RING-THEORETIC PROPERTIES

OF SEMIGROUP RINGS

By

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CHAPTER 1

INTRODUCTION

This survey is designed to summarize the known results of ring-theoretic properties of group rings and their generalization, semigroup rings. The first paper to look at many of these properties was Ian Connell's [15] "On the Group Ring" in 1963. Since then the study of group rings and their generalizations has grown considerably. There have been two books by Donald Passman [61,62] on group rings. These principally explore group rings when the coefficient ring is a field. A book on commutative group rings by Gregory Karpilovsky [43] and a book on commutative semigroup rings by Robert Gilmer [26] have also been published recently. Due to the difficulty in working with group (semigroup) rings in general most authors impose restrictions such as commutativity on the structures.

The object of this paper is to summarize the results on selected ring-theoretic topics in the most general possible terms. The format will be to give results on group rings followed by generalizations to semigroup rings. Occasionally proofs of some of the theorems will be given. The proofs that are included have been chosen to be instructive and representative of typical proofs encountered in the study of semigroup rings. In these proofs it will be attempted to rely on previous definitions and theorems. Unfortunately some proofs require results from later sections and so sometimes results that have not be covered yet will be used.

The Introduction contains the basic definitions that are employed in subsequent chapters. Throughout this paper S will denote a semigroup, T a monoid (a semigroup with an identity element), G a group, R a ring with identity, and K a field. Unless otherwise noted Z will be understood to be the set of integers and Q the rational numbers.

Definition 1.1. A <u>semigroup</u> is a nonempty set S together with a binary operation on S which is

(a) associative: $a(bc) = (ab)c$ for all $a, b, c \in S$.

A <u>monoid</u> is a semigroup T which contains a

(b) (two-sided) identity element $e \in G$ such that $ae = ea = a$ for all $a \in T$.

A <u>group</u> is a monoid G such that

(c) for every $a \in G$ there exists a (two-sided) inverse element $a^{-1} \in G$ such that $a^{-1}a = aa^{-1} = e$.

A semigroup S is said to be <u>commutative</u> if its binary operation is

(d) commutative: $ab = ba$ for all $a,b \in G$.

The term <u>abelian</u> is used when referring to groups that are commutative.

Definition 1.2. A <u>ring with identity</u> is a nonempty set R together with two binary operations (usually denoted as addition (+) and multiplication (·)) such that:

(a) (R,+) is an abelian group;

(b) (R,·) is a monoid;

(c) $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ (left and right distributive laws).

A division ring D is a ring R such that

(d) $(R,\cdot)$ is a group.

A field K is a division ring D such that

(e) $(D,\cdot)$ is an abelian group.

(Note that in this work it is assumed that $(R,\cdot)$ is a monoid (i.e., that R has a multiplicative identity).

Definition 1.3. Let R be a ring and G a group. Let RG be the additive abelian group $\sum_{g \varepsilon G} R$ (one copy of R for each $g \varepsilon G$). An element $x = \{r_g\}_{g\varepsilon G}$ of RG has only finitely many nonzero coordinates, say $r_{g_1},\ldots,r_{g_n}$ $(g_i \varepsilon G)$. Denote x by the formal sum $\sum_{i=1}^{n} r_{g_i} g_i$ where some of the r_{g_i} may be zero. In this notation, addition in the group RG is given by:

$$\sum_{i=1}^{n} r_{g_i} g_i + \sum_{i=1}^{n} s_{g_i} g_i = \sum_{i=1}^{n} (r_{g_i} + s_{g_i}) g_i \;;$$

(by inserting zero coefficients if necessary to assure the two formal sums involve exactly the same indices $g_1,\ldots,g_n$). Define multiplication in RG by

$$\left(\sum_{i=1}^{n} r_{g_i} g_i\right)\left(\sum_{j=1}^{m} s_{g_j} g_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} (r_{g_i} s_{g_j})(g_i g_j);$$

With these operation RG is a ring, called the group ring of G over R. R is also called the coefficient ring of RG.

Definition 1.4. Let R be a ring and S a semigroup. The semigroup ring, R[S], of S over R is defined as above with the group G being replaced by the semigroup S everywhere. A semigroup ring with S replaced by a monoid T is called a monoid ring, R[T]. A monoid ring with T the monoid of nonnegative integers under additions is called a polynomial ring in one indeterminate and is written R[x].

One of the problems in working with semigroup rings is that while R is always assumed to have a two-sided identity the semigroup ring R[S] does not necessarily have one. With monoid rings the situation is much nicer since R[T] has $1_R \cdot 1_T$ as its identity. The distinction between rings with and without an identity should be kept in mind since the presence of an identity can drastically alter some results.

Some definitions that pertain to rings, groups and semigroups follow.

Definition 1.5. Let R be a ring. If there is a least positive integer n such that na = 0 for all a ε R, then R is said to have characteristic n. (Notation: char R = n). If no such n exists R is said to have characteristic zero. Whenever char R = p is used it will be assumed that p is prime.

Definition 1.6. A ring R is said to be a K-algebra if there exists a field K contained in R.

Definition 1.7. Let R be a ring. The Jacobson radical J(R) of R is the intersection of all maximal left (right) ideals of R.

Definition 1.8. An ideal P in a ring R is said to be prime if P $\neq$ R and for any ideals A, B in R, AB $\leqq$ P implies A $\leqq$ P or B $\leqq$ P.

Definition 1.9. The prime radical or the lower nil radical N(R) of a ring R is the intersection of all prime ideals of R. The upper nil radical U(R) is defined to the be the unique largest nil ideal of R.

Definition 1.10. The order of a group G is the cardinal number of the set G. The order of g ε G is the order of the subgroup $\{g, g^2, g^3, \ldots\}$. An element in a group is said to be a torsion element if it has finite order. A group is said to be a torsion group if every element in G is of finite order. If every nonidentity element of G has infinite order, then G is torsion-free.

Definition 1.11. A group G is said to be locally finite if every finitely generated subgroup of G is finite.

Every locally finite group is a torsion group, but there exist torsion groups which are not locally finite [35]. For an abelian group, the two are equivalent.

Definition 1.12. Let N be a subgroup of a group G. If $aNa^{-1} = N$ for all a ε G ($aNa = \{ana^{-1} \mid n \in N\}$) then N is said to be normal in G. A group G is solvable if there exists a sequence

$$\{1\} = G_0 < G_1 < \cdots < G_n = G$$

of subgroups of G, with G_{i-1} normal in G_i, such that G_i/G_{i-1} is abelian for $1 \leqq i \leqq n$. A group G is locally solvable if every finitely generated subgroup is solvable. Suppose G is a solvable group with a sequence of subgroups as defined above. G is nilpotent if it is solvable and G/G_{i-1} is abelian for $1 \leqq i \leqq n$. A group G is locally nilpotent if every finitely generated subgroup is nilpotent. A group in which every element has order a nonnegative power of some fixed prime p is called a p-group. (A p-group is nilpotent and hence solvable.)

Definition 1.13. Let F be a free commutative semigroup with basis X. The cardinal number $|X|$ of X is called the rank of F. If S is a commutative semigroup, then the unique cardinal number that represents the size of any maximal linearly independent subset of S is the rank of S.

Definition 1.14. Let S be a multiplicative semigroup and let $s \in S$. Define $\langle s \rangle$ to be the set $\{s, s^2, s^3, \ldots\}$. The element s is said to be periodic if $\langle s \rangle$ is finite, and aperiodic if $\langle s \rangle$ is infinite. Similarly the semigroup S is periodic if it consists entirely of periodic elements and aperiodic if it consists entirely of aperiodic elements, and nonperiodic if not every element of S is periodic. S is cyclic if $S = \langle s \rangle$ for some $s \in S$. For the case in which S is a group the terms periodic and nonperiodic are respectively equivalent to torsion and nontorsion.

One important type of semigroup are the cancellative semigroups. The commutative cancellative semigroups have the enviable property that they may be embedded in a group.

Definition 1.15. A semigroup S is <u>cancellative</u> if either ab = ac or ba = ca for a, b, c ε S implies b = c.

Theorem 1.16 [26]. Every commutative cancellative semigroup may be embedded in a group.

Definition 1.17. Let S be a semigroup. A <u>subsemigroup</u> of S is a nonempty subset of S which is closed under the induced multiplication. If it is a group, then we shall call it a <u>subgroup</u> of S. Note that the identity of the subgroup may not be that of S even if the latter exists. A <u>left (right) ideal</u> I of a semigroup S is a nonempty subset of S such that $SI \subseteqq I$ ($IS \subseteqq I$). If I is both a left and right ideal, I is called an <u>ideal</u>. (Note that the only left or right ideal of a group G is G since given any nonempty subset I of G, GI = IG = G).

Definition 1.18 [26]. If S is a semigroup, the homomorphisms defined on S and the homomorphic images of S play an important role in determining the structure of S. An equivalent and convenient way of considering homomorphisms on S is through the notion of a congruence on S, defined as follows. A <u>congruence</u> on S is an equivalence relation on S that is compatible with the semigroup operation. Theorem 1.19 states the basic relationships between homomorphisms and congruences.

Theorem 1.19 [26]. Let S be an additive semigroup.

(a) If ~ is a congruence on S, then for s ε S, denote by <u>[s]</u> the equivalence class of s under ~ for each s ε S, and let $S/\sim = \{[s] \mid s \in S\}$. Then S/~ is an abelian semigroup under the operation [a] + [b]

= [a + b], and the mapping f: S → S/~ defined by f(s) = [s] is a homomorphism of S onto S/~; moreover, $f(s_1) = s_2$ if and only if $s_1 \sim s_2$. The semigroup S/~ is called the factor semigroup of S with respect to ~.

(b) Conversely, if h: S → T is a homomorphism of S onto T, define the relation ρ on S by aρb if h(a) = h(b). Then ρ is a congruence on S and the semigroups S/ρ and T are isomorphic under the mapping [s] → h(s), where [s] denotes the equivalence class of s ε S under ρ.

Definition 1.20. An element of a multiplicative semigroup S is nilpotent in case there is a natural number n such that $x^n = 0$. A subset A of S is nilpotent in case there is an integer n > 0 such that $x_1 x_2 \cdots x_n = 0$ for every sequence $x_1, x_2, \ldots, x_n$ in A. Also, A is nil in case each of its elements is nilpotent. Thus, every nilpotent subset of S is certainly nil; but there are nil subsets that are not nilpotent. A subset A of S is left T-nilpotent (T stands for transfinite) in case for every sequence $a_1, a_2, \ldots$ in A there is an n such that $a_1 \cdots a_n = 0$. The subset A is right T-nilpotent in case for each $a_1, a_2, \ldots$ in A, $a_n \cdots a_1 = 0$ for some n. (Note that the word nilpotent has two different meanings. When nilpotent is used to refer to a group, Definition 1.12 applies. When nilpotent is used in any other context, Definition 1.20 applies.)

Definition 1.21. An element e in a multiplicative semigroup S is idempotent if $e^2 = e$.

Definition 1.22. Let x be an element of S. If xs = sx = x for any s in S, then x is said to be a zero element for S. If xs = sx = s for any s in S, then x is said to be an identity element for S. Since a zero and an identity element can be adjoined to any semigroup, S^0 will denote S V {θ}, and S^1 will denote S V {1}, where θ and 1 represent a zero and an identity element respectively. If S already has a zero element or an identity, then $S = S^0$ or $S = S^1$ respectively. (The notation V used here denotes the union operation; when Λ is encountered it will indicate the intersection operation.)

Definition 1.23. Suppose the semigroup S has a zero θ. It is sometimes useful to identify the zero of the semigroup with the zero of the semigroup ring. To accomplish this the contracted semigroup ring, $(R[S])_0$, is defined to be the ring R[S]/Rθ.

There are many nice results for commutative semigroup (group) rings. One thing that is particularly convenient is that the coefficient ring and semigroup (group) are commutative if and only if the semigroup ring is commutative.

Theorem 1.24. R[S] (RG) is commutative if and only if R is commutative and S (G) is commutative (abelian).

A couple of examples which explore some of the above concepts will now be given.

Example 1.25. Let R be a ring with identity. Let

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ \theta = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\cdot$	e_{11}	e_{12}	e_{22}	θ
e_{11}	e_{11}	e_{12}	θ	θ
e_{12}	θ	θ	e_{12}	θ
e_{22}	θ	θ	e_{22}	θ
θ	θ	θ	θ	θ

Let $S = \{e_{11}, e_{12}, e_{22}, \theta\}$. So $R[S] = Re_{11} + Re_{12} + Re_{22} + R\theta$. Suppose $R[S]$ has an identity $1_{R[S]}$. Let us determine what the identity must be. Let $s_1e_{11} + s_2e_{12} + s_3e_{22} + s_4\theta \in R[S]$ and suppose the identity is $r_1e_{11} + r_2e_{12} + r_3e_{22} + r_4\theta$. Therefore

$$(s_1e_{11} + s_2e_{12} + s_3e_{22} + s_4\theta)(r_1e_{11} + r_2e_{12} + r_3e_{22} + r_4\theta)$$
$$= s_1e_{11} + s_2e_{12} + s_3e_{22} + s_4\theta,$$

and so $s_1r_1 = s_1$, $s_1r_2+s_2r_3 = s_2$, $s_3r_3 = s_3$. Therefore $r_1 = r_3 = 1_R$ and $r_2 = 0_R$.

$$(s_1e_{11} + s_2e_{12} + s_3e_{22} + s_4\theta)(e_{11} + e_{22} + r_4\theta)$$
$$= s_1e_{11} + s_2e_{12} + s_3e_{22} + s_4\theta$$

implies $s_1 + s_2 + s_3 + s_4 + s_4 + s_1r_4 + s_2r_4 + s_3r_4 + s_4r_4 = s_4$. Hence $r_4 = -1$ and a simple calculation shows that $e_{11} + e_{22} - \theta$ is a two-sided identity.

Now $(R[S])_0 = (Re_{11} + Re_{12} + Re_{22} + R\theta) / R\theta$
$$= Re_{11} + Re_{12} + Re_{22}$$

By the above reasoning $(R[S])_0$ has $e_{11} + e_{22}$ as a two-sided identity.

Example 1.26. Let R be a ring with identity and the e_{ij} as defined above. Let $S = \{e_{12}, e_{22}, \theta\}$. So $R[S] = Re_{12} + Re_{22} + R\theta$. Suppose $R[S]$ has an identity $1_{R[S]}$. Again, let us determine what the identity must be. Let $s_2e_{12} + s_3e_{22} + s_4\theta \in R[S]$ and suppose the identity is $r_2e_{12} + r_3e_{22} + r_4\theta$. Therefore

$$(s_2e_{12} + s_3e_{22} + s_4\theta)(r_2e_{12} + r_3e_{22} + r_4\theta) = s_2e_{12} + s_3e_{22} + s_4\theta,$$

and so $s_2r_3 = s_2$ and $s_3r_3 = s_3$ imply $r_3 = 1_R$.

$(s_2e_{12} + s_3e_{22} + s_4\theta)(r_2e_{12} + e_{22} + r_4\theta) = s_2e_{12} + s_3e_{22} + s_4\theta$

implies $s_2r_2 + s_2r_4 + s_3r_2 + s_3r_4 + s_4r_2 + s_4 + s_4r_4 = s_4$.

Therefore $r_2 = -r_4$ and $r_2e_{12} + e_{22} - r_2\theta$ is a right identity for any r_2. $r_2e_{12} + e_{22} - r_2\theta$ not a two-sided identity for R[S] since

$(r_2e_{12} + e_{22} - r_2\theta)(s_2e_{12} + s_3e_{22} + s_4\theta)$

$$= r_2s_3e_{12} + s_3e_{22} + (s_2+s_4-r_2s_3)\theta.$$

Similarly $(R[S])_0 = Re_{12} + Re_{22}$ has $r_2e_{12} + e_{22}$ as a right identity for any r_2 and no two-sided identity.

Below are the proofs of some theorems, under certain simplifying constrains, that are representative of the proofs of several broad classes of theorems. The definitions of the properties described in this section (such as noetherian, artinian, etc.) can be found later in this work as well as many other theorems relating to these properties.

Let P be a ring property and suppose that a semigroup ring R[S] has this property. The first type of theorem considers whether this implies that R has property P.

Theorem 1.27. Let P be a property inherited by factor rings (epimorphic images). If R[S] has property P, then R has property P.

Proof. A ring epimorphism ρ: R[S] $\to$ R may be defined by

$$\rho\left(\sum_{i=1}^{n} r_i s_i\right) = \sum_{i=1}^{n} r_i.$$

Definition 1.28. The epimorphism ρ defined in Theorem 1.27 is called the <u>augmentation map</u> on R[S]. The kernel of this map is called the <u>augmentation ideal, ωS</u>.

The augmentation ideal $\omega S = \text{Ker } \rho = \{\Sigma r_i s_i \mid \Sigma r_i = 0\}$

$= \text{the ideal generated by } \{rs_1 - rs_2 \mid r \in R \text{ and } s_1, s_2 \in S\}$

One of the things that makes the augmentation ideal an important tool in the analysis of group rings and semigroup rings is that by using ωS, R is a factor ring of R[S] (since ρ being an epimorphism gives $R[S]/\omega S = R[S]/\text{Ker } \rho \cong R$). Examples of properties inherited by factor rings are those of being noetherian, artinian, perfect, semi-perfect, semisimple, local, semilocal, and von Neumann regular.

The next type of theorem is the converse of the first type. It is concerned with properties of R being passed to R[S].

Theorem 1.29. Let P be left noetherian or left artinian and let T be a finite monoid. If R has property P, then R[T] has property P.

Proof. If $t \in T$, then a ring monomorphism $\iota: R \to R[T]$ may be defined by $\iota(r) = r \cdot 1_T$. Consider the left-module structure of monoid rings. First, it is obvious that ${}_R R$ and ${}_{R[T]}R[T]$. Also the maps ρ and ι cause R[T] to be a left R-module and R to be a left R[T]-module. Observe that ${}_R R[T]$ is a free R-module with free basis $\{t \mid t \in T\}$. Since T is finite, ${}_R R[T]$ is finitely generated. For an arbitary module M let $\Gamma(M)$ be the set of all submodules of M. Since $R < R[T]$,

$\Gamma({}_{R[T]}R[T]) \leqq \Gamma({}_R R[T])$.

Suppose R is left artinian. This implies ${}_R R$ is left artinian. T finite implies that ${}_R R[T]$ is left artinian and so ${}_{R[T]}R[T]$ is artinian (thus R[T] is left artinian). The same type of reasoning shows that R[T] is left noetherian if R is.

The next type of problem looks at properties that are transmitted from polynomial rings to semigroup rings.

Theorem 1.30. Let R be a ring and T a finitely generated commutative monoid. If R is left noetherian, then R[T] is left noetherian.

Proof. Let T be generated by n elements; say $T = \langle t_1, \ldots t_n \rangle$. There exists a natural epimorphism from $R[x_1, \ldots, x_n]$ to R[T] by mapping x_n to t_n. If R is left noetherian, then by the Hilbert Basis Theorem, so is $R[x_1, \ldots x_n]$. Since the noetherian condition is preserved under epimorphisms, R[T] is also left noetherian.

The result of Theorem 1.30 is also true if T is replaced by a finitely generated abelian group G. If G is generated by $\{g_1, \ldots, g_n\}$, then as a monoid, G is generated by $\{g_1, \ldots, g_n, -g_1, \ldots, -g_n\}$. The preferred epimorphism from a polynomial ring to RG is the natural map from $R[x_1, \ldots, x_{2n}]$ to $R\langle g_1, \ldots, g_n, -g_1, \ldots, -g_n \rangle$.

One of the best-known theorems dealing with group rings is due to Maschke. Maschke [50] was concerned with semisimple group rings when the coefficient ring is a field and proved a forerunner of the following theorem in 1898.

Theorem 1.31 [31] (Maschke). Suppose G is a finite group, K a field, and either char K = 0 or char K = p with $p \nmid |G|$. Then KG is semisimple.

Proof. Let ${}_{KG}N \leqq {}_{KG}M$. KG is semisimple if there exists a ${}_{KG}F$ such that $M = N \oplus F$. Since ${}_{K}M$ is semisimple and ${}_{K}N \leqq {}_{K}M$ there exists ${}_{K}V$ such that $M = N \oplus V$. So given any $x \in M$ there exist unique $y \in N$ and $z \in V$ such that $x = y + z$. Define an idempotent K-homomorphism f: $M \to N$ by $f(x) = y$. Observe that $f(M) = N$ and ${}_{K}M = {}_{K}f(M) \oplus {}_{K}(1-f)M$. Suppose that there exists an idempotent K-homomorphism g: $M \to N$ such that $xg(m) = g(xm)$ for $m \in M$, $x \in G$. Then $xg(m) = g(xm) \in g(M)$ and ${}_{KG}g(M) \leqq {}_{KG}M$. Similarly ${}_{KG}(1-g)(M) \leqq {}_{KG}M$ and the intersection of g(M) and (1-g)M is 0. Thus ${}_{KG}M = {}_{KG}g(M) \oplus {}_{KG}(1-g)(M)$. The goal now is to construct some suitable function g: $M \to N$.

Define g: $M \to M$ by $g(m) = |G|^{-1} \sum_{x} x\, f(x^{-1}m)$ where $x \in G$. The claim is that this g works. Let y be an element of G that is fixed.

$$\begin{aligned} yg(y^{-1}m) &= y\ |G|^{-1} \Sigma\ x\ f(x^{-1}y^{-1}m) \\ &= y\ |G|^{-1} \Sigma\ x\ f((yx)^{-1}m) \\ &= |G|^{-1} \Sigma\ yx\ f((yx)^{-1}m). \end{aligned}$$

In this sum, yx ranges over all the elements of G as x does. Therefore $yg(y^{-1}m) = g(m)$ and so $g(y^{-1}m) = y^{-1}g(m)$.

Let S be a semigroup that is not a monoid and let T be the monoid obtained by adjoining an identity to S. Another type of problem looks at when R[S] has property P implies that R[T] has property P.

Theorem 1.32. If R[S] is left noetherian, then R[T] is left noetherian.

Proof. R[S] becomes an R[T]-module by multiplication; i.e.,

$R[T]\cdot R[S] = R[S]$

As R[T]-modules

$$0 \to {}_{R[T]}R[S] \to {}_{R[T]}R[T] \to {}_{R[T]}R \to 0$$

is a short exact sequence where the map R[T]→R takes 1_T to 1 and R[T] acts on R via $(\Sigma\, r_i s_i + r1_T)\cdot x = rx$. To show R[T] is left noetherian it is sufficient to show that ${}_{R[T]}R[S]$ and ${}_{R[T]}R$ are noetherian. Let ι: R[S] → R[T] be the natural R[T]-module monomorphism defined by $\iota(\Sigma\, r_i s_i) = \Sigma\, r_i t_i$ $(t_i = s_i)$. This induces a map U_ι from the category ${}_{R[T]}\underline{M}$ of left R[T]-modules to the category ${}_{R[S]}\underline{M}$ of left R[S]-modules, given by $U_\iota(N) = N$ as a set, and for $x \in R[S]$ and $n \in N$, $x\cdot n = \iota(x)\cdot n$. Therefore, given any ${}_{R[T]}N$,

$$\Gamma({}_{R[T]}N) \leqq \Gamma({}_{R[S]}U_\iota(N))$$

so

$$\Gamma({}_{R[T]}R[S]) \leqq \Gamma({}_{R[S]}U_\iota(R[S])) = \Gamma({}_{R[S]}R[S])$$

Thus ${}_{R[S]}R[S]$ noetherian implies ${}_{R[T]}R[S]$ is noetherian. Let ρ: R[T] → R be the natural R[T]-module epimorphism defined by

$$\rho(\Sigma\, r_i t_i) = \rho(r_1 1_T + \sum_{t\neq 1} r_i t_i) = r_1.$$

This induces a map U^ρ: ${}_R\underline{M} \to {}_{R[T]}\underline{M}$, given by $U_\rho(N) = N$ as a set, and for $x \in R[S]$ and $n \in N$, $x\cdot n = \rho(x)\cdot n$. Therefore, given any ${}_RN$,

$$\Gamma({}_{R[T]}N) = \Gamma({}_RU^\rho(N))$$

so

$$\Gamma({}_RR) = \Gamma({}_{R[T]}U^\rho(R)) = \Gamma({}_{R[T]}R)$$

That R is noetherian follows from Theorem 1.27, and we thus see that ${}_R R$ is noetherian implies ${}_{R[T]}R$ is noetherian. Using the short exact sequence above results in R[T] being noetherian.

It is frequently easier to prove results for a semigroup with a zero element. Since a zero can be adjoined to any semigroup, it is necessary determine when properties of R[S] are passed to $R[S^0]$ and vice versa. If P is perfect, regular, or semisimple then R[S] has property P if and only if $R[S^0]$ has property P. A related problem looks at the relationship between R[S] and $(R[S])_0$.

Theorem 1.33 [75]. R[S] is von Neumann regular if and only if $R[S^0]$ is von Neumann regular.
The following lemma is needed in the proof.

Lemma 1.34 [75]. Let I be a two-sided ideal of a ring U. Then U is regular if and only if U/I and I are regular.

Proof of Theorem 1.33. Observe that R[S] regular and $R[S^0]$ each imply that R is regular (since R is a factor ring of each). Let I = $R\theta \cong R$ and U = $R[S^0]$. Theorem 1.33 follows by applying Lemma 1.34.

Theorem 1.35 [75]. Suppose S has a zero element. R[S] is von Neumann regular if and only if $(R[S])_0$ is von Neumann regular.

Proof. Apply the proof of the previous theorem.

CHAPTER 2

FINITENESS CONDITIONS

Two of the best-known properties of rings are the chain conditions on submodules. Connell [15] was able to find results for artinian and noetherian group rings. The structure imposed by artinian group rings is particularly nice since the group (semigroup) is forced to be finite.

Definition 2.1. A module M is artinian in case the lattice of all submodules of M satisfies the descending chain condition. A ring R is left (right) artinian in case the left (right) module $_RR$ (R_R) is an artinian module. The ring is artinian in case it is both left and right artinian.

Theorem 2.2 [15]. RG is left (right) artinian if and only if R is left (right) artinian and G is finite.

This result can be generalized in a couple of ways. The first way looks at weakening the group structure to a monoid or semigroup. Zel'manov showed that the above result is true if the group is replaced by a monoid.

Theorem 2.3 [78]. R[T] is left (right) artinian if and only if R is left (right) artinian and T is finite.

If the monoid is replaced by a semigroup, part of the results are lost.

Example 2.4. R commutative artinian and S commutative finite does not necessarily imply that R[S] is artinian since R[S] need not have an identity. Let S = {a,b} be the commutative semigroup with multiplication such that the product of any two elements is b (so $a^2 = ab = ba = b^2 = b$). Let Q be the ring of rational numbers (an artinian ring), and let Z be the set of integers, and consider Q[S].

$$\{na + qb \mid n\varepsilon Z, q\varepsilon Q\} > \{2na + qb \mid n\varepsilon Z, q\varepsilon Q\} > \{4na + qb \mid n\varepsilon Z, q\varepsilon Q\} > \cdots$$

is an infinite descending chain so Q[S] is not artinian.

Any semigroup without an identity can be made into a monoid by adjoining an identity element to the semigroup.

Theorem 2.5. Let S be a semigroup that is not a monoid. If R[S] is left (right) artinian, then $R[S^1]$ is left (right) artinian.

Proof. See the proof of Theorem 1.32.

Corollary 2.6 [78]. If R[S] is left (right) artinian, then R is left (right) artinian and S is finite.

Another way to generalize Connell's theorem on artinian group rings is to weaken the ring structure. One generalization of artinian rings are noetherian rings, and these are considered next.

Definition 2.7. A module M is <u>noetherian</u> in case the lattice of all submodules of M satisfies the ascending chain condition. A ring R is <u>left (right) noetherian</u> in case the left (right) module $_RR$ (R_R) is a noetherian module. The ring is <u>noetherian</u> in case it is both left and right noetherian.

Theorem 2.8 (Hopkins) [5]. R is left (right) artinian if and only if R is left (right) noetherian, R/J(R) is semisimple, and J(R) is nilpotent.

As Connell observed, a group ring being artinian forces a much stronger structure on the group than does being noetherian.

Theorem 2.9 [15]. Let R be a ring and G a group.

(a) If R is left (right) noetherian and G is finite, then RG is left (right) noetherian.

(b) If RG is left (right) noetherian, then R is left (right) noetherian and G has the maximum condition on subgroups.

(c) If G is abelian, then RG is left (right) noetherian if and only if R is left (right) noetherian and G is finitely generated.

The maximum condition on subgroups implies that G is finitely generated, but as Example 2.10 shows the two are not equivalent.

Example 2.10 [39]. Let G be the multiplicative group generated by the real matrices,

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Consider the subgroups of G,

$$I_n = \{\begin{pmatrix} 1 & m\cdot 2^{-k} \\ 0 & 1 \end{pmatrix} \mid 0 \leqq k \leqq n, \text{ and } m, n, \text{ and } k \text{ are nonnegative integers}\}.$$

Since $I_0 < I_1 < \cdots$ is an infinite ascending chain of subgroups, G does not have the maximum condition on subgroups.

There is a classic theorem for noetherian polynomial rings. Hilbert [36] proved the following theorem when the coefficient ring is a field near the end of the 19th century.

Theorem 2.11 (Hilbert Basis Theorem) [40]. If R is a left noetherian ring, then so is R[x].

Theorem 1.30. Let R be a ring and T a finitely generated commutative monoid. If R is left noetherian, then R[T] is left noetherian.

Theorem 2.12. If R[S] is left noetherian, then R is left noetherian and S has the maximum condition on left ideals.

Proof. By Theorem 1.27 R must be left noetherian. Let $H_1 < H_2 < \cdots$ be an infinite ascending chain of left ideals of S. Let ωH be the left ideal of R[S] generated by $\{rh_1 - rh_2 \mid r \in R \text{ and } h_1, h_2 \in H\}$, where H is a left ideal of S. Therefore $\omega H_1 < \omega H_2 < \cdots$ is an infinite ascending chain of left ideals of R[S] and so it must terminate. Therefore the original chain terminates and S has the maximum condition.

Theorem 2.13 [26]. Let R be a commutative ring and T a commutative monoid. The monoid ring R[T] is noetherian if and only if R is noetherian and T is finitely generated.

It would be nice if the result of Theorem 2.13 was true if the coefficient ring is noncommutative.

Question 2.14. If T is commutative, then does R[T] left noetherian imply that T is finitely generated?

If the monoid in Theorem 2.13 is replaced by a semigroup, part of the results are lost.

Example 2.15. Consider Example 2.4 with $S = \{a,b\}$ and multiplication defined by $a^2 = ab = ba = b^2 = b$.

$$\{na + qb \mid n\varepsilon Z, q\varepsilon Q\} < \{\tfrac{1}{2}na + qb \mid n\varepsilon Z, q\varepsilon Q\} < \{\tfrac{1}{4}na + qb \mid n\varepsilon Z, q\varepsilon Q\} < \cdots$$

is an infinite ascending chain of ideals in Q[S] so Q[S] is not noetherian even though Q is noetherian.

Theorem 1.32. If R[S] is left (right) noetherian, then $R[S^1]$ is left (right) noetherian.

Corollary 2.16 [26]. Let R be a commutative ring and S a commutative semigroup. If R[S] is noetherian, then R is noetherian and S is finitely generated.

Yet another generalization of artinian rings are perfect rings. In these rings the structure on J(R) has been weakened from being nilpotent to being T-nilpotent. Recall Definition 1.20 that a subset A of a multiplicative semigroup S is <u>left T-nilpotent</u> in case for every sequence $a_1, a_2, \ldots$ in A there is an n such that $a_1 \cdots a_n = 0$. The subset A is <u>right T-nilpotent</u> in case for each $a_1, a_2, \ldots$ in A, $a_n \cdots a_1 = 0$ for some n.

Definition 2.17. A ring R is <u>left (right) perfect</u> if R/J(R) is semisimple and J(R) is left (right) T-nilpotent.

Woods [76] was able to completely determine perfect group rings and find a result analogous to Connell's theorem for artinian group rings.

Theorem 2.18 (Woods) [76]. RG is left (right) perfect if and only if R is left (right) perfect and G is finite.

Unfortunately the result of Woods' Theorem doesn't hold for semigroup rings. There have been two different directions for proving results for perfect semigroup rings. The first approach is by Domanov [17]. Although Domanov completely characterizes perfect semigroup ring, his results require that some new terms be defined.

Definition 2.19 [13,17]. A semigroup S with zero θ is called <u>0-simple</u> if it has no nonzero proper ideals and $S^2 \neq \{\theta\}$.

Definition 2.20 [13,17]. A 0-simple semigroup is <u>completely 0-simple</u> if it has a minimal nonzero left ideal and a minimal nonzero right ideal.

Definition 2.21 [69]. A ring R is said to satisfy a <u>polynomial identity</u> if there exists a polynomial in non-commuting variables $x_1,\ldots,x_d$ of the form

$\sum_{\sigma \varepsilon S_d} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(d)}$, where the coefficients α_σ are ±1, such that

$\sum_{\sigma \varepsilon S_d} \alpha_\sigma r_{\sigma(1)} \cdots r_{\sigma(d)} = 0$ for all choices of $r_1,\ldots,r_d \;\varepsilon\; R$ (S_d denotes the symmetric group on d letters).

If S does not have a zero element (identity), then one can be adjoined to S and the resulting semigroup rings are perfect if and only if the original semigroup ring was perfect.

Theorem 2.22 [17]. $R[S]$ is left (right) perfect if and only if $R[S^1]$ is left (right) perfect if and only if $R[S^0]$ is left (right) perfect.

Theorem 2.23 [17]. $R[S^0]$ is left (right) perfect if and only if S^0 has a finite series of ideals $S^0 = F_n > F_{n-1} > \cdots > F_1 > F_0 = \{\theta\}$ such that for any $i = 1,2,\ldots,n$:

(a) F_i/F_{i-1} is either 0-simple or locally nilpotent;

(b) $R[F_i/F_{i-1}]$ is left (right) perfect.

Theorem 2.24 [17]. Let S be a locally nilpotent semigroup. The semigroup ring $R[S]$ is left perfect if and only if R is left perfect and S is left T-nilpotent.

Theorem 2.25 [17]. If S is 0-simple but not completely 0-simple, then its semigroup ring is not left perfect.

Theorem 2.26 [13]. Any periodic (in particular, any finite) 0-simple semigroup is completely 0-simple.

Theorem 2.27 [17]. Let S be a completely 0-simple semigroup and B the subring of R generated by the identity. The semigroup ring $R[S]$ is left perfect if and only if:

(a) R is left perfect;

(b) the subgroups of S are finite;

(c) $B[S]$ satisfies a polynomial identity.

Thus Domanov completely characterizes perfect semigroup rings. More recently, Okninski [58] has found another characterization of many perfect semigroup rings. First he obtains results for semigroup rings whose coefficient ring is a field, and then extends these results to K-algebras.

Definition 2.28. $E(S) = \{\text{idempotents of } S\}$. If $e \in E(S)$, then $S_e = \{g \in eSe \mid g \text{ is invertible in } eSe\}$. The elements $e, f \in S$ are said to be <u>p-equivalent</u> if for any $g \in S$ the following statement holds:

> $ege \in S_e$ if and only if $efge \in S_e$ and $egfe \in S_e$, and if these are elements of S_e, then ege, efge, egfe belong to the same coset of a normal p-subgroup in S_e.

Theorem 2.29 [58]. Let K be a field with char K = p. Then K[S] is left perfect if and only if

(a) S is periodic,

(b) S has d.c.c. on left principal subgroups,

(c) S has no infinite subgroups,

(d) $E(S) = \bigvee_{i=1}^{s} E_i$ for some disjoint subsets E_i of mutually p-equivalent idempotents.

Theorem 2.30 [58]. Let R be a K-algebra. R[S] is left perfect if and only if so are the rings R and K[S].

A class of rings that are left and right perfect and contains all left and right artinian rings is the class of semiprimary rings.

Definition 2.31. A ring R is <u>semiprimary</u> if R/J(R) is semisimple and J(R) is nilpotent.

Theorem 2.32 [70]. RG is semiprimary if and only if R is semiprimary and G is finite.

It would be nice to know more about semiprimary semigroup rings.

Question 2.33. What does R[S] (R[T]) semiprimary imply about S (T)?

In the case of perfect rings the Jacobson radical was required to be T-nilpotent. If J(R) is T-nilpotent, then J(R) is nil which implies that idempotents lift modulo J(R). It is this last condition that forms the basis for a generalization of perfect rings, the semiperfect rings.

Definition 2.34. A ring R is <u>semiperfect</u> if R/J(R) is semisimple and idempotents lift modulo J(R).

One difference between semiperfect rings and some of the other rings examined in this chapter is that R semiperfect and G finite does not imply RG is semiperfect.

Example 2.35 [77]. Let $R = \{ \frac{a}{b} \mid a,b \in Z$ and 7 does not divide b$\}$. Thus $J(R) = \{ \frac{7a}{b} \mid a,b \in Z$ and 7 does not divide b$\}$ and $R/J(R) \cong Z_7$ which is a field. Therefore R is semiperfect. Let $C_3 = \{1,\omega,\omega^2\}$, where ω is the cube root of 1. By Theorem 2.44 if suffices to consider lifting idempotents modulo $J(R)C_3$.

$$(R/J(R))C_3 \cong Z_7[x] \,/\, (x^3 - 1)$$
$$\cong Z_7[x] \,/\, \{(x - 1)(x - 2)(x - 4)\}$$
$$RC_3 \cong R[x] \,/\, (x^3 - 1)$$
$$\cong R[x] \,/\, \{(x - 1)(x^2 + x + 1)\}$$

Since $x^2 + x + 1$ is irreducible over R, a result of Azumaya [7] shows that there is an idempotent that does not lift. Explicitly, let

$A = \{6\omega^2 + 3\omega + 5,\ 5\omega^2 + 5\omega + 5,\ 3\omega^2 + 6\omega + 5\}$ and

$B = \{\frac{1}{3}(\omega^2 + \omega + 1),\ \frac{-1}{3}(\omega^2 + \omega - 2)\}$.

All idempotents of $(R/J(R))C_3$ are elements of A or a sum of some of the elements of A. Similarly B may be used to create all of the idempotents of RC_3. The idempotent $5\omega^2 + 5\omega + 5$ lifts to $\frac{1}{3}(\omega^2 + \omega + 1)$, but the other two elements of A do not lift to idempotents of RC_3. (However $(6\omega^2 + 3\omega + 5) + (3\omega^2 + 6\omega + 5) = 2\omega^2 + 2\omega + 3$ lifts to $\frac{-1}{3}(\omega^2 + \omega - 2)$.)

Now consider semiperfect rings when the coefficient ring is a field.

Theorem 2.36 [71]. Let K be an algebraically closed, noncountable field with char K = p. If KG is semiperfect then G is a finite extension of a p-group.

Theorem 2.37 [29]. Let K be a field with char K = p and G a locally finite or locally solvable group. The group algebra KG is semiperfect if and only if G is a finite extension of a p-group.

Theorem 2.38 [77]. If RG is semiperfect, then R is semiperfect, G is a torsion group, and there are no infinite chains of finite subgroups of G whose orders are units in R.

It is natural to ask if RG semiperfect implies that G is locally finite. Burgess was able to show that this is not necessarily true.

Example 2.39 [8]. Herstein [35, Chapter 8] gives an example of a p-group G that is generated by three elements but is not locally finite. In the particular example if K is a field of characteristic p, then every element of ωG is nilpotent and so ωG is a nil ideal. Thus $\omega G < J(KG)$. Now $0 = J(K) \cong J(KG/\omega G)$. Theorem 7.10 gives $J(KG) = \omega G$ so $KG/J(KG) = KG/\omega G \cong K$ which is field. Therefore KG is semiperfect.

If the group G is assumed to be abelian, Burgess [8] was able to get a nice result that uses the Weddberburn-Artin Theorem.

Definition 2.40. Let $\underline{\mathrm{Mat}_n(R)}$ denote the ring of square matrices with n rows and columns with entries from R.

Theorem 2.41 [39] (Wedderburn-Artin). Let R be a semisimple ring. There exists division rings $D_1,\ldots,D_t$ and positive integers $n_1,\ldots,n_t$ such that $R \cong \mathrm{Mat}_{n_1} D_1 \times \cdots \times \mathrm{Mat}_{n_t} D_t$.

Theorem 2.42 [8]. If RG is semiperfect and G is abelian, then either

(a) R is semiperfect and G is finite or

(b) R is semiperfect and $G \cong G_p \times H$ where G_p is an infinite p-group, H is finite, p does not divide the order of H, and each of the division rings associated with the semisimple ring R/J(R) by the Wedderburn-Artin Theorem is of characteristic p.

Theorem 2.43 [8]. If R is commutative, G is abelian, $G \cong H \times G_p$ where G_p is an infinite p-group, H is finite, p does not divide the order of H, RH is semiperfect, and each of the fields associated with the semisimple ring RH/J(RH) by the Wedderburn-Artin Theorem is of characteristic p, then RG is semiperfect.

Theorem 2.44 [8]. Let R be semiperfect and let G be finite. If idempotents can be lifted modulo J(R)G, then RG is semiperfect. If RG is commutative the converse is true.

Two corollaries for semiperfect semigroup rings that are proven in Chapter 3 follow.

Corollary 2.45. Let R be a commutative ring of characteristic zero and S a finite cancellative commutative semigroup. If R[S] is semiperfect, then S is a finite group.

Corollary 2.46. Let K be a field with char K = 0 and S a cancellative commutative semigroup. K[S] is semiperfect if and only if S is a finite group.

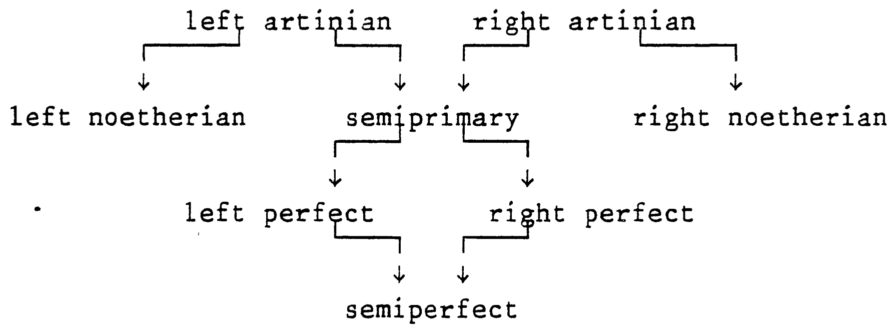

Figure 1. Implication Diagram for Finiteness Conditions

CHAPTER 3

LOCAL AND SEMILOCAL

Definition 3.1. A ring R is <u>local</u> if R/J(R) is a division ring.

Some properties of local rings were encountered in Chapter 2 since local rings are an example of semiperfect rings. The two are closely related for commutative rings.

Theorem. 3.2 [5]. A commutative ring is semiperfect if and only if it is isomorphic to the finite direct product of local rings.

As one would expect the commutative local group rings are particularly well-behaved.

Theorem 3.3 [32]. If R and G are commutative, then RG is local if only if R is local, G is a p-group and $p \in J(R)$.

Note that R local and G finite does not necessarily imply RG is local.

Example 3.4. Let C be the field of complex numbers. Since $C/J(C) = C/0 = C$ is a division ring, C is local. The group ring CZ_2 isn't local, however. Elements of the group ring may be written as $c_1 + c_2x$. Since $J(CZ_2) = 0$ and the nonzero element $1 + x$ has no inverse, CZ_2 isn't local.

Nicholson [54] generalized Theorem 3.3 to noncommutative group rings.

Theorem 3.5 [54]. Let R be a ring and G a group.

(a) If RG is local, then R is local, G is a p-group and $p \in J(R)$.

(b) (Partial converse.) If R is local, G is a locally finite p-group and $p \in J(R)$, then RG is local.

(c) If G is abelian, then RG is local if and only if R is local, G is a p-group and $p \in J(R)$.

Theorem 3.6 [26,33]. Let R be a commutative ring and T a commutative monoid. R[T] is local if and only if R is local, T is a p-group and $p \in J(R)$. (Note that T is forced to be a p-group.)

Hardy and Shores [33] originally proved Theorem 3.6 using the assumption that T was cancellative. Gilmer [26] was able to remove this restriction.

Theorem 3.7 [57]. Let K be a field.

(a) If char K = 0, then K[S] is local if and only if S is locally finite and $eSe = \{e\}$ for any idempotent $e \in S$.

(b) Assume S is locally finite and char K = p. Then K[S] is local if and only if eSe is a p-group for any idempotent $e \in S$.

A generalization of the local (and semiperfect) rings are the semilocal rings.

Definition 3.8. A ring R is <u>semilocal</u> if R/J(R) is a semisimple ring. (Thus R is semiperfect if and only if R is semilocal and idempotents lift modulo J(R).)

The close connection between semilocal and semiperfect rings becomes even more apparent for commutative semigroup rings. Applying Theorem 7.12 gives the following theorem.

Theorem 3.9. Let RG be commutative. If idempotents lift modulo J(R)G (e.g. a ring with J(R) = 0), then RG is semilocal if and only if RG is semiperfect.

It is probable that for arbitrary KG, semilocal and semiperfect are equivalent. If Conjecture 7.17 is true, then

Conjecture 3.10. KG is semilocal if and only if KG is semiperfect.

Theorem 3.11 [77]. If RG is semilocal, then G is torsion.

Theorem 3.12 [77]. Let R be a ring such that char(R/J(R)) = 0 (e.g. let R be a field of characteristic zero). RG is semilocal if and only if R is semilocal and G is finite.

Theorem 3.13 [61,71]. Let K be an algebraically closed nondenumerable field and let G be a group. Then KG is semilocal if and only if either

(a) char K = 0 and G is finite, or

(b) char K = p and G has a normal p-subgroup P of finite index with $\omega P = J(KP)$.

Theorem 3.14 [49,57,61]. Let K be a field and G a group.

(a) If char K = 0, then KG is semilocal if and only if G is finite.

(b) If char K = p and G is locally finite, then KG is semilocal if and only if G contains a p-subgroup of finite index.

Theorem 3.15 [8]. If R[S] is semilocal, then D[S]/J(D[S]) is artinian for any division ring D associated with R through the Wedderburn-Artin Theorem.

Proof. If R[S] is semilocal, then (R/J(R))[S] is semilocal.

(Since $\dfrac{(R/J(R))[S]}{J((R/J(R))[S])} \cong \dfrac{R[S]\ /\ (J(R)[S])}{J(R[S])\ /\ (J(R)[S])} \cong R[S]\ /\ J(R[S])$.)

If R[S] is semilocal, then R is semilocal, so that R/J(R) is semisimple. By the Wedderburn-Artin Theorem

$$R/J(R) \cong Mat_{n_1} D_1 \times \cdots \times Mat_{n_t} D_t$$

$$(R/J(R))[S] \cong (Mat_{n_1} D_1 \times \cdots \times Mat_{n_t} D_t)[S]$$

$$\cong Mat_{n_1} D_1[S] \times \cdots \times Mat_{n_t} D_t[S]$$

Observe that $Mat_{n_i} D_i[S]$ is factor ring of (R/J(R))[S]. Since (R/J(R))[S] is semilocal and factor rings of semilocal rings are semilocal, $Mat_{n_i} D_i[S]$ is semilocal. $Mat_{n_i} D_i[S]$ is semilocal if and only if $D_i[S]$ is semilocal, so $D_i[S]$ is semilocal. If $D_i[S]$ is semilocal, then $D_i[S]/J(D_i[S])$ is semisimple.

Theorem 3.16 [43,65]. If R and G are commutative, then RG is semilocal if and only if one of the following two conditions hold:

(a) R is semilocal and G is finite

(b) R is semilocal, $R/J(R)$ is of prime characteristic $p > 0$, $G \cong G_p \times H$ where G_p is an infinite p-group and H is a finite group whose order is not divisible by p.

Proof. (Only the ($\rightarrow$) direction will be proven.) Applying Theorem 3.15 gives $K_iG/J(K_iG)$ is artinian, where K_i is a field.

Case 1: K a field of characteristic zero.
By Corollary 7.13 $J(KG) = 0$, so $KG/J(KG) = KG$ is artinian. Corollary 2.6 implies that G must be finite.

Case 2: K a field of prime characteristic p.
Here $G \cong G_p \times H$, where G_p is an infinite p-group and H has no elements of order p (the case where G_p is finite is included in part (a)). Since $KG \cong K(G_p \times H) \cong KG_p \times KH$, KH is a factor ring of KG and so KH is semilocal. As in case 1, $J(KH) = 0$, so $KH/J(KH) = KH$ is artinian and thus H must be finite.

Corollary 3.17 [56,77]. If R[S] is semilocal, then R is semilocal and S is periodic. If $R/J(R)$ is not torsion, then the semigroup S must be locally finite.

Related to Theorem 3.16 is the following theorem.

Theorem 3.18. Let R be a commutative ring of characteristic zero and S a finite cancellative commutative semigroup. If R[S] is semilocal, then S is a finite group.

Proof. By Theorem 3.15 $K[S]/J(K[S])$ is artinian for each field associated with R through the Wedderburn-Artin theorem. By Theorem 7.25 $J(K[S]) = 0$ and so K[S] is artinian and hence S is finite by

Corollary 2.6. Since any cancellative commutative finite semigroup is a group, S is a group.

Corollary 2.45 follows immediately from Theorem 3.18.

Corollary 3.19. Let K be a field with char K = 0 and S a cancellative commutative semigroup. K[S] is semilocal if and only if S is a finite group.

Proof. (→) Theorem 3.18.

(←) Theorem 3.14.

Corollary 2.46. Let K be a field with char K = 0 and S a cancellative commutative semigroup. K[S] is semiperfect if and only if S is a finite group.

Proof. (→) Corollary 3.19.

(←) Theorem 2.44.

In [57] Okninski obtains some nice results for semigroup rings when the coefficient ring is a field. He then extends these results to K-algebras. He uses the following theorem as the starting point for his investigations.

Theorem 3.20 [49,56,57]. Let K be a field, S a semigroup, and assume that K[S] is semilocal. Then

(a) S is torsion

(b) S is locally finite if char K = 0.

Theorem 3.21 [57]. Let char K = 0. Then K[S] is semilocal if and only if

(a) S is locally finite and there exists $N \geqq 1$ such that S has no subgroup of order exceeding N, and

(b) $E(S) = \bigvee_{i=1}^{s} E_i$ for some disjoint subsemigroup E_i with the property that if $e,f \in E_i$ and $g \in S$ then ege is invertible in eSe if and only if so are the elements efge and egfe in which case, necessarily ege = efge = egfe.

Theorem 3.22 [57]. Let char K = p and let S be locally finite. Then K[S] is semilocal if and only if

(a) there exists $N \geqq 1$ such that any subgroup in S has a p-subgroup in S of index not exceeding N, and

(b) $E(S) = \bigvee_{i=1}^{s} E_i$ for disjoint sets E_i with the property that if $e,f \in E_i$ and $g \in G$, then ege is invertible in eSe if and only if so are efge and egfe - and then ege, efge and egfe are in the same coset of some normal p-subgroup in the subgroup of invertible elements in eSe.

Corollary 3.23 [57]. Let S be commutative. Then K[S] is semilocal if and only if S is torsion with at most finitely many idempotents, and

(a) S has no infinite subgroups if char K = 0, and

(b) any subgroup in S contains a p-subgroup of finite index if char K = p.

Theorem 3.24 [57]. Let R be a K-algebra. Assume that S is locally finite. Then R[S] is semilocal if and only if so are R and K[S].

Example 3.25. The polynomial ring $R[x]$ is never semilocal (and hence is never semiperfect) since

$$R[x]/J(R[x]) = R[x]/M[x] \quad \text{where } M = J(R[x]) \cap R \quad \text{by Theorem 7.8}$$

$$\cong (R/M)[x] \quad \text{which is never artinian by Theorem 2.3.}$$

Local
↓
Semiperfect
↓
Semilocal

Figure 2. Implication Diagram for Local and Related Rings

CHAPTER 4

VON NEUMANN REGULAR

Definition 4.1. An element b of a ring R is (von Neumann) regular if there exists $x \in R$ such that $bxb = b$. If every element of R is regular, then R is said to be a regular ring. Similarly, an element s of a semigroup S is regular if there exists $t \in S$ such that $sts = s$.

Theorem 4.2 [15]. RG is regular if and only if

(a) R is regular.

(b) G is locally finite.

(c) the order of every finite subgroup of G is a unit in R.

Example 4.3. R[x] is never regular.

If R[x] is regular, then given $1 \cdot x^1 \in R[x]$ there exists $y \in R[x]$ such that $xyx = x$. Now $y = y_n x^n + \ldots + y_0$. So $xyx = y_n x^{n+2} + \ldots + y_0 x^2 = x$ which is a contradiction.

Theorem 4.4 [75]. Let S be a semigroup with a zero element θ and suppose that $S \neq \{\theta\}$.

(a) R[S] is regular if and only if $(R[S])_0$ is regular.

(b) If $(R[S])_0$ is regular, then both R and S are regular.

Since a zero can be adjoined to any semigroup without affecting regularity results, the following theorems assume the semigroup S has a zero.

Theorem 4.5 [28]. Let R be a commutative ring and let S be a commutative semigroup. The ring R[S] is regular if and only if the following three conditions are satisfied.

(a) R is regular.

(b) S is a union of torsion groups.

(c) each prime p that divides the order of an element of S is a unit of R.

Theorem 4.6 [75]. Let S be any semigroup. If R[S] is regular, then

(a) R is regular,

(b) every subgroup of S is locally finite, and

(c) the order of every finite subgroup of S is a unit in R.

Theorem 4.7 [75]. Let S be an inverse semigroup, i.e., a regular semigroup in which idempotents commute. R[S] is regular if

(a) R is regular,

(b) every finite subset of S is contained in a finite inverse semigroup, and

(c) the order of every finite subgroup of S is a unit in R.

Theorem 4.8 [75]. Let S be an inverse semigroup which is a union of groups G_α, $\alpha \in \Omega$. Then R[S] is regular if and only if

(a) R is regular,

(b) for every $\alpha \in \Omega$, G_α is locally finite, and

(c) for every $\alpha \in \Omega$, the order of every finite subgroup of G_α is a unit in R.

CHAPTER 5

SIMPLE AND SEMISIMPLE

Definition 5.1. A ring R is _simple_ in case 0 and R are the only ideals of R.

Theorem 5.2. If R[S] is simple, then R is simple.

Proof. Apply Theorem 1.27.

Theorem 5.3 [39]. A commutative ring R is simple if and only R is a field.

Theorem 5.4 [15]. RG is a field if and only if R is a field and $G = \{1\}$.

Definition 5.5. A module M is _simple_ in case $M \neq 0$ and it has no nontrivial submodules. A module M is _semisimple_ if it is the direct sum of some set of simple modules. A ring R is _semisimple_ if $_R R$ is semisimple.

Since semisimple rings are artinian, a semisimple semigroup ring R[S] must have S finite. Unfortunately semisimple semigroup rings have a more complex structure than artinian semigroup rings. One of the best known theorems for group rings addresses the problem of semisimple group rings and is due to H. Maschke. The following theorem (and its proof) are found in the Introduction.

Theorem 1.31 [31] (Maschke) Suppose G is a finite group, K a field, and either char K = 0 or char K = p with $p \nmid |G|$. Then KG is semisimple.

Theorem 5.6 [15]. A ring R is semisimple if and only if R is regular and left or right noetherian.

Theorem 5.7 [15] (The Generalized Maschke Theorem). RG is semisimple if and only if

(a) R is semisimple,

(b) G is finite, and

(c) the order of G is a unit in R

Proof. Combine Theorems 2.9, 4.2, and 5.6.

Theorem 5.8. Let R be a commutative ring and S a commutative semigroup with a zero element. If R[S] is semisimple, then

(a) R is semisimple,

(b) S is a union of torsion groups,

(c) each prime p that divides the order of an element of S is a unit of R, and

(d) S is finite.

If S is a monoid, the converse is true.

Proof. Combine Theorems 2.13, 2.16, 4.5, and 5.6.

Definition 5.9 [10]. Let G be a group. Let G^0 be the semigroup obtained by adjoining a zero element to G. Let P be an $n \times m$ matrix with entries in G^0. Then the <u>Rees matrix semigroup $M^0(G;m,n;P)$</u> is defined to be the set of all $m \times n$ matrices with entries in G^0 such that at most one entry is nonzero. The multiplication is defined by $A \cdot B = APB$ for A,B in $M^0(G;m,n;P)$. Note that the zero matrix is the zero element. Let $\underline{M(G;m,n;P)}$ mean $M^0(G;m,n;P) - \theta$, where θ denotes the zero matrix.

Theorem 5.10 [10]. Let S be a semigroup with a zero element θ. The following are equivalent.

(a) R[S] is semisimple.

(b) R is semisimple and S is finite with principal series

$$S = S_1 > S_2 > \cdots > S_n > S_{n+1} = \{\theta\}$$

such that $S_i/S_{i+1} \cong M^0(G_i;m_i,m_i;P_i)$ where G_i is a subgroup of S_i/S_{i+1} with its order invertible in R and where P_i is invertible over RG_i for $i = 1,2,...n-1$.

(c) R is semisimple and $R[S] \cong (\mathop{X}_{i=1}^{n-1} \mathrm{Mat}_{m_i}(RG_i)) \times R$.

If S has no zero element, then one may adjoin a zero element to S without affecting the semisimplicity of R[S].

Theorem 5.11 [10]. Let S be a semigroup without a zero element. Let θ be a zero element. The following are equivalent.

(a) R[S] is semisimple.

(b) R is semisimple and S is finite with principal series

$$S = S_1 > S_2 > \cdots > S_n > S_{n+1} = \{\theta\}$$

such that $S_i/S_{i+1} \cong M^0(G_i;m_i,m_i;P_i)$ for $i = 1,2,...n-1$

$\cong M(G_i;m_i,m_i;P_i)$ for $i = n$,

where G_i is a subgroup of S_i/S_{i+1} with its order invertible in R and where P_i is invertible over RG_i.

(c) R is semisimple and $R[S] \cong \overset{n}{\underset{i=1}{X}} \mathrm{Mat}_{m_i}(RG_i)$ where G_i is a finite group with its order invertible in R.

Example 5.12. R[x] is never semisimple.

CHAPTER 6

INDECOMPOSABLE

Definition 6.1. A ring R is <u>indecomposable</u> if it is not expressible as a nontrivial internal direct sum of ideals.

Theorem 6.2 [5]. A ring R is an indecomposable ring if and only if 1 is the only nonzero central idempotent of R.

Definition 6.3. A commutative ring R with identity $1_R \neq 0$ and no zero divisors is called an <u>integral domain</u>.

Theorem 6.4 [14]. Let R be an integral domain and let G be a finite group of order n. RG is indecomposable if and only if any prime p dividing the order of G is a nonunit in R.

Theorem 6.5 [43]. Let R be a commutative ring and G an abelian group. Then RG is indecomposable if and only if R is indecomposable and G does not contain p-elements whenever p is a unit of R.

Theorem 6.6 [26]. For a unitary commutative ring R and a commutative monoid T, R[T] is indecomposable if and only if R is indecomposable and the idempotents of R[T] are those of R.

Theorem 6.7 [26]. For a unitary commutative ring R and a commutative monoid T, the ring R[T] is indecomposable if and only if R is indecomposable, the set of periodic elements of T is a subgroup of T, and the order of each nonzero periodic element of T is a nonunit of R.

Theorem 6.8 [28]. Let R be a commutative ring and let S be an additive commutative semigroup. The ring R[S] is indecomposable if and only if R is indecomposable, the set of periodic elements of S is a subgroup of S, and the order of each periodic element of S is a nonunit of R.

Corollary 6.9 [28]. Let R be a commutative ring and let S be an additive commutative semigroup. If R has prime characteristic p, then R[S] is indecomposable if and only if R is indecomposable and the set G of periodic elements of S is a p-group.

CHAPTER 7

JACOBSON AND NIL RADICALS

Recall the following three definitions from Chapter 1.

Definition 1.7. Let R be a ring. The Jacobson radical J(R) of R is the intersection of all maximal left (right) ideals of R.

Definition 1.8. An ideal P in a ring R is said to be prime if P $\neq$ R and for any ideals A, B in R, AB $\leqq$ P implies A $\leqq$ P or B $\leqq$ P.

Definition 1.9. The prime radical or the lower nil radical N(R) of a ring R is the intersection of all prime ideals of R. The upper nil radical U(R) is the unique largest nil ideal of R.

Theorem 7.1 [5]. N(R) $\leqq$ U(R) $\leqq$ J(R).

Theorem 7.2 [61]. Let S be a ring, and let R be a subring with the same 1. Suppose that as left R-modules R is a direct summand of S. Then (J(S) $\wedge$ R) $\leqq$ J(R).

Corollary 7.3. (J(RG) $\wedge$ R) $\leqq$ J(R) and (J(R[T]) $\wedge$ R) $\leqq$ J(R).

Theorem 7.4 [15]. (J(RG) $\wedge$ R) $\leqq$ J(R). There is equality if either R is artinian or if G is locally finite.

Theorem 7.5 [15]. Let H be a subgroup of G. Then $J(RG) \wedge RH \leqq J(RH)$.

Theorem 7.6 [61]. Let K be a field with char K = p, and let H be a normal subgroup of G with G/H abelian. Let K be a field, and suppose, further, that G/H has no elements of order p in case K has characteristic p. Then

$J(KG) = (J(KG) \wedge KH)\cdot KG \leqq J(KH)\cdot KG$.

Corollary 7.7 [61]. Let G be an abelian group with no elements of order p in case K has characteristic p. Then $J(KG) = 0$.

Proof. Let H = 0 in Theorem 7.6.

Theorem 7.8 [2]. $J(R[x]) = M[x]$, where $M = (J(R[x]) \wedge R)$ is a nil ideal in R. If R is a commutative ring, then $M = U(R)$.

Theorem 7.9 [61]. Let H be a nonidentity subgroup of G. If $\omega H \leqq J(KG)$, then $\omega H = J(KH)$, K is a field of characteristic p for some prime p, and H is a p-group.

Theorem 7.10 [61]. Let G be a locally finite p-group, and let K be a field of characteristic p. Then $J(KG) = \omega G$.

[44] Let J(RG) be the Jacobson radical of the group ring RG of an abelian group G over a commutative ring R. If R is a field of characteristic 0, or a subdirect product of such, a result due to Amitsur [1] and Villamayor [73,74] provides necessary and sufficient conditions satisfied by R and G which force $J(RG) = 0$. This result was extended to arbitrary rings by Connell [15]. Karpilovsky [44] was

able to provide a complete description of J(RG) in terms of R and G when both are commutative and this is given in Theorem 7.12.

Definition 7.11 [44]. Let $G_p = \{g \in G \mid g$ has order a power of $p\}$, P = the set of primes which are orders of elements of G, and <u>$\langle U \rangle$</u> = the ideal of RG generated by a subset U of RG.

$N_p = \{r \in R \mid pr \in N(R)\}$,

$J_p = \{r \in R \mid pr \in J(R)\}$, where $p \in P$.

Theorem 7.12 [44]. Let G be an abelian group and R a commutative ring. Then

$$J(RG) = \begin{cases} J(R)G + \langle r(g - 1) \mid g \in G_p,\ r \in J_p(R) \text{ for some } p \in P\rangle & \text{if } G \text{ is torsion} \\ N(R)G + \langle r(g - 1) \mid g \in G_p,\ r \in N_p(R) \text{ for some } p \in P\rangle & \text{otherwise} \end{cases}$$

Corollary 7.13 [15,44]. Let R and G be commutative.

(a) If G is torsion, then J(RG) = 0 if and only if J(R) = 0 and any prime that is the order of an element of G is not a zero divisor of R.

(b) If G is not torsion, then J(RG) = 0 if and only if N(R) = 0 and any prime that is the order of an element of G is not a zero division of R.

Corollary 7.14 [44]. Let G be an abelian group and R a commutative ring. J(RG) is nil if and only if one of the following two conditions hold: (a) G is not torsion or (b) G is torsion and J(R) is nil.

Corollary 7.14 gives the interesting result that if RG is commutative and J(R) is nil, then J(RG) is nil.

It will be shown that J(KG) = N(KG) for torsion or commutative groups, but it is not true in general that J(KG) = N(KG) (although this equality may be true for finitely generated groups). Passman [61] was led to define an analog of the nil radical, namely the N^*-radical, to handle groups that are not necessarily finitely generated.

Definition 7.15 [60]. Let $N^* = \{\alpha \in R \mid \alpha S$ is nilpotent for all finitely generated subrings $S \leqq R\}$.

Theorem 7.16 [61]. Let R be a K-algebra with 1. Then $N^*(R)$ is a nil ideal of R, and $N(R) \leqq N^*(R) \leqq J(R)$. If R is finitely generated as an algebra, then $N^*(R) = N(R)$.

Conjecture 7.17 [60]. Let G be a group and let K be a field. Then $J(KG) = N^*(KG)$.

Definition 7.18 [52]. A commutative semigroup S is said to be <u>separative</u> if given any x, y ε S, $x^2 = xy = y^2$ implies that x = y. Let p be a prime. A commutative semigroup S is said to be <u>p-separative</u> if given any x, y ε S, $x^p = y^p$ implies that x = y.

Definition 7.19 [53]. A <u>semilattice congruence</u> on a semigroup S is a congruence ρ such that S/ρ is a semilattice (a commutative semigroup of idempotents). The classes of such a congruence are subsemigroups of S. It can be shown that S admits a least congruence whose corresponding factor group is separative and we shall call S <u>semican-</u>

cellative if and only if this factor group is cancellative. Recall that S is nonperiodic if and only if it contains an element x such that $x^r \neq x^s$ for all distinct positive integers r and s.

Theorem 7.20 [53]. Let R be an integral domain and let G be an abelian group. Suppose, further, that if char R is a prime p then G has no p-elements. Then

(a) N(RG) = 0,

(b) J(RG) = 0 if G is nonperiodic.

Theorem 7.21 [53]. If R is a commutative ring and S is a commutative semigroup which admits a semilattice congruence whose classes are nonperiodic and semicancellative, then J(R[S]) is nil.

Corollary 7.22 [53]. If R is commutative ring and S is a cancellative commutative semigroup which is not a torsion abelian group, then J(R[S]) is nil.

Corollary 7.23 [53]. If R is a commutative ring and S is a commutative semigroup without idempotents, then J(R[S]) is nil.

Theorem 7.24 [52]. Let S be a commutative semigroup and K a field with char K = 0. Then the following statements are equivalent.

(a) J(K[S]) = 0.

(b) N(K[S]) = 0.

(c) S is separative.

Corollary 7.25 [52]. Let S be a cancellative commutative semigroup and K a field with char K = 0. Then J(K[S]) = 0.

Proof. Every cancellative semigroup is separable. Apply Theorem 7.24.

Theorem 7.26 [52]. Let S be a commutative semigroup and K a field with char K = p. Then the following statements are equivalent.

(a) $J(K[S]) = 0$.

(b) $N(K[S]) = 0$.

(c) S is p-separative.

Definition 7.27 [52]. For an arbitrary semigroup S, congruence ρ on S and field K we denote by $I(\rho)$ the subspace of K[S] spanned by the set $\{x-y \mid x\rho y\}$. ($I(\rho)$ is an ideal of K[S] and is zero if and only if ρ is the identity congruence on S.)

Theorem 7.28 [52]. Let K be a field and let S be a commutative semigroup. Then

$$J(K[S]) = N(K[S]) = I(\rho),$$

where

$$\rho = \begin{cases} \text{the least separative congruence on } S \text{ if char } K = 0, \\ \text{the least } p\text{-separative congruence on } S \text{ if char } K = p. \end{cases}$$

Theorem 7.29 [43]. Let G be an abelian group, R a commutative ring, and let G_p, N_p, and P be defined as in Definition 7.11. Then

$$N(RG) = N(R)G + \langle r(g - 1) \mid g \in G_p,\ r \in N_p(R) \text{ for some } p \in P\rangle$$

Corollary 7.30. If G is an abelian group that is not a torsion group and R is a commutative ring, then $J(RG) = N(RG)$.

Proof. Use Theorems 7.12 and 7.29.

Definition 7.31 [26]. Let a be an element of a semigroup S and ρ a congruence on S. Let [a] denote the equivalence class of $a \in S$ under ρ.

Theorem 7.32 [26]. Assume that S is an additive commutative semigroup and that M is a multiplicative semigroup of positive integers. For a, b $\in$ S, define $a \sim b$ to mean that ma = mb for some m $\in$ M. Then $\sim$ is a congruence on S, and if m[a] = m[b] for some [a],[b] $\in S/\sim$ and some $m \in M$, then [a] = [b].

Definition 7.33 [26]. If $M = Z^+$ in Theorem 7.32, the semigroup $S/\sim$ is torsion-free and $\sim$ is the smallest congruence ρ on S such that S/ρ is torsion-free. If $M = \{p^i\}_{i=0}^{\infty}$ is the set of powers of a prime p, the congruence is denoted by $\sim_p$ in this case and is referred to as p-equivalence. If $\sim_p$ is the identity congruence on S, we say that S is p-torsion-free. Another congruence encountered is that of asymptotic equivalence, defined by setting $a \sim b$ if there exists $K \in Z^+$ such that ka = kb for each $k \geq K$. We say that S is free of asymptotic torsion if distinct elements of S are not asymptotically equivalent.

Theorem 7.34 [26]. Assume that R is a commutative ring of prime characteristic p with nilradical N and S is commutative. Then N[S] is the nilradical of R[S] if and only if S is p-torsion-free.

Definition 7.35 [26]. Let R be a commutative ring, S a commutative semigroup, and $\sim$ a congruence on S. Let I be the ideal generated by $\{rs_1 - rs_2 \mid r \in R, s_1,s_2 \in S, \text{ and } s_1 \sim s_2\}$. The ideal I is called the kernel ideal of the congruence $\sim$.

Theorem 7.36 [26]. Let R be a commutative ring and let S be a commutative semigroup. Let N be the nilradical of R. In order that N[S] should be the nilradical of R[S], the following two conditions are necessary and sufficient.

(a) S is free of asymptotic torsion.

(b) If p is a prime such that S in not p-torsion-free, then p is not a zero divisor in R/N.

Corollary 7.37 [26]. Let R be a commutative ring and let S be a torsion-free commutative semigroup. Let N be the nilradical of R. Then N(R[S]) = N[S].

Theorem 7.38 [59]. Let R be a commutative ring with identity of characteristic p^n for some prime p. Then the nilradical of of R[S] is the ideal N[S] + I where N is the nilradical of R and where I is the kernel ideal of the p-congruence on S.

Theorem 7.39 [59]. Let R be a commutative ring with identity having nonzero characteristic and let S be an commutative semigroup with zero. Then the nilradical of R[S] is

$$N[S] + \sum_{i=1}^{v} R_{p_i}\{x^a - x^b \mid a \underset{p_i}{\sim} b\}$$

where $\{p_1, p_2, \ldots, p_v\}$ is the set of distinct prime divisors of char R, and $R_p = \{x \in R \mid p^n x = 0$ for some positive integer $n\}$.

Theorem 7.40 [26]. Assume that R is a unitary commutative ring with nilradical N and that T is a nonzero torsion-free aperiodic or torsion-free cancellative commutative monoid. Then J(R[T]) = N(R[T]) = N[T].

Theorem 7.41 [66]. If T is a finite monoid, then $J(R) = J(R[T]) \wedge R$ and $N(R) = N(R[T]) \wedge R$.

Theorem 7.42 [42]. If S is a torsion-free commutative semigroup of rank n, then $J(R[S]) = (J(R[x_1,\ldots x_n]) \wedge R)[S]$ and $N(R[S]) = (N(R[x_1,\ldots x_n]) \wedge R)[S]$.

Corollary 7.43 [46]. If G is a torsion-free abelian group of rank n, then $J(RG) = (J(R[x_1,\ldots x_n]) \wedge R)G$

CHAPTER 8

PRIME AND PRIMITIVE RINGS

Definition 8.1. Let R be a ring and let M be a left R-module. If $\{a \in R \mid ax = 0 \text{ for all } x \in M\} = 0$, then M is said to be a _faithful_ left R-module. If N is a right R-module and $\{a \in R \mid xa = 0 \text{ for all } x \in N\} = 0$, then N is said to be a _faithful_ right R-module.

Definition 8.2. A ring R is said to be _prime_ if every nonzero left ideal is faithful. (Or equivalently if 0 is a prime ideal.)

Recall (Definition 6.3) that a commutative ring R with identity $1_R \neq 0$ and no zero divisors is called an _integral domain_.

Theorem 8.3 [5]. A commutative ring is prime if and only if it is an integral domain.

Theorem 8.4 [15]. RG is an integral domain if and only if R is an integral domain and G is abelian torsion-free.

Theorem 8.5 [26]. Let R be a ring and let S be a semigroup. R[S] is an integral domain if and only if R is an integral domain and S is torsion-free and cancellative.

Theorem 8.6 [15]. The group ring RG is prime if and only if R is prime and G has no nontrivial finite normal subgroups.

Definition 8.7. A ring R is <u>semiprime</u> if $N(R) = 0$. (If R is commutative, a semiprime ring is often referred to as a <u>reduced</u> ring.)

Theorem 8.8 [43]. Let R be a commutative ring and G an abelian group. Then RG is semiprime if and only the following two conditions hold:

(a) R is semiprime.

(b) For all $p \in P$, p is not a zero divisor in R where P is the set of primes which are orders of elements of G.

Theorem 8.9 [15,63]. Let R be a commutative ring. Then RG is semiprime if and only if R is semiprime and the order of every finite normal subgroup of G is not a zero divisor in R.

Definition 8.10 [12]. A semigroup S is <u>Archimedean</u> if for all x, $y \in S$, there are positive integers m,n such that x divides y^m and y divides x^n.

Recall (Theorem 1.16) that every cancellative commutative semigroup may be imbedded in a group.

Theorem 8.11 [12,26]. Every commutative monoid is uniquely expressible as a semilattice of Archimedean semigroups, namely: $S = \vee\{S_\alpha | \alpha \in Y\}$ where η is the congruence on S defined by $x\eta y$ if and only if each of x and y divides a power of the other; then $Y = S/\eta$ and for each $\alpha \in Y$, S_α is simply the equivalence class α. If in addition S is separative, each S_α is cancellative and $S \leqq G = \vee\{G_\alpha | \alpha \in Y\}$, where G_α is the quotient group of S_α.

Theorem 8.12 [12]. Let R be a commutative ring and S a commutative monoid. The semigroup ring R[S] is semiprime if and only if the following conditions hold:

(a) R is semiprime,

(b) S is separative, and

(c) for any $\alpha \in Y$ and $x \in$ the torsion subgroup of G_α, the order of x is not a zero divisor in R.

Theorem 8.13 [26]. Let R be a commutative ring and S a commutative semigroup. Then semigroup ring R[S] is semiprime if and only if

(a) R is semiprime,

(b) S is free of asymptotic torsion, and

(c) If p is a prime such that S is not p-torsion-free, then p is not a zero divisor in R.

Corollary 8.14 [26]. Assume that R is an integral domain of characteristic zero and that S is a commutative semigroup. Then R[S] is semiprime if and only if S is free of asymptotic torsion.

Corollary 8.15 [26]. If R is a nonzero integral domain with char R = p, and S is a commutative semigroup, then R[S] is semiprime if and only if S is p-torsion-free.

Definition 8.16. A ring R is said to be <u>left (right) primitive</u> if there exists a simple faithful left (right) R-module. R is <u>primitive</u> if it is both left and right primitive.

Recall (Definition 1.2) that a ring D with identity 1_D in which every nonzero element is a unit is called a <u>division ring</u>. A <u>field</u> is a commutative division ring.

Theorem 8.17 [15]. RG is a division ring (field) if and only if R is a division ring (field) and G = {1}.

Theorem 8.18 [39]. A commutative ring R is primitive if and only if R is a field.

Example 8.19. Let R be a commutative ring. Since R[x] is never a field, R[x] is never primitive.

In general primitive group rings are more difficult to work with than prime group rings. As would be expected the most results are known for group rings over a field K. Until E. Formanek and R. L. Snider's paper [24] it was an open question as to whether KG could be primitive when G was infinite.

Theorem 8.20 [67]. If KG is left or right primitive, then G has no abelian subgroup of finite index.

Definition 8.21 [22]. A ring R is <u>locally left (right) artinian</u> if $R = \bigvee_{i=1}^{\infty} R_i$ where each R_i is a left (right) artinian ring, $R_1 \leqq R_2 \leqq R_3 \leqq \cdots$, and the R_i have a common identity.

Example 8.22 [22]. If R is a left (right) artinian ring and G is a countable group, then RG is locally left (right) artinian.

Theorem 8.23 [22]. Let R be a locally left (right) artinian ring with J(R) = 0. Then R is primitive if and only if R is prime.

Corollary 8.24 [22]. Let R be a left artinian ring and let G be a countable locally finite group such that J(RG) = 0. Then RG is primitive if and only if RG is prime.

Corollary 8.25 [22,24]. Let G be a countable locally finite group and let K be a field with char K = 0, or char K = p if G has no elements of order p. Then KG is primitive if and only if KG is prime.

Corollary 8.26 [24]. Suppose G is a countable locally finite group and K is a field with char K = 0, or char K = p if G has no elements of order p. Then KG is primitive if and only if G has no finite normal subgroups.

Theorem 8.27 [23]. Let R be a domain and G a free group on at least two generators such that $|G| \geq |R|$. Then RG is primitive.

A result of the above theorem is that RG can be primitive even if R is not primitive.

Example 8.28. Let Z be the ring of integers and let G be the free group generated by the real numbers. By the above theorem ZG is primitive. Note that Z is not primitive since there are no simple faithful Z-modules.

A case when RG primitive implies that R is primitive is the following:

Theorem 8.29 [48]. If R is regular and RG is primitive, then R is primitive.

Hodges [38] has given an example to show that R[x] primitive does not imply that R is primitive.

Theorem 8.30 [23]. Let $G = A * B$ be a free product of nontrivial groups (except $G = Z_2 * Z_2$) and let R be a domain such that $|G| \geq |R|$. Then RG is primitive.

A class of rings that is intermediate between the prime and primitive rings are the weakly primitive rings.

Definition 8.31 [55]. If R is a ring an R-module M is called <u>compressible</u> when it can be embedded in each of its nonzero submodules; M is called <u>monoform</u> if each partial endomorphism $N \to M$, $N \leqq M$, is either zero or monic. The ring R is called <u>left (right) weakly primitive</u> if it has a faithful monoform compressible left (right) R-module.

Example 8.32. The ring of integers Z is left and right weakly primitive since Z is a faithful monoform compressible left and right R-module.

In Theorem 8.33 Lawrence obtains a result closely related to Theorem 8.30.

Theorem 8.33 [48]. For any ring R the following are equivalent.

(a) R is left (right) weakly primitive.

(b) If $G = A * B$ is a free products of groups A and B, $|A| = \infty$, $|B| > 1$, with $|G| \geqq |R|$, then the group ring is left (right) primitive.

Theorem 8.34 [55]. If R is left (right) weakly primitive and G is an abelian group, then the group ring RG is left (right) weakly primitive if and only if G is torsion-free.

Theorem 8.35 [55]. If R is a left (right) weakly primitive ring, then $R[x_1,\ldots,x_n]$ is also a left (right) weakly primitive ring.

Definition 8.36. A ring R is <u>semiprimitive</u> in case $J(R) = 0$.

Three corollaries from Chapter 7 are worth repeating in light of the above definition.

Corollary 7.13 [15,44]. Let R and G be commutative.

(a) If G is torsion, then RG is semiprimitive if and only if R is semiprimitive and the order of every finite subgroup of G is regular in R.

(b) If G is not torsion, then RG is semiprimitive if and only if R is semiprime and the order of every finite subgroup of G is regular in R.

Corollary 7.7 [61]. Let G be an abelian group with no elements of order p in case char $K = p$. Then KG is semiprimitive.

Corollary 7.25 [52]. Let S be a cancellative commutative semigroup and char $K = 0$. Then $K[S]$ is semiprimitive.

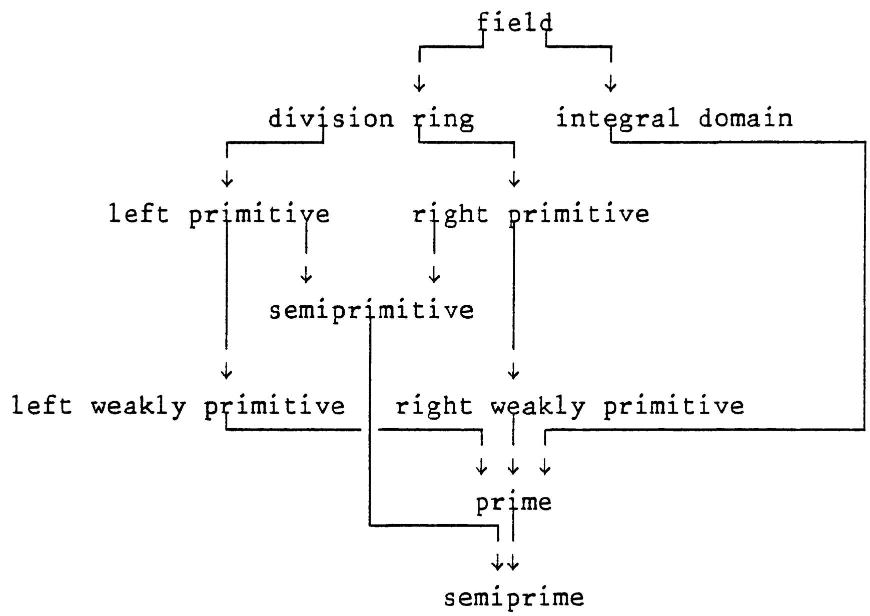

Figure 3. Implication Diagram for Prime, Primitive and Related Rings

CHAPTER 9

PROJECTIVE AND INJECTIVE MODULES

Definition 9.1. A module P over a ring R is said to be <u>projective</u> if given any diagram of R-module homomorphisms

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow{\scriptstyle f} & & \\ A & \xrightarrow{g} & B & \rightarrow & 0 \end{array}$$

with bottom row exact (that is, g an epimorphism), there exists an R-module homomorphism h: P $\rightarrow$ A such that the diagram

$$\begin{array}{ccccc} P & = & P & & \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} & & \\ A & \xrightarrow{g} & B & \rightarrow & 0 \end{array}$$

is commutative (that is, $gh = f$).

By applying [5] Exercise 10, p.261, the following results are obtained.

Theorem 9.2.

(a) If ${}_{R[T]}M$ is projective, then ${}_{R}M$ is projective.

(b) If T is finite and ${}_{R[T]}M$ is finitely generated, then ${}_{R}M$ is finitely generated.

(c) If ${}_{R[T]}R$ is projective and ${}_{R}M$ is (finitely generated) projective, then ${}_{R[T]}M$ is (finitely generated) projective.

Theorem 9.3 [43]. Let R be a commutative ring, G be a finite abelian group of order n where n is a unit of R, and P be a left RG-module. If ${}_R P$ is projective, then ${}_{RG} P$ is projective.

Definition 9.4. A module J over a ring R is said to be <u>injective</u> if given any diagram of R-module homomorphisms

$$\begin{array}{ccccc} 0 & \to & A & \xrightarrow{g} & B \\ & & {\scriptstyle f}\downarrow & & \\ & & J & & \end{array}$$

with top row exact (that is, g a monomorphism), there exists an R-module homomorphism h: B → J such that the diagram

$$\begin{array}{ccccc} 0 & \to & A & \xrightarrow{g} & B \\ & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} \\ & & J & = & J \end{array}$$

is commutative (that is, hg = f).

By applying [5] Exercise 10, p.261, the following results are obtained.

Theorem 9.5.

(a) If ${}_{R[T]}M$ injective, then ${}_R M$ is injective.

(b) If $R_{R[T]}$ is projective and ${}_R M$ is injective, then ${}_{R[T]}M$ is injective.

Theorem 9.6 [21,61]. The principal KG-module KG/ωG ≅ K is injective if and only if G is locally finite with no elements of order p if char K = p.

Definition 9.7. A ring is <u>left (right) self-injective</u> if ${}_R R$ (R_R) is injective.

Theorem 9.8 [65]. RG is left (right) self-injective if and only if R is left (right) self-injective and G is finite.

Many of the ring properties studied imply that the ring is self-injective. By Theorem 9.8 self-injective group rings force a tight structure on the group. What is needed is an analogous result for semigroup (monoid) rings.

Question 9.9. What does R[S] (R[T]) self-injective imply about S (T)?

Definition 9.10. (Villamayor) A ring R is a <u>left V-ring</u> in case the following equivalent conditions are satisfied:

(a) Each simple left R-module is injective.

(b) Each left ideal is the intersection of maximal left ideals.

(c) For all ${}_R M$ the intersection of all maximal submodules is zero.

Theorem 9.11 [19]. Let R be a commutative ring. R is a V-ring if and only if R is von Neumann regular.

Theorem 9.12 [21,34,61]. Let G be a countable group. Then KG is a V-ring if and only if KG is regular and G has an abelian subgroup of finite index.

Definition 9.13. A left R-module G is a <u>generator</u> in case for every ${}_R M$ there is a set A and an R-module epimorphism

$G^{(A)} \to M \to 0$

where $G^{(A)}$ is the direct sum of A copies of G.

Definition 9.14. A ring R is left <u>(finitely) pseudo-Frobenius</u>, <u>(F)PF</u> if every (finitely generated) faithful left R-module is a generator.

Theorem 9.15 [20]. Every PF ring is semiperfect self-injective, and every commutative self-injective ring is FPF.

Theorem 9.16 [20]. Let G be a finite group and R a ring such that there is an (R,RG)-bimodule isomorphism $\mathrm{Hom}_R(RG,R) \cong RG$. If R is left (right) PF, then RG is left (right) PF.

Theorem 9.17 [20]. Let G be a finite group. If R is commutative and injective, then RG is injective and FPF (both sides).

Theorem 9.18 [20]. If R is self-injective left (right) FPF, and G is a finite group such that the order of G is a unit in R, then RG is left (right) FPF.

Example 9.19 [20]. If G is finite, R left FPF need not imply that RG is FPF. Z is left FPF, but ZG is not left FPF given any finite group $G \neq 1$.

Question 9.20 [20]. Let G be finite and $|G|$ be a unit in R. Does R left FPF imply that RG is left FPF?

CHAPTER 10

MORITA DUALITY

Definition 10.1. Let ${}_RU_S$ be a bimodule. Then the pair of contravariant additive functors

$\mathrm{Hom}_R(\_,{}_RU_S): {}_R\underline{M} \to \underline{M}_S$ and $\mathrm{Hom}_S(\_,{}_RU_S): \underline{M}_S \to {}_R\underline{M}$

is called the U-dual. For brevity write

$(\ )^* = \mathrm{Hom}(\_,{}_RU_S)$

to denote either of these functors. The module M^* is said to be the U-dual of M and the map f^* is called the U-dual of f. Also M^{**} and f^{**} are called the double dual of M and f, respectively. For each M in ${}_R\underline{M}$ or $\underline{M}_S$

$[\sigma_M(m)](\gamma) = \gamma(m) \qquad (m \in M,\ \gamma \in M^*)$

defines the evaluation map

$\sigma_M: M \to M^{**}$

A module M is said to be U-reflexive in case σ_M is an isomorphism.

Definition 10.2. A bimodule ${}_RU_S$ defines a Morita duality in case

(a) ${}_RR$ and S_S are U-reflexive;

(b) Every submodule and every factor module of a U-reflexive module is U-reflexive.

Definition 10.3. We say that the ring R has duality if there exists a ring S and bimodule ${}_RU_S$ satisfying the above definition. If, in addition, R=S then R has <u>self-duality</u>. [5] If R=S=U then R is said to be a <u>cogenerator ring</u>.

Definition 10.4. A left R-module C is a <u>cogenerator</u> in case each left R-module M can be embedded in a product of copies of C

$0 \to M \to C^A$.

Theorem 10.5 [5]. Let R and S be rings. Then for a bimodule ${}_RU_S$ the following statements are equivalent:

(a) ${}_RU_S$ defines a Morita duality;

(b) Every factor module of ${}_RR$, S_S, ${}_RU$ and U_S is U-reflexive;

(c) $R \cong \mathrm{End}(U_S)$, $S \cong \mathrm{End}({}_RU)$, and ${}_RU$ and U_S are injective cogenerators.

Theorem 10.6 [5]. R is a cogenerator ring if and only if ${}_RR$ and R_R are injective cogenerators.

Theorem 10.7. If RG is a cogenerator ring, then G is finite.

Proof. RG a cogenerator ring implies that RG is self-injective. Theorem 9.8 implies that G is finite.

Question 10.8. If RG has duality (or self-duality), must G be finite? If RG has duality, then RG is semiperfect. Unfortunately this isn't a tight enough structure to guarantee that G is finite. Another property that duality implies is the linearly compact condition. Group rings with the linearly compact condition have apparently not been studied by any authors however.

Theorem 10.9 [72]. Let R be a subring of V such that ${}_RV$ is finitely generated by elements that centralize R. If ${}_RC$ induces a Morita duality between left R-modules and right End(${}_RC$)-modules then ${}_VW$ induces a Morita duality between left V-modules and right End(${}_VW$)-modules, where ${}_VW = \mathrm{Hom}_R({}_RV, {}_RC)$.

$$ {}_R\underline{M} \quad \underline{\text{duality}} \quad M_{\mathrm{End}({}_RC)} \quad \rightarrow \quad {}_V\underline{M} \quad \underline{\text{duality}} \quad M_{\mathrm{End}(\mathrm{Hom}_R(V_V, C)} $$

Example 10.10. Let T be a finite monoid and R a ring. If ${}_RC$ induces a Morita duality between left R-modules and right End(${}_RC$)-modules then ${}_{R[T]}W$ induces a Morita duality between left R[T]-modules and right End(${}_{R[T]}W$)-modules, where ${}_{R[T]}W = \mathrm{Hom}_R({}_RR[T], {}_RC)$.

Theorem 10.11 [72]. If R is commutative, then R has a Morita duality if and only if R has self-duality.

Theorem 10.12 [72]. If R is commutative and T is finite, then R has a Morita duality if and only if R[T] has self-duality.

Fuller and Haack [25] employed Theorem 10.9 to determine a necessary condition for a duality between two rings to imply a duality between their semigroup rings.

Definition 10.13 [25]. A semigroup ring R[S] is <u>unital</u> in case it has an identity element $1 \in R[S]$ such that the embedding $r \rightarrow r1$ ($r \in R$) defined a (necessarily unital) ring homomorphism $R \rightarrow R[S]$, i.e., $(r_1r_2)1 = (r_11)(r_21)$.

Theorem 10.14 [25]. Let S be a finite semigroup and suppose $V = (R[S])_0$ is unital. If ${}_RC$ induces a Morita duality between left R-modules and right $End({}_RC)$-modules, then ${}_VW$ induces a Morita duality between $(R[S])_0$ and $(End({}_RC)[S])_0$, where ${}_VW = Hom_R(V_V, C)$. Hence, if R has self-duality, then so does $(R[S])_0$.

$$ {}_R\underline{M} \ \underline{\text{duality}} \ M_{End({}_RC)} \leftrightarrow (R[S])_0\underline{M} \ \underline{\text{duality}} \ M_{End(Hom_R((R[S])_0,C))} $$

Definition 10.15. A ring R for which the ${}_RR_R$-dual $Hom_R(\_,R)$ defines a duality between the category of finitely generated left and right modules over R is called a <u>quasi-Frobenius (QF) ring</u>.

Theorem 10.16 [3]. A ring R is QF if and only if R is left or right artinian and left self-injective. (QF rings are artinian cogenerator rings and every QF ring is PF.)

Theorem 10.17. RG is QF if and only if R is QF and G is finite.

Proof. Use Theorems 2.2 and 9.8.

Definition 10.18. Let R be a finite dimensional algebra over a field K. Then R is called a <u>Frobenius algebra</u> if ${}_RR_K \cong Hom_K({}_KR_R, K)$.

Theorem 10.19 [5]. Let K be a field and G a finite group. Then the group algebra R = KG is Frobenius.

Theorem 10.20 [5]. If R is a Frobenius algebra over K, then R is both left and right self-injective.

Corollary 10.21 [5]. KG is a Frobenius algebra if and only if G is finite.

Proof. By Theorems 9.8, 10.19 and 10.20.

Definition 10.22. A ring is called a <u>left (right) QF-3</u> if it has a (unique) minimal faithful left (right) module.

Definition 10.23 [45]. Let $V > R$ be a ring extension. We say that V is a <u>left QF extension</u> of R if ${}_RV$ is finitely generated projective and if ${}_VV_R$ is isomorphic to a direct summand of a direct sum of a finite number of copies of ${}_V(\mathrm{Hom}_R({}_RV, {}_RR))_R$.

Definition 10.24 [45]. A bimodule ${}_RM_R$ is said to be <u>generated by normalizing elements</u> if there is a set $\{m_i \mid i \in I\} < M$ and $Rm_i = m_iR$ is free as both a left and right module on $\{m_i\}$.

Theorem 10.25 [45]. Let $V > R$ be a ring extension such that ${}_RV$ and V_R are finitely generated projective. Suppose that V is finitely generated over R by normalizing elements. If V is left QF-3, then R is left QF-3. The converse is true if $V > R$ is a left or right QF extension.

Corollary 10.26 [45]. A group ring RG of a ring R with a finite group G is left QF-3 if and only if R is left QF-3.

Corollary 10.27. Let T be a finite monoid. If $R[T]$ is left QF-3, then R is left QF-3.

Theorem 10.28 [4]. R is noetherian and left QF-3 if and only if R is an artinian QF-3 ring.

Corollary 10.29. RG is noetherian and left QF-3 if and only if R is an artinian QF-3 ring and G is finite.

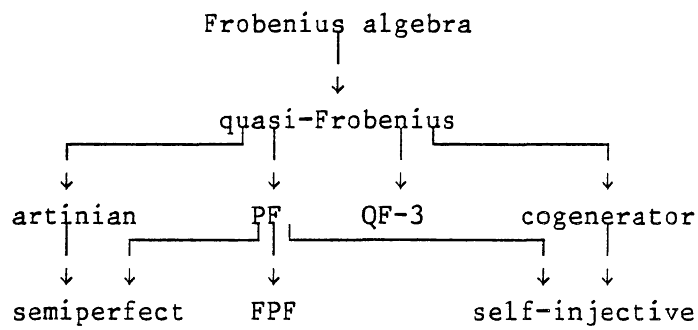

Figure 4. Implication Diagram for Quasi-Frobenius and Related Rings

CHAPTER 11

HEREDITARY AND SEMIHEREDITARY

Definition 11.1. A ring is said to be <u>left (right) hereditary</u> provided that every left (right) ideal is projective.

Definition 11.2. A ring is said to be <u>left (right) semihereditary</u> provided that every finitely generated left (right) ideal is projective.

Theorem 11.3 [68]. Every semisimple ring is hereditary, while every regular ring is semihereditary.

[47] Hereditary and semihereditary rings have been the subject of considerable study. Many interesting examples of these rings arise as group rings or semigroup rings. Recently Chouinard, Hardy, and Shores [12] have completely determined the commutative semihereditary monoid rings. Goursaud and Valette [30] have given results concerning hereditary and semihereditary group rings, while Dicks [16] has completely determined the hereditary group rings.

Theorem 11.4 [30]. Suppose R is a ring, and G is a finite group. RG is hereditary (semihereditary) if and only if R is hereditary (semihereditary) and the order of G is invertible in R.

Theorem 11.5 [30]. Suppose G is a nilpotent infinite group, and R is a ring. RG is hereditary if and only if at least one of the following conditions holds:

(a)(1) G is an extension of a finite group H by Z,

(2) RH is semisimple;

(b)(1) G is countable and locally finite,

(2) RG is regular,

(3) R is countable.

Definition 11.6 [16]. For any ring R a group G <u>has no R-torsion</u> if the order of every finite subgroup of G is invertible in R.

Definition 11.7 [16]. By a <u>graph</u>, X, we understand a system consisting of: a nonempty set, V(X), whose elements are called <u>vertices</u> of X, a set, E(X), whose elements are called the <u>edges</u> of X, and an incidence map (ι,τ): $E(X) \to V(X) \times V(X)$. For any edge e of X, ιe and τe are called the <u>initial</u> and <u>terminal</u> vertices of e, respectively. Let us fix a connected graph X. By a <u>connected graph of groups</u>, ψ, we understand two families of groups $\{G_e \mid e \in E(X)\}$ and $\{H_v \mid v \in V(X)\}$ indexed by the edges and the vertices, together with families of group homomorphisms $\{\psi_{\iota e}: G_e \to H_{\iota e}\}$ and $\{\psi_{\tau e}: G_e \to H_{\tau e}\}$. Each of these homomorphisms have domain an edge group and codomain the group corresponding to the initial or terminal vertex of the edge. The homomorphisms $\psi_{\iota e}: G_e \to H_{\iota e}$ will be denoted $g \to g^{\iota e}$, and similarly for τ. Since X is connected we can find a <u>spanning tree</u>, that is, a subgraph with the same vertex set and with a minimal edge set so that the subgraph is still connected. For any spanning tree T of X, and any graph of groups ψ, consider the class F of triples $(K,\{\phi_v^K: H_v \to K \mid v \in$

$V(X)\}$, $\{q^K(e) \varepsilon K | e \varepsilon E(X)\})$ such that K is a group, each ϕ_v is a group homomorphism, $q^K(e)^{-1} \cdot g^{\iota e} \cdot q^K(e) = g^{\tau e}$ for all $g \varepsilon G_e$ with $q^K(e) = 1$ if e is an edge of T. The <u>fundamental group</u> $\pi = \pi(\psi, T)$ of ψ with respect to T, is an element of F such that given any other element K of ψ, there is a unique group homomorphism $\gamma: \pi \rightarrow K$ such that $\phi_v^K = \gamma\phi_v^{\pi}$ for all $v \varepsilon V(X)$ and $q^K(e) = \gamma(q^{\pi}(e))$ for all $e \varepsilon E(X)$.

Theorem 11.8 [16]. The group ring RG is left hereditary if and only if one (or more) of the following holds:

(a) R is semisimple and G is the fundamental group of a connected graph of finite groups with no R-torsion;

(b) Every left ideal of R is countably generated, R is von Neumann regular, and G is a countable locally finite group with no R-torsion;

(c) R is left hereditary and G is a finite group with no R-torsion.

Definition 11.9 [11]. Recall Definition 1.15 that a monoid T is <u>cancellative</u> if either ab = ac or ba = ca for a, b, c ε T implies b = c. A monoid T is <u>weakly cancellative</u> if (i) ab = a or ba = b implies b = 1, and (ii) aub = ab with u a unit implies u = 1. Clearly cancellative monoids are weakly cancellative. The monoid T is <u>partially free</u> if it is the free product of a free group with a free monoid.

Theorem 11.10 [11]. Suppose $R \neq 0$ is a ring with identity and suppose $T \neq 1$ is a monoid. Then the following are equivalent:

(a) The monoid ring R[T] is left and right hereditary and T is weakly cancellative and torsion-free.

(b) R is semisimple and T is partially free.

Now semihereditary group rings will be analyzed.

Theorem 11.11 [30]. Let R be a commutative ring. Suppose G is an abelian group with a torsion subgroup H. RG is semihereditary if and only if G/H is isomorphic to a subgroup of the field of rational numbers and RH is regular.

Corollary 11.12 [30]. Let R be a commutative ring. Suppose G is a locally nilpotent group with a torsion subgroup H. RG is semihereditary if and only if R is regular, G/H is isomorphic to a subgroup of the rational numbers, and RH is regular.

Definition 11.13. An <u>arithmetical ring</u> is a commutative ring with identity whose ideals form a distributive lattice with respect to intersections and sums.

The following theorem is an extension of Theorem 11.11.

Theorem 11.14 [33]. Let R be a commutative ring and let T be a commutative cancellative monoid. The semigroup ring R[T] is semihereditary if and only if R[T] is arithmetical, R is semihereditary and the order of every torsion element of T is a unit in R.

Definition 11.15 [12]. Recall Theorem 8.11 on the decomposition of monoids into Archimedean semigroups. Let S be a separative commutative monoid with Archimedean decomposition $V\{S_\alpha | \alpha \in Y\}$ (so S_α is cancellative) and suppose the quotient group G_α of S_α has rank $\leqq 1$. The $\phi_{\beta,\alpha}$: $G_\beta \to G_\alpha$ induce maps $S_\beta \to G_\alpha$. Let t(H) be the torsion subgroup of the group H. For any homomorphism ϕ: H $\to$ K of abelian groups, we let ϕ^{red}: $H/t(H) \to K/t(K)$ be the map induced by ϕ on the corresponding torsion-free groups.

Theorem 11.16 [12]. Let R be a commutative ring and let T be a monoid with Archimedean decomposition $T = \bigvee_{\alpha \varepsilon Y} S_\alpha$. Then R[T] is semihereditary if and only if the following conditions hold:

(a) R[T] is semiprime arithmetical.

(b) For every $t \varepsilon T$, $\{x \varepsilon R \mid tx = 0\}$ is finitely generated.

(c) For every $\alpha \varepsilon Y$, the ideal $Y_{(\alpha)} = \{\beta \varepsilon Y | \beta \leqq \alpha$ and $\phi_{\alpha,\beta}{}^{red}$ is trivial$\}$ is finitely generated or empty.

(d) R is semihereditary.

For a ring that is not necessarily commutative, Kuzmanovich and Teply have found several nice results.

Theorem 11.17 [47]. Let $T = \bigvee_{\alpha \varepsilon Y} G_\alpha$ be a semilattice of torsion groups G_α, and let R be a nonzero ring. Then R[T] is left semihereditary if and only if the following conditions hold.

(a) The ring R is left semihereditary.

(b) Each G_α is locally finite.

(c) The order of each element of T is a unit in R.

Corollary 11.18 [47]. Let T be a commutative periodic monoid, and let R be a nonzero ring. Then R[T] is left semihereditary if and only if the following conditions hold.

(a) The ring R is left semihereditary.

(b) T is a semilattice of groups.

(c) The order of each element of T is a unit in R.

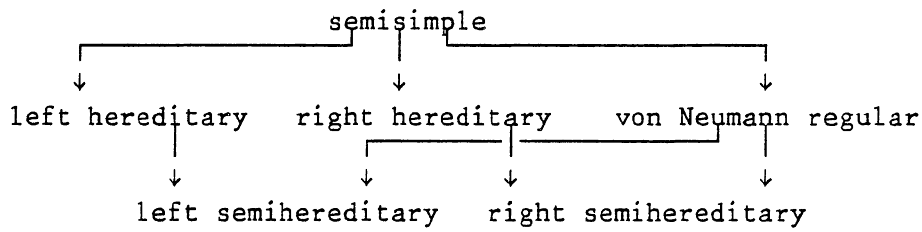

Figure 5. Implication Diagram for Hereditary and Related Rings

CHAPTER 12

GLOBAL DIMENSION

Another way to view some of the above conditions relates to the various dimensions of rings. The first of these is the global dimension of a ring.

Definition 12.1. A projective resolution of a module N is an exact sequence

$$\cdots \to P_{n+1} \to P_n \to \cdots \to P_1 \to P_0 \to N \to 0$$

where each P_n is projective. If N is a left R-module, then pd(N) $\leqq$ n (pd abbreviates projective dimension) if there is a projective resolution

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to N \to 0$$

If no such finite resolution exists, define pd(N) = ∞; otherwise, if n is the least such integer, define pd(N) = n. If R is a ring, its left projective global dimension, lpD(R), is defined by

$$\mathrm{lpD}(R) = \sup\{\mathrm{pd}(N) \mid N \in {}_R\underline{M}\}$$

Similarly a left injective global dimension, liD(R), can be defined. Since these two coincide for a ring R, one defines the left global dimension, lD(R), as the common value of lpD(R) and liD(R). If one considers right R-modules, he may define the right global dimension, rD(R).

Theorem 12.2 [68]. lD(R) = 0 if and only if R is semisimple.

Theorem 12.3 [68]. If R is quasi-Frobenius, then $lD(R) = 0$ or ∞.

Theorem 12.4 [68]. $lD(R) \leqq 1$ if and only if R is left hereditary.

Theorem 12.5 [68]. For any ring R, $lD(R[x]) = lD(R) + 1$.

Corollary 12.6 [19] (Hilbert Syzygy Theorem). For any ring $R \neq 0$, the global dimension of the polynomial ring $R[x_1,\ldots,x_n]$ in n indeterminates is given by

$$lD(R[x_1,\ldots,x_n]) = lD(R) + n.$$

Corollary 12.7 [19,68]. R is semisimple if and only if R[x] is left or right hereditary.

Theorem 12.8 [61]. $lD(KG) = pd({}_{KG}(KG/\omega G)) = pd({}_{KG}K)$.

Theorem 12.9 [61]. Let KG be given, and let H be a normal subgroup of G. If KH and K(G/H) have finite global dimension, then so does KG, and we have

$$lD(KG) \leqq lD(KH) + lD(K(G/H)).$$

Theorem 12.10 [16]. Let $cd_R G$ denote the projective RG-dimension of R viewed as a left RG-module with trivial G-action, and cen(R) denote the center of R. Then $lD(RG) \leqq lD(R) + cd_{cen(R)}G$.

Theorem 12.11 [37]. For any ring R and free semigroup S, the semigroup ring R[S] has

$$lD(R[S]) = lD(R) + 1$$

Definition 12.12. A flat resolution of a module N is an exact sequence

$$\cdots \to F_{n+1} \to F_n \to \cdots \to F_1 \to P_0 \to N \to 0$$

in which each F_n is flat. If N is a left R-module, then $fd(N) \leqq n$ (fd abbreviates flat dimension) if there is a flat resolution

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to N \to 0$$

If no such finite resolution exists, define $fd(N) = \infty$; otherwise, if n is the least such integer, define $fd(N) = n$. The left weak global dimension, $lwD(R)$, of a ring R is defined by

$$lwD(R) = \sup\{fd(N) \mid N \in {}_R\underline{M}\}$$

Similarly a right weak global dimension, $rwD(R)$, can be defined. Since these two coincide for a ring R, one defines the weak global dimension, $wD(R)$, as the common value of $lwD(R)$ and $rwD(R)$.

Theorem 12.13 [68]. $wD(R) = 0$ if and only if R is von Neumann regular.

Theorem 12.14 [68]. The class of rings R with $wD(R) \leqq 1$ contains all left or right semihereditary rings. In particular commutative rings R with $wD(R) \leqq 1$ are semihereditary.

Unfortunately there exists rings of weak global dimension one which are not semihereditary.

Example 12.15. Let K be a field of characteristic p, S_0 be an infinite torsion group such that the orders of the elements of S_0 are not divisible by p, and let Z_+ denote the semigroup of positive integers. Let S be the semigroup created by taking the disjoint union of S_0 and Z_+ with $s_0 + n = n$ for $s_0 \in S_0$ and $n \in Z_+$. Chouinard, Hardy, and Shores [12] show that K[S] has weak global dimension one, but it is not semihereditary.

Theorem 12.16 [18]. Let G be an abelian group. Then in order that wD(RG) be finite it is necessary and sufficient that the following three conditions be satisfied:

(a) qR = R whenever q is the order of a torsion element of G;

(b) $wD(R) < \infty$;

(c) rank $G < \infty$.

Theorem 12.17 [18]. Let G be an abelian group and suppose that qR = R whenever q is the order of a torsion element of G. Then

wD(RG) = wD(R) + rank G

Corollary 12.18 [6,18]. Let G be an abelian group and let D be a division ring with char D = 0. Then

wD(DG) = rank G.

Corollary 12.19 [6,18]. Let G be an abelian group and let D be a division ring with char D = p. Then wD(DG) is finite if and only if G is of finite rank and p does not divide the order of any torsion element in G. Moreover, when p does not divide the order of any torsion element in G, then

wD(DG) = rank G.

Theorem 12.20 [41]. For an ring R, wD(R[x]) = wD(R) + 1.

Corollary 12.21 [9,41,51]. For a commutative ring R, R is von Neumann regular if and only if R[x] is semihereditary.

Corollary 12.22 [41,64]. If R[x] is either left or right semi-hereditary, then R is a von Neumann regular ring.

Definition 12.23. Let R be a commutative ring. The <u>Krull dimension</u> of R, written <u>Dim(R)</u>, is defined as the supremum of the lengths of chains of prime ideals in R (possibly infinite). If Dim(R) is finite, then it is equal to the length of the longest prime chain in R. For example, any field has dimension 0, while any principal ideal domain distinct from a field has dimension 1.

Theorem 12.24 [43]. Let R be a commutative noetherian ring. Then

$$\mathrm{Dim}(R[x_1,\ldots,x_m]) = \mathrm{Dim}(R) + m$$

Theorem 12.25 [43]. Let R be a commutative ring and Z the group of integers. Then Dim(RZ) = Dim(R[x]).

Theorem 12.26 [27]. Let R be a commutative ring and G an abelian group of rank α. Then the following properties hold:

(a) If $\alpha = 0$, then Dim(RG) = Dim(R). If $\alpha > 0$, then Dim(RG) $\geqq$ Dim(R) + 1.

(b) Suppose that both Dim(R) and α are finite and R is noetherian. Then Dim(RG) = Dim(R) + α.

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