Topology Vol. 27, No. 1, pp. 27 35, 1988
Printed in Great Britain.

0040 9383 88 $3.00 + .00
Pergamon Press plc

CONTRACTIBLE OPEN 3-MANIFOLDS WHICH ARE NOT COVERING SPACES

ROBERT MYERS*

(*Received in revised form* 15 *January* 1987)

§1. INTRODUCTION

SUPPOSE M is a closed, aspherical 3-manifold. Then the universal covering space $\tilde{M}$ of M is a contractible open 3-manifold. For all "known" such M, i.e. M a Haken manifold [15] or a manifold with a geometric structure in the sense of Thurston [14], $\tilde{M}$ is homeomorphic to R^3. One suspects that this is always the case. This contrasts with the situation in dimension $n > 3$, in which Davis [2] has shown that there are closed, aspherical n-manifolds whose universal covering spaces are not homeomorphic to R^n.

This paper addresses the simpler problem of finding examples of contractible open 3-manifolds which do not cover closed, aspherical 3-manifolds. As pointed out by McMillan and Thickstun [11] such examples must exist, since by an earlier result of McMillan [10] there are uncountably many contractible open 3-manifolds but there are only countably many closed 3-manifolds and therefore only countably many contractible open 3-manifolds which cover closed 3-manifolds. Unfortunately this argument does not provide any *specific* such examples.

The first example of a contractible open 3-manifold not homeomorphic to R^3 was given by Whitehead in 1935 [16]. It is a certain monotone union of solid tori, as are the later examples of McMillan [10] mentioned above. These examples are part of a general class of contractible open 3-manifolds called *genus one Whitehead manifolds*. In this paper it is proven that none of these manifolds can cover a closed 3-manifold. In fact a stronger result is obtained: genus one Whitehead manifolds admit no non-trivial, fixed point free, properly discontinuous group actions. Thus they cannot non-trivially cover even another non-compact 3-manifold.

There is some disagreement as to the proper definition of proper discontinuity. If X is a manifold, G is a group of homeomorphisms of X, and $x \in X$, let G_x be the isotropy subgroup of G at x, i.e. $G_x = \{g \in G | g(x) = x\}$. G acts *properly discontinuously* on X if (i) for each point $x \in X$ there is an open neighborhood U of x such that $U \cap g(U) = \varnothing$ for every $g \in G \backslash G_x$ and (ii) a condition on G_x which is in dispute. Some authors require that each G_x be trivial (see [9], [13]). Under this definition the phrase "fixed point free" is redundant and G acts properly discontinuously if and only if the projection $X \to X/G$ is a regular covering map. Other authors may merely require that each G_x be finite (see [3]). This allows G to have elements of finite order with fixed points, such as those occurring in Kleinian groups.

The second definition is of course implied by the first; it in turn implies that for every compact subset C of X the set $\{g \in G | C \cap g(C) \neq \varnothing\}$ is finite. This last condition is the working

*This research was supported in part by NSF Grant DMS 8403571.

definition used in this paper. It will be shown that given any homeomorphism g of a genus one Whitehead manifold W there is a compact subset C of W such that $C \cap g^k(C) \neq \emptyset$ for infinitely many values of k. Thus if g were an element of a properly discontinuous group G of homeomorphisms of W, then g would have finite order, which, by passing to a power of g, one may assume to be a prime p. Then since W is contractible g must have a fixed point. This can be seen in two ways. First note that g extends to a homeomorphism of order p of the one-point compactification $W \cup \{\infty\}$ of W having ∞ as a fixed point. Since this space is a compact mod p Čech cohomology sphere, Smith theory [1] implies that its fixed point set is also a mod p Čech cohomology sphere. Since it is non-empty at least one of its fixed points must be in W. Alternatively, one can observe that if g had no fixed points, then the projection $W \to W/\langle g \rangle$ would be a covering map. Hence $W/\langle g \rangle$ would be a finite dimensional $K(Z_p, 1)$. This contradicts the fact that finite groups have infinite cohomological dimension [4].

We shall work throughout in the PL category. The terminology in this paper follows that of [6] or [7]. In particular if F is a surface in a 3-manifold M, then $\sigma_F(M)$ denotes the manifold obtained by splitting M along F. A compact 3-manifold M is *simple* if it is irreducible, has incompressible boundary, and every incompressible torus in M is boundary-parallel. A simple 3-manifold M is *atoroidal* if every incompressible annulus in M is boundary-parallel.

The original example of Whitehead is easier to deal with than most genus one Whitehead manifolds because the closure of the region between any two successive solid tori in its defining sequence is atoroidal. Such manifolds and sequences will be termed *excellent*. The proof of the theorem is much easier to follow for excellent genus one Whitehead manifolds, the proof in the general case being cluttered by details about characteristic Seifert submanifolds. The paper has therefore been organized so that one may on a first reading confine oneself to the case of excellent manifolds and sequences, postponing consideration of the "very good" sequences which arise in the general case.

Section 2 develops basic facts about genus one Whitehead manifolds and their defining sequences. Section 3 gives conditions under which a homeomorphism of a genus one Whitehead manifold can be isotoped so as to preserve the defining sequence. Section 4 presents the key insight underlying the proof: the track of a sufficiently complicated simple closed curve under a sufficiently long ambient isotopy must pass through the core of the manifold. Section 5 assembles these ideas into the proof of the theorem.

§2. GENUS ONE WHITEHEAD SEQUENCES

A sequence $\{V_n\}_{n=0}^{\infty}$ of solid tori is a *genus one Whitehead sequence* if $V_n \subseteq Int(V_{n+1})$ and the inclusion map $V_n \to V_{n+1}$ is null-homotopic for each $n \geq 0$. Let $W = \cup_{n=0}^{\infty} V_n$. For each $n \geq 0$, let $T_n = \partial V_n$, let $W_n = W \setminus Int(V_n)$, and let J_n be a core of V_n. For each $n \geq 1$, let $X_n = V_n \setminus Int(V_{n-1})$. For $q > p \geq 0$ let $Y_{p,q} = V_q \setminus Int(V_p)$.

A genus one Whitehead sequence $\{V_n\}$ is *good* if ∂X_n is incompressible in X_n for each $n \geq 1$. In this case W is called a *genus one Whitehead manifold*.

LEMMA 2.1. *Let $\{V_n\}$ be a genus one Whitehead sequence. The following are equivalent:*
1. *$\{V_n\}$ is good.*
2. *For each $n \geq 0$, J_n is not contained in a 3-cell in W.*
3. *For each $n \geq 1$, X_n is irreducible.*

Proof. 1⇒2: Suppose some J_n is contained in a 3-cell B in W. Then B lies in some V_m, $m > n$. A meridional disk D for V_m can be chosen missing B. Put D in general position with respect to all the T_k, $n < k < m$, and assume the number of intersection curves to be minimal.

Then no such curve bounds a disk on a T_k. (Otherwise an innermost such disk could be used to replace the disk on D having the same boundary; after an isotopy one would have a new meridional disk for V_m having fewer intersection curves with the T_k, contradicting minimality.) An innermost disk on D must therefore compress ∂X_j for some $j, n < j \leq m$.

$2 \Rightarrow 3$: Suppose some X_n is reducible. Let S be a 2-sphere in X_n which does not bound a 3-cell in X_n. S does bound a 3-cell B in V_n, so B must contain J_{n-1}.

$3 \Rightarrow 1$: Suppose some X_n has compressible boundary. The manifold obtained by splitting X_n along a compressing disk has two boundary components, one of which is a 2-sphere. This 2-sphere, when isotoped into $Int(X_n)$, separates the components of ∂X_n and so does not bound a 3-cell in X_n. □

This lemma implies that genus one Whitehead manifolds are not homeomorphic to R^3. For such manifolds standard arguments show that all the $Y_{p,q}$ and W_n are irreducible and that all the T_k contained in them are incompressible.

A good genus one Whitehead sequence and the associated genus one Whitehead manifold are *excellent* if X_n^* is atoroidal for each $n \geq 1$. Whitehead's example [16] is excellent. The genus one examples of McMillan [10] are not excellent.

LEMMA 2.2. *Let $\{V_n\}$ be an excellent genus one Whitehead sequence. If T is an incompressible torus in $Y_{p,q}$, then T is isotopic to some T_n, $p \leq n \leq q$, via an isotopy with support in $Y_{p,q}$, if $p < n < q$ and support in a regular neighborhood of $Y_{p,q}$ if $n = p$ or $n = q$.*

Proof. Put T in general position with respect to $\cup_{n=p}^{q} T_n$. Then use the irreducibility of $Y_{p,q}$ to remove those intersection curves which bound disks on T. Next use the fact that every incompressible annulus in X_n is boundary-parallel to remove the remaining intersection curves. Finally isotop T to one of the T_n using the fact that every incompressible torus in X_n is boundary-parallel. □

The remainder of this section deals with genus one Whitehead manifolds which are not excellent and may be skipped on a first reading.

We shall need the following well-known facts about Seifert fibered spaces in 3-manifolds. See [7], [8], and [12] for reference.

Let V be an unknotted solid torus in S^3 with core J. Let K be a simple closed curve in ∂V. If K is knotted in S^3, then its exterior is a *torus knot space*. If K goes around V longitudinally at least twice, then the exterior of $J \cup K$ is a *cable space*. A manifold homeomorphic to $S^1 \times P_n$, where P_n is a disk with n holes, $n \geq 1$, is an *n-fold composing space*. A 1-fold composing space is homeomorphic to $S^1 \times S^1 \times I$ and is called a *shell*. A surface in a Seifert fibered space is *vertical* if it is a union of fibers.

LEMMA 2.3. *Let S be a torus knot space, cable space, or composing space.*

1. *S admits a Seifert fibration having two, one, or no exceptional fibers and decomposition surface a disk, annulus, or P_n, respectively. This Seifert fibration is unique up to isotopy unless S is a shell.*
2. *Every incompressible torus in S is isotopic to a vertical torus; it is boundary parallel unless S is an n-fold composing space with $n \geq 3$.*
3. *Every incompressible, non-boundary-parallel annulus in S is isotopic to vertical annulus unless S is a shell, in which case it is isotopic to $J \times I$ for some simple closed curve J in $S^1 \times S^1$.* □

LEMMA 2.4. *Suppose X is a compact sub-manifold of the exterior C of a non-trivial knot in S^3 with ∂X a union of tori incompressible in C. Let S be a compact submanifold of X such that ∂S*

is a union of tori, $S \cap \partial X$ is a union of components of ∂S, and ∂S is incompressible in X.

1. *If S is a Seifert fibered space, then S is a torus knot space, cable space, or composing space.*
2. *If S contains an incompressible non-boundary-parallel annulus but no incompressible, non-boundary-parallel torus, then S is Seifert fibered.*
3. *If S is an n-fold composing space, $n \geq 2$, and T is a component of ∂S, then T bounds a unique solid torus in S^3, and each Seifert fiber in T is a meridian of this solid torus.*
4. *Suppose S' is another such submanifold meeting S only in a common boundary component T. If S and S' are Seifert fibered and induce isotopic fibrations of T, then S, S', and $S \cup S'$ are composing spaces.* □

LEMMA 2.5. *Let X be as in Lemma 2.4. Then there exists a 2-manifold F in X, unique up to isotopy, such that*

1. *The components of F are incompressible tori.*
2. *No component of F is boundary-parallel.*
3. *No two components of F are parallel.*
4. *Each component of $\sigma_F(X)$ is either atoroidal or Seifert fibered.*
5. *If S and S' are Seifert fibered components of $\sigma_F(X)$ meeting in a common boundary component T, then the fibrations of T induced by S and S' are not isotopic.*
6. *Every incompressible torus in X is isotopic to a component of F, is isotopic to a vertical torus in a Seifert fibered component of $\sigma_F(X)$, or is boundary-parallel.*

Proof. F is the *canonical 2-manifold* in the Splitting Theorem of [8]. It can be obtained from the boundary F' of the characteristic Seifert pair of X by deleting the components of ∂X and one boundary component of each shell in $\sigma_{F'}(X)$. It can also be obtained directly from the two previous lemmas, as follows.

By Haken's Finiteness Theorem [5] there is a compact 2-manifold F' in X whose components are incompressible, non-boundary-parallel, pairwise non-parallel tori such that every incompressible non-boundary-parallel torus in X disjoint from F' is parallel to a component of F'. It follows that every component S of $\sigma_{F'}(X)$ is simple. If S is not atoroidal then it is a torus knot space, cable space, or composing space. If S_1 and S_2 are two such components meeting in a common boundary component T' on which they induce isotopic fibrations then delete T' from F'. Deleting all such tori from F' gives a 2-manifold F which has properties 1–5. These properties and the previous lemma imply property 6 and the isotopy uniqueness of F: Let T be an incompressible torus in X which is in general position with respect to F. Each component S of $\sigma_F(X)$ is irreducible; this allows one to remove those intersection curves which bound disks. The existence of any boundary-parallel annuli among the components of $F \cap S$ also allows one to remove intersection curves. Suppose some component A of $F \cap S$ is not boundary-parallel. Then it is isotopic to a vertical annulus in a Seifert fibration of S. There is a component S' of $\sigma_F(X)$ whose intersection with T contains an annulus A' having at least one boundary component in common with A. A' must be boundary-parallel in S'; otherwise it would be isotopic to a vertical annulus in a Seifert fibration of S', which is impossible since the Seifert fibrations induced by S and S' on a common boundary component are not isotopic. Therefore T can be isotoped into a component of $\sigma_F(X)$ and the results follow by Lemma 2.4. □

LEMMA 2.6. *Suppose V and V' are solid tori with boundaries T and T', respectively, and that $V' \subseteq Int(V)$. Suppose S is a Seifert fibered submanifold of $V \setminus Int(V')$ with $T \cup T' \subseteq \partial S$ and ∂S incompressible in $V \setminus Int(V')$. Then $V' \to V$ is not null-homotopic.*

Proof. Choose a knotted embedding of V in S^3. Then the incompressibility of ∂S in $V\backslash Int(V')$ and of T in $S^3\backslash Int(V)$ implies that of T' in $S^3\backslash Int(V')$. Therefore V' is knotted in S^3. Let $C=S^3\backslash Int(V')$ and $X=V\backslash Int(V')$. Then X and S satisfy the hypotheses of Lemma 2.4. Hence S is either a cable space or a composing space. Note that there is a vertical annulus A joining T to T' in S.

Suppose S is a cable space. If the fibers of T' are not meridians of V', then the Seifert fibration of S extends to a Seifert fibration of V with decomposition surface a disk and (since V is a solid torus) one exceptional fiber. Therefore V is a standard fibered solid torus and the core of V' is homotopic to a regular fiber, hence $V'\to V$ is not null-homotopic. If the fibers of T' are meridians of V', then the union of A and a meridional disk for V' is a meridional disk for V which meets a core of V' exactly once. Hence $V'\to V$ is in fact a homotopy equivalence.

Suppose S is a composing space. Then the fibers of T' are meridians of V' and the result follows as above. □

Suppose $\{V_n\}$ is a good genus one Whitehead sequence. Let F_n be the canonical 2-manifold of X_n. $\{V_n\}$ is *very good* if (i) no component of F_n bounds a solid torus V' in W with $V_{n-1}\to V'$ and $V'\to V_n$ both null-homotopic (so the sequence is in a sense maximal) and (ii) the component of $\sigma_{F_n}(X_n)$ containing T_n is atoroidal (the "top" piece of X_n is not Seifert fibered).

Lemma 2.7. *Every genus one Whitehead manifold admits a very good genus one Whitehead sequence.*

Proof. Let $\{V_n\}$ be a good genus one Whitehead sequence for W. Form the union of $\{T_n\}$ with the set of those components of the F_n which bound solid tori in W. The solid tori bounded by the elements of this set form a sequence $\{Q_k\}_{k=0}^{\infty}$, with $Q_0=T_0$ and $Q_k\subseteq Q_{k+1}$ for $k\geq 0$. Define the new Whitehead sequence $\{V'_m\}$ to be a certain subsequence of $\{Q_k\}$, as follows. Let $V'_0=Q_0$. If V'_m has been defined to be Q_i, let V'_{m+1} be the next Q_j for which $Q_i\to Q_j$ is null-homotopic. This new sequence clearly satisfies (i). If it violated (ii), then there would be a Q_s, $i<s<j$ such that $Q_j\backslash Int(Q_s)$ contained a Seifert fibered space with $\partial Q_s\cup\partial Q_j$ in its boundary. By Lemma 2.6 $Q_s\to Q_j$ would not be null-homotopic, forcing $Q_i\to Q_s$ to be null-homotopic, in contradiction to the choice of Q_j. □

Lemma 2.8. *Let $\{V_n\}$ be a very good genus one Whitehead sequence. Then the canonical 2-manifold of $Y_{p,q}$ is the union of the canonical 2-manifolds F_n of X_n, $p+1\leq n\leq q$, and the T_n, $p<n<q$.*

Proof. This set clearly satisfies properties 1–4 of Lemma 2.5. Property 5 of the lemma follows from property (ii) of the definition of very good. As shown in the proof of Lemma 2.5 properties 1–5 suffice to characterize the canonical 2-manifold. □

§3. THE SHIFT LEMMA

Let W be a genus one Whitehead manifold with good genus one Whitehead sequence $\{V_n\}$. This section investigates the extent to which homeomorphisms of W can be isotoped so as to preserve this sequence.

The first lemma is needed only for very good sequences which are not excellent and can be skipped on a first reading.

LEMMA 3.1. *Suppose* $\{V_n\}$ *is very good. Let* f: $W \to W$ *be a homeomorphism. Suppose for some* $k>0$ *and* $j>i\geq 0$ *that* $f(V_0)\subseteq V_i \subseteq f(V_k) \subseteq V_j$. *Then* $f(T_k)$ *is isotopic to some* T_m, $i\leq m\leq j$, *via an isotopy with support in* $Y_{i,j}$ *if* $i<m<j$ *and support in a regular neighborhood of* $Y_{i,j}$ *if* $m=i$ *or* $m=j$.

Proof. Let F be the canonical 2-manifold of $Y_{i,j}$. By Lemma 2.5 there are three possibilities:

Case 1. $f(T_k)$ is isotopic in $Y_{i,j}$ to a component T of F. If T were a component of the canonical 2-manifold F_n of some X_n, $i<n\leq j$, then property (i) of the definition of very good would be violated, so T must be one of the T_m, $i<m<j$, and we are done.

Case 2. $f(T_k)$ is isotopic in $Y_{i,j}$ to a vertical torus in a Seifert fibered component S of $\sigma_F(Y_{i,j})$. Denote the composition consisting of f followed by this isotopy again by f. By Cases 1 and 3 we may assume that $f(T_k)$ is not boundary-parallel in S. Then S is a composing space which is split into two components by $f(T_k)$. Let S_0 be the component contained in $f(V_k)$. Exactly one component T of ∂S_0 other than $f(T_k)$ bounds a solid torus V in W. Now $f^{-1}(T)$ is an incompressible torus in $Y_{0,k}$. Let F' be the canonical 2-manifold of $Y_{0,k}$. Then $f^{-1}(T)$ is isotopic in $Y_{0,k}$ to some T_n, $0<n<k$, to a component of some F_n, $0<n\leq k$, or to a vertical torus in a Seifert fibered component of $\sigma_{F'}(Y_{0,k})$ or is boundary-parallel in $Y_{0,k}$. These are all impossible by Lemma 2.6, the definition of very good, and the fact that $f^{-1}(S_0)$ is Seifert fibered.

Case 3. $f(T_k)$ is boundary-parallel in $Y_{i,j}$. Then we are done. □

LEMMA 3.2. *Suppose* $\{V_n\}$ *is excellent or very good. Let* f: $W\to W$ *be a homeomorphism such that* $f(Y_{p,q})=Y_{r,s}$. *Then* $q-p=r-s$ *and* f *is isotopic to a homeomorphism* f' *such that* $f'(T_{p+i})=T_{r+i}$ *for* $0\leq i\leq q-p$. *This isotopy can be chosen with support in* $Y_{p,q}$.

Proof. The proof is by induction on $q-p$.

Suppose $q-p=1$. Then $s-r=1$. If $\{V_n\}$ is excellent this follows from the fact that $Y_{p,q}=X_q$ is atoroidal. If $\{V_n\}$ is very good this follows from the fact that if $s-r>1$ then $f^{-1}(T_{r+1})$ violates part (i) of the definition of very good. Therefore no isotopy is necessary in this case.

Suppose $q-p>1$. Then $f(T_{p+1})$ is isotopic to some T_m with $r\leq m\leq s$. If $\{V_n\}$ is excellent this follows from Lemma 2.2. If $\{V_n\}$ is very good this follows by Lemma 3.1 with $i=r, j=s$, and $k=p+1$. Since T_{p+1} is not parallel to T_p or T_q, $r<m<q$ and the isotopy has support in $Y_{p,q}$. The result now follows by the inductive hypothesis. □

LEMMA 3.3. (THE SHIFT LEMMA) *Suppose* $\{V_n\}$ *is excellent or very good. Let* g: $W\to W$ *be a homeomorphism. Then there exist integers* $N\geq 1$ *and* $s\geq 1-N$ *and a homeomorphism* h: $W\to W$ *isotopic to* g *such that* $h(V_n)=V_{n+s}$ *for all* $n\geq N$.

Proof. First choose a $p_0>0$ such that $(V_0\cup g(V_0))\subseteq V_{p_0}$. Next choose an $N_0>0$ such that $V_{p_0}\subseteq g(V_{N_0})$. Then choose a $q_0>p_0$ such that $g(V_{N_0})\subseteq V_{q_0}$. $g(T_{N_0})$ is incompressible in $g(W_0)$ and so is incompressible in W_{p_0} and hence in Y_{p_0,q_0}. It can therefore be isotoped to some T_{M_0}, $p_0\leq M_0\leq q_0$, via an isotopy fixed outside a regular neighborhood of Y_{p_0,q_0} in W. This follows from Lemma 2.2 if $\{V_n\}$ is excellent and from Lemma 3.1 if $\{V_n\}$ is very good.

Let $p_1=q_0+1$. Choose an $N_1>N_0$ such that $V_{p_1}\subseteq g(V_{N_1})$ and a $q_1>p_1$ such that $g(V_{N_1})\subseteq V_{q_1}$. $g(T_{N_1})$ is incompressible in Y_{p_1,q_1} and so, as above, is isotopic to some T_{M_1}, $p_1\leq M_1\leq q_1$, via an isotopy fixed outside a regular neighborhood of Y_{p_1,q_1} in W.

Continuing in this fashion one obtains a sequence of isotopies with disjoint supports (and therefore a single isotopy) whose composition with g results in a homeomorphism g' such that $g'(T_{N_k})=T_{M_k}$ for some increasing sequences of integers $\{N_k\}$ and $\{M_k\}$. So $g'(Y_{N_k,N_{k+1}})=Y_{M_k,M_{k+1}}$. By Lemma 3.2 $M_{k+1}-M_k=N_{k+1}-N_k$ and there is an isotopy with support in $Int(g'(Y_{N_k,N_{k+1}}))$ taking $g'(T_i)$ to T_j for $N_k<i<N_{k+1}$ and $j=i+M_k-N_k$. The composition of this isotopy with g' gives the desired homeomorphism g, with $N=N_0$ and $s=M_0-N_0$. □

§4. THE LONG ISOTOPY LEMMA

Let W be a genus one Whitehead manifold with good genus one Whitehead sequence $\{V_n\}$. In this section it will be shown that sufficiently complicated simple closed curves in W cannot be "isotoped to infinity" without hitting V_0 infinitely often.

LEMMA 4.1. (THE LONG ISOTOPY LEMMA) *Suppose $\{V_n\}$ is excellent or very good. Let J be a simple closed curve in W which does not lie in a 3-cell. Suppose G: $W\times[a,b]\rightarrow W$ is an isotopy with $g_a=G|(W\times\{a\})$ and $g_b=G|(W\times\{b\})$. If $g_a(J)\subseteq V_m$ and $g_b(J)\subseteq W_{m+1}$, then $G(J\times[a,b])\cap J_0\neq\varnothing$.*

Proof. Let $J_a=g_a(J)$, $J_b=g_b(J)$, and $K=G(J\times[a,b])$. We reduce to the special case in which one end of the isotopy is the identity, as follows. Define F: $W\times[a,b]\rightarrow W$ by $F(x,t)=G(g_a^{-1}(x),t)$. Then F is an isotopy with $F|(W\times\{a\})=id_W$, $F(J_a\times\{b\})=J_b$, and $F(J_a\times[a,b])=K$.

Assume $K\cap J_0=\varnothing$. Then we may assume $K\cap V_0=\varnothing$. Let N be a regular neighborhood of K in W_0 By the Covering Isotopy Theorem [17] there is an isotopy F': $W\times[a,b]\rightarrow W$ with support in N such that $F'|(J_a\times[a,b])=F|(J_a\times[a,b])$. Let $f=F'|(W\times\{b\})$. Then $f|V_0=id_{V_0}$ and $f(J_a)=J_b$.

It will now be shown that the torus $f(T_m)$ is isotopic in the complement of $V_0\cup J_b$ to a torus disjoint from T_{m+1}. The result of following f by this isotopy will again be denoted by f.

First put $f(T_m)$ in general position with respect to all the T_n. Suppose D is an innermost disk on $f(T_m)$ with ∂D a component of $f(T_m)\cap T_{m+1}$. Then, as usual, $\partial D=\partial D'$ for a disk D' in T_{m+1} and $D\cup D'$ bounds a 3-cell B in W_0. Since J, and hence J_b, does not lie in a 3-cell in W one can isotop D across B, without moving J_b, so as to remove ∂D from the intersection. Continuing in this fashion all intersection curves which bound disks on $f(T_m)$ can be removed. Assume that the intersection is still non-empty.

If $\{V_n\}$ is excellent then since each X_n is atoroidal each component of $f(T_m)\cap Y_{0,m+1}$ is an annulus in X_{m+1} which is parallel to an annulus in T_{m+1}. Thus there is an isotopy with support in a regular neighborhood of X_{m+1} which removes the remaining intersection curves. Since J_b lies outside V_{m+1} it is not moved by the isotopy and the claim is proven.

If $\{V_n\}$ is very good then put $f(T_m)$ in general position with respect to the canonical 2-manifold F of $Y_{0,m+1}$. Let Q be the component of $\sigma_F(Y_{0,m+1})$ containing T_{m+1}. Since Q is atoroidal each component of $f(T_m)\cap Y_{0,m+1}$ is an annulus in Q which is parallel to an annulus in T_{m+1}. The claim now follows as above.

Thus there is a homeomorphism f: $W\rightarrow W$ such that $f(V_0)=V_0$ and, since J_b is outside V_{m+1}, $V_{m+1}\subseteq f(V_m)$. Choose j such that $f(V_m)\subseteq V_j$. By Lemma 2.2 (if $\{V_n\}$ is excellent) or Lemma 3.1 (if $\{V_n\}$ is very good) $f(T_m)$ is isotopic to some T_k, $m+1\leq k\leq j$, via an isotopy which has support in a regular neighborhood of $Y_{m+1,j}$ and is therefore fixed on V_0. Thus (again calling the new homeomorphism f) one has that $f(Y_{0,m})=Y_{0,k}$ with $k>m$, in contradiction to Lemma 3.2. □

§5. THE NON-EXISTENCE OF GROUP ACTIONS

THEOREM 5.1. *Genus one Whitehead manifolds admit no non-trivial, fixed point free, properly discontinuous group actions, thus cannot be non-trivial covering spaces.*

Proof. Let W be a genus one Whitehead manifold with excellent or very good Whitehead sequence $\{V_n\}$. It will be shown that for every homeomorphism g of W there exists a compact subset C of W such that $C \cap g^k(C) \neq \emptyset$ for infinitely many values of k. We may assume that for every $N \geq 0$ and $n \geq 0$ there exists a $K > 0$ such that $g^k(V_N) \subseteq W_n$ for all $k \geq K$, otherwise we would clearly have our set C. The argument will be easier to follow if it is divided into two cases, depending on whether or not g is isotopic to the identity.

Special Case. g is isotopic to the identity. Suppose F: $W \times [0, 1] \to W$ is an isotopy with $F(x, 0) = x$ and $F(x, 1) = g(x)$ for all $x \in W$. In this case it will be shown that $F(J_0 \times [0, 1])$ is the desired compact set C. Define G: $W \times [0, \infty) \to W$ by $G(x, t) = g^k(F(x, t-k))$ for $x \in W$ and $k \leq t \leq k+1$. G is a "very long isotopy" which interpolates the sequence of homeomorphisms $\{g^k\}$, $k \geq 0$. Note that $G(J_0 \times [k, k+1]) = g^k(C)$.

Suppose $g^p(J_0) \subseteq V_n$. Then for some $q > p$, $g^q(J_0) \subseteq W_{n+1}$. $G|(W \times [p, q])$ is an isotopy with $G(J_0 \times \{p\}) = g^p(J_0)$ and $G(J_0 \times \{q\}) = g^q(J_0)$. By the Long Isotopy Lemma $G(J_0 \times [p, q]) \cap J_0 \neq \emptyset$. Since $G(J_0 \times [p, q]) = \cup_{k=p}^{q-1} g^k(C)$, $g^k(C) \cap J_0 \neq \emptyset$ and hence $g^k(C) \cap C \neq \emptyset$ for some $k, p \leq k \leq q-1$. Repeating the above argument for arbitrarily high values of p establishes the claim.

General Case. g is not necessarily isotopic to the identity. By the Shift Lemma there exist integers $N \geq 1$ and $s \geq 1 - N$ and a homeomorphism h isotopic to g such that $h(V_n) = V_{n+s}$ for all $n \geq N$. Since g^{-1} and h^{-1} are isotopic we may assume, replacing g by g^{-1} if necessary, that $s \geq 0$. Let F: $W \times [0, 1] \to W$ be an isotopy such that $F(x, 0) = h(x)$ and $F(x, 1) = g(x)$ for all $x \in W$. In this case it will be shown that $F(V_N \times [0, 1])$ is the desired compact set C. Define $H: W \times [0, \infty) \to W$ by $H(x, t) = g^k(F(h^{-k}(x), t-k))$ for $x \in W$ and $k \leq t \leq k+1$. H interpolates the sequence of homeomorphisms $\{h, g, g^2 h^{-1}, \ldots, g^k h^{1-k}, \ldots\}$. Note that $h^{-1}(V_N) \subseteq V_N$. This implies that

$$H(J_N \times [k, k+1]) \subseteq H(V_N \times [k, k+1]) = g^k(F(h^{-k}(V_N) \times [0, 1])) \subseteq g^k(C).$$

Suppose $g^p h^{1-p}(J_N) \subseteq V_n$. For some $q > p$, $g^q(V_N) \subseteq W_{n+1}$. Then (again since $h^{-1}(V_N) \subseteq V_N$)

$$g^q h^{1-q}(J_N) \subseteq g^q h^{1-q}(V_N) \subseteq g^q(V_N) \subseteq W_{n+1}.$$

By the Long Isotopy Lemma $H(J_N \times [p, q]) \cap J_0 \neq \emptyset$. Since

$$H(J_N \times [p, q]) = \bigcup_{k=p}^{q-1} H(J_N \times [k, k+1]) \subseteq \bigcup_{k=p}^{q-1} g^k(C)$$

one sees that $g^k(C) \cap J_0 \neq \emptyset$ for some $k, p \leq k \leq q-1$. Since $J_0 \subseteq V_0 \subseteq V_N \subseteq h(V_N) \subseteq C$, one has that $g^k(C) \cap C \neq \emptyset$. Repeating this argument for arbitrarily high values of p completes the proof. □

REFERENCES

1. G. BREDON: *Introduction to Compact Transformation Groups*, Academic Press, New York (1972).
2. M. W. DAVIS: Groups generated by reflections and aspherical manifolds not covered by Euclidean spaces, *Ann. of Math.* **117** (1983), 293–324.
3. H. M. FARKAS and I. KRA: *Riemann Surfaces*, Springer, New York (1980).
4. K. W. GRUENBERG: Cohomological Topics in Group Theory, *Lecture Notes in Mathematics*, Springer, New York **143** (1980).

5. W. HAKEN: Some results on surfaces in 3-manifolds, *Studies in Modern Topology*, Math. Assoc. Am., Prentice Hall, New Jersey (1968), 34–98.
6. J. HEMPEL: 3-manifolds, *Ann. of Math. Studies*, Princeton University Press, London **86** (1976).
7. W. JACO: Lectures on 3-manifold topology, *CBMS Lecture Series, Am. Math. Soc.* **43** (1980).
8. W. JACO and P. B. SHALEN: Seifert fibered spaces in 3-manifolds, *Memoirs Am. Math. Soc.* **220** (1979).
9. W. S. MASSEY: *Algebraic Topology: An Introduction*, Harcourt Brace and World, New York (1967).
10. D. R. MCMILLAN, JR.: Some contractible open 3-manifolds, *Trans. Am. Math. Soc.* **102** (1962), 373–382.
11. D. R. MCMILLAN, JR. and T. L. THICKSTUN: Open 3-manifolds and the Poincaré conjecture, *Topology* **19** (1980), 313–320.
12. R. MYERS: Companionship of knots and the Smith Conjecture, *Trans. Am. Math. Soc.* **259** (1980), 1–32.
13. E. H. SPANIER: *Algebraic Topology*, McGraw-Hill, U.K. (1966).
14. W. P. THURSTON: Three dimensional manifolds, Kleinian groups, and hyperbolic geometry, *Bull. A. M. S.* **6** (1982), 357–381.
15. F. WALDHAUSEN: On irreducible 3-manifolds which are sufficiently large, *Ann. Math.* **87** (1968), 56–88.
16. J. H. C. WHITEHEAD: A certain open manifold whose group is unity, *Q. Jl. Math.* **6** (1935), 268–279.
17. E. C. ZEEMAN: Seminar on combinatorial topology, (notes), I.H.E.S. and Univ. of Warwick, (1963–66).

Department of Mathematics,
Oklahoma State University
Stillwater, OK 74078
U.S.A.