \partial M$ and a disk $V_{2}^{\prime}$ in $F_{n+1}$ along an arc in $F_{n+1} \cap \partial M$. A halfdisk push of $V_{0}$ past $V_{1}^{\prime}$ across the ball $B_{0}$ bounded by $V_{0} \cup V_{0}^{\prime}$ replaces $W_{0}$ and $V_{0}$ by a partial disk $W_{1}$ in $X_{n+2}$ of order at least two. If $Z_{1}$ is some other vertex in $X$ whose image under the boundary slide lies in $B_{0}$, then it must be the center of a falling star, and so the halfdisk push replaces the falling star by a vertex in $X_{n+2}$. Now suppose $J$ is a simple closed curve. Then $W_{0}$ is an annulus one of whose boundary components is $J=\partial V_{0} \subseteq F_{n+1}$. So $V_{0}$ is $\partial$-parallel in $X$ to a disk $V_{0}^{\prime}$ in $F_{n+1}$ across a 3 -ball $B_{0}$. A disk push of $V_{0}$ across $B_{0}$ past $V_{0}^{\prime}$ replaces $W_{0}$ and $V_{0}$ by a disk $W_{1}$ in $X_{n+2}$. Note that since $Z_{0}$ has order at least two $W_{0}$ must have been obtained by joining at least two distinct vertices in $X_{n+2}$. It follows that $W_{1}$ has order at least two. If $Z_{1}$ is some other vertex in $X$ whose image under the boundary slide lies in $B_{0}$, then again it must be the center of a falling star which is replaced by a vertex in $X_{n+2}$ by the disk push. Thus in all cases $P$ is put back in $n_{0}$-standard position. Finally consider a band unfolding. This move is the concatenation of a boundary slide and a halfdisk push. An analysis similar to that above provides isotopies which return $P$ to $n_{0}$-standard position and reduce the number of vertices in $\Gamma_{n+2}$.

Thus given any falling vertex $Z$ in $X$, there is andient isotopy of $P$ in $M$ supported in $X^{+}$which collapses the falling star with center $Z$ to a vertex in $X_{n+2}$. The only other possible effects of this isotopy on $\Gamma$ are to collapse other such falling
stars with centers in $X$ in a similar fashion and to amalgamate vertices in $X_{n+2}$ which are adjacent to vertices in $X$, thereby reducing the orders of these vertices, but not reducing them to one. The number of vertices in each of $\Gamma \cap X$ and $\Gamma \cap X_{n+2}$ is reduced. $\Gamma$ is unchanged outside $X \cup X_{n+2}$.

Define a sequence $n_{0}<n_{1}<n_{2}<\cdots$ by choosing $n_{i+1}>n_{i}+1$ so that if $T$ is any falling tree descending from $F_{n_{i+1}}$, then $T \cap F_{n_{i}+1}=\emptyset$. Every falling tree in $C_{n_{0}}-\operatorname{Int} C_{0}$ is a falling vertex. By the discussion above there is an isotopy supported in $C_{n_{0}+1}-\operatorname{Int} C_{0}$ which eliminates them all. This isotopy may create new falling vertices descending from $F_{n_{0}+1}$, but it creates no new falling trees. It may reduce the order of $D_{0}$ but it does not eliminate it since $D_{0} \cap C_{0} \neq \emptyset$.

Suppose $i>0$. Let $k$ be the smallest index for which there is a falling tree $T$ descending from $F_{n_{i}}$ such that $T \cap F_{k} \neq \emptyset$. Then $k>n_{i-1}+1$, and $T \cap X_{k}$ consists of falling vertices. There is an isotopy supported in $X_{k} \cup X_{k+1}$ which eliminates them and creates no new falling trees descending from $F_{n_{i}}$. Continuing in this fashion there is an isotopy supported in $\operatorname{Int} C_{n_{i}+1}-C_{n_{i-1}+1}$ which eliminates all falling trees descending from $F_{n_{i}}$. Since these isotopies have disjoint compact supports they give a single ambient isotopy of $P$ in $M$ after which each $\Gamma_{n_{i}}$ is connected.

Finally there is an isotopy supported in $\left(\operatorname{Int} C_{n_{i}}\right)-C_{n_{i-1}}$ which removes all falling trees descending from $F_{n}$ for $n_{i-1}<n<n_{i}$ and creates no new falling trees. Again this yields a single ambient isotopy of $P$ in $M$ after which $P$ is in monotone position.

Theorem 4.3. Let $M$ be a connected, irreducible, non-compact 3-manifold which admits a nice exhaustion. Let $P$ be a proper partial plane in $M$. Then $\partial P$ lies in a single component $E$ of $\partial M$, and there is a collar $H$ on $E$ in $M$ such that $P \subseteq$ Int $H$.

Proof. By Lemma 4.2 we may assume that $P$ is in $n_{0}$-monotone position with respect to a nice exhaustion $C$.

Suppose $P$ meets at least two components of $\partial M$. Then for some $n$ there is a component $\alpha$ of $P \cap F_{n}$ which has one endpoint in a boundary plane $E$ and the other in another boundary plane $E^{\prime}$. Let $Z$ be the patch in $X_{n+1}$ containing $\alpha$. Then $\partial Z$ bounds a disk $Z^{\prime}$ in $\partial X_{n+1}$ by the $\partial$-irreducibility of $X_{n+1}$. But this is impossible because the simple closed curve $E \cap F_{n}$ meets $\partial Z$ transversely in a single point since $Z \cap F_{n}=\alpha$. Thus $\partial P$ lies in a single component $E$ of $\partial M$.

We now claim that every component $\alpha$ of $P \cap F_{n}$ is parallel in $F_{n}$ to an arc $\beta$ in $E \cap F_{n}$. For let $Z$ be the patch in $X_{n+1}$ containing $\alpha$. Then $\partial Z$ bounds a disk $Z^{\prime}$ in $\partial X_{n+1}$. The components of $F_{n} \cap E \cap Z^{\prime}$ are proper arcs in $Z^{\prime}$. If the endpoints of $\alpha$ lie on distinct such arcs $\beta_{1}$ and $\beta_{2}$, then the other endpoints of $\beta_{1}$ and $\beta_{2}$ must lie on components of $Z \cap F_{n}$ other than $\alpha$, contradicting the fact that $\partial_{-} Z=\alpha$. So the endpoints of $\alpha$ are joined by an $\operatorname{arc} \beta$. Then $\alpha \cup \beta$ bounds a disk $W$ in $Z^{\prime}$. If $W$ does not lie in $F_{n}$, then $W$ must contain a component of $F_{n+1}$, contradicting the positive genus assumption on $C$. Thus $\alpha$ is parallel to $\beta$ across the disk $W$ in $F_{n}$.

For each component $\alpha$ of $P \cap F_{n}$ let $W(\alpha)$ be the disk in $F_{n}$ across which $\alpha$ is parallel to an $\operatorname{arc} \beta$ in $F_{n} \cap E$. Let $\mathcal{W}_{n}$ be the set of maximal such disks, i.e. those $W(\alpha)$ which do not lie in $\operatorname{Int}_{F_{n}} W\left(\alpha^{\prime}\right)$ for some $\alpha^{\prime}$. The $W(\alpha)$ are unique and the elements of $\mathcal{W}_{n}$ are disjoint since otherwise some component of $F_{n}$ would be a disk.

Let $G_{n}=E \cap C_{n}$ and $A_{n}=E \cap X_{n}$.
Consider $\partial D_{0}$. It has not been assumed that $C_{0}$ is $\partial$-irreducible. Nevertheless, $\partial D_{0}$ does bound a disk $D_{0}^{\prime}$ in $\partial C_{0}$. For let $\mathcal{W}_{0}=\left\{W\left(\alpha_{0,1}\right), \ldots, W\left(\alpha_{0, k_{0}}\right)\right\}$. Then $G_{0}^{+}=G_{0} \cup W\left(\alpha_{0,1}\right) \cup \cdots \cup W\left(\alpha_{0, k_{0}}\right)$ is a disk in $\partial C_{0}$ which contains $\partial D_{0}$, and the result follows.

Now since $M$ is irreducible $D_{0}$ is parallel across a 3 -ball $B_{0}$ in $C_{0}$ to a disk $D_{0}^{\prime}$ in $\partial C_{0}$. There is a regular neighborhood $H_{0}$ of $G_{0}^{+}$in $C_{0}$ such that $B_{0} \subseteq$ Int $H_{0}$. Then $G_{0}^{*}=F r_{C_{0}} H_{0}$ is a proper disk in $C_{0}$, and there is a product structure $G_{0} \times[0,1]$ on $H_{0}$ such that $G_{0} \times\{0\}=G_{0}, G_{0} \times\{1\}=G_{0}^{*}$, and $L_{0}=\left(\partial G_{0}\right) \times[0,1] \subseteq F_{0}$.

Let $\mathcal{W}_{1}=\left\{W\left(\alpha_{1,1}\right), \ldots, W\left(\alpha_{1, k_{1}}\right)\right\}$. Then $A_{1}^{+}=A_{1} \cup L_{0} \cup W\left(\alpha_{1,1}\right) \cup \cdots \cup$ $W\left(\alpha_{1, k_{1}}\right)$ is an annulus in $\partial X_{1}$ which contains $\partial Z$ for each patch $Z$ in $X_{1}$. Each such $Z$ is parallel across a 3 -ball $B(Z)$ in $X_{1}$ to a disk $Z^{\prime}$ in $A_{1}^{+}$. There is a regular neighborhood $H_{1}$ of $A_{1}^{+}$in $X_{1}$ such that each $B(Z)$ is contained in Int $H_{1}$ and $H_{1} \cap F_{0}=H_{0} \cap F_{0}$. Then $A_{1}^{*}=F r_{X_{1}} H_{1}$ is a proper annulus in $X_{1}$, and there is a product structure $A_{1} \times[0,1]$ on $H_{1}$ such that $A_{1} \times\{0\}=A_{1}, A_{1} \times\{1\}=A_{1}^{*}$, and $\left(\partial A_{1}\right) \times[0,1]=L_{0} \cup L_{1}$, where $L_{1}$ is an annulus in $F_{1}$ and the product structures on $L_{0}$ induced by those on $H_{0}$ and $H_{1}$ agree.

We now continue this process, constructing for each $n$ an $H_{n}=A_{n} \times[0,1]$ with $A_{n} \times\{0\}=A_{n}, A_{n} \times\{1\}=A_{n}^{*}$, a proper annulus in $X_{n},\left(\partial A_{n}\right) \times[0,1]=L_{n-1} \cup L_{n}$, where $L_{n}$ is an annulus in $F_{n}$ and the product structures on $L_{n-1}$ induced by those on $H_{n-1}$ and $H_{n}$ agree such that $\left(P \cap X_{n}\right) \subseteq$ Int $H_{n}$. We then let $H=\cup_{n \geq 0} H_{n}$.

## 5. Strong Aplanarity: The General Case

A partial plane system is a surface $\mathcal{P}$ each component of which is a partial plane. Thus an aplanar 3 -manifold $M$ is strongly aplanar if and only if for every proper partial plane system $\mathcal{P}$ in $M$ there is a collar $H$ on $\partial M$ such that Int $H$ contains $\mathcal{P}$. The goal of this section is to show that this is the case if $M$ is irreducible and admits a nice exhaustion.

Lemma 5.1. Suppose $M$ is a connected, irreducible, non-compact 3-manifold which is not homeomorphic to $\mathbf{R}^{2} \times[0,1)$. Let $E$ be a component of $\partial M$ which is a plane and has the property that for every proper partial plane $P$ in $M$ with $\partial P$ in $E$ there is a collar $H$ on $E$ in $M$ such that Int $H$ contains $P$. Then for every proper partial plane system $\mathcal{P}$ in $M$ with $\partial \mathcal{P}$ in $E$ there is a collar $H^{\prime}$ on $E$ in $M$ such that Int $H^{\prime}$ contains $\mathcal{P}$.

Proof. There is a collection $\left\{\beta_{i, j}\right\}$ of disjoint arcs in $E$ whose union is end-proper in $E$ such that $\beta_{i, j} \cap \mathcal{P}=\partial \beta_{i, j}$ and the union $K$ of $\mathcal{P}$ with these arcs is simply connected. The notation is chosen so that $\beta_{i, j}$ joins components $P_{i}$ and $P_{j}$ of $\mathcal{P}$ for some choice of $i<j$. One way to see this is as follows. Let $\left\{D_{n}\right\}$ be an exhaustion for $E$ chosen so that each $D_{n}$ is a disk and $\partial \mathcal{P}$ and $\cup \partial D_{n}$ are in general position. We may assume that $\partial D_{0}$ meets at least two components of $\mathcal{P}$. There is then an arc $\beta_{0,1}$ in $\partial D_{0}$ whose interior misses $\mathcal{P}$ and whose endpoints lie in different components $P_{0}$ and $P_{1}$ of $\mathcal{P}$. Let $K_{1}=P_{0} \cup \beta_{0,1} \cup P_{1}$. If $\partial D_{0}$ meets other components of $\mathcal{P}$, then there is an arc $\beta_{i, 2}$ in $\partial D_{0}$ whose interior is disjoint from $K_{1} \cup \mathcal{P}$ such that one endpoint lies in a third component $P_{2}$ of $\mathcal{P}$ and the other endpoint lies in $P_{i}$,
where $i=0$ or 1 . If this endpoint of $\beta_{i, 2}$ is an endpoint of $\beta_{0,1}$, then we isotop $\beta_{i, 2}$ slightly in $E$ to make the two arcs disjoint, keeping this endpoint of $\beta_{i, 2}$ in $P_{i}$. We then let $K_{2}=K_{1} \cup \beta_{i, 2} \cup P_{2}$. We continue in this fashion until we have a simply connected 2-complex $K_{m_{0}}$ containing all those components of $\mathcal{P}$ which meet $\partial D_{0}$. We then adjoin arcs in $\partial D_{1}$ and the components of $\mathcal{P}$ which meet $\partial D_{1}$ but do not lie in $K_{m_{0}}$ to obtain a 2-complex $K_{m_{1}}$. Continuing in this manner we inductively construct the desired 2-complex $K$.

Let $B_{i, j}$ be a regular neighborhood of $\beta_{i, j}$ in the 3-manifold obtained by splitting $M$ along $\mathcal{P}$. Thus $B_{i, j}$ has the form $D_{i, j} \times[0,1]$, where $D_{i, j}$ is a halfdisk, $\left(\partial_{0} D_{i, j}\right) \times$ $[0,1]=B_{i, j} \cap E=B_{i, j} \cap \partial M, B_{i, j} \cap \mathcal{P}=D_{i, j} \times\{0,1\}, B_{i, j} \cap P_{i}=D_{i, j} \times\{0\}$, and $B_{i, j} \cap P_{j}=D_{i, j} \times\{1\}$. Identify $D_{i, j}$ with $D_{i, j} \times\{1 / 2\}$. We now form the "band sum" $P$ of the components of $\mathcal{P}$ along the $\beta_{i, j}$ by deleting all the $\operatorname{Int}_{\mathcal{P}}\left(B_{i, j} \cap \mathcal{P}\right)$ and then adjoining all the bands $\left(\partial_{1} D_{i, j}\right) \times[0,1]$. Then $P$ is a proper partial plane in $M$ and so lies in a collar $H=E \times[0,1]$ on $E$ in $M$ with $E=E \times\{0\}$. Let $E^{*}=E \times\{1\}$.

Let $M^{\prime}$ be the 3 -manifold obtained by splitting $M$ along $P$. Thus $\partial M^{\prime}$ contains two copies of $P$ whose identification recovers $M$. Let $M^{\prime \prime}$ be the component of $M^{\prime}$ containing $E^{*}$. Then $M^{\prime \prime}$ is a connected, irreducible, non-compact 3-manifold which is not homeomorphic to $\mathbf{R}^{3}$. Let $\mathcal{D}$ be the union of those $D_{i, j}$ which lie in $M^{\prime \prime}$. Then $E^{*}$ and $\mathcal{D}$ are proper incompressible surfaces in $M^{\prime \prime}$ no component of which is a 2-sphere. If $E^{*}$ is trivial in $M^{\prime \prime}$, then it is trivial in $M$ and hence $M$ is homeomorphic to $\mathbf{R}^{2} \times[0,1)$, a contradiction. Therefore by Corollary 2.2 there is an ambient isotopy of $E^{*}$ in $M^{\prime \prime}$, fixed on $\partial M^{\prime \prime}$, which takes $E^{*}$ to a plane disjoint from $\mathcal{D}$. It follows that there is an ambient isotopy of $E^{*}$ in $M$, fixed on $P \cup \partial M$, which takes $E^{*}$ to a plane disjoint from all the $B_{i, j}$ and thus disjoint from $\mathcal{P}$. The image $H^{\prime}$ of $H$ under this isotopy is a collar on $E$ in $M$ such that Int $H^{\prime}$ contains $\mathcal{P}$.

Lemma 5.2. Suppose $M$ is a connected, irreducible, non-compact 3-manifold. Let $E$ be a component of $\partial M$ which is a plane. Let $H$ be a collar on $E$ in $M$. Suppose $\mathcal{P}$ is a proper partial plane system in $M$ such that $\mathcal{P}=\mathcal{P}_{0} \cup \mathcal{P}_{1}$, where $\mathcal{P}_{0}$ and $\mathcal{P}_{1}$ are unions of components of $\mathcal{P}$ such that $\mathcal{P}_{0} \subseteq H$ and $\mathcal{P}_{1} \cap E=\emptyset$. Then there is an ambient isotopy of $\mathcal{P}_{1}$ in $M$, fixed on $\mathcal{P}_{0} \cup \partial M$, taking $\mathcal{P}_{1}$ to a surface $\mathcal{P}_{1}^{\prime}$ such that $\mathcal{P}_{1}^{\prime} \cap H=\emptyset$.

Proof. We may assume that $\mathcal{P}_{1} \neq \emptyset$. Let $H=E \times[0,1]$ with $E=E \times\{0\}$ and $E^{*}=E \times\{1\}$. Let $M^{\prime}$ be the 3 -manifold obtained by splitting $M$ along $\mathcal{P}_{0}$. Let $M^{\prime \prime}$ be the component of $M^{\prime}$ containing $E^{*}$. Then $M^{\prime \prime}$ is a connected, irreducible, non-compact 3-manifold which is not homeomorphic to $\mathbf{R}^{3} . \mathcal{P}_{1}$ and $E^{*}$ are proper incompressible surfaces in $M^{\prime \prime}$ having no 2-sphere components. No component of $\mathcal{P}_{1}$ is a plane. If $E^{*}$ is trivial in $M^{\prime \prime}$, then it is trivial in $M$ and so $M$ is homeomorphic to $\mathbf{R}^{2} \times[0,1)$. However, $\mathcal{P}_{1} \neq \emptyset$ and $\mathcal{P}_{1} \cap E=\emptyset$ implies that $\partial M$ has at least two components, a contradiction. Thus by Corollary 2.2 there is an ambient isotopy of $\mathcal{P}_{1}$, fixed on $\partial M^{\prime \prime}$, which takes $\mathcal{P}_{1}$ to a surface $\mathcal{P}_{1}^{\prime}$ such that $\mathcal{P}_{1}^{\prime}$ and $E^{*}$ are disjoint. The desired conclusion then follows.

Theorem 5.3. Let $M$ be a connected, irreducible, non-compact 3-manifold which admits a nice exhaustion. Then $M$ is strongly aplanar.

Proof. $M$ is aplanar by Theorem 3.4. $\partial M$ has components $E_{1}, \ldots, E_{\nu}$; each $E_{i}$ is a plane. Let $\mathcal{P}$ be a proper partial plane system in $M$. By Theorem $4.3 \mathcal{P}=\mathcal{P}_{1} \cup$ $\cdots \cup \mathcal{P}_{\nu}$, where $\mathcal{P}_{i}$ is the union of those components $P$ of $\mathcal{P}$ such that $\partial P \subseteq E_{i}$. We induct on the number of nonempty $\mathcal{P}_{i}$. If $\mathcal{P}$ meets only one boundary component, say $E_{1}$, then we apply Theorem 4.3 and Lemma 5.1 to get a collar $H_{1}$ on $E_{1}$ in $M$ such that $\mathcal{P} \subseteq I n t H_{1}$ and then take arbitrary collars on the remaining $E_{i}$ in $M$ - Int $H_{1}$. Suppose that the theorem is true for those proper partial plane systems meeting at most $k$ of the $\nu$ boundary components. Let $\mathcal{P}$ be a proper partial plane system meeting $k+1$ of them, say $E_{1}, \ldots, E_{k+1}$. By Theorem 4.3 and Lemma 5.1 there is a collar $H_{k+1}=E_{k+1} \times[0,1]$ on $E_{k+1}$ in $M$ with $E_{k+1}=E_{k+1} \times\{0\}$ such that $\mathcal{P}_{k+1} \subseteq$ Int $H_{k+1}$. Let $M^{\prime}=M-$ Int $H_{k+1}$. By Lemma 5.2 there is an ambient isotopy of $\mathcal{P}$ in $M$, fixed on $\mathcal{P}_{k+1} \cup \partial M$, which takes $\mathcal{P}_{1} \cup \cdots \cup \mathcal{P}_{k}$ to $\mathcal{P}_{1}^{\prime} \cup \cdots \cup \mathcal{P}_{k}^{\prime}$ lying in $M^{\prime}-\left(E_{k+1} \times\{1\}\right)$. Since $M^{\prime}$ is homeomorphic to $M$ the inductive hypothesis gives a collar on $\partial M^{\prime}$ in $M^{\prime}$ whose interior in $M^{\prime}$ contains $\mathcal{P}_{1}^{\prime} \cup \cdots \cup \mathcal{P}_{k}^{\prime}$. It follows that there is a collar on $\partial M$ in $M$ whose interior in $M$ contains $\mathcal{P}$.

## 6. Attaching Boundary Planes

In this section we show how to attach boundary planes to a connected, open 3 -manifold $U$ with finitely many ends to obtain non-compact 3 -manifolds $M$ with certain properties. In each case we construct $M$ by choosing a finite set of disjoint rays $[0, \infty)$ end-properly embedded in $U$ and then removing the interiors of disjoint regular neighborhoods of these rays. It is easily seen that $U$ is homeomorphic to int $M$ via a homeomorphism which takes the end of $U$ determined by a ray in $U$ to the end of $M$ determined by the corresponding boundary plane of $M$. The rays will be chosen using an exhaustion of $U$ so as to obtain a nice exhaustion for $M$.

We will need some results and terminology from [14]. A compact, connected 3 -manifold $X$ is excellent if it is $\mathbf{P}^{2}$-irreducible and $\partial$-irreducible, it is not a 3ball, it contains a 2 -sided, proper, incompressible surface, and every connected, proper, incompressible surface of zero Euler characteristic in $X$ is $\partial$-parallel. (So in particular $X$ is anannular and atoroidal.) Let $\lambda$ be a proper 1-manifold in a 3-manifold $Y$. The exterior of $\lambda$ in $Y$ is the closure of the complement of a regular neighborhood of $\lambda$ in $Y$. Suppose $Y$ is compact. If the exterior of $\lambda$ is excellent, then we say that $\lambda$ itself is excellent. It follows from Thurston's work [12] that $\lambda$ is excellent if and only if its exterior is hyperbolic. Theorem 1.1 of [14] states, among other things, that if $\kappa$ is any proper 1-manifold in a compact, connected 3-manifold $Y$ such that $\kappa$ meets every 2 -sphere in $\partial Y$ at least twice and every projective plane in $\partial Y$ at least once, then $\kappa$ is homotopic rel $\partial \kappa$ to an excellent 1-manifold $\lambda$.
Theorem 6.1. Let $1 \leq \mu<\infty$, and for $1 \leq i \leq \mu$ let $1 \leq \nu_{i}<\infty$. Let $U$ be a connected, open 3-manifold with $\mu$ ends. Then there is a non-compact 3-manifold $M$ which has the following properties.
(1) $U$ is homeomorphic to int $M$.
(2) $\partial M$ is a disjoint union of planes $E^{i, j}, 1 \leq i \leq \mu, 1 \leq j \leq \nu_{i}$.
(3) For $1 \leq i \leq \mu$ the image of the end $e^{i}$ of $U$ under the homeomorphism and inclusion induced maps $\varepsilon(U) \rightarrow \varepsilon($ int $M) \rightarrow \varepsilon(M)$ is the image of $\varepsilon\left(\cup_{j=1}^{\nu_{i}} E^{i, j}\right)$ under the inclusion induced map $\varepsilon(\partial M) \rightarrow \varepsilon(M)$.
(4) $M$ is connected, eventually end-irreducible, eventually $\mathbf{P}^{2}$-irreducible, anannular at infinity, and totally acylindrical, and is not almost compact.
(5) If $U$ is irreducible, then $M$ is strongly aplanar.

Proof. Let $\left\{K_{n}\right\}$ be an exhaustion for $U$. Denote $M-K_{n}$ by $V_{n}$ and $K_{n+1}-$ Int $K_{n}$ by $Y_{n+1}$. By passing to a subsequence of $\left\{K_{n}\right\}$ we may assume that each $V_{n}$ has $\mu$ components $V_{n}^{i}, 1 \leq i \leq \mu$, with $V_{n}^{i} \supseteq V_{n+1}^{i}$. Let $Y_{n+1}^{i}=\left(C l V_{n}^{i}\right) \cap Y_{n+1}$. By passing to a subsequence and attaching 1-handles to $\operatorname{Fr} K_{n}$ in $C l V_{n}$ we may assume that each $Y_{n+1}^{i}$ is connected, each component of $\operatorname{Fr} K_{n}$ has negative Euler characteristic, and each orientable component of $\operatorname{Fr} K_{n}$ has positive genus. In particular $\partial Y_{n+1}$ contains no 2 -spheres or projective planes.

By Theorem 1.1 of [14] there exist disjoint proper $\operatorname{arcs} \alpha_{n+1}^{i, j}, 1 \leq j \leq \nu_{i}$, in $Y_{n+1}^{i}$ such that $\alpha_{n+1}^{i, j}$ runs from $\operatorname{Fr} K_{n}$ to $\operatorname{Fr} K_{n+1}, \alpha_{n+1}^{i, j} \cap \operatorname{Fr} K_{n+1}=\alpha_{n+2}^{i, j} \cap \operatorname{Fr} K_{n+1}$, and the exterior $X_{n+1}^{i}$ of $\cup_{j=1}^{\nu_{i}} \alpha_{n+1}^{i, j}$ in $Y_{n+1}^{i}$ is excellent. Then $\alpha^{i, j}=\cup_{n=1}^{\infty} \alpha_{n}^{i, j}$ is a ray which is proper in $V_{0}$ and end-proper in $U$.

Let $X_{n+1}=\cup_{i=1}^{\mu} X_{n+1}^{i}$. We may assume that $X_{n+1} \cap F r K_{n+1}=X_{n+2} \cap \operatorname{Fr} K_{n+1}$. Let $M=K_{0} \cup \cup_{n=1}^{\infty} X_{n}$. Note that $C l_{Y_{n}}\left(Y_{n}-X_{n}\right)$ consists of disjoint 3-balls $N_{n}^{i, j}$ which are regular neighborhoods of $\alpha_{n}^{i, j}$, respectively. Let $N^{i, j}=\cup_{n=1}^{\infty} N_{n}^{i, j}$. Then $E^{i, j}=F r_{U} N^{i, j}$ is a proper plane in $U$, and $\partial M$ is the disjoint union of these planes.

Let $H^{i, j}$ be disjoint regular neighborhoods of $N^{i, j}$ in $U$. Then $H^{i, j}$ and $H^{i, j}-N^{i, j}$ are both homeomorphic to $\mathbf{R}^{2} \times[0,1)$, and thus there is a homeomorphism from $U$ to int $M$ with the required properties.

Finally let $C_{0}=K_{0}$ and $C_{n+1}=C_{n} \cup X_{n+1}$. Then $\left\{C_{n}\right\}$ is a nice exhaustion for $M$, and so by Lemma 1.2, Lemma 1.3, Theorem 3.4, Theorem 3.5, and Theorem 5.3 the 3 -manifold $M$ has all the stated properties.

We now turn to the modifications of the basic construction of Theorem 6.1 which yield the 3-manifolds with the additional properties mentioned in the introduction. We will need some preliminary results.

Lemma 6.2. Let $R$ be a compact, connected 3-manifold. Let $S$ be a compact, proper, 2-sided surface in $R$. Let $R^{\prime}$ be the 3-manifold obtained by splitting $R$ along $S$. Let $S_{1}$ and $S_{2}$ be the two copies of $S$ in $\partial R^{\prime}$ which are identified to obtain $R$. If each component of $R^{\prime}$ is excellent, $S_{1} \cup S_{2}$ and $\left(\partial R^{\prime}\right)-\operatorname{int}\left(S_{1} \cup S_{2}\right)$ are incompressible in $R^{\prime}$, and each component of $S$ has negative Euler characteristic, then $R$ is excellent.

Proof. This is Lemma 2.1 of [14].
We first strengthen our construction so that it withstands the removal of some (but not all) boundary planes from each end; the basic technical device employed is called a "poly-excellent tangle." Let $n \geq 1$. An $n$-tangle is a proper 1-manifold $\lambda$ in a 3 -ball such that $\lambda$ has $n$ components and each of these components is an arc.

For $1 \leq k \leq n$ a $k$-subtangle of an $n$-tangle $\lambda$ is a union of $k$ components of $\lambda$. An $n$-tangle $\lambda$ is poly-excellent if for each $1 \leq k \leq n$ each $k$-subtangle of $\lambda$ is excellent.

Theorem 6.3. For all $n \geq 1$ poly-excellent $n$-tangles exist.
The proof of this theorem is given in the Appendix.
Lemma 6.4. Let $T$ be a solid torus or a solid Klein bottle, and let $1 \leq \nu<\infty$. Then there exist disjoint proper arcs $\rho_{1}, \ldots, \rho_{\nu}$ in $T$ such that for every nonempty subset $\left\{j_{1}, \ldots, j_{k}\right\}$ of $\{1, \ldots, \nu\}$ the 1-manifold $\rho_{j_{1}} \cup \cdots \cup \rho_{j_{k}}$ is excellent in $T$.

Proof. Let $G$ be a meridional disk in $T$. Let $B$ be the 3-ball obtained by splitting $T$ along $G$, and let $G_{1}$ and $G_{2}$ be the disks in $\partial B$ which are identified to obtain $T$.

By Theorem $6.3 B$ contains a poly-excellent $3 \nu$-tangle $\lambda$. Divide the components of $\lambda$ into three groups $\left\{\beta_{j}\right\},\left\{\gamma_{j}\right\}$, and $\left\{\delta_{j}\right\}$, where $1 \leq j \leq \nu$. Isotop $\lambda$ so that $\beta_{j}$ and $\gamma_{j}$ run from int $G_{1}$ to $\partial B-\left(G_{1} \cup G_{2}\right), \delta_{j}$ runs from int $G_{2}$ to itself, and under the identification of $G_{1}$ and $G_{2}$ one has that $\left(\beta_{j} \cup \gamma_{j}\right) \cap \operatorname{int} G_{1}$ is identified with $\delta_{j} \cap \operatorname{int} G_{2}$. Let $\rho_{j}^{\prime}=\beta_{j} \cup \gamma_{j} \cup \delta_{j}$, and let $\rho_{j}$ be the image of $\rho_{j}^{\prime}$ in $T$.

Let $\left\{j_{1}, \ldots, j_{k}\right\}$ be a nonempty subset of $\{1, \ldots, \nu\}$. Let $R$ be the exterior of $\rho_{j_{1}} \cup \cdots \cup \rho_{j_{k}}$ in $T$, and let $R^{\prime}$ be the exterior of $\rho_{j_{1}}^{\prime} \cup \cdots \cup \rho_{j_{k}}^{\prime}$ in $B$. We may assume that these exteriors have been chosen so that $S=R \cap G$ is a disk with $2 k$ holes and $R^{\prime}$ is the 3 -manifold obtained by splitting $R$ along $S$. Let $S_{1}$ and $S_{2}$ be the copies of $S$ in $\partial R^{\prime}$ which are identified to obtain $R$. Each component of $\partial R^{\prime}-\operatorname{int}\left(S_{1} \cup S_{2}\right)$ is an annulus. Since $\partial R^{\prime}$ is incompressible in $R^{\prime}$ and no component of $S_{1} \cup S_{2}$ or of $\partial R^{\prime}-\operatorname{int}\left(S_{1} \cup S_{2}\right)$ is a disk each of these surfaces is incompressible in $R^{\prime}$. Since $R^{\prime}$ is excellent and $S$ has negative Euler characteristic Lemma 6.2 implies that $R$ is excellent.

Theorem 6.5. Let $\mu, \nu_{i}$, and $U$ be as in Theorem 6.1. Then there is a 3-manifold $M$ having all the properties listed in that theorem such that if $1 \leq \nu_{i}^{\prime} \leq \nu_{i}$, then every 3-manifold $\widehat{M}$ obtained from $M$ by deleting for each $i$ any $\nu_{i}-\nu_{i}^{\prime}$ of the $E^{i, j}$ from $\partial M$ has all these properties.
Proof. Let $\left\{K_{n}\right\}, V_{n}, V_{n}^{i}, Y_{n+1}$, and $Y_{n+1}^{i}$ be as in the proof of Theorem 6.1. Let $\xi_{n+1}^{i}$ be a proper arc in $Y_{n+1}^{i}$ running from $\operatorname{Fr} K_{n}$ to $\operatorname{Fr} K_{n+1}$ chosen so that $\xi_{n+1}^{i} \cap \operatorname{Fr} K_{n+1}=\xi_{n+2}^{i} \cap \operatorname{Fr} K_{n+1}$. Let $Z_{n+1}^{i}$ be the exterior of $\xi_{n+1}^{i}$ in $Y_{n+1}^{i}$. We may assume that $Z_{n+1}^{i} \cap \operatorname{Fr} K_{n+1}=Z_{n+2}^{i} \cap \operatorname{Fr} K_{n+1}$. Then $\partial Z_{n+1}^{i}$ contains no 2-spheres or projective planes because each component of $\operatorname{Fr} K_{n} \cup \operatorname{Fr} K_{n+1}$ has negative Euler characteristic. By Theorem 1.1 of [14] there is an excellent proper arc $\eta_{n+1}^{i}$ in $Z_{n+1}^{i}$ with $\partial \eta_{n+1}^{i}$ in the open annulus $\left(\partial Z_{n+1}^{i}\right) \cap\left(i n t Y_{n+1}^{i}\right)$. Let $L_{n+1}^{i}$ be the exterior of $\eta_{n+1}^{i}$ in $Z_{n+1}^{i}$. Then $T_{n+1}^{i}=Y_{n+1}^{i}-I n t_{Y_{n+1}^{i}} L_{n+1}^{i}$ is a solid torus or solid Klein bottle which meets $\partial Y_{n+1}^{i}$ in a disk $D_{n}^{i}$ in $\operatorname{Fr} K_{n}$ and a disk $D_{n+1}^{i}$ in Fr $K_{n+1}$.

By Lemma 6.4 there is a set of disjoint proper $\operatorname{arcs} \alpha_{n+1}^{i, j}, 1 \leq j \leq \nu_{i}$, in $T_{n+1}^{i}$ with $\alpha_{n+1}^{i, j}$ running from int $D_{n}^{i}$ to int $D_{n+1}^{i}$ such that for every nonempty subset the union of its elements is excellent in $T_{n+1}^{i}$. We then proceed to construct $M$ as in the proof of Theorem 6.1, with the same notation as defined there.

Now suppose that for each $i$ we have deleted a set of $\nu_{i}-\nu_{i}^{\prime}$ boundary planes $E^{i, j}$ from the $i^{t h}$ end of $M$ so as to obtain $\widehat{M}$. This 3 -manifold is homeomorphic to the one obtained by ignoring the corresponding rays $\alpha^{i, j}$ in our construction. Hence this corresponds to ignoring the $\operatorname{arcs} \alpha_{n+1}^{i, j}$. Let $\widehat{X}_{n+1}^{i}$ be the exterior of the remaining arcs in $Y_{n+1}^{i}$, and let $R_{n+1}^{i}$ be the exterior of these arcs in $T_{n+1}^{i}$. Then $\widehat{X}_{n+1}^{i}=L_{n+1}^{i} \cup R_{n+1}^{i}$ and the surface $S_{n+1}^{i}=L_{n+1}^{i} \cap R_{n+1}^{i}$ is a torus or Klein bottle from which the interiors of two disjoint disks have been removed. $\partial L_{n+1}^{i}-i n t S_{n+1}^{i}$ has two components obtained by removing int $D_{n}^{i}$ and int $D_{n+1}^{i}$ from components of $\operatorname{Fr} K_{n}$ and $\operatorname{Fr} K_{n+1}$ respectively. $\partial R_{n+1}^{i}-i n t S_{n+1}^{i}$ is a connected, orientable surface of genus $\nu_{i}^{\prime}-1$ with two boundary components. Thus $S_{n+1}^{i}, \partial R_{n+1}^{i}-i n t S_{n+1}^{i}$, and each component of $\partial L_{n+1}^{i}-i n t S_{n+1}^{i}$ have negative Euler characteristic. It then follows from Lemma 6.2 that $\widehat{X}_{n+1}^{i}$ is excellent. We let $\widehat{X}_{n+1}=\cup_{i=1}^{\mu} \widehat{X}_{n+1}^{i}$, $\widehat{C}_{0}=C_{0}$, and $\widehat{C}_{n+1}=\widehat{C}_{n} \cup \widehat{X}_{n+1}$. We then have a nice exhaustion $\left\{\widehat{C}_{n}\right\}$ of $\widehat{M}$.

We now further modify the construction so as to ensure the non-existence of homeomorphisms which carry one boundary plane to another or reverse orientation, as well as to produce an uncountable collection of pairwise non-homeomorphic 3manifolds having the same given interior and distribution of boundary planes among the ends.

A classical knot space $Q$ is a space homeomorphic to the exterior of a nontrivial knot in $S^{3}$. If $Q$ is embedded in a 3-manifold $X$ and $\partial Q$ is incompressible in $X$, then we say that $Q$ is incompressibly embedded in $X$. The idea of the proof is to associate to each boundary plane $E$ an infinite sequence of disjoint classical knot spaces whose union lies in int $M$ and is end-properly embedded in $M$. These knot spaces are not incompressibly embedded in $M$. However, for any "sufficiently large" compact subset $K$ of $M$ which meets $E$ all but finitely many of them will be incompressibly embedded in $M-K$. On the other hand at most finitely many will be incompressibly embedded in the complement of a compact subset which does not meet $E$. Furthermore the knot spaces will be chosen so that they characterize the plane with which they are associated. A useful analogy is that of a string of lights associated to each plane, with different planes corresponding to different colors. The removal of certain compact subsets meeting a collection of planes then turns on all but finitely many lights in the strings associated to these planes.

We first need a couple of preliminary technical lemmas.
Lemma 6.6. Let $T$ be a solid torus or solid Klein bottle, and let $1 \leq \nu<\infty$. Then there exist disjoint solid tori $T_{1}, \ldots, T_{\nu}$ in int $T$, disjoint compressing disks $D_{1}, \ldots, D_{\nu}$ for $\partial T_{1}, \ldots \partial T_{\nu}$ in $T-\operatorname{int}\left(T_{1} \cup \cdots \cup T_{\nu}\right)$, and disjoint proper arcs $\rho_{1}, \ldots, \rho_{\nu}$ in $T-\left(T_{1} \cup \cdots \cup T_{\nu}\right)$ such that $D_{i} \cap \rho_{j}=\emptyset$ for $i \neq j$ and for every nonempty subset $\left\{j_{1}, \ldots, j_{k}\right\}$ of $\{1, \ldots, \nu\}$ the 1-manifold $\rho_{j_{1}} \cup \cdots \cup \rho_{j_{k}}$ is excellent in $T-\operatorname{int}\left(T_{j_{1}} \cup \cdots \cup T_{j_{k}}\right)$.
Proof. Let $G, B, G_{1}$, and $G_{2}$ be as in the proof of Lemma 6.4. Let $W_{1}, \ldots, W_{\nu}$ be disjoint disks in int $G$. Let $W_{i, j}$ be the copy of $W_{j}$ in $G_{i}$ whose image under the identification of $G_{1}$ with $G_{2}$ is $W_{j}$. Let $T_{1}, \ldots, T_{\nu}$ be disjoint regular neighborhoods of $\partial W_{1}, \ldots, \partial W_{\nu}$ in $T$, chosen so that $A_{j}=T_{j} \cap G$ is a regular neighborhood of
$\partial W_{j}$ in $G$. Then $T_{j}$ is split by $A_{j}$ into solid tori $T_{1, j}$ and $T_{2, j}$ which are regular neighborhoods of $\partial W_{1, j}$ and $W_{2, j}$, respectively, in $B$. Let $A_{i, j}=F r_{B} T_{i, j}$. Let $T^{*}=T-\operatorname{int}\left(T_{1} \cup \cdots \cup T_{\nu}\right)$ and $B^{*}=B-\operatorname{Int} t_{B}\left(T_{1,1} \cup \cdots \cup T_{1, \nu} \cup T_{2,1} \cup \cdots \cup T_{2, \nu}\right)$. Then $B^{*}$ is a 3-ball which meets $G_{i}$ in $\nu$ disks $D_{i, j} \subseteq \operatorname{int} W_{i, j}$ and a disk with $\nu$ holes $H_{i}$. Let $D_{j}$ be the image of $D_{i, j}$ under the identification. Then the $D_{j}$ are disjoint compressing disks for $\partial T_{j}$ in $T^{*}$.

By Theorem 6.3 $B^{*}$ contains a poly-excellent $4 \nu$-tangle $\lambda$. Divide the components of $\lambda$ into four groups $\left\{\beta_{j}\right\},\left\{\gamma_{j}\right\},\left\{\delta_{j}\right\}$, and $\left\{\omega_{j}\right\}, 1 \leq j \leq \nu$. Isotop $\lambda$ so that $\beta_{j}$ runs from $\partial B-\left(G_{1} \cup G_{2}\right)$ to int $D_{1, j}, \gamma_{j}$ runs from int $D_{2, j}$ to itself, $\delta_{j}$ runs from int $D_{1, j}$ to int $H_{1}$, and $\omega_{j}$ runs from int $H_{2}$ to $\partial B-\left(G_{1} \cup G_{2}\right)$. Do this so that under the identification we have $\left(\beta_{j} \cup \delta_{j}\right) \cap D_{1, j}$ identified with $\gamma_{j} \cap D_{2, j}$ and we have $\delta_{j} \cap H_{1}$ identified with $\omega_{j} \cap H_{2}$. Let $\rho_{j}^{\prime}=\beta_{j} \cup \gamma_{j} \cup \delta_{j} \cup \omega_{j}$. Let $\rho_{j}$ be the image of $\rho_{j}^{\prime}$ under the identification.

Now let $\left\{j_{1}, \ldots, j_{k}\right\}$ be a nonempty subset of $\{1, \ldots, \nu\}$. Let $R_{0}$ be the exterior of $\rho_{j_{1}} \cup \cdots \cup \rho_{j_{k}}$ in $T-\operatorname{int}\left(T_{j_{1}} \cup \cdots \cup T_{j_{k}}\right), R_{0}^{\prime}$ the exterior of $\rho_{j_{1}}^{\prime} \cup \cdots \cup \rho_{j_{k}}^{\prime}$ in $B$, and $R_{0}^{*}$ the exterior of $\rho_{j_{1}}^{\prime} \cup \cdots \cup \rho_{j_{k}}^{\prime}$ in $B^{*}$. We may assume these exteriors are chosen so that $R_{0}^{\prime}$ is the union of $R_{0}^{*}$ with all those $T_{i, j}$ for which $j \in\left\{j_{1}, \ldots, j_{k}\right\}$. Since $A_{j}$ is parallel to $A_{i, j}$ across $T_{i, j}$ we have that $R_{0}^{*}$ is homeomorphic to $R_{0}^{\prime}$ and is therefore excellent. We may further assume that $R_{0}^{*}$ is obtained by splitting $R_{0}$ along the surface $S$ consisting of the $k$ disks with two holes $R_{0} \cap D_{j_{r}}$ and the disk with $2 k$ holes $R_{0} \cap\left(G-\left(D_{j_{1}} \cup \cdots \cup D_{j_{k}}\right)\right)$. Let $S_{1}$ and $S_{2}$ be the two copies of $S$ in $\partial R_{0}^{*}$. Then the components of $\partial R_{0}^{*}-\operatorname{int}\left(S_{1} \cup S_{2}\right)$ are $4 k$ annuli and a disk with $2 k+1$ holes. Since $\partial R_{0}^{*}$ is incompressible in $R_{0}^{*}$ it follows that $S_{1} \cup S_{2}$ and $\partial R_{0}^{*}-\operatorname{int}\left(S_{1} \cup S_{2}\right)$ are incompressible in $R_{0}^{*}$. Thus by Lemma 6.2 we have that $R_{0}$ is excellent.

Lemma 6.7. Let $T$ be a solid torus or solid Klein bottle. Let $J_{1}, \ldots, J_{\nu}$ be excellent $k n o t s$ in $S^{3}$. Then there are disjoint classical knot spaces $Q_{1}, \ldots, Q_{\nu}$ in int $T$ and disjoint proper arcs $\rho_{1}, \ldots, \rho_{\nu}$ in $T$-int $\left(Q_{1} \cup \cdots \cup Q_{\nu}\right)$ such that $Q_{j}$ is homeomorphic to the exterior of $J_{j}$ in $S^{3}$, there are disjoint 3-balls $B_{j}$ in int $T$ such that $Q_{j} \subseteq B_{j}$ and $B_{i} \cap \rho_{j}=\emptyset$ for $i \neq j$. and for every nonempty subset $\left\{j_{1}, \ldots, j_{k}\right\}$ of $\{1, \ldots, \nu\}$
(1) the exterior $R$ of the 1-manifold $\rho_{j_{1}} \cup \cdots \cup \rho_{j_{k}}$ in $T$ is $\mathbf{P}^{2}$-irreducible, $\partial$ irreducible, and anannular,
(2) each $Q_{j_{r}}$ is incompressibly embedded in $R$, and
(3) given any classical knot space $Q$ incompressibly embedded in int $R$ there is an ambient isotopy of $Q$ in $R$, fixed on $\partial R$, which takes $Q$ to some $Q_{j_{r}}$.

Proof. Let $T_{j}, D_{j}$, and $\rho_{j}$ be as in Lemma 6.6. Let $T^{*}=T-\operatorname{int}\left(T_{1} \cup \cdots \cup T_{\nu}\right)$. Let $Q_{j}$ be the exterior of $J_{j}$ in $S^{3}$. Form $T_{0}$ by gluing $Q_{1} \cup \cdots \cup Q_{\nu}$ to $T^{*}$ by identifying $\partial Q_{j}$ with $\partial T_{j}$ so that a meridian of $J_{j}$ is identified with $\partial D_{j}$. The union $B_{j}$ of $Q_{j}$ with a regular neighborhood of $D_{j}$ in $T^{*}$ is then a 3-ball, and $T_{0}$ is again a solid torus or solid Klein bottle.

Let $R_{0}$ be the exterior of $\rho_{j_{1}} \cup \cdots \cup \rho_{j_{k}}$ in $T-\operatorname{int}\left(T_{j_{1}} \cup \cdots T_{j_{k}}\right)$. Then $R=$ $R_{0} \cup Q_{j_{1}} \cup \cdots \cup Q_{j_{k}}$ is the exterior of $\rho_{j_{1}} \cup \cdots \cup \rho_{j_{k}}$ in $T_{0}$. Since $R_{0}$ and the $Q_{j_{r}}$ are $\mathbf{P}^{2}$-irreducible, $\partial$-irreducible, anannular, and atoroidal and $\partial R$ is not a torus one can apply standard general position and isotopy arguments to show that each $\partial Q_{j_{r}}$
is incompressible in $R$ and that every incompressible torus in $R$ is isotopic to some $\partial Q_{j_{r}}$. The result follows after changing the name of $T_{0}$ to $T$.

Theorem 6.8. Let $\mu, \nu_{i}$, and $U$ be as in Theorem 6.5.
(1) There is a 3-manifold $M$ having all the properties listed in Theorem 6.5 such that $M$ admits no self homeomorphisms which take one boundary plane to another or reverse orientation.
(2) Each $\widehat{M}$ as in Theorem 6.5 also has all these properties (including (1)), and distinct $\widehat{M}$ are not homeomorphic.
(3) There are uncountably many pairwise non-homeomorphic such $M$, and if $M$ and $N$ are two of these manifolds which are not homeomorphic, then for every pair of associated manifolds $\widehat{M}$ and $\widehat{N}$ we have that $\widehat{M}$ and $\widehat{N}$ are not homeomorphic.

Proof. Let $\mathcal{J}$ be a countably infinite set of excellent knots in $S^{3}$ whose exteriors are pairwise non-homeomorphic and admit no orientation reversing self homeomorphisms. An example of such a set is the collection of all non-trivial twist knots other than the trefoil and figure eight knots [16]. Let $\mathcal{S}$ be the set of all triples $(i, j, n)$ where $1 \leq i \leq \mu, 1 \leq j \leq \nu_{i}$, and $n \geq 1$. Index $\mathcal{J}$ by choosing a bijection with $\mathcal{S} \times\{0,1\}$. Denote the indexed knot by $J(i, j, n, p)$, where $p \in\{0,1\}$, and its exterior by $Q(i, j, n, p)$.

Let $\varphi: \mathcal{S} \rightarrow\{0,1\}$. We will associate a 3 -manifold $M$ to $\varphi$ as follows. Let $K_{n}$, $L_{n+1}^{i}, T_{n+1}^{i}$, and $D_{n}^{i}$ be as in the proof of Theorem 6.5. We apply Lemma 6.7 to $T_{n+1}^{i}$ together with the knots $J(i, j, n+1, \varphi(i, j, n+1)), 1 \leq j \leq \nu_{i}$ to get disjoint proper arcs $\alpha_{n+1}^{i, j}$ running from $D_{n}^{i}$ to $D_{n+1}^{i}$ having the properties stated for the $\rho_{j}$ in the lemma. We then carry out the rest of the construction in the proof of Theorem 6.5 to get $M$.

Suppose $\widehat{M}$ is obtained by deleting boundary planes as in Theorem 6.5. Then $\widehat{M}$ has an exhaustion $\left\{\widehat{C}_{n}\right\}$ with $\widehat{V}_{n}^{i}$ the $i^{t h}$ component of $\widehat{M}-\widehat{C}_{n}, \widehat{V}_{n}^{i}=\cup_{q=n}^{\infty} \widehat{X}_{q+1}^{i}$, and $\widehat{X}_{q+1}^{i}=L_{q+1}^{i} \cup R_{q+1}^{i}$, where $R_{q+1}^{i}$ is the exterior of $\alpha_{q+1}^{i, j_{1}} \cup \cdots \cup \alpha_{q+1}^{i, j_{k}}$ in $T_{q+1}^{i}$.

Standard general position and isotopy arguments show that $\widehat{X}_{q+1}^{i}=L_{q+1}^{i} \cup$ $R_{q+1}^{i}$ is $\mathbf{P}^{2}$-irreducible, $\partial$-irreducible, and anannular. It follows that $\left\{\widehat{C}_{n}\right\}$ is a nice exhaustion for $\widehat{M}$. Arguments of this type also show that each $Q\left(i, j_{r}, q+\right.$ $\left.1, \varphi\left(i, j_{r}, q+1\right)\right)$ in our construction is incompressibly embedded in $\widehat{V}_{n}^{i}$ whenever $q \geq n$ and that every classical knot space $Q$ which is incompressibly embedded in $\widehat{V}_{n}^{i}$ can be ambiently isotoped into $\widehat{X}_{q+1}^{i}$ for some $q \geq n$, then into $R_{q+1}^{i}$, and hence by Lemma 6.7 to some $Q\left(i, j_{r}, q+1, \varphi\left(i, j_{r}, q+1\right)\right)$ in our construction.

Now suppose we have another function $\psi: \mathcal{S} \rightarrow\{0,1\}$. Denote the two resulting $M$ by $M[\varphi]$ and $M[\psi]$, and distinguish the various submanifolds arising in their construction by similar notation. Let $\widehat{M}[\psi]$ be obtained by deleting all but one boundary plane from each end of $M[\psi]$. Suppose $g: \widehat{M}[\psi] \rightarrow N$ is a homeomorphism, where $N$ is obtained by deleting some boundary planes from $M[\varphi]$. Since $g$ induces a bijection $\varepsilon(\widehat{M}[\psi]) \rightarrow \varepsilon(N)$ and $\widehat{M}[\psi]$ has exactly one boundary plane
per end, so does $N$. Hence it must be some $\widehat{M}[\varphi]$ obtained by deleting all but one boundary plane from each end of $M[\varphi]$.

Fix $i$, and let $E^{i, j}[\psi]$ be the single boundary plane of the $i^{t h}$ end of $\widehat{M}[\psi]$. Then $g\left(E^{i, j}[\psi]\right)=E^{s, t}[\varphi]$ for some $1 \leq s \leq \mu$ and $1 \leq t \leq \nu_{s}$. Choose $n>0$ such that $g\left(\widehat{C}_{0}[\psi]\right) \subseteq \operatorname{Int} \widehat{C}_{n}[\varphi]$. Then choose $m>0$ such that $\widehat{C}_{n}[\varphi] \subseteq \operatorname{Int} g\left(\widehat{C}_{m}[\psi]\right)$. Let $q \geq m$, and let $Q$ be the copy of $Q(i, j, q+1, \psi(i, j, q+1))$ embedded in $\widehat{M}[\psi]$ by our construction. Then $Q$ lies in $\widehat{V}_{m}^{i}[\psi]$, and $\partial Q$ is incompressible in $\widehat{V}_{0}^{i}[\psi]$. Therefore $g(Q)$ lies in $g\left(\widehat{V}_{m}^{i}[\psi]\right)$ and thus in the larger set $\widehat{V}_{n}^{s}[\varphi]$, and $g(\partial Q)$ in incompressible in $g\left(\widehat{V}_{0}^{i}[\psi]\right)$ and thus in the smaller set $\widehat{V}_{n}^{s}[\varphi]$. Hence $g(Q)$ is isotopic to the copy of some $Q(s, t, r+1, \varphi(s, t, r+1))$ embedded in $\widehat{M}[\varphi]$ by our construction. Therefore $s=i, t=j, r=q$, and $\varphi(i, j, q+1)=\varphi(i, j, q+1)$.

Now suppose $h: M[\psi] \rightarrow M[\varphi]$ is a homeomorphism. By restricting $h$ to $\widehat{M}[\psi]$ as above we see that $h$ must take the $i^{t h}$ end of $M[\psi]$ to the $i^{t h}$ end of $M[\varphi]$ and the $j^{t h}$ boundary plane of the $i^{t h}$ end of $M[\psi]$ to the $j^{t h}$ boundary plane of the $i^{t h}$ end of $M[\varphi]$. Moreover, there exists $m>0$ such that $\psi(i, j, q+1)=\varphi(i, j, q+1)$ for all $q \geq m$. Thus for fixed $i$ and $j$ we get two infinite sequences of zeros and ones which agree after a finite number of terms. This property defines an equivalence relation on the set of all such sequences, which is uncountable, such that each equivalence class is countable. Therefore the set of all equivalence classes is uncountable, and so the set of homeomorphism classes of the $M[\varphi]$ is uncountable.

Taking $\psi=\varphi$ we get that $h$ must take each boundary plane to itself and some $Q$ to itself. Since these classical knot spaces admit no orientation reversing homeomorphisms neither does $M[\varphi]$.

Clearly these considerations apply to all the $\widehat{M}[\psi]$ and $\widehat{M}[\varphi]$ obtained by deleting boundary planes as in Theorem 6.5, and so we are done.

## Appendix: Poly-excellent Tangles

Recall that an $n$ component tangle in a 3-ball is poly-excellent if for every $k$ with $1 \leq k \leq n$ each of its $k$ component subtangles is excellent, i.e. has hyperbolic exterior. In this appendix we prove Theorem 6.3, which asserts the existence of poly-excellent $n$-tangles for all $n \geq 1$. The case $n=1$ is trivial since we can choose any proper arc in a 3-ball having exterior homeomorphic to the exterior of an excellent knot in $S^{3}$, for example the figure eight knot. So we may assume $n \geq 2$.

We shall make use of an excellent $n$-tangle $\lambda$ in a 3 -ball $B$, defined for $n \geq 2$, called the true lover's $n$-tangle. This $n$-tangle is defined and its basic properties are proven in section 4 (pages $275-281$ ) of [13]. See figures 1 and 2 of that paper. Each component $\lambda_{j}$ of $\lambda$ is a trefoil knotted arc. Two distinct components $\lambda_{j}$ and $\lambda_{i}$ are linked in $B$ if and only if $|j-i|=1$. In fact, $B=B_{1} \cup \cdots \cup B_{2 n-1}$, where each $B_{p}$ is a 3-ball, $B_{p} \cap B_{p+1}$ is a disk, $B_{p} \cap B_{q}=\emptyset$ for $|p-q|>1, \lambda_{1} \subseteq B_{1} \cup B_{2}$, $\lambda_{j} \subseteq B_{2 j-2} \cup B_{2 j-1} \cup B_{2 j}$ for $1<j<n$, and $\lambda_{n} \subseteq B_{2 n-2} \cup B_{2 n-1}$. Moreover $\partial \lambda_{j} \subseteq \partial B_{2 j-1}$ for all $j$. See figure 3 of [13].

Now $\lambda$ is excellent (Proposition 4.1 of [13]) but is not poly-excellent. However, it has the property that for $2 \leq k \leq n$ each $k$-tangle consisting of $k$ consecutive
components of $\lambda$ is excellent. This property is not stated explicitly in [13], but it follows directly from the proof of Proposition 4.1 of that paper.

The basic idea behind the proof of Theorem 6.3 is to stack up several copies of $\lambda$, joining the bottom endpoints of one copy to the top endpoints of the copy beneath it, to obtain a new $n$-tangle $\theta$. The endpoints are to be joined using braids, so that each component of $\theta$ consists of segments which are components of the copies of $\lambda$ and may have different indices. This is to be done so that given any subset of $\{1, \ldots, n\}$ any two components of $\theta$ with indices in the subset will have segments with consecutive indices in some copy of $\lambda$. The exterior of the resulting subtangle $\widehat{\theta}$ of $\theta$ is then to be analyzed using Lemma 6.2.

Unfortunately, this basic idea does not work. This can be seen by considering the case where $\widehat{\theta}$ has exactly one component. It then meets each disk between adjacent 3 -balls in a single point, from which it follows that its exterior in not anannular. It also meets each of these 3-balls in a trefoil knotted arc, which is not excellent.

Fortunately, there is a modification of the basic idea which does work. The $n$ tangle $\theta$ will be constructed so that, among other things, each of its components doubles back twice at each level so that it meets each intermediate disk three times and meets each 3-ball in an excellent tangle. This requires us to use copies of the true lover's tangle having more than $n$ components, which accounts for much of the complication in the following argument.

Proof of Theorem 6.3. As noted above we may assume $n \geq 2$. We take as the 3 -ball containing $\theta$ the set

$$
B=\left\{(x, y, z): 0 \leq x \leq 9 n+1,-1 \leq y \leq 1,0 \leq z \leq n^{2}-n+1\right\}
$$

We regard $x$ and $y$ as increasing from left to right and from back to front, respectively, and $z$ as increasing in the downward direction. For $0 \leq p \leq n^{2}-n+1$, $1 \leq q \leq 3 n$, and $1 \leq j \leq n$, let $H_{p}=[0,9 n+1] \times[-1,1] \times\{p\}, x_{p, q}=(3 q-1,0, p)$, $a_{p, j}=x_{p, 3 j-2}, b_{p, j}=x_{p, 3 j-1}$, and $c_{p, j}=x_{p, 3 j}$.

Let $m=\left(n^{2}-n\right) / 2$. We now define certain subsets of $B$.
First suppose $0 \leq i \leq m$.

$$
\begin{gathered}
B_{i}=[0,9 n+1] \times[-1,1] \times[2 i, 2 i+1] \\
B_{i, j}=[9 j-9,9 j+1] \times[-1,1] \times[2 i, 2 i+1], 1 \leq j \leq n \\
N_{i, j}=[9 j, 9 j+1] \times[-1,1] \times[2 i, 2 i+1], 0 \leq j \leq n
\end{gathered}
$$

Next suppose $1 \leq i \leq m$.

$$
\begin{gathered}
C_{i}=[0,9 n+1] \times[-1,1] \times[2 i-1,2 i] \\
C_{i, j}=[9 j-9,9 j+1] \times[-1,1] \times[2 i-1,2 i], 1 \leq j \leq n \\
K_{i, j}=[9 j, 9 j+1] \times[-1,1] \times[2 i-1,2 i], 0 \leq j \leq n
\end{gathered}
$$

Thus $B$ is a stack of rectangular solids, starting with $B_{0}$ on the top and then alternating in the pattern $C_{i}, B_{i}, C_{i+1}, B_{i+1}$ until concluding with $B_{m}$ on the bottom. We have $C_{i} \cap B_{i}=H_{2 i}$ and $B_{i} \cap C_{i+1}=H_{2 i+1}$. Each $B_{i}$ consists of $n$ rectangular solids $B_{i, j}$ which are disjoint except for the pairs $B_{i, j}$ and $B_{i, j+1}$, whose overlap is the rectangular solid $N_{i, j}$. A similar pattern holds for the $C_{i, j}$ and $K_{i, j}$. We will refer to the $B_{i, j}$ and $C_{i, j}$ as blocks. Let $N_{i}=N_{i, 0} \cup \cdots \cup N_{i, n}$ and $K_{i}=K_{i, 0} \cup \cdots \cup K_{i, n}$. Let $B_{i, j}^{\prime}$ be the closure in $B$ of $B_{i, j}-N_{i}$. We define $C_{i, j}^{\prime}$ in a similar way. The $B_{i, j}^{\prime}$ and $C_{i, j}^{\prime}$ are called bricks.

Let $\Lambda_{0}$ be a copy of the true lover's $2 n$-tangle in $B_{0}$ with components $\lambda_{0,1}, \ldots$, $\lambda_{0,2 n}$. For $1 \leq j \leq n$ let $\alpha_{0, j}=\lambda_{0,2 j-1}, \gamma_{0, j}=\lambda_{0,2 j}$, and $\Lambda_{0, j}=\alpha_{0, j} \cup \gamma_{0, j}$. Isotop $\Lambda_{0}$ so that $\Lambda_{0, j}$ is a 2-tangle in $B_{0, j}, \alpha_{0, j}$ runs from $a_{0, j}$ to $a_{1, j}$, and $\gamma_{0, j}$ runs from $b_{1, j}$ to $c_{1, j}$. The existence of this isotopy and the similar isotopies required below follows from the description of $\lambda$ given earlier in this appendix.

For $1 \leq i \leq m-1$ let $\Lambda_{i}$ be a copy of the true lover's $3 n$-tangle in $B_{i}$ with components $\lambda_{i, 1}, \ldots, \lambda_{i, 3 n}$. For $1 \leq j \leq n$ let $\delta_{i, j}=\lambda_{i, 3 j-2}, \alpha_{i, j}=\lambda_{i, 3 j-1}$, $\gamma_{i, j}=\lambda_{i, 3 j}$, and $\Lambda_{i, j}=\delta_{i, j} \cup \alpha_{i, j} \cup \gamma_{i, j}$. Isotop $\Lambda_{i}$ so that $\Lambda_{i, j}$ is a 3-tangle in $B_{i, j}$, $\delta_{i, j}$ runs from $a_{2 i, j}$ to $b_{2 i, j}, \alpha_{i, j}$ runs from $c_{2 i, j}$ to $a_{2 i+1, j}$, and $\gamma_{i, j}$ runs from $b_{2 i+1, j}$ to $c_{2 i+1, j}$. (For $n=2$ this piece of the construction does not occur.)

Let $\Lambda_{m}$ be a copy of the true lover's $2 n$-tangle in $B_{m}$ with components $\lambda_{m, 1}, \ldots$, $\lambda_{m, 2 n}$. For $1 \leq j \leq n$ let $\delta_{m, j}=\lambda_{m, 2 j-1}, \alpha_{m, j}=\lambda_{m, 2 j}$, and $\Lambda_{m, j}=\delta_{m, j} \cup \alpha_{m, j}$. Isotop $\Lambda_{m}$ so that $\Lambda_{m, j}$ is a 2-tangle in $B_{m, j}, \delta_{m, j}$ runs from $a_{2 m, j}$ to $b_{2 m, j}$, and $\alpha_{m, j}$ runs from $c_{2 m, j}$ to $c_{2 m+1, j}$.

Let $\mathcal{B}_{3 n}$ be the Artin braid group on $3 n$ strings; let $\sigma_{1}, \ldots, \sigma_{3 n-1}$ be the standard generators for $\mathcal{B}_{3 n}$. (See [1].) For $1 \leq i \leq m$, given an element $\beta_{i}$ of $\mathcal{B}_{3 n}$, we interpret it as a geometric braid in $C_{i}$, i.e. it consists of $3 n$ disjoint proper arcs in $C_{i}$ such that the $q^{t h}$ arc runs from $x_{2 i-1, q}$ to some $x_{2 i, r}$ and meets each horizontal plane in a single point. We follow the convention that as one reads a word in the generators of $\mathcal{B}_{3 n}$ from left to right the geometric braid goes downward.

Thus we can associate to each sequence $\beta_{1}, \ldots, \beta_{m}$ of elements of $\mathcal{B}_{3 n}$ a proper 1 -manifold $\theta$ in $B$ by taking the union of the $\beta_{i}, 1 \leq i \leq m$, and the $\Lambda_{i}, 0 \leq i \leq m$. For $1 \leq j \leq n-1$, let

$$
\Sigma_{j}=\sigma_{3 j} \sigma_{3 j-1} \sigma_{3 j+1} \sigma_{3 j-2} \sigma_{3 j} \sigma_{3 j+2} \sigma_{3 j-1} \sigma_{3 j+1} \sigma_{3 j}
$$

If one partitions the $3 n$ strings into $n$ consecutive groups of 3 consecutive strings, then $\Sigma_{j}$ is obtained by crossing the $j^{t h}$ group in front of the $(j+1)^{s t}$ group and is thus the analogue of the $j^{t h}$ standard generator of $\mathcal{B}_{n}$. Note that if $\beta_{i}=\Sigma_{j}$, then we may assume that the strings numbered $3 j-2$ through $3 j+3$ lie in $C_{i, j} \cup C_{i, j+1}$ and that all other strings are vertical.

We now let the sequence $\beta_{1}, \ldots, \beta_{m}$ be

$$
\Sigma_{1}, \ldots, \Sigma_{n-1}, \Sigma_{1}, \ldots, \Sigma_{n-2}, \ldots, \Sigma_{1}, \Sigma_{2}, \Sigma_{1}
$$

Note that if the $\Sigma_{j}$ were the generators of $\mathcal{B}_{n}$, then the element $\Delta$ of $\mathcal{B}_{n}$ determined by the word corresponding to this sequence would be a half twist of the entire set of $n$ strings. It is easily checked that $\theta$ is an $n$-tangle in $B$ whose $j^{\text {th }}$ component $\theta_{j}$
runs from $a_{0, j}$ to $c_{2 m+1, n+1-j}$, and $B_{i} \cap \theta_{j}$ is some $\Lambda_{i, \varphi(i, j)}$. Moreover, given any $j^{\prime}>j$, there is some $i$ such that $\varphi\left(i, j^{\prime}\right)=\varphi(i, j)+1$.

Now let $J_{0}=\left\{j_{1}, \ldots, j_{k}\right\}$ be a non-empty subset of $\{1, \ldots, n\}$. Let $\widehat{\theta}=\theta_{j_{1}} \cup$ $\cdots \cup \theta_{j_{k}}$. Given any 3 -manifold $M$ in $B$ such that $\widehat{\theta} \cap M$ is a proper 1-manifold in $M$, we denote the exterior of $\widehat{\theta} \cap M$ in $M$ by $M^{*}$.

The proof that $\widehat{\theta}$ is excellent is based on a simple strategy which may be somewhat obscured by the deluge of notation which follows. $B_{0}$ contains a finite collection of disjoint 3-balls each of which intersects $\widehat{\theta}$ in an excellent tangle. As one moves down in $B$ these 3 -balls expand downwards, following $\widehat{\theta}$ and maintaining the same tangle type. As this occurs one notes that the "empty space" above and immediately around each of these 3 -balls consists of disjoint 3 -balls each of which meets exactly one of these 3-balls in a single disk. They can therefore be adjoined to these 3balls without changing the tangle type. Eventually one may encounter a crossing $\Sigma_{t}$ which involves strings emanating from two different 3-balls. At this point the 3 -balls come together and are attached to a 3 -ball just below $\Sigma_{t}$ whose intersection with $\widehat{\theta}$ is an excellent tangle. The result is a new 3 -ball whose intersection with $\widehat{\theta}$ is, by application of Lemma 6.2, an excellent tangle. One then notices that the "empty space" in $B$ above and immediately around this new 3 -ball again can be adjoined to the new 3 -ball without changing the tangle type. This process of downward expansion, amalgamation, and adjunction is then continued until it has engulfed all of $B$.

For $0 \leq i \leq m$ let $J_{i}$ be the set of those $j \in\{1, \ldots, n\}$ such that $\widehat{\theta} \cap B_{i, j} \neq \emptyset$, and let $\widehat{B}_{i}$ be the union of those $B_{i, j}$. Let $T_{i}=J_{0} \cup \cdots \cup J_{i}$. For $1 \leq i \leq m$ let $I_{i}$ be the set of those $j \in\{1, \ldots, n\}$ such that $\widehat{\theta} \cap C_{i, j} \neq \emptyset$, and let $\widehat{C}_{i}$ be the union of those $C_{i, j}$. Let $S_{i}=I_{1} \cup \cdots \cup I_{i}$.

We consider how these sets change when $i$ increases by one. We have that $\beta_{i+1}=\Sigma_{t}$ for some $1 \leq t \leq n-1$.

Case 1. $t, t+1 \in J_{i}$. Then $J_{i+1}=I_{i+1}=J_{i}$, and so $T_{i+1}=T_{i}$.
Case 2. $t, t+1 \notin J_{i}$. Again $J_{i+1}=I_{i+1}=J_{i}$, and $T_{i+1}=T_{i}$.
Case 3. $t \in J_{i}, t+1 \notin J_{i}$. Then $I_{i+1}=J_{i} \cup\{t+1\}, J_{i+1}=I_{i+1}-\{t\}$, and $T_{i+1}=T_{i} \cup\{t+1\}$.

Case 4. $t \notin J_{i}, t+1 \in J_{i}$. Then $I_{i+1}=J_{i} \cup\{t\}, J_{i+1}=I_{i+1}-\{t+1\}$, and $T_{i+1}=T_{i} \cup\{t\}$.

Note that by induction on $i$ it follows that $S_{i}=T_{i}$ for $1 \leq i \leq m$. It is also easily checked that $T_{m}=\{1, \ldots, n\}$.

For $0 \leq i \leq m$ let $R_{i}$ be the union of all the $B_{r, j}$ and $C_{s, j}$ such that $0 \leq r \leq i$, $1 \leq s \leq i$, and $j \in T_{i}$. The components of $R_{i}$ are, in a sense, the "minimal" rectangular solids whose union contains $\widehat{B}_{0} \cup \widehat{C}_{1} \cup \widehat{B}_{1} \cup \cdots \cup \widehat{C}_{i} \cup \widehat{B}_{i}$. Note that $R_{0}=\widehat{B}_{0}$ and $R_{m}=B$. To prove the theorem it suffices to prove by induction on $i$ that each component of $R_{i}^{*}$ is excellent.

Note that each component $W$ of $\widehat{B}_{i}$ is a 3 -ball which meets $\widehat{\theta}$ in $w$ consecutive components of $\Lambda_{i}$ for some $w \geq 2$. Moreover, there is a homeomorphism from $W$ to $B_{i}$ which is fixed on $\widehat{\theta} \cap W$. It follows that $W^{*}$ is excellent. In particular each component of $R_{0}^{*}$ is excellent.

Now suppose the components of $R_{i}^{*}$ are excellent. Let $P=R_{i} \cup \widehat{C}_{i+1} \cup \widehat{B}_{i+1}$. Let $U$ be the union of all the $B_{r, j}$ and $C_{s, j}$ such that $0 \leq r \leq i, 1 \leq s \leq i$, and $j \in T_{i+1}-T_{i}$. So $U$ is the union of all the blocks of $R_{i+1}$ which lie above a block of $P$ but are not contained in $P$. Let $Q=P \cup U$. Let $L$ be the union of all the $B_{i+1, j}$ and $C_{i+1, j}$ such that $j \in T_{i+1}-J_{i+1}$. We have that $L$ is the union of all the blocks of $R_{i+1}$ which lie below a block of $P$ but do not lie in $P$. Then $R_{i+1}=Q \cup L$. We will prove in succession that the components of $P^{*}, Q^{*}$, and $R_{i+1}^{*}$ are excellent. Recall that $\beta_{i+1}=\Sigma_{t}$.

Case 1. $t, t+1 \in J_{i}$. Then $\left(\widehat{C}_{i+1} \cup \widehat{B}_{i+1}\right)^{*}$ is homeomorphic to $\widehat{B}_{i+1}^{*}$ and so has all components excellent. The components of its intersection with $R_{i}^{*}$ are horizontal surfaces with negative Euler characteristics whose complements in the boundaries of both 3-manifolds have no components with closure a disk. Thus by Lemma 6.2 we have that each component of $P^{*}$ is excellent. Since $T_{i+1}=T_{i}$ we have $U=\emptyset$ and $L=\emptyset$, and so $R_{i+1}=Q=P$, and we are done.

Case 2. $t, t+1 \notin J_{i}$. This case is similar to Case 1.
Case 3. $t \in J_{i}, t+1 \notin J_{i}$. Note that since $T_{i+1}-T_{i}=\{t+1\}$ we have that $U=B_{0, t+1} \cup C_{1, t+1} \cup B_{1, t+1} \cup \cdots \cup C_{i, t+1} \cup B_{i, t+1}$ if $i>0$, and $U=B_{0, t+1}$ if $i=0$, and thus is the vertical stack of blocks above $C_{i+1, t+1}$. Let $U^{\prime}$ denote the union of the corresponding bricks, while $U_{t}=N_{0, t} \cup K_{1, t} \cup N_{1, t} \cup \cdots \cup K_{i, t} \cup N_{i, t}$ if $i>0$ and $U_{t}=N_{0, t}$ if $i=0$, and $U_{t+1}$ is the corresponding union with the index $t+1$ in place of $t$. Note that $L$ includes the block $B_{i+1, t}$ directly under $C_{i+1, t}$. Let $X$ be the component of $R_{i}$ containing $B_{i, t}$. Let $Y$ be the component of $\widehat{C}_{i+1} \cup \widehat{B}_{i+1}$ containing $B_{i+1, t+1}$. Denote the components of $P, Q$, and $R_{i+1}$ containing $X$ by $P_{X}, Q_{X}$, and $R_{i+1, X}$, with similar notation for the components of $P^{*}, Q^{*}$, and $R_{i+1}^{*}$ containing $X^{*}$.

Subcase (a). $t+2 \notin J_{i}$. Then each component of $R_{i}$ meets a single component of $\widehat{C}_{i+1} \cup \widehat{B}_{i+1}$, and vice versa. For components of $R_{i}$ other than $X$ the situation is as in Case 1. Let $Z=C_{i+1, t} \cup C_{i+1, t+1} \cup B_{i+1, t+1}$. Then $Z^{*}$ is homeomorphic to $B_{i+1, t+1}^{*}$ and so is excellent.

Suppose $t-1 \notin J_{i}$. Then $Y=Z, P_{X}^{*}=X^{*} \cup Z^{*}, Q_{X}^{*}=P_{X}^{*} \cup U^{\prime} \cup U_{t+1}$, and $R_{i+1, X}^{*}=Q_{X}^{*} \cup B_{i+1, t}^{\prime} \cup N_{i+1, t-1}$. We have that $X^{*} \cap Z^{*}$ is the surface $B_{i, t}^{*} \cap C_{i+1, t}^{*}$, which is easily seen to satisfy the requirements of Lemma 6.2 , and thus $P_{X}^{*}$ is excellent. $U^{\prime} \cup U_{t+1}$ is a 3-ball which meets $P_{X}^{*}$ in a disk; thus $Q_{X}^{*}$ is homeomorphic to $P_{X}^{*}$. We have that $B_{i+1, t}^{\prime} \cup N_{i+1, t-1}$ is a 3 -ball which meets $Q_{X}^{*}$ in a disk, and so $R_{i+1}^{*}$ is homeomorphic to $Q_{X}^{*}$, and we are done.

Suppose $t-1 \in J_{i}$. Let $\widehat{Y}$ be the closure in $B$ of $Y-Z$. Then $P_{X}^{*}=X^{*} \cup$ $\widehat{Y}^{*} \cup Z^{*} \cup N_{i+1, t-1}, Q_{X}^{*}=P_{X}^{*} \cup U^{\prime} \cup U_{t+1}$, and $R_{i+1, X}^{*}=Q_{X}^{*} \cup B_{i+1, t}^{\prime}$. Now $\widehat{Y}^{*}$ is homeomorphic to $W^{*}$, where $W$ is the component of $\widehat{B}_{i+1}$ containing $B_{i+1, t-1}$, and so is excellent. $\widehat{Y}^{*}$ meets $X^{*}$ along a surface satisfying the hypotheses of Lemma 6.2, and so $X^{*} \cup \widehat{Y}^{*}$ is excellent. Lemma 6.2 also implies that $X^{*} \cup \widehat{Y}^{*} \cup Z^{*}$ is excellent. This manifold meets the 3 -ball $N_{i+1, t-1}$ in a disk and so is homeomorphic to $P_{X}^{*}$. As before $Q_{X}^{*}$ is homeomorphic to $P_{X}^{*}$. The 3 -ball $B_{i+1, t}^{\prime}$ meets $Q_{X}^{*}$ in a disk, and so $R_{i+1, X}^{*}$ is homeomorphic to $Q_{X}^{*}$.

Subcase (b). $t+2 \in J_{i}$. Let $V$ be the component of $R_{i}$ containing $B_{i, t+2}$. For
components of $R_{i}$ other than $X$ or $V$ the situation is as in Case 1 . Let $W$ be the component of $\widehat{B}_{i+1}$ containing $B_{i+1, t+1}$. So $W=B_{i+1, t+1} \cup \cdots \cup B_{i+1, t+r}$ for some $r>1$. Let $Z=C_{i+1, t} \cup W \cup C_{i+1, t+1} \cup \cdots \cup C_{i+1, t+r}$. Then $Z^{*}$ is homeomorphic to $W^{*}$ and so is excellent.

Suppose $t-1 \notin J_{i}$. Then $Y=Z, P_{X}^{*}=X^{*} \cup Z^{*} \cup V^{*}, Q_{X}^{*}=P_{X}^{*} \cup U^{\prime}$, and $R_{i+1, X}^{*}=Q_{X}^{*} \cup B_{i+1, t}^{\prime} \cup N_{i+1, t-1}$. Lemma 6.2 implies that $P_{X}^{*}$ is excellent. $U^{\prime}$ is a 3 -ball which meets $P_{X}^{*}$ in a disk, and so $Q_{X}^{*}$ is homeomorphic to $P_{X}^{*}$. Since $B_{i+1, t}^{\prime} \cup N_{i+1, t-1}$ is a 3 -ball which meets $Q_{X}^{*}$ in a disk we have that $R_{i+1, X}^{*}$ is homeomorphic to $Q_{X}^{*}$.

Suppose $t-1 \in J_{i}$. Let $\widehat{Y}$ be the closure in $B$ of $Y-Z$. Then $P_{X}^{*}=X^{*} \cup$ $\widehat{Y}^{*} \cup Z^{*} \cup V^{*} \cup N_{i+1, t-1}, Q_{X}^{*}=P_{X}^{*} \cup U^{\prime}$, and $R_{i+1, X}^{*}=Q_{X}^{*} \cup B_{i+1, t}^{\prime}$. Successive applications of Lemma 6.2 show that $X^{*} \cup \widehat{Y}^{*} \cup Z^{*} \cup V^{*}$ is excellent. Since this manifold meets the 3 -ball $N_{i+1, t-1}$ in a disk it is homeomorphic to $P_{X}^{*}$. For the same reasons $P_{X}^{*}$ is homeomorphic to $Q_{X}^{*}$, which is homeomorphic to $R_{i+1, X}^{*}$.

Case 4. $t \notin J_{i}, t+1 \in J_{i}$. This case is similar to Case 3.

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