NONLINEAR STATIC ANALYSIS OF SHELLS OF REVOLUTION BY DYNAMIC RELAXATION METHOD

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LIST OF SYMBOLS

a	representative length or radius of the shell of revolution
A_ϕ', A_w'	left hand side of Equation 3.6; residuals in the 'ϕ' and 'z' directions respectively
b'	highest eigen value of the stiffness matrix of displacements
D_o	$\dfrac{E_\theta h_o^3}{12(1-\nu_{\phi\theta}\nu_{\theta\phi})}$; $\dfrac{E\ h_o^3}{12(1-\nu^2)}$ for isotropy
E_ϕ, E_θ	elastic moduli
G	shear modulus
h	thickness of shell at any point
h_o	representative thickness of shell
k_θ, k_ϕ, k_w	damping factors in the θ, ϕ, and z directions
m_θ, m_ϕ, m_w	mass/unit area in the θ, ϕ, and z directions
M_θ, M_ϕ	bending moments/unit length
$M_{\theta\phi}$, $M_{\phi\theta}$	twisting moments/unit length
N_θ, N_ϕ	in-plane normal forces/unit length
$N_{\theta\phi}$, $N_{\phi\theta}$	in-plane shear forces/unit length
q_θ, q_ϕ, q	load/unit area in the θ, ϕ, and z directions
Q_θ, Q_ϕ	transverse shear forces/unit length
r_θ, r_ϕ	radii of curvature of shell
r	distance from the axis of revolution to a point on shell
t	time
t_o	representative time = $a^2\sqrt{m_w/D_o}$

Δt	time increment
u_θ, u_ϕ, w	displacements in the circumferential, meridional, and transverse directions
$\dot{u}_\phi$, $\dot{w}$	velocities
$\ddot{u}_\phi$, $\ddot{w}$	accelerations
θ , ϕ , z	circumferential, meridional, and transverse coordinates
$\Delta\phi$	increment in angle along meridian
ϵ_θ°, ϵ_ϕ°	in plane strains in the reference surface of shell
$\gamma_{\phi\theta}^\circ$	shear strain in the reference surface of shell
κ_θ, κ_ϕ	change in curvatures of reference surface of shell
$\kappa_{\theta\phi}$, $\kappa_{\phi\theta}$	components of twist (defined in Equation A.5)
τ	twist of reference surface of shell
$\nu_{\theta\phi}$, $\nu_{\phi\theta}$	Poisson's ratios
ω	angular frequency of fundamental or significant mode

Note: All primed variables are non-dimensional

CHAPTER I

INTRODUCTION

1.1 General

Linear small displacement analysis is not sufficiently accurate for the prediction of displacements of thin plate and shell structures, when the loads acting on the structure produce displacements that are of the order of the thickness of the structure. When the displacements are of the order of the thickness of the structure, the strain-displacement and the equilibrium equations become nonlinear while the stress-strain relations remain linear. This is referred to as geometric nonlinearity or large displacement conditions.

There have been numerous investigations on the geometrically nonlinear analysis of plates and the behavior of plates under large displacements is well known. The geometrically nonlinear analysis of shells has also been investigated, but to a lesser extent than plate structures. Closed form solutions for the geometrically nonlinear analysis of shells are few. Most of the previous work has been done, using the finite difference or finite element discretization and by solving the resulting algebraic equations by incremental, iterative or initial value procedures. In these procedures, the loads are increased in small steps and large numbers of simultaneous equations are solved for each load step, to trace the load displacement path.

An alternative finite difference technique called Dynamic Relaxation was developed in 1960. Using this method the nonlinear equations can be solved in a single load increment and solution of simultaneous equations can be avoided. In the dynamic relaxation technique, the solution to the nonlinear problem is obtained by considering the equivalent dynamic problem with viscous damping introduced to damp the oscillations. The equivalent dynamic problem is solved by an explicit finite difference integration scheme. If the damping, which is artificial, is sufficient, the oscillations will die out and the static solution will be obtained.

The merits of the dynamic relaxation method are:

1. Once the governing equations are set up the whole procedure is simple and easy for programming on a digital computer.

2. Compared to the conventional finite difference and the finite element formulation, in the present dynamic relaxation method, there is no need to store large coefficient(stiffness) matrices. The finite difference coefficients at a node are generated when the governing equations are applied at that node and they are not stored. Consequently the computer storage requirements for the dynamic relaxation procedure are far less than the other procedures.

3. Dynamic relaxation is particularly suited for non-

linear problems. Linear and nonlinear problems can be solved by the same basic procedure with only slight modifications. In the usual finite difference and the finite element procedures, the nonlinear load displacement equations are solved in small load increments. In the dynamic relaxation technique the nonlinear displacements are obtained in a single load increment. Solution of simultaneous equations can be avoided.

4. Various types of boundary conditions, loadings, variation in geometry, thickness and material properties of the structure can all be incorporated without difficulty.

1.2 Purpose and Scope

The objective of this study is to determine the geometrically nonlinear static displacements and stress resultants in shells of revolution, using the dynamic relaxation method, thereby establishing the suitability of the method for such problems.

The geometrically nonlinear static analysis of shells of revolution involves the solution of a highly complex system of partial differential equations satisfying equilibrium, compatibility and boundary conditions. An additional objective of this thesis is to show the ease with which such a complex system of equations can be solved by the dynamic relaxation method. The ease of the solution procedure and the related programming will be evident by this study. A further objective is to derive the nonlinear equilibrium equations for shells of revolution from the general strain displacement relations, using the principle of minimum total potential energy. These equations will be more general than those presented in previous work.

The nonlinear displacements and stress resultants in axysymmetric spherical shells with different end opening angles are determined. Nonlinear displacements and stress resultants for general shells of revolution, namely an elliptic and a parabolic shell of revolution, are also obtained. Two computer programs are developed.

(a) A program for the geometrically nonlinear static analysis of an axysymmetric spherical shell of revolution.

(b) A program for the geometrically nonlinear static analysis of an axysymmetric general shell of revolution.

CHAPTER II

LITERATURE REVIEW

In this chapter a brief discussion on the solution methods for geometrically nonlinear static structural analysis and the review papers on this topic are given. This is followed by the references to the literature on the geometrically nonlinear static analysis of shells of revolution. Finally the application of dynamic relaxation technique to static structural problems is reviewed.

2.1 Geometrically Nonlinear Structural Analysis

Most of the previous work on the formulation of the governing equations for geometrically nonlinear structural analysis problems has been done using the finite difference or the finite element approach. The resulting equations have been solved using a number of procedures which can be classified as (a) Incremental procedures, (b) Iterative procedures, or (c) Initial value procedures. In the incremental procedure the load is applied in small increments so that the structure can be assumed to respond linearly during each increment of load. For each increment of load, increments of displacements and corresponding increments of stress are computed. These incremental quantities are used to compute various corrective stiffness matrices which serve to take into account the deformed geometry of the structure. A subsequent increment of load is applied and the process continued until the desired number of load increments have been applied. The net effect is to solve a sequence of linear problems wherein the stiffness properties are recomputed based on the current geometry prior to each load increment. Since equilibrium is not satisfied at any load level this procedure exhibits a shift of the solution from the true solution. Self correcting forms of the incremental procedure have been developed, by which, equilibrium correction is done after a certain number of load steps. For structures requiring many degrees of freedom the updating of the incremental stiffness matrix plus the solution of the new coefficient matrix at each load step, becomes excessively time consuming.

The perturbation method is another incremental type of procedure. In this method the incremental displacements are expanded in a Taylor's series with respect to some load parameters and about some known or assumed equilibrium state. Equations are obtained in the form,

$$\{q\}_{i+1} = \{q\}_i + \{\Delta\dot{q}\}_i\Delta\overline{P} + 0.5\{\Delta\ddot{q}\}_i\Delta\overline{P}^2 + \ldots$$

where the dot denotes the derivative with respect to the the load parameter, $\overline{P}$. $\{\Delta\dot{q}\}$, $\{\Delta\ddot{q}\}$ etc are path derivatives and i denotes the load increment index. The terms in the

Taylor's series are obtained through the solution of several sets of linear equations equal in number to the number of terms retained in the expansion. Once the displacements are obtained at a particular load value, the whole process is repeated to obtain the displacements at the next load value. This procedure may deviate from the true solution since errors will tend to accumulate and the amount of deviation is dependent upon the load step size and the number of terms retained in the expansion. This procedure may become time consuming because of the numerous evaluations of the path derivatives.

The iterational approach to solve the governing nonlinear equations has been used by many investigators. Starting with an initial estimate to the displacement solution the nonlinear effects are estimated and a set of linearized equations are solved to obtain an improved solution. This solution is back-substituted into the equations and the iterations continued until convergence of successive iterations is obtained. The success of the method depends to a large extent upon the accuracy of the initial estimate of the displacements. The Newton Raphson iteration procedure has been most popular. This procedure is extremely accurate and usually converges very rapidly for realistic initial estimate of the solution. Its chief drawback is the excessive computational effort required to form the coefficient matrix and invert it at each iterational cycle. In the modified Newton Raphson procedure the coefficient matrix is held constant for a number of iterations and then updated after the convergence rate has begun to deteriorate.

The initial value approach treats the loads and displacements as a function of some load parameter $\overline{P}$ such that $\{Q\} = \overline{P}\{\overline{Q}\}$. By differentiating the equilibrium equations, $[K]\{q\} = \{Q\}$ with respect to $\overline{P}$, a set of differential equations is obtained in the form

$$[\overline{K}]\{dq/d\overline{P}\} = \{\overline{Q}\}$$

where $[\overline{K}]$ is a nonlinear stiffness matrix dependent upon displacements, $\{q\}$, and $\{\overline{Q}\}$ is a vector of scaled or normalized generalized forces. Values of $\{q\}$ at any load $\overline{P}$ can be obtained by numerical integration from a known initial displacement state. If the simple Euler method is used for the integration then the incremental approach is obtained. More accurate integration schemes such as Runge Kutta method or the predictor corrector method may be used to reduce the deviation which is prominent with the Euler integration. The newest development in solution procedures is the self correcting initial value formulation in which equilibrium correction is included.

A survey of the solution methods for geometrically nonlinear structural analysis mentioned above and an evaluation of the relative merits of these methods are presented in (1, 2, 3, 4). These papers give an extensive list of references on procedures and solution methods for geometrically nonlinear structural analysis.

2.2 Geometrically Nonlinear Static Analysis of Shells

The earlier attempts to solve the geometrically nonlinear static analysis of shells were by the power series and finite difference methods. The power series method was used by Simons (5), Reiss (6), and Weinitschke (7). The finite difference method has been used to investigate the large deflections of spherical shells by Archer (8), Famili and Archer (9), Mescall (10), Wilson and Spier (11), Ball (12), and Perrone and Kao (13). Large deflection of orthotropic stiffened shells of revolution has been studied by Bushnell (14) using the finite difference method.

Leicester (15) has used a truncated Fourier series in displacements to solve the governing nonlinear equations of finite deformation of doubly curved shells. Thurston (16) has used a numerical integration procedure for the solution of the nonlinear axysymmetrical bending of shallow spherical shells. Kalinins and Lestingi (17) have used a multisegment method to solve the nonlinear analysis of symmetric shells of revolution. Mason et al. (18) have used the method of influence coefficients to solve the geometrically nonlinear arbitrary shell of revolution problem.

The finite element formulation of the governing equations for the geometrically nonlinear analysis of shells has been popular. Stricklin, Haisler, MacDougall and Stebbins (19) have used the matrix displacement method for the nonlinear analysis of shells of revolution. Popov and Yaghmai

(20) have used the finite element method to solve the nonlinear static analysis of axysymmetrically loaded thin shells of revolution. The large deflection of shallow shells has been investigated by Batoz et al. (21), Bregan and Clough (22). The large deflection of shallow cylindrical shells has been studied by Gass and Taborok (23), and Brebbia and Connor (24).

2.3 Dynamic Relaxation Technique

Dynamic relaxation technique was conceived in 1960 by Day (25). Day has given an outline of the method and illustrated its use by the analysis of a portal frame, a flat plate under lateral load, a skew slab under lateral load, and a thick cylinder under internal pressure. Otter (26) used this method for the calculation of the stresses in a prestressed concrete reactor pressure vessel. Otter, Casell, and Hobbs (27) applied this method for the calculation of stresses in an arch dam subjected to hydrostatic, temperature, and gravity loading.

Extensive application of dynamic relaxation to the small deflection (28) and large deflection (29, 30) analysis of plates has been done by Rushton. Post buckling of rectangular plates (31) has also been studied by Rushton. Alwar and Rao (32, 33) have analyzed the large deflection of skew plates. Circular plates have been analyzed by Murthy and Sherbourne (34). Tapered annular circular plates have been studied by Turvey (35). Frieze, Hobbs, and Dowling (36) have applied dynamic relaxation to the large deflection elasto-plastic analysis of plates. Post buckling of laminated plates has been investigated by Turvey and Wittrick (37).

The first application of dynamic relaxation to the analysis of shells has been done by Casell, Kinsey, and Sefton (38). They have analyzed a cylindrical arch dam using small deflection theory of shells. Casell (39) has analyzed a doubly curved shell under small deflection. Apart from these two papers, there is no work reported especially on the large deflection analysis of shells using dynamic relaxation. The motivation for the present study has been partly due to this fact. Brew (40), Wood (41), Brunce (42), Alwar (43), Casell (44), Papadrakakis (45), and Underwood (46) have attempted to improve the dynamic relaxation method.

CHAPTER III

METHOD OF ANALYSIS

3.1 Governing Equations for the Geometrically Nonlinear Static Analysis of Shells of Revolution

The governing equations required in this analysis are the nonlinear equations of motion, the stress-strain equations and the nonlinear strain-displacement relations for a shell of revolution. The general nonlinear equations of equilibrium for a shell of revolution for large displacements but small strains are presented in Appendix A and they are based on the previous work (47, 48). The nonlinear equations of equilibrium have been developed by the principle of minimum potential energy and they have been derived from more general strain-displacement relations without the simplifying assumptions made by other investigators. The general stress-strain relations, the nonlinear strain displacement equations, and the assumptions of the shell theory used in this analysis are also presented in Appendix A.

3.1.1 Axysymmetric Shell of Revolution

As this study is concerned with axysymmetric shells of revolution, the pertinent equations for this type of shell are presented in this subsection. Figure 1(a) represents an element of a shell of revolution. Figures 1(b) and 1(c) represent the positive directions for the stress resultants acting on an element of the shell. For a shell with axysymmetric geometry material properties and loads, the general equations presented in Appendix A can be reduced to the following equations by noting that the shear stress resultants $N_{\phi\theta}$ and $N_{\theta\phi}$, twisting moments $M_{\phi\theta}$ and $M_{\theta\phi}$, the circumferential displacement u_θ, the in plane shear strain $\gamma_{\phi\theta}$, the twist τ including its components $\kappa_{\phi\theta}$ and $\kappa_{\theta\phi}$, and all derivatives with respect to 'θ' are zero. The equilibrium equation in the 'θ' direction vanishes. The following equations of motion are then obtained by including the inertia and damping terms.

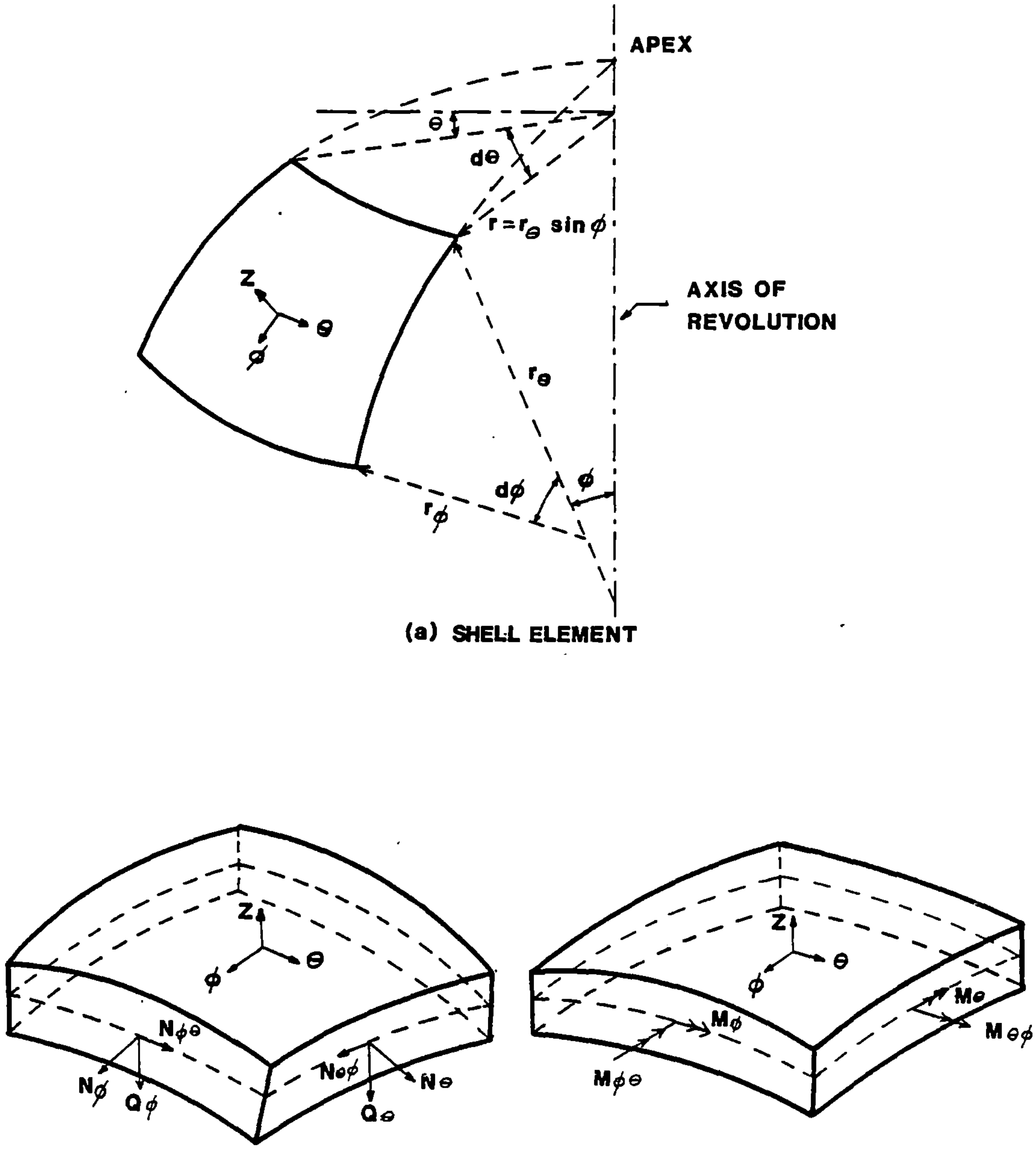
Nonlinear equations of motion

'ϕ' direction:

$$\frac{\partial N_\phi}{r_\phi \partial \phi} + \frac{\cos\phi}{r}(N_\phi - N_\theta) + Q_\phi\left(\frac{1}{r_\phi} + \kappa_\phi\right) - N_\phi \frac{\beta_\phi}{r_\phi} + \frac{Q_\phi \beta_\phi}{r}\cos\phi + \frac{\partial Q_\phi}{r_\phi \partial \phi}\beta_\phi + q_\phi = m_\phi \frac{\partial^2 u_\phi}{\partial t^2} + k_\phi \frac{\partial u_\phi}{\partial t} \tag{3.1a}$$

'z' direction:

$$\frac{\partial^2 M_\phi}{r_\phi^2 \partial \phi^2} + \frac{2\cos\phi}{r}\frac{\partial M_\phi}{r_\phi \partial \phi} - \frac{\cos\phi}{r}\frac{\partial M_\theta}{r_\phi \partial \phi} - \frac{\sin\phi}{r\, r_\phi}(M_\phi - M_\theta) - \frac{1}{r_\phi^2}\frac{\partial r_\phi}{\partial \phi}\frac{\partial M_\phi}{r_\phi \partial \phi} - N_\phi\left(\frac{1}{r_\phi} + \kappa_\phi\right) - \frac{N_\theta}{r_\theta} - N_\phi \frac{\beta_\phi \cos\phi}{r} - \frac{Q_\phi \beta_\phi}{r_\phi} - \frac{\partial N_\phi}{r_\phi \partial \phi}\beta_\phi - q = m_w \frac{\partial^2 w}{\partial t^2} + k_w \frac{\partial w}{\partial t} \tag{3.1b}$$



(a) SHELL ELEMENT

(b) IN PLANE AND SHEAR STRESS RESULTANTS (FORCES)

(c) BENDING AND TWISTING STRESS RESULTANTS (MOMENTS)

Figure 1. Element of Shell of Revolution and Stress Resultants

where

$$Q_\phi = (M_\phi - M_\theta)\frac{\cos\phi}{r} + \frac{\partial M_\phi}{r_\phi \partial\phi}$$

$$\kappa_\phi = \frac{1}{r_\phi}\frac{\partial}{\partial\phi}\left(\frac{u_\phi}{r_\phi} - \frac{\partial w}{r_\phi \partial\phi}\right)$$

$$\kappa_\theta = \frac{\cos\phi}{r}\left(\frac{u_\phi}{r_\phi} - \frac{\partial w}{r_\phi \partial\phi}\right)$$

$$\beta_\phi = \left(\frac{u_\phi}{r_\phi} - \frac{\partial w}{r_\phi \partial\phi}\right)$$

Stress resultant-strain equations

The linear stress resultant-strain equations are:

$$N_\phi = \frac{E_\phi h}{(1 - \nu_{\phi\theta}\nu_{\theta\phi})}(\epsilon_\phi^\circ + \nu_{\phi\theta}\epsilon_\theta^\circ)$$

$$N_\theta = \frac{E_\theta h}{(1 - \nu_{\phi\theta}\nu_{\theta\phi})}(\epsilon_\theta^\circ + \nu_{\theta\phi}\epsilon_\phi^\circ)$$

$$M_\phi = \frac{E_\phi h^3}{12(1 - \nu_{\phi\theta}\nu_{\theta\phi})}(\kappa_\phi + \nu_{\phi\theta}\kappa_\theta) \tag{3.2}$$

$$M_\theta = \frac{E_\theta h^3}{12(1 - \nu_{\phi\theta}\nu_{\theta\phi})}(\kappa_\theta + \nu_{\theta\phi}\kappa_\phi)$$

Strain-displacement equations

$$\epsilon_\phi^\circ = \left(\frac{\partial u_\phi}{r_\phi \partial\phi} + \frac{w}{r_\phi}\right) + 0.5\left\{\left(\frac{\partial u_\phi}{r_\phi \partial\phi} + \frac{w}{r_\phi}\right)^2 + \left(\frac{u_\phi}{r_\phi} - \frac{\partial w}{r_\phi \partial\phi}\right)^2\right\}$$

$$\epsilon_\theta^\circ = \left(\frac{u_\phi \cos\phi + w\sin\phi}{r}\right) + 0.5\left(\frac{u_\phi \cos\phi}{r} + \frac{w\sin\phi}{r}\right)^2 \tag{3.3}$$

$$\kappa_\phi = \frac{1}{r_\phi}\frac{\partial}{\partial\phi}\left(\frac{u_\phi}{r_\phi} - \frac{\partial w}{r_\phi \partial\phi}\right)$$

$$\kappa_\theta = \frac{\cos\phi}{r}\left(\frac{u_\phi}{r_\phi} - \frac{\partial w}{r_\phi \partial\phi}\right)$$

3.1.2 Non Dimensional Equations of Motion

Using the non dimensional relations (A-8) given in Appendix A, the equations of motion (3.1) are converted to non dimensional form and shown below.

1. 'ϕ' direction:

$$\frac{\partial N'_\phi}{r'_\phi \partial\phi} + (N'_\phi - N'_\theta)\frac{\cos\phi}{r'} + \{(M'_\phi - M'_\theta)\frac{\cos\phi}{r'} + \frac{\partial M'_\phi}{r'_\phi \partial\phi}\}$$
$$\{\frac{1}{r'_\phi} + (\frac{h_o}{a})^2 \frac{\partial u'_\phi}{r'^2_\phi \partial\phi} - \frac{h_o}{a}\frac{\partial^2 w'}{r'^2_\phi \partial\phi^2} + \frac{1}{r'^2_\phi}\frac{\partial r'_\phi}{\partial\phi}(\frac{h_o}{a}\frac{\partial w'}{r'_\phi\partial\phi}$$
$$- (\frac{h_o}{a})^2\frac{u'_\phi}{r'_\phi})\} + \{\frac{\partial^2 M'_\phi}{r'^2_\phi\partial\phi^2} + \frac{2\cos\phi}{r'}\frac{\partial M'_\phi}{r'_\phi\partial\phi} - \frac{\cos\phi}{r'}\frac{\partial M'_\theta}{r'_\phi\partial\phi}$$
$$- \frac{\sin\phi}{r' r'_\phi}(M'_\phi - M'_\theta) - \frac{1}{r'^2_\phi}\frac{\partial r'_\phi}{\partial\phi}\frac{\partial M'_\phi}{r'_\phi\partial\phi} - \frac{N'_\phi}{r'_\phi}\}(\frac{h_o^2}{a^2}\frac{u'_\phi}{r'_\phi}$$
$$- \frac{h_o}{a}\frac{\partial w'}{r'_\phi\partial\phi}) + q'_\phi = m'_\phi\frac{\partial^2 u'_\phi}{\partial t'^2} + k'_\phi\frac{\partial u'_\phi}{\partial t'} \tag{3.4a}$$

2. 'z' direction:

$$\frac{\partial^2 M'_\phi}{r'^2_\phi\partial\phi^2} + \frac{2\cos\phi}{r'}\frac{\partial M'_\phi}{r'_\phi\partial\phi} - \frac{\cos\phi}{r'}\frac{\partial M'_\theta}{r'_\phi\partial\phi} - \frac{\sin\phi}{r' r'_\phi}(M'_\phi - M'_\theta)$$
$$- \frac{1}{r'^2_\phi}\frac{\partial r'_\phi}{\partial\phi}\frac{\partial M'_\phi}{r'_\phi\partial\phi} - N'_\phi\{\frac{1}{r'_\phi} + (\frac{h_o}{a})^2\frac{1}{r'_\phi}\frac{\partial u'_\phi}{r'_\phi\partial\phi}$$
$$- \frac{h_o}{a}\frac{\partial^2 w'}{r'^2_\phi\partial\phi^2} + \frac{1}{r'^2_\phi}\frac{\partial r'_\phi}{\partial\phi}[\frac{h_o}{a}\frac{\partial w'}{r'_\phi\partial\phi} - (\frac{h_o}{a})^2\frac{u'_\phi}{r'_\phi}]\}$$
$$- \frac{N'_\theta}{r'_\theta} - \{\frac{1}{r'_\phi}(\frac{M'_\phi\cos\phi}{r'} - \frac{M'_\theta\cos\phi}{r'} + \frac{\partial M'_\phi}{r'_\phi\partial\phi}) + \frac{\partial N'_\phi}{r'_\phi\partial\phi} + \frac{N'_\phi\cos\phi}{r'}\}$$
$$(\frac{h_o^2}{a^2}\frac{u'_\phi}{r'_\phi} - \frac{h_o}{a}\frac{\partial w'}{r'_\phi\partial\phi}) - q = m'_w\frac{\partial^2 w'}{\partial t'^2} + k'_w\frac{\partial w'}{\partial t'} \tag{3.4b}$$

where

$$m'_{\phi} = \left(\frac{h_o}{a}\right)\frac{m_{\phi}}{m_W} \quad ; \quad k'_{\phi} = \frac{h_o}{a}\,\frac{a^2\,k_{\phi}}{\sqrt{D_o\,m'_W}}$$

$$m'_W = 1 \quad ; \quad k'_W = \frac{a^2\,k_W}{\sqrt{D_o\,m'_W}}$$

3.1.3 Non Dimensional Stress-Resultant Displacement Equations

From Equations (3.2), (3.3) and (A-8) the non dimensional stress resultant-displacement equations are obtained as,

$$N'_{\phi} = 12h'\frac{E_{\phi}}{E_{\theta}}\Big[\left(\frac{a}{h_o}\right)\frac{\partial u'_{\phi}}{r'_{\phi}\partial\phi} + \left(\frac{a}{h_o}\right)^2\frac{w'}{r'_{\phi}} + 0.5\,\frac{a}{h_o}\left\{\frac{h_o}{a}\,\frac{\partial u'_{\phi}}{r'_{\phi}\partial\phi} + \frac{w'}{r'_{\phi}}\right\}^2$$

$$+\ 0.5\,\frac{a}{h_o}\left\{\frac{\partial w'}{r'_{\phi}\partial\phi} - \frac{h_o}{a}\,\frac{u'_{\phi}}{r'_{\phi}}\right\}^2 + \nu_{\phi\theta}\Big\{\frac{a}{h_o}\,\frac{u'_{\phi}}{r'}\cos\phi$$

$$+\left(\frac{a}{h_o}\right)^2\frac{w'\sin\phi}{r'} + 0.5\left(\frac{a}{h_o}\right)\left(\frac{h_o}{a}\,\frac{u'_{\phi}\cos\phi}{r'} + \frac{w'\sin\phi}{r'}\right)^2\Big\}\Big]$$

$$N'_{\theta} = 12h'\Big[\left(\frac{a}{h_o}\right)\frac{u'_{\phi}\cos\phi}{r'} + \left(\frac{a}{h_o}\right)^2\frac{w'\sin\phi}{r'} + 0.5\left(\frac{a}{h_o}\right)$$

$$\left(\frac{h_o}{a}\,\frac{u'_{\phi}\cos\phi}{r'} + \frac{w'\sin\phi}{r'}\right)^2 + \nu_{\theta\phi}\Big\{\left(\frac{a}{h_o}\right)\frac{\partial u'_{\phi}}{r'_{\phi}\partial\phi} + \left(\frac{a}{h_o}\right)^2\frac{w'}{r'_{\phi}}$$

$$+\ 0.5\left(\frac{a}{h_o}\right)\left(\frac{h_o}{a}\,\frac{\partial u'_{\phi}}{r'_{\phi}\partial\phi} + \frac{w'}{r'_{\phi}}\right)^2 + 0.5\left(\frac{a}{h_o}\right)\left(\frac{\partial w'}{r'_{\phi}\partial\phi} - \left(\frac{h_o}{a}\right)\frac{u'_{\phi}}{r'_{\phi}}\right)^2\Big\}\Big]$$

$$M'_\phi = \frac{E_\phi}{E_\theta} h'^3 [(\frac{h_o}{a}) \frac{1}{r'_\phi} \frac{\partial u'_\phi}{r'_\phi \partial\phi} - \frac{\partial^2 w'}{r'^2_\phi \partial\phi^2} \tag{3.5}$$

$$+ \frac{\partial r'_\phi}{r'^2_\phi \partial\phi} (\frac{1}{r'_\phi} \frac{\partial w'}{\partial\phi} - (\frac{h_o}{a}) \frac{u'_\phi}{r'_\phi}) + \gamma_{\phi\theta} \cos\phi$$

$$\{ (\frac{h_o}{a}) \frac{u'_\phi}{r' r'_\phi} - \frac{1}{r' r'_\phi} \frac{\partial w'}{\partial\phi} \}]$$

$$M'_\theta = h'^3 [\cos\phi \{ (\frac{h_o}{a}) \frac{u'_\phi}{r' r'_\phi} - \frac{1}{r' r'_\phi} \frac{\partial w'}{\partial\phi} \} + \gamma_{\theta\phi} \{ \frac{h_o}{a} \frac{\partial u'_\phi}{r'_\phi r'_\phi \partial\phi}$$

$$- \frac{\partial^2 w'}{r'^2_\phi \partial\phi^2} + \frac{1}{r'^2_\phi} \frac{\partial r'_\phi}{\partial\phi} (\frac{\partial w'}{r'_\phi \partial\phi} - (\frac{h_o}{a}) \frac{u'_\phi}{r'_\phi}) \}]$$

3.2 Dynamic Relaxation Procedure for the Axysymmetric Shell of Revolution

The dynamic relaxation procedure for the axysymmetric shell of revolution is outlined in this section. The nonlinear equations of motion (3.4) and the stress resultant-displacement equations (3.5) are converted to finite difference equations using central difference approximations for the spatial and time derivatives. Figure 2 shows the typical finite difference grid used for the spatial and time variables. Suffix 'i' refers to nodal point and 'j' refers to time node.

The finite difference equations of motion are given by,

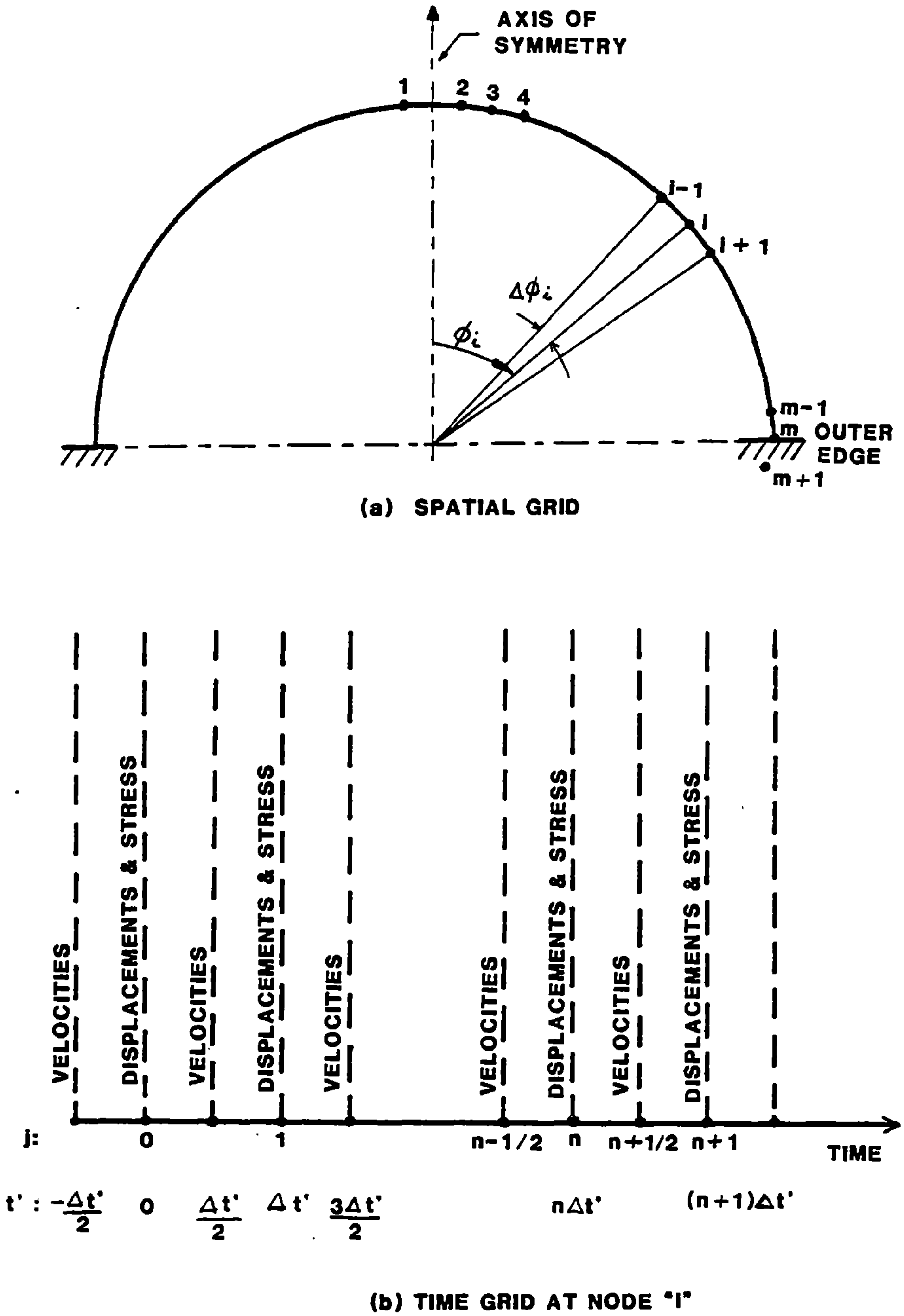


Figure 2. Finite Difference Grid for Spatial and Time Variables

'ϕ' direction:

$$\frac{(N'_{\phi\,i+1} - N'_{\phi\,i-1})^j}{2r'_{\phi i}\Delta\phi_i} + (N'_{\phi\,i} - N'_{\theta\,i})\frac{\cos\phi_i}{r'_i}$$

$$+ \left\{\frac{(M'_{\phi i} - M'_{\theta i})^j\cos\phi_i}{r'_i} + \frac{(M'_{\phi\,i+1} - M'_{\phi\,i-1})^j}{2r'_{\phi i}\Delta\phi_i}\right\}$$

$$\left\{\frac{1}{r'_{\phi i}} + \left(\frac{h_o}{a}\right)^2\frac{(u'_{\phi\,i+1} - u'_{\phi\,i-1})^j}{2r'^2_{\phi i}\;\Delta\phi_i} - \frac{h_o}{a}\frac{(w'_{i+1} - 2w'_i + w'_{i-1})^j}{(r'_{\phi i}\Delta\phi_i)^2}\right.$$

$$\left. + \frac{1}{r'^2_{\phi i}}\frac{(r'_{\phi\,i+1} - r'_{\phi\,i-1})}{2\,\Delta\phi_i}\left(\frac{h_o}{a}\frac{(w'_{i+1} - w'_{i-1})^j}{2r'_{\phi i}\Delta\phi_i} - \left(\frac{h_o}{a}\right)^2\frac{u'_{\phi i,j}}{r'_{\phi i}}\right)\right\}$$

$$+\left\{\frac{(M'_{\phi\,i+1} - 2M'_{\phi\,i} + M'_{\phi\,i-1})^j}{(r'_{\phi i}\Delta\phi_i)^2} + \frac{2\cos\phi_i}{r'_i}\frac{(M'_{\phi\,i+1} - M'_{\phi\,i-1})^j}{2r'_{\phi i}\Delta\phi_i}\right.$$

$$- \frac{\cos\phi_i}{r'_i}\frac{(M'_{\theta\,i+1} - M'_{\theta\,i-1})^j}{2r'_{\phi i}\Delta\phi_i} - \frac{\sin\phi_i}{r'_i r'_{\phi i}}(M'_{\phi i} - M'_{\theta i})^j$$

$$\left. - \frac{1}{r'^2_{\phi i}}\frac{(r'_{\phi\,i+1} - r'_{\phi\,i-1})}{2\,\Delta\phi_i}\frac{(M'_{\phi\,i+1} - M'_{\phi\,i-1})^j}{2r'_{\phi i}\Delta\phi_i} - \frac{N'_{\phi i,j}}{r'_{\phi i}}\right\}$$

$$\left\{\frac{h_o^2}{a^2}\frac{u'_{\phi i,j}}{r'_{\phi i}} - \frac{h_o}{a}\frac{(w'_{i+1} - w'_{i-1})^j}{2r'_{\phi i}\Delta\phi_i}\right\} + q'_{\phi i}$$

$$= m'_\phi\frac{(\dot u'_{\phi i,j+1/2} - \dot u'_{\phi i,j-1/2})}{\Delta t'} + k'_\phi\frac{(\dot u'_{\phi i,j+1/2} + \dot u'_{\phi i,j-1/2})}{2} \quad (3.6a)$$

'z' direction:

$$\frac{(M'_{\phi_{i+1}}-2M'_{\phi_i}+M'_{\phi_{i-1}})^j}{(r'_{\phi_i}\Delta\phi_i)^2}+\frac{2\cos\phi_i}{r'_i}\frac{(M'_{\phi_{i+1}}-M'_{\phi_{i-1}})^j}{2r'_{\phi_i}\Delta\phi_i}$$

$$-\frac{\cos\phi_i}{r'_i}\frac{(M'_{\theta_{i+1}}-M'_{\theta_{i-1}})^j}{2r'_{\phi_i}\Delta\phi_i}-\frac{\sin\phi_i}{r'_i r'_{\phi_i}}(M'_{\phi_i}-M'_{\theta_i})^j$$

$$-\frac{1}{r'^2_{\phi_i}}\frac{(r'_{\phi_{i+1}}-r'_{\phi_{i-1}})}{2\Delta\phi_i}\frac{(M'_{\phi_{i+1}}-M'_{\phi_{i-1}})^j}{2r'_{\phi_i}\Delta\phi_i}-N'_{\phi_{i,j}}\{\frac{1}{r'_{\phi_i}}$$

$$+(\frac{h_o}{a})^2\frac{1}{r'_{\phi_i}}\frac{(u'_{\phi_{i+1}}-u'_{\phi_{i-1}})^j}{2r'_{\phi_i}\Delta\phi_i}-\frac{h_o}{a}\frac{(w'_{i+1}-2w'_i+w'_{i-1})^j}{(r'_{\phi_i}\Delta\phi_i)^2}$$

$$+\frac{1}{r'^2_{\phi_i}}\frac{(r'_{\phi_{i+1}}-r'_{\phi_{i-1}})}{2\Delta\phi_i}[\frac{h_o}{a}\frac{(w'_{i+1}-w'_{i-1})^j}{2r'_{\phi_i}\Delta\phi_i}-(\frac{h_o}{a})^2\frac{u'_{\phi_{i,j}}}{r'_{\phi_i}}]\}$$

$$-\frac{N'_{\theta_{i,j}}}{r'_{\theta_i}}-\{\frac{1}{r'_{\phi_i}}(\frac{M'_{\phi_{i,j}}\cos\phi_i}{r'_i}-\frac{M'_{\theta_{i,j}}\cos\phi_i}{r'_i}+\frac{(M'_{\phi_{i+1}}-M'_{\phi_{i-1}})^j}{2r'_{\phi_i}\Delta\phi_i})$$

$$+\frac{(N'_{\phi_{i+1}}-N'_{\phi_{i-1}})^j}{2r'_{\phi_i}\Delta\phi_i}+\frac{N'_{\phi_{i,j}}\cos\phi_i}{r'_i}\}\{\frac{h_o^2}{a^2}\frac{u'_{\phi_{i,j}}}{r'_{\phi_i}}$$

$$-\frac{h_o}{a}\frac{(w'_{i+1}-w'_{i-1})^j}{2r'_{\phi_i}\Delta\phi_i}\}-q'_i=m'_w\frac{(\dot{w}'_{i,j+1/2}-\dot{w}'_{i,j-1/2})}{\Delta t'}$$

$$+k'_w\frac{(\dot{w}'_{i,j+1/2}+\dot{w}'_{i,j-1/2})}{2} \qquad (3.6b)$$

where the dot indicates derivative with respect to time, $\Delta t'$ = increment in time, $\Delta\phi$= increment in meridional angle; j refers to time node; i refers to spatial node; i can take values from 1, 2, . . . m; j can take values from 0, 1, 2, . . n. The displacements and stress resultants are defined at times 0, $\Delta t'$, $2\Delta t'$, . . . $j\Delta t'$, while the velocities are defined at times $\Delta t'/2$, $(3/2)\Delta t'$, . . . $(j+1/2)\Delta t'$.

The accelerations $(\partial^2 u'_\phi/\partial t'^2)$ and $(\partial^2 w'/\partial t'^2)$ in Equations (3.4) have been replaced by the following central difference expressions:

$$\frac{\partial^2 u'_{\phi i,j}}{\partial t'^2} = \frac{(\dot{u}'_{\phi i,j+1/2} - \dot{u}'_{\phi i,j-1/2})}{\Delta t'}$$
$$\frac{\partial^2 w'_{i,j}}{\partial t'^2} = \frac{(\dot{w}'_{i,j+1/2} - \dot{w}'_{i,j-1/2})}{\Delta t'} \tag{3.7}$$

For the velocities $(\partial u'_{\phi}/\partial t')$ and $(\partial w'/\partial t')$ the average values are used as follows:

$$\frac{\partial u'_{\phi i,j}}{\partial t'} = \frac{1}{2}(\dot{u}'_{\phi i,j+1/2} + \dot{u}'_{\phi i,j-1/2})$$
$$\frac{\partial w'_{i,j}}{\partial t'} = \frac{1}{2}(\dot{w}'_{i,j+1/2} + \dot{w}'_{i,j-1/2}) \tag{3.8}$$

For convenience, the left hand sides of Equations (3.6) are denoted by $A'_{\phi i,j}$ and $A'_{w i,j}$ respectively. By rearranging Equations (3.6), the velocities at time $(j+1/2)\Delta t'$, in the 'ϕ' and 'z' directions are obtained in terms of the velocities, displacements, and stress resultants at the previous time interval as follows.

$$\dot{u}'_{\phi i,j+1/2} = \frac{1}{\left(1+\dfrac{0.5k'_{\phi}\Delta t'}{m'_{\phi}}\right)}\left\{\left(1 - \frac{0.5k'_{\phi}\Delta t'}{m'_{\phi}}\right)\dot{u}'_{\phi i,j-1/2} + A'_{\phi i,j}\frac{\Delta t'}{m'_{\phi}}\right\}$$

$$\dot{w}'_{i,j+1/2} = \frac{1}{\left(1+\frac{0.5k'_W\Delta t'}{m'_W}\right)}\left\{\left(1 - \frac{0.5k'_W\Delta t'}{m'_W}\right)\dot{w}'_{i,j-1/2} + A'_{Wi,j}\frac{\Delta t'}{m'_W}\right\} \tag{3.9}$$

where i = 1, 2, . . . m and j = 0, 1, 2, . . . n. The displacements at time (j+1)Δt' are obtained from,

$$\begin{aligned} u'_{\phi i,j+1} &= u'_{\phi i,j} + \Delta t'\dot{u}'_{\phi i,j+1/2} \\ w'_{i,j+1} &= w'_{i,j} + \Delta t'\dot{w}'_{i,j+1/2} \end{aligned} \tag{3.10}$$

where i = 1, 2, . . . m and j = 0, 1, 2, . . . n. The stress resultants at time (j+1)Δt' are obtained from the finite difference form of Equations (3.5) which are given below

$$\begin{aligned} N'_{\phi i,j+1} = 12h'\frac{E_\phi}{E_\theta}\Bigg[& \left(\frac{a}{h_o}\right)\frac{(u'_{\phi i+1} - u'_{\phi i-1})_{j+1}}{2r'_{\phi i}\Delta\phi_i} + \left(\frac{a}{h_o}\right)^2\frac{w'_{i,j+1}}{r'_{\phi i}} \\ & + 0.5\frac{a}{h_o}\left(\frac{h_o}{a}\frac{(u'_{\phi i+1} - u'_{\phi i-1})_{j+1}}{2r'_{\phi i}\Delta\phi_i} + \frac{w'_{i,j+1}}{r'_{\phi i}}\right)^2 \\ & + 0.5\frac{a}{h_o}\left(\frac{(w'_{i+1} - w'_{i-1})_{j+1}}{2r'_{\phi i}\Delta\phi_i} - \left(\frac{h_o}{a}\right)\frac{u'_{\phi i,j+1}}{r'_{\phi i}}\right)^2 \\ & +\nu_{\phi\theta}\Big\{\left(\frac{a}{h_o}\right)\frac{u'_{\phi i,j+1}\cos\phi_i}{r'_i} + \left(\frac{a}{h_o}\right)^2\frac{w'_{i,j+1}\sin\phi_i}{r'_i} \\ & + 0.5\frac{a}{h_o}\left(\left(\frac{h_o}{a}\right)\frac{u'_{\phi i,j+1}\cos\phi_i}{r'_i} + \frac{w'_{i,j+1}\sin\phi_i}{r'_i}\right)^2\Big\}\Bigg] \end{aligned}$$

$$N'_{\theta i,j+1} = 12h'[(\frac{a}{h_o})\frac{u'_{\phi i,j+1}\cos\phi_i}{r'_i} + (\frac{a}{h_o})^2 \frac{w'_{i,j+1}\sin\phi_i}{r'_i}$$

$$+ 0.5(\frac{a}{h_o})(\frac{h_o}{a}\frac{u'_{\phi i,j+1}\cos\phi_i}{r'_i} + \frac{w'_{i,j+1}\sin\phi_i}{r'_i})^2$$

$$+\nu_{\theta\phi}\{\frac{a}{h_o}\frac{(u'_{\phi i+1} - u'_{\phi i-1})}{2r'_{\phi i}\Delta\phi_i} + (\frac{a}{h_o})^2\frac{w'_{i,j+1}}{r'_{\phi i}}$$

$$+ 0.5\frac{a}{h_o}(\frac{h_o}{a}\frac{(u'_{\phi i+1} - u'_{\phi i-1})_{j+1}}{2r'_{\phi i}\Delta\phi_i} + \frac{w'_{i,j+1}}{r'_{\phi i}})^2$$

$$+ 0.5\frac{a}{h_o}(\frac{(w'_{i+1} - w'_{i-1})_{j+1}}{2r'_{\phi i}\Delta\phi_i} - \frac{h_o}{a}\frac{u'_{\phi i,j+1}}{r'_{\phi i}})^2\}]$$

(3.11)

$$M'_{\phi i,j+1} = h'^3\frac{E_\phi}{E_\theta}[\frac{h_o}{a}\frac{1}{r'_{\phi i}}\frac{(u'_{\phi i+1} - u'_{\phi i-1})_{j+1}}{2r'_{\phi i}\Delta\phi_i}$$

$$- \frac{(w'_{i+1} - 2w'_i + w'_{i-1})_{j+1}}{(r'_{\phi i}\Delta\phi_i)^2} + \frac{r'_{\phi i+1} - r'_{\phi i-1}}{2r'^2_{\phi i}\Delta\phi_i}$$

$$(\frac{(w'_{i+1} - w'_{i-1})_{j+1}}{2r'_{\phi i}\Delta\phi_i} - \frac{h_o}{a}\frac{u'_{\phi i,j+1}}{r'_{\phi i}}) + \nu_{\phi\theta}\cos\phi_i$$

$$\{\frac{h_o}{a}\frac{u'_{\phi i,j+1}}{r'_i r'_{\phi i}} - \frac{1}{r'_i r'_{\phi i}}\frac{(w'_{i+1} - w'_{i-1})_{j+1}}{2\Delta\phi_i}\}]$$

$$M'_{\theta i,j+1} = h'^3[\cos\phi_i\{\frac{h_o}{a}\frac{u'_{\phi i,j+1}}{r'_i r'_{\phi i}} - \frac{1}{r'_i r'_{\phi i}}\frac{(w'_{i+1} - w'_{i-1})_{j+1}}{2\Delta\phi_i}\}$$

$$+\nu_{\theta\phi}\{\frac{h_o}{a}\frac{1}{r'_{\phi i}}\frac{(u'_{\phi i+1} - u'_{\phi i-1})_{j+1}}{2r'_{\phi i}\Delta\phi_i} - \frac{(w'_{i+1} - 2w'_i + w'_{i-1})_{j+1}}{(r'_{\phi i}\Delta\phi_i)^2}$$

$$+ \frac{r'_{\phi i+1} - r'_{\phi i-1}}{2r'^2_{\phi i}\Delta\phi_i}(\frac{(w'_{i+1} - w'_{i-1})_{j+1}}{2r'_{\phi i}\Delta\phi_i} - \frac{h_o}{a}\frac{u'_{\phi i,j+1}}{r'_{\phi i}})\}]$$

where $i = 1, 2, \ldots m$ and $j = 0, 1, 2, \ldots n$.

Equations (3.9), (3.10), and (3.11) are used to propagate the solution with respect to time. The details of the solution procedure are explained in the next chapter. The finite difference equations (3.9) to (3.11) are applicable only for a uniform spatial mesh. In this study a non uniform spatial finite difference mesh has been used because of accuracy requirements near the edges of the shell where the displacements and stress resultants may vary very rapidly. A finer mesh is used near the edge of the shell and a coarser mesh in the interior of the shell for computational economy. Hence the equations derived in this section have to be modified using the following non uniform spatial finite difference approximations (50) for the derivatives.

$$\frac{\partial F_i}{\partial s} = \frac{F_{i+1} - F_i(1-\alpha^2) - \alpha^2 F_{i-1}}{\alpha(1+\alpha)\Delta s_i}$$

$$\frac{\partial^2 F_i}{\partial s^2} = \frac{2}{\alpha(1+\alpha)} \frac{F_{i+1} - (1+\alpha)F_i + \alpha F_{i-1}}{\Delta s_i^2} \qquad (3.12)$$

where $F = \{u'_\phi \; ; \; w' ; \; N'_\phi \; ; \; N'_\theta \; ; \; M'_\phi \; ; \; M'_\theta \}$

$\alpha = (\Delta s_{i+1}/\Delta s_i)$

Δs = non uniform mesh size

CHAPTER IV

SOLUTION OF EQUATIONS

The solution procedure for the dynamic relaxation equations (3.9) to (3.11) is explained in this chapter. The integration of these equations is started with the initial conditions for the velocities and displacements at the nodal points and the velocities and displacements at the next time interval are obtained. The calculation of velocities and displacements at subsequent time intervals is continued until a steady value is reached which is the static solution to the problem. Boundary conditions prescribed at the edges of the shell are then explained. Finally the parameters which aid the convergence to the static solution, namely, the time increment, damping factors and the mass densities are discussed.

4.1 Initial Conditions

The initial conditions are the displacements and velocities at all nodes are zero at time equal to zero. ($t' = 0$)

$$u'_{\phi\, i,o} \,;\; w'_{i,o} = 0 \qquad (4.1a)$$

$$\dot{u}'_{\phi\, i,o} \,;\; \dot{w}'_{i,o} = 0 \qquad (4.1b)$$

Since the displacements are zero, the stress resultants are all zero at time $t' = 0$. Hence, $A'_{\phi i,0}$ and $A'_{w i,0}$ are also equal to zero at $t' = 0$. From conditions (4.1b) and Equation (3.8)

$$\begin{aligned} \dot{u}'_{\phi i,-1/2} &= -\dot{u}'_{\phi i,1/2} \\ \dot{w}'_{i,-1/2} &= -\dot{w}'_{i,1/2} \end{aligned} \tag{4.2}$$

Initial conditions (4.1a) and (4.2) are used to start the solution procedure and thereby obtain the velocities at time $t' = \Delta t'/2$ from Equation (3.9), from which the displacements at time $t' = \Delta t'$ are obtained from Equation (3.10), and the stress resultants at time $t' = \Delta t'$ are obtained from Equation (3.11).

4.2 Cyclic Solution of the Dynamic Relaxation Equations

The stress resultants obtained at time $\Delta t'$ are substituted into Equation (3.9) to obtain the velocities at time $(3/2)\Delta t'$. These velocities are used to obtain the displacements and stress resultants at time $2\Delta t'$ from Equations (3.10) and (3.11), respectively. These repetitive substitutions are continued until the displacements and stress resultants reach a constant value which is the static solution to the problem.

The following Table illustrates the cyclic solution procedure of the dynamic relaxation method.

TABLE I

SEQUENCE OF CALCULATIONS IN THE DYNAMIC RELAXATION PROCEDURE

Set displacements and stress resultants equal to zero at time t' = 0.
From initial velocities equal to zero at t'=0, velocities at $-(1/2)\Delta t'$ = velocities at $+(1/2)\Delta t'$
Calculate velocities at all nodes at $t' = \Delta t'/2$
Calculate displacements and stress resultants at $t'=\Delta t'$
Calculate velocities at $t' = (3/2)\Delta t'$
Calculate displacements and stress resultants at $t'=2\Delta t'$
- -
- -
Calculate velocities at $t'=(j+1/2)\Delta t'$
Calculate displacement and stress resultants at $t'=(j+1)\Delta t'$
- -
- -
Calculations are stopped when the variation in displacements and stress resultants with respect to time are small

4.3 Boundary Conditions

The following type of boundary conditions have been considered in the analysis of the shell of revolution. The shell may have an opening in the center and could be simply supported or clamped at the edges as shown in Figure 3.

4.3.1 Symmetry Boundary Conditions

If the shell has no opening in the center, symmetry boundary conditons have to be applied at the apex point of the shell. The apex point is a singular point because the radial distance from the axis of the shell is zero. The dynamic relaxation equations cannot be applied at the apex because of this singularity. To avoid this difficulty a grid point is chosen as close to the apex as possible and symmetry boundary conditions are applied at this point. At this point the boundary conditions are,

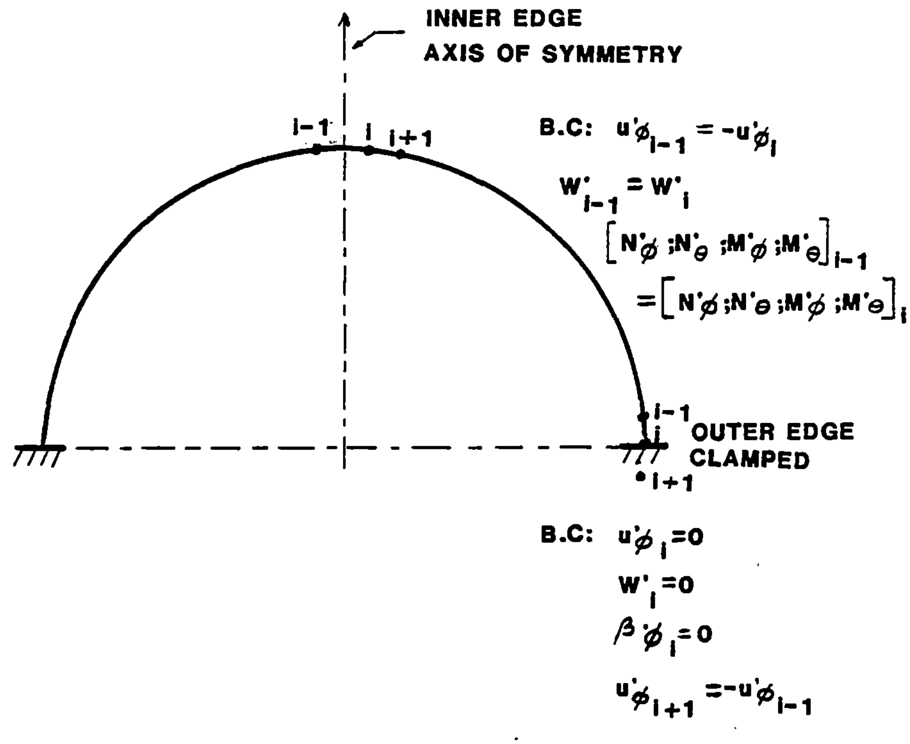
$$\begin{aligned}
&\text{(a)}\quad u'_{\phi\, i-1} = -u'_{\phi\, i} \\
&\qquad\ w'_{i-1} = w'_{i} \\
&\text{(b)}\quad N'_{\phi\, i-1} = N'_{\phi\, i} \\
&\qquad\ N'_{\theta\, i-1} = N'_{\theta\, i} \\
&\qquad\ M'_{\phi\, i-1} = M'_{\phi\, i} \\
&\qquad\ M'_{\theta\, i-1} = M'_{\theta\, i}
\end{aligned} \tag{4.3}$$

where i refers to the point near the apex.

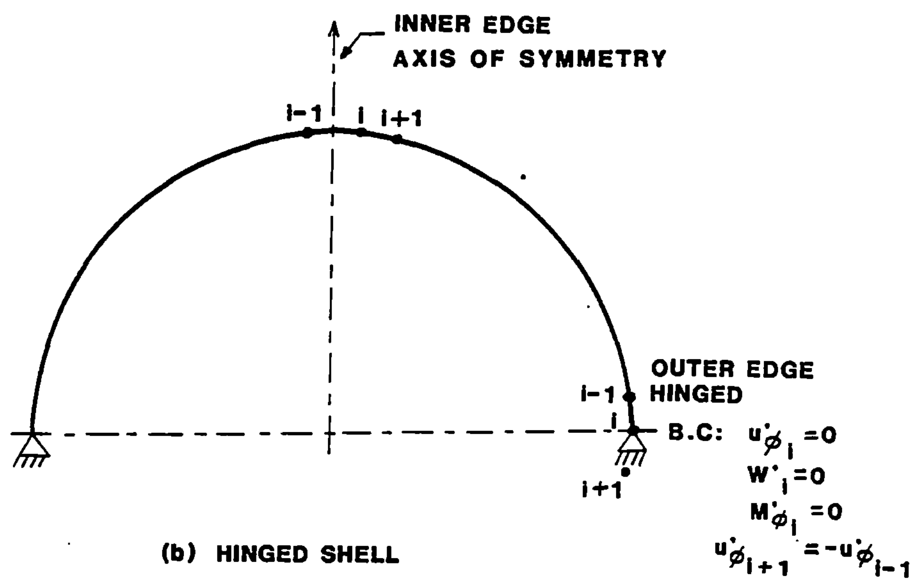
4.3.2 Clamped Boundary Conditions

At the clamped boundary node i,

$$\begin{aligned}
&\text{(a)}\quad u'_{\phi\, i} = 0 \\
&\text{(b)}\quad w'_{i} = 0 \\
&\text{(c)}\quad \beta'_{\phi\, i} = 0 \quad \text{i.e.}\ \left(\frac{u'_\phi}{r'_\phi} - \frac{\partial w'}{r'_\phi\, \partial \phi}\right)_i = 0
\end{aligned} \tag{4.4}$$



(a) CLAMPED SHELL



(b) HINGED SHELL

Figure 3. Boundary Conditions for a Spherical Shell of Revolution

$$\text{Since } u'_{\phi_i} = 0, \ \left(\frac{\partial w'}{r'_\phi \partial \phi}\right)_i = 0$$

From which, $w'_{i+1} = w'_{i-1}$

(d) $u'_{\phi_{i+1}} = -u'_{\phi_{i-1}}$.

This is a linear approximation for the value of u'_ϕ at the fictitious point i+1. This is required for evaluating N'_ϕ , N'_Θ , M'_ϕ , and M'_Θ at the clamped boundary node, i.

4.3.3 Hinged Boundary Conditions

At the hinged boundary node i,

(a) $u'_{\phi_i} = 0$

(b) $w'_i = 0$

(c) $M'_{\phi_i} = 0$, i.e.

$$\left(\frac{h_o}{a}\right) \frac{1}{r'_\phi} \frac{\partial u'_\phi}{r'_\phi \partial \phi} - \frac{\partial^2 w'}{r'^2_\phi \partial \phi^2} + \frac{\partial r'_\phi}{r'^2_\phi \partial \phi} \left(\frac{\partial w}{r'_\phi \partial \phi} - \frac{h_o}{a} \frac{u'_\phi}{r'_\phi} \right)$$

$$+ \nu_{\phi\Theta} \cos\phi \left\{ \frac{h_o}{a} \frac{u'_\phi}{r' r'_\phi} - \frac{1}{r' r'_\phi} \frac{\partial w'}{\partial \phi} \right\} = 0 \quad \text{at node i} \qquad (4.5)$$

This equation can be written in finite difference form and rearranged to give w'_{i+1} in terms of w'_{i-1}, $u'_{\phi_{i+1}}$ & $u'_{\phi_{i-1}}$. $u'_{\phi_{i+1}}$ is known from condition (d).

(d) $u'_{\phi_{i+1}} = -u'_{\phi_{i-1}}$.

As for the clamped edge, this is a linear approximation for u'_ϕ at the fictitious point i+1, necessary for the calculation of M'_Θ , N'_ϕ , and N'_Θ at the hinged edge nodal point, i.

The dynamic relaxation Equations (3.9) and (3.10) need not be applied at the clamped or hinged edge node point since the displacements u'_ϕ and w' and the velocities $\dot{u}'_\phi$ and $\dot{w}'$ are known to be zero at these nodes. To evaluate N'_ϕ, N'_θ, M'_ϕ, and M'_θ at the clamped or hinged edge the derivative $\partial u'_\phi / r'_\phi \partial \phi$ is required, which is found either by using the fictitious value $u'_{\phi\, i+1} = -u'_{\phi\, i-1}$ obtained by linear interpolation or by using higher order backward/forward difference formulae involving known values of functions at interior points. The following third order backward/forward difference formulae are used for $\partial u'_\phi / r'_\phi \partial \phi$.

Backward difference:

$$\frac{\partial u'_\phi}{r'_\phi \partial \phi} = \left(\frac{11}{6} u'_{\phi i} - 3u'_{\phi i-1} + 1.5u'_{\phi i-2} - 0.33u'_{\phi i-3}\right) \frac{1}{r'_{\phi i} \Delta \phi}$$

Forward difference:

$$\frac{\partial u'_\phi}{r'_\phi \partial \phi} = \left(\frac{-11}{6} u'_{\phi i} + 3u'_{\phi i+1} - 1.5u'_{\phi i+2} + 0.33u'_{\phi i+3}\right) \frac{1}{r'_{\phi i} \Delta \phi}$$

These higher order backward/forward difference formulae will give a better accuracy than the linear approximation formula.

4.4 Convergence to Static Solution

The convergence of the dynamic relaxation solution to the static solution of the problem is governed by three parameters which are,

(a) The time increment $\Delta t'$

(b) The damping factors k'_ϕ and k'_w

(c) The masses/area $m'_{\emptyset}$ and m'_{W}

4.4.1 Time Increment

For rapid convergence to the static solution of the finite difference equations, the largest possible time increment $\Delta t'$ should be used. The size of the time increment is governed by the numerical stability criterion of the dynamic relaxation procedure. If an increment larger than the critical time increment is used, the successive iterations lead to results which diverge. The optimal time increment (38) is related to the highest eigenvalue of the coefficient stiffness matrix and is given by the following equation

$$\frac{\Delta t'^2}{m'} \leq \frac{4}{b'} \tag{4.6}$$

where m' is the mass/unit area and b' is the highest eigenvalue of the finite difference coefficient stiffness matrix. To determine b' accurately, existing methods for the extraction of the eignenvalues of a coefficient stiffness matrix can be used. Several methods exist but they all require storage of the coefficient matrix which has been avoided in the present dynamic relaxation formulation. For nonlinear problems the eigenvalues will vary with the load.

Approximate values of b' are found (38) by using the Gershgorin's theorem which states that: For a symmetric matrix an upper bound of b' may be found from the largest absolute sum of the coefficients of the rows of the stiffness matrix.

$$|b'| \leq \max \sum_{k=1}^{n} |S_{ik}|$$

were S_{ik} are the coefficients of the finite difference stiffness matrix. In this study the upper bound on b' is determined as follows. The absolute numerical values of the coefficients of N'_ϕ , N'_θ , M'_ϕ and M'_θ for every node are determined from the absolute sum of the coefficients of the terms on the right hand sides of Equation (3.11). These numerical values are then used in Equation (3.6), without the mass, damping and load terms, to determine the absolute sum of the coefficients of a row of the stiffness matrix of the nodal displacements. Similar sums of the coefficients are determined for all rows of the stiffness matrix of displacements. The largest of these sums of the coefficients of the rows of the stiffness matrix, gives the bounding value of b' to be used in Equation (4.6) to determine an estimate of the largest time increment.

In this study the preliminary value of the time increment has been calculated by neglecting the nonlinear terms in the governing equations. This initial estimate of the time increment was sufficient to ensure numerical stability of the solution procedure for the load range considered. This initial estimate of the time increment may have to be adjusted suitably as the load is increased further as otherwise it could lead to either slow convergence or numerical instability. The calculation of the time increment has been shown in Appendix E, neglecting the effect of nonlinear terms in the governing equations.

4.4.2 Damping Factors

A further requirement for an economical solution is that the damping factors k'_ϕ and k'_w should have a nearly critical damping value, so that the number of iterations for convergence to the static solution is a minimum. It is not necessary to do a rigorous analysis to find the exact critical damping values. A good approximation to the critical damping is given by

$$k'_{cr} = 2m'\omega' \tag{4.7}$$

where ω' is the angular frequency of the lowest or most significant mode of vibration of the system. Though Equation (4.7) is valid only for the free damped vibration of a single degree of freedom system, it has been found by Rushton (28) that the same relation can be used to give the approximate critical damping values for the vibration of continuous systems such as beams and plates. Critical damping of the most significant mode is sufficient to damp out the vibrations of all the other modes also.

Two procedures can be used to find the critical damping value. In the first procedure, the dynamic relaxation calculations are performed with damping factors k'_ϕ and k'_w set equal to zero. The variation of the displacements of a node or the sum of the squares of the velocities of all nodes (which is a measure of the total kinetic energy), with respect to time could be obtained. This represents the free oscillations of the system and the angular frequency ω' can be obtained as

$$\omega' = 2\pi f' = \frac{2\pi}{n\Delta t'} \quad (\text{as } f' = \frac{1}{T'} = \frac{1}{n\Delta t'}) \tag{4.8}$$

where f' is the cyclic frequency and n denotes the number of iterations for one complete cycle of the most significant mode.

The critical damping factor is then obtained by substituting (4.8) into (4.7) to give

$$k'_{cr} = \frac{4\pi m'}{n\Delta t'} \tag{4.9}$$

For the axysymmetric shell problem, the critical damping factors, $k'_{\phi cr}$ and k'_{Wcr} are given by

$$\begin{aligned} k'_{\phi cr} &= \frac{4\pi m'_{\phi}}{n_{\phi}\Delta t'} \\ k'_{Wcr} &= \frac{4\pi m'_{W}}{n_{W}\Delta t'} \end{aligned} \tag{4.10}$$

where n_{ϕ} and n_W are the number of time steps or iterations for one complete cycle of the most significant mode of vibration in the ϕ and z directions respectively. The calculations are restarted with these values of k'_{ϕ} and k'_W and continued until convergence to static solution.

In the second procedure, the frequencies of free vibrations of shells of revolution available in the literature, can be used to determine the critical damping factors. There are two types of oscillations namely the inplane or membrane and bending or transverse oscillations for a shell vibration problem. Usually the lowest membrane frequency is higher than the lowest bending frequency. The values of the lowest few membrane and bending frequencies for spherical shells with different end conditions are available (49, 51, 52). From these references the most significant frequencies of vibration in the in-plane and transverse directions have been calculated and used to estimate the approximate critical damping factors, from Equation (4.7).

The damping factors calculated for spherical shells could be adapted as trial values for other general shells of revolutions. They may not be near critical values for these type of shells and hence convergence to the static solution may be slower. Neverthless these estimated critical damping factors were found to be sufficient to damp the ocillations and obtain the static solution quite efficiently.

In lieu of these procedures, one has to perform an eigenvalue analysis of the coefficient stiffness matrix to obtain the minimum eigenvalue which is related to the lowest frequency. The eigenvalue determination for the present dynamic relaxation procedure is not possible, because coefficient matrices are not stored. In this study, the second procedure outlined, has been adapted to find the approximate critical damping factors.

4.4.3 Fictitious Mass Densities

The actual masses/unit area of the shell in the coordinate directions are not required for the analysis, as the actual vibration response of the shell is not of interest. The masses can be suitably chosen and so they are fictitious. Rushton (54) has suggested that the mass densities in the in-plane and transverse directions could be chosen such that the fundamental or significant frequencies in the two directions become nearly equal. This would enable the oscillations in both directions to be damped out and reach the static values at the same time. Cassell (38) has adjusted the mass at each nodal point and in each coordinate direction such that the time increment $\Delta t'$ calculated from Equation (4.6) is a constant for every nodal point. $\Delta t'$ is typically taken to have a unit value. This has the effect of optimizing the time increment and hence reducing the time of convergence. In the present study, mass densities at all nodes are taken to be unity and no adjustment of mass densities has been done.

4.5 Checks for Convergence

To ensure that the static solution has been reached the following convergence checks are made.

4.5.1 Check for Residuals of the Static Equilibrium Equations

If the value of the displacements and stress resultants are the exact static values, then they must satisfy the static equations of equilibrium. Hence an effective check for convergence is to substitute the values of the stresses and displacements at the end of each iteration into the finite difference form of the static equilibrium equations (left hand side of Equation (3.6)) and find the residuals. If the residuals are smaller than a prescribed limit, it can be concluded that convergence has been achieved and the iterations can be stopped. This is the most important condition for ensuring convergence.

4.5.2 Check for Convergence of Stress Resultants and Displacements

The change in displacements and stress resultants after every iteration can be checked. If the change is smaller than a prescribed value then the displacements and stress resultants have converged. The stress resultants converge much later than the displacements. If, only the accuracy of displacements is required it would be sufficient to check convergence of displacements alone.

CHAPTER V

COMPUTER PROGRAMS

The dynamic relaxation procedure for the nonlinear static analysis of shells of revolution explained in Chapters III and IV has been programmed in FORTRAN IV and run on the Oklahoma State University IBM 3081K computer system. Two programs have been written. The program NSDRSHELL is applicable to the analysis of spherical shells. A listing of this program and selected output of results are presented in Appendix B. Program NSDRGSHELL is applicable to the analysis of a general shell of revolution. A listing and selected output for this program is given in Appendix D. In this chapter, a flow chart for the programs and a brief description of the programs are given.

5.1 Program Capability

Program NSDRSHELL can analyze the linear or nonlinear displacements and stress resultants in a spherical shell with axysymmetric loading. The shell could have a central opening and either hinged or clamped boundary conditions could be prescribed at the edges of the shell. Symmetric boundary conditions can be specified at the apex of the shell for a shell without a central opening. The material of the shell can be orthotropic with isotropy as a special case. Uniformly distributed external or internal pressure loads can be prescribed. Program NSDRGSHELL has similar capabilities as the previous program except that it is applicable to a shell of revolution with general meridional shape. In this study elliptic and parabolic meridional shapes have been considered.

5.2 Description of Programs

The flow chart for the programs is shown in Figure 4. A guide to input data and description of the important variables in the programs are given in Appendix D.

5.2.1 Input Data

The program commences with the prescription of the input variables required for execution of the program. The main input variables are the material properties, mass densities, damping factors, time increment, iteration control parameters, the non uniform finite difference mesh sizes, the intensity of the uniformly distributed loads, and the type of boundary conditions at the edges of the shell.

5.2.2 Finite Difference Parameters

The typical finite difference mesh used in the program is shown in Figures 5 and 6. A non uniform mesh, with finer spacing near the inner and outer edge of the shell is used. The reference node points and mesh spacing variables are also shown in the figures. A description of the variables shown in the figures is given in Appendix D.

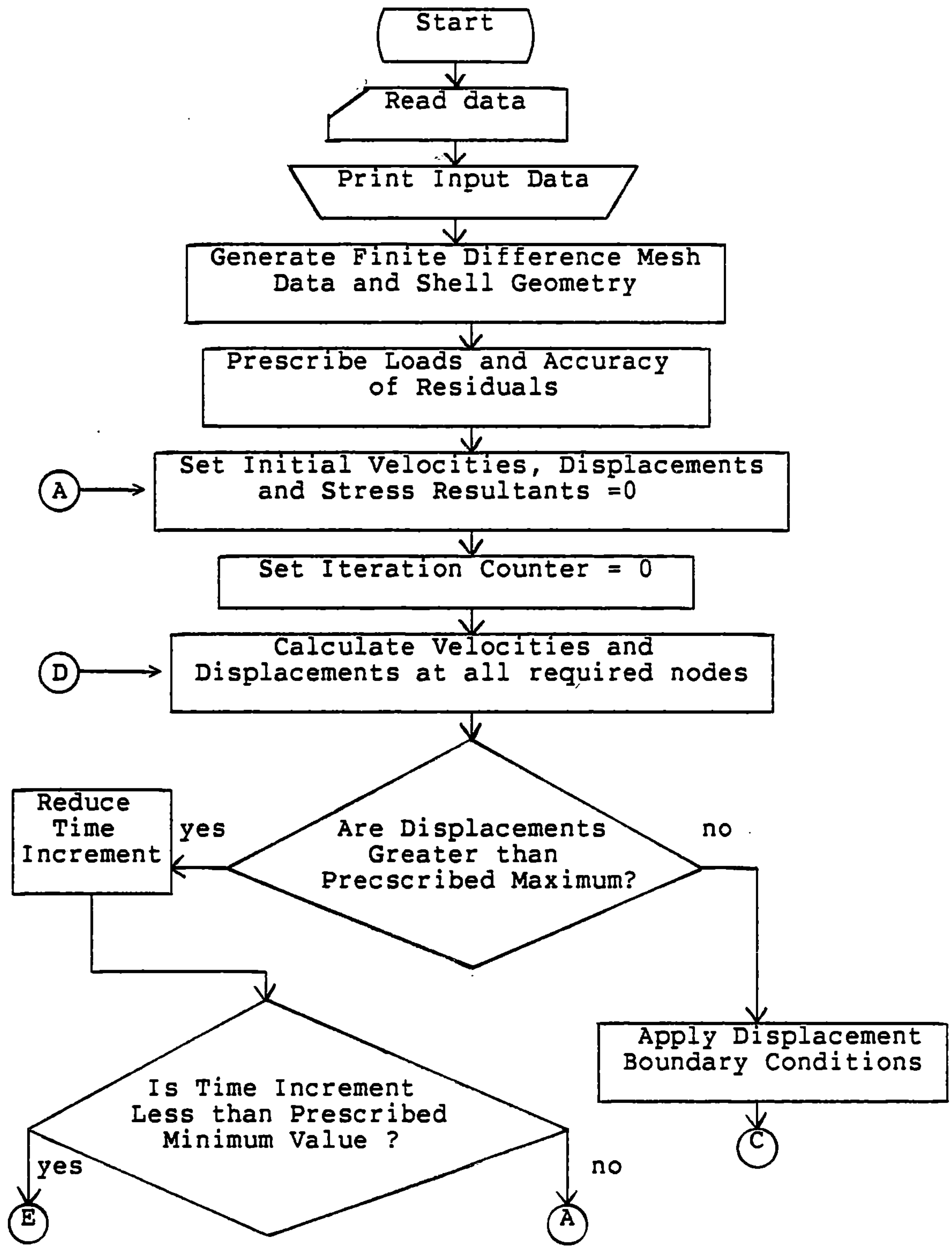


Figure 4. Summary Flowchart for Programs

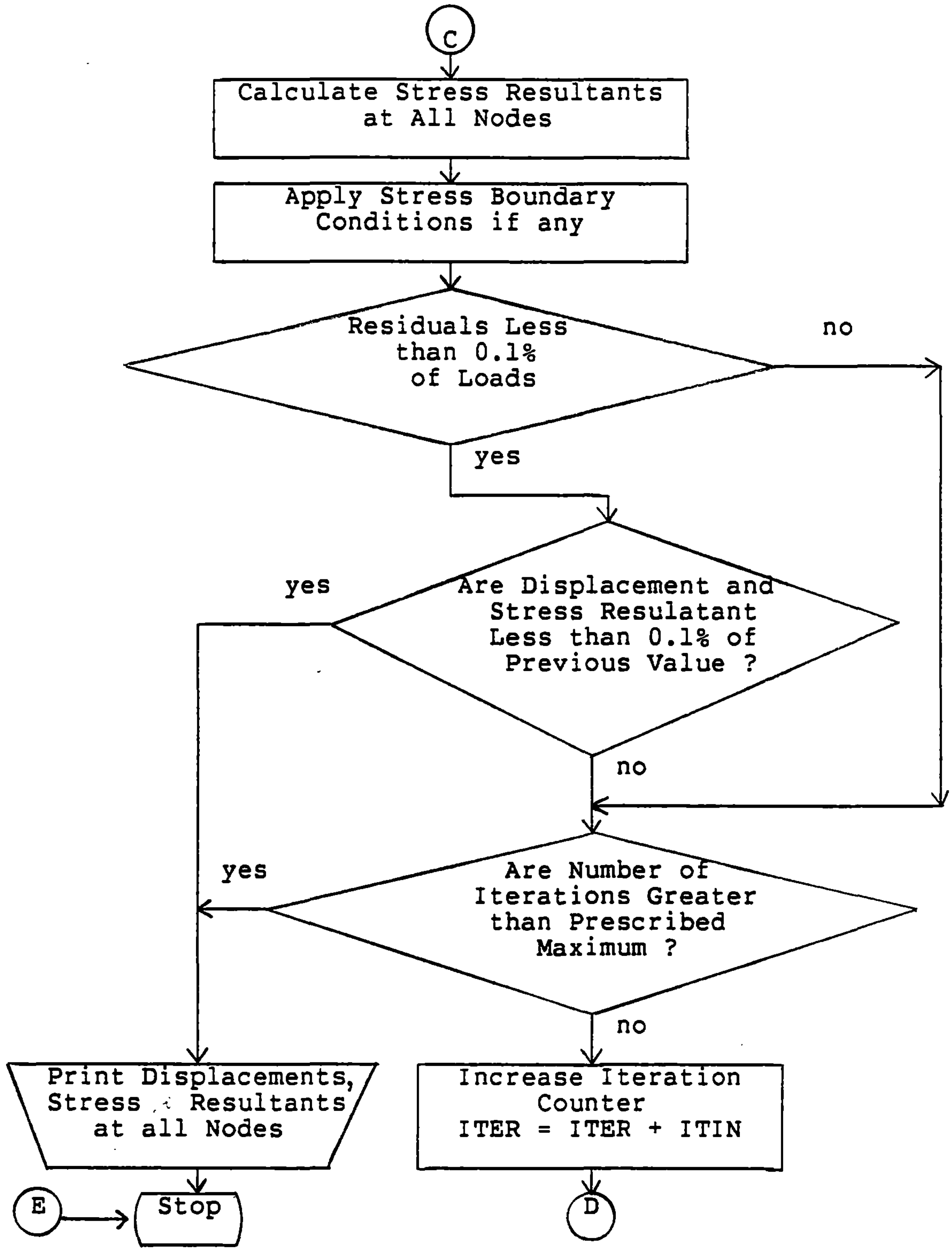


Figure 4. (continued)

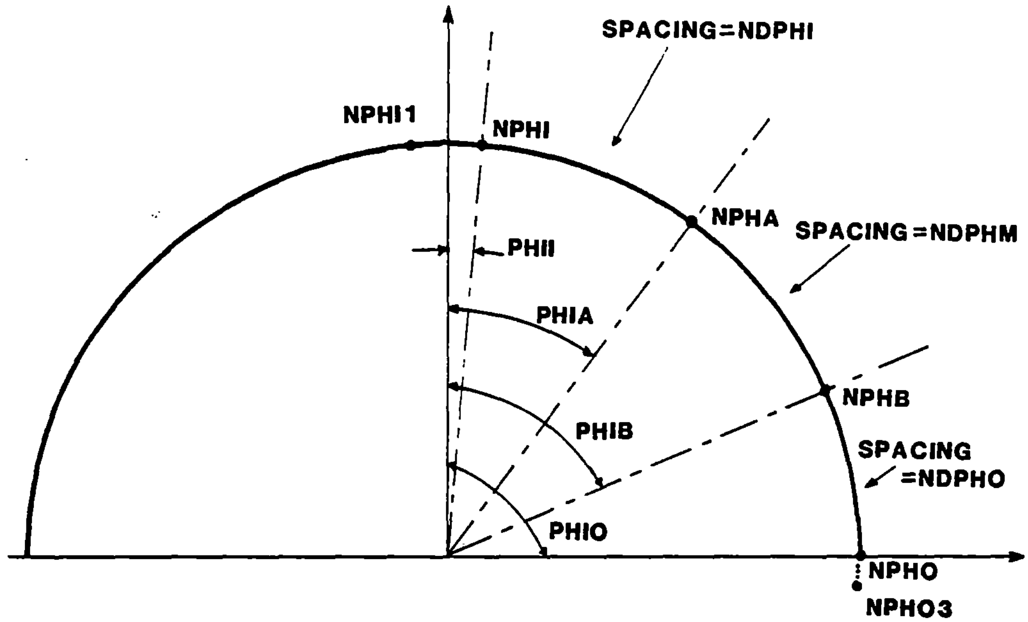


Figure 5. Finite Difference Mesh Parameters for a Spherical Shell of Revolution

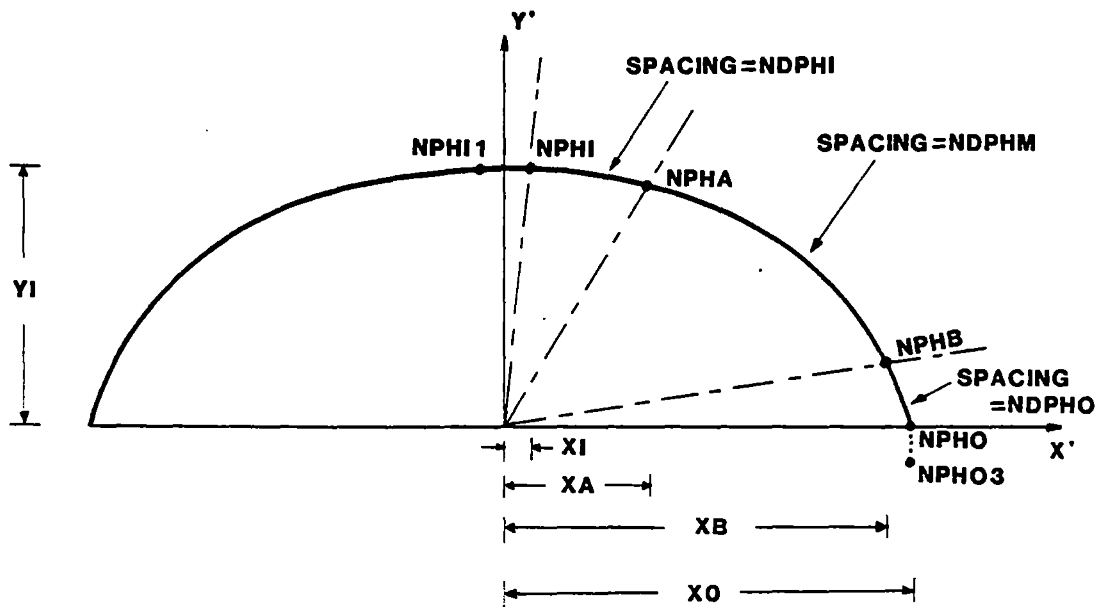


Figure 6. Finite Difference Mesh Parameters for a General Shell of Revolution

5.2.3 Geometry of Shell of Revolution

The geometry of the shell of revolution is generated by the program. The axial distance r', the radii of curvature of the shell r'_ϕ and r'_θ and the meridional angle ϕ at any node point are determined from the given equations for the particular type of shell. By prescribing the equations for the axial distance r', radii of curvature r'_ϕ and r'_θ and the meridional angle ϕ in program NSDRGSHELL, shells of general meridional shapes can be analyzed.

5.2.4 Loads on the Shell

Uniform load can be prescribed at all the node points.

5.2.5 Calculation of Velocities and Displacements

Starting from the node at the inner edge of the shell the velocities and displacements at each node in the ϕ and z directions are calculated using Equations (3.9) and (3.10). The uniform central difference expressions in these equations have to be replaced by corresponding non uniform finite difference approximations as given by Equation (3.12). These equations are not applied at nodes where the displacements are prescribed or known, for example at a clamped or hinged edge node. Immediately after calculating the displacements at any node a check on the displacements is performed. If the displacements exceed the maximum prescribed limits, numerical instability is indicated and the time increment has to be reduced by a suitable factor. If the time increment is smaller than a prescribed minimum value the computations are stopped. If not, the integration is restarted from the initial values with the reduced time increment. After calculating the displacements at all the required nodes, the displacement boundary conditions are applied.

5.2.6 Displacement Boundary Conditions

The displacement boundary conditions are discussed in section 4.3. The following type of boundary conditions can be applied at the edge of the shell (a) clamped (b) hinged (c) symmetry boundary conditions.

5.2.7 Stress Resultants

The stress resultants N'_ϕ , N'_θ , M'_ϕ , and M'_θ are calculated using Equations (3.11). Non uniform finite difference approximations given by Equation (3.12) are used, in place of the uniform finite difference expressions shown in Equation (3.11). After calculating the stress resultants at all the nodes the stress resultant boundary conditions if any, are applied.

5.2.8 Checks for Convergence to Static Solution

The values of the residuals of the static equilibrium portion of the Equation (3.6) (left hand sides of Equation (3.6)) are checked to be less than 0.1% of the maximum applied load. The change in displacements and stress resultants are also checked to be less than 0.1% of the values at the previous time step.

5.2.9 Output Information

The input variables are printed first. The calculated static displacements u'_{ϕ} , w', the stress resultants N'_{ϕ} , N'_{θ} , M'_{ϕ} , M'_{θ} at a selected node are then printed at certain specified iteration intervals. If the displacements diverge, the time increment is reduced by a factor of 0.5 and the iterations are restarted. If convergence is reached all the displacements and stress resultants are printed. The values of the residuals are also printed. If convergence is not reached within a specified numbers of iterations or if the time step becomes smaller than a prescribed limit the iterations are stopped after printing the values of the displacements and stress resultants at the last iteration.

CHAPTER VI

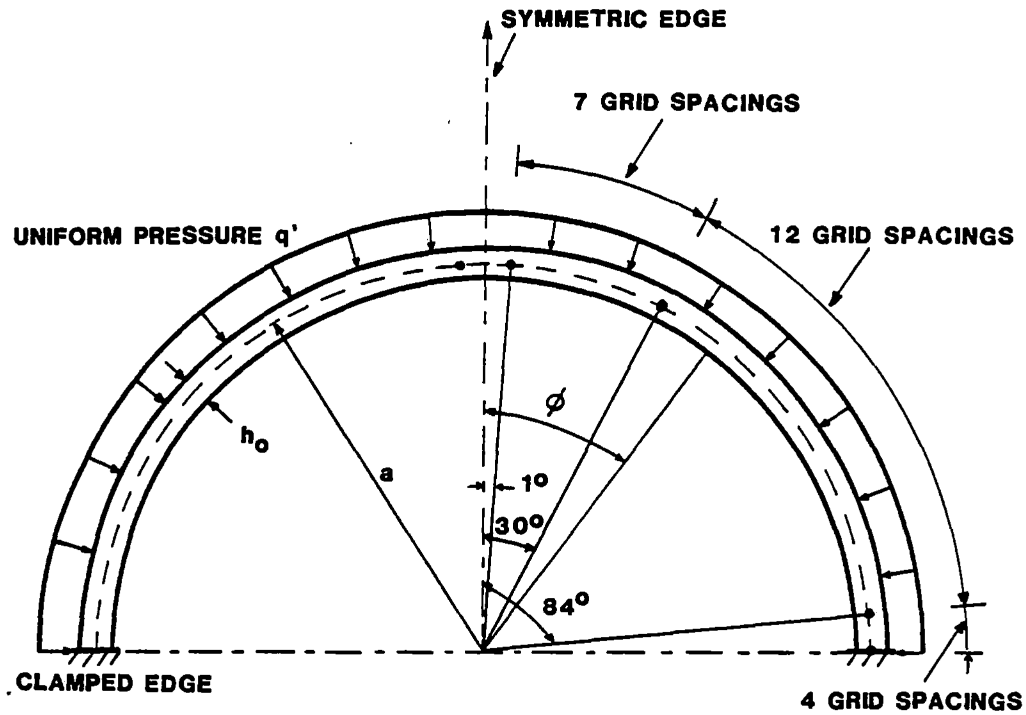
EXAMPLE PROBLEMS AND RESULTS

The programs developed in this study have been applied to some example problems and the validity of the procedure and the programs is first established in sections 6.1 and 6.2. The results for the nonlinear static analysis of spherical shells with various outer opening angles are presented in section 6.3.

6.1 Program NSDRSHELL

6.1.1 Hemispherical Shell under Uniform External Pressure

Figure 7 shows the non uniform finite difference grid used for analyzing the hemispherical shell under uniform external pressure q' = 1. Due to symmetry only one half of the section of the shell need be analyzed. Symmetry boundary conditions are applied at the apex. The apex point, being a point of singularity, is not included in the finite difference grid. Clamped boundary conditions are applied at the outer edge. a/h_0 is chosen to be equal to 100. The spacing near the clamped edge is made finer to account for the edge effects which are significant. The mass densities, damping factors, time increment and the material constants



$k'_\phi = 75$; $k'_w = 300$

$m'_\phi = 1$; $m'_w = 1$

$\Delta t' = 0.2 \times 10^{-3}$

$a/h_o = 100$; $a = 1$

$E_\phi = E_\theta = 2.5 \times 10^6$

$\nu_\phi = \nu_\theta = 0.25$

$G = 1.0 \times 10^6$

$q' = 1.0$

Figure 7. Non Uniform Finite Difference Grid for a Clamped Hemispherical Shell under External Pressure

chosen are also indicated in Figure 7.

The circumferential moments, meridional moments and the circumferential stress resultants in the shell have been plotted as a function of the meridional angle ϕ, in Figures 8, 9, and 10. The comparison between the analytical results given in (49) and the dynamic relaxation results are good. The fine mesh chosen near the clamped edge is sufficient to account for the edge effects quite accurately. A printout of the results of this problem is given in Appendix B, after the listing of the program NSDRSHELL.

6.1.2. A Shallow Shell under Uniform External Pressure

To verify the nonlinear solution, a shallow spherical shell with a semiopening angle of 19.38° as shown in Figure 11 has been analyzed. The shell is subjected to a uniform external pressure of $q' = 0.86 \times 10^5$ which is near the buckling load for the shell. The finite difference grid adapted and the iteration parameters used are shown in Figure 11. In Figure 12 the normal displacements have been plotted as a function of the radial distance from the axis of revolution of the shell. The comparison with the results given in (13) is shown in Figure 12 and it is found to be satisfactory. The comparison of the dynamic relaxation solution using the nonlinear equilibrium equations derived in this study and the results obtained by the same technique using the nonlinear equations given in reference (47) is also shown in Figure 12. There is very little difference between the two

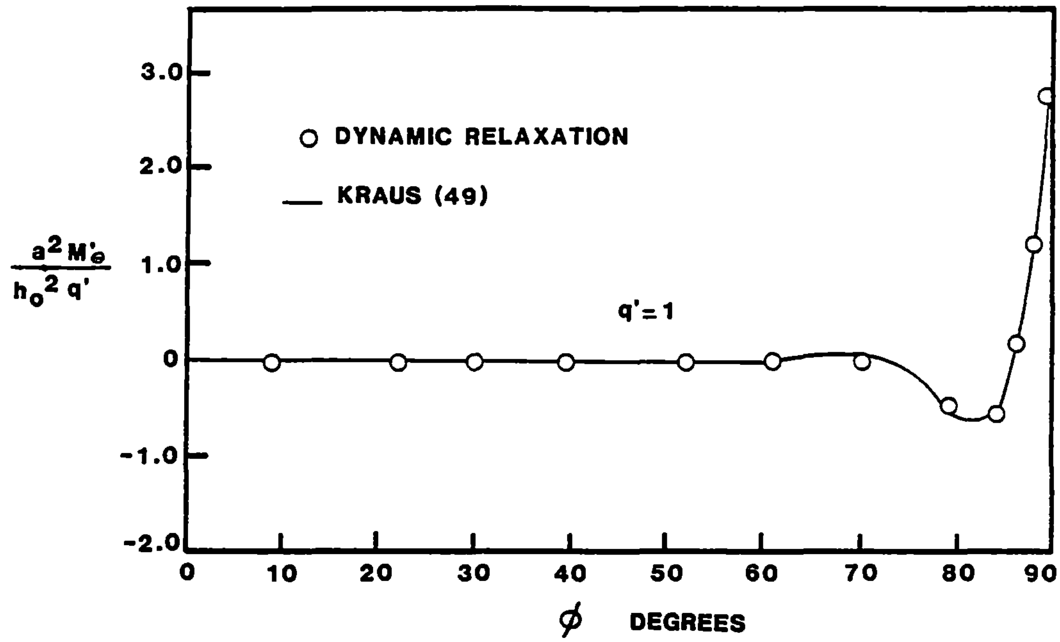


Figure 8. Circumferential Moments in a Clamped Hemispherical Shell under External Pressure

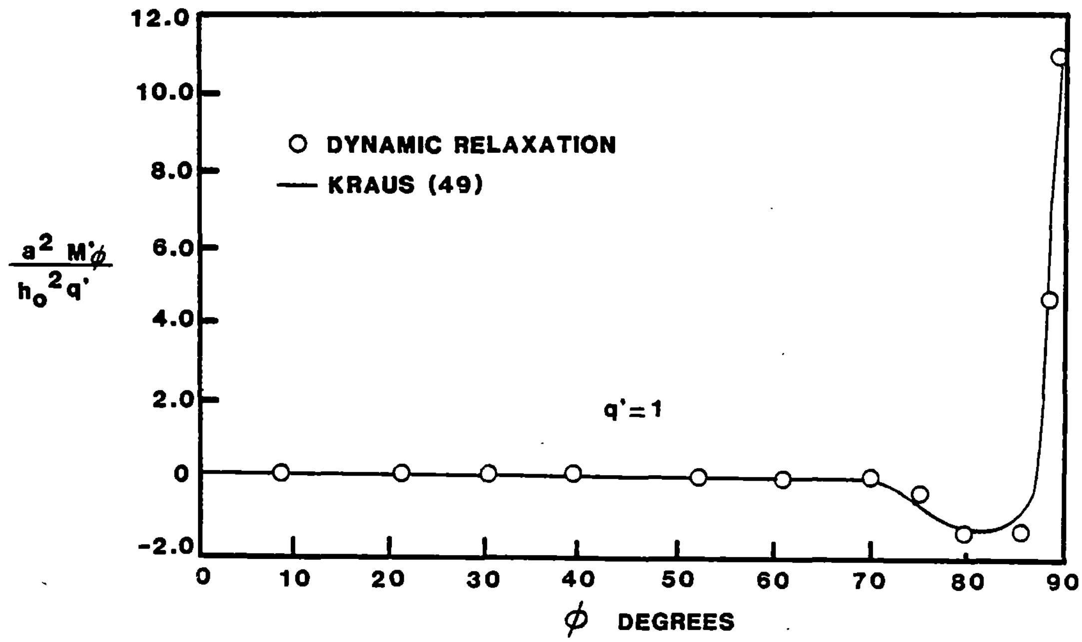


Figure 9. Meridional Moments in a Clamped Hemispherical Shell under External Pressure

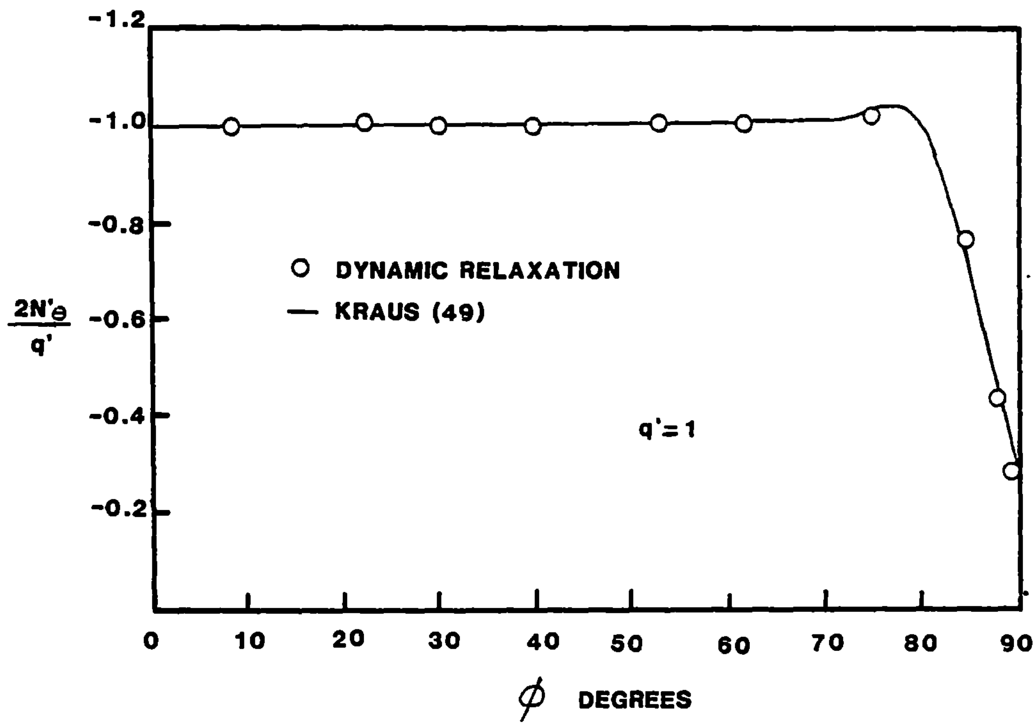
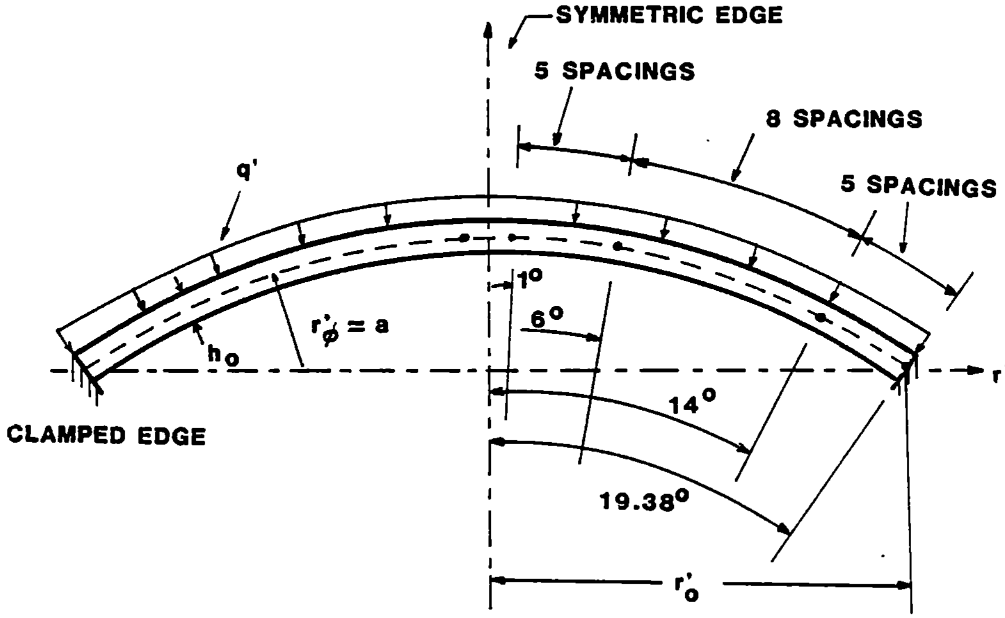


Figure 10. Circumferential Forces in a Clamped Hemispherical Shell under External Pressure



$k'_\phi = 400$; $k'_w = 750$

$m'_\phi = 1$; $m'_w = 1$

$\Delta t' = 0.8 \times 10^{-4}$

$a/h_o = 100$; $a = 1$

$E_\phi = E_\theta = 1 \times 10^7$

$\nu_\phi = \nu_\theta = 0.33$

$G = 0.385 \times 10^7$

$q' = 0.86 \times 10^5$

Figure 11. Finite Difference Grid for a Clamped Shallow (19.38°) Spherical Shell under Uniform External Pressure

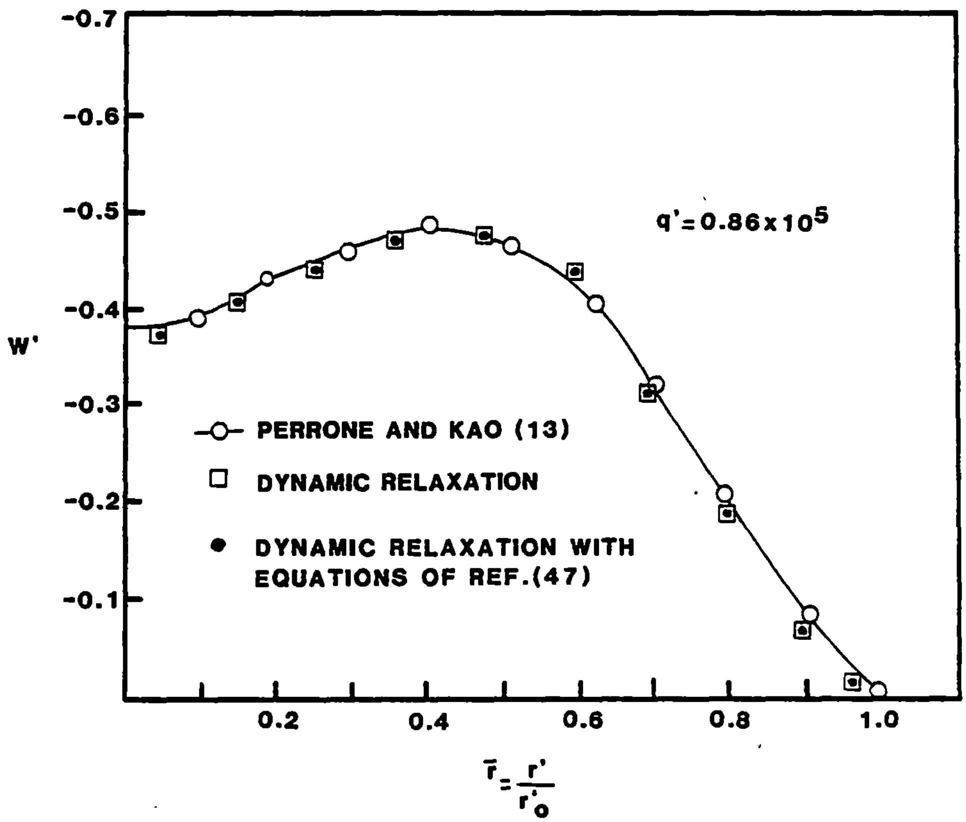


Figure 12. Comparison of the Normal Displacements in a Shallow (19.38°) Spherical Shell under uniform External Pressure

results, showing that for thin shells the equations of reference (47) are sufficiently accurate. This comparison verifies the nonlinear solution obtained by the dynamic relaxation procedure used in this study. A printout of the results for this problem is given in Appendix B.

6.2 Program NSDRGSHELL

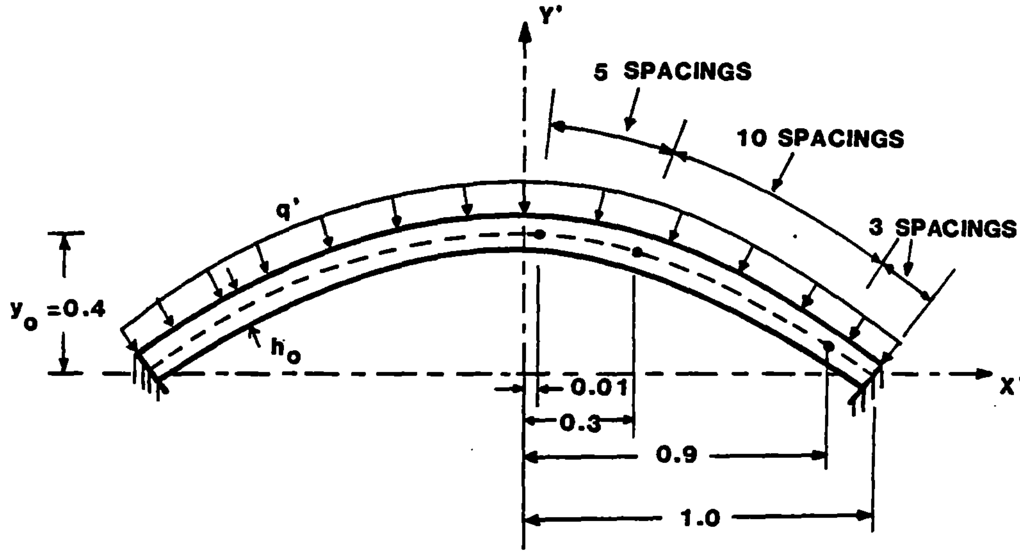
This program has been applied to the analysis of (a) parabolic shell of revolution (b) elliptic shell of revolution

6.2.1 Parabolic Shell of Revolution

A clamped parabolic shell of revolution subjected to uniform external pressure, shown in Figure 13, has been analyzed. The finite difference grid and the iteration parameters chosen are indicated in Figure 13. The equation of the meridian of the shell is given by $y' = y_o(1-x'^2)$. The stress resultants N'_ϕ and N'_θ have been plotted in Figure 14. The values of N'_ϕ and N'_θ at the apex compare very well with the analytical value at the apex which is -0.625, calculated from membrane shell theory. No comparisons were available for the values at the clamped edge and at other points on the shell. A printout of the results for the parabolic shell is given in Appendix C, after the listing of the program NSDRGSHELL.

6.2.2 Elliptic Shell of Revolution

Figure 15 shows a clamped elliptic shell of revolution



EQUATION OF MERIDIAN : $y' = y_o (1 - x'^2)$

$k'_\phi = 75$; $k'_w = 300$

$\Delta t' = 0.5 \times 10^{-3}$

$m'_\phi = m'_w = 1.0$

$a/h_o = 100$

$E_\phi = E_\theta = 2.5 \times 10^6$

$\nu_\phi = \nu_\theta = 0.25$

$G = 1 \times 10^6$

Figure 13. Finite Difference Grid for a Clamped Parabolic Shell of Revolution

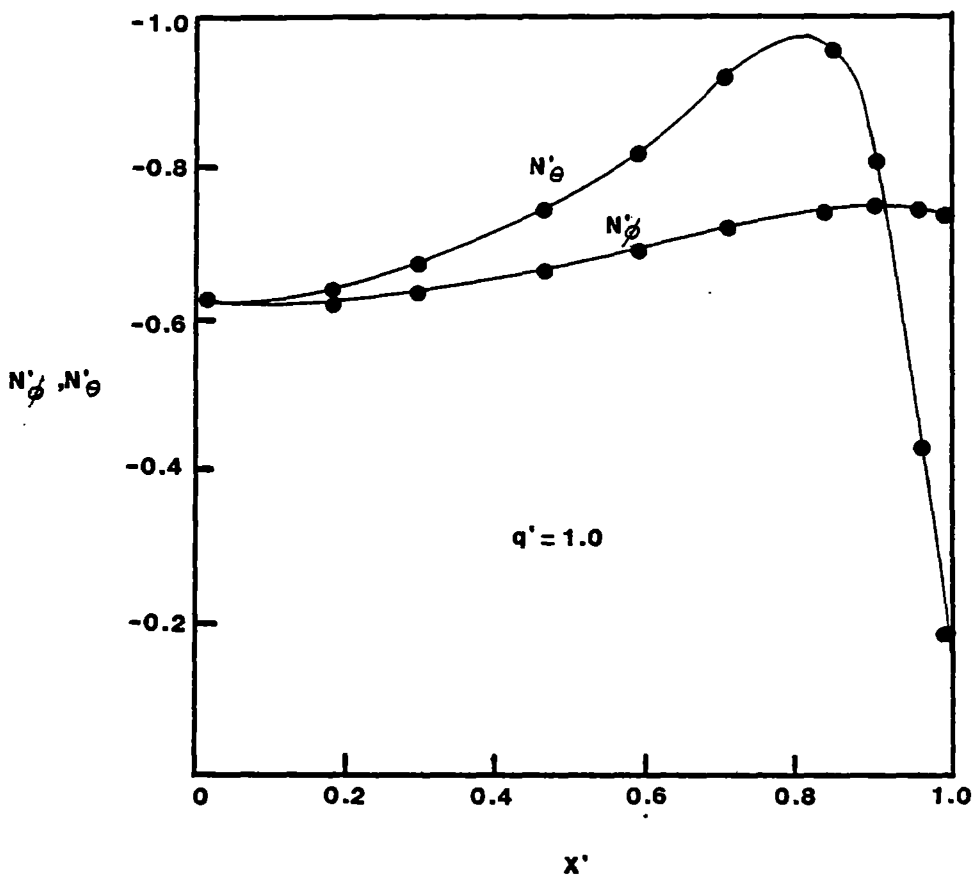
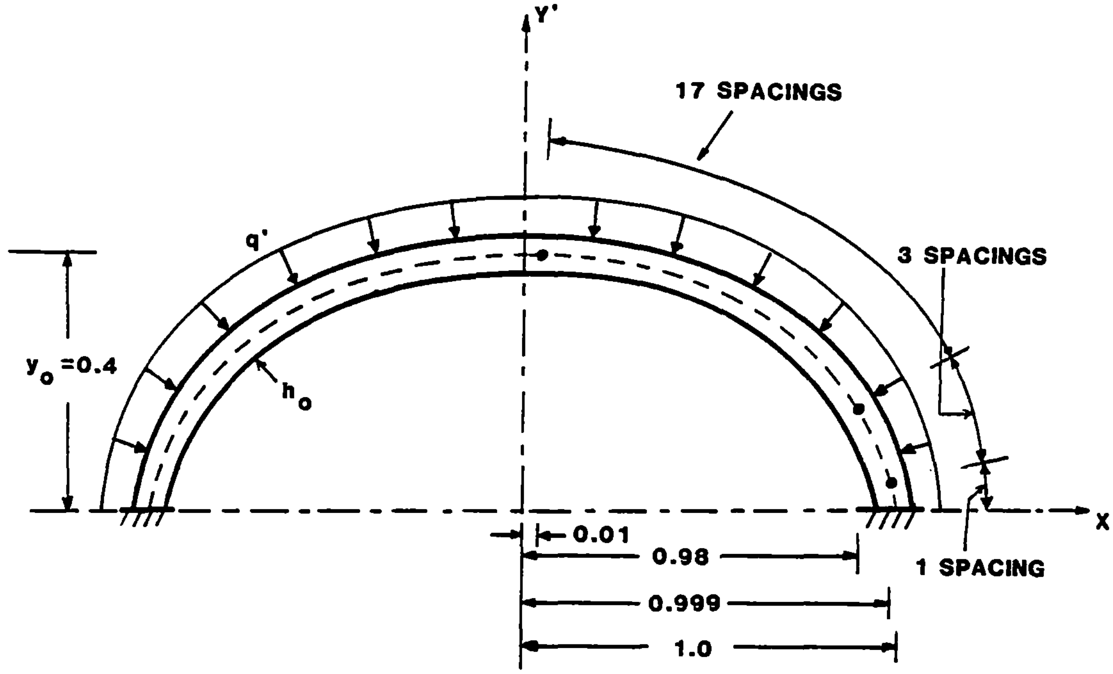


Figure 14. In Plane Stress Resultants in a Clamped Parabolic Shell of Revolution under Uniform External Pressure



EQUATION OF MERIDIAN $y' = y_o \sqrt{1 - x'^2}$

$k'_\phi = 45$, $k'_w = 200$

$\Delta t' = 0.2 \times 10^{-3}$

$m'_\phi = m'_w = 1$

$a/h_o = 100$

$E_\phi = E_\theta = 2.5 \times 10^6$

$\nu_\phi = \nu_\theta = 0.25$

$G = 1 \times 10^6$

Figure 15. Finite Difference Grid for a Clamped Elliptic Shell of Revolution under Uniform External Pressure

under external pressure. The finite difference grid and iteration parameters are also indicated in the figure. The equation of the meridian of the shell is given by $y' = y_o\sqrt{(1-x'^2)}$. The stress resultants N'_ϕ and N'_θ have been plotted in Figure 16. The analytical values of N'_ϕ and N'_θ at the apex of the shell obtained from membrane theory of shells which is -1.25, compare very well with the dynamic relaxation values. Again no comparisons were available for the values at other points on the shell. A printout of the results for the elliptic shell is given in Appendix C.

6.3 Nonlinear Static Analysis of Spherical Shells of Revolution

The main purpose of this study was to investigate the behavior of the shell of revolution when the loads are large and the displacements are of the order of the thickness of the shell. The nonlinear effects have been studied by analyzing clamped spherical shells with various half outer opening angles, subjected to uniform external or internal pressure.

Results for a deep shell (90°), semi deep shell (45°), and a shallow shell (15°) are shown in Figures 17 to 22. Figure 17 shows the variation of the transverse and meridional displacements near the apex of the 90° shell subjected to uniform external pressure. The linear solution obtained by the same procedure is also shown in the figure. In Figure 18 the displacements near the apex of the 90°

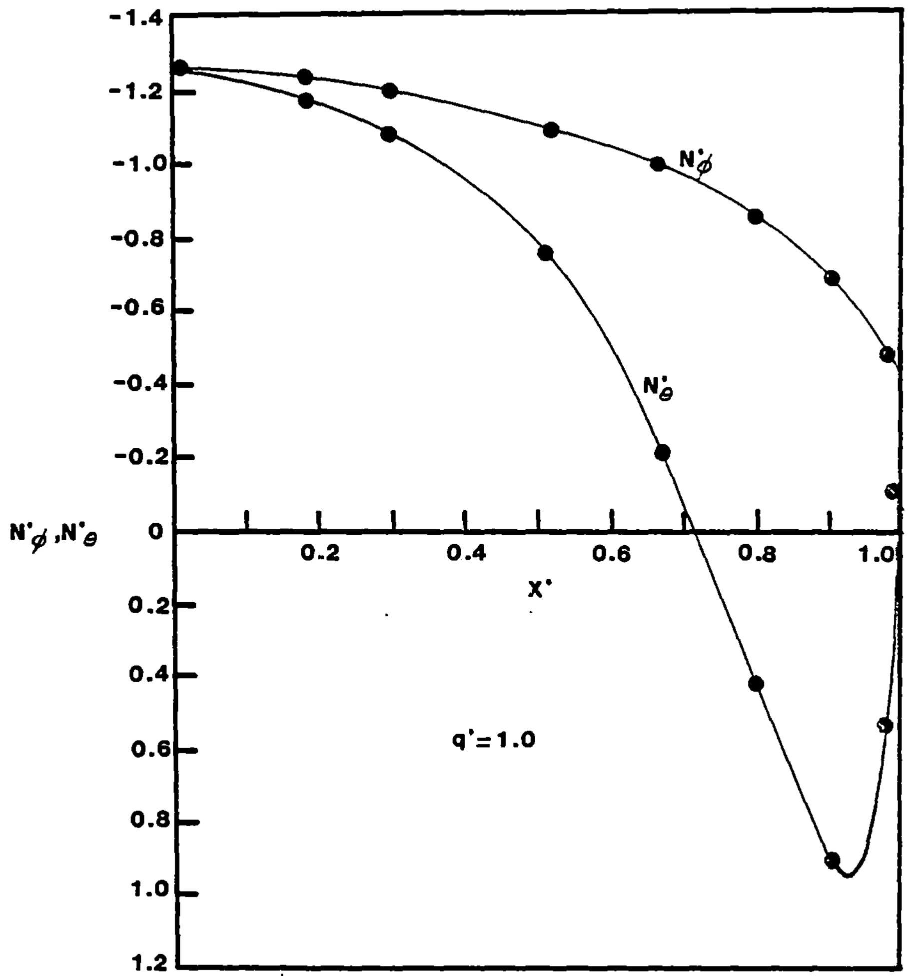


Figure 16. In Plane Stress Resultants in a Clamped Elliptic Shell of Revolution under Uniform External Pressure

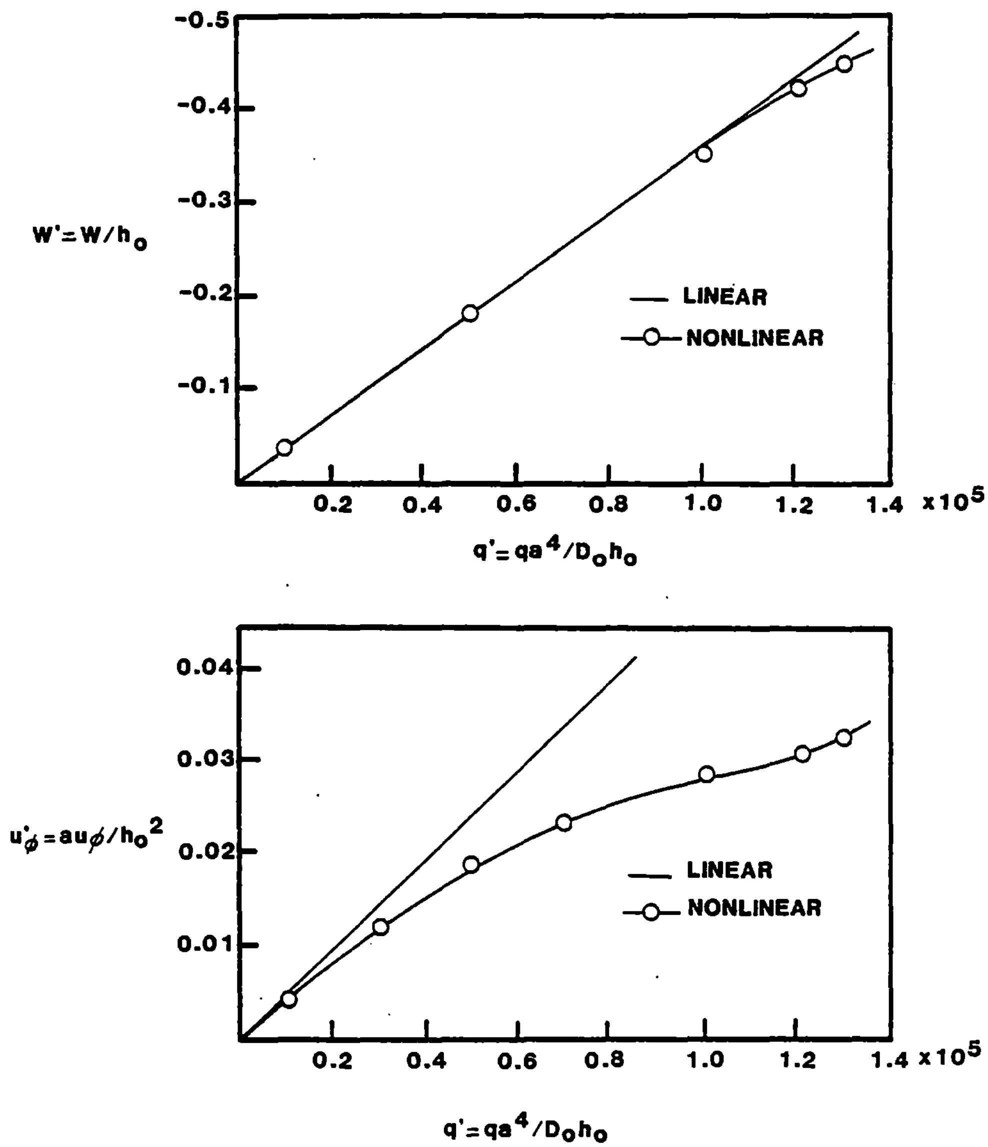


Figure 17. Normal and Meridional Displacements Near the Apex of a Deep Spherical Shell (ϕ=90°) under External Pressure

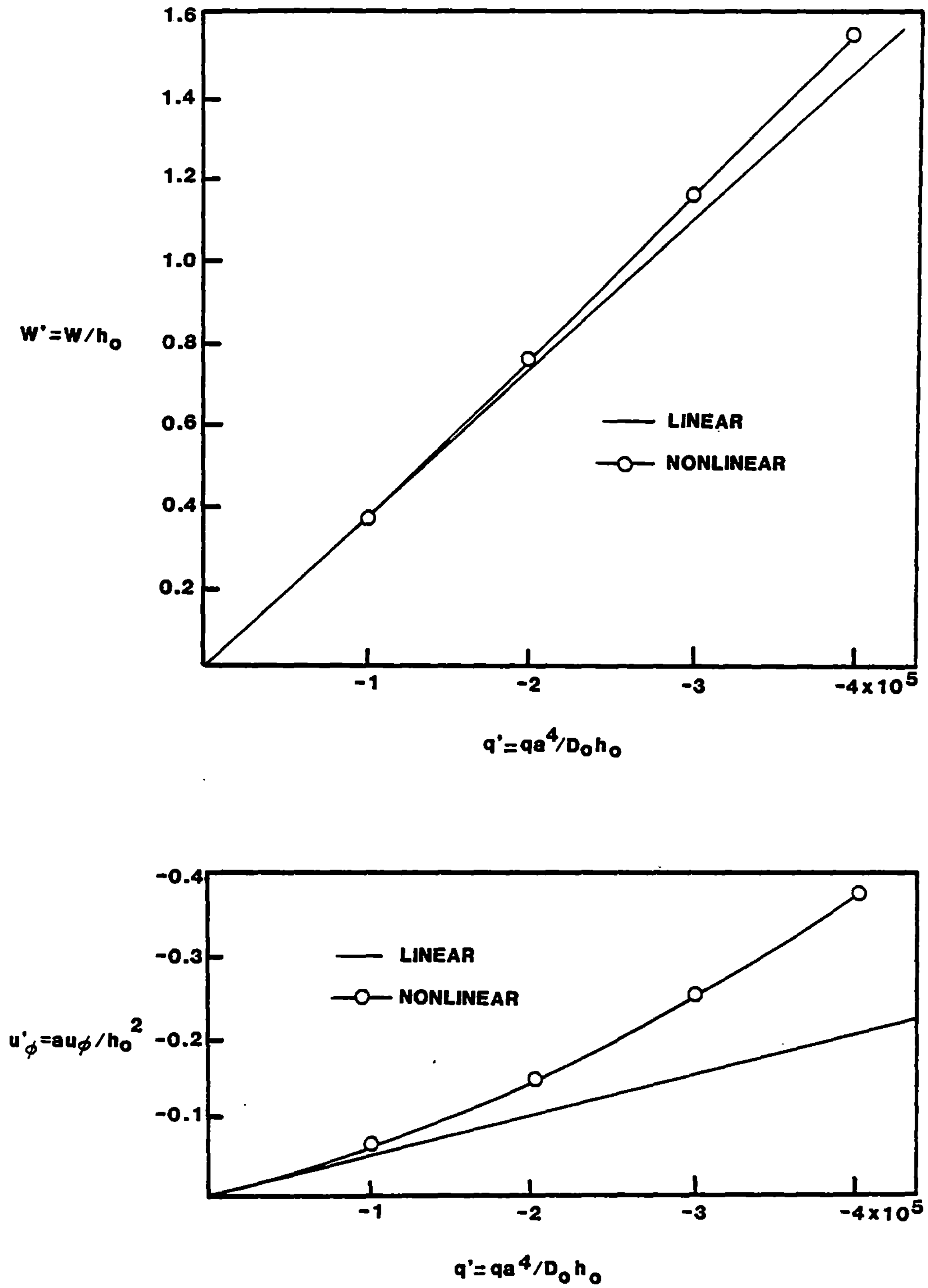


Figure 18. Normal and Meridional Displacements Near the Apex of a Deep Spherical Shell (ϕ=90°) under Internal Pressure

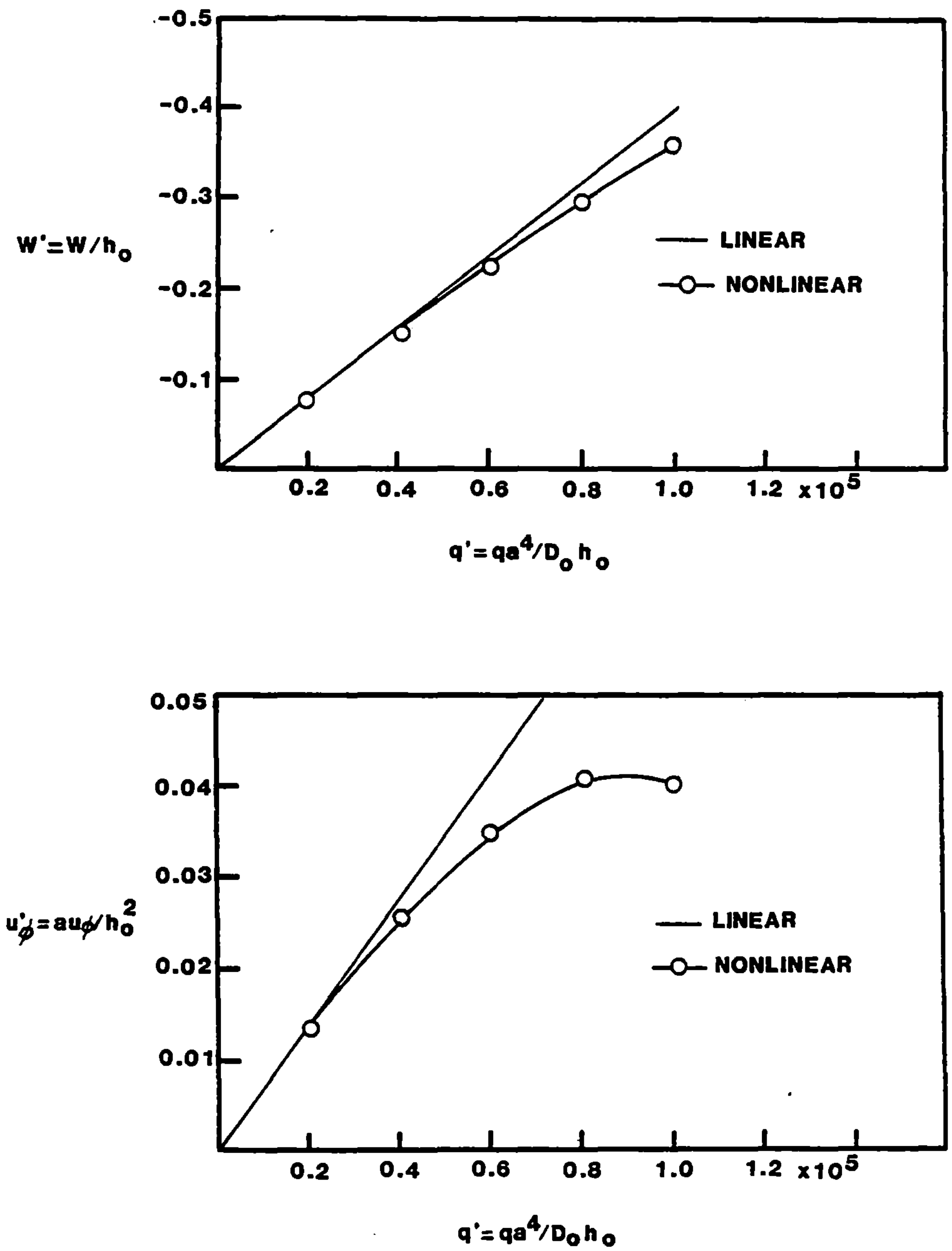
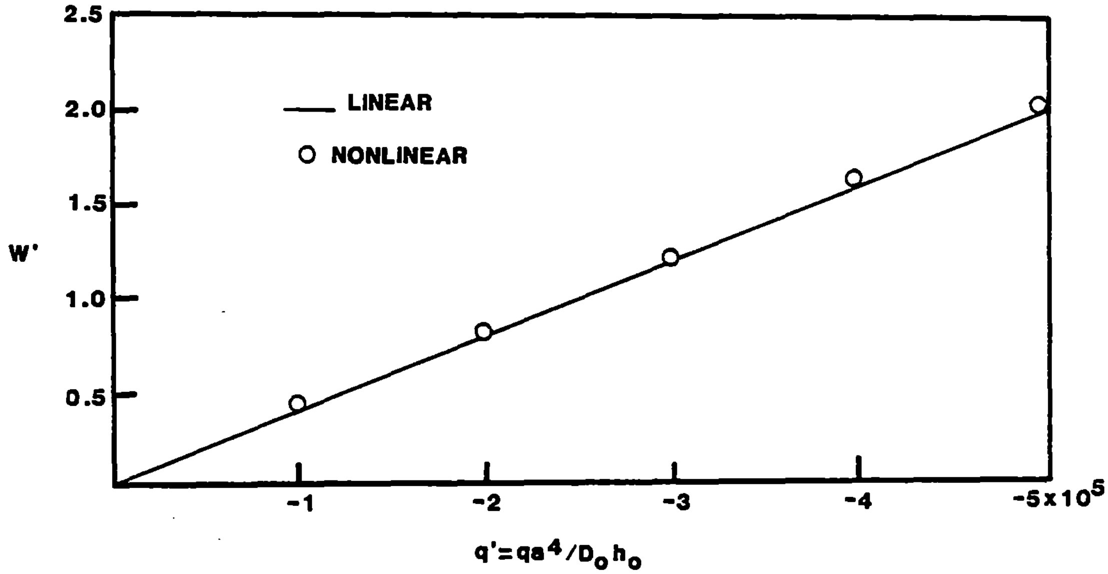


Figure 19. Normal and Meridional Displacements Near the Apex of a Semideep Spherical Shell (ϕ=45°) under External Pressure



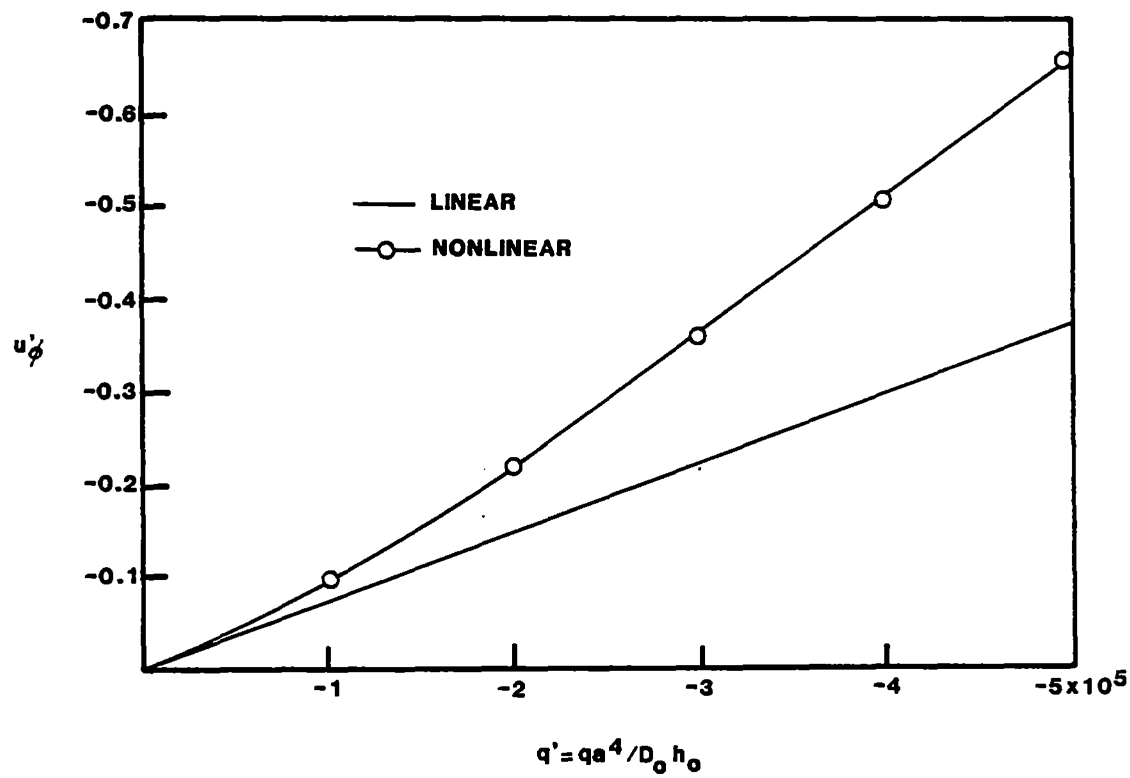


Figure 20. Normal and Meridional Displacements Near the Apex of a Semideep Spherical Shell (∅=45°) under Internal Pressure

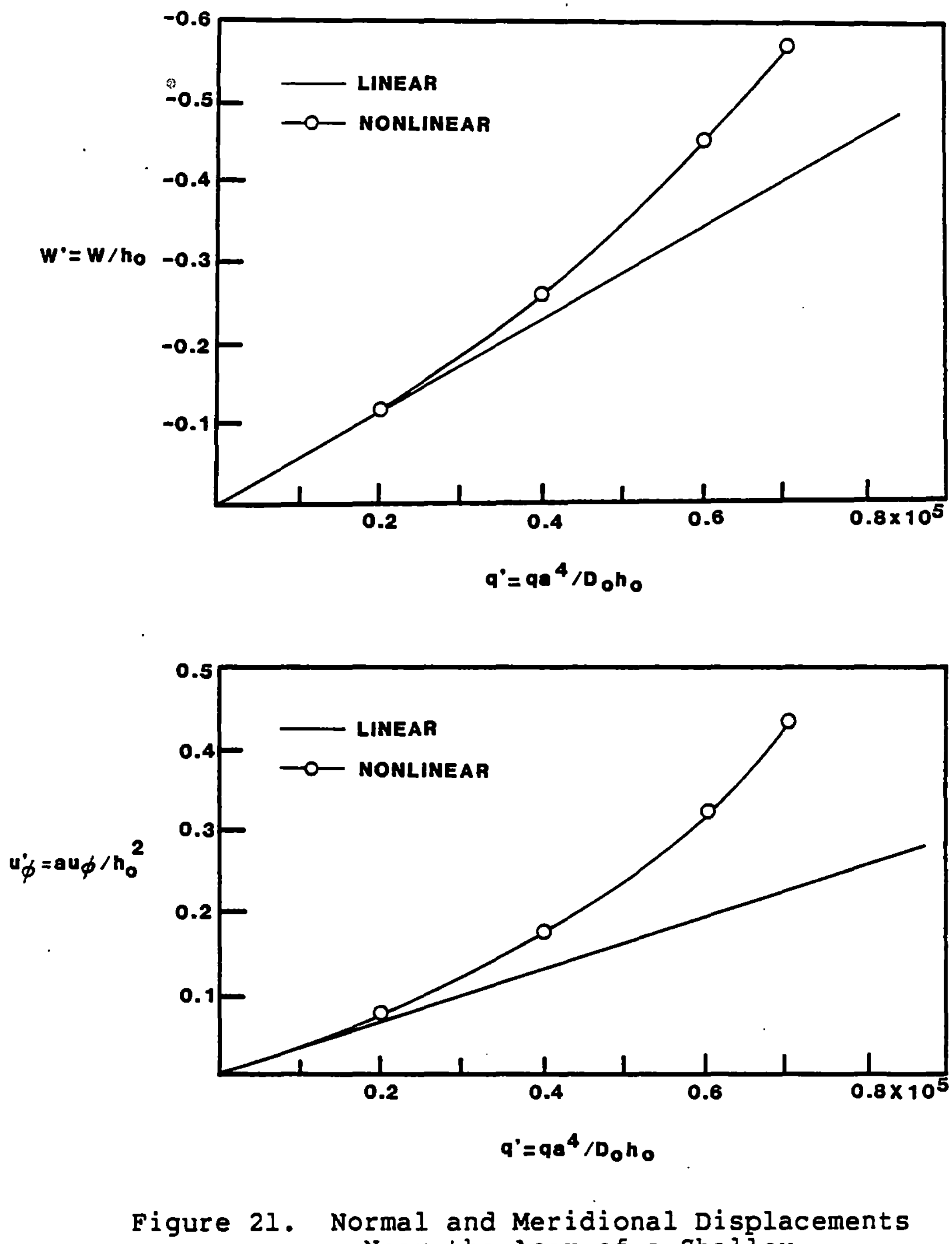


Figure 21. Normal and Meridional Displacements Near the Apex of a Shallow Spherical Shell (ϕ=15°) under External Pressure

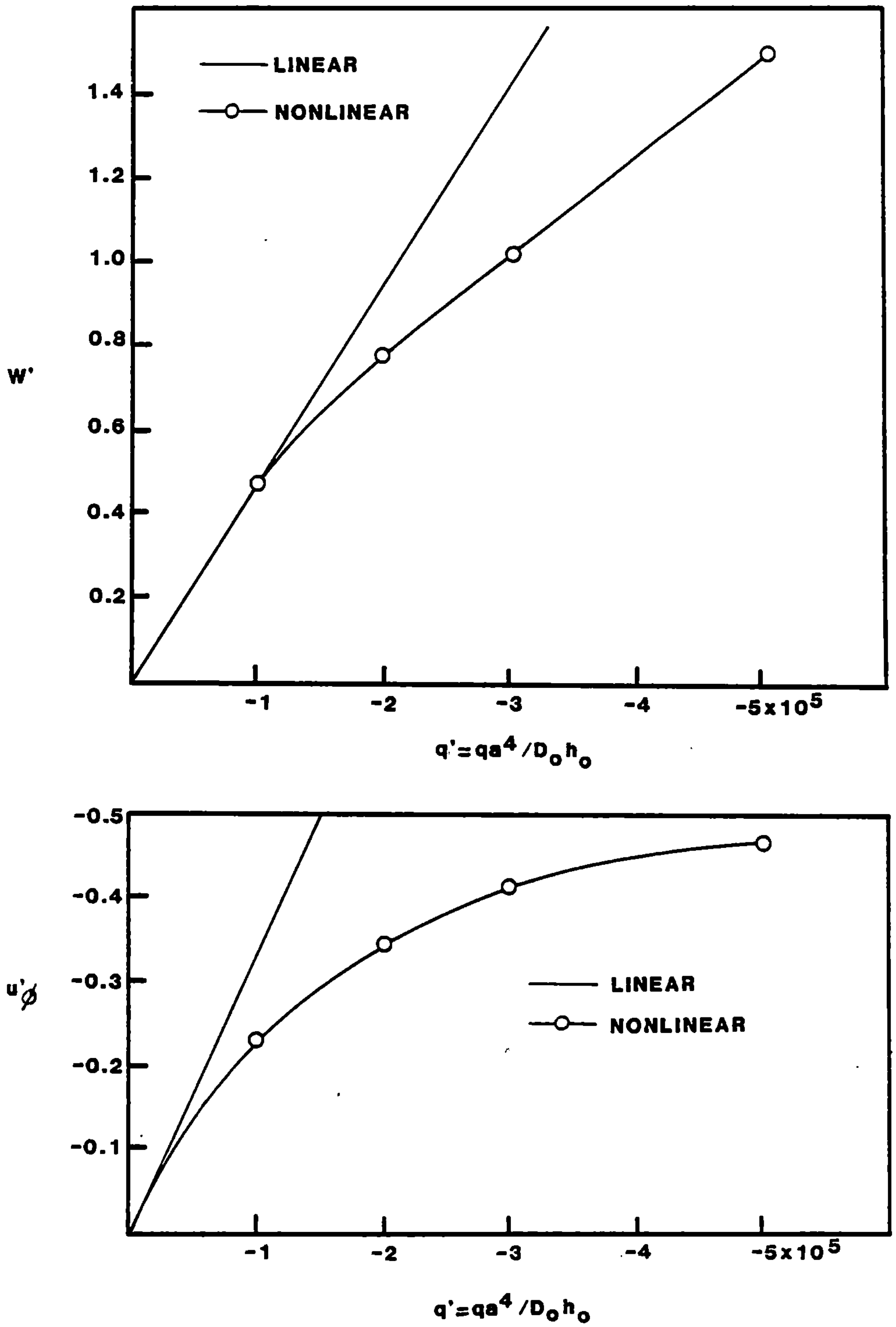


Figure 22. Normal and Meridional Displacements Near the Apex of a Shallow Spherical Shell (Ø=15°) under Internal Pressure

shell, for internal presssure, are shown. The difference in the variation of the normal and meridional displacement with load, for external and internal pressure can be observed from Figures 17 and 18. The transverse displacements are almost linear throughout the load range for the deep shell.

Similar results are shown for a semi deep (45°) shell in Figures 19 and 20. The meridional displacements exhibit greater non linearity than the transverse displacements for both the 90° and 45° shell.

Figures 21 and 22 give the results for a 15° shallow shell. The meridional and transverse displacements of the shallow shell exhibit greater nonlinearity than the corresponding displacements of the deep and semi deep shells. In a shallow shell the decrease in stiffness with increase in external pressure and increase in stiffness with increase in internal pressure can be observed. This effect has been brought out by the nonlinear analysis.

CHAPTER VII

SUMMARY AND CONCLUSIONS

7.1 Summary

The dynamic relaxation method has been applied to the geometrically nonlinear static analysis of some shells of revolution. The governing nonlinear equations for the geometrically nonlinear static analysis of shells of revolution have been formulated first. The nonlinear equilibrium equations for the shell of revolution have been derived using more general strain displacement relations than those used by previous authors. The principle of minimum potential energy has been used to develop the nonlinear equilibrium equations. The equations presented in this study are new and not given in any other previous work. These equations have been written in nondimensional form and then converted to finite difference equations using nonuniform finite difference expressions for the spatial variation and a uniform finite difference grid for the variation in time. By a stepwise integration procedure the displacements and stress resultants in the shell of revolution, have been calculated.

Computer programs for the nonlinear static analysis of spherical as well as shells of general meridional shape have been developed. The programs have been verified by application and comparison with shell problems whose solutions are available in literature. Spherical shells with half opening angles of 90°, 45°, and 15° subjected to axysymmetric internal and external pressure have been analyzed. The nonlinear displacements and stress resultants have been evaluated for a wide range of loading.

7.2 Conclusions

Dynamic Relaxation is a suitable alternative procedure for the nonlinear static analysis of shells. The displacements can be obtained in a single load step, without the need to solve large sets of simultaneous equations at each load step, as in some of the other well known procedures. By this procedure a complex system of equations has been solved with ease and the related programming has been simple. The solution obtained by this procedure compares well with other nonlinear solutions. The nonlinear solution obtained by the equations presented in this study do not differ significantly from the solution obtained by the nonlinear equations of reference (47), as only thin shells have been considered in this study.

Shallow spherical shells exhibit a greater degree of non linearity of displacements and stress resultants in comparison with deep spherical shells.

The behavior of a shallow spherical shell of revolution under inward and outward uniform load is different as the load becomes large. For a shallow spherical shell there is a decrease in stiffness with increase in external pressure and an increase in stiffness with increase in internal pressure. This is valid for the load range considered in this study.

7.3 Suggestions for Further Work

In this study, the finite difference discretization of the shell has been employed. Lynch, Kelsey, and Saxe (53) have used the finite element discretization of the spatial variables, instead of finite differences, for the dynamic relaxation procedure and have demonstrated this approach by application to plane stress problems of plates having discontinuities in the form of circular and elliptic holes. It would be worthwhile to use finite element discretization with dynamic relaxation integration procedure for the nonlinear static analysis of shells. Though this would necessitate matrix operations and storage of large coefficient matrices, the advantage of this procedure is that the maximum and minimum eigenvalues of the stiffness matrix can be calculated more accurately by an eigenvalue analysis and thereby a better estimate of the time increment and critical damping factors can be obtained. This in turn would reduce the number of iterations required for the convergence to the static solution.

The dynamic relaxation solution for the shell problems, does not converge beyond the load range considered in this study. Beyond these loads, the solutions diverge due to numerical instability or exhibit indefinite ocillations. Adjustment of the time increment and damping factors did not help in attaining convergence. The reasons for this have to be investigated and suitable methods have to be found to overcome this problem if the load-displacement behavior beyond the load range considered in this study is of interest. It is possible that the maximum loads may be in the vicinity of the critical buckling load of the shell where bifurcation or snap-through is taking place. The convergence to the equilibrium path beyond the bifurcation or snap-through point has to be investigated.

The programs developed in this study can be modified for general type of loads. A more general program which can analyze any shell of revolution with general loading and various boundary conditions, has to be developed. Automatic estimation of the iteration parameters namely the time increment and damping factors has to be incorporated in the program.

Dynamic relaxation could be applied to the combined material and geometric nonlinear analysis of shells of revolution.

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APPENDIX A

GOVERNING EQUATIONS FOR A SHELL OF REVOLUTION

In this Appendix the assumptions of the shell theory used and the governing equations for the geometrically nonlinear structural analysis of shells of revolution are presented.

Assumptions of Shell Theory

Love's first order approximation shell theory is used with the following assumptions.

1. The shell thickness is negligibly small in comparison to the radius of curvature of the middle surface.
2. Linear elements normal to the unstrained middle surface remain straight during deformation and suffer no extensions.
3. Normals to the undeformed middle surface remain normal to the deformed middle surface.
4. The components of stress normal to the middle surface are small compared to other components of stress and may be neglected in the stress-strain relationships.
5. Strains are small but displacements and rotations are large.
6. Higher order terms in curvature-displacement relation are neglected.

Nonlinear Equations of Equilibrium

The nonlinear equilibrium equations have been derived using principle of minimum potential energy and a Lagrangian coordinate system based on (47, 48).

'ϕ' direction:

$$\frac{\partial N_{\phi\theta}}{r\,\partial\theta} + \frac{\partial N_{\phi}}{r_{\phi}\partial\phi} + (N_{\phi} - N_{\theta})\frac{\cos\phi}{r} - \frac{N_{\phi}\beta_{\phi}}{r_{\phi}} - \frac{N_{\phi\theta}\beta_{\theta}}{r_{\phi}}$$

$$+ Q_{\phi}(\frac{1}{r_{\phi}} + K_{\phi}) + Q_{\theta}K_{\theta\phi} + Q_{\phi}\beta_{\phi}\frac{\cos\phi}{r} + \frac{\partial Q_{\phi}}{r_{\phi}\partial\phi}\beta_{\phi} \qquad (A.1)$$

$$+ \frac{\partial Q_{\theta}}{r\,\partial\theta}\beta_{\phi} + q_{\phi} = 0$$

'θ' direction:

$$\frac{\partial N_{\phi\theta}}{r_{\phi}\partial_{\phi}} + \frac{\partial N_{\theta}}{r\,\partial\theta} + \frac{2N_{\phi\theta}\cos\phi}{r} - \frac{N_{\theta}\beta_{\theta}}{r_{\theta}} - \frac{N_{\theta\phi}\beta_{\phi}}{r_{\theta}}$$

$$+ Q_{\theta}(\frac{1}{r_{\theta}} + K_{\theta}) + Q_{\phi}K_{\phi\theta} + \frac{Q_{\phi}\beta_{\theta}\cos\phi}{r} + \frac{\partial Q_{\theta}}{r\,\partial\theta}\beta_{\theta} \qquad (A.2)$$

$$+ \frac{\partial Q_{\phi}}{r_{\phi}\partial\phi}\beta_{\theta} + q_{\theta} = 0$$

'z' direction:

$$\frac{Q_{\phi}\cos\phi}{r} + \frac{\partial Q_{\phi}}{r_{\phi}\partial\phi} + \frac{\partial Q_{\theta}}{r\,\partial\theta} - \frac{Q_{\phi}\beta_{\phi}}{r_{\phi}} - \frac{Q_{\theta}\beta_{\theta}}{r_{\theta}} - N_{\phi}(\frac{1}{r_{\phi}} + K_{\phi})$$

$$- \frac{N_{\theta}}{r_{\theta}} - \frac{N_{\theta}\partial\beta_{\theta}}{r\,\partial\theta} - N_{\phi}\beta_{\phi}\frac{\cos\phi}{r} - N_{\theta\phi}\tau - \frac{\partial N_{\phi}}{r_{\phi}\partial\phi}\beta_{\phi} \qquad (A.3)$$

$$- \frac{\partial N_{\theta\phi}}{r_{\phi}\partial\phi}\beta_{\theta} - \frac{\partial N_{\theta}}{r\,\partial\theta}\beta_{\theta} - \frac{\partial N_{\theta\phi}}{r\partial\theta}\beta_{\phi} - q = 0$$

Moment about ϕ axis:

$$\frac{\partial M_{\theta}}{r\,\partial\theta} + \frac{\partial M_{\phi\theta}}{r_{\phi}\partial\phi} + \frac{2M_{\phi\theta}\cos\phi}{r} - Q_{\theta} = 0 \qquad (A.4)$$

Moment about θ axis:

$$\frac{\partial M_{\theta\phi}}{r\partial\theta} + \frac{\partial M_{\phi}}{r_{\phi}\partial\phi} + (M_{\phi} - M_{\theta})\frac{\cos_{\phi}}{r} - Q_{\phi} = 0 \tag{A.5}$$

where $\beta_{\phi} = (\frac{u_{\phi}}{r_{\phi}} - \frac{\partial w}{r_{\phi}\partial_{\phi}})$

$$\beta_{\theta} = (\frac{u_{\theta}}{r_{\theta}} - \frac{\partial w}{r\partial_{\theta}})$$

$$K_{\phi} = \frac{1}{r_{\phi}}\frac{\partial}{\partial\phi}(\frac{u_{\phi}}{r_{\phi}} - \frac{\partial w}{r_{\phi}\partial_{\phi}})$$

$$K_{\theta} = \frac{1\partial}{r\partial\theta}(\frac{u_{\theta}}{r_{\theta}} - \frac{\partial w}{r\partial\theta}) + \frac{\cos\phi}{r}(\frac{u_{\phi}}{r_{\phi}} - \frac{\partial w}{r_{\phi}\partial\phi})$$

$$K_{\phi\theta} = \frac{1}{r_{\phi}}\frac{\partial}{\partial\phi}(\frac{u_{\theta}}{r_{\theta}} - \frac{\partial w}{r\partial\theta})$$

$$K_{\theta\phi} = \frac{1\partial}{r\partial\theta}(\frac{u_{\phi}}{r_{\phi}} - \frac{\partial w}{r_{\phi}\partial\phi}) - \frac{\cos\phi}{r}(\frac{u_{\theta}}{r_{\theta}} - \frac{\partial w}{r\partial\theta})$$

$$\tau = K_{\phi\theta} + K_{\theta\phi}$$

Substituting in Equations (A.1), (A.2), and (A.3) for Q_{θ} and Q_{ϕ} obtained from (A.4) and (A.5), the above equations can be reduced to three equilibrium equations in the ϕ, θ, and z directions respectively.

Stress Resultant Strain Equations

The linear stress resultant-strain equations are (present study is concerned only with geometrical nonlinearity)

$$N_{\phi} = \frac{E_{\phi}h}{(1 - \nu_{\phi\theta}\nu_{\theta\phi})}(\epsilon_{\phi}^{\circ} + \nu_{\phi\theta}\epsilon_{\theta}^{\circ})$$

$$N_\theta = \frac{E_\theta h}{(1 - \nu_{\phi\theta}\nu_{\theta\phi})}(\epsilon_\theta^\circ + \nu_{\theta\phi}\epsilon_\phi^\circ)$$

$$N_{\phi\theta} = N_{\theta\phi} = Gh\,\gamma_{\phi\theta}^\circ \qquad \text{(A.6)}$$

$$M_\phi = \frac{E_\phi h^3}{12(1 - \nu_{\phi\theta}\nu_{\theta\phi})}(\kappa_\phi + \nu_{\phi\theta}\kappa_\theta)$$

$$M_\theta = \frac{E_\theta h^3}{12(1 - \nu_{\phi\theta}\nu_{\theta\phi})}(\kappa_\theta + \nu_{\theta\phi}\kappa_\phi)$$

$$M_{\phi\theta} = M_{\theta\phi} = Gh^3\tau/12$$

Strain-Displacement Relations

The nonlinear strain-displacement relations are,

$$\epsilon_\phi^\circ = \left(\frac{\partial u_\phi}{r_\phi\partial\phi} + \frac{w}{r_\phi}\right) + \frac{1}{2}\left\{\left(\frac{\partial u_\phi}{r_\phi\partial\phi} + \frac{w}{r_\phi}\right)^2 + \left(\frac{\partial u_\theta}{r_\phi\partial\phi}\right)^2 + \left(\frac{u_\phi}{r_\phi} - \frac{\partial w}{r_\phi\partial\phi}\right)^2\right\}$$

$$\epsilon_\theta^\circ = \left(\frac{\partial u_\theta}{r\partial\theta} + \frac{u_\phi\cos\phi}{r} + \frac{w\sin\phi}{r}\right) + \frac{1}{2}\left\{\left(\frac{\partial u_\theta}{r\partial\theta} + \frac{u_\phi\cos\phi}{r} + \frac{w\sin\phi}{r}\right)^2 + \left(\frac{\partial u_\phi}{r\partial\theta} - \frac{u_\theta\cos\phi}{r}\right)^2 + \left(\frac{u_\theta}{r_\theta} - \frac{\partial w}{r\partial\theta}\right)^2\right\}$$

$$\gamma_{\phi\theta}^\circ = \left(\frac{\partial u_\theta}{r_\phi\partial\phi} + \frac{\partial u_\phi}{r\partial\theta} - \frac{u_\theta\cos\phi}{r}\right) + \left(\frac{\partial u_\phi}{r_\phi\partial\phi} + \frac{w}{r_\phi}\right)\left(\frac{\partial u_\phi}{r\partial\theta} - \frac{u_\theta\cos\phi}{r}\right) + \left(\frac{\partial u_\theta}{r\partial\theta} + \frac{u_\phi\cos\phi}{r} + \frac{w\sin\phi}{r}\right)\left(\frac{\partial u_\theta}{r_\phi\partial\phi}\right) + \left(\frac{u_\phi}{r_\phi} - \frac{\partial w}{r_\phi\partial\phi}\right)\left(\frac{u_\theta}{r_\theta} - \frac{\partial w}{r\partial\theta}\right)$$

(A.7)

$$\kappa_\phi = \frac{1}{r_\phi}\frac{\partial}{\partial\phi}\left(\frac{u_\phi}{r_\phi} - \frac{\partial w}{r_\phi\partial\phi}\right)$$

$$\kappa_\theta = \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{u_\theta}{r_\theta} - \frac{\partial w}{r\partial\theta}\right) + \frac{\cos\phi}{r}\left(\frac{u_\phi}{r_\phi} - \frac{\partial w}{r_\phi\partial\phi}\right)$$

$$\tau = \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{u_\phi}{r_\phi} - \frac{\partial w}{r_\phi \partial\phi}\right) + \frac{1\,\partial}{r_\phi\partial\phi}\left(\frac{u_\theta}{r_\theta} - \frac{\partial w}{r\,\partial\theta}\right)$$
$$- \frac{\cos\phi}{r}\left(\frac{u_\theta}{r_\theta} - \frac{\partial w}{r\,\partial\theta}\right)$$

Equations (A.6) and (A.7) can be combined to obtain the nonlinear stress resultant-displacement equations.

Non Dimensional Relations

The nonlinear equilibrium equations and the nonlinear stress resultant displacement equations can be converted to non dimensional form using the following relations. The non dimensional variables are denoted by primes.

$$h = h_o h'; \; r_\phi = a r'_\phi \; ; \; r_\theta = a r'_\theta \; ; \; r = a r'$$
$$w = h_o w'; \quad u_\phi = \frac{h_o^2 u'_\phi}{a}; \quad u_\theta = \frac{h_o^2 u'_\theta}{a}$$
$$N_\phi = \frac{D_o h_o N'_\phi}{a^3}; \quad N_\theta = \frac{D_o h_o N'_\theta}{a^3}; \quad N_{\phi\theta} = \frac{D_o h_o N'_{\phi\theta}}{a^3}$$
$$M_\phi = \frac{D_o h_o M'_\phi}{a^2}; \quad M_\theta = \frac{D_o h_o M'_\theta}{a^2}; \quad M_{\phi\theta} = \frac{D_o h_o M'_{\phi\theta}}{a^2} \qquad \text{(A.8)}$$
$$q_\phi = \frac{D_o h_o q'_\phi}{a^4}; \quad q_\theta = \frac{D_o h_o q'_\theta}{a^4}; \quad q = \frac{D_o h_o q'}{a^4}$$
$$t = t_o t'$$

where $t_o = a^2\sqrt{\dfrac{m_W}{D_o}}$ and $D_o = \dfrac{E_\theta h_o^3}{12(1 - \nu_{\phi\theta}\nu_{\theta\phi})}$

APPENDIX B

PROGRAM NSDRSHELL - LISTING AND SELECTED OUTPUT

```
C
C
C
C
C *    PROGRAM FOR THE NON LINEAR STATIC ANALYSIS OF SHELLS OF
C *    REVOLUTION BY DYNAMIC RELAXATION - SPHERICAL SHELL
C *    PROGRAM 'NSDRSHELL'
C *    __ AXYSYMMETRIC LOADING
C *    __ NON LINEAR/LINEAR STRAIN-DISPLACEMENT RELATIONS
C *    __ NON LINEAR/LINEAR EQUILIBRIUM EQUATIONS
C *    __ WITH/WITHOUT CENTRAL OPENING
C *    __ LINEAR ISOTROPIC/ORTHOTROPIC STRESS-STRAIN RELATIONS
C *
C
C
        IMPLICIT REAL *8 (A-H,O-Z)
        REAL *4 CHAR (17)
        DIMENSION  ENTH(50),ENPH(50),EMTH(50),EMPH(50),
     &             APH(50),AW(50),UPH(50),W(50),VPH(50),VW(50),
     &             RTH(50),RPH(50),R(50),
     &             VPHS(50),VWS(50),PUPH(50),PW(50), ENG(50),
     &             QPH(50),QW(50),
     &             H(50),APHI(50), APHD(50),
     &             STPH(50),STTH(50),
     &             PEMPH(50), PEMTH(50),PENPH(50),PENTH(50)
C
         SIN (PHI) = DSIN(PHI)
         COS (PHI) = DCOS(PHI)
C
C*    INPUT DATA CARDS
C
         READ, EPH, ETH, PRPH, PRTH, G
         READ, ISHELL,IOSHEL
         READ, ICASE, ISTCHK,NDFCHK
         READ, DPH, DW
         READ, DFPH, DFW, DELT
         READ, IMAX, ITIN, DELTM, DMAX
         READ, NDPHI,NDPHM, NDPHO,NPHI,IP
         READ, PHII, PHIA, PHIB,PHIO,HAO
         READ, IBCI, IBCO
         READ,FLDW, FLDPH
         READ (5,500) (CHAR(I),I=1,16)
C
C*     INPUT DATA FORMAT
C
 500     FORMAT (16 A4)
C
C
C*     WRITE PROBLEM TITLE AND INPUT DATA
C
          WRITE (6,600)
          WRITE (6,605) (CHAR(I), I=1,16)
          WRITE (6,610) EPH, ETH, PRPH, PRTH, G
          WRITE (6,620) DPH, DW
          WRITE (6,630) DFPH,DFW,DELT
          WRITE (6,640) DELTM, DMAX, IMAX, ITIN
          WRITE (6,650) NDPHI,NDPHM, NDPHO,NPHI
          WRITE (6,660) ISHELL,IOSHEL , ICASE, IBCI, IBCO,ISTCHK,NDFCHK
          WRITE (6,645) PHII,PHIA, PHIB,PHIO,HAO
          WRITE (6,655) FLDW, FLDPH
C
C*      OUTPUT FORMAT
C
 600     FORMAT ('1', 60X, 'RESULTS'//25X, 'NON LINEAR STATIC',
     &' ANALYSIS OF SHELLS OF REVOLUTION BY DYNAMIC RELAXATION'//)
 605     FORMAT (5X, 'SHELL TYPE:', 5X, 4A4//16X,  A4, 1X, 'LINEAR',
```

```
     &        ' EQUILIBRIUM AND STRAIN-DISPLACEMENT'//16X, 5A4,1X,
     &      'STRESS-STRAIN'//16X, 'INNER EDGE',3X,3A4//16X, 'OUTER EDGE',
     &    3X,3A4//)
 610       FORMAT(5X,'MATERIAL CONSTANTS'//10X,'EPH=',E15.5//10X ,
     &'ETH=',E15.5 //10X,'PRPH=',F10.5//10X,'PRTH=',F10.5// 10X,'G=',
     & E15.5//)
 620       FORMAT (5X,'DENSITIES'//10X,'DPH=',F10.5//10X,'DW=',F10.5//)
 630       FORMAT(5X,'ITERATION FACTORS'//10X,'DFPH=',E15.5//10X,
     & 'DFW=', E15.5//10X,'DELT=',E15.5//)
 640       FORMAT(5X,'ITERATION CONTROL PARAMETERS'//10X,'DELTMAX=',
     & E15.5,10X,'DMAX=',E15.5//10X,'IMAX=',I6,10X,'ITIN=',I5//)
 650       FORMAT(5X,'FINITE DIFFERENCE MESH GENERATION AND LOCATION',
     &' PARAMETERS ' //10X,'NDPHI=',I3,10X,'NDPHM=',I3//10X,'NDPHO=',
     &I3,10X,'NPHI=',I3//)
 660       FORMAT(5X,'OTHER CONTROL PARAMETERS'//10X,'ISHELL=',I3,
     & 10X,'IOSHEL =',I3//10X,'ICASE=',I3//10X,'IBCI=',I3,10X,
     & 'IBCO=',I3// 10X,'ISTCHK=',I3//10X,'NDFCHK=',I3//)
 645       FORMAT(5X,'SHELL GEOMETRY'//10X,'PHII=',F10.5//10X,
     &  'PHIA=',F10.5//10X,'PHIB=',F10.5//10X,'PHIO=',F10.5//10X,
     &'HAO=',F10.5//)
 655       FORMAT(5X,'LOAD DATA'//10X,'FLDW=',E15.5//10X,'FLDPH=',
     &    E15.5//)
C
C*    FINITE DIFFERENCE MESH PARAMETERS
C
           PHII=PHII*22./(7.*180.)
           PHIA=PHIA*22./(7.*180.)
           PHIB=PHIB*22./(7.*180.)
           PHIO=PHIO*22./(7.*180.)
           XPHI=(PHIA-PHII)/NDPHI
           XPHM=(PHIB-PHIA)/NDPHM
           XPHO=(PHIO-PHIB)/NDPHO
           NPHA=NPHI+NDPHI
           NPHB=NPHA+NDPHM
           NPHO=NPHB+NDPHO
           NPHI1 = NPHI - 1
           NPHI3 = NPHI + 1
           NPHO1 = NPHO - 1
           NPHO3 = NPHO + 1
           IF (IBCI .GT. 2) NA = NPHI
           IF (IBCI .LE. 2) NA = NPHI + 1
           IF (IBCO .GT. 2) NB = NPHO
           IF (IBCO .LE. 2) NB = NPHO - 1
C
C*     GENERATE R, RTH, RPH, APHI: SHELL GEOMETRY
C
           HOA = 1./HAO
           DO 400 I = NPHI1, NPHO3
            RTH(I) = 1.
            RPH(I) = 1.
             H(I) = 1.
 400       CONTINUE
           DO 410 I = NPHI1, NPHO
           IF (I.LT.NPHI) GO TO 1030
           IF (I.EQ.NPHI) GO TO 1040
           IF (I.LE.NPHA) GO TO 1050
           IF (I.LE.NPHB) GO TO 1060
           IF (I.GT.NPHB) GO TO 1070
 1030      IF (IOSHEL.EQ.1) GO TO 1035
            APHI(I) = 0.
            GO TO 1080
 1035       APHI(I)=(PHII-XPHI)
            GO TO 1080
 1040       APHI(I)=PHI1
            GO TO 1080
 1050       APHI(I)=APHI(I-1)+XPHI
```

```
      GO TO 1080
 1060 APHI(I)=APHI(I-1)+XPHM
      GO TO 1080
 1070 APHI(I)=APHI(I-1)+XPHO
 1080 R(I)=RTH(I)*DSIN(APHI(I))
 410  CONTINUE
      DO 415 I = NPHI, NPHO
       APHD(I) =APHI(I) * 180.*7./22.
 415  CONTINUE
C
C*    PRESCRIBE LOADING
C
      DO 420 I = NPHI, NPHO
       QPH(I) = FLDPH
       QW(I) =  FLDW
 420  CONTINUE
C
C*    PRESCRIBE ACCURACY FOR RESIDUAL CHECK
C
      FACT = 1000.
      ACCW  = DABS(FLDW)/FACT
      ACCPH = ACCW
C
C*    CALCULATE CONSTANTS
C
 900  B1 =DELT/DPH
      B2 = 1. +(.5 *DFPH* B1)
      B3 = 1.-(.5 * DFPH* B1)
      C1 = DELT/DW
      C2 = 1.+(.5 * DFW * C1)
      C3 = 1.-(.5 * DFW * C1)
C
C*   STORE INITIAL VALUES
C
      DO 430 I = NPHI1, NPHO3
       ENTH(I) = 0.
       ENPH(I) = 0.
       EMTH(I) = 0.
       EMPH(I) = 0.
        UPH(I) = 0.
         W(I) = 0.
        VPH(I) = 0.
         VW(I) = 0.
       VPHS(I) = 0.
        VWS(I) = 0.
        ENG(I) = 0.
 430  CONTINUE
C
C*   SET ITERATION COUNTER EQUAL TO ZERO
C
      ITER = 0
C
C*   CALCULATE VELOCITIES AND DISPLACEMENTS
C
      DO 800 J = 1, IMAX
          EW = 0.
      DO 440 I = NA, NB
        PHI = APHI(I)
        IF =I+1
        IB =I-1
        DELS=RPH(I)*(APHI(I)-APHI(I-1))
        IF(I.EQ.NPHO) GO TO 75
        DELS1=RPH(I)*(APHI(I+1)-APHI(I))
        IF ((I.EQ.NPHI).AND.(IOSHEL.NE.1))DELS=DELS*2.
        IF((I.EQ.NPHI).AND.(IOSHEL.EQ.1))DELS=DELS1
        GO TO 110
```

```
 75        IF(IBCO.EQ.4) GO TO 80
           DELS1=DELS
           GO TO 110
 80        DELS1=RPH(I)*(11./7.-PHIO)*2.
C
C*    CALCULATION OF VELOCITIES AND DISPLACEMENTS IN THE TRANSVERSE
C*      DIRECTION
C
 110      ALPHA = DELS1/ DELS
          ALPHA1 = 1. - ALPHA* ALPHA
          ALPHA2 = 1. + ALPHA
          ALPHA3 = 2./(ALPHA* ALPHA2)
          ALPHA4 =ALPHA* ALPHA2
          Z1 = (EMPH(IF)- ALPHA2* EMPH(I) + ALPHA* EMPH(IB))/(DELS
     &    * DELS) * ALPHA3
          Z2 = (EMPH(I) - EMTH(I)) *DSIN(PHI)/( R(I)* RPH(I))
          Z3 = 2. * DCOS(PHI)/R(I)*(EMPH(IF)- ALPHA1*EMPH(I)- ALPHA*
     &    ALPHA * EMPH(IB))/(ALPHA4* DELS)
          Z4 = DCOS(PHI)/R(I) * (EMTH(IF)-ALPHA1* EMTH(I)- ALPHA*
     &    ALPHA* EMTH(IB))/(ALPHA4* DELS)
          Z41 =1./RPH(I)*(RPH(IF)-ALPHA1*RPH(I)-ALPHA*ALPHA*RPH(IB))/
     &    (ALPHA4*DELS)*(EMPH(IF)-ALPHA1*EMPH(I)-ALPHA*ALPHA*EMPH(IB)
     &    )/(ALPHA4*DELS)
           Y41 = 1./RPH(I)
           Y42 = HOA* HOA* 1./RPH(I)* (UPH(IF)- UPH(I)*ALPHA1- ALPHA
     &     * ALPHA* UPH(IB))/(ALPHA4* DELS)
           Y43 = HOA* (W(IF)-ALPHA2*'W(I)+ ALPHA*W(IB))/(DELS*DELS)
     &     * ALPHA3
           Y44 = 1./RPH(I) * (RPH(IF)- ALPHA1*RPH(I)-ALPHA*ALPHA*
     &     RPH(IB))/(ALPHA4*DELS ) * (HOA*(W(IF)-ALPHA1*W(I)-ALPHA*
     &     ALPHA*W(IB))/(ALPHA4*DELS) -HOA*HOA* UPH(I)/RPH(I))
           Y4 = Y41 + Y42 - Y43 +Y44
           IF (ICASE .EQ. 1) Y4 = Y41
           Z5 =  ENPH(I)*Y4
           Z6 = ENTH(I)/RTH(I)
           Y1 = (ENPH(IF)-ENPH(I)*ALPHA1- ALPHA*ALPHA*ENPH(IB))
     &     /(ALPHA4*DELS)
           Y2 = (ENPH(I)- ENTH(I))* DCOS(PHI)/R(I)
           Y31 = EMPH(I)* DCOS(PHI)/R(I)
           Y32 = (EMPH(IF)-EMPH(I)*ALPHA1-ALPHA*ALPHA*EMPH(IB))/
     &     (ALPHA4* DELS)
           Y33 = EMTH(I)* DCOS(PHI)/R(I)
           Y3 = Y31 + Y32 -Y33
           Z7=Y3/RPH(I)+Y1+ENPH(I)*DCOS(PHI)/R(I)
           BETA =HOA*HOA*UPH(I)/RPH(I) -HOA*(W(IF)-W(I)*ALPHA1
     &     -ALPHA*ALPHA*W(IB))/(ALPHA4*DELS)
           Y6=(Z1-Z2+Z3-Z4-Z41-ENPH(I)/RPH(I))
           IF (ICASE .EQ. 1) BETA= O.
           AW(I) =Z1-Z2 +Z3- Z4-Z41- Z5 - Z6 -Z7*BETA -QW(I)
           PW(I) = W(I)
           VW(I) = 1./C2 * (C3 * VW(I) + C1* AW(I))
           IF(J.EQ.1) VW(I)=DELT/(2.*DW)*AW(I)
           W(I) = W(I) + VW(I) * DELT
           IF (NDFCHK .NE. 1) GO TO 115
           VWS(I)= VW(I) * VW(I)
 115       IF(DABS(W(I))-DMAX) 120, 130, 130
C
C*    CALCULATION OF VELOCITIES AND DISPLACEMENTS IN THE MERIDIONAL
C*       DIRECTION
C
 120       APH(I) =Y1+ Y2 +Y3 * Y4 +Y6*BETA+ QPH(I)
           PUPH(I) = UPH(I)
           VPH(I) = 1./B2 * (B3 * VPH(I) + B1 * APH(I))
           IF(J.EQ.1) VPH(I)=DELT/(2.*DPH)*APH(I)
           UPH(I) = UPH(I) + VPH(I) * DELT
           IF (NDFCHK .NE. 1)GO TO 125
```

```
          VPHS(I) = VPH(I) * VPH(I)
          ENG(I) = VWS(I) + VPHS(I)
          EW = EW + ENG(I)
  125     IF (DABS (UPH(I))- DMAX) 440, 130, 130
C
C*   CHECK IF DISPLACEMENTS DIVERGE.IF SO,PRINT THE DISPLACEMENTS,
C*   REDUCE THE TIME INCREMENT AND RESTART THE D.R. INTEGERATION.
C
  130     WRITE (6, 665) J, DELT
          WRITE (6, 670) I, UPH(I), W(I), VPH(I), VW(I), APH(I), AW(I)
          WRITE (6,735)
          WRITE (6, 680) (K, UPH(K), W(K), ENPH(K), ENTH(K), EMPH(K),
     &     EMTH(K),APHD(K), K= NPHI, NPHO)
          DELT = DELT* .5
          IF (DELT.LE. DELTM) GO TO 1000
          GO TO 900
  440     CONTINUE
C
C
  665     FORMAT (5X, 'NUMERICAL INSTABILITY AT ITERATION NO. =',
     &    I5, 5X, 'WITH DELT=', E15.5//)
  670     FORMAT (5X, I3, 5X, 6E15.5 //)
  680     FORMAT ((5X, I3, 5X, 7E15.5)/)
C
C*   DISPLACEMENT BOUNDARY CONDITIONS
C*    BOUNDARY CONDITIONS ON INNER EDGE
C
         GO TO (135, 140,145 ,160), IBCI
C*    FIXED.BOUNDARY CONDITIONS
  135    W(NPHI -1) = W(NPHI + 1)
         UPH(NPHI -1) = -UPH(NPHI + 1)
         GO TO 175
C*    HINGED BOUNDARY CONDITIONS
  140    UPH(NPHI-1) = -UPH(NPHI+1)
         DELPH = XPHI
         PO = (RPH(NPHI+1)-RPH(NPHI-1))/(4.*RPH(NPHI)*RPH(NPHI)
     &   * RPH(NPHI)* DELPH * DELPH) - PRPH*DCOS(APHI(NPHI))*1./(2.
     &   * R(NPHI)*RPH(NPHI) * DELPH)
         P1 = PO - 1./(RPH(NPHI) * RPH(NPHI) * DELPH * DELPH)
         P2 = HOA * 1./(2. * RPH(NPHI) * RPH(NPHI) * DELPH)
         P3 = PO + 1./(RPH(NPHI) * RPH(NPHI) * DELPH * DELPH)
         W(NPHI -1) = P2/P3 * (UPH(NPHI +1) - UPH(NPHI -1)) + P1/P3
     &   * W(NPHI +1)
         GO TO 175
C*    FREE BOUNDARY CONDITIONS
  145    GO TO 175
C*    SYMMETRY BOUNDARY CONDITIONS
  160    W(NPHI-1) = W(NPHI)
         UPH(NPHI-1) = -UPH(NPHI)
         GO TO 175
C*   BOUNDARY CONDITIONS ON OUTER  EDGE
  175    GO TO (180, 190,195,196),IBCO
C*    FIXED BOUNDARY CONDITIONS
  180    W(NPHO+1) = W(NPHO -1)
         UPH(NPHO +1) = -UPH(NPHO-1)
         GO TO 200
C*    HINGED BOUNDARY CONDITIONS
  190    UPH(NPHO+1) = -UPH(NPHO-1)
         DELPH = XPHO
         PO = (RPH(NPHO+1) - RPH(NPHO-1))/(4.* RPH(NPHO)*RPH(NPHO)
     &   * RPH(NPHO) * DELPH* DELPH)- PRPH*DCOS(APHI(NPHO))*1./(2.*
     &   R(NPHO)*RPH(NPHO) * DELPH)
         P1 = PO - 1./(RPH(NPHO)*RPH(NPHO)* DELPH* DELPH)
         P2 = HOA * 1./(RPH(NPHO) * RPH(NPHO) * DELPH)
         P3 = PO + 1./(RPH(NPHO) * RPH(NPHO) * DELPH * DELPH)
         W(NPHO+1) = P3/P1 * W(NPHO-1)-P2/P1 * (UPH(NPHO+1)-UPH(NPHO
```

```
     &     -1))
           GO TO 200
C*    FREE BOUNDARY CONDITIONS
 195       GO TO 200
C*    SYMMETRY BOUNDARY CONDITIONS
 196        W(NPHO+1) = W(NPHO)
            UPH(NPHO+1) = -UPH(NPHO)
C
C*  CALCULATION OF STRESS RESULTANTS
C
 200       DO 450 I = NPHI, NPHO
              PHI = APHI(I)
              IF = I+1
              IB = I-1
              DELS=RPH(I)*(APHI(I)-APHI(I-1))
              IF(I.EQ.NPHO) GO TO 210
              DELS1=RPH(I)*(APHI(I+1)-APHI(I))
              IF((I.EQ.NPHI).AND.(IOSHEL.NE.1))DELS=DELS*2.
              IF((I.EQ.NPHI).AND.(IOSHEL.EQ.1))DELS=DELS1
              GO TO 250
 210          IF(IBCO.EQ.4)GO TO 220
              DELS1=DELS
              GO TO 250
 220          DELS1=RPH(I)*(11./7.-PHIO) *2.
 250          ALPHA = DELS1/DELS
              ALPHA1 =1.- ALPHA* ALPHA
              ALPHA2 = 1. + ALPHA
              ALPHA3 = 2./(ALPHA* ALPHA2)
              ALPHA4 = ALPHA * ALPHA2
              F1 = (UPH(IF)-ALPHA1* UPH(I)-ALPHA*ALPHA* UPH(IB))/
     &        (ALPHA4*DELS)
              F2 = W(I)/RPH(I)
              F22 = UPH(I)/RPH(I)
              F3 = (W(IF)- ALPHA1*W(I)-ALPHA*ALPHA*W(IB))/(ALPHA4*DELS)
              F4 = W(I)*DSIN(PHI)/R(I)
              F5 = UPH(I)*DCOS(PHI)/R(I)
              F6 = (W(IF)-ALPHA2* W(I)+ALPHA* W(IB))/(DELS*DELS)*ALPHA3
              F7 = 1./RPH(I)* (RPH(IF)-ALPHA1* RPH(I)-ALPHA*ALPHA*RPH(IB)
     &        )/(ALPHA4*DELS)
              IF ((IBCI .LE. 2) .AND.(I .EQ. NPHI))F1=(-11./6.*UPH(NPHI)
     &        +3.*UPH(NPHI+1)-1.5*UPH(NPHI+2)+.33*UPH(NPHI+3))/DELS
              IF ((IBCO .LE.2) .AND.(I .EQ. NPHO)) F1=(11./6.*UPH(NPHO)
     &         -3.*UPH(NPHO-1)+1.5*UPH(NPHO-2)-.33*UPH(NPHO-3))/DELS
              F21 = HOA * F1+ F2
              F31 = F3- HOA* F22
              F41 = HOA * F5 + F4
              IF (ICASE .NE. 1) GO TO 260
              F21 = 0.
              F31 = 0.
              F41 = 0.
 260          STPH(I) = HAO* F1 + HAO*HAO*F2+.5*HAO*F21*F21+.5*
     &        HAO*F31*F31
              STTH(I) = HAO*F5 + HAO*HAO*F4 + .5*HAO*F41*F41
              PENPH(I) = ENPH(I)
              PENTH(I) = ENTH(I)
              ENPH(I) = 12.*H(I)* EPH/ETH* (STPH(I)+ PRPH*STTH(I) )
              ENTH(I) = 12. * H(I) * (STTH(I) + PRTH* STPH(I))
              CPH = HOA* F1/RPH(I) -F6 + F7* (F3 -HOA*F22)
              CTH =DCOS(PHI)* (HOA* F22/R(I) -F3/R(I) )
              PEMPH(I) = EMPH(I)
              PEMTH(I) = EMTH(I)
              EMPH(I) = EPH/ETH * (H(I)**3)* (CPH + PRPH* CTH)
              EMTH(I) = (H(I)* * 3) * (CTH + PRTH*CPH)
 450         CONTINUE
C
C*    STRESS BOUNDARY CONDITIONS
```

```
C      BOUNDARY CONDITIONS ON INNER EDGE
            GO TO (290,290,275,280),IBCI
C*      FREE BOUNDARY CONDITIONS
 275        GO TO 290
C*      SYMMETRY BOUNDARY CONDITIONS
 280          ENPH(NPHI-1) = ENPH(NPHI)
              ENTH(NPHI-1) = ENTH(NPHI)
              EMPH(NPHI-1) = EMPH(NPHI)
              EMTH(NPHI-1) = EMTH(NPHI)
C*     BOUNDARY CONDITION ON OUTER EDGE
 290          GO TO (300,300,295,296),IBCO
C*      FREE  BOUNDARY CONDITIONS
 295          GO TO 300
C      SYMMETRY BOUNDARY CONDITIONS
 296          ENPH(NPHO+1) = ENPH(NPHO)
              ENTH(NPHO+1) = ENTH(NPHO)
              EMPH(NPHO+1) = EMPH(NPHO)
              EMTH(NPHO+1) = EMTH(NPHO)
C
C*     CHECK FOR CONVERGENCE
C        CHECK RESIDUAL OF EQUILIBRIUM EQUATIONS
 300          DO 460 I = NA ,NB
                 IF (DABS(APH(I)).GE. ACCPH) GO TO 310
                 IF (DABS(AW(I)) .GE. ACCW) GO TO 310
 460          CONTINUE
C*       CHECK CONVERGENCE OF DISPLACEMENTS
              DO 470 I = NA ,NB
                DIFUPH = (UPH(I) - PUPH(I))/PUPH(I)
                DIFW = (W(I) - PW(I))/PW(I)
                IF (DABS(DIFUPH) .GT. .001) GO TO 310
                IF (DABS(DIFW) .GT. .001) GO TO 310
 470          CONTINUE
C*       CHECK CONVERGENCE OF STRESS RESULTANTS
                IF (ISTCHK .NE. 1) GO TO 350
                DO 475 I = NPHI, NPHO
                DIENPH = (ENPH(I)-PENPH(I))/PENPH(I)
                DIENTH = (ENTH(I)-PENTH(I))/PENTH(I)
                DIEMPH = (EMPH(I)-PEMPH(I))/PEMPH(I)
                DIEMTH = (EMTH(I)- PEMTH(I))/PEMTH(I)
                IF(DABS(DIENPH) .GT. .001) GO TO 310
                IF (DABS(DIENTH) .GT. .001) GO TO 310
                IF (DABS(DIEMPH) .GT. .001) GO TO 310
                IF (DABS(DIEMTH) .GT. .001) GO TO 310
 475            CONTINUE
C
                GO TO 350
 310          IF (J- ITER) 800, 330, 330
 330          IF (NDFCHK .NE. 1) GO TO 340
              IF (J .EQ. 1) WRITE (6,705)
              WRITE (6,700) J
              WRITE (6, 710) IP,UPH(IP),W(IP), VPHS(IP),VWS(IP),ENG(IP),EW
              GO TO 345
 340          IF (J .EQ. 1) WRITE (6,715)
              WRITE (6,700) J
              WRITE (6,720) IP,UPH(IP),W(IP),ENPH(IP),ENTH(IP),EMPH(IP),
     &        EMTH(IP), APH(IP), AW(IP)
 345          ITER = ITER + ITIN
 800         CONTINUE
             WRITE (6,725) J, DELT
             GO TO 355
 350          WRITE (6,730) J, DELT
 355          WRITE (6,735)
              WRITE (6,740) (IY,UPH(IY),W(IY),ENPH(IY),ENTH(IY),EMPH(IY),
     &        EMTH(IY),APHD(IY),IY= NPHI, NPHO)
              WRITE (6,750)
              WRITE (6,745) (IY,APH(IY),AW(IY),IY=NPHI,NPHO1)
```

```
        WRITE (6,755)
700     FORMAT ( //5X,'ITERATION NO.=', I5/)
705     FORMAT (7X,'I',12X,'UPH',14X,'W',12X,'VPHS',11X,'VWS',11X,
   &    'ENGI',11X,'EW'//)
710     FORMAT (5X,I3, 5X, 6E15.5//)
715     FORMAT (4X,'I',10X,'UPH',14X,'W',12X,'ENPH',11X,'ENTH',11X,
   &    'EMPH',11X,'EMTH',11X,'APH',13X,'AW'//)
720     FORMAT (2X,I3,2X,8E15.5/)
725     FORMAT (//5X,'CONVERGENCE NOT REACHED AT ITERATION NO.=',
   &    I5,5X,'DELT=',E15.5//)
730     FORMAT (//5X,'CONVERGENCE REACHED AT ITERATION NO.=',
   &    I5,5X,'DELT=',E15.5//)
735     FORMAT (7X,'I',12X,'UPH',14X,'W',12X,'ENPH',11X,'ENTH',
   &    11X,'EMPH',11X,'EMTH',11X,'APHD'//)
740     FORMAT ((5X,I3,5X, 7E15.5)/)
745     FORMAT ((5X,I3,5X,2E15.5)/)
750     FORMAT (//5X,'RESIDUALS'//20X,'APH',13X,'AW'//)
755     FORMAT (1H1)
1000    STOP
        END
```

RESULTS

NON LINEAR STATIC ANALYSIS OF SHELLS OF REVOLUTION BY DYNAMIC RELAXATION

SHELL TYPE: HEMISPHERICAL

NON LINEAR EQUILIBRIUM AND STRAIN-DISPLACEMENT

LINEAR ISOTROPIC STRESS-STRAIN

INNER EDGE SYMMETRICAL

OUTER EDGE FIXED

MATERIAL CONSTANTS

EPH= 0.25000D 07

ETH= 0.25000D 07

PRPH= 0.25000

PRTH= 0.25000

G= 0.10000D 07

DENSITIES

DPH= 1.00000

DW= 1.00000

ITERATION FACTORS

DFPH= 0.75000D 02

DFW= 0.30000D 03

DELT= 0.20000D-03

ITERATION CONTROL PARAMETERS

DELTMAX= 0.20000D-03 DMAX= 0.10000D 04

IMAX= 1500 ITIN= 50

FINITE DIFFERENCE MESH GENERATION AND LOCATION PARAMETERS

NDPHI= 7 NDPHM= 12

NDPHO= 4 NPHI= 2

OTHER CONTROL PARAMETERS

```
     ISHELL=  1          IOSHEL =  2
     ICASE=  2
     IBCI=  4          IBCO=  1
     ISTCHK=  0
     NDFCHK=  0

SHELL GEOMETRY
     PHII=    1.00000
     PHIA=  30.00000
     PHIB=  84.00000
     PHIO=  90.00000
     HAO= 100.00000

LOAD DATA
     FLDW=    0.10000D 01
     FLDPH=    0.00000D 00
```

I	UPH	W	ENPH	ENTH	EMPH	EMTH	APH	AW
ITERATION NO.= 1								
2	0.00000D 00	-0.20000D-07	-0.30000D-02	-0.30000D-02	0.14927D-20	-0.28250D-20	0.00000D 00	-0.10000D 01
ITERATION NO.= 50								
2	0.21539D-12	-0.31177D-05	-0.46766D 00	-0.46766D 00	0.54256D-10	0.54232D-10	0.42001D-08	-0.42141D-01
ITERATION NO.= 100								
2	0.52900D-12	-0.34481D-05	-0.51722D 00	-0.51722D 00	0.14702D-10	0.14662D-10	0.34745D-06	0.39474D-01
ITERATION NO.= 150								
2	0.40877D-09	-0.33713D-05	-0.50566D 00	-0.50566D 00	0.12723D-07	0.12713D-07	0.74086D-04	0.11401D-01

```
ITERATION NO.=  200
2   0.92206D-08  -0.33402D-05  -0.50021D 00  -0.50023D 00   0.18516D-06   0.18504D-06   0.51616D-03   0.31173D-03

ITERATION NO.=  250
2   0.56441D-07  -0.33585D-05  -0.49889D 00  -0.49892D 00   0.74534D-06   0.74507D-06   0.13807D-02  -0.20406D-02

ITERATION NO.=  300
2   0.17671D-06  -0.34240D-05  -0.49840D 00  -0.49841D 00   0.12183D-05   0.12181D-05   0.16963D-02  -0.30765D-02

ITERATION NO.=  350
2   0.34154D-06  -0.35203D-05  -0.49869D 00  -0.49870D 00   0.77273D-06   0.77256D-06   0.63345D-03  -0.26038D-02

ITERATION NO.=  400
2   0.45397D-06  -0.35916D-05  -0.49975D 00  -0.49974D 00  -0.29354D-06  -0.29347D-06  -0.63224D-03  -0.58136D-03

ITERATION NO.=  450
2   0.46904D-06  -0.36039D-05  -0.50029D 00  -0.50029D 00  -0.42072D-06  -0.42063D-06  -0.47371D-03   0.52086D-03

ITERATION NO.=  500
2   0.44794D-06  -0.35903D-05  -0.50006D 00  -0.50006D 00   0.41202D-07   0.41180D-07   0.18609D-03   0.18155D-03

ITERATION NO.=  550
2   0.45061D-06  -0.35901D-05  -0.49979D 00  -0.49980D 00   0.36253D-06   0.36247D-06   0.31536D-03  -0.43141D-03

ITERATION NO.=  600
2   0.46864D-06  -0.36013D-05  -0.49993D 00  -0.49993D 00  -0.55757D-08  -0.55744D-08  -0.73768D-04  -0.14240D-03

ITERATION NO.=  650
2   0.47473D-06  -0.36051D-05  -0.49999D 00  -0.49999D 00   0.18080D-07   0.18077D-07   0.12169D-05  -0.27315D-04
```

ITERATION NO.= 700

2	0.47761D-06	-0.36067D-05	-0.49997D 00	-0.49998D 00	0.76800D-07	0.76787D-07	0.27643D-04	-0.65533D-04

CONVERGENCE REACHED AT ITERATION NO.= 706 DELT= 0.20000D-03

I	UPH	W	ENPH	ENTH	EMPH	EMTH	APHD
2	0.47801D-06	-0.36069D-05	-0.49997D 00	-0.49998D 00	0.72074D-07	0.72062D-07	0.10000D 01
3	0.24586D-05	-0.36061D-05	-0.50001D 00	-0.49997D 00	0.50656D-07	0.59684D-07	0.51429D 01
4	0.44214D-05	-0.36040D-05	-0.49989D 00	-0.50002D 00	-0.73330D-07	-0.10325D-08	0.92857D 01
5	0.63893D-05	-0.36001D-05	-0.50011D 00	-0.49995D 00	0.82547D-07	0.33596D-07	0.13429D 02
6	0.82656D-05	-0.35953D-05	-0.49979D 00	-0.50006D 00	-0.16790D-06	-0.42848D-07	0.17571D 02
7	0.10208D-04	-0.35881D-05	-0.50023D 00	-0.49990D 00	0.20850D-06	0.56549D-07	0.21714D 02
8	0.11919D-04	-0.35808D-05	-0.49967D 00	-0.50010D 00	-0.22565D-06	-0.56158D-07	0.25857D 02
9	0.13832D-04	-0.35709D-05	-0.50036D 00	-0.49988D 00	0.35487D-06	0.98959D-07	0.30000D 02
10	0.15413D-04	-0.35609D-05	-0.49943D 00	-0.50024D 00	-0.49607D-06	-0.12658D-06	0.34500D 02
11	0.17497D-04	-0.35464D-05	-0.50056D 00	-0.49981D 00	0.59301D-06	0.15452D-06	0.39000D 02
12	0.18676D-04	-0.35343D-05	-0.49928D 00	-0.50030D 00	-0.66464D-06	-0.16811D-06	0.43500D 02
13	0.20765D-04	-0.35167D-05	-0.50071D 00	-0.49979D 00	0.64835D-06	0.16475D-06	0.48000D 02
14	0.21452D-04	-0.35021D-05	-0.49915D 00	-0.50027D 00	-0.98485D-06	-0.25842D-06	0.52500D 02
15	0.23545D-04	-0.34804D-05	-0.50083D 00	-0.49956D 00	0.98013D-06	0.23989D-06	0.57000D 02
16	0.23631D-04	-0.34638D-05	-0.49901D 00	-0.50001D 00	0.53110D-06	0.15347D-06	0.61500D 02
17	0.25734D-04	-0.34496D-05	-0.50095D 00	-0.50044D 00	0.61187D-05	0.16596D-05	0.66000D 02
18	0.25342D-04	-0.34723D-05	-0.49896D 00	-0.50530D 00	0.32873D-05	0.10416D-05	0.70500D 02
19	0.28248D-04	-0.35140D-05	-0.50134D 00	-0.51216D 00	-0.25044D-04	-0.63057D-05	0.75000D 02
20	0.28418D-04	-0.33989D-05	-0.49940D 00	-0.50143D 00	-0.13410D-03	-0.34638D-04	0.79500D 02
21	0.29428D-04	-0.24593D-05	-0.50112D 00	-0.39849D 00	-0.24499D-03	-0.63328D-04	0.84000D 02
22	0.25441D-04	-0.18120D-05	-0.50064D 00	-0.32677D 00	-0.14528D-03	-0.38249D-04	0.85500D 02
23	0.19413D-04	-0.10666D-05	-0.50073D 00	-0.24405D 00	0.72079D-04	0.16695D-04	0.87000D 02
24	0.10555D-04	-0.37295D-06	-0.50060D 00	-0.16680D 00	0.46377D-03	0.11546D-03	0.88500D 02
25	0.00000D 00	0.00000D 00	-0.50131D 00	-0.12533D 00	0.10832D-02	0.27081D-03	0.90000D 02

RESIDUALS

	APH	AW
2	0.20955D-04	-0.62321D-04
3	0.10118D-03	-0.53813D-04
4	0.11247D-03	-0.36753D-04
5	0.28579D-04	-0.23610D-04
6	0.19565D-04	-0.21831D-04
7	-0.32295D-04	-0.27370D-04
8	0.84695D-05	-0.27941D-04
9	-0.69826D-04	-0.20626D-04
10	-0.88152D-04	-0.10905D-04
11	-0.45284D-05	-0.92538D-05
12	0.12799D-03	-0.10491D-05
13	0.14406D-04	0.17868D-04
14	0.78444D-04	0.12354D-04
15	0.12150D-03	0.88126D-05
16	-0.16087D-03	0.22897D-04
17	0.60829D-04	0.38247D-04
18	-0.50385D-03	0.28419D-05
19	-0.68791D-03	0.38811D-04
20	0.33153D-03	0.19511D-03
21	-0.45800D-03	-0.13034D-03
22	-0.33471D-03	-0.13049D-03
23	-0.98678D-03	-0.90874D-04
24	-0.48397D-03	-0.37148D-04

RESULTS

NON LINEAR STATIC ANALYSIS OF SHELLS OF REVOLUTION BY DYNAMIC RELAXATION

SHELL TYPE: HEMISPHERICAL

NON LINEAR EQUILIBRIUM AND STRAIN-DISPLACEMENT

LINEAR ISOTROPIC STRESS-STRAIN

INNER EDGE SYMMETRICAL

OUTER EDGE FIXED

MATERIAL CONSTANTS

EPH= 0.10000D 08

ETH= 0.10000D 08

PRPH= 0.33000

PRTH= 0.33000

G= 0.38500D 07

DENSITIES

DPH= 1.00000

DW= 1.00000

ITERATION FACTORS

DFPH= 0.40000D 03

DFW= 0.75000D 03

DELT= 0.80000D-04

ITERATION CONTROL PARAMETERS

DELTMAX= 0.80000D-04 DMAX= 0.10000D 04

IMAX= 1500 ITIN= 50

FINITE DIFFERENCE MESH GENERATION AND LOCATION PARAMETERS

NDPHI= 5 NDPHM= 8

NDPHO= 5 NPHI= 2

OTHER CONTROL PARAMETERS

ISHELL= 1 IOSHEL = 2

ICASE= 2

IBCI= 4 IBCO= 1

ISTCHK= 0

NDFCHK= 0

SHELL GEOMETRY

PHII= 1.00000

PHIA= 6.00000

PHIB= 14.00000

PHIO= 19.38000

HAO= 100.00000

LOAD DATA

FLDW= 0.86000D 05

FLDPH= 0.00000D 00

I	UPH	W	ENPH	ENTH	EMPH	EMTH	APH	AW
ITERATION NO.= 1								
2	0.00000D 00	-0.27520D-03	-0.43922D 02	-0.43922D 02	0.21731D-15	0.78313D-16	0.00000D 00	-0.86000D 05
ITERATION NO.= 50								
2	-0.25812D-02	-0.22804D 00	-0.36456D 05	-0.36545D 05	-0.12108D 02	-0.12108D 02	0.37103D 05	0.58244D 04
ITERATION NO.= 100								
2	-0.59283D-01	-0.24077D 00	-0.43471D 05	-0.43682D 05	0.92378D 01	0.92357D 01	0.15098D 05	0.25019D 05
ITERATION NO.= 150								
2	0.26556D-01	-0.22910D 00	-0.33818D 05	-0.34007D 05	0.35791D 02	0.35788D 02	-0.28702D 05	-0.11956D 05

```
 ITERATION NO.=  200
2       0.54665D-01   -0.27720D 00   -0.38805D 05   -0.39067D 05    0.35374D 02    0.36371D 02   -0.74082D 04   -0.49546D 04

 ITERATION NO.=  250
2       0.85639D-01   -0.30412D 00   -0.40227D 05   -0.40516D 05    0.33828D 02    0.33824D 02   -0.21993D 04   -0.33500D 04

 ITERATION NO.=  300
2       0.10629D 00   -0.32202D 00   -0.41173D 05   -0.41476D 05    0.32755D 02    0.32752D 02    0.19268D 04   -0.27493D 04

 ITERATION NO.=  350
2       0.12274D 00   -0.33546D 00   -0.41834D 05   -0.42123D 05    0.31479D 02    0.31475D 02    0.19503D 04   -0.19502D 04

 ITERATION NO.=  400
2       0.13463D 00   -0.34525D 00   -0.42329D 05   -0.42604D 05    0.30472D 02    0.30469D 02    0.92665D 03   -0.13631D 04

 ITERATION NO.=  450
2       0.14339D 00   -0.35223D 00   -0.42655D 05   -0.42922D 05    0.29726D 02    0.29723D 02    0.51258D 03   -0.10368D 04

 ITERATION NO.=  500
2       0.14962D 00   -0.35738D 00   -0.42913D 05   -0.43175D 05    0.29143D 02    0.29139D 02    0.67009D 03   -0.75765D 03

 ITERATION NO.=  550
2       0.15428D 00   -0.36115D 00   -0.43095D 05   -0.43353D 05    0.28702D 02    0.28699D 02    0.51849D 03   -0.55355D 03

 ITERATION NO.=  600
2       0.15773D 00   -0.36392D 00   -0.43228D 05   -0.43481D 05    0.28372D 02    0.28369D 02    0.24936D 03   -0.40301D 03

 ITERATION NO.=  650
2       0.16023D 00   -0.36595D 00   -0.43327D 05   -0.43578D 05    0.28127D 02    0.28124D 02    0.18115D 03   -0.29428D 03
```

```
ITERATION NO.=  700
2       0.16207D 00   -0.36745D 00   -0.43400D 05   -0.43649D 05    0.27946D 02    0.27943D 02    0.17780D 03   -0.21700D 03

ITERATION NO.=  750
2       0.16344D 00   -0.36855D 00   -0.43453D 05   -0.43701D 05    0.27812D 02    0.27809D 02    0.98967D 02   -0.15980D 03

ITERATION NO.=  800
2       0.16445D 00   -0.36937D 00   -0.43492D 05   -0.43739D 05    0.27712D 02    0.27709D 02    0.30358D 02   -0.11735D 03

ITERATION NO.=  850
2       0.16518D 00   -0.36997D 00   -0.43522D 05   -0.43768D 05    0.27639D 02    0.27636D 02    0.37849D 02   -0.86731D 02

ITERATION NO.=  900
2       0.16572D 00   -0.37041D 00   -0.43544D 05   -0.43789D 05    0.27585D 02    0.27582D 02    0.55770D 02   -0.64358D 02

ITERATION NO.=  950
2       0.16612D 00   -0.37074D 00   -0.43560D 05   -0.43805D 05    0.27544D 02    0.27541D 02    0.35177D 02   -0.47490D 02

ITERATION NO.= 1000
2       0.16643D 00   -0.37098D 00   -0.43571D 05   -0.43816D 05    0.27515D 02    0.27512D 02    0.10188D 02   -0.34920D 02

CONVERGENCE REACHED AT ITERATION NO.= 1033     DELT=    0.80000D-04
```

I	UPH	W	ENPH	ENTH	EMPH	EMTH	APHD
2	0.16657D 00	-0.37110D 00	-0.43578D 05	-0.43822D 05	0.27499D 02	0.27496D 02	0.10000D 01
3	0.33940D 00	-0.38051D 00	-0.43942D 05	-0.44758D 05	0.23879D 02	0.25552D 02	0.20000D 01
4	0.53299D 00	-0.39518D 00	-0.44385D 05	-0.45987D 05	0.17847D 02	0.22000D 02	0.30000D 01
5	0.74845D 00	-0.41343D 00	-0.44997D 05	-0.47566D 05	0.96564D 01	0.17083D 02	0.40000D 01
6	0.99770D 00	-0.43301D 00	-0.45654D 05	-0.49125D 05	-0.27322D 00	0.11028D 02	0.50000D 01
7	0.12768D 01	-0.45121D 00	-0.46367D 05	-0.50507D 05	-0.11263D 02	0.41634D 01	0.60000D 01

8	0.15870D 01	-0.46503D 00	-0.47014D 05	-0.51367D 05	-0.22429D 02	-0.30757D 01	0.70000D 01
9	0.19114D 01	-0.47148D 00	-0.47590D 05	-0.51524D 05	-0.32670D 02	-0.10152D 02	0.80000D 01
10	0.22329D 01	-0.46783D 00	-0.47974D 05	-0.50732D 05	-0.40806D 02	-0.16471D 02	0.90000D 01
11	0.25186D 01	-0.45204D 00	-0.48173D 05	-0.48916D 05	-0.45684D 02	-0.21436D 02	0.10000D 02
12	0.27361D 01	-0.42299D 00	-0.48086D 05	-0.46011D 05	-0.46374D 02	-0.24525D 02	0.11000D 02
13	0.28487D 01	-0.38081D 00	-0.47753D 05	-0.42122D 05	-0.42278D 02	-0.25348D 02	0.12000D 02
14	0.28290D 01	-0.32702D 00	-0.47102D 05	-0.37393D 05	-0.33245D 02	-0.23706D 02	0.13000D 02
15	0.26627D 01	-0.26456D 00	-0.46221D 05	-0.32114D 05	-0.19553D 02	-0.19593D 02	0.14000D 02
16	0.23208D 01	-0.19239D 00	-0.44970D 05	-0.26197D 05	-0.35868D 00	-0.12605D 02	0.15076D 02
17	0.18381D 01	-0.12180D 00	-0.43561D 05	-0.20614D 05	0.22313D 02	-0.33678D 01	0.16152D 02
18	0.12550D 01	-0.60590D-01	-0.41984D 05	-0.16008D 05	0.47090D 02	0.75762D 01	0.17228D 02
19	0.62895D 00	-0.17157D-01	-0.40444D 05	-0.13149D 05	0.72507D 02	0.18602D 02	0.18304D 02
20	0.00000D 00	0.00000D 00	-0.38956D 05	-0.12855D 05	0.96895D 02	0.31975D 02	0.19380D 02

RESIDUALS

	APH	AW
2	0.18403D 02	-0.28699D 02
3	0.35021D 02	-0.27460D 02
4	0.46196D 02	-0.25449D 02
5	0.60430D 02	-0.22725D 02
6	0.78670D 02	-0.19350D 02
7	0.85859D 02	-0.15479D 02
8	0.83273D 02	-0.11345D 02
9	0.85573D 02	-0.71864D 01
10	0.85980D 02	-0.32475D 01
11	0.73981D 02	0.19990D 00
12	0.59099D 02	0.29046D 01
13	0.49385D 02	0.47074D 01
14	0.37963D 02	0.55752D 01
15	0.24174D 02	0.56660D 01
16	0.14423D 02	0.47257D 01
17	0.97171D 01	0.33276D 01
18	0.62131D 01	0.17854D 01
19	0.32665D 01	0.53799D 00

APPENDIX C

PROGRAM NSDRGSHELL - LISTING

AND SELECTED OUTPUT

```
C
C
C
C
C *   PROGRAM FOR THE NONLINEAR STATIC ANALYSIS OF SHELLS OF
C *   REVOLUTION BY DYNAMIC RELAXATION -GENERAL SHELL
C *   PROGRAM 'NSDRGSHELL'
C *   __ AXYSYMMETRIC LOADING
C *   __ NON LINEAR/LINEAR STRAIN-DISPLACEMENT RELATIONS
C *   __ NON LINEAR/LINEAR EQUILIBRIUM EQUATIONS
C *   __ WITH/WITHOUT CENTRAL OPENING
C *   __ LINEAR ISOTROPIC/ORTHOTROPIC STRESS-STRAIN RELATIONS
C *
C
C
      IMPLICIT REAL *8 (A-H,O-Z)
      REAL *4 CHAR (17)
      DIMENSION  ENTH(50),ENPH(50),EMTH(50),EMPH(50),
     &           APH(50),AW(50),UPH(50),W(50),VPH(50),VW(50),
     &           RTH(50),RPH(50),R(50),
     &           VPHS(50),VWS(50),PUPH(50),PW(50), ENG(50)
     &           QPH(50),QW(50),
     &           H(50),APHN(50), APHO(50), YY(50),
     &           STPH(50),STTH(50),
     &           PEMPH(50), PEMTH(50),PENPH(50),PENTH(50)
C
C
C*    INPUT DATA CARDS
C
       READ, EPH, ETH, PRPH, PRTH, G
       READ, ISHELL,IOSHEL
       READ, ICASE ,ISTCHK,NDFCHK
       READ, DPH, DW
       READ, DFPH, DFW, DELT
       READ, IMAX, ITIN, DELTM, DMAX
       READ, NDPHI, NDPHM, NDPHO, NPHI,IP
       READ, XI,XA,XB,XO,YI,HAO
       READ, IBCI, IBCO
       READ, FLDW, FLDPH
       READ (5,500) (CHAR(I),I=1,16)
C
C*     INPUT DATA FORMAT
C
 500   FORMAT (16 A4)
C
C
C*     WRITE PROBLEM TITLE AND INPUT DATA
C
        WRITE (6,600)
        WRITE (6,605) (CHAR(I), I=1,16)
        WRITE (6,610) EPH, ETH, PRPH, PRTH, G
        WRITE (6,620) DPH, DW
        WRITE (6,630) DFPH,DFW,DELT
        WRITE (6,640) DELTM, DMAX, IMAX, ITIN
        WRITE (6,650) NDPHI, NDPHM, NDPHO,NPHI
        WRITE (6,660) ISHELL,IOSHEL , ICASE, IBCI, IBCO,ISTCHK,NDFCHK
        WRITE (6,645) XI,XA,XB,XO,YI,HAO
        WRITE (6,655) FLDW, FLDPH
C
C*      OUTPUT FORMAT
C
 600   FORMAT ('1', 60X, 'RESULTS'//25X, 'NON LINEAR STATIC',
     & ' ANALYSIS OF SHELLS OF REVOLUTION BY DYNAMIC RELAXATION'//)
 605   FORMAT (5X, 'SHELL TYPE:', 5X, 4A4//16X, A4, 1X, 'LINEAR',
     &  ' EQUILIBRIUM AND STRAIN-DISPLACEMENT'//16X, 5A4,1X,
     &  'STRESS-STRAIN'//16X, 'INNER EDGE',3X,3A4//16X, 'OUTER EDGE',
```

```
     &     3X,3A4//)
 610       FORMAT(5X,'MATERIAL CONSTANTS'//10X,'EPH=',E15.5//10X ,
     & 'ETH=',E15.5 //10X,'PRPH=',F10.5//10X,'PRTH=',F10.5// 10X,'G=',
     &  E15.5//)
 620       FORMAT (5X,'DENSITIES'//10X,'DPH=',F10.5//10X,'DW=',F10.5//)
 630       FORMAT(5X,'ITERATION FACTORS'//10X,'DFPH=',E15.5//10X,
     & 'DFW=', E15.5//10X,'DELT=',E15.5//)
 640       FORMAT(5X,'ITERATION CONTROL PARAMETERS'//10X,'DELTMAX=',
     & E15.5,10X,'DMAX=',E15.5//10X,'IMAX=',I6,10X,'ITIN=',I5//)
 650       FORMAT(5X,'FINITE DIFFERENCE MESH GENERATION AND LOCATION',
     &' PARAMETERS ' //10X,'NDPHI=',I3,10X,'NDPHM=',I3//10X,'NDPHO=',
     &  I3,10X,'NPHI=',I3//)
 660       FORMAT(5X,'OTHER CONTROL PARAMETERS'//10X,'ISHELL=',I3,
     & 10X,'IOSHEL =',I3//10X,'ICASE=',I3//10X,'IBCI=',I3,10X,
     & 'IBCO=',I3//10X,'ISTCHK=',I3//10X,'NDFCHK=',I3//)
 645       FORMAT(5X,'SHELL GEOMETRY'//10X,' XI =',F10.5//10X,
     &  ' XA =',F10.5//10X,' XB=',F10.5//10X,' XO=',F10.5//10X,
     &' YI=',F10.5//10X,'HAO=',F10.5//)
 655       FORMAT (5X,'LOAD DATA '//10X,'FLDW=', E15.5//10X,
     &  'FLDPH=', E15.5//)
C
C    FINITE DIFFERENCE MESH PARAMETERS
C
           XPHI =(XA-XI)/NDPHI
           XPHM =(XB-XA)/NDPHM
           XPHO =(XO-XB)/NDPHO
           NPHA = NPHI+ NDPHI
           NPHB = NPHA + NDPHM
           NPHO = NPHB + NDPHO
           NPHI1 = NPHI - 1
           NPHI3 = NPHI + 1
           NPHO1 = NPHO - 1
           NPHO3 = NPHO + 1
           IF (IBCI .GT. 2) NA = NPHI
           IF (IBCI .LE. 2) NA = NPHI + 1
           IF (IBCO .GT. 2) NB = NPHO
           IF (IBCO .LE. 2) NB = NPHO - 1
C
C*    GENERATE R, RTH, RPH, APHI
C
             HOA = 1./HAO
            DO 400 I =NPHI, NPHO
              H(I) =1.
 400        CONTINUE
            DO 415 I = NPHI1, NPHO3
           IF (I .LT. NPHI) GO TO 1030
           IF (I .EQ. NPHI) GO TO 1040
           IF (I .LE. NPHA) GO TO 1050
           IF (I .LE. NPHB) GO TO 1060
           IF (I .GT. NPHB) GO TO 1070
 1030       IF (IOSHEL .EQ. 1) GO TO 1035
            R(I) = O.
            GO TO 1080
 1035       R(I) = XI - XPHI
            GO TO 1080
 1040       R(I) = XI
            GO TO 1080
 1050       R(I) = R(I-1) + XPHI
            GO TO 1080
 1060       R(I) = R(I-1) + XPHM
            GO TO 1080
 1070       R(I) = R(I-1) + XPHO
 1080       IF(ISHELL.EQ.2) GO TO 1075
            IF((I.EQ.NPHO3).AND.(R(I).GT..999))GO TO 417
 1075       IF (ISHELL.EQ.3) YY(I) =YI* DSQRT(1.-R(I)*R(I))
            IF (ISHELL.EQ.2) YY(I) = YI -YI* R(I)*R(I)
```

```
          IF (I .GE. NPHI) GO TO 1085
          IF (IOSHEL .NE.1) GO TO 415
 1085     IF (ISHELL.EQ.2) GO TO 1100
          IF (R(I).LE. .999) GO TO 1100
             APHN(I) = 11./7.
             GO TO 1110
 1100     IF(ISHELL.EQ.3) APHN(I)= DATAN(R(I)*YI/DSQRT(1.-R(I)*R(I)))
          IF(ISHELL.EQ.2) APHN(I) = DATAN(2.*YI*R(I))
 1110        PHIN = APHN(I)
             RTH(I) = R(I)/DSIN(PHIN)
            IF(ISHELL.EQ.3) RPH(I) = YI* YI *(RTH(I)**3)
           IF(ISHELL.EQ.2)RPH(I) = RTH(I)/(DCOS(PHIN)*DCOS(PHIN))
 415       CONTINUE
 417      IF (IOSHEL .EQ. 1) GO TO 416
          RPH(NPHI-1) = RPH(NPHI)
          RTH(NPHI-1) = RTH(NPHI)
 416      IF(ISHELL.EQ.2)GO TO 419
          IF(R(NPHO3).LE..999)GO TO 419
          IF(IBCO.EQ.4)GO TO 418
          RPH( NPHO+1) = RPH(NPHO-1)
          RTH(NPHO +1) = RTH(NPHO-1)
          GO TO 419
 418      RPH(NPHO+1) = RPH(NPHO)
          RTH(NPHO+1) = RTH(NPHO)
 419      CONTINUE
C
C*     PRESCRIBE LOADING
C
         DO 420 I = NPHI, NPHO
          QPH(I) = FLDPH
          QW(I) = FLDW
 420     CONTINUE
C
C*    PRESCRIBE ACCURACY FOR RESIDUAL CHECK
C
         FACT = 1000.
         ACCW  =DABS(FLDW)/FACT
         ACCPH = ACCW
C
C*    CALCULATE CONSTANT EXPRESSIONS
C
 900     B1 =DELT/DPH
         B2 = 1. +(.5 *DFPH* B1)
         B3 = 1.-(.5 * DFPH* B1)
         C1 = DELT/DW
         C2 = 1.+(.5 * DFW * C1)
         C3 = 1.-(.5 * DFW * C1)
C
C*    STORE INITIAL VALUES
C
         DO 430 I = NPHI1, NPHO3
          ENTH(I) = O.
          ENPH(I) = O.
          EMTH(I) = O.
          EMPH(I) = O.
           UPH(I) = O.
            W(I) = O.
           VPH(I) = O.
            VW(I) = O.
          VPHS(I) = O.
           VWS(I) = O.
           ENG(I) = O.
 430     CONTINUE
C
C*    SET ITERATION COUNTER EQUAL TO ZERO
C
```

```
          ITER = 0
C
C*    CALCULATE VELOCITIES AND DISPLACEMENTS
C
          DO 800 J = 1, IMAX
              EW = 0.
          DO 440 I = NA, NB
            PHIN = APHN(I)
            IF =I+1
            IB =I-1
            DELX = R(I)-R(I-1)
            DELY = YY(I-1)-YY(I)
            DELS = DSQRT(DELX*DELX+DELY*DELY)
            IF (I .EQ. NPHO) GO TO 75
            DELX1 =R(I+1) - R(I)
            DELY1 = YY(I) -YY(I+1)
            DELS1 = DSQRT(DELX1*DELX1 +DELY1*DELY1)
            IF ((I .EQ. NPHI) .AND. (IOSHEL .NE. 1)) DELS = DELS*2.
            IF ((I .EQ. NPHI) .AND. (IOSHEL .EQ. 1)) DELS = DELS1
            GO TO 110
 75         IF (IBCO .EQ. 4) GO TO 80
            DELS1 = DELS
            GO TO 110
 80         DELX1 = 1.-XO
            DELY1 = YY(I)
            DELS1 = DSQRT(DELX1*DELX1 + DELY1*DELY1) * 2.
C
C*    CALCULATION OF VELOCITIES AND DISPLACEMENTS IN THE TRANSVERSE
C*      DIRECTION
C
 110      ALPHA = DELS1/ DELS
          ALPHA1 = 1. - ALPHA* ALPHA
          ALPHA2 = 1. + ALPHA
          ALPHA3 = 2./(ALPHA* ALPHA2)
          ALPHA4 =ALPHA* ALPHA2
          Z1 = (EMPH(IF)- ALPHA2* EMPH(I) + ALPHA* EMPH(IB))/(DELS
     &    * DELS) * ALPHA3
          Z2 = (EMPH(I) - EMTH(I)) * DSIN(PHIN)/( R(I)* RPH(I))
          Z3 = 2. *DCOS(PHIN)/R(I)*(EMPH(IF)- ALPHA1*EMPH(I)- ALPHA*
     &    ALPHA * EMPH(IB))/(ALPHA4* DELS)
          Z4 = DCOS(PHIN)/R(I) * (EMTH(IF)-ALPHA1* EMTH(I)- ALPHA*
     &    ALPHA* EMTH(IB))/(ALPHA4* DELS)
          Z41 = 1./RPH(I)*(RPH(IF)-ALPHA1*RPH(I)-ALPHA*ALPHA*RPH(IB))
     &    /(ALPHA4*DELS)*(EMPH(IF)-ALPHA1*EMPH(I)-ALPHA*ALPHA*EMPH(IB)
     &    )/(ALPHA4*DELS)
           Y41 = 1./RPH(I)
           Y42 = HOA* HOA* 1./RPH(I)* (UPH(IF)- UPH(I)*ALPHA1- ALPHA
     &     * ALPHA* UPH(IB))/(ALPHA4* DELS)
           Y43 = HOA* (W(IF)-ALPHA2* W(I)+ ALPHA*W(IB))/(DELS*DELS)
     &     * ALPHA3
           Y44 = 1./RPH(I) * (RPH(IF)- ALPHA1*RPH(I)-ALPHA*ALPHA*
     &     RPH(IB))/(ALPHA4*DELS ) * (HOA*(W(IF)-ALPHA1*W(I)-ALPHA*
     &     ALPHA*W(IB))/(ALPHA4*DELS) -HOA*HOA* UPH(I)/RPH(I))
           Y4 = Y41 + Y42 - Y43 +Y44
           IF (ICASE .EQ. 1) Y4 = Y41
           Z5 =  ENPH(I)*Y4
           Z6 = ENTH(I)/RTH(I)
          Y1 = (ENPH(IF)-ENPH(I)*ALPHA1- ALPHA*ALPHA*ENPH(IB))
     &    /(ALPHA4*DELS)
          Y2 = (ENPH(I)- ENTH(I))* DCOS(PHIN)/R(I)
          Y31 = EMPH(I)* DCOS(PHIN)/R(I)
          Y32 = (EMPH(IF)-EMPH(I)*ALPHA1-ALPHA*ALPHA*EMPH(IB))/
     &    (ALPHA4* DELS)
          Y33 = EMTH(I)* DCOS(PHIN)/R(I)
          Y3 = Y31 + Y32 -Y33
           Z7=Y3/RPH(I)+Y1+ENPH(I)*DCOS(PHIN)/R(I)
```

```
            Y6=Z1-Z2+Z3-Z4-Z41-ENPH(I)/RPH(I)
            BETA=HOA*HOA*UPH(I)/RPH(I)-HOA*(W(IF)-W(I)*ALPHA1
     &      -ALPHA*ALPHA*W(IB))/(ALPHA4*DELS)
            IF (ICASE .EQ. 1) BETA = O.
            AW(I) =Z1-Z2 +Z3- Z4- Z41-Z5 - Z6-Z7*BETA -QW(I)
            PW(I) = W(I)
            VW(I) = 1./C2 * (C3 * VW(I) + C1* AW(I))
            IF(J.EQ.1) VW(I)=DELT/(2.*DW)*AW(I)
            W(I) = W(I) + VW(I) * DELT
            IF (NDFCHK .NE.1)GO TO 115
            VWS(I)= VW(I) * VW(I)
 115        IF(DABS(W(I))-DMAX) 120, 130, 130
C
C*     CALCULATION OF VELOCITIES AND DISPLACEMENTS IN THE MERIDIONAL
C*        DIRECTION
C
 120        APH(I) =Y1+ Y2 +Y3 * Y4+Y6*BETA + QPH(I)
            PUPH(I) = UPH(I)
            VPH(I) = 1./B2 * (B3 * VPH(I) + B1 * APH(I))
            IF(J.EQ.1) VPH(I)=DELT/(2.*DPH)*APH(I)
            UPH(I) = UPH(I) + VPH(I) * DELT
            IF (NDFCHK .NE.1) GO TO 125
            VPHS(I) = VPH(I) * VPH(I)
            ENG(I)  = VWS(I) + VPHS(I)
            EW = EW + ENG(I)
 125        IF (DABS (UPH(I))- DMAX) 440, 130, 130
 130        WRITE (6, 665) J, DELT
            WRITE (6, 670) I, UPH(I), W(I), VPH(I), VW(I), APH(I), AW(I)
            WRITE (6,735)
            WRITE (6, 680) (K, UPH(K), W(K), ENPH(K), ENTH(K), EMPH(K),
     &       EMTH(K), K= NPHI, NPHO)
            DELT = DELT* .5
            IF (DELT.LE. DELTM) GO TO 1000
            GO TO 900
 440        CONTINUE
 665        FORMAT (5X, 'NUMERICAL INSTABILITY AT ITERATION NO. =',
     &      I5, 5X, 'WITH DELT=', E15.5//)
 670        FORMAT (5X, I3, 5X, 6E15.5 //)
 680        FORMAT ((5X, I3, 5X, 6E15.5)/)
C
C*    DISPLACEMENT BOUNDARY CONDITIONS
C*     BOUNDARY CONDITIONS ON INNER EDGE
           GO TO (135, 140,145 ,160), IBCI
C*     FIXED BOUNDARY CONDITIONS
 135       W(NPHI -1) = W(NPHI + 1)
           UPH(NPHI -1) = -UPH(NPHI + 1)
           GO TO 175
C*     HINGED BOUNDARY CONDITIONS
 140       UPH(NPHI-1) = -UPH(NPHI+1)
           DELX = R(NPHI+1)- R(NPHI)
           DELY = YY(NPHI) - YY(NPHI+1)
           DELS = DSQRT(DELX *DELX +DELY *DELY)
           PO = (RPH(NPHI+1)-RPH(NPHI-1))/(4.* DELS* DELS)
     &     - PRPH*DCOS(APHN(NPHI))* 1./(2.* R(NPHI)* DELS)
           P1 = PO - 1./(DELS * DELS)
           P2 = HOA * 1./(2. * RPH(NPHI) * DELS)
           P3 = PO + 1./(DELS * DELS)
           W(NPHI -1) = P2/P3 * (UPH(NPHI +1) - UPH(NPHI -1)) + P1/P3
     &     * W(NPHI +1)
           GO TO 175
C*     FREE BOUNDARY CONDITIONS
 145       GO TO 175
C*     SYMMETRY BOUNDARY CONDITIONS
 160       W(NPHI-1) = W(NPHI)
           UPH(NPHI-1) = -UPH(NPHI)
           GO TO 175
```

```
C*    BOUNDARY CONDITIONS ON OUTER  EDGE
 175      GO TO (180, 190,195,196 ), IBCO
C*    FIXED BOUNDARY CONDITIONS
 180      W(NPHO+1) = W(NPHO -1)
          UPH(NPHO +1) = -UPH(NPHO-1)
          GO TO 200
C*    HINGED BOUNDARY CONDITIONS
 190      UPH(NPHO+1) = -UPH(NPHO-1)
          DELX = R(NPHO) -R(NPHO-1)
          DELY = YY(NPHO-1) - YY(NPHO)
          DELS = DSQRT(DELX *DELX + DELY*DELY)
          PO = (RPH(NPHO+1) - RPH(NPHO-1))/(4.* DELS*DELS)
     &    -PRPH* DCOS(APHN(NPHO))*1./(2.*R(NPHO)*DELS)
          P1 = PO - 1./(DELS * DELS)
          P2 = HOA * 1./(RPH(NPHO) * DELS)
          P3 = PO + 1./(DELS * DELS)
          W(NPHO+1) = P3/P1 * W(NPHO-1)-P2/P1 * (UPH(NPHO+1)-UPH(NPHO
     &     -1))
          GO TO 200
C*   FREE BOUNDARY CONDITIONS
 195      GO TO 200
C*   SYMMETRY BOUNDARY CONDITIONS
 196      W(NPHO+1) = W(NPHO)
          UPH(NPHO+1) =-UPH(NPHO)
C
C*  CALCULATION OF STRESS RESULTANTS
C
 200      DO 450 I = NPHI, NPHO
             PHIN = APHN(I)
             IF = I+1
             IB = I-1
             DELX = R(I) - R(I-1)
             DELY = YY(I-1) - YY(I)
             DELS = DSQRT(DELX*DELX + DELY*DELY)
             IF (I .EQ. NPHO) GO TO 210
             DELX1 = R(I+1) - R(I)
             DELY1 = YY(I) - YY(I+1)
             DELS1 = DSQRT(DELX1*DELX1 + DELY1*DELY1)
             IF (( I .EQ. NPHI) .AND. (IOSHEL .NE. 1)) DELS = 2.*DELS
             IF (( I .EQ. NPHI) .AND. (IOSHEL .EQ. 1))DELS = DELS1
             GO TO 250
 210         IF (IBCO .EQ. 4) GO TO 220
              DELS1 = DELS
              GO TO 250
 220          DELX1 = 1.- XO
              DELY1 = YY(I)
              DELS1 = DSQRT(DELX1*DELX1 + DELY1*DELY1)* 2.
 250         ALPHA = DELS1/DELS
             ALPHA1 =1.- ALPHA* ALPHA
             ALPHA2 = 1. + ALPHA
             ALPHA3 = 2./(ALPHA* ALPHA2)
             ALPHA4 = ALPHA * ALPHA2
             F1 = (UPH(IF)-ALPHA1* UPH(I)-ALPHA*ALPHA* UPH(IB))/
     &       (ALPHA4*DELS)
             F2 = W(I)/RPH(I)
             F22 = UPH(I)/RPH(I)
             F3 = (W(IF)- ALPHA1*W(I)-ALPHA*ALPHA*W(IB))/(ALPHA4*DELS)
             F4 = W(I)*DSIN(PHIN)/R(I)
             F5 = UPH(I)* DCOS(PHIN)/R(I)
             F6 = (W(IF)-ALPHA2* W(I)+ALPHA* W(IB))/(DELS*DELS)*ALPHA3
             F7 = 1./RPH(I)* (RPH(IF)-ALPHA1* RPH(I)-ALPHA*ALPHA*RPH(IB)
     &       )/(ALPHA4*DELS)
 251         F21 = HOA * F1+ F2
             F31 = F3- HOA* F22
             F41 = HOA * F5 + F4
             IF (ICASE .NE. 1) GO TO 260
```

```
            F21 = O.
            F31 = O.
            F41 = O.
 260        STPH(I) = HAO* F1 + HAO*HAO*F2+.5*HAO*F21*F21+.5*
     &      HAO*F31*F31
            STTH(I) = HAO*F5 + HAO*HAO*F4 + .5*HAO*F41*F41
            PENPH(I) = ENPH(I)
            PENTH(I) = ENTH(I)
            ENPH(I) = 12.*H(I)* EPH/ETH* (STPH(I)+ PRPH*STTH(I) )
            ENTH(I) = 12. * H(I) * (STTH(I) + PRTH* STPH(I))
            CPH = HOA* F1/RPH(I) -F6 + F7* (F3 -HOA*F22)
            CTH = DCOS(PHIN)* (HOA* F22/R(I) -F3/R(I) )
            PEMPH(I) = EMPH(I)
            PEMTH(I) = EMTH(I)
            EMPH(I) = EPH/ETH * (H(I)**3)* (CPH + PRPH* CTH)
            EMTH(I) = (H(I)* * 3) * (CTH + PRTH*CPH)
 450       CONTINUE
C
C*    STRESS BOUNDARY CONDITIONS
C*    BOUNDARY CONDITIONS ON INNER EDGE
           GO TO (290,290,275,280),IBCI
C*    FREE BOUNDARY CONDITIONS
 275      GO TO 290
C*    SYMMETRY BOUNDARY CONDITIONS
 280       ENPH(NPHI-1) = ENPH(NPHI)
           ENTH(NPHI-1) = ENTH(NPHI)
           EMPH(NPHI-1) = EMPH(NPHI)
           EMTH(NPHI-1) = EMTH(NPHI)
C*    BOUNDARY CONDITION ON OUTER EDGE
 290       GO TO ( 300,300,295,296),IBCO
C*    FREE BOUNDARY CONDITIONS
 295       GO TO 300
C*    SYMMETRY BOUNDARY CONDITIONS
 296       ENPH(NPHO+1) = ENPH(NPHO)
           ENTH(NPHO+1) = ENTH(NPHO)
           EMPH(NPHO+1) = EMPH(NPHO)
           EMTH(NPHO+1) = EMTH(NPHO)
C
C*    CHECK FOR CONVERGENCE
C*    CHECK FOR RESIDUALS OF EQUILIBRIUM EQUATIONS
 300       DO 460 I = NA ,NB
              IF (DABS(APH(I)).GE. ACCPH) GO TO 310
              IF (DABS(AW(I)) .GE. ACCW) GO TO 310
 460       CONTINUE
C*    CHECK FOR CONVERGENCE OF DISPLACEMENTS
           DO 470 I = NA,NB
             DIFUPH = (UPH(I) - PUPH(I))/PUPH(I)
             DIFW = (W(I) - PW(I))/PW(I)
             IF (DABS(DIFUPH) .GT. .001) GO TO 310
             IF (DABS(DIFW) .GT. .001) GO TO 310
 470       CONTINUE
           IF (ISTCHK .NE. 1) GO TO 350
C*    CHECK FOR CONVERGENCE OF STRESS RESULTANTS
             DO 475 I = NPHI, NPHO
             DIENPH = (ENPH(I)-PENPH(I))/PENPH(I)
             DIENTH = (ENTH(I)-PENTH(I))/PENTH(I)
             DIEMPH = (EMPH(I)-PEMPH(I))/PEMPH(I)
             DIEMTH = (EMTH(I)- PEMTH(I))/PEMTH(I)
             IF(DABS(DIENPH) .GT. .001) GO TO 310
             IF (DABS(DIENTH) .GT. .001) GO TO 310
             IF (DABS(DIEMPH) .GT. .001) GO TO 310
             IF (DABS(DIEMTH) .GT. .001) GO TO 310
 475         CONTINUE
C
           GO TO 350
 310       IF (J- ITER) 800, 330, 330
```

```
330       IF (NDFCHK .NE. 1) GO TO 340
          IF (J .EQ. 1) WRITE (6,705)
          WRITE (6, 700) J
          WRITE (6, 710)IP,UPH(IP). W(IP). VPHS(IP).VWS(IP).ENG(IP).EW
          GO TO 345
340       IF (J .EQ.1) WRITE (6,715)
          WRITE (6,700) J
          WRITE (6,720) IP.UPH(IP).W(IP).ENPH(IP).ENTH(IP).EMPH(IP).
    &     EMTH(IP),APH(IP),AW(IP)
345       ITER = ITER + ITIN
800      CONTINUE
          WRITE (6,725) J, DELT
          GO TO 355
350       WRITE (6,730) J, DELT
355       WRITE (6,735)
          WRITE (6,740) (IY,UPH(IY),W(IY),ENPH(IY),ENTH(IY),EMPH(IY),
    &     EMTH(IY),R(IY),YY(IY),IY= NPHI, NPHO)
          WRITE (6,750)
          WRITE (6,745) (IY,APH(IY),AW(IY),IY=NPHI,NPHO1)
          WRITE (6,755)
700       FORMAT ( //5X,'ITERATION NO.=', I5/)
705       FORMAT (7X,'I',12X,'UPH',14X,'W',12X,'VPHS',11X,'VWS',11X,
    &     'EVPH',11X,'EW'//)
710       FORMAT (5X,I3, 5X, 6E15.5/)
715       FORMAT (4X,'I',10X,'UPH',14X,'W',12X,'ENPH',11X,'ENTH',11X,
    &     'EMPH',11X,'EMTH',11X,'APH',13X,'AW'//)
720       FORMAT (2X,I3,2X,8E15.5/)
725       FORMAT (//5X,'CONVERGENCE NOT REACHED AT ITERATION NO.=',
    &     I5,5X,'DELT=',E15.5//)
730       FORMAT (//5X,'CONVERGENCE REACHED AT ITERATION NO.=',
    &     I5,5X,'DELT=',E15.5//)
735       FORMAT (4X,'I',12X,'UPH',14X,'W',12X,'ENPH',11X,'ENTH',
    &     11X,'EMPH',11X,'EMTH',11X,' R  ',10X,'YY'//)
740       FORMAT ((2X,I3,5X, 8E15.5)/)
745       FORMAT ((5X,I3,5X,2E15.5)/)
750       FORMAT (//5X,'RESIDUALS'//20X,'APH',13X,'AW'//)
755       FORMAT (1H1)
1000      STOP
          END
```

RESULTS

NON LINEAR STATIC ANALYSIS OF SHELLS OF REVOLUTION BY DYNAMIC RELAXATION

SHELL TYPE: PARABOLICAL

NON LINEAR EQUILIBRIUM AND STRAIN-DISPLACEMENT

LINEAR ISOTROPIC STRESS-STRAIN

INNER EDGE SYMMETRICAL

OUTER EDGE FIXED

MATERIAL CONSTANTS

EPH= 0.25000D 07

ETH= 0.25000D 07

PRPH= 0.25000

PRTH= 0.25000

G= 0.10000D 07

DENSITIES

DPH= 1.00000

DW= 1.00000

ITERATION FACTORS

DFPH= 0.75000D 02

DFW= 0.30000D 03

DELT= 0.50000D-03

ITERATION CONTROL PARAMETERS

DELTMAX= 0.50000D-03 DMAX= 0.10000D 04

IMAX= 2000 ITIN= 50

FINITE DIFFERENCE MESH GENERATION AND LOCATION PARAMETERS

NDPHI= 5 NDPHM= 10

NDPHO= 3 NPHI= 2

OTHER CONTROL PARAMETERS

ISHELL= 2 IOSHEL = 2

ICASE= 2

IBCI= 4 IBCO= 1

ISTCHK= 0

NDFCHK= 0

SHELL GEOMETRY

XI = 0.01000

XA = 0.30000

XB= 0.90000

XO= 1.00000

YI= 0.40000

HAO= 100.00000

LOAD DATA

FLDW= 0.10000D 01

FLDPH= 0.00000D 00

I	UPH	W	ENPH	ENTH	EMPH	EMTH	APH	AW
ITERATION NO.= 1								
2	0.00000D 00	-0.12500D-06	-0.14999D-01	-0.14999D-01	0.00000D 00	0.00000D 00	0.00000D 00	-0.10000D 01
ITERATION NO.= 50								
2	-0.29752D-07	-0.52511D-05	-0.63449D 00	-0.63456D 00	0.21856D-04	0.21858D-04	0.22269D-02	0.18969D-01
ITERATION NO.= 100								
2	0.50310D-06	-0.58446D-05	-0.62583D 00	-0.62585D 00	0.16182D-04	0.16183D-04	-0.21193D-02	0.65189D-02
ITERATION NO.= 150								
2	0.21125D-06	-0.54625D-05	-0.62370D 00	-0.62377D 00	0.26443D-04	0.26445D-04	0.24230D-02	-0.27927D-02

ITERATION NO.= 200

2	0.25417D-06	-0.55241D-05	-0.62469D 00	-0.62473D 00	0.21694D-04	0.21696D-04	-0.44318D-03	-0.55017D-03

ITERATION NO.= 250

2	0.26250D-06	-0.55373D-05	-0.62501D 00	-0.62506D 00	0.21559D-04	0.21561D-04	-0.13938D-04	0.19512D-04

ITERATION NO.= 300

2	0.26428D-06	-0.55396D-05	-0.62502D 00	-0.62507D 00	0.21741D-04	0.21743D-04	0.53115D-04	0.10550D-04

CONVERGENCE REACHED AT ITERATION NO.= 306 DELT= 0.50000D-03

I	UPH	W	ENPH	ENTH	EMPH	EMTH	R	YY
2	0.26437D-06	-0.55394D-05	-0.62499D 00	-0.62504D 00	0.21680D-04	0.21682D-04	0.10000D-01	0.39996D 00
3	0.18043D-05	-0.55782D-05	-0.62611D 00	-0.62802D 00	0.21721D-04	0.21744D-04	0.68000D-01	0.39815D 00
4	0.34364D-05	-0.56756D-05	-0.62829D 00	-0.63477D 00	0.21041D-04	0.21513D-04	0.12600D 00	0.39365D 00
5	0.52801D-05	-0.58313D-05	-0.63216D 00	-0.64533D 00	0.19978D-04	0.21069D-04	0.18400D 00	0.38646D 00
6	0.73353D-05	-0.60452D-05	-0.63665D 00	-0.65983D 00	0.18099D-04	0.20284D-04	0.24200D 00	0.37657D 00
7	0.98173D-05	-0.63149D-05	-0.64331D 00	-0.67768D 00	0.16268D-04	0.19410D-04	0.30000D 00	0.36400D 00
8	0.12595D-04	-0.66518D-05	-0.65018D 00	-0.70000D 00	0.13781D-04	0.18269D-04	0.36000D 00	0.34816D 00
9	0.16047D-04	-0.70444D-05	-0.65996D 00	-0.72522D 00	0.13011D-04	0.17596D-04	0.42000D 00	0.32944D 00
10	0.19500D-04	-0.74971D-05	-0.66881D 00	-0.75443D 00	0.13528D-04	0.17439D-04	0.48000D 00	0.30784D 00
11	0.23918D-04	-0.80203D-05	-0.68147D 00	-0.78726D 00	0.18711D-04	0.18900D-04	0.54000D 00	0.28336D 00
12	0.28066D-04	-0.86461D-05	-0.69228D 00	-0.82715D 00	0.23537D-04	0.20787D-04	0.60000D 00	0.25600D 00
13	0.33968D-04	-0.94092D-05	-0.70862D 00	-0.87480D 00	0.18115D-04	0.20058D-04	0.66000D 00	0.22576D 00
14	0.39602D-04	-0.10301D-04	-0.72243D 00	-0.93032D 00	-0.35206D-04	0.50106D-05	0.72000D 00	0.19264D 00
15	0.47977D-04	-0.11073D-04	-0.74322D 00	-0.97259D 00	-0.18349D-03	-0.40825D-04	0.78000D 00	0.15664D 00
16	0.53170D-04	-0.10948D-04	-0.75751D 00	-0.94813D 00	-0.43936D-03	-0.12618D-03	0.84000D 00	0.11776D 00
17	0.53384D-04	-0.84349D-05	-0.77032D 00	-0.75450D 00	-0.56516D-03	-0.18882D-03	0.90000D 00	0.76000D-01
18	0.43195D-04	-0.55857D-05	-0.76536D 00	-0.55244D 00	-0.21699D-03	-0.11404D-03	0.93333D 00	0.51556D-01

19	0.26034D-04	-0.22327D-05	-0.75554D 00	-0.32387D 00	0.69659D-03	0.12324D-03	0.96667D 00	0.26222D-01
20	0.00000D 00	0.00000D 00	-0.73662D 00	-0.18415D 00	0.24802D-02	0.62005D-03	0.10000D 01	0.13878D-15

RESIDUALS

	APH	AW
2	-0.51859D-04	-0.27927D-04
3	-0.34995D-03	-0.75800D-05
4	-0.51232D-03	0.24512D-04
5	-0.50122D-03	0.43180D-04
6	-0.27521D-03	0.45258D-04
7	-0.11026D-03	0.44425D-04
8	-0.11203D-03	0.38324D-04
9	0.41608D-05	0.33091D-05
10	-0.22901D-03	-0.30979D-04
11	0.61384D-04	-0.28990D-04
12	-0.49841D-03	0.42348D-05
13	-0.14173D-03	0.10077D-04
14	0.55425D-03	-0.40551D-04
15	-0.84267D-03	-0.22651D-04
16	-0.56722D-03	0.72105D-04
17	0.46215D-03	0.54936D-05
18	-0.42398D-03	-0.29507D-04
19	-0.85054D-03	-0.29933D-04

RESULTS

NON LINEAR STATIC ANALYSIS OF SHELLS OF REVOLUTION BY DYNAMIC RELAXATION

SHELL TYPE: PARABOLICAL

NON LINEAR EQUILIBRIUM AND STRAIN-DISPLACEMENT

LINEAR ISOTROPIC   STRESS-STRAIN

INNER EDGE   SYMMETRICAL

OUTER EDGE   FIXED

MATERIAL CONSTANTS

EPH=   0.25000D 07

ETH=   0.25000D 07

PRPH=   0.25000

PRTH=   0.25000

G=   0.10000D 07

DENSITIES

DPH=   1.00000

DW=   1.00000

ITERATION FACTORS

DFPH=   0.75000D 02

DFW=   0.30000D 03

DELT=   0.50000D-03

ITERATION CONTROL PARAMETERS

DELTMAX=   0.50000D-03   DMAX=   0.10000D 04

IMAX= 1200   ITIN=   50

FINITE DIFFERENCE MESH GENERATION AND LOCATION PARAMETERS

NOPHI= 5   NOPHM= 10

NOPHO= 3   NPHI= 2

OTHER CONTROL PARAMETERS

ISHELL= 2 IOSHEL = 2

ICASE= 2

IBCI= 4 IBCO= 1

ISTCHK= 0

NDFCHK= 0

SHELL GEOMETRY

XI = 0.01000

XA = 0.30000

XB= 0.90000

XO= 1.00000

YI= 0.40000

HAO= 100.00000

LOAD DATA

FLDW= -0.100000D 06

FLDPH= 0.000000D 00

I	UPH	W	ENPH	ENTH	EMPH	EMTH	APH	AW
ITERATION NO.= 1								
2	0.00000D 00	0.12500D-01	0.15000D 04	0.15000D 04	0.15313D-14	0.61295D-14	0.00000D 00	0.10000D 06
ITERATION NO.= 50								
2	0.89170D-02	0.53623D 00	0.65816D 05	0.65824D 05	-0.19474D 01	-0.19475D 01	-0.17234D 02	-0.42033D 04
ITERATION NO.= 100								
2	-0.15719D 00	0.72974D 00	0.64163D 05	0.64133D 05	0.57974D 01	0.57978D 01	0.58383D 04	-0.99726D 04
ITERATION NO.= 150								
2	-0.45205D-01	0.58743D 00	0.63819D 05	0.63838D 05	-0.51157D 01	-0.51161D 01	-0.28170D 04	0.44942D 04

ITERATION NO.= 200

2	-0.54919D-01	0.600950 00	0.63999D 05	0.64007D 05	-0.24568D 01	-0.24570D 01	-0.49241D 03	0.18113D 02

ITERATION NO.= 250

2	-0.60422D-01	0.60654D 00	0.63848D 05	0.63854D 05	-0.18631D 01	-0.18632D 01	0.26524D 03	-0.36561D 03

ITERATION NO.= 300

2	-0.59808D-01	0.605860 00	0.63857D 05	0.63863D 05	-0.206800 01	-0.20682D 01	0.75811D 02	-0.31805D 02

ITERATION NO.= 350

2	-0.59040D-01	0.604950 00	0.63863D 05	0.63870D 05	-0.21623D 01	-0.21625D 01	-0.29569D 02	0.62648D 02

CONVERGENCE REACHED AT ITERATION NO.= 395 DELT= 0.50000D-03

I	UPH	W	ENPH	ENTH	EMPH	EMTH	R	YY
2	-0.58958D-01	0.60488D 00	0.63868D 05	0.63874D 05	-0.21277D 01	-0.21279D 01	0.10000D-01	0.39896D 00
3	-0.40200D 00	0.60864D 00	0.64012D 05	0.64161D 05	-0.21264D 01	-0.21309D 01	0.68000D-01	0.39815D 00
4	-0.74819D 00	0.61803D 00	0.64114D 05	0.64830D 05	-0.20222D 01	-0.20895D 01	0.12600D 00	0.39365D 00
5	-0.11284D 01	0.63293D 00	0.64654D 05	0.65803D 05	-0.19532D 01	-0.20518D 01	0.18400D 00	0.38646D 00
6	-0.14897D 01	0.65342D 00	0.64808D 05	0.67254D 05	-0.16746D 01	-0.19420D 01	0.24200D 00	0.37657D 00
7	-0.19536D 01	0.67897D 00	0.65802D 05	0.68868D 05	-0.15599D 01	-0.18657D 01	0.30000D 00	0.36400D 00
8	-0.23306D 01	0.71086D 00	0.65952D 05	0.71100D 05	-0.10808D 01	-0.16747D 01	0.36000D 00	0.34816D 00
9	-0.29416D 01	0.74706D 00	0.67508D 05	0.73283D 05	-0.89508D 00	-0.15461D 01	0.42000D 00	0.32944D 00
10	-0.32727D 01	0.78750D 00	0.67541D 05	0.76044D 05	-0.74330D-01	-0.12251D 01	0.48000D 00	0.30784D 00
11	-0.40319D 01	0.82941D 00	0.69620D 05	0.78385D 05	0.44772D 00	-0.84601D 00	0.54000D 00	0.28336D 00
12	-0.42015D 01	0.87101D 00	0.69402D 05	0.81100D 05	0.20584D 01	-0.32403D 00	0.60000D 00	0.25600D 00
13	-0.50643D 01	0.90524D 00	0.71985D 05	0.82593D 05	0.36524D 01	0.38359D 00	0.66000D 00	0.22576D 00
14	-0.48088D 01	0.92411D 00	0.71248D 05	0.83561D 05	0.74464D 01	0.18103D 01	0.72000D 00	0.19264D 00
15	-0.55616D 01	0.90733D 00	0.74097D 05	0.81171D 05	0.12404D 02	0.37748D 01	0.78000D 00	0.15664D 00

16	-0.43172D 01	0.825400 00	0.72247D 05	0.75064D 05	0.21772D 02	0.72497D 01	0.84000D 00	0.11776D 00
17	-0.44684D 01	0.61978D 00	0.74674D 05	0.59477D 05	0.29535D 02	0.10741D 02	0.80000D 00	0.76000D-01
18	-0.32119D 01	0.42729D 00	0.73116D 05	0.46025D 05	0.20242D 02	0.91834D 01	0.93333D 00	0.51556D-01
19	-0.24861D 01	0.19091D 00	0.73032D 05	0.29567D 05	-0.30626D 02	-0.37687D 01	0.96667D 00	0.26222D-01
20	0.00000D 00	0.00000D 00	0.70548D 05	0.17637D 05	-0.21205D 03	-0.53013D 02	0.10000D 01	0.13878D-15

RESIDUALS

	APH	AW
2	0.28914D 01	-0.14193D 02
3	0.18014D 02	-0.74228D 01
4	-0.30872D 02	0.34265D 01
5	-0.71686D 02	0.83212D 01
6	-0.97258D 02	0.33864D 01
7	0.12834D 02	-0.48478D 01
8	0.52542D 02	-0.59060D 01
9	0.79990D 01	0.33173D 00
10	-0.47315D 02	0.57776D 01
11	-0.74010D 02	0.32068D 01
12	0.33324D 02	-0.39478D 01
13	0.79833D 02	-0.41087D 01
14	-0.23930D 02	0.25332D 01
15	-0.97489D 02	0.39813D 01
16	-0.19329D 02	0.16012D 00
17	0.59830D 02	0.16586D 01
18	0.66024D 02	0.18013D 01
19	0.34106D 02	0.10613D 01

RESULTS

NON LINEAR STATIC ANALYSIS OF SHELLS OF REVOLUTION BY DYNAMIC RELAXATION

SHELL TYPE: ELLIPTICAL

NON LINEAR EQUILIBRIUM AND STRAIN-DISPLACEMENT

LINEAR ISOTROPIC STRESS-STRAIN

INNER EDGE SYMMETRICAL

OUTER EDGE FIXED

MATERIAL CONSTANTS

EPH= 0.25000D 07

ETH= 0.25000D 07

PRPH= 0.25000

PRTH= 0.25000

G= 0.10000D 07

DENSITIES

DPH= 1.00000

DW= 1.00000

ITERATION FACTORS

DFPH= 0.45000D 02

DFW= 0.20000D 03

DELT= 0.20000D-03

ITERATION CONTROL PARAMETERS

DELTMAX= 0.20000D-03 DMAX= 0.10000D 04

IMAX= 2000 ITIN= 50

FINITE DIFFERENCE MESH GENERATION AND LOCATION PARAMETERS

NDPHI= 17 NDPHM= 3

NDPHO= 1 NPHI= 2

OTHER CONTROL PARAMETERS

ISHELL= 3 IOSHEL = 2

ICASE= 2

IBCI= 4 IBCO= 1

ISTCHK= 0

NDFCHK= 0

SHELL GEOMETRY

XI = 0.01000

XA = 0.98000

XB= 0.99900

XO= 1.00000

YI= 0.40000

HAO= 100.00000

LOAD DATA

FLDW= 0.10000D 01

FLDPH= 0.00000D 00

I	UPH	W	ENPH	ENTH	EMPH	EMTH	APH	AW
ITERATION NO.= 1								
2	0.00000D 00	-0.20000D-07	-0.12001D-02	-0.12001D-02	0.37618D-20	0.94045D-21	0.00000D 00	-0.10000D 01
ITERATION NO.= 50								
2	-0.33325D-06	-0.19901D-04	-0.12442D 01	-0.12441D 01	-0.31261D-04	-0.31256D-04	-0.21204D-01	-0.24849D-01
ITERATION NO.= 100								
2	-0.13367D-05	-0.21674D-04	-0.15008D 01	-0.15010D 01	-0.11135D-03	-0.11134D-03	0.80012D-01	0.21172D 00
ITERATION NO.= 150								
2	0.42505D-05	-0.20197D-04	-0.57513D 00	-0.57452D 00	0.84698D-04	0.84686D-04	-0.23828D 00	-0.53896D 00

```
ITERATION NO.=  200
2       0.28867D-05  -0.30794D-04  -0.14135D 01  -0.14144D 01  -0.80160D-04  -0.80151D-04   0.44223D 00   0.90431D-01

ITERATION NO.=  250
2       0.48157D-05  -0.35923D-04  -0.143300 01  -0.14331D 01  -0.44907D-03  -0.44900D-03   0.70199D 00   0.48306D 00

ITERATION NO.=  300
2       0.70711D-05  -0.39703D-04  -0.13212D 01  -0.13215D 01  -0.17788D-03  -0.17785D-03   0.42645D 00   0.16679D 00

ITERATION NO.=  350
2       0.88458D-05  -0.42822D-04  -0.12425D 01  -0.12425D 01  -0.98912D-04  -0.98898D-04   0.14188D 00   0.16236D-01

ITERATION NO.=  400
2       0.97849D-05  -0.44744D-04  -0.12171D 01  -0.12170D 01  -0.59634D-04  -0.59625D-04  -0.37557D-02  -0.57056D-01

ITERATION NO.=  450
2       0.10204D-04  -0.45811D-04  -0.12183D 01  -0.12182D 01  -0.80663D-04  -0.80651D-04  -0.60641D-01  -0.58182D-01

ITERATION NO.=  500
2       0.10153D-04  -0.46016D-04  -0.12384D 01  -0.12382D 01  -0.92653D-04  -0.92639D-04  -0.49726D-01  -0.36027D-01

ITERATION NO.=  550
2       0.99777D-05  -0.45872D-04  -0.12560D 01  -0.12558D 01  -0.12281D-03  -0.12279D-03  -0.28543D-01  -0.11297D-02

ITERATION NO.=  600
2       0.977600-05  -0.45404D-04  -0.12582D 01  -0.12580D 01  -0.11798D-03  -0.11796D-03  -0.13891D-01   0.19113D-02

ITERATION NO.=  650
2       0.959050-05  -0.44977D-04  -0.12604D 01  -0.12602D 01  -0.12794D-03  -0.12792D-03  -0.63145D-02   0.11274D-01
```

ITERATION NO.= 700

2 0.94162D-05 -0.44507D-04 -0.12583D 01 -0.12582D 01 -0.11869D-03 -0.11867D-03 -0.22895D-02 0.49465D-02

ITERATION NO.= 750

2 0.92936D-05 -0.44167D-04 -0.12563D 01 -0.12561D 01 -0.11970D-03 -0.11968D-03 -0.17975D-02 0.48456D-02

ITERATION NO.= 800

2 0.92083D-05 -0.43917D-04 -0.12541D 01 -0.12540D 01 -0.11696D-03 -0.11694D-03 -0.87650D-03 0.14823D-02

ITERATION NO.= 850

2 0.91562D-05 -0.43767D-04 -0.12529D 01 -0.12528D 01 -0.11795D-03 -0.11793D-03 -0.68693D-05 0.20399D-02

ITERATION NO.= 900

2 0.91328D-05 -0.43684D-04 -0.12514D 01 -0.12513D 01 -0.11735D-03 -0.11733D-03 -0.28648D-04 0.54050D-03

ITERATION NO.= 950

2 0.91257D-05 -0.43651D-04 -0.12505D 01 -0.12504D 01 -0.11666D-03 -0.11664D-03 -0.13043D-03 -0.30648D-03

ITERATION NO.= 1000

2 0.91327D-05 -0.43660D-04 -0.12500D 01 -0.12499D 01 -0.11716D-03 -0.11714D-03 -0.35232D-03 -0.73921D-03

ITERATION NO.= 1050

2 0.91430D-05 -0.43683D-04 -0.12498D 01 -0.12497D 01 -0.11652D-03 -0.11651D-03 0.25467D-04 -0.10233D-02

ITERATION NO.= 1100

2 0.91582D-05 -0.43724D-04 -0.12500D 01 -0.12499D 01 -0.11739D-03 -0.11737D-03 0.14566D-03 -0.46905D-03

ITERATION NO.= 1150

2 0.91708D-05 -0.43758D-04 -0.12501D 01 -0.12500D 01 -0.11688D-03 -0.11687D-03 0.29969D-03 -0.57617D-03

ITERATION NO.= 1200

2	0.91824D-05	-0.43791D-04	-0.12504D 01	-0.12502D 01	-0.11742D-03	-0.11740D-03	0.77408D-04	-0.22635D-03

ITERATION NO.= 1250

2	0.91892D-05	-0.43810D-04	-0.12505D 01	-0.12504D 01	-0.11704D-03	-0.11702D-03	0.60371D-05	-0.40666D-03

CONVERGENCE REACHED AT ITERATION NO.= 1259 DELT= 0.20000D-03

I	UPH	W	ENPH	ENTH	EMPH	EMTH	R	YY
2	0.91903D-05	-0.43813D-04	-0.12505D 01	-0.12504D 01	-0.11711D-03	-0.11709D-03	0.10000D-01	0.39998D 00
3	0.61635D-04	-0.43599D-04	-0.12478D 01	-0.12440D 01	-0.11771D-03	-0.11706D-03	0.67059D-01	0.39910D 00
4	0.11436D-03	-0.43069D-04	-0.12441D 01	-0.12265D 01	-0.11908D-03	-0.11698D-03	0.12412D 00	0.39691D 00
5	0.16715D-03	-0.42225D-04	-0.12316D 01	-0.11996D 01	-0.12291D-03	-0.11748D-03	0.18118D 00	0.39338D 00
6	0.22116D-03	-0.41065D-04	-0.12241D 01	-0.11603D 01	-0.12617D-03	-0.11761D-03	0.23824D 00	0.38848D 00
7	0.27463D-03	-0.39595D-04	-0.12010D 01	-0.11127D 01	-0.13512D-03	-0.11919D-03	0.29529D 00	0.38216D 00
8	0.33098D-03	-0.37805D-04	-0.11900D 01	-0.10500D 01	-0.14387D-03	-0.12077D-03	0.35235D 00	0.37435D 00
9	0.38536D-03	-0.35693D-04	-0.11555D 01	-0.97945D 00	-0.16573D-03	-0.12614D-03	0.40941D 00	0.36494D 00
10	0.44502D-03	-0.33220D-04	-0.11410D 01	-0.88875D 00	-0.18115D-03	-0.13314D-03	0.46647D 00	0.35381D 00
11	0.49988D-03	-0.30346D-04	-0.10936D 01	-0.78673D 00	-0.24352D-03	-0.14839D-03	0.52353D 00	0.34080D 00
12	0.56265D-03	-0.26953D-04	-0.10743D 01	-0.65267D 00	-0.30301D-03	-0.16705D-03	0.58059D 00	0.32568D 00
13	0.61448D-03	-0.22918D-04	-0.10101D 01	-0.49430D 00	-0.39962D-03	-0.19698D-03	0.63765D 00	0.30813D 00
14	0.67436D-03	-0.18021D-04	-0.98235D 00	-0.27857D 00	-0.47297D-03	-0.22128D-03	0.69471D 00	0.28772D 00
15	0.70709D-03	-0.12155D-04	-0.89294D 00	-0.15164D-01	-0.53356D-03	-0.23970D-03	0.75176D 00	0.26377D 00
16	0.73537D-03	-0.52868D-05	-0.85038D 00	0.32910D 00	-0.39473D-03	-0.19758D-03	0.80882D 00	0.23522D 00
17	0.68638D-03	0.18523D-05	-0.72255D 00	0.68955D 00	-0.32616D-04	-0.80360D-04	0.86588D 00	0.20010D 00
18	0.57229D-03	0.76335D-05	-0.67225D 00	0.97945D 00	0.93956D-03	0.21894D-03	0.92294D 00	0.15398D 00
19	0.15478D-03	0.60612D-05	-0.48425D 00	0.57981D 00	0.13726D-02	0.37576D-03	0.98000D 00	0.79599D-01
20	0.10582D-03	0.49109D-05	-0.47865D 00	0.44372D 00	0.95332D-03	0.27003D-03	0.98633D 00	0.65905D-01
21	0.53942D-04	0.32843D-05	-0.46498D 00	0.25764D 00	0.17344D-03	0.67959D-04	0.99267D 00	0.48354D-01

22	0.66329D-05	0.63398D-06	-0.46118D 00	-0.43513D-01	-0.19841D-02	-0.49040D-03	0.99900D 00	0.17884D-01
23	0.00000D 00	0.00000D 00	-0.44437D 00	-0.11109D 00	-0.39752D-02	-0.99380D-03	0.10000D 01	0.78850D-08

RESIDUALS

	APH	AW
2	-0.48197D-04	-0.39126D-03
3	-0.33988D-03	-0.29906D-03
4	-0.22663D-03	-0.18844D-03
5	-0.36202D-04	-0.16817D-03
6	-0.26767D-03	-0.19938D-03
7	-0.26674D-03	-0.20923D-03
8	-0.15438D-03	-0.19451D-03
9	-0.14907D-04	-0.17220D-03
10	-0.23510D-03	-0.13268D-03
11	-0.50459D-03	-0.10268D-03
12	-0.25804D-03	-0.11520D-03
13	-0.12294D-03	-0.12840D-03
14	-0.11014D-03	-0.78106D-04
15	-0.24584D-03	0.83069D-05
16	-0.69851D-03	0.26818D-04
17	-0.26012D-04	0.22656D-04
18	-0.15198D-03	0.18616D-04
19	-0.71377D-03	-0.78335D-04
20	0.29106D-03	-0.44578D-04
21	0.40881D-03	-0.43820D-05
22	0.77993D-03	0.39746D-05

RESULTS

NON LINEAR STATIC ANALYSIS OF SHELLS OF REVOLUTION BY DYNAMIC RELAXATION

SHELL TYPE: ELLIPTICAL

NON LINEAR EQUILIBRIUM AND STRAIN-DISPLACEMENT

LINEAR ISOTROPIC   STRESS-STRAIN

INNER EDGE   SYMMETRICAL

OUTER EDGE   FIXED

MATERIAL CONSTANTS

EPH=   0.25000D 07

ETH=   0.25000D 07

PRPH=   0.25000

PRTH=   0.25000

G=   0.10000D 07

DENSITIES

DPH=   1.00000

DW=   1.00000

ITERATION FACTORS

DFPH=   0.45000D 02

DFW=   0.20000D 03

DELT=   0.20000D-03

ITERATION CONTROL PARAMETERS

DELTMAX=   0.20000D-03   DMAX=   0.10000D 04

IMAX= 2000   ITIN=   50

FINITE DIFFERENCE MESH GENERATION AND LOCATION PARAMETERS

NDPHI= 17   NDPHM= 3

NDPHO= 1   NPHI= 2

OTHER CONTROL PARAMETERS

ISHELL= 3   IOSHEL = 2

ICASE= 2

IBCI= 4   IBCO= 1

ISTCHK= 0

NDFCHK= 0

SHELL GEOMETRY

XI = 0.01000

XA = 0.98000

XB= 0.99900

XO= 1.00000

YI= 0.40000

HAO= 100.00000

LOAD DATA

FLDW= -0.100000D 05

FLDPH= 0.000000D 00

I	UPH	W	ENPH	ENTH	EMPH	EMTH	APH	AW
ITERATION NO.= 1								
2	0.00000D 00	0.20000D-03	0.12001D 02	0.12001D 02	-0.18484D-16	0.18491D-16	0.00000D 00	0.10000D 05
ITERATION NO.= 50								
2	0.34278D-02	0.19943D 00	0.12486D 05	0.12486D 05	0.22156D 00	0.22153D 00	0.21519D 03	0.25045D 03
ITERATION NO.= 100								
2	0.13291D-01	0.21339D 00	0.14803D 05	0.14804D 05	0.11807D 01	0.11805D 01	-0.83906D 03	-0.22865D 04
ITERATION NO.= 150								
2	-0.41440D-01	0.19888D 00	0.57260D 04	0.57205D 04	-0.11662D 01	-0.11681D 01	0.25149D 04	0.58287D 04

```
ITERATION NO.=  200
2     -0.31230D-01    0.30413D 00    0.13566D 05    0.13569D 05    0.17178D 01    0.17176D 01   -0.42035D 04   -0.22239D 04

ITERATION NO.=  250
2     -0.53060D-01    0.34954D 00    0.13018D 05    0.13019D 05    0.27459D 01    0.27454D 01   -0.47628D 04   -0.35628D 04

ITERATION NO.=  300
2     -0.76098D-01    0.38995D 00    0.11985D 05    0.11988D 05    0.48198D 00    0.48191D 00   -0.15147D 04    0.40832D 03

ITERATION NO.=  350
2     -0.90751D-01    0.42335D 00    0.11794D 05    0.11794D 05    0.63654D 00    0.63644D 00    0.19312D 02    0.71659D 03

ITERATION NO.=  400
2     -0.96965D-01    0.44098D 00    0.11922D 05    0.11920D 05    0.82503D 00    0.82491D 00    0.44524D 03    0.61769D 03

ITERATION NO.=  450
2     -0.98053D-01    0.44686D 00    0.12112D 05    0.12110D 05    0.10022D 01    0.10020D 01    0.41549D 03    0.33613D 03

ITERATION NO.=  500
2     -0.96345D-01    0.44490D 00    0.12251D 05    0.12249D 05    0.10943D 01    0.10941D 01    0.17568D 03    0.12369D 03

ITERATION NO.=  550
2     -0.93503D-01    0.44019D 00    0.12394D 05    0.12393D 05    0.11558D 01    0.11557D 01   -0.41131D 02   -0.10647D 03

ITERATION NO.=  600
2     -0.91499D-01    0.43492D 00    0.12379D 05    0.12377D 05    0.12457D 01    0.12455D 01   -0.22157D 02   -0.15375D 03

ITERATION NO.=  650
2     -0.89325D-01    0.42874D 00    0.12334D 05    0.12333D 05    0.11831D 01    0.11829D 01    0.10133D 02   -0.95285D 02
```

```
ITERATION NO.=  700
2     -0.87727D-01    0.42412D 00    0.12296D 05    0.12294D 05    0.11715D 01    0.11713D 01    0.34991D 02   -0.49924D 02

ITERATION NO.=  750
2     -0.86505D-01    0.42068D 00    0.12273D 05    0.12272D 05    0.11325D 01    0.11323D 01    0.13925D 02   -0.19168D 02

ITERATION NO.=  800
2     -0.85904D-01    0.41894D 00    0.12259D 05    0.12257D 05    0.11653D 01    0.11651D 01    0.18313D 01   -0.33733D 02

ITERATION NO.=  850
2     -0.85542D-01    0.41778D 00    0.12244D 05    0.12242D 05    0.11329D 01    0.11328D 01   -0.14126D 02   -0.70996D 01

ITERATION NO.=  900
2     -0.85543D-01    0.41760D 00    0.12232D 05    0.12231D 05    0.11466D 01    0.11464D 01   -0.66803D 01   -0.50051D 01

ITERATION NO.=  950
2     -0.85641D-01    0.41768D 00    0.12223D 05    0.12221D 05    0.11266D 01    0.11264D 01   -0.37326D 01    0.17808D 02

ITERATION NO.= 1000
2     -0.85852D-01    0.41819D 00    0.12221D 05    0.12220D 05    0.11385D 01    0.11384D 01   -0.50105D-01    0.10388D 02

ITERATION NO.= 1050
2     -0.86067D-01    0.41873D 00    0.12222D 05    0.12221D 05    0.11333D 01    0.11332D 01   -0.15779D 01    0.13485D 02

ITERATION NO.= 1100
2     -0.86278D-01    0.41932D 00    0.12225D 05    0.12224D 05    0.11409D 01    0.11408D 01   -0.96029D 00    0.81937D 01

ITERATION NO.= 1150
2     -0.86435D-01    0.41975D 00    0.12228D 05    0.12227D 05    0.11401D 01    0.11399D 01   -0.11994D 00    0.58221D 01
```

ITERATION NO.= 1200

2	-0.86528D-01	0.42004D 00	0.12231D 05	0.12230D 05	0.11418D 01	0.11416D 01	0.19200D 00	0.28229D 01

ITERATION NO.= 1250

2	-0.86575D-01	0.42019D 00	0.12233D 05	0.12232D 05	0.11435D 01	0.11433D 01	0.37502D 00	0.10096D 01

ITERATION NO.= 1300

2	-0.86578D-01	0.42022D 00	0.12234D 05	0.12233D 05	0.11430D 01	0.11428D 01	-0.37365D 00	-0.45361D 00

ITERATION NO.= 1350

2	-0.86572D-01	0.42022D 00	0.12235D 05	0.12234D 05	0.11443D 01	0.11441D 01	0.13697D 00	-0.17548D 01

CONVERGENCE REACHED AT ITERATION NO.= 1400 DELT= 0.20000D-03

I	UPH	W	ENPH	ENTH	EMPH	EMTH	R	YY
2	-0.86552D-01	0.42017D 00	0.12235D 05	0.12234D 05	0.11424D 01	0.11423D 01	0.10000D-01	0.39998D 00
3	-0.58046D 00	0.41808D 00	0.12207D 05	0.12170D 05	0.11512D 01	0.11437D 01	0.67059D-01	0.39910D 00
4	-0.10772D 01	0.41291D 00	0.12172D 05	0.11997D 05	0.11728D 01	0.11472D 01	0.12412D 00	0.39691D 00
5	-0.15744D 01	0.40464D 00	0.12047D 05	0.11731D 05	0.12248D 01	0.11597D 01	0.18118D 00	0.39338D 00
6	-0.20836D 01	0.39322D 00	0.11974D 05	0.11337D 05	0.12773D 01	0.11716D 01	0.23824D 00	0.38848D 00
7	-0.25866D 01	0.37865D 00	0.11743D 05	0.10858D 05	0.13865D 01	0.11991D 01	0.29529D 00	0.38216D 00
8	-0.31176D 01	0.36076D 00	0.11635D 05	0.10221D 05	0.14900D 01	0.12256D 01	0.35235D 00	0.37435D 00
9	-0.36263D 01	0.33951D 00	0.11287D 05	0.94983D 04	0.17040D 01	0.12832D 01	0.40941D 00	0.36494D 00
10	-0.41854D 01	0.31453D 00	0.11140D 05	0.85666D 04	0.19210D 01	0.13449D 01	0.46647D 00	0.35381D 00
11	-0.46911D 01	0.28555D 00	0.10658D 05	0.75219D 04	0.23487D 01	0.14676D 01	0.52353D 00	0.34080D 00
12	-0.52723D 01	0.25171D 00	0.10462D 05	0.61700D 04	0.27860D 01	0.15999D 01	0.58059D 00	0.32568D 00
13	-0.57384D 01	0.21229D 00	0.98132D 04	0.46154D 04	0.35395D 01	0.18216D 01	0.63765D 00	0.30813D 00
14	-0.62867D 01	0.16571D 00	0.95428D 04	0.25596D 04	0.40807D 01	0.19868D 01	0.69471D 00	0.28772D 00
15	-0.65705D 01	0.11139D 00	0.86597D 04	0.13489D 03	0.46344D 01	0.21331D 01	0.75176D 00	0.26377D 00

16	-0.68400D 01	0.48897D-01	0.82731D 04	-0.29878D 04	0.35102D 01	0.17733D 01	0.80882D 00	0.23522D 00
17	-0.63839D 01	-0.15780D-01	0.70441D 04	-0.62226D 04	0.51614D 00	0.78659D 00	0.86588D 00	0.20010D 00
18	-0.53674D 01	-0.69195D-01	0.66211D 04	-0.88997D 04	-0.83394D 01	-0.19312D 01	0.92294D 00	0.15398D 00
19	-0.14582D 01	-0.56951D-01	0.47810D 04	-0.53902D 04	-0.13100D 02	-0.35753D 01	0.98000D 00	0.79599D-01
20	-0.99868D 00	-0.46319D-01	0.47279D 04	-0.41308D 04	-0.92849D 01	-0.26159D 01	0.98633D 00	0.65905D-01
21	-0.51206D 00	-0.31138D-01	0.45946D 04	-0.23957D 04	-0.20402D 01	-0.74085D 00	0.99267D 00	0.48354D-01
22	-0.65509D-01	-0.60834D-02	0.45595D 04	0.45088D 03	0.18552D 02	0.45843D 01	0.99900D 00	0.17884D-01
23	0.00000D 00	0.00000D 00	0.43895D 04	0.10974D 04	0.38150D 02	0.95375D 01	0.10000D 01	0.78850D-08

RESIDUALS

	APH	AW
2	0.17970D 00	-0.73247D 00
3	-0.10160D 01	-0.72556D 00
4	-0.24429D 00	-0.82310D 00
5	0.17677D 01	-0.10833D 01
6	0.32433D 01	-0.12170D 01
7	0.40662D 01	-0.10503D 01
8	0.45084D 01	-0.80493D 00
9	0.45212D 01	-0.73488D 00
10	0.40021D 01	-0.77037D 00
11	0.48914D 01	-0.78485D 00
12	0.71909D 01	-0.77135D 00
13	0.91111D 01	-0.66909D 00
14	0.84173D 01	-0.39917D 00
15	0.65441D 01	-0.82571D-01
16	0.63321D 01	0.27656D-01
17	0.86822D 01	0.18720D 00
18	0.60376D 01	0.42225D 00
19	-0.23356D 01	-0.30218D 00
20	0.18238D 01	-0.18256D 00
21	0.47307D 00	0.19603D-01
22	-0.35199D 01	-0.31708D-01

APPENDIX D

GUIDE TO INUPUT DATA AND DESCRIPTION
OF VARIABLES IN PROGRAMS NSDRSHELL
AND NSDRGSHELL

Input Data Cards

The following data cards are read using formatless READ statements. The last data card which is a description of the problem uses alphanumeric format.

1. EPH, ETH, PRPH, PRTH, G

1 80

EPH = Elastic modulus in the ϕ direction

ETH = Elastic modulus in the θ direction

PRPH = Poisson's ratio in the ϕ direction

PRTH = Poisson's ratio in the θ direction

G = Shear modulus

2. ISHELL, IOSHELL

1 80

ISHELL = 1, Hemispherical shell of revolution

= 2, Parabolic shell of revolution

= 3, Elliptic shell of revolution

IOSHELL = 1, Shell with central opening

$\neq$ 1, Shell without central opening

3. ICASE, ISTCHK, NDFCHK

1 80

ICASE = 1, Linear analysis

$\neq$ 1, Nonlinear analysis

ISTCHK = 1, Check for convergence of stress resultants

≠ 1, No check for convergence of stress resultants

NDFCHK = 1, Undamped analysis (Damping factors = 0)

≠ 1, Damped analysis

4. DPH, DW

1 80

DPH = Non dimensional mass/unit area in the ∅ direction

DW = Non dimensional mass/unit area in the z direction

5. DFPH, DFW, DELT

1 80

DFPH = Non dimensional damping factor in the ∅ direction

DFW = Non dimensional damping factor in the z direction

DELT = Non dimensional time increment

6. IMAX, ITIN, DELTM, DMAX

1 80

IMAX = Maximum number of iterations

ITIN = Iteration interval for printing the displacements and stress resultants at any selected node

DELTM = Minimum time increment

DMAX = Prescribed displacement limit for checking numerical instability

7. NDPHI, NDPHM, NDPHO, NPHI, IP

1 ... 80

NDPHI = Number of mesh spacings near the inner edge of shell

NDPHM = Number of mesh spacings at the interior of shell

NDPHO = Number of mesh spacing near the outer edge of shell

NPHI = Reference number of the node at the inner edge of shell (Figure 5 or 6)

IP = Reference number of the node where displacements and stress resultants are to be printed at specified iteration intervals

8. PHII, PHIA, PHIB, PHIO, HAO; or
XI, XA, XB, XO, YI, HAO (See Figures 5 and 6)

1 ... 80

PHII = Inner opening angle of the shell in degrees

PHIA, PHIB = Intermediate angles which are arbitary (degrees)

PHIO = Outer opening angle of the shell in degrees

XI = Inner opening axial distances (non dimensional)

XA, XB = Arbitary distances (non dimensional)

XO = Outer opening axial distances (non dimensional)

YI = Ordinate at the inner opening (non dimensional)

HAO = a/h_o, ratio of representative length to representative thickness of shell

9. IBCI, IBCO

1 80

IBCI = 1, Clamped boundary condition at inner edge
2, Hinged boundary condition at inner edge
4, Symmetry boundary condition at inner edge

IBCO = 1, Clamped boundary condition at outer edge
2, Hinged boundary condition at outer edge
4, Symmetry boundary condition at outer edge

10. FLDW, FLDPH

1 80

FLDW = Non dimensional load/unit area in the z direction

FLDPH = Non dimensional load/unit area in the ϕ direction

11. CHAR(I)

4A4	1A4	5A4	3A4	3A4	
1	17	21	41	53	65 80

Columns 1-16: Type of Shell, e.g. 'HEMISPHERICAL'
17-20: 'NON '
21-40: 'LINEAR ISOTROPIC '
41-52: Type of boundary condition at inner edge

e.g. 'SYMMETRIC'

53-64: Type of boundary condition at outer edge

e.g. 'FIXED '

Description of Other Variables

UPH(I) = Meridional displacements

W(I) = Normal displacements

ENPH(I) = In plane stress resultants in ϕ direction

ENTH(I) = In plane stress resultants in θ direction

EMPH(I) = Moment stress resultants in ϕ direction

EMTH(I) = Moment stress resultants in θ direction

APH(I), AW(I) = Residuals of static equilibrium equations in the ϕ and z directions

VPH(I), VW(I) = Velocities in the ϕ and z directions

R(I) = Distance from axis of revolution to the nodal point

YY(I) = Ordinates of the meridian of the shell

RPH(I), RTH(I) = Principal radii of curvature r'_ϕ and r'_θ respectively at any nodal point on shell

APHI(I) or APHN(I) = Meridional angle at any point on shell

H(I) = Thickness of shell at any nodal point

QPH(I), QW(I) = Non dimensional loads in the meridional and normal directions, respectively

ACCW, ACCPH = Prescribed accuracies for residuals of equilibrium equations in the z and ϕ directions respectively

NPHO = Reference number of the node at the outer edge

APPENDIX E

CALCULATION OF TIME INCREMENT

This Appendix shows the procedure for calculating the time increment $\Delta t'$. The time increment has been calculated using only the linear terms in the governing equations. This gives a good estimate of the time increment to start with and for larger loads the time increment can be suitably modified by a multiplying factor, according to the type of nonlinearity. For stiffening structures, the time increment has to be reduced. For softening type of structures, time increment has to be increased.

Calculation of Time Increment

Governing Equations for Linear Axysymmetric Shell of Revolution

From Equations (3.5) the linear stress resultant displacement equations are,

$$N'_\phi = 12h'\frac{E_\phi}{E_\theta}\left[\frac{a}{h_o}\frac{\partial u'_\phi}{r'_\phi\,\partial\phi} + \left(\frac{a}{h_o}\right)^2\frac{w'}{r'_\phi} + \nu_{\phi\theta}\left\{\frac{a}{h_o}\frac{u'_\phi\cos\phi}{r'} + \left(\frac{a}{h_o}\right)^2\frac{w'\sin\phi}{r'}\right\}\right]$$

$$N'_\theta = 12h'\left[\frac{a}{h_o}\frac{u'_\phi\cos\phi}{r'} + \left(\frac{a}{h_o}\right)^2\frac{w'\sin\phi}{r'} + \nu_{\theta\phi}\left\{\frac{a}{h_o}\frac{\partial u'_\phi}{r'_\phi\,\partial\phi} + \left(\frac{a}{h_o}\right)^2\frac{w'}{r'_\phi}\right\}\right]$$

$$M'_\phi = \frac{E_\phi h'^3}{E_\theta}\left[\frac{h_o}{a}\frac{1}{r'_\phi}\frac{\partial u'_\phi}{r'_\phi\,\partial\phi} - \frac{\partial^2 w'}{r'^2_\phi\,\partial\phi^2} + \frac{\partial r'_\phi}{r'^2_\phi\,\partial\phi}\left(\frac{\partial w'}{r'_\phi\,\partial\phi} - \frac{h_o}{a}\frac{u'_\phi}{r'_\phi}\right) + \nu_{\phi\theta}\cos\phi\left\{\frac{h_o}{a}\frac{u'_\phi}{r'r'_\phi} - \frac{1}{r'r'_\phi}\frac{\partial w'}{\partial\phi}\right\}\right] \quad \text{(E.1)}$$

$$M'_\theta = h'^3[\cos\phi\{\frac{h_o}{a}\frac{u'_\phi}{r' r'_\phi} - \frac{1}{r' r'_\phi}\frac{\partial w'}{\partial \phi}\} + \nu_{\theta\phi}\{\frac{h_o}{a}\frac{1}{r'_\phi}\frac{\partial u'_\phi}{r'_\phi \partial \phi}$$

$$-\frac{\partial^2 w'}{r'^2_\phi \partial \phi^2} + \frac{1}{r'^2_\phi}\frac{\partial r'_\phi}{\partial \phi}(\frac{\partial w'}{r'_\phi \partial \phi} - \frac{h_o}{a}\frac{u'_\phi}{r'_\phi})\}]$$

The linear equilibrium equations are obtained from Equations (3.4) (omitting inertia and damping terms)

$$\frac{\partial N'_\phi}{r'_\phi \partial \phi} + (N'_\phi - N'_\theta)\frac{\cos\phi}{r'} + (M'_\phi - M'_\theta)\frac{\cos\phi}{r' r'_\phi}$$

$$+ \frac{1}{r'_\phi}\frac{\partial M'_\phi}{r'_\phi \partial \phi'} + q'_\phi = 0$$

$$\frac{\partial^2 M'_\phi}{r'^2_\phi \partial \phi^2} - \frac{\sin\phi}{r' r'_\phi}(M'_\phi - M'_\theta) + \frac{2\cos\phi}{r'}\frac{\partial M'_\phi}{r'_\phi \partial \phi} \tag{E.2}$$

$$- \frac{1}{r'^2_\phi}\frac{\partial r'_\phi}{\partial \phi}\frac{\partial M'_\phi}{r'_\phi \partial \phi} - \frac{\cos\phi}{r'}\frac{\partial M'_\theta}{r'_\phi \partial \phi} - \frac{N'_\phi}{r'_\phi} - \frac{N'_\theta}{r'_\theta} - q' = 0$$

The coefficients of N'_ϕ, N'_θ, M'_ϕ, and M'_θ at any node i are obtained from Equation (E.1) as,

$$|\text{Coeff. of } N'_{\phi i}| = 12h'\frac{E_\phi}{E_\theta}[\frac{a}{h_o}\frac{1}{r'_{\phi i}\Delta\phi_i} + (\frac{a}{h_o})^2\frac{1}{r'_{\phi i}}$$

$$+ \nu_{\phi\theta}\{\frac{a}{h_o}\frac{\cos\phi_i}{r'_i} + (\frac{a}{h_o})^2\frac{\sin\phi_i}{r'_i}\}]$$

$$|\text{Coeff. of } N'_{\theta i}| = 12h'[\frac{a}{h_o}\frac{\cos\phi_i}{r'_i} + (\frac{a}{h})^2\frac{\sin\phi_i}{r'_i}$$

$$+ \nu_{\theta\phi}\{\frac{a}{h_o}\frac{1}{r'_{\phi i}\Delta\phi_i} + (\frac{a}{h_o})^2\frac{1}{r'_{\phi i}}\}]$$

$$|\text{Coeff. of } M'_{\phi_i}| = \frac{E_\phi h'^3}{E_\theta}\Big[\frac{h_o}{a}\frac{1}{r'_{\phi_i}}\frac{1}{r'_{\phi_i}\Delta\phi_i} + \frac{4}{(r'_{\phi_i}\Delta\phi_i)^2}$$

$$+ \frac{1}{r'^2_{\phi_i}}\frac{(r'_{\phi_{i+1}} - r'_{\phi_{i-1}})}{2\,\Delta\phi_i}\Big(\frac{1}{r'_{\phi_i}\Delta\phi_i} + \frac{h_o}{a}\frac{1}{r'_{\phi_i}}\Big)$$

$$+ \nu_{\phi\theta}\cos\phi_i\Big\{\frac{h_o}{a}\frac{1}{r'_i\, r'_{\phi_i}} + \frac{1}{r'_i}\frac{1}{r'_{\phi_i}\Delta\phi_i}\Big\}\Big] \tag{E.3}$$

$$|\text{Coeff. of } M'_{\theta_i}| = h'^3\Big[\cos\phi_i\Big\{\frac{h_o}{a}\frac{1}{r'_i\, r'_{\phi_i}} + \frac{1}{r'_i\, r'_{\phi_i}\Delta\phi_i}\Big\}$$

$$+ \nu_{\theta\phi}\Big\{\frac{h_o}{a}\frac{1}{r'_{\phi_i}}\frac{1}{r'_{\phi_i}\Delta\phi_i} + \frac{4}{(r'_{\phi_i}\Delta\phi_i)^2}$$

$$+ \frac{1}{r'^2_{\phi_i}}\frac{(r'_{\phi_{i+1}} - r'_{\phi_{i-1}})}{2\,\Delta\phi_i}\Big(\frac{1}{r'_{\phi_i}\Delta\phi_i} + \frac{h_o}{a}\frac{1}{r'_{\phi_i}}\Big)\Big\}\Big]$$

The sum of the coefficients of a row of the finite difference stiffness matrix obtained from Equation (E.2) is given by,

'ϕ' direction:

$$b_{\phi_i} = |\text{coeff. of } N'_{\phi_i}|\Big(\frac{1}{r'_{\phi_i}\Delta\phi_i} + \frac{\cos\phi_i}{r'_i}\Big)$$

$$+ |\text{coeff. of } N'_{\theta_i}|\Big(\frac{\cos\phi_i}{r'_i}\Big)$$

$$+ |\text{coeff. of } M'_{\phi_i}|\Big(\frac{1}{r'^2_{\phi_i}\Delta\phi_i} + \frac{\cos\phi_i}{r'_i\, r'_{\phi_i}}\Big)$$

$$+ |\text{coeff. of } M'_{\theta_i}|\Big(\frac{\cos\phi_i}{r'_i\, r'_{\phi_i}}\Big)$$

'z' direction:

$$b_{W_i} = |\text{coeff. of } N'_{\phi_i}|\left(\frac{1}{r'_{\phi_i}}\right) \tag{E.4}$$

$$+ |\text{coeff. of } N'_{\theta_i}|\left(\frac{1}{r'_{\theta_i}}\right)$$

$$+ |\text{coeff. of } M'_{\phi_i}|\left\{\frac{4}{(r'_{\phi_i}\Delta\phi_i)^2} + \frac{\sin\phi_i}{r'_i r'_{\phi_i}} + \frac{2\cos\phi_i}{r'_i(r'_{\phi_i}\Delta\phi_i)}\right.$$

$$\left. + \frac{1}{r'^2_{\phi_i}}\,\frac{(r'_{\phi_{i+1}} - r'_{\phi_{i-1}})}{2\Delta\phi_i}\left(\frac{1}{r'_{\phi_i}\Delta\phi_i}\right)\right\}$$

$$+ |\text{coeff. of } M'_{\theta_i}|\left\{\frac{\sin\phi_i}{r'_i\, r'_{\phi_i}} + \frac{\cos\phi_i}{r'_i\,(r'_{\phi_i}\Delta\phi_i)}\right\}$$

The largest values of b_{ϕ_i} and b_{W_i} are used to determine the critical time increments in the ϕ and z directions respectively from,

$$\Delta t_\phi \le 2\sqrt{\frac{m_\phi}{b_{\phi_{max}}}}$$

$$\Delta t_W \le 2\sqrt{\frac{m_W}{b_{W_{max}}}} \tag{E.5}$$

The smaller of Δt_ϕ or Δt_W is chosen as the time increment to be used in the dynamic relaxation procedure.

2

VITA

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Doctor of Philosophy

Thesis: NONLINEAR STATIC ANALYSIS OF SHELLS OF REVOLUTION BY DYNAMIC RELAXATION

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