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AN ALGEBRAIC DETERMINATION OF CLOSED ORIENTABLE 3-MANIFOLDS

BY

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ABSTRACT. Associated with each polyhedral simple closed curve j in a closed, orientable 3-manifold M is the fundamental group of the complement of j in M, $\pi_1(M - j)$. The set, $\mathcal{K}(M)$, of knot groups of M is the set of groups $\pi_1(M - j)$ as j ranges over all polyhedral simple closed curves in M. We prove that two closed, orientable 3-manifolds M and N are homeomorphic if and only if $\mathcal{K}(M) = \mathcal{K}(N)$. We refine the set of knot groups to a subset $\mathcal{F}(M)$ of fibered knot groups of M and modify the above proof to show that two closed, orientable 3-manifolds M and N are homeomorphic if and only if $\mathcal{F}(M) = \mathcal{F}(N)$.

Associated with each polyhedral simple closed curve j in a closed orientable 3-manifold M is the fundamental group of the complement of j in M, $\pi_1(M - j)$. Hence, any closed orientable 3-manifold M has associated with it a set of groups $\mathcal{K}(M)$ defined to be precisely the groups $\pi_1(M - j)$ as j ranges over all polyhedral simple closed curves in M. The set of groups $\mathcal{K}(M)$ is the set of *knot-groups* of M.

It was proposed by R. H. Fox at the Princeton Bicentennial Conference of 1946 [7, p. 24] that certain 3-manifolds may be distinguished by their knot-groups. In fact, Fox used this method [8] to reprove the PL-classification of lens spaces, established earlier by K. Reidemeister [17]. E. J. Brody [3] extended the methods of Fox to establish a topological classification of lens spaces without reference to the Hauptvermutung, as well as a topological classification of the connected sum of two lens spaces.

In this paper we prove that closed orientable 3-manifolds are topologically determined by their knot-groups. This is our Theorem 6.1 which states that two closed orientable 3-manifolds M and N are homeomorphic if and only if $\mathcal{K}(M) = \mathcal{K}(N)$.

At the Georgia Topology Conference in 1969, A. C. Conner announced that if N is a homotopy 3-sphere and $\mathcal{K}(N) = \mathcal{K}(S^3)$, then N is homeomorphic to S^3. The following year Conner announced, via an abstract in the *Notices* [5], the result that we prove in Theorem 6.1. He also circulated a

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manuscript [6] in which he claimed to prove the theorem, but only in the case that N is a homotopy 3-sphere and $\mathcal{K}(N) = \mathcal{K}(S^3)$. Our proof is modeled on this theorem of Conner as reworked by J. Simon in [19] and [20].

The second paper of Simon [20] refines the claim of Conner for homotopy spheres to the consideration of fibered knots in homotopy spheres. A polyhedral simple closed curve j in the closed, orientable 3-manifold M is a *fibered-knot in* M if M admits an open book decomposition with binding j [16]. It follows from the work of [9] and [16] that any closed, orientable 3-manifold contains fibered-knots. Hence, analogous to the set of knot groups of a closed orientable 3-manifold, each closed, orientable 3-manifold M has associated with it a set of groups $\mathcal{F}(M)$ defined to be precisely the groups $\pi_1(M - j)$ as j ranges over all fibered-knots in M. The set of groups $\mathcal{F}(M)$ is the set of *fibered-knot groups of* M. Simon [20] proved that if N is a homotopy 3-sphere and if $\mathcal{F}(N) = \mathcal{F}(S^3)$, then N is homeomorphic to S^3.

In §7 we prove that the result of Simon extends to arbitrary closed, orientable 3-manifolds. Our Theorem 7.1 states that two closed, orientable 3-manifolds M and N are homeomorphic if and only if $\mathcal{F}(M) = \mathcal{F}(N)$.

A result similar to our Theorem 6.1 has recently been announced by Harry Row (Abstract No. 77T-G75, Notices Amer. Math. Soc. **24** (1977), p. A-398).

1. Preliminaries. Throughout, we will work in the PL-category. If M is a manifold we use ∂M and Int M for the boundary of M and the interior of M, respectively. If X is a space and Y is a subspace of X, then $^-(Y)$ is the closure of Y in X. If X is a polyhedron and Y is a subpolyhedron of X, then $U(Y)$ is homeomorphic to an open derived neighborhood of Y in some sufficiently small subdivision of X where Y is a full subcomplex.

If M is a 3-manifold, F is a two-sided surface embedded in M and G is either a two-sided surface embedded in M or a surface embedded in ∂M, we say that F is *parallel to* G in M if there exists an embedding H: $F \times I \to M$ such that $h|F \times \{0\}$: $F \times \{0\} \to F$ is a homeomorphism, $h|F \times \{1\}$: $F \times \{1\} \to G$ is a homeomorphism and $h|\partial F \times I$: $\partial F \times I \to \partial M$ is an embedding. A mapping

$$f\colon (S^1 \times I, S^1 \times \partial I) \to (M, \partial M)$$

is *essential* if f induces injections on both $\pi_1(S^1 \times I) \to \pi_1(M)$ and $\pi_1(S^1 \times I, S^1 \times \partial I) \to \pi_1(M, \partial M)$. Our uses of the notions of incompressible 2-manifold in the 3-manifold M, an irreducible 3-manifold, a ∂-incompressible 3-manifold and a sufficiently-large 3-manifold are standard; e.g., see [21]. Notice that if M is an irreducible 3-manifold with incompressible boundary and f: $(S^1 \times I, S^1 \times \partial I) \to (M, \partial M)$ induces an injection of $\pi_1(S^1 \times I) \to \pi_1(M)$ and f is not essential, then f is homotopic (rel$(S^1 \times \partial I)$) to a mapping f': $S^1 \times I \to M$ such that $f'(S^1 \times I) \subset \partial M$. If A is an annulus properly embedded in M, then A is *essential in* M if the inclusion map $A \hookrightarrow M$ is essential. Hence, if M is irreducible and A is a properly embedded incompressible annulus in M and ∂M is incompressible, then either A is parallel to an annulus in ∂M or A is essential in M.

If $Y \subset X$ and $Y' \subset X'$ and $h\colon Y \to Y'$ is a homeomorphism, then $X \cup_h X'$ is the identification space obtained from the disjoint union of X and X' via the identification $y = h(y), y \in Y$. If M_1 and M_2 are closed, oriented 3-manifolds, $B_i \subset M_i$ is a 3-cell and $h\colon \partial B_1 \to \partial B_2$ is an orientation reversing homeomorphism, then $M_1 \# M_2$ is the manifold $^-(M_1 - B_1) \cup_h {}^-(M_2 - B_2)$. The manifold $M_1 \# M_2$ is called the *connected sum* of M_1 and M_2. The manifold M is *prime* if $M = M_1 \# M_2$ always implies that either M_1 or M_2 is homeomorphic to S^3. It is known that any closed, orientable 3-manifold M admits a decomposition $M = M_1 \# \cdots \# M_n$ where each M_i is prime $(1 \leq i \leq n)$ and that this decomposition is unique up to order and homeomorphism of the factors **[14]**, **[15]**.

We use a minimal amount of information about Seifert fibered spaces. For our purposes **[10]** and **[12]** are satisfactory references.

2. Essential annuli in knot-manifolds. A compact, orientable 3-manifold L is called a *knot-manifold* if ∂L is connected, incompressible and homeomorphic to $S^1 \times S^1$. A consequence of the definition, which will be used later without explicit reference, is that a homotopy solid torus is *not* a knot-manifold.

In this section we study essential annuli embedded in knot-manifolds and make the appropriate definitions which generalize the concept of composite knots, prime knots and cabled knots of the classical theory.

A knot-manifold L is a *composite knot-manifold* if there exists an essential annulus A embedded in L such that A separates L into two components L_1 and L_2 where $^-(L_i)$ $(i = 1, 2)$ is a knot-manifold. A knot-manifold L is a *generalized composite knot-manifold* if either L is a composite knot-manifold or there exists an annulus A embedded in L such that A does not separate L. The knot-manifold L is a *prime knot-manifold* if it is not a generalized composite knot-manifold.

A knot-manifold L is a *cabled knot-manifold* if there exists an essential annulus A embedded in L such that A separates L into two components where the closure of one is a homotopy solid torus (this does not exclude the closures of both being homotopy solid tori). Notice that if L is a cabled knot-manifold, A is an essential annulus embedded in L and T is a component of $L - A$ such that $^-(T)$ is a homotopy solid torus, then the inclusion induced homomorphism $\pi_1(A) \hookrightarrow \pi_1^-(T)$ is injective; however, it is not surjective. This follows from the fact that A is essential in L. Also, recall that a cabled knot-manifold in S^3 is prime (a proof of this fact appears in **[18]** or can be constructed from the proof of our Lemma 2.4). This fact is not true for cabled knot-manifolds as we have defined here.

2.1. LEMMA. *If L is an irreducible knot-manifold and A and A' are essential annuli embedded in L, then either*

(i) *L is a twisted I-bundle over a Klein bottle, or*

(ii) *there exists an ambient isotopy h_t $(0 \leq t \leq 1)$ of L such that $h_1(\partial A') \cap \partial A = \varnothing$, and each component, if any, of $h_1(A') \cap A$ is a noncontractible simple closed curve.*

PROOF. Assume that L is not a twisted I-bundle over a Klein bottle.

Now, let h_t $(0 \leq t \leq 1)$ be an ambient isotopy of L such that $h_1(A')$ is transverse to A and that among all such ambient isotopies of L, the number of components of $h_1(A') \cap A$ is minimal. We shall prove that h_t $(0 \leq t \leq 1)$ is the desired isotopy by proving (under the assumption that L is not a twisted I-bundle over a Klein bottle) that either $h_1(A') \cap A = \varnothing$ or each component of $h_1(A') \cap A$ is a simple closed curve which is not contractible in either $h_1(A')$ or A. We prove this by establishing three claims.

Claim (a). *$h_1(A') \cap A$ does not contain a simple closed curve J which is contractible in either $h_1(A')$ or A.*

Suppose that there exists a simple closed curve J contained in $h_1(A') \cap A$ such that J is contractible in (say) A. Then J bounds a disk D in A and there is no loss of generality in assuming that $D \cap h_1(A') = J$. Since both A and $h_1(A')$ are essential (so, $\pi_1(h_1(A')) \hookrightarrow \pi_1(L)$ and $\pi_1(A) \hookrightarrow \pi_1(L)$ are injective) it follows that J is contractible in $h_1(A')$ as well as in A. Hence J bounds a disk D' in $h_1(A')$. Since L is irreducible and $D \cap D' = J$, $D \cup D'$ is a 2-sphere and bounds a 3-cell in L; hence we can construct an ambient isotopy of L moving $h_1(A')$ and reducing the number of components of $h_1(A') \cap A$. This contradicts our assumption about the minimality of the number of components of $h_1(A') \cap A$.

We say that a spanning arc α in an annulus A is *essential* if $A - \alpha$ is connected; otherwise we say that the spanning arc is *inessential.*

Claim (b). *$h_1(A') \cap A$ does not contain a spanning arc α which is inessential in either $h_1(A')$ or A.*

Suppose that there exists a spanning arc α in $h_1(A') \cap A$ which is inessential in (say) A. Then α is parallel in A to an arc β in ∂A. It follows that $\alpha \cup \beta$ bounds a disk D in A and there is no loss of generality in assuming that $D \cap h_1(A') = \alpha$. Since both A and $h_1(A')$ are essential annuli, it follows that α is parallel in $h_1(A')$ to an arc β' in $\partial h_1(A')$. Hence, $\alpha \cup \beta'$ bounds a disk D' in $h_1(A')$ and $D \cup D'$ is a properly embedded disk in L with $\partial(D \cup D') = \beta \cup \beta'$. Since L is ∂-incompressible it follows that $\beta \cup \beta'$ bounds a disk Δ in

∂L. We now use the fact that L is irreducible and that $D \cup D' \cup \Delta$ is a 2-sphere (we use here the fact that no intersections occur as in Claim (a)) to construct an ambient isotopy of L moving $h_1(A')$ and reducing the number of components of the intersection with A. This contradicts our assumption about the minimality of the number of components of $h_1(A') \cap A$.

Claim (c). *$h_1(A') \cap A$ does not contain a spanning arc α which is an essential spanning arc in either $h_1(A')$ or A.*

Since no intersections as in Claim (a) or Claim (b) can occur and we are assuming that L is not a twisted I-bundle over a Klein bottle, then either our conclusion is satisfied or each component of $A \cap h_1(A')$ is an essential spanning arc in either A or $h_1(A')$. Hence, either conclusion (ii) is satisfied or we have that each component of $A \cap h_1(A')$ is a spanning arc which is essential in both A and $h_1(A')$.

Let us assume that each component of $A \cap h_1(A')$ is a spanning arc which is essential in both A and $h_1(A')$.

Each component of the closure of $h_1(A') - U(A \cap h_1(A'))$ is a disk; furthermore, such a disk may be given the structure of a product $I \times I$ where $\partial I \times I \subset A$ and $I \times \partial I \subset \partial L$. Let D denote the closure of a component of $h_1(A') - U(A \cap h_1(A'))$ and let L' denote the closure of the component of $L - U(A)$ containing D. Then D is a properly embedded disk in L'.

If ∂D is contractible in $\partial L'$, then ∂D bounds a disk D' in $\partial L'$ and it follows that L' has two components. Now, by our above observation as to the intersection of D with A and the fact that L is irreducible and A is incompressible, which implies that L' is irreducible, we can construct an ambient isotopy of L reducing the number of components of $h_1(A') \cap A$.

Since we are assuming that the components of $h_1(A') \cap A$ are essential spanning arcs, it follows from the preceding that if D is the closure of a component of $h_1(A') - U(A \cap h_1(A'))$ and L' is the closure of the component of $L - U(A)$ containing D, then D is a properly embedded disk in L' and ∂D is not contractible in $\partial L'$. Now, $\partial L'$ is a torus and L' is irreducible. It follows that L' is a solid torus. We consider two cases depending on whether or not A separates L.

If A separates L, let L_1 and L_2 denote the closures of the components of $L - U(A)$. Each of L_1 and L_2 is a solid torus; furthermore, L_i contains an annulus $A_i \subset \partial L_i$ $(i = 1, 2)$ such that $\pi_1(A_i) \to \pi_1(L_i)$ is injective and

$$L = L_1 \cup_{A_1 = A_2} L_2.$$

Let D_i $(i = 1, 2)$ denote the closure of a component of $h_1(A') - U(A \cap h_1(A'))$ such that $D_i \subset L_i$. By our above observations about the structure of D_i, it follows that $\partial D_i \cap A_i$ consists of two arcs and therefore $\pi_1(A_i)$ has index two in $\pi_1(L_i)$. Hence, L is a twisted I-bundle over a Klein bottle. This contradicts our first assumption about L.

If A does not separate L, let L' denote the closure of $L - U(A)$. Then L' is a solid torus and L' contains two annuli A_0 and A_1 in $\partial L'$ such that L is obtained from L' by identifying A_0 to A_1. Let D be the closure of a component of $h_1(A') - U(A \cap h_1(A'))$ in L'. Again by our above observations about the structure of D, it follows that $\partial D \cap A_i$ consists of a single arc and therefore L' has the structure of a product $S^1 \times I \times I$ with $A_0 = S^1 \times I \times \{0\}$ and $A_1 = S^1 \times I \times \{1\}$. It follows that L is an annulus bundle over S^1. However, L is a knot-manifold and therefore must have connected boundary. It follows that L is a twisted I-bundle over a Klein bottle. Again this contradicts our first assumption about L.

This completes the proof of Lemma 2.1.

Recall that a twisted I-bundle over a Klein bottle admits the structure of a Seifert fibered space with decomposition surface a disk and having two singular fibers. In our next lemma the exceptional case is a Seifert fibered space with decomposition surface a disk and having three singular fibers.

2.2. LEMMA. *If L is an irreducible, prime knot-manifold and A and A' are essential annuli embedded in L, then either*

(i) *L admits the structure of a Seifert fibered space with decomposition surface a disk and having three singular fibers, or*

(ii) *there exists an ambient isotopy h_t $(0 \leq t \leq 1)$ of L such that $h_1(A') \cap A = \varnothing$.*

PROOF. Assume that L does not admit the structure of a Seifert fiber space with decomposition surface a disk and having three singular fibers.

As in the proof of Lemma 2.1, let h_t $(0 \leq t \leq 1)$ be an ambient isotopy of L such that the number of components of $h_1(A') \cap A$ is minimal. We shall prove that under these assumptions $h_1(A') \cap A = \varnothing$.

It follows from Lemma 2.1 that either (i) L is a twisted I-bundle over a Klein bottle, or (ii) we may assume that if $h_1(A') \cap A \neq \varnothing$, then each component of $h_1(A') \cap A$ is a simple closed curve J which is not contractible in either A or $h_1(A')$.

However, case (i) is not a possibility since we have assumed that L is a prime knot-manifold and a twisted I-bundle over a Klein bottle is not prime (it contains a nonseparating essential annulus). Hence, we may assume that case (ii) holds.

If $h_1(A') \cap A \neq \varnothing$ and J is a component of $h_1(A') \cap A$, then it follows from the argument in Claim (a) of the proof of Lemma 2.1 that J is a noncontractible simple closed curve in both A and $h_1(A')$. Since A is an essential annulus in the prime knot-manifold L, it follows that A separates L

into two submanifolds and the closure of one is a solid torus. Let T and L' denote the closures of the components of $L - A$ where we choose notation so that T is a solid torus. Furthermore, we have that $\pi_1(A) \hookrightarrow \pi_1(T)$ is not surjective.

Each component of $h_1(A') \cap T$ must be an annulus. Furthermore, if C is such a component, then $\partial C \cap \partial h_1(A') = \varnothing$ (otherwise, we could reduce the number of components of $A \cap h_1(A')$ via an ambient isotopy of L since each annulus in T is parallel to an annulus in ∂T).

Let X denote the closure of a component of $h_1(A') - (A \cap h_1(A'))$ such that $\partial X \cap \partial h_1(A') \neq \varnothing$ (i.e., X is an "outer most" component of $h_1(A') - (A \cap h_1(A')))$. It follows from the above that $X \subset L'$. Furthermore, X is a properly embedded annulus in L'.

2.3. *Claim. X is not essential in L'.*

PROOF. Since L is prime the annulus X must separate L' (otherwise, from $X \cup A$ we could construct an annulus not separating L).

Now suppose that X is essential in L' and let L_1' and L_2' denote the closures of the components of $L' - X$. If one of the L_i' $(i = 1, 2)$ is not a solid torus (say L_2'), then $T \cup L_1'$ is a knot-manifold and L_2' is a knot-manifold. It follows that L is not prime. This contradicts our hypothesis. Hence, we may assume that both L_1' and L_2' are solid tori. However, if X is essential in L', then L admits the structure of a Seifert fiber space with decomposition surface a disk and having three singular fibers (L is a union of three solid tori meeting pairwise in annuli). This contradicts our first assumption. This contradiction establishes our claim that X is not essential in L'.

Now, since X is not essential in L' we can construct an ambient isotopy of L moving X into T and thereby reducing the number of components of $A \cap h_1(A')$. This contradicts our minimality assumption on the number of components of $A \cap h_1(A')$ up to ambient isotopy of L.

It follows that $A \cap h_1(A') = \varnothing$ and this establishes the proof of Lemma 2.2.

Notice in the exceptional case of Lemma 2.1 (Case (i)) that there exists essential annuli A and A' embedded in L such that no ambient isotopy of L moves $\partial A'$ off of ∂A. Similarly, in the exceptional case of Lemma 2.2 (Case (i)) there exist essential annuli A and A' embedded in L such that no ambient isotopy of L moves A' off of A.

Our next lemma is analogous to the situation for classical cabled knots in S^3. However, in the general case we must hypothesize that the knot-manifold is prime.

2.4. LEMMA. *If L is an irreducible, prime, cabled knot-manifold, then either*

(i) *L admits the structure of a Seifert fiber space with decomposition surface a disk and having three singular fibers, or*

(ii) *there exists a unique (up to ambient isotopy of L) essential annulus embedded in L.*

PROOF. Suppose that L does not admit the structure of a Seifert fiber space with decomposition surface a disk and having three singular fibers.

By hypothesis, the knot-manifold L is cabled; hence L contains an essential annulus. Let A and A' be essential annuli embedded in L. Since L is irreducible and prime, it follows from Lemma 2.2 that we may assume that $A \cap A' = \varnothing$. Furthermore, both A and A' separate L. Let T and L' denote the closures of the components of $L - A$ with notation chosen so that T is a solid torus (at least one of the manifolds T or L' must be a solid torus since L is prime and irreducible). We consider the two cases where we have either $A' \subset T$ or $A' \subset L'$.

Case (a). $A' \subset T$. Since A' is incompressible, it follows that A' is parallel in T to an annulus in ∂T. However, $A' \cap A = \varnothing$ and A' is essential in L. It follows that A' is parallel to A in T. Therefore, we may construct an ambient isotopy of L taking A' into A.

Case (b). $A' \subset L'$. Since A' separates L, A' separates L'. Let T' and L'' denote the closures of the components of $L' - A'$ in L' where notation is chosen so that T' is a solid torus (since L is prime at least one of the manifolds T' or L'' must be a solid torus).

Suppose that $A \subset \partial T'$. Since A is essential in L, $\pi_1(A) \hookrightarrow \pi_1(T)$ is not onto; similarly if L'' is a solid torus, then $\pi_1(A') \hookrightarrow \pi_1(L'')$ is not onto. It follows that either T' admits the structure of a product $S^1 \times I \times I$ with $A = S^1 \times I \times \{0\}$ and $A' = S^1 \times I \times \{1\}$ or L admits the structure of a Seifert fiber space with decomposition surface a disk and having three singular fibers (otherwise, L would be the union of the two knot-manifolds $T \cup T'$ and L'' and therefore would not be prime). By our own first assumption the latter situation cannot happen. Hence, there exists an ambient isotopy of L taking A' onto A.

Suppose that $A \subset \partial L''$. If L'' is a solid torus, then we argue as above replacing L'' with T'. Therefore, we may assume that L'' is not a solid torus.

Now, since A' is essential in L, it follows that $\pi_1(A') \hookrightarrow \pi_1(T')$ is not surjective. Let X be an annulus in ∂L such that $X \cap A = \partial X \cap \partial A$ is a component of each, and $X \cap A' = \partial X \cap \partial A'$ is a component of each. Let T'' denote a small product neighborhood of X in L'' and let L''' denote the closure of $L'' - T''$ in L''. Then L''' is homeomorphic to L'' and L is the union of the two knot-manifolds L''' and $T \cup T'' \cup T'$. This contradicts the hypothesis that L is prime.

This completes the proof of Lemma 2.4.

The next lemma is well known.

2.5. LEMMA. *Let L be a twisted I-bundle over a Klein bottle. Then*

(i) *every essential torus embedded in L is parallel to ∂L, and* (ii) *there exist precisely two essential annuli embedded in L up to ambient isotopy of L.*

Let L' be a knot-manifold and let T be a solid torus. Let $f: S^1 \times I \to \partial L'$ be an embedding such that

$$f_*: \pi_1(S^1 \times I) \to \pi_1(L')$$

is injective. Let $g: S^1 \times I \to \partial T$ be an embedding such that

$$g_*: \pi_1(S^1 \times I) \to \pi_1(T)$$

is injective, but not surjective. Let $h = f \circ g^{-1}$. The manifold $L = T \cup_h L'$ is said to be a *cabled knot-manifold about the knot-manifold L'*.

2.6. LEMMA. *Let L' be an irreducible knot-manifold. If L is a cabled knot-manifold about the knot-manifold L', then L is irreducible and $\partial L'$ is an incompressible torus in the knot-manifold L.*

PROOF. It is standard that L is irreducible. Hence, we need only prove that $\partial L'$ is incompressible in L.

To the contrary, suppose that $\partial L'$ is not incompressible in L. Then there exists a disk $D \subset L$ such that $D \cap \partial L' = \partial D$ is a noncontractible simple closed curve in $\partial L'$. By assumption L' is a knot-manifold and therefore $\partial L'$ is incompressible in L'. It follows that $D \cap L' = D \cap \partial L' = \partial D$. However, $L = T \cup L'$ where T is a solid torus and $T \cap L' = \partial T \cap \partial L' = A$ is an annulus. Since $\partial D \subset L'$ and $D \subset T$, it follows that $\partial D \subset A$. However, $\pi_1(A) \hookrightarrow \pi_1(T)$ is injective by hypothesis. The only possibility is that ∂D is contractible in A. This contradicts our assumption that ∂D is noncontractible in $\partial L'$. Hence $\partial L'$ must be incompressible in L.

2.7. LEMMA. *Let L' be an irreducible knot-manifold. If L is a cabled knot-manifold about the knot-manifold L', then L is not homeomorphic to a twisted I-bundle over a Klein bottle.*

PROOF. By Lemma 2.6, the torus $\partial L'$ is incompressible in L. If L is homeomorphic to a twisted I-bundle over a Klein bottle, then by Lemma 2.5, $\partial L'$ is parallel to ∂L. This would contradict the assumption that L is a cabled knot-manifold about the knot-manifold L'.

3. Surgery on composite knot-manifolds. Let K be a knot-manifold and let T be a solid torus. Suppose that $h: \partial T \to \partial K$ is a homeomorphism. The manifold $M = T \cup_h K$ is said to be *obtained by surgery on the knot-manifold K*. Notice that M is a closed, orientable 3-manifold. Furthermore, if D is a meridianal disk in T, i.e., if D is a disk properly embedded in T and ∂D is not contractible in ∂T, then for $\alpha = \partial D$, the homeomorphism type of M is completely determined by $h(\alpha)$. That is, if h': $\partial T \to \partial K$ is a homeomorphism and $h'(\alpha)$ is homotopic in ∂K to $h(\alpha)$, then the manifold $M' = T \cup_{h'} K$ is homeomorphic to M. Hence, if K is a knot-manifold and β is a simple closed curve in ∂K which is not contractible in ∂K and h: $\partial T \to \partial K$ is a homeomorphism such that $h(\alpha)$ is homotopic in ∂K to β, where α is as above, then we say that $M = T \cup_h K$ is *obtained by surgery along* β.

In the proof of Theorem 6.1 we shall have a 3-manifold M which we know has been obtained by surgery on the knot-manifold K. However, we shall need to know exactly (up to homotopy) which simple closed curve β in ∂K determines the surgery. We give a condition in Proposition 3.3 which is satisfactory for our needs. We then generalize to knot-manifolds the well-known result that a classical composite knot space has Property P [2].

First we make a definition which is a slight generalization of the notion in §2 of cabling a knot-manifold. Let K' be any 3-manifold with nonempty boundary. Let T be a solid torus. Let f: $S^1 \times I \to \partial K'$ be an embedding such that f_*: $\pi_1(S^1 \times I) \to \pi_1(K')$ is injective and let g: $S^1 \times I \to \partial T$ be an embedding such that g_*: $\pi_1(S^1 \times I) \to \pi_1(T)$ is injective. Let $\beta = f(S^1 \times \{1/2\})$. Then β is a simple closed curve in $\partial K'$ and β is not contractible in K'. Let $h = f \circ g^{-1}$. The manifold $K = T \cup_h K'$ is said to be *obtained from* K' *by adding a root along* β (or simply obtained from K' by adding a root). If g_*: $\pi_1(S^1 \times I) \to \pi_1(T)$ is not surjective we say that the *root is nontrivial*; otherwise, we say that the *root is trivial*. In this terminology, if K' is a knot-manifold and K is a cabled knot-manifold about the knot-manifold K', then K is obtained from K' by adding a nontrivial root.

3.1. LEMMA. *Let K' be an orientable 3-manifold with incompressible boundary. Suppose that K is obtained from K' by adding a root along a noncontractible simple closed curve in $\partial K'$. Then ∂K is incompressible. Furthermore, if K' is irreducible, then K is irreducible.*

PROOF. Since K is obtained from K' by adding a root along a noncontractible simple closed curve in $\partial K'$, there exists an annulus A properly embedded in K such that A separates K into two components with closures T and K' where T is a solid torus. Furthermore, A is incompressible in both T and K', and therefore, A is incompressible in K.

It follows from [21] that K is irreducible if K' is irreducible. Also, if the root is trivial, then K is homeomorphic to K' and hence by hypothesis ∂K is incompressible. Therefore, we may assume, in what follows, that the root is nontrivial.

Suppose that ∂K is compressible. Then there exists a disk $D \subset K$ such that $D \cap \partial K = \partial D$ is a noncontractible simple closed curve in ∂K. Now, consider all such disks in K which are transverse to A, and let D be one such that

$D \cap A$ has a minimal number of components. We shall prove that $D \cap A = \varnothing$ which leads to a contradiction of the existence of D consistent with $\partial K'$ incompressible. We prove this by establishing two claims.

Claim (a). *$D \cap A$ does not contain a component which is a simple closed curve.*

Suppose that J is a simple closed curve in $D \cap A$. Since J is contractible in D (and hence in K) and A is incompressible, it follows that J bounds a disk in A. There is no loss in assuming that J bounds a disk $D' \subset A$ where $D' \cap D = J$. It is now easy to construct a disk $E \subset K$ such that $\partial E = \partial D$ and $E \cap A$ contains fewer components than $D \cap A$. This contradicts our minimality assumption on the number of components of $A \cap D$ and establishes Claim (a).

Claim (b). *$D \cap A$ does not contain a component which is a spanning arc.*

By Claim (a) each component of $D \cap A$ is a spanning arc or $D \cap A = \varnothing$. Hence, suppose that α is a component of $D \cap A$ such that there exists an arc $\beta \subset \partial D$ with $\alpha \cap \beta = \partial\alpha \cap \partial\beta = \partial\alpha$ and $\alpha \cup \beta$ bounds a disk $D' \subset D$ where $D' \cap A = \alpha$.

Suppose that $D' \subset T$. We now use the fact that the root is nontrivial to conclude that $\alpha \cup \beta$ bounds a disk in ∂T. We use this disk to construct a new disk E in K such that $E \cap A$ has the same number of components as $D \cap A$; however, it exchanges a spanning arc component for a simple closed curve component. By Claim (a), E, and therefore D, is not a minimal disk. This contradicts our minimality assumption on the number of components of $A \cap D$.

Suppose that $D' \subset K'$. By assumption $\partial K'$ is incompressible in K'; hence, $\alpha \cup \beta$ bounds a disk in $\partial K'$. The argument now duplicates the preceding argument. This completes the proof of Lemma 3.1.

If K is a composite knot-manifold, then there exists an essential annulus A embedded in K such that the closure of each component of $K - A$ is a knot-manifold. If K_i $(i = 1, 2)$ denotes the closure of a component of $K - A$, we shall say that K is a *composite of the knot-manifolds K_1 and K_2 along A.*

3.2. Proposition. *Suppose that K is a composite of the knot-manifolds K_1 and K_2 along A. Suppose that M is obtained by surgery on the knot-manifold K. Let $F_i = \partial K_i$ $(i = 1, 2)$ and let β denote a component of ∂A. If either F_1 or F_2 is compressible in M, then both F_1 and F_2 are compressible in M and M is obtained by surgery along β.*

Proof. Since M is obtained by surgery on the knot-manifold K, there exists a solid torus T and a homeomorphism h: $\partial T \to K$ such that $M = T \cup_h K$. Suppose that F_1 is compressible in M. If M is not obtained by surgery along β, then $K_2 \cup T$ is obtained from K_2 by adding a root along the noncontractible simple closed curve β. Since K_2 is a knot-manifold, ∂K_2 is incompressible. It follows from Lemma 3.1 that $\partial(T \cup K_2)$ is incompressible in $(T \cup K_2)$. However, in M, we have $\partial(T \cup K_2) = \partial K_1$ and since K_1 is a knot-manifold ∂K_1 is incompressible in K_1. It follows that M is obtained from the union $(T \cup K_2) \cup K_1$ where $\partial(T \cup K_2) = \partial K_1$ is incompressible in each factor and therefore incompressible in M. This contradicts our hypothesis; therefore, the manifold M is obtained by surgery along β.

If $F_2 = \partial K_2$ is compressible in M, the argument is exactly symmetric. This completes the proof.

Let M be a closed 3-manifold and let X be a compact 3-manifold. An embedding $h\colon X \to M$ is an *injective embedding* if $h_*\colon \pi_1(X) \to \pi_1(M)$ is injective. If there exists an injective embedding $h\colon X \to M$, then we say that X *injectively embeds in* M.

3.3. Proposition. *Suppose that K is a composite of the knot-manifolds K_1 and K_2 along A. Suppose that K can be embedded in the closed, orientable 3-manifold M so that ${}^-(M - K)$ is irreducible and that K_2 cannot be injectively embedded in M. Let β denote a component of ∂A. Then ${}^-(M - K)$ is a solid torus and M is obtained by surgery along β.*

Proof. Since K can be embedded in M, we assume that $K \subset M$. We first prove that M is obtained by surgery on the knot-manifold K; i.e., that ${}^-(M - K)$ is a solid torus.

Let $F = \partial K$. If $\pi_1(F) \to \pi_1(M)$ is injective, then $\pi_1(K) \to \pi_1(M)$ is injective; and hence $\pi_1(K_2) \to \pi_1(M)$ is injective. It follows from the fact that K_2 cannot be injectively embedded in M that F is compressible in M. Thus there exists a disk $D \subset M$ such that $D \cap K = D \cap \partial K$ is a noncontractible curve in ∂K. Since ${}^-(M - K)$ is irreducible, it follows that ${}^-(M - K)$ is a solid torus, say T.

If M is not obtained by surgery along β, then it follows from Proposition 3.2 that $\partial(K_2)$ is incompressible in M. Hence $\pi_1(K_2) \to \pi_1(M)$ is injective. This contradiction establishes the proof of Proposition 3.3.

The significance of Proposition 3.3 will be more evident after the results of the next section.

Our next result is our generalization to knot-manifolds of the notion of Property P. It is not used in the sequel; however, it has independent interest.

3.4. Proposition. *Suppose that K is a composite of the irreducible knot-manifolds K_1 and K_2 along A. Suppose that M is obtained by surgery on the knot-manifold K. Let β denote a component of ∂A. Then either*

(i) *M is irreducible and sufficiently-large, or*

(ii) *M is obtained by surgery along β.*

PROOF. Since M is obtained by surgery on the knot-manifold K, there exists a solid torus T and a homeomorphism h: $\partial T \to \partial K$ such that $M = T \cup_h K$. Suppose that M is not both irreducible and sufficiently-large.

We shall prove that both F_1 and F_2 ($F_i = \partial K_i$ ($i = 1, 2$)) are compressible in M. The argument is symmetric; hence, we shall prove that F_1 is compressible in M.

If F_1 is incompressible in M, then the surgery could not be along β. Hence, $T \cup K_1$ is obtained from K_1 by adding a root along the noncontractible simple closed curve β. It follows from Lemma 3.1 that $T \cup K_1$ is irreducible and that $\partial(T \cup K_1)$ is incompressible in $T \cup K_1$. In M we have $\partial(T \cup K_1) = \partial(K_2)$. Hence, M is a union of two irreducible 3-manifolds joined along their incompressible boundaries. It follows that M is irreducible and that $\partial(T \cup K_1) = \partial(K_2)$ is an incompressible surface in M. Hence, M is also sufficiently-large. This contradicts our assumption about M. The only possibility left is that F_1 is compressible in M. Now, by Proposition 3.2 M is obtained by surgery along β. This completes the proof of Proposition 3.4.

4. Torus knot spaces in closed, orientable 3-manifolds. Let (p, q) be a coprime pair of positive integers with $1 < q < p$. Let $T_{p,q}$ denote the (p, q)-torus knot space in S^3, i.e. $T_{p,q}$ is the complement in S^3 of an open tubular neighborhood of a (p, q)-torus knot. In this section we prove that if M is a closed 3-manifold, then there exists at most a finite number of pairs of coprime positive integers (p, q) with $1 < q < p$ such that $T_{p,q}$ can be injectively embedded in M. We believe that it is probably true that there exists at most a finite number of such pairs (p, q) such that there exists an injection of $\pi_1(T_{p,q})$ into $\pi_1(M)$. However, in trying to prove this result one confronts the problem of irreducible 3-manifolds which are not sufficiently-large and this prevents the application of Theorem 4.2. Actually, in the sequel we do not really need more than the fact that when M is closed, irreducible and sufficiently-large, then there exists at most a finite number of pairs (p, q) such that $\pi_1(T_{p,q})$ injects into $\pi_1(M)$. However, to prove the weaker result here then requires more work (additional cases) for the proof of Theorem 6.1.

It has been pointed out to us by the referee that the main result that we need from this section is that if M is a closed, orientable 3-manifold, then there exists a classical, fibered knot space which does not injectively embed in M. This is an obvious corollary to our work. However, the referee informed us of a relatively easy proof (due to H. Row) that is sufficient for this result. In view of this we advise the reader that there does exist an easier route to a proof of Theorem 6.1. On the other hand our results in this section do introduce the reader to characteristic Seifert manifolds. We also raise the question of how many classical knot groups can be embedded in the fundamental group of a closed, orientable 3-manifold; and it is not too difficult to extend our methods to prove that if *M is a closed, orientable 3-manifold, then there exists at most a finite number of classical knot spaces which injectively embed in M.*

4.1. LEMMA. *Let S be a Seifert fiber space. Then there exists at most a finite number of pairs of coprime positive integers (p, q) with $1 < q < p$ such that $\pi_1(T_{p,q})$ injects into $\pi_1(S)$.*

PROOF. Suppose that $\pi_1(T_{p,q})$ injects into $\pi_1(S)$. Let ζ denote the center of $\pi_1(T_{p,q})$. Let C denote the normal subgroup of $\pi_1(S)$ corresponding to the fiber (C is well defined up to conjugacy) [12]. Since $\pi_1(T_{p,q})$ does not contain an abelian subgroup of index ≤ 2, it follows from Lemma 2.4 of [12] that ζ injects into C; and in fact, the image of $\pi_1(T_{p,q})$ in $\pi_1(S)$ meets C in precisely the image of ζ.

Now, $\pi_1(S)/C$ is isomorphic to one of the following groups:

(i) $F * \mathbf{Z}_{\alpha_1} * \cdots * \mathbf{Z}_{\alpha_r}$ where F is a free group and $\mathbf{Z}_{\alpha_i}$ is the finite cyclic group of order α_i,

(ii) $(F * \mathbf{Z}_{\alpha_1} * \cdots * \mathbf{Z}_{\alpha_r})/\langle \Pi_1^n[x_i, y_i] = \Pi_1^r z_i\rangle$ where F is the free group on $\{x_1, y_1, \ldots, x_n, y_n\}$, $[x_i, y_i] = x_i^{-1}y_i^{-1}x_iy_i$ and $\mathbf{Z}_{\alpha_i}$ is the finite cyclic group of order α_i generated by z_i,

(iii) $(F * \mathbf{Z}_{\alpha_1} * \cdots * \mathbf{Z}_{\alpha_r})/\langle \Pi_1^n x_i^2 = \Pi_1^r z_i\rangle$ where F is the free group on $\{x_1, \ldots, x_n\}$ and $\mathbf{Z}_{\alpha_i}$ is the finite cyclic group of order α_i generated by z_i. Both the integer n and the integers $r, \alpha_1, \ldots, \alpha_r$ are homotopy invariants of S. From the above arguments, if $\pi_1(T_{p,q})$ injects into $\pi_1(S)$, then $\pi_1(T_{p,q})/\zeta$ is isomorphic to $\mathbf{Z}_p * \mathbf{Z}_q$ injects into $\pi_1(S)/C$. Hence p divides some α_i $(1 \leq i \leq r)$ and q divides some α_j $(1 \leq j \leq r)$. There are only finitely many pairs of positive integers (p, q), $1 < q < p$, that satisfy this condition. This establishes Lemma 4.1.

If M is a 3-manifold and S is a Seifert fiber space, then a mapping f: $S \to M$ is *essential* if f_* is injective. The Seifert fiber space S is *degenerate* if $\pi_1(S)$ is cyclic (this includes $\pi_1(S) = 1$). Otherwise, the Seifert fiber space S is *nondegenerate*.

Let M be a closed, irreducible and sufficiently-large 3-manifold. The compact, codimension zero submanifold $\mathbb{S}$ of M is a *characteristic Seifert manifold for M* if

(a) each component of $\mathbb{S}$ is a Seifert fiber space,

(b) each component of $\partial\mathbb{S}$ is an incompressible torus in M (in particular, $\mathbb{S}$ is injectively embedded in M),

(c) each essential mapping of a nondegenerate Seifert fiber space into M can be homotoped into $\mathbb{S}$, and

(d) no proper collection of components of $\mathbb{S}$ satisfies (a)–(c).

The following theorem is proved in [11].

4.2. THEOREM. *If M is a closed orientable irreducible and sufficiently-large 3-manifold, then a characteristic Seifert manifold for M exists and is unique up to ambient isotopy of M.*

4.3. THEOREM. *Let M be a closed, orientable irreducible and sufficiently-large 3-manifold. Then there exists at most a finite number of pairs of coprime positive integers (p, q) with $1 < q < p$ such that $\pi_1(T_{p,q})$ injects into $\pi_1(M)$.*

PROOF. Let $\mathbb{S}$ be the characteristic Seifert manifold of M. If $\pi_1(T_{p,q})$ injects into $\pi_1(M)$ then there exists a mapping f: $T_{p,q} \to M$ such that f_* is injective. Since $T_{p,q}$ is a nondegenerate Seifert fiber space and f is an essential map of $T_{p,q}$ into M, there must exist a component S of $\mathbb{S}$ and a mapping f': $T_{p,q} \to S$ such that f' is homotopic to f in M. Hence,

$$f'_*: \pi_1(T_{p,q}) \to \pi_1(S)$$

is injective. By Lemma 4.1 there exists at most a finite number of pairs of coprime integers (p, q) with $1 < q < p$ such that $\pi_1(T_{p,q})$ injects into $\pi_1(S)$. Since $\mathbb{S}$ has only a finite number of components, the conclusion of Theorem 4.3 follows.

4.4. LEMMA. *Let M be a closed orientable 3-manifold and suppose that $M = M_1 \# \cdots \# M_n$ is a prime decomposition of M. If $T_{p,q}$ injectively embeds in M, then $T_{p,q}$ injectively embeds in M_i for some i $(1 \leqslant i \leqslant n)$. (Hence, for that i, the closed, orientable 3-manifold M_i is irreducible and sufficiently-large.)*

PROOF. Suppose that $T_{p,q}$ injectively embeds in M. Then we may assume that $T_{p,q} \subset M$ and $\pi_1(T_{p,q}) \to \pi_1(M)$ is injective. Hence $\partial T_{p,q}$ is an incompressible torus in M. It is standard to get a prime decomposition of M as $M = M'_1 \# \cdots \# M'_m$ such that $\partial T_{p,q} \subset M'_j$ for some j $(1 \leqslant j \leqslant m)$; e.g., the proof of Theorem 4 of [23]. Since $T_{p,q}$ is irreducible, it follows that $T_{p,q} \subset M'_j$. However, by uniqueness of the prime decomposition of M up to order and homeomorphism, it follows that $m = n$ and M'_j is homeomorphic to M_i for some i $(1 \leqslant i \leqslant n)$. This completes the proof of Lemma 4.4.

4.5. THEOREM. *Let M be a closed orientable 3-manifold. Then there exists at most a finite number of pairs of coprime positive integers (p, q) with $1 < q < p$ such that $T_{p,q}$ injectively embeds in M.*

PROOF. The proof follows immediately from Theorem 4.3 and Lemma 4.4 using the parenthetical statement at the end of the statement of Lemma 4.4.

4.6. REMARK. Theorem 4.5 extends to all compact orientable 3-manifolds. To prove Theorem 4.5 in the bounded case requires a stronger relative version of Theorem 4.2 which was proven in [12]. This relative version allows

I-bundles as well as Seifert fiber spaces to appear and then the I-bundles disappear since $\pi_1(T_{p,q})$ does not inject into the fundamental group of an I-bundle. Also, if $\pi_1(T_{p,q})$ injects into $\pi_1(M)$ and $M = M_1 \# \cdots \# M_n$ then $\pi_1(T_{p,q})$ injects into $\pi_1(M_i)$ for some i $(1 < i < n)$. However, we do not get the parenthetical statement as in our Lemma 4.4 since indeed there are irreducible 3-manifolds N which are not sufficiently-large and $\pi_1(T_{p,q})$ injects into $\pi_1(N)$.

5. Deforming homotopy equivalences of knot-manifolds. In this section we prove that homotopy equivalences between certain types of knot-manifolds can "almost" be deformed to homeomorphisms. Our theorem is a special case of a theorem announced by K. Johannson in [13]. Since the proof is rather elementary and the proof of Johannson's result is not yet in print, we include a proof. The word "almost" will become evident after the statement of our theorem.

The reader will observe that the method of proof of Theorem 5.1 does not use the standard "binding ties" argument most often employed in such situations. This is for a good reason; while it works very easily if one is dealing with homotopy spheres, we could not apply it directly in the generality needed for our work.

5.1. THEOREM. *Let K and L be irreducible knot-manifolds. Suppose that L is a prime knot-manifold which is cabled about the knot-manifold L' and that L is not a Seifert fiber space with decomposition surface a disk having three singular fibers. If f: $K \to L$ is a homotopy equivalence, then K is a knot-manifold which is cabled about the knot-manifold K' and there exists a map f': $K \to L$ such that*

(i) *f' is homotopic to f,*
(ii) *$f'|K'$: $K' \to L'$ is a homeomorphism, and*
(iii) *$f'|\,{}^-(K - K') \to {}^-(L - L')$ is a homotopy equivalence.*

PROOF. Since f: $K \to L$ is a homotopy equivalence, there exists a map g: $L \to K$ which is a homotopy inverse to f. Now, it follows from Lemma 2.4 that up to ambient isotopy of L there exists a unique essential annulus A embedded in L. Using standard techniques, there is no loss in assuming that each component of $f^{-1}(A)$ is an essential annulus embedded in K. Furthermore, if $\mathfrak{A}' = f^{-1}(A)$, we may also assume that each component of $g^{-1}(\mathfrak{A}')$ is an essential annulus embedded in L. Let $\mathfrak{A} = g^{-1}(\mathfrak{A}')$. Then it follows (again from Lemma 2.4) that each component of $\mathfrak{A}$ is parallel in L to A. Hence, the components of $\mathfrak{A}$ may be indexed as $A_1, \ldots, A_k$ such that A_i separates L into two components the closures of which are L_i and T_i where L_i is homeomorphic to L' and T_i is a solid torus and $A_1, \ldots, A_{i-1}$ are contained in L_i while $A_{i+1}, \ldots, A_k$ are contained in T_i $(1 < i < k)$. Furthermore, we may assume that $A_1 = A$ (after possbily a small deformation of f). Hence $L_1 = L'$.

Let K' denote the closure of the component of $K - \mathfrak{A}'$ such that $g(L') \subset K'$. Since L is not a Seifert fiber space with decomposition surface a disk and having three singular fibers and L is cabled about the knot-manifold L', it follows that L' is not a Seifert fiber space with decomposition surface a disk and having two singular fibers; hence, L' is not a twisted I-bundle over a Klein bottle. Furthermore, since L' is a knot-manifold, L' is not homeomorphic to either $S^1 \times I \times I$ or $S^1 \times S^1 \times I$. Now, since $g|L'$ induces a monomorphism on fundamental groups and each A_i is incompressible, it follows by **[21]** that K' is not a twisted I-bundle over a Klein bottle or homeomorphic to $S^1 \times S^1 \times I$.

Let T be a component of $\partial K'$. Then T is a torus. Furthermore, $T \cap \mathfrak{A}' \neq \varnothing$. Let B be the closure of a component of $T - \mathfrak{A}'$; then B is an annulus. The mapping $f|B$ maps B into L' such that $f|\partial B \to \partial A$. Now, $f|B$ induces an injection of $\pi_1(B)$ into $\pi_1(L')$. If $f|B$ is essential (i.e., $f|B$ induces an injection of $\pi_1(B, \partial B)$ into $\pi_1(L', \partial L')$), then it follows from the relative version of the Annulus Theorem **[4]** or **[22]** that there exists an essential annulus B' embedded in L' such that $\partial B' \cap \partial A = \varnothing$. This contradicts Lemma 2.4 which says that each annulus B' in L' with $\partial B' \cap \partial A = \varnothing$ is parallel into $\partial L'$. It follows that $f|B$ is not essential. Since L' is a knot-manifold, $\partial L'$ is incompressible. Also, by assumption L' is irreducible. Thus there exists a deformation of f fixed off of a small neighborhood of B taking B into $\partial L'$.

By the preceding we may assume that $f|K'\colon (K', \partial K') \to (L', \partial L')$. Now, $f|K'$ induces an injection of $\pi_1(K')$ into $\pi_1(L')$ and thus $f|\partial K'\colon \partial K' \to \partial L'$ induces an injection of $\pi_1(\partial K')$ into $\pi_1(\partial L')$. It follows that $f|\partial K'$ may be deformed to a covering map of $\partial K'$ into $\partial L'$. This deformation may be performed on the map $f\colon K \to L$ by possibly having to "flip" f along certain components of $\mathfrak{A}' \cap K'$. Therefore, we may assume that $f|K'\colon (K', \partial K') \to (L', \partial L')$ induces an injection of $\pi_1(K')$ into $\pi_1(L')$ and that $f|\partial K'\colon \partial K' \to \partial L'$ is a covering map. Since K' is not an I-bundle, it follows from **[21]** that f may be deformed only on $\mathrm{Int}(K')$ to a mapping $f'\colon K \to L$ such that $f'|K'\colon K' \to L'$ is a covering map.

Since $f'|K'\colon K' \to L'$ is a covering map and L' contains no essential annulus C such that $\partial C \cap \partial A = \varnothing$, it follows that if $A' = \mathfrak{U}' \cap \partial K'$ (A' is a collection of annuli), then K' contains no essential annulus C' such that $\partial C' \cap \partial A' = \varnothing$.

By symmetry of the argument, we may now assume that there exists a mapping $g'\colon L \to K$ which is homotopic to g and $g'|L'\colon L' \to K'$ is a covering map.

Let $h = f' \circ g'$. Then $h\colon L \to L$ is homotopic to the identity on L and $h|L'\colon L' \to L'$ is a covering map. The map $h|L'$ is homotopic, as a map of L' into L, to the identity mapping of L' into L'. However, it is not difficult to see that the homotopy between $h|L'$ and the identity on L' through L may actually be taken as a homotopy between $h|L'$ and the identity on L' through L' (for example, pass to the covering space of L corresponding to $\pi_1(L')$). Hence, $\deg(h|L') = 1$. However,

$$\deg(h|L') = \deg(f'|K')\deg(g'|L').$$

It follows that the covering map $f'|K'$ is actually a homeomorphism. All parts of the conclusion to Theorem 5.1 now follow.

6. Algebraic determination of closed orientable 3-manifolds. In this section we prove our main theorem, which is that closed, orientable 3-manifolds are topologically determined by their knot groups. In the next section we refine this theorem to the class of fibered knot-groups.

6.1. THEOREM. *Let M and N be closed, orientable 3-manifolds. Then $\mathcal{K}(M) = \mathcal{K}(N)$ if and only if M is homeomorphic to N.*

PROOF. It follows from Theorem 4.5 that there exists a coprime pair of positive integers (p, q) with $1 < q < p$ such that $T_{p,q}$ *does not* injectively embed in M.

Let j_0 be a simple closed curve in N such that $N - j_0$ is irreducible. The existence of such a simple closed curve is well known **[1]** or **[16]**. We may assume that j_0 is a full subcomplex of some triangulation $\mathcal{T}$ of N. Let e denote an edge of j_0 and let $E = \mathrm{st}(e, \mathcal{T})$ be the star of e in $\mathcal{T}$. Since j_0 is a full subcomplex of $\mathcal{T}$, $E \cap j_0 = e$. Furthermore, E is a 3-cell and the ball pair (e, E) is unknotted. Let e_1 denote an arc in E such that $e_1 \cap \partial E = e \cap \partial E$ and the ball pair (e_1, E) is knotted such that $E - U(e_1)$ is homeomorphic to $T_{p,q}$.

If $\pi_1(N - j_0) \not\approx \mathbf{Z}$, let $j_1 = (j_0 - e) \cup e_1$. If $\pi_1(N - j_0) \approx \mathbf{Z}$, let $j_1' = (j_0 - e) \cup e_1$. Then let (e', E') be a ball pair as (e, E) above where e' is an edge of j_1' and (e_1', E') is a ball pair as (e_1, E). Let $j_1 = (j_1' - e') \cup e_1'$. Let $L' = N - U(j_1)$. Then L' is an irreducible knot-manifold. Furthermore, L' is a composite of the knot-manifolds L_0 and $T_{p,q}$ along an annulus A where L_0 is homeomorphic to N minus a tubular neighborhood of j_0 or j_1' as the case may be. Furthermore, if β denotes the homotopy class in $\partial L'$ of a component of ∂A, then N is obtained by surgery along β.

Since $T_{p,q}$ cannot be injectively embedded in the twisted I-bundle over a Klein bottle for any coprime pair of positive integers (p, q), it follows from Lemma 2.1 that if A' is any essential annulus in L', then each component of $\partial A'$ is homotopic in $\partial L'$ to β.

Let β determine a framing of $\partial L'$ and let γ be an (r, s)-curve in this framing, where (r, s) is a coprime pair of integers with $1 < r < s$. Let j be the simple closed curve obtained by moving γ slightly into $U(j_1)$. Then j is a nontrivial cable knot about j_1, i.e., if $L = N - U(j)$, then L is a cable knot-manifold about L' and if T is the solid torus such that $L = L' \cup T$, where

$$L' \cap T = \partial L' \cap \partial T$$

is the annulus C, then a component of ∂C is not homotopic on $\partial L'$ to β.

6.2. *Claim. L is an irreducible, prime knot-manifold which is cabled about the knot-manifold L' and L is not a Seifert fibered space with decomposition surface a disk having three singular fibers.*

PROOF. We only need to prove that L is a prime knot-manifold. All of the other conditions follow from the above construction of L.

We have that $L = L' \cup T$ where T is a solid torus, $L' \cap T = \partial L' \cap \partial T = C$ is an annulus, $\pi_1(C) \to \pi_1(T)$ and $\pi_1(C) \to \pi_1(L')$ are injective and $\pi_1(C) \to \pi_1(T)$ is not surjective. Suppose that C' is an essential annulus embedded in L. It follows from Lemma 2.1 and the fact that L is not a twisted I-bundle over a Klein bottle that we may assume that $\partial C' \cap \partial C = \varnothing$.

Under the above conditions, if there does not exist an ambient isotopy h_t $(0 < t < 1)$ of L such that $h_1(C') \cap C = \varnothing$, then from the union of C' and C we can construct an essential annulus C'' in L such that $C'' \cap C = \varnothing$ and C'' is not parallel in L to C. In any case either there exists an ambient isotopy h_t $(0 < t < 1)$ of L taking C' onto C or there exists an essential annulus C'' in L such that $C'' \cap C = \varnothing$ and C'' is not parallel in L to C. We shall show that this latter situation cannot occur.

This is certainly clear if $C'' \subset T$. If $C'' \subset L'$ and C'' is essential in L and not parallel into C, then C'' is essential in L'. Hence each component of $\partial C''$ is homotopic in $\partial L'$ to β. This contradicts our construction of L as a nontrivial cable knot-manifold about L' in a framing determined by β. This contradiction establishes our Claim 6.2.

By assumption there exists a simple closed curve K in M such that $\pi_1(M - k)$ is isomorphic to $\pi_1(N - j)$. Let $\tilde{K} = M - U(k)$. Since L is irreducible and $\pi_1(L)$ is not cyclic, $\pi_1(L)$ is neither infinite cyclic nor a nontrivial free product. Since $\pi_1(\tilde{K})$ is isomorphic to $\pi_1(L)$, $\partial\tilde{K}$ is incompressible and $\tilde{K}$ is a knot-manifold.

Now, it may be the case that $\tilde{K}$ is not irreducible. However, by the above arguments about $\pi_1(\tilde{K})$ it follows that at the very worst $\tilde{K} = K \# \Sigma$ where K is irreducible and Σ is a homotopy 3-sphere. Since $\pi_1(K) \approx \pi_1(L)$ and both manifolds are irreducible and have infinite fundamental groups there exists a homotopy equivalence $f\colon K \to L$. It follows from Theorem 5.1 that K is a cabled knot-manifold about the knot-manifold K' and that f can be deformed to a map $f'\colon K \to L$ such that $f'|L'\colon K' \to L'$ is a homeomorphism and $f'|\,^{-}(K - K')\colon\ ^{-}(K - K') \to\ ^{-}(L - L')$ is a homotopy equivalence.

The knot-manifold K' is a composite of the irreducible knot-manifolds K_0 and $T_{p,q}$ along A' where K_0 is homeomorphic to L_0. Let M' denote the manifold such that $M = M' \# \Sigma$. If the closure of $M' - K'$ is irreducible, then it follows from Proposition 3.3 that if β' is the homotopy class in $\partial K'$ of a component of $\partial A'$, then M' is obtained by surgery along β'. Hence $f'|K'$ can be extended to a homeomorphism of M' onto N.

If the closure of $M' - K'$ is not irreducible, then it is a connected sum of a lens space $L(m, n)$ and a solid torus T. Let $M'' = K' \cup T$. Then again by Proposition 3.3 M'' is obtained by surgery along β'. Hence, $f'|K'$ can be extended to a homeomorphism of M'' onto N.

In any case we have proved that either

(i) M is homeomorphic to N, or

(ii) M is homeomorphic to $N \# \Sigma$ where Σ is a homotopy 3-sphere, or

(iii) M is homeomorphic to $N \# L(m, n)$ where $L(m, n)$ is a lens space, or

(iv) M is homeomorphic to $N \# \Sigma \# L(m, n)$ where Σ and $L(m, n)$ are as in case (ii) and (iii), respectively.

However, symmetry of the argument gives us that either

(i′) N is homeomorphic to M, or

(ii′) N is homeomorphic to $M \# \Lambda$ where Λ is a homotopy 3-sphere, or

(iii′) N is homeomorphic to $M \# L(m', n')$ where $L(m', n')$ is a lens space, or

(iv′) N is homeomorphic to $M \# \Lambda \# L(m', n')$ where Λ and $L(m', n')$ are as in (ii′) and (iii′), respectively.

It now follows from the uniqueness of the prime decomposition of a closed 3-manifold that the only possibilities that can happen are (i) and (i′).

Hence, M is homeomorphic to N and this completes the proof of Theorem 6.1.

7. Fibered knot-groups and closed, orientable 3-manifolds. In this section we refine Theorem 6.1 to the consideration of fibered-knots. This is analogous to the work of J. Simon in **[20]** for homotopy spheres.

Recall that if (p, q) is a coprime pair of positive integers, then the (p, q)-torus knot in S^3 is a fibered knot.

7.1. THEOREM. *Let M and N be closed orientable 3-manifolds. Then $\mathfrak{F}(M) = \mathfrak{F}(N)$ if and only if M is homeomorphic to N.*

PROOF. Let j_0 be a fibered-knot in N such that $\pi_1(N - j_0) \not\approx \mathbf{Z}$. As remarked in the introduction a closed 3-manifold N contains a fibered-knot. To obtain the additional condition that $\pi_1(N - j_0) \not\approx \mathbf{Z}$ we may use the adjustment we did in the proof of Theorem 6.1 using the fact that a (p, q)-torus knot is fibered.

Now, beginning with j_0 we then build a simple closed curve j in N as in the proof of Theorem 6.1. It follows precisely as in [20] that j is a fibered knot in N.

Let k be a fibered-knot in M such that $\pi_1(M - k) \approx \pi_1(N - j)$. Since $M - k$ fibers over S^1 with fiber a noncompact surface, it follows that $M - k$ is irreducible.

We proceed precisely as in the proof of Theorem 6.1 only now since $M - k$ is irreducible, we conclude that either

(i) M is homeomorphic to N, or

(ii) M is homeomorphic to $N \# L(m, n)$ where $L(m, n)$ is a lens space.

We again use the symmetry of the argument to conclude that either

(i′) N is homeomorphic to M, or

(ii′) N is homeomorphic to $M \# L(m', n')$ where $L(m', n')$ is a lens space.

As in the proof of Theorem 6.1, the uniqueness of the prime decomposition of a closed 3-manifold excludes all possibilities except (i) and (i′).

This completes the proof.

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