REPRESENTATIONS OF THE $p$-ADIC GSpin$_4$ AND GSpin$_6$ AND THE ADJOINT L-FUNCTION

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Abstract. We prove a conjecture of B. Gross and D. Prasad about determination of generic $L$-packets in terms of the analytic properties of the adjoint $L$-function for $p$-adic general even spin groups of semi-simple ranks 2 and 3. We also explicitly write the adjoint $L$-function for each $L$-packet in terms of the local Langlands $L$-functions for the general linear groups.

1. Introduction

In this article, we provide further details on the local $L$-packets for the non-Archimedean split general spin groups GSpin$_4$ and GSpin$_6$, following our earlier work [AC17]. We then use our explicit description of these $L$-packets to prove a conjecture of B. Gross and D. Prasad [Gr22, GP92] determining which of the $L$-packets are “generic” (i.e., contain an irreducible representation with a Whittaker model) in terms of the analytic properties at $s = 1$ of the adjoint $L$-function of the packet. We also write the adjoint $L$-function for each $L$-packet in terms of the local Langlands $L$-functions of the general linear groups. In addition to details about our results that we provide here, that the adjoint $L$-functions have a significant role in the Gan-Gross-Prasad conjectures, we expect that our results in this paper would be helpful in that direction as well. Particularly striking is the generalization of the Gan-Gross-Prasad to the non-tempered case [GGP20] where the relevant adjoint $L$-function does have a pole at $s = 1$.

Let $F$ be a $p$-adic field of characteristic zero. Denote by $W_F$ the Weil group of $F$ and let $W'_F = W_F \times SL_2(\mathbb{C})$ be the Weil-Deligne group of $F$. Let $G$ be a connected, reductive, linear algebraic group over $F$. The local Langlands Conjecture (LLC) predicts a surjective, finite-to-one map $L$ from the set $\text{Irr}(G)$ of equivalence classes of irreducible, smooth, complex representations of $G(F)$ to the set $\Phi(G)$ of $G$-conjugacy classes of $L$-parameters of $G(F)$, i.e., admissible homomorphisms $\phi : W'_F \to \hat{L}G$. Here, $\hat{L}G$ denotes the $L$-group of $G$ with $\hat{G} = {}^L G^0$ its connected component, i.e., the complex dual of $G$ [Bor79]. Among other properties, the map $L$ is supposed to preserve the local $L$-, $\epsilon$-, and $\gamma$-factors. Moreover, the (finite) fibers $\Pi_\phi$, for $\phi \in \Phi(G)$, of the map $L$ are called the $L$-packets of $G$ and their structures are expected to be controlled by certain finite subgroups of $\hat{G}$.

Consider the split general spin groups $G = \text{GSpin}_4$ and $G = \text{GSpin}_6$, of type $D_2 = A_1 \times A_2$ and $D_3 = A_3$ respectively, whose algebraic structure we review in Section 2.3. We constructed most of the $L$-packets for these two groups in [AC17] and proved that they satisfy the expected properties of preservation of the local factors and their internal structure. We review and complete the construction of these $L$-packets. In particular, using the classification of representations of $GL_n$, we give more explicit descriptions of the $L$-packets for GSpin$_4$ and GSpin$_6$ in terms of given representations of $GL_2 \times GL_2$ and $GL_4 \times GL_1$, respectively. As a byproduct, we are able to give the criteria for determining the size of the $L$-packets for GSpin$_4$ and GSpin$_6$ (see Sections 4 and 5).

The known cases of the LLC for the $p$-adic groups include $GL_n$ [HT01, Hen00, Sch13]; $SL_n$ [GK82]; non-quasi-split $F$-inner forms of $GL_n$ and $SL_n$ [HS12, ABPS16]; GSpin$_4$ and Sp$_4$ [GT11, GT19]; non-quasi-split $F$-inner form GSp$_{1,1}$ of GSp$_4$ [GT14]; Sp$_{2n}$, SO$_n$, and quasi-split SO$_{2n}$ [Art13; U$_n$ [Rog90, Mok15]; non quasi-split $F$-inner forms of $U_n$ [Rog90, KMSW14]; non-quasi-split $F$-inner form Sp$_{1,1}$ of Sp$_4$ [Cho17]; GSpin$_4$, GSpin$_6$ and their inner forms [AC17]; GSp$_{2n}$ and GO$_{2n}$ [Xu18].

Going back to the case of general $G$, assume that $\rho$ is a finite-dimensional complex representation of $\hat{G}$. When LLC is known, one can define the local Langlands $L$-functions $L(s, \pi, \rho) = L(s, \rho \circ \phi)$.
for each \( \pi \in \Pi_\phi \). Here, the \( L \)-factors on the right hand side are the Artin local factors associated to the given representation of \( W'_p \).

B. Gross and D. Prasad conjectured (in the generality of quasi-split groups) that the local \( L \)-packet \( \Pi_\phi(G) \) is generic if and only if the adjoint \( L \)-function \( L(s, \text{Ad} \circ \phi) \) is regular at \( s = 1 \) [GP92, Conj. 2.6]. Here, \( \text{Ad} \) denotes the adjoint representation of \( \hat{G} \) on the dual Lie algebra \( \hat{g} \) of \( G \). (Note that in the body of this paper we use \( \text{Ad} \) exclusively for the restriction of the adjoint representation to the derived group of \( \hat{G} \) to distinguish it from the full adjoint \( L \)-function, which would have an extra factor of the \( L \)-function for the trivial character when \( \hat{g} \) has a one-dimensional center.)

We prove the above conjecture for the groups \( \text{GSpin}_4 \) and \( \text{GSpin}_6 \) as a consequence of our construction of the \( L \)-packets for these groups. In fact, we prove the conjecture for a larger class of groups \( G = G'_{m,n} \), which are given as subgroups of \( \text{GL}_m \times \text{GL}_n \) satisfying a certain determinant equality (2.6). We are able to work in the slightly larger generality because, as in the construction of the \( L \)-packets, we use the approach of restricting representations from \( \text{GL}_m(F) \times \text{GL}_n(F) \) to the subgroup \( G \).

Moreover, we also give the adjoint \( L \)-function in all cases explicitly in terms of local Langlands \( L \)-functions of the general linear groups. While we are able to prove the Gross-Prasad conjecture already without the explicit knowledge of the adjoint \( L \)-function, the explicit description of the adjoint \( L \)-function certainly also verifies the conjecture and we include it here since it may lead to other number theoretic or representation theoretic results.

Finally, we take this opportunity to correct a few inaccuracies in [AC17]. They do not affect the main results in that paper and fix some errors in our description of the \( L \)-packets. The details are given in Section 6.

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2. Preliminaries

2.1. Local Langlands Correspondence (LLC). Let \( p \) be a prime number and let \( F \) be a \( p \)-adic field of characteristic zero, i.e., a finite extension of \( \mathbb{Q}_p \). We fix an algebraic closure \( \bar{F} \) of \( F \). Denote the ring of integers of \( F \) by \( \mathcal{O}_F \) and its unique maximal ideal by \( \mathcal{P}_F \). Moreover, let \( q \) denote the cardinality of the residue field \( \mathcal{O}_F/\mathcal{P}_F \) and fix a uniformizer \( \varpi \) with \( |\varpi|_F = q^{-1} \). Also, let \( W_F \) denote the Weil group of \( F \), \( W'_p \) the Weil-Deligne group of \( F \), and \( \Gamma \) the absolute Galois group \( \text{Gal}(\bar{F}/F) \). Throughout the paper, we will use the notation \( \nu(\cdot) = |\cdot|_F \).

Let \( G \) be a connected, reductive, linear algebraic group over \( F \). Fixing \( \Gamma \)-invariant splitting data we define the \( L \)-group of \( G \) as a semi-direct product \( L(G) := \hat{G} \rtimes \Gamma \), where \( \hat{G} = L(G^0) \) denotes the connected component of the \( L \)-group of \( G \), i.e., the complex dual of \( G \) (see [Bor79, §2]).

LLC (still conjectural in this generality) asserts that there is a surjective, finite-to-one map from the set \( \text{Irr}(G) \) of isomorphism classes of irreducible smooth complex representations of \( G(F) \) to the set \( \Phi(G) \) of \( \hat{G} \)-conjugacy classes of \( L \)-parameters, i.e., admissible homomorphisms \( \varphi : W'_F \rightarrow L(G) \).

Given \( \varphi \in \Phi(G) \), its fiber \( \Pi_\varphi(G) \), which is called an \( L \)-packet for \( G \), is expected to be controlled by a certain finite group living in the complex dual group \( \hat{G} \). Furthermore, for \( \pi \in \Pi_\varphi(G) \) and \( \rho \) a finite dimensional algebraic representation of \( L(G) \) one defines the local factors

\[
L(s, \pi, \rho) = L(s, \rho \circ \phi), \tag{2.1}
\]

\[
\epsilon(s, \pi, \rho, \psi) = \epsilon(s, \rho \circ \phi, \psi), \tag{2.2}
\]

\[
\gamma(s, \pi, \rho, \psi) = \gamma(s, \rho \circ \phi, \psi). \tag{2.3}
\]

provided that LLC is known for the case in question. Here, the factors on the right are Artin factors.

2.2. The Adjoint \( L \)-Function. What we recall in this subsection holds for \( G \) quasi-split ([GP92, §2]). However, for simplicity we will take \( G \) to be split over \( F \) since the groups we are working with in this article are split. When \( G \) is split over \( F \), we may replace the \( L \)-group \( L(G) \) by its connected component
Let \( \hat{G} = \mathcal{L} G^0 \). Take \( \rho \) to be the adjoint action of \( \hat{G} \) on its Lie algebra. Then we obtain the adjoint \( L \)-function \( L(s, \pi, \text{Ad}_{\hat{G}}) = L(s, \text{Ad}_{\hat{G}} \circ \phi) \) for all \( \pi \in \Pi_\rho(G) \). The following is a conjecture of D. Gross and D. Prasad (see [GP92, Conj. 2.6]).

**Conjecture 2.1.** \( \Pi_\rho(G) \) contains a generic member if and only if \( L(s, \text{Ad}_{\hat{G}} \circ \phi) \) is regular at \( s = 1 \). (Equivalently, \( \pi \) is generic if and only if \( L(s, \pi, \text{Ad}_{\hat{G}}) \) is regular at \( s = 1 \).)

The conjecture is known in many cases in which the LLC is known. To mention a few, it was verified for \( GL_n \) by B. Gross and D. Prasad [GP92], for \( GSp_4 \) in [GT11] and, for non-supercuspidals, in [AS08], and for SO and Sp groups, it follows from the work of Arthur on endoscopic classification [Art13]. We will verify this conjecture for the small rank split groups \( GSpin_4 \) and \( GSpin_6 \).

### 2.3. The Groups \( GSpin_4 \) and \( GSpin_6 \)

We gave detailed information about the structure of these two groups (as well as their inner forms) in [AC17, §2]. For now we just recall the incidental isomorphisms

\[
GSpin_4 \cong \{ (g_1, g_2) \in GL_2 \times GL_2 : \det g_1 = \det g_2 \} \tag{2.4}
\]

\[
GSpin_6 \cong \{ (g_1, g_2) \in GL_1 \times GL_4 : g_1^2 = \det g_2 \}. \tag{2.5}
\]

While our main interests in this article are the split general spin groups \( GSpin_4 \) and \( GSpin_6 \), for the purposes of Conjecture 2.1 it is no more difficult, and perhaps also more natural, to consider a slightly more general setup as follows.

Fix integers \( m, n \geq 1 \) and \( r, s \geq 1 \) and assume that \( \gcd(r, s) = 1 \). Define

\[
G = G_{m,n}^{r,s} := \{ (g, h) \in GL_m \times GL_n \mid (\det g)^r = (\det h)^s \} \tag{2.6}
\]

**Proposition 2.2.** The group \( G_{m,n}^{r,s} \) is a split, connected, reductive, linear algebraic group over \( F \).

**Proof.** Let \( X = (X_{ij}) \) and \( Y = (Y_{kl}) \) be \( m \times m \) and \( n \times n \) matrices, respectively. It is clear that \( G_{m,n}^{r,s} \) is an almost direct product of \( SL_m \times SL_n \) and a torus, is reductive. The only issue that requires justification is that the polynomial \( f(X, Y) = (\det X)^r - (\det Y)^s \) is irreducible in \( F[X_{ij}, Y_{kl}] \) if and only if \( d = \gcd(r, s) = 1 \). It is clear that if \( d > 1 \), then \( f \) is reducible since it would be divisible by \( (\det X)^{(r/d)} - (\det Y)^{(s/d)} \). It remains to show that if \( d = 1 \), then \( f(X, Y) \) is irreducible. This assertion should be easy to see via elementary arguments considering the polynomials in a possible factorization of \( f \). However, we prove it below as a special case of a more general fact.

Assume that \( f(x, y) \) is an (arbitrary) irreducible polynomial in \( F[x, y] \). Let

\[
p(x_1, x_2, \ldots, x_a) \in F[x_1, x_2, \ldots, x_a] \quad \text{and} \quad p(y_1, y_2, \ldots, y_b) \in F[y_1, y_2, \ldots, y_b]
\]

be two polynomials such that \( p - \alpha \) and \( q - \alpha \) are irreducible for all constants \( \alpha \). Then, \( f(p, q) \) is irreducible in \( F[x_1, x_2, \ldots, x_a, y_1, y_2, \ldots, y_b] \).

Our Proposition would clearly follow from the above assertion since \( (\det -\alpha) \) is always an irreducible polynomial and it is well-known that the two-variable polynomial \( x^r - y^s \) is irreducible in \( F[x, y] \) provided that \( d = \gcd(r, s) = 1 \).

To prove the assertion above, we proceed as follows. By base extension to an algebraic closure we may assume, without loss of generality, that \( F \) is algebraically closed.

Let \( A \) be the subscheme of \( \text{Spec} \ F[x_1, x_2, \ldots, x_a, y_1, y_2, \ldots, y_b] \) defined by \( f(p, q) \), and let \( B \) be the subscheme of \( \text{Spec} \ F[x, y] \) defined by \( x^r - y^s \). The latter is irreducible since \( x^r - y^s \) is an irreducible polynomial by our assumption that \( d = 1 \). There is a natural map \( A \to B \) which has irreducible (geometric) fibers. The result now follows from the following claim.

Claim: Let \( g : A \to B \) be an open morphism of schemes of finite type over an algebraically closed field \( F \) such that the (geometric) fibers of \( g \) are irreducible and \( B \) is irreducible. Then \( A \) is irreducible.

To see the claim let \( U \) be an open in \( A \). We want to show that for any other open \( V \), we have that \( U \cap V \) is nonempty. Since \( B \) is irreducible and \( g \) is open, we have that \( g(U) \cap g(V) \) is nonempty so there is a fiber \( F_0 \) of \( g \) such that \( F_0 \cap U \) and \( F_0 \cap V \) are nonempty. Hence, by irreducibility of \( F_0 \), they have a nonempty intersection in \( F_0 \). In particular, \( U \cap V \) is nonempty, which gives the claim.

It only remains to check that the map \( A \to B \) above is open. In fact, it is flat since it is a base extension of the cartesian product of two flat morphisms \( p : \text{Spec} \ F[x_1, \ldots, x_a] \to \text{Spec} \ F[x] \) and \( q : \text{Spec} \ F[y_1, \ldots, y_b] \to \text{Spec} \ F[y] \). (Here, we are using the fact that \( \text{Spec} \ F[x] \) is a curve.) This finishes the proof. \( \square \)
Since the subgroup $\{z^{-r}I_m, z^sI_n \in \mathbb{C}^\times \}$ we similarly write $\text{Ad}$ for the adjoint action of $\text{GL}_m$ on its Lie algebra $\mathfrak{gl}_m$. Recall by (2.8) that we have a natural map

$$pr = pr_{m,n}^r : \text{GL}_m(\mathbb{C}) \times \text{GL}_n(\mathbb{C}) \rightarrow \hat{G}.$$  

Then we have

$$\phi_x = pr \circ (\phi_m \otimes \phi_n).$$  

Since the subgroup $\{z^{-r}I_m, z^sI_n \in \mathbb{C}^\times \}$ is central in $\text{GL}_m(\mathbb{C}) \times \text{GL}_n(\mathbb{C})$ the following diagram commutes.

Note that the adjoint action $\text{Ad}_m$ of $\text{GL}_m(\mathbb{C})$ on $\mathfrak{gl}_m(\mathbb{C})$ preserves the trace, and similarly for $n$, so we obtain a right downward arrow by simply restricting any automorphism to the set of those pairs satisfying the trace equality in (2.10). We have

$$L(s, 1_{F^\times})L(s, \pi, \text{Ad}) \cdot L(s, 1_{F^\times}) = L(s, \pi, \text{Ad}\hat{G}) \cdot L(s, 1_{F^\times})$$

$$= L(s, \text{Ad}\hat{G} \circ \phi_{m \odot n}) \cdot L(s, 1_{F^\times})$$

$$= L(s, (\text{Ad}_m \otimes \text{Ad}_n) \circ (\phi_{m \odot n}))$$

$$= L(s, \text{Ad}_m \circ \phi_{m \odot n})L(s, \text{Ad}_n \circ \phi_{n})$$

$$= L(s, \pi_{m, \text{Ad}_m})L(s, \pi_{n, \text{Ad}_n})$$

$$= L(s, 1_{F^\times})^2 L(s, \pi, \text{Ad}) L(s, \pi, \text{Ad}).$$
Therefore, we obtain the more convenient equality
\[
L(s, \pi, \text{Ad}) = L(s, \pi_m, \text{Ad})L(s, \pi_n, \text{Ad}),
\] (2.15)
which holds thanks to our choice of the notation Ad. In Section 3.2 this relation helps verify Conjecture 2.1 for the groups of interest to us.

3. Genericity and The Conjecture of B. Gross and D. Prasad

3.1. Restriction of Generic Representations. Let us write $\square^D$ for the group $\text{Hom}(\square, \mathbb{C}^\times)$ of all continuous characters on a topological group $\square$. Denote by $\square_{\text{der}}$ the derived group of $\square$. Let $G$ and $\bar{G}$ be connected, reductive, linear, algebraic groups over $F$ satisfying the property that
\[
G_{\text{der}} = \bar{G}_{\text{der}} \subseteq G \subseteq \bar{G}.
\] (3.1)
For any connected, reductive, linear, algebraic group $\square$ over $F$, we write $\text{Irr}_{sc}(\square)$ and $\text{Irr}_{eq}(\square)$ for the set of equivalence classes of supercuspidal and essentially square-integrable representations of $\square(F)$, respectively.

Assume $\bar{G}$ and $G$ to be $F$-split. Let $\tilde{B}$ be a Borel subgroup of $\bar{G}$ with Levi decomposition $\tilde{B} = \bar{T}\tilde{U}$. Then $B = \tilde{B} \cap G$ is a Borel subgroup of $G$ with $B = TU$. Note that $T = \tilde{T} \cap G$ and $\tilde{U} = U$. Let $\psi$ be a generic character of $U(F)$. From [Tad92, Proposition 2.8] we know that given a $\psi$-generic irreducible representation $\tilde{\sigma}$ of $\tilde{G}(F)$ we have a unique $\psi$-generic $\sigma$ of $G(F)$ such that
\[
\sigma \mapsto \text{Res}_{\tilde{G}}^G(\tilde{\sigma}).
\]
The generic character associated with $\sigma$ is not unique though.

**Proposition 3.1.** Each generic character associated with $\sigma$ is determined up to the action of $\tilde{T}(F)/T(F)$.

**Proof.** We let $\tilde{\sigma} \in \text{Irr}(\tilde{G})$ be $\psi$-generic. Then there is a unique $\psi$-generic $\sigma_\psi \in \Pi_\tilde{\psi}(G)$. On the other hand, for each $\sigma \in \Pi_\tilde{\psi}(G)$ there exists $t \in \tilde{T}(F)/T(F) \cong \tilde{G}/G(F)$ such that $\sigma = t^* \sigma_\psi$, where $t^* \sigma_\psi(g) = \sigma(t^{-1}gt)$. This implies that $\sigma$ is $t^*\psi$-generic. Here $t^*\psi$ is defined as $t^*\psi(u) = \psi(t^{-1}ut)$. \qed

**Remark 3.2.** We say $\sigma \in \text{Irr}(G)$, resp. $\tilde{\sigma} \in \text{Irr}(\tilde{G})$, is generic if it is $\psi$-generic with respect to some generic character $\psi$. With this notation, $\sigma \in \text{Irr}(G)$ is generic if and only if is $\tilde{\sigma} \in \text{Irr}(\tilde{G})$.

3.2. Criterion for Genericity. In this section we verify Conjecture 2.1 for the small rank general spin groups we are considering in this article.

**Theorem 3.3.** Let $G = G_{m,n}^{r,s}$ be the group defined in (2.6). Let $\pi$ be an irreducible admissible representation of $G(F)$. Then $\pi$ is generic if and only if $L(s, \pi, \text{Ad})$ is regular at $s = 1$.

**Proof.** Given $\pi$ there exist irreducible admissible representations $\pi_m$ of $GL_m(F)$ and $\pi_n$ of $GL_n(F)$ such that $\pi$ is a subrepresentation of the restriction to $G(F)$ of $\pi_m \otimes \pi_n$ as in (2.9). Now, $\pi$ is generic if and only if both $\pi_m$ and $\pi_n$ are generic. By the truth of Conjecture 2.1 for the general linear groups, the latter is equivalent to both $L(s, \pi_m, \text{Ad})$ and $L(s, \pi_n, \text{Ad})$ being regular at $s = 1$. Hence, by (2.15) and the fact that neither of the $L$-functions can have a zero at $s = 1$, we have that $\pi$ is generic if and only if $L(s, \pi, \text{Ad})$ is regular at $s = 1$. This proves the theorem. \qed

As we observed in Section 2.3, the split groups $G_{4}^{r,s}$ and $G_{6}^{r,s}$ are special cases of $G_{m,n}^{r,s}$. Therefore, we have the following.

**Corollary 3.4.** Conjecture 2.1 holds for the groups $G_{4}^{r,s}$ and $G_{6}^{r,s}$.

4. Representations of $G_{4}^{r,s}$

In this section we list all the irreducible representations of $G_{4}^{r,s}(F)$ and then calculate their associated adjoint $L$-function explicitly. To this end, we give the nilpotent matrix associated to their parameter in each case.

4.1. The Representations.
4.1.1. Classification of representations of $\text{GSpin}_4$. Following [AC17], we have
\[
1 \rightarrow \text{GSpin}_4(F) \rightarrow \text{GL}_2(F) \times \text{GL}_2(F) \rightarrow F^\times \rightarrow 1. \tag{4.1}
\]
Recall that
\[
\text{GSpin}_4(F) \cong \{(g_1, g_2) \in \text{GL}_2(F) \times \text{GL}_2(F) : \det g_1 = \det g_2\}, \tag{4.2}
\]
\[
\text{L} \text{GSpin}_4 = \text{GSpin}_4 = \text{GSO}_4(\mathbb{C}) \cong (\text{GL}_2(\mathbb{C}) \times \text{GL}_2(\mathbb{C}))/\{(z^{-1}, z) : z \in \mathbb{C}^\times\}, \tag{4.3}
\]
and
\[
1 \rightarrow \mathbb{C}^\times \rightarrow \text{GL}_2(\mathbb{C}) \times \text{GL}_2(\mathbb{C}) \xrightarrow{pr_4} \text{GSpin}_4 \rightarrow 1. \tag{4.4}
\]
When convenient, we view $\text{GSO}_4$ as the group similitude orthogonal $4 \times 4$ matrices with respect to the anti-diagonal matrix
\[
J = J_4 = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}. \tag{4.5}
\]
The Lie algebra of this group is also defined with respect to $J$ and an element $X$ in this Lie algebra satisfies
\[
^tXJ + JX = 0.
\]

4.1.2. Construction of the $L$-packets of $\text{GSpin}_4$ (recalled from [AC17]). Given $\sigma \in \text{Irr}(\text{GSpin}_4)$ we have a lift $\tilde{\sigma} \in \text{Irr}(\text{GL}_2 \times \text{GL}_2)$ such that
\[
\sigma \mapsto \text{Res}_{\text{GSpin}_4}^{\text{GL}_2 \times \text{GL}_2}(\tilde{\sigma}). \tag{4.6}
\]
It follows from the LLC for $GL_n$ [HT01, Hen00, Sch13] that there is a unique $\bar{\varphi}_{\tilde{\sigma}} \in \Phi(\text{GL}_2 \times \text{GL}_2)$ corresponding to the representation $\tilde{\sigma}$. We now have a surjective, finite-to-one map
\[
\mathcal{L}_4 : \text{Irr}(\text{GSpin}_4) \rightarrow \Phi(\text{GSpin}_4) \tag{4.7}
\]
\[
\sigma \mapsto pr_4 \circ \bar{\varphi}_{\tilde{\sigma}},
\]
which does not depend on the choice of the lifting $\tilde{\sigma}$. Then, for each $\varphi \in \Phi(\text{GSpin}_4)$, all inequivalent irreducible constituents of $\bar{\varphi}$ constitutes the $L$-packet
\[
\Pi_\varphi(\text{GSpin}_4) := \Pi_{\bar{\varphi}}(\text{GSpin}_4) = \left\{ \sigma \mid \sigma \mapsto \text{Res}_{\text{GSpin}_4}^{\text{GL}_2 \times \text{GL}_2}(\tilde{\sigma}) \right\} \cong \mathbb{C}^\times. \tag{4.8}
\]
Here, $\tilde{\sigma}$ is the member in the singleton $\Pi_{\bar{\varphi}}(\text{GL}_2 \times \text{GL}_2)$ and $\bar{\varphi} \in \Phi(\text{GL}_2 \times \text{GL}_2)$ is such that $pr_4 \circ \bar{\varphi} = \varphi$. We note that the construction does not depends on the choice of $\bar{\varphi}$, due to the LLC for $\text{GL}_2$, [GK82, Lemma 2.4], [Tad92, Corollary 2.5], and [HS12, Lemma 2.2]. Further details can be found in [AC17, Section 5.1].

4.1.3. The $L$-parameters of $\text{GL}_2$. We recall the generic representations of $\text{GL}_2(F)$ in this paragraph. We refer to [Wed08, Kud94, GR10] for details. Let $\chi : F^\times \rightarrow \mathbb{C}^\times$ denote a continuous quasi-character of $F^\times$. By Zelevinski ([Ze80, Theorem 9.7] or [Kud94, Theorem 2.3.1]) we know that the generic representations of $\text{GL}_2$ are: the supercuspidals, $\text{St} \otimes (\chi \circ \det)$ where $\text{St}$ denotes the Steinberg representation, and normally induced representations $i_{\text{GL}_1 \times \text{GL}_1}^{\text{GL}_2}(\chi_1 \otimes \chi_2)$ with $\chi_1 \neq \chi_2 \nu^\pm 1$. The only non-generic representation is $\chi \circ \det$.

4.2. Generic Representations of $\text{GSpin}_4$. Following [AC17, Section 5.3], given $\varphi \in \Phi(\text{GSpin}_4)$, fix the lift
\[
\bar{\varphi} = \bar{\varphi}_1 \otimes \bar{\varphi}_2 \in \Phi(\text{GL}_2 \times \text{GL}_2)
\]
with $\bar{\varphi}_i \in \Phi(\text{GL}_2)$ such that $\varphi = pr_4 \circ \bar{\varphi}$. Let
\[
\tilde{\sigma} = \tilde{\sigma}_1 \boxplus \tilde{\sigma}_2 \in \Pi_{\bar{\varphi}}(\text{GL}_2 \times \text{GL}_2)
\]
be the unique member such that $\{\tilde{\sigma}_i\} = \Pi_{\bar{\varphi}_i}(\text{GL}_2)$.
Recall the notation
\[
I_{\text{GSpin}_4}(\tilde{\sigma}) := \left\{ \chi \in (\text{GL}_2(F) \times \text{GL}_2(F)/\text{GSpin}_4(F))^D \mid \tilde{\sigma} \otimes \chi \cong \tilde{\sigma} \right\}.
\]
Then we have
\[
\Pi_\varphi(\text{GSpin}_4) \xrightarrow{\sim} I_{\text{GSpin}_4}(\tilde{\sigma}), \tag{4.8}
\]
and we recall that, by [AC17, Proposition 5.7], we have
\[
I^{G\text{Spin}_4}(\tilde{\sigma}) = \begin{cases} 
I^{SL_2}(\tilde{\sigma}_1), & \text{if } \tilde{\sigma}_2 \cong \tilde{\sigma}_1\eta \text{ for some } \eta \in (F^x)^D; \\
I^{SL_2}(\tilde{\sigma}_2) \cap I^{SL_2}(\tilde{\sigma}_2), & \text{if } \tilde{\sigma}_2 \not\cong \tilde{\sigma}_1\eta \text{ for any } \eta \in (F^x)^D.
\end{cases}
\] (4.9)

4.2.1. Irreducible Parameters. Let \( \varphi \in \Phi(G\text{Spin}_4) \) be irreducible. Then \( \bar{\varphi}, \bar{\varphi}_1, \) and \( \bar{\varphi}_2 \) are all irreducible. By Section 3.1, we have the following.

**Proposition 4.1.** Let \( \varphi \in \Phi(G\text{Spin}_4) \) be irreducible. Then every member in \( \Pi_{2}(G\text{Spin}_4) \) is supercuspidal and generic.

To study the internal structure of \( \Pi_{2}(G\text{Spin}_4) \), by (4.8), we need to know the structure of \( I^{G\text{Spin}_4}(\tilde{\sigma}) \), as we now recall from [AC17].

\textbf{gmnt-(a)} When \( \tilde{\sigma}_2 \cong \tilde{\sigma}_1\eta \) for some \( \eta \in (F^x)^D \), we have
\[
I^{G\text{Spin}_4}(\tilde{\sigma}) \cong \begin{cases} 
\{1\}, & \text{if } \tilde{\varphi}_1 \text{ (and hence also } \tilde{\varphi}_2) \text{ is primitive or non-trivial on } SL_2(\mathbb{C}); \\
\mathbb{Z}/2\mathbb{Z}, & \text{if } \tilde{\varphi}_1 \text{ (and hence also } \tilde{\varphi}_2) \text{ is dihedral w.r.t. one quadratic extension; } \\
(\mathbb{Z}/2\mathbb{Z})^2, & \text{if } \tilde{\varphi}_1 \text{ (and hence also } \tilde{\varphi}_2) \text{ is dihedral w.r.t. three quadratic extensions.}
\end{cases}
\]

\textbf{gmnt-(b)} When \( \tilde{\sigma}_2 \not\cong \tilde{\sigma}_1\eta \) for any \( \eta \in (F^x)^D \), then by (4.9) we have
\[
I^{G\text{Spin}_4}(\tilde{\sigma}) \cong \{1\} \text{ or } \mathbb{Z}/2\mathbb{Z}.
\]

Since \( \tilde{\sigma}_2 \not\cong \tilde{\sigma}_1\eta \) for any \( \eta \in (F^x)^D \), the case of both \( \tilde{\varphi}_1 \) and \( \tilde{\varphi}_2 \) being dihedral w.r.t. three quadratic extensions is excluded. Thus, we have the following list:
- If at least one of \( \tilde{\varphi}_i \) is primitive, then \( I^{G\text{Spin}_4}(\tilde{\sigma}) \cong \{1\} \).
- If both are dihedral, then \( I^{G\text{Spin}_4}(\tilde{\sigma}) \cong \mathbb{Z}/2\mathbb{Z} \).

From [AC17, Proposition 2.1], we recall the identification
\[
\Delta' = \{ \beta_1' = f_{11}' - f_{12}', \beta_2' = f_{21}' - f_{22}' \},
\] (4.10)

using the notation \( f_{ij}, \) and \( f_{ij}', 1 \leq i, j \leq 2 \), for the usual \( \mathbb{Z} \)-basis of characters and cocharacters of \( GL_2 \times GL_2 \) and \( \beta_1, \beta_2 \) denote the simple roots of \( G\text{Spin}_4 \). We can use this identification to relate the nilpotent matrices associated to the parameters of \( GL_2 \times GL_2 \) and \( G\text{Spin}_4 \), respectively.

For both (a) and (b) above, we have
\[
N_{GL_2(\mathbb{C}) \times GL_2(\mathbb{C})} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ (4.10) } N_{G\text{SO}_4(\mathbb{C})} = 0_{4 \times 4}.
\]

**Remark 4.2.** We note that case (b) above was mentioned, less precisely, in [AC17, Remark 5.10].

4.2.2. Reducible Parameters. If \( \varphi \in \Phi(G\text{Spin}_4) \) is reducible, then at least one \( \tilde{\varphi}_i \) must be reducible. Since the number of irreducible constituents in \( Res_{SL_2}(\tilde{\sigma}_i) \) is at most 2, we have \( I^{SL_2}(\tilde{\sigma}_i) \cong \{1\} \), or \( \mathbb{Z}/2\mathbb{Z} \). This implies that
\[
I^{G\text{Spin}_4}(\tilde{\sigma}) \cong \{1\} \text{ or } \mathbb{Z}/2\mathbb{Z}.
\]

If \( \tilde{\varphi}_i \) is reducible and generic, then \( \tilde{\sigma}_i \) is either the Steinberg representation twisted by a character or an irreducibly induced representation from the Borel subgroup of \( GL_2 \). We make case-by-case arguments as follows.

\textbf{gmnt-(i)} Note that the Steinberg representation of \( GL_2 \times GL_2 \) is of the form \( St_{GL_2} \boxtimes St_{GL_2} \). We have
\[
Res_{G\text{Spin}_4}^{GL_2 \times GL_2}(St_{GL_2} \boxtimes St_{GL_2}) = St_{G\text{Spin}_4}
\] (4.11)

and
\[
Res_{G\text{Spin}_4}^{GL_2 \times GL_2}(St_{GL_2} \otimes \chi_1 \boxtimes St_{GL_2} \otimes \chi_2) = St_{G\text{Spin}_4} \otimes \chi
\]

for some \( \chi \). We have \( I^{G\text{Spin}_4}(\tilde{\sigma}) \cong \{1\} \) as \( I^{G}(St_G) \cong \{1\} \). Thus, by (4.9), the \( L \)-packet remains a singleton and the restriction is irreducible.

- To determine \( \chi \), we use the required properties of \( \chi_1, \chi_2 \). Using
\[
T = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \left| \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \right| ab = cd \right\},
\] (4.12)

we have \( \chi_1(ab) = \chi_2(cd) \iff \chi_1 = \chi_2 \). Denote \( \chi_1 = \chi_2 \) by \( \chi \).
For (4.11), we have
\[
N_{\text{GL}_2(\mathbb{C}) \times \text{GL}_2(\mathbb{C})} = \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix} \quad (4.10) \quad N_{\text{GSO}_4(\mathbb{C})} = \begin{pmatrix}
0 & 1 & 1 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]


**ii)** Next we consider
\[
\text{Res}_{\text{Spin}_4}^{\text{GL}_2 \times \text{GL}_2} \left( \iota_{\text{GL}_2}^i \chi_1 \otimes \chi_2 \boxtimes \text{St}_{\text{GL}_2} \otimes \chi \right).
\]
(4.13)

By (4.9), the fact that \( \bar{\sigma}_2 \not\equiv \bar{\sigma}_1 \bar{\eta} \) for any \( \bar{\eta} \in (F^\times)^D \), and since \( I^G(\text{St}_G) \cong \{1\} \), it follows that
\[
I^G_{\text{Spin}_4}(\bar{\sigma}) \cong \{1\}.
\]

Thus, the \( L \)-packet remains a singleton and the restriction (4.13) is irreducible.

- To describe the restriction (4.13), we proceed similarly as above. We have
\[
\chi_1(a)\chi_2(b) = \chi(ab) = \chi(ab) \iff \chi_1\chi^{-1}(a) = \chi_2^{-1}(b)
\]
Specializing to \( a = b \) and \( c = d \) in the center, we have
\[
\chi_1\chi_2 = 1
\]
For (4.13), we have
\[
N_{\text{GL}_2(\mathbb{C}) \times \text{GL}_2(\mathbb{C})} = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix} \quad (4.10) \quad N_{\text{GSO}_4(\mathbb{C})} = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]


**iii)** We consider
\[
\text{Res}_{\text{Spin}_4}^{\text{GL}_2 \times \text{GL}_2} \left( \iota_{\text{GL}_1 \times \text{GL}_1}^i \chi_1 \otimes \chi_2 \boxtimes \iota_{\text{GL}_1 \times \text{GL}_1}^i \chi_3 \otimes \chi_4 \right) = \iota_T^{\text{Spin}_4} \left( \chi_1 \otimes \chi_2 \chi_3 \otimes \chi_1 \chi_2 \chi_3^{-1} \right).
\]
Here, \( \chi_1 \not\equiv \chi_2 \nu^{\pm 1} \) and \( \chi_3 \not\equiv \chi_4 \nu^{\pm 1} \). Note that by (4.9) this induced representation may be irreducible or consist of two irreducible inequivalent constituents. We have
\[
N_{\text{GL}_2(\mathbb{C}) \times \text{GL}_2(\mathbb{C})} = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix} \quad (4.10) \quad N_{\text{GSO}_4(\mathbb{C})} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

**iv)** Given a supercuspidal \( \bar{\sigma} \in \text{Irr}(\text{GL}_2) \), we consider
\[
\text{Res}_{\text{Spin}_4}^{\text{GL}_2 \times \text{GL}_2} (\bar{\sigma} \boxtimes \text{St}_{\text{GL}_2} \otimes \chi).
\]
(4.14)

Since \( I^G(\text{St}_G) \cong \{1\} \), due to (4.9), the restriction (4.14) is irreducible. We then have
\[
N_{\text{GL}_2(\mathbb{C}) \times \text{GL}_2(\mathbb{C})} = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix} \quad (4.10) \quad N_{\text{GSO}_4(\mathbb{C})} = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

**v)** Given supercuspidal \( \bar{\sigma} \in \text{Irr}(\text{GL}_2) \), we next consider
\[
\text{Res}_{\text{Spin}_4}^{\text{GL}_2 \times \text{GL}_2} (\bar{\sigma} \boxtimes \iota_{\text{GL}_1 \times \text{GL}_1}^i \chi_1 \otimes \chi_2).
\]
Note from (4.9) that this may be irreducible or consist of two irreducible inequivalent constituents. We have
\[
N_{\text{GL}_2(\mathbb{C}) \times \text{GL}_2(\mathbb{C})} = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix} \quad (4.10) \quad N_{\text{GSO}_4(\mathbb{C})} = 0_{4 \times 4}.
\]
4.3. Non-Generic Representations of GSpin$_4$. If $\sigma \in \text{Irr}(\text{GSpin}_4)$ is non-generic, then $\sigma$ is of the form

$$\text{Res}_{\text{GSpin}_4}^{\text{GL}_2 \times \text{GL}_2} ((\chi \circ \text{det}) \boxtimes \tilde{\sigma}),$$

(4.15)

with $\tilde{\sigma} \in \text{Irr}((\text{GL}_2))$. Note this restriction is irreducible due to (4.9), and that as $\chi \circ \text{det}$ is non-generic, so is the restriction $\sigma$ for any $\tilde{\sigma} \in \text{Irr}((\text{GL}_2))$.

For $\tilde{\sigma} = \text{St} \in \text{Irr}((\text{GL}_2))$, we have

$$N_{\text{GL}_2(\mathbb{C}) \times \text{GL}_2(\mathbb{C})} = \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right),$$

and otherwise we have

$$N_{\text{GL}_2(\mathbb{C}) \times \text{GL}_2(\mathbb{C})} = \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right).$$

We summarize the above information about the representations of GSpin$_4$ in Table 1.

4.4. Computation of the Adjoint $L$-function for GSpin$_4$. We now give explicit expressions for the adjoint $L$-function for each of the representations of GSpin$_4$. We start by recalling that the adjoint $L$-functions of the representations $\tilde{\sigma} \in \text{Irr}(\text{GL}_2)$ are as follows.

$$L(s, \tilde{\sigma}, \text{Ad}) = \begin{cases} L(s)^2 L(s, \chi_1 \chi_2^{-1}) L(s, \chi_1^{-1} \chi_2), & \text{if } \tilde{\sigma} = i_{\text{GL}_2 \times \text{GL}_2}^\chi (\chi_1 \boxtimes \chi_2) \text{ with } \chi_1 \chi_2^{-1} \neq \nu^\pm 1; \\
L(s)L(s + 1), & \text{if } \tilde{\sigma} = \text{St}_{\text{GL}_2} \otimes \chi; \\
L(s)L(s, \tilde{\sigma}, \text{Sym}^2 \otimes \omega^{-1}_\tilde{\sigma}), & \text{if } \tilde{\sigma} \text{ is supercuspidal;} \\
L(s)^2 L(s - 1)L(s + 1), & \text{if } \tilde{\sigma} = \chi \circ \text{det}.
\end{cases}$$

Here, $L(s) = L(s, 1_{F^\times})$. Recall our choice of notation

$$L(s, \tilde{\sigma}, \text{Ad}_2) = L(s)L(s, \tilde{\sigma}, \text{Ad}).$$

Combining with (2.14), Sections 4.2.1 and 4.2.2, we have the following.

\textbf{gnt-(a)&(b)} Given a supercuspidal $\sigma \in \text{Irr}(\text{GSpin}_4)$, we recall that

$$\sigma \subset \text{Res}_{\text{GSpin}_4}^{\text{GL}_2 \times \text{GL}_2} (\tilde{\sigma}_1 \boxtimes \tilde{\sigma}_2)$$

for some supercuspidal $\tilde{\sigma}_1 \boxtimes \tilde{\sigma}_2 \in \text{Irr}(\text{GL}_2 \times \text{GL}_2)$. By (2.15) we have

$$L(s, \sigma, \text{Ad}) = L(s, \tilde{\sigma}_1, \text{Sym}^2 \otimes \omega^{-1}_{\tilde{\sigma}_1}) L(s, \tilde{\sigma}_2, \text{Sym}^2 \otimes \omega^{-1}_{\tilde{\sigma}_2}).$$

\textbf{gnt-(i)} Given

$$\sigma = \text{St}_{\text{GSpin}_4} \otimes \chi \in \text{Irr}(\text{GSpin}_4),$$

by (2.15) we have

$$L(s, \sigma, \text{Ad}) = L(s + 1)^2.$$ 

\textbf{gnt-(ii)} Given $\sigma \in \text{Irr}(\text{GSpin}_4)$ such that

$$\sigma = \text{Res}_{\text{GSpin}_4}^{\text{GL}_2 \times \text{GL}_2} \left( i_{\text{GL}_2 \times \text{GL}_2}^\chi (\chi_1 \otimes \chi_2) \boxtimes \text{St}_{\text{GL}_2} \otimes \chi \right),$$

by (2.15) we have

$$L(s, \sigma, \text{Ad}) = L(s)L(s, \chi_1 \chi_2^{-1}) L(s, \chi_1^{-1} \chi_2) L(s + 1).$$

\textbf{gnt-(iii)} Given $\sigma \in \text{Irr}(\text{GSpin}_4)$ such that

$$\sigma \subset \text{Res}_{\text{GSpin}_4}^{\text{GL}_2 \times \text{GL}_2} \left( i_{\text{GL}_2 \times \text{GL}_2}^\chi (\chi_1 \otimes \chi_2) \boxtimes i_{\text{GL}_2 \times \text{GL}_2}^\chi (\chi_3 \otimes \chi_4) \right)$$

by (2.15) we have

$$L(s, \sigma, \text{Ad}) = L(s)^2 L(s, \chi_1 \chi_2^{-1}) L(s, \chi_1^{-1} \chi_2) L(s, \chi_3 \chi_4^{-1}) L(s, \chi_3^{-1} \chi_4).$$
Given \( \sigma \in \text{Irr}(GSpin_4) \) such that
\[
\sigma = \text{Res}_{GSpin_4}^{GL_2 \times GL_2} (\bar{\sigma} \boxtimes \text{St}_{GL_2} \otimes \chi)
\]
by (2.15) we have
\[
L(s, \sigma, \text{Ad}) = L(s, \bar{\sigma}_2, \text{Sym}^2 \otimes \omega_{\bar{\sigma}_2}^{-1}) L(s + 1).
\]

For (v) Given \( \sigma \in \text{Irr}(GSpin_4) \) such that
\[
\sigma \subset \text{Res}_{GSpin_4}^{GL_2 \times GL_2} \left( \bar{\sigma} \boxtimes \text{St}_{GL_2} \otimes (\chi_1 \otimes \chi_2) \right)
\]
by (2.15) we have
\[
L(s, \sigma, \text{Ad}) = L(s) L(s, \bar{\sigma}_2, \text{Sym}^2 \otimes \omega_{\bar{\sigma}_2}^{-1}) L(s, \chi_1 \chi_2^{-1}) L(s, \chi_1^{-1} \chi_2).
\]

For \( \text{nongnr} \) given a non-generic \( \sigma \in \text{Irr}(GSpin_4) \), from (4.15), we recall that
\[
\sigma = \text{Res}_{GSpin_4}^{GL_2 \times GL_2} (\chi \circ \det \boxtimes \bar{\sigma})
\]
and by (2.15) we have
\[
L(s, \sigma, \text{Ad}) = L(s) L(s - 1) L(s + 1) L(s, \bar{\sigma}, \text{Ad}).
\]

We summarize the explicit computations above in Table 2.

5. Representations of \( \text{GSpin}_6 \)

We now list all the representations of \( \text{GSpin}_6(F) \) and then calculate their associated adjoint \( L \)-function explicitly. Again, we do this explicit calculation by finding the \( 6 \times 6 \) nilpotent matrix in the complex dual group \( \text{GSO}_6(\mathbb{C}) \) in each case that is associated with the parameter of the representation.

5.1. The Representations.

5.1.1. Classification of representations of \( \text{GSpin}_6 \).

Again, following [AC17], we have
\[
1 \longrightarrow \text{GSpin}_6(F) \longrightarrow GL_4(F) \times GL_4(F) \longrightarrow F^{	imes} \longrightarrow 1.
\]
Recall that
\[
\text{GSpin}_6(F) \cong \{(g_1, g_2) \in GL_1(F) \times GL_4(F) : g_1^2 = \det g_2\},
\]
\[
\hat{L}\text{GSpin}_6 = \text{GSpin}_6 = \text{GSO}_6(\mathbb{C}) \cong (GL_1(\mathbb{C}) \times GL_4(\mathbb{C})) / \{(z^{-2}, z) : z \in \mathbb{C}^{	imes}\},
\]
and
\[
1 \longrightarrow \mathbb{C}^{	imes} \longrightarrow GL_1(\mathbb{C}) \times GL_4(\mathbb{C}) \xrightarrow{pr_6} \hat{\text{GSpin}}_6 \longrightarrow 1.
\]

Just as the rank two case, here too we view \( \text{GSO}_6 \) as the group similitude orthogonal \( 6 \times 6 \) matrices with respect to the analogous \( 6 \times 6 \), anti-diagonal, matrix \( J = J_6 \) as in (4.5), and similarly define its Lie algebra with respect to \( J \).

5.1.2. Construction of the \( L \)-packets of \( \text{GSpin}_6 \) (recalled from [AC17]).

Given \( \sigma \in \text{Irr}(\text{GSpin}_6) \) we have a lift \( \bar{\sigma} \in \text{Irr}(GL_1 \times GL_4) \) such that
\[
\sigma \hookrightarrow \text{Res}_{GL_1 \times GL_4}^{GL_1 \times GL_4}(\bar{\sigma}).
\]
It follows from the LLC for \( GL_n \) [HT01, Hen00, Sch13] that there is a unique \( \varphi_{\bar{\sigma}} \in \Phi(GL_1 \times GL_4) \) corresponding to the representation \( \bar{\sigma} \). We now have a surjective, finite-to-one map
\[
L_{0} : \text{Irr}(\text{GSpin}_6) \longrightarrow \Phi(\text{GSpin}_6)
\]
\[
\sigma \longrightarrow pr_6 \circ \varphi_{\bar{\sigma}},
\]
which does not depend on the choice of the lifting \( \bar{\sigma} \). Then, for each \( \varphi \in \Phi(\text{GSpin}_6) \), all inequivalent irreducible constituents of \( \bar{\sigma} \) constitutes the \( L \)-packet
\[
\Pi_{\varphi}(\text{GSpin}_6) := \Pi_{\bar{\varphi}}(\text{GSpin}_6) \bigg/ \cong,
\]
where \( \bar{\varphi} \) is the unique member of \( \Pi_{\varphi}(GL_1 \times GL_4) \) and \( \varphi \in \Phi(GL_1 \times GL_4) \) is such that \( pr_6 \circ \bar{\varphi} = \varphi \). We note that the construction does not depends on the choice of \( \bar{\varphi} \). Further details can be found in [AC17, Section 6.1].
Following [AC17, Section 6.3], given $\varphi \in \Phi(\mathrm{GSpin}_6)$, fix the lift
$$\tilde{\varphi} = \tilde{\eta} \otimes \tilde{\varphi}_0 \in \Phi(\mathrm{GL}_1 \times \mathrm{GL}_4)$$
with $\tilde{\varphi}_0 \in \Phi(\mathrm{GL}_4)$ such that $\varphi = \text{pr}_0 \circ \tilde{\varphi}$. Let
$$\tilde{\sigma} = \tilde{\eta} \boxtimes \tilde{\sigma}_0 \in \Pi_{\tilde{\varphi}}(\mathrm{GL}_1 \times \mathrm{GL}_4)$$
be the unique member such that $\{\tilde{\sigma}_0\} = \Pi_{\tilde{\varphi}_0}(\mathrm{GL}_4)$.

Recall that $I_{\mathrm{GSpin}_6}(\tilde{\sigma}) := \left\{ \tilde{x} \in \left( \mathrm{GL}_4(F) \times \mathrm{GL}_4(F)/\mathrm{GSpin}_6(F) \right)^D : \tilde{\sigma} \boxtimes \tilde{x} \cong \tilde{\sigma} \right\}$. Then we have
$$\Pi_{\tilde{\varphi}}(\mathrm{GSpin}_6) \xrightarrow{\text{5.7}} I_{\mathrm{GSpin}_6}(\tilde{\sigma}),$$
and by [AC17, Lemma 6.5 and Proposition 6.6] we have
$$I_{\mathrm{GSpin}_6}(\tilde{\sigma}) \cong \{ \tilde{x} \in I_{\mathrm{SL}_4}(\tilde{\sigma}_0) : \tilde{x}^2 = 1_{F^\times} \}$$
and any $\tilde{x} \in I_{\mathrm{GSpin}_6}(\tilde{\sigma})$ is of the form
$$\tilde{x} = (\tilde{x}' - 2 \boxtimes \tilde{x}'$$
for some $\tilde{x}' \in (F^\times)^D$.

### 5.2. Generic Representations of $\mathrm{GSpin}_6$

Thanks to the group structure (5.2) and the relation of generic representations in Section 3.1, in order to classify the generic representations of $\mathrm{GSpin}_6$, it suffices to classify the generic representations of $\mathrm{GL}_4$.

Here are two key facts from the GL theory.

- Recall from [Zel80, Theorem 9.7] and [Kud94, Theorem 2.3.1] that a generic representation of $\mathrm{GL}_4$ is of the form
$$I_{M_t}(\sigma_b)$$
where $M_t$ runs through any $F$-Levi subgroup of $\mathrm{GL}_4$ (including $\mathrm{GL}_4$ itself) and $\sigma_b$ is any essentially square-integrable representation of $M_t$.

- For their $L$-parameters, we note from [Kud94, §5.2] that the generic representations of $\mathrm{GL}_4$ have Langlands parameters (i.e., 4-dimensional Weil-Deligne representations $(\rho, N)$) of the form
$$(\rho_1 \otimes \text{sp}(r_1)) \otimes \ldots \otimes (\rho_t \otimes \text{sp}(r_t))$$
with $t \leq 4$, where $\rho_i$’s are irreducible and no two segments are linked.

#### 5.2.1. Irreducible Parameters

Let $\varphi \in \Phi(\mathrm{GSpin}_6)$ be irreducible. Then $\tilde{\varphi}$ and $\tilde{\varphi}_0$ are also irreducible. By Section 3.1, we have the following.

**Proposition 5.1.** Let $\varphi \in \Phi(\mathrm{GSpin}_6)$ be irreducible. Every member in $\Pi_{\varphi}(\mathrm{GSpin}_6)$ is supercuspidal and generic.

To see the internal structure of $\Pi_{\varphi}(\mathrm{GSpin}_6)$, we need, by (5.7), to know the detailed structure of $I_{\mathrm{GSpin}_6}(\tilde{\sigma})$ as follows.

- **gnc- (a)** Given $\sigma \in \text{Irr}_{\text{sc}}(\mathrm{GSpin}_6)$, we have
$$\tilde{\sigma} = \tilde{\sigma}_0 \boxtimes \tilde{\eta} \in \text{Irr}_{\text{sc}}(\mathrm{GL}_1 \times \mathrm{GL}_4).$$
From [AC17, Proposition 2.1], we recall the identification:
$$\Delta^\vee = \{ \beta^\vee_1 = f_2^* - f_3^*, \beta^\vee_2 = f_1^* - f_2^*, \beta^\vee_3 = f_3^* - f_4^* \}.$$

Using the notation $f_{ij}$ and $f_{ij}^*$, $1 \leq i,j \leq 4$, for the usual $\mathbb{Z}$-basis of characters and cocharacters of $\mathrm{GL}_4$. Also, $\{\beta_1, \beta_2, \beta_3\}$ are the simple roots of $\mathrm{GSpin}_6$.

We have
$$N_{\mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_4(\mathbb{C})} = (0_{4 \times 4}, 0) \xrightarrow{\text{5.10}} N_{\mathrm{GSO}_6(\mathbb{C})} = 0_{6 \times 6}.$$
5.2.2. Reducible Parameters. When $\varphi_0$ is not irreducible, we have proper parabolic inductions. An exhaustive list of $F$-Levi subgroups $M$ of $GSpin_6$ (up to isomorphism) is as follows.

- $M \cong GL_1 \times GL_1 \times GL_1 \times GL_1 = \tilde{M} \cap GSpin_6$, where $\tilde{M} = (GL_1 \times GL_1 \times GL_1 \times GL_1) \times GL_1$.
- $M \cong GL_2 \times GL_1 \times GL_1 = \tilde{M} \cap GSpin_6$, where $\tilde{M} = (GL_2 \times GL_1 \times GL_1) \times GL_1$.
- $M \cong GL_3 \times GL_1 = \tilde{M} \cap GSpin_6$, where $\tilde{M} = (GL_3 \times GL_1) \times GL_1$. (Note: The factor $GL_1$ of $M$ is $GSpin_6$ by convention.)
- $M \cong GL_1 \times GSpin_4 = \tilde{M} \cap GSpin_6$, where $\tilde{M} = (GL_1 \times GL_2) \times GL_1$.
- $M \cong GSpin_6 = \tilde{M} \cap GSpin_6$, where $\tilde{M} = GL_4 \times GL_1$.

(Note that $M \cong GL_2 \times GL_2$ does not occur on this list.) We now consider each case and, by abuse of notation, conflate algebraic groups and their $F$-points.

**gnr-(I)** $M \cong GL_1 \times GL_1 \times GL_1 \times GL_1$ and $\tilde{M} = (GL_1 \times GL_1 \times GL_1 \times GL_1) \times GL_1$.

Given $\chi_i \in (F^\times)^D$ we consider

$$i_M^{GSpin_6}(\chi_1 \boxtimes \chi_2 \boxtimes \chi_3 \boxtimes \chi_4).$$

(5.11)

Write $\chi_1 \boxtimes \chi_2 \boxtimes \chi_3 \boxtimes \chi_4 = (\tilde{x}_1 \boxtimes \tilde{x}_2 \boxtimes \tilde{x}_3 \boxtimes \tilde{x}_4 \boxtimes \tilde{\eta})|_M$ with $\tilde{x}_i, \tilde{\eta} \in (F^\times)^D$ so that

$$\tilde{x}_1 \tilde{x}_2 \tilde{x}_3 \tilde{x}_4 = \tilde{\eta}^2.$$ 

Then we have the following relations

$$\chi_1 = \tilde{x}_1, \chi_2 = \tilde{x}_2, \chi_3 = \tilde{x}_3, \chi_4 = \eta^2(\tilde{x}_2 \tilde{x}_3 \tilde{x}_4)^{-1}. $$

(5.12)

By Section 3.1, we know that the representation (5.11) is generic if and only if its lift

$$i_M^{GL_4 \times GL_1}(\tilde{x}_1 \boxtimes \tilde{x}_2 \boxtimes \tilde{x}_3 \boxtimes \tilde{x}_4 \boxtimes \tilde{\eta})$$

(5.13)

is generic if and only if

$$i_M^{GL_4 \times GL_1 \times GL_1 \times GL_1}(\tilde{x}_1 \boxtimes \tilde{x}_2 \boxtimes \tilde{x}_3 \boxtimes \tilde{x}_4)$$

(5.14)

is generic. By the classification of the generic representations of $GL_n$ ([Zel80, Theorem 9.7] and [Kud94, Theorem 2.3.1]), this amounts to (5.14) being irreducible. By [Kud94, Theorem 2.1.1] and [BZ77, Zel80], the necessary and sufficient condition for this to occur is that there is no pair $i, j$ with $i \neq j$ such that

$$\tilde{x}_i = \nu \tilde{x}_j.$$ 

We have

$$N_{GL_4(C) \times GL_1(C)} = (0 \times 4, 0) \overset{(5.10)}{\leftrightarrow} N_{GSO_6(C)} = 0_{6 \times 6}.$$ 

gnr-(II) $M \cong GL_2 \times GL_1 \times GL_1$ and $\tilde{M} = (GL_2 \times GL_1 \times GL_1) \times GL_1$.

Given $\sigma_0 \in \text{Irr}_{\text{eq}}(GL_2)$ and $\chi_1, \chi_2 \in (F^\times)^D$, we consider

$$i_M^{GSpin_6}(\sigma_0 \boxtimes \chi_1 \boxtimes \chi_2).$$

(5.15)

Write $\sigma_0 \boxtimes \chi_1 \boxtimes \chi_2 = (\tilde{\sigma}_0 \boxtimes \tilde{x}_1 \boxtimes \tilde{x}_2 \boxtimes \tilde{\eta})|_M$ with $\tilde{\sigma}_0 \in \text{Irr}_{\text{eq}}(GL_2), \tilde{x}_i, \tilde{\eta} \in (F^\times)^D$.

Given $(g, h_1, h_2, h_3) \in \tilde{M}$ with $\det(gh_1h_2) = h_3^2$,

- if we set $(g, h_1, h_3) \in \tilde{M}$, we have

$$\tilde{\sigma}_0(g)\tilde{x}_1(h_1)\tilde{x}_2(h_2)\tilde{\eta}(h_3) = \tilde{\sigma}_0(g)\tilde{x}_1(h_1)\tilde{x}_2(\det g^{-1}h_1^{-1}h_3^{-1})\tilde{\eta}(h_3) = (\tilde{\sigma}_0\tilde{x}_2^{-1} \circ \det)(g)(\tilde{x}_1\tilde{x}_2^{-1})(h_1)(\tilde{x}_2^{-1}\tilde{\eta})(h_3)$$

$$= \sigma(g)\chi_1(h_1)\chi_2(h_3).$$

Then we have

$$\tilde{\sigma}_0 = \sigma_0\tilde{x}_2, \quad \tilde{x}_1 = \chi_1\tilde{x}_2, \quad \tilde{\eta} = \chi_2\tilde{x}_2^{-2}.$$
If we set \((g, h_2, h_3) \in M\), we have
\[
\tilde{\sigma}_0(g)\chi_1(h_1)\tilde{\chi}_2(h_2)\tilde{\eta}(h_3) = \tilde{\sigma}_0(g)\chi_1(\det g^{-1}h_2^{-1}h_3^2)\tilde{\chi}_2(h_2)\tilde{\eta}(h_3) = (\tilde{\sigma}_0\chi_1^{-1} \circ \det)(g)(\tilde{\chi}_2\chi_1^{-1})(h_2)(\tilde{\chi}_1\tilde{\eta})(h_3) = \sigma(g)\chi_1(h_2)\chi_2(h_3).
\]

Then we have
\[
\tilde{\sigma}_0 = \sigma_0\chi_1, \tilde{\chi}_2 = \chi_2\tilde{\chi}_1, \tilde{\eta} = \chi_1\tilde{\chi}_1^{-2}.
\]

As before, the representation (5.15) is generic if and only if its lift
\[
i_{M}^{iGL_4 \times GL_1}(\sigma_0 \boxtimes \tilde{\chi}_1 \boxtimes \tilde{\chi}_2 \boxtimes \tilde{\eta})
\]
is generic if and only if
\[
i_{M}^{iGL_4 \times iGL_2 \times GL_1 \times GL_1}(\sigma_0 \boxtimes \tilde{\chi}_1 \boxtimes \tilde{\chi}_2)
\]
is generic. Again by the classification of the generic representations of GL_n this amounts to (5.18) being irreducible. Hence, we must have
\[
\tilde{\chi}_1 \neq \nu^{\pm 1}\tilde{\chi}_2.
\]
In other words, given \((g, h_1, h_2, h_3) \in \tilde{M} \) with \(\det(gh_1h_2) = h_3^2\),

- if we set \((g, h_1, h_3) \in M\), then
  \[
  \chi_1 \neq \nu^{\pm 1};
  \]
- if we set \((g, h_2, h_3) \in M\), then
  \[
  \chi_2 \neq \nu^{\pm 1}.
  \]

We have the following two cases. If \(\sigma_0\) is supercuspidal, then
\[
N_{GL_4(\mathbb{C}) \times GL_1(\mathbb{C})} = (0_{4 \times 4}, 0) \overset{(5.10)}{\iff} N_{GSO_6(\mathbb{C})} = 0_{6 \times 6}.
\]

If \(\sigma_0\) is non-supercuspidal, then
\[
N_{GL_4(\mathbb{C}) \times GL_1(\mathbb{C})} = \left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, 0 \right) \overset{(5.10)}{\iff} N_{GSO_6(\mathbb{C})} = \left[ \begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right].
\]

\textbf{gnt-(III)} \(M \cong GL_3 \times GL_1\) and \(\tilde{M} = (GL_3 \times GL_1) \times GL_1\).

Given \(\sigma_0 \in \text{Irr}_{\text{res}}(GL_3)\) and \(\chi \in (F^\times)^D\), we consider
\[
i_{M}^{iGSpin_6}(\sigma_0 \boxtimes \chi).
\]

Write \(\sigma_0 \boxtimes \chi = (\sigma_0 \boxtimes \tilde{\chi} \boxtimes \tilde{\eta})|_{\tilde{M}}\) with \(\sigma_0 \in \text{Irr}_{\text{res}}(GL_3)\), \(\tilde{\chi}, \tilde{\eta} \in (F^\times)^D\).

Given \((g, h_1, h_2) \in \tilde{M}\) with \(\det(gh_1) = h_2^2\), if we set \((g, h_2) \in M\), then we have
\[
\tilde{\sigma}_0(g)\tilde{\chi}(h_1)\tilde{\eta}(h_2) = \tilde{\sigma}_0(g)(\det g^{-1}h_2^{-1}h_2^2)\tilde{\eta}(h_2) = (\tilde{\sigma}_0\chi_1^{-1} \circ \det)(g)(\tilde{\chi}_2\chi_1^{-1})(h_2)(\tilde{\chi}_1\tilde{\eta})(h_2) = \sigma(g)\chi(h_2).
\]

Then, we have
\[
\tilde{\sigma}_0 = \sigma_0\tilde{\chi} \quad \text{and} \quad \tilde{\eta} = \chi_2\tilde{\chi}_1^{-2}.
\]

As before, (5.19) is generic if and only if its lift
\[
i_{M}^{iGL_4 \times GL_1}(\tilde{\sigma}_0 \boxtimes \tilde{\chi} \boxtimes \tilde{\eta})
\]
is generic if and only if
\[
i_{M}^{iGL_4 \times GL_1}(\tilde{\sigma}_0 \boxtimes \tilde{\chi})
\]
is generic. This amounts to (5.22) being irreducible as before, which is always true since $\tilde{\sigma}_0$ is an essentially square integrable representation of $GL_3$. Note that by the classification of essentially square-integrable representations of $GL_3$ ([Kud94, Proposition 1.1.2]), $\tilde{\sigma}_0$ must be either supercuspidal or the unique subrepresentation of

$$i_{GL_3}^{GL_1 \times GL_1} \left( \nu \chi \boxtimes \chi \boxtimes \nu^{-1} \chi \right)$$

with any $\chi \in (F^\times)^D$.

We have the following two cases. If $\sigma_0$ is supercuspidal, then

$$N_{GL_4(\mathbb{C}) \times GL_1(\mathbb{C})} = (0_{4 \times 4}, 0) \overset{(5.10)}{=} N_{GSO_6(\mathbb{C})} = 0_{6 \times 6}.$$  

If $\sigma_0$ is the non-supercuspidal, unique, subrepresentation of (5.23), then

$$N_{GL_4(\mathbb{C}) \times GL_1(\mathbb{C})} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad 0 \overset{(5.10)}{=} N_{GSO_6(\mathbb{C})} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Given $\sigma_0 \in \text{Irr}_{eq}(GSpin_4)$ and $\chi \in (F^\times)^D$ we consider

$$i_{M}^{GSpin_4} (\chi \boxtimes \sigma_0).$$

Write $\chi \boxtimes \sigma_0 \subset (\tilde{\sigma}_1 \boxtimes \tilde{\sigma}_2 \boxtimes \tilde{\eta}) |_{M}$ with $\tilde{\sigma}_1 \in \text{Irr}_{eq}(GL_2), \tilde{\eta} \in (F^\times)^D$.

As before, (5.24) is generic if and only if its lift

$$i_{M}^{GL_4 \times GL_1} (\tilde{\sigma}_1 \boxtimes \tilde{\sigma}_2 \boxtimes \tilde{\eta})$$

is generic if and only if

$$i_{GL_4 \times GL_2}^{GL_1} (\tilde{\sigma}_1 \boxtimes \tilde{\sigma}_2) \overset{(5.25)}{=} \frac{\chi \boxtimes \sigma_0}{\sigma_0}$$

is generic. This amounts to (5.26) being irreducible. Thus, we must have

$$\tilde{\sigma}_1 \neq \nu^\pm \tilde{\sigma}_2.$$  

We have several cases to consider. If $\sigma_0$ is supercuspidal (so are $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$), then

$$N_{GL_4(\mathbb{C}) \times GL_1(\mathbb{C})} = (0_{4 \times 4}, 0) \overset{(5.10)}{=} N_{GSO_6(\mathbb{C})} = 0_{6 \times 6}.$$  

If $\sigma_0$ is non-supercuspidal, then for supercuspidal $\tilde{\sigma}_1$ and non-supercuspidal $\tilde{\sigma}_2$ we have

$$N_{GL_4(\mathbb{C}) \times GL_1(\mathbb{C})} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{(5.10)}{=} N_{GSO_6(\mathbb{C})} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

for non-supercuspidal $\tilde{\sigma}_1$ and supercuspidal $\tilde{\sigma}_2$ we have

$$N_{GL_4(\mathbb{C}) \times GL_1(\mathbb{C})} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad 0 \overset{(5.10)}{=} N_{GSO_6(\mathbb{C})} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

and so on.
and for non-supercuspidal $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ we have

$$N_{GL_4(C) \times GL_4(C)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, 0$$

$$N_{GSO_4(C)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

\textbf{gnt-(V)} $M \cong GSpin_6$ and $\tilde{M} = GL_4 \times GL_1$. Given $\sigma \in \text{Irr}_{res}(GSpin_6) \setminus \text{Irr}_{sc}(GSpin_6)$, we consider

$$\sigma \subset (\tilde{\sigma} \boxtimes \tilde{\eta})|_M$$

with $\tilde{\sigma} \in \text{Irr}_{res}(GL_4) \setminus \text{Irr}_{sc}(GL_4), \tilde{\eta} \in (F^\times)^D$. Here, we note that $\varphi \in \Phi(GSpin_6)$ is not irreducible and neither $\tilde{\sigma}$ nor $\sigma$ is supercuspidal. It is clear that $\sigma$ is generic as $\tilde{\sigma} \boxtimes \tilde{\eta}$ is. By the classification of essentially square-integrable representations of $GL_4$ ([Kud94, Proposition 1.1.2]), $\tilde{\sigma}$ must be the unique subrepresentation of either

$$i_{GL_4 \times GL_4 \times GL_4}^{GL_4} \left( \nu^{1/2} \tilde{\chi} \boxtimes \nu^{1/2} \tilde{\chi} \boxtimes \nu^{-1/2} \tilde{\chi} \boxtimes \nu^{-3/2} \tilde{\chi} \right)$$

with any $\tilde{\chi} \in (F^\times)^D$ (i.e., $\tilde{\sigma} = \text{St}_{GL_4} \otimes \tilde{\chi}$), or of

$$i_{GL_4 \times GL_2}^{GL_4} \left( \nu^{1/2} \tilde{\tau} \boxtimes \nu^{-1/2} \tilde{\tau} \right)$$

with any $\tilde{\tau} \in \text{Irr}_{sc}(GL_2)$.

Now, for (5.27) we have

$$N_{GL_4(C) \times GL_4(C)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, 0$$

$$N_{GSO_6(C)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

and for (5.28) we have

$$N_{GL_4(C) \times GL_4(C)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, 0$$

$$N_{GSO_6(C)} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(We note, cf. [Tat79, (4.1.5)], that $N_{GL_4(C)}$ is of the form $O_{2 \times 2} \otimes I_{2 \times 2} + \left[ \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right] \otimes I_{2 \times 2}$.)

\subsection*{5.3. Non-Generic Representations of $GSpin_6$.}

Using the transitivity of the parabolic induction and the classification of generic representations of $GL_n$ ([Ze180, Theorem 9.7] and [Kud94, Theorem 2.3.1]), the non-generic representations of $GSpin_6$ are as follows.

\textbf{nongen-(A)} $M \cong GL_1 \times GL_1 \times GL_1 \times GL_1$ and $\tilde{M} = (GL_1 \times GL_1 \times GL_1 \times GL_1) \times GL_1$.

Given $\chi_4 \in (F^\times)^D$, by Section 3.1 and using (5.12), the representation (5.11) contains a non-generic constituent if and only if the same is true for

$$i_{M}^{GL_4 \times GL_1} (\tilde{\chi}_1 \otimes \tilde{\chi}_2 \otimes \tilde{\chi}_3 \otimes \tilde{\chi}_4 \otimes \tilde{\eta})$$

if and only if

$$i_{GL_4 \times GL_1}^{GL_4} (\tilde{\chi}_1 \otimes \tilde{\chi}_2 \otimes \tilde{\chi}_3 \otimes \tilde{\chi}_4)$$

(5.30)
contains a non-generic constituent. This amounts to (5.30) being reducible. As before, the necessary and sufficient condition for this to occur is that there is some pair \( i, j \) with \( i \neq j \) such that \( \chi_i = \nu \chi_j \).

By the Langlands classification and the description of constituents of the parabolic induction (see [Zel80, Theorem 7.1], [Rod82, Theorem 7.1], and [Kud94, Theorems 2.1.1 §5.1.1]), each constituent can be described as a Langlands quotient, denoted by \( Q(...) \), as follows.

The first case is when there is only one pair, say \( \chi_1 = \nu^{1/2} \chi \) and \( \chi_2 = \nu^{-1/2} \chi \) for some \( \chi \in (F^\times)^D \) while \( \chi_3 \neq \nu^{\pm 1} \chi_j \) for \( j \neq 3 \) and \( \chi_4 \neq \nu^{\pm 1} \chi_j \) for \( j \neq 4 \). Then we have the non-generic constituent

\[
Q \left[ [\nu^{1/2} \chi], [\nu^{-1/2} \chi], [\chi_3], [\chi_4] \right],
\]

(5.31)

which is the Langlands quotient of

\[
i_{GL_2 \times GL_1} \left( Q \left[ [\nu^{1/2} \chi], [\nu^{-1/2} \chi] \right] \otimes \chi_3 \otimes \chi_4 \right) = i_{GL_2 \times GL_1 \times GL_4} ((\chi \circ \det) \otimes \chi_3 \otimes \chi_4).
\]

We have

\[
N_{GL_4(C) \times GL_4(C)} = (0_{4 \times 4}, 0) \quad N_{GSO_6(C)} = 0_{6 \times 6}.
\]

Note that the other constituent of this induced representation, which is generic, is

\[
Q \left[ [\nu^{-1/2} \chi], [\nu^{1/2} \chi], [\chi_3], [\chi_4] \right] = i_{GL_2 \times GL_1 \times GL_4} \left( Q \left[ [\nu^{-1/2} \chi], [\nu^{1/2} \chi] \right] \otimes \chi_3 \otimes \chi_4 \right)
\]

\[
= i_{GL_2 \times GL_1 \times GL_4} ((\chi \otimes \chi) \otimes \chi_3 \otimes \chi_4).
\]

The next case is when there are two pairs, say \( \chi_1 = \nu \chi, \chi_2 = \chi, \) and \( \chi_3 = \nu^{-1} \chi \) for some \( \chi \in (F^\times)^D \) and \( \chi_4 \neq \nu^{\pm 1} \chi_i \) for \( i = 1, 2, 3 \). Then we have the following three non-generic constituents:

\[
Q \left[ [\nu \chi], [\chi, [\nu^{-1} \chi], [\chi_4] \right] = i_{GL_3 \times GL_1} ((\chi \circ \det) \otimes \chi_3 \otimes \chi_4);
\]

(5.32)

\[
Q \left[ [\nu \chi], [\chi, [\nu^{-1} \chi], [\chi_4] \right];
\]

(5.33)

\[
Q \left[ [\nu \chi], [\chi, [\nu^{-1} \chi], [\chi_4] \right] .
\]

(5.34)

For (5.32) we have

\[
N_{GL_4(C) \times GL_4(C)} = (0_{4 \times 4}, 0) \quad N_{GSO_6(C)} = 0_{6 \times 6},
\]

for (5.33) we have

\[
N_{GL_4(C) \times GL_4(C)} = \left( \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, 0 \right) \quad N_{GSO_6(C)} = \left[ \begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right],
\]

(5.10)

and for (5.34) we have

\[
N_{GL_4(C) \times GL_4(C)} = \left( \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, 0 \right) \quad N_{GSO_6(C)} = \left[ \begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].
\]

(5.10)

Finally, in the case where we have three pairs we are in the situation of (5.27). Then we have the following seven non-generic constituents:

\[
Q \left[ [\nu^{3/2} \chi], [\nu^{1/2} \chi], [\nu^{-1/2} \chi], [\nu^{-3/2} \chi] \right] = \chi \circ \det;
\]

(5.35)

\[
Q \left[ [\nu^{1/2} \chi], [\nu^{1/2} \chi], [\nu^{-1/2} \chi], [\nu^{-3/2} \chi] \right];
\]

(5.36)
(5.37)
\[ Q\left(\{\nu^{3/2}, \nu^{-1/2}, \nu^{1/2}, \nu^{-3/2}\}\right); \]

(5.38)
\[ Q\left(\{\nu^{3/2}, \nu^{-1/2}, \nu^{-3/2}, \nu^{-1/2}\}\right); \]

(5.39)
\[ Q\left(\{\nu^{1/2}, \nu^{3/2}, \nu^{-3/2}, \nu^{-1/2}\}\right); \]

(5.40)
\[ Q\left(\{\nu^{-1/2}, \nu^{1/2}, \nu^{-3/2}, \nu^{3/2}\}\right); \]

(5.41)
\[ Q\left(\{\nu^{3/2}, \nu^{-1/2}, \nu^{-3/2}, \nu^{1/2}\}\right). \]

For (5.35) we have
\[ N_{GL_4(C) \times GL_4(C)} = (0_{4 \times 4}, 0) \]

(5.10)
\[ \Rightarrow \quad N_{GSO_6(C)} = 0_{6 \times 6}. \]

for (5.36) we have
\[ N_{GL_4(C) \times GL_4(C)} = \left(\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}\right) \]

(5.10)
\[ \Rightarrow \quad N_{GSO_6(C)} = \left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}\right), \]

for (5.37) we have
\[ N_{GL_4(C) \times GL_4(C)} = \left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}\right) \]

(5.10)
\[ \Rightarrow \quad N_{GSO_6(C)} = \left(\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}\right). \]

for (5.38) we have
\[ N_{GL_4(C) \times GL_4(C)} = \left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{array}\right) \]

(5.10)
\[ \Rightarrow \quad N_{GSO_6(C)} = \left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}\right), \]

for (5.39) we have
\[ N_{GL_4(C) \times GL_4(C)} = \left(\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{array}\right) \]

(5.10)
\[ \Rightarrow \quad N_{GSO_6(C)} = \left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}\right), \]

for (5.40) we have
\[ N_{GL_4(C) \times GL_4(C)} = \left(\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}\right) \]

(5.10)
\[ \Rightarrow \quad N_{GSO_6(C)} = \left(\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}\right). \]
and for (5.41) we have

$$N_{GL_4(C) \times GL_1(C)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

with non-generic

$$N_{GSO_6(C)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$ 

**nongrn-(B)** $M \cong GL_2 \times GL_1 \times GL_1$ and $\tilde{M} = (GL_2 \times GL_1 \times GL_1) \times GL_1$.

Given $\sigma_0 \in \text{Irr}(GL_2)$ and $\chi_1, \chi_2 \in (F^\times)^D$, we consider

$$i_M^{\text{GSpin}_4}(\sigma_0 \boxtimes \chi_1 \boxtimes \chi_2).$$

Write

$$\sigma_0 \boxtimes \chi_1 \boxtimes \chi_2 = (\sigma_0 \boxtimes \tilde{\chi}_1 \boxtimes \tilde{\chi}_2 \boxtimes \tilde{\eta}) |_M$$

with $\sigma_0 \in \text{Irr}(GL_2)$ and $\tilde{\chi}_1, \tilde{\eta} \in (F^\times)^D$. By (5.16), it follows that (5.42) contains a non-generic constituent if and only if its lift

$$i_M^{GL_2 \times GL_1}(\sigma_0 \boxtimes \tilde{\chi}_1 \boxtimes \tilde{\chi}_2)$$

contains a non-generic constituent if and only if

$$i_M^{GL_2 \times GL_1}(\sigma_0 \boxtimes \tilde{\chi}_1 \boxtimes \tilde{\chi}_2)$$

does. Recalling **nongrn-(A)**, it is sufficient to consider the case of $\sigma_0 \in \text{Irr}(GL_2)$, $\tilde{\chi}_1 = \nu^{1/2} \tilde{\chi}$, and $\tilde{\chi}_2 = \nu^{-1/2} \tilde{\chi}$ for $\tilde{\chi} \in (F^\times)^D$, where the segment $\Delta_{\tilde{\sigma}_0}$ of $\tilde{\sigma}_0$ does not precede either $\tilde{\chi}_1$ or $\tilde{\chi}_2$. We then have the following sole non-generic constituent:

$$Q(\Delta_{\tilde{\sigma}_0}, [\nu^{1/2} \tilde{\chi}], [\nu^{-1/2} \tilde{\chi}]).$$

We have

$$N_{GL_4(C) \times GL_1(C)} = (0_{4 \times 4}, 0)$$

and $N_{GSO_6(C)} = 0_{6 \times 6}$.

**nongrn-(C)** $M \cong GL_3 \times GL_1$ and $\tilde{M} = (GL_3 \times GL_1) \times GL_1$.

Given a non-generic $\sigma_0 \in \text{Irr}(GL_3)$ and any $\chi \in (F^\times)^D$, we consider

$$i_M^{\text{GSpin}_4}(\sigma_0 \boxtimes \chi).$$

Write

$$\sigma_0 \boxtimes \chi = (\sigma_0 \boxtimes \tilde{\chi} \boxtimes \tilde{\eta}) |_M$$

with non-generic $\sigma_0 \in \text{Irr}(GL_3)$ and $\tilde{\chi}, \tilde{\eta} \in (F^\times)^D$. As in (5.20) we have

$$\sigma_0 = \sigma_0 \tilde{\chi}, \quad \tilde{\eta} = \chi_2 \tilde{\chi}^{-2}.$$ 

As before, (5.46) contains a non-generic constituent if and only if its lift

$$i_M^{GL_3 \times GL_1}(\sigma_0 \boxtimes \tilde{\chi})$$

also contains one if and only if

$$i_M^{GL_1}(\sigma_0 \boxtimes \tilde{\chi})$$

does. To have a non-generic $\sigma_0$ of $GL_3(F)$, the irreducible representation $\tilde{\sigma}_0$ must be some constituent in a reducible induction. This case has been covered in **nongrn-(A)** and (B) above.

**nongrn-(D)** $M \cong GL_4 \times \text{GSpin}_4$ and $\tilde{M} = (GL_2 \times GL_2) \times GL_1$.

Given a non-generic $\sigma_0 \in \text{Irr}(\text{GSpin}_4)$, by Section 4.3, we know that it must be of the form

$$\text{Res}_{\text{GSpin}_4}^{GL_2 \times GL_2}(\chi \circ \text{det} \boxtimes \tilde{\sigma})$$

for $\tilde{\sigma} \in \text{Irr}(GL_2)$. For $\eta \in (F^\times)^D$, the induced representation

$$i_M^{\text{GSpin}_4}(\chi \circ \text{det} \boxtimes \tilde{\sigma} \boxtimes \eta)$$

(5.49)
contains a non-generic constituent if and only if so does
$$i_{GL_1 \times GL_1}^G((\chi \circ \det) \boxtimes \tilde{\sigma}),$$
which is always the case. Therefore, if \( \tilde{\sigma} \) is supercuspidal, then
$$N_{GL_1(\mathbb{C}) \times GL_1(\mathbb{C})} = (0_{4 \times 4}, 0).$$
If \( \tilde{\sigma} \) is non-supercuspidal, then it suffices to consider the case \( \tilde{\sigma} = \text{St}_{GL_2} \otimes \eta \) with \( \eta \in (F^\times)^D \) since the other case has been covered in \text{ngnr}(A). Thus, we have
$$N_{GL_4(\mathbb{C}) \times GL_1(\mathbb{C})} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$
\text{ngnr}(E) \( M \cong GSpin_6 \) and \( \tilde{M} = GL_4 \times GL_1 \).
Given a non-generic \( \sigma \in \text{Irr}(GSpin_6) \), it must be of the form
$$\text{Res}_{GSpin_6}^{GL_4 \times GL_1}(\tilde{\chi}, \tilde{\eta}) = \chi \circ \det,$$
for some \( \tilde{\chi}, \tilde{\eta} \in (F^\times)^D \). This is the case \( Q([\nu^{3/2} \tilde{\chi}], [\nu^{1/2} \tilde{\chi}], [\nu^{-1/2} \tilde{\chi}], [\nu^{-3/2} \tilde{\chi}]) \) in \text{ngnr}(A).

5.4. Computation of the Adjoint L-function for \( GSpin_4 \). We now give explicit expressions for the adjoint \( L \)-function of each of the representations of \( GSpin_6(F) \). Recall that if we have a parameter \( (\phi, N) \) with \( N \) a nilpotent matrix on the vector space \( V \), then its adjoint \( L \)-function is
$$L(s, \phi, \text{Ad}) = \det (1 - q^{-s} \text{Ad}(\phi)|V_N^\perp),$$
where \( V_N = \text{ker}(N) \), \( V^\perp \) the vectors fixed by the inertia group, and \( V_N^\perp = V^\perp \cap V_N \). Below for the cases where \( N \) is non-zero, we write \( \text{ker}(\text{Ad}(N)) \) and we use \( L_\alpha \) to denote the root group associated with the root \( \alpha \).

We now consider each case. Using (2.14) and Sections 5.2, and 5.3, we have the following.
\text{ngnr}(a) Given \( \sigma \in \text{Irr}_{nc}(GSpin_6) \), we have \( \tilde{\sigma} = \tilde{\sigma}_0 \boxtimes \tilde{\eta} \in \text{Irr}_{nc}(GL_4 \times GL_1) \). Then
$$L(s, 1_{F^\times}, L(s, \sigma, \text{Ad}) = L(s, \tilde{\sigma}_0, \text{Ad}^{\text{GL}_4}_{\tilde{M}})$$
or
$$L(s, \sigma, \text{Ad}) = L(s, \tilde{\sigma}_0, \text{Ad}).$$
\text{ngnr}(I) Given \( M \cong GL_1 \times GL_1 \times GL_1 \times GL_1 \) and \( \tilde{M} = (GL_1 \times GL_1 \times GL_1 \times GL_1) \times GL_1 \), we recall
$$i_{GL_1 \times GL_1 \times GL_1 \times GL_1}^{GL_4 \times GL_1 \times GL_1}(\tilde{\chi}_1 \boxtimes \tilde{\chi}_2 \boxtimes \tilde{\chi}_3 \boxtimes \tilde{\chi}_4)$$
must be irreducible. Thus, given \( \sigma \in \text{Irr}(GSpin_6) \) such that
$$\sigma = i_{M}^{GSpin_6}(\tilde{\chi}_1 \boxtimes \tilde{\chi}_2 \boxtimes \tilde{\chi}_3 \boxtimes \tilde{\chi}_4),$$
we have
$$L(s, \sigma, \text{Ad}) = (s)^3 \prod_{i \neq j} L(s, \tilde{\chi}_i \tilde{\chi}_j^{-1}).$$
\text{ngnr}(II) Given \( M \cong GL_2 \times GL_1 \times GL_1 \) and \( \tilde{M} = (GL_2 \times GL_1 \times GL_1) \times GL_1 \), for \( \sigma_0 \in \text{Irr}_{eq}(GL_2) \) and \( \chi_1, \chi_2 \in (F^\times)^D \), we have an irreducible induced representation
$$\sigma = i_{M}^{GSpin_6}(\sigma_0 \boxtimes \chi_1 \boxtimes \chi_2) = \text{Res}_{GSpin_6}^{GL_2 \times GL_1 \times GL_1}(i_{GL_2 \times GL_1 \times GL_1}^{GL_4 \times GL_1}(\sigma_0 \boxtimes \tilde{\chi}_1 \boxtimes \tilde{\chi}_2 \boxtimes \tilde{\eta})), $$
for some \( \tilde{\sigma}_0 \in \text{Irr}_{eq}(GL_2) \), and \( \tilde{\chi}_1, \tilde{\eta} \in (F^\times)^D \). For supercuspidal \( \tilde{\sigma}_0 \) we have
$$L(s, \sigma, \text{Ad}) = L(s)^2 L(s, \tilde{\sigma}_0, \text{Ad}) L(s, \tilde{\sigma}_0 \times \tilde{\chi}_1^{-1}) L(s, \tilde{\sigma}_0 \times \tilde{\chi}_2^{-1}) L(s, \tilde{\sigma}_0 \times \tilde{\chi}_3^{-1}) L(s, \tilde{\sigma}_0 \times \tilde{\chi}_4^{-1}).$$
For non-supercuspidal $\tilde{\sigma}_0 \in \text{Irr}(\text{GL}_2)$, i.e., $\sigma_0 = \text{St}_{\text{GL}_2} \otimes \tilde{\chi}$ for some $\tilde{\chi} \in (F^\times)^D$, we have
\[
\ker \left( \text{ad} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) = \left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{bmatrix}, \begin{bmatrix} L_{f_1-f_2}, L_{f_1-f_3}, L_{f_2-f_4}, L_{f_3-f_4}, L_{f_4-f_2}, L_{f_4-f_3} \end{bmatrix} \right\}.
\]
\[(5.51)\]

It follows that
\[
L(s, \sigma, \text{Ad}) = L(s)^2 L(s+1)L(s+1, \tilde{\chi}^{-1})L(s+1, \tilde{\chi}_2^{-1}) \cdot L(s, \tilde{\chi}_1^{-1})L(s, \tilde{\chi}_2^{-1})L(s, \tilde{\chi}_3^{-1}).
\]

\text{gnc-(III)} Given $\text{M} \cong \text{GL}_3 \times \text{GL}_1$ and $\widehat{\text{M}} = (\text{GL}_3 \times \text{GL}_1) \times \text{GL}_1$, for $\sigma_0 \in \text{Irr}_\text{eq} (\text{GL}_3)$ and $\chi \in (F^\times)^D$, we have an irreducible induced representation
\[
\sigma = i_M^{\text{GSpin}_3} (\sigma_0 \otimes \chi) = \text{Res}_{G_{\text{Spin}_3} \times \text{GL}_1} \left( i_{\text{GL}_3 \times \text{GL}_1} (\sigma_0 \otimes \tilde{\chi} \otimes \tilde{\eta}) \right),
\]

for $\tilde{\sigma}_0 \in \text{Irr}_\text{eq} (\text{GL}_3)$ and $\tilde{\chi}, \tilde{\eta} \in (F^\times)^D$. If $\tilde{\sigma}_0 \in \text{Irr}_\text{eq} (\text{GL}_3)$ is supercuspidal, then we have

\[
L(s, \sigma, \text{Ad}) = L(s) L(s, \tilde{\sigma}_0, \text{Ad}) L(s, \tilde{\sigma}_0 \times \tilde{\chi}) L(s, \tilde{\sigma}_0 \times \tilde{\chi}^\vee) L(s, \tilde{\sigma}_0 \times \tilde{\chi}^{-1}) L(s, \tilde{\sigma}_0 \times \tilde{\chi}^{-2}).
\]

For non-supercuspidal $\tilde{\sigma}_0 \in \text{Irr}_\text{eq} (\text{GL}_3)$, i.e., $\sigma_0 = \text{St}_{\text{GL}_3} \otimes \tilde{\chi}_0$ for some $\tilde{\chi}_0 \in (F^\times)^D$, we have

\[
\ker \left( \text{ad} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) = \left\{ \begin{bmatrix} a & c & 0 & 0 \\ 0 & a & c & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{bmatrix}, \begin{bmatrix} L_{f_1-f_2}, L_{f_1-f_3}, L_{f_2-f_4}, L_{f_3-f_4}, L_{f_4-f_2}, L_{f_4-f_3} \end{bmatrix} \right\}.
\]
\[(5.52)\]

It follows that
\[
L(s, \sigma, \text{Ad}) = L(s) L(s+1) L(s+1, \tilde{\chi}_0^{-1}) L(s+1, \tilde{\chi}_1^{-1}) L(s+1, \tilde{\chi}_2^{-1}) L(s+1, \tilde{\chi}_3^{-1}).
\]

\text{gnc-(IV)} Given $\text{M} \cong \text{GL}_1 \times \text{GSpin}_4$ and $\widehat{\text{M}} = (\text{GL}_2 \times \text{GL}_2) \times \text{GL}_1$, we have the representation $(5.24)$
\[
\sigma = i_M^{\text{GSpin}_4} (\chi \otimes \sigma_0)
\]

with $\sigma_0 \in \text{Irr}_\text{eq} (\text{GSpin}_4)$, and $\chi \in (F^\times)^D$. We have the irreducible $i_M^{\text{GL}_2 \times \text{GL}_2} (\tilde{\sigma}_1 \otimes \tilde{\sigma}_2)$ as in $(5.26)$, where $\chi \otimes \sigma_0 \subset (\tilde{\sigma}_1 \otimes \tilde{\sigma}_2) \otimes (\tilde{\eta} \otimes \tilde{\eta}) |_{\text{M}}$ with $\tilde{\sigma}_1 \in \text{Irr}_\text{eq} (\text{GL}_2)$, $\tilde{\eta} \in (F^\times)^D$. Thus, if $\sigma_0$ is supercuspidal (and hence so are $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$) we have

\[
L(s, \sigma, \text{Ad}) = L(s) L(s, \tilde{\sigma}_1, \text{Ad}) L(s, \tilde{\sigma}_2, \text{Ad}) L(s, \tilde{\sigma}_1 \times \tilde{\sigma}_2^\vee) L(s, \tilde{\sigma}_1 \times \tilde{\sigma}_1).
\]

If $\sigma_0$ is non-supercuspidal, with $\tilde{\sigma}_1$ supercuspidal and $\tilde{\sigma}_2$ non-supercuspidal, i.e., $\tilde{\sigma}_2 = \text{St}_{\text{GL}_2} \otimes \tilde{\chi}$ for some $\tilde{\chi} \in (F^\times)^D$, we have

\[
\ker \left( \text{ad} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) = \left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{bmatrix}, \begin{bmatrix} L_{f_1-f_2}, L_{f_2-f_4}, L_{f_3-f_4}, L_{f_4-f_1}, L_{f_4-f_2}, L_{f_4-f_3} \end{bmatrix} \right\},
\]
\[(5.53)\]

and it then follows that
\[
L(s, \sigma, \text{Ad}) = L(s) L(s+1) L(s, \tilde{\sigma}_1, \text{Ad}) L(s + \frac{1}{2} \tilde{\sigma}_1 \otimes \tilde{\chi}) L(s + \frac{1}{2} \tilde{\sigma}_1 \times \tilde{\chi}^{-1}).
\]

If $\sigma_0$ is non-supercuspidal, with $\tilde{\sigma}_1$ non-supercuspidal and $\tilde{\sigma}_2$ supercuspidal, i.e., $\tilde{\sigma}_1 = \text{St}_{\text{GL}_2} \otimes \tilde{\chi}$ for some $\tilde{\chi} \in (F^\times)^D$, then $\ker(\text{ad}(N))$ is as in $(5.51)$ and we have
\[
L(s, \sigma, \text{Ad}) = L(s) L(s+1) L(s, \tilde{\sigma}_2, \text{Ad}) L(s + \frac{1}{2} \tilde{\sigma}_2 \otimes \tilde{\chi}) L(s + \frac{1}{2} \tilde{\sigma}_2 \times \tilde{\chi}^{-1}).
\]
If both $\tilde{\sigma}_{1}$ and $\tilde{\sigma}_{2}$ are non-supercuspidal, i.e., $\tilde{\sigma}_{i} = \text{St}_{2} \otimes \tilde{\chi}_{i}$ with $\tilde{\chi}_{1}, \tilde{\chi}_{2} \in (F^{*})^{D}$ satisfying $\tilde{\chi}_{1} \neq \tilde{\chi}_{2}^{\nu+1}$, we have

$$\ker \left( \text{ad} \begin{bmatrix} 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \end{bmatrix} \right) = \left\langle \begin{bmatrix} a & 0 & c & 0 \\
0 & a & 0 & c \\
d & 0 & b & 0 \\
0 & d & 0 & b \end{bmatrix}, L_{f_{1}-f_{2}}, L_{f_{1}-f_{4}}, L_{f_{3}-f_{2}}, L_{f_{3}-f_{4}} \right\rangle,$$  

and it follows that

$$L(s, \sigma, \text{Ad}) = L(s)L(s+1)^{2}L(s+1, \tilde{\chi}_{1}\tilde{\chi}_{2}^{-1})L(s+1, \tilde{\chi}_{1}^{-1}\tilde{\chi}_{2})L(s, \tilde{\chi}_{1}\tilde{\chi}_{2}^{-1}).$$

**gln**: Given $M \cong \text{GL}_{4} \times \text{GSpin}_{4}$ and $\tilde{M} = (\text{GL}_{2} \times \text{GL}_{2}) \times \text{GL}_{1}$, we consider $\sigma \in \text{Irr}_{\text{reg}}(\text{GSpin}_{4})$ and $\tilde{\sigma} \in \text{Irr}_{\text{reg}}(\text{GL}_{4})$ and $\tilde{\eta} \in (F^{*})^{D}$ such that $\sigma \subset (\tilde{\sigma} \boxtimes \tilde{\eta})|_{M}$. Then, $\tilde{\sigma}$ must be either (5.27) or (5.28).

For (5.27) (i.e., $\tilde{\sigma} = \text{St}_{\text{GL}_{4}} \otimes \tilde{\chi}$), we have

$$\ker \left( \text{ad} \begin{bmatrix} 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \end{bmatrix} \right) = \left\langle \begin{bmatrix} a & b & c & 0 \\
a & 0 & b & c \\
0 & a & 0 & b \\
0 & 0 & a & b \end{bmatrix}, L_{f_{1}-f_{4}} \right\rangle,$$  

and it follows that

$$L(s, \sigma, \text{Ad}) = L(s+3)L(s+2)L(s+1).$$

For (5.28) (i.e., $\tilde{\sigma} \in \text{Irr}_{\text{rc}}(\text{GL}_{2})$), we have

$$\ker \left( \text{ad} \begin{bmatrix} 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{bmatrix} \right) = \left\langle \begin{bmatrix} a & c & 0 & 0 \\
d & b & 0 & 0 \\
0 & 0 & a & c \\
0 & 0 & d & b \end{bmatrix}, L_{f_{1}-f_{3}}, L_{f_{1}-f_{4}}, L_{f_{3}-f_{2}}, L_{f_{3}-f_{4}} \right\rangle,$$  

and it follows that

$$L(s, \sigma, \text{Ad}) = L(s, \tilde{\sigma}, \text{Ad})L(s, \tilde{\sigma} \times \tilde{\sigma}^{\vee}).$$

**nongnrt**: For $Q([\nu^{1/2}]_{\tilde{\chi}}, [\nu^{-1/2}]_{\tilde{\chi}}, [\tilde{\chi}_{3}], [\tilde{\chi}_{4}])$ (5.31), we have

$$L(s, \sigma, \text{Ad}) = L(s)^{2}L(s+1)L(s-1)L(s, \tilde{\chi}_{3}\tilde{\chi}_{4}^{-1})L(s, \tilde{\chi}_{3}^{-1}\tilde{\chi}_{4})$$

$$\prod_{i=3}^{4} \left( L(s+1/2, \tilde{\chi}_{i}^{-1})L(s-1/2, \tilde{\chi}_{i}^{-1})L(s-1/2, \tilde{\chi}_{i}^{-1})L(s+1/2, \tilde{\chi}_{i}^{-1}) \right)$$

For $Q([\nu \tilde{\chi}], [\tilde{\chi}], [\nu^{-1}]_{\tilde{\chi}}, [\tilde{\chi}_{4}])$ (5.32), we have

$$L(s, \sigma, \text{Ad}) = L(s)^{3}L(s+1)^{2}L(s-2) \prod_{t=0,1,-1} \left( L(s+t, \tilde{\chi}_{4}^{-1})L(s+t, \tilde{\chi}_{4}^{-1}) \right),$$

For $Q([\tilde{\chi}, \nu \tilde{\chi}], [\nu^{-1} \tilde{\chi}], [\tilde{\chi}_{4}])$ (5.33), we have $\ker(\text{ad}(N))$ as in (5.51) and

$$L(s, \sigma, \text{Ad}) = L(s)^{2}L(s-1)^{2}L(s-2) \prod_{t=-1,0} L(s+t, \tilde{\chi}_{4}^{-1}) \prod_{t=\pm 1} L(s+t, \tilde{\chi}_{4}^{-1}).$$

For $Q([\nu \tilde{\chi}], [\tilde{\chi}, \nu^{-1} \tilde{\chi}], [\tilde{\chi}_{4}])$ (5.34), since

$$\ker \left( \text{ad} \begin{bmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{bmatrix} \right) = \left\langle \begin{bmatrix} a & 0 & 0 & 0 \\
0 & b & 0 & 0 \\
0 & 0 & b & 0 \\
0 & 0 & 0 & c \end{bmatrix}, L_{f_{1}-f_{3}}, L_{f_{1}-f_{4}}, L_{f_{2}-f_{1}}, L_{f_{2}-f_{3}}, L_{f_{2}-f_{4}}, L_{f_{3}-f_{1}}, L_{f_{3}-f_{2}} \right\rangle,$$  

we have

$$L(s, \sigma, \text{Ad}) = L(s)^{2}L(s+2)L(s-1)L(s+1) \prod_{t=0,1} L(s+t, \tilde{\chi}_{4}^{-1}) \prod_{t=\pm 1} L(s+t, \tilde{\chi}_{4}^{-1}).$$
For $Q([\nu^{3/2} \chi], [\nu^{-1/2} \chi], [\nu^{-3/2} \chi])$ (5.35), we have

$$L(s, \sigma, Ad) = L(s)^3 L(s+1)^3 L(s-1)^3 L(s+2)^2 L(s-2)^2 L(s+3)L(s-3).$$

For $Q([\nu^{3/2} \chi], [\nu^{1/2} \chi], [\nu^{-3/2} \chi])$ (5.36), we have ker(ad($N$)) is as in (5.51) and

$$L(s, \sigma, Ad) = L(s)^2 L(s-1)^2 L(s+1)^2 L(s-2)L(s+2)L(s-3).$$

For $Q([\nu^{3/2} \chi], [\nu^{-1/2} \chi], [\nu^{1/2} \chi], [\nu^{-3/2} \chi])$ (5.37), we have ker(ad($N$)) is as in (5.57) and

$$L(s, \sigma, Ad) = L(s)^2 L(s+1)^2 L(s-1)^2 L(s+2)L(s-2)L(s-3).$$

For $Q([\nu^{3/2} \chi], [\nu^{1/2} \chi], [\nu^{-3/2} \chi], [\nu^{-1/2} \chi])$ (5.38), we have ker(ad($N$)) is as in (5.53) and

$$L(s, \sigma, Ad) = L(s)^2 L(s+1)^2 L(s-1)^2 L(s+2)L(s-2)L(s-3).$$

For $Q([\nu^{-1/2} \chi], [\nu^{1/2} \chi], [\nu^{3/2} \chi], [\nu^{-3/2} \chi])$ (5.39), we have ker(ad($N$)) is as in (5.54) and

$$L(s, \sigma, Ad) = L(s)^2 L(s+1)^2 L(s-1)^2 L(s+2)L(s-2)L(s-3).$$

Finally, for $Q([\nu^{3/2} \chi], [\nu^{-3/2} \chi], [\nu^{-1/2} \chi], [\nu^{1/2} \chi])$ (5.41), since

$$\ker \left( \text{ad} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) = \left\langle \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & 0 & b & c \\ 0 & 0 & 0 & b \end{bmatrix}, L_{f_1-f_4}, L_{f_2-f_1}, L_{f_2-f_4} \right\rangle,$$ (5.58)

we have

$$L(s, \sigma, Ad) = L(s)L(s+1)L(s-1)L(s-2)L(s-3).$$

**nongr-(B)** For $Q([\Delta_{\bar{s}}], [\nu^{1/2} \chi], [\nu^{-1/2} \chi])$ (5.45), with say $[\Delta_{\bar{s}}] = \varepsilon_{GL_2 \times GL_1}^{GL_1} (\bar{\eta}_1, \bar{\eta}_2)$, $\bar{\eta}_1 \bar{\eta}_2^{-1} \neq \nu^\pm 1$ we have

$$L(s, \sigma, Ad) = L(s+1)^3 L(s-1)L(s, \bar{\eta}_1 \bar{\eta}_2^{-1})L(s, \bar{\eta}_1^{-1} \bar{\eta}_2) \prod_{i=1,2} \left( \frac{L(s - 1/2, \bar{\eta}_i \chi^{-1})}{L(s + 1/2, \bar{\eta}_i \chi^{-1})}L(s + 1/2, \bar{\eta}_i^{-1} \chi)L(s - 1/2, \bar{\eta}_i^{-1} \chi) \right).$$

**nongr-(C)** As mentioned before, all the possibilities in this case were covered in (A) and (B) above.

**nongr-(D)** For (5.49) with $\bar{\sigma}$ supercuspidal, we have

$$L(s, \sigma, Ad) = L(s)^2 L(s+1)L(s-1)L(s, \sigma, Ad) \prod_{i=1,2} \left( \frac{L(s - 1/2, \sigma \times \chi^{-1})}{L(s + 1/2, \sigma \times \chi^{-1})}L(s + 1/2, \sigma \times \chi)L(s - 1/2, \sigma \times \chi) \right).$$

For (5.49) with non-supercuspidal $\bar{\sigma} = St_{GL_2} \oplus \eta$, $\eta \in (F^\times)^D$ we have ker(ad($N$)) as in (5.53) and

$$L(s, \sigma, Ad) = L(s)^2 L(s+1)^2 L(s-1)L(s, \chi \eta^{-1})L(s + 1, \chi \eta^{-1})L(s + 1, \chi^{-1} \eta)L(s, \chi^{-1} \eta).$$

Recall that the remaining possibilities in this case were already covered in (A) above.

**nongr-(E)** Finally, as mentioned before, all the possibilities in this case we also covered in (A).

6. **Correction to [AC17]**

We take this opportunity to correct the following errors in our earlier work [AC17]. They do not affect the main results in that paper.
6.1. Proposition 5.5 and 6.4.

- Change “1,2,4,8, if $p = 2$” to “1,2,4,8,..., 2$^{[F:Q_2]}+2$, if $p = 2$.” We have $2^{[F:Q_p]+2}$ due to the fact that $\left| F^*/(F^*)^2 \right| = 2^{[F:Q_2]+2}$.
- For Proposition 5.5, using [GP92, Corollary 7.7], it follows that the case of $p = 2$ is bounded by $\left| (Z/2Z)^{1-1} \right| = 8$. Here 4 is coming from $GSpin_4 \cong GSO(4,\mathbb{C})$.
- For Proposition 6.4, using [GP92, Corollary 7.7], it follows that the case of $p = 2$ is bounded by $\left| (Z/2Z)^{6-1} \right| = 32$. Here 6 is coming from $GSpin_6 \cong GSO(6,\mathbb{C})$.

6.2. Remark 5.11.

- The formula (5.13) should read as follows:
  \[ \left| \Pi_{\varphi}(GSpin_4) \right| = \left| \Pi_{\varphi}(GSpin_4^{1,1}) \right| = 4, \quad \left| \Pi_{\varphi}(GSpin_4^{2,1}) \right| = 1. \tag{5.13} \]
  Also, in the following sentence change “in which case the multiplicity is 2” to “in which case the multiplicity 2 could also occur”. We thank Hengfei Lu [Lu20] for bringing this error to our attention.
- In addition, it is more accurate that we use ‘not irreducible’ rather than ‘reducible’ in this Remark since one may have indecomposable parameters. Alternatively, we may write $\tilde{\varphi}_i|_{W_F}$ is reducible. Thus, at the beginning the Remark, change “When $\tilde{\varphi}_i$ is reducible,” to “When $\tilde{\varphi}_i$ is not irreducible,”.

References


[Zel80] A. V. Zelevinsky. Induced representations of reductive $p$-adic groups. II. On irreducible representations of $GL(n)$.

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*Email address: kchoiy@siu.edu*
Table 1. Representations of $\text{GSpin}_4(F)$

<table>
<thead>
<tr>
<th>$\text{Res}^{\text{GL}_2 \times \text{GL}<em>2}</em>{\text{GSpin}_4}$ of</th>
<th>$L$-packet Structure</th>
<th>generic</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) $\bar{\sigma}_1 \otimes \bar{\sigma}_2$, $\bar{\sigma}_2 \cong \bar{\sigma}_1 \bar{\eta}, \bar{\sigma}<em>1 \in \text{Irr}</em>{\text{eq}}(\text{GL}_2)$</td>
<td>${1}, \mathbb{Z}/2\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^2$</td>
<td>●</td>
</tr>
<tr>
<td>(b) $\bar{\sigma}_1 \boxtimes \bar{\sigma}_2$, $\bar{\sigma}_2 \not\cong \bar{\sigma}_1 \bar{\eta}, \bar{\sigma}<em>1 \in \text{Irr}</em>{\text{eq}}(\text{GL}_2)$</td>
<td>${1}, \mathbb{Z}/2\mathbb{Z}$</td>
<td>●</td>
</tr>
<tr>
<td>(i) $(\text{St}_{\text{GL}<em>2} \boxtimes \text{St}</em>{\text{GL}_2}) = \text{St}^{\text{GSpin}_4}$ (irreducible)</td>
<td>${1}$</td>
<td>●</td>
</tr>
<tr>
<td>(ii) $(\iota^{\text{GL}<em>2}</em>{\text{GL}<em>4} \chi</em>{\text{GL}_4 \times \text{GL}<em>4}(\chi_1 \otimes \chi_2) \boxtimes \text{St}</em>{\text{GL}_2} \otimes \chi)$ (irreducible)</td>
<td>${1}$</td>
<td>●</td>
</tr>
<tr>
<td>(iii) $(\iota^{\text{GL}<em>2}</em>{\text{GL}<em>4} \chi</em>{\text{GL}_4 \times \text{GL}_4}(\chi_1 \otimes \chi_2)) \boxtimes \chi \not\equiv \nu^{\pm 1} \chi_1, \chi_3 \not\equiv \nu^{\pm 1} \chi_4$</td>
<td>${1}, \mathbb{Z}/2\mathbb{Z}$</td>
<td>●</td>
</tr>
<tr>
<td>(iv) $(\bar{\sigma} \boxtimes \text{St}_{\text{GL}<em>2} \otimes \chi)$, $\bar{\sigma} \in \text{Irr}</em>{\text{eq}}(\text{GL}_2)$ (irreducible)</td>
<td>${1}$</td>
<td>●</td>
</tr>
<tr>
<td>(v) $(\bar{\sigma} \boxtimes \iota^{\text{GL}<em>2}</em>{\text{GL}<em>4} \chi</em>{\text{GL}_4 \times \text{GL}<em>4}(\chi_1 \otimes \chi_2))$, $\bar{\sigma} \in \text{Irr}</em>{\text{eq}}(\text{GL}_2)$</td>
<td>${1}, \mathbb{Z}/2\mathbb{Z}$</td>
<td>●</td>
</tr>
<tr>
<td>nongnr $(\chi \circ \det \boxtimes \bar{\sigma})$, $\bar{\sigma} \in \text{Irr}_{\text{eq}}(\text{GL}_2)$ (irreducible)</td>
<td>${1}$</td>
<td>●</td>
</tr>
</tbody>
</table>

Table 2. The adjoint $L$-function $L(s, \sigma, \text{Ad})$ for $\text{GSpin}_4$

<table>
<thead>
<tr>
<th>$L(s, \sigma, \text{Ad})$</th>
<th>$\text{ord}_{s=1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)$&amp;$(b) $L(s, \bar{\sigma}<em>1, \text{Sym}^2 \otimes \omega</em>{\bar{\sigma}_2}^{-1})L(s, \bar{\sigma}<em>2, \text{Sym}^2 \otimes \omega</em>{\bar{\sigma}_2}^{-1})$</td>
<td>0</td>
</tr>
<tr>
<td>(i) $L(s+1)^2$</td>
<td>0</td>
</tr>
<tr>
<td>(ii) $L(s)L(s+1)L(s, \chi_1 \chi_2^{-1})L(s, \tilde{\chi}_1^{-1} \chi_2)$</td>
<td>0</td>
</tr>
<tr>
<td>(iii) $L(s)^2L(s, \chi_1 \chi_2^{-1})L(s, \chi_1 \tilde{\chi}_2)\chi_1 \not\equiv \nu^{\pm 1} \tilde{\chi}_2$</td>
<td>0</td>
</tr>
<tr>
<td>(iv) $L(s+1)L(s, \bar{\sigma}<em>2, \text{Sym}^2 \otimes \omega</em>{\bar{\sigma}_2}^{-1}$</td>
<td>0</td>
</tr>
<tr>
<td>(v) $L(s)L(s, \chi_1 \chi_2^{-1})L(s, \chi_1^{-1} \chi_2)L(s, \bar{\sigma}<em>2, \text{Sym}^2 \otimes \omega</em>{\bar{\sigma}_2}^{-1})$</td>
<td>0</td>
</tr>
<tr>
<td>nongnr $L(s-1)L(s)L(s+1)L(s, \bar{\sigma}, \text{Ad})$</td>
<td>$1 + \text{ord}_{s=1} L(s, \bar{\sigma}, \text{Ad})$</td>
</tr>
</tbody>
</table>

Table 3. Representations of $\text{GSpin}_6(F)$

<table>
<thead>
<tr>
<th>$\text{Res}^{\text{GL}_4 \times \text{GL}<em>1}</em>{\text{GSpin}_6}$ of</th>
<th>generic</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) $\bar{\sigma}_0 \boxtimes \bar{\eta}$, $\bar{\sigma}<em>0 \in \text{Irr}</em>{\text{eq}}(\text{GL}_4)$</td>
<td>●</td>
</tr>
<tr>
<td>(I) $\iota^{\text{GL}_4 \times \text{GL}<em>1}</em>{\text{GL}_4 \times \text{GL}_1 \times \text{GL}_1 \times \text{GL}<em>1} \chi</em>{\text{GL}_4 \times \text{GL}_4 \times \text{GL}_4}(\tilde{\chi}_1 \boxtimes \tilde{\chi}_2 \boxtimes \tilde{\chi}_3 \boxtimes \tilde{\chi}_4 \boxtimes \bar{\eta})$, $\tilde{\chi}_i \not\equiv \nu \tilde{\chi}_j$</td>
<td>●</td>
</tr>
<tr>
<td>(II) $\iota^{\text{GL}_4 \times \text{GL}<em>1}</em>{\text{GL}_4 \times \text{GL}_4 \times \text{GL}<em>4} \chi</em>{\text{GL}_4 \times \text{GL}_4 \times \text{GL}_4}(\tilde{\sigma}_0 \boxtimes \tilde{\chi}_1 \boxtimes \tilde{\chi}_2 \boxtimes \bar{\eta})$, $\tilde{\sigma}<em>0 \in \text{Irr}</em>{\text{eq}}(\text{GL}_2)$, $\tilde{\chi}_1 \not\equiv \nu^{\pm 1} \tilde{\chi}_2$</td>
<td>●</td>
</tr>
<tr>
<td>(III) $\iota^{\text{GL}_4 \times \text{GL}<em>1}</em>{\text{GL}_4 \times \text{GL}_4 \times \text{GL}<em>4} \chi</em>{\text{GL}_4 \times \text{GL}_4 \times \text{GL}_4}(\tilde{\sigma}_0 \boxtimes \tilde{\chi} \boxtimes \bar{\eta})$, $\tilde{\sigma}<em>0 \in \text{Irr}</em>{\text{eq}}(\text{GL}_3)$</td>
<td>●</td>
</tr>
<tr>
<td>(IV) $\iota^{\text{GL}_4 \times \text{GL}<em>1}</em>{\text{GL}_4 \times \text{GL}_4 \times \text{GL}<em>4} \chi</em>{\text{GL}_4 \times \text{GL}_4 \times \text{GL}_4}(\tilde{\sigma}_1 \boxtimes \tilde{\sigma}_2 \boxtimes \bar{\eta})$, $\tilde{\sigma}<em>1 \in \text{Irr}</em>{\text{eq}}(\text{GL}_2)$, $\tilde{\sigma}_1 \not\equiv \nu^{\pm 1} \tilde{\sigma}_2$</td>
<td>●</td>
</tr>
<tr>
<td>(V) $(\tilde{\sigma} \boxtimes \bar{\eta})$, $\tilde{\sigma} \in \text{Irr}_{\text{eq}}(\text{GL}<em>4) \setminus \text{Irr}</em>{\text{eq}}(\text{GL}_4)$</td>
<td>●</td>
</tr>
<tr>
<td>(A) $\iota^{\text{GL}_4 \times \text{GL}<em>1}</em>{\text{GL}_4 \times \text{GL}_4 \times \text{GL}_4 \times \text{GL}<em>4} \chi</em>{\text{GL}_4 \times \text{GL}_4 \times \text{GL}_4 \times \text{GL}_4}(\tilde{\chi}_1 \boxtimes \tilde{\chi}_2 \boxtimes \tilde{\chi}_3 \boxtimes \tilde{\chi}_4 \boxtimes \bar{\eta})$, $\tilde{\chi}_i = \nu \tilde{\chi}_j$</td>
<td>●</td>
</tr>
<tr>
<td>(B) $\iota^{\text{GL}_4 \times \text{GL}<em>1}</em>{\text{GL}_4 \times \text{GL}_4 \times \text{GL}_4 \times \text{GL}<em>4} \chi</em>{\text{GL}_4 \times \text{GL}_4 \times \text{GL}_4 \times \text{GL}_4}(\tilde{\sigma}_0 \boxtimes \tilde{\chi}_1 \boxtimes \tilde{\chi}_2 \boxtimes \bar{\eta})$, $\tilde{\sigma}<em>0 \not\in \text{Irr}</em>{\text{eq}}(\text{GL}_2)$, or $\tilde{\chi}_1 = \nu^{\pm 1} \tilde{\chi}_2$</td>
<td>●</td>
</tr>
<tr>
<td>(C) $\iota^{\text{GL}_4 \times \text{GL}<em>1}</em>{\text{GL}_4 \times \text{GL}_4 \times \text{GL}_4 \times \text{GL}<em>4} \chi</em>{\text{GL}_4 \times \text{GL}_4 \times \text{GL}_4 \times \text{GL}_4}(\tilde{\sigma}_0 \boxtimes \tilde{\chi} \boxtimes \bar{\eta})$, non-generic $\tilde{\sigma}_0 \in \text{Irr}(\text{GL}_3)$</td>
<td>●</td>
</tr>
<tr>
<td>(D) $\iota^{\text{GL}_4 \times \text{GL}<em>1}</em>{\text{GL}_4 \times \text{GL}_4 \times \text{GL}_4 \times \text{GL}<em>4} \chi</em>{\text{GL}_4 \times \text{GL}_4 \times \text{GL}_4 \times \text{GL}_4}(\chi \circ \det \boxtimes \tilde{\sigma} \boxtimes \bar{\eta})$, $\tilde{\sigma} \in \text{Irr}(\text{GL}_2)$</td>
<td>●</td>
</tr>
<tr>
<td>(E) $(\tilde{\chi} \circ \det \boxtimes \bar{\eta})$, $\tilde{\sigma} \in \text{Irr}_{\text{eq}}(\text{GL}<em>4) \setminus \text{Irr}</em>{\text{eq}}(\text{GL}_4)$</td>
<td>●</td>
</tr>
</tbody>
</table>
Table 4. The adjoint $L$-function $L(s, \sigma, Ad)$ for $\text{GSpin}_6$

<table>
<thead>
<tr>
<th>Condition</th>
<th>Expression</th>
<th>Ordinary $s$-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>$\sigma \in \text{Irr}(\text{GSpin}_6(F))$ determined by</td>
<td>$L(s, \sigma, Ad)$</td>
</tr>
<tr>
<td>(I)</td>
<td>$\sigma_0 \in \text{Irr}_{sc}(\text{GL}_4)$</td>
<td>$L(s, \sigma_0, Ad)$</td>
</tr>
<tr>
<td>(II)</td>
<td>$\sigma_0 \in \text{Irr}_{sc}(\text{GL}_2)$</td>
<td>$L(s)^2 L(s, \sigma_0, Ad)L(s, \sigma_0 \times \chi_1^{-1})$</td>
</tr>
<tr>
<td>(III)</td>
<td>$\sigma_0 \in \text{Irr}_{sc}(\text{GL}_3)$</td>
<td>$L(s)(L(s, \sigma_0, Ad)L(s, \sigma_0 \times \chi_1^{-1}))$</td>
</tr>
<tr>
<td>(IV)</td>
<td>$\tau \in \text{Irr}_{sc}(\text{GL}_2)$</td>
<td>$L(s)(\nu \sigma_0 \times \chi_1^{-1})$</td>
</tr>
<tr>
<td>(A)</td>
<td>$Q \left([\nu^{1/2} \chi], [\nu^{-1/2} \chi], [\chi_3], [\chi_4]\right)$</td>
<td>$L(s-1)(L(s+1)^2(L(s, \chi_3 \chi_4^{-1})L(s, \chi_3^{-1} \chi_4))$</td>
</tr>
<tr>
<td>(B)</td>
<td>$Q \left([i^{GL}_B(\bar{n}_1 \otimes \bar{n}_2)], [\chi^{1/2}], [\chi^{1/2}]; n_1 \neq \nu^{11}, \bar{n}_2 \neq \nu^{11}\right)$</td>
<td>$L(s-1)(L(s+1)^2(L(s, \eta \eta_2^{-1})L(s, \eta_1^{-1} \eta_2))$</td>
</tr>
</tbody>
</table>

For each expression, the ordinary $s$-value is indicated.