ATTACHING BOUNDARY PLANES TO IRREDUCIBLE OPEN 3-MANIFOLDS

Robert Myers

ABSTRACT. Given any connected, open 3-manifold U having finitely many ends, a non-compact 3-manifold M is constructed having the following properties: the interior of M is homeomorphic to U; the boundary of M is the disjoint union of finitely many planes; M is not almost compact; M is eventually end-irreducible; there are no proper, incompressible embeddings of $S^1 \times \mathbf{R}$ in M; every compact subset of M is contained in a larger compact subset whose complement is an annular; there is a compact subset of M whose complement is \mathbf{P}^2 -irreducible.

If U is irreducible it also has the following two properties: every proper, non-trivial plane in M is boundary-parallel; every proper surface in M each component of which has non-empty boundary and is non-compact and simply connected lies in a collar on ∂M .

This construction can be chosen so that M admits no homeomorphisms which take one boundary plane to another or reverse orientation. For the given U there are uncountably many non-homeomorphic such M.

Two auxiliary results may be of independent interest. First, general conditions are given under which infinitely many "trivial" compact components of the intersection of two proper, non-compact surfaces in an irreducible 3-manifold can be removed by an ambient isotopy. Second, n component tangles in a 3-ball are constructed such that every non-empty union of components of the tangle has hyperbolic exterior.

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INTRODUCTION

Suppose U is an open 3-manifold and M is a non-compact 3-manifold each of whose boundary components is homeomorphic to \mathbf{R}^2 . If U is homeomorphic to the interior of M, then one may say that M is obtained by **attaching boundary planes** to U. The fact that this can be done in different ways is most dramatically illustrated by the existence of 3-manifolds with interior homeomorphic to \mathbf{R}^3 and boundary homeomorphic to \mathbf{R}^2 which are not homeomorphic to $\mathbf{R}^2 \times [0, \infty)$. The first such example was constructed by Fox and Artin [9]. Tucker [18] later gave a different method for constructing such examples and showed that planes were the only surfaces which could be, in a sense explained below, "bad" boundary components of 3-manifolds.

A non-compact 3-manifold M is **almost compact** if there is a compact 3manifold Q and a closed subset K of ∂Q such that M is homeomorphic to Q - K. The examples mentioned above fail to be almost compact. Tucker [18] showed that if one attaches a connected surface with finitely generated fundamental group to a \mathbf{P}^2 -irreducible, almost compact 3-manifold in such a way that it becomes an incompressible boundary component, then either the new 3-manifold is almost compact or the surface is a plane. Scott and Tucker [17] gave other examples of 3-manifolds which are not almost compact but have almost compact interiors, including an example with boundary consisting of two disjoint planes such that the complement of each plane is homeomorphic to $\mathbf{R}^2 \times [0, 1)$ and an example with boundary a single plane whose complement is homeomorphic to $S^1 \times \mathbf{R}^2$. It can be shown that the latter example contains no proper non-separating planes.

This paper gives a general procedure for attaching boundary planes to U to obtain an M having special properties which may be very different from those of U. The open 3-manifold U is required to be connected and irreducible and to have finitely many ends. It is not assumed to be orientable or \mathbf{P}^2 -irreducible. One attaches any finite number of boundary planes to U subject to the restriction that one attaches at least one plane to each end. The resulting 3-manifold M is not almost compact. However, it has an important property which is shared by those almost compact 3-manifolds whose boundaries are finite disjoint unions of planes, namely eventual end-irreducibility. This property, introduced by Brown [8], ensures that the ends of the 3-manifold, while perhaps not tame, are at least not excessively wild, in the sense that they can be analyzed using incompressible surface theory. This has proven to be a fruitful concept in the study of non-compact 3-manifolds. See [2, 3, 4, 6, 7, 8]. The next section includes discussions of ends and of eventual end-irreducibility.

The most important special properties considered in this paper concern certain embeddings of surfaces. A surface S embedded in a 3-manifold M is **proper** if $S \cap \partial M = \partial S$ and $S \cap C$ is compact for every compact subset C of M. A proper surface S with $\partial S = \emptyset$ is ∂ -**parallel** if some component of M - S has closure homeomorphic to $S \times [0, 1]$ with $S = S \times \{0\}$ and $S \times \{1\}$ a component of ∂M ; it is **end-parallel** or **trivial** if some component of M - S has closure homeomorphic to $S \times [0, \infty)$ with $S = S \times \{0\}$. A 3-manifold M is **aplanar** if every proper plane in M is either ∂ -parallel or end-parallel; it is **acylindrical** if the same is true for every proper incompressible cylinder $S^1 \times \mathbf{R}$. It is **totally acylindrical** if it contains no proper incompressible cylinders. The boundary planes will be attached so that M is aplanar and totally acylindrical. In particular one can take an irreducible one ended open 3-manifold which contains non-trivial planes or nontrivial incompressible cylinders, such as the interior of a cube with handles or $S^3 - K$ for K a torus knot, cable knot, or composite knot, and create an aplanar, totally acylindrical 3-manifold by attaching a single boundary plane.

It will be shown that M has two further embedding properties. It is strongly aplanar, meaning that in addition to being aplanar it has the property that given any proper surface \mathcal{P} such that each component of \mathcal{P} is non-compact, is simply connected, and has non-empty boundary, there exists a collar on ∂M containing \mathcal{P} . It is also **an annular at infinity** in the sense that for every compact subset K of M there is a compact subset L of M containing K such that M - L is an annular. Note for comparison that $\mathbf{R}^2 \times [0,1)$ is a planar but not strongly a planar, while $\mathbf{R}^2 \times [0,1]$ is a planar but neither strongly a planar nor an annular at infinity. (See [15].) These two properties are involved in the study of "plane sums" of noncompact 3-manifolds, i.e. 3-manifolds obtained by gluing together a collection of 3-manifolds whose boundary components are planes along these planes. In [15] it will be shown that (subject to some mild additional hypotheses) if the summands are irreducible, strongly aplanar, and anannular at infinity, then the image of each gluing plane is non-trivial in the sum, and every non-trivial plane in the sum is ambient isotopic to one of these planes. This result is then used to investigate a non-compact analogue of the connected sum called the "end sum." The present paper provides the mechanism for generating the relevant examples.

By a modification of the basic construction M can be built so that, in addition to the previous properties, it admits no orientation reversing self-homeomorphisms, there are no self-homeomorphisms taking one boundary plane to another, and there are uncountably many pairwise non-homeomorphic such M having the same number of boundary planes per end of U. Moreover all these properties hold for the 3manifolds obtained by deleting any collection of boundary planes from M subject to the restriction that there remains at least one boundary plane attached to each end. These properties are also relevant to the study of plane sums and end sums.

Although irreducible 3-manifolds are the main objects of interest in this paper it is worth noting that some of our results generalize to the reducible case. In fact the only properties of M which require the irreducibility of U are aplanarity and strong aplanarity. Moreover M can be constructed so as to be **eventually** \mathbf{P}^2 **irreducible** in the sense that there is a compact subset of M whose complement is \mathbf{P}^2 -irreducible. One can, for example, create a totally acylindrical, eventually \mathbf{P}^2 -irreducible 3-manifold by attaching two boundary planes to the product of a closed, connected surface with \mathbf{R} , including the cases when the surface is S^2 or \mathbf{P}^2 .

Two auxiliary results in this paper may be of independent interest. First, we give general conditions under which infinite collections of "trivial" intersection curves of two non-compact proper surfaces in an irreducible 3-manifold can be removed by an ambient isotopy. By a "trivial" intersection curve we mean a simple closed curve which bounds disks on both surfaces or a proper arc which is ∂ -parallel on both surfaces. These results are used both in the present paper and in [15]. Second, we prove the existence of "poly-excellent tangles." A poly-excellent *n*-tangle is the union of *n* disjoint proper arcs in a 3-ball such that the union of any non-empty collection of its components has hyperbolic exterior. This result is required in our modification of the basic construction.

The paper is organized as follows. Section 1 contains background material and discusses exhaustions of non-compact 3-manifolds. In particular it introduces the concept of a "nice" exhaustion. It is readily seen that a 3-manifold M with a nice exhaustion is not almost compact but is eventually end-irreducible, eventually \mathbf{P}^2 -irreducible, and anannular at infinity. Section 2 gives conditions under which "trivial" intersections of non-compact surfaces can be removed by an ambient isotopy. Section 3 reformulates some work of Winters [19, 20] to show that a 3-manifold with a nice exhaustion is totally acylindrical and, if it is irreducible, is aplanar. Sections 4 and 5 show that an irreducible 3-manifold with a nice exhaustion is strongly aplanar. Section 6 shows how to construct M from U so that M has a nice exhaustion; it also describes the modification of this basic construction and proves the additional properties listed above. The proof of the existence of poly-excellent tangles has a different flavor from the rest of the paper and is given in an appendix so as not to disrupt the main line of the argument.

1. Preliminaries

We shall work throughout in the PL category. An *m*-manifold M may or may not have boundary but is assumed to be second countable. ∂M and *int* M denote the manifold theoretic boundary and interior of M, respectively. Let A be a subset of M. The topological boundary, interior, and closure of A in M are denoted by $Fr_M A$, $Int_M A$, and $Cl_M A$, respectively, with the subscript deleted when M is clear from the context. All isotopies of A in M will be ambient. A is **bounded** if Cl Ais compact. M is **open** if $\partial M = \emptyset$ and no component of M is compact.

A surface is a 2-manifold; no assumptions are made about its being connected or compact or having a boundary.

A map $f : M \to N$ of manifolds is ∂ -proper if $f^{-1}(\partial N) = \partial M$. It is endproper if preimages of compact sets are compact. It is proper if it has both these properties. These terms are applied to a submanifold if its inclusion map has the corresponding property.

Let S be a proper codimension one submanifold of the m-manifold M. Suppose S' is either another such submanifold such that $int S \cap int S' = \emptyset$ or is an end-proper submanifold of ∂M . Assume that $\partial S = \partial S'$. Then S and S' are **parallel** if some component of $M - (S \cup S')$ has closure homeomorphic to $S \times [0, 1]$ with $S \times \{0\} = S$ and $S \times \{1\} = S'$ when $\partial S = \emptyset$, while $((\partial S) \times [0, 1]) \cup (S \times \{1\}) = S'$ when $\partial S \neq \emptyset$. The product $S \times [0, 1]$ is a **parallelism** between S and S'. When $S' \subseteq \partial M$ one says that S is ∂ -parallel. We say that S is end-parallel or trivial if some component of M - S has closure homeomorphic to $S \times [0, \infty)$ with $S = S \times \{0\}$.

Infinite sequences, unless indicated otherwise, will be indexed by the set of nonnegative integers. An exhausting sequence $C = \{C_n\}$ for a non-compact *m*-manifold M is a sequence $C_0 \subseteq C_1 \subseteq C_2 \subseteq \cdots$ of compact subsets of M whose union is M. A sequence $V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots$ of open subsets of M is an end sequence associated to C if each V_n is a component of $M - C_n$. Two end sequences $\{V_n\}$ and $\{W_p\}$ associated to exhausting sequences C and K for M are cofinal if for every n there is a p such that $V_n \supseteq W_p$ and for every p there is a q such that $W_p \supseteq V_q$. Cofinality is an equivalence relation on end sequences of M. The equivalence classes are called the ends of M. The set of all ends of M is denoted by $\varepsilon(M)$. An endproper map $M \to N$ induces a well defined function $\varepsilon(M) \to \varepsilon(N)$. If ∂M has no compact components, then the inclusion map induces a well defined bijection $\varepsilon(int M) \to \varepsilon(M)$.

An exhausting sequence C for a connected, non-compact m-manifold M is an **exhaustion** for M if each C_n is a compact, connected m-manifold, $C_n \cap \partial M$ is either empty or an (m-1)-manifold, $C_n \subseteq Int C_{n+1}$, and $M - C_n$ has no bounded components. Connected non-compact m-manifolds always have exhaustions. Given an exhaustion C for M and a subsequence $\{n_k\}$ of the non-negative integers, let $C'_k = C_{n_k}$. Then C' is also an exhaustion for M and will be called a **subexhaustion** of C.

The reader is referred to [10] or [11] for basic 3-manifold topology, including the definition of incompressible surface. We adopt the conventions of [11] that every proper disk in a 3-manifold M is incompressible and that a proper 2-sphere is compressible if and only if it bounds a 3-ball. M is **irreducible** if every 2-sphere in M is compressible; it is **P**²-**irreducible** if it contains no 2-sided projective planes. It is ∂ -**irreducible** if ∂M is incompressible in M. It is **anannular** if every proper incompressible annulus in M is ∂ -parallel. It is **atoroidal** if every proper incompressible torus in M is ∂ -parallel.

A partial disk is a pair $(D, \partial_0 D)$, where D is a disk and $\partial_0 D$ is a non-empty finite union of disjoint arcs in ∂D ; the **order** of $(D, \partial_0 D)$ is the number of these arcs. A **halfdisk** is a partial disk of order one; a **band** is a partial disk of order two. $\partial D - int \partial_0 D$ is denoted by $\partial_1 D$. A partial disk may be denoted by D when $\partial_0 D$ is clear from the context. Suppose D is a partial disk contained in a surface S. Then D is **proper** in S if $D \cap \partial S = \partial_0 D$. If D is a proper partial disk in S such that no component of $S - Int_S D$ is a proper halfdisk D' in S with $\partial_1 D' = D \cap D'$, then D is **well embedded** in S.

A proper surface S in M is ∂ -incompressible if it is not a ∂ -parallel disk and whenever D is a halfdisk in M such that $D \cap \partial M = \partial_0 D$ and $D \cap S = \partial_1 D$, one has that $\partial_1 D$ is ∂ -parallel in S.

We now give some terminology for some standard isotopies which will be used later. Suppose S and T are end-proper surfaces in an irreducible 3-manifold M. Suppose S and T are in general position and J is a simple closed curve component of $int S \cap int T$ which bounds a disk D on S and a disk G on T. Then J is **innermost** on S if $D \cap T = J$. In this case there is a 3-ball B in M with $\partial B = D \cup G$. Let B^+ be a regular neighborhood of B in M. There is an ambient isotopy of S in Msupported in B^+ which carries D to G and then off G into $B^+ - B$. This is called a **disk push** of D across B past G. Let S^* and D^* be the images of S and D, respectively, under this isotopy. Then S^* is in general position with respect to T and $((S^* - D^*) \cap T) \subseteq ((S - D) \cap T)$ and $D^* \cap T = \emptyset$.

Now suppose that S and T are also ∂ -proper in M and that there is an endproper surface R in ∂M such that $\partial S \cup \partial T$ lies in *int* R and R is incompressible in M. Suppose α is a component of $S \cap T$ which is an arc such that $\alpha = \partial_1 D = \partial_1 G$, where D and G are proper halfdisks in S and T, respectively. Then α is **innermost** on S if $D \cap T = \alpha$. In this case $\partial_0 D \cup \partial_0 G = \partial D'$ for a disk D' in *int* R, and there is a 3-ball B in M such that $\partial B = D \cup G \cup D'$. Let B^+ be a regular neighborhood of B in M. There is an ambient isotopy of S in M supported in B^+ which carries D to G and then off G into $B^+ - B$. This is called a **halfdisk push** of D across Bpast G. The images S^* and D^* then satisfy the same conditions as for a disk push.

A partial plane P is a non-compact simply connected 2-manifold with $\partial P \neq \emptyset$. When ∂P has exactly one component P is called a **halfplane**. We next give criteria for a proper plane or halfplane to be trivial. Note that a proper halfplane is trivial if and only if it is ∂ -parallel.

Lemma 1.1. Let M be a connected, irreducible, non-compact 3-manifold.

- (1) A proper plane P in M is trivial if and only if there exist sequences $\{D_n\}$ and $\{D'_n\}$ of disks in M such that $\{D_n\}$ is an exhaustion for P, $D'_n \cap P = \partial D_n$, and $\cup D'_n$ is end-proper in M.
- (2) A proper halfplane P in M is trivial if and only if there exist sequences $\{D_n\}$ and $\{D'_n\}$ of halfdisks in M such that D_n is proper in P, $\{D_n\}$ is an exhaustion for P, $D'_n \cap P = \partial_1 D_n = \partial_1 D'_n$, $D'_n \cap \partial M = \partial_0 D'_n$, $\cup D'_n$ is end-proper in M, and $\partial_0 D_n \cup \partial_0 D'_n$ bounds a disk in ∂M .

Proof. Necessity is obvious in both cases.

(1) Given any compact subset K of M there is an n such that D_n contains $K \cap P$ and $D'_m \cap K = \emptyset$ for all $m \ge n$. If K is a simple closed curve which meets Ptransversely in a single point, then K meets the 2-sphere $D_n \cup D'_n$ transversely in a single point, contradicting the fact that $D_n \cup D'_n$ bounds a 3-ball in M. Thus P must separate M into two components with closures X and Y. By passing to a subsequence we may assume that all D'_n lie in X. Then for each compact subset K of X there is an n such that K lies in the 3-ball in X bounded by $D_n \cup D'_n$. It follows that X is homeomorphic to $\mathbf{R}^2 \times [0, 1)$.

(2) The proof is similar to that of (1) and is left to the reader. \Box

Let M be a connected, non-compact 3-manifold. It is said to be **eventually** end-irreducible if it contains a compact subset J such that for every compact subset K containing J there is a compact subset L containing K such that every loop in M - L which is null-homotopic in M - J must be null-homotopic in M - K.

Let C be an exhaustion for a connected, non-compact 3-manifold M. Denote $Fr C_n$ by $F_n, C_{n+1} - Int C_n$ by X_{n+1} , and $\bigcup_{n \ge k} F_n$ by \mathcal{F}_k . The exhaustion is **good** if for each $n \ge 0$ one has that $F_n \cup F_{n+1}$ is incompressible in X_{n+1}, X_{n+1} is \mathbf{P}^2 -irreducible, no component of \mathcal{F}_0 is a disk, and no component of X_{n+1} has the form $F \times [0, 1]$, where $F \times \{0\}$ and $F \times \{1\}$ are components of F_n and F_{n+1} respectively. Standard arguments show that in this case \mathcal{F}_0 is incompressible in $M - Int C_0$

and has no 2-sphere or projective plane components and that $M - Int C_0$ is \mathbf{P}^2 irreducible. It is easily seen that every subexhaustion of a good exhaustion is good. Note that M itself need not be \mathbf{P}^2 -irreducible or even irreducible.

Lemma 1.2. If the connected, non-compact 3-manifold M admits a good exhaustion, then M is eventually end-irreducible and eventually \mathbf{P}^2 -irreducible and is not almost compact.

Proof. The first property is well known; its proof will be sketched for completeness. Let $J = C_0$. Suppose K is compact and contains J. There is an n > 0 such that $K \subseteq C_n$. Let $L = C_{n+1}$. Then the incompressibility of F_n in $M - Int C_0$ implies that any null-homotopy in M - J of a loop in M - L can be cut off on F_n so as to obtain a null-homotopy in M - K.

The \mathbf{P}^2 -irreducibility of $M - Int C_0$ implies that M is eventually \mathbf{P}^2 -irreducible.

To show that M is not almost compact it suffices to show that the fundamental group of some component of $M - C_0$ is not finitely generated. In fact, this is the case for every component V of $M - C_0$. We may assume that $H_1(V)$ is finitely generated. Then for all sufficiently large n one has that the intersection of each component of $V \cap X_{n+1}$ with each component of $V \cap X_{n+2}$ is connected. Since no component of X_{n+1} has the form $F \times [0, 1]$ described above it follows from Theorem 10.2 of [10] that for every component Y of X_{n+1} and component S of FrY we have that $\pi_1(S) \to \pi_1(Y)$ is a non-surjective monomorphism. It then follows that $\pi_1(V)$ is an infinite non-trivial free product with amalgamation, hence is not finitely generated. \Box

Now suppose M is a connected 3-manifold with a finite number $\mu > 0$ of ends whose boundary consists of a finite number $\nu \ge 0$ of disjoint planes E^i . An exhaustion C for M is **nice** if $C_n \cap \partial M$ consists of a single disk in each E^i , X_{n+1} is \mathbf{P}^2 -irreducible, ∂ -irreducible, and anannular, each component of F_n has negative Euler characteristic, each orientable component of F_n has positive genus, and $M - Int C_n$ has μ components for all $n \ge 0$. Note again that M need not be \mathbf{P}^2 -irreducible or irreducible.

Lemma 1.3. Let C be a nice exhaustion for the connected, non-compact 3-manifold M. Then the following conditions hold.

- (1) C is a good exhaustion for M.
- (2) $X_{n+1} \cap \partial M$ consists of one annulus in each component of ∂M and is incompressible in X_{n+1} .
- (3) $M C_0$, $M Int C_0$, and, for $n \ge 1$, $C_n Int C_0$ are \mathbf{P}^2 -irreducible, ∂ -irreducible, and anannular.
- (4) In $M Int C_0$ one has that $\partial M Int(C_0 \cap \partial M)$ is incompressible and F_n is ∂ -incompressible.
- (5) Every subexhaustion of C is nice.
- (6) M is an annular at infinity.

Proof. Clearly $\{C_n \cap E^i\}$ is an exhaustion for E^i by concentric disks, and so $X_{n+1} \cap E^i$ is an annulus with one boundary component in ∂F_n and the other in ∂F_{n+1} .

Suppose D is a compressing disk for $F_n \cup F_{n+1}$ in X_{n+1} . Since X_{n+1} is ∂ -irreducible there is a disk D' in ∂X_{n+1} with $\partial D' = \partial D$ such that D' must contain some annulus $X_{n+1} \cap E_i$ and hence some component of $F_n \cup F_{n+1}$, contradicting the fact that no component of \mathcal{F}_0 is a planar surface. Thus $F_n \cup F_{n+1}$ is incompressible in X_{n+1} . If some component of X_{n+1} is a product $F \times [0, 1]$ with $F \times \{0\}$ in F_n and $F \times \{1\}$ in F_{n+1} , then since F has negative Euler characteristic there is an incompressible product annulus in $F \times [0, 1]$ which is not ∂ -parallel , contradicting the fact that X_{n+1} is an annular. Thus C satisfies (1) and (2).

Now suppose D is a compressing disk for the boundary of $Y_n = C_n - Int C_0$, where n > 1. Let $F = F_1 \cup \cdots \cup F_{n-1}$. If $D \cap F = \emptyset$, then D lies in X_1 or X_n , say X_1 . Isotop D so that ∂D lies in F_0 . Then $\partial D = \partial D'$ for a disk D' in ∂X_1 . If D' does not lie in ∂Y_n , then it must contain a component of F_1 , contradicting the positive genus condition. Thus we may assume that $D \cap F \neq \emptyset$. Suppose D does not meet both F_0 and F_n ; say it misses F_n . Then D can be isotoped so that ∂D lies in F_0 . By the incompressibility of the F_i and the irreducibility of the X_i we may, if necessary, apply a finite sequence of disk pushes to D so that $D \cap F = \emptyset$. Thus we may assume that D meets both F_0 and F_n . By a disk push argument similar to that just given one may assume that $D \cap F$ contains no simple closed curves. Assume further that D has been isotoped so that among all such D the number of components of $D \cap F$ is minimal. Now $D \cap F$ splits D into partial disks. Among these is a halfdisk H which meets ∂D in $\partial_0 H$ and F in $\partial_1 H$. We may assume $H \subseteq X_1$, and so $\partial_1 H \subseteq F_1$. Then $\partial H = \partial H'$ for a disk H' in ∂X_1 . This implies that $\partial(\partial_1 H)$ lies in a single component J of ∂F_1 , and so $J \cap H'$ is an arc which splits H' into a disk H'_1 in F_1 and a disk H'_0 in $\partial X_1 - int F_1$. Since X_1 is irreducible $H \cup H'$ bounds a ball B in X_1 , and so there is a halfdisk push of H across B past H'_1 which removes at least $\partial_1 H$ from the intersection, thereby contradicting minimality. Thus Y_n is ∂ -irreducible. The \mathbf{P}^2 -irreducibility of Y_n and $M - Int C_0$ follows from that of the X_j together with the incompressibility of the F_i . This implies that $M - C_0$ is \mathbf{P}^2 -irreducible as well.

Suppose D is a compressing disk for $\partial(M - Int C_0)$. Then for some n one has $D \subseteq (Y_n - F_n)$, and so $\partial D = \partial D'$ for a disk D' in ∂Y_n . If D' does not lie in $\partial(M - Int C_0)$, then it must contain a component of F_n . But this contradicts the positive genus condition, and so $M - Int C_0$ is ∂ -irreducible. The positive genus condition applied to F_0 now implies that $\partial M - Int(C_0 \cap \partial M)$ is incompressible in $M - Int C_0$ and hence in $M - C_0$.

Suppose D is a ∂ -compressing halfdisk in $M - Int C_0$ for some F_n . So $D \cap F_n = \partial_1 D$ and $D \cap \partial (M - Int C_0) = \partial_0 D$. Now either $D \subseteq Y_n$ or $D \cap Y_n = \partial_1 D$. In the first case the ∂ -irreducibility of Y_n implies that $\partial D = \partial D'$ for a disk D' in ∂Y_n . It follows that $\partial(\partial_1 D)$ lies in a component J of ∂F_n and that $J \cap D'$ is an arc which splits D' into a disk in F_n and a disk in $\partial Y_n - int F_n$; thus $\partial_1 D$ is ∂ -parallel in F_n . In the second case D lies in some $Y' = C_{n+k} - Int C_n$, and one applies the same argument to this manifold.

We have now established (4) and the portions of (3) not dealing with annuli. So suppose A is an incompressible proper annulus in Y_n . Assume that the number of components of $A \cap F$ is minimal. Then a disk push argument shows that it

contains no simple closed curve components which bound disks on A. Suppose α is a component of $A \cap F$ which is an arc. If α is ∂ -parallel in A, then we may assume that α is innermost on A, hence $\alpha = \partial_1 D$ for a proper halfdisk D on A such that $D \cap F = \alpha$. Since F is ∂ -incompressible in $M - Int C_0$ one has that α is ∂ -parallel in F, and so $\alpha = \partial_1 D'$ for a proper halfdisk D' in F. Then the disk $D \cup D'$ is ∂ -parallel in Y_n via a 3-ball B. Thus there is a halfdisk push of D across B past D' which removes at least α from $A \cap F$, contradicting minimality. Thus α is a spanning arc in A. Since each F_i separates Y_n there must be two such arcs α and α' such that $\alpha \cup \alpha' = \partial_1 D$ for a proper band D in A such that $D \cap F = \alpha \cup \alpha'$. Now D lies in some X_i , and so $\partial D = \partial D'$ for a disk D' in ∂X_i . Now $D \cup D'$ bounds a 3-ball B in X_i . Then $D' \cap F$ consists of one or two disks. If it is a single disk D'', then D' is the union of D'' and two disks in ∂Y_n . There is an isotopy (a **band push**) which moves the disk D across B past D'' which removes at least $\alpha \cup \alpha'$ from $A \cap F$, contradicting minimality. If $D' \cap F$ consists of two disks then there are ∂ -compressing halfdisks for A in Y_n , and hence A is ∂ -parallel in Y_n . (See Lemma 2.2 of [14].) Thus we may assume $A \cap F$ contains no spanning arcs of A. Suppose J is a simple closed curve component of $A \cap F$ which is non-contractible in A. Then A is ∂ -parallel in A via an annulus A_0 in A; we may assume that $A_0 \cap F = J$. Then A_0 is a proper incompressible annulus in some X_j . Since X_j is an annular A_0 is parallel in X_i to an annulus A_1 in ∂X_i with $\partial A_1 = \partial A_0$. The parallelism T is a solid torus with $\partial T = A_0 \cup A_1$. There is a component A_2 of $A_1 \cap F$ which is an annulus one of whose boundary components is J. Then there is an isotopy supported in a regular neighborhood of T which moves A_0 across T and past A_2 , thereby removing at least J from $A \cap F$ and thereby contradicting minimality. Thus we assume that $A \cap F = \emptyset$. Therefore A lies in some X_i and is parallel in X_i to an annulus A' in ∂X_i whose boundary misses that of F. By the positive genus condition $A' \cap F = \emptyset$, and so $A' \subseteq \partial Y_n$. Thus Y_n is an annular.

Now suppose A is an incompressible proper annulus in $M - Int C_0$. Then for some n one has $A \subseteq (Y_n - F_n)$, and so A is parallel to an annulus A' in ∂Y_n . If A' does not lie in $\partial (M - Int C_0)$, then it must contain a component of F_n . But this contradicts the positive genus condition, and so $M - Int C_0$ is an annular. The positive genus condition applied to F_0 implies that $M - C_0$ is an annular. This completes the proof of (3).

Clearly the properties we have proven for Y_n also hold for $C_n - Int C_m$ whenever n > m. It follows from this that every subexhaustion of C is nice, thus establishing (5).

Finally we note that given any compact subset J of M we can choose an m > 0such that $J \subseteq C_m$. Then by (5) and (3) we have that $M - Int C_m$ is an annular. Suppose A is an incompressible proper annulus in $M - C_m$. Then A is parallel in $M - Int C_m$ to an annulus A' in $\partial(M - Int C_m)$. Since F_m is incompressible in $M - Int C_m$ and has no disk or annulus components A' must lie in $\partial(M - C_m)$. Thus M is an annular at infinity. \Box

2. Removing Trivial Intersections

Given two compact, proper, incompressible surfaces in an irreducible 3-manifold

it is a standard result that one can ambiently isotop one of the surfaces so that the two surfaces are in general position and no simple closed curve component of their intersection is trivial, i.e. bounds a disk in one (and hence both) of the surfaces. This isotopy consists of a finite series of disk pushes. In the case of non-compact surfaces one is faced with the possibility of an infinite series of disk pushes, which might not converge to an ambient isotopy. This section gives, in a slightly more general setting, conditions under which the isotopy can be accomplished, treating as well the removal of ∂ -parallel arcs in ∂ -incompressible surfaces.

Proposition 2.1. Let M be a connected, irreducible, non-compact 3-manifold which is not homeomorphic to \mathbb{R}^3 . Let \mathcal{P} and \mathcal{Q} be proper surfaces in M which are in general position. Let \mathcal{J} be a union of simple closed curve components of $\mathcal{P} \cap \mathcal{Q}$. Assume that the following conditions are satisfied.

- (1) No component of \mathcal{P} or of \mathcal{Q} is a 2-sphere.
- (2) Each component J of \mathcal{J} bounds a disk D(J) on \mathcal{P} and a disk G(J) on \mathcal{Q} .
- (3) There is no infinite sequence $\{J_m\}$ of distinct components of \mathcal{J} such that either $D(J_m) \subseteq int D(J_{m+1})$ for all m or $G(J_m) \subseteq int G(J_{m+1})$ for all m, i.e. there is no infinite nesting on \mathcal{P} or on \mathcal{Q} among the components of \mathcal{J} .

Then there is an ambient isotopy of \mathcal{P} in M, fixed on ∂M , which takes \mathcal{P} to a surface \mathcal{P}' such that \mathcal{P}' and \mathcal{Q} are in general position and $(\mathcal{P}' \cap \mathcal{Q}) \subseteq (\mathcal{P} \cap \mathcal{Q}) - \mathcal{J}$. Moreover, the isotopy is fixed on $\mathcal{P}' \cap \mathcal{Q}$.

Proof. Since neither \mathcal{P} nor \mathcal{Q} contains 2-spheres the disks D(J) and G(J) are unique. Call D(J) maximal if there is no D(J') such that $D(J) \subseteq int D(J')$. These maximal disks are disjoint and their union contains all the D(J). Let $\{D(J_i)\}$ be the set of all maximal disks. Let D_i be a regular neighborhood of $D(J_i)$ in \mathcal{P} chosen so that $D_i \cap \mathcal{Q} = D(J_i) \cap \mathcal{Q}$ and distinct D_i are disjoint. Let $\mathcal{D} = \bigcup D_i$. Similar remarks apply to the maximal disks on \mathcal{Q} , yielding $\mathcal{G} = \bigcup G_j$. It suffices to ambiently isotop \mathcal{P} to \mathcal{P}' such that, denoting the image of \mathcal{D} under the isotopy by \mathcal{D}' , we have that \mathcal{P}' and \mathcal{Q} are in general position, $\mathcal{D}' \cap \mathcal{Q} = \emptyset$, $((\mathcal{P}' - \mathcal{D}') \cap \mathcal{Q})$ is a union of components of $((\mathcal{P} - \mathcal{D}) \cap \mathcal{Q})$, and the isotopy is fixed on $((\mathcal{P}' - \mathcal{D}') \cap \mathcal{Q})$.

Choose an exhaustion C for M such that for each $n \ge 0$ and each m > n one has that C_n and $C_m - Int C_n$ are irreducible. This is possible because M is irreducible and is not homeomorphic to \mathbb{R}^3 . We may assume that $\mathcal{G} \cap \mathcal{F}_0 = \emptyset$ because the G_j are disjoint disks in the interior of M. By passing to a subexhaustion of C we may also assume that $Int C_{n+1}$ contains all those D_i which meet F_n . Let $Y_0 = Int C_1$, and let $Y_n = (Int C_{n+1}) - C_{n-1}$ for $n \ge 1$. Thus if D_i meets F_n , then $D_i \subseteq Y_n$.

Suppose n is even, D_i meets F_n , and D_i meets \mathcal{G} . There is a disk D in D_i such that $D \cap \mathcal{G} = \partial D$. Note that ∂D need not be a component of \mathcal{J} and that int D need not be disjoint from \mathcal{Q} . Let D_+ be a regular neighborhood of D in \mathcal{P} chosen so that $D_+ \cap \mathcal{Q} = D \cap \mathcal{Q}$. Let G be the disk on \mathcal{G} bounded by ∂D . For a subset A of M let A^* denote the image of A under the disk push of D past G across the 3-ball B in Y_n bounded by $D \cup G$. We may assume that ∂D_+ lies on ∂B^+ , where B^+ is the regular neighborhood of B supporting the isotopy. Then \mathcal{P}^* and \mathcal{Q} are

in general position, $(\mathcal{D}^* - D^*_+) \cap \mathcal{Q}$ and $((\mathcal{P}^* - \mathcal{D}^*) \cap \mathcal{Q})$ are unions of components of, respectively, $(\mathcal{D} - D_+) \cap \mathcal{Q}$ and $((\mathcal{P} - \mathcal{D}) \cap \mathcal{Q})$, and $D^*_+ \cap \mathcal{Q} = \emptyset$.

Now suppose K is a component of $\mathcal{P} \cap \mathcal{Q}$. If K lies outside B, then we may assume that the isotopy is fixed on K, and so K is a component of $\mathcal{P}^* \cap \mathcal{Q}$. If K meets B, then it is a component of $D \cap \mathcal{Q}$ or of $\mathcal{P} \cap G$, and so it is removed by the isotopy. Since all the components of $\mathcal{P}^* \cap \mathcal{Q}$ are components of $\mathcal{P} \cap \mathcal{Q}$, we have that the isotopy is fixed on $\mathcal{P}^* \cap \mathcal{Q}$.

We next consider the effect of the isotopy on other disks D_k . Suppose $D_k \cap F_n = \emptyset$. If $D_k \cap B = \emptyset$, then we may assume that $D_k^* = D_k$. If $D_k \cap B \neq \emptyset$, then $D_k \cap G \neq \emptyset$. The isotopy might move $D_k \cap B$ across some portion of F_n , but since $(D_k \cap B)^*$ lies within a regular neighborhood of G and $G \cap F_n = \emptyset$ we may assume that $D_k^* \cap F_n = \emptyset$. Thus the number of disks D_k^* in \mathcal{P}^* which meet F_n is no greater than the number of disks D_k in \mathcal{P} which meet F_n . Therefore after performing a finite sequence of these isotopies we may assume that if $D_i \cap F_n \neq \emptyset$, then $D_i \cap \mathcal{G} = \emptyset$. Since these isotopies are supported in Y_n we may do this simultaneously for all even n.

Now for n odd we still have that if $D_i \cap F_n \neq \emptyset$, then $D_i \subseteq Y_n$. Thus if D_i is a component of \mathcal{D} such that $D_i \cap \mathcal{G} \neq \emptyset$, then $D_i \subseteq Y_n$ for some odd n. By performing a finite sequence of disk pushes in each such Y_n we obtain the desired surface \mathcal{P}' . \Box

Corollary 2.2. Let M be a connected, irreducible, non-compact 3-manifold which is not homeomorphic to \mathbb{R}^3 . Let \mathcal{P} and \mathcal{Q} be proper, incompressible surfaces in Msuch that no component of \mathcal{P} or of \mathcal{Q} is a 2-sphere or a trivial plane. Assume that there do not exist plane components P of \mathcal{P} and Q of \mathcal{Q} on both of which there is infinite nesting among the components of $P \cap Q$. Then there is an ambient isotopy of \mathcal{P} in M, fixed on ∂M , which takes \mathcal{P} to a surface \mathcal{P}' such that \mathcal{P}' and \mathcal{Q} are in general position and no simple closed curve component of $\mathcal{P}' \cap \mathcal{Q}$ bounds a disk on \mathcal{P}' or on \mathcal{Q} . This isotopy is fixed on $\mathcal{P}' \cap \mathcal{Q}$.

Proof. Let \mathcal{J} be the set of all simple closed curve components of $\mathcal{P} \cap \mathcal{Q}$ which bound disks on \mathcal{P} (or equivalently on \mathcal{Q} .) Infinite nesting on \mathcal{P} or on \mathcal{Q} among the elements of \mathcal{J} implies by Lemma 1.1 (1) that either one of these surfaces has a component which is a trivial plane or there are two plane components P and Q as above. \Box

We now consider the removal of trivial arcs of intersection.

Proposition 2.3. Let M be a connected, irreducible, non-compact 3-manifold which has non-empty boundary and is not homeomorphic to $\mathbb{R}^2 \times [0,1)$. Let \mathcal{P} and \mathcal{Q} be proper surfaces in M which are in general position. Let \mathcal{A} be a union of components of $\mathcal{P} \cap \mathcal{Q}$, each of which is an arc. Let R be an end-proper surface in ∂M . Assume that the following conditions are satisfied.

- (1) No component of \mathcal{P} or of \mathcal{Q} is a 2-sphere or a disk.
- (2) Each component α of \mathcal{A} is ∂ -parallel in \mathcal{P} across a halfdisk $D(\alpha)$ and is ∂ -parallel in \mathcal{Q} across a halfdisk $G(\alpha)$.
- (3) There is no infinite sequence $\{\alpha_m\}$ of distinct components of \mathcal{A} such that either $D(\alpha_m) \subseteq Int_{\mathcal{P}}D(\alpha_{m+1})$ for all m or $G(\alpha_m) \subseteq Int_{\mathcal{Q}}G(\alpha_{m+1})$ for all

m, i.e. there is no infinite nesting on \mathcal{P} or on \mathcal{Q} among the components of \mathcal{A} .

- (4) $\partial \mathcal{P} \cup \partial \mathcal{Q}$ lies in int R.
- (5) R is incompressible in M.
- (6) Each component J of \mathcal{J} bounds a disk G(J) in \mathcal{Q} , where \mathcal{J} is the union of all those simple closed curve components of $\mathcal{P} \cap \mathcal{Q}$ which lie in some $D(\alpha)$.
- (7) There is no infinite nesting on \mathcal{Q} among the components of \mathcal{J} .

Then there is an ambient isotopy of \mathcal{P} in M, fixed on $(\partial M) - \operatorname{int} R$ which takes \mathcal{P} to a surface \mathcal{P}' such that \mathcal{P}' and \mathcal{Q} are in general position and $(\mathcal{P}' \cap \mathcal{Q}) \subseteq (\mathcal{P} \cap \mathcal{Q}) - (\mathcal{A} \cup \mathcal{J})$. Moreover, the isotopy is fixed on $\mathcal{P}' \cap \mathcal{Q}$.

Proof. Since neither \mathcal{P} nor \mathcal{Q} has a disk component the halfdisks $D(\alpha)$ and $G(\alpha)$ are unique. As before call a halfdisk $D(\alpha)$ maximal if there is no $D(\alpha')$ such that $D(\alpha) \subseteq Int_{\mathcal{P}}D(\alpha')$. By hypothesis each $D(\alpha)$ lies in some maximal halfdisk and these maximal halfdisks are disjoint. Let $\{D(\alpha_i)\}$ be the set of maximal halfdisks. Let D_i be a regular neighborhood of $D(\alpha_i)$ in \mathcal{P} chosen so that $D_i \cap \mathcal{Q} = D(\alpha_i) \cap \mathcal{Q}$ and distinct D_i are disjoint. Let $\mathcal{D} = \bigcup D_i$. The maximal halfdisks in \mathcal{Q} yield $\mathcal{G} = \bigcup G_j$. It suffices to ambiently isotop \mathcal{P} to \mathcal{P}' such that, denoting the image of \mathcal{D} under the isotopy by \mathcal{D}' , we have that \mathcal{P}' and \mathcal{Q} are in general position, $\mathcal{D}' \cap \mathcal{G} = \emptyset$, $((\mathcal{P}' - \mathcal{D}') \cap \mathcal{Q})$ is a union of components of $((\mathcal{P} - \mathcal{D}) \cap \mathcal{Q})$, and the isotopy is fixed on $((\mathcal{P}' - \mathcal{D}') \cap \mathcal{Q})$.

Since $\mathcal{J} \subseteq \mathcal{D}$ there is no infinite nesting on \mathcal{P} among the components of \mathcal{J} . Thus by Proposition 2.1 we may assume $\mathcal{J} = \emptyset$.

As before we can choose an exhaustion C for M such that for each n > 0 and each m > n one has that C_n and $C_m - Int C_n$ are irreducible. Since the G_i are disjoint disks each meeting ∂M in a single arc we may assume that $\mathcal{G} \cap \mathcal{F}_0 = \emptyset$. We may also assume that $\partial \mathcal{F}_0$ and ∂R are in general position, and that if D_i meets F_n , then $D_i \subseteq Int C_{n+1}$. Note that since M is not homeomorphic to $\mathbf{R}^2 \times [0,1)$ we may assume that no C_n lies in a 3-ball in M which meets ∂M in a single disk. We claim that $C_n \cap R$ and $(C_m - Int C_n) \cap R$ are incompressible in C_n and $(C_m - Int C_n)$ respectively. If D is a compressing disk for $C_n \cap R$ in C_n , then $\partial D = \partial D'$ for a disk D' in R. If D' does not lie in $C_n \cap R$, then it must meet $M - C_n$. Since M is irreducible $D \cup D'$ bounds a 3-ball containing a component of $M - C_n$, contradicting the fact that $M - C_n$ has no bounded components. If D is a compressing disk for $(C_m - Int C_n) \cap R$ in $C_m - Int C_n$, then $\partial D = \partial D'$ for a disk D' in R which must meet C_n or $M - C_m$. Let B be the 3-ball in M bounded by $D \cup D'$. Then D' does not meet C_n , for otherwise C_n would lie in B. So D' meets $M - C_m$ and hence a component of $M - C_m$ is contained in B, contradicting the fact that $M - C_m$ has no bounded components.

We now proceed as in the proof of Proposition 2.1. We let $Y_0 = Int C_1$ and $Y_n = (Int C_{n+1}) - C_{n-1}$ for $n \ge 1$. Let $R_n = Y_n \cap R$. Suppose D_i meets both F_n and \mathcal{G} . The role of an innermost disk is now given to an innermost halfdisk, i.e. a proper halfdisk D in \mathcal{P} which lies in D_i such that $\partial_1 D = D_i \cap \mathcal{G}$, although $\partial_1 D$ need not be a component of \mathcal{A} and $Int_{\mathcal{P}}D$ need not be disjoint from \mathcal{Q} . There is a unique proper halfdisk G in \mathcal{Q} such that G lies in \mathcal{G} and $\partial_1 G = \partial_1 D$. Then $D \cup G$

is a proper disk in Y_n with $\partial(D \cup G)$ in R_n . Since R_n is incompressible in Y_n there is a disk D' in R_n with $\partial D' = \partial(D \cup G)$. Since Y_n is irreducible $D \cup G \cup D'$ bounds a 3-ball B in Y_n . A halfdisk push of D across B past G removes at least $\partial_1 D$ from the intersection, adds no new components of intersection, either fixes or removes each component of $\mathcal{P} \cap \mathcal{Q}$, and does not increase the number of D_k meeting F_n . As before we remove all intersections of D_i with \mathcal{G} for those D_i which meet F_n for neven and then remove all remaining intersections of \mathcal{D} with \mathcal{G} . \Box

Corollary 2.4. Let M be a connected, irreducible, non-compact 3-manifold which has non-empty boundary and is not homeomorphic to $\mathbb{R}^2 \times [0,1)$. Let \mathcal{P} and \mathcal{Q} be proper, incompressible, ∂ -incompressible surfaces in M such that no component of \mathcal{P} or of \mathcal{Q} is a 2-sphere, a disk, a trivial plane, or a trivial halfplane. Assume that there do not exist components P of \mathcal{P} and \mathcal{Q} of \mathcal{Q} which are either both planes or both halfplanes on both of which there is infinite nesting among the components of $P \cap Q$. Suppose R is an end-proper surface in ∂M which is incompressible in Mand whose interior contains $\partial \mathcal{P} \cup \partial \mathcal{Q}$. Then there is an ambient isotopy of \mathcal{P} in M, fixed on $(\partial M) - \operatorname{int} R$, which takes \mathcal{P} to a surface \mathcal{P}' such that \mathcal{P}' and \mathcal{Q} are in general position, no simple closed curve component of $\mathcal{P}' \cap \mathcal{Q}$ bounds a disk on \mathcal{P}' or on \mathcal{Q} , and no component of $\mathcal{P}' \cap \mathcal{Q}$ is an arc which is ∂ -parallel in \mathcal{P}' or in \mathcal{Q} . This isotopy is fixed on $\mathcal{P}' \cap \mathcal{Q}$.

Proof. First apply Corollary 2.2 to remove all trivial simple closed curve components of $\mathcal{P} \cap \mathcal{Q}$. Then let \mathcal{A} be the set of all those components of $\mathcal{P} \cap \mathcal{Q}$ which are ∂ -parallel arcs in \mathcal{P} (or equivalently in \mathcal{Q} .) Infinite nesting among the components of \mathcal{A} implies by Lemma 1.1 (2) that either one of these surfaces has a component which is a trivial halfplane or there are halfplane components P and Q as above. \Box

3. Aplanarity and total acylindricality

The goal of this section is to show that a connected, non-compact 3-manifold which possesses a nice exhaustion must be totally acylindrical and, if it is irreducible, must be aplanar. These results are basically due to Winters and are contained, either explicitly or implicitly, in his thesis [19], where they sometimes appear in a slightly different form and context. We include proofs of them here for several reasons. First, the paper [20] containing the relevant portions of [19] has not yet been published, and so giving proofs here will make the argument of the present paper more complete. Second, the proof given here of Lemmas 3.1 and 3.2 is somewhat different from that of the corresponding Lemma X.1 of [19] in that it applies the general machinery for removing trivial intersections developed in the previous section of this paper rather than the direct arguments of [19]; this remark also applies to the proof of Theorem 3.5. Third, Lemma 3.3 and Theorems 3.4 and 3.5 are not stated explicitly in [19] in the forms we shall need, although their proofs either are contained in or can be easily deduced from the proofs of Lemmas II.2, II.3, and XII.3 of [19]. Finally, the terminology and the organization of the proof of aplanarity given in this section establish the background and conceptual framework for the more difficult analysis of partial planes in the next section.

In the first three lemmas M is a connected non-compact 3-manifold and C is an exhaustion for M. Whenever J is a simple closed curve on a plane P, let D(J) be the disk on P bounded by J. Let $n_0 > 0$. A proper plane P is in n_0 -standard **position** with respect to C if P is in general position with respect to \mathcal{F}_0 , $P \cap \mathcal{F}_{n_0}$ is a sequence $\{J_m\}$ of simple closed curves such that $\{D(J_m)\}$ is an exhaustion for $P, D(J_0)$ is a proper disk in C_{n_0} , and $(P \cap C_0) \subseteq int D(J_0)$. Note that if this is the case and $n_1 > n_0$, then P is also in n_1 -standard position with respect to C.

Lemma 3.1. Suppose C is a good exhaustion for M and P is a proper plane in M. Then for some $n_0 > 0$ one has that P is ambient isotopic to a plane which is in n_0 -standard position with respect to C.

Proof. First ambiently isotop P so that it is in general position with respect to \mathcal{F}_0 . Since P is proper there is a disk D in P which contains $P \cap C_0$ in its interior. (If $P \cap C_0 = \emptyset$, let D be any disk in P.) Since C is an exhaustion and D is compact there is an $n_0 > 0$ such that $D \subseteq Int C_{n_0}$. There is then a component J of $P \cap F_{n_0}$ such that $D \subseteq int D(J)$. Let J_0 be the innermost such component of $P \cap F_{n_0}$, i.e. there is no component J of $P \cap F_{n_0}$ such that $D \subseteq int D(J) \subseteq int D(J_0)$.

Now let J be a component of $D(J_0) \cap F_{n_0}$ other than J_0 . Assume that J is innermost on P among such curves. Then $D(J) \cap F_{n_0} = J$. Moreover $D(J) \cap D = \emptyset$, and so D(J) lies in $M - Int C_0$. Since F_{n_0} is incompressible in $M - Int C_0$ there is a disk D' in F_{n_0} bounded by J. Since $M - Int C_0$ is irreducible the 2-sphere $D(J) \cup D'$ bounds a 3-ball B in $M - Int C_0$, and so one can perform a disk push of D(J)across B past D'. This ambient isotopy of P is supported in $M - Int C_0$, removes Jfrom the intersection, and adds no new components. We continue performing such isotopies until $D(J_0)$ becomes a proper disk in C_{n_0} .

Suppose J is a component of $P \cap F_{n_0}$ such that D(J) does not contain $D(J_0)$. Then D(J) lies in $M - Int C_0$ and so a sequence of disk pushes similar to that described above removes all such J from the intersection.

Let $Y_{n_0} = M - Int C_{n_0}$ and $\mathcal{P} = P \cap Y_{n_0}$. Then Y_{n_0} is irreducible and ∂ irreducible and the set of components of \mathcal{P} consists of a half-cylinder (homeomorphic to $S^1 \times [0, \infty)$) and possibly a finite collection of annuli, all of which are proper in Y_{n_0} . Let $\mathcal{Q} = \mathcal{F}_{n_0+1}$. Let \mathcal{J} be the union of all those components J of $\mathcal{P} \cap \mathcal{Q}$ such that D(J) does not contain $D(J_0)$. Then Y_{n_0} , \mathcal{P} , \mathcal{Q} , and \mathcal{J} satisfy the hypotheses of Proposition 2.1, and so there is an ambient isotopy in Y_{n_0} , fixed on F_{n_0} , which removes \mathcal{J} from the intersection and adds no new components to it. Since the isotopy is fixed on those intersection curves which are not removed they all bound disks which contain $D(J_0)$. The isotopy thus extends by the identity isotopy on C_{n_0} to an ambient isotopy in M which carries P to a plane P' which is in n_0 -standard position with respect to C. \Box

A proper plane P is in **non-trivial** n_0 -standard position with respect to an exhaustion C if it is in n_0 -standard position with respect to C and no component of $P \cap \mathcal{F}_{n_0}$ bounds a disk in \mathcal{F}_{n_0} .

Lemma 3.2. Let C be a good exhaustion for M, and let P be a non-trivial proper plane in M. Suppose M is irreducible. Then for some $n_0 > 0$ one has that P is ambient isotopic to a plane which is in non-trivial n_0 -standard position with respect to C.

Proof. By Lemma 3.1 we may assume that P is in k-standard position with respect to C for some k > 0. Then $P \cap \mathcal{F}_k$ is a nested sequence $\{J_m\}$ of simple closed curves on P. If there is an increasing sequence $\{m_i\}$ such that J_{m_i} bounds a disk $D'(J_m)$ in \mathcal{F}_k , then by Lemma 1.1 (1) we have that P is trivial in M.

Thus no such sequence exists, and so there is an $m_0 \ge 0$ such that for all $m \ge m_0$ J_m does not bound a disk in \mathcal{F}_k . There is then an $m_1 \ge m_0$ and an $n_0 \ge k$ such that $J_{m_1} \subseteq F_{n_0}$ and $D(J_{m_1}) \subseteq C_{n_0}$. Then P is in non-trivial n_0 -standard position with respect to C. \Box

A proper plane P in M is in n_0 -monotone position with respect to an exhaustion C if it is in n_0 -standard position with respect to C and for $n \ge n_0$ one has that $P \cap X_{n+1}$ is an annulus with one boundary component in F_n and the other in F_{n+1} . We apply the adjective "non-trivial" in the same sense as above. Again P is in n_1 -monotone position for all $n_1 > n_0$. We now assume that M is a connected non-compact 3-manifold having finitely many ends and finitely many boundary components and that each boundary component is a plane.

Lemma 3.3. Suppose C is a nice exhaustion for M and P is a non-trivial proper plane in M. Assume that M is irreducible. Then for some $n_0 > 0$ one has that P is ambient isotopic to a plane which is in non-trivial n_0 -monotone position with respect to C.

Proof. By Lemma 3.2 we may assume that P is in non-trivial n_0 -standard position with respect to C for some $n_0 > 0$. Then there is an exhaustion $\{D_m\}$ of P with each D_m a disk such that $(P \cap C_0) \subseteq int D_0$ and D_0 is a proper disk in C_{n_0} . Moreover, since ∂D_m does not bound a disk in \mathcal{F}_{n_0} and \mathcal{F}_{n_0} is incompressible in $M - Int C_0$, one has that each annulus $D_{m+1} - int D_m$ is incompressible in the X_{n+1} containing it. Denote ∂D_m by J_m .

Let $m_0 = 0$. Let m_1 be the smallest index for which $J_{m_1} \subseteq F_{n_0+1}$ and $(P \cap C_{n_0}) \subseteq D_{m_1}$. Assume m_1, \ldots, m_r have been defined. Let m_{r+1} be the smallest index for which $J_{m_{r+1}} \subseteq F_{n_0+r+1}$ and $(P \cap C_{n_0+r}) \subseteq D_{m_{r+1}}$. Then $\{D_{m_r}\}$ is an exhaustion for P. Let $A_{r+1} = D_{m_{r+1}} - int D_{m_r}$. Suppose $(int A_{r+1}) \cap \mathcal{F}_{n_0} \neq \emptyset$. Call any component of this set a **redundant intersection**. Since each F_n separates M the redundant intersections occur in pairs, with each component of a pair lying in the same component of \mathcal{F}_{n_0} and the pair forming the boundary of an annulus contained in A_{r+1} . Call such an annulus a **redundant annulus**. There is then a redundant annulus A which is innermost in the sense that its interior misses \mathcal{F}_{n_0} . Then A is a proper incompressible annulus in some X_{n+1} , and $\partial A \subseteq F_n$ or $\partial A \subseteq F_{n+1}$.

Assume $\partial A \subseteq F_n$. Then A is parallel in X_{n+1} to an annulus A' in ∂X_{n+1} . Suppose A' does not lie in F_n . By the incompressibility of F_n , F_{n+1} , and $X_{n+1} \cap \partial M$ in X_{n+1} the components of $A' \cap F_n$, $A' \cap F_{n+1}$ and $A' \cap \partial M$ must be annuli. $A' \cap F_{n+1} = \emptyset$ since its components would be components of F_{n+1} , which, since Cis nice, has no annulus components. Since F_n has no annulus components $A' \cap F_n$ has exactly two components, each of which is a collar on a boundary component of F_n . Thus $A' \cap \partial M$ is an annulus whose boundary lies entirely in F_n , again contradicting the fact that C is nice. Therefore A' lies in F_n and so there is an ambient isotopy supported in a regular neighborhood of X_{n+1} which reduces the number of redundant intersections. A similar argument holds for $\partial A \subseteq F_{n+1}$. Note that this process may move some redundant annuli and remove others. But it introduces no new redundant annuli and leaves any remaining redundant annuli lying in the union of the same set of X_{k+1} 's.

Thus any one redundant annulus can be removed by an ambient isotopy of P supported in $M - Int C_0$. However, since there may be infinitely many redundant annuli one must avoid performing an infinite sequence of these isotopies which fails to converge to an ambient isotopy of M. We proceed as follows.

Choose $n_1 > n_0$ so that if A is any redundant annulus with $\partial A \subseteq F_{n_0}$, then A lies in $(Int C_{n_1}) - C_0$, and if H is any redundant annulus with $\partial H \subseteq F_{n_1}$, then H lies in $M - C_{n_0}$. Now suppose $n_0, n_1, \ldots, n_i, i > 0$, have been chosen. Choose $n_{i+1} > n_i$ so that if A is any redundant annulus with $A \subseteq F_{n_i}$, then A lies in $(Int C_{n_{i+1}}) - C_{n_{i-1}}$, and if H is any redundant annulus with $\partial H \subseteq F_{n_{i+1}}$, then H lies in $M - C_{n_i}$.

By an ambient isotopy supported in $(Int C_{n_1}) - C_0$ remove all redundant intersections with F_{n_0} . For each even i > 0 perform an ambient isotopy supported in $(Int C_{n_{i+1}}) - C_{n_{i-1}}$ which removes all redundant intersections with F_{n_i} . Since they have disjoint compact supports these isotopies give a single ambient isotopy supported in $M - C_0$. Now for each odd $i \ge 1$ perform an ambient isotopy which removes all redundant intersections with F_{n_i} . Again this gives a single ambient isotopy supported in $M - C_0$.

One now has no redundant intersections with the F_{n_i} . For each i > 0 perform an ambient isotopy supported in $(Int C_{n_i}) - C_{n_{i-1}}$ which removes all redundant intersections with those F_j with $n_{i-1} < j < n_i$. This completes the removal of all redundant intersections and puts P in non-trivial n_0 -monotone position with respect to C. \Box

Theorem 3.4. Let M be a connected, irreducible, non-compact 3-manifold which has a nice exhaustion. Then M is a planar.

Proof. Suppose P is a nontrivial proper plane in M, and C is a nice exhaustion for M. By Lemma 3.3 we may assume that P is in non-trivial n_0 -monotone position with respect to C. For simplicity of notation we shall reindex so that $n_0 = 0$. Then $P \cap C_n$ is a disk D_n , $\{D_n\}$ is an exhaustion of P, and $P \cap X_{n+1} = D_{n+1} - int D_n$ is an annulus A_{n+1} with one boundary component in F_n and the other in F_{n+1} . Moreover $J_n = \partial D_n$ does not bound a disk in F_n .

It follows that each A_{n+1} is parallel in X_{n+1} to an annulus A'_{n+1} in ∂X_{n+1} . Since \mathcal{F}_0 has no disk components and is incompressible in $M - Int C_0$ one has that each component of $A'_{n+1} \cap (F_n \cup F_{n+1})$ is an annulus. Since \mathcal{F}_0 has no annulus components this intersection consists of collars G_{n+1} and H_{n+1} , respectively, on J_n and J_{n+1} in A'_{n+1} . Let $R_{n+1} = A'_{n+1} \cap \partial M$. Then $A'_{n+1} = G_{n+1} \cup R_{n+1} \cup H_{n+1}$. The parallelism between A_{n+1} and A'_{n+1} defines an embedding of $S^1 \times [0, 1] \times [0, 1]$ in X_{n+1} , with $S^1 \times [0, 1] \times \{0\} = A_{n+1}, S^1 \times [0, 1] \times \{1\} = R_{n+1}, S^1 \times \{0\} \times I =$ G_{n+1} , and $S^1 \times \{1\} \times I = H_{n+1}$. Consider the analogous situation in X_{n+2} . If $G_{n+2} \neq H_{n+1}$, then $G_{n+2} \cup H_{n+1}$ is a component of F_{n+1} , contradicting the fact that \mathcal{F}_0 has no annulus components. Therefore $G_{n+2} = H_{n+1}$, all the R_{n+1} lie in the same component E_i of ∂M , and so one can fit together the embeddings of $S^1 \times [0,1] \times [0,1]$ to get an end proper embedding of $S^1 \times [1,\infty) \times [0,1]$ in $M - Int C_0$ with $S^1 \times [1,\infty) \times \{0\} = P - int D_0, S^1 \times [1,\infty) \times \{1\} = E_i - (E_i \cap C_0)$, and $S^1 \times \{1\} \times [0,1] = G_1$. Now $D_0 \cup G_1$ is a proper disk in M whose boundary is that of $E_i \cap C_0$. Since M is irreducible the union of these disks bounds a 3-ball in M which can be used to extend the product structure to obtain a parallelism between P and E_i . \Box

Now recall that M is totally acylindrical if it admits no proper incompressible embeddings of a cylinder $S^1 \times \mathbf{R}$.

Theorem 3.5. Let M be a connected, non-compact 3-manifold which has a nice exhaustion. Then M is totally acylindrical.

Proof. Suppose $S = S^1 \times \mathbf{R}$ is a proper, incompressible cylinder in M. Let C be a nice exhaustion for M. Put S in general position with respect to \mathcal{F}_0 . Since S is proper there exist $a, b \in \mathbf{R}, a < b$, such that the annulus $A = S^1 \times [a, b]$ contains $S \cap C_0$. (If $S \cap C_0 = \emptyset$, then choose a < b arbitrarily.) There is then an $n_0 > 0$ such that $A \subseteq Int C_{n_0}$. There are components J_0^+ and J_0^- of $S \cap F_{n_0}$, neither of which bounds a disk on S, such that A is contained in the interior of the annulus A_0 on S bounded by $J_0^+ \cup J_0^-$. We may assume that A_0 is an innermost such annulus, i.e. there are no components J^+ , J^- of $S \cap F_{n_0}$ which are non-contractible on S such that the annulus bounded by $J^+ \cup J^-$ contains A and is contained in, but does not equal, the annulus A_0 .

We now proceed as in the proof of Lemma 3.1. All the components of $A_0 \cap F_{n_0}$ other than J^+ and J^- bound disks in A_0 which miss A and lie in $M - Int C_0$. We perform a finite sequence of disk pushes which remove these curves. We then isotop the two half-cylinders composing $S - Int A_0$ to remove all the other components of $S \cap F_{n_0}$ which bound disks on S.

Let $Y_{n_0} = M - Int C_{n_0}$ and $\mathcal{P} = S \cap Y_{n_0}$. Then \mathcal{P} consists of two half-cylinders and possibly a finite collection of annuli, all of which are proper in Y_{n_0} . Let $\mathcal{Q} = \mathcal{F}_{n_0+1}$. Let \mathcal{J} be the union of all those components of $\mathcal{P} \cap \mathcal{Q}$ which bound disks on S and hence on \mathcal{P} . Since Y_{n_0} is irreducible Proposition 2.1 gives an ambient isotopy of S in M which removes \mathcal{J} from the intersection, adds no new components, and is fixed on those which are not removed.

Now each $X \cap X_{n+1}$ is a union of disjoint annuli. Since S is proper this intersection is non-empty for all sufficiently large n and in fact must contain an annulus A'running from F_n to F_{n+1} . Since X_{n+1} is an annular A' is parallel to an annulus A''in ∂X_{n+1} . Now A'' must contain the annulus $E \cap X_{n+1}$ for some component E of ∂M . Since E is a plane it follows that S is compressible in M, a contradiction. \Box

4. Strong Aplanarity: The Special Case

Recall that a partial plane P is a non-compact, simply-connected 2-manifold with non-empty boundary. In this section we show that given any proper partial plane P in an irreducible 3-manifold M which has a nice exhaustion there exists a collar on ∂M which contains P. This is a strong condition on M since such a P cannot meet distinct components of ∂M . One should note, however, that P need not be ∂ -parallel. In the next section this result will be extended to proper surfaces each of whose components is a partial plane.

The general line of argument will be similar to that of the previous section. Instead of working with simple closed curves and the disks they bound we shall work with finite collections of proper arcs which cut off disks on P. Specifically, let α be the union of finitely many disjoint proper arcs $\alpha^1, \ldots, \alpha^k$ in a partial plane P. If for some proper partial disk D in P one has $\alpha = \partial_1 D$, then α is called a **bounding arc system** in P, and D is denoted by $D(\alpha)$.

A proper partial plane P is in n_0 -standard position with respect to an exhaustion C if P is in general position with respect to \mathcal{F}_0 , each component of $P \cap \mathcal{F}_{n_0}$ is an arc, there is a sequence $\{D_m\}$ of well embedded partial disks in P which is an exhaustion for P, $(P \cap C_0) \subseteq Int_P D_0$, D_0 is a proper disk in C_{n_0} , and $P \cap \mathcal{F}_{n_0} = \bigcup_{m>0} Fr_P D_m$. In this case it is in n_1 -standard position for all $n_1 > n_0$.

Lemma 4.1. Let C be a nice exhaustion for M, and let P be a proper partial plane in M. Then for some $n_0 > 0$ one has that P is ambient isotopic to a partial plane which is in n_0 -standard position with respect to C.

Proof. First ambiently isotop P so that it is in general position with respect to \mathcal{F}_0 . Since $\partial P \neq \emptyset$ and P is proper $\partial M \neq \emptyset$. Since P is proper there is a disk D in P such that $(P \cap C_0) \subseteq Int_P D$ and $D \cap \partial P \neq \emptyset$. (If $P \cap C_0 = \emptyset$, let D be any disk in P with $D \cap \partial P \neq \emptyset$.) Since C is an exhaustion and D is compact there is an $n_0 > 0$ such that $D \subseteq Int C_{n_0}$. There is a unique component L of $P \cap C_{n_0}$ such that $D \subseteq Int_P L$. Then L is a planar surface whose boundary consists of one simple closed curve K which meets ∂P and possibly some other simple closed curves K_j which do not meet ∂P . Let $\beta = K \cap F_{n_0}$. Then β is a bounding arc system in P such that $D(\beta)$ is the union of L with the disks in P bounded by the K_j .

If $D(\beta)$ is not well embedded in P, then for some components β^i of β there exist proper halfdisk components D^i of $P - Int_P D(\beta)$ such that $\partial_1 D^i = \beta^i$. Delete these β^i from β to obtain a new arc system α . Then α is a bounding arc system such that $D(\alpha)$ is well embedded in P, $\alpha \subseteq (P \cap F_{n_0})$, and $D \subseteq Int_P D(\alpha)$. Note that $D(\alpha)$ need not lie in C_{n_0} . We shall next isotop P so that afterwards $D(\alpha)$ does lie in C_{n_0} . To simplify the notation denote $D(\alpha)$ by D_0 .

Suppose J is a simple closed curve component of $D_0 \cap F_{n_0}$. We may assume that J is innermost on P among such curves. Since $D \cap \partial P \neq \emptyset$ one has that $D(J) \cap D = \emptyset$, and so D(J) lies in $M - Int C_0$. Since $M - Int C_0$ is irreducible and F_{n_0} is incompressible in $M - Int C_0$, there is a disk push which removes J from the intersection and adds no new components. Continue in this fashion until all such Jare removed.

Now suppose γ is a component of $(Int D_0) \cap F_{n_0}$ which is a proper arc in D_0 . We may assume that γ is innermost on P among such arcs. Hence there is a proper halfdisk H in P with $\partial_1 H = \gamma = H \cap F_{n_0}$. We may further assume that H does not contain D, and so H lies in $M - Int C_0$. Since F_{n_0} is ∂ -incompressible in $M - Int C_0$ there is a proper halfdisk H' in F_{n_0} with $\partial_1 H' = \gamma$. Since $\partial M - Int(C_0 \cap \partial M)$ is incompressible in $M - Int C_0$, the simple closed curve $\partial_0 H \cup \partial_0 H'$ bounds a disk H'' in $\partial M - Int(C_0 \cap \partial M)$. Hence by the irreducibility of $M - Int C_0$ there is a halfdisk push of H past H' across the ball bounded by $H \cup H' \cup H''$ which removes γ from the intersection and adds no new components. We continue in this fashion until D_0 becomes a proper disk in C_{n_0} .

We now consider components of $P \cap F_{n_0}$ which do not lie in D_0 . As above we first remove all such components which are simple closed curves and then remove all those which are ∂ -parallel arcs in P. Note that afterwards $P \cap F_{n_0}$ splits P into finitely many partial disks and partial planes.

Let $Y_{n_0} = M - Int C_{n_0}$. We now have that $P \cap C_{n_0}$ consists of D_0 and possibly a finite number of other disks. $P \cap Y_{n_0}$ consists of a finite number of partial planes and perhaps finitely many disks D_j , each of which meets F_{n_0} . Let \mathcal{P} be the union of the partial planes, and let $\mathcal{Q} = \mathcal{F}_{n_0+1}$. By an isotopy fixed on ∂Y_{n_0} we remove all simple closed curve intersections of the D_j with \mathcal{Q} . Then by an isotopy fixed on F_{n_0} we remove all those components of $(\cup D_j) \cap \mathcal{Q}$ which are arcs which are ∂ -parallel on P across halfdisks which do not contain D_0 .

Now $\partial D_j = \partial D'_j$ for a disk D'_j in ∂Y_{n_0} such that $D_j \cup D'_j = \partial B_j$ for a 3-ball B_j in Y_{n_0} . Let R' be the union of F_{n_0} and the D'_j . Let $R = (\partial Y_{n_0}) - int R'$. Since each component of \mathcal{P} is non-compact we have that \mathcal{P} lies outside the union of the B_j and thus $\mathcal{P} \cap \mathcal{Q} \cap \partial Y_{n_0}$ lies in R. We claim that R is incompressible in Y_{n_0} . For suppose G is a compressing disk. Then $\partial G = \partial G'$ for a disk G' in ∂Y_{n_0} . Since no component of F_{n_0} is a planar surface G' must lie in $(\partial Y_{n_0}) - F_{n_0}$. But then int G'must contain some D'_j , and so D_j cannot meet F_{n_0} , a contradiction.

Now let \mathcal{J} be the union of all the simple closed curve components of $\mathcal{P} \cap \mathcal{Q}$. Then Y_{n_0} , \mathcal{P} , \mathcal{Q} , and \mathcal{J} satisfy the hypotheses of Proposition 2.1, and so \mathcal{J} can be removed from the intersection by an isotopy which is fixed on those components of $\mathcal{P} \cap \mathcal{Q}$ which are not removed. Next let \mathcal{A} be the union of all the components of $\mathcal{P} \cap \mathcal{Q}$ which are arcs that are ∂ -parallel in \mathcal{P} across a halfdisk which does not contain D_0 . Each such component is ∂ -parallel in \mathcal{P} to an arc which does not meet F_{n_0} . Then Y_{n_0} , \mathcal{P} , \mathcal{Q} , R, and \mathcal{A} satisfy the hypotheses of Proposition 2.3, and so \mathcal{A} can be removed from the intersection by an isotopy fixed on the components which are not removed. Note that since these isotopies are fixed on R' we may assume that they are fixed on the union of F_{n_0} and the B_j . We thus get an ambient isotopy of P in M which puts it into n_0 -standard position with respect to C. \Box

Let P be in n_0 -standard position with respect to C; let $\{D_m\}$ be the corresponding exhaustion for P. For each $m \ge 0$ each component Z of $D_{m+1} - Int_P D_m$ is a partial disk and will be called a **patch**. Z lies in X_{n+1} for some $n \ge n_0$ or in $C_{n_0} - Int C_0$. In the first case let $\partial_+ Z = \partial_1 Z \cap F_{n+1}$ and $\partial_- Z = \partial_1 Z \cap F_n$; in the second case let $\partial_+ Z = \partial_1 Z$ and $\partial_- Z = \emptyset$. The set of all patches together with D_0 forms the vertex set of a locally finite graph Γ in which the edges are all the components of all the $Fr_P D_m$; two vertices are joined by an edge whenever the two disks meet along the corresponding arc. Since P is simply connected Γ is a tree. A subgraph of Γ will often be identified with the submanifold of P consisting of the union of its vertices. For $n \ge n_0$ let $\Gamma_n = \Gamma \cap C_n$. If T is a component of Γ_n , then T is a finite tree and $Fr_PT \subseteq F_n$. If T does not contain D_0 , then T is called a **falling tree** with **frontier** in F_n . We also say that the falling tree T **descends** from F_n . A falling tree which is the star of some vertex Z is called a **falling star** with **falling vertex** or **center** Z. Every falling tree contains a falling star.

P is in n_0 -monotone position with respect to *C* if it is in n_0 -standard position with respect to *C* and for each patch *Z* one has that $\partial_- Z$ is a single arc. This is equivalent to each Γ_n being connected and hence to there being no falling trees.

Lemma 4.2. Let C be a nice exhaustion for M, and let P be a proper partial plane in M. Then for some $n_0 > 0$ one has that P is ambient isotopic to a partial plane which is in n_0 -monotone position with respect to C.

Proof. Use Lemma 4.1 to put P in n_0 -standard position. By choosing n_0 sufficiently large we may assume that $D_0 \cap C_0 \neq \emptyset$. We first describe isotopies supported in 3-balls which reduce the number of edges of a falling star and which eliminate falling stars with two edges. We then examine the effect of these isotopies on other portions of P and describe further isotopies which may be needed to keep P in n_0 -standard position. The concatenation of these isotopies gives an isotopy with compact support which eliminates a falling star but may create new ones. We then show how to organize these isotopies so as to eliminate all falling trees.

Suppose Z is the center of a falling star. Then $\partial_1 Z \subseteq F_{n+1}$ for some $n \ge n_0 - 1$, and Z lies in X, where X is X_{n+1} if $n \ge n_0$ and $C_{n_0} - Int C_0$ if $n = n_0 - 1$. Let X^+ be the union of X and a collar on F_{n+1} in X_{n+2} . Since X is irreducible and ∂ -irreducible there is a disk Z' in ∂X to which Z is ∂ -parallel across a 3-ball B in X. Let A be a component of $X \cap \partial M$ which meets Z. Then A is an annulus which meets F_{n+1} in a simple closed curve K. Since $Z \cap F_n = \emptyset$ each component β of $A \cap \partial Z$ is an arc which is ∂ -parallel in A to an arc β' in K. Let β be an innermost such component, i.e. $\beta \cup \beta'$ bounds a disk G in A such that $G \cap Z = \beta$.

Suppose Z has order at least three. Let B_1 be a regular neighborhood of G in X^+ . There is an ambient isotopy of P in M supported in B_1 which moves β to β' and then past β' into $int(X_{n+2} \cap \partial M)$. Such an isotopy (regardless of the order of Z) is called a **boundary slide** of β past β' . The endpoints of β must lie on distinct arcs γ_1 , γ_2 of $\partial_1 Z$; otherwise they would be joined by a single arc γ in $\partial_1 Z$ and so one would have $\partial Z = \beta \cup \gamma$, contradicting the fact that P is in n_0 -standard position. Now γ_1 and γ_2 are components of $\partial_- Z_1$ and $\partial_- Z_2$, respectively, for distinct patches Z_1 and Z_2 in X_{n+2} . The boundary slide replaces Z_1 and Z_2 by a new patch W obtained by joining them by a band. W has order one less than the sum of the orders of Z_1 and Z_2 ; since these orders are at least two W has order at least three. The boundary slide replaces Z by a partial disk V with order one less than that of Z; hence V has order at least two.

Suppose Z has order two. Then the endpoints of the arc $\gamma_1 \cup \beta \cup \gamma_2$ are joined by an arc δ in some component A^* of $X \cap \partial M$, and the union of these two arcs is $\partial Z'$. Moreover δ is ∂ -parallel in A^* to an arc δ' in $A^* \cap F_{n+1}$ across a disk G^* .

Suppose $G \cap G^* = \emptyset$. Then $\beta' \cup \gamma_1 \cup \delta' \cup \gamma_2$ bounds a disk H in F_{n+1} , and $Z' = G \cup H \cup G^*$. Let B_2 be a regular neighborhood in X^+ of B. There is an

ambient isotopy of P in M supported in B_2 which carries Z to H and then past H into $Int X_{n+2}$. This **band push** replaces Z, Z_1 , and Z_2 by a patch Y in X_{n+2} whose order is two less than the sum of the orders of Z_1 and Z_2 and hence is at least two.

Suppose $G \cap G^* \neq \emptyset$. Then $A^* = A$, $G \subseteq Int_A G^*$, and $\beta' \subseteq int \delta'$. Moreover each γ_i is ∂ -parallel in F_{n+1} across a disk H_i to an arc δ'_i in K, $H_1 \cap H_2 = \emptyset$, $Z' = H_1 \cup (G^* - Int_{G^*}G) \cup H_2$, and $\delta' = \delta'_1 \cup \beta' \cup \delta'_2$. Let B_3 be a regular neighborhood in X^+ of $B \cup G$. There is an ambient isotopy of P in M supported in B_3 which consists of a boundary slide of β past β' , followed by a boundary slide of δ past δ' , followed by a disk push which replaces Z, Z_1 , and Z_2 by a patch Y in X_{n+2} of order at least two. This is a **band unfolding**.

Now suppose Z_0 is some other vertex of $\Gamma \cap X$ which meets the ball B_i (i = 1, 2, 2)or 3) supporting one of these isotopies. For a band push we have $Z_0 \subseteq B_2$, and so Z_0 is the center of a falling star. The isotopy replaces this star by a new vertex in X_{n+2} of order at least two, and so P is still in n_0 -standard position. Now consider a boundary slide. Let $D = \beta' \times [0,1]$ be a disk in F_{n+1} with $\beta' \times \{0\} = \beta'$, and $\beta' \times \{0,1\} \subseteq (\gamma_1 \cup \gamma_2)$. We may assume that $D \subseteq B_1$ and that $Z_0 \cap D$ consists of product arcs. This isotopy deletes each of the arcs of $Z_0 \cap (G \cup D)$ and joins its endpoints by an arc in D. It joins together the vertices in X_{n+2} adjacent to Z_0 by bands corresponding to the new arcs in D to get a 2-manifold W_0 ; it also replaces Z_0 by a disk V_0 . Let $J = W_0 \cap V_0$. Then $J \subseteq \partial V_0$. If J has at least two components, then each of these components is an arc, and V_0 and each component of W_0 is a partial disk of order at least two. If J has exactly one component, then it is an arc or a simple closed curve. Suppose J is an arc. Then W_0 is a partial disk of order at least three. If V_0 has order one, then $\partial V_0 = \partial V'_0$ for a disk V'_0 in ∂X such that V'_0 is the union of a disk V'_1 in $X \cap \partial M$ and a disk V'_2 in F_{n+1} along an arc in $F_{n+1} \cap \partial M$. A halfdisk push of V_0 past V'_1 across the ball B_0 bounded by $V_0 \cup V'_0$ replaces W_0 and V_0 by a partial disk W_1 in X_{n+2} of order at least two. If Z_1 is some other vertex in X whose image under the boundary slide lies in B_0 , then it must be the center of a falling star, and so the halfdisk push replaces the falling star by a vertex in X_{n+2} . Now suppose J is a simple closed curve. Then W_0 is an annulus one of whose boundary components is $J = \partial V_0 \subseteq F_{n+1}$. So V_0 is ∂ -parallel in X to a disk V'_0 in F_{n+1} across a 3-ball B_0 . A disk push of V_0 across B_0 past V'_0 replaces W_0 and V_0 by a disk W_1 in X_{n+2} . Note that since Z_0 has order at least two W_0 must have been obtained by joining at least two distinct vertices in X_{n+2} . It follows that W_1 has order at least two. If Z_1 is some other vertex in X whose image under the boundary slide lies in B_0 , then again it must be the center of a falling star which is replaced by a vertex in X_{n+2} by the disk push. Thus in all cases P is put back in n_0 -standard position. Finally consider a band unfolding. This move is the concatenation of a boundary slide and a halfdisk push. An analysis similar to that above provides isotopies which return P to n_0 -standard position and reduce the number of vertices in Γ_{n+2} .

Thus given any falling vertex Z in X, there is an ambient isotopy of P in M supported in X^+ which collapses the falling star with center Z to a vertex in X_{n+2} . The only other possible effects of this isotopy on Γ are to collapse other such falling

stars with centers in X in a similar fashion and to amalgamate vertices in X_{n+2} which are adjacent to vertices in X, thereby reducing the orders of these vertices, but not reducing them to one. The number of vertices in each of $\Gamma \cap X$ and $\Gamma \cap X_{n+2}$ is reduced. Γ is unchanged outside $X \cup X_{n+2}$.

Define a sequence $n_0 < n_1 < n_2 < \cdots$ by choosing $n_{i+1} > n_i + 1$ so that if T is any falling tree descending from $F_{n_{i+1}}$, then $T \cap F_{n_i+1} = \emptyset$. Every falling tree in $C_{n_0} - Int C_0$ is a falling vertex. By the discussion above there is an isotopy supported in $C_{n_0+1} - Int C_0$ which eliminates them all. This isotopy may create new falling vertices descending from F_{n_0+1} , but it creates no new falling trees. It may reduce the order of D_0 but it does not eliminate it since $D_0 \cap C_0 \neq \emptyset$.

Suppose i > 0. Let k be the smallest index for which there is a falling tree T descending from F_{n_i} such that $T \cap F_k \neq \emptyset$. Then $k > n_{i-1}+1$, and $T \cap X_k$ consists of falling vertices. There is an isotopy supported in $X_k \cup X_{k+1}$ which eliminates them and creates no new falling trees descending from F_{n_i} . Continuing in this fashion there is an isotopy supported in $Int C_{n_i+1} - C_{n_{i-1}+1}$ which eliminates all falling trees descending from F_{n_i} . Since these isotopies have disjoint compact supports they give a single ambient isotopy of P in M after which each Γ_{n_i} is connected.

Finally there is an isotopy supported in $(Int C_{n_i}) - C_{n_{i-1}}$ which removes all falling trees descending from F_n for $n_{i-1} < n < n_i$ and creates no new falling trees. Again this yields a single ambient isotopy of P in M after which P is in monotone position. \Box

Theorem 4.3. Let M be a connected, irreducible, non-compact 3-manifold which admits a nice exhaustion. Let P be a proper partial plane in M. Then ∂P lies in a single component E of ∂M , and there is a collar H on E in M such that $P \subseteq Int H$.

Proof. By Lemma 4.2 we may assume that P is in n_0 -monotone position with respect to a nice exhaustion C.

Suppose P meets at least two components of ∂M . Then for some n there is a component α of $P \cap F_n$ which has one endpoint in a boundary plane E and the other in another boundary plane E'. Let Z be the patch in X_{n+1} containing α . Then ∂Z bounds a disk Z' in ∂X_{n+1} by the ∂ -irreducibility of X_{n+1} . But this is impossible because the simple closed curve $E \cap F_n$ meets ∂Z transversely in a single point since $Z \cap F_n = \alpha$. Thus ∂P lies in a single component E of ∂M .

We now claim that every component α of $P \cap F_n$ is parallel in F_n to an arc β in $E \cap F_n$. For let Z be the patch in X_{n+1} containing α . Then ∂Z bounds a disk Z' in ∂X_{n+1} . The components of $F_n \cap E \cap Z'$ are proper arcs in Z'. If the endpoints of α lie on distinct such arcs β_1 and β_2 , then the other endpoints of β_1 and β_2 must lie on components of $Z \cap F_n$ other than α , contradicting the fact that $\partial_- Z = \alpha$. So the endpoints of α are joined by an arc β . Then $\alpha \cup \beta$ bounds a disk W in Z'. If W does not lie in F_n , then W must contain a component of F_{n+1} , contradicting the positive genus assumption on C. Thus α is parallel to β across the disk W in F_n .

For each component α of $P \cap F_n$ let $W(\alpha)$ be the disk in F_n across which α is parallel to an arc β in $F_n \cap E$. Let \mathcal{W}_n be the set of maximal such disks, i.e. those $W(\alpha)$ which do not lie in $Int_{F_n}W(\alpha')$ for some α' . The $W(\alpha)$ are unique and the elements of \mathcal{W}_n are disjoint since otherwise some component of F_n would be a disk. Let $G_n = E \cap C_n$ and $A_n = E \cap X_n$.

Consider ∂D_0 . It has not been assumed that C_0 is ∂ -irreducible. Nevertheless, ∂D_0 does bound a disk D'_0 in ∂C_0 . For let $\mathcal{W}_0 = \{W(\alpha_{0,1}), \ldots, W(\alpha_{0,k_0})\}$. Then $G_0^+ = G_0 \cup W(\alpha_{0,1}) \cup \cdots \cup W(\alpha_{0,k_0})$ is a disk in ∂C_0 which contains ∂D_0 , and the result follows.

Now since M is irreducible D_0 is parallel across a 3-ball B_0 in C_0 to a disk D'_0 in ∂C_0 . There is a regular neighborhood H_0 of G_0^+ in C_0 such that $B_0 \subseteq Int H_0$. Then $G_0^* = Fr_{C_0}H_0$ is a proper disk in C_0 , and there is a product structure $G_0 \times [0, 1]$ on H_0 such that $G_0 \times \{0\} = G_0$, $G_0 \times \{1\} = G_0^*$, and $L_0 = (\partial G_0) \times [0, 1] \subseteq F_0$.

Let $\mathcal{W}_1 = \{W(\alpha_{1,1}), \ldots, W(\alpha_{1,k_1})\}$. Then $A_1^+ = A_1 \cup L_0 \cup W(\alpha_{1,1}) \cup \cdots \cup W(\alpha_{1,k_1})$ is an annulus in ∂X_1 which contains ∂Z for each patch Z in X_1 . Each such Z is parallel across a 3-ball B(Z) in X_1 to a disk Z' in A_1^+ . There is a regular neighborhood H_1 of A_1^+ in X_1 such that each B(Z) is contained in $Int H_1$ and $H_1 \cap F_0 = H_0 \cap F_0$. Then $A_1^* = Fr_{X_1}H_1$ is a proper annulus in X_1 , and there is a product structure $A_1 \times [0, 1]$ on H_1 such that $A_1 \times \{0\} = A_1$, $A_1 \times \{1\} = A_1^*$, and $(\partial A_1) \times [0, 1] = L_0 \cup L_1$, where L_1 is an annulus in F_1 and the product structures on L_0 induced by those on H_0 and H_1 agree.

We now continue this process, constructing for each n an $H_n = A_n \times [0, 1]$ with $A_n \times \{0\} = A_n, A_n \times \{1\} = A_n^*$, a proper annulus in $X_n, (\partial A_n) \times [0, 1] = L_{n-1} \cup L_n$, where L_n is an annulus in F_n and the product structures on L_{n-1} induced by those on H_{n-1} and H_n agree such that $(P \cap X_n) \subseteq Int H_n$. We then let $H = \bigcup_{n>0} H_n$. \Box

5. Strong Aplanarity: The General Case

A partial plane system is a surface \mathcal{P} each component of which is a partial plane. Thus an aplanar 3-manifold M is strongly aplanar if and only if for every proper partial plane system \mathcal{P} in M there is a collar H on ∂M such that Int H contains \mathcal{P} . The goal of this section is to show that this is the case if M is irreducible and admits a nice exhaustion.

Lemma 5.1. Suppose M is a connected, irreducible, non-compact 3-manifold which is not homeomorphic to $\mathbb{R}^2 \times [0, 1)$. Let E be a component of ∂M which is a plane and has the property that for every proper partial plane P in M with ∂P in E there is a collar H on E in M such that Int H contains P. Then for every proper partial plane system \mathcal{P} in M with $\partial \mathcal{P}$ in E there is a collar H' on E in M such that Int H'contains \mathcal{P} .

Proof. There is a collection $\{\beta_{i,j}\}$ of disjoint arcs in E whose union is end-proper in E such that $\beta_{i,j} \cap \mathcal{P} = \partial \beta_{i,j}$ and the union K of \mathcal{P} with these arcs is simply connected. The notation is chosen so that $\beta_{i,j}$ joins components P_i and P_j of \mathcal{P} for some choice of i < j. One way to see this is as follows. Let $\{D_n\}$ be an exhaustion for E chosen so that each D_n is a disk and $\partial \mathcal{P}$ and $\cup \partial D_n$ are in general position. We may assume that ∂D_0 meets at least two components of \mathcal{P} . There is then an arc $\beta_{0,1}$ in ∂D_0 whose interior misses \mathcal{P} and whose endpoints lie in different components P_0 and P_1 of \mathcal{P} . Let $K_1 = P_0 \cup \beta_{0,1} \cup P_1$. If ∂D_0 meets other components of \mathcal{P} , then there is an arc $\beta_{i,2}$ in ∂D_0 whose interior is disjoint from $K_1 \cup \mathcal{P}$ such that one endpoint lies in a third component P_2 of \mathcal{P} and the other endpoint lies in P_i , where i = 0 or 1. If this endpoint of $\beta_{i,2}$ is an endpoint of $\beta_{0,1}$, then we isotop $\beta_{i,2}$ slightly in E to make the two arcs disjoint, keeping this endpoint of $\beta_{i,2}$ in P_i . We then let $K_2 = K_1 \cup \beta_{i,2} \cup P_2$. We continue in this fashion until we have a simply connected 2-complex K_{m_0} containing all those components of \mathcal{P} which meet ∂D_0 . We then adjoin arcs in ∂D_1 and the components of \mathcal{P} which meet ∂D_1 but do not lie in K_{m_0} to obtain a 2-complex K_{m_1} . Continuing in this manner we inductively construct the desired 2-complex K.

Let $B_{i,j}$ be a regular neighborhood of $\beta_{i,j}$ in the 3-manifold obtained by splitting M along \mathcal{P} . Thus $B_{i,j}$ has the form $D_{i,j} \times [0,1]$, where $D_{i,j}$ is a halfdisk, $(\partial_0 D_{i,j}) \times [0,1] = B_{i,j} \cap E = B_{i,j} \cap \partial M$, $B_{i,j} \cap \mathcal{P} = D_{i,j} \times \{0,1\}$, $B_{i,j} \cap P_i = D_{i,j} \times \{0\}$, and $B_{i,j} \cap P_j = D_{i,j} \times \{1\}$. Identify $D_{i,j}$ with $D_{i,j} \times \{1/2\}$. We now form the "band sum" P of the components of \mathcal{P} along the $\beta_{i,j}$ by deleting all the $Int_{\mathcal{P}}(B_{i,j} \cap \mathcal{P})$ and then adjoining all the bands $(\partial_1 D_{i,j}) \times [0,1]$. Then P is a proper partial plane in M and so lies in a collar $H = E \times [0,1]$ on E in M with $E = E \times \{0\}$. Let $E^* = E \times \{1\}$.

Let M' be the 3-manifold obtained by splitting M along P. Thus $\partial M'$ contains two copies of P whose identification recovers M. Let M'' be the component of M' containing E^* . Then M'' is a connected, irreducible, non-compact 3-manifold which is not homeomorphic to \mathbb{R}^3 . Let \mathcal{D} be the union of those $D_{i,j}$ which lie in M''. Then E^* and \mathcal{D} are proper incompressible surfaces in M'' no component of which is a 2-sphere. If E^* is trivial in M'', then it is trivial in M and hence M is homeomorphic to $\mathbb{R}^2 \times [0, 1)$, a contradiction. Therefore by Corollary 2.2 there is an ambient isotopy of E^* in M'', fixed on $\partial M''$, which takes E^* to a plane disjoint from \mathcal{D} . It follows that there is an ambient isotopy of E^* in M, fixed on $P \cup \partial M$, which takes E^* to a plane disjoint from all the $B_{i,j}$ and thus disjoint from \mathcal{P} . The image H' of H under this isotopy is a collar on E in M such that Int H' contains \mathcal{P} . \Box

Lemma 5.2. Suppose M is a connected, irreducible, non-compact 3-manifold. Let E be a component of ∂M which is a plane. Let H be a collar on E in M. Suppose \mathcal{P} is a proper partial plane system in M such that $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1$, where \mathcal{P}_0 and \mathcal{P}_1 are unions of components of \mathcal{P} such that $\mathcal{P}_0 \subseteq H$ and $\mathcal{P}_1 \cap E = \emptyset$. Then there is an ambient isotopy of \mathcal{P}_1 in M, fixed on $\mathcal{P}_0 \cup \partial M$, taking \mathcal{P}_1 to a surface \mathcal{P}'_1 such that $\mathcal{P}'_1 \cap H = \emptyset$.

Proof. We may assume that $\mathcal{P}_1 \neq \emptyset$. Let $H = E \times [0, 1]$ with $E = E \times \{0\}$ and $E^* = E \times \{1\}$. Let M' be the 3-manifold obtained by splitting M along \mathcal{P}_0 . Let M'' be the component of M' containing E^* . Then M'' is a connected, irreducible, non-compact 3-manifold which is not homeomorphic to \mathbf{R}^3 . \mathcal{P}_1 and E^* are proper incompressible surfaces in M'' having no 2-sphere components. No component of \mathcal{P}_1 is a plane. If E^* is trivial in M'', then it is trivial in M and so M is homeomorphic to $\mathbf{R}^2 \times [0, 1)$. However, $\mathcal{P}_1 \neq \emptyset$ and $\mathcal{P}_1 \cap E = \emptyset$ implies that ∂M has at least two components, a contradiction. Thus by Corollary 2.2 there is an ambient isotopy of \mathcal{P}_1 , fixed on $\partial M''$, which takes \mathcal{P}_1 to a surface \mathcal{P}'_1 such that \mathcal{P}'_1 and E^* are disjoint. The desired conclusion then follows. \Box

Theorem 5.3. Let M be a connected, irreducible, non-compact 3-manifold which admits a nice exhaustion. Then M is strongly aplanar.

Proof. M is a planar by Theorem 3.4. ∂M has components E_1, \ldots, E_{ν} ; each E_i is a plane. Let \mathcal{P} be a proper partial plane system in M. By Theorem 4.3 $\mathcal{P} = \mathcal{P}_1 \cup$ $\cdots \cup \mathcal{P}_{\nu}$, where \mathcal{P}_i is the union of those components P of \mathcal{P} such that $\partial P \subseteq E_i$. We induct on the number of nonempty \mathcal{P}_i . If \mathcal{P} meets only one boundary component, say E_1 , then we apply Theorem 4.3 and Lemma 5.1 to get a collar H_1 on E_1 in M such that $\mathcal{P} \subseteq Int H_1$ and then take arbitrary collars on the remaining E_i in $M-Int H_1$. Suppose that the theorem is true for those proper partial plane systems meeting at most k of the ν boundary components. Let \mathcal{P} be a proper partial plane system meeting k + 1 of them, say E_1, \ldots, E_{k+1} . By Theorem 4.3 and Lemma 5.1 there is a collar $H_{k+1} = E_{k+1} \times [0,1]$ on E_{k+1} in M with $E_{k+1} = E_{k+1} \times \{0\}$ such that $\mathcal{P}_{k+1} \subseteq Int H_{k+1}$. Let $M' = M - Int H_{k+1}$. By Lemma 5.2 there is an ambient isotopy of \mathcal{P} in M, fixed on $\mathcal{P}_{k+1} \cup \partial M$, which takes $\mathcal{P}_1 \cup \cdots \cup \mathcal{P}_k$ to $\mathcal{P}'_1 \cup \cdots \cup \mathcal{P}'_k$ lying in $M' - (E_{k+1} \times \{1\})$. Since M' is homeomorphic to M the inductive hypothesis gives a collar on $\partial M'$ in M' whose interior in M' contains $\mathcal{P}'_1 \cup \cdots \cup \mathcal{P}'_k$. It follows that there is a collar on ∂M in M whose interior in M contains \mathcal{P} . \Box

6. Attaching Boundary Planes

In this section we show how to attach boundary planes to a connected, open 3-manifold U with finitely many ends to obtain non-compact 3-manifolds M with certain properties. In each case we construct M by choosing a finite set of disjoint rays $[0, \infty)$ end-properly embedded in U and then removing the interiors of disjoint regular neighborhoods of these rays. It is easily seen that U is homeomorphic to *int* M via a homeomorphism which takes the end of U determined by a ray in Uto the end of M determined by the corresponding boundary plane of M. The rays will be chosen using an exhaustion of U so as to obtain a nice exhaustion for M.

We will need some results and terminology from [14]. A compact, connected 3-manifold X is **excellent** if it is \mathbf{P}^2 -irreducible and ∂ -irreducible, it is not a 3ball, it contains a 2-sided, proper, incompressible surface, and every connected, proper, incompressible surface of zero Euler characteristic in X is ∂ -parallel. (So in particular X is anannular and atoroidal.) Let λ be a proper 1-manifold in a 3-manifold Y. The **exterior** of λ in Y is the closure of the complement of a regular neighborhood of λ in Y. Suppose Y is compact. If the exterior of λ is excellent, then we say that λ itself is **excellent**. It follows from Thurston's work [12] that λ is excellent if and only if its exterior is hyperbolic. Theorem 1.1 of [14] states, among other things, that if κ is any proper 1-manifold in a compact, connected 3-manifold Y such that κ meets every 2-sphere in ∂Y at least twice and every projective plane in ∂Y at least once, then κ is homotopic rel $\partial \kappa$ to an excellent 1-manifold λ .

Theorem 6.1. Let $1 \le \mu < \infty$, and for $1 \le i \le \mu$ let $1 \le \nu_i < \infty$. Let U be a connected, open 3-manifold with μ ends. Then there is a non-compact 3-manifold M which has the following properties.

(1) U is homeomorphic to int M.

- (2) ∂M is a disjoint union of planes $E^{i,j}$, $1 \leq i \leq \mu$, $1 \leq j \leq \nu_i$.
- (3) For $1 \leq i \leq \mu$ the image of the end e^i of U under the homeomorphism and inclusion induced maps $\varepsilon(U) \to \varepsilon(int M) \to \varepsilon(M)$ is the image of $\varepsilon(\cup_{i=1}^{\nu_i} E^{i,j})$ under the inclusion induced map $\varepsilon(\partial M) \to \varepsilon(M)$.
- (4) M is connected, eventually end-irreducible, eventually \mathbf{P}^2 -irreducible, anannular at infinity, and totally acylindrical, and is not almost compact.
- (5) If U is irreducible, then M is strongly aplanar.

Proof. Let $\{K_n\}$ be an exhaustion for U. Denote $M - K_n$ by V_n and $K_{n+1} - Int K_n$ by Y_{n+1} . By passing to a subsequence of $\{K_n\}$ we may assume that each V_n has μ components V_n^i , $1 \le i \le \mu$, with $V_n^i \supseteq V_{n+1}^i$. Let $Y_{n+1}^i = (Cl V_n^i) \cap Y_{n+1}$. By passing to a subsequence and attaching 1-handles to $Fr K_n$ in $Cl V_n$ we may assume that each Y_{n+1}^i is connected, each component of $Fr K_n$ has negative Euler characteristic, and each orientable component of $Fr K_n$ has positive genus. In particular ∂Y_{n+1} contains no 2-spheres or projective planes.

By Theorem 1.1 of [14] there exist disjoint proper arcs $\alpha_{n+1}^{i,j}$, $1 \leq j \leq \nu_i$, in Y_{n+1}^i such that $\alpha_{n+1}^{i,j}$ runs from $Fr K_n$ to $Fr K_{n+1}$, $\alpha_{n+1}^{i,j} \cap Fr K_{n+1} = \alpha_{n+2}^{i,j} \cap Fr K_{n+1}$, and the exterior X_{n+1}^i of $\bigcup_{j=1}^{\nu_i} \alpha_{n+1}^{i,j}$ in Y_{n+1}^i is excellent. Then $\alpha^{i,j} = \bigcup_{n=1}^{\infty} \alpha_n^{i,j}$ is a ray which is proper in V_0 and end-proper in U.

Let $X_{n+1} = \bigcup_{i=1}^{\mu} X_{n+1}^{i}$. We may assume that $X_{n+1} \cap Fr K_{n+1} = X_{n+2} \cap Fr K_{n+1}$. Let $M = K_0 \cup \bigcup_{n=1}^{\infty} X_n$. Note that $Cl_{Y_n}(Y_n - X_n)$ consists of disjoint 3-balls $N_n^{i,j}$ which are regular neighborhoods of $\alpha_n^{i,j}$, respectively. Let $N^{i,j} = \bigcup_{n=1}^{\infty} N_n^{i,j}$. Then $E^{i,j} = Fr_U N^{i,j}$ is a proper plane in U, and ∂M is the disjoint union of these planes.

Let $H^{i,j}$ be disjoint regular neighborhoods of $N^{i,j}$ in U. Then $H^{i,j}$ and $H^{i,j}-N^{i,j}$ are both homeomorphic to $\mathbf{R}^2 \times [0, 1)$, and thus there is a homeomorphism from U to *int* M with the required properties.

Finally let $C_0 = K_0$ and $C_{n+1} = C_n \cup X_{n+1}$. Then $\{C_n\}$ is a nice exhaustion for M, and so by Lemma 1.2, Lemma 1.3, Theorem 3.4, Theorem 3.5, and Theorem 5.3 the 3-manifold M has all the stated properties. \Box

We now turn to the modifications of the basic construction of Theorem 6.1 which yield the 3-manifolds with the additional properties mentioned in the introduction. We will need some preliminary results.

Lemma 6.2. Let R be a compact, connected 3-manifold. Let S be a compact, proper, 2-sided surface in R. Let R' be the 3-manifold obtained by splitting R along S. Let S_1 and S_2 be the two copies of S in $\partial R'$ which are identified to obtain R. If each component of R' is excellent, $S_1 \cup S_2$ and $(\partial R') - int(S_1 \cup S_2)$ are incompressible in R', and each component of S has negative Euler characteristic, then Ris excellent.

Proof. This is Lemma 2.1 of [14]. \Box

We first strengthen our construction so that it withstands the removal of some (but not all) boundary planes from each end; the basic technical device employed is called a "poly-excellent tangle." Let $n \ge 1$. An *n*-tangle is a proper 1-manifold λ in a 3-ball such that λ has *n* components and each of these components is an arc.

For $1 \leq k \leq n$ a k-subtangle of an *n*-tangle λ is a union of k components of λ . An *n*-tangle λ is **poly-excellent** if for each $1 \leq k \leq n$ each k-subtangle of λ is excellent.

Theorem 6.3. For all $n \ge 1$ poly-excellent n-tangles exist.

The proof of this theorem is given in the Appendix.

Lemma 6.4. Let T be a solid torus or a solid Klein bottle, and let $1 \leq \nu < \infty$. Then there exist disjoint proper arcs $\rho_1, \ldots, \rho_{\nu}$ in T such that for every nonempty subset $\{j_1, \ldots, j_k\}$ of $\{1, \ldots, \nu\}$ the 1-manifold $\rho_{j_1} \cup \cdots \cup \rho_{j_k}$ is excellent in T.

Proof. Let G be a meridional disk in T. Let B be the 3-ball obtained by splitting T along G, and let G_1 and G_2 be the disks in ∂B which are identified to obtain T.

By Theorem 6.3 *B* contains a poly-excellent 3ν -tangle λ . Divide the components of λ into three groups $\{\beta_j\}$, $\{\gamma_j\}$, and $\{\delta_j\}$, where $1 \leq j \leq \nu$. Isotop λ so that β_j and γ_j run from $int G_1$ to $\partial B - (G_1 \cup G_2)$, δ_j runs from $int G_2$ to itself, and under the identification of G_1 and G_2 one has that $(\beta_j \cup \gamma_j) \cap int G_1$ is identified with $\delta_j \cap int G_2$. Let $\rho'_j = \beta_j \cup \gamma_j \cup \delta_j$, and let ρ_j be the image of ρ'_j in *T*.

Let $\{j_1, \ldots, j_k\}$ be a nonempty subset of $\{1, \ldots, \nu\}$. Let R be the exterior of $\rho_{j_1} \cup \cdots \cup \rho_{j_k}$ in T, and let R' be the exterior of $\rho'_{j_1} \cup \cdots \cup \rho'_{j_k}$ in B. We may assume that these exteriors have been chosen so that $S = R \cap G$ is a disk with 2k holes and R' is the 3-manifold obtained by splitting R along S. Let S_1 and S_2 be the copies of S in $\partial R'$ which are identified to obtain R. Each component of $\partial R' - int(S_1 \cup S_2)$ is an annulus. Since $\partial R'$ is incompressible in R' and no component of $S_1 \cup S_2$ or of $\partial R' - int(S_1 \cup S_2)$ is a disk each of these surfaces is incompressible in R'. Since R' is excellent and S has negative Euler characteristic Lemma 6.2 implies that R is excellent. \Box

Theorem 6.5. Let μ , ν_i , and U be as in Theorem 6.1. Then there is a 3-manifold M having all the properties listed in that theorem such that if $1 \leq \nu'_i \leq \nu_i$, then every 3-manifold \widehat{M} obtained from M by deleting for each i any $\nu_i - \nu'_i$ of the $E^{i,j}$ from ∂M has all these properties.

Proof. Let $\{K_n\}$, V_n , V_n^i , Y_{n+1} , and Y_{n+1}^i be as in the proof of Theorem 6.1. Let ξ_{n+1}^i be a proper arc in Y_{n+1}^i running from $Fr K_n$ to $Fr K_{n+1}$ chosen so that $\xi_{n+1}^i \cap Fr K_{n+1} = \xi_{n+2}^i \cap Fr K_{n+1}$. Let Z_{n+1}^i be the exterior of ξ_{n+1}^i in Y_{n+1}^i . We may assume that $Z_{n+1}^i \cap Fr K_{n+1} = Z_{n+2}^i \cap Fr K_{n+1}$. Then ∂Z_{n+1}^i contains no 2-spheres or projective planes because each component of $Fr K_n \cup Fr K_{n+1}$ has negative Euler characteristic. By Theorem 1.1 of [14] there is an excellent proper arc η_{n+1}^i in Z_{n+1}^i with $\partial \eta_{n+1}^i$ in the open annulus $(\partial Z_{n+1}^i) \cap (int Y_{n+1}^i)$. Let L_{n+1}^i be the exterior of η_{n+1}^i in Z_{n+1}^i . Then $T_{n+1}^i = Y_{n+1}^i - Int_{Y_{n+1}^i} L_{n+1}^i$ is a solid torus or solid Klein bottle which meets ∂Y_{n+1}^i in a disk D_n^i in $Fr K_n$ and a disk D_{n+1}^i in $Fr K_{n+1}$.

By Lemma 6.4 there is a set of disjoint proper arcs $\alpha_{n+1}^{i,j}$, $1 \leq j \leq \nu_i$, in T_{n+1}^i with $\alpha_{n+1}^{i,j}$ running from $int D_n^i$ to $int D_{n+1}^i$ such that for every nonempty subset the union of its elements is excellent in T_{n+1}^i . We then proceed to construct M as in the proof of Theorem 6.1, with the same notation as defined there. Now suppose that for each i we have deleted a set of $\nu_i - \nu'_i$ boundary planes $E^{i,j}$ from the i^{th} end of M so as to obtain \widehat{M} . This 3-manifold is homeomorphic to the one obtained by ignoring the corresponding rays $\alpha^{i,j}$ in our construction. Hence this corresponds to ignoring the arcs $\alpha_{n+1}^{i,j}$. Let \widehat{X}_{n+1}^i be the exterior of the remaining arcs in Y_{n+1}^i , and let R_{n+1}^i be the exterior of these arcs in T_{n+1}^i . Then $\widehat{X}_{n+1}^i = L_{n+1}^i \cup R_{n+1}^i$ and the surface $S_{n+1}^i = L_{n+1}^i \cap R_{n+1}^i$ is a torus or Klein bottle from which the interiors of two disjoint disks have been removed. $\partial L_{n+1}^i - int S_{n+1}^i$ has two components obtained by removing $int D_n^i$ and $int D_{n+1}^i$ from components of $Fr K_n$ and $Fr K_{n+1}$ respectively. $\partial R_{n+1}^i - int S_{n+1}^i$ is a connected, orientable surface of genus $\nu'_i - 1$ with two boundary components. Thus S_{n+1}^i , $\partial R_{n+1}^i - int S_{n+1}^i$, and each component of $\partial L_{n+1}^i - int S_{n+1}^i$ have negative Euler characteristic. It then follows from Lemma 6.2 that \widehat{X}_{n+1}^i is excellent. We let $\widehat{X}_{n+1} = \bigcup_{i=1}^{\mu} \widehat{X}_{n+1}^i$, $\widehat{Q}_0 = C_0$, and $\widehat{C}_{n+1} = \widehat{C}_n \cup \widehat{X}_{n+1}$. We then have a nice exhaustion $\{\widehat{C}_n\}$ of \widehat{M} .

We now further modify the construction so as to ensure the non-existence of homeomorphisms which carry one boundary plane to another or reverse orientation, as well as to produce an uncountable collection of pairwise non-homeomorphic 3manifolds having the same given interior and distribution of boundary planes among the ends.

A classical knot space Q is a space homeomorphic to the exterior of a nontrivial knot in S^3 . If Q is embedded in a 3-manifold X and ∂Q is incompressible in X, then we say that Q is **incompressibly embedded** in X. The idea of the proof is to associate to each boundary plane E an infinite sequence of disjoint classical knot spaces whose union lies in *int* M and is end-properly embedded in M. These knot spaces are not incompressibly embedded in M. However, for any "sufficiently large" compact subset K of M which meets E all but finitely many of them will be incompressibly embedded in M - K. On the other hand at most finitely many will be incompressibly embedded in the complement of a compact subset which does not meet E. Furthermore the knot spaces will be chosen so that they characterize the plane with which they are associated. A useful analogy is that of a string of lights associated to each plane, with different planes corresponding to different colors. The removal of certain compact subsets meeting a collection of planes then turns on all but finitely many lights in the strings associated to these planes.

We first need a couple of preliminary technical lemmas.

Lemma 6.6. Let T be a solid torus or solid Klein bottle, and let $1 \leq \nu < \infty$. Then there exist disjoint solid tori T_1, \ldots, T_{ν} in int T, disjoint compressing disks D_1, \ldots, D_{ν} for $\partial T_1, \ldots \partial T_{\nu}$ in $T - int(T_1 \cup \cdots \cup T_{\nu})$, and disjoint proper arcs $\rho_1, \ldots, \rho_{\nu}$ in $T - (T_1 \cup \cdots \cup T_{\nu})$ such that $D_i \cap \rho_j = \emptyset$ for $i \neq j$ and for every nonempty subset $\{j_1, \ldots, j_k\}$ of $\{1, \ldots, \nu\}$ the 1-manifold $\rho_{j_1} \cup \cdots \cup \rho_{j_k}$ is excellent in $T - int(T_{j_1} \cup \cdots \cup T_{j_k})$.

Proof. Let G, B, G_1 , and G_2 be as in the proof of Lemma 6.4. Let W_1, \ldots, W_{ν} be disjoint disks in *int* G. Let $W_{i,j}$ be the copy of W_j in G_i whose image under the identification of G_1 with G_2 is W_j . Let T_1, \ldots, T_{ν} be disjoint regular neighborhoods of $\partial W_1, \ldots, \partial W_{\nu}$ in T, chosen so that $A_j = T_j \cap G$ is a regular neighborhood of

 ∂W_j in G. Then T_j is split by A_j into solid tori $T_{1,j}$ and $T_{2,j}$ which are regular neighborhoods of $\partial W_{1,j}$ and $W_{2,j}$, respectively, in B. Let $A_{i,j} = Fr_B T_{i,j}$. Let $T^* = T - int(T_1 \cup \cdots \cup T_{\nu})$ and $B^* = B - Int_B(T_{1,1} \cup \cdots \cup T_{1,\nu} \cup T_{2,1} \cup \cdots \cup T_{2,\nu})$. Then B^* is a 3-ball which meets G_i in ν disks $D_{i,j} \subseteq int W_{i,j}$ and a disk with ν holes H_i . Let D_j be the image of $D_{i,j}$ under the identification. Then the D_j are disjoint compressing disks for ∂T_j in T^* .

By Theorem 6.3 B^* contains a poly-excellent 4ν -tangle λ . Divide the components of λ into four groups $\{\beta_j\}$, $\{\gamma_j\}$, $\{\delta_j\}$, and $\{\omega_j\}$, $1 \leq j \leq \nu$. Isotop λ so that β_j runs from $\partial B - (G_1 \cup G_2)$ to $int D_{1,j}$, γ_j runs from $int D_{2,j}$ to itself, δ_j runs from $int D_{1,j}$ to $int H_1$, and ω_j runs from $int H_2$ to $\partial B - (G_1 \cup G_2)$. Do this so that under the identification we have $(\beta_j \cup \delta_j) \cap D_{1,j}$ identified with $\gamma_j \cap D_{2,j}$ and we have $\delta_j \cap H_1$ identified with $\omega_j \cap H_2$. Let $\rho'_j = \beta_j \cup \gamma_j \cup \delta_j \cup \omega_j$. Let ρ_j be the image of ρ'_j under the identification.

Now let $\{j_1, \ldots, j_k\}$ be a nonempty subset of $\{1, \ldots, \nu\}$. Let R_0 be the exterior of $\rho_{j_1} \cup \cdots \cup \rho_{j_k}$ in $T - int(T_{j_1} \cup \cdots \cup T_{j_k})$, R'_0 the exterior of $\rho'_{j_1} \cup \cdots \cup \rho'_{j_k}$ in B, and R_0^* the exterior of $\rho'_{j_1} \cup \cdots \cup \rho'_{j_k}$ in B^* . We may assume these exteriors are chosen so that R'_0 is the union of R_0^* with all those $T_{i,j}$ for which $j \in \{j_1, \ldots, j_k\}$. Since A_j is parallel to $A_{i,j}$ across $T_{i,j}$ we have that R_0^* is homeomorphic to R'_0 and is therefore excellent. We may further assume that R_0^* is obtained by splitting R_0 along the surface S consisting of the k disks with two holes $R_0 \cap D_{j_r}$ and the disk with 2k holes $R_0 \cap (G - (D_{j_1} \cup \cdots \cup D_{j_k}))$. Let S_1 and S_2 be the two copies of S in ∂R_0^* . Then the components of $\partial R_0^* - int(S_1 \cup S_2)$ are 4k annuli and a disk with 2k + 1 holes. Since ∂R_0^* is incompressible in R_0^* it follows that $S_1 \cup S_2$ and $\partial R_0^* - int(S_1 \cup S_2)$ are incompressible in R_0^* . Thus by Lemma 6.2 we have that R_0 is excellent. \Box

Lemma 6.7. Let T be a solid torus or solid Klein bottle. Let J_1, \ldots, J_{ν} be excellent knots in S^3 . Then there are disjoint classical knot spaces Q_1, \ldots, Q_{ν} in int T and disjoint proper arcs $\rho_1, \ldots, \rho_{\nu}$ in $T-int(Q_1 \cup \cdots \cup Q_{\nu})$ such that Q_j is homeomorphic to the exterior of J_j in S^3 , there are disjoint 3-balls B_j in int T such that $Q_j \subseteq B_j$ and $B_i \cap \rho_j = \emptyset$ for $i \neq j$. and for every nonempty subset $\{j_1, \ldots, j_k\}$ of $\{1, \ldots, \nu\}$

- (1) the exterior R of the 1-manifold $\rho_{j_1} \cup \cdots \cup \rho_{j_k}$ in T is \mathbf{P}^2 -irreducible, ∂ -irreducible, and an annular,
- (2) each Q_{i_r} is incompressibly embedded in R, and
- (3) given any classical knot space Q incompressibly embedded in int R there is an ambient isotopy of Q in R, fixed on ∂R , which takes Q to some Q_{i_r} .

Proof. Let T_j , D_j , and ρ_j be as in Lemma 6.6. Let $T^* = T - int(T_1 \cup \cdots \cup T_{\nu})$. Let Q_j be the exterior of J_j in S^3 . Form T_0 by gluing $Q_1 \cup \cdots \cup Q_{\nu}$ to T^* by identifying ∂Q_j with ∂T_j so that a meridian of J_j is identified with ∂D_j . The union B_j of Q_j with a regular neighborhood of D_j in T^* is then a 3-ball, and T_0 is again a solid torus or solid Klein bottle.

Let R_0 be the exterior of $\rho_{j_1} \cup \cdots \cup \rho_{j_k}$ in $T - int(T_{j_1} \cup \cdots T_{j_k})$. Then $R = R_0 \cup Q_{j_1} \cup \cdots \cup Q_{j_k}$ is the exterior of $\rho_{j_1} \cup \cdots \cup \rho_{j_k}$ in T_0 . Since R_0 and the Q_{j_r} are \mathbf{P}^2 -irreducible, ∂ -irreducible, anannular, and atoroidal and ∂R is not a torus one can apply standard general position and isotopy arguments to show that each ∂Q_{j_r}

is incompressible in R and that every incompressible torus in R is isotopic to some ∂Q_{i_r} . The result follows after changing the name of T_0 to T. \Box

Theorem 6.8. Let μ , ν_i , and U be as in Theorem 6.5.

- (1) There is a 3-manifold M having all the properties listed in Theorem 6.5 such that M admits no self homeomorphisms which take one boundary plane to another or reverse orientation.
- (2) Each M as in Theorem 6.5 also has all these properties (including (1)), and distinct \widehat{M} are not homeomorphic.
- (3) There are uncountably many pairwise non-homeomorphic such M, and if M and N are two of these manifolds which are not homeomorphic, then for every pair of associated manifolds \widehat{M} and \widehat{N} we have that \widehat{M} and \widehat{N} are not homeomorphic.

Proof. Let \mathcal{J} be a countably infinite set of excellent knots in S^3 whose exteriors are pairwise non-homeomorphic and admit no orientation reversing self homeomorphisms. An example of such a set is the collection of all non-trivial twist knots other than the trefoil and figure eight knots [16]. Let \mathcal{S} be the set of all triples (i, j, n) where $1 \leq i \leq \mu$, $1 \leq j \leq \nu_i$, and $n \geq 1$. Index \mathcal{J} by choosing a bijection with $\mathcal{S} \times \{0, 1\}$. Denote the indexed knot by J(i, j, n, p), where $p \in \{0, 1\}$, and its exterior by Q(i, j, n, p).

Let $\varphi : S \to \{0, 1\}$. We will associate a 3-manifold M to φ as follows. Let K_n , L_{n+1}^i, T_{n+1}^i , and D_n^i be as in the proof of Theorem 6.5. We apply Lemma 6.7 to T_{n+1}^i together with the knots $J(i, j, n+1, \varphi(i, j, n+1)), 1 \leq j \leq \nu_i$ to get disjoint proper arcs $\alpha_{n+1}^{i,j}$ running from D_n^i to D_{n+1}^i having the properties stated for the ρ_j in the lemma. We then carry out the rest of the construction in the proof of Theorem 6.5 to get M.

Suppose \widehat{M} is obtained by deleting boundary planes as in Theorem 6.5. Then \widehat{M} has an exhaustion $\{\widehat{C}_n\}$ with \widehat{V}_n^i the i^{th} component of $\widehat{M} - \widehat{C}_n$, $\widehat{V}_n^i = \bigcup_{q=n}^{\infty} \widehat{X}_{q+1}^i$, and $\widehat{X}_{q+1}^i = L_{q+1}^i \cup R_{q+1}^i$, where R_{q+1}^i is the exterior of $\alpha_{q+1}^{i,j_1} \cup \cdots \cup \alpha_{q+1}^{i,j_k}$ in T_{q+1}^i .

Standard general position and isotopy arguments show that $\widehat{X}_{q+1}^i = L_{q+1}^i \cup R_{q+1}^i$ is \mathbf{P}^2 -irreducible, ∂ -irreducible, and anannular. It follows that $\{\widehat{C}_n\}$ is a nice exhaustion for \widehat{M} . Arguments of this type also show that each $Q(i, j_r, q + 1, \varphi(i, j_r, q + 1))$ in our construction is incompressibly embedded in \widehat{V}_n^i whenever $q \geq n$ and that every classical knot space Q which is incompressibly embedded in \widehat{V}_n^i and hence by Lemma 6.7 to some $Q(i, j_r, q + 1, \varphi(i, j_r, q + 1))$ in our construction.

Now suppose we have another function $\psi : S \to \{0, 1\}$. Denote the two resulting M by $M[\varphi]$ and $M[\psi]$, and distinguish the various submanifolds arising in their construction by similar notation. Let $\widehat{M}[\psi]$ be obtained by deleting all but one boundary plane from each end of $M[\psi]$. Suppose $g : \widehat{M}[\psi] \to N$ is a homeomorphism, where N is obtained by deleting some boundary planes from $M[\varphi]$. Since g induces a bijection $\varepsilon(\widehat{M}[\psi]) \to \varepsilon(N)$ and $\widehat{M}[\psi]$ has exactly one boundary plane

per end, so does N. Hence it must be some $M[\varphi]$ obtained by deleting all but one boundary plane from each end of $M[\varphi]$.

Fix *i*, and let $E^{i,j}[\psi]$ be the single boundary plane of the i^{th} end of $\widehat{M}[\psi]$. Then $g(E^{i,j}[\psi]) = E^{s,t}[\varphi]$ for some $1 \leq s \leq \mu$ and $1 \leq t \leq \nu_s$. Choose n > 0 such that $g(\widehat{C}_0[\psi]) \subseteq Int \widehat{C}_n[\varphi]$. Then choose m > 0 such that $\widehat{C}_n[\varphi] \subseteq Int g(\widehat{C}_m[\psi])$. Let $q \geq m$, and let Q be the copy of $Q(i, j, q+1, \psi(i, j, q+1))$ embedded in $\widehat{M}[\psi]$ by our construction. Then Q lies in $\widehat{V}_m^i[\psi]$, and ∂Q is incompressible in $\widehat{V}_0^i[\psi]$. Therefore g(Q) lies in $g(\widehat{V}_m^i[\psi])$ and thus in the larger set $\widehat{V}_n^s[\varphi]$, and $g(\partial Q)$ in incompressible in $g(\widehat{V}_0^i[\psi])$ and thus in the smaller set $\widehat{V}_n^s[\varphi]$. Hence g(Q) is isotopic to the copy of some $Q(s, t, r+1, \varphi(s, t, r+1))$ embedded in $\widehat{M}[\varphi]$ by our construction. Therefore s = i, t = j, r = q, and $\varphi(i, j, q+1) = \varphi(i, j, q+1)$.

Now suppose $h: M[\psi] \to M[\varphi]$ is a homeomorphism. By restricting h to $\widehat{M}[\psi]$ as above we see that h must take the i^{th} end of $M[\psi]$ to the i^{th} end of $M[\varphi]$ and the j^{th} boundary plane of the i^{th} end of $M[\psi]$ to the j^{th} boundary plane of the i^{th} end of $M[\psi]$. Moreover, there exists m > 0 such that $\psi(i, j, q + 1) = \varphi(i, j, q + 1)$ for all $q \ge m$. Thus for fixed i and j we get two infinite sequences of zeros and ones which agree after a finite number of terms. This property defines an equivalence relation on the set of all such sequences, which is uncountable, such that each equivalence class is countable. Therefore the set of all equivalence classes is uncountable, and so the set of homeomorphism classes of the $M[\varphi]$ is uncountable.

Taking $\psi = \varphi$ we get that h must take each boundary plane to itself and some Q to itself. Since these classical knot spaces admit no orientation reversing homeomorphisms neither does $M[\varphi]$.

Clearly these considerations apply to all the $M[\psi]$ and $M[\varphi]$ obtained by deleting boundary planes as in Theorem 6.5, and so we are done. \Box

APPENDIX: POLY-EXCELLENT TANGLES

Recall that an n component tangle in a 3-ball is poly-excellent if for every k with $1 \leq k \leq n$ each of its k component subtangles is excellent, i.e. has hyperbolic exterior. In this appendix we prove Theorem 6.3, which asserts the existence of poly-excellent n-tangles for all $n \geq 1$. The case n = 1 is trivial since we can choose any proper arc in a 3-ball having exterior homeomorphic to the exterior of an excellent knot in S^3 , for example the figure eight knot. So we may assume $n \geq 2$.

We shall make use of an excellent *n*-tangle λ in a 3-ball *B*, defined for $n \geq 2$, called the true lover's *n*-tangle. This *n*-tangle is defined and its basic properties are proven in section 4 (pages 275-281) of [13]. See figures 1 and 2 of that paper. Each component λ_j of λ is a trefoil knotted arc. Two distinct components λ_j and λ_i are linked in *B* if and only if |j - i| = 1. In fact, $B = B_1 \cup \cdots \cup B_{2n-1}$, where each B_p is a 3-ball, $B_p \cap B_{p+1}$ is a disk, $B_p \cap B_q = \emptyset$ for |p - q| > 1, $\lambda_1 \subseteq B_1 \cup B_2$, $\lambda_j \subseteq B_{2j-2} \cup B_{2j-1} \cup B_{2j}$ for 1 < j < n, and $\lambda_n \subseteq B_{2n-2} \cup B_{2n-1}$. Moreover $\partial \lambda_j \subseteq \partial B_{2j-1}$ for all *j*. See figure 3 of [13].

Now λ is excellent (Proposition 4.1 of [13]) but is not poly-excellent. However, it has the property that for $2 \leq k \leq n$ each k-tangle consisting of k consecutive

components of λ is excellent. This property is not stated explicitly in [13], but it follows directly from the proof of Proposition 4.1 of that paper.

The basic idea behind the proof of Theorem 6.3 is to stack up several copies of λ , joining the bottom endpoints of one copy to the top endpoints of the copy beneath it, to obtain a new *n*-tangle θ . The endpoints are to be joined using braids, so that each component of θ consists of segments which are components of the copies of λ and may have different indices. This is to be done so that given any subset of $\{1, \ldots, n\}$ any two components of θ with indices in the subset will have segments with consecutive indices in some copy of λ . The exterior of the resulting subtangle $\hat{\theta}$ of θ is then to be analyzed using Lemma 6.2.

Unfortunately, this basic idea does not work. This can be seen by considering the case where $\hat{\theta}$ has exactly one component. It then meets each disk between adjacent 3-balls in a single point, from which it follows that its exterior in not anannular. It also meets each of these 3-balls in a trefoil knotted arc, which is not excellent.

Fortunately, there is a modification of the basic idea which does work. The *n*-tangle θ will be constructed so that, among other things, each of its components doubles back twice at each level so that it meets each intermediate disk three times and meets each 3-ball in an excellent tangle. This requires us to use copies of the true lover's tangle having more than *n* components, which accounts for much of the complication in the following argument.

Proof of Theorem 6.3. As noted above we may assume $n \ge 2$. We take as the 3-ball containing θ the set

$$B = \{(x, y, z) : 0 \le x \le 9n + 1, -1 \le y \le 1, 0 \le z \le n^2 - n + 1\}.$$

We regard x and y as increasing from left to right and from back to front, respectively, and z as increasing in the downward direction. For $0 \le p \le n^2 - n + 1$, $1 \le q \le 3n$, and $1 \le j \le n$, let $H_p = [0, 9n + 1] \times [-1, 1] \times \{p\}$, $x_{p,q} = (3q - 1, 0, p)$, $a_{p,j} = x_{p,3j-2}$, $b_{p,j} = x_{p,3j-1}$, and $c_{p,j} = x_{p,3j}$.

Let $m = (n^2 - n)/2$. We now define certain subsets of *B*. First suppose $0 \le i \le m$.

$$\lim_{t \to 0} \sup_{t \to 0} \inf_{t \to 0} \inf_{i$$

$$B_i = [0, 9n + 1] \times [-1, 1] \times [2i, 2i + 1]$$
$$B_{i,j} = [9j - 9, 9j + 1] \times [-1, 1] \times [2i, 2i + 1], 1 \le j \le n$$
$$N_{i,j} = [9j, 9j + 1] \times [-1, 1] \times [2i, 2i + 1], 0 \le j \le n$$

Next suppose $1 \le i \le m$.

$$C_i = [0, 9n + 1] \times [-1, 1] \times [2i - 1, 2i]$$
$$C_{i,j} = [9j - 9, 9j + 1] \times [-1, 1] \times [2i - 1, 2i], 1 \le j \le n$$
$$K_{i,j} = [9j, 9j + 1] \times [-1, 1] \times [2i - 1, 2i], 0 \le j \le n$$

Thus B is a stack of rectangular solids, starting with B_0 on the top and then alternating in the pattern C_i , B_i , C_{i+1} , B_{i+1} until concluding with B_m on the bottom. We have $C_i \cap B_i = H_{2i}$ and $B_i \cap C_{i+1} = H_{2i+1}$. Each B_i consists of n rectangular solids $B_{i,j}$ which are disjoint except for the pairs $B_{i,j}$ and $B_{i,j+1}$, whose overlap is the rectangular solid $N_{i,j}$. A similar pattern holds for the $C_{i,j}$ and $K_{i,j}$. We will refer to the $B_{i,j}$ and $C_{i,j}$ as **blocks**. Let $N_i = N_{i,0} \cup \cdots \cup N_{i,n}$ and $K_i = K_{i,0} \cup \cdots \cup K_{i,n}$. Let $B'_{i,j}$ be the closure in B of $B_{i,j} - N_i$. We define $C'_{i,j}$ in a similar way. The $B'_{i,j}$ and $C'_{i,j}$ are called **bricks**.

Let Λ_0 be a copy of the true lover's 2*n*-tangle in B_0 with components $\lambda_{0,1}, \ldots, \lambda_{0,2n}$. For $1 \leq j \leq n$ let $\alpha_{0,j} = \lambda_{0,2j-1}, \gamma_{0,j} = \lambda_{0,2j}$, and $\Lambda_{0,j} = \alpha_{0,j} \cup \gamma_{0,j}$. Isotop Λ_0 so that $\Lambda_{0,j}$ is a 2-tangle in $B_{0,j}, \alpha_{0,j}$ runs from $a_{0,j}$ to $a_{1,j}$, and $\gamma_{0,j}$ runs from $b_{1,j}$ to $c_{1,j}$. The existence of this isotopy and the similar isotopies required below follows from the description of λ given earlier in this appendix.

For $1 \leq i \leq m-1$ let Λ_i be a copy of the true lover's 3n-tangle in B_i with components $\lambda_{i,1}, \ldots, \lambda_{i,3n}$. For $1 \leq j \leq n$ let $\delta_{i,j} = \lambda_{i,3j-2}, \alpha_{i,j} = \lambda_{i,3j-1},$ $\gamma_{i,j} = \lambda_{i,3j}$, and $\Lambda_{i,j} = \delta_{i,j} \cup \alpha_{i,j} \cup \gamma_{i,j}$. Isotop Λ_i so that $\Lambda_{i,j}$ is a 3-tangle in $B_{i,j}$, $\delta_{i,j}$ runs from $a_{2i,j}$ to $b_{2i,j}, \alpha_{i,j}$ runs from $c_{2i,j}$ to $a_{2i+1,j}$, and $\gamma_{i,j}$ runs from $b_{2i+1,j}$ to $c_{2i+1,j}$. (For n = 2 this piece of the construction does not occur.)

Let Λ_m be a copy of the true lover's 2*n*-tangle in B_m with components $\lambda_{m,1}, \ldots, \lambda_{m,2n}$. For $1 \leq j \leq n$ let $\delta_{m,j} = \lambda_{m,2j-1}, \alpha_{m,j} = \lambda_{m,2j}$, and $\Lambda_{m,j} = \delta_{m,j} \cup \alpha_{m,j}$. Isotop Λ_m so that $\Lambda_{m,j}$ is a 2-tangle in $B_{m,j}, \delta_{m,j}$ runs from $a_{2m,j}$ to $b_{2m,j}$, and $\alpha_{m,j}$ runs from $c_{2m,j}$ to $c_{2m+1,j}$.

Let \mathcal{B}_{3n} be the Artin braid group on 3n strings; let $\sigma_1, \ldots, \sigma_{3n-1}$ be the standard generators for \mathcal{B}_{3n} . (See [1].) For $1 \leq i \leq m$, given an element β_i of \mathcal{B}_{3n} , we interpret it as a geometric braid in C_i , i.e. it consists of 3n disjoint proper arcs in C_i such that the q^{th} arc runs from $x_{2i-1,q}$ to some $x_{2i,r}$ and meets each horizontal plane in a single point. We follow the convention that as one reads a word in the generators of \mathcal{B}_{3n} from left to right the geometric braid goes downward.

Thus we can associate to each sequence β_1, \ldots, β_m of elements of \mathcal{B}_{3n} a proper 1-manifold θ in B by taking the union of the β_i , $1 \leq i \leq m$, and the Λ_i , $0 \leq i \leq m$. For $1 \leq j \leq n-1$, let

$$\Sigma_{j} = \sigma_{3j}\sigma_{3j-1}\sigma_{3j+1}\sigma_{3j-2}\sigma_{3j}\sigma_{3j+2}\sigma_{3j-1}\sigma_{3j+1}\sigma_{3j}.$$

If one partitions the 3n strings into n consecutive groups of 3 consecutive strings, then Σ_j is obtained by crossing the j^{th} group in front of the $(j+1)^{st}$ group and is thus the analogue of the j^{th} standard generator of \mathcal{B}_n . Note that if $\beta_i = \Sigma_j$, then we may assume that the strings numbered 3j - 2 through 3j + 3 lie in $C_{i,j} \cup C_{i,j+1}$ and that all other strings are vertical.

We now let the sequence β_1, \ldots, β_m be

$$\Sigma_1,\ldots,\Sigma_{n-1},\Sigma_1,\ldots,\Sigma_{n-2},\ldots,\Sigma_1,\Sigma_2,\Sigma_1.$$

Note that if the Σ_j were the generators of \mathcal{B}_n , then the element Δ of \mathcal{B}_n determined by the word corresponding to this sequence would be a half twist of the entire set of *n* strings. It is easily checked that θ is an *n*-tangle in *B* whose j^{th} component θ_j runs from $a_{0,j}$ to $c_{2m+1,n+1-j}$, and $B_i \cap \theta_j$ is some $\Lambda_{i,\varphi(i,j)}$. Moreover, given any j' > j, there is some *i* such that $\varphi(i,j') = \varphi(i,j) + 1$.

Now let $J_0 = \{j_1, \ldots, j_k\}$ be a non-empty subset of $\{1, \ldots, n\}$. Let $\hat{\theta} = \theta_{j_1} \cup \cdots \cup \theta_{j_k}$. Given any 3-manifold M in B such that $\hat{\theta} \cap M$ is a proper 1-manifold in M, we denote the exterior of $\hat{\theta} \cap M$ in M by M^* .

The proof that $\hat{\theta}$ is excellent is based on a simple strategy which may be somewhat obscured by the deluge of notation which follows. B_0 contains a finite collection of disjoint 3-balls each of which intersects θ in an excellent tangle. As one moves down in B these 3-balls expand downwards, following θ and maintaining the same tangle type. As this occurs one notes that the "empty space" above and immediately around each of these 3-balls consists of disjoint 3-balls each of which meets exactly one of these 3-balls in a single disk. They can therefore be adjoined to these 3balls without changing the tangle type. Eventually one may encounter a crossing Σ_t which involves strings emanating from two different 3-balls. At this point the 3-balls come together and are attached to a 3-ball just below Σ_t whose intersection with θ is an excellent tangle. The result is a new 3-ball whose intersection with θ is, by application of Lemma 6.2, an excellent tangle. One then notices that the "empty space" in B above and immediately around this new 3-ball again can be adjoined to the new 3-ball without changing the tangle type. This process of downward expansion, amalgamation, and adjunction is then continued until it has engulfed all of B.

For $0 \leq i \leq m$ let J_i be the set of those $j \in \{1, \ldots, n\}$ such that $\widehat{\theta} \cap B_{i,j} \neq \emptyset$, and let \widehat{B}_i be the union of those $B_{i,j}$. Let $T_i = J_0 \cup \cdots \cup J_i$. For $1 \leq i \leq m$ let I_i be the set of those $j \in \{1, \ldots, n\}$ such that $\widehat{\theta} \cap C_{i,j} \neq \emptyset$, and let \widehat{C}_i be the union of those $C_{i,j}$. Let $S_i = I_1 \cup \cdots \cup I_i$.

We consider how these sets change when *i* increases by one. We have that $\beta_{i+1} = \Sigma_t$ for some $1 \le t \le n-1$.

Case 1. $t, t+1 \in J_i$. Then $J_{i+1} = I_{i+1} = J_i$, and so $T_{i+1} = T_i$.

Case 2. $t, t+1 \notin J_i$. Again $J_{i+1} = I_{i+1} = J_i$, and $T_{i+1} = T_i$.

Case 3. $t \in J_i$, $t+1 \notin J_i$. Then $I_{i+1} = J_i \cup \{t+1\}$, $J_{i+1} = I_{i+1} - \{t\}$, and $T_{i+1} = T_i \cup \{t+1\}$.

Case 4. $t \notin J_i$, $t+1 \in J_i$. Then $I_{i+1} = J_i \cup \{t\}$, $J_{i+1} = I_{i+1} - \{t+1\}$, and $T_{i+1} = T_i \cup \{t\}$.

Note that by induction on *i* it follows that $S_i = T_i$ for $1 \le i \le m$. It is also easily checked that $T_m = \{1, \ldots, n\}$.

For $0 \leq i \leq m$ let R_i be the union of all the $B_{r,j}$ and $C_{s,j}$ such that $0 \leq r \leq i$, $1 \leq s \leq i$, and $j \in T_i$. The components of R_i are, in a sense, the "minimal" rectangular solids whose union contains $\hat{B}_0 \cup \hat{C}_1 \cup \hat{B}_1 \cup \cdots \cup \hat{C}_i \cup \hat{B}_i$. Note that $R_0 = \hat{B}_0$ and $R_m = B$. To prove the theorem it suffices to prove by induction on ithat each component of R_i^* is excellent.

Note that each component W of \widehat{B}_i is a 3-ball which meets $\widehat{\theta}$ in w consecutive components of Λ_i for some $w \geq 2$. Moreover, there is a homeomorphism from W to B_i which is fixed on $\widehat{\theta} \cap W$. It follows that W^* is excellent. In particular each component of R_0^* is excellent.

Now suppose the components of R_i^* are excellent. Let $P = R_i \cup \widehat{C}_{i+1} \cup \widehat{B}_{i+1}$. Let U be the union of all the $B_{r,j}$ and $C_{s,j}$ such that $0 \leq r \leq i, 1 \leq s \leq i$, and $j \in T_{i+1} - T_i$. So U is the union of all the blocks of R_{i+1} which lie above a block of P but are not contained in P. Let $Q = P \cup U$. Let L be the union of all the $B_{i+1,j}$ and $C_{i+1,j}$ such that $j \in T_{i+1} - J_{i+1}$. We have that L is the union of all the blocks of R_{i+1} which lie below a block of P but do not lie in P. Then $R_{i+1} = Q \cup L$. We will prove in succession that the components of P^* , Q^* , and R_{i+1}^* are excellent. Recall that $\beta_{i+1} = \Sigma_t$.

Case 1. $t, t+1 \in J_i$. Then $(\widehat{C}_{i+1} \cup \widehat{B}_{i+1})^*$ is homeomorphic to \widehat{B}_{i+1}^* and so has all components excellent. The components of its intersection with R_i^* are horizontal surfaces with negative Euler characteristics whose complements in the boundaries of both 3-manifolds have no components with closure a disk. Thus by Lemma 6.2 we have that each component of P^* is excellent. Since $T_{i+1} = T_i$ we have $U = \emptyset$ and $L = \emptyset$, and so $R_{i+1} = Q = P$, and we are done.

Case 2. $t, t+1 \notin J_i$. This case is similar to Case 1.

Case 3. $t \in J_i$, $t+1 \notin J_i$. Note that since $T_{i+1} - T_i = \{t+1\}$ we have that $U = B_{0,t+1} \cup C_{1,t+1} \cup B_{1,t+1} \cup \cdots \cup C_{i,t+1} \cup B_{i,t+1}$ if i > 0, and $U = B_{0,t+1}$ if i = 0, and thus is the vertical stack of blocks above $C_{i+1,t+1}$. Let U' denote the union of the corresponding bricks, while $U_t = N_{0,t} \cup K_{1,t} \cup N_{1,t} \cup \cdots \cup K_{i,t} \cup N_{i,t}$ if i > 0 and $U_t = N_{0,t}$ if i = 0, and U_{t+1} is the corresponding union with the index t+1 in place of t. Note that L includes the block $B_{i+1,t}$ directly under $C_{i+1,t}$. Let X be the component of R_i containing $B_{i,t}$. Let Y be the component of $\widehat{C}_{i+1} \cup \widehat{B}_{i+1}$ containing $K_{i+1,t+1}$. Denote the components of P, Q, and R_{i+1} containing X by P_X, Q_X , and $R_{i+1,X}$, with similar notation for the components of P^*, Q^* , and R_{i+1}^*

Subcase (a). $t + 2 \notin J_i$. Then each component of R_i meets a single component of $\widehat{C}_{i+1} \cup \widehat{B}_{i+1}$, and vice versa. For components of R_i other than X the situation is as in Case 1. Let $Z = C_{i+1,t} \cup C_{i+1,t+1} \cup B_{i+1,t+1}$. Then Z^* is homeomorphic to $B_{i+1,t+1}^*$ and so is excellent.

Suppose $t-1 \notin J_i$. Then Y = Z, $P_X^* = X^* \cup Z^*$, $Q_X^* = P_X^* \cup U' \cup U_{t+1}$, and $R_{i+1,X}^* = Q_X^* \cup B'_{i+1,t} \cup N_{i+1,t-1}$. We have that $X^* \cap Z^*$ is the surface $B_{i,t}^* \cap C_{i+1,t}^*$, which is easily seen to satisfy the requirements of Lemma 6.2, and thus P_X^* is excellent. $U' \cup U_{t+1}$ is a 3-ball which meets P_X^* in a disk; thus Q_X^* is homeomorphic to P_X^* . We have that $B'_{i+1,t} \cup N_{i+1,t-1}$ is a 3-ball which meets Q_X^* in a disk, and so R_{i+1}^* is homeomorphic to Q_X^* , and we are done.

Suppose $t - 1 \in J_i$. Let \widehat{Y} be the closure in B of Y - Z. Then $P_X^* = X^* \cup \widehat{Y}^* \cup Z^* \cup N_{i+1,t-1}$, $Q_X^* = P_X^* \cup U' \cup U_{t+1}$, and $R_{i+1,X}^* = Q_X^* \cup B'_{i+1,t}$. Now \widehat{Y}^* is homeomorphic to W^* , where W is the component of \widehat{B}_{i+1} containing $B_{i+1,t-1}$, and so is excellent. \widehat{Y}^* meets X^* along a surface satisfying the hypotheses of Lemma 6.2, and so $X^* \cup \widehat{Y}^*$ is excellent. Lemma 6.2 also implies that $X^* \cup \widehat{Y}^* \cup Z^*$ is excellent. This manifold meets the 3-ball $N_{i+1,t-1}$ in a disk and so is homeomorphic to P_X^* . As before Q_X^* is homeomorphic to P_X^* .

Subcase (b). $t+2 \in J_i$. Let V be the component of R_i containing $B_{i,t+2}$. For

components of R_i other than X or V the situation is as in Case 1. Let W be the component of \hat{B}_{i+1} containing $B_{i+1,t+1}$. So $W = B_{i+1,t+1} \cup \cdots \cup B_{i+1,t+r}$ for some r > 1. Let $Z = C_{i+1,t} \cup W \cup C_{i+1,t+1} \cup \cdots \cup C_{i+1,t+r}$. Then Z^* is homeomorphic to W^* and so is excellent.

Suppose $t-1 \notin J_i$. Then Y = Z, $P_X^* = X^* \cup Z^* \cup V^*$, $Q_X^* = P_X^* \cup U'$, and $R_{i+1,X}^* = Q_X^* \cup B_{i+1,t}' \cup N_{i+1,t-1}$. Lemma 6.2 implies that P_X^* is excellent. U' is a 3-ball which meets P_X^* in a disk, and so Q_X^* is homeomorphic to P_X^* . Since $B_{i+1,t}' \cup N_{i+1,t-1}$ is a 3-ball which meets Q_X^* in a disk we have that $R_{i+1,X}^*$ is homeomorphic to Q_X^* .

Suppose $t - 1 \in J_i$. Let \hat{Y} be the closure in B of Y - Z. Then $P_X^* = X^* \cup \hat{Y}^* \cup Z^* \cup V^* \cup N_{i+1,t-1}$, $Q_X^* = P_X^* \cup U'$, and $R_{i+1,X}^* = Q_X^* \cup B'_{i+1,t}$. Successive applications of Lemma 6.2 show that $X^* \cup \hat{Y}^* \cup Z^* \cup V^*$ is excellent. Since this manifold meets the 3-ball $N_{i+1,t-1}$ in a disk it is homeomorphic to P_X^* . For the same reasons P_X^* is homeomorphic to Q_X^* , which is homeomorphic to $R_{i+1,X}^*$.

Case 4. $t \notin J_i$, $t+1 \in J_i$. This case is similar to Case 3. \Box

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Department of Mathematics, Oklahoma State University, Stillwater, OK 74078-0613, USA

E-mail address: myersr@math.okstate.edu