REPRESENTATIONS OF THE p-ADIC GSpin_4 AND GSpin_6 AND THE ADJOINT L-FUNCTION

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Abstract. We prove a conjecture of B. Gross and D. Prasad about determination of generic L-packets in terms of the analytic properties of the adjoint L-function for p-adic general even spin groups of semi-simple ranks 2 and 3. We also explicitly write the adjoint L-function for each L-packet in terms of the local Langlands L-functions for the general linear groups.

1. Introduction

In this article, we provide further details on the local L-packets for the non-Archimedean split general spin groups GSpin_4 and GSpin_6, following our earlier work [AC17]. We then use our explicit description of these L-packets to prove a conjecture of B. Gross and D. Prasad [Gr22, GP92] determining which of the L-packets are "generic" (i.e., contain an irreducible representation with a Whittaker model) in terms of the analytic properties at $s = 1$ of the adjoint L-function of the packet. We also write the adjoint L-function for each L-packet in terms of the local Langlands L-functions of the general linear groups. In addition to details about the representations that our results provide, given that the adjoint L-functions have a significant role in the Gan-Gross-Prasad conjectures, we expect that our results in this paper would be helpful in that direction as well. Particularly striking is the generalization of the Gan-Gross-Prasad to the non-tempered case [GGP20] where the relevant adjoint L-function does have a pole at $s = 1$.

Let F be a p-adic field of characteristic zero. Denote by W_F the Weil group of F and let $W_F' = W_F \times \mathrm{SL}_2(\mathbb{C})$ be the Weil-Deligne group of F. Let G be a connected, reductive, linear algebraic group over F. The local Langlands Conjecture (LLC) predicts a surjective, finite-to-one map $\mathcal{L}$ from the set $\mathrm{Irr}(G)$ of equivalence classes of irreducible, smooth, complex representations of $G(F)$ to the set $\Phi(G)$ of $\widehat{G}$-conjugacy classes of L-parameters of $G(F)$, i.e., admissible homomorphisms $\phi : W_F' \longrightarrow {}^L G$. Here, ${}^L G$ denotes the L-group of G with $\widehat{G} = {}^L G^0$ its connected component, i.e., the complex dual of G [Bor79]. Among other properties, the map $\mathcal{L}$ is supposed to preserve the local L-, ϵ-, and γ-factors. Moreover, the (finite) fibers Π_ϕ, for $\phi \in \Phi(G)$, of the map $\mathcal{L}$ are called the L-packets of G and their structures are expected to be controlled by certain finite subgroups of $\widehat{G}$.

Consider the split general spin groups $G = \mathrm{GSpin}_4$ and $G = \mathrm{GSpin}_6$, of type $D_2 = A_1 \times A_2$ and $D_3 = A_3$ respectively, whose algebraic structure we review in Section 2.3. We constructed most of the L-packets for these two groups in [AC17] and proved that they satisfy the expected properties of preservation of the local factors and their internal structure. We review and complete the construction of these L-packets. In particular, using the classification of representations of GL_n, we give more explicit descriptions of the L-packets for GSpin_4 and GSpin_6 in terms of given representations of $\mathrm{GL}_2 \times \mathrm{GL}_2$ and $\mathrm{GL}_4 \times GL_1$, respectively. As a byproduct, we are able to give the criteria for determining the size of the L-packets for GSpin_4 and GSpin_6 (see Sections 4 and 5).

The known cases of the LLC for the p-adic groups include GL_n[HT01, Hen00, Sch13]; SL_n [GK82]; non-quasi-split F-inner forms of GL_n and SL_n [HS12, ABPS16]; GSp_4 and Sp_4 [GT11, GT10]; non-quasi-split F-inner form $\mathrm{GSp}_{1,1}$ of GSp_4 [GT14]; $\mathrm{Sp}_{2n}, \mathrm{SO}_n$, and quasi-split SO_{2n}^* [Art13]; U_n [Rog90, Mok15]; non quasi-split F-inner forms of U_n [Rog90, KMSW14]; non-quasi-split F-inner form $\mathrm{Sp}_{1,1}$ of Sp_4 [Cho17]; $\mathrm{GSpin}_4, \mathrm{GSpin}_6$ and their inner forms [AC17]; GSp_{2n} and GO_{2n} [Xu18].

Going back to the case of general G, assume that ρ is a finite-dimensional complex representation of ${}^L G$. When LLC is known, one can define the local Langlands L-functions

$$L(s, \pi, \rho) = L(s, \rho \circ \phi)$$

for each $\pi \in \Pi_\phi$. Here, the L-factors on the right hand side are the Artin local factors associated to the given representation of W'_F.

B. Gross and D. Prasad conjectured (in the generality of quasi-split groups) that the local L-packet $\Pi_\phi(G)$ is generic if and only if the adjoint L-function $L(s, \mathrm{Ad} \circ \phi)$ is regular at $s = 1$ [GP92, Conj. 2.6]. Here, Ad denotes the adjoint representation of ${}^LG$ on the dual Lie algebra $\widehat{\mathfrak{g}}$ of $\widehat{G}$. (Note that in the body of this paper we use Ad exclusively for the restriction of the adjoint representation to the derived group of $\widehat{\mathfrak{g}}$ to distinguish it from the full adjoint L-function, which would have an extra factor of the L-function for the trivial character when $\widehat{\mathfrak{g}}$ has a one-dimensional center.)

We prove the above conjecture for the groups GSpin_4 and GSpin_6 as a consequence of our construction of the L-packets for these groups. In fact, we prove the conjecture for a larger class of groups $G = G^{r,s}_{m,n}$, which are given as subgroups of $\mathrm{GL}_m \times \mathrm{GL}_n$ satisfying a certain determinant equality (2.6). We are able to work in the slightly larger generality because, as in the construction of the L-packets, we use the approach of restricting representations from $\mathrm{GL}_m(F) \times \mathrm{GL}_n(F)$ to the subgroup G.

Moreover, we also give the adjoint L-function in all cases explicitly in terms of local Langlands L-functions of the general linear groups. While we are able to prove the Gross-Prasad conjecture already without the explicit knowledge of the adjoint L-function, the explicit description of the adjoint L-function certainly also verifies the conjecture and we include it here since it may lead to other number theoretic or representation theoretic results.

Finally, we take this opportunity to correct a few inaccuracies in [AC17]. They do not affect the main results in that paper and fix some errors in our description of the L-packets. The details are given in Section 6.

Acknowledgements. We are grateful to Behrang Noohi and Ralf Schmidt for helpful discussions. We also thank B. Gross for his interest in this paper and clarifying the history of his conjecture and the context in which it was made.

K. Choiy was supported by a gift from the Simons Foundation (#840755).

2. Preliminaries

2.1. **Local Langlands Correspondence (LLC).** Let p be a prime number and let F be a p-adic field of characteristic zero, i.e., a finite extension of $\mathbb{Q}_p$. We fix an algebraic closure $\bar{F}$ of F. Denote the ring of integers of F by $\mathcal{O}_F$ and its unique maximal ideal by $\mathcal{P}_F$. Moreover, let q denote the cardinality of the residue field $\mathcal{O}_F/\mathcal{P}_F$ and fix a uniformizer ϖ with $|\varpi|_F = q^{-1}$. Also, let W_F denote the Weil group of F, W'_F the Weil-Deligne group of F, and Γ the absolute Galois group $\mathrm{Gal}(\bar{F}/F)$. Throughout the paper, we will use the notation $\nu(\cdot) = |\cdot|_F$.

Let G be a connected, reductive, linear algebraic group over F. Fixing Γ-invariant splitting data we define the L-group of G as a semi-direct product ${}^LG := \widehat{G} \rtimes \Gamma$, where $\widehat{G} = {}^LG^0$ denotes the connected component of the L-group of G, i.e., the complex dual of G (see [Bor79, §2]).

LLC (still conjectural in this generality) asserts that there is a surjective, finite-to-one map from the set $\mathrm{Irr}(G)$ of isomorphism classes of irreducible smooth complex representations of $G(F)$ to the set $\Phi(G)$ of $\widehat{G}$-conjugacy classes of L-parameters, i.e., admissible homomorphisms $\varphi : W'_F \longrightarrow {}^LG$.

Given $\varphi \in \Phi(G)$, its fiber $\Pi_\varphi(G)$, which is called an L-packet for G, is expected to be controlled by a certain finite group living in the complex dual group $\widehat{G}$. Furthermore, for $\pi \in \Pi_\varphi(G)$ and ρ a finite dimensional algebraic representation of ${}^LG$ one defines the local factors

$$L(s, \pi, \rho) = L(s, \rho \circ \phi), \tag{2.1}$$

$$\epsilon(s, \pi, \rho, \psi) = \epsilon(s, \rho \circ \phi, \psi), \tag{2.2}$$

$$\gamma(s, \pi, \rho, \psi) = \gamma(s, \rho \circ \phi, \psi). \tag{2.3}$$

provided that LLC is known for the case in question. Here, the factors on the right are Artin factors.

2.2. **The Adjoint L-Function.** What we recall in this subsection holds for G quasi-split ([GP92, §2]). However, for simplicity we will take G to be split over F since the groups we are working with in this article are split. When G is split over F, we may replace the L-group ${}^LG$ by its connected component

$\widehat{G} = {}^L G^0$. Take ρ to be the adjoint action of $\widehat{G}$ on its Lie algebra. Then we obtain the adjoint L-function $L(s, \pi, \mathrm{Ad}_{\widehat{G}}) = L(s, \mathrm{Ad}_{\widehat{G}} \circ \phi)$ for all $\pi \in \Pi_\varphi(G)$. The following is a conjecture of D. Gross and D. Prasad (see [GP92, Conj. 2.6]).

Conjecture 2.1. *$\Pi_\varphi(G)$ contains a generic member if and only if $L(s, \mathrm{Ad}_{\widehat{G}} \circ \phi)$ is regular at $s = 1$. (Equivalently, π is generic if and only if $L(s, \pi, \mathrm{Ad}_{\widehat{G}})$ is regular at $s = 1$.)*

The conjecture is known in many cases in which the LLC is known. To mention a few, it was verified for GL_n by B. Gross and D. Prasad [GP92], for GSp_4 in [GT11] and, for non-supercuspidals, in [AS08], and for SO and Sp groups, it follows from the work of Arthur on endoscopic classification [Art13]. We will verify this conjecture for the small rank split groups GSpin_4 and GSpin_6.

2.3. **The Groups GSpin_4 and GSpin_6.** We gave detailed information about the structure of these two groups (as well as their inner forms) in [AC17, §2.2]. For now we just recall the incidental isomorphisms

$$\mathrm{GSpin}_4 \cong \{(g_1, g_2) \in \mathrm{GL}_2 \times \mathrm{GL}_2 : \det g_1 = \det g_2\} \tag{2.4}$$

$$\mathrm{GSpin}_6 \cong \left\{(g_1, g_2) \in \mathrm{GL}_1 \times \mathrm{GL}_4 : g_1^2 = \det g_2\right\}. \tag{2.5}$$

While our main interests in this article are the split general spin groups GSpin_4 and GSpin_6, for the purposes of Conjecture 2.1 it is no more difficult, and perhaps also more natural, to consider a slightly more general setup as follows.

Fix integers $m, n \geq 1$ and $r, s \geq 1$ and assume that $\gcd(r, s) = 1$. Define

$$G = G^{r,s}_{m,n} := \{(g, h) \in \mathrm{GL}_m \times \mathrm{GL}_n \mid (\det g)^r = (\det h)^s\} \tag{2.6}$$

Proposition 2.2. *The group $G^{r,s}_{m,n}$ is a split, connected, reductive, linear algebraic group over F.*

Proof. Let $X = (X_{ij})$ and $Y = (Y_{kl})$ be $m \times m$ and $n \times n$ matrices, respectively. It is clear that $G^{r,s}_{m,n}$, being an almost direct product of $SL_m \times \mathrm{SL}_n$ and a torus, is reductive. The only issue that requires justification is that the polynomial $f(X, Y) = (\det X)^r - (\det Y)^s$ is irreducible in $F[X_{ij}, Y_{kl}]$ if and only if $d = \gcd(r, s) = 1$. It is clear that if $d > 1$, then f is reducible since it would be divisible by $(\det X)^{(r/d)} - (\det Y)^{(s/d)}$. It remains to show that if $d = 1$, then $f(X, Y)$ is irreducible. This assertion should be easy to see via elementary arguments considering the polynomials in a possible factorization of f. However, we prove it below as a special case of a more general fact.

Assume that $f(x, y)$ is an (arbitrary) irreducible polynomial in $F[x, y]$. Let

$$p(x_1, x_2, \ldots, x_a) \in F[x_1, x_2, \ldots, x_a] \quad \text{and} \quad p(y_1, y_2, \ldots, y_b) \in F[y_1, y_2, \ldots, y_b]$$

be two polynomials such that $p - \alpha$ and $q - \alpha$ are irreducible for all constants α. Then, $f(p, q)$ is irreducible in $F[x_1, x_2, \ldots, x_a, y_1, y_2, \ldots, y_b]$.

Our Proposition would clearly follow from the above assertion since $(\det - \alpha)$ is always an irreducible polynomial and it is well-known that the two-variable polynomial $x^r - y^s$ is irreducible in $F[x, y]$ provided that $d = \gcd(r, s) = 1$.

To prove the assertion above, we proceed as follows. By base extension to an algebraic closure we may assume, without loss of generality, that F is algebraically closed.

Let A be the subscheme of $\operatorname{Spec} F[x_1, x_2, \ldots, x_a, y_1, y_2, \ldots, y_b]$ defined by $f(p, q)$, and let B be the subscheme of $\operatorname{Spec} F[x, y]$ defined by $x^r - y^s$. The latter is irreducible since $x^r - y^s$ is an irreducible polynomial by our assumption that $d = 1$. There is a natural map $A \to B$ which has irreducible (geometric) fibers. The result now follows from the following claim.

Claim: Let $g : A \to B$ be an open morphism of schemes of finite type over an algebraically closed field F such that the (geometric) fibers of g are irreducible and B is irreducible. Then A is irreducible.

To see the claim let U be an open in A. We want to show that for any other open V, we have that $U \cap V$ is nonempty. Since B is irreducible and g is open, we have that $g(U) \cap g(V)$ is nonempty so there is a fiber F_0 of g such that $F_0 \cap U$ and $F_0 \cap V$ are nonempty. Hence, by irreducibility of F_0, they have a nonempty intersection in F_0. In particular, $U \cap V$ is nonempty, which gives the claim.

It only remains to check that the map $A \to B$ above is open. In fact, it is flat since it is a base extension of the cartesian product of two flat morphisms $p : \operatorname{Spec} F[x_1, ..., x_a] \to \operatorname{Spec} F[x]$ and $q : \operatorname{Spec} F[y_1, ..., y_b] \to \operatorname{Spec} F[y]$. (Here, we are using the fact that $\operatorname{Spec} F[x]$ is a curve.) This finishes the proof. □

Of particular interest to us in this paper are the cases

- $m = n = 2$ and $r = s = 1$, when $G = \mathrm{GSpin}_4$, and
- $m = 1$, $n = 4$ and $r = 2$, $s = 1$, when $G = \mathrm{GSpin}_6$.

The (connected) L-group of G is

$$ {}^L G^{r,s\,0}_{m,n} = \widehat{G} \cong (\mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}))/\{(z^{-r} I_m, z^s I_n) : z \in \mathbb{C}^\times\} \tag{2.7} $$

and we have the exact sequence

$$ 1 \longrightarrow \{(z^{-r} I_m, z^s I_n) : z \in \mathbb{C}^\times\} \cong \mathbb{C}^\times \longrightarrow \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \xrightarrow{pr^{r,s}_{m,n}} \widehat{G^{r,s}_{m,n}} \longrightarrow 1. \tag{2.8} $$

2.4. **Computation of the Adjoint L-Function for G.** Let π be an irreducible admissible representation of $G(F)$. There exist irreducible admissible representations π_m and π_n of $\mathrm{GL}_m(F)$ and $\mathrm{GL}_n(F)$, respectively, such that

$$ \pi \hookrightarrow \mathrm{Res}^{\mathrm{GL}_m(F)\times \mathrm{GL}_n(F)}_{G(F)} (\pi_m \otimes \pi_n). \tag{2.9} $$

Let $\mathrm{Ad}_{\widehat{G}}$ denote the adjoint action of $\widehat{G}$ on its Lie algebra

$$ \widehat{\mathfrak{g}} = \{(X,Y) \in \mathfrak{gl}_m(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C}) \mid r\,\mathrm{tr}(X) = s\,\mathrm{tr}(Y)\}. \tag{2.10} $$

In what follows, let us write

$$ \mathrm{Ad}_{\widehat{G}} = \mathrm{triv} \oplus \mathrm{Ad} \tag{2.11} $$

and for $i \in \{m,n\}$ we similarly write $\mathrm{Ad}_i = \mathrm{Ad}_{\widehat{GL_i}} = \mathrm{triv} \oplus \mathrm{Ad}$, where Ad here denotes the action of $\mathrm{GL}_i(\mathbb{C})$ on the space of traceless $i \times i$ complex matrices $\mathfrak{s}l_i(\mathbb{C})$.

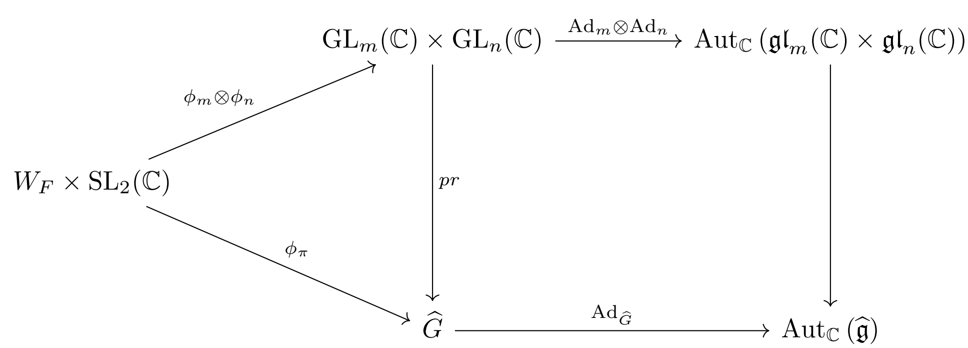
Let $\phi_\pi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}$ be the L-parameter of π and let $\phi_i : W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_i(\mathbb{C})$, $i = m, n$, be the L-parameter of π_i. Recall by (2.8) that we have a natural map

$$ pr = pr^{r,s}_{m,n} : \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \longrightarrow \widehat{G}. \tag{2.12} $$

Then we have

$$ \phi_\pi = pr \circ (\phi_m \otimes \phi_n). \tag{2.13} $$

Since the subgroup $\{(z^{-r} I_m, z^s I_n) : z \in \mathbb{C}^\times\}$ is central in $\mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ the following diagram commutes.



Note that the adjoint action Ad_m of $\mathrm{GL}_m(\mathbb{C})$ on $\mathfrak{g}l_m(\mathbb{C})$ preserves the trace, and similarly for n, so we obtain a right downward arrow by simply restricting any automorphism to the set of those pairs satisfying the trace equality in (2.10). We have

$$ \begin{aligned} L(s, 1_{F^\times}) L(s, \pi, \mathrm{Ad}) \cdot L(s, 1_{F^\times}) &= L(s, \pi, \mathrm{Ad}_{\widehat{G}}) \cdot L(s, 1_{F^\times}) \\ &= L(s, \mathrm{Ad}_{\widehat{G}} \circ \phi_\pi) \cdot L(s, 1_{F^\times}) \\ &= L\left(s, (\mathrm{Ad}_m \otimes \mathrm{Ad}_n) \circ (\phi_m \otimes \phi_n)\right) \\ &= L(s, \mathrm{Ad}_m \circ \phi_m) L(s, \mathrm{Ad}_n \circ \phi_n) \\ &= L(s, \pi_m, \mathrm{Ad}_m) L(s, \pi_n, \mathrm{Ad}_n) \\ &= L(s, 1_{F^\times})^2 L(s, \pi_m, \mathrm{Ad}) L(s, \pi_n, \mathrm{Ad}). \end{aligned} \tag{2.14} $$

Therefore, we obtain the more convenient equality

$$L(s,\pi,\mathrm{Ad}) = L(s,\pi_m,\mathrm{Ad})L(s,\pi_n,\mathrm{Ad}), \tag{2.15}$$

which holds thanks to our choice of the notation Ad. In Section 3.2 this relation helps verify Conjecture 2.1 for the groups of interest to us.

3. Genericity and The Conjecture of B. Gross and D. Prasad

3.1. **Restriction of Generic Representations.** Let us write $\square^D$ for the group $\mathrm{Hom}(\square,\mathbb{C}^\times)$ of all continuous characters on a topological group $\square$. Dente by $\square_{\mathrm{der}}$ the derived group of $\square$. Let G and $\widetilde{G}$ be connected, reductive, linear, algebraic groups over F satisfying the property that

$$G_{\mathrm{der}} = \widetilde{G}_{\mathrm{der}} \subseteq G \subseteq \widetilde{G}. \tag{3.1}$$

For any connected, reductive, linear, algebraic group $\square$ over F, we write $\mathrm{Irr}_{\mathrm{sc}}(\square)$ and $\mathrm{Irr}_{\mathrm{esq}}(\square)$ for the set of equivalence classes of supercuspidal and essentially square-integrable representations of $\square(F)$, respectively.

Assume $\widetilde{G}$ and G to be F-split. Let $\widetilde{B}$ be a Borel subgroup of $\widetilde{G}$ with Levi decomposition $\widetilde{B} = \widetilde{T}\widetilde{U}$. Then $B = \widetilde{B} \cap G$ is a Borel subgroup of G with $B = TU$. Note that $T = \widetilde{T} \cap G$ and $\widetilde{U} = U$. Let ψ be a generic character of $U(F)$. From [Tad92, Proposition 2.8] we know that given a ψ-generic irreducible representation $\widetilde{\sigma}$ of $\widetilde{G}(F)$ we have a unique ψ-generic σ of $G(F)$ such that

$$\sigma \hookrightarrow \mathrm{Res}_G^{\widetilde{G}}(\widetilde{\sigma}).$$

The generic character associated with σ is not unique though.

Proposition 3.1. *Each generic character associated with σ is determined up to the action of $\widetilde{T}(F)/T(F)$.*

Proof. We let $\widetilde{\sigma} \in \mathrm{Irr}(\widetilde{G})$ be ψ-generic. Then there is a unique ψ-generic $\sigma_\psi \in \Pi_{\widetilde{\sigma}}(G)$. On the other hand, for each $\sigma \in \Pi_{\widetilde{\sigma}}(G)$ there exists $t \in \widetilde{T}(F)/T(F) \cong \widetilde{G}/G(F)$ such that $\sigma = {}^t\sigma_\psi$, where ${}^t\sigma_\psi(g) = \sigma(t^{-1}gt)$. This implies that σ is ${}^t\psi$-generic. Here ${}^t\psi$ is defined as ${}^t\psi(u) = \psi(t^{-1}ut)$. □

Remark 3.2. We say $\sigma \in \mathrm{Irr}(G)$, resp. $\widetilde{\sigma} \in \mathrm{Irr}(\widetilde{G})$, is generic if it is ψ-generic with respect to some generic character ψ. With this notation, $\sigma \in \mathrm{Irr}(G)$ is generic if and only if is $\widetilde{\sigma} \in \mathrm{Irr}(\widetilde{G})$.

3.2. **Criterion for Genericity.** In this section we verify Conjecture 2.1 for the small rank general spin groups we are considering in this article.

Theorem 3.3. *Let $G = G_{m,n}^{r,s}$ be the group defined in* (2.6)*. Let π be an irreducible admissible representation of $G(F)$. Then π is generic if and only if $L(s,\pi,\mathrm{Ad})$ is regular at $s = 1$.*

Proof. Given π there exist irreducible admissible representations π_m of $\mathrm{GL}_m(F)$ and π_n of $\mathrm{GL}_n(F)$ such that π is a subrepresentation of the restriction to $G(F)$ of $\pi_m \otimes \pi_n$ as in (2.9). Now, π is generic if and only if both π_m and π_n are generic. By the truth of Conjecture 2.1 for the general linear groups, the latter is equivalent to both $L(s,\pi_m,\mathrm{Ad})$ and $L(s,\pi_n,\mathrm{Ad})$ being regular at $s = 1$. Hence, by (2.15) and the fact that neither of the L-functions can have a zero at $s = 1$, we have that π is generic if and only if $L(s,\pi,\mathrm{Ad})$ is regular at $s = 1$. This proves the theorem. □

As we observed in Section 2.3, the split groups GSpin_4 and GSpin_6 are special cases of $G_{m,n}^{r,s}$. Therefore, we have the following.

Corollary 3.4. *Conjecture 2.1 holds for the groups* GSpin_4 *and* GSpin_6.

4. Representations of GSpin_4

In this section we list all the irreducible representations of $\mathrm{GSpin}_4(F)$ and then calculate their associated adjoint L-function explicitly. To this end, we give the nilpotent matrix associated to their parameter in each case.

4.1. **The Reprsentations.**

4.1.1. *Classification of representations of* GSpin_4. Following [AC17], we have

$$1 \longrightarrow \mathrm{GSpin}_4(F) \longrightarrow \mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \longrightarrow F^\times \longrightarrow 1. \tag{4.1}$$

Recall that

$$\mathrm{GSpin}_4(F) \cong \{(g_1, g_2) \in \mathrm{GL}_2(F) \times \mathrm{GL}_2(F) : \det g_1 = \det g_2\}, \tag{4.2}$$

$$^L\mathrm{GSpin}_4 = \widehat{\mathrm{GSpin}_4} = \mathrm{GSO}_4(\mathbb{C}) \cong (\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}))/\{(z^{-1}, z) : z \in \mathbb{C}^\times\}, \tag{4.3}$$

and

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) \xrightarrow{pr_4} \widehat{\mathrm{GSpin}_4} \longrightarrow 1. \tag{4.4}$$

When convenient, we view GSO_4 as the group similitude orthogonal 4×4 matrices with respect to the anti-diagonal matrix

$$J = J_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{4.5}$$

The Lie algebra of this group is also defined with respect to J and an element X in this Lie algebra satisfies

$$^tXJ + JX = 0.$$

4.1.2. *Construction of the L-packets of* GSpin_4 *(recalled from* [AC17]*).* Given $\sigma \in \mathrm{Irr}(\mathrm{GSpin}_4)$ we have a lift $\widetilde{\sigma} \in \mathrm{Irr}(\mathrm{GL}_2 \times \mathrm{GL}_2)$ such that

$$\sigma \hookrightarrow \mathrm{Res}^{\mathrm{GL}_2 \times \mathrm{GL}_2}_{\mathrm{GSpin}_4}(\widetilde{\sigma}).$$

It follows form the LLC for GL_n [HT01, Hen00, Sch13] that there is a unique $\widetilde{\varphi}_{\widetilde{\sigma}} \in \Phi(\mathrm{GL}_2 \times \mathrm{GL}_2)$ corresponding to the representation $\widetilde{\sigma}$. We now have a surjective, finite-to-one map

$$\begin{array}{rcl} \mathcal{L}_4 : \mathrm{Irr}(\mathrm{GSpin}_4) & \longrightarrow & \Phi(\mathrm{GSpin}_4) \\ \sigma & \longmapsto & pr_4 \circ \widetilde{\varphi}_{\widetilde{\sigma}}, \end{array} \tag{4.6}$$

which does not depend on the choice of the lifting $\widetilde{\sigma}$. Then, for each $\varphi \in \Phi(\mathrm{GSpin}_4)$, all inequivalent irreducible constituents of $\widetilde{\sigma}$ constitutes the L-packet

$$\Pi_\varphi(\mathrm{GSpin}_4) := \Pi_{\widetilde{\sigma}}(\mathrm{GSpin}_4) = \left\{\sigma \,\middle|\, \sigma \hookrightarrow \mathrm{Res}^{\mathrm{GL}_2 \times \mathrm{GL}_2}_{\mathrm{GSpin}_4}(\widetilde{\sigma})\right\} \Big/ \cong. \tag{4.7}$$

Here, $\widetilde{\sigma}$ is the member in the singleton $\Pi_{\widetilde{\varphi}}(\mathrm{GL}_2 \times \mathrm{GL}_2)$ and $\widetilde{\varphi} \in \Phi(\mathrm{GL}_2 \times \mathrm{GL}_2)$ is such that $pr_4 \circ \widetilde{\varphi} = \varphi$. We note that the construction does not depends on the choice of $\widetilde{\varphi}$, due to the LLC for GL_2, [GK82, Lemma 2.4], [Tad92, Corollary 2.5], and [HS12, Lemma 2.2]. Further details can be found in [AC17, Section 5.1].

4.1.3. *The L-parameters of* GL_2. We recall the generic representations of $\mathrm{GL}_2(F)$ in this paragraph. We refer to [Wed08, Kud94, GR10] for details. Let $\chi : F^\times \to \mathbb{C}^\times$ denote a continuous quasi-character of $F^\times$. By Zelevinski ([Zel80, Theorem 9.7] or [Kud94, Theorem 2.3.1]) we know that the generic representations of GL_2 are: the supercuspidals, $\mathsf{St} \otimes (\chi \circ \det)$ where St denotes the Steinberg representation, and normally induced representations $i^{\mathrm{GL}_2}_{\mathrm{GL}_1 \times \mathrm{GL}_1}(\chi_1 \otimes \chi_2)$ with $\chi_1 \neq \chi_2 \nu^{\pm 1}$. The only non-generic representation is $\chi \circ \det$.

4.2. **Generic Representations of** GSpin_4**.** Following [AC17, Section 5.3], given $\varphi \in \Phi(\mathrm{GSpin}_4)$, fix the lift

$$\widetilde{\varphi} = \widetilde{\varphi}_1 \otimes \widetilde{\varphi}_2 \in \Phi(\mathrm{GL}_2 \times \mathrm{GL}_2)$$

with $\widetilde{\varphi}_i \in \Phi(\mathrm{GL}_2)$ such that $\varphi = pr_4 \circ \widetilde{\varphi}$. Let

$$\widetilde{\sigma} = \widetilde{\sigma}_1 \boxtimes \widetilde{\sigma}_2 \in \Pi_{\widetilde{\varphi}}(\mathrm{GL}_2 \times \mathrm{GL}_2)$$

be the unique member such that $\{\widetilde{\sigma}_i\} = \Pi_{\widetilde{\varphi}_i}(\mathrm{GL}_2)$.

Recall the notation

$$I^{\mathrm{GSpin}_4}(\widetilde{\sigma}) := \left\{\chi \in (\mathrm{GL}_2(F) \times \mathrm{GL}_2(F)/\mathrm{GSpin}_4(F))^D \,\middle|\, \widetilde{\sigma} \otimes \chi \cong \widetilde{\sigma}\right\}.$$

Then we have

$$\Pi_\varphi(\mathrm{GSpin}_4) \overset{1-1}{\longleftrightarrow} I^{\mathrm{GSpin}_4}(\widetilde{\sigma}), \tag{4.8}$$

and we recall that, by [AC17, Proposition 5.7], we have

$$I^{\mathrm{GSpin}_4}(\widetilde{\sigma}) = \begin{cases} I^{\mathrm{SL}_2}(\widetilde{\sigma}_1), & \text{if } \widetilde{\sigma}_2 \cong \widetilde{\sigma}_1\widetilde{\eta} \text{ for some } \widetilde{\eta} \in (F^\times)^D; \\ I^{\mathrm{SL}_2}(\widetilde{\sigma}_1) \cap I^{\mathrm{SL}_2}(\widetilde{\sigma}_2), & \text{if } \widetilde{\sigma}_2 \not\cong \widetilde{\sigma}_1\widetilde{\eta} \text{ for any } \widetilde{\eta} \in (F^\times)^D. \end{cases} \tag{4.9}$$

4.2.1. *Irreducible Parameters.* Let $\varphi \in \Phi(\mathrm{GSpin}_4)$ be irreducible. Then $\widetilde{\varphi}$, $\widetilde{\varphi}_1$, and $\widetilde{\varphi}_2$ are all irreducible. By Section 3.1, we have the following.

Proposition 4.1. *Let $\varphi \in \Phi(\mathrm{GSpin}_4)$ be irreducible. Then every member in $\Pi_\varphi(\mathrm{GSpin}_4)$ is supercuspidal and generic.*

To study the internal structure of $\Pi_\varphi(\mathrm{GSpin}_4)$, by (4.8), we need to know the structure of $I^{\mathrm{GSpin}_4}(\widetilde{\sigma})$, as we now recall from [AC17].

gnr-(a) When $\widetilde{\sigma}_2 \cong \widetilde{\sigma}_1\widetilde{\eta}$ for some $\widetilde{\eta} \in (F^\times)^D$, we have

$$I^{\mathrm{GSpin}_4}(\widetilde{\sigma}) \cong \begin{cases} \{1\}, & \text{if } \widetilde{\varphi}_1 \text{ (and hence also } \widetilde{\varphi}_2) \text{ is primitive or non-trivial on } \mathrm{SL}_2(\mathbb{C}); \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } \widetilde{\varphi}_1 \text{ (and hence also } \widetilde{\varphi}_2) \text{ is dihedral w.r.t. one quadratic extension;} \\ (\mathbb{Z}/2\mathbb{Z})^2, & \text{if } \widetilde{\varphi}_1 \text{ (and hence also } \widetilde{\varphi}_2) \text{ is dihedral w.r.t. three quadratic extensions.} \end{cases}$$

gnr-(b) When $\widetilde{\sigma}_2 \not\cong \widetilde{\sigma}_1\widetilde{\eta}$ for any $\widetilde{\eta} \in (F^\times)^D$, then by (4.9) we have

$$I^{\mathrm{GSpin}_4}(\widetilde{\sigma}) \cong \{1\} \text{ or } \mathbb{Z}/2\mathbb{Z}.$$

Since $\widetilde{\sigma}_2 \not\cong \widetilde{\sigma}_1\widetilde{\eta}$ for any $\widetilde{\eta} \in (F^\times)^D$, the case of both $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ being diredral w.r.t. three quadratic extensions is excluded. Thus, we have the following list:

- If at least one of $\widetilde{\varphi}_i$ is primitive, then $I^{\mathrm{GSpin}_4}(\widetilde{\sigma}) \cong \{1\}$.
- If both are dihedral, then $I^{\mathrm{GSpin}_4}(\widetilde{\sigma}) \cong \mathbb{Z}/2\mathbb{Z}$.

From [AC17, Proposition 2.1], we recall the identification

$$\Delta^\vee = \{\beta_1^\vee = f_{11}^* - f_{12}^*, \beta_2^\vee = f_{21}^* - f_{22}^*\}, \tag{4.10}$$

using the notation f_{ij} and f_{ij}^*, $1 \leq i, j \leq 2$, for the usual $\mathbb{Z}$-basis of characters and cocharacters of $\mathrm{GL}_2 \times \mathrm{GL}_2$ and β_1, β_2 denote the simple roots of GSpin_4. We can use this identification to relate the nilpotent matrices associated to the parameters of $\mathrm{GL}_2 \times \mathrm{GL}_2$ and GSpin_4, respectively.

For both (a) and (b) above, we have

$$N_{\mathrm{GL}_2(\mathbb{C})\times\mathrm{GL}_2(\mathbb{C})} = \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) \overset{(4.10)}{\Longleftrightarrow} N_{\mathrm{GSO}_4(\mathbb{C})} = 0_{4\times 4}.$$

Remark 4.2. We note that case (b) above was mentioned, less precisely, in [AC17, Remark 5.10].

4.2.2. *Reducible Parameters.* If $\varphi \in \Phi(\mathrm{GSpin}_4)$ is reducible, then at least one $\widetilde{\varphi}_i$ must be reducible. Since the number of irreducible constituents in $\mathrm{Res}^{\mathrm{GL}_2}_{\mathrm{SL}_2}(\widetilde{\sigma}_i)$ is at most 2, we have $I^{\mathrm{SL}_2}(\widetilde{\sigma}_i) \cong \{1\}$, or $\mathbb{Z}/2\mathbb{Z}$. This implies that

$$I^{\mathrm{GSpin}_4}(\widetilde{\sigma}) \cong \{1\}, \text{ or } \mathbb{Z}/2\mathbb{Z}.$$

If $\widetilde{\varphi}_i$ is reducible and generic, then $\widetilde{\sigma}_i$ is either the Steinberg representation twisted by a character or an irreducibly induced representation from the Borel subgroup of GL_2. We make case-by-case arguments as follows.

gnr-(i) Note that the Steinberg representation of $\mathrm{GL}_2 \times \mathrm{GL}_2$ is of the form $\mathsf{St}_{\mathrm{GL}_2} \boxtimes \mathsf{St}_{\mathrm{GL}_2}$. We have

$$\mathrm{Res}^{\mathrm{GL}_2\times\mathrm{GL}_2}_{\mathrm{GSpin}_4}(\mathsf{St}_{\mathrm{GL}_2} \boxtimes \mathsf{St}_{\mathrm{GL}_2}) = \mathsf{St}_{\mathrm{GSpin}_4} \tag{4.11}$$

and

$$\mathrm{Res}^{\mathrm{GL}_2\times\mathrm{GL}_2}_{\mathrm{GSpin}_4}(\mathsf{St}_{\mathrm{GL}_2} \otimes \chi_1 \boxtimes \mathsf{St}_{\mathrm{GL}_2} \otimes \chi_2) = \mathsf{St}_{\mathrm{GSpin}_4} \otimes \chi$$

for some χ. We have $I^{\mathrm{GSpin}_4}(\widetilde{\sigma}) \cong \{1\}$ as $I^G(\mathsf{St}_G) \cong \{1\}$. Thus, by (4.9), the L-packet remains a singleton and the restriction is irreducible.

- To determine χ, we use the required properties of χ_1, χ_2. Using

$$T = \left\{ \left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} \right) \middle| \, ab = cd \right\}, \tag{4.12}$$

we have $\chi_1(ab) = \chi_2(cd) \Leftrightarrow \chi_1 = \chi_2$. Denote $\chi_1 = \chi_2$ by χ.

For (4.11), we have

$$N_{\mathrm{GL}_2(\mathbb{C})\times\mathrm{GL}_2(\mathbb{C})} = \left(\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right) \overset{(4.10)}{\Longleftrightarrow} N_{\mathrm{GSO}_4(\mathbb{C})} = \begin{bmatrix}0 & 1 & 1 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$\mathfrak{gnr}$-(ii) Next we consider

$$\mathrm{Res}^{\mathrm{GL}_2\times\mathrm{GL}_2}_{\mathrm{GSpin}_4}\left(i^{\mathrm{GL}_2}_{\mathrm{GL}_1\times\mathrm{GL}_1}(\chi_1\otimes\chi_2)\boxtimes \mathsf{St}_{\mathrm{GL}_2}\otimes\chi\right). \tag{4.13}$$

By (4.9), the fact that $\widetilde{\sigma}_2 \ncong \widetilde{\sigma}_1\widetilde{\eta}$ for any $\widetilde{\eta}\in (F^\times)^D$, and since $I^G(\mathsf{St}_G)\cong\{1\}$, it follows that

$$I^{\mathrm{GSpin}_4}(\widetilde{\sigma})\cong\{1\}.$$

Thus, the L-packet remains a singleton and the restriction (4.13) is irreducible.

- To describe the restriction (4.13), we proceed similarly as above. We have

$$\chi_1(a)\chi_2(b)=\chi(cd)=\chi(ab) \ \Leftrightarrow\ \chi_1\chi^{-1}(a)=\chi_2^{-1}\chi(b)$$

  Specializing to $a=b$ and $c=d$ in the center, we have

$$\chi_1\chi_2\chi^{-2}=1$$

For (4.13) , we have

$$N_{\mathrm{GL}_2(\mathbb{C})\times\mathrm{GL}_2(\mathbb{C})} = \left(\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right) \overset{(4.10)}{\Longleftrightarrow} N_{\mathrm{GSO}_4(\mathbb{C})} = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

$\mathfrak{gnr}$-(iii) We consider

$$\mathrm{Res}^{\mathrm{GL}_2\times\mathrm{GL}_2}_{\mathrm{GSpin}_4}\left(i^{\mathrm{GL}_2}_{\mathrm{GL}_1\times\mathrm{GL}_1}(\chi_1\otimes\chi_2)\boxtimes i^{\mathrm{GL}_2}_{\mathrm{GL}_1\times\mathrm{GL}_1}(\chi_3\otimes\chi_4)\right) = i_T^{\mathrm{GSpin}_4}\left(\chi_1\otimes\chi_2, \chi_3\otimes\chi_1\chi_2\chi_3^{-1}\right).$$

Here, $\chi_1\neq\chi_2\nu^{\pm1}$ and $\chi_3\neq\chi_4\nu^{\pm1}$. Note that by (4.9) this induced representation may be irreducible or consist of two irreducible inequivalent constituents. We have

$$N_{\mathrm{GL}_2(\mathbb{C})\times\mathrm{GL}_2(\mathbb{C})} = \left(\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\right) \overset{(4.10)}{\Longleftrightarrow} N_{\mathrm{GSO}_4(\mathbb{C})} = \begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

$\mathfrak{gnr}$-(iv) Given a supercuspidal $\widetilde{\sigma}\in\mathrm{Irr}(\mathrm{GL}_2)$, we consider

$$\mathrm{Res}^{\mathrm{GL}_2\times\mathrm{GL}_2}_{\mathrm{GSpin}_4}\left(\widetilde{\sigma}\boxtimes \mathsf{St}_{\mathrm{GL}_2}\otimes\chi\right). \tag{4.14}$$

Since $I^G(\mathsf{St}_G)\cong\{1\}$, due to (4.9), the restriction (4.14) is irreducible. We then have

$$N_{\mathrm{GL}_2(\mathbb{C})\times\mathrm{GL}_2(\mathbb{C})} = \left(\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right) \overset{(4.10)}{\Longleftrightarrow} N_{\mathrm{GSO}_4(\mathbb{C})} = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

$\mathfrak{gnr}$-(v) Given supercuspidal $\widetilde{\sigma}\in\mathrm{Irr}(\mathrm{GL}_2)$, we next consider

$$\mathrm{Res}^{\mathrm{GL}_2\times\mathrm{GL}_2}_{\mathrm{GSpin}_4}\left(\widetilde{\sigma}\boxtimes i^{\mathrm{GL}_2}_{\mathrm{GL}_1\times\mathrm{GL}_1}(\chi_1\otimes\chi_2)\right).$$

Note from (4.9) that this may be irreducible or consist of two irreducible inequivalent constituents. We have

$$N_{\mathrm{GL}_2(\mathbb{C})\times\mathrm{GL}_2(\mathbb{C})} = \left(\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\right) \overset{(4.10)}{\Longleftrightarrow} N_{\mathrm{GSO}_4(\mathbb{C})} = 0_{4\times4}.$$

4.3. **Non-Generic Representations of** GSpin_4**.** If $\sigma \in \mathrm{Irr}(\mathrm{GSpin}_4)$ is non-generic, then σ is of the form

$$\mathrm{Res}^{\mathrm{GL}_2\times\mathrm{GL}_2}_{\mathrm{GSpin}_4}\left((\chi\circ\det)\boxtimes\widetilde{\sigma}\right), \tag{4.15}$$

with $\widetilde{\sigma}\in\mathrm{Irr}(\mathrm{GL}_2)$. Note this restriction is irreducible due to (4.9), and that as $\chi\circ\det$ is non-generic, so is the restriction σ for any $\widetilde{\sigma}\in\mathrm{Irr}(\mathrm{GL}_2)$.

For $\widetilde{\sigma}=\mathsf{St}\in\mathrm{Irr}(\mathrm{GL}_2)$, we have

$$N_{\mathrm{GL}_2(\mathbb{C})\times\mathrm{GL}_2(\mathbb{C})}=\left(\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix}\right)\overset{(4.10)}{\Longleftrightarrow} N_{\mathrm{GSO}_4(\mathbb{C})}=\begin{bmatrix}0&0&1&0\\0&0&0&-1\\0&0&0&0\\0&0&0&0\end{bmatrix},$$

and otherwise we have

$$N_{\mathrm{GL}_2(\mathbb{C})\times\mathrm{GL}_2(\mathbb{C})}=\left(\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&0\end{bmatrix}\right)\overset{(4.10)}{\Longleftrightarrow} N_{\mathrm{GSO}_4(\mathbb{C})}=0_{4\times4}.$$

We summarize the above information about the representations of GSpin_4 in Table 1.

4.4. **Computation of the Adjoint L-function for** GSpin_4**.** We now give explicit expressions for the adjoint L-function for each of the representations of $\mathrm{GSpin}_4(F)$. We start by recalling that the adjoint L-functions of the representations $\widetilde{\sigma}\in\mathrm{Irr}(\mathrm{GL}_2)$ are as follows.

$$L(s,\widetilde{\sigma},\mathrm{Ad}_2)=\begin{cases}L(s)^2L(s,\chi_1\chi_2^{-1})L(s,\chi_1^{-1}\chi_2), & \text{if } \widetilde{\sigma}=i^{\mathrm{GL}_2}_{\mathrm{GL}_1\times\mathrm{GL}_1}(\chi_1\boxtimes\chi_2) \text{ with } \chi_1\chi_2^{-1}\neq\nu^{\pm1};\\ L(s)L(s+1), & \text{if } \widetilde{\sigma}=\mathsf{St}_{\mathrm{GL}_2}\otimes\chi;\\ L(s)L(s,\widetilde{\sigma},\mathrm{Sym}^2\otimes\omega_{\widetilde{\sigma}}^{-1}), & \text{if } \widetilde{\sigma} \text{ is supercuspidal};\\ L(s)^2L(s-1)L(s+1), & \text{if } \widetilde{\sigma}=\chi\circ\det.\end{cases}$$

Here, $L(s)=L(s,1_{F^\times})$. Recall our choice of notation

$$L(s,\widetilde{\sigma},\mathrm{Ad}_2)=L(s)L(s,\widetilde{\sigma},\mathrm{Ad}).$$

Combining with (2.14), Sections 4.2.1 and 4.2.2, we have the following.

$\mathfrak{gnr}$-(a)&(b) Given a supercuspidal $\sigma\in\mathrm{Irr}(\mathrm{GSpin}_4)$, we recall that

$$\sigma\subset\mathrm{Res}^{\mathrm{GL}_2\times\mathrm{GL}_2}_{\mathrm{GSpin}_4}(\widetilde{\sigma}_1\boxtimes\widetilde{\sigma}_2)$$

for some supercuspidal $\widetilde{\sigma}_1\boxtimes\widetilde{\sigma}_2\in\mathrm{Irr}(\mathrm{GL}_2\times\mathrm{GL}_2)$. By (2.15) we have

$$L(s,\sigma,\mathrm{Ad})=L(s,\widetilde{\sigma}_1,\mathrm{Sym}^2\otimes\omega_{\widetilde{\sigma}_1}^{-1})L(s,\widetilde{\sigma}_2,\mathrm{Sym}^2\otimes\omega_{\widetilde{\sigma}_2}^{-1}).$$

$\mathfrak{gnr}$-(i) Given

$$\sigma=\mathsf{St}_{\mathrm{GSpin}_4}\otimes\chi\in\mathrm{Irr}(\mathrm{GSpin}_4),$$

by (2.15) we have

$$L(s,\sigma,\mathrm{Ad})=L(s+1)^2.$$

$\mathfrak{gnr}$-(ii) Given $\sigma\in\mathrm{Irr}(\mathrm{GSpin}_4)$ such that

$$\sigma=\mathrm{Res}^{\mathrm{GL}_2\times\mathrm{GL}_2}_{\mathrm{GSpin}_4}\left(i^{\mathrm{GL}_2}_{\mathrm{GL}_1\times\mathrm{GL}_1}(\chi_1\otimes\chi_2)\boxtimes\mathsf{St}_{\mathrm{GL}_2}\otimes\chi\right),$$

by (2.15) we have

$$L(s,\sigma,\mathrm{Ad})=L(s)L(s,\chi_1\chi_2^{-1})L(s,\chi_1^{-1}\chi_2)L(s+1).$$

$\mathfrak{gnr}$-(iii) Given $\sigma\in\mathrm{Irr}(\mathrm{GSpin}_4)$ such that

$$\sigma\subset\mathrm{Res}^{\mathrm{GL}_2\times\mathrm{GL}_2}_{\mathrm{GSpin}_4}\left(i^{\mathrm{GL}_2}_{\mathrm{GL}_1\times\mathrm{GL}_1}(\chi_1\otimes\chi_2)\boxtimes i^{\mathrm{GL}_2}_{\mathrm{GL}_1\times\mathrm{GL}_1}(\chi_3\otimes\chi_4)\right)$$

by (2.15) we have

$$L(s,\sigma,\mathrm{Ad})=L(s)^2L(s,\chi_1\chi_2^{-1})L(s,\chi_1^{-1}\chi_2)L(s,\chi_3\chi_4^{-1})L(s,\chi_3^{-1}\chi_4).$$

$\mathfrak{gnr}$-(iv) Given $\sigma \in \mathrm{Irr}(\mathrm{GSpin}_4)$ such that

$$\sigma = \mathrm{Res}^{\mathrm{GL}_2\times\mathrm{GL}_2}_{\mathrm{GSpin}_4}(\widetilde{\sigma} \boxtimes \mathsf{St}_{\mathrm{GL}_2} \otimes \chi)$$

by (2.15) we have

$$L(s,\sigma,\mathrm{Ad}) = L(s,\widetilde{\sigma}_2,\mathrm{Sym}^2 \otimes \omega_{\widetilde{\sigma}_2}^{-1})L(s+1).$$

$\mathfrak{gnr}$-(v) Given $\sigma \in \mathrm{Irr}(\mathrm{GSpin}_4)$ such that

$$\sigma \subset \mathrm{Res}^{\mathrm{GL}_2\times\mathrm{GL}_2}_{\mathrm{GSpin}_4}\left(\widetilde{\sigma} \boxtimes i^{\mathrm{GL}_2}_{\mathrm{GL}_1\times\mathrm{GL}_1}(\chi_1\otimes\chi_2)\right)$$

by (2.15) we have

$$L(s,\sigma,\mathrm{Ad}) = L(s)L(s,\widetilde{\sigma}_2,\mathrm{Sym}^2 \otimes \omega_{\widetilde{\sigma}_2}^{-1})L(s,\chi_1\chi_2^{-1})L(s,\chi_1^{-1}\chi_2).$$

$\mathfrak{nongnr}$ Given a non-generic $\sigma \in \mathrm{Irr}(\mathrm{GSpin}_4)$, from (4.15), we recall that

$$\sigma = \mathrm{Res}^{\mathrm{GL}_2\times\mathrm{GL}_2}_{\mathrm{GSpin}_4}(\chi\circ\det \boxtimes \widetilde{\sigma})$$

and by (2.15) we have

$$L(s,\sigma,\mathrm{Ad}) = L(s)L(s-1)L(s+1)L(s,\widetilde{\sigma},\mathrm{Ad}).$$

We summarize the explicit computations above in Table 2.

5. Representations of GSpin_6

We now list all the representations of $\mathrm{GSpin}_6(F)$ and then calculate their associated adjoint L-function explicitly. Again, we do this explicit calculation by finding the 6×6 nilpotent matrix in the complex dual group $\mathrm{GSO}_6(\mathbb{C})$ in each case that is associated with the parameter of the representation.

5.1. **The Represenations.**

5.1.1. *Classification of representations of* GSpin_6. Again, following [AC17], we have

$$1 \longrightarrow \mathrm{GSpin}_6(F) \longrightarrow \mathrm{GL}_1(F)\times\mathrm{GL}_4(F) \longrightarrow F^\times \longrightarrow 1. \tag{5.1}$$

Recall that

$$\mathrm{GSpin}_6(F) \cong \left\{(g_1,g_2)\in \mathrm{GL}_1(F)\times\mathrm{GL}_4(F) : g_1^2 = \det g_2\right\}, \tag{5.2}$$

$${}^L\mathrm{GSpin}_6 = \widehat{\mathrm{GSpin}_6} = \mathrm{GSO}_6(\mathbb{C}) \cong (\mathrm{GL}_1(\mathbb{C})\times\mathrm{GL}_4(\mathbb{C}))/\{(z^{-2},z) : z\in\mathbb{C}^\times\}, \tag{5.3}$$

and

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \mathrm{GL}_1(\mathbb{C})\times\mathrm{GL}_4(\mathbb{C}) \xrightarrow{pr_6} \widehat{\mathrm{GSpin}_6} \longrightarrow 1. \tag{5.4}$$

Just as the rank two case, here too we view GSO_6 as the group similitude orthogonal 6×6 matrices with respect to the analogous 6×6, anti-diagonal, matrix $J = J_6$ as in (4.5), and similarly define its Lie algebra with respect to J.

5.1.2. *Construction of the L-packets of* GSpin_6 *(recalled from* [AC17]*).* Given $\sigma\in\mathrm{Irr}(\mathrm{GSpin}_6)$ we have a lift $\widetilde{\sigma}\in\mathrm{Irr}(\mathrm{GL}_1\times\mathrm{GL}_4)$ such that

$$\sigma \hookrightarrow \mathrm{Res}^{\mathrm{GL}_1\times\mathrm{GL}_4}_{\mathrm{GSpin}_6}(\widetilde{\sigma}).$$

It follows from the LLC for GL_n [HT01, Hen00, Sch13] that there is a unique $\widetilde{\varphi}_{\widetilde{\sigma}}\in\Phi(\mathrm{GL}_1\times\mathrm{GL}_4)$ corresponding to the representation $\widetilde{\sigma}$. We now have a surjective, finite-to-one map

$$\begin{aligned} \mathcal{L}_6 : \mathrm{Irr}(\mathrm{GSpin}_6) &\longrightarrow \Phi(\mathrm{GSpin}_6) \\ \sigma &\longmapsto pr_6\circ\widetilde{\varphi}_{\widetilde{\sigma}}, \end{aligned} \tag{5.5}$$

which does not depend on the choice of the lifting $\widetilde{\sigma}$. Then, for each $\varphi\in\Phi(\mathrm{GSpin}_6)$, all inequivalent irreducible constituents of $\widetilde{\sigma}$ constitutes the L-packet

$$\Pi_\varphi(\mathrm{GSpin}_6) := \Pi_{\widetilde{\sigma}}(\mathrm{GSpin}_6) = \left\{\sigma : \sigma\hookrightarrow \mathrm{Res}^{\mathrm{GL}_1\times\mathrm{GL}_4}_{\mathrm{GSpin}_6}(\widetilde{\sigma})\right\}\Big/\cong, \tag{5.6}$$

where $\widetilde{\sigma}$ is the unique member of $\Pi_{\widetilde{\varphi}}(\mathrm{GL}_1\times\mathrm{GL}_4)$ and $\widetilde{\varphi}\in\Phi(\mathrm{GL}_1\times\mathrm{GL}_4)$ is such that $pr_6\circ\widetilde{\varphi}=\varphi$. We note that the construction does not depends on the choice of $\widetilde{\varphi}$. Further details can be found in [AC17, Section 6.1].

Following [AC17, Section 6.3], given $\varphi \in \Phi(\mathrm{GSpin}_6)$, fix the lift

$$\widetilde{\varphi} = \widetilde{\eta} \otimes \widetilde{\varphi}_0 \in \Phi(\mathrm{GL}_1 \times \mathrm{GL}_4)$$

with $\widetilde{\varphi}_0 \in \Phi(\mathrm{GL}_4)$ such that $\varphi = pr_6 \circ \widetilde{\varphi}$. Let

$$\widetilde{\sigma} = \widetilde{\eta} \boxtimes \widetilde{\sigma}_0 \in \Pi_{\widetilde{\varphi}}(\mathrm{GL}_1 \times \mathrm{GL}_4)$$

be the unique member such that $\{\widetilde{\sigma}_0\} = \Pi_{\widetilde{\varphi}_0}(\mathrm{GL}_4)$.

Recall that

$$I^{\mathrm{GSpin}_6}(\widetilde{\sigma}) := \left\{ \widetilde{\chi} \in \Big(\mathrm{GL}_1(F) \times \mathrm{GL}_4(F)/\mathrm{GSpin}_6(F) \Big)^D : \widetilde{\sigma} \otimes \widetilde{\chi} \cong \widetilde{\sigma} \right\}.$$

Then we have

$$\Pi_\varphi(\mathrm{GSpin}_6) \overset{1-1}{\longleftrightarrow} I^{\mathrm{GSpin}_6}(\widetilde{\sigma}), \tag{5.7}$$

and by [AC17, Lemma 6.5 and Proposition 6.6] we have

$$I^{\mathrm{GSpin}_6}(\widetilde{\sigma}) \cong \{ \widetilde{\chi} \in I^{\mathrm{SL}_4}(\widetilde{\sigma}_0) : \widetilde{\chi}^2 = 1_{F^\times} \} \tag{5.8}$$

and any $\widetilde{\chi} \in I^{\mathrm{GSpin}_6}(\widetilde{\sigma})$ is of the form

$$\widetilde{\chi} = (\widetilde{\chi}')^{-2} \boxtimes \widetilde{\chi}',$$

for some $\widetilde{\chi}' \in (F^\times)^D$.

5.2. **Generic Representations of** GSpin_6**.** Thanks to the group structure (5.2) and the relation of generic representations in Section 3.1, in order to classify the generic representations of GSpin_6, it suffices to classify the generic representations of GL_4.

Here are two key facts from the GL theory.

- Recall from [Zel80, Theorem 9.7] and [Kud94, Theorem 2.3.1] that a generic representation of GL_4 is of the form
$$i_{M_\flat}^{GL_4}(\sigma_\flat)$$
where $M_\flat$ runs through any F-Levi subgroup of GL_4 (including GL_4 itself) and $\sigma_\flat$ is any essentially square-integrable representation of $M_\flat$.
- For their L-parameters, we note from [Kud94, §5.2] that the generic representations of GL_4 have Langlands parameters (i.e., 4-dimensional Weil-Deligne representations (ρ, N)) of the form
$$(\rho_1 \otimes sp(r_1)) \otimes .. \otimes (\rho_t \otimes sp(r_t))$$
with $t \leq 4$, where ρ_i's are irreducible and no two segments are linked.

5.2.1. *Irreducible Parameters.* Let $\varphi \in \Phi(\mathrm{GSpin}_6)$ be irreducible. Then $\widetilde{\varphi}$ and $\widetilde{\varphi}_0$ are also irreducible. By Section 3.1, we have the following.

Proposition 5.1. *Let $\varphi \in \Phi(\mathrm{GSpin}_6)$ be irreducible. Every member in $\Pi_\varphi(\mathrm{GSpin}_6)$ is supercuspidal and generic.*

To see the internal structure of $\Pi_\varphi(\mathrm{GSpin}_6)$, we need, by (5.7), to know the detailed structure of $I^{\mathrm{GSpin}_6}(\widetilde{\sigma})$ as follows.

$\mathfrak{gnr}$-(a) Given $\sigma \in \mathrm{Irr}_{\mathrm{sc}}(\mathrm{GSpin}_6)$, we have

$$\widetilde{\sigma} = \widetilde{\sigma}_0 \boxtimes \widetilde{\eta} \in \mathrm{Irr}_{\mathrm{sc}}(\mathrm{GL}_4 \times \mathrm{GL}_1). \tag{5.9}$$

From [AC17, Proposition 2.1], we recall the identification:

$$\Delta^\vee = \{ \beta_1^\vee = f_2^* - f_3^*, \beta_2^\vee = f_1^* - f_2^*, \beta_3^\vee = f_3^* - f_4^* \}. \tag{5.10}$$

using the notation f_{ij} and f_{ij}^*, $1 \leq i,j \leq 4$, for the usual $\mathbb{Z}$-basis of characters and cocharacters of GL_4. Also, $\{\beta_1, \beta_2, \beta_3\}$ are the simple roots of GSpin_6.

We have

$$N_{\mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})} = (0_{4\times 4}, 0) \overset{(5.10)}{\Longleftrightarrow} N_{\mathrm{GSO}_6(\mathbb{C})} = 0_{6\times 6}.$$

5.2.2. *Reducible Parameters.* When $\widetilde{\varphi}_0$ is not irreducible, we have proper parabolic inductions. An exhaustive list of F-Levi subgroups $\mathbf{M}$ of GSpin_6 (up to isomorphism) is as follows.

- $\mathbf{M} \cong \mathrm{GL}_1 \times GL_1 \times \mathrm{GL}_1 \times \mathrm{GL}_1 = \widetilde{\mathbf{M}} \cap \mathrm{GSpin}_6$, where $\widetilde{\mathbf{M}} = (\mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{GL}_1) \times \mathrm{GL}_1$.
- $\mathbf{M} \cong \mathrm{GL}_2 \times \mathrm{GL}_1 \times \mathrm{GL}_1 = \widetilde{\mathbf{M}} \cap \mathrm{GSpin}_6$, where $\widetilde{\mathbf{M}} = (\mathrm{GL}_2 \times \mathrm{GL}_1 \times \mathrm{GL}_1) \times \mathrm{GL}_1$.
- $\mathbf{M} \cong \mathrm{GL}_3 \times \mathrm{GL}_1 = \widetilde{\mathbf{M}} \cap \mathrm{GSpin}_6$, where $\widetilde{\mathbf{M}} = (\mathrm{GL}_3 \times \mathrm{GL}_1) \times \mathrm{GL}_1$. (Note: The factor GL_1 of $\mathbf{M}$ is GSpin_0 by convention.)
- $\mathbf{M} \cong \mathrm{GL}_1 \times \mathrm{GSpin}_4 = \widetilde{\mathbf{M}} \cap \mathrm{GSpin}_6$, where $\widetilde{\mathbf{M}} = (\mathrm{GL}_2 \times \mathrm{GL}_2) \times \mathrm{GL}_1$.
- $\mathbf{M} \cong \mathrm{GSpin}_6 = \widetilde{\mathbf{M}} \cap \mathrm{GSpin}_6$, where $\widetilde{\mathbf{M}} = \mathrm{GL}_4 \times \mathrm{GL}_1$.

(Note that $\mathbf{M} \cong \mathrm{GL}_2 \times \mathrm{GL}_2$ does not occur on this list.) We now consider each case and, by abuse of notation, conflate algebraic groups and their F-points.

$\mathfrak{gnr}$-(I) $\mathbf{M} \cong \mathrm{GL}_1 \times GL_1 \times \mathrm{GL}_1 \times \mathrm{GL}_1$ and $\widetilde{\mathbf{M}} = (\mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{GL}_1) \times \mathrm{GL}_1$.
Given $\chi_i \in (F^\times)^D$ we consider

$$i_M^{\mathrm{GSpin}_6}(\chi_1 \boxtimes \chi_2 \boxtimes \chi_3 \boxtimes \chi_4). \tag{5.11}$$

Write $\chi_1 \boxtimes \chi_2 \boxtimes \chi_3 \boxtimes \chi_4 = (\widetilde{\chi}_1 \boxtimes \widetilde{\chi}_2 \boxtimes \widetilde{\chi}_3 \boxtimes \widetilde{\chi}_4 \boxtimes \widetilde{\eta})|_M$ with $\widetilde{\chi}_i, \widetilde{\eta} \in (F^\times)^D$ so that

$$\widetilde{\chi}_1\widetilde{\chi}_2\widetilde{\chi}_3\widetilde{\chi}_4 = \widetilde{\eta}^2.$$

Then we have the following relations

$$\chi_1 = \widetilde{\chi}_1,\ \chi_2 = \widetilde{\chi}_2,\ \chi_3 = \widetilde{\chi}_3,\ \chi_4 = \widetilde{\eta}^2(\widetilde{\chi}_2\widetilde{\chi}_3\widetilde{\chi}_4)^{-1}. \tag{5.12}$$

By Section 3.1, we know that the representation (5.11) is generic if and only if its lift

$$i_{\widetilde{M}}^{\mathrm{GL}_4 \times GL_1}(\widetilde{\chi}_1 \boxtimes \widetilde{\chi}_2 \boxtimes \widetilde{\chi}_3 \boxtimes \widetilde{\chi}_4 \boxtimes \widetilde{\eta}) \tag{5.13}$$

is generic if and only if

$$i_{\mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{GL}_1}^{GL_4}(\widetilde{\chi}_1 \boxtimes \widetilde{\chi}_2 \boxtimes \widetilde{\chi}_3 \boxtimes \widetilde{\chi}_4) \tag{5.14}$$

is generic. By the classification of the generic representations of GL_n ([Zel80, Theorem 9.7] and [Kud94, Theorem 2.3.1]), this amounts to (5.14) being irreducible. By [Kud94, Theorem 2.1.1] and [BZ77, Zel80], the necessary and sufficient condition for this to occur is that there is no pair i, j with $i \neq j$ such that

$$\widetilde{\chi}_i = \nu\widetilde{\chi}_j.$$

We have

$$N_{\mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})} = (0_{4\times4}, 0) \overset{(5.10)}{\Longleftrightarrow} N_{\mathrm{GSO}_6(\mathbb{C})} = 0_{6\times6}$$

$\mathfrak{gnr}$-(II) $\mathbf{M} \cong \mathrm{GL}_2 \times \mathrm{GL}_1 \times \mathrm{GL}_1$ and $\widetilde{\mathbf{M}} = (\mathrm{GL}_2 \times \mathrm{GL}_1 \times \mathrm{GL}_1) \times \mathrm{GL}_1$.
Given $\sigma_0 \in \mathrm{Irr}_{\mathrm{esq}}(\mathrm{GL}_2)$ and $\chi_1, \chi_2 \in (F^\times)^D$, we consider

$$i_M^{\mathrm{GSpin}_6}(\sigma_0 \boxtimes \chi_1 \boxtimes \chi_2). \tag{5.15}$$

Write $\sigma_0 \boxtimes \chi_1 \boxtimes \chi_2 = (\widetilde{\sigma}_0 \boxtimes \widetilde{\chi}_1 \boxtimes \widetilde{\chi}_2 \boxtimes \widetilde{\eta})|_M$ with $\widetilde{\sigma}_0 \in \mathrm{Irr}_{\mathrm{esq}}(\mathrm{GL}_2), \widetilde{\chi}_i, \widetilde{\eta} \in (F^\times)^D$.
Given $(g, h_1, h_2, h_3) \in \widetilde{M}$ with $\det(gh_1h_2) = h_3^2$,

- if we set $(g, h_1, h_3) \in M$, we have

$$\begin{aligned}\widetilde{\sigma}_0(g)\widetilde{\chi}_1(h_1)\widetilde{\chi}_2(h_2)\widetilde{\eta}(h_3) &= \widetilde{\sigma}_0(g)\widetilde{\chi}_1(h_1)\widetilde{\chi}_2(\det g^{-1}h_1^{-1}h_3^2)\widetilde{\eta}(h_3)\\ &= (\widetilde{\sigma}_0\widetilde{\chi}_2^{-1} \circ \det)(g)(\widetilde{\chi}_1\widetilde{\chi}_2^{-1})(h_1)(\widetilde{\chi}_2^2\widetilde{\eta})(h_3)\\ &= \sigma(g)\chi_1(h_1)\chi_2(h_3).\end{aligned}$$

Then we have

$$\widetilde{\sigma}_0 = \sigma_0\widetilde{\chi}_2,\ \widetilde{\chi}_1 = \chi_1\widetilde{\chi}_2,\ \widetilde{\eta} = \chi_2\widetilde{\chi}_2^{-2}.$$

- If we set $(g, h_2, h_3) \in M$, we have

$$\begin{aligned}\widetilde{\sigma}_0(g)\widetilde{\chi}_1(h_1)\widetilde{\chi}_2(h_2)\widetilde{\eta}(h_3) &= \widetilde{\sigma}_0(g)\widetilde{\chi}_1(\det g^{-1}h_2^{-1}h_3^2)\widetilde{\chi}_2(h_2)\widetilde{\eta}(h_3)\\ &= (\widetilde{\sigma}_0\widetilde{\chi}_1^{-1}\circ\det)(g)(\widetilde{\chi}_2\widetilde{\chi}_1^{-1})(h_2)(\widetilde{\chi}_1^2\widetilde{\eta})(h_3)\\ &= \sigma(g)\chi_1(h_2)\chi_2(h_3).\end{aligned}$$

Then we have

$$\widetilde{\sigma}_0 = \sigma_0\widetilde{\chi}_1,\ \widetilde{\chi}_2 = \chi_2\widetilde{\chi}_1,\ \widetilde{\eta} = \chi_1\widetilde{\chi}_1^{-2}. \tag{5.16}$$

As before, the representation (5.15) is generic if and only if its lift

$$i_{\widetilde{M}}^{\mathrm{GL}_4\times GL_1}(\widetilde{\sigma}_0\boxtimes\widetilde{\chi}_1\boxtimes\widetilde{\chi}_2\boxtimes\widetilde{\eta}) \tag{5.17}$$

is generic if and only if

$$i_{\mathrm{GL}_2\times\mathrm{GL}_1\times\mathrm{GL}_1}^{GL_4}(\widetilde{\sigma}_0\boxtimes\widetilde{\chi}_1\boxtimes\widetilde{\chi}_2) \tag{5.18}$$

is generic. Again by the classification of the generic representations of GL_n this amounts to (5.18) being irreducible. Hence, we must have

$$\widetilde{\chi}_1 \neq \nu^{\pm1}\widetilde{\chi}_2.$$

In other words, given $(g, h_1, h_2, h_3) \in \widetilde{M}$ with $\det(gh_1h_2) = h_3^2$,

- if we set $(g, h_1, h_3) \in M$, then

$$\chi_1 \neq \nu^{\pm1};$$

- if we set $(g, h_2, h_3) \in M$, then

$$\chi_2 \neq \nu^{\pm1}.$$

We have the following two cases. If σ_0 is supercuspidal, then

$$N_{\mathrm{GL}_4(\mathbb{C})\times\mathrm{GL}_1(\mathbb{C})} = (0_{4\times4}, 0) \overset{(5.10)}{\Longleftrightarrow} N_{\mathrm{GSO}_6(\mathbb{C})} = 0_{6\times6}.$$

If σ_0 is non-supercuspidal, then

$$N_{\mathrm{GL}_4(\mathbb{C})\times\mathrm{GL}_1(\mathbb{C})} = \left(\begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, 0\right) \overset{(5.10)}{\Longleftrightarrow} N_{\mathrm{GSO}_6(\mathbb{C})} = \begin{bmatrix}0&0&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}.$$

$\mathfrak{gnr}$-(III) $\mathbf{M} \cong \mathrm{GL}_3\times\mathrm{GL}_1$ and $\widetilde{\mathbf{M}} = (\mathrm{GL}_3\times\mathrm{GL}_1)\times\mathrm{GL}_1$.

Given $\sigma_0 \in \mathrm{Irr}_{\mathrm{esq}}(\mathrm{GL}_3)$ and $\chi \in (F^\times)^D$, we consider

$$i_M^{\mathrm{GSpin}_6}(\sigma_0\boxtimes\chi). \tag{5.19}$$

Write $\sigma_0\boxtimes\chi = (\widetilde{\sigma}_0\boxtimes\widetilde{\chi}\boxtimes\widetilde{\eta})|_M$ with $\widetilde{\sigma}_0 \in \mathrm{Irr}_{\mathrm{esq}}(\mathrm{GL}_3), \widetilde{\chi}, \widetilde{\eta} \in (F^\times)^D$.

Given $(g, h_1, h_2) \in \widetilde{M}$ with $\det(gh_1) = h_2^2$, if we set $(g, h_2) \in M$, then we have

$$\begin{aligned}\widetilde{\sigma}_0(g)\widetilde{\chi}(h_1)\widetilde{\eta}(h_2) &= \widetilde{\sigma}_0(g)\widetilde{\chi}(\det g^{-1}h_2^2)\widetilde{\eta}(h_2)\\ &= (\widetilde{\sigma}_0\widetilde{\chi}^{-1}\circ\det)(g)(\widetilde{\chi}^2\widetilde{\eta})(h_2)\\ &= \sigma(g)\chi(h_2).\end{aligned} \tag{5.20}$$

Then, we have

$$\widetilde{\sigma}_0 = \sigma_0\widetilde{\chi} \quad \text{and} \quad \widetilde{\eta} = \chi_2\widetilde{\chi}^{-2}.$$

As before, (5.19) is generic if and only if its lift

$$i_{\widetilde{M}}^{\mathrm{GL}_4\times GL_1}(\widetilde{\sigma}_0\boxtimes\widetilde{\chi}\boxtimes\widetilde{\eta}) \tag{5.21}$$

is generic if and only if

$$i_{\mathrm{GL}_3\times\mathrm{GL}_1}^{GL_4}(\widetilde{\sigma}_0\boxtimes\widetilde{\chi}) \tag{5.22}$$

is generic. This amounts to (5.22) being irreducible as before, which is always true since $\widetilde{\sigma}_0$ is an essentially square integrable representation of GL_3. Note that by the classification of essentially square-integrable representations of GL_3 ([Kud94, Proposition 1.1.2]), $\widetilde{\sigma}_0$ must be either supercuspidal or the unique subrepresentation of

$$i^{\mathrm{GL}_3}_{\mathrm{GL}_1\times\mathrm{GL}_1\times\mathrm{GL}_1}\left(\nu\chi\boxtimes\chi\boxtimes\nu^{-1}\chi\right) \tag{5.23}$$

with any $\chi\in(F^\times)^D$.

We have the following two cases. If σ_0 is supercuspidal, then

$$N_{\mathrm{GL}_4(\mathbb{C})\times\mathrm{GL}_1(\mathbb{C})}=(0_{4\times4},0)\overset{(5.10)}{\Longleftrightarrow}N_{\mathrm{GSO}_6(\mathbb{C})}=0_{6\times6}.$$

If σ_0 is the non-supercuspidal, unique, subrepresentation of (5.23), then

$$N_{\mathrm{GL}_4(\mathbb{C})\times\mathrm{GL}_1(\mathbb{C})}=\left(\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix},0\right)\overset{(5.10)}{\Longleftrightarrow}N_{\mathrm{GSO}_6(\mathbb{C})}=\begin{bmatrix}0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&-1\\0&0&0&0&0&0\end{bmatrix}.$$

$\mathfrak{gnr}$-(IV) $\mathbf{M}\cong\mathrm{GL}_1\times\mathrm{GSpin}_4$ and $\widetilde{\mathbf{M}}=(\mathrm{GL}_2\times\mathrm{GL}_2)\times\mathrm{GL}_1$.

Given $\sigma_0\in\mathrm{Irr}_{\mathrm{esq}}(\mathrm{GSpin}_4)$ and $\chi\in(F^\times)^D$ we consider

$$i_M^{\mathrm{GSpin}_6}(\chi\boxtimes\sigma_0). \tag{5.24}$$

Write $\chi\boxtimes\sigma_0\subset(\widetilde{\sigma}_1\boxtimes\widetilde{\sigma}_2\boxtimes\widetilde{\eta})|_M$ with $\widetilde{\sigma}_i\in\mathrm{Irr}_{\mathrm{esq}}(\mathrm{GL}_2),\widetilde{\eta}\in(F^\times)^D$.

As before, (5.24) is generic if and only if its lift

$$i_{\widetilde{M}}^{\mathrm{GL}_4\times GL_1}(\widetilde{\sigma}_1\boxtimes\widetilde{\sigma}_2\boxtimes\widetilde{\eta}) \tag{5.25}$$

is generic if and only if

$$i^{GL_4}_{\mathrm{GL}_2\times\mathrm{GL}_2}(\widetilde{\sigma}_1\boxtimes\widetilde{\sigma}_2) \tag{5.26}$$

is generic. This amounts to (5.26) being irreducible. Thus, we must have

$$\widetilde{\sigma}_1\neq\nu^{\pm1}\widetilde{\sigma}_2.$$

We have several cases to consider. If σ_0 is supercuspidal (so are $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$), then

$$N_{\mathrm{GL}_4(\mathbb{C})\times\mathrm{GL}_1(\mathbb{C})}=(0_{4\times4}0)\overset{(5.10)}{\Longleftrightarrow}N_{\mathrm{GSO}_6(\mathbb{C})}=0_{6\times6}.$$

If σ_0 is non-supercuspidal, then for supercuspidal $\widetilde{\sigma}_1$ and non-supercuspidal $\widetilde{\sigma}_2$ we have

$$N_{\mathrm{GL}_4(\mathbb{C})\times\mathrm{GL}_1(\mathbb{C})}=\left(\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix},0\right)\overset{(5.10)}{\Longleftrightarrow}N_{\mathrm{GSO}_6(\mathbb{C})}=\begin{bmatrix}0&0&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix};$$

for non-supercuspidal $\widetilde{\sigma}_1$ and supercuspidal $\widetilde{\sigma}_2$ we have

$$N_{\mathrm{GL}_4(\mathbb{C})\times\mathrm{GL}_1(\mathbb{C})}=\left(\begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},0\right)\overset{(5.10)}{\Longleftrightarrow}N_{\mathrm{GSO}_6(\mathbb{C})}=\begin{bmatrix}0&0&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix};$$

and for non-supercuspidal $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ we have

$$N_{\mathrm{GL}_4(\mathbb{C})\times\mathrm{GL}_1(\mathbb{C})} = \left(\begin{bmatrix} 0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix}, 0\right) \overset{(5.10)}{\Longleftrightarrow} N_{\mathrm{GSO}_6(\mathbb{C})} = \begin{bmatrix} 0&0&0&0&0&0\\0&0&1&1&0&0\\0&0&0&0&-1&0\\0&0&0&0&-1&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}.$$

$\mathfrak{gnr}$-(V) $\mathbf{M} \cong \mathrm{GSpin}_6$ and $\widetilde{\mathbf{M}} = \mathrm{GL}_4 \times \mathrm{GL}_1$.

Given $\sigma \in \mathrm{Irr}_{\mathrm{esq}}(\mathrm{GSpin}_6) \setminus \mathrm{Irr}_{\mathrm{sc}}(\mathrm{GSpin}_6)$, we consider

$$\sigma \subset (\widetilde{\sigma} \boxtimes \widetilde{\eta})|_M$$

with $\widetilde{\sigma} \in \mathrm{Irr}_{\mathrm{esq}}(\mathrm{GL}_4) \setminus \mathrm{Irr}_{\mathrm{sc}}(\mathrm{GL}_4), \widetilde{\eta} \in (F^\times)^D$. Here, we note that $\varphi \in \Phi(\mathrm{GSpin}_6)$ is not irreducible and neither $\widetilde{\sigma}$ nor σ is supercuspidal. It is clear that σ is generic as $\widetilde{\sigma} \boxtimes \widetilde{\eta}$ is. By the classification of essentially square-integrable representations of GL_4 ([Kud94, Proposition 1.1.2]), $\widetilde{\sigma}$ must be the unique subrepresentation of either

$$i^{\mathrm{GL}_4}_{\mathrm{GL}_1\times\mathrm{GL}_1\times\mathrm{GL}_1\times\mathrm{GL}_1}\left(\nu^{3/2}\widetilde{\chi} \boxtimes \nu^{1/2}\widetilde{\chi} \boxtimes \nu^{-1/2}\widetilde{\chi} \boxtimes \nu^{-3/2}\widetilde{\chi}\right) \tag{5.27}$$

with any $\widetilde{\chi} \in (F^\times)^D$ (i.e., $\widetilde{\sigma} = \mathsf{St}_{\mathrm{GL}_4} \otimes \widetilde{\chi}$), or of

$$i^{\mathrm{GL}_4}_{\mathrm{GL}_2\times\mathrm{GL}_2}\left(\nu^{1/2}\widetilde{\tau} \boxtimes \nu^{-1/2}\widetilde{\tau}\right) \tag{5.28}$$

with any $\widetilde{\tau} \in \mathrm{Irr}_{\mathrm{sc}}(\mathrm{GL}_2)$.

Now, for (5.27) we have

$$N_{\mathrm{GL}_4(\mathbb{C})\times\mathrm{GL}_1(\mathbb{C})} = \left(\begin{bmatrix} 0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}, 0\right) \overset{(5.10)}{\Longleftrightarrow} N_{\mathrm{GSO}_6(\mathbb{C})} = \begin{bmatrix} 0&1&0&0&0&0\\0&0&1&1&0&0\\0&0&0&0&-1&0\\0&0&0&0&-1&0\\0&0&0&0&0&-1\\0&0&0&0&0&0\end{bmatrix};$$

and for (5.28) we have

$$N_{\mathrm{GL}_4(\mathbb{C})\times\mathrm{GL}_1(\mathbb{C})} = \left(\begin{bmatrix} 0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}, 0\right) \overset{(5.10)}{\Longleftrightarrow} N_{\mathrm{GSO}_6(\mathbb{C})} = \begin{bmatrix} 0&0&1&1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&-1\\0&0&0&0&0&-1\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}.$$

(We note, cf. [Tat79, (4.1.5)], that $N_{\mathrm{GL}_4(\mathbb{C})}$ is of the form $O_{2\times 2} \otimes I_{2\times 2} + \begin{bmatrix} 0&1\\0&0\end{bmatrix} \otimes I_{2\times 2}$.)

5.3. **Non-Generic Representaions of** GSpin_6**.** Using the transitivity of the parabolic induction and the classification of generic representations of GL_n, ([Zel80, Theorem 9.7] and [Kud94, Theorem 2.3.1]), the non-generic representations of GSpin_6 are as follows.

$\mathfrak{nongnr}$-(A) $\mathbf{M} \cong \mathrm{GL}_1 \times GL_1 \times \mathrm{GL}_1 \times \mathrm{GL}_1$ and $\widetilde{\mathbf{M}} = (\mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{GL}_1) \times \mathrm{GL}_1$.

Given $\chi_i \in (F^\times)^D$, by Section 3.1 and using (5.12), the representation (5.11) contains a non-generic constituent if and only if the same is true for

$$i^{\mathrm{GL}_4\times GL_1}_{\widetilde{M}}(\widetilde{\chi}_1 \boxtimes \widetilde{\chi}_2 \boxtimes \widetilde{\chi}_3 \boxtimes \widetilde{\chi}_4 \boxtimes \widetilde{\eta}) \tag{5.29}$$

if and only if

$$i^{GL_4}_{\mathrm{GL}_1\times\mathrm{GL}_1\times\mathrm{GL}_1\times\mathrm{GL}_1}(\widetilde{\chi}_1 \boxtimes \widetilde{\chi}_2 \boxtimes \widetilde{\chi}_3 \boxtimes \widetilde{\chi}_4) \tag{5.30}$$

contains a non-generic constituent. This amounts to (5.30) being reducible. As before, the necessary and sufficient condition for this to occur is that there is some pair i, j with $i \neq j$ such that $\widetilde{\chi}_i = \nu\widetilde{\chi}_j$.

By the Langlands classification and the description of constituents of the parabolic induction (see [Zel80, Theorem 7.1], [Rod82, Theorem 7.1], and [Kud94, Theorems 2.1.1 §5.1.1]), each constituent can be described as a Langlands quotient, denoted by $Q(...)$, as follows.

The first case is when there is only one pair, say $\widetilde{\chi}_1 = \nu^{1/2}\widetilde{\chi}$ and $\widetilde{\chi}_2 = \nu^{-1/2}\widetilde{\chi}$ for some $\widetilde{\chi} \in (F^\times)^D$ while $\widetilde{\chi}_3 \neq \nu^{\pm 1}\widetilde{\chi}_j$ for $j \neq 3$ and $\widetilde{\chi}_4 \neq \nu^{\pm 1}\widetilde{\chi}_j$ for $j \neq 4$. Then we have the non-generic constituent

$$Q\left([\nu^{1/2}\widetilde{\chi}], [\nu^{-1/2}\widetilde{\chi}], [\widetilde{\chi}_3], [\widetilde{\chi}_4]\right), \tag{5.31}$$

which is the Langlands quotient of

$$i^{GL_4}_{\mathrm{GL}_2\times\mathrm{GL}_1\times\mathrm{GL}_1}\left(Q\left([\nu^{1/2}\widetilde{\chi}], [\nu^{-1/2}\widetilde{\chi}]\right) \boxtimes \widetilde{\chi}_3 \boxtimes \widetilde{\chi}_4\right) = i^{GL_4}_{\mathrm{GL}_2\times\mathrm{GL}_1\times\mathrm{GL}_1}\left((\widetilde{\chi}\circ\det) \boxtimes \widetilde{\chi}_3 \boxtimes \widetilde{\chi}_4\right).$$

We have

$$N_{\mathrm{GL}_4(\mathbb{C})\times\mathrm{GL}_1(\mathbb{C})} = (0_{4\times4}, 0) \overset{(5.10)}{\Longleftrightarrow} N_{\mathrm{GSO}_6(\mathbb{C})} = 0_{6\times6}.$$

Note that the other constituent of this induced representation, which is generic, is

$$\begin{aligned} Q\left([\nu^{-1/2}\widetilde{\chi}, \nu^{1/2}\widetilde{\chi}], [\widetilde{\chi}_3], [\widetilde{\chi}_4]\right) &= i^{GL_4}_{\mathrm{GL}_2\times\mathrm{GL}_1\times\mathrm{GL}_1}\left(Q\left([\nu^{-1/2}\widetilde{\chi}, \nu^{1/2}\widetilde{\chi}]\right) \boxtimes \widetilde{\chi}_3 \boxtimes \widetilde{\chi}_4\right) \\ &= i^{GL_4}_{\mathrm{GL}_2\times\mathrm{GL}_1\times\mathrm{GL}_1}\left((\mathsf{St}\otimes\widetilde{\chi}) \boxtimes \widetilde{\chi}_3 \boxtimes \widetilde{\chi}_4\right). \end{aligned}$$

The next case is when there are two pairs, say $\widetilde{\chi}_1 = \nu\widetilde{\chi}$, $\widetilde{\chi}_2 = \widetilde{\chi}$, and $\widetilde{\chi}_3 = \nu^{-1}\widetilde{\chi}$ for some $\widetilde{\chi} \in (F^\times)^D$ and $\widetilde{\chi}_4 \neq \nu^{\pm1}\widetilde{\chi}_i$ for $i = 1, 2, 3$. Then we have the following three non-generic constituents:

$$Q\left([\nu\widetilde{\chi}], [\widetilde{\chi}], [\nu^{-1}\widetilde{\chi}], [\widetilde{\chi}_4]\right) = i^{GL_4}_{\mathrm{GL}_3\times\mathrm{GL}_1}((\widetilde{\chi}\circ\det) \boxtimes \widetilde{\chi}_3 \boxtimes \widetilde{\chi}_4); \tag{5.32}$$

$$Q\left([\widetilde{\chi}, \nu\widetilde{\chi}], [\nu^{-1}\widetilde{\chi}], [\widetilde{\chi}_4]\right); \tag{5.33}$$

$$Q\left([\nu\widetilde{\chi}], [\widetilde{\chi}, \nu^{-1}\widetilde{\chi}], [\widetilde{\chi}_4]\right). \tag{5.34}$$

For (5.32) we have

$$N_{\mathrm{GL}_4(\mathbb{C})\times\mathrm{GL}_1(\mathbb{C})} = (0_{4\times4}, 0) \overset{(5.10)}{\Longleftrightarrow} N_{\mathrm{GSO}_6(\mathbb{C})} = 0_{6\times6},$$

for (5.33) we have

$$N_{\mathrm{GL}_4(\mathbb{C})\times\mathrm{GL}_1(\mathbb{C})} = \left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, 0\right) \overset{(5.10)}{\Longleftrightarrow} N_{\mathrm{GSO}_6(\mathbb{C})} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and for (5.34) we have

$$N_{\mathrm{GL}_4(\mathbb{C})\times\mathrm{GL}_1(\mathbb{C})} = \left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, 0\right) \overset{(5.10)}{\Longleftrightarrow} N_{\mathrm{GSO}_6(\mathbb{C})} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Finally, in the case where we have three pairs we are in the situation of (5.27). Then we have the following seven non-generic constituents:

$$Q\left([\nu^{3/2}\widetilde{\chi}], [\nu^{1/2}\widetilde{\chi}], [\nu^{-1/2}\widetilde{\chi}], [\nu^{-3/2}\widetilde{\chi}]\right) = \widetilde{\chi}\circ\det; \tag{5.35}$$

$$Q\left([\nu^{1/2}\widetilde{\chi}, \nu^{3/2}\widetilde{\chi}], [\nu^{-1/2}\widetilde{\chi}], [\nu^{-3/2}\widetilde{\chi}]\right); \tag{5.36}$$

$$Q\left([\nu^{3/2}\widetilde{\chi}],[\nu^{-1/2}\widetilde{\chi},\nu^{1/2}\widetilde{\chi}],[\nu^{-3/2}\widetilde{\chi}]\right); \tag{5.37}$$

$$Q\left([\nu^{3/2}\widetilde{\chi}],[\nu^{1/2}\widetilde{\chi}],[\nu^{-3/2}\widetilde{\chi},\nu^{-1/2}\widetilde{\chi}]\right); \tag{5.38}$$

$$Q\left([\nu^{1/2}\widetilde{\chi},\nu^{3/2}\widetilde{\chi}],[\nu^{-3/2}\widetilde{\chi},\nu^{-1/2}\widetilde{\chi}]\right); \tag{5.39}$$

$$Q\left([\nu^{-1/2}\widetilde{\chi},\nu^{1/2}\widetilde{\chi},\nu^{3/2}\widetilde{\chi}],[\nu^{-3/2}\widetilde{\chi}]\right); \tag{5.40}$$

$$Q\left([\nu^{3/2}\widetilde{\chi}],[\nu^{-3/2}\widetilde{\chi},\nu^{-1/2}\widetilde{\chi},\nu^{1/2}\widetilde{\chi}]\right). \tag{5.41}$$

For (5.35) we have

$$N_{\mathrm{GL}_4(\mathbb{C})\times\mathrm{GL}_1(\mathbb{C})} = (0_{4\times 4}, 0) \overset{(5.10)}{\Longleftrightarrow} N_{\mathrm{GSO}_6(\mathbb{C})} = 0_{6\times 6},$$

for (5.36) we have

$$N_{\mathrm{GL}_4(\mathbb{C})\times\mathrm{GL}_1(\mathbb{C})} = \left(\begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},0\right) \overset{(5.10)}{\Longleftrightarrow} N_{\mathrm{GSO}_6(\mathbb{C})} = \begin{bmatrix}0&0&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix},$$

for (5.37) we have

$$N_{\mathrm{GL}_4(\mathbb{C})\times\mathrm{GL}_1(\mathbb{C})} = \left(\begin{bmatrix}0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix},0\right) \overset{(5.10)}{\Longleftrightarrow} N_{\mathrm{GSO}_6(\mathbb{C})} = \begin{bmatrix}0&1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&-1\\0&0&0&0&0&0\end{bmatrix},$$

for (5.38) we have

$$N_{\mathrm{GL}_4(\mathbb{C})\times\mathrm{GL}_1(\mathbb{C})} = \left(\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix},0\right) \overset{(5.10)}{\Longleftrightarrow} N_{\mathrm{GSO}_6(\mathbb{C})} = \begin{bmatrix}0&0&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix},$$

for (5.39) we have

$$N_{\mathrm{GL}_4(\mathbb{C})\times\mathrm{GL}_1(\mathbb{C})} = \left(\begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix},0\right) \overset{(5.10)}{\Longleftrightarrow} N_{\mathrm{GSO}_6(\mathbb{C})} = \begin{bmatrix}0&0&0&0&0&0\\0&0&1&1&0&0\\0&0&0&0&-1&0\\0&0&0&0&-1&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix},$$

for (5.40) we have

$$N_{\mathrm{GL}_4(\mathbb{C})\times\mathrm{GL}_1(\mathbb{C})} = \left(\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix},0\right) \overset{(5.10)}{\Longleftrightarrow} N_{\mathrm{GSO}_6(\mathbb{C})} = \begin{bmatrix}0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&-1\\0&0&0&0&0&0\end{bmatrix},$$

and for (5.41) we have

$$N_{\mathrm{GL}_4(\mathbb{C})\times\mathrm{GL}_1(\mathbb{C})} = \left(\begin{bmatrix} 0&0&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&0&0&0 \end{bmatrix}, 0 \right) \overset{(5.10)}{\Longleftrightarrow} N_{\mathrm{GSO}_6(\mathbb{C})} = \begin{bmatrix} 0&1&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&-1&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&-1\\ 0&0&0&0&0&0 \end{bmatrix}.$$

$\mathfrak{nongnr}$-(B) $\mathbf{M} \cong \mathrm{GL}_2 \times \mathrm{GL}_1 \times \mathrm{GL}_1$ and $\widetilde{\mathbf{M}} = (\mathrm{GL}_2 \times \mathrm{GL}_1 \times \mathrm{GL}_1) \times \mathrm{GL}_1$.

Given $\sigma_0 \in \mathrm{Irr}(\mathrm{GL}_2)$ and $\chi_1, \chi_2 \in (F^\times)^D$, we consider

$$i_M^{\mathrm{GSpin}_6}(\sigma_0 \boxtimes \chi_1 \boxtimes \chi_2). \tag{5.42}$$

Write

$$\sigma_0 \boxtimes \chi_1 \boxtimes \chi_2 = (\widetilde{\sigma}_0 \boxtimes \widetilde{\chi}_1 \boxtimes \widetilde{\chi}_2 \boxtimes \widetilde{\eta})|_M$$

with $\widetilde{\sigma}_0 \in \mathrm{Irr}(\mathrm{GL}_2)$ and $\widetilde{\chi}_i, \widetilde{\eta} \in (F^\times)^D$. By (5.16), it follows that (5.42) contains a non-generic constituent if and only if its lift

$$i_{\widetilde{M}}^{\mathrm{GL}_4 \times GL_1}(\widetilde{\sigma}_0 \boxtimes \widetilde{\chi}_1 \boxtimes \widetilde{\chi}_2 \boxtimes \widetilde{\eta}) \tag{5.43}$$

contains a non-generic constituent if and only if

$$i_{\mathrm{GL}_2 \times \mathrm{GL}_1 \times \mathrm{GL}_1}^{GL_4}(\widetilde{\sigma}_0 \boxtimes \widetilde{\chi}_1 \boxtimes \widetilde{\chi}_2) \tag{5.44}$$

does. Recalling $\mathfrak{nongnr}$-(A), it is sufficient to consider the case of $\widetilde{\sigma}_0 \in \mathrm{Irr}(\mathrm{GL}_2)$, $\widetilde{\chi}_1 = \nu^{1/2}\widetilde{\chi}$, and $\widetilde{\chi}_2 = \nu^{-1/2}\widetilde{\chi}$ for $\widetilde{\chi} \in (F^\times)^D$, where the segment $\Delta_{\widetilde{\sigma}_0}$ of $\widetilde{\sigma}_0$ does not precede either $\widetilde{\chi}_1$ or $\widetilde{\chi}_2$. We then have the following sole non-generic constituent:

$$Q([\Delta_{\widetilde{\sigma}_0}], [\nu^{1/2}\widetilde{\chi}], [\nu^{-1/2}\widetilde{\chi}]). \tag{5.45}$$

We have

$$N_{\mathrm{GL}_4(\mathbb{C})\times\mathrm{GL}_1(\mathbb{C})} = (0_{4\times4}, 0) \overset{(5.10)}{\Longleftrightarrow} N_{\mathrm{GSO}_6(\mathbb{C})} = 0_{6\times6}.$$

$\mathfrak{nongnr}$-(C) $\mathbf{M} \cong \mathrm{GL}_3 \times \mathrm{GL}_1$ and $\widetilde{\mathbf{M}} = (\mathrm{GL}_3 \times \mathrm{GL}_1) \times \mathrm{GL}_1$.

Given a non-generic $\sigma_0 \in \mathrm{Irr}(\mathrm{GL}_3)$ and any $\chi \in (F^\times)^D$, we consider

$$i_M^{\mathrm{GSpin}_6}(\sigma_0 \boxtimes \chi). \tag{5.46}$$

Write

$$\sigma_0 \boxtimes \chi = (\widetilde{\sigma}_0 \boxtimes \widetilde{\chi} \boxtimes \widetilde{\eta})|_M$$

with non-generic $\widetilde{\sigma}_0 \in \mathrm{Irr}(\mathrm{GL}_3)$ and $\widetilde{\chi}, \widetilde{\eta} \in (F^\times)^D$. As in (5.20) we have

$$\widetilde{\sigma}_0 = \sigma_0\widetilde{\chi}, \quad \text{and} \quad \widetilde{\eta} = \chi_2\widetilde{\chi}^{-2}.$$

As before, (5.46) contains a non-generic constituent if and only if its lift

$$i_{\widetilde{M}}^{\mathrm{GL}_4 \times GL_1}(\widetilde{\sigma}_0 \boxtimes \widetilde{\chi} \boxtimes \widetilde{\eta}) \tag{5.47}$$

also contains one if and only if

$$i_{\mathrm{GL}_3 \times \mathrm{GL}_1}^{GL_4}(\widetilde{\sigma}_0 \boxtimes \widetilde{\chi}) \tag{5.48}$$

does. To have a non-generic $\widetilde{\sigma}_0$ of $\mathrm{GL}_3(F)$, the irreducible representation $\widetilde{\sigma}_0$ must be some constituent in a reducible induction. This case has been covered in $\mathfrak{nongnr}$-(A) and (B) above.

$\mathfrak{nongnr}$-(D) $\mathbf{M} \cong \mathrm{GL}_1 \times \mathrm{GSpin}_4$ and $\widetilde{\mathbf{M}} = (\mathrm{GL}_2 \times \mathrm{GL}_2) \times \mathrm{GL}_1$.

Given a non-generic $\sigma_0 \in \mathrm{Irr}(\mathrm{GSpin}_4)$, by Section 4.3, we know that it must be of the form

$$\mathrm{Res}_{\mathrm{GSpin}_4}^{\mathrm{GL}_2 \times \mathrm{GL}_2}((\chi \circ \det) \boxtimes \widetilde{\sigma})$$

for $\widetilde{\sigma} \in \mathrm{Irr}(\mathrm{GL}_2)$. For $\eta \in (F^\times)^D$, the induced representation

$$i_M^{\mathrm{GSpin}_6}((\chi \circ \det) \boxtimes \widetilde{\sigma} \boxtimes \eta) \tag{5.49}$$

contains a non-generic constituent if and only if so does

$$i^{\mathrm{GL}_4}_{\mathrm{GL}_2\times\mathrm{GL}_2}((\chi\circ\det)\boxtimes\widetilde{\sigma}),$$

which is always the case. Therefore, if $\widetilde{\sigma}$ is supercuspidal, then

$$N_{\mathrm{GL}_4(\mathbb{C})\times\mathrm{GL}_1(\mathbb{C})}=(0_{4\times4},0)\overset{(5.10)}{\Longleftrightarrow}N_{\mathrm{GSO}_6(\mathbb{C})}=0_{6\times6}.$$

If $\widetilde{\sigma}$ is non-supercuspidal, then it suffices to consider the case $\widetilde{\sigma}=\mathsf{St}_{\mathrm{GL}_2}\otimes\eta$ with $\eta\in(F^\times)^D$ since the other case has been covered in $\mathfrak{nongnr}$-(A). Thus, we have

$$N_{\mathrm{GL}_4(\mathbb{C})\times\mathrm{GL}_1(\mathbb{C})}=\left(\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix},0\right)\overset{(5.10)}{\Longleftrightarrow}N_{\mathrm{GSO}_6(\mathbb{C})}=\begin{bmatrix}0&0&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}.$$

$\mathfrak{nongnr}$-(E) $\mathbf{M}\cong\mathrm{GSpin}_6$ and $\widetilde{\mathbf{M}}=\mathrm{GL}_4\times\mathrm{GL}_1$.

Given a non-generic $\sigma\in\mathrm{Irr}(\mathrm{GSpin}_6)$, it must be of the form

$$\mathrm{Res}^{\mathrm{GL}_4\times\mathrm{GL}_1}_{\mathrm{GSpin}_6}(\widetilde{\chi}\circ\det\boxtimes\widetilde{\eta})=\chi\circ\det,\tag{5.50}$$

for some $\widetilde{\chi},\widetilde{\eta}\in(F^\times)^D$. This is the case $Q([\nu^{3/2}\widetilde{\chi}],[\nu^{1/2}\widetilde{\chi}],[\nu^{-1/2}\widetilde{\chi}],[\nu^{-3/2}\widetilde{\chi}])$ in $\mathfrak{nongnr}$-(A).

5.4. **Computation of the Adjoint L-function for** GSpin_6**.** We now give explicit expressions for the adjoint L-function of each of the representations of $\mathrm{GSpin}_6(F)$. Recall that if we have a parameter (ϕ,N) with N a nilpotent matrix on the vector space V, then its adjoint L-function is

$$L(s,\phi,\mathrm{Ad})=\det\left(1-q^{-s}\mathrm{Ad}(\phi)|V_N^I\right)^{-1},$$

where $V_N=\ker(N)$, V^I the vectors fixed by the inertia group, and $V_N^I=V^I\cap V_N$. Below for the cases where N is non-zero, we write $\ker(\mathrm{Ad}(N))$ and we use L_α to denote the root group associated with the root α.

We now consider each case. Using (2.14) and Sections 5.2, and 5.3, we have the following.

$\mathfrak{gnr}$-(a) Given $\sigma\in\mathrm{Irr}_{\mathrm{sc}}(\mathrm{GSpin}_6)$, we have $\widetilde{\sigma}=\widetilde{\sigma}_0\boxtimes\widetilde{\eta}\in\mathrm{Irr}_{\mathrm{sc}}(\mathrm{GL}_4\times\mathrm{GL}_1)$. Then

$$L(s,1_{F^\times})L(s,\sigma,\mathrm{Ad})=L(s,\widetilde{\sigma}_0,\mathrm{Ad}_{\widehat{GL_4}})$$

or

$$L(s,\sigma,\mathrm{Ad})=L(s,\widetilde{\sigma}_0,\mathrm{Ad}).$$

$\mathfrak{gnr}$-(I) Given $\mathbf{M}\cong\mathrm{GL}_1\times GL_1\times\mathrm{GL}_1\times\mathrm{GL}_1$ and $\widetilde{\mathbf{M}}=(\mathrm{GL}_1\times\mathrm{GL}_1\times\mathrm{GL}_1\times\mathrm{GL}_1)\times\mathrm{GL}_1$, we recall

$$i^{GL_4}_{\mathrm{GL}_1\times\mathrm{GL}_1\times\mathrm{GL}_1\times\mathrm{GL}_1}(\widetilde{\chi}_1\boxtimes\widetilde{\chi}_2\boxtimes\widetilde{\chi}_3\boxtimes\widetilde{\chi}_4)$$

must be irreducible. Thus, given $\sigma\in\mathrm{Irr}(\mathrm{GSpin}_6)$ such that

$$\sigma=i_M^{\mathrm{GSpin}_6}(\widetilde{\chi}_1\boxtimes\widetilde{\chi}_2\boxtimes\widetilde{\chi}_3\boxtimes\widetilde{\chi}_4),$$

we have

$$L(s,\sigma,\mathrm{Ad})=L(s)^3\prod_{i\neq j}L(s,\widetilde{\chi}_i\widetilde{\chi}_j^{-1}).$$

$\mathfrak{gnr}$-(II) Given $\mathbf{M}\cong\mathrm{GL}_2\times\mathrm{GL}_1\times\mathrm{GL}_1$ and $\widetilde{\mathbf{M}}=(\mathrm{GL}_2\times\mathrm{GL}_1\times\mathrm{GL}_1)\times\mathrm{GL}_1$, for $\sigma_0\in\mathrm{Irr}_{\mathrm{esq}}(\mathrm{GL}_2)$ and $\chi_1,\chi_2\in(F^\times)^D$, we have an irreducible induced representation

$$\sigma=i_M^{\mathrm{GSpin}_6}(\sigma_0\boxtimes\chi_1\boxtimes\chi_2)=\mathrm{Res}^{\mathrm{GL}_4\times\mathrm{GL}_1}_{\mathrm{GSpin}_6}\left(i^{GL_4}_{\mathrm{GL}_2\times\mathrm{GL}_1\times\mathrm{GL}_1}(\widetilde{\sigma}_0\boxtimes\widetilde{\chi}_1\boxtimes\widetilde{\chi}_2\boxtimes\widetilde{\eta})\right),$$

for some $\widetilde{\sigma}_0\in\mathrm{Irr}_{\mathrm{esq}}(\mathrm{GL}_2)$, and $\widetilde{\chi}_i,\widetilde{\eta}\in(F^\times)^D$. For supercuspidal $\widetilde{\sigma}_0$ we have

$$\begin{aligned}L(s,\sigma,\mathrm{Ad})\;=\;&L(s)^2L(s,\widetilde{\sigma}_0,\mathrm{Ad})L(s,\widetilde{\sigma}_0\times\widetilde{\chi}_1^{-1})L(s,\widetilde{\sigma}_0^\vee\times\widetilde{\chi}_1)\\&L(s,\widetilde{\sigma}_0\times\widetilde{\chi}_2^{-1})L(s,\widetilde{\sigma}_0^\vee\times\widetilde{\chi}_2)L(s,\widetilde{\chi}_1\widetilde{\chi}_2^{-1})L(s,\widetilde{\chi}_2\widetilde{\chi}_1^{-1}).\end{aligned}$$

For non-supercuspidal $\widetilde{\sigma}_0 \in \mathrm{Irr}(\mathrm{GL}_2)$, i.e., $\sigma_0 = \mathsf{St}_{\mathrm{GL}_2} \otimes \widetilde{\chi}$ for some $\widetilde{\chi} \in (F^\times)^D$, we have

$$\ker\left(\mathrm{ad}\begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\right) = \left\langle \begin{bmatrix}a&0&0&0\\0&a&0&0\\0&0&b&0\\0&0&0&c\end{bmatrix}, L_{f_1-f_2}, L_{f_1-f_3}, L_{f_1-f_4}, L_{f_3-f_2}, L_{f_3-f_4}, L_{f_4-f_2}, L_{f_4-f_3}\right\rangle. \tag{5.51}$$

It follows that

$$\begin{aligned} L(s,\sigma,\mathrm{Ad}) &= L(s)^2 L(s+1) L(s+1, \widetilde{\chi}\widetilde{\chi}_1^{-1}) L(s+1, \widetilde{\chi}\widetilde{\chi}_2^{-1}) \\ &\cdot L(s, \widetilde{\chi}^{-1}\widetilde{\chi}_1) L(s, \widetilde{\chi}^{-1}\widetilde{\chi}_2) L(s, \widetilde{\chi}_1\widetilde{\chi}_2^{-1}) L(s, \widetilde{\chi}_2\widetilde{\chi}_1^{-1}). \end{aligned}$$

$\mathfrak{gnr}$-(III) Given $\mathbf{M} \cong \mathrm{GL}_3 \times \mathrm{GL}_1$ and $\widetilde{\mathbf{M}} = (\mathrm{GL}_3 \times \mathrm{GL}_1) \times \mathrm{GL}_1$, for $\sigma_0 \in \mathrm{Irr}_{\mathrm{esq}}(\mathrm{GL}_3)$ and $\chi \in (F^\times)^D$, we have an irreducible induced representation

$$\sigma = i_M^{\mathrm{GSpin}_6}(\sigma_0 \boxtimes \chi) = \mathrm{Res}^{\mathrm{GL}_4 \times \mathrm{GL}_1}_{\mathrm{GSpin}_6}\left(i^{GL_4 \times \mathrm{GL}_1}_{\mathrm{GL}_3 \times \mathrm{GL}_1 \times \mathrm{GL}_1}(\widetilde{\sigma}_0 \boxtimes \widetilde{\chi} \boxtimes \widetilde{\eta})\right),$$

for $\widetilde{\sigma}_0 \in \mathrm{Irr}_{\mathrm{esq}}(\mathrm{GL}_3)$ and $\widetilde{\chi}, \widetilde{\eta} \in (F^\times)^D$. If $\widetilde{\sigma}_0 \in \mathrm{Irr}_{\mathrm{esq}}(\mathrm{GL}_3)$ is supercuspidal, then we have

$$L(s,\sigma,\mathrm{Ad}) = L(s) L(s, \widetilde{\sigma}_0, \mathrm{Ad}) L(s, \widetilde{\sigma}_0 \times \widetilde{\chi}^{-1}) L(s, \widetilde{\sigma}_0^\vee \times \widetilde{\chi}).$$

For non-supercuspidal $\widetilde{\sigma}_0 \in \mathrm{Irr}_{\mathrm{esq}}(\mathrm{GL}_3)$, i.e., $\sigma_0 = \mathsf{St}_{\mathrm{GL}_3} \otimes \widetilde{\chi}_0$ for some $\widetilde{\chi}_0 \in (F^\times)^D$, we have

$$\ker\left(\mathrm{ad}\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\right) = \left\langle \begin{bmatrix}a&c&0&0\\0&a&c&0\\0&0&a&0\\0&0&0&b\end{bmatrix}, L_{f_1-f_3}, L_{f_1-f_4}, L_{f_4-f_3}\right\rangle. \tag{5.52}$$

It follows that

$$L(s,\sigma,\mathrm{Ad}) = L(s) L(s+1) L(s+2) L(s+1, \widetilde{\chi}\widetilde{\chi}_0^{-1}) L(s+1, \widetilde{\chi}^{-1}\widetilde{\chi}_0).$$

$\mathfrak{gnr}$-(IV) Given $\mathbf{M} \cong \mathrm{GL}_1 \times \mathrm{GSpin}_4$ and $\widetilde{\mathbf{M}} = (\mathrm{GL}_2 \times \mathrm{GL}_2) \times \mathrm{GL}_1$, we have the representation (5.24)

$$\sigma = i_M^{\mathrm{GSpin}_6}(\chi \boxtimes \sigma_0)$$

with $\sigma_0 \in \mathrm{Irr}_{\mathrm{esq}}(\mathrm{GSpin}_4)$, and $\chi \in (F^\times)^D$. We have the irreducible $i^{GL_4}_{\mathrm{GL}_2 \times \mathrm{GL}_2}(\widetilde{\sigma}_1 \boxtimes \widetilde{\sigma}_2)$ as in (5.26), where $\chi \boxtimes \sigma_0 \subset (\widetilde{\sigma}_1 \boxtimes \widetilde{\sigma}_2 \boxtimes \widetilde{\eta})|_M$ with $\widetilde{\sigma}_i \in \mathrm{Irr}_{\mathrm{esq}}(\mathrm{GL}_2), \widetilde{\eta} \in (F^\times)^D$. Thus, if σ_0 is supercuspidal (and hence so are $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$) we have

$$L(s,\sigma,\mathrm{Ad}) = L(s) L(s, \widetilde{\sigma}_1, \mathrm{Ad}) L(s, \widetilde{\sigma}_2, \mathrm{Ad}) L(s, \widetilde{\sigma}_1 \times \widetilde{\sigma}_2^\vee) L(s, \widetilde{\sigma}_1^\vee \times \widetilde{\sigma}_1).$$

If σ_0 is non-supercuspidal, with $\widetilde{\sigma}_1$ supercuspidal and $\widetilde{\sigma}_2$ non-supercuspidal, i.e., $\widetilde{\sigma}_2 = \mathsf{St}_{\mathrm{GL}_2} \otimes \widetilde{\chi}$ for some $\widetilde{\chi} \in (F^\times)^D$, we have

$$\ker\left(\mathrm{ad}\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix}\right) = \left\langle \begin{bmatrix}a&0&0&0\\0&b&0&0\\0&0&c&0\\0&0&0&c\end{bmatrix}, L_{f_1-f_2}, L_{f_1-f_4}, L_{f_2-f_1}, L_{f_2-f_4}, L_{f_3-f_1}, L_{f_3-f_2}, L_{f_3-f_4}\right\rangle, \tag{5.53}$$

and it then follows that

$$L(s,\sigma,\mathrm{Ad}) = L(s) L(s+1) L(s, \widetilde{\sigma}_1, \mathrm{Ad}) L(s+\frac{1}{2}, \widetilde{\sigma}_1^\vee \times \widetilde{\chi}) L(s+\frac{1}{2}, \widetilde{\sigma}_1 \times \widetilde{\chi}^{-1}).$$

If σ_0 is non-supercuspidal, with $\widetilde{\sigma}_1$ non-supercuspidal and $\widetilde{\sigma}_2$ supercuspidal, i.e., $\widetilde{\sigma}_1 = \mathsf{St}_{\mathrm{GL}_2} \otimes \widetilde{\chi}$ for some $\widetilde{\chi} \in (F^\times)^D$, then $\ker(\mathrm{ad}(N))$ is as in (5.51) and we have

$$L(s,\sigma,\mathrm{Ad}) = L(s) L(s+1) L(s, \widetilde{\sigma}_2, \mathrm{Ad}) L(s+\frac{1}{2}, \widetilde{\sigma}_2^\vee \times \widetilde{\chi}) L(s+\frac{1}{2}, \widetilde{\sigma}_2 \times \widetilde{\chi}^{-1}).$$

If both $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ are non-supercuspidal, i.e., $\widetilde{\sigma}_i = \mathsf{St}_{\mathrm{GL}_2} \otimes \widetilde{\chi}_i$ with $\widetilde{\chi}_1, \widetilde{\chi}_2 \in (F^\times)^D$ satisfying $\widetilde{\chi}_1 \neq \widetilde{\chi}_2 \nu^{\pm 1}$, we have

$$\ker\left(\mathrm{ad}\begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix}\right) = \left\langle \begin{bmatrix}a&0&c&0\\0&a&0&c\\d&0&b&0\\0&d&0&b\end{bmatrix}, L_{f_1-f_2}, L_{f_1-f_4}, L_{f_3-f_2}, L_{f_3-f_4}\right\rangle, \tag{5.54}$$

and it follows that

$$L(s,\sigma,\mathrm{Ad}) = L(s)L(s+1)^2 L(s+1,\widetilde{\chi}_1\widetilde{\chi}_2^{-1})L(s+1,\widetilde{\chi}_1^{-1}\widetilde{\chi}_2)L(s,\widetilde{\chi}_1^{-1}\widetilde{\chi}_2)L(s,\widetilde{\chi}_1\widetilde{\chi}_2^{-1}).$$

$\mathfrak{gnr}$-(V) Given $\mathbf{M} \cong \mathrm{GL}_1 \times \mathrm{GSpin}_4$ and $\widetilde{\mathbf{M}} = (\mathrm{GL}_2 \times \mathrm{GL}_2) \times \mathrm{GL}_1$, we consider $\sigma \in \mathrm{Irr}_{\mathrm{esq}}(\mathrm{GSpin}_6)$ and $\widetilde{\sigma} \in \mathrm{Irr}_{\mathrm{esq}}(\mathrm{GL}_4)$ and $\widetilde{\eta} \in (F^\times)^D$ such that $\sigma \subset (\widetilde{\sigma} \boxtimes \widetilde{\eta})|_M$. Then, $\widetilde{\sigma}$ must be either (5.27) or (5.28).

For (5.27) (i.e., $\widetilde{\sigma} = \mathsf{St}_{\mathrm{GL}_4} \otimes \widetilde{\chi}$), we have

$$\ker\left(\mathrm{ad}\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}\right) = \left\langle \begin{bmatrix}a&b&c&0\\0&a&b&c\\0&0&a&b\\0&0&0&a\end{bmatrix}, L_{f_1-f_4}\right\rangle, \tag{5.55}$$

and it follows that

$$L(s,\sigma,\mathrm{Ad}) = L(s+3)L(s+2)L(s+1).$$

For (5.28) (i.e., $\widetilde{\tau} \in \mathrm{Irr}_{\mathrm{sc}}(\mathrm{GL}_2)$), we have

$$\ker\left(\mathrm{ad}\begin{bmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}\right) = \left\langle \begin{bmatrix}a&c&0&0\\d&b&0&0\\0&0&a&c\\0&0&d&b\end{bmatrix}, L_{f_1-f_3}, L_{f_1-f_4}, L_{f_2-f_3}, L_{f_2-f_4}\right\rangle, \tag{5.56}$$

and it follows that

$$L(s,\sigma,\mathrm{Ad}) = L(s,\widetilde{\tau},\mathrm{Ad})L(s,\widetilde{\tau}\times\widetilde{\tau}^\vee).$$

$\mathfrak{nongnr}$-(A) For $Q([\nu^{1/2}\widetilde{\chi}],[\nu^{-1/2}\widetilde{\chi}],[\widetilde{\chi}_3],[\widetilde{\chi}_4])$ (5.31), we have

$$\begin{aligned} L(s,\sigma,\mathrm{Ad}) &= L(s)^3 L(s+1)L(s-1)L(s,\widetilde{\chi}_3\widetilde{\chi}_4^{-1})L(s,\widetilde{\chi}_3^{-1}\widetilde{\chi}_4) \\ &\prod_{i=3,4}\left(L(s+\frac{1}{2},\widetilde{\chi}\widetilde{\chi}_i^{-1})L(s-\frac{1}{2},\widetilde{\chi}^{-1}\widetilde{\chi}_i)L(s-\frac{1}{2},\widetilde{\chi}\widetilde{\chi}_i^{-1})L(s+\frac{1}{2},\widetilde{\chi}^{-1}\widetilde{\chi}_i)\right)\end{aligned}$$

For $Q\left([\nu\widetilde{\chi}],[\widetilde{\chi}],[\nu^{-1}\widetilde{\chi}],[\widetilde{\chi}_4]\right)$ (5.32), we have

$$L(s,\sigma,\mathrm{Ad}) = L(s)^3L(s+1)^2L(s-1)^2L(s+2)L(s-2)\prod_{t=0,1,-1}\left(L(s+t,\widetilde{\chi}\widetilde{\chi}_4^{-1})L(s+t,\widetilde{\chi}^{-1}\widetilde{\chi}_4)\right),$$

For $Q([\widetilde{\chi},\nu\widetilde{\chi}],[\nu^{-1}\widetilde{\chi}],[\widetilde{\chi}_4])$ (5.33), we have $\ker(\mathrm{ad}(N))$ as in (5.51) and

$$L(s,\sigma,\mathrm{Ad}) = L(s)^2L(s-1)^2L(s-2)\prod_{t=-1,0}L(s+t,\widetilde{\chi}\widetilde{\chi}_4^{-1})\prod_{t=\pm1}L(s+t,\widetilde{\chi}^{-1}\widetilde{\chi}_4).$$

For $Q\left([\nu\widetilde{\chi}],[\widetilde{\chi},\nu^{-1}\widetilde{\chi}],[\widetilde{\chi}_4]\right)$ (5.34), since

$$\ker\left(\mathrm{ad}\begin{bmatrix}0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\right) = \left\langle \begin{bmatrix}a&0&0&0\\0&b&0&0\\0&0&b&0\\0&0&0&c\end{bmatrix}, L_{f_1-f_3}, L_{f_1-f_4}, L_{f_2-f_1}, L_{f_2-f_3}, L_{f_2-f_4}, L_{f_4-f_1}, L_{f_4-f_3}\right\rangle, \tag{5.57}$$

we have

$$L(s,\sigma,\mathrm{Ad}) = L(s)^2L(s+2)L(s-1)L(s+1)\prod_{t=0,1}L(s+t,\widetilde{\chi}\widetilde{\chi}_4^{-1})\prod_{t=\pm1}L(s+t,\widetilde{\chi}^{-1}\widetilde{\chi}_4).$$

For $Q([\nu^{3/2}\widetilde{\chi}],[\nu^{1/2}\widetilde{\chi}],[\nu^{-1/2}\widetilde{\chi}],[\nu^{-3/2}\widetilde{\chi}])$ (5.35), we have

$$L(s,\sigma,\mathrm{Ad})=L(s)^3L(s+1)^3L(s-1)^3L(s+2)^2L(s-2)^2L(s+3)L(s-3).$$

For $Q\left([\nu^{1/2}\widetilde{\chi},\nu^{3/2}\widetilde{\chi}],[\nu^{-1/2}\widetilde{\chi}],[\nu^{-3/2}\widetilde{\chi}]\right)$ (5.36), we have $\ker(\mathrm{ad}(N))$ is as in (5.51) and

$$L(s,\sigma,\mathrm{Ad})=L(s)^2L(s-1)^2L(s+1)^2L(s-2)L(s+2)L(s-3).$$

For $Q([\nu^{3/2}\widetilde{\chi}],[\nu^{-1/2}\widetilde{\chi},\nu^{1/2}\widetilde{\chi}],[\nu^{-3/2}\widetilde{\chi}])$ (5.37), we have $\ker(\mathrm{ad}(N))$ is as in (5.57) and

$$L(s,\sigma,\mathrm{Ad})=L(s)^2L(s+1)^2L(s+2)L(s-1)^2L(s-3)L(s-2).$$

For $Q([\nu^{3/2}\widetilde{\chi}],[\nu^{1/2}\widetilde{\chi}],[\nu^{-3/2}\widetilde{\chi},\nu^{-1/2}\widetilde{\chi}])$ (5.38), we have $\ker(\mathrm{ad}(N))$ is as in (5.53) and

$$L(s,\sigma,\mathrm{Ad})=L(s)^2L(s+1)^2L(s-1)^2L(s-2)L(s+2)L(s-3).$$

For $Q([\nu^{1/2}\widetilde{\chi},\nu^{3/2}\widetilde{\chi}],[\nu^{-3/2}\widetilde{\chi},\nu^{-1/2}\widetilde{\chi}])$ (5.39), we have $\ker(\mathrm{ad}(N))$ is as in (5.54) and

$$L(s,\sigma,\mathrm{Ad})=L(s)L(s-1)^2L(s+1)L(s+2)L(s-2)L(s-3).$$

For $Q([\nu^{-1/2}\widetilde{\chi},\nu^{1/2}\widetilde{\chi},\nu^{3/2}\widetilde{\chi}],[\nu^{-3/2}\widetilde{\chi}])$ (5.40), we have $\ker(\mathrm{ad}(N))$ is as in (5.52) and

$$L(s,\sigma,\mathrm{Ad})=L(s)L(s-1)L(s-2)L(s+1)L(s-3).$$

Finally, for $Q\left([\nu^{3/2}\widetilde{\chi}],[\nu^{-3/2}\widetilde{\chi},\nu^{-1/2}\widetilde{\chi},\nu^{1/2}\widetilde{\chi}]\right)$ (5.41), since

$$\ker\left(\mathrm{ad}\begin{bmatrix}0&0&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}\right)=\left\langle\begin{bmatrix}a&0&0&0\\0&b&c&0\\0&0&b&c\\0&0&0&b\end{bmatrix},L_{f_1-f_4},L_{f_2-f_1},L_{f_2-f_4}\right\rangle, \tag{5.58}$$

we have

$$L(s,\sigma,\mathrm{Ad})=L(s)L(s+1)L(s-1)L(s-2)L(s-3).$$

$\mathfrak{nongnr}$-(B) For $Q([\Delta_{\widetilde{\sigma}_0}],[\nu^{1/2}\widetilde{\chi}],[\nu^{-1/2}\widetilde{\chi}])$ (5.45), with say $[\Delta_{\widetilde{\sigma}_0}]=i^{\mathrm{GL}_2}_{\mathrm{GL}_1\times\mathrm{GL}_1}(\widetilde{\eta}_1\boxtimes\widetilde{\eta}_2),\ \widetilde{\eta}_1\widetilde{\eta}_2^{-1}\neq\nu^{\pm1}$ we have

$$\begin{aligned}L(s,\sigma,\mathrm{Ad}) &= L(s)^3L(s+1)L(s-1)L(s,\widetilde{\eta}_1\widetilde{\eta}_2^{-1})L(s,\widetilde{\eta}_1^{-1}\widetilde{\eta}_2)\\ &\quad\prod_{i=1,2}\left(L(s-\frac{1}{2},\widetilde{\eta}_i\widetilde{\chi}^{-1})L(s+\frac{1}{2},\widetilde{\eta}_i\widetilde{\chi}^{-1})L(s+\frac{1}{2},\widetilde{\eta}_i^{-1}\widetilde{\chi})L(s-\frac{1}{2},\widetilde{\eta}_i^{-1}\widetilde{\chi})\right).\end{aligned}$$

$\mathfrak{nongnr}$-(C) As mentioned before, all the possibilities in this case were covered in (A) and (B) above.

$\mathfrak{nongnr}$-(D) For (5.49) with $\widetilde{\sigma}$ supercuspidal, we have

$$\begin{aligned}L(s,\sigma,\mathrm{Ad}) &= L(s)^2L(s+1)L(s-1)L(s,\sigma,\mathrm{Ad})\\ &\quad L(s-\frac{1}{2},\sigma\times\chi^{-1})L(s+\frac{1}{2},\sigma\times\chi^{-1})L(s-\frac{1}{2},\sigma^\vee\times\chi)L(s+\frac{1}{2},\sigma^\vee\times\chi),\end{aligned}$$

For (5.49) with non-supercuspidal $\widetilde{\sigma}=\mathsf{St}_{\mathrm{GL}_2}\otimes\eta,\ \eta\in(F^\times)^D$ we have $\ker(\mathrm{ad}(N))$ as in (5.53) and

$$L(s,\sigma,\mathrm{Ad}) = L(s)^2L(s+1)^2L(s-1)L(s,\chi\eta^{-1})L(s+1,\chi\eta^{-1})L(s+1,\chi^{-1}\eta)L(s,\chi^{-1}\eta).$$

Recall that the remaining possibilities in this case were already covered in (A) above.

$\mathfrak{nongnr}$-(E) Finally, as mentioned before, all the possibilities in this case we also covered in (A).

6. Correction to [AC17]

We take this opportunity to correct the following errors in our earlier work [AC17]. They do not affect the main results in that paper.

6.1. Proposition 5.5 and 6.4.

- Change "1,2,4,8, if $p = 2$" to "1,2,4,8,..., $2^{[F:\mathbb{Q}_2]+2}$, if $p = 2$." We have $2^{[F:\mathbb{Q}_p]+2}$ due to the fact that $\left|F^\times/(F^\times)^2\right| = 2^{[F:\mathbb{Q}_2]+2}$.
- For Proposition 5.5, using [GP92, Corollary 7.7], it follows that the case of $p = 2$ is bounded by $|(\mathbb{Z}/2\mathbb{Z})^{4-1}| = 8$. Here 4 is coming from $\widehat{\mathrm{GSpin}_4} = \mathrm{GSO}(4, \mathbb{C})$.
- For Proposition 6.4, using [GP92, Corollary 7.7], it follows that the case of $p = 2$ is bounded by $|(\mathbb{Z}/2\mathbb{Z})^{6-1}| = 32$. Here 6 is coming from $\widehat{\mathrm{GSpin}_6} = \mathrm{GSO}(6, \mathbb{C})$.

6.2. Remark 5.11.

- The formula (5.13) should read as follows:
$$\Big|\Pi_\varphi\left(\mathrm{GSpin}_4\right)\Big| = \Big|\Pi_\varphi(\mathrm{GSpin}_4^{1,1})\Big| = 4, \qquad \Big|\Pi_\varphi\left(\mathrm{GSpin}_4^{2,1}\right)\Big| = 1. \tag{5.13}$$
Also, in the following sentence change "in which case the multiplicity is 2" to "in which case the multiplicity 2 could also occur". We thank Hengfei Lu [Lu20] for bringing this error to our attention.
- In addition, it is more accurate that we use 'not irreducible' rather than 'reducible' in this Remark since one may have indecomposable parameters. Alternatively, we may write $\widetilde{\varphi}_i|_{W_F}$ is reducible. Thus, at the beginning the Remark, change "When $\widetilde{\varphi}_i$ is reducible," to "When $\widetilde{\varphi}_i$ is not irreducible,".

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TABLE 1. Representations of $\mathrm{GSpin}_4(F)$

	$\mathrm{Res}^{\mathrm{GL}_2\times\mathrm{GL}_2}_{\mathrm{GSpin}_4}$ of	L-packet Structure	generic
(a)	$(\widetilde{\sigma}_1 \boxtimes \widetilde{\sigma}_2), \quad \widetilde{\sigma}_2 \cong \widetilde{\sigma}_1\widetilde{\eta}, \widetilde{\sigma}_i \in \mathrm{Irr}_{\mathrm{sc}}(\mathrm{GL}_2)$	$\{1\}, \mathbb{Z}/2\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^2$	•
(b)	$(\widetilde{\sigma}_1 \boxtimes \widetilde{\sigma}_2), \quad \widetilde{\sigma}_2 \not\cong \widetilde{\sigma}_1\widetilde{\eta}, \widetilde{\sigma}_i \in \mathrm{Irr}_{\mathrm{sc}}(\mathrm{GL}_2)$	$\{1\}, \mathbb{Z}/2\mathbb{Z}$	•
(i)	$(\mathsf{St}_{\mathrm{GL}_2} \boxtimes \mathsf{St}_{\mathrm{GL}_2}) = \mathsf{St}_{\mathrm{GSpin}_4}$ (irreducible)	$\{1\}$	•
(ii)	$(i^{\mathrm{GL}_2}_{\mathrm{GL}_1\times\mathrm{GL}_1}(\chi^{\mathrm{GL}_2}_{\mathrm{GL}_1\times\mathrm{GL}_1}(\chi_1 \otimes \chi_2) \boxtimes \mathsf{St}_{\mathrm{GL}_2} \otimes \chi)$ (irreducible)	$\{1\}$	•
(iii)	$(i^{\mathrm{GL}_2}_{\mathrm{GL}_1\times\mathrm{GL}_1}(\chi_1 \otimes \chi_2) \boxtimes i^{\mathrm{GL}_2}_{\mathrm{GL}_1\times\mathrm{GL}_1}(\chi_3 \otimes \chi_4)), \chi_1 \neq \nu^{\pm1\chi_2}, \chi_3 \neq \nu^{\pm1}\chi_4$	$\{1\}, \mathbb{Z}/2\mathbb{Z}$	•
(iv)	$(\widetilde{\sigma} \boxtimes \mathsf{St}_{\mathrm{GL}_2} \otimes \chi), \quad \widetilde{\sigma} \in \mathrm{Irr}_{\mathrm{sc}}(\mathrm{GL}_2)$ (irreducible)	$\{1\}$	•
(v)	$(\widetilde{\sigma} \boxtimes i^{\mathrm{GL}_2}_{\mathrm{GL}_1\times\mathrm{GL}_1}(\chi_1 \otimes \chi_2)), \quad \widetilde{\sigma} \in \mathrm{Irr}_{\mathrm{sc}}(\mathrm{GL}_2)$	$\{1\}, \mathbb{Z}/2\mathbb{Z}$	•
$\mathfrak{nongnr}$	$(\chi \circ \det \boxtimes \widetilde{\sigma}), \quad \widetilde{\sigma} \in \mathrm{Irr}(\mathrm{GL}_2)$ (irreducible)	$\{1\}$	

TABLE 2. The adjoint L-function $L(s, \sigma, \mathrm{Ad})$ for GSpin_4

	$L(s, \sigma, \mathrm{Ad})$	$\mathrm{ord}_{s=1}$
(a)&(b)	$L(s, \widetilde{\sigma}_1, \mathrm{Sym}^2 \otimes \omega^{-1}_{\widetilde{\sigma}_1}) L(s, \widetilde{\sigma}_2, \mathrm{Sym}^2 \otimes \omega^{-1}_{\widetilde{\sigma}_2})$	0
(i)	$L(s+1)^2$	0
(ii)	$L(s)L(s+1)L(s, \chi_1\chi_2^{-1})L(s, \chi_1^{-1}\chi_2)$	0
(iii)	$L(s)^2 L(s, \chi_1\chi_2^{-1})L(s, \chi_1^{-1}\chi_2)L(s, \chi_3\chi_4^{-1})L(s, \chi_3^{-1}\chi_4)$	0
(iv)	$L(s+1)L(s, \widetilde{\sigma}_2, \mathrm{Sym}^2 \otimes \omega^{-1}_{\widetilde{\sigma}_2})$	0
(v)	$L(s)L(s, \chi_1\chi_2^{-1})L(s, \chi_1^{-1}\chi_2)L(s, \widetilde{\sigma}_2, \mathrm{Sym}^2 \otimes \omega^{-1}_{\widetilde{\sigma}_2})$	0
$\mathfrak{nongnr}$	$L(s-1)L(s)L(s+1)L(s, \widetilde{\sigma}, \mathrm{Ad})$	$1 + \mathrm{ord}_{s=1} L(s, \widetilde{\sigma}, \mathrm{Ad})$

TABLE 3. Representations of $\mathrm{GSpin}_6(F)$

	$\mathrm{Res}^{\mathrm{GL}_4\times\mathrm{GL}_1}_{\mathrm{GSpin}_6}$ of	generic
(a)	$(\widetilde{\sigma}_0 \boxtimes \widetilde{\eta}), \quad \widetilde{\sigma}_0 \in \mathrm{Irr}_{\mathrm{sc}}(\mathrm{GL}_4)$	•
(I)	$i^{\mathrm{GL}_4\times GL_1}_{(\mathrm{GL}_1\times\mathrm{GL}_1\times\mathrm{GL}_1\times\mathrm{GL}_1)\times\mathrm{GL}_1}(\widetilde{\chi}_1 \boxtimes \widetilde{\chi}_2 \boxtimes \widetilde{\chi}_3 \boxtimes \widetilde{\chi}_4 \boxtimes \widetilde{\eta}), \quad \widetilde{\chi}_i \neq \nu\widetilde{\chi}_j$	•
(II)	$i^{\mathrm{GL}_4\times GL_1}_{(\mathrm{GL}_2\times\mathrm{GL}_1\times\mathrm{GL}_1)\times\mathrm{GL}_1}(\widetilde{\sigma}_0 \boxtimes \widetilde{\chi}_1 \boxtimes \widetilde{\chi}_2 \boxtimes \widetilde{\eta}), \quad \widetilde{\sigma}_0 \in \mathrm{Irr}_{\mathrm{esq}}(\mathrm{GL}_2), \widetilde{\chi}_1 \neq \nu^{\pm1}\widetilde{\chi}_2$	•
(III)	$i^{\mathrm{GL}_4\times GL_1}_{(\mathrm{GL}_3\times\mathrm{GL}_1)\times\mathrm{GL}_1}(\widetilde{\sigma}_0 \boxtimes \widetilde{\chi} \boxtimes \widetilde{\eta}), \quad \widetilde{\sigma}_0 \in \mathrm{Irr}_{\mathrm{esq}}(\mathrm{GL}_3)$	•
(IV)	$i^{\mathrm{GL}_4\times GL_1}_{(\mathrm{GL}_2\times\mathrm{GL}_2)\times\mathrm{GL}_1}(\widetilde{\sigma}_1 \boxtimes \widetilde{\sigma}_2 \boxtimes \widetilde{\eta}), \quad \widetilde{\sigma}_i \in \mathrm{Irr}_{\mathrm{esq}}(\mathrm{GL}_2), \widetilde{\sigma}_1 \neq \nu^{\pm1}\widetilde{\sigma}_2$	•
(V)	$(\widetilde{\sigma} \boxtimes \widetilde{\eta}), \quad \widetilde{\sigma} \in \mathrm{Irr}_{\mathrm{esq}}(\mathrm{GL}_4) \setminus \mathrm{Irr}_{\mathrm{sc}}(\mathrm{GL}_4)$	•
(A)	$i^{\mathrm{GL}_4\times GL_1}_{(\mathrm{GL}_1\times\mathrm{GL}_1\times\mathrm{GL}_1\times\mathrm{GL}_1)\times\mathrm{GL}_1}(\widetilde{\chi}_1 \boxtimes \widetilde{\chi}_2 \boxtimes \widetilde{\chi}_3 \boxtimes \widetilde{\chi}_4 \boxtimes \widetilde{\eta}), \quad \widetilde{\chi}_i = \nu\widetilde{\chi}_j$	
(B)	$i^{\mathrm{GL}_4\times GL_1}_{(\mathrm{GL}_2\times\mathrm{GL}_1\times\mathrm{GL}_1)\times\mathrm{GL}_1}(\widetilde{\sigma}_0 \boxtimes \widetilde{\chi}_1 \boxtimes \widetilde{\chi}_2 \boxtimes \widetilde{\eta}), \quad \widetilde{\sigma}_0 \notin \mathrm{Irr}_{\mathrm{esq}}(\mathrm{GL}_2)$, or $\widetilde{\chi}_1 = \nu^{\pm1}\widetilde{\chi}_2$	
(C)	$i^{\mathrm{GL}_4\times GL_1}_{(\mathrm{GL}_3\times\mathrm{GL}_1)\times\mathrm{GL}_1}(\widetilde{\sigma}_0 \boxtimes \widetilde{\chi} \boxtimes \widetilde{\eta})$, non-generic $\widetilde{\sigma}_0 \in \mathrm{Irr}(\mathrm{GL}_3)$	
(D)	$i^{\mathrm{GL}_4\times GL_1}_{(\mathrm{GL}_2\times\mathrm{GL}_2)\times\mathrm{GL}_1}((\chi \circ \det) \boxtimes \widetilde{\sigma} \boxtimes \widetilde{\eta}), \quad \widetilde{\sigma} \in \mathrm{Irr}(\mathrm{GL}_2)$	
(E)	$(\widetilde{\chi} \circ \det \boxtimes \widetilde{\eta}), \quad \widetilde{\sigma} \in \mathrm{Irr}_{\mathrm{esq}}(\mathrm{GL}_4) \setminus \mathrm{Irr}_{\mathrm{sc}}(\mathrm{GL}_4)$	

TABLE 4. The adjoint L-function $L(s,\sigma,\mathrm{Ad})$ for GSpin_6

	$\sigma \in \mathrm{Irr}(\mathrm{GSpin}_6(F))$ determined by	$L(s,\sigma,\mathrm{Ad})$	$\mathrm{ord}_{s=1}$
(a)	(5.9) $\widetilde{\sigma}_0 \in \mathrm{Irr}_{\mathrm{sc}}(\mathrm{GL}_4)$	$L(s,\widetilde{\sigma}_0,\mathrm{Ad})$	0
(I)	(5.14) $\widetilde{\chi}_1 \boxtimes \widetilde{\chi}_2 \boxtimes \widetilde{\chi}_3 \boxtimes \widetilde{\chi}_4 \boxtimes \widetilde{\eta}$	$L(s)^3 \prod_{i\neq j} L(s,\widetilde{\chi}_i\widetilde{\chi}_j^{-1})$	0
(II)	(5.18) $\widetilde{\sigma}_0 \in \mathrm{Irr}_{\mathrm{sc}}(\mathrm{GL}_2)$	$L(s)^2 L(s,\widetilde{\sigma}_0,\mathrm{Ad})L(s,\widetilde{\sigma}_0\times\widetilde{\chi}_1^{-1})L(s,\widetilde{\sigma}_0^\vee\times\widetilde{\chi}_1)$ $L(s,\widetilde{\sigma}_0\times\widetilde{\chi}_2^{-1})L(s,\widetilde{\sigma}_0^\vee\times\widetilde{\chi}_2)L(s,\widetilde{\chi}_1\widetilde{\chi}_2^{-1})L(s,\widetilde{\chi}_2\widetilde{\chi}_1^{-1})$	0
(II)	(5.18) $\widetilde{\sigma}_0 = \mathsf{St}_{\mathrm{GL}_2}\otimes\widetilde{\chi}$	$L(s)^2L(s+1)L(s+1,\widetilde{\chi}\widetilde{\chi}_1^{-1})L(s+1,\widetilde{\chi}\widetilde{\chi}_2^{-1})$ $L(s,\widetilde{\chi}^{-1}\widetilde{\chi}_1)L(s,\widetilde{\chi}^{-1}\widetilde{\chi}_2)L(s,\widetilde{\chi}_1\widetilde{\chi}_2^{-1})L(s,\widetilde{\chi}_2\widetilde{\chi}_1^{-1})$	0
(III)	(5.22) $\widetilde{\sigma}_0 \in \mathrm{Irr}_{\mathrm{sc}}(\mathrm{GL}_3)$	$L(s)L(s,\widetilde{\sigma}_0,\mathrm{Ad})L(s,\widetilde{\sigma}_0\times\widetilde{\chi}^{-1})L(s,\widetilde{\sigma}_0^\vee\times\widetilde{\chi})$	0
(III)	(5.22) $\widetilde{\sigma}_0 = \mathsf{St}_{\mathrm{GL}_3}\otimes\widetilde{\chi}_0$	$L(s)L(s+1)L(s+2)L(s+1,\widetilde{\chi}\widetilde{\chi}_0^{-1})L(s+1,\widetilde{\chi}^{-1}\widetilde{\chi}_0)$	0
(IV)	(5.26) $\widetilde{\sigma}_i \in \mathrm{Irr}_{\mathrm{sc}}(\mathrm{GL}_2)$	$L(s)L(s,\widetilde{\sigma}_1,\mathrm{Ad})L(s,\widetilde{\sigma}_2,\mathrm{Ad})$ $L(s,\widetilde{\sigma}_1\times\widetilde{\sigma}_2^\vee)L(s,\widetilde{\sigma}_1^\vee\times\widetilde{\sigma}_1)$	0
(IV)	(5.26) $\widetilde{\sigma}_1 \in \mathrm{Irr}_{\mathrm{sc}}(\mathrm{GL}_2), \widetilde{\sigma}_2 = \mathsf{St}_{\mathrm{GL}_2}\otimes\widetilde{\chi}$	$L(s)L(s+1)L(s,\widetilde{\sigma}_1,\mathrm{Ad})$ $L(s+\frac{1}{2},\widetilde{\sigma}_1^\vee\times\widetilde{\chi})L(s+\frac{1}{2},\widetilde{\sigma}_1\times\widetilde{\chi}^{-1})$	0
(IV)	(5.26) $\widetilde{\sigma}_2 \in \mathrm{Irr}_{\mathrm{sc}}(\mathrm{GL}_2), \widetilde{\sigma}_1 = \mathsf{St}_{\mathrm{GL}_2}\otimes\widetilde{\chi}$	$L(s)L(s+1)L(s,\widetilde{\sigma}_2,\mathrm{Ad})$ $L(s+\frac{1}{2},\widetilde{\sigma}_2^\vee\times\widetilde{\chi})L(s+\frac{1}{2},\widetilde{\sigma}_2\times\widetilde{\chi}^{-1})$	0
(IV)	(5.26) $\widetilde{\sigma}_1 = \mathsf{St}_{\mathrm{GL}_2}\otimes\widetilde{\chi}_1\widetilde{\sigma}_2 = \mathsf{St}_{\mathrm{GL}_2}\otimes\widetilde{\chi}_2$	$L(s)L(s+1)^2L(s,\widetilde{\chi}_1^{-1}\widetilde{\chi}_2)L(s,\widetilde{\chi}_1\widetilde{\chi}_2^{-1})$ $L(s+1,\widetilde{\chi}_1\widetilde{\chi}_2^{-1})L(s+1,\widetilde{\chi}_1^{-1}\widetilde{\chi}_2)$	0
(V)	(5.27) $\widetilde{\sigma} = \mathsf{St}_{\mathrm{GL}_4}\otimes\widetilde{\chi}$	$L(s+1)L(s+2)L(s+3)$	0
(V)	(5.28) $\widetilde{\sigma} = \Delta[\nu^{1/2},\nu^{-1/2}], \widetilde{\tau}\in\mathrm{Irr}_{\mathrm{sc}}(\mathrm{GL}_2)$	$L(s,\widetilde{\tau},\mathrm{Ad})L(s,\widetilde{\tau}\times\widetilde{\tau}^\vee)$	0
(A)	(5.31) $Q\left([\nu^{1/2}\widetilde{\chi}],[\nu^{-1/2}\widetilde{\chi}],[\widetilde{\chi}_3],[\widetilde{\chi}_4]\right)$	$L(s-1)L(s)^3L(s+1)L(s,\widetilde{\chi}_3\widetilde{\chi}_4^{-1})L(s,\widetilde{\chi}_3^{-1}\widetilde{\chi}_4)$ $\prod_{i=3,4}\begin{pmatrix} L(s+\frac{1}{2},\widetilde{\chi}\widetilde{\chi}_i^{-1})L(s-\frac{1}{2},\widetilde{\chi}^{-1}\widetilde{\chi}_i) \\ L(s-\frac{1}{2},\widetilde{\chi}\widetilde{\chi}_i^{-1})L(s+\frac{1}{2},\widetilde{\chi}^{-1}\widetilde{\chi}_i)\end{pmatrix}$	≥ 1
(A)	(5.32) $Q\left([\nu\widetilde{\chi}],[\widetilde{\chi}],[\nu^{-1}\widetilde{\chi}],[\widetilde{\chi}_4]\right)$	$L(s-2)L(s-1)^2L(s)^3L(s+1)^2L(s+2)$ $\prod_{t=-1,0,1}\left(L(s+t,\widetilde{\chi}\widetilde{\chi}_4^{-1})L(s+t,\widetilde{\chi}^{-1}\widetilde{\chi}_4)\right)$	≥ 2
(A)	(5.33) $Q\left([\widetilde{\chi},\nu\widetilde{\chi}],[\nu^{-1}\widetilde{\chi}],[\widetilde{\chi}_4]\right)$	$L(s-2)L(s-1)^2L(s)^2$ $\prod_{t=-1,0} L(s+t,\widetilde{\chi}\widetilde{\chi}_4^{-1}) \prod_{t=-1,1} L(s+t,\widetilde{\chi}^{-1}\widetilde{\chi}_4)$	≥ 2
(A)	(5.34) $Q\left([\nu\widetilde{\chi}],[\widetilde{\chi},\nu^{-1}\widetilde{\chi}],[\widetilde{\chi}_4]\right)$	$L(s-1)L(s)^2L(s+1)L(s+2)$ $\prod_{t=0,1} L(s+t,\widetilde{\chi}\widetilde{\chi}_4^{-1}) \prod_{t=-1,1} L(s+t,\widetilde{\chi}^{-1}\widetilde{\chi}_4)$	≥ 1
(A)	(5.35) $Q\left([\nu^{3/2}\widetilde{\chi}],[\nu^{1/2}\widetilde{\chi}],[\nu^{-1/2}\widetilde{\chi}],[\nu^{-3/2}\widetilde{\chi}]\right)$	$L(s-3)L(s-2)^2L(s-1)^3L(s)^3$ $L(s+1)^3L(s+2)^2L(s+3)$	3
(A)	(5.36) $Q\left([\nu^{1/2}\widetilde{\chi},\nu^{3/2}\widetilde{\chi}],[\nu^{-1/2}\widetilde{\chi}],[\nu^{-3/2}\widetilde{\chi}]\right)$	$L(s-3)L(s-2)L(s-1)^2L(s)^2L(s+1)^2L(s+2)$	2
(A)	(5.37) $Q\left([\nu^{3/2}\widetilde{\chi}],[\nu^{-1/2}\widetilde{\chi},\nu^{1/2}\widetilde{\chi}],[\nu^{-3/2}\widetilde{\chi}]\right)$	$L(s-3)L(s-2)L(s-1)^2L(s)^2L(s+1)^2L(s+2)$	2
(A)	(5.38) $Q\left([\nu^{3/2}\widetilde{\chi}],[\nu^{1/2}\widetilde{\chi}],[\nu^{-3/2}\widetilde{\chi},\nu^{-1/2}\widetilde{\chi}]\right)$	$L(s-3)L(s-2)L(s-1)^2L(s)^2L(s+1)^2L(s+2)$	2
(A)	(5.39) $Q\left([\nu^{1/2}\widetilde{\chi},\nu^{3/2}\widetilde{\chi}],[\nu^{-3/2}\widetilde{\chi},\nu^{-1/2}\widetilde{\chi}]\right)$	$L(s-3)L(s-2)L(s-1)^2L(s)L(s+1)L(s+2)$	2
(A)	(5.40) $Q\left([\nu^{-1/2}\widetilde{\chi},\nu^{1/2}\widetilde{\chi},\nu^{3/2}\widetilde{\chi}],[\nu^{-3/2}\widetilde{\chi}]\right)$	$L(s-3)L(s-2)L(s-1)L(s)L(s+1)$	1
(A)	(5.41) $Q\left([\nu^{3/2}\widetilde{\chi}],[\nu^{-3/2}\widetilde{\chi},\nu^{-1/2}\widetilde{\chi},\nu^{1/2}\widetilde{\chi}]\right)$	$L(s-3)L(s-2)L(s-1)L(s)L(s+1)$	1
(B)	(5.45) $Q\left([i_B^{\mathrm{GL}_2}(\widetilde{\eta}_1\boxtimes\widetilde{\eta}_2)],[\widetilde{\chi}\nu^{1/2}],[\widetilde{\chi}\nu^{-1/2}]\right)$, $\widetilde{\eta}_1\widetilde{\eta}_2^{-1}\neq\nu^{\pm 1}$	$L(s-1)L(s)^3L(s+1)L(s,\widetilde{\eta}_1\widetilde{\eta}_2^{-1})L(s,\widetilde{\eta}_1^{-1}\widetilde{\eta}_2)$ $\prod_{t=\pm\frac{1}{2}}\prod_{i=1,2}\left(L(s+t,\widetilde{\eta}_i\widetilde{\chi}^{-1})L(s+t,\widetilde{\eta}_i^{-1}\widetilde{\chi})\right)$	≥ 1
(B)	(5.45) (others covered in (A))		
(C)	(5.48) (covered in (A) and (B))		
(D)	(5.49) with $\widetilde{\sigma}\in\mathrm{Irr}_{\mathrm{sc}}(\mathrm{GL}_2)$	$L(s-1)L(s)^2L(s+1)L(s,\sigma,\mathrm{Ad})$ $\prod_{t=\pm\frac{1}{2}}\left(L(s+t,\sigma\times\chi^{-1})L(s+t,\sigma^\vee\times\chi)\right)$	1
(D)	(5.49) with $\widetilde{\sigma} = \mathsf{St}_{\mathrm{GL}_2}\otimes\eta$	$L(s-1)L(s)^2L(s+1)^2$ $L(s,\chi\eta^{-1})L(s+1,\chi\eta^{-1})L(s+1,\chi^{-1}\eta)L(s,\chi^{-1}\eta)$	≥ 1
(D)	(5.49) (others covered in (A))		
(E)	(5.50) (covered in (A))		