SPATIAL EQUILIBRIUM THEORY AND PRICE DISCRIMINATION IN THE SPATIAL MARKET

By

DAVID ALLAN PYLES

Bachelor of Science
Texas Tech University
Lubbock, Texas
1980

Master of Science
Oklahoma State University
Stillwater, Oklahoma
1983

Submitted to the Faculty of the
Graduate College of the
Oklahoma State University
in partial fulfillment of
the requirements for
the Degree of
DOCTOR OF PHILOSOPHY
December, 1986

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Thesis Approved:

Leo V. Blakley
Thesis Adviser

Michael R. Edgmand

Daniel S. Tilley

James G. Plaxico

Norman N. Durham
Dean of the Graduate College

PREFACE

This study considers spatial equilibrium theory and the practice of price discrimination in the spatial market. The first objective of the study was the development of mathematical means to the determination of optimal price and quantity vectors for a discriminator operating in a spatial market. The second objective was the development of a trade policy whereby the discriminator could influence the dynamics of the market such that it would converge upon and equilibrate at the optimal price and quantity vectors. Since the price discrimination models are derivatives of the ordinary spatial equilibrium model, spatial equilibrium theory is a major topic in this work. Also, since all economic theory of this work is expressed in terms of nonlinear programming models, one chapter is dedicated to the presentation of nonlinear programming theory.

The topic was originally suggested by Dr. L.V. Blakley. It was felt that the present deterioration in the U.S balance of trade demanded further investigations into international trade policy. Particular concern was had for policies which could improve the trade situation for agricultural commodities. The original intents included the measurement of potential gains from a discriminatory pricing policy in U.S. international wheat trade. However, since the theory pertaining to such policy proved to be more involved than anticipated, the study was confined to theoretical investigation.

I extend gratitude to my adviser, Dr. L.V. Blakley, for suggesting this topic and for the degree of latitude that was granted in this and other graduate work. A better adviser could not have been had. I am appreciative to Dr. J.S. Plaxico for his efforts, which were more than would be expected of a committee member. Also, Dr. Plaxico encouraged academic training which proved essential to the dynamic analyses in this work. I also extend thanks to Dr. Dan Tilley and Dr. Michael Edgeman for their contributions.

I am appreciative to the faculty and staff of the Department of Agricultural Economics of this university for its efforts toward an outstanding academic program. I extend unending thanks to all American taxpayers for their financial contributions, which are too frequently unnoticed, but which have made this and all other of my academic endeavors possible.

Finally, I dedicate this work to my parents, who are to be credited to the fullest extent. Indeed, I reckon this and all other academic accomplishments as being no less theirs than mine.

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CHAPTER I

INTRODUCTION

Whenever the concept of price discrimination is encountered in economic literature, it is almost always found under the assumptions that the discriminator is a pure monopolist, and that the various markets into which it sells are or can be perfectly separated. However, it is evident that these assumptions are unnecessarily restrictive. Pure monopoly is seldomly if ever observed, and the incidence of perfect market separation is equally infrequent. Yet few would question that price discrimination does actually occur in practice.

The possibilities of gains from price discrimination in the absence of these assumptions is illustrated by the spatial market. Here, it is assumed that the aggregate market is composed of several regional markets with possibly numerous buyers and sellers in each. It is also assumed that at least some of the regional markets are at liberty to trade with one another, but that there are nonzero costs of transporting the product between regions. Now, consider a spatial market consisting of three regional markets that are situated on a line. Thus, the market map might appear as follows:

A <--------- B ---------> C

where A, B, and C represent the three regional markets. Suppose that interregional trade occurs in the directions of the arrows shown. Moreover, suppose B incurs a $1 per-unit transportation charge on shipments to A and on shipments to C, and that the costs of shipping between A and C are $2 per-unit.

In an ordinary spatial equilibrium, the price differences between trading markets will be exactly equal to the per-unit costs of transporting the product between regions. Hence, in the market described above, the price in A and C will be exactly $1 greater than the price in B. Now, it is apparent that the spatial equilibrium prices and quantities are not necessarily optimal insofar as the maximization of B's export revenue is concerned. If the excess demand in A is more elastic than the excess demand in C, then B can increase its export revenue with a proper price reduction to A and price increase to C. Moreover, B is capable of making such price adjustments to a limited extent, for it can charge prices to A and C differing by as much as $2 without inducing arbitrage. Thus, the partial degree of market separation in the spatial model can permit successful price discrimination. However, observe that B is not necessarily a monopolist, nor are its markets perfectly separated.

The primary objectives of this study are the determination of optimal price and quantity vectors for a revenue-maximizing discriminator or cooperative of discriminators operating in a spatial market, and the formulation of a trade policy whereby the discriminator or cooperative of discriminators may impose such price and quantity vectors upon the market. Hence, concern is directed toward problems such as the one described above. Consideration is also directed toward those cases where a group of regions exercises discrimination in a cooperative fashion. There are basically two questions to be addressed. First, what are the optimal price and quantity vectors? Second, after having determined the optimal vectors, how does the discriminator then cause the actual quantities and prices in the market to equal the chosen values?

As all economic models in this work are formulated as nonlinear programming problems, the second chapter is dedicated to a thorough and rigorous derivation of general nonlinear programming theory. As the price discrimination models are derivatives of the ordinary spatial equilibrium model, pursuit of the objectives necessitates a thorough development of ordinary spatial equilibrium theory. Spatial equilibrium theory is developed in the third chapter. In the fourth chapter, various price discrimination models are constructed. The chapter considers the case of a single discriminating region, and the case where several regions exercise price discrimination in a cooperative fashion. The chapter begins with the construction of nonlinear programming models having solutions equal to the optimal price and quantity vectors. It is then shown how that the discriminator may influence the dynamic adjustment mechanism of the spatial market such that the market converges to and equilibrates at such vectors. In the fifth chapter, some hypothetical market configurations are constructed, and a spatial equilibrium is calculated for each configuration. It is then assumed that one or more of the regions in the market practices price discrimination. The models are solved again under this assumption, and comparisons are made between earned revenue in the former and latter situations.

CHAPTER II

NONLINEAR PROGRAMMING THEORY[1]

In this chapter, nonlinear optimization theory necessary to subsequent chapters is developed. Primary concern is directed toward deriving necessary and sufficient conditions for solutions to the problem:

$$\begin{aligned} &\text{maximize}(\mathbf{x}): && f(\mathbf{x}) \\ &\text{subject to:} && \mathbf{G(x)} \geq \mathbf{0} \\ & && \mathbf{x} \in X \end{aligned}$$

where: $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$ $\qquad \mathbf{G(x)} = \begin{bmatrix} g_1(\mathbf{x}) \\ g_2(\mathbf{x}) \\ \cdot \\ \cdot \\ \cdot \\ g_m(\mathbf{x}) \end{bmatrix}$

$f(\mathbf{x})$ is called the "objective function." The condition, $\mathbf{G(x)} \geq \mathbf{0}$, is called the "functional constraint," while the condition, $\mathbf{x} \in X$ is called the "set constraint." The set constraint usually serves to establish the general domain of the functions involved in the problem. In practice, X is generally taken to be the n-dimensional euclidean space, hereafter denoted by R^n.

The relation, $\geq$, and associated relations are interpreted as follows: 1) $\mathbf{z} \geq \mathbf{0}$ implies that all components of $\mathbf{z}$ are nonnegative. 2) $\mathbf{z} > \mathbf{0}$ implies that all components of $\mathbf{z}$ are positive, and 3) $\mathbf{z} \geq \mathbf{0}$ implies that $\mathbf{z} \geqq \mathbf{0}$, but $\mathbf{z} \neq \mathbf{0}$. Accordingly, for any two vectors, $\mathbf{z}_1$ and $\mathbf{z}_2$: 1) $\mathbf{z}_1 \geqq \mathbf{z}_2$ implies $\mathbf{z}_1 - \mathbf{z}_2 \geqq \mathbf{0}$. 2) $\mathbf{z}_1 \geq \mathbf{z}_2$ implies $\mathbf{z}_1 - \mathbf{z}_2 \geq \mathbf{0}$, and 3) $\mathbf{z}_1 > \mathbf{z}_2$ implies $\mathbf{z}_1 - \mathbf{z}_2 > \mathbf{0}$. The relations: $\leqq$, $\leq$, and $<$ are similarly defined.

The emphasis of the discussion is upon the "Kuhn-Tucker" optimality conditions. For the latter problem, these conditions are:

$$\mathbf{f}_{\mathbf{x}}(\mathbf{x}) + \mathbf{G}_{\mathbf{x}}(\mathbf{x})\lambda = \mathbf{0}$$

$$\lambda' \mathbf{G}(\mathbf{x}) = 0$$

$$\lambda \geqq \mathbf{0}$$

where $\mathbf{f}_{\mathbf{x}}(\mathbf{x})$ is the gradient of $f(\mathbf{x})$ and $\mathbf{G}_{\mathbf{x}}(\mathbf{x})$ is an nxm matrix whose ith column is the gradient of $g_i(\mathbf{x})$, hereafter denoted by $\nabla \mathbf{g}_i(\mathbf{x})$. The vector, λ, is commonly called the vector of "Lagrangian multipliers." Note that the second condition implies that if $\lambda_i > 0$, then $g_i(\mathbf{x}) = 0$. That is, if the Lagrangian multiplier is positive, then the corresponding constraint is "binding" or "active." Such relationships are called "complementary slackness" relations. Accordingly, the variables involved are said to be "complementary."

The most important conclusions to be drawn in this chapter concerning the Kuhn-Tucker conditions are summarized in the following theorems:

Theorem (Kuhn-Tucker Necessary Conditions): Let X be a nonempty open set in R^n, and let $f(\mathbf{x}):X \rightarrow R^1$ and $\mathbf{G(x)}:X \rightarrow R^m$. Consider the problem to maximize $f(\mathbf{x})$ subject to $\mathbf{x} \in X$ and $\mathbf{G(x)} \geq \mathbf{0}$. Let $\mathbf{G}'(\mathbf{x}) = [\hat{\mathbf{G}}'(\mathbf{x}),\tilde{\mathbf{G}}'(\mathbf{x})]$ where $\tilde{\mathbf{G}}(\mathbf{x})$ is affine. Let $\bar{\mathbf{x}}$ be a local optimal solution to the problem, and let:

$$\hat{I} = \{i: \hat{g}_i(\bar{x}) = 0\}$$
$$\tilde{I} = \{i: \tilde{g}_i(\bar{x}) = 0\}$$

Suppose that $f(\mathbf{x})$ and $\mathbf{G(x)}$ are differentiable at $\bar{\mathbf{x}}$. Moreover, suppose that the $\nabla\hat{\mathbf{g}}_i(\bar{\mathbf{x}})$ are linearly independent for $i \in \hat{I}$, then there exists $(\lambda_1,\lambda_2,,,\lambda_m)$ such that:

$$\mathbf{f_x}(\bar{x}) + \Sigma_{i=1}^m \lambda_i \nabla \mathbf{g}_i(\bar{x}) = \mathbf{0}$$
$$\lambda_i g_i(\bar{x}) = 0; \qquad i = 1,2,,,m$$
$$\lambda_i \geq 0; \qquad i = 1,2,,,m$$

Thus, under the stated assumption of the theorem, local optimal solutions imply Kuhn-Tucker points. As a global optimal solution is also a local optimal solution, it may be concluded that if the various assumptions hold, then the Kuhn-Tucker conditions are necessary conditions to the global optima.

The next theorem cites conditions under which Kuhn-Tucker points imply global optima:

Theorem (Kuhn-Tucker Sufficient Conditions): Let X be a nonempty convex set in R^n, and let $f(\mathbf{x}):X \rightarrow R^1$ and $\mathbf{G(x)}:X \rightarrow R^m$. Consider the problem to maximize $f(\mathbf{x})$ subject to $\mathbf{G(x)} \geq \mathbf{0}$ and $\mathbf{x} \in X$. Let $\bar{\mathbf{x}}$ be a feasible solution, and suppose that there exists $(\lambda_1, \lambda_2,,, \lambda_m)$ such that:

$$\mathbf{f_x(\bar{x})} + \Sigma_{i=1}^{m} \lambda_i \nabla \mathbf{g_1(\bar{x})} = \mathbf{0}$$
$$\lambda_i g_i(\bar{\mathbf{x}}) = 0; \qquad i = 1,2,,,m$$
$$\lambda_i \geq 0; \qquad i = 1,2,,,m$$

Let $I = \{i : g_i(\bar{\mathbf{x}}) = 0\}$, and let $g_i(\mathbf{x})$ for $i \in I$ be quasiconcave at $\bar{\mathbf{x}}$ with respect to points in the feasible region. Moreover, let $f(\mathbf{x})$ be pseudoconcave (strictly pseudoconcave) at $\bar{\mathbf{x}}$ with respect to points in the feasible region, then $\bar{\mathbf{x}}$ is a global optimal solution (unique global optimal solution) to the maximization problem.

After establishing the above, attention is then directed toward the Lagrangian saddle point characterization of the maximization problem. It is shown that under certain assumptions, the solutions to the Kuhn-Tucker conditions may be formulated as the "saddle points" of the "Lagrangian." The Lagrangian to the particular maximization problem at hand is the function:

$$l(\mathbf{x},\boldsymbol{\lambda}) = f(\mathbf{x}) + \boldsymbol{\lambda}' \mathbf{G(x)}; \qquad (\mathbf{x},\boldsymbol{\lambda}) \in X \otimes R_+^m$$

where R_+^m denotes the nonnegative orthant of the m-dimensional euclidean space. $\boldsymbol{\lambda}$ is called the vector of "Lagrangian multipliers" in this context also. $(\bar{\mathbf{x}},\bar{\boldsymbol{\lambda}})$ is said to be a "saddle point" of $l(\mathbf{x},\boldsymbol{\lambda})$ if:

$$l(\mathbf{x},\bar{\boldsymbol{\lambda}}) \leq l(\bar{\mathbf{x}},\bar{\boldsymbol{\lambda}}) \leq l(\bar{\mathbf{x}},\boldsymbol{\lambda}); \qquad \forall\ (\mathbf{x},\boldsymbol{\lambda}) \in X \otimes R_+^m$$

Note that at the saddle point, $l(\mathbf{x},\bar{\lambda})$ is maximized subject to $\mathbf{x} \in X$, and $l(\bar{\mathbf{x}},\lambda)$ is minimized subject to $\lambda \in R^m_+$; hence the term, "saddle point."

In subsequent chapters, the primary interest in saddle points is their relation to the Kuhn-Tucker conditions. In this chapter, the relation is formerly established with a proof of the following theorem:

Theorem: Let X be a nonempty open set in R^n, and let $f(\mathbf{x}):X \to R^1$ and $\mathbf{G}(\mathbf{x}):X \to R^m$. Let $l(\mathbf{x},\lambda) = f(\mathbf{x}) + \lambda'\mathbf{G}(\mathbf{x})$ and suppose that $(\bar{\mathbf{x}},\bar{\lambda}) \in X \otimes R^m_+$ satisfies the saddle point relation:

$$l(\mathbf{x},\bar{\lambda}) \leq l(\bar{\mathbf{x}},\bar{\lambda}) \leq l(\bar{\mathbf{x}},\lambda); \qquad \forall\ (\mathbf{x},\lambda) \in X \otimes R^m_+$$

Further, suppose that $f(\mathbf{x})$ and $\mathbf{G}(\mathbf{x})$ are differentiable at $\bar{\mathbf{x}}$, then $\bar{\mathbf{x}}$ is feasible; moreover, $(\bar{\mathbf{x}},\bar{\lambda})$ satisfies the Kuhn-Tucker conditions:

$$\mathbf{f}_{\mathbf{x}}(\mathbf{x}) + \mathbf{G}_{\mathbf{x}}(\mathbf{x})\lambda = \mathbf{0}$$
$$\lambda'\mathbf{G}(\mathbf{x}) = 0$$
$$\lambda \geq \mathbf{0}$$

Conversely, let $\bar{\mathbf{x}}$ be feasible, and suppose that $(\bar{\mathbf{x}},\bar{\lambda})$ satisfies the Kuhn-Tucker conditions. Let $I = \{i: g_i(\bar{\mathbf{x}}) = 0\}$. Moreover, let X be convex, and let $f(\mathbf{x})$ and $g_i(\mathbf{x})$ for $i \in I$ be concave at $\bar{\mathbf{x}}$, then $(\bar{\mathbf{x}},\bar{\lambda})$ solves the saddle point relation.

Hence, under the assumptions of the theorem, saddle points in the Lagrangian are one-to-one with the solutions to the Kuhn-Tucker conditions.

The final topic considered is the case where a nonnegativity requirement for $\mathbf{x}$ is included in the constraints to the maximization problem. Hence, the general form of the problem becomes:

$$\begin{aligned} &\text{maximize}(\mathbf{x}): && f(\mathbf{x}) \\ &\text{subject to:} && \mathbf{G}^*(\mathbf{x}) \geq \mathbf{0} \\ & && \mathbf{x} \in X \end{aligned}$$

where $\mathbf{G}^*(\mathbf{x}) = [\mathbf{G}(\mathbf{x}), \mathbf{x}]'$. It is shown that in such cases, an alternative statement of the Kuhn-Tucker conditions is:

$$\mathbf{f_x}(\mathbf{x}) + \mathbf{G_x}(\mathbf{x})\boldsymbol{\lambda} \leq \mathbf{0}; \qquad \mathbf{x}'[\mathbf{f_x}(\mathbf{x}) + \mathbf{G_x}(\mathbf{x})\boldsymbol{\lambda}] = 0$$

$$\boldsymbol{\lambda}'\mathbf{G}(\mathbf{x}) = 0$$

$$\boldsymbol{\lambda} \geq \mathbf{0}$$

The next four sections develop the mathematical groundwork necessary to the formulation of the above theory, and to other quantative analysis throughout this work. The fifth section treats Kuhn-Tucker theory. The sixth section is concerned with Lagrangian saddle points. In the seventh section, the general theory is applied to those cases where nonnegativity in $\mathbf{x}$ is an explicit constraint upon the maximization problem.

2.1 Topological Concepts

The following definitions are frequently used in connection with sets. Each employs the "ϵ-neighborhood" concept. The ϵ-neighborhood of $\bar{\mathbf{x}}$ is the set:

$$N_\epsilon(\bar{\mathbf{x}}) = \{\mathbf{x}: |\mathbf{x} - \bar{\mathbf{x}}| < \epsilon\}$$

where $|\mathbf{x} - \bar{\mathbf{x}}|$ denotes the euclidean distance from $\mathbf{x}$ to $\bar{\mathbf{x}}$.

2.1.1 Definition: Let X be a nonempty set in R^n. A point, $\mathbf{x}$, is said to be in the "closure" of X, denoted by cl X, if $X \cap N_\epsilon(\mathbf{x}) \neq \emptyset$ for every $\epsilon > 0$. If X = cl X, then X is said to be a "closed set." $\mathbf{x}$ is said to be in the "interior" of X, denoted by int X, if $N_\epsilon(\mathbf{x}) \subset X$ for some $\epsilon > 0$. If X = int X, then X is said to be an "open set." $\mathbf{x}$ is said to be on the "boundary" of X, denoted by ∂X, if $N_\epsilon(\mathbf{x})$ contains at least one point in X and one point not in X for every $\epsilon > 0$. X is said to be "bounded" if there exists $\gamma \in R^1$ such that $|\mathbf{x}| \leq \gamma$ for every $\mathbf{x} \in X$. X is said to be "compact" if it is both bounded and closed.

Consider the set:

$$X = \{(x_1, x_2): x_1^2 + x_2^2 \leq 1\}$$

Geometrically, X is the set of all points on and within a unit circle centered at the origin. X is closed, that is, X = cl X. The interior of X consists of all points inside the circle, so:

$\text{int } X = \{(x_1,x_2): x_1^2 + x_2^2 < 1\}$

The boundary of X is the circle itself, or:

$\partial X = \{(x_1,x_2): x_1^2 + x_2^2 = 1\}$

X is clearly bounded; moreover, since X is closed, it is also compact.

The following is a list of topological properties of open, closed, and compact sets in R^n. The verification of these properties can be found in most topology texts:[2]

1) The intersection of a finite number of open sets is open.

2) The union of open sets is open.

3) The intersection of closed sets is closed.

4) The union of a finite number of closed sets is closed.

5) The intersection of a compact set and closed set is compact.

6) The union of a finite number of compact sets is compact.

Oftentimes, sets are defined by the inverse images of functions. For example, the set, $\{\mathbf{x} \in R^n: g(\mathbf{x}) \geq 0\}$, is defined by the inverse image of $g(\mathbf{x})$. The next theorem proves useful toward determining whether such sets are open or closed:

2.1.2 Theorem: Let X be a nonempty set in R^n, and let $f(\mathbf{x}):X \to Y$ be continuous on X. Furthermore, let $S \subset Y$, then:

1) If S is open, then $f^{-1}(S)$ is open.

2) If S is closed, then $f^{-1}(S)$ is closed.

The following definitions are frequently used in connection with the minimization and maximization of functions over sets:

2.1.3 Definition: Let X be a nonempty set in R^n, and let $f(\mathbf{x}):X \to R^1$. The "infimum" of $f(\mathbf{x})$, denoted by inf $f(\mathbf{x})$, is the largest real number ι satisfying $f(\mathbf{x}) \geq \iota$ for all $\mathbf{x} \in X$, that is, inf $f(\mathbf{x})$ is the "greatest lower bound" of $f(\mathbf{x})$. If $f(\mathbf{x})$ has no lower bound, then inf $f(\mathbf{x}) = -\infty$. If there exits $\mathbf{x}_\iota \in X$ such that $f(\mathbf{x}_\iota) = \iota$, then $\mathbf{x}_\iota$ is said to be a "minimum point," and $f(\mathbf{x}_\iota)$ is said to be the "minimum" of $f(\mathbf{x})$. The "supremum" of $f(\mathbf{x})$, denoted by sup $f(\mathbf{x})$, is the smallest real number σ satisfying $f(\mathbf{x}) \leq \sigma$ for all $\mathbf{x} \in X$, that is, sup $f(\mathbf{x})$ is the "least upper bound" of $f(\mathbf{x})$. If $f(\mathbf{x})$ has no upper bound, then sup $f(\mathbf{x}) = \infty$. If there exists $\mathbf{x}_\sigma \in X$ such that $f(\mathbf{x}_\sigma) = \sigma$, then $\mathbf{x}_\sigma$ is said to be a "maximum point," and $f(\mathbf{x}_\sigma)$ is said to be the "maximum" of $f(\mathbf{x})$.

Consider $f(x) = x$ on $x \in [0,1)$. Clearly, inf $f(x) = 0$, and sup $f(x) = 1$. As $f(x)$ attains it infimum at $x = 0$, zero is a minimum point of $f(x)$ on $x \in [0,1)$; however, as $f(x)$ does not attain its supremum, then the function has no maximum on the stated domain. This example illustrates that a function may not have a minimum or maximum over a set that is not closed. Now, consider $f(x) = 1 - e^{-x}$ on $x \in [0,\infty)$. The infimum of the function is zero and is attained at $x = 0$. The supremum of the function is one, but the supremum is not attained at any x; consequently, the function has no maximum. This illustrates that a function may not have a minimum or maximum over an unbounded set. Finally, consider $f(x) = 1/x$ on $x \in [0,1]$. The supremum of this function is ∞; however, as such supremum is clearly unattainable, the function has no maximum. This illustrates that a function may not have a minimum or maximum if it has points of discontinuity on the domain.

The next theorem is a famous result due to Weierstrass, and is proven in many topology texts. The theorem establishes conditions under which a function must have a minimum and maximum.

2.1.4 Theorem (Weierstrass Theorem): Let X be a nonempty compact set in R^n, and let $f(\mathbf{x}):X \to R^1$ be a continuous function on X, then $f(\mathbf{x})$ is bounded; moreover, $f(\mathbf{x})$ attains unto both inf $f(\mathbf{x})$ and sup $f(\mathbf{x})$, or equivalently, $f(\mathbf{x})$ has both a minimum and a maximum on X.

2.2 Convex Sets

One of the most important concepts in mathematical programming theory is the convex set. Such sets are formerly defined as follows:

2.2.1 Definition: Let X be set in R^n. X is said to be "convex" if $\mathbf{x}_1, \mathbf{x}_2 \in X$ implies $\alpha \mathbf{x}_1 + (1-\alpha)\mathbf{x}_2 \in X$ for each $\alpha \in (0,1)$. The null set, $\emptyset$, and sets consisting of a single point are also classified as convex sets.

Geometrically speaking, a set is convex if the line segment connecting any two points in the set lies completely within the set. A commonly encountered example of a convex set is the "half space." A half space consists of all points lying on either of the two sides of a hyperplane. Hence, the hyperplane, $\mathbf{a}'\mathbf{x} = \beta$, defines four half spaces; two of which are $X_g = \{\mathbf{x}\colon \mathbf{a}'\mathbf{x} \geq \beta\}$ and $X_l = \{\mathbf{x}\colon \mathbf{a}'\mathbf{x} \leq \beta\}$. The other two are simply the open variants of these. To demonstrate the convexity of X_g, suppose $\mathbf{x}_1, \mathbf{x}_2 \in X_g$, then $\mathbf{a}'\mathbf{x}_1 \geq \beta$, and $\mathbf{a}'\mathbf{x}_2 \geq \beta$, which implies:

$$\mathbf{a}'[\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2] \geq \alpha\beta + (1-\alpha)\beta = \beta$$

Thus, $\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2 \in X_g$ for all α, and particularly for $\alpha \in (0,1)$.

The following lemma is an immediate consequence of the definition of convexity:

2.2.2 Lemma: Let X_1 and X_2 be convex sets in R^n, then:

1) $X_1 \cap X_2$ is convex.

2) $X_1 + X_2 = \{\mathbf{x}_1 + \mathbf{x}_2\colon \mathbf{x}_1 \in X_1,\ \mathbf{x}_2 \in X_2\}$ is convex.

3) $X_1 - X_2 = \{\mathbf{x}_1 - \mathbf{x}_2\colon \mathbf{x}_1 \in X_1,\ \mathbf{x}_2 \in X_2\}$ is convex.

Proof: To prove part one, suppose $\mathbf{x}_1, \mathbf{x}_2 \in X_1 \cap X_2$. Since $\mathbf{x}_1 \in X_1$, and $\mathbf{x}_2 \in X_1$, then by convexity of X_1, $\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2 \in X_1$ for $\alpha \in (0,1)$. But similar reasoning leads to the conclusion that $\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2 \in X_2$; consequently, $\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2 \in X_1 \cap X_2$.

For part two, suppose $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2 \in X_1+X_2$ and $\bar{\mathbf{x}} = \bar{\mathbf{x}}_1 + \bar{\mathbf{x}}_2 \in X_1+X_2$, then:

$$\alpha\mathbf{x} + (1-\alpha)\bar{\mathbf{x}} = [\alpha\mathbf{x}_1 + (1-\alpha)\bar{\mathbf{x}}_1] + [\alpha\mathbf{x}_2 + (1-\alpha)\bar{\mathbf{x}}_2]$$

Since X_1 is convex, the first term on the right is in X_1 for $\alpha \in (0,1)$. Similarly, the second term is in X_2; consequently, $\alpha\mathbf{x} + (1-\alpha)\bar{\mathbf{x}} \in X_1+X_2$, which proves the proposition. The proof of part three is similar.

Consider the set $X = \{\mathbf{x}: \mathbf{A}\mathbf{x} \geq \mathbf{b}\}$ where $\mathbf{b}$ is an m vector and $\mathbf{A}$ is an mxn matrix. Observe that X is simply the intersection of the m half spaces defined by the rows of $\mathbf{A}$ and $\mathbf{b}$. As these half spaces are convex, and as the intersection of convex sets is also convex, then X is convex. This result may also be easily proven directly. Such sets are called "polyhedral sets."

The following theorem and corollaries formerly affirm some rather intuitive results concerning the interiors and closures of convex sets:

2.2.3 Theorem: Let X be a convex set in R^n with a nonempty interior. Let $\bar{\mathbf{x}}_1 \in \text{cl } X$ and $\bar{\mathbf{x}}_2 \in \text{int } X$, then:

$$\bar{\mathbf{x}} = \alpha\bar{\mathbf{x}}_1 + (1-\alpha)\bar{\mathbf{x}}_2 \in \text{int } X; \qquad \forall\ \alpha \in (0,1)$$

Proof: Since $\bar{\mathbf{x}}_2 \in \text{int } X$, there exists an $\epsilon > 0$ such that $N_\epsilon(\bar{\mathbf{x}}_2) \in X$. It will be shown that:

$$\{\mathbf{x}: |\mathbf{x} - \bar{\mathbf{x}}| < (1-\alpha)\epsilon\} \subset X$$

That is, there is an $(1-\alpha)\epsilon$-neighborhood of $\bar{\mathbf{x}}$ contained in X; therefore, $\bar{\mathbf{x}} \in \text{int } X$ by definition. First, choose any $\mathbf{x}$ satisfying $|\mathbf{x} - \bar{\mathbf{x}}| < (1-\alpha)\epsilon$. Now, since $\bar{\mathbf{x}}_1 \in \text{cl } X$, then for any $\delta > 0$, there exists $\mathbf{x}_1 \in X$ such that $|\mathbf{x}_1 - \bar{\mathbf{x}}_1| < \delta$. Let:

$$\mathbf{x}_2 = (\mathbf{x} - \alpha\mathbf{x}_1)/(1-\alpha) \tag{2.1}$$

Using the triangular inequality (See APPENDIX), it may be confirmed that:

$$|\mathbf{x}_2 - \bar{\mathbf{x}}_2| = \left|\frac{\mathbf{x} - \alpha\mathbf{x}_1}{1-\alpha} - \frac{\bar{\mathbf{x}} - \alpha\bar{\mathbf{x}}_1}{1-\alpha}\right| \le$$

$$\frac{1}{1-\alpha}(|\mathbf{x} - \bar{\mathbf{x}}| + \alpha|\bar{\mathbf{x}}_1 - \mathbf{x}_1|) < \frac{1}{1-\alpha}|\mathbf{x} - \bar{\mathbf{x}}| + \frac{\alpha\delta}{1-\alpha} < \epsilon$$

where the last inequality holds since $|\mathbf{x} - \bar{\mathbf{x}}| < (1-\alpha)\epsilon$ by assumption and since δ may be chosen arbitrarily small. Thus, $\mathbf{x}_2$ is in an ϵ-neighborhood of $\bar{\mathbf{x}}_2$, and consequently, $\mathbf{x}_2 \in X$. Now, (2.1) implies that $\mathbf{x} = \alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2$. It has been shown that $\mathbf{x}_2 \in X$. By assumption, $\mathbf{x}_1 \in X$. By convexity of X, it follows that $\mathbf{x} \in X$, and subsequently, $\{\mathbf{x}: |\mathbf{x} - \bar{\mathbf{x}}| < (1-\alpha)\epsilon\} \subset X$, which completes the proof.

Corollary 1: Let X be a convex set in R^n with nonempty interior, then int X is convex.

Corollary 2: Let X be a convex set in R^n, then cl X is convex.

Proof: If int $X = \emptyset$, then necessarily, $X = \partial X$ and cl $X = \partial X$, so cl $X = X$, and consequently, the corollary holds trivially. Suppose int $X \neq \emptyset$. Let $\mathbf{x}_1, \mathbf{x}_2 \in$ cl X and $\mathbf{x} \in$ int X. Using the theorem, $\lambda\mathbf{x} + (1-\lambda)\mathbf{x}_1 \in$ int X for $\lambda \in (0,1)$, and subsequently:

$$\alpha[\lambda\mathbf{x} + (1-\lambda)\mathbf{x}_1] + (1-\alpha)\mathbf{x}_2 \in \text{int } X; \qquad \forall\ \alpha \in (0,1)$$

Upon taking the limit of the above as $\lambda \to 0$, one obtains $\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2 \in$ cl X, which completes the proof.

2.3 Hyperplane Separation

In this section, several intuitive but critical theorems concerning convex sets are stated and proven. These theorems are then used to ascertain the existence or nonexistence of solutions to certain linear systems.

All results of this section are consequences of the following lemma:

2.3.1 Lemma: Let X be a nonempty convex set in R^n, and let $\mathbf{y} \notin X$, then there exists a nonzero $\mathbf{a} \in R^n$ such that $\mathbf{a}'\mathbf{x} > \mathbf{a}'\mathbf{y}$ for every $\mathbf{x} \in$ cl X.

Proof: Let $f(\mathbf{x}) = |\mathbf{x} - \mathbf{y}|$. It is apparent that there exists a $\iota > 0$ such that $\iota = \inf\{f(\mathbf{x}): \mathbf{x} \in X\}$. If $f(\mathbf{x})$ attains its infimum, then it clearly must do so at a point in $X^* = \text{cl } X \cap \{\mathbf{x}: |\mathbf{x} - \mathbf{y}| \leq \epsilon\}$ where $\epsilon \geq \iota$. As X^* is compact, and as $f(\mathbf{x})$ is continuous, then it known by the Weierstrass theorem (Theorem 2.1.4) that $f(\mathbf{x})$ does in fact have a minimum on X^*, and consequently, on cl X. Let $\bar{\mathbf{x}}$ be a minimum point, and let $\mathbf{x}$ be any other point in cl X. Since cl X is convex, $\alpha\mathbf{x} + (1-\alpha)\bar{\mathbf{x}} \in$ cl X for $\alpha \in (0,1)$; subsequently:

$$|\alpha\mathbf{x} + (1-\alpha)\bar{\mathbf{x}} - \mathbf{y}|^2 \geq |\bar{\mathbf{x}} - \mathbf{y}|^2$$

Expand the left-hand side to obtain:

$$\alpha^2|\mathbf{x} - \bar{\mathbf{x}}|^2 + |\bar{\mathbf{x}} - \mathbf{y}|^2 + 2\alpha(\mathbf{x} - \bar{\mathbf{x}})'(\bar{\mathbf{x}} - \mathbf{y}) \geq |\bar{\mathbf{x}} - \mathbf{y}|^2$$

which implies:

$$\alpha|\mathbf{x} - \bar{\mathbf{x}}|^2 + 2(\mathbf{x} - \bar{\mathbf{x}})'(\bar{\mathbf{x}} - \mathbf{y}) \geq 0$$

Upon taking the limit of the above as $\alpha \to 0$, one obtains:

$$(\mathbf{x} - \bar{\mathbf{x}})'(\bar{\mathbf{x}} - \mathbf{y}) \geq 0$$

which implies:

$$(\bar{\mathbf{x}} - \mathbf{y})'\mathbf{x} \geq (\bar{\mathbf{x}} - \mathbf{y})'(\bar{\mathbf{x}} - \mathbf{y} + \mathbf{y}) = \iota^2 + (\bar{\mathbf{x}} - \mathbf{y})'\mathbf{y}$$

Let $\mathbf{a} = (\bar{\mathbf{x}} - \mathbf{y})$ to complete the proof.

The theorem simply asserts that for any convex set and a single point isolated from the closure of the set, there exists a hyperplane passing through the point and having all of the closure of the set in one of its open half spaces. The particular hyperplane constructed in the proof is the hyperplane passing through $\mathbf{y}$ and perpendicular to $(\bar{\mathbf{x}} - \mathbf{y})$.

The next theorem is a consequence of the lemma:

2.3.2 Theorem (Supporting Hyperplane Theorem): Let X be a nonempty convex set in R^n, and let $\bar{\mathbf{x}} \in \partial X$, then there exists a nonzero $\mathbf{a} \in R^n$ such that $\mathbf{a}'\mathbf{x} \geq \mathbf{a}'\bar{\mathbf{x}}$ for every $\mathbf{x} \in \mathrm{cl}\ X$.

Proof: Since $\bar{\mathbf{x}} \in \partial X$, one may construct a sequence, $\{\mathbf{y}_k\}$, not in X, satisfying $\lim_{k\to\infty} \mathbf{y}_k = \bar{\mathbf{x}}$. By the last lemma, for every such $\mathbf{y}_k$ there exists a nonzero $\mathbf{a}_k$ satisfying:

$$\mathbf{a}_k'\mathbf{x} > \mathbf{a}_k'\mathbf{y}_k; \qquad \forall\ \mathbf{x} \in \mathrm{cl}\ X$$

Without loss of generality, it may be assumed that the $\mathbf{a}_k$ are normalized so that $|\mathbf{a}_k| = 1$. Hence, the $\mathbf{a}_k$ are bounded and must therefore possess a limit as $\mathbf{y}_k \to \bar{\mathbf{x}}$. Let $\lim_{k\to\infty} \mathbf{a}_k = \mathbf{a}$. Upon taking the limit of the above as $k \to \infty$, one obtains:

$$\mathbf{a}'\mathbf{x} \geq \mathbf{a}'\bar{\mathbf{x}}; \qquad \forall\ \mathbf{x} \in \mathrm{cl}\ X$$

which was to be shown.

The supporting hyperplane theorem asserts that for any $\bar{\mathbf{x}}$ on the boundary of a convex set X, there exits a hyperplane passing through $\bar{\mathbf{x}}$ and having all of X in one of its closed half spaces.

The following corollary is a generalization of Lemma 2.3.1:

Corollary: Let X be a nonempty convex set in R^n, and suppose $\mathbf{y} \notin \mathrm{int}\ X$, then there exits a nonzero $\mathbf{a} \in R^n$ such that $\mathbf{a}'\mathbf{x} \geq \mathbf{a}'\mathbf{y}$ for every $\mathbf{x} \in \mathrm{cl}\ X$.

Proof: If $\mathbf{y} \notin \mathrm{cl}\ X$, then the corollary follows directly from Lemma 2.3.1. If $\mathbf{y} \in \partial X$, then the corollary follows directly from the supporting hyperplane theorem.

With use of the latter corollary, the important separating hyperplane theorem may now be established:

2.3.3 Theorem (Separating Hyperplane Theorem): Let X_1 and X_2 be nonempty convex sets in R^n satisfying $X_1 \cap X_2 = \emptyset$, then there exists a nonzero $\mathbf{a} \in R^n$ such that $\mathbf{a}'\mathbf{x}_1 \geq \mathbf{a}'\mathbf{x}_2$ for every $\mathbf{x}_1 \in \mathrm{cl}\ X_1$ and $\mathbf{x}_2 \in \mathrm{cl}\ X_2$.

Proof: Let $X = X_1 - X_2 = \{\mathbf{x}_1 - \mathbf{x}_2 : \mathbf{x}_1 \in X_1,\ \mathbf{x}_2 \in X_2\}$. By Lemma 2.2.2, X is convex. Furthermore, $\mathbf{0} \notin X$, for otherwise, $X_1 \cap X_2$ would not be empty. Subsequently, by the last corollary, there exists a nonzero $\mathbf{a} \in R^n$ such that $\mathbf{a}'\mathbf{x} \geq \mathbf{a}'\mathbf{0} = 0$ for every $\mathbf{x} \in \mathrm{cl}\ X$. But by the definition of X, this implies:

$$\mathbf{a}'\mathbf{x}_1 \geq \mathbf{a}'\mathbf{x}_2; \qquad \forall\ \mathbf{x}_1 \in \mathrm{cl}\ X_1,\ \forall\ \mathbf{x}_2 \in \mathrm{cl}\ X_2$$

Thus, for any two nonintersecting convex sets, there exits a hyperplane having one set in each of its closed half spaces.

The theorems thus derived bear important implications for the existence of solutions in certain linear systems. The next two theorems are concerned with some of these implications:

2.3.4 Theorem (Farkas' Theorem): Let $\mathbf{A} \in R^{m\times n}$ and $\mathbf{c} \in R^n$, then exactly one of the following systems has a solution:

system 1: $\mathbf{Ax} \leq \mathbf{0}$ and $\mathbf{c}'\mathbf{x} > 0$; $\mathbf{x} \in R^n$

system 2: $\mathbf{A}'\mathbf{y} = \mathbf{c}$; $\mathbf{y} \in R^m_+$

Proof: Suppose that $\bar{\mathbf{y}}$ solves system two, and that $\bar{\mathbf{x}}$ satisfies $\mathbf{A}\bar{\mathbf{x}} \leq \mathbf{0}$, then $\mathbf{c}'\bar{\mathbf{x}} = \bar{\mathbf{y}}'\mathbf{A}\bar{\mathbf{x}} \leq \mathbf{0}$; hence, system one has no solution. Now, suppose system two has no solution, and let $X = \{\mathbf{x}: \mathbf{x} = \mathbf{A}'\mathbf{y}, \mathbf{y} \in R^m_+\}$. Since $\mathbf{c} \notin X$, then by Lemma 2.3.1, there exits a nonzero $\bar{\mathbf{x}} \in R^n$ such that:

$$\mathbf{y}'\mathbf{A}\bar{\mathbf{x}} < \mathbf{c}'\bar{\mathbf{x}}; \qquad \forall\ \mathbf{y} \in R^m_+$$

Since the components of $\mathbf{y}$ may be arbitrarily large, the latter inequality implies that $\mathbf{A}\bar{\mathbf{x}} \leq \mathbf{0}$. Moreover, since $\mathbf{y}$ may be equal to $\mathbf{0}$, it follows that $\mathbf{c}'\bar{\mathbf{x}} > 0$. Thus, system one has a solution.

2.3.5 Theorem (Gordan's Theorem): Let $\mathbf{A} \in R^{n\times m}$, then exactly one of the following systems has a solution:

system 1: $\mathbf{Ax} < \mathbf{0}$; $\mathbf{x} \in R^n$

system 2: $\mathbf{A}'\mathbf{y} = \mathbf{0}$; $\mathbf{y} \in R^m_+$, $\mathbf{y} \neq \mathbf{0}$

Proof: Suppose $\bar{\mathbf{x}}$ is a solution to system one and that $\bar{\mathbf{y}}$ is a solution to system two, then $\mathbf{A}\bar{\mathbf{x}} < \mathbf{0}$ and $\bar{\mathbf{y}} \geq \mathbf{0}$ imply $\bar{\mathbf{y}}'\mathbf{A}\bar{\mathbf{x}} < 0$. But, this contradicts the fact that $\mathbf{A}'\bar{\mathbf{y}} = \mathbf{0}$; hence, both systems cannot simultaneously have solutions.

Now, suppose system one has no solution and consider the following sets:

$Z_1 = \{\mathbf{z}: \mathbf{z} = \mathbf{A}\mathbf{x},\ \mathbf{x} \in R^n\}$

$Z_2 = \{\mathbf{z}: \mathbf{z} < \mathbf{0}\}$

As Z_1 and Z_2 are nonempty convex sets satisfying $Z_1 \cap Z_2 = \emptyset$, then by the separating hyperplane theorem, there exists a nonzero $\bar{\mathbf{y}} \in R^m$ such that:

$$\bar{\mathbf{y}}'\mathbf{A}\mathbf{x} \geq \bar{\mathbf{y}}'\mathbf{z}; \qquad \forall\ \mathbf{x} \in R^n,\ \forall\ \mathbf{z} \in \mathrm{cl}\ Z_2$$

Since the components of $\mathbf{z}$ may be arbitrarily small, it follows that $\bar{\mathbf{y}} \geq \mathbf{0}$. Moreover, as $\mathbf{0} \in \mathrm{cl}\ Z_2$, the latter inequality implies that $\bar{\mathbf{y}}'\mathbf{A}\mathbf{x} \geq \mathbf{0}$ for every $\mathbf{x} \in R^n$. Therefore, upon setting $\mathbf{x} = -\mathbf{A}'\bar{\mathbf{y}}$, it follows that $-|\mathbf{A}'\bar{\mathbf{y}}|^2 \geq 0$, which implies that $\mathbf{A}'\bar{\mathbf{y}} = \mathbf{0}$; hence system two has a solution.

2.4 Concave Functions

One of the most essential concepts in nonlinear programming theory is the concavity property in real functions. Several variants of concavity could be discussed; however, this section treats only the most essential forms. The most essential and most frequently encountered form of concavity is "simple concavity," which is formerly defined as follows:

2.4.1 Definition: Let X be a nonempty convex set in R^n, and let $f(\mathbf{x}): X \to R^1$. Moreover, let $\bar{\mathbf{x}} \in X$. $f(\mathbf{x})$ is said to be "concave" at $\bar{\mathbf{x}}$ if for each $\mathbf{x} \in X$ and $\alpha \in (0,1)$:

$$f[\alpha\mathbf{x} + (1-\alpha)\bar{\mathbf{x}}] \geq \alpha f(\mathbf{x}) + (1-\alpha)f(\bar{\mathbf{x}})$$

$f(\mathbf{x})$ is said to be "strictly concave" at $\bar{\mathbf{x}}$ if strict inequality holds in the above relation for all $\mathbf{x} \neq \bar{\mathbf{x}}$. $f(\mathbf{x})$ is said to be concave (strictly concave) on X if it is concave (strictly concave) for every $\bar{\mathbf{x}} \in X$.

Geometrically, these definitions imply that a function is concave at $\bar{\mathbf{x}}$ if it lies on or above any cord connecting $f(\bar{\mathbf{x}})$ and any other point on the surface of the function. If the function is strictly concave, then it is strictly above any such cord at all points other than the end points.

$f(\mathbf{x})$ is said to be convex (strictly convex) if $-f(\mathbf{x})$ is concave (strictly concave). In the remainder of this section, only concave functions are explicitly considered; however, the foregoing results may be easily modified to accomodate convex functions as well.

The next theorem is one of the most useful results in mathematical programming. Before stating the theorem, some definitions are needful:

2.4.2 Definition: Let X be a nonempty set in R^n, and let $f(\mathbf{x}):X \to R^1$. Consider the problem to maximize $f(\mathbf{x})$ subject to $\mathbf{x} \in X$. Any $\mathbf{x} \in X$ is said to be a "feasible point." Accordingly, X is called the "feasible region." If $\bar{\mathbf{x}} \in X$, and $f(\bar{\mathbf{x}}) \geq f(\mathbf{x})$ for every $\mathbf{x} \in X$, then $\bar{\mathbf{x}}$ is said to be a "global optimal solution" or simply an "optimal solution." If $\bar{\mathbf{x}} \in X$ and there exist an ϵ-neighborhood, $N_\epsilon(\bar{\mathbf{x}})$, about $\bar{\mathbf{x}}$ such that $f(\bar{\mathbf{x}}) \geq f(\mathbf{x})$ for every $\mathbf{x} \in N_\epsilon(\bar{\mathbf{x}}) \cap X$, then $\bar{\mathbf{x}}$ is said to be a "local optimal solution."

2.4.3 Theorem (Local-Global Theorem): Let X be a nonempty convex set in R^n, and let $f(\mathbf{x}):X \to R^1$. Consider the problem to maximize $f(\mathbf{x})$ subject to $\mathbf{x} \in X$. Suppose $\bar{\mathbf{x}}$ is a local optimal solution to the problem, then:

1) If $f(\mathbf{x})$ is concave at $\bar{\mathbf{x}}$, then $\bar{\mathbf{x}}$ is a global optimal solution.

2) If $f(\mathbf{x})$ is strictly concave at $\bar{\mathbf{x}}$, then $\bar{\mathbf{x}}$ is a unique global optimal solution.

Proof: To prove part one, suppose there exists an $\hat{\mathbf{x}} \in X$ such that $f(\hat{\mathbf{x}}) > f(\bar{\mathbf{x}})$, then by concavity of $f(\mathbf{x})$ at $\bar{\mathbf{x}}$, it follows that for $\alpha \in (0,1)$:

$$f[\alpha\hat{\mathbf{x}} + (1-\alpha)\bar{\mathbf{x}}] \geq \alpha f(\hat{\mathbf{x}}) + (1-\alpha)f(\bar{\mathbf{x}}) > f(\bar{\mathbf{x}})$$

But for α sufficiently small, $\alpha\hat{\mathbf{x}} + (1-\alpha)\bar{\mathbf{x}} \in N_\varepsilon(\bar{\mathbf{x}})$ so that the above inequality contradicts the local optimality of $\bar{\mathbf{x}}$.

To prove part two, suppose that $\bar{\mathbf{x}}$ and $\hat{\mathbf{x}}$ are both global optimal solutions. By the strict concavity of $f(\mathbf{x})$ at $\bar{\mathbf{x}}$:

$$f(.5\bar{\mathbf{x}} + .5\hat{\mathbf{x}}) > .5f(\bar{\mathbf{x}}) + .5f(\hat{\mathbf{x}}) = f(\bar{\mathbf{x}})$$

which contradicts the global optimality of $\bar{\mathbf{x}}$.

In nonlinear programming problems, the feasible region, X, used in the above theorem is defined by the constraints. Frequently, the constraints will be such that X is convex. If this is the case, then the latter theorem will prove useful.

The next two theorems may be frequently used to identify constraint sets defining convex feasible regions. First, the following definitions must be established:

2.4.4 Definition: Let X be a set in R^n, and let $f(\mathbf{x}):X \to R^1$, then:

1) The set $\{\mathbf{x} \in X: f(\mathbf{x}) \geq \beta\}$ is called an "upper set."

2) The set $\{\mathbf{x} \in X: f(\mathbf{x}) > \beta\}$ is called a "strict upper set."

3) The set $\{\mathbf{x} \in X: f(\mathbf{x}) \leq \beta\}$ is called a "lower set."

4) The set $\{\mathbf{x} \in X: f(\mathbf{x}) < \beta\}$ is called a "strict lower set."

5) The set $\{\mathbf{x} \in X: f(\mathbf{x}) = \beta\}$ is called a "level set."

2.4.5 Theorem: Let X be a convex set in R^n, and let $f(\mathbf{x}):X \to R^1$ be concave on X, then:

1) The upper set, $S = \{\mathbf{x} \in X: f(\mathbf{x}) \geq \beta\}$, is convex.

2) The strict upper set, $S' = \{\mathbf{x} \in X: f(\mathbf{x}) > \beta\}$, is convex.

Proof: To prove part one, suppose $\mathbf{x}_1, \mathbf{x}_2 \in S$. Thus, $f(\mathbf{x}_1) \geq \beta$ and $f(\mathbf{x}_2) \geq \beta$. Since $f(\mathbf{x})$ is concave on X, then for $\alpha \in (0,1)$:

$$f[\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2] \geq \alpha f(\mathbf{x}_1) + (1-\alpha)f(\mathbf{x}_2) \geq \alpha\beta + (1-\alpha)\beta = \beta$$

Hence, $\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2 \in S$. The proof of part two is similar.

A vector-valued function is said to be concave if all of its component functions are concave. Using this definition, the latter result may be easily extended to include vector-valued functions with the following theorem:

2.4.6 Theorem: Let X be a convex set in R^n, and let $\mathbf{F}(\mathbf{x}):X \to R^m$ be concave on X. Let $\mathbf{b} \in R^m$, then:

1) The upper set, $S = \{\mathbf{x} \in X: \mathbf{F}(\mathbf{x}) \geq \mathbf{b}\}$, is convex.

2) The strict upper set, $S' = \{\mathbf{x} \in X: \mathbf{F}(\mathbf{x}) > \mathbf{b}\}$, is convex.

Proof: The sets, S and S', are simply the intersections of the m upper sets or strict upper sets defined by the component functions of $\mathbf{F}(\mathbf{x})$ and the corresponding components of $\mathbf{b}$. As each of these sets is convex by the last theorem, and as the intersection of convex sets is convex by Lemma 2.2.2, it follows that S and S' are convex.

If $f(\mathbf{x})$ is a differentiable function, then the following theorems affirm some useful properties of concavity, and even offer an alternative definition of concavity over a convex set:

2.4.7 Theorem: Let X be a nonempty open[3] convex set in R^n, and let $f(\mathbf{x}):X \to R^1$ be concave at $\bar{\mathbf{x}} \in X$. If $f(\mathbf{x})$ is differentiable at $\bar{\mathbf{x}}$, then:

$$f(\mathbf{x}) \leq f(\bar{\mathbf{x}}) + \mathbf{f}'_{\mathbf{x}}(\bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}}); \qquad \forall\ \mathbf{x} \in X$$

Proof: Since $f(\mathbf{x})$ is concave at $\bar{\mathbf{x}}$, then for $\alpha \in (0,1)$:

$$f[\alpha\mathbf{x} + (1-\alpha)\bar{\mathbf{x}}] \geq \alpha f(\mathbf{x}) + (1-\alpha)f(\bar{\mathbf{x}}); \qquad \forall\ \mathbf{x} \in X$$

which implies:

$$\frac{f[\bar{\mathbf{x}} + \alpha(\mathbf{x} - \bar{\mathbf{x}})] - f(\bar{\mathbf{x}})}{\alpha} \geq f(\mathbf{x}) - f(\bar{\mathbf{x}}); \qquad \forall\ \mathbf{x} \in X$$

Upon taking the limit as $\alpha \to 0$, the above becomes:

$$\mathbf{f}'_{\mathbf{x}}(\bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}}) \geq f(\mathbf{x}) - f(\bar{\mathbf{x}}); \qquad \forall\ \mathbf{x} \in X$$

or:

$$f(\mathbf{x}) \leq f(\bar{\mathbf{x}}) + \mathbf{f}'_{\mathbf{x}}(\bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}}); \qquad \forall\ \mathbf{x} \in X$$

which was to be shown.

Corollary: Let X be a nonempty open convex set in R^n, and let $f(\mathbf{x}):X \to R^1$ be strictly concave at $\bar{\mathbf{x}} \in X$. If $f(\mathbf{x})$ is differentiable at $\bar{\mathbf{x}}$, then:

$$f(\mathbf{x}) < f(\bar{\mathbf{x}}) + \mathbf{f}'_{\mathbf{x}}(\bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}}); \qquad \forall\ \mathbf{x} \in X,\ \mathbf{x} \neq \bar{\mathbf{x}}$$

Proof: It is known from the theorem that:

$$f(\mathbf{x}) \leq f(\bar{\mathbf{x}}) + \mathbf{f}'_{\mathbf{x}}(\bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}}); \qquad \forall\ \mathbf{x} \in X$$

Suppose that there is an $\hat{\mathbf{x}} \in X$, not equal to $\bar{\mathbf{x}}$, such that:

$$f(\hat{\mathbf{x}}) = f(\bar{\mathbf{x}}) + \mathbf{f}'_{\mathbf{x}}(\bar{\mathbf{x}})(\hat{\mathbf{x}} - \bar{\mathbf{x}})$$

Multiply the latter by α and subtract from the former, and set $\mathbf{x} = \alpha\hat{\mathbf{x}} + (1-\alpha)\bar{\mathbf{x}}$ to obtain:

$$f[\alpha\hat{\mathbf{x}} + (1-\alpha)\bar{\mathbf{x}}] \leq \alpha f(\hat{\mathbf{x}}) + (1-\alpha)f(\bar{\mathbf{x}})$$

But, this contradicts the strict concavity of $f(\mathbf{x})$ at $\bar{\mathbf{x}}$, and the proof is complete.

Geometrically, the last theorem implies that a function that is concave at a point lies everywhere on or beneath the tangent plane at that point. The corollary implies that in the case of strict concavity, the function lies strictly beneath any such tangent plane except at the point of tangency.

It should be observed that the last theorem and corollary deal with concavity at a point. The next theorem and corollary deal with concavity over a convex set:

2.4.8 Theorem: Let X be a nonempty open convex set in R^n, and let $f(\mathbf{x}): X \to R^1$ be differentiable on X, then $f(\mathbf{x})$ is concave on X if and only if:

$$f(\mathbf{x}) \leq f(\bar{\mathbf{x}}) + \mathbf{f}'_{\mathbf{x}}(\bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}}); \qquad \forall\ \mathbf{x}, \bar{\mathbf{x}} \in X \tag{2.2}$$

Proof: The assertion that concavity implies the latter relation is proven in the last theorem. It remains to show that the latter relation implies concavity on X. Thus, suppose that (2.2) holds, then for any $\mathbf{x}_1, \mathbf{x}_2 \in X$:

$$f(\mathbf{x}_1) \leq f(\bar{\mathbf{x}}) + \mathbf{f}'_{\mathbf{x}}(\bar{\mathbf{x}})(\mathbf{x}_1 - \bar{\mathbf{x}}); \qquad \forall\ \mathbf{x}_1, \bar{\mathbf{x}} \in X$$

$$f(\mathbf{x}_2) \leq f(\bar{\mathbf{x}}) + \mathbf{f}'_{\mathbf{x}}(\bar{\mathbf{x}})(\mathbf{x}_2 - \bar{\mathbf{x}}); \qquad \forall\ \mathbf{x}_2, \bar{\mathbf{x}} \in X$$

Multiply the first relation by α and the second by $(1-\alpha)$ and add the products to obtain:

$$\alpha f(\mathbf{x}_1) + (1-\alpha) f(\mathbf{x}_2) \leq f(\bar{\mathbf{x}}) + \mathbf{f}'_{\mathbf{x}}(\bar{\mathbf{x}})[\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2 - \bar{\mathbf{x}}]$$

for every $\mathbf{x}_1, \mathbf{x}_2, \bar{\mathbf{x}} \in X$. Now, set $\bar{\mathbf{x}} = \alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2$ to produce:

$$\alpha f(\mathbf{x}_1) + (1-\alpha) f(\mathbf{x}_2) \leq f[\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2]; \qquad \forall\ \mathbf{x}_1, \mathbf{x}_2 \in X$$

which proves that (2.2) implies concavity on X.

Corollary: Let X be a nonempty open convex set in R^n, and let $f(\mathbf{x}):X \to R^1$ be differentiable on X, then $f(\mathbf{x})$ is strictly concave on X if and only if for distinct $\mathbf{x},\bar{\mathbf{x}} \in X$:

$$f(\mathbf{x}) < f(\bar{\mathbf{x}}) + \mathbf{f}_{\mathbf{x}}'(\bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}})$$

Proof: By the theorem, it is known that $f(\mathbf{x})$ is concave on X if and only if:

$$f(\mathbf{x}) \leq f(\bar{\mathbf{x}}) + \mathbf{f}_{\mathbf{x}}'(\bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}}); \qquad \forall\ \mathbf{x},\bar{\mathbf{x}} \in X \tag{2.3}$$

Suppose there is an $\hat{\mathbf{x}} \in X$, not equal to $\bar{\mathbf{x}}$, such that:

$$f(\hat{\mathbf{x}}) = f(\bar{\mathbf{x}}) + \mathbf{f}_{\mathbf{x}}'(\bar{\mathbf{x}})(\hat{\mathbf{x}} - \bar{\mathbf{x}})$$

It follows from this equality and the strict concavity of $f(\mathbf{x})$ that:

$$f[\alpha\hat{\mathbf{x}} + (1-\alpha)\bar{\mathbf{x}}] > \alpha f(\hat{\mathbf{x}}) + (1-\alpha)f(\bar{\mathbf{x}}) =$$
$$f(\bar{\mathbf{x}}) + \alpha\mathbf{f}_{\mathbf{x}}'(\bar{\mathbf{x}})(\hat{\mathbf{x}} - \bar{\mathbf{x}})$$

Upon setting $\mathbf{x} = \alpha\hat{\mathbf{x}} + (1-\alpha)\bar{\mathbf{x}}$, the latter may be written:

$$f(\mathbf{x}) > f(\bar{\mathbf{x}}) + \mathbf{f}_{\mathbf{x}}'(\bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}})$$

But, this inequality contradicts (2.3), and the proof is complete.

A vector-valued function is said to be differentiable if all of its component functions are differentiable. The last two theorems and corollaries extend without alteration to include all differentiable vector-valued functions. For example, if X is a nonempty open convex set in R^n, then a differentiable function, $\mathbf{F(x)}:X \to R^m$, is concave on X if and only if:

$$\mathbf{F(x)} \leq \mathbf{F(\bar{x})} + \mathbf{F}_{\mathbf{x}}'(\bar{x})(x - \bar{x}); \qquad \forall\ \mathbf{x},\bar{\mathbf{x}} \in X$$

where $\mathbf{F_x(\bar{x})}$ is an nxm matrix whose ith column is the gradient of the ith component function. Here, strict inequality holds in the case of strict concavity and distinct $\mathbf{x}$ and $\bar{\mathbf{x}}$.

The following theorem provides perhaps the most useful means of determining concavity:

2.4.9 Theorem: Let X be a nonempty open convex set in R^n, and let $f(\mathbf{x}):X \to R^1$ be twice differentiable on X, then $f(\mathbf{x})$ is concave (strictly concave) on X if and only if the Hessian matrix, $\mathbf{H(x)}$, is negative semidefinite (negative definite) over all $\mathbf{x} \in X$.

Proof: Suppose $f(\mathbf{x})$ is concave on X and let $\bar{\mathbf{x}} \in X$. Since X is open, then $\bar{\mathbf{x}} + \lambda\mathbf{x} \in X$ for λ sufficiently small. From Theorem 2.4.8, it follows that:

$$f(\bar{\mathbf{x}} + \lambda\mathbf{x}) \leq f(\bar{\mathbf{x}}) + \lambda\mathbf{f}_{\mathbf{x}}'(\bar{\mathbf{x}})\mathbf{x}$$

Using the definition of differentiability (See APPENDIX), it may be concluded that:

$$f(\bar{\mathbf{x}} + \lambda\mathbf{x}) = f(\bar{\mathbf{x}}) + \lambda\mathbf{f}_{\mathbf{x}}'(\bar{\mathbf{x}})\mathbf{x} + (1/2)\lambda^2\mathbf{x}'\mathbf{H}(\bar{\mathbf{x}})\mathbf{x} + \lambda^2|\mathbf{x}|^2\omega(\bar{\mathbf{x}},\lambda\mathbf{x})$$

where $\omega(\bar{\mathbf{x}},\lambda\mathbf{x}) \to 0$ as $\lambda \to 0$. Subtract the former from the latter to obtain:

$$(1/2)\lambda^2\mathbf{x}'\mathbf{H}(\bar{\mathbf{x}})\mathbf{x} + \lambda^2|\mathbf{x}|^2\omega(\bar{\mathbf{x}},\lambda\mathbf{x}) \leq 0$$

Divide the latter by λ^2, and let $\lambda \to 0$ to produce $\mathbf{x}'\mathbf{H}(\bar{\mathbf{x}})\mathbf{x} \leq 0$.

Conversely, suppose that $\mathbf{H}(\mathbf{x})$ is negative semidefinite at every point in X, and let $\mathbf{x},\bar{\mathbf{x}} \in X$. Using Taylor's theorem, $f(\mathbf{x})$ may be expressed as:

$$f(\mathbf{x}) = f(\bar{\mathbf{x}}) + \mathbf{f}_{\mathbf{x}}'(\bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}}) + (1/2)(\mathbf{x} - \bar{\mathbf{x}})'\mathbf{H}(\hat{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}})$$

where $\hat{\mathbf{x}} = \alpha\bar{\mathbf{x}} + (1-\alpha)\mathbf{x}$ for some $\alpha \in (0,1)$; consequently, $\hat{\mathbf{x}} \in X$. As $\mathbf{H}(\hat{\mathbf{x}})$ is negative semidefinite by assumption, it is known that $(\mathbf{x} - \bar{\mathbf{x}})'\mathbf{H}(\hat{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}}) \leq 0$; therefore, the latter inequality implies that:

$$f(\mathbf{x}) \leq f(\bar{\mathbf{x}}) + \mathbf{f}_{\mathbf{x}}'(\bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}})$$

But from Theorem 2.4.8, this inequality implies that $f(\mathbf{x})$ is concave on X. The proof for the strictly concave case is obtained by replacing the inequalities above with strict inequalities.

There are other forms of concavity that are less restrictive than simple concavity inasmuch as simple concavity proves to be a special case of these forms. However, these generalized forms share many desirable properties with simple concavity. One of the most common of these forms is defined as follows:

2.4.10 Definition: Let X be a nonempty convex set in R^n, and let $f(\mathbf{x}):X \to R^1$. Moreover, let $\bar{\mathbf{x}} \in X$. $f(\mathbf{x})$ is said to be "quasiconcave" at $\bar{\mathbf{x}}$ if for every $\mathbf{x} \in X$ and $\alpha \in (0,1)$, the following inequality holds:

$$f[\alpha\mathbf{x} + (1-\alpha)\bar{\mathbf{x}}] \geq \min[f(\mathbf{x}), f(\bar{\mathbf{x}})]$$

$f(\mathbf{x})$ is said to be "strictly quasiconcave" at $\bar{\mathbf{x}}$ if the above holds with strict inequality for distinct $f(\mathbf{x})$ and $f(\bar{\mathbf{x}})$. $f(\mathbf{x})$ is said to be quasiconcave (strictly quasiconcave) on X if it is quasiconcave (strictly quasiconcave) for every $\bar{\mathbf{x}} \in X$.

Geometrically, these definitions imply that a function is quasiconcave at $\bar{\mathbf{x}}$ if it lies on or above the lowest endpoint of any cord connecting $f(\bar{\mathbf{x}})$ and any other point on the surface of the function. If the function is strictly quasiconcave, then it lies strictly above any such cord that is not perfectly horizontal. Note that every concave function is also quasiconcave, and that every strictly concave function is strictly quasiconcave. The function, $f(\mathbf{x})$, is said to be "quasiconvex" ("strictly quasiconvex") if $-f(\mathbf{x})$ is quasiconcave (strictly quasiconcave).

In Theorem 2.4.5, it was shown that the upper sets and strict upper sets of a concave function are convex. The next theorem and proof demonstrate that this property also holds for quasiconcave functions:

2.4.11 Theorem: Let X be a convex set in R^n, and let $f(\mathbf{x}): X \to R^1$. $f(\mathbf{x})$ is quasiconcave on X if and only if the set, $S_\gamma = \{\mathbf{x} \in X: f(\mathbf{x}) \geq \gamma\}$, is convex for each $\gamma \in R^1$.

Proof: Suppose $f(\mathbf{x})$ is quasiconcave on X and let $\mathbf{x}_1, \mathbf{x}_2 \in S_\gamma$. Observe that $\min[f(\mathbf{x}_1), f(\mathbf{x}_2)] \geq \gamma$. Since $f(\mathbf{x})$ is quasiconcave, then for any $\alpha \in (0,1)$:

$$f[\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2] \geq \min[f(\mathbf{x}_1), f(\mathbf{x}_2)] \geq \gamma$$

which demonstrates that quasiconcavity on X implies convexity in S_γ.

Conversely, suppose that S_γ is convex for every γ. Set $\gamma = \min[f(\mathbf{x}_1), f(\mathbf{x}_2)]$, so $f(\mathbf{x}_1) \geq \gamma$ and $f(\mathbf{x}_2) \geq \gamma$; subsequently, $\mathbf{x}_1, \mathbf{x}_2 \in S_\gamma$. The convexity of S_γ imples:

$$f[\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2] \geq \gamma = \min[f(\mathbf{x}_1), f(\mathbf{x}_2)]; \qquad \forall\ \alpha \in (0,1)$$

which shows that convexity of S_γ implies quasiconcavity of $f(\mathbf{x})$ on X.

Though strict concavity implies concavity, it does not follow that strict quasiconcavity implies quasiconcavity. This is illustrated by the following function:

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

Observe that $f(x)$ is strictly quasiconcave on R^1; however, it is not quasiconcave, for at $\bar{x} = 1$ and $x = -1$, $f(\bar{x}) = 0$ and $f(x) = 0$, but:

$$f[.5x + .5\bar{x}] = f(0) = 1 > \min[f(x), f(\bar{x})]$$

However, the next theorem affirms that strict quasconcavity implies quasiconcavity under continuity:

2.4.12 Theorem: Let X be a nonempty convex set in R^n, and let $f(\mathbf{x}): X \to R^1$ be continuous and strictly quasiconcave on X, then $f(\mathbf{x})$ is quasiconcave on X.

Proof: Let $\mathbf{x}, \bar{\mathbf{x}} \in X$. It needs to be shown that if $f(\mathbf{x}) = f(\bar{\mathbf{x}})$, then:

$$f[\alpha\mathbf{x} + (1-\alpha)\bar{\mathbf{x}}] \geq \min[f(\mathbf{x}), f(\bar{\mathbf{x}})] = f(\bar{\mathbf{x}})$$

To the contrary, suppose that $f(\mathbf{x}) = f(\bar{\mathbf{x}})$, but that:

$$f[\alpha\mathbf{x} + (1-\alpha)\bar{\mathbf{x}}] < f(\bar{\mathbf{x}}); \qquad \text{for some } \alpha \in (0,1)$$

Using the continuity of $f(\mathbf{x})$, it may be concluded that there exists a $\lambda \in (0,1)$ such that:

$$f(\mathbf{x}) < f[\lambda\mathbf{x} + (1-\lambda)\bar{\mathbf{x}}] < f(\bar{\mathbf{x}})$$

which contradicts the fact the $f(\mathbf{x}) = f(\bar{\mathbf{x}})$, and the proof is complete.

The following definition gives yet another variant of concavity that proves useful in nonlinear programming theory:

2.4.13 Definition: Let X be a nonempty open convex set in R^n, and let $f(\mathbf{x}): X \to R^1$. Moreover let $\bar{\mathbf{x}} \in X$, and let $f(\mathbf{x})$ be differentiable at $\bar{\mathbf{x}}$. $f(\mathbf{x})$ is said to be "pseudoconcave" at $\bar{\mathbf{x}}$ if it is true that for every $\mathbf{x} \in X$, $\mathbf{f}'_{\mathbf{x}}(\bar{x})(x - \bar{x}) \leq 0$ implies $f(\mathbf{x}) \leq f(\bar{x})$, or equivalently, $f(\mathbf{x}) > f(\bar{x})$ implies $\mathbf{f}'_{\mathbf{x}}(\bar{x})(x - \bar{x}) > 0$. $f(\mathbf{x})$ is said to be "strictly pseudoconcave" at $\bar{x}$ if $\mathbf{f}'_{\mathbf{x}}(\bar{x})(x - \bar{x}) \leq 0$ implies $f(\mathbf{x}) < f(\bar{x})$, or equivalently, $f(\mathbf{x}) \geq f(\bar{x})$ implies $\mathbf{f}'_{\mathbf{x}}(\bar{x})(x - \bar{x}) > 0$. $f(\mathbf{x})$ is said to be pseudoconcave (strictly pseudoconcave) on X if it is differentiable on X and if it is pseudoconcave (strictly pseudoconcave) at every $\bar{\mathbf{x}} \in X$.

Geometrically, pseudoconcavity means that if a tangent plane approximation of $f(\mathbf{x})$ from $f(\bar{x})$ indicates that $f(\mathbf{x}) \leq f(\bar{x})$, then this is indeed the case. Strict pseudoconcavity means that if such tangent plane approximation indicates that $f(\mathbf{x}) \leq f(\bar{x})$, then in fact, $f(\mathbf{x}) < f(\bar{x})$. Observe that the concept of pseudoconcavity is relevent only where $f(\mathbf{x})$ is differentiable. Also, observe that Theorem 2.4.8 and corollary imply that every concave function is pseudoconcave, and that every strictly concave function is strictly pseudoconcave. Also, the function, $f(\mathbf{x})$, is said to be "pseudoconvex" ("strictly pseudoconvex") if $-f(\mathbf{x})$ is pseudoconcave (strictly pseudoconcave).

The next theorem restates the local-global theorem (Theorem 2.4.3) in terms of pseudoconcavity.

2.4.14 Theorem: Let X be a nonempty open set in R^n, and let $f(\mathbf{x}): X \rightarrow R^1$. Moreover, let S be a nonempty convex subset of X, and consider the problem to maximize $f(\mathbf{x})$ subject to $\mathbf{x} \in S$. Let $\bar{\mathbf{x}}$ be a local optimal solution to the problem, and let $f(\mathbf{x})$ be differentiable at $\bar{\mathbf{x}}$, then:

1) If $f(\mathbf{x})$ is pseudoconcave at $\bar{\mathbf{x}}$, then $\bar{\mathbf{x}}$ is a global optimal solution.

2) If $f(\mathbf{x})$ is strictly pseudoconcave at $\bar{\mathbf{x}}$, then $\bar{\mathbf{x}}$ is a unique global optimal solution.

Proof: To prove part one, suppose there exists $\hat{\mathbf{x}} \in S$ such that $f(\hat{\mathbf{x}}) > f(\bar{\mathbf{x}})$, then by pseudoconcavity of $f(\mathbf{x})$ at $\bar{\mathbf{x}}$ it follows that:

$$f'_{\mathbf{x}}(\bar{\mathbf{x}})(\hat{\mathbf{x}} - \bar{\mathbf{x}}) > 0$$

By convexity of S, $\mathbf{x} = \alpha\hat{\mathbf{x}} + (1-\alpha)\bar{\mathbf{x}} \in S$ for $\alpha \in (0,1)$; moreover, $\mathbf{x} \in N_{\epsilon}(\bar{\mathbf{x}})$ for α sufficiently small. It will be shown that there exists a $\delta > 0$ such that $f(\mathbf{x}) > f(\bar{\mathbf{x}})$ for all $\alpha \in (0,\delta)$, thus contradicting the local optimality of $\bar{\mathbf{x}}$. By the differentiability of $f(\mathbf{x})$ at $\bar{\mathbf{x}}$, it is known that:

$$f(\mathbf{x}) = f(\bar{\mathbf{x}}) + \alpha f'_{\mathbf{x}}(\bar{\mathbf{x}})(\hat{\mathbf{x}} - \bar{\mathbf{x}}) + \alpha|\hat{\mathbf{x}} - \bar{\mathbf{x}}|\omega[\bar{\mathbf{x}},\alpha(\hat{\mathbf{x}}-\bar{\mathbf{x}})]$$

where $\omega[\bar{\mathbf{x}},\alpha(\hat{\mathbf{x}}-\bar{\mathbf{x}})] \to 0$ as $\alpha \to 0$. The latter implies:

$$\frac{f(\mathbf{x}) - f(\bar{\mathbf{x}})}{\alpha} = f'_{\mathbf{x}}(\bar{\mathbf{x}})(\hat{\mathbf{x}} - \bar{\mathbf{x}}) + |\hat{\mathbf{x}} - \bar{\mathbf{x}}|\omega[\bar{\mathbf{x}},\alpha(\hat{\mathbf{x}} - \bar{\mathbf{x}})] \tag{2.4}$$

Let $\alpha \to 0$. There exists a $\delta > 0$ such that the right-hand side of the latter equation is positive for $\alpha \in (0,\delta)$, so for such α, $f(\mathbf{x}) > f(\bar{\mathbf{x}})$, which proves part one.

To prove part two, suppose there exists $\hat{x} \in S$ such that $f(\hat{x}) = f(\bar{x})$, then by strict pseudoconcavity of $f(\mathbf{x})$ at $\bar{\mathbf{x}}$, this implies:

$$f'_{\mathbf{x}}(\bar{x})(\hat{x} - \bar{x}) > 0$$

After following exactly the same steps in the proof of part one, it may be concluded that (2.4) contradicts the local optimality of $\bar{\mathbf{x}}$.

The next theorem and proof show that pseudoconcavity is a special case of both quasiconcavity and strict quasiconcavity.

2.4.15 Theorem: Let X be a nonempty open convex set in R^n, and let $f(\mathbf{x}):X \to R^1$ be pseudoconcave over X, then $f(\mathbf{x})$ is both quasiconcave and strictly quasiconcave over X.

Proof: It will be shown that $f(\mathbf{x})$ is strictly quasiconcave on X. Quasiconcavity on X will then follow from continuity over X and Theorem 2.4.12. Let $\mathbf{x}_1, \mathbf{x}_2 \in X$, and suppose $f(\mathbf{x}_1) < f(\mathbf{x}_2)$. Let $\bar{\mathbf{x}} = \alpha \mathbf{x}_1 + (1-\alpha)\mathbf{x}_2$ and suppose that for some $\alpha \in (0,1)$:

$$f(\bar{\mathbf{x}}) < \min[f(\mathbf{x}_1), f(\mathbf{x}_2)] = f(\mathbf{x}_1)$$

By pseudoconcavity of $f(\mathbf{x})$, this implies:

$$f'_{\mathbf{x}}(\bar{x})(\mathbf{x}_1 - \bar{x}) > 0$$

Noting the definition of $\bar{\mathbf{x}}$, this implies:

$$\frac{-(1-\alpha)f'_x(\bar{x})(x_2 - \bar{x})}{\alpha} > 0$$

By pseudoconcavity of $f(x)$, this implies $f(x_2) \leq f(\bar{x})$, thus:

$$f(x_2) \leq f(\bar{x}) < f(x_1)$$

But, this contradicts the fact the $f(x_1) < f(x_2)$, and the proof is complete.

2.5 Kuhn-Tucker Optimality Conditions

Consider the problem:

maximize(x): $f(x)$

subject to: $\mathbf{G(x)} \geq \mathbf{0}$

$\mathbf{x} \in X$

where $f(x):X \to R^1$ and $\mathbf{G(x)}:X \to R^m$. Let $\bar{x}$ be a solution to the above. In this section, it is first shown that if: 1) X is open. 2) $f(x)$ and $\mathbf{G(x)}$ are differentiable at $\bar{x}$, and 3) $\mathbf{G(x)}$ satisfies certain regularity conditions at $\bar{x}$, then there exists $\bar{\lambda}$ such that $(\bar{\lambda},\bar{x})$ will solve the following set of conditions:

$$f_x(x) + G_x(x)\lambda = 0 \tag{2.5}$$

$$\lambda' G(x) = 0 \tag{2.6}$$

$$\lambda \geq 0 \tag{2.7}$$

The above conditions were first introduced by Kuhn and Tucker (1951), and have since been further developed by numerous mathematicians. Much attention has been directed toward determining regularity conditions upon $\mathbf{G(x)}$ that are sufficient to ensure the existence of $\bar{\lambda}$ such that the Kuhn-Tucker conditions are solvable. The vector, $\bar{\lambda}$, is commonly known as the vector of "Lagrangian multipliers." Likewise, the emphasis here is upon regularity conditions, or "constraint qualifications," as they are commonly called. After treating the necessity of the Kuhn-Tucker conditions, it is then shown that under certain concavity assumptions, such conditions are also sufficient for the optimal solutions to the problem above.

The necessary conditions herein developed for the solutions to the problem above are actually necessary for all local optimal solutions. However, as a global optimum is a local optimum, then necessary conditions to the latter also prove necessary to the former.

The foregoing theorem establishes necessary conditions to unconstrained local maxima:

2.5.1 Theorem: Let $f(\mathbf{x}):R^n \to R^1$ be differentiable at $\bar{\mathbf{x}}$. If there is a vector $\mathbf{d}$ such that $\mathbf{f}_{\mathbf{x}}'(\bar{\mathbf{x}})\mathbf{d} > 0$, then there exists a $\delta > 0$ such that $f(\bar{\mathbf{x}} + \lambda\mathbf{d}) > f(\bar{\mathbf{x}})$ for each $\lambda \in (0,\delta)$.

Proof: Using the differentiability of $f(\mathbf{x})$ at $\bar{\mathbf{x}}$:

$$f(\bar{\mathbf{x}} + \lambda\mathbf{d}) = f(\bar{\mathbf{x}}) + \lambda\mathbf{f}_{\mathbf{x}}'(\bar{\mathbf{x}})\mathbf{d} + \lambda|\mathbf{d}|\omega(\bar{\mathbf{x}},\lambda\mathbf{d})$$

The latter implies:

$$\frac{f(\bar{x} + \lambda d) - f(\bar{x})}{\lambda} = f'_x(\bar{x})d + |d|\omega(\bar{x},\lambda d)$$

Since $f'_x(\bar{x})d > 0$, and since $\omega(\bar{x},\lambda d) \to 0$ as $\lambda \to 0$, there exists a $\delta > 0$ such that $f'_x(\bar{x})d + |d|\omega(\bar{x},\lambda d) > 0$ for all $\lambda \in (0,\delta)$, which implies $f(\bar{x} + \lambda d) - f(\bar{x}) > 0$.

Any vector, **z**, satisfying $f(\bar{x} + \lambda z) > f(\bar{x})$ is said to be an "ascent direction" of $f(x)$ at $\bar{x}$. By the latter theorem, any **d** such that $f'_x(\bar{x})d > 0$ is an ascent direction. If D_a denotes the set of all such **d** for the point $\bar{x}$, then the theorem clearly implies that a necessary condition for $\bar{x}$ to be a local maximum is that $D_a = \emptyset$. Of course, this can only be true if $f'_x(\bar{x}) = 0$, which is the familiar first-order condition for unconstrained maximization.

The next theorem uses the last result to obtain necessary conditions for constrained maximization problems:

2.5.2 Theorem: Let X be a nonempty open set in R^n, and let $f(x):X \to R^1$ and $G(x):X \to R^m$. Consider the problem to maximize $f(x)$ subject to $G(x) \geq 0$ and $x \in X$. Let $\bar{x}$ be a local optimal solution, and let $f(x)$ and $G(x)$ be differentiable at $\bar{x}$. Furthermore, let $I = \{i: g_i(\bar{x}) = 0\}$, then $D_a \cap D'_f = \emptyset$, where:

$$D_a = \{d: f'_x(\bar{x})d > 0\}$$
$$D_f = \{d: \nabla g'_i(\bar{x})d > 0 \quad \forall\ i \in I\}$$

and where $\nabla g_i(\bar{x})$ denotes the gradient of $g_i(x)$ at $\bar{x}$.

Proof: Suppose $D_a \cap D_f' \neq \emptyset$ and let $\mathbf{d} \in D_a \cap D_f'$. Since X is open, there exists a δ_1 such that:

$$\bar{\mathbf{x}} + \lambda\mathbf{d} \in X; \qquad \forall\ \lambda \in (0,\delta_1)$$

Since the $g_i(\mathbf{x})$ are continuous at $\bar{\mathbf{x}}$, and since $g_i(\bar{\mathbf{x}}) > 0$ for $i \notin I$, then there exists a δ_2 such that:

$$g_i(\bar{\mathbf{x}} + \lambda\mathbf{d}) > 0; \qquad \forall\ \lambda \in (0,\delta_2) \text{ and } \forall\ i \notin I$$

Also, since the $g_i(\mathbf{x})$ are differentiable at $\bar{\mathbf{x}}$, then by Theorem 2.5.1, there exists a δ_3 such that:

$$g_i(\bar{\mathbf{x}} + \lambda\mathbf{d}) > g_i(\bar{\mathbf{x}}); \qquad \forall\ \lambda \in (0,\delta_3) \text{ and } \forall\ i \in I$$

Finally, from Theorem 2.5.1, it is known that there exists a δ_4 such that:

$$f(\bar{\mathbf{x}} + \lambda\mathbf{d}) > f(\bar{\mathbf{x}}); \qquad \forall\ \lambda \in (0,\delta_4)$$

Now, let $\delta = \min(\delta_1,\delta_2,\delta_3,\delta_4)$. It is apparent from the above that $f(\mathbf{x})$ can be feasibly increased by movement to $\bar{\mathbf{x}} + \lambda\mathbf{d}$ for any $\lambda \in (0,\delta)$, but this contradicts the fact that $\bar{\mathbf{x}}$ is a local optimal solution.

The constraints satisfying $g_i(\bar{\mathbf{x}}) = 0$ are said to be "active" at $\bar{\mathbf{x}}$. The last theorem leads to the following result due to Fritz John (1948), which also utilizes the concept of active constraints:

2.5.3 Theorem (Fritz John Necessary Conditions): Let X be a nonempty open set in R^n, and let $f(\mathbf{x}):X \to R^1$ and

$\mathbf{G(x)}: X \to R^m$. Consider the problem to maximize $f(\mathbf{x})$ subject to $\mathbf{G(x)} \geq \mathbf{0}$ and $\mathbf{x} \in X$. Let $\bar{\mathbf{x}}$ be a local optimal solution, and let $f(\mathbf{x})$ and $\mathbf{G(x)}$ be differentiable at $\bar{\mathbf{x}}$. Furthermore, let $I = \{i: g_i(\bar{\mathbf{x}}) = 0\}$, then there exist nonnegative scalers, μ_0 and $(\mu_1, \mu_2,,, \mu_m)$, not all zero, such that:

$$\mu_0 \mathbf{f_x}(\bar{\mathbf{x}}) + \Sigma_{i=1}^m \mu_i \nabla \mathbf{g_i}(\bar{\mathbf{x}}) = \mathbf{0}$$

$$\mu_i g_i(\bar{\mathbf{x}}) = 0; \qquad i = 1,2,,,m$$

Proof: Since $\bar{\mathbf{x}}$ is a local optimal solution, then by the last theorem, there does not exist a vector, $\mathbf{d}$, such that $\mathbf{f}'_\mathbf{x}(\bar{\mathbf{x}})\mathbf{d} > 0$ and $\nabla \mathbf{g}'_\mathbf{i}(\bar{\mathbf{x}})\mathbf{d} > 0$ for $i \in I$. Let $\mathbf{A}$ be the matrix whose rows are $\mathbf{f}'_\mathbf{x}(\bar{\mathbf{x}})$ and $\nabla \mathbf{g}'_\mathbf{i}(\bar{\mathbf{x}})$ for every $i \in I$. The last theorem implies that the system, $\mathbf{Ad} > \mathbf{0}$, is inconsistent; subsequently, by Gordon's theorem (Theorem 2.3.5), there exists a $\bar{\mathbf{y}} \geq \mathbf{0}$ such that $\mathbf{A}'\bar{\mathbf{y}} = \mathbf{0}$. Set μ_0 and μ_i for $i \in I$ equal to their corresponding components in $\bar{\mathbf{y}}$, and set $\mu_i = 0$ for $i \notin I$ to complete the proof.

Note that the second of the Fritz John conditions implies that if $g_i(\bar{\mathbf{x}}) > 0$ then $\mu_i = 0$. Such variables are said to be "complementary." Accordingly, the relation is commonly called a "complementary slackness" relation.

The Fritz John conditions in conjunction with a constraint qualification lead immediately to the famous Kuhn-Tucker necessary conditions:

2.5.4 Theorem (Kuhn-Tucker Necessary Conditions): Let X be a nonempty open set in R^n, and let $f(\mathbf{x}): X \to R^1$ and

$\mathbf{G(x)}: X \to R^m$. Consider the problem to maximize $f(\mathbf{x})$ subject to $\mathbf{G(x)} \geq \mathbf{0}$ and $\mathbf{x} \in X$. Let $\bar{\mathbf{x}}$ be a local optimal solution, and let $f(\mathbf{x})$ and $\mathbf{G(x)}$ be differentiable at $\bar{\mathbf{x}}$. Furthermore, let $I = \{i : g_i(\bar{\mathbf{x}}) = 0\}$, and let $\nabla \mathbf{g_i}(\bar{\mathbf{x}})$ for $i \in I$ be linearly independent, then there exists $(\lambda_1, \lambda_2,,, \lambda_m)$ satisfying:

$$\mathbf{f_x}(\bar{\mathbf{x}}) + \Sigma_{i=1}^m \lambda_i \nabla \mathbf{g_i}(\bar{\mathbf{x}}) = \mathbf{0}$$

$$\lambda_i g_i(\bar{\mathbf{x}}) = 0; \qquad i = 1,2,,,m$$

$$\lambda_i \geq 0; \qquad i = 1,2,,,m$$

Proof: Since X is open, and since $f(\mathbf{x})$ and $\mathbf{G(x)}$ are differentiable at $\bar{\mathbf{x}}$, it is known from the former theorem that there exist nonnegative scalers, μ_0 and $(\mu_1, \mu_2,,, \mu_m)$, not all zero, such that:

$$\mu_0 \mathbf{f_x}(\bar{\mathbf{x}}) + \Sigma_{i=1}^m \mu_i \nabla \mathbf{g_i}(\bar{\mathbf{x}}) = \mathbf{0}$$

$$\mu_i g_i(\bar{\mathbf{x}}) = 0; \qquad i = 1,2,,,m$$

Moreover, it may be concluded that $\mu_0 \neq 0$, for otherwise, the first condition would contradict the assumption that the $\nabla \mathbf{g_i}(\bar{\mathbf{x}})$ are linearly independent for $i \in I$. Let $\lambda_i = \mu_i / \mu_0$ to complete the proof.

Hence, under the stated assumptions, the Kuhn-Tucker conditions are necessary conditions for the solutions to the maximization problem. It will be observed that the scaler variant of these conditions as stated in the latter theorem is equivalent to the matrix variant as stated in conditions (2.5) through (2.7).

The constraint qualification described in the previous theorem is known as the "linear independence" qualification. While the Kuhn-Tucker conditions follow immediately upon the assumption of this qualification, the linear independence requirement is unnecessarily restrictive. For example, it may be shown that the Kuhn-Tucker conditions are always necessary conditions when X is open and when the constraints are affine, even when the gradients of the active constraints are not linearly independent. Numerous other constraint qualifications have been proposed in nonlinear programming literature; nearly all of which are less restrictive than the linear independence requirement. By "less restrictive" is meant that the constraint qualification specifies a broader range of circumstances under which it may be concluded that local optima imply Kuhn-Tucker points.

One of the most general constraint qualifications utilizes a concept known as the "cone of tangents," which is formerly defined with the following:

2.5.5 Definition: Let S be a nonempty set in R^n. S is said to be a "cone" if $\mathbf{x} \in S$ implies that $\lambda\mathbf{x} \in S$ for all $\lambda \geq 0$.

2.5.6 Definition: Let S be a nonempty set in R^n, and let $\bar{\mathbf{x}} \in \mathrm{cl}\ S$. The "cone of tangents" of S at $\bar{\mathbf{x}}$ is the set, D_t, consisting of all $\mathbf{d}$ such that:

$$\mathbf{d} = \lim_{k\to\infty} \alpha(\mathbf{x}_k - \bar{\mathbf{x}})/|\mathbf{x}_k - \bar{\mathbf{x}}|$$

for all $\alpha \geq 0$ and all sequences, $\{x_k\}$, in S satisfying $\lim_{k\to\infty} x_k = \bar{x}$.

Hence, the cone of tangents of S at $\bar{x}$ is the set of all directions from which $\bar{x}$ may be approached from within S. Observe that if $\bar{x} \in \text{int } S$, then $\bar{x}$ may be so approached from all directions; thus, $D_t = R^n$. Also, note that if for some $\delta > 0$, $\bar{x} + \lambda\mathbf{d} \in S$ for all $\lambda \in (0,\delta)$, then necessarily $\mathbf{d} \in D_t$, as may be seen by simply setting $x_k = \bar{x} + (\delta/k)\mathbf{d}$ in the definition above.

The next two theorems establish the general constraint qualification:

2.5.7 Theorem: Let X be a nonempty open set in R^n, and let $f(\mathbf{x}):X \to R^1$. Consider the problem to maximize $f(\mathbf{x})$ subject to $\mathbf{x} \in S$ where S is a nonempty subset of X. Let $\bar{\mathbf{x}}$ be a local optimal solution to the problem. Moreover, suppose that $f(\mathbf{x})$ is differentiable at $\bar{x}$, then $D_a \cap D_t = \emptyset$, where $D_a = \{\mathbf{d}: \mathbf{f}'_{\mathbf{x}}(\bar{x})\mathbf{d} > 0\}$, and D_t is the cone of tangents of S at $\bar{x}$.

Proof: Let $\{x_k\}$ be any sequence in S satisfying $\lim_{k\to\infty} x_k = \bar{x}$, and let $\mathbf{d} = \lim_{k\to\infty} (x_k - \bar{x})/|x_k - \bar{x}|$. Observe that $\mathbf{d} \in D_t$. By the differentiability of $f(\mathbf{x})$ at $\bar{x}$:

$$f(x_k) - f(\bar{x}) = \mathbf{f}'_{\mathbf{x}}(\bar{x})(x_k - \bar{x}) + |x_k - \bar{x}|\omega(\bar{x}, x_k - \bar{x})$$

where $\omega(\bar{x}, x_k - \bar{x}) \to 0$ as $x_k \to \bar{x}$. Since $\bar{x}$ is a local optimal solution, then for sufficiently large k, the above implies:

$$f_x'(\bar{x})(x_k - \bar{x}) + |x_k - \bar{x}|\omega(\bar{x}, x_k - \bar{x}) \leq 0$$

Divide the latter by $|x_k - \bar{x}|$ and take the limit as $k \to \infty$ to produce $f_x'(\bar{x})\mathbf{d} \leq 0$. Hence, $\mathbf{d} \in D_t$ implies that $f_x'(\bar{x})\mathbf{d} \leq 0$, which completes the proof.

2.5.8 Theorem (Kuhn-Tucker Necessary Conditions): Let X be a nonempty open set in R^n, and let $f(\mathbf{x}): X \to R^1$ and $\mathbf{G(x)}: X \to R^m$. Consider the problem to maximize $f(\mathbf{x})$ subject to $\mathbf{x} \in X$ and $\mathbf{G(x)} \geq \mathbf{0}$. Let $\bar{x}$ be a local optimal solution, and let $I = \{i: g_i(\bar{x}) = 0\}$. Suppose that $f(\mathbf{x})$ and $\mathbf{G(x)}$ are differentiable at $\bar{x}$. Further, suppose that $D_t = D_f$ where $D_f = \{\mathbf{d}: \nabla g_i'(\bar{x})\mathbf{d} \geq 0 \quad \forall\ i \in I\}$, and D_t is the cone of tangents of the feasible region at $\bar{x}$, then there exists $(\lambda_1, \lambda_2, , , \lambda_m)$ satisfying:

$$\mathbf{f_x}(\bar{x}) + \Sigma_{i=1}^{m} \lambda_i \nabla \mathbf{g_i}(\bar{x}) = \mathbf{0}$$
$$\lambda_i g_i(\bar{x}) = 0; \qquad i = 1, 2, , , m$$
$$\lambda_i \geq 0; \qquad i = 1, 2, , , m$$

Proof: Since $\bar{x}$ is a local optimal solution, then by the former theorem, $D_a \cap D_t = \emptyset$, where $D_a = \{\mathbf{d}: f_x'(\bar{x})\mathbf{d} > 0\}$. As $D_t = D_f$ by assumption, it follows that $D_a \cap D_f = \emptyset$. Hence, the following system has no solution:

$f_x'(\bar{x})d > 0$

$\nabla g_i'(\bar{x})d \geq 0; \qquad \forall\ i \in I$

Let **A** be a matrix whose rows consist of the $-\nabla g_i'(\bar{x})$ for $i \in I$, and let $\mathbf{c} = \mathbf{f_x}(\bar{x})$. From Farkas' theorem (Theorem 2.3.4), there exists $\bar{y} \geq 0$ such that $\mathbf{A}'\bar{y} = \mathbf{c}$. Set λ_i for $i \in I$ equal to the corresponding components in $\bar{y}$, and set $\lambda_i = 0$ for $i \notin I$ to complete the proof.

The constraint qualification, $D_t = D_f$, used in the theorem above is due to Abadie (1967), and is commonly known as the "Abadie constraint qualification." The next theorem shows that all affine constraints satisfy the Abadie constraint qualification. Subsequently, if X is open, it may be concluded that the Kuhn-Tucker conditions are always necessary conditions to the solutions of the maximization problem if **G(x)** is affine.

2.5.9 Theorem: Let X be a nonempty open set in R^n, and let $\mathbf{A} \in R^{m\times n}$ and $\mathbf{b} \in R^m$. Moreover, let $S = \{\mathbf{x} \in X\colon \mathbf{Ax} \geq \mathbf{b}\}$. Suppose $\bar{\mathbf{x}} \in S$ satisfies $\mathbf{A}_1\bar{\mathbf{x}} = \mathbf{b}_1$ and $\mathbf{A}_2\bar{\mathbf{x}} > \mathbf{b}_2$, where $\mathbf{A}' = (\mathbf{A}_1', \mathbf{A}_2')$ and $\mathbf{b}' = (\mathbf{b}_1', \mathbf{b}_2')$, then $D_t = D_f$, where $D_f = \{\mathbf{d}\colon \mathbf{A}_1\mathbf{d} \geq \mathbf{0}\}$ and D_t is the cone of tangents of S at $\bar{\mathbf{x}}$.

Proof: If $\mathbf{A}_1$ is vacuous, then $D_f = R^n$. Futhermore, $\bar{\mathbf{x}} \in \text{int } S$, which implies that $D_t = R^n$; hence, $D_t = D_f$, and the theorem holds. Suppose $\mathbf{A}_1$ is not vacuous, and let $\{\mathbf{x}_K\}$ be any sequence in S satisfying $\lim_{K\to\infty} \mathbf{x}_K = \bar{\mathbf{x}}$, then:

$$\mathbf{A}_1(\mathbf{x}_K - \bar{\mathbf{x}}) \geq \mathbf{b}_1 - \mathbf{b}_1 = \mathbf{0}$$

Divide the above by $|\mathbf{x}_k - \bar{\mathbf{x}}|$, and take the limit as $k \to \infty$ to obtain $\mathbf{A}_1\mathbf{d} \geq \mathbf{0}$ where $\mathbf{d} \in D_t$. But this implies $\mathbf{d} \in D_f$, and subsequently, $D_t \subset D_f$. Now, suppose $\mathbf{d} \in D_f$, that is, $\mathbf{A}_1\mathbf{d} \geq \mathbf{0}$. Since $\mathbf{A}_2\bar{\mathbf{x}} > \mathbf{b}_2$, and since X is open, there is a $\delta > 0$ such that both $\mathbf{A}_2(\bar{\mathbf{x}} + \lambda\mathbf{d}) > \mathbf{b}_2$ and $\bar{\mathbf{x}} + \lambda\mathbf{d} \in X$ for all $\lambda \in (0,\delta)$. Also, $\mathbf{A}_1(\bar{\mathbf{x}} + \lambda\mathbf{d}) \geq \mathbf{b}_1$ for all $\lambda \geq 0$. Hence, $\bar{\mathbf{x}} + \lambda\mathbf{d} \in S$ for each $\lambda \in (0,\delta)$, which implies that $\mathbf{d} \in D_t$, so $D_f \subset D_t$. As $D_t \subset D_f$ and $D_f \subset D_t$, then necessarily, $D_t = D_f$, which was to be shown.

The foregoing lemma shows that under differentiability assumptions, $D_t \subset D_f$ in all cases; consequently, the Abadie constraint qualification is know to hold if it can be shown that $D_f \subset D_t$.

2.5.10 Lemma: Let X be a nonempty open set in R^n, and let $\mathbf{G}(\mathbf{x}): X \to R^m$. Moreover, let $S = \{\mathbf{x} \in X: \mathbf{G}(\mathbf{x}) \geq \mathbf{0}\}$ and let $\bar{\mathbf{x}} \in S$. Suppose that $g_i(\bar{\mathbf{x}}) = 0$ for $i \in I$, and suppose that the $g_i(\mathbf{x})$ are differentiable at $\bar{\mathbf{x}}$ for $i \in I$, then $D_t \subset D_f$, where $D_f = \{\mathbf{d}: \nabla\mathbf{g}_i'(\bar{\mathbf{x}})\mathbf{d} \geq 0 \quad \forall\ i \in I\}$ and D_t is the cone of tangents of S at $\bar{\mathbf{x}}$.

Proof: Let $\{\mathbf{x}_k\}$ be any sequence in S satisfying $\lim_{k\to\infty} \mathbf{x}_k = \bar{\mathbf{x}}$. Using the differentiability of the $g_i(\mathbf{x})$ at $\bar{\mathbf{x}}$ for $i \in I$, it may be concluded that:

$$g_i(\mathbf{x}_k) = \nabla\mathbf{g}_i'(\bar{\mathbf{x}})(\mathbf{x}_k - \bar{\mathbf{x}}) + |\mathbf{x}_k - \bar{\mathbf{x}}|\omega(\bar{\mathbf{x}},\mathbf{x}_k-\bar{\mathbf{x}}); \quad \forall\ i \in I$$

Divide these expressions by $|\mathbf{x}_k - \bar{\mathbf{x}}|$ and take the limit as $k \to \infty$ with the result that:

$$\nabla\mathbf{g}_i'(\bar{\mathbf{x}})\mathbf{d} = \lim_{k\to\infty} g_i(\mathbf{x}_k)/|\mathbf{x}_k - \bar{\mathbf{x}}| \geq 0; \qquad \forall\ i \in I$$

where $\mathbf{d} \in D_t$. Hence, $\mathbf{d} \in D_t$ implies $\mathbf{d} \in D_f$, which completes the proof.

The next theorem confirms that the linear independence constraint qualification is a special case of the Abadie constraint qualification:

2.5.11 Theorem: Let X be a nonempty open set in R^n, and let $\mathbf{G}(\mathbf{x})$:$X \to R^m$. Let $S = \{\mathbf{x} \in X\text{: } \mathbf{G}(\mathbf{x}) \geq \mathbf{0}\}$ and let $\bar{\mathbf{x}} \in S$. Moreover, let $I = \{i\text{: } g_i(\bar{\mathbf{x}}) = 0\}$. Suppose that the $g_i(\mathbf{x})$ are differentiable at $\bar{\mathbf{x}}$ for $i \in I$, and that the $\nabla\mathbf{g}_i(\bar{\mathbf{x}})$ are linearly independent for $i \in I$, then $D_f = D_t$ where $D_f = \{\mathbf{d}\text{: } \nabla\mathbf{g}_i'(\bar{\mathbf{x}})\mathbf{d} \geq 0 \quad \forall\ i \in I\}$ and D_t is the cone of tangents of S at $\bar{\mathbf{x}}$.

Proof: Let $\mathbf{A}'$ be a matrix whose columns consist of the $\nabla\mathbf{g}_i(\bar{\mathbf{x}})$ for $i \in I$. As a consequence of the linear independence assumption, $\mathbf{A}'\bar{\mathbf{y}} = \mathbf{0}$ has no solution; consequently, by Gordon's theorem (Theorem 2.3.5), there exists a $\hat{\mathbf{d}}$ such that $\mathbf{A}\hat{\mathbf{d}} > \mathbf{0}$, or equivalently:

$$\nabla\mathbf{g}_i'(\bar{\mathbf{x}})\hat{\mathbf{d}} > 0; \qquad \forall\ i \in I$$

Let $\mathbf{z}_k = (1/k)\hat{\mathbf{d}} + (1 - 1/k)\mathbf{d}$ for any $\mathbf{d} \in D_f$ and any $k > 0$. Without loss of generality, it may be assumed that $|\mathbf{d}| = 1$. Observe that:

$$\nabla\mathbf{g}_i'(\bar{\mathbf{x}})\mathbf{z}_k > 0; \qquad \forall\ i \in I \text{ and } k > 0$$

By Theorem 2.5.1 and the openness of X, it may be concluded that there exists a $\lambda_k > 0$ such that $\bar{\mathbf{x}} + \lambda_k \mathbf{z}_k \in S$. Without loss of generality, the λ_k may be chosen such that $\lim_{k\to\infty} \lambda_k = 0$. Let $\mathbf{x}_k = \bar{\mathbf{x}} + \lambda_k \mathbf{z}_k$, and observe that $\lim_{k\to\infty} \mathbf{x}_k = \bar{\mathbf{x}}$. Finally, consider:

$$\lim_{k\to\infty} (\mathbf{x}_k - \bar{\mathbf{x}})/|\mathbf{x}_k - \bar{\mathbf{x}}| = \lim_{k\to\infty} \mathbf{z}_k/|\mathbf{z}_k| = \mathbf{d}$$

Hence, $\mathbf{d} \in D_t$ by definition, and subsequently, $D_f \subset D_t$. $D_t \subset D_f$ follows from Lemma 2.5.10; thus, $D_f = D_t$, which was to be shown.

The next theorem develops the Kuhn-Tucker necessary conditions in terms that prove essential in a subsequent chapter:

2.5.12 Theorem (Kuhn-Tucker Necessary Conditions): Let X be a nonempty open set in R^n, and let $f(\mathbf{x}):X \to R^1$ and $\mathbf{G}(\mathbf{x}):X \to R^m$. Consider the problem to maximize $f(\mathbf{x})$ subject to $\mathbf{x} \in X$ and $\mathbf{G}(\mathbf{x}) \geq \mathbf{0}$. Let $\mathbf{G}'(\mathbf{x}) = [\hat{\mathbf{G}}'(\mathbf{x}),\tilde{\mathbf{G}}'(\mathbf{x})]$ where $\tilde{\mathbf{G}}(\mathbf{x})$ is affine. Let $\bar{\mathbf{x}}$ be a local optimal solution to the problem, and let:

$$\hat{I} = \{i: \hat{g}_i(\bar{\mathbf{x}}) = 0\}$$
$$\tilde{I} = \{i: \tilde{g}_i(\bar{\mathbf{x}}) = 0\}$$

Suppose that $f(\mathbf{x})$ and $\mathbf{G}(\mathbf{x})$ are differentiable at $\bar{\mathbf{x}}$. Moreover, suppose that the $\nabla\hat{\mathbf{g}}_i(\bar{\mathbf{x}})$ are linearly independent for $i \in \hat{I}$, then there exists $(\lambda_1,\lambda_2,,,\lambda_m)$ such that:

$$f_x(\bar{x}) + \Sigma_{i=1}^{m} \lambda_i \nabla g_i(\bar{x}) = 0$$

$$\lambda_i g_i(\bar{x}) = 0; \qquad i = 1,2,,,m$$

$$\lambda_i \geq 0; \qquad i = 1,2,,,m$$

Proof: Define the following sets:

$$\hat{D}_f = \{d: \nabla\hat{g}_i'(\bar{x})d \geq 0 \text{ for } i \in \hat{I}\}$$

$$\tilde{D}_f = \{d: \nabla\tilde{g}_i'(\bar{x})d \geq 0 \text{ for } i \in \tilde{I}\}$$

$$D_f = \{d: \nabla g_i'(\bar{x})d \geq 0 \text{ for } i \in \hat{I}\cup\tilde{I}\}$$

$$\hat{S} = \{\mathbf{x} \in X: \hat{\mathbf{G}}(\mathbf{x}) \geq \mathbf{0}\}$$

$$\tilde{S} = \{\mathbf{x} \in X: \tilde{\mathbf{G}}(\mathbf{x}) \geq \mathbf{0}\}$$

$$S = \{\mathbf{x} \in X: \mathbf{G}(\mathbf{x}) \geq \mathbf{0}\}$$

Let $\hat{D}_t$, $\tilde{D}_t$, and D_t be the cones of tangents corresponding to $\hat{S}$, $\tilde{S}$, and S, respectively. By Theorem 2.5.11, $\hat{D}_f = \hat{D}_t$. By Theorem 2.5.9, $\tilde{D}_f = \tilde{D}_t$. Hence:

$$D_f = \hat{D}_f \cap \tilde{D}_f = \hat{D}_t \cap \tilde{D}_t = D_t$$

Therefore, the proposition follows from the Theorem 2.5.8.

The next theorem cites conditions under which Kuhn-Tucker points imply global optima:

2.5.13 Theorem (Kuhn-Tucker Sufficient Conditions): Let X be a nonempty convex set in R^n, and let $f(\mathbf{x}):X \to R^1$ and $\mathbf{G}(\mathbf{x}):X \to R^m$. Consider the problem to maximize $f(\mathbf{x})$ subject to $\mathbf{G}(\mathbf{x}) \geq \mathbf{0}$ and $\mathbf{x} \in X$. Let $\bar{\mathbf{x}}$ be a feasible solution, and suppose that there exists $(\lambda_1,\lambda_2,,,\lambda_m)$ such that:

$$f_{\mathbf{x}}(\bar{x}) + \Sigma_{i=1}^{m}\lambda_i \nabla g_1(\bar{x}) = 0$$

$$\lambda_i g_i(\bar{x}) = 0; \qquad i = 1,2,,,m$$

$$\lambda_i \geq 0; \qquad i = 1,2,,,m$$

Let $I = \{i : g_i(\bar{x}) = 0\}$, and let $g_i(x)$ for $i \in I$ be quasiconcave at $\bar{x}$ with respect to points in the feasible region. Moreover, let $f(\mathbf{x})$ be pseudoconcave (strictly pseudoconcave) at $\bar{x}$ with respect to points in the feasible region, then $\bar{x}$ is a global optimal solution (unique global optimal solution) to the maximization problem:

Proof: First, suppose that $f(\mathbf{x})$ is pseudoconcave at $\bar{x}$. Let $\hat{x}$ be any other feasible solution to the problem, then for $i \in I$, $g_i(\hat{x}) \geq g_i(\bar{x}) = 0$. By quasiconcavity of $g_i(\mathbf{x})$ at $\bar{x}$, it follows that for $i \in I$:

$$g_i[\alpha\hat{x} + (1-\alpha)\bar{x}] = g_i[\bar{x} + \alpha(\hat{x} - \bar{x})] \geq 0; \qquad \forall\ \alpha \in (0,1)$$

This implies that the $g_i(x)$ for $i \in I$ do not decrease with a movement from $\bar{x}$ in the direction of $(\hat{x} - \bar{x})$; therefore, by Theorem 2.5.1:

$$\nabla g_1'(\bar{x})(\hat{x} - \bar{x}) \geq 0; \qquad \forall\ i \in I$$

Multiply these terms by their corresponding λ_i in the Kuhn-Tucker conditions and sum over $i \in I$ to obtain:

$$[\Sigma_{i\in I}\ \lambda_i \nabla g_1(\bar{x})]'(\hat{x} - \bar{x}) \geq 0$$

But, since $f_{\mathbf{x}}(\bar{x}) + \Sigma_{i\in I}\ \lambda_i \nabla g_1(\bar{x}) = 0$, the latter implies that:

$$f'_{\mathbf{x}}(\bar{x})(\hat{x} - \bar{x}) \leq 0$$

Consequently, by the pseudoconcavity of $f(\mathbf{x})$ at $\bar{\mathbf{x}}$, this implies $f(\hat{\mathbf{x}}) \leq f(\bar{\mathbf{x}})$. The proof for the strictly pseudoconcave case is accomplished by replacing the inequality in the latter with strict inequality.

2.6 Lagrangian Saddle Point Characterization

Consider the problem:

$$\begin{aligned} \text{maximize}(\mathbf{x}): \quad & f(\mathbf{x}) \\ \text{subject to:} \quad & \mathbf{G}(\mathbf{x}) \geq \mathbf{0} \\ & \mathbf{x} \in X \end{aligned}$$

where $f(\mathbf{x}):X \rightarrow R^1$, and $\mathbf{G}(\mathbf{x}):X \rightarrow R^m$. Oftentimes it becomes convenient to formulate the solutions to such problems in terms of the "saddle points" of the associated "Lagrangian." The Lagrangian to the above problem is the function:

$$l(\mathbf{x},\lambda) = f(\mathbf{x}) + \lambda'\mathbf{G}(\mathbf{x}); \qquad (\mathbf{x},\lambda) \in X \otimes R^m_+$$

$(\bar{\mathbf{x}},\bar{\lambda})$ is said to be a saddle point of $l(\mathbf{x},\lambda)$ if:

$$l(\mathbf{x},\bar{\lambda}) \leq l(\bar{\mathbf{x}},\bar{\lambda}) \leq l(\bar{\mathbf{x}},\lambda); \qquad \forall\ (\mathbf{x},\lambda) \in X \otimes R^m_+$$

Thus, at the saddle point, $l(\mathbf{x},\bar{\lambda})$ is maximized with respect to $\mathbf{x}$ subject to $\mathbf{x} \in X$, and $l(\bar{\mathbf{x}},\lambda)$ is minimized with respect to λ subject to $\lambda \in R^m_+$. The justification of the term "saddle point" should now be apparent.

There is considerable body of theory dealing with Lagrangians and with Lagrangian saddle points; however, the only concern here is the relation between such saddle points and the solutions to the Kuhn-Tucker conditions. It is shown that under certain conditions, the saddle points in the Lagrangian are one-to-one with the solutions to the Kuhn-Tucker conditions.

Observe that the Kuhn-Tucker conditions can be written in terms of the derivatives of the Lagrangian as:

$$l_x(x,\lambda) = f_x(x) + G_x(x)\lambda = 0$$
$$\lambda' l_\lambda(\bar{x},\bar{\lambda}) = \lambda' G(x) = 0$$
$$\lambda \geq 0$$

The following theorem establishes a connection between the solutions to the Kuhn-Tucker conditions and the solutions to the saddle point relation:

2.6.1 Theorem: Let X be a nonempty open set in R^n, and let $f(x):X \to R^1$ and $G(x):X \to R^m$. Let $l(x,\lambda) = f(x) + \lambda' G(x)$, and suppose that $(\bar{x},\bar{\lambda}) \in X \otimes R^m_+$ satisfies the saddle point relation:

$$l(x,\bar{\lambda}) \leq l(\bar{x},\bar{\lambda}) \leq l(\bar{x},\lambda); \qquad \forall\ (x,\lambda) \in X \otimes R^m_+$$

Further, suppose that $f(x)$ and $G(x)$ are differentiable at $\bar{x}$, then $\bar{x}$ is feasible; moreover, $(\bar{x},\bar{\lambda})$ satisfies the Kuhn-Tucker conditions:

$$f_x(x) + G_x(x)\lambda = 0 \tag{2.8}$$
$$\lambda' G(x) = 0 \tag{2.9}$$
$$\lambda \geq 0$$

Conversely, let $\bar{x}$ be feasible, and suppose that $(\bar{x},\bar{\lambda})$ satisfies the Kuhn-Tucker conditions. Let $I = \{i: g_i(\bar{x}) = 0\}$. Moreover, let X be convex, and let $f(x)$ and $g_i(x)$ for $i \in I$ be concave at $\bar{x}$, then $(\bar{x},\bar{\lambda})$ solves the saddle point relation.

Proof: Suppose that $(\bar{x},\bar{\lambda})$ satisfies the saddle point relation. The right-hand inequality in this relation implies:

$$f(\bar{x}) + \bar{\lambda}'G(\bar{x}) \leq f(\bar{x}) + \lambda'G(\bar{x}); \qquad \forall\ \lambda \in R^m_+$$

Since the components of λ may be arbitrarily large, then the above implies $G(\bar{x}) \geq 0$; hence, $\bar{x}$ is feasible. Since the components of λ may be zero, then necessarily $\bar{\lambda}'G(\bar{x}) = 0$; hence, (2.9) holds. Using this result, the left-hand inequality in the saddle point relation becomes:

$$f(x) + \bar{\lambda}'G(x) \leq f(\bar{x}) + \bar{\lambda}'G(\bar{x}); \qquad \forall\ x \in X$$

Thus, $\bar{x}$ maximizes $f(x) + \bar{\lambda}'G(x)$ subject to $x \in X$. Since X is open, $\bar{x} \in \text{int } X$, and since $f(x)$ and $G(x)$ are differentiable at $\bar{x}$, then the last inequality implies:

$$f_x(\bar{x}) + G_x(\bar{x})\bar{\lambda} = 0$$

Hence, (2.8) holds, and consequently, saddle points imply Kuhn-Tucker points.

Conversely, suppose that $(\bar{x},\bar{\lambda})$ satisfies the Kuhn-Tucker conditions. Since $f(x)$ and $g_i(x)$ for $i \in I$ are concave at $\bar{x}$, then by Theorem 2.4.7:

$f(x) \leq f_x'(\bar{x})(x - \bar{x})$; $\forall\ x \in X$

$g_i(x) \leq g_i(\bar{x}) + \nabla g_i'(\bar{x})(x - \bar{x})$; $\forall\ x \in X$ and $\forall\ i \in I$

Multiply each of the inequalities in the latter by their respective $\bar{\lambda}_i$ and add the products to the first relation to produce:

$$f(x) + \Sigma_{i \in I}\ \bar{\lambda}_i g_i(x) \leq f(\bar{x}) + [f_x(\bar{x}) + \Sigma_{i \in I}\ \bar{\lambda}_i \nabla g_i(\bar{x})]' (x - \bar{x}); \quad \forall\ x \in X$$

Substitution of condition (2.8) into the above yields:

$$f(x) + \Sigma_{i \in I}\ \bar{\lambda}_i g_i(x) \leq f(\bar{x})$$

Note that the Kuhn-Tucker conditions require: $\bar{\lambda}_i = 0$ for $i \notin I$, $\lambda \geq \mathbf{0}$, and $\bar{\lambda}'G(\bar{x}) = 0$. Hence, the above may be extended to:

$$f(x) + \bar{\lambda}'G(x) \leq f(\bar{x}) + \bar{\lambda}'G(\bar{x}) \leq f(\bar{x}) + \lambda'G(\bar{x})$$

for every $(x,\lambda) \in X \otimes R_+^m$, and the proof is complete.

Observe that the left-hand inequality in the saddle point relation implies:

$$f(x) + \bar{\lambda}'G(x) \leq f(\bar{x}); \quad \forall\ x \in X$$

Since $\bar{\lambda} \geq \mathbf{0}$ and $G(x) \geq \mathbf{0}$ for all feasible x, then this inequality implies that any $\bar{x}$ satisfying the saddle point relation must be an optimal solution to the problem to maximize $f(x)$ subject to $G(x) \geq \mathbf{0}$ and $x \in X$. Hence, solutions to the saddle point relation always imply solutions to the maximization problem. To establish the converse, a linkage may be established through the Kuhn-Tucker conditions wherein it is shown: 1) Optimal solutions to the maximization problem imply solutions to the Kuhn-Tucker conditions, and 2) solutions to the Kuhn-Tucker conditions imply solutions to the saddle point relation. The first link is established in the former section with the various theorems demonstrating the necessity of the Kuhn-Tucker conditions. The second link is established in the latter theorem under that assumptions that $f(\mathbf{x})$ and the active $g_i(\mathbf{x})$ are concave at the Kuhn-Tucker point.

2.7 Kuhn Tucker Conditions Under Explicit Nonnegativity

In this section, an alternative statement of the Kuhn-Tucker conditions is derived for cases where the condition, $\mathbf{x} \geq \mathbf{0}$, is imposed as a constraint upon the problem. Thus, the general problem to be considered is:

$$\begin{aligned} &\text{maximize}(\mathbf{x}): && f(\mathbf{x}) \\ &\text{subject to:} && \mathbf{G}^*(\mathbf{x}) \geq \mathbf{0} \\ & && \mathbf{x} \in X \end{aligned}$$

where $\mathbf{G}^*(\mathbf{x}) = [\mathbf{G}(\mathbf{x}), \mathbf{x}]'$, and where $f(\mathbf{x}): X \to R^1$ and $\mathbf{G}(\mathbf{x}): X \to R^m$. Assuming that X is nonempty and open, and that $f(\mathbf{x})$ and $G(\mathbf{x})$ are differentiable over X, the Kuhn-Tucker conditions are:

$$\mathbf{f_x}(x) + \mathbf{G}_{\mathbf{x}}^{*}(x)\lambda^{*} = \mathbf{0}$$

$$(\lambda^{*})'\mathbf{G}^{*}(\mathbf{x}) = 0$$

$$\lambda^{*} \geq \mathbf{0}$$

With the partition $\lambda^{*} = (\lambda,\lambda_{x})'$, the above may be written:

$$\mathbf{f_x}(x) + \mathbf{G_x}(x)\lambda + \lambda_{x} = \mathbf{0}$$

$$\lambda'\mathbf{G}(\mathbf{x}) = 0$$

$$\lambda_{x}'\mathbf{x} = 0$$

$$(\lambda,\lambda_{x}) \geq \mathbf{0}$$

However, the conditions, $\lambda_{x} \geq \mathbf{0}$ and $\lambda_{x}'\mathbf{x} = 0$, may be completely incorporared into the first condition to obtain:

$$\mathbf{f_x}(x) + \mathbf{G_x}(x)\lambda \leq \mathbf{0}; \qquad \mathbf{x}'[\mathbf{f_x}(x) + \mathbf{G_x}(x)\lambda] = 0$$

$$\lambda'\mathbf{G}(\mathbf{x}) = 0$$

$$\lambda \geq \mathbf{0}$$

which are the Kuhn-Tucker conditions under explicit nonnegativity in $\mathbf{x}$.

There are frequently advantages to stating the Kuhn-Tucker conditions in these terms. For example, some solution algorithms inherently maintain nonnegativity in $\mathbf{x}$; consequently, provisions for such a restriction in the functional constraint would be redundant. Also, this statement fits well within some theoretical contexts, as will be seen in the next chapter.

FOOTNOTES

[1]The material in this chapter was compiled from several sources; however, the largest percentage of the material was taken from the unsurpassed work of Mokhtar S. Bazaraa and C.M. Shetty (1979).

[2]Excellent sources for these results and other material in this section include: Lipschutz (1965), Munkres (1975), and Kuratowski (1962).

[3]The openness of X is assumed here to achieve compatability with the differentiability assumption.

CHAPTER III

SPATIAL EQUILIBRIUM THEORY

In this chapter, the concept of spatial equilibrium is presented and discussed. The analysis begins with a basic partial equilibrium situation. It is demonstrated that under general conditions, the equilibrium price and quantity vectors may be formulated as the solutions to a nonlinear programming problem. Second, some of the general properties of partial spatial equilibria are derived. Third, it is shown how that many of the common barriers to trade may be incorporated into the basic model. Fourth, the model is reformulated in terms of price-dependent excess demands. Fifth, the partial equilibria results are extended to accomodate the multiproduct or general equilibrium case. Finally, the stability properties of the spatial equilibrium model are analyzed.

The foregoing theory was first presented by Enke (1951). Later, Samuelson (1952) formulated the Enke problem in terms of a mathematical programming model. Several more recent authors have contibuted to the development and extension of the theory, including: Smith (1963), Takayama and Judge (1964, 1970, and 1971), and Silberberg (1970).

3.1 The Basic Partial Equilibrium Model

Consider an aggregate market for a single commodity that is composed of n spatially separated regional markets. Suppose that the regional markets are at liberty to trade the product with one another, but that there are nonzero costs of transferring the product between regions. Moreover, assume that the following static conditions hold:

1) The commodity is homogeneous within and across the regional markets.
2) The commodity is of uniform price within any regional market.
3) The ith regional market is characterized by: a demand function, $d_i(p_i)$; a supply function, $s_i(p_i)$, and an excess demand function, $e_i(p_i) = d_i(p_i) - s_i(p_i)$, where p_i is the regional price.
4) The ith region may ship an arbitrary quantity, x_{ij}, to the jth region at the constant per-unit transportation rate, t_{ij}.

Let n_i denote the net imports of region i. Let $\dot{p}_i$ and $\dot{x}_{ij}$ denote time derivatives, and suppose that the following dynamic conditions hold:[1]

5 a) $\dot{p}_i > 0$ if and only if $e_i(p_i) - n_i > 0$.

b) $\dot{p}_i < 0$ if and only if $e_i(p_i) - n_i < 0$ and $p_i > 0$.

6 a) $\dot{x}_{ij} > 0$ only if $t_{ij} - (p_j - p_i) < 0$.

b) If $t_{ij} - (p_j - p_i) < 0$ for some j, then $\dot{x}_{ij} > 0$ for at least one such j.

c) $\dot{x}_{ij} < 0$ if and only if $t_{ij} - (p_j - p_i) > 0$ and $x_{ij} > 0$.

The second condition is implied by intraregional product homogeneity and perfect product mobility within regional bounds. Condition three necessitates conditions one and two, for it is assumed that supply and demand are functions of a single regional price, and that such functions are invariant with respect to origin or destination of product. Condition four is implied by perfectly elastic supply of transportation services.

The rationale for the dynamic conditions is as follows: When demand exceeds supply, or equivalently, when deficits occur, prices are bid upwards, hence condition 5a. When a state of excess supply exists, or equivalently, when surpluses occur, sellers reduce prices in order to clear the market, hence condition 5b. If price in j exceeds price in i by more than the costs of transporting from i to j, then profits may be had by shipping from i to j, hence conditions 6a and 6b. If the price difference between two regions does not cover the costs of transporting between the regions, then losses are incurred on present shipments, hence condition 6c. Conditions 6a and 6b account for the possibility that some profitable trade routes might be temporarily unresponsive if there exists other routes yielding greater profits. On the other hand, condition 6c assumes that all active trade flows inflicting losses are promptly reduced.

Observe that these adjustment rules imply that if the initial p_i and x_{ij} are nonnegative, then all subsequent p_i and x_{ij} are also nonnegative; thus, the system directing dynamic adjustment is positive. $x_{ij} < 0$ is disallowed because of the way in which the term is defined. Shipments from j to i are measured by positive x_{ji}; not by negative x_{ij}. Of course, x_{ij} is measured as flow per unit of time; thus, $\dot{x}_{ij}$ measures rate of change in flow.

Equilibrium is attained where $\dot{p}_i = 0$ and $\dot{x}_{ij} = 0$ for every i and j. Conditions five and six imply that equilibrium occurs if and only if:

$$e_i(p_i) - n_i \leq 0, \text{ if } < \text{ then } p_i = 0; \qquad \forall\ i \tag{3.1}$$

$$t_{ij} - (p_j - p_i) \geq 0, \text{ if } > \text{ then } x_{ij} = 0; \qquad \forall\ i,j \tag{3.2}$$

$$p_i \geq 0,\ x_{ij} \geq 0; \qquad \forall\ i,j \tag{3.3}$$

The latter conditions may be expediently expressed in matrix notation as:

$$\mathbf{E(p)} - \mathbf{Nx} \leqq \mathbf{0}; \qquad \mathbf{p}'[\mathbf{E(p)} - \mathbf{Nx}] = 0 \tag{3.4}$$

$$\mathbf{t} - \mathbf{N}'\mathbf{p} \geqq \mathbf{0}; \qquad \mathbf{x}'(\mathbf{t} - \mathbf{N}'\mathbf{p}) = 0 \tag{3.5}$$

$$(\mathbf{p},\mathbf{x}) \geqq \mathbf{0} \tag{3.6}$$

Here, $\mathbf{E(p)}$ is a vector-valued function of dimension n having the $e_i(p_i)$ for components. $\mathbf{p}$ is the n-dimensional vector of prices. $\mathbf{x}$, $\mathbf{t}$ and $\mathbf{N}$ are defined as follows:

$$\mathbf{x} = \left[\begin{array}{c} x_{12} \\ x_{13} \\ \vdots \\ x_{1n} \\ \hline x_{21} \\ x_{23} \\ \vdots \\ x_{2n} \\ \hline \vdots \\ \hline x_{n1} \\ x_{n2} \\ \vdots \\ x_{n,n-1} \end{array}\right] \qquad \mathbf{t} = \left[\begin{array}{c} t_{12} \\ t_{13} \\ \vdots \\ t_{1n} \\ \hline t_{21} \\ t_{23} \\ \vdots \\ t_{2n} \\ \hline \vdots \\ \hline t_{n1} \\ t_{n2} \\ \vdots \\ t_{n,n-1} \end{array}\right]$$

$$\mathbf{N} = \left[\begin{array}{cc|cc|cc|c|cc} -1 & -1 \ldots -1 & 1 & 0\ 0 \ldots 0 & 1 & 0\ 0 \ldots 0 & \ldots & 1 & 0\ 0 \ldots 0 \\ 1 & 0 \ldots 0 & -1 & -1 \ldots -1 & 0 & 1\ 0 \ldots 0 & \ldots & 0 & 1\ 0 \ldots 0 \\ 0 & 1\ 0 \ldots 0 & 0 & 1\ 0 \ldots 0 & -1 & -1 \ldots -1 & \ldots & 0 & 0\ 1\ 0 \ldots 0 \\ & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 \ldots 1 & 0 & 0 \ldots 1 & 0 & 0 \ldots 1 & \ldots & -1 & -1 \ldots -1 \end{array}\right]$$

The first partition of **x** contains exports by region one to the other regions. Likewise, the second partition contains the exports of region two, and so on. The x_{ii} terms are not included in the **x** vector. Their inclusion is needless since this analysis is conducted in terms of excess demands. Of course, these terms may be easily recovered with use of the regional demand functions once prices and net imports are determined. The **t** vector is identical in construction to **x**.

The **N** matrix is a most useful and expediting instrument to this analysis. It may be confirmed that:

$$\mathbf{N}\mathbf{x} = \begin{bmatrix} x_{21} + x_{31} + \ldots + x_{n1} & - x_{12} - x_{13} - \ldots - x_{1n} \\ x_{12} + x_{32} + \ldots + x_{n2} & - x_{21} - x_{23} - \ldots - x_{2n} \\ \vdots & \\ x_{1n} + x_{2n} + \ldots + x_{n-1,n} & - x_{n1} - x_{n2} - \ldots - x_{n,n-1} \end{bmatrix} = \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_n \end{bmatrix}$$

Thus, $\mathbf{Nx}$ is equal to the vector of net imports. Also, it may be confirmed that:

$$\mathbf{N}'\mathbf{p} = \begin{bmatrix} p_2 - p_1 \\ p_3 - p_1 \\ \vdots \\ p_n - p_1 \\ \hline p_1 - p_2 \\ p_3 - p_2 \\ \vdots \\ p_n - p_2 \\ \hline \vdots \\ \hline p_1 - p_n \\ p_2 - p_n \\ \vdots \\ p_{n-1} - p_n \end{bmatrix} \tag{3.7}$$

Hence, $\mathbf{N}'\mathbf{p}$ is the vector of price differentials measuring gains or losses before transportation charges for all possible schemes of trade. The rationale for the usage of $\mathbf{N}$ in (3.5) should now be apparent. Thus, $\mathbf{N}$ not only facilitates the transition of (3.1) to its matrix variant in (3.4), but also the transition of (3.2) to its matrix variant in (3.5).

A convenient attribute of the above model is that if the $e_i(p_i)$ are integrable, then the equilibria values for $\mathbf{p}$ and $\mathbf{x}$ may be formulated as the Kuhn-Tucker points of a nonlinear programming problem. Note that the spatial equilibrium conditions in (3.4) through (3.6) bear resemblance to the Kuhn-Tucker relations of the previous chapter inasmuch as inequality conditions are required in conjunction with complementary slackness relations. Indeed, it may be confirmed that these conditions are rendered as the Kuhn-Tucker conditions to the following problem:

maximize($\mathbf{p}$): $f(\mathbf{p})$

subject to: $\mathbf{t} - \mathbf{N}'\mathbf{p} \geq \mathbf{0}$

$\mathbf{p} \geq \mathbf{0}$

where:

$$f(\mathbf{p}) = \Sigma_{i=1}^{n} \int e_i(p_i)dp_i$$

and where $\mathbf{x}$ serves as the vector of Lagrangian multipliers on the first constraint. Here the set constraint is simply $\mathbf{p} \in R^n$. Henceforth, the set constraint will be ignored with the understanding that it is always taken to be the euclidean space. Upon observing that the gradient of $f(\mathbf{p})$ is $\mathbf{E}(\mathbf{p})$, it should be apparent that the Kuhn-Tucker conditions to the above problem are precisely conditions (3.4) through (3.6). Also, note that the Lagrangian to this problem is constructed as:

$$l(\mathbf{p},\mathbf{x}) = f(\mathbf{p}) + \mathbf{x}'(\mathbf{t} - \mathbf{N}'\mathbf{p}); \qquad (\mathbf{p},\mathbf{x}) \geq \mathbf{0}$$

Observe that the necessity and/or sufficiency of the Kuhn-Tucker conditions is of no great interest in the spatial equilibrium problem, for it is the Kuhn-Tucker conditions themselves that are of immediate concern. As shown above, the spatial market is in equilibrium if and only if the Kuhn-Tucker conditions to the above problem are satisfied. Therefore, any solution to the Kuhn-Tucker conditions is satisfactory for present purposes, even if it does not correspond to an optimal solution, either local or global, to the programming problem.

However, it will be noted that the programming problem does have many desirable properties. First, if $\mathbf{E}(\mathbf{p})$ is continuous at some $\bar{\mathbf{p}}$, then $f(\mathbf{p})$ will be differentiable at $\bar{\mathbf{p}}$. Second, the constraints are linear. Using Theorem 2.5.12, the differentiability of $f(\mathbf{p})$ and the linearity of the constraints ensures that the Kuhn-Tucker conditions are indeed necessary conditions to the local optimal solutions. Moreover, $f(\mathbf{p})$ is typically concave. Using Theorem 2.4.9, $f(\mathbf{p})$ is concave on R^n if its Hessian matrix is negative semidefinite on the same. The Hessian matrix of $f(\mathbf{p})$ is:

$$\mathbf{H}(\mathbf{p}) = \begin{bmatrix} \partial e_1/\partial p_1 & 0 & . & . & 0 \\ 0 & \partial e_2/\partial p_2 & 0 & . & 0 \\ . & 0 & . & & . \\ . & . & & . & 0 \\ 0 & 0 & . & 0 & \partial e_n/\partial p_n \end{bmatrix}$$

If the $\partial e_i(p_i)/\partial p_i < 0$, as is usually the case, then the Hessian matrix is indeed negative definite, and consequently, by the same theorem, $f(\mathbf{p})$ is strictly concave on R^n. As the constraints are also concave, then by Theorem 2.5.13, the Kuhn-Tucker conditions are sufficient for the global optima, and by Theorem 2.6.1, the Kuhn-Tucker points are one-to-one with the saddle points of the Lagrangian.

The concavity or differentiability of $f(\mathbf{p})$ over negative prices is of no real concern. Observe that as the $e_i(p_i)$ are irrelevant on $p_i < 0$, these functions may be defined in any way one chooses over negative prices. However, even this is not necessary. Theorem 2.5.12 (Kuhn-Tucker necessary conditions) requires differentiability only at the local optimal solutions, and Theorem 2.5.13 (Kuhn-Tucker sufficient conditions) merely requires that the Kuhn-Tucker point be concave with respect to the other points in the feasible region. As for Theorem 2.6.1, the set constraint could be defined here as $\mathbf{p} \in R^n_+$; consequently, if $f(\mathbf{p})$ is differentiable at all saddle points over nonnegative prices[2], and if it is concave at all Kuhn-Tucker points with respect to to all other points in R^n_+ , then the Kuhn-Tucker points are one-to-one with the saddle points of the Lagrangian over nonnegative $\mathbf{p}$.

Unfortunately, since the feasible region to the spatial equilibrium problem is not bounded, the Weierstrass theorem (Theorem 2.1.4) does not guarantee the existence of a solution.[3] Indeed, it is not difficult to construct market configurations for which no equilibrium exists. For example, if the excess demand functions are equal to constants summing to some value greater than zero, then an equilibrium solution cannot exist.

The programming problem considered above was developed in a rather peculiar fashion. Typically, one first specifies the objective function and constraints, and then derives the appropriate Kuhn-Tucker conditions. Here, a set of spatial equilibrium conditions was viewed as a set of Kuhn-Tucker conditions, and then a programming problem yielding these conditions was found. The contrived objective function, $f(\mathbf{p})$, is of interest only in that it serves to render the spatial equilibrium criteria as a set of Kuhn-Tucker conditions. Geometrically, $f(\mathbf{p})$ is some constant plus the sum of the areas lying to the left of the excess demand functions and beneath their respective price lines. However, it is not apparent that any profound inference concerning market behavior is to be drawn from this observation. Hence, no interpretive significance is attached to the function. It should be observed that $f(\mathbf{p})$ is similar to the "quasi-indirect welfare function" of Takayama and Judge (1971). Other formulations of the spatial equilibrium problem also utilize such instrumental functions. These include the "quasi welfare" function of the same authors and the "net social payoff" function of Samuelson (1952).

An advantage of the approach taken here is that spatial equilbria calculation can usually be accomplished using general nonlinear programming algorithms. In particular, if the excess demand functions are linear, then $f(\mathbf{p})$ will be quadratic so that any one of several quadratic programming routines may be used. Also, the spatial equilibrium problem is made accessible to a large body of mathematical programming theory. For example, since the constraints are quasiconcave, then it is known from Theorem 2.5.13 that the equilibrium price vector is uniquely determined if $f(\mathbf{p})$ is strictly pseudoconcave. Also, it will shortly be seen that the mathematical programming formulation greatly facilitates the examination of the stability of the model.

3.2 Properties of Spatial Equilibria

As noted before, if $f(\mathbf{p})$ is strictly pseudoconcave, then it is known from Theorem 2.5.13 that an equilibrium solution is unique with respect to $\mathbf{p}$. As shown in the previous section, $f(\mathbf{p})$ is strictly concave on R^n if $\partial e_i(p_i)/\partial p_i < 0$ for every i. Thus, if the excess demands have negative slopes, then it may be concluded that an equilibrium price vector is unique.

Though an equilibrium price vector is unique when $f(\mathbf{p})$ is strictly concave, this is not necessarily the case for $\mathbf{x}$. It should be observed that $\mathbf{x}$ serves as a vector of Lagrangian multipliers to the programming problem, and that strict concavity in the objective function does not imply uniqueness for such variables. Consider an aggregate market consisting of three regional markets that are situated on a line. Thus, the market map appears as follows:

A ---------> B --------> C

Suppose that a spatial equilibrium exists wherein A ships to C. Moreover, suppose that the costs of shipping A to C are equal to the sum of the costs of shipping from A to B and from B to C. It should be apparent that the quantities from A that are ultimately destined to C may be shipped either directly to C, or from A to B and then from B to C, or in any combination of these two schemes. Prices and net imports are the same in all scenarios; consequently, the spatial equilibrium conditions are satisfied by any one of an infinite number of trading arrangements. However, each arrangement clearly implies a distinct $\mathbf{x}$. This conclusion clearly holds regardless of the forms of the excess demands; therefore, strict concavity in $f(\mathbf{p})$ does not imply uniqueness in $\mathbf{x}$. This argument can also be illustrated with other market configurations.

Spatial equilibria are economically efficient inasmuch as interregional trade necessary to equilibrate the market is accomplished at minimal transportation costs.[4] In the previous section, the conditions for spatial equilibrium were found to be:

$$\mathbf{E}(\mathbf{p}) - \mathbf{N}\mathbf{x} \leq \mathbf{0}; \qquad \mathbf{p}'[\mathbf{E}(\mathbf{p}) - \mathbf{N}\mathbf{x}] = 0 \tag{3.8}$$

$$\mathbf{t} - \mathbf{N}'\mathbf{p} \geq \mathbf{0}; \qquad \mathbf{x}'(\mathbf{t} - \mathbf{N}'\mathbf{p}) = 0 \tag{3.9}$$

$$(\mathbf{p},\mathbf{x}) \geq \mathbf{0} \tag{3.10}$$

Now, consider the programming model:

$$\begin{aligned}
\text{maximize}(\mathbf{p},\mathbf{x}):\quad & \mathbf{p}'[\mathbf{E}(\mathbf{p}) - \mathbf{N}\mathbf{x}] - \mathbf{x}'(\mathbf{t} - \mathbf{N}'\mathbf{p}) \\
\text{subject to:}\quad & \mathbf{E}(\mathbf{p}) - \mathbf{N}\mathbf{x} \leq \mathbf{0} \\
& \mathbf{t} - \mathbf{N}'\mathbf{p} \geq \mathbf{0} \\
& (\mathbf{p},\mathbf{x}) \geq \mathbf{0}
\end{aligned}$$

Here, the objective function has been constructed from the complementary slackness conditions in (3.8) and (3.9). Let $(\bar{\mathbf{p}},\bar{\mathbf{x}})$ be any spatial equilibrium solution, and observe that the objective function to the problem is equal to zero when evaluated at $(\bar{\mathbf{p}},\bar{\mathbf{x}})$. Also, observe that this objective function can be no greater than zero over the feasible region; hence, $(\bar{\mathbf{p}},\bar{\mathbf{x}})$ is a solution to the problem. Moreover, it is apparent that any solution to the problem must also be a spatial equilibrium. Now, given an equilibrium price of $\bar{\mathbf{p}}$, any corresponding equilibrium $\mathbf{x}$ may be found by fixing $\mathbf{p}$ at $\bar{\mathbf{p}}$ in the above problem, and by then solving the problem for the optimal $\mathbf{x}$. After eliminating constants from the objective function, and after deleting the automatic constraints, the problem becomes:

$$\begin{aligned}
\text{maximize}(\mathbf{x}):\quad & -\mathbf{t}'\mathbf{x} \\
\text{subject to:}\quad & \mathbf{E}(\bar{\mathbf{p}}) - \mathbf{N}\mathbf{x} \leq \mathbf{0} \\
& \mathbf{x} \geq \mathbf{0}
\end{aligned}$$

from which it may be seen that the equilibria $\mathbf{x}$ minimize $\mathbf{t}'\mathbf{x}$ for given quantities of excess demands.

A property satisfied by a typical spatial market is that there always exists an equilibrium wherein no transshipments occur, or equivalently, there will always exist an equilibrium wherein no region simultaneously imports and exports. This property is valid if the following condition concerning transportation rates holds:

$$t_{ij} \leq t_{ik} + t_{jk}; \qquad \forall\ i,j,k \tag{3.11}$$

That is, the costs of shipping indirectly are not less than the costs of shipping directly. Henceforth, this relation shall be referred to as the "triangular inequality of **t**." It is difficult to imagine a situation in which this condition would not hold. Indeed, if the t_{ij} are measured from least cost routes, then the condition must hold. Now, suppose there exists an equilibrium solution in which i ships to k, which in turn ships to j. Since equilibrium prevails, it is known that $t_{ik} = p_k - p_i$, and that $t_{kj} = p_j - p_k$. Substitution of these equalities into (3.11) yields:

$$t_{ij} \leq p_j - p_i$$

If the latter holds with strict inequality, then potential exists for profitable trade so that the market could not be in equilibrium as supposed. Thus, $t_{ij} = p_j - p_i$. But if this be the case, then it is possible to reroute shipments without disrupting the equilibrium conditions and such that no transshipments occur. Observe that if the triangular inequality of **t** holds with strict inequality, then it may be concluded that there is no equilibrium wherein transshipments occur.

Under all circumstances, it may be concluded that there exists an equilibrium solution in which no two exporters share two importers. Consider the following market configuration:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \\ D & C \arrow[l] \arrow[u] \end{tikzcd}$$

Here, A and C both export to D and B. However, a second equilibrium solution may be obtained by transferring an exported unit from A to B into D, and by transferring an exported unit from C to D into B. Such rearrangement does not effect prices or net imports; consequently, the equilibrium conditions are not disrupted. The transfers may be repeated until at least one of the trade flows is reduced to zero. Hence, an equilibrium solution must exist wherein no two exporters share two importers.

As a consequence of the last observation and the observation concerning transshipments, one should expect a typical spatial equilibrium to have very few trade flows, or equivalently, that the equilibrium $\mathbf{x}$ vector largely consists of zero components. In fact, there will always exist a spatial equilibrium having not more than n - 1 trade flows.[5] This assertion is verified with use of the following theorem:

3.2.1 Theorem: Let $\mathbf{A}$ be an nxm matrix having rank, ρ. Let $\mathbf{b}$ be an m vector and consider the system:

$\mathbf{Ax} = \mathbf{b}$

$\mathbf{x} \geq 0$

Suppose that $\bar{\mathbf{x}}$ solves the system, then there exists an $\hat{\mathbf{x}} \geq 0$ of dimension less than or equal to ρ such that $\hat{\mathbf{A}}\hat{\mathbf{x}} = \mathbf{b}$ where the columns of $\hat{\mathbf{A}}$ consist of a set of linearly independent columns from $\mathbf{A}$.

Proof: Let $\mathbf{A}$ be partitioned as $\mathbf{A} = (\mathbf{a}_1, \mathbf{a}_2, , , \mathbf{a}_n)$. Since $\bar{\mathbf{x}}$ solves the system:

$$\bar{x}_1 \mathbf{a}_1 + \bar{x}_2 \mathbf{a}_2 + \ldots + \bar{x}_n \mathbf{a}_n = \mathbf{b}$$

Without loss of generality, it may be assumed that only the first l components of $\bar{\mathbf{x}}$ are nonzero; subsequently:

$$\bar{x}_1 \mathbf{a}_1 + \bar{x}_2 \mathbf{a}_2 + \ldots + \bar{x}_l \mathbf{a}_l = \mathbf{b}$$

If $(\mathbf{a}_1, \mathbf{a}_2, , , \mathbf{a}_l)$ are linearly independent, then the theorem holds immediately. Suppose $(\mathbf{a}_1, \mathbf{a}_2, , , \mathbf{a}_l)$ are linearly dependent, then there exists $(c_1, c_2, , , c_l)$, not all zero, such that:

$$c_1 \mathbf{a}_1 + c_2 \mathbf{a}_2 + \ldots + c_l \mathbf{a}_l = 0$$

Without loss of generality, it may be assumed that at least one $c_i > 0$. Multiply the latter relation by λ and subtract the product from the former relation to produce:

$$(\bar{x}_1 - \lambda c_1)\mathbf{a}_1 + (\bar{x}_2 - \lambda c_2)\mathbf{a}_2 + \ldots + (\bar{x}_l - \lambda c_l)\mathbf{a}_l = \mathbf{b}$$

Hence, a second solution is obtained. Moreover, if one chooses $\lambda = \min(\bar{x}_i/c_i : c_i > 0)$, then the second solution is nonnegative and consists of not more than l − 1 nonzero components. The process may be repeated until a positive solution involving only linearly independent columns of **A** is obtained. Let $\hat{\mathbf{x}}$ equal the terminal solution to complete the proof.

Corollary: Let **A** be an nxm matrix having rank, ρ. Let **b** be an m vector, and consider the system:

$$\mathbf{Ax} = \mathbf{b}$$
$$\mathbf{x} \geqq \mathbf{0}$$

Let $\bar{\mathbf{x}}$ solve the system, then there exists an $\tilde{\mathbf{x}}$ which also solves the system and which has not more that ρ positive components.

The fact that there exists a spatial equilibrium having not more than n − 1 trade flows follows from the fact that the matrix, **N**, has rank equal to n − 1. From (3.7), it may be seen that the matrix, **N**′, is a linear transformation that calculates all possible differences between the components of an n-dimensional vector. As there are only n − 1 ways in which such differences can be independently taken, then **N**′ must be of rank n − 1. Of course, this implies that **N** has rank n − 1 also. Observe that **x** enters spatial equilibrium conditions [conditions (3.8) through (3.10)] only through the term **Nx**. Suppose $\bar{\mathbf{x}}$ is an equilibrium solution, then by the last corollary, there exists an $\tilde{\mathbf{x}}$ having not more that $n - 1$ positive components such that $\mathbf{N}\tilde{\mathbf{x}} = \mathbf{N}\bar{\mathbf{x}}$. Thus, $\tilde{\mathbf{x}}$ is also an equilibrium $\mathbf{x}$.

Under usual circumstances, spatial equilibrium prices will be greater than zero. A sufficient condition ensuring positive prices is:

$$e_i(0) > 0; \quad \forall\ i$$

That is, each regional excess demand is greater than zero when evaluated at zero price. In each region for which this condition holds, the commodity is said to be "desirable." Now suppose the commodity is desirable in all regions, and that there is a spatial equilibrium in which $p_j = 0$. It is known from condition (3.8) that:

$$e_j(0) \leq n_j$$

and from the desirability assumption it may be concluded:

$$0 < e_j(0) \leq n_j$$

Hence, any region with zero price must be a net importer. Suppose that region j imports from region i. From condition (3.9), p_i must satisfy:

$$t_{ij} = 0 - p_i$$

But, since $t_{ij} > 0$, this requires $p_i < 0$, which cannot hold. Hence, if the commodity is desirable in every region, then equilibrium prices must be greater than zero.

3.3 Implementation of Trade Barriers

An advantageous feature of the presented model is that it allows the incorporation of several commonly enforced trade restrictions. Some of these restrictions are discussed here, and the compensating modifications to the model are explained.

A common restriction to trade is the specific tariff, which is a per-unit tax levied by an importing region on imports. Suppose region j imposes an α_{ij} specific tariff against imports from region i. The effect of such tariff upon the equilibrium solution is in no respect different from an increase in the per-unit transportation charge on shipments from i to j, t_{ij}. Therefore, compensation for the tariff is accomplished simply by adding the tariff rate to the transportation charge. If α is the vector of specific tariff rates, then the appropriate Lagrangian would be:

$$l(\mathbf{p},\mathbf{x}) = f(\mathbf{p}) + \mathbf{x}'(t + \alpha - \mathbf{N}'\mathbf{p}); \qquad (\mathbf{p},\mathbf{x}) \geq 0$$

A specific export subsidy is a per-unit subsidy paid by an exporting region to exporters. Suppose region i pays a δ_{ij} per-unit subsidy on exports to region j. The effect of such subsidy is the same as an equivalent reduction in the transportation charge, t_{ij}. Let δ denote the vector of specific subsidies, then the following Lagrangian incorporates both specific tariffs and specific export subsidies:

$$l(\mathbf{p},\mathbf{x}) = f(\mathbf{p}) + \mathbf{x}'(\mathbf{t} + \alpha - \delta - \mathbf{N}'\mathbf{p}); \qquad (\mathbf{p},\mathbf{x}) \geq \mathbf{0}$$

Tariffs are frequently assessed as a percentage-of-value rather than on a per-unit basis. In such cases, the tariff is said to be an "ad-valorem" tariff. Here, it is assumed that the tariff is applied to the c.i.f. price (delivered price). Also, it is assumed that a common tax rate is applied to imports of all origins. Suppose region i imposes a γ_i percent tariff on imports. If p_i is the "border price," then the "internal price" is $(1 + \gamma_i)p_i$. The quantity of excess demand is derived from this internal price. Border prices must still be such that profitable trade is not possible. Thus, the following Lagrangian is implied:

$$l(\mathbf{p},\mathbf{x}) = f[(\mathbf{I} + \Gamma)\mathbf{p}] + \mathbf{x}'(\mathbf{t} - \mathbf{N}'\mathbf{p}); \qquad (\mathbf{p},\mathbf{x}) \geq \mathbf{0}$$

Here, $\mathbf{I}$ is the nxn identity matrix, and Γ is an nxn diagonal matrix with the tariff rates situated along the diagonal. $\mathbf{p}$ is now interpreted as the vector of border prices. Internal prices are given by $(\mathbf{I} + \Gamma)\mathbf{p}$. It should be observed that the Kuhn-Tucker conditions for the above will require:

$$t_{ij} - (p_j - p_i) \geq 0, \text{ if } > \text{ then } x_{ij} = 0; \qquad \forall\ i,j$$

Thus, if $x_{ij} > 0$, then $p_j = p_i + t_{ij}$. That is, the border price in j is equal to the c.i.f or delivered price for shipments from i. Let $\hat{p}_j = (1 + \gamma_j)p_j$ be the internal price in j. If $x_{ij} > 0$, then it follows that:

$$\hat{p}_j = (1 + \gamma_j)(p_i + t_{ij})$$

which shows that the tariff assumed here is effectively applied to both the commodity and the services required to transport it. If the tariff were in fact assessed only against f.o.b price, then the approach outlined here may still be employed; however, the t_{ij} should be replaced in the Lagrangian with $t_{ij}/(1 + \gamma_j)$ so that the effect of the tariff on the transportation charge is cancelled.

An ad valorem subsidy is a percentage-of-value subsidy paid by an exporting region to exporters. It is assumed that the subsidy is based upon f.o.b. price, and that a common subsidy rate applies to exports to all destinations. Suppose a subsidy of π_i percent is paid by region i. Let p_i be the border price and $\hat{p}_i$ be the internal price in region i, then $\hat{p}_i = p_i/(1 - \pi_i)$. In a spatial equilibrium, border prices must still be such that possibilities for profitable trade are removed. Hence, the following Lagrangian for the spatial equilibrium problem is implied:

$$l(\mathbf{p},\mathbf{x}) = f[(\mathbf{I} - \mathbf{\Pi})^{-1}\mathbf{p})] + \mathbf{x}'(\mathbf{t} - \mathbf{N}'\mathbf{p}); \qquad (\mathbf{p},\mathbf{x}) \geq \mathbf{0}$$

where: $\mathbf{I}$ is the nxn identity matrix; $\mathbf{\Pi}$ is an nxn diagonal matrix with the subsidy rates on the diagonal, and $\mathbf{p}$ is the vector of border prices. The constraints will reqiure:

$$t_{ij} - (p_j - p_i) \geq 0; \qquad \forall\ i,j$$

In terms of the internal price, $\hat{p}_i$, this becomes:

$$t_{ij} - \pi_i\hat{p}_i - (p_j - \hat{p}_i) \geq 0; \qquad \forall\ i,j$$

From this relation, it is apparent that possibilities for profitable trade are indeed removed.

A Lagrangian incorporating both ad valorem tariffs and subsidies is given by:

$$l(\mathbf{p},\mathbf{x}) = f[(\mathbf{I} + \Gamma)(\mathbf{I} - \Pi)^{-1}\mathbf{p}] + \mathbf{x}'(\mathbf{t} - \mathbf{N}'\mathbf{p}); \qquad (\mathbf{p},\mathbf{x}) \geq \mathbf{0}$$

However, the model assumes that no one region imposes both a tariff and a subsidy. As the region is either an importer or an exporter, but not both, then either the subsidy is of force, or the tariff is of force, but not both.

The model will also accomodate variable levies. By a "variable levy" is meant a scheme wherein a region sets a target price, and then implements tariffs or subsidies to attain and sustain the target. The programming model may be modified to account for the variable levy without explicit consideration of the tariffs or subsidies involved. Suppose region i sets a target price of $\hat{p}_i$, then the variable levy is incorporated simply by fixing the internal price in region i at this level. With internal price fixed, excess demand becomes a constant at $e_i(\hat{p}_i)$. The equilibria should be solved using this constant as the excess demand function for the ith region. The solution value for p_i will be the border price.

Import quotas may be incorporated if the quota pertains to the sum of imports from all origins. Supposing this to be the case for the ith region, then the regional excess demand becomes kinked at the quota. Let the quota be fixed at $\hat{q}_i$, and suppose $e_i(\hat{p}_i) = \hat{q}_i$, then the excess demand function must be redefined as:

$$\hat{e}_i(p_i) = \begin{cases} e_i(p_i); & p_i \geq \hat{p}_i \\ \hat{q}_i; & p_i < \hat{p}_i \end{cases}$$

where p_i is the border price.

To determine the spatial equilibria with quotas imposed, one must determine the relevant sections of the excess demand functions. The model should first be solved using the $e_i(p_i)$ for excess demands. The solution should then be examined to see if any quotas are exceeded. If so, then it is known that the equilibria occur in the lower segments of the excess demand functions for those regions in which quotas are being violated. Thus, the $\hat{q}_i$ should be entered as the excess demands for these regions, and the model should be solved again. In the second solution, the enforced quotas will cause shipments to be diverted into other regions of the model; consequently, quotas not exceeded in the first solution could possibly be exceeded in the second. In this case, the process should be repeated until an equilibrium is obtained. An equilibrium p_i will be both the border price and the internal price for every region not having a quota, and for regions whose quota is nonbinding. For regions having binding quotas, the p_i will be the border prices, and the internal prices will be the $\hat{p}_i$.

If an ad valorem tariff and a quota are simultaneously imposed, then the modifications described above may be employed in combination. Suppose region i imposes a quota of $\hat{q}_i$ and an ad valorem tariff of γ_i. Moreover, let $e_i(\hat{p}_i) = \hat{q}_i$. The solution value of p_i will be the border price. If the quota is nonbinding, then $(1 + \gamma_i)p_i$ will be the internal price. If the quota is binding, then the internal price will be $\hat{p}_i$.

3.4 Price Dependent Formulation

Until now, the spatial model has been formulated using quantity-dependent excess demands. Here, the model is reconstructed using price-dependent functions. The primary advantage of the price-dependent model is its compatability with a large variety of trade policies, particularly those involving restrictions upon quantities.

The equilibrium conditions for the quantity-dependent model were found to be:

$$\mathbf{E(p)} - \mathbf{Nx} \leq \mathbf{0}; \qquad \mathbf{p}'[\mathbf{E(p)} - \mathbf{Nx}] = 0 \tag{3.12}$$

$$\mathbf{t} - \mathbf{N}'\mathbf{p} \geq \mathbf{0}; \qquad \mathbf{x}'(\mathbf{t} - \mathbf{N}'\mathbf{p}) = 0$$

$$(\mathbf{p},\mathbf{x}) \geq \mathbf{0}$$

Now, let $\mathbf{q} = \mathbf{E(p)}$. If the $e_i(p_i)$ are monotonic on $p_i \geq 0$, then $\mathbf{p}$ may be solved as $\mathbf{p} = \mathbf{E}^{-1}(\mathbf{q})$, where $\mathbf{E}^{-1}(\mathbf{q})$ is a vector-valued function whose components are the $e_i^{-1}(q_i)$. Also, condition (3.12) may be expressed as:

$$\mathbf{q} - \mathbf{Nx} \leq \mathbf{0}; \qquad \mathbf{p}'(\mathbf{q} - \mathbf{Nx}) = 0$$

With this observation, it should be apparent that the former set of conditions is equivalent to:

$$\mathbf{Nx} - \mathbf{q} \geq \mathbf{0}; \qquad \mathbf{p}'(\mathbf{Nx} - \mathbf{q}) = 0$$
$$\mathbf{t} - \mathbf{N}'\mathbf{p} \geq \mathbf{0}; \qquad \mathbf{x}'(\mathbf{t} - \mathbf{N}'\mathbf{p}) = 0$$
$$\mathbf{E}^{-1}(\mathbf{q}) - \mathbf{p} = \mathbf{0} \tag{3.13}$$
$$(\mathbf{p},\mathbf{x}) \geq \mathbf{0}$$

Suppose the $e_i^{-1}(q_i)$ are integrable and define:

$$f^{-1}(\mathbf{q}) = \Sigma_{i=1}^{n} \int e_i^{-1}(q_i)dq_i$$

$f^{-1}(\mathbf{q})$ is not, strictly speaking, the inverse of $f(\mathbf{p})$ as formerly defined; however, this notation is chosen for the sake of consistency and clarity. It may be confirmed that the latter conditions are the Kuhn-Tucker conditions to the following problem:

$$\begin{array}{ll} \text{maximize}(\mathbf{q},\mathbf{x}): & f^{-1}(\mathbf{q}) - \mathbf{t}'\mathbf{x} \\ \text{subject to:} & \mathbf{Nx} - \mathbf{q} \geq \mathbf{0} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

where the Lagrangian to the problem is constructed as:

$$l(\mathbf{p},\mathbf{x},\mathbf{q}) = f^{-1}(\mathbf{q}) - \mathbf{t}'\mathbf{x} + \mathbf{p}'(\mathbf{Nx} - \mathbf{q}); \qquad (\mathbf{p},\mathbf{x}) \geq \mathbf{0}$$

As before, the necessity and/or sufficiency of the Kuhn-Tucker conditions is of no concern insofar as equilibrium determination is concerned. The Kuhn-Tucker points are one-to-one with the equilibria regardless of whether such points imply or are implied by the optima of the above problem. However, the programming model does have several desirable properties. First, if $\mathbf{E}^{-1}(\mathbf{q})$ is continuous at some $\bar{\mathbf{q}}$, then $f^{-1}(\mathbf{q})$ is differentiable at $\bar{\mathbf{q}}$. Second, the constraints are linear. Hence, by Theorem 2.5.12, the Kuhn-Tucker conditions are known to be necessary to the optimal solutions. Third, if the excess demands have negative slopes, then it may be easily confirmed that the Hessian matrix of the objective function is negative definite, and consequently, the objective function is strictly concave. As the constraints are also concave, Theorem 2.5.13 guarantees that the Kuhn-Tucker conditions are also sufficient for the global optima, and Theorem 2.6.1 guarantees that the Kuhn-Tucker points are one-to-one with the saddle points of the Lagrangian.

All of the formerly discussed trade restrictions may be incorporated into this model and in much the same fashion as before. In particular, specific tariffs and specific subsidies are treated exactly the same. An ad valorem tariff by region i may be incorporated by replacing $e_i^{-1}(q_i)$ in (3.13) with $e_i^{-1}(q_i)/(1 + \gamma_i)$, where γ_i is the tariff rate. The solution values for the p_i will be the border prices, as before. The internal prices will be the $\hat{p}_i = e_i^{-1}(q_i) = (1 + \gamma_i)p_i$. The treatment of ad valorem subsidies is similar.

A variety of restrictions may be imposed upon $\mathbf{x}$, including both quotas upon total imports and quotas upon imports of a specific origin or combination of origins. Suppose the restrictions upon $\mathbf{x}$ are comprehended in:

$$\hat{\mathbf{q}} - \mathbf{Q}\mathbf{x} \geq \mathbf{0}$$

A spatial equilibrium solution satisfying these conditions may be solved using the Lagrangian:

$$l(\mathbf{p},\mathbf{x},\mathbf{q},\boldsymbol{\mu}) = f^{-1}(\mathbf{q}) - \mathbf{t}'\mathbf{x} + \mathbf{p}'(\mathbf{N}\mathbf{x} - \mathbf{q}) + \boldsymbol{\mu}'(\hat{\mathbf{q}} - \mathbf{Q}\mathbf{x})$$

on $(\mathbf{p},\mathbf{x},\boldsymbol{\mu}) \geq \mathbf{0}$. Here, the solution value for $\mathbf{p}$ will be the equilibrium vector of internal prices. In this formulation, the optimal $\mathbf{Q}'\boldsymbol{\mu}$ becomes a vector of per-unit tariff and/or subsidy equivalents to the quotas.

A variable levy may also be implemented in the form of a restriction upon $\mathbf{x}$. Suppose region i installs a variable levy with target price, $\hat{p}_i$. Let $\hat{q}_i = e_i^{-1}(\hat{p}_i)$. The variable levy may be incorporated by including a constraint requiring the net imports of i to equal $\hat{q}_i$. The solution will yield $p_i = \hat{p}_i$; thus, p_i will be the internal price of region i.

An ad valorem tariff and quota may be simultaneously imposed by implementing the above modifications in combination. The internal prices become $\hat{\mathbf{p}} = \mathbf{E}^{-1}(\mathbf{q})$; however, the border prices are indeterminate without specific knowledge of the construction of $\mathbf{Q}$.

3.5 General Equilibrium

The partial equilibrium model is easily extended to accomodate general equilibrium problems. Suppose there are n countries and m commodities. The aggregate market for each individual commodity is assumed to satisfy the same static and dynamic conditions as before; however, it is now recognized that the domestic excess demands are functions the m internal prices corresponding to each commodity. Thus, if p_j^i denotes the price of the ith commodity in the jth region, then the excess demand for the same takes the form:

$$e_j^i = e_j^i(p_j^1,p_j^2,,,p_j^m).$$

It will be shown that the general spatial equilibrium model is effectively m partial equilibrium models that are connected through the arguments of the excess demands.

Let $\mathbf{p}^i = (p_1^i,p_2^i,,,p_n^i)$ be the vector of prices for the ith commodity. Note that $\mathbf{p}^i$ is analogous to $\mathbf{p}$ in the previous sections. Let $\mathbf{p}$ now be the vector of all prices in all regions. That is:

$$\mathbf{p} = \begin{bmatrix} \mathbf{p}^1 \\ \mathbf{p}^2 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{p}^m \end{bmatrix}$$

Let $\mathbf{x}^i$ be the trade vector of the ith commodity. $\mathbf{x}^i$ is analogous to $\mathbf{x}$ of the previous sections. Let $\mathbf{t}^i$ be the vector of transportation rates for the ith commodity. $\mathbf{t}^i$ is analogous to $\mathbf{t}$ of the previous sections. Let $\mathbf{E}^i(\mathbf{p})$ be the vector of excess demands for the ith commodity. Again, $\mathbf{E}^i(\mathbf{p})$ is analogous to $\mathbf{E}(\mathbf{p})$ of the previous sections. Now, redefine $\mathbf{x}, \mathbf{t}$, and $\mathbf{E}(\mathbf{p})$ as:

$$\mathbf{x} = \begin{bmatrix} x^1 \\ x^2 \\ \cdot \\ \cdot \\ \cdot \\ x^m \end{bmatrix} \qquad \mathbf{t} = \begin{bmatrix} t^1 \\ t^2 \\ \cdot \\ \cdot \\ \cdot \\ t^m \end{bmatrix} \qquad \mathbf{E}(\mathbf{p}) = \begin{bmatrix} E^1(p) \\ E^2(p) \\ \cdot \\ \cdot \\ \cdot \\ E^m(p) \end{bmatrix}$$

Finally, define $\mathbf{N}$ exactly as before and let:

$$\hat{\mathbf{N}} = \begin{bmatrix} N & 0 & \cdot & \cdot & 0 \\ 0 & N & 0 & \cdot & 0 \\ \cdot & 0 & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & 0 \\ 0 & 0 & \cdot & 0 & N \end{bmatrix}$$

Using these definition, the conditions for general spatial equilibrium may be briefly expressed as:

$$\mathbf{E}(\mathbf{p}) - \hat{\mathbf{N}}\mathbf{x} \leq \mathbf{0}; \qquad \mathbf{p}'[\mathbf{E}(\mathbf{p}) - \hat{\mathbf{N}}\mathbf{x}] = 0 \tag{3.14}$$

$$\mathbf{t} - \hat{\mathbf{N}}'\mathbf{p} \geq \mathbf{0}; \qquad \mathbf{x}'(\mathbf{t} - \hat{\mathbf{N}}'\mathbf{p}) = 0 \tag{3.15}$$

$$(\mathbf{p}, \mathbf{x}) \geq \mathbf{0} \tag{3.16}$$

Here, the inequality relations are:

$$\begin{bmatrix} E^1(p) \\ E^2(p) \\ \cdot \\ \cdot \\ E^m(p) \end{bmatrix} - \begin{bmatrix} N & 0 & \cdot & \cdot & 0 \\ 0 & N & 0 & \cdot & 0 \\ \cdot & 0 & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & 0 \\ 0 & 0 & \cdot & 0 & N \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ \cdot \\ \cdot \\ x^m \end{bmatrix} \leq 0$$

$$\begin{bmatrix} t^1 \\ t^2 \\ \cdot \\ \cdot \\ t^m \end{bmatrix} - \begin{bmatrix} N' & 0 & \cdot & \cdot & 0 \\ 0 & N' & 0 & \cdot & 0 \\ \cdot & 0 & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & 0 \\ 0 & 0 & \cdot & 0 & N' \end{bmatrix} \begin{bmatrix} p^1 \\ p^2 \\ \cdot \\ \cdot \\ p^m \end{bmatrix} \geq 0$$

From these, it may be observed that the set of general equilibrium conditions is much like m sets of partial equilibrium conditions stacked on top of each other. The m "partial equilibrium" models are linked only through the arguments of the excess demand functions.

The general equilibrium problem could be formulated as a programming problem if a function, $f(\mathbf{p})$, could be found having gradient $\mathbf{E}(\mathbf{p})$. Were this the case, then the appropriate programming model would be:

$$\begin{aligned} &\text{maximize}(\mathbf{p}): && f(\mathbf{p}) \\ &\text{subject to:} && \mathbf{t} - \hat{\mathbf{N}}'\mathbf{p} \geq \mathbf{0} \\ &&& \mathbf{p} \geq \mathbf{0} \end{aligned}$$

and the Lagrangian would be constructed as:

$$l(\mathbf{p},\mathbf{x}) = f(\mathbf{p}) + \mathbf{x}'(\mathbf{t} - \hat{\mathbf{N}}'\mathbf{p}); \qquad (\mathbf{p},\mathbf{x}) \geq \mathbf{0}$$

In general, $f(\mathbf{p})$ will not be easy to determine. Of course, knowledge of this function is not essential to the determination of the spatial equilibria. The equilibria may still be found by solving (3.14) through (3.16) as though they were a set of Kuhn-Tucker conditions, and $\hat{\mathbf{E}}(\mathbf{p})$ may be treated as though it were the gradient of a function being maximized.

It should be apparent that the Hessian matrix of $f(\mathbf{p})$ is simply the Jacobian matrix of $\mathbf{E}(\mathbf{p})$. By Theorem 2.4.9, it may be concluded that $f(\mathbf{p})$ is concave (strictly concave) if and only if the Hessian matrix is negative semidefinite (negative definite). Let p_i be the vector of m commodity prices for the ith region, and let $\mathbf{E_i}(\mathbf{p}_i)$ be the vector of m excess demands for the same. With a proper rearrangement of rows and columns in the Hessian matrix of $f(\mathbf{p})$, one obtains a block-diagonal matrix with blocks corresponding to the Jacobian matrices of the $\mathbf{E_i}(\mathbf{p}_i)$. Hence, it may be concluded that the Hessian matrix is negative semidefinite (negative definite) if and only if the n Jacobians of the $\mathbf{E_i}(\mathbf{p}_i)$ are negative semidefinite (negative definite).

Some of the properties of partial equilibria also pertain to general equilibria. In particular, if $f(\mathbf{p})$ is strictly concave, then a general equilibrium price vector is unique. If the costs of shipping indirectly are not less than the costs of shipping directly, then it remains that there must exist a spatial equilibrium wherein no transshipments occur. It also remains that in all cases, there must exist an equilibrium wherein no two exporters share two importers. The proof of the last two assertions is in no respect different from the proofs presented for the partial equilibrium model. Also, as the matrix, $\hat{\mathbf{N}}$, is of rank $m(n - 1)$, then with use of Theorem 3.2.1, it may be concluded that there exists an equilibrium having not more than $m(n - 1)$ trade flows. Moreover, upon disassembling the disassembling the block diagonal structure of conditions (3.14) through (3.16), it may be seen that the same theorem implies that there exists an equilibrium having not more than n - 1 trade flows for any one commodity.

If $\mathbf{E}^{-1}(\mathbf{p})$ exists, then a price-dependent formulation of the general equilibrium model may also be constructed. The steps of derivation are exactly those taken with the partial equilibrium model.

Trade restrictions may be implemented in both the quantity-dependent and price-dependent versions of the general equilibrium model. The installation of such restrictions is accomplished in the same manner as in the partial equilibrium models. It should be observed that the price-dependent version of the general equilibrium model will allow restrictions not only upon quantities of a single commodity, but also upon combinations of commodities.

3.6 Stability[6]

Until now, the discussion has dealt with the determination of equilibrium points and the properties characterizing such points. However, it has not been established that the dynamics of the model are such that these equilibrium points are stable. If there is no inherent tendency of the market to converge toward the equilibria, then such points are indeed of little significance, and knowledge of such points or of their properties is of little if any practical value. For this reason, the question of stability is now addressed. Attention is centered upon the partial equilibrium model of the first section; however, it will shortly become apparent that the logic applies equally well toward establishing the stability of the general equilibrium model also.

It will be recalled that the dynamic assumptions of the partial equilibrium model are:

5 a) $\dot{p}_i > 0$ if and only if $e_i(p_i) - n_i > 0$.

b) $\dot{p}_i < 0$ if and only if $e_i(p_i) - n_i < 0$.

6 a) $\dot{x}_{ij} > 0$ only if $t_{ij} - (p_j - p_i) < 0$.

b) if $t_{ij} - (p_j - p_i) < 0$ for some j, then $\dot{x}_{ij} > 0$ for at least one such j.

c) $\dot{x}_{ij} < 0$ if and only if $t_{ij} - (p_j - p_i) > 0$ and $x_{ij} > 0$.

It was noted that $(\mathbf{p},\mathbf{x})$ satisfies the above conditions if and only if:

$$l_{\mathbf{p}}(\mathbf{p},\mathbf{x}) = \mathbf{E}(\mathbf{p}) - \mathbf{N}\mathbf{x} \leq \mathbf{0}; \qquad \mathbf{p}'[\mathbf{E}(\mathbf{p}) - \mathbf{N}\mathbf{x}] = 0$$
$$l_{\mathbf{x}}(\mathbf{p},\mathbf{x}) = \mathbf{t} - \mathbf{N}'\mathbf{p} \geq \mathbf{0}; \qquad \mathbf{x}'(\mathbf{t} - \mathbf{N}'\mathbf{p}) = 0$$
$$(\mathbf{p},\mathbf{x}) \geq \mathbf{0}$$

It was then noted that if $f(\mathbf{p})$ is differentiable at all the Lagrangian saddle points, and concave at all points satisfying the above, then the spatial equilibrium solutions are one-to-one with the saddle points of the Lagrangian. Recall that the Lagrangian is:

$$l(\mathbf{p},\mathbf{x}) = f(\mathbf{p}) + \mathbf{x}'(\mathbf{t} - \mathbf{N}'\mathbf{p}); \qquad (\mathbf{p},\mathbf{x}) \geq \mathbf{0}$$

Unfortunately, the dynamic specifications above are alone insufficient to ensure convergence to the equilibrium points, or to the saddle points of $l(\mathbf{p},\mathbf{x})$. However, under more restrictive but plausible specifications, global stability with respect to the saddle points may be established.

In particular, suppose that prices and interregional trade flows adjust according to the following system:

$$\dot{p}_i = \begin{cases} 0 & \text{if } p_i = 0 \text{ and } e_i(p_i) - n_i < 0 \\ a_i[e_i(p_i) - n_i] & \text{otherwise} \end{cases}$$

$$\dot{x}_{ij} = \begin{cases} 0 & \text{if } x_{ij} = 0 \text{ and } t_{ij} - (p_j - p_i) > 0 \\ -b_{ij}[t_{ij} - (p_j - p_i)] & \text{otherwise} \end{cases}$$

where the a_i and b_{ij} are positive constants. Thus, the rate of price adjustment is proportional to the deficit, unless such adjustment would lead to a negative price. Accordingly, the rate of adjustment in interregional trade flow is proportional to profits, unless such adjustment would lead to a negative flow. Observe that this system may be expressed in terms of the Lagrangian as:

$$\dot{p}_i = \begin{cases} 0 & \text{if } p_i = 0 \text{ and } \partial l/\partial p_i < 0 \\ a_i \partial l(\mathbf{p},\mathbf{x})/\partial p_i & \text{otherwise} \end{cases}$$

$$\dot{x}_{ij} = \begin{cases} 0 & \text{if } x_{ij} = 0 \text{ and } \partial l/\partial x_{ij} > 0 \\ -b_{ij} \partial l(\mathbf{p},\mathbf{x})/\partial x_{ij} & \text{otherwise} \end{cases}$$

The foregoing theorem and proof show that the above process does in fact converge upon a saddle point of $l(\mathbf{p},\mathbf{x})$ if this function is linear in $\mathbf{x}$ and strictly concave in $\mathbf{p}$. The function is clearly linear in $\mathbf{x}$; moreover, it is strictly concave in $\mathbf{p}$ if $f(\mathbf{p})$ is strictly concave, and it has been shown that $f(\mathbf{p})$ is strictly concave if the excess demands have negative slopes. The theorem is a generalization of theorems presented by Arrow, Hurwicz, and Uzawa (1958) in connection with gradient-method optimization algorithms.[7] The theorem is presented here in general terms rather than in terms of the present model.

3.6.1 Theorem: Let $l(\mathbf{x},\mathbf{y}):R^n \oplus R^m \rightarrow R^1$ be linear in $\mathbf{y}$ and strictly concave and differentiable in $\mathbf{x}$ over all $\mathbf{x} \in R^n_+$. Moreover, let $(\bar{\mathbf{x}},\bar{\mathbf{y}}) \in R^n_+ \oplus R^m_+$ be a saddle point of $l(\mathbf{x},\mathbf{y})$. Consider the system:

$$\dot{x}_i = \begin{cases} 0 & \text{if } x_i = 0 \text{ and } \partial l/\partial x_i < 0 \\ a_i \partial l(\mathbf{x},\mathbf{y})/\partial x_i & \text{otherwise} \end{cases} \tag{3.17}$$

$$\dot{y}_i = \begin{cases} 0 & \text{if } y_i = 0 \text{ and } \partial l/\partial y_i > 0 \\ -b_i \partial l(\mathbf{x},\mathbf{y})/\partial y_i & \text{otherwise} \end{cases} \tag{3.18}$$

where the a_i and b_i are positive constants. From any initial t_0 and $(\mathbf{x}_0,\mathbf{y}_0)$ in $R^n_+ \oplus R^m_+$, $(\mathbf{x},\mathbf{y})$ satisfies $\lim_{t\to\infty} [\mathbf{x}(t),\mathbf{y}(t)] = (\bar{\mathbf{x}},\hat{\mathbf{y}})$ where $(\bar{\mathbf{x}},\hat{\mathbf{y}})$ is a saddle point of $l(\mathbf{x},\mathbf{y})$ on $R^n_+ \oplus R^m_+$.

Proof: As $l(\mathbf{x},\mathbf{y})$ is strictly concave in $\mathbf{x}$ and linear in $\mathbf{y}$, then with use of the corollary to Theorem 2.4.8, it may be concluded that:

$$l(\bar{\mathbf{x}},\mathbf{y}) < l(\mathbf{x},\mathbf{y}) + l_{\mathbf{x}}'(\mathbf{x},\mathbf{y})(\bar{\mathbf{x}} - \mathbf{x}); \qquad \mathbf{x} \in R_+^n,\ \mathbf{x} \neq \bar{\mathbf{x}}$$
$$l(\mathbf{x},\bar{\mathbf{y}}) = l(\mathbf{x},\mathbf{y}) + l_{\mathbf{y}}'(\mathbf{x},\mathbf{y})(\bar{\mathbf{y}} - \mathbf{y})$$

As $(\bar{\mathbf{x}},\bar{\mathbf{y}})$ is a saddle point of $l(\mathbf{x},\mathbf{y})$, then:

$$l(\mathbf{x},\bar{\mathbf{y}}) \leq l(\bar{\mathbf{x}},\bar{\mathbf{y}}) \leq l(\bar{\mathbf{x}},\mathbf{y}); \qquad \forall\ (\mathbf{x},\mathbf{y}) \in R_+^n \oplus R_+^m$$

The last three relations imply:

$$(\bar{\mathbf{x}} - \mathbf{x})' l_{\mathbf{x}} - (\bar{\mathbf{y}} - \mathbf{y})' l_{\mathbf{y}} \geq l(\bar{\mathbf{x}},\mathbf{y}) - l(\mathbf{x},\bar{\mathbf{y}}) \geq 0$$

for all $(\mathbf{x},\mathbf{y}) \in R_+^n \oplus R_+^m$, and where the first inequality is strict if $\mathbf{x} \neq \bar{\mathbf{x}}$; hence:

$$(\bar{\mathbf{x}} - \mathbf{x})' l_{\mathbf{x}} - (\bar{\mathbf{y}} - \mathbf{y})' l_{\mathbf{y}} \geq 0, \text{ if } \mathbf{x} \neq \bar{\mathbf{x}}, \text{ then } > 0. \qquad (3.19)$$

for all $(\mathbf{x},\mathbf{y}) \in R_+^n \oplus R_+^m$. Now, let:

$$\mathbf{A} = \begin{bmatrix} a_1 & 0 & . & . & 0 \\ 0 & a_2 & 0 & . & 0 \\ . & 0 & . & & . \\ . & . & & . & 0 \\ 0 & 0 & . & 0 & a_n \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b_1 & 0 & . & . & 0 \\ 0 & b_2 & 0 & . & 0 \\ . & 0 & . & & . \\ . & . & & . & 0 \\ 0 & 0 & . & 0 & b_m \end{bmatrix}$$

$$\delta_x^i = \begin{cases} 1 & \text{if } x_i = 0 \text{ and } \partial l/\partial x_i < 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\delta_y^i = \begin{cases} 1 & \text{if } y_i = 0 \text{ and } \partial l/\partial y_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\Delta_x = \begin{bmatrix} \delta_x^1 & 0 & . & . & 0 \\ 0 & \delta_x^2 & 0 & . & 0 \\ . & 0 & . & & . \\ . & . & & . & 0 \\ 0 & 0 & . & 0 & \delta_x^n \end{bmatrix} \quad \Delta_y = \begin{bmatrix} \delta_y^1 & 0 & . & . & 0 \\ 0 & \delta_y^2 & 0 & . & 0 \\ . & 0 & . & & . \\ . & . & & . & 0 \\ 0 & 0 & . & 0 & \delta_y^m \end{bmatrix}$$

Finally, let:

$$D(x,y) = 1/2[(x - \bar{x})'A^{-1}(x - \bar{x}) + (y - \bar{y})'B^{-1}(y - \bar{y})]$$

Note that A^{-1} and B^{-1} are positive definite and symmetric. Also note that the system in (3.17) and (3.18) can be written in terms of the matrices above as:

$$\dot{x} = A[I - \Delta_x]l_x$$
$$\dot{y} = -B[I - \Delta_y]l_y$$

Consider:

$$\dot{D} = (x - \bar{x})'A^{-1}\dot{x} + (y - \bar{y})'B^{-1}\dot{y}$$

Substitution of the former relations into the latter yields:

$$\dot{D} = (x - \bar{x})'l_x - (y - \bar{y})'l_y + \bar{x}'\Delta_x l_x - \bar{y}'\Delta_y l_y \qquad (3.20)$$

where $x'\Delta_x = 0$ and $y'\Delta_y = 0$ have been used. Now, from the definitions of Δ_x and Δ_y and from (3.19), it may be concluded that $\dot{D} \leq 0$, and is strictly less than zero if $x \neq \bar{x}$. As $D \geq 0$, and as $\dot{D} \leq 0$, it follows that D must converge upon a limit, D^*, as $t \to \infty$. That is:

$$\lim_{t\to\infty} D = D^*$$

Therefore, (x,y) must converge to a limit cycle[8], $[\hat{x}(r),\hat{y}(r)]$ satisfying $D[\hat{x}(r),\hat{y}(r)] = D^*$ for all r.

Now, at all points along the limit cycle, $\dot{D} = 0$; consequently, $\hat{\mathbf{x}}(r) = \bar{\mathbf{x}}$ for all r.

Hence, it is known that $\mathbf{x}$ does converge to $\bar{\mathbf{x}}$. It remains to show that $\mathbf{y}$ converges to $\hat{\mathbf{y}}$ such that $(\bar{\mathbf{x}},\hat{\mathbf{y}})$ is a saddle point of $l(\mathbf{x},\mathbf{y})$. It will be shown that every point on $[\bar{\mathbf{x}},\hat{\mathbf{y}}(r)]$ is a saddle point of $l(\mathbf{x},\mathbf{y})$; consequently, as saddle points are equilibrium points, the limit cycle must in fact consist of only one point.

At all points on the limit cycle, $\dot{\mathbf{x}} = 0$; consequently, it is known from the definition of $\dot{\mathbf{x}}$ in (3.17) that $[\bar{\mathbf{x}},\hat{\mathbf{y}}(r)]$ must satisfy:

$$l_{\mathbf{x}}[\bar{\mathbf{x}},\hat{\mathbf{y}}(r)] \leq 0; \qquad \bar{\mathbf{x}}' l_{\mathbf{x}}[\bar{\mathbf{x}},\hat{\mathbf{y}}(r)] = 0$$

for all r. Moreover, as $\dot{D} = 0$ at all points on the limit cycle, it is known from (3.20) that $[\bar{\mathbf{x}},\hat{\mathbf{y}}(r)]$ must satisfy:

$$[\hat{\mathbf{y}}(r) - \bar{\mathbf{y}}]' l_{\mathbf{y}} - \bar{\mathbf{y}}' \Delta_{\mathbf{y}} l_{\mathbf{y}} = 0; \qquad \forall\ r$$

However, as $l(\mathbf{x},\mathbf{y})$ is linear in $\mathbf{y}$, it follows that $l_{\mathbf{y}}$ is a function of $\mathbf{x}$ only; consequently, $l_{\mathbf{y}}$ is constant at $l_{\mathbf{y}}(\bar{\mathbf{x}})$. Therefore, the above becomes:

$$[\hat{\mathbf{y}}(r) - \bar{\mathbf{y}}]' l(\bar{\mathbf{x}}) - \bar{\mathbf{y}}' \Delta_{\mathbf{y}} l_{\mathbf{y}}(\bar{\mathbf{x}}); \qquad \forall\ r$$

But $\bar{\mathbf{y}}$ is complementary to $l_{\mathbf{y}}(\bar{\mathbf{x}})$ by assumption, so this reduces to:

$$\hat{\mathbf{y}}'(r) l_{\mathbf{y}}(\bar{\mathbf{x}}) = 0; \qquad \forall\ r$$

Also, by assumption:

$$l_y(\bar{x}) \geq 0$$

Finally, it is known that $\hat{y}(r)$ cannot exit the nonnegative orthant, for the adjustment process forbids otherwise. Thus, summarizing, it may be said that for all r, $[\bar{x},\hat{y}(r)]$ satisfies:

$$l_x[\bar{x},\hat{y}(r)] \leq 0; \qquad \bar{x}' l_x[\bar{x},\hat{y}(r)] = 0$$
$$l_y(\bar{x}) \geq 0; \qquad \hat{y}'(r) l_y(\bar{x}) = 0$$
$$[\bar{x},\hat{y}(r)] \in R_+^n \oplus R_+^m$$

But, these are precisely the Kuhn-Tucker conditions for saddle points in $l(\mathbf{x},\mathbf{y})$. It follows that the limit cycle must consist of the single point, $(\bar{x},\hat{y})$, that is also a saddle point of $l(\mathbf{x},\mathbf{y})$ on $R_+^n \oplus R_+^m$.

FOOTNOTES

[1]The dynamic perspective of this work is largely due to the encouragements of Dr. J.S. Plaxico.

[2]As the set of nonnegative prices is closed, and as differentiability cannot be defined on closed sets, a more accurate requirement here is that $f(\mathbf{p})$ be differentiable over some open set containing all nonnegative prices. Accordingly, the set constraint in Theorem 2.6.1 should actually be defined as some open set containing the nonnegative orthant.

[3]Takayama and Judge (1971) assert that a solution does exist; however, this conclusion is based on a peculiar and erroneous corollary which they append to the Weierstrass theorem on page 13. Here, it is said: "As a corollary to this theorem we can prove: $f(\mathbf{x})$ defined on a closed set attains a maximum (minimum) if it is bounded from above (below)." Note that this corollary is clearly contradicted by $f(x) = 1 - (1/x)$ on $x \in [1,\infty)$.

[4]This observation was originally emphasized by Samuelson (1952).

[5]It appears that this observation was first made by Silberberg (1970).

[6]Silberberg (1970) also presents a proof of stability, but under the unrealistic assumption that interregional trade flow adjustment is instantaneous.

[7]Here, the primary modification of the Arrow, Hurwicz, and Uzawa theorems is in the generalization of the distance function, D.

[8]A "limit cycle" is a closed curve towards which the state vector of a system converges.

CHAPTER IV

PRICE DISCRIMINATION IN THE SPATIAL MARKET

In this chapter, the practice of price discrimination in the spatial market is considered. Mathematical programming models are developed dealing with discrimination by a single region, and by a cooperative consisting of a group of regions. In both cases, models are developed for the maximization of net export revenue and for the maximization of total net revenue. In all cases, discrimination is exercised in a single product; hence, the analysis is partial. Here and hereafter, the term, "net revenue," refers to total revenue less transportation charges. By "total net revenue" is meant the sum of net revenues on foreign sales and domestic sales.

The first section considers the single-discriminator case, while the second section treats the cooperative discrimination models. In each section, mathematical programming models having solutions equal to the optimal price and quantity vectors are developed. It is then shown that the solutions to these problems may be found as the solutions to the associated Kuhn-Tucker conditions. Also, it is shown that in some cases, the Kuhn-Tucker conditions are sufficient conditions for the optima. Finally, in the third section, a trade policy is formulated whereby the discriminator may impose the optimal price and quantity vectors upon the spatial market.

4.1 Discrimination by a Single Region

In this section, the case of a single region practicing price discrimination in a single product is considered. First, the general nature of the discrimination problem is examined. Second, a model is developed wherein the discriminator maximizes net revenue on exports. Thus in the first model, it is assumed that discrimination is practiced only in the export market. Finally, a model is developed wherein the discriminator maximizes the sum of net revenues from the export market and the domestic market. Here, it is assumed that discrimination is exercised not only in the export market, but also between the export market and the domestic market.

Suppose that a single region in a spatial market seeks to price and allocate its exports such that net revenue from trade is maximized. Obviously, in determining the optimal allocation and pricing scheme, the discriminator must consider the responses of competing sellers to its policy. Moreover, two types of competition must be considered; namely, that of arbitragers, and the direct competition of other producers. Here, the implications of both types of competition are examined under the assumption that all buyers and sellers in the export market behave in accordance with the rules of ordinary spatial adjustment. These rules are mathematically expressed in assumptions five and six of the spatial model of the previous chapter. In words, the adjustment processes are such that: 1) deficits cause price increases, 2) surpluses cause price decreases, 3) interregional trade flows increase when profits from such trade are forthcoming, and 4) interregional trade flows decrease when such trade results in losses.

The possibilities of arbitrage will limit the extent to which prices set by the discriminator may differ. Were the discriminator to set two regional prices at values differing by more than the per-unit costs of transporting between the regions, then its direct shipments to the region of higher price would eventually be terminated. Instead, this region would be supplied by arbitragers via transshipments through the region of lower price.

Suppose region i is the discriminator, and that i has set prices of $\bar{p}_j$ and $\bar{p}_k$ on its exports to regions j and k. Arbitragers in j will supply buyers in k at any price in excess of $\bar{p}_j + t_{jk}$, for this sum represents the arbitragers' total costs in the purchase and transport of a commodity unit. At this point, it is assumed that the discriminator requires that all its sales be shipped to the region of purchase. Consequently, it is not possible for an arbitrager in j to arrange a trade wherein it ships directly from the discriminator to region k. That is, if the discriminator's sales are to be arbitraged, then they must first be shipped to the region of the arbitrager and from there to the region of the arbitrager's buyer. Now, if the discriminator sets $\bar{p}_k > \bar{p}_j + t_{jk}$, then buyers in k will turn to arbitragers in j for their foreign supplies. The arbitragers will continue to purchase from the discriminator at the fixed price, $\bar{p}_j$, and resell these quantities to k until the price in k is reduced to $p_k = \bar{p}_j + t_{jk}$, at which point, arbitrage is no longer profitable. Thus, if the discriminator sets any $\bar{p}_k > \bar{p}_j + t_{jk}$, then $\bar{p}_k$ will not in fact be the realized price in k, but rather, arbitrage will cause the actual price to be $p_k = \bar{p}_j + t_{jk}$.

Under general conditions, it can be said that the discriminator is never advantaged by allowing its shipments to be arbitraged. This will always be true if:

$$t_{ik} \leq t_{ij} + t_{jk}; \qquad \forall\ j,k \tag{4.1}$$

It will be observed that this same inequality was assumed in the spatial equilibrium model of the previous chapter. Now, if the discriminator's shipments are being arbitraged through j to k, then as noted above, the price in k will be $\bar{p}_j + t_{jk}$. For each unit transshipped, the discriminator nets $\bar{p}_j - t_{ij}$. Suppose the discriminator bypasses the arbitragers by reducing $\bar{p}_k$ to $\bar{p}_j + t_{jk}$. k would then purchase the same quantity as before, but directly from the discriminator. The discriminator would then net

$(\bar{p}_j + t_{jk}) - t_{ik}$ per unit. The difference between the latter rate and the former rate is:

$$(\bar{p}_j + t_{jk}) - t_{ik} - (\bar{p}_j - t_{ij}) = t_{ij} + t_{jk} - t_{ik} \geq 0$$

where the inequality follows from (4.1). This shows that the discriminator's total revenue at prices inducing arbitrage can always be equaled or exceeded with a price adjustment causing arbitrage to cease.

It should be apparent that no price discrimination scheme can succeed if arbitrage is allowed through the region of discriminator itself. With arbitragers operating within the region of the discriminator, prices and trade flows invariably degenerate to values dictated by ordinary spatial equilibrium. Thus, the discriminator must prohibit arbitrage through its own region.

In addition to arbitrage, the discriminator must concern itself with the reaction of competing producers. Whereas arbitragers can inflict revenue losses when prices are set with excessive disparity, competing producers can inflict losses when prices are set at improper levels. Were the discriminator to set a regional price too high, its share of the market's imports could be significantly if not completely lost to such competition. On the other hand, a reduction in regional price in the presence of competition could result in an insupportable level of demand for the discriminator's exports. Of course, improperly fixed prices could lead to such consequences even in the absence of competition; however, the presence of competition will generally cause the consequences to be more severe.

To illustrate the response of competition to discriminator price policy, consider the market configuration below:

$$
\begin{array}{ccccccccc}
 & & A & & & & & & \\
 & & \downarrow & & & & & & \\
B & \longrightarrow & C & \longleftarrow & D & \longrightarrow & E & \longleftarrow & F
\end{array}
$$

Assume that the market is initially at a spatial equilibrium and that the equilibrium is such that trade occurs in the directions of the arrows shown. However, suppose that trade between A and the other region is prohibited by large transportation charges. Suppose that D is the discriminator, and that it implements its scheme by increasing its price to C and by reducing its price to E.

When D reduces its price to E, E will initially respond by diverting all of its excess demand towards D. That is, F will be underpriced, and subsequently, quantities formerly obtained from F will now be sought from D. However, this will cause a surplus in F, and consequently, producers in the same will match the reduced price of D in order to clear the market. At the reduced price, excess demand in E will be increased, but excess supply in F will be reduced; subsequently, shipments from F to E will be reduced as well. Thus, if the discriminator is to support its reduced price, it must not only accomodate the increased demand in E, but must also replace shipments formerly obtained by E from F. This illustrates that a price reduction may prompt considerable increase in demand for one's exports in the presence of competition. Indeed, large price reductions may prove insupportable.

Consider the increase in price offered to C. Initially, C will respond by diverting all of its excess demand towards B; however, deficits in B will cause its price to be bid upwards until it matches the increased price of D. At the increased price, the excess demand of C is reduced; however, the excess supply of B is increased so that shipments from B to C will rise. Thus, exports from D to C will be reduced not only as a result of the reduced excess demand in C, but also because B will capture a greater percentage of C's market. D is further supplanted in C if its price increase is sufficiently large to allow A to overcome the transportation barrier. The discriminator must then compete with both A and B, which will generally imply that any further increases in price will cause even greater reductions in exports. This illustrates that in the presence of competition, price increases may lead to rapid and accelerating loss of the discriminator's export market.

It might appear that potential gains from price discrimination in a typical spatial market must be small with such volatility in export demand. However, it is not the absolute sensitivities of demands that counts toward successful discrimination, but rather, it is the relative sensitivities of the demands for those markets in which the discriminator lowers price to those markets for which the price is increased. For example, while it may be true that a price increase to C results in a large reduction in demand for the discriminator's exports, it may also be true that a very small price reduction to E results in a more than offsetting increase in export demand. That is, shipments to E may be even more price sensitive than shipments to C. Hence, the quantities that D cannot sell to C because of the increased price to the same may be disposed in E at a very small price reduction. The net result could be a considerable increase in revenue. As a practical matter, one could probably expect the presence of arbitragers to impose more severe limitations to gains from price discrimination than the presence of competing producers.

It remains to mathematically formulate the problem described above. To this end, the terms, "market" and "aggregate market," shall henceforth refer to the collection of all regions other than the discriminator. Thus, the discriminator is treated as a distinct entity. Accordingly, n is now the number of all regions other than the discriminator. The discriminator's export volumn to region i is denoted by y_i. Of course, the discriminator is assumed to be strictly an exporter; hence, $y_i \geq 0$.

The aggregate market is assumed to satisfy all of the conditions of an ordinary spatial market. In addition to these, certain assumption are made of the discriminator and its relation to the market. The market is assumed to satisfy the following static conditions:

1) The commodity is homogeneous within and across the regional markets.
2) The commodity is of uniform price within any regional market.
3) The ith regional market is characterized by: a demand function, $d_i(p_i)$; a supply function, $s_i(p_i)$, and an excess demand function, $e_i(p_i) = d_i(p_i) - s_i(p_i)$, where p_i is the regional price.
4) The ith region may ship an arbitrary quantity, x_{ij}, to the jth region at the constant per-unit transportation rate, t_{ij}.

To these, add the following static assumptions pertaining to the discriminator:

5) The product of the discriminator and the product of the aggregate market are homogeneous.
6) Sales by the discriminator are required to be shipped to the region of purchase.
7) Arbitragers are forbidden to operate within the region of the discriminator.
8) The discriminator may ship an arbitrary quantity, y_i, to region i at the constant per-unit transportation rate, r_i.

The dynamic assumptions are:

5 a) $\dot{p}_i > 0$ if and only if $e_i(p_i) - n_i - y_i > 0$.

b) $\dot{p}_i < 0$ if and only if $e_i(p_i) - n_i - y_i < 0$ and $p_i > 0$.

6 a) $\dot{x}_{ij} > 0$ only if $t_{ij} - (p_j - p_i) < 0$.

b) If $t_{ij} - (p_j - p_i) < 0$ for some j, then $\dot{x}_{ij} > 0$ for at least one such j.

c) $\dot{x}_{ij} < 0$ if and only if $t_{ij} - (p_j - p_i) > 0$ and $x_{ij} > 0$.

where the n_i measure net imports from all regions other than the discriminator. For the moment, the y_i are taken as given constants; consequently, no adjustment rules for these are yet specified.

The dynamic assumptions imply that for given y_i, the market is at a spatial equilibrium if and only if:

$$e_i(p_i) - n_i - y_i \le 0, \text{ if } < \text{ then } p_i = 0; \qquad \forall\ i$$
$$t_{ij} - (p_j - p_i) \ge 0, \text{ if } > \text{ then } x_{ij} = 0; \qquad \forall\ i,j$$
$$p_i \ge 0,\ x_{ij} \ge 0; \qquad \forall\ i,j$$

These conditions may be expressed in matrix notation as:

$$\mathbf{E(p)} - \mathbf{Nx} - \mathbf{y} \leqq \mathbf{0}; \qquad \mathbf{p}'[\mathbf{E(p)} - \mathbf{Nx} - \mathbf{y}] = 0 \tag{4.2}$$

$$\mathbf{t} - \mathbf{N}'\mathbf{p} \geqq \mathbf{0}; \qquad \mathbf{x}'(\mathbf{t} - \mathbf{N}'\mathbf{p}) = 0 \tag{4.3}$$

$$(\mathbf{p},\mathbf{x}) \geqq \mathbf{0} \tag{4.4}$$

where **y** is the discriminator's export vector, and where **p**, **E(p)**, **x**, and **N** are defined as in section 3.1 except that components corresponding to the region of the discriminator are not included.

Typically, there will be an infinite number of $(\mathbf{p},\mathbf{x},\mathbf{y})$ that will qualify as equilibria. The discriminator has some degree over control over equilibrium determination inasmuch as it can control the components of $\mathbf{y}$, and in that it can select its own offer prices. The objective here is to determine the particular equilibria or equilibrium that renders maximal net revenue on exports. Hence, the problem could be informally stated as follows:

maximize$(\mathbf{p},\mathbf{x},\mathbf{y})$: discriminator's net export revenue

subject to: the market is in spatial equilibrium

The equilibrium requirement is imposed because of the static nature of the problem and the fact that nonequilibrium points are unsustained.

The discriminator's net revenue on exports is measured by $f(\mathbf{p},\mathbf{y}) = (\mathbf{p} - \mathbf{r})'\mathbf{y}$, where $\mathbf{r}$ is the vector of transportation rates for shipments from the discriminator. For given $\mathbf{y}$, the conditions for equilibrium are summarized in (4.2) through (4.4). In addition to these, it is assumed that the sum of the discriminator's exports cannot exceed some constant, say $\bar{y}$. Let $\mathbf{u} = (1,1,,,1)'$, then this condition may be expressed as $\bar{y} - \mathbf{u}'\mathbf{y} \geq 0$. Therefore, the problem to maximize discriminator net export revenue may be formerly stated as:

Problem One

$$\text{maximize}(\mathbf{p},\mathbf{x},\mathbf{y}): \quad (\mathbf{p} - \mathbf{r})'\mathbf{y}$$

$$\text{subject to:} \quad \mathbf{E}(\mathbf{p}) - \mathbf{N}\mathbf{x} - \mathbf{y} \leq \mathbf{0} \tag{4.5}$$

$$\mathbf{t} - \mathbf{N}'\mathbf{p} \geq \mathbf{0} \tag{4.6}$$

$$\bar{y} - \mathbf{u}'\mathbf{y} \geq 0 \tag{4.7}$$

$$\mathbf{p}'[\mathbf{E}(\mathbf{p}) - \mathbf{N}\mathbf{x} - \mathbf{y}] = 0 \tag{4.8}$$

$$\mathbf{x}'(\mathbf{t} - \mathbf{N}'\mathbf{p}) = 0 \tag{4.9}$$

$$(\mathbf{p},\mathbf{x},\mathbf{y}) \geq \mathbf{0} \tag{4.10}$$

Let S be the set of all $(\mathbf{p},\mathbf{x},\mathbf{y})$ satisfying the first, second, and last constraints. That is:

$$S = \{(\mathbf{p},\mathbf{x},\mathbf{y}) \geq \mathbf{0}: \mathbf{E}(\mathbf{p}) - \mathbf{N}\mathbf{x} - \mathbf{y} \leq \mathbf{0},\ \mathbf{t} - \mathbf{N}'\mathbf{p} \geq \mathbf{0}\}$$

Observe that the complementary slackness relations in (4.8) and (4.9) satisfy:

$$\mathbf{p}'[\mathbf{E}(\mathbf{p}) - \mathbf{N}\mathbf{x} - \mathbf{y}] \leq 0; \qquad \forall\ (\mathbf{p},\mathbf{x},\mathbf{y}) \in S$$

$$-\mathbf{x}'(\mathbf{t} - \mathbf{N}'\mathbf{p}) \leq 0; \qquad \forall\ (\mathbf{p},\mathbf{x},\mathbf{y}) \in S$$

Upon summing these two expressions, one obtains:

$$\mathbf{p}'\mathbf{E}(\mathbf{p}) - \mathbf{t}'\mathbf{x} - \mathbf{p}'\mathbf{y} \leq 0; \qquad \forall\ (\mathbf{p},\mathbf{x},\mathbf{y}) \in S$$

where equality holds if and only if both of the complementary slackness conditions hold. Hence, given that $(\mathbf{p},\mathbf{x},\mathbf{y}) \in S$, that is, given that (4.5), (4.6) and (4.10) are satisfied, then (4.8) and (4.9) are both satisfied with either of the following conditions:

$$\mathbf{p}'\mathbf{E}(\mathbf{p}) - \mathbf{t}'\mathbf{x} - \mathbf{p}'\mathbf{y} = 0 \tag{4.11}$$

or:

$$\mathbf{p}'\mathbf{E}(\mathbf{p}) - \mathbf{t}'\mathbf{x} - \mathbf{p}'\mathbf{y} \geq 0 \tag{4.12}$$

Thus, the two constraints requiring the complementary slackness conditions may be collapsed into one constraint.

If $\mathbf{E}(\mathbf{p})$ is continuous on the nonnegative orthant, and if there exists a $\mathbf{p}_0$ such that $\mathbf{p}'\mathbf{E}(\mathbf{p}) < 0$ for all $\mathbf{p} > \mathbf{p}_0$, then the Weierstrass theorem (Theorem 2.1.4) guarantees the existence of a solution to Problem One provided that the feasible region is nonempty. Observe that the objective function is continuous if $\mathbf{E}(\mathbf{p})$ is continuous. If the constraints are continuous on the nonnegative orthant, then with use of Theorem 2.1.3, it may be confirmed that the feasible region is closed. The nonnegativity requirement imposes a lower bound upon all variables. From (4.7), there is clearly an upper bound on $\mathbf{y}$. Condition (4.6) implies that if one p_i is increased without bound, then all p_i must be increased without bound; hence, if there exists a $\mathbf{p}_0$ such that $\mathbf{p}'\mathbf{E}(\mathbf{p}) < 0$ for all $\mathbf{p} > \mathbf{p}_0$ then condition (4.12) imposes an upper bound on $\mathbf{p}$. As the set of all feasible $\mathbf{p}$ is closed and bounded, and since $\mathbf{E}(\mathbf{p})$ is continuous, then $\mathbf{p}'\mathbf{E}(\mathbf{p})$ attains a maximum over the feasible region; consequently, from condition (4.12) it may be concluded that $\mathbf{x}$ is bounded. Therefore, the feasible region is compact; thus, it follows that if the feasible region is nonempty, then the Weierstrass theorem affirms the existence of a solution.

Rather than imposing the complementary slackness conditions as constraints upon the problem, first consider the possibilty of incorporating (4.11) into the objective function through a penalty function of the form:

$$\rho(\mathbf{p},\mathbf{x},\mathbf{y},\alpha) = \alpha[\mathbf{p}'\mathbf{E}(\mathbf{p}) - \mathbf{t}'\mathbf{x} - \mathbf{p}'\mathbf{y}]$$

where α is some positive constant. Observe that $\rho \leq 0$ on the set, S. The problem is now expressed as:

Problem Two

$$\begin{array}{ll} \text{maximize}(\mathbf{p},\mathbf{x},\mathbf{y}): & (\mathbf{p} - \mathbf{r})'\mathbf{y} + \alpha[\mathbf{p}'\mathbf{E}(\mathbf{p}) - \mathbf{t}'\mathbf{x} - \mathbf{p}'\mathbf{y}] \\ \text{subject to:} & \mathbf{E}(\mathbf{p}) - \mathbf{N}\mathbf{x} - \mathbf{y} \leq \mathbf{0} \\ & \mathbf{t} - \mathbf{N}'\mathbf{p} \geq \mathbf{0} \\ & \bar{y} - \mathbf{u}'\mathbf{y} \geq 0 \\ & (\mathbf{p},\mathbf{x},\mathbf{y}) \geq \mathbf{0} \end{array}$$

Suppose that π is the optimal value of Problem One. Since an optimal solution to Problem One is feasible to Problem Two, then for any α, the optimal value to Problem Two must be greater than or equal π. Hence, if there exists an α sufficiently large that the optimal solutions to Problem Two are such that the penalty function is equal to zero, then as such solutions are feasible to Problem One, they must also be optimal to Problem One. Moreover, it is apparent that at such α, optimal solutions to Problem One are also optimal to Problem Two.

It can be shown that $\rho \to 0$ as $\alpha \to \infty$, that is, the penalty function does in fact approach zero with increasing

α. However, to prove that the generated sequence of solutions actually converges to a solution of Problem One, it must be shown that $\alpha\rho \to 0$ as $\alpha \to \infty$. To demonstrate this, it must generally be shown that the sequence of solutions is contained within a compact subset of the feasible region to Problem Two. The proof follows immediately if the feasible region is itself compact, and generally, this is the way in which convergence must be proven. It may be confirmed that the feasible region to Problem Two is not compact; consequently, convergence cannot be guaranteed. Nonetheless, in practice, convergence is typically observed, and generally for finite values of α.

With minor modifications, the last problem may be extended to cover the case where discrimination is not only exercised in the export market, but also between the export market and the domestic market. Let p denote the price in the region of the discriminator, and let d(p) and e(p) be the demand and excess demand for the same. The present objective is the maximization of the sum of net revenues for domestic sales and foreign sales; subsequently, the appropriate programming model is:

$$\text{maximize}(p,\mathbf{p},\mathbf{x},\mathbf{y}): \quad p\,d(p) + (\mathbf{p} - \mathbf{r})'\mathbf{y} + \alpha[\mathbf{p}'\mathbf{E}(\mathbf{p}) - \mathbf{t}'\mathbf{x} - \mathbf{p}'\mathbf{y}]$$

$$\text{subject to:} \quad \mathbf{E}(\mathbf{p}) - \mathbf{N}\mathbf{x} - \mathbf{y} \leq \mathbf{0}$$

$$\mathbf{t} - \mathbf{N}'\mathbf{p} \geq \mathbf{0}$$

$$-e(p) - \mathbf{u}'\mathbf{y} \geq 0$$

$$(p,\mathbf{p},\mathbf{x},\mathbf{y}) \geq \mathbf{0}$$

which differs from the last problem only in that domestic revenue has been added to the objective function, and $\bar{y}$ has been replaced with the excess supply function of the discriminator. Indeed, the former problem is the special case of this problem where $d(p) = 0$ and $-e(p) = \bar{y}$.

Note that this model possesses some of the features of the traditional price discrimination problem. In particular, the discriminator's region is perfectly separated from the regions in the market. Also, with such separation, the discriminator effectively becomes a monopolist to buyers in its own region.

Let the Lagrangian to the latter problem be constructed as:

$$l(p,\mathbf{p},\mathbf{x},\mathbf{y},\boldsymbol{\mu},\boldsymbol{\lambda},\gamma) = pd(p) + (\mathbf{p} - \mathbf{r})'\mathbf{y} + \alpha[\mathbf{p}'\mathbf{E}(\mathbf{p}) - \mathbf{t}'\mathbf{x} - \mathbf{p}'\mathbf{y}]$$
$$- \boldsymbol{\mu}'[\mathbf{E}(\mathbf{p}) - \mathbf{N}\mathbf{x} - \mathbf{y}] + \boldsymbol{\lambda}'(\mathbf{t} - \mathbf{N}'\mathbf{p}) - \gamma[e(p) + \mathbf{u}'\mathbf{y}]$$

on $(\mathbf{p},\mathbf{x},\mathbf{y},\boldsymbol{\mu},\boldsymbol{\lambda},\gamma) \geq \mathbf{0}$. The corresponding Kuhn-Tucker conditions are:

$$l_p = d(p) + pd_p(p) - \gamma e_p(p) \leq 0; \qquad pl_p = 0$$
$$l_{\mathbf{p}} = (1 - \alpha)\mathbf{y} + \alpha\mathbf{E}(\mathbf{p}) + \mathbf{E}_{\mathbf{p}}(\mathbf{p})(\alpha\mathbf{p} - \boldsymbol{\mu}) - \mathbf{N}\boldsymbol{\lambda} \leq \mathbf{0}; \qquad \mathbf{p}'l_{\mathbf{p}} = 0$$
$$l_{\mathbf{x}} = -\alpha\mathbf{t} + \mathbf{N}'\boldsymbol{\mu} \leq \mathbf{0}; \qquad \mathbf{x}'l_{\mathbf{x}} = 0$$
$$l_{\mathbf{y}} = \mathbf{p} - \mathbf{r} - \alpha\mathbf{p} + \boldsymbol{\mu} - \gamma\mathbf{u} \leq \mathbf{0}; \qquad \mathbf{y}'l_{\mathbf{y}} = 0$$
$$l_{\boldsymbol{\mu}} = -\mathbf{E}(\mathbf{p}) + \mathbf{N}\mathbf{x} + \mathbf{y} \geq \mathbf{0}; \qquad \boldsymbol{\mu}'l_{\boldsymbol{\mu}} = 0$$
$$l_{\boldsymbol{\lambda}} = \mathbf{t} - \mathbf{N}'\mathbf{p} \geq \mathbf{0}; \qquad \boldsymbol{\lambda}'l_{\boldsymbol{\lambda}} = 0$$
$$l_{\gamma} = -e(p) - \mathbf{u}'\mathbf{y} \geq 0; \qquad \gamma l_{\gamma} = 0$$

As shown in Theorem 2.5.12, the Kuhn-Tucker conditions are necessary for the optimal solutions if the gradients of the active nonlinear constraints are linearly independent at such solutions. If the excess demand functions are linear, then all the constraints are linear; consequently, the necessity of the Kuhn-Tucker conditions holds automatically. However, suppose $\mathbf{E(p)}$ and $e(p)$ are nonlinear. It may be confirmed that the columns of the following matrix comprise the gradients of all the nonlinear constraints to the latter problem:

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & | -e_p(p) \\ -\mathbf{E_p(p)} & | \ \mathbf{0}^p \\ & | \\ \mathbf{N}' & | \ \mathbf{0} \\ \mathbf{I} & | \ \mathbf{u} \end{bmatrix} \qquad (4.13)$$

where $\mathbf{I}$ is the nxn identity matrix, and $\mathbf{E_p(p)}$ is a diagonal matrix whose ith column is the gradient of $e_i(p_i)$. The first partition of $\mathbf{A}$ contains the gradients in $-\mathbf{E(p)} + \mathbf{Nx} + \mathbf{y}$, and the second partition contains the gradient of $-e(p) - \mathbf{u'y}$. It is apparent that the columns in $\mathbf{A}$ are linearly independent over all $(p,\mathbf{p},\mathbf{x},\mathbf{y})$ if $e_p(p) \neq 0$. Now, suppose $e_p(p) = 0$, and suppose there exists $\mathbf{c} = (\mathbf{c}_1, c_2)$, not equal to zero, such that $\mathbf{Ac} = \mathbf{0}$. With this being the case, the third row partition of the above implies:

$$\mathbf{N}'\mathbf{c}_1 = \mathbf{0}$$

It may be confirmed that the construction of $\mathbf{N}'$ is such that this can occur only if all of the components of $\mathbf{c}_1$ are equal to a single constant. However, the second row partion then implies:

$$\mathbf{E_p}(\mathbf{p})\mathbf{c}_1 = \mathbf{0}$$

If the components $\mathbf{c}_1$ are indeed equal to a constant, and if the slope of at least one regional excess demand is unequal to zero, then the latter equation implies that $\mathbf{c}_1 = \mathbf{0}$. But, if this be the case, then the last row partition implies:

$$uc_2 = \mathbf{0}$$

which can occur only if $c_2 = 0$. But, this contradicts the assumption that $\mathbf{c} \neq \mathbf{0}$; consequently, the gradients of all nonlinear constraints to the problem are linearly independent if the slopes of the excess demands do not all simultaneously go to zero.

An interesting case occurs when $\alpha = 1$ is sufficient to drive the penalty function to zero. At $\alpha = 1$, the objective function reduces to:

$$f(p,\mathbf{p},\mathbf{x},\mathbf{y}) = pd(p) + \mathbf{p}'\mathbf{E}(\mathbf{p}) - \mathbf{t}'\mathbf{x} - \mathbf{r}'\mathbf{y}$$

If $f(p,\mathbf{p},\mathbf{x},\mathbf{y})$ is pseudoconcave over the feasible region, and if the constraints are quasiconcave over the same, then by Theorem 2.5.13, the Kuhn-Tucker conditions are sufficient conditions for the optimal solutions. It may be confirmed that all of these conditions hold if the excess demands are linear with nonpositive slopes. However, $f(p,\mathbf{p},\mathbf{x},\mathbf{y})$ can never be pseudoconcave if $\alpha \neq 1$.

Now, consider the case where the complementary slackness conditions are imposed by including condition (4.12) as a constraint to the problem. The problem then becomes:

$$
\begin{array}{ll}
\text{maximize}(p,\mathbf{p},\mathbf{x},\mathbf{y}): & pd(p) + (\mathbf{p} - \mathbf{r})'\mathbf{y} \\
\text{subject to:} & \mathbf{E}(\mathbf{p}) - \mathbf{N}\mathbf{x} - \mathbf{y} \leq \mathbf{0} \\
 & \mathbf{t} - \mathbf{N}'\mathbf{p} \geq \mathbf{0} \\
 & -e(p) - \mathbf{u}'\mathbf{y} \geq 0 \\
 & \mathbf{p}'\mathbf{E}(\mathbf{p}) - \mathbf{t}'\mathbf{x} - \mathbf{p}'\mathbf{y} \geq 0 \\
 & (p,\mathbf{p},\mathbf{x},\mathbf{y}) \geq \mathbf{0}
\end{array}
$$

Again, using Theorem 2.5.12, the Kuhn-Tucker conditions to this problem are known to be necessary for the optimal solutions if the gradients of the active nonlinear constraints are linearly independent at such solutions. The columns of the following matrix comprise the gradients of all the nonlinear constraints:

$$
\mathbf{A} = \left[\begin{array}{c|c|c}
\mathbf{0} & -e_p(p) & 0 \\
-\mathbf{E_p}(\mathbf{p}) & \mathbf{0} & \mathbf{E}(\mathbf{p}) + \mathbf{E_p}(\mathbf{p})\mathbf{p} - \mathbf{y} \\
 & & \\
\mathbf{N}' & \mathbf{0} & -\mathbf{t} \\
\mathbf{I} & -\mathbf{u} & -\mathbf{p}
\end{array}\right]
$$

The first two partitions are the same as in (4.13). The last partition contains the gradient of $\mathbf{p}'\mathbf{E}(\mathbf{p}) - \mathbf{t}'\mathbf{x} - \mathbf{p}'\mathbf{y}$. Suppose there is a nonzero $\mathbf{c} = (c_1,c_2,c_3)$ such that $\mathbf{Ac} = \mathbf{0}$, then using the third row partition of $\mathbf{A}$, it may be concluded:

$$\mathbf{N}'\mathbf{c}_1 - \mathbf{t}\mathbf{c}_3 = \mathbf{0}$$

Note that the components in each column of $\mathbf{N}'$ sum to zero. Hence, upon summing the rows of the above, one obtains:

$$0 - (\Sigma_{i,j}\ t_{ij})c_3 = 0$$

But, this implies $c_3 = 0$, and as noted before, the columns in the first two partitions are linearly independent over all $(p,\mathbf{p},\mathbf{x},\mathbf{y})$ if the slopes of the excess demands do not all simultaneously go to zero. Hence, provided that the slopes do not simultaneously go to zero, then $\mathbf{Ac} = \mathbf{0}$ implies $\mathbf{c} = \mathbf{0}$.

Observe that as the solutions to all the problems of this section are consistent with spatial equilibrium in the market, then all of the properties of such equilibria discussed in the previous chapter pertain here also. It has been shown that if the triangular inequality of $\mathbf{t}$ holds, then there must exist a solution wherein the discriminator's exports are not transshipped; thus, it remains that there must exist an equilibrium wherein no region simultaneously imports and exports. By the same reasoning used in the former chapter, it may be shown that there must exist an equilibrium wherein no two exporters, including the discriminator, share two importers.

It was shown in the previous chapter that if the commodity is desirable in every region of the market, then in an ordinary spatial equilibrium, prices must be positive. In the present models, desirability does not ensure positive prices; however, it does ensure another important property; namely, that the equilibria must be such that market excess demands are exactly equal to net imports. From the complementary slackness condition, $p_i[e_i(p_i) - n_i - y_i] = 0$, it may be seen that if $p_i > 0$, then the result must hold. Now, suppose $p_i = 0$, then by the desirability assumption, region i must be an importer. However, as shown in the previous chapter, it cannot import from other regions in the market; consequently, it must import from the discriminator. Now, if $y_i > e_i(0)$, then y_i could be reduced such that $y_i = e_i(0)$ without affecting total revenue; however, transportation expenses would be reduced, and consequently, net revenue would be increased. Thus, $y_i > e_i(0)$ cannot be optimal, and consequently, under the desirability assumption, market excess demands must equal net imports.

It should be apparent that all the trade restrictions discussed in the former chapter may also be incorporated into the discrimination models. However, it must be assumed that the trade policies of the various regions are invariant with respect to the actions of the discriminator. If this were not the case, then the excess demand functions could not be safely regarded as being stable.

4.2 Discrimination by a Cooperative

It is likely that a single region could accomplish greater gains through price discrimination if it could persuade otherwise competing sellers to cooperate in a joint price discrimination scheme. Moreover, it is certain that several price discriminators can accomplish greater total gains working jointly than on an independent basis. In this section, models are developed wherein it assumed that a group of regions exercise price discrimination cooperatively with the objective of maximizing net revenue to the cooperative.

The assumptions of the foregoing models are exactly the same as those in the models of the previous section; however, statements that are there made of a single discriminator are now made of the cooperative. The terms, "market" and "aggregate market," shall now refer to all regions other than those contained in the cooperative.

Suppose there are n regions in the market and m regions in the cooperative. Let y_{ij} denote shipments from the ith cooperative member to the jth region in the market. Accordingly, let r_{ij} denote the per-unit transportation charge for shipments from the ith cooperative member to the jth region in the market, and define the following matrices:

$$\mathbf{y}_i = \begin{bmatrix} y_{i1} \\ y_{i2} \\ \cdot \\ \cdot \\ y_{in} \end{bmatrix} \qquad \mathbf{r}_i = \begin{bmatrix} r_{i1} \\ r_{i2} \\ \cdot \\ \cdot \\ r_{in} \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ y_m \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \\ . \\ . \\ r_m \end{bmatrix}$$

$$\mathbf{C} = \left[\begin{array}{ccccc|ccccc|ccc|ccccc} 1 & 0 & . & . & 0 & 1 & 0 & . & . & 0 & . & . & . & 1 & 0 & . & . & 0 \\ 0 & 1 & 0 & . & 0 & 0 & 1 & 0 & . & 0 & . & . & . & 0 & 1 & 0 & . & 0 \\ . & 0 & . & & . & . & 0 & . & & . & & . & & . & 0 & . & & . \\ . & . & & . & 0 & . & . & & . & 0 & & . & & . & . & & . & 0 \\ 0 & 0 & . & 0 & 1 & 0 & 0 & . & 0 & 1 & . & . & . & 0 & 0 & . & 0 & 1 \end{array}\right]$$

$$\mathbf{U} = \left[\begin{array}{ccccc|ccccc|ccc|ccccc} 1 & 1 & 1 & . & 1 & 0 & 0 & . & . & 0 & . & . & . & 0 & 0 & . & . & 0 \\ 0 & 0 & . & . & 0 & 1 & 1 & 1 & . & 1 & . & . & . & 0 & 0 & . & . & 0 \\ 0 & 0 & . & . & . & 0 & 0 & . & . & . & & . & & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & & . & & 0 & 0 & . & . & 0 \\ 0 & 0 & . & . & 0 & 0 & 0 & . & . & 0 & . & . & . & 1 & 1 & 1 & . & 1 \end{array}\right]$$

Finally, let $\bar{y}_i$ be the exportable quantity in the ith cooperative member and let $\bar{\mathbf{y}} = (\bar{y}_1, \bar{y}_2, , , \bar{y}_m)'$.

With these constructions, the net export revenue to the cooperative is given by $f(\mathbf{p},\mathbf{y}) = (\mathbf{C}'\mathbf{p} - \mathbf{r})'\mathbf{y}$. The spatial equilibrium conditions for given $\mathbf{y}$ become:

$$\mathbf{E}(\mathbf{p}) - \mathbf{N}\mathbf{x} - \mathbf{C}\mathbf{y} \leq \mathbf{0}; \qquad \mathbf{p}'[\mathbf{E}(\mathbf{p}) - \mathbf{N}\mathbf{x} - \mathbf{C}\mathbf{y}] = 0$$

$$\mathbf{t} - \mathbf{N}'\mathbf{p} \geq \mathbf{0}; \qquad \mathbf{x}'(\mathbf{t} - \mathbf{N}'\mathbf{p}) = 0$$

$$(\mathbf{p},\mathbf{x}) \geq \mathbf{0}$$

where $\mathbf{p}$, $\mathbf{E}(\mathbf{p})$, $\mathbf{N}$, and $\mathbf{x}$ are defined as before except that components corresponding to the discriminating regions are not included. The constraint limiting cooperative member shipments is $\bar{\mathbf{y}} - \mathbf{U}\mathbf{y} \geq \mathbf{0}$. Thus, the problem to maximize cooperative net export revenue subject to the condition that the market be in spatial equilibrium may be formerly stated as:

$$\begin{aligned} \text{maximize}(\mathbf{p},\mathbf{x},\mathbf{y}): \quad & (\mathbf{C}'\mathbf{p} - \mathbf{r})'\mathbf{y} \\ \text{subject to:} \quad & \mathbf{E}(\mathbf{p}) - \mathbf{N}\mathbf{x} - \mathbf{C}\mathbf{y} \leq \mathbf{0} \\ & \mathbf{t} - \mathbf{N}'\mathbf{p} \geq \mathbf{0} \\ & \bar{\mathbf{y}} - \mathbf{U}\mathbf{y} \geq \mathbf{0} \\ & \mathbf{p}'[\mathbf{E}(\mathbf{p}) - \mathbf{N}\mathbf{x} - \mathbf{C}\mathbf{y}] = 0 \\ & \mathbf{x}'(\mathbf{t} - \mathbf{N}'\mathbf{p}) = 0 \\ & (\mathbf{p},\mathbf{x},\mathbf{y}) \geq \mathbf{0} \end{aligned}$$

If $\mathbf{E}(\mathbf{p})$ is continuous, and if there exists $\mathbf{p}_0$ such that $\mathbf{p}'\mathbf{E}(\mathbf{p}) < 0$ for all $\mathbf{p} > \mathbf{p}_0$, then using the arguments of the previous section, it may be seen that the feasible region is compact. Consequently, since the objective function is continuous, the Weierstrass theorem (Theorem 2.1.4) ensures the existence of a solution if the feasible region is nonempty.

Observe that if S denotes the set of all $(\mathbf{p},\mathbf{x},\mathbf{y})$ satisfying the first, second and last constraints, then the complementary slackness conditions in the fourth and fifth constraints must satisfy:

$$\mathbf{p}'[\mathbf{E}(\mathbf{p}) - \mathbf{N}\mathbf{x} - \mathbf{C}\mathbf{y}] \leq 0; \qquad \forall\ (\mathbf{p},\mathbf{x},\mathbf{y}) \in S$$

$$-\mathbf{x}'(\mathbf{t} - \mathbf{N}'\mathbf{p}) \leq 0; \qquad \forall\ (\mathbf{p},\mathbf{x},\mathbf{y}) \in S$$

Upon adding these two conditions, one obtains

$$\mathbf{p}'\mathbf{E}(\mathbf{p}) - \mathbf{t}'\mathbf{x} - \mathbf{p}'\mathbf{C}\mathbf{y} \leq 0; \qquad \forall\ (\mathbf{p},\mathbf{x},\mathbf{y}) \in S$$

where equality holds if and only if the complementary slackness conditions both hold. Thus, given that

$(\mathbf{p},\mathbf{x},\mathbf{y}) \in S$, the fourth and fifth constraints are both implied by either of the following conditions:

$$\mathbf{p}'\mathbf{E}(\mathbf{p}) - \mathbf{t}'\mathbf{x} - \mathbf{p}'\mathbf{C}\mathbf{y} = 0 \tag{4.14}$$

or:

$$\mathbf{p}'\mathbf{E}(\mathbf{p}) - \mathbf{t}'\mathbf{x} - \mathbf{p}'\mathbf{C}\mathbf{y} \geq 0 \tag{4.15}$$

As before, the complementary slackness conditions may be imposed either by substituting (4.15) for the fourth and fifth constraints in the latter problem, or by the incorporation of (4.14) into the objective function through a penalty function. If the penalty function technique is chosen, then the model becomes:

$$\begin{aligned}
\text{maximize}(\mathbf{p},\mathbf{x},\mathbf{y}): \quad & (\mathbf{C}'\mathbf{p} - \mathbf{r})'\mathbf{y} + \alpha[\mathbf{p}'\mathbf{E}(\mathbf{p}) - \mathbf{t}'\mathbf{x} - \mathbf{p}'\mathbf{C}\mathbf{y}] \\
\text{subject to:} \quad & \mathbf{E}(\mathbf{p}) - \mathbf{N}\mathbf{x} - \mathbf{C}\mathbf{y} \leq \mathbf{0} \\
& \mathbf{t} - \mathbf{N}'\mathbf{p} \geq \mathbf{0} \\
& \bar{\mathbf{y}} - \mathbf{U}\mathbf{y} \geq \mathbf{0} \\
& (\mathbf{p},\mathbf{x},\mathbf{y}) \geq \mathbf{0}
\end{aligned}$$

As before, the model may be extended with little difficulty to maximize the total net revenue of the cooperative. Let $\hat{\mathbf{p}}$ denote the vector of prices for the regions in the cooperative, and let $\hat{\mathbf{D}}(\hat{\mathbf{p}})$ and $\hat{\mathbf{E}}(\hat{\mathbf{p}})$ denote the vectors of demands and excess demands for the cooperative. The problem may then be formally stated as:

$$\text{maximize}(\hat{p},p,x,y): \quad \hat{p}'\hat{D}(\hat{p}) + (C'p - r)'y + \alpha[p'E(p) - t'x - p'Cy]$$

$$\text{subject to:} \quad E(p) - Nx - Cy \leq 0$$

$$t - N'p \geq 0$$

$$-\hat{E}(\hat{p}) - Uy \geq 0$$

$$(\hat{p},p,x,y) \geq 0$$

It may be confirmed that the former model is the special case of this model where $\hat{D}(\hat{p}) = 0$ and $\hat{E}(\hat{p}) = -\bar{y}$. Indeed, even the problems of the previous section may be considered as special cases of this problem where the cooperative consists of only one region.

Supposing that the excess demands are nonlinear, the gradients of the nonlinear constraints in the latter model are given by the columns of:

$$A = \left[\begin{array}{c|c} 0 & -\hat{E}_p(\hat{p}) \\ -E_p(p) & 0 \\ N' & 0 \\ C' & -U' \end{array}\right]$$

It may be confirmed that A has full column rank if the slopes of the excess demands do not all simultaneously go to zero. Provided that this does not occur, then by Theorem 2.5.12, the Kuhn-Tucker conditions are necessary conditions for the solutions to last problem.

Suppose $\alpha = 1$ is sufficient to drive the penalty function to zero, then at $\alpha = 1$, the objective function to the last problem becomes:

$$f(\hat{\mathbf{p}},\mathbf{p},\mathbf{x},\mathbf{y}) = \hat{\mathbf{p}}'\hat{\mathbf{D}}(\hat{\mathbf{p}}) + \mathbf{p}'\mathbf{E}(\mathbf{p}) - \mathbf{t}'\mathbf{x} - \mathbf{r}'\mathbf{y}$$

If $f(\hat{\mathbf{p}},\mathbf{p},\mathbf{x},\mathbf{y})$ is pseudoconcave over the feasible region, and if the constraints are quasiconcave over the same, then by Theorem 2.5.13, the Kuhn-Tucker conditions are sufficient for the optima. All of these conditions will hold if the demands and excess demands are linear. However, as before, the objective function can never be pseudoconcave if $\alpha \neq 1$.

If the complementary slackness conditions are enforced by the inclusion of (4.15) as a constraint, then the problem becomes:

$$\begin{array}{ll}
\text{maximize}(\hat{\mathbf{p}},\mathbf{p},\mathbf{x},\mathbf{y}): & \hat{\mathbf{p}}'\hat{\mathbf{D}}(\hat{\mathbf{p}}) + (\mathbf{C}'\mathbf{p} - \mathbf{r})'\mathbf{y} \\
\text{subject to:} & \mathbf{E}(\mathbf{p}) - \mathbf{N}\mathbf{x} - \mathbf{C}\mathbf{y} \leq \mathbf{0} \\
 & \mathbf{t} - \mathbf{N}'\mathbf{p} \geq \mathbf{0} \\
 & -\hat{\mathbf{E}}(\hat{\mathbf{p}}) - \mathbf{U}\mathbf{y} \geq \mathbf{0} \\
 & \mathbf{p}'\mathbf{E}(\mathbf{p}) - \mathbf{t}'\mathbf{x} - \mathbf{p}'\mathbf{C}\mathbf{y} \geq 0 \\
 & (\hat{\mathbf{p}},\mathbf{p},\mathbf{x},\mathbf{y}) \geq \mathbf{0}
\end{array}$$

Assuming that the excess demands are nonlinear, then the gradients of the nonlinear constraints are contained in:

$$\mathbf{A} = \left[\begin{array}{l|l|l}
\mathbf{0} & -\hat{\mathbf{E}}_{\mathbf{p}}(\hat{\mathbf{p}}) & \mathbf{0} \\
-\mathbf{E}_{\mathbf{p}}(\mathbf{p}) & \mathbf{0} & \mathbf{E}(\mathbf{p}) + \mathbf{E}_{\mathbf{p}}(\mathbf{p})\mathbf{p} - \mathbf{y} \\
\mathbf{N}' & \mathbf{0} & -\mathbf{t} \\
\mathbf{C}' & -\mathbf{U}' & -\mathbf{p}
\end{array}\right]$$

Using the same reasoning as that of the previous section, it may be confirmed that any $\mathbf{c} = (c_1,c_2,c_3)$ such that $\mathbf{Ac} = \mathbf{0}$ must have $c_3 = 0$, but it has already been noted that the first two partitions of this matrix are linearly independent if the slopes of the excess demands do not all simultaneously go to zero.

4.3 Policy Implementation

In the former sections, programming models were developed having solutions equal to the optimal price and quantity vectors for a discriminator or group of discriminators operating in a spatial market. These solutions were constrained to be consistent with spatial equilibrium in that given the discriminator's export vector, $\mathbf{y}$, the optimal $\mathbf{p}$ and $\mathbf{x}$ vectors had to be such that the market would be in spatial equilibrium. Now, suppose that the optimal vectors are $\bar{\mathbf{p}}$, $\bar{\mathbf{x}}$, and $\bar{\mathbf{y}}$. The equilibrium constraints that were imposed in the determination of these vectors ensure that if $\bar{\mathbf{p}}$, $\bar{\mathbf{x}}$, and $\bar{\mathbf{y}}$ were simultaneously realized, then the market would in fact be in equilibrium. However, there is no assurance as of yet that such vectors will actually be realized. Thus, it remains to determine if there is a trade policy whereby the discriminator can impose its chosen price and quantity vectors upon the market.

Of course, the equilibrium value of $\mathbf{x}$ is of no consequence to the discriminator's net revenue; therefore, any equilibrium of the form, $(\bar{\mathbf{p}},\hat{\mathbf{x}},\bar{\mathbf{y}})$, is optimal. To demonstrate that a particular trade policy will cause the market to converge upon a $(\bar{\mathbf{p}},\hat{\mathbf{x}},\bar{\mathbf{y}})$, it must first be shown that such policy forbids all equilibrium points other than those of this form. That is, if the natural dynamic adjustment process is such that more than one equilibrium exists, then the trade policy of the discriminator must influence this process so that the only equilibria that exist are those involving $\bar{\mathbf{p}}$ and $\bar{\mathbf{y}}$. Second, it must be shown that given the discriminator's policy, the market does converge to equilibrium. In this section, a policy is developed that satisfies both conditions under general assumptions. The foregoing derivations treat the single discriminator model of the first section; however, the theory extends to cover the cooperative discrimination model with almost no modification.

It will be recalled that the dynamic assumptions of the discrimination model are:

5 a) $\dot{p}_i > 0$ if and only if $e_i(p_i) - n_i - y_i > 0$.

b) $\dot{p}_i < 0$ if and only if $e_i(p_i) - n_i - y_i < 0$ and $p_i > 0$.

6 a) $\dot{x}_{ij} > 0$ only if $t_{ij} - (p_j - p_i) < 0$.

b) If $t_{ij} - (p_j - p_i) < 0$ for some j, then $\dot{x}_{ij} > 0$ for at least one such j.

c) $\dot{x}_{ij} < 0$ if and only if $t_{ij} - (p_j - p_i) > 0$ and $x_{ij} > 0$.

These adjustment rules say nothing of $\dot{\mathbf{y}}$, for until now, it has been sufficient to take $\mathbf{y}$ as given. Subsequently, suppose that $\mathbf{y}$ adjusts according to the following rules:

7 a) $\dot{y}_i > 0$ if and only if $\bar{p}_i - p_i < 0$.

b) $\dot{y}_i < 0$ if and only if $\bar{p}_i - p_i > 0$ and $y_i > 0$.

where it is assumed that the discriminator has fixed its offered prices at $\bar{\mathbf{p}}$. These adjustment rules could be somewhat further generalized as were the rules for $\mathbf{x}$; however, as such generalizations do not effect the equilibria of the system, they are neglected here. Also, the fact that this adjustment process could produce insupportable values of $\mathbf{y}$ is ignored for the moment. Provisions excluding such possibilities will be made shortly.

Assumptions five through seven imply that the system is at equilibrium if and only if:

$$e_i(p_i) - n_i - y_i \leq 0, \text{ if } < \text{ then } p_i = 0; \qquad \forall\ i$$
$$t_{ij} - (p_j - p_i) \geq 0, \text{ if } > \text{ then } x_{ij} = 0; \qquad \forall\ i,j$$
$$\bar{p}_i - p_i \geq 0, \text{ if } > \text{ then } y_i = 0; \qquad \forall\ i$$
$$p_i \geq 0,\ x_{ij} \geq 0,\ y_i \geq 0; \qquad \forall\ i,j$$

The matrix variants of these are:

$$\mathbf{E}(\mathbf{p}) - \mathbf{N}\mathbf{x} - \mathbf{y} \leq \mathbf{0}; \qquad \mathbf{p}'[\mathbf{E}(\mathbf{p}) - \mathbf{N}\mathbf{x}\ \mathbf{y}] = 0 \tag{4.16}$$
$$\mathbf{t} - \mathbf{N}'\mathbf{p} \geq \mathbf{0}; \qquad \mathbf{x}'(\mathbf{t} - \mathbf{N}'\mathbf{p}) = 0 \tag{4.17}$$
$$\bar{\mathbf{p}} - \mathbf{p} \geq \mathbf{0}; \qquad \mathbf{y}'(\bar{\mathbf{p}} - \mathbf{p}) = 0 \tag{4.18}$$
$$(\mathbf{p},\mathbf{x},\mathbf{y}) \geq \mathbf{0} \tag{4.19}$$

It may be confirmed that these conditions are the Kuhn-Tucker conditions to the following problem:

$$\begin{aligned} \text{maximize}(\mathbf{p}): \quad & f(\mathbf{p}) \\ \text{subject to:} \quad & \mathbf{t} - \mathbf{N}'\mathbf{p} \geq \mathbf{0} \\ & \bar{\mathbf{p}} - \mathbf{p} \geq \mathbf{0} \\ & \mathbf{p} \geq \mathbf{0} \end{aligned}$$

where:

$$f(\mathbf{p}) = \Sigma_{i=1}^{n} \int e_i(p_i)dp_i$$

and where the Lagrangian is constructed as:

$$l(\mathbf{p},\mathbf{x},\mathbf{y}) = f(\mathbf{p}) + \mathbf{x}'(\mathbf{t} - \mathbf{N}'\mathbf{p}) + \mathbf{y}'(\bar{\mathbf{p}} - \mathbf{p}); \qquad (\mathbf{p},\mathbf{x},\mathbf{y}) \geq \mathbf{0}$$

It has been shown that if the $e_i(p_i)$ have negative slopes, then $f(\mathbf{p})$ is strictly concave. Henceforth, it shall be assumed that this is the case; moreover, it is assumed that $f(\mathbf{p})$ is everywhere differentiable. Consequently, since the constraints are concave, Theorem 2.5.13 determines that any solution to the Kuhn-Tucker conditions corresponds to a global optimal solution that is unique with respect to $\mathbf{p}$. Moreover, by Theorem 2.6.1, it may be concluded that the Kuhn-Tucker points are one-to-one with the saddle points in the Lagrangian. Now, observe that $(\bar{\mathbf{p}},\bar{\mathbf{x}},\bar{\mathbf{y}})$ is the solution to a mathematical programming problem whose constraints require satisfaction of conditions (4.16) through (4.19); hence, $(\bar{\mathbf{p}},\bar{\mathbf{x}},\bar{\mathbf{y}})$, solves the problem above, and the uniqueness property guarantees that $\bar{\mathbf{p}}$ is the only equilibrium price. Therefore, it may be concluded that the discriminator can in fact force market equilibrium prices to equal to its chosen price vector. It may do so by simply fixing its own vector of offered prices at $\bar{\mathbf{p}}$.

Though the equilibrium price is uniquely determined at $\bar{\mathbf{p}}$, there may be any number of $\mathbf{x}$ and $\mathbf{y}$ that solve the Kuhn-Tucker conditions to the problem above, and consequently, $(\bar{\mathbf{p}},\bar{\mathbf{x}},\bar{\mathbf{y}})$ is not necessarily a unique equilibrium solution. Thus, a policy wherein the discriminator merely fixes its offer prices at $\bar{\mathbf{p}}$ is insufficient to guarantee convergence of $\mathbf{y}$ to $\bar{\mathbf{y}}$. However, suppose $\bar{\mathbf{p}} > \mathbf{0}$. That is, there are no zero prices in the discriminator's optimal price vector. If such is the case, then it may be concluded that at any equilibrium solution, $(\bar{\mathbf{p}},\hat{\mathbf{x}},\hat{\mathbf{y}})$, condition (4.16) holds with equality. That is:

$$\mathbf{E}(\bar{\mathbf{p}}) - \mathbf{N}\hat{\mathbf{x}} - \hat{\mathbf{y}} = \mathbf{0}$$

Sum the rows of the latter system to obtain:

$$\Sigma\; e_i(\bar{p}_i) = \Sigma\; \hat{y}_i$$

where the fact that the columns of $\mathbf{N}$ sum to zero has been used. Now, the right-hand side of the latter equation is unique to all equilibria; consequently, it may be concluded that the sum of the discriminator's exports is the same regardless of the particular equilibrium that occurs. Moreover, as $\bar{\mathbf{y}}$ is an equilibrium $\mathbf{y}$, then it may be concluded that any equilibrium $\hat{\mathbf{y}}$ must satisfy:

$$\Sigma\; \hat{y}_i = \Sigma\; \bar{y}_i$$

Now, suppose that in conjunction with fixing its offer prices at $\bar{\mathbf{p}}$, the discriminator were to impose an export quota requiring $y_i \leq \bar{y}_i$ for every i. If $\hat{\mathbf{y}}$ were at an equilibrium to the system under such a quota, then it must be true that $\hat{\mathbf{y}}$ satisfies both of the following conditions:

$$\Sigma \hat{y}_i = \Sigma \bar{y}_i$$
$$\hat{y}_i \leq \bar{y}_i ; \qquad \forall \; i$$

But, these together imply that $\hat{\mathbf{y}} = \bar{\mathbf{y}}$. Thus, by fixing its offer prices at $\bar{\mathbf{p}}$, and by imposing an export quota requiring $\mathbf{y} \leq \bar{\mathbf{y}}$, the discriminator can prohibit all equilibria other than those of the form $(\bar{\mathbf{p}},\hat{\mathbf{x}},\bar{\mathbf{y}})$.

Observe that if the commodity is desirable in every region of the market, then the conclusions of the latter analysis holds even if some $p_i = 0$. It was shown in the previous section, that under the desirability assumption, the discriminator will be the sole supplier to any region having zero price; moreover, if $\bar{p}_i = 0$, then $\bar{y}_i = e_i(0)$. If $\hat{y}_i$ is the realized equilibrium value, then condition (4.16) ensures that $\hat{y}_i$ cannot be less that $e_i(0)$. On the other hand, if the discriminator imposes an export quota, then $\hat{y}_i$ cannot be greater than $\bar{y}_i = e_i(0)$; hence, $\hat{y}_i = e_i(0)$. Thus, the export quota itself ensures that equality will hold in condition (4.16), so the above analysis is still valid.

It remains to show that the market price and quantity vectors will actually converge to $(\bar{\mathbf{p}},\hat{\mathbf{x}},\bar{\mathbf{y}})$. If it is assumed that the discriminator can prohibit all other equilibria with use of an export quota, then it is sufficient to show that the system directing market adjustment is stable when the discriminator's quota is enforced.

The dynamic adjustment rules in assumptions five through seven are not sufficient to guarantee stability; however, it can be shown that the foregoing special case of these assumptions is stable. Suppose that:

$$\dot{p}_i = \begin{cases} 0 & \text{if } p_i = 0 \text{ and } e_i(p_i) - n_i - y_i < 0 \\ a_i[e_i(p_i) - n_i - y_i] & \text{otherwise} \end{cases}$$

$$\dot{x}_{ij} = \begin{cases} 0 & \text{if } x_{ij} = 0 \text{ and } t_{ij} - (p_j - p_i) > 0 \\ -b_{ij}[t_{ij} - (p_j - p_i)] & \text{otherwise} \end{cases}$$

$$\dot{y}_i = \begin{cases} 0 & \text{if } y_i = 0 \text{ and } \bar{p}i - p_i > 0 \\ 0 & \text{if } y_i = \bar{y}_i \text{ and } \bar{p}_i - p_i < 0 \\ -c_i(\bar{p}_i - p_i) & \text{otherwise} \end{cases}$$

where the a_i, b_{ij}, and c_i are positive constants. Thus, the rate of price adjustment is proportional to the deficit, unless such adjustment would lead to a negative price. The rate of market interregional trade flow adjustment is proportional to profits, unless such adjustment would lead to a negative flow, and the rate of discriminator export flow adjustment is proportional to the discriminator's discount, unless such adjustment would lead to negative export flow or quota violation. Observe that this system may be expressed in terms of the Lagrangian as:

$$\dot{p}_i = \begin{cases} 0 & \text{if } p_i = 0 \text{ and } \partial l(\mathbf{p},\mathbf{x},\mathbf{y})/\partial p_i < 0 \\ a_i \partial l(\mathbf{p},\mathbf{x},\mathbf{y})/\partial p_i & \text{otherwise} \end{cases}$$

$$\dot{x}_{ij} = \begin{cases} 0 & \text{if } x_{ij} = 0 \text{ and } \partial l(\mathbf{p},\mathbf{x},\mathbf{y})/\partial x_{ij} > 0 \\ -b_{ij} \partial l(\mathbf{p},\mathbf{x},\mathbf{y})/\partial x_{ij} & \text{otherwise} \end{cases}$$

$$\dot{y}_i = \begin{cases} 0 & \text{if } y_i = 0 \text{ and } \partial l(\mathbf{p},\mathbf{x},\mathbf{y})/\partial y_i > 0 \\ 0 & \text{if } y_i = \bar{y}_i \text{ and } \partial l(\mathbf{p},\mathbf{x},\mathbf{y})/\partial y_i < 0 \\ -c_i \partial l(\mathbf{p},\mathbf{x},\mathbf{y})/\partial y_i & \text{otherwise} \end{cases}$$

The following theorem shows that the above process converges upon a saddle point of $l(\mathbf{p},\mathbf{x},\mathbf{y})$ if this function is strictly concave and differentiable in $\mathbf{p}$, and if the process is initiated at some $\mathbf{y} \leq \bar{\mathbf{y}}$. Again, $l(\mathbf{p},\mathbf{x},\mathbf{y})$ is strictly concave in $\mathbf{p}$ if the excess demands have negative slopes. Since the saddle points are one-to-one with the equilibrium points under the concavity and differentiability assumptions, then it may be concluded that the model is indeed stable. The theorem is stated in general terms rather than in terms of the problem above:

4.3.1 Theorem: Let $l(\mathbf{x},\mathbf{y},\mathbf{z}):R^n \oplus R^m \oplus R^l \to R^1$ be linear in $\mathbf{y}$ and $\mathbf{z}$, and strictly concave and differentiable in $\mathbf{x}$ over all $\mathbf{x} \in R^n_+$. Moreover, let $(\bar{\mathbf{x}},\bar{\mathbf{y}},\bar{\mathbf{z}}) \in R^n_+ \oplus R^m_+ \oplus R^l_+$ be a saddle point of $l(\mathbf{x},\mathbf{y},\mathbf{z})$. Consider the system:

$$\dot{x}_i = \begin{cases} 0 & \text{if } x_i = 0 \text{ and } \partial l(\mathbf{x},\mathbf{y},\mathbf{z})/\partial x_i < 0 \\ a_i \partial l(\mathbf{x},\mathbf{y},\mathbf{z})/\partial x_i & \text{otherwise} \end{cases} \tag{4.20}$$

$$\dot{y}_i = \begin{cases} 0 & \text{if } y_i = 0 \text{ and } \partial l(\mathbf{x},\mathbf{y},\mathbf{z})/\partial y_i > 0 \\ -b_i \partial l(\mathbf{x},\mathbf{y},\mathbf{z})/\partial y_i & \text{otherwise} \end{cases} \tag{4.21}$$

$$\dot{z}_i = \begin{cases} 0 & \text{if } z_i = 0 \text{ and } \partial l(\mathbf{x},\mathbf{y},\mathbf{z})/\partial z_i > 0 \\ 0 & \text{if } z_i = \bar{z}_i \text{ and } \partial l(\mathbf{x},\mathbf{y},\mathbf{z})/\partial z_i < 0 \\ -c_i \partial l(\mathbf{x},\mathbf{y},\mathbf{z})/\partial z_i & \text{otherwise} \end{cases} \tag{4.22}$$

where the a_i, b_i, and c_i are positive constants. From any intial t_0 and $(\mathbf{x}_0,\mathbf{y}_0,\mathbf{z}_0) \in R^n_+ \oplus R^m_+ \oplus R^l_+$ such that $\mathbf{z}_0 \leq \bar{\mathbf{z}}$, $(\mathbf{x},\mathbf{y},\mathbf{z})$ satisfies $\lim_{t\to\infty} [\mathbf{x}(t),\mathbf{y}(t),\mathbf{z}(t)] = (\bar{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})$ where $(\bar{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})$ is a saddle point of $l(\mathbf{x},\mathbf{y},\mathbf{z})$ on $R^n_+ \oplus R^m_+ \oplus R^l_+$.

Proof: As $l(\mathbf{x},\mathbf{y},\mathbf{z})$ is strictly concave in $\mathbf{x}$ and linear in $\mathbf{y}$ and $\mathbf{z}$, then with use of the corollary to Theorem 2.4.8, it may be confirmed that:

$$l(\bar{\mathbf{x}},\mathbf{y},\mathbf{z}) < l(\mathbf{x},\mathbf{y},\mathbf{z}) + l'_{\mathbf{x}}(\mathbf{x},\mathbf{y},\mathbf{z})(\bar{\mathbf{x}} - \mathbf{x}); \qquad \mathbf{x} \neq \bar{\mathbf{x}}$$

$$l(\mathbf{x},\bar{\mathbf{y}},\bar{\mathbf{z}}) = l(\mathbf{x},\mathbf{y},\mathbf{z}) + l'_{\mathbf{y}}(\mathbf{x},\mathbf{y},\mathbf{z})(\bar{\mathbf{y}} - \mathbf{y}) + l'_{\mathbf{z}}(\mathbf{x},\mathbf{y},\mathbf{z})$$
$$(\bar{\mathbf{z}} - \mathbf{z})$$

for every $(\mathbf{x},\mathbf{y},\mathbf{z})$ such that $\mathbf{x} \in R^n_+$. As $(\bar{\mathbf{x}},\bar{\mathbf{y}},\bar{\mathbf{z}})$ is a saddle point of $l(\mathbf{x},\mathbf{y},\mathbf{z})$), then:

$$l(\mathbf{x},\bar{\mathbf{y}},\bar{\mathbf{z}}) \leq l(\bar{\mathbf{x}},\bar{\mathbf{y}},\bar{\mathbf{z}}) \leq l(\bar{\mathbf{x}},\mathbf{y},\mathbf{z})$$

for every $(\mathbf{x},\mathbf{y},\mathbf{z}) \in R^n_+ \oplus R^m_+ \oplus R^l_+$. The last three relations imply:

$$(\bar{\mathbf{x}} - \mathbf{x})' l_{\mathbf{x}} - (\bar{\mathbf{y}} - \mathbf{y})' l_{\mathbf{y}} - (\bar{\mathbf{z}} - \mathbf{z})' l_{\mathbf{z}} \geq 0;$$
$$\text{if } \mathbf{x} \neq \bar{\mathbf{x}}, \text{ then } > 0 \qquad (4.23)$$

Now, let:

$$\mathbf{A} = \begin{bmatrix} a_1 & 0 & . & . & 0 \\ 0 & a_2 & 0 & . & 0 \\ . & 0 & . & & . \\ . & . & & . & 0 \\ 0 & 0 & . & 0 & a_n \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b_1 & 0 & . & . & 0 \\ 0 & b_2 & 0 & . & 0 \\ . & 0 & . & & . \\ . & . & & . & 0 \\ 0 & 0 & . & 0 & b_m \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} c_1 & 0 & . & . & 0 \\ 0 & c_2 & 0 & . & 0 \\ . & 0 & . & & . \\ . & . & & . & 0 \\ 0 & 0 & . & 0 & c_l \end{bmatrix}$$

$$\delta_x^i = \begin{cases} 1 & \text{if } x_i = 0 \text{ and } \partial l/\partial x_i < 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\delta_y^i = \begin{cases} 1 & \text{if } y_i = 0 \text{ and } \partial l/\partial y_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\delta_z^i = \begin{cases} 1 & \text{if } z_i = 0 \text{ and } \partial l/\partial z_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\bar{\delta}_z^i = \begin{cases} 1 & \text{if } z_i = \bar{z}_i \text{ and } \partial l/\partial z_i < 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{\Delta_x} = \begin{bmatrix} \delta_x^1 & 0 & . & . & 0 \\ 0 & \delta_x^2 & 0 & . & 0 \\ . & 0 & . & & . \\ . & . & & . & 0 \\ 0 & 0 & . & 0 & \delta_x^n \end{bmatrix} \qquad \mathbf{\Delta_y} = \begin{bmatrix} \delta_y^1 & 0 & . & . & 0 \\ 0 & \delta_y^2 & 0 & . & 0 \\ . & 0 & . & & . \\ . & . & & . & 0 \\ 0 & 0 & . & 0 & \delta_y^m \end{bmatrix}$$

$$\mathbf{\Delta_z} = \begin{bmatrix} \delta_z^1 & 0 & . & . & 0 \\ 0 & \delta_z^2 & 0 & . & 0 \\ . & 0 & . & & . \\ . & . & & . & 0 \\ 0 & 0 & . & 0 & \delta_z^l \end{bmatrix} \qquad \mathbf{\Delta_z} = \begin{bmatrix} \bar{\delta}_z^1 & 0 & . & . & 0 \\ 0 & \bar{\delta}_z^2 & 0 & . & 0 \\ . & 0 & . & & . \\ . & . & & . & 0 \\ 0 & 0 & . & 0 & \bar{\delta}_z^l \end{bmatrix}$$

Finally, let:

$$D(\mathbf{x},\mathbf{y},\mathbf{z}) = 1/2[(\mathbf{x} - \bar{\mathbf{x}})'\mathbf{A}^{-1}(\mathbf{x} - \bar{\mathbf{x}}) + (\mathbf{y} - \bar{\mathbf{y}})'\mathbf{B}^{-1}(\mathbf{y} - \bar{\mathbf{y}}) + (\mathbf{z} - \bar{\mathbf{z}})'\mathbf{C}^{-1}(\mathbf{z} - \bar{\mathbf{z}})]$$

Note that $\mathbf{A}^{-1}$, $\mathbf{B}^{-1}$, and $\mathbf{C}^{-1}$ are positive definite and symmetric. Also note that the system in (4.20) through (4.22) can be written in terms of the matrices above as:

$$\dot{\mathbf{x}} = \mathbf{A}[I - \Delta_{\mathbf{x}}]\mathbf{1}_{\mathbf{x}}$$
$$\dot{\mathbf{y}} = -\mathbf{B}[I - \Delta_{\mathbf{y}}]\mathbf{1}_{\mathbf{y}}$$
$$\dot{\mathbf{z}} = -\mathbf{C}[I - \Delta_{\mathbf{z}} - \Delta_{\mathbf{z}}]\mathbf{1}_{\mathbf{z}}$$

Consider:

$$\dot{D} = (\mathbf{x} - \bar{\mathbf{x}})'\mathbf{A}^{-1}\,\dot{\mathbf{x}} + (\mathbf{y} - \bar{\mathbf{y}})'\mathbf{B}^{-1}\,\dot{\mathbf{y}} + (\mathbf{z} - \bar{\mathbf{z}})'\mathbf{C}^{-1}\,\dot{\mathbf{z}}$$

Substitution of the former relations into the latter yields:

$$\dot{D} = (\mathbf{x} - \bar{\mathbf{x}})'\mathbf{1}_{\mathbf{x}} - (\mathbf{y} - \bar{\mathbf{y}})'\mathbf{1}_{\mathbf{y}} - (\mathbf{z} - \bar{\mathbf{z}})'\mathbf{1}_{\mathbf{z}} + \bar{\mathbf{x}}'\Delta_{\mathbf{x}}\mathbf{1}_{\mathbf{x}} - \bar{\mathbf{y}}'\Delta_{\mathbf{y}}\mathbf{1}_{\mathbf{y}} - \bar{\mathbf{z}}'\Delta_{\mathbf{z}}\mathbf{1}_{\mathbf{z}} \tag{4.24}$$

where the following substitutions have been used:

$$\mathbf{x}'\Delta_{\mathbf{x}} = 0$$

$$\mathbf{y}'\Delta_{\mathbf{y}} = 0$$

$$\mathbf{z}'\Delta_{\mathbf{z}} = 0$$

$$(\mathbf{z} - \bar{\mathbf{z}})'\Delta_{\mathbf{z}} = 0$$

Now, from the definitions of $\Delta_{\mathbf{x}}$, $\Delta_{\mathbf{y}}$, and $\Delta_{\mathbf{z}}$, and from (4.23), it may be concluded that $\dot{D} \leq 0$, and is strictly less than zero if $\mathbf{x} \neq \bar{\mathbf{x}}$. As $D \geq 0$, and as $\dot{D} \leq 0$, it follows that D must converge upon a limit, D^*, as $t \to \infty$. That is:

$$\lim_{t\to\infty} D = D^*$$

Therefore, $(\mathbf{x},\mathbf{y},\mathbf{z})$ must converge to a limit cycle, $[\hat{\mathbf{x}}(r),\hat{\mathbf{y}}(r),\hat{\mathbf{z}}(r)]$, satisfying $D[\hat{\mathbf{x}}(r),\hat{\mathbf{y}}(r),\hat{\mathbf{z}}(r)] = D^*$ for all r. As $\dot{D} = 0$ at all points on the limit cycle, then necessarily $\hat{\mathbf{x}}(r) = \bar{\mathbf{x}}$.

Hence, it is known that $\mathbf{x}$ does converge to $\bar{\mathbf{x}}$. It remains to show that $\mathbf{y}$ and $\mathbf{z}$ converge to $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ such that $(\bar{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})$ is a saddle point of $l(\mathbf{x},\mathbf{y},\mathbf{z})$. It will be shown that every point on the limit cycle, $[\bar{\mathbf{x}},\hat{\mathbf{y}}(r),\hat{\mathbf{z}}(r)]$, is a saddle point of $l(\mathbf{x},\mathbf{y},\mathbf{z})$; consequently, as saddle points are equilibrium points, the limit cycle must in fact consist of only one point.

At all points on the limit cycle, $\dot{\mathbf{x}} = 0$; consequently, it is known from the definition of $\dot{\mathbf{x}}$ in (4.20) that for all r, $[\bar{\mathbf{x}},\hat{\mathbf{y}}(r),\hat{\mathbf{z}}(r)]$ must satisfy:

$$l_{\mathbf{x}}[\bar{\mathbf{x}},\hat{\mathbf{y}}(r),\hat{\mathbf{z}}(r)] \leq 0; \qquad \bar{\mathbf{x}}' l_{\mathbf{x}}[\bar{\mathbf{x}},\hat{\mathbf{y}}(r),\hat{\mathbf{z}}(r)] = 0$$

Moreover, as $\dot{D} = 0$ on the limit cycle, it is known from (4.24) that for all r, $[\bar{\mathbf{x}},\hat{\mathbf{y}}(r),\hat{\mathbf{z}}(r)]$ must satisfy:

$$-(\mathbf{y} - \bar{\mathbf{y}})' l_{\mathbf{y}} - (\mathbf{z} - \bar{\mathbf{z}})' l_{\mathbf{z}} - \bar{\mathbf{y}}'\Delta_{\mathbf{y}} l_{\mathbf{y}} - \bar{\mathbf{z}}'\Delta_{\mathbf{z}} l_{\mathbf{z}} = 0$$

However, as $l(\mathbf{x},\mathbf{y},\mathbf{z})$ is linear in $\mathbf{y}$ and $\mathbf{z}$, it follows that $l_{\mathbf{y}}$ and $l_{\mathbf{z}}$ are functions of $\mathbf{x}$ only; consequently, on the limit cycle, the above becomes:

$$-(\mathbf{y} - \bar{\mathbf{y}})' l_{\mathbf{y}}(\bar{\mathbf{x}}) - (\mathbf{z} - \bar{\mathbf{z}})' l_{\mathbf{z}}(\bar{\mathbf{x}}) - \bar{\mathbf{y}}' \Delta_{\mathbf{y}} l_{\mathbf{y}}(\bar{\mathbf{x}}) - \bar{\mathbf{z}}' \Delta_{\mathbf{z}} l_{\mathbf{z}}(\bar{\mathbf{x}}) = 0$$

By assumption, $\bar{\mathbf{y}}$ is complementary to $l_{\mathbf{y}}(\bar{\mathbf{x}})$, and $\bar{\mathbf{z}}$ is complementary to $l_{\mathbf{z}}(\bar{\mathbf{x}})$, so this reduces to:

$$-\hat{\mathbf{y}}'(r) l_{\mathbf{y}}(\bar{\mathbf{x}}) - \hat{\mathbf{z}}'(r) l_{\mathbf{z}}(\bar{\mathbf{x}}) = 0$$

Also, by assumption, $l_{\mathbf{y}}(\bar{\mathbf{x}}) \geq \mathbf{0}$ and $l_{\mathbf{z}}(\bar{\mathbf{x}}) \geq \mathbf{0}$. As the adjustment process requires $\hat{\mathbf{y}}(r) \geq \mathbf{0}$ and $\hat{\mathbf{z}}(r) \geq \mathbf{0}$, then the latter equation can hold only if both terms are equal to zero.

Thus, summarizing, it may be said that $[\bar{\mathbf{x}},\hat{\mathbf{y}}(r),\hat{\mathbf{z}}(r)]$ satisfies:

$$l_{\mathbf{x}}[\bar{\mathbf{x}},\hat{\mathbf{y}}(r),\hat{\mathbf{z}}(r)] \leq \mathbf{0}; \qquad \bar{\mathbf{x}}' l_{\mathbf{x}}[\bar{\mathbf{x}},\hat{\mathbf{y}}(r),\hat{\mathbf{z}}(r)] = 0$$
$$l_{\mathbf{y}}(\bar{\mathbf{x}}) \geq \mathbf{0}; \qquad \hat{\mathbf{y}}'(r) l_{\mathbf{y}}(\bar{\mathbf{x}}) = 0$$
$$l_{\mathbf{z}}(\bar{\mathbf{x}}) \geq \mathbf{0}; \qquad \hat{\mathbf{z}}'(r) l_{\mathbf{y}}(\bar{\mathbf{x}}) = 0$$
$$[\bar{\mathbf{x}},\hat{\mathbf{y}}(r),\hat{\mathbf{z}}(r)] \in R^n_+ \oplus R^m_+ \oplus R^l_+$$

But, these are precisely the Kuhn-Tucker conditions for saddle points in $l(\mathbf{x},\mathbf{y},\mathbf{z})$. It follows that the limit cycle must consist of the single point, $(\bar{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})$, that is also a saddle point of $l(\mathbf{x},\mathbf{y},\mathbf{z})$ on $R^n_+ \oplus R^m_+ \oplus R^l_+$.

Thus, if either $\bar{\mathbf{p}} > \mathbf{0}$ or the commodity is desirable in every regional market, then the discriminator can prohibit all equilibria other than those involving $\bar{\mathbf{p}}$ and $\bar{\mathbf{y}}$ by fixing its offer prices at $\bar{\mathbf{p}}$ and by imposing a quota requiring exports to be less than or equal to $\bar{\mathbf{y}}$. Moreover, if such policy is enforced, and if the excess demands all have negative slopes, then the market will indeed equilibrate at some $(\bar{\mathbf{p}},\hat{\mathbf{x}},\bar{\mathbf{y}})$.

CHAPTER V

SOLVING THE SPATIAL EQUILIBRIUM AND PRICE DISCRIMINATION PROBLEMS

In this chapter, the determination of solutions for spatial equilibrium problems and price discrimination problems is considered. The first section contains a general discussion of conventional nonlinear programming algorithms. Also, a specific algorithm capable of solving all nonlinear programming problems of the previous chapters is presented and validated. In the second section, several hypothetical spatial equilibrium problems and price discrimination problems are constructed and solved.

5.1 Solution Algorithms

Numerous algorithms have been designed for solving nonlinear programming problems. As might be expected, each algorithm requires its own set of assumptions, and the relative performances of the various algorithms will depend upon the general structure of the problem to be solved. Common assumptions upon which many algorithms depend are: 1) a quadratic objective function, 2) a concave objective function, 3) linear constraints, 4) concave constraints, and/or 5) constraints with linearly independent gradients.

Also, nearly all algorithms will require differentiability in both the objective function and the constraints. Oftentimes, constraint characteristics such as concavity, linear independence in the gradients, and differentiability are required only of those constraints that become active in the course of the algorithmic process.

The spatial equilibrium problem will satisfy most the criteria above. It will be recalled that the quantity-dependent, partial equilibrium variant of this problem was:

$$\begin{aligned} \text{maximize}(\mathbf{p}): \quad & f(\mathbf{p}) \\ \text{subject to:} \quad & \mathbf{t} - \mathbf{N}'\mathbf{p} \geq \mathbf{0} \\ & \mathbf{p} \geq \mathbf{0} \end{aligned}$$

$$\text{where:} \quad f(\mathbf{p}) = \Sigma_{i=1}^{n} \int e_i(p_i)dp_i$$

The objective function is nearly always differentiable. The constraints are linear, and the objective function will nearly always be concave if not strictly concave. Moreover, if the excess demands are linear, as is commonly the case, then the objective function will be quadratic.

As a consequence of these characteristics, the spatial equilibrium problem can usually be solved with any one of a large number of algorithms. One possible difficulty with this problem is that the constraints are not linearly independent; however, under usual circimstances, there will be few if any linearly independent combinations of constraints that can be feasibly active at the same time.

Observe that the same conclusions generally hold for the price-dependent problem and the general equilibrium problems also.

If the excess demands are linear, then the simplex-based quadratic programming routines will probably be among the best algorithms for solving the spatial equilibrium problem. Such routines include the well-known Wolfe algorithm (1959), or Lemke's complementary pivoting algorithm (1968).[1] These algorithms virtually guarantee convergence to an exact solution in a finite number of iterations, whereas other algorithms are asymptotically convergent as a rule, and are much more subject to algorithmic break-down. The biggest difficulty with the simplex routines is that with increasing problem size, the reqiured computer storage grows at a rapid and increasing rate. Also, the simplex methods are subject to cumulative computational error, which will often necessitate the inclusion of a reinversion subroutine for large problems.

In the case of nonlinear excess demands, several algorithms could be used for the spatial equilibrium problem. Perhaps one of the best of these is the Zoutendijk algorithm (1960) discussed below. However, for extremely large problems, with either linear or nonlinear excess demands, the gradient method should be considered. This method is commonly discussed in nonlinear programming texts.

The primary difficulty with the price discrimination problems is the imposition of the complementary slackness constraints. These constraints should probably be handled with the penalty function technique, in which case, the problem becomes:

$$\text{maximize}(\hat{\mathbf{p}},\mathbf{p},\mathbf{x},\mathbf{y}): \quad \hat{\mathbf{p}}'\hat{\mathbf{D}}(\hat{\mathbf{p}}) + (\mathbf{C}'\mathbf{p} - \mathbf{r})'\mathbf{y} + \alpha[\mathbf{p}'\mathbf{E}(\mathbf{p}) - \mathbf{t}'\mathbf{x} - \mathbf{p}'\mathbf{C}\mathbf{y}]$$

$$\text{subject to:} \quad \mathbf{E}(\mathbf{p}) - \mathbf{N}\mathbf{x} - \mathbf{C}\mathbf{y} \leq 0$$

$$\mathbf{t} - \mathbf{N}'\mathbf{p} \geq 0$$

$$-\hat{\mathbf{E}}(\hat{\mathbf{p}}) - \mathbf{U}\mathbf{y} \geq 0$$

$$(\hat{\mathbf{p}},\mathbf{p},\mathbf{x},\mathbf{y}) \geq 0$$

The attributes of the problem largely hinge upon the required size of the penalty parameter, α. As shown in the previous chapter, when the penalty parameter is set to unity, the objective function collapses into a simple expression that is very possibly concave, and is definately concave if the excess demands are linear and with nonpositive slopes. Moreover, if the excess demands are linear, and if the penalty parameter is equal to one, then the constraints are also linear, and the objective function is quadratic.

However, the objective function is not concave (nor pseudoconcave) for any value of the penalty parameter other than unity. Consequently, if $\alpha > 1$ is required, then the Kuhn-Tucker conditions are no longer sufficient for the global optima. If the excess demands are nonlinear, then the constraints to the discrimination problem will also be nonlinear; moreover, there are no a-priori theoretical grounds for expecting the constraints to be concave. Also, the gradients of the constraints are not linearly independent, although it remains improbable that a feasible active combination of the constraints should have linearly dependent gradients.

Hence, the discrimination problem is potentially one having few desirable features. Indeed, if the excess demands are nonlinear, and if $\alpha > 1$ is required, then about the only desirable characteristic that the problem can possibly have is differentiability. Consequently, most of the conventional solution algorithms can handle this problem only over a very narrow range of scenarios.

The best possible scenario occurs when the excess demands are linear and $\alpha = 1$ is a sufficient penalty parameter. In this case, the problem reduces to a quadratic programming problem; consequently, the reliable Wolfe algorithm or Lemke algorithm may be used. Again, a primary advantage to these procedures is that an exact solution is rendered in a finite number of steps. However, these algorithms will require extremely large amounts of computer storage for large problems.

One of the most powerful solution algorithms is the one due to Zoutendijk (1960). The algorithm is capable of solving the spatial equilibrium problem and all variants of the price discrimination problem as well. The only major assumption necessary to qualify the procedure is differentiability in the objective function and the constraints. The algorithm is as follows:

Zoutendijk Algorithm

Problem: Maximize $f(\mathbf{x})$ subject to $\mathbf{G(x)} \geq \mathbf{0}$ and $\mathbf{x} \in R^n$, where $f(\mathbf{x}):R^n \rightarrow R^1$ and $\mathbf{G(x)}:R^n \rightarrow R^m$.

Assumptions: $f(\mathbf{x})$ and $\mathbf{G(x)}$ are differentiable over the feasible region, or $\{\mathbf{x} \in R^n: \mathbf{G(x)} \geq \mathbf{0}\}$.

Initialization Step: Choose $\mathbf{x}_1$ such that $\mathbf{G(x_1)} \geq \mathbf{0}$. Let $k = 1$ and go to main step.

Main Step

1) Let $I = \{i: g_i(\mathbf{x}_k) = 0\}$, and solve the following problem:

maximize$(\mathbf{d},z)$: z

subject to:

$$\mathbf{f}_\mathbf{x}'(\mathbf{x}_k)\mathbf{d} - z \geq 0 \tag{5.1}$$

$$\nabla \mathbf{g}_i'(\mathbf{x}_k)\mathbf{d} - z \geq 0; \qquad \forall\ i \in I \tag{5.2}$$

$$d_i \in [-1,1]; \qquad \forall\ i \tag{5.3}$$

Let $(\mathbf{d}_k, z_k)$ be an optimal solution. If $z_k = 0$, then stop; $\mathbf{x}_k$ is a Fritz John point. If $z_k > 0$, then go to step two.

2) Let λ_k be an optimal solution to the following problem:

maximize(λ_k): $f(\mathbf{x}_k + \lambda_k \mathbf{d}_k)$

subject to: $\lambda \in [0, \lambda^*]$

where $\lambda^* = \sup\{\lambda: g_i(\mathbf{x}_k + \lambda_k \mathbf{d}_k) \geq 0 \quad \forall\ i\}$. Next, let $\mathbf{x}_{k+1} = \mathbf{x}_k + \lambda_k \mathbf{d}_k$. Replace k with $k+1$ and go to step one.

Observe that the $\mathbf{x}_k$ are feasible at every iteration, and that $f(\mathbf{x}_k)$ is strictly increasing in k. Also, observe that if the optimal solution to the problem in step one has $z = 0$, then the following system has no solution:

$$\mathbf{f}_{\mathbf{x}}'(\mathbf{x}_k)\mathbf{d} > 0$$
$$\nabla \mathbf{g}_i'(\mathbf{x}_k)\mathbf{d} > 0; \qquad \forall\ i \in I$$

The fact that $\mathbf{x}_k$ is a Fritz John point then follows immediately from Gordan's theorem (Theorem 2.3.5).

Step one of the latter algorithm may be accomplished with an ordinary simplex routine. However, the problem must first be expressed in terms of nonnegative variables. This may be accomplished with a simple coordinate translation where $\mathbf{d}$ is replaced in (5.1) and (5.2) with $(\mathbf{d}^* - \mathbf{u})$, where $\mathbf{u} = (1,1,,,1)'$. Also, (5.3) is replaced with $\mathbf{0} \leq \mathbf{d}^* \leq 2\mathbf{u}$. One should solve the modified problem for $\mathbf{d}^*$, and then set $\mathbf{d} = \mathbf{d}^* - \mathbf{u}$.

The rate of convergence in the Zoutendijk algorithm can be slow if at certain iterations there are constraints that are nearly but not exactly binding. Such constraints force λ^* to be small, thus limiting the step length. Consequently, the algorithm might be improved by redefining the set of active constraints such that $I = \{i: g_i(\mathbf{x}_k) < \epsilon\}$ for some small $\epsilon > 0$. The generated $\mathbf{d}_k$ will then allow greater step lengths. As a practical matter, this must be done anyway since computational errors will generally prevent exact equality even where it should occur.

If the constraints are linear, then the Zoutendijk algorithm may be modified such that termination occurs at a Kuhn-Tucker point. The modified algorithm is as follows:

Zoutendijk Algorithm (Linear Constraints)

Problem: Maximize $f(\mathbf{x})$ subject to $\mathbf{b} - \mathbf{A}\mathbf{x} \geq \mathbf{0}$ and $\mathbf{x} \in R^n$ where $f(\mathbf{x}):R^n \rightarrow R^1$, $\mathbf{b} \in R^m$, and $\mathbf{A} \in R^{m \times n}$.

Assumptions: $f(\mathbf{x})$ is differentiable over the feasible region, or $\{\mathbf{x} \in R^n: \mathbf{b} - \mathbf{A}\mathbf{x} \geq \mathbf{0}\}$.

Initialization Step: Choose $\mathbf{x}_1$ such that $\mathbf{b} - \mathbf{A}\mathbf{x}_1 \geq \mathbf{0}$. Let $k = 1$ and go to main step.

Main Step

1) For given $\mathbf{x}_k$, partition $\mathbf{b}$ and $\mathbf{A}$ into $(\mathbf{b}_1', \mathbf{b}_2')$ and $(\mathbf{A}_1', \mathbf{A}_2')$ such that $\mathbf{A}_1 \mathbf{x}_k = \mathbf{b}_1$ and $\mathbf{A}_2 \mathbf{x}_k < \mathbf{b}_2$. Solve the problem:

$$
\begin{array}{ll}
\text{maximize}(\mathbf{d}): & \mathbf{f}_{\mathbf{x}}'(\mathbf{x}_k)\mathbf{d} \\
\text{subject to:} & \mathbf{A}_1 \mathbf{d} \leq \mathbf{0} \\
 & d_i \in [-1,1]; \quad \forall\ i
\end{array}
$$

If the optimal value of the above problem is zero, then stop; $\mathbf{x}_k$ is a Kuhn-Tucker point. Otherwise, let $\mathbf{d}_k$ denote the optimal solution and go to step two.

2) Let λ_k be an optimal solution to the following problem:

$$
\begin{array}{ll}
\text{maximize}(\lambda_k): & f(\mathbf{x}_k + \lambda_k \mathbf{d}_k) \\
\text{subject to:} & \lambda_k \in [0, \lambda^*]
\end{array}
$$

where $\lambda^* = \sup\{\lambda: \mathbf{A}_1(\mathbf{x}_k + \lambda\mathbf{d}_k) \leq \mathbf{0}\}$. Let $\mathbf{x}_{k+1} = \mathbf{x}_k + \lambda_k\mathbf{d}_k$. Replace k with k+1 and go to step one.

As before, the generated sequence, $\{\mathbf{x}_k\}$, is feasible for every k, and $f(\mathbf{x}_k)$ is strictly increasing in k. Note that if the optimal value to the problem in step one is zero, then the following system has no solution:

$$\mathbf{f}_x'(\mathbf{x}_k)\mathbf{d} > 0$$

$$\mathbf{A}_1\mathbf{d} \leq \mathbf{0}; \qquad \forall\ i \in I$$

The fact that $\mathbf{x}_k$ is a Kuhn-Tucker point then follows from Farkas' theorem (Theorem 2.3.4).

The Zoutendijk algorithms are not only versatile, but can also solve the spatial equilibrium and price discrimination problems with less computer storage than most other routines. This follows from the fact that the algorithm employs only active constraints, which are apt to be relatively few in number, and the fact that the binding constraints to these problems can be easily assembled into computer-usable matrices as the algorithm proceeds. That is, it is not necessary to store the entire system of constraints in matrix format, but rather, the constraints can be constructed from the basic data as they become active, and can then be discarded upon becoming inactive.

Convergence in the Zoutendijk algorithms is not guaranteed. The algorithm is subject to "jamming," which is a phenomenon where the generated step lengths tend towards zero as a nonoptimal point is approached. Problems for which the algorithm does not converge have been contrived by Wolfe (1972); however, such counterexamples are difficult to construct.[2]

Finally, note that if the complementary slackness conditions are imposed with a constraint, then the model becomes:

$$
\begin{array}{ll}
\text{maximize}(\hat{\mathbf{p}},\mathbf{p},\mathbf{x},\mathbf{y}): & \hat{\mathbf{p}}'\hat{\mathbf{D}}(\hat{\mathbf{p}}) + (\mathbf{C}'\mathbf{p} - \mathbf{r})'\mathbf{y} \\
\text{subject to:} & \mathbf{E}(\mathbf{p}) - \mathbf{N}\mathbf{x} - \mathbf{C}\mathbf{y} \leq 0 \\
 & \mathbf{t} - \mathbf{N}'\mathbf{p} \geq 0 \\
 & -\hat{\mathbf{E}}(\hat{\mathbf{p}}) - \mathbf{U}\mathbf{y} \geq 0 \\
 & \mathbf{p}'\mathbf{E}(\mathbf{p}) - \mathbf{t}'\mathbf{x} - \mathbf{p}'\mathbf{C}\mathbf{y} \geq 0 \\
 & (\hat{\mathbf{p}},\mathbf{p},\mathbf{x},\mathbf{y}) \geq 0
\end{array}
$$

The objective function of this model can never be quadratic, or concave, nor can the constraints be linear or concave. Consequently, the Zoutendijk algorithm or an algorithm of equal flexibility must be used to solve the problem.

5.2 Example Problems

In this section, several spatial equilibrium and price discrimination problems are constructed and solved. Linear excess demands are assumed; consequently, in all cases but one, Lemke's complementary pivoting algorithm is used to find the solutions. However, a case is considered where a penalty parameter greater than unity is needed in a price discrimination problem. Here, the Zoutendijk method is employed.

Consider the spatial market described by the excess demand functions and transportation cost matrix in Table 5.1, which is situated at the end of the chapter. The market consists of seven regions. Assuming that the transportation charge per unit of distance is the same for all shipment routes, the market map appears as follows:

(1).		.(4)
(2).	(7) .	.(5)
(3).		.(6)

A set of spatial equilibrium prices and trade flows for this market are recorded in Table 5.2. Observe that regions three, five, and seven are the exporters at the equilibrium. Region seven is the largest exporter, having almost 50 percent of the interregional trade market.

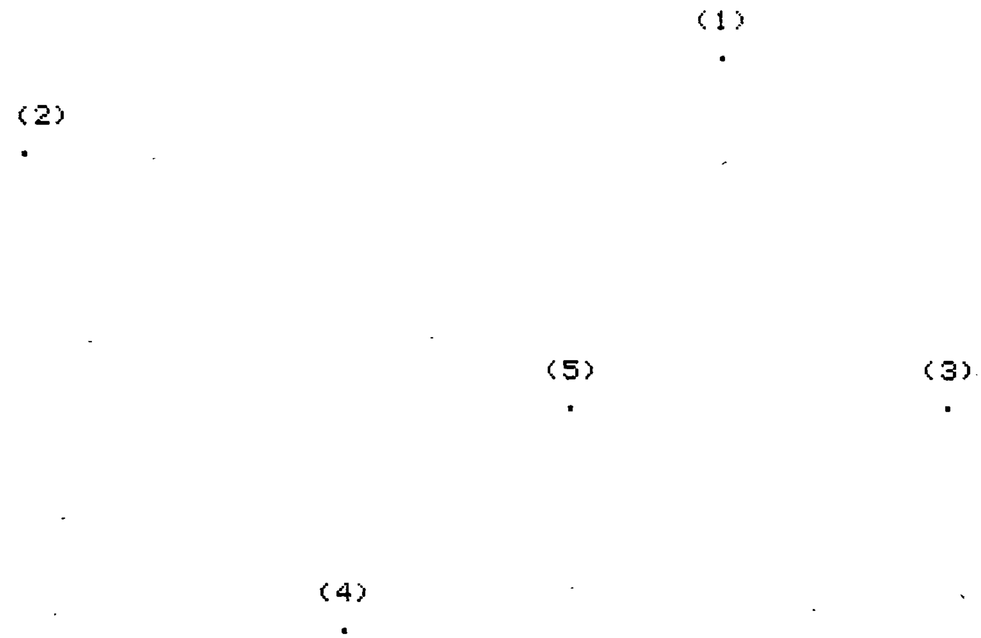
Suppose that region seven adopts a price discrimination policy wherein it seeks to maximize net export revenue. Observe that region seven's excess demand function is simply $e_7(p_7) = -100$. It is assumed that $d_7(p_7) = 0$. Therefore, the discrimination model maximizes export revenue to 100 exportable units. The optimal price and quantity vectors are recorded in Table 5.3. By comparing this table with Table 5.2, it may be seen that the discrimination policy increases region seven's net export revenue by 151.07, which represents a 3.57 percent improvement over ordinary spatial equilibrium. Comparisons between the spatial and discrimination equilibria are also recorded in Table 5.5.

As a group, the three markets to the right of the discriminator are more elastic than the three markets on the left by virtue of the steeper slope of region five's excess demand. Hence, the discriminator would be expected to lower prices to the markets on its right and raise prices to the markets on its left. From Table 5.3, it may be seen that this is precisely the case. Prices on the right are uniformly lower than under ordinary spatial equilibrium, and prices on the left are uniformly higher.

Now, suppose that region seven and region five exercise discrimination cooperatively such that net export revenue to the cooperative is maximized. It is assumed that the cooperative agreement is such that region five's potential export volumn is fixed at 74.97 units, which was the quantity exported by region five in the former model. Thus, the demands and excess demands for the cooperative are: $d_5(p_5) = 0$, $e_5(p_5) = -74.97$, $d_7(p_7) = 0$, and $e_7(p_7) = -100$. The optimal price and quantity vectors for this scheme are recorded in Table 5.4.

A comparison between the cooperative policy and the previously considered policies is made in Table 5.5. Note that the net export revenue of region seven is lower under the cooperative agreement than when it practices discrimination independently. Of course, the regions in the cooperative could enter into a revenue sharing agreement wherein net revenues would not necessarily equal direct receipts. However, observe that if region five were to compensate region seven so as to restore its net export revenue to the level earned under independent discrimination, then region five's net export revenue would be reduced to 3471.86. This value is less than region five's earnings in the ordinary spatial equilibrium model, but greater than its earnings when region seven independently discriminates. This illustrates that an independent discriminator might not be advantaged by a cooperative agreement if such agreement has no revenue sharing provision. It also illustrates that a region in the market might be willing to join a cooperative, but not to increase its net revenue over the ordinary spatial equilibrium level, but to avoid the detrimental impacts of the other discriminators upon its own trade.

In both of the discrimination models above, a penalty parameter equal to unity is sufficient to drive the penalty function to zero. Apparently, it is difficult to construct a hypothetical market for which this is not the case. After considerable but fruitless effort to systematically construct a counterexample, a randomizing algorithm was written, and market configurations were randomly generated until a counterexample was found. The resulting market consists of five regions and is discribed by the excess demands and transportation cost matrix in Table 5.6. Assuming that the transportation charge per unit of distance is the same along all routes, the market map appears as follows:



A spatial equilibrium solution for this market is recorded in Table 5.7. Under the assumption that region five practices independent discrimination, the solution of the Lemke algorithm for $\alpha = 1$ is recorded in Table 5.8. Note that $e_5(p_5) = -1250$. Region five's demand function is set at $d_5(p_5) = 0$; hence, the model attempts to maximize region five's net export revenue. Observe that the penalty function is equal to −162.93 for this solution; consequently, α must be increased to attain feasibility. Upon setting $\alpha = 10$, and solving the model with the Zoutendijk algorithm, the solution recorded in Table 5.9 is obtained after 32 iterations. The discrimination policy increases region five's net export revenue by 5857.52 over the ordinary spatial equilibrium level, which represents a 14.63 percent improvement.

TABLE 5.1

SPECIFICATIONS FOR MARKET I

Excess Demand Functions

region	intercept	slope
1	100.00	-1.00
2	100.00	-1.00
3	10.00	-1.00
4	100.00	-1.00
5	10.00	-2.00
6	100.00	-1.00
7	-100.00	0.00

Transportation Cost Matrix

region	1	2	3	4	5	6	7
1	0.00	3.00	6.00	8.00	8.54	10.00	5.00
2	3.00	0.00	3.00	8.54	8.00	8.54	4.00
3	6.00	3.00	0.00	10.00	8.54	8.00	5.00
4	8.00	8.54	10.00	0.00	3.00	6.00	5.00
5	8.54	8.00	8.54	3.00	0.00	3.00	4.00
6	10.00	8.54	8.00	6.00	3.00	0.00	5.00
7	5.00	4.00	5.00	5.00	4.00	5.00	0.00

TABLE 5.2

SPATIAL EQUILIBRIUM (MARKET I)

Trade Matrix

region	1	2	3	4	5	6	7	exports
1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
3	0.00	33.29	0.00	0.00	0.00	0.00	0.00	33.26
4	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
5	0.00	0.00	0.00	25.86	0.00	52.71	0.00	78.59
6	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
7	52.71	20.43	0.00	26.86	0.00	0.00	0.00	100.00
imports:	52.71	53.71	0.00	52.71	0.00	52.71	0.00	

Prices, Net Imports, and Net Revenue

region	price	net imports	net revenue
1	47.28571	52.71429	-2492.63
2	46.28571	53.71429	-2486.20
3	43.28571	-33.28571	1440.80
4	47.28571	52.71429	-2492.63
5	44.28571	-78.57143	3479.59
6	47.28571	52.71429	-2492.63
7	42.28571	-100.00000	4228.57

TABLE 5.3

EQUILIBRIUM UNDER DISCRIMINATION BY REGION SEVEN (MARKET I)

Trade Matrix

region	1	2	3	4	5	6	7	exports
1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
3	0.00	35.02	0.00	0.00	0.00	0.00	0.00	35.02
4	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
5	0.00	0.00	0.00	54.52	0.00	20.45	0.00	74.97
6	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
7	48.98	16.95	0.00	0.00	0.00	34.07	0.00	100.00
imports:	48.98	51.97	0.00	54.52	0.00	54.52	0.00	

Prices, Net Imports, and Net Revenue

region	price	net imports	net revenue
1	51.02286	48.97714	-2498.95
2	48.02286	51.97715	-2496.09
3	45.02286	-35.02286	1576.83
4	45.48286	54.51714	-2479.60
5	42.48286	-74.96571	3184.76
6	45.48286	54.51714	-2479.60
7	43.79637[a]	-100.00000	4379.64

[a]average net revenue

TABLE 5.4

EQUILIBRIUM UNDER JOINT DISCRIMINATION BY REGIONS FIVE AND SEVEN (MARKET I)

Trade Matrix

region	1	2	3	4	5	6	7	exports
1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
3	0.00	31.61	0.00	0.00	0.00	0.00	0.00	31.61
4	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
5	0.00	0.00	0.00	24.58	0.00	50.39	0.00	74.97
6	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
7	52.39	23.79	0.00	23.82	0.00	0.00	0.00	100.00
imports:	52.39	55.39	0.00	48.39	0.00	50.39	0.00	

Prices, Net Imports, and Net Revenue

region	price	net imports	net revenue
1	47.60600	48.97714	-2331.61
2	44.60600	51.97715	-2318.49
3	41.60600	-35.02286	1457.16
4	51.60600	54.51714	-2813.41
5	47.26162[a]	-74.97000	3543.20
6	49.60600	54.51714	-2704.38
7	43.08296[a]	-100.00000	4308.30

[a]average net revenue

TABLE 5.5

COMPARISON OF POLICIES

	spatial	reg. 7 disc.	coop disc.
Region Five			
exports	78.59	74.97	74.97
avg. net rev.	44.29	42.48	47.26
net revenue	3479.59	3184.76	3543.20
Region Seven			
exports	100.00	100.00	100.00
avg. net rev.	42.29	43.80	43.08
net revenue	4228.57	4379.64	4308.30
Cooperative			
exports	178.59	174.97	174.97
avg. net rev.	43.16	43.23	44.87
net revenue	7708.16	7564.40	7851.50

TABLE 5.6

SPECIFICATIONS FOR MARKET II

Excess Demand Functions

region	intercept	slope
1	633.38	-9.85
2	276.09	-8.76
3	961.26	-3.41
4	332.53	-5.82
5	-1250.00	0.00

Transportation Cost Matrix

region	1	2	3	4	5
1	0.00	6.87	5.06	7.83	4.66
2	6.87	0.00	8.98	6.39	6.26
3	5.06	8.98	0.00	5.59	2.88
4	7.83	6.39	5.59	0.00	3.39
5	4.66	6.26	2.88	3.39	0.00

TABLE 5.7

SPATIAL EQUILIBRIUM (MARKET II)

Trade Matrix

region	1	2	3	4	5	exports
1	0.00	0.00	0.00	0.00	0.00	0.00
2	0.00	0.00	0.00	0.00	0.00	0.00
3	0.00	0.00	0.00	0.00	0.00	0.00
4	0.00	0.00	0.00	0.00	0.00	0.00
5	271.98	0.00	851.64	126.38	0.00	1250.00
imports:	271.98	0.00	851.64	126.38	0.00	

Prices, Net Imports, and Net Revenue

region	price	net imports	net revenue
1	36.69062	271.97737	-9979.02
2	31.51712	0.00000	0.00
3	34.91062	851.64065	-29731.30
4	35.42062	126.38198	-4476.53
5	32.03062	-1250.00000	40038.28

TABLE 5.8

SOLUTION TO DISCRIMINATION PROBLEM WITH VIOLATED PENALTY FUNCTION (MARKET II)

Trade Matrix

region	1	2	3	4	5	exports
1	0.00	0.00	0.00	0.00	0.00	0.00
2	55.23	0.00	0.00	0.00	0.00	55.23
3	0.00	0.00	0.00	0.00	0.00	0.00
4	0.00	0.00	0.00	0.00	0.00	0.00
5	166.99	0.00	814.30	92.68	0.00	1073.03
imports:	222.22	0.00	814.30	92.68	0.00	

Prices, Net Imports, and Net Revenue

region	price	net imports	net revenue
1	41.74184	222.22285	-9275.99
2	37.82184	-55.22934	2088.88
3	46.80184	814.30221	-38110.84
4	41.21184	92.67708	-3819.39
5	42.33189[a]	-1073.97280	45463.30

Value of Penalty Function: -162.92656

[a]average net revenue

TABLE 5.9

EQUILIBRIUM UNDER DISCRIMINATION BY REGION FIVE (MARKET II)

Trade Matrix

region	1	2	3	4	5	exports
1	0.00	0.00	0.00	0.00	0.00	0.00
2	0.00	51.98	0.00	0.00	0.00	51.98
3	0.00	0.00	0.00	0.00	0.00	0.00
4	0.00	0.00	0.00	0.00	0.00	0.00
5	225.87	0.00	763.48	94.83	0.00	1084.19
imports:	225.87	0.00	763.48	94.83	0.00	

Prices, Net Imports, and Net Revenue

region	price	net imports	net revenue
1	41.37126	225.87310	-9344.65
2	37.45126	-51.98303	1946.83
3	46.43126	763.48282	-35449.47
4	40.84126	94.83387	-3873.13
5	42.33189[a]	-1084.18980	45895.80

[a]average net revenue

FOOTNOTES

[1]For a discussion of the Wolfe algorithm, see Bazaraa and Shetty (1979), Sposito (1975), or Martos (1975). The Lemke algorithm is thoroughly developed in Bazaraa and Shetty.

[2]For a more thorough treatment of the convergence properties of the Zoutendijk algorithm, or of algorithms in general, see Bazaraa and Shetty (1979) or Luenberger (1984).

CHAPTER VI

SUMMARY

Though price discrimination models are generally constructed under the assumptions of monopoly and perfect market separability, gainful price discrimination can be practiced when neither of these assumptions hold. In particular, a region operating in spatial market can typically increase its net revenue through a price discrimination scheme. The possibilities for successful price discrimination result from the fact that the regions in a spatial market are partially separated by the nonzero costs of transporting between the regions. Also, perfect separation may be achieved between the discriminator's own region and the other regions in the market. Consequently, gains can be had through discrimination within the export market and through discrimination between the export market and the domestic market.

The discriminator must consider the response of competition when determining its optimal price and quantity vectors. Improperly set prices may result in loss of the discriminator's market shares to competing producers in other regions, or may induce detrimental arbitrage of the discriminator's own supplies. Such possibilities are avoided by imposing certain constraints upon the selection of the price and quantity vectors. In particular, the choice set is confined to those price and quantity vectors that are consistent with spatial equilibrium. The discriminator is capable of a certain degree of control over the determination of spatial equilibrium points inasmuch as it can control its own exports and its own offer prices. Hence, the discriminator typically has an infinite number of spatial equilibrium points from which to choose. It then chooses the particular equilibria or equilibrium rendering maximal net revenue.

The optimal price and quantity vectors may be formulated as the solutions to a nonlinear programming problem. In particular, the discriminator's revenue function is maximized subject to constraints requiring that the chosen vectors be in accord with spatial equilibrium. Among the imposed spatial equilibrium conditions, there are certain complementary slackness conditions. These conditions may be imposed as constraints upon the problem; however, the constraints are nonlinear and nonconcave. A possibly better approach is to enforce such constraints with a penalty function. In either case, the resulting nonlinear programming problem is such that the Kuhn-Tucker conditions are necessary conditions for the optimal solutions in all but very unlikely circumstances. However, the Kuhn-Tucker conditions are sufficient for the optima only when the penalty function approach is taken, and when a penalty parameter equal to unity is sufficiently large to achieve feasibility.

The models described above are easily generalized to accomodate a group of discriminators operating in a cooperative fashion. In these models, it is assumed that the objective is the maximization of net revenue to the cooperative. The resulting programming models are of the same character as those pertaining to the single-discriminator cases. Indeed, the single-discriminator models are special cases of the cooperative models where the cooperative consists of only one region.

Under certain assumptions, the discriminator may influence the dynamic adjustment mechanism of the spatial market such that the market will in fact converge upon the chosen price and quantity vectors. These assumptions are that the excess demands are differentiable and have strictly negative slopes, and that either the commodity is desirable in every market, or the optimal prices are strictly greater than zero. If such is the case, then the discriminator may impose its chosen vectors upon the market simply by fixing its offer prices at the optimal prices, and by imposing an export quota such that the exports to any one region cannot exceed the optimal export level.

Unfortunately, the nonlinear programming problems do not necessarily possess many of the properties required by most conventional solution algorithms. As before noted, certain of the constraints are nonlinear and nonconcave. If such constraints are enforced with a penalty function, then the properties of the resulting problem depend upon the required size of the penalty parameter. If the excess demands are linear, and if a penalty parameter equal to unity is sufficient to achieve feasibility, then the problem reduces to a quadratic programming problem. If the excess demands are nonlinear, then the objective function is very possibly concave if the penalty parameter is equal to one. However, the objective function is neither quadratic nor concave (nor pseudoconcave) for any value of the penalty parameter other than unity. Also, if the excess demands are nonlinear, then the constraints will be nonlinear; moreover, there is little reason to expect the constraints to be concave.

However, the Zoutendijk optimization algorithm only requires that the objective function and constraints be differentiable. Consequently, this algorithm should handle the discrimination problems under most circumstances. Another attribute of the algorithm is that it requires less computer storage than most other routines. Also, simplex-based quadratic programming routines may be used in some cases, and probably should be used where possible. However, a possible difficulty with these routines is their large computer storage requirements.

There are several areas in which further theoretical research into the price discrimination problem is needed. In particular, the implications of retaliation against the discrimination scheme is in need of further study. In its present form, retaliatory trade policies can be incorporated into the model, and the optimal solutions under a given system of trade policies can then be found. However, the model does not anticipate the adjustments that might be made in such policies in response to the discriminator's actions. A more realistic model would be one in which mathematical provisions were made for the retaliatory behavior of importers and exporters in the market.

The possibilities of a more efficient solution algorithm for the discrimination problem need to be investigated. The conventional solution algorithms will require tremendous and possibly prohibitive quantities of computer storage for problems involving numerous regions. It is perhaps possible that a less demanding algorithm could be designed by exploiting the peculiar characteristics of the problem.

In this study, the comparative static properties of the discrimination models have been neglected. Since the constraints used in the models are not equality constraints, a comparative static analysis would require the assumption that the static adjustments in the exogenous variables would not alter the set of active constraints. With this assumption, a model lending itself to comparative static analysis could be constructed by discarding the inactive constraints and by treating the active constraints as equality constraints. In situations where this modified model would be reasonable, a comparative static analysis might be of interest.

The most useful applications of the model would occur in the formulation of international trade policy. For example, the international markets for several agricultural commodities are reasonably described by the assumptions of the model. Empirical studies investigating the possibilities of gains to the United States from a discriminatory policy in certain agricultural products appear to be warranted.

Much recent discussion has centered upon the possibilities of a cartel arrangement in the international wheat market involving the United States and other major wheat exporters. This market appears to be a prime example of one in which a discriminatory policy could prove gainful, particularly if such policy could be cooperatively exercised by several wheat exporters. The potential for gains from discrimination in the wheat market is furthered by the fact that it is likely that some countries would be willing to enter into an arrangement with a discriminator wherein it would be agreed that imports from the discriminator would not be arbitraged. Such arrangements would give the discriminator greater liberty to divert shipments from relatively inelastic markets. Since there is much question as to whether a wheat cartel could effectively control production, it is possible that greater gains could be reasonably expected from discriminatory pricing than from quantity control.

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APPENDIX

MISCELLANEOUS THEOREMS AND DEFINITIONS

Theorem (Schwartz Inequality): Let $\mathbf{x}_1, \mathbf{x}_2 \in R^n$, then

$$|\mathbf{x}_1' \mathbf{x}_2| \leq |\mathbf{x}_1| |\mathbf{x}_2|$$

Corollary (Triangular Inequality): Let $\mathbf{x}_1, \mathbf{x}_2 \in R^n$, then:

$$|\mathbf{x}_1 + \mathbf{x}_2| \leq |\mathbf{x}_1| + |\mathbf{x}_2|$$

Definition (Limit of a Function): Let X be a nonempty subset of R^n, and let $f(\mathbf{x}): X \to R^1$. $f(\mathbf{x})$ is said to approach the "limit," l, as $\mathbf{x}$ approaches $\mathbf{a}$, denoted as $\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = l$, if for every $\epsilon > 0$ there is a $\delta > 0$ such that:

$$|f(\mathbf{x}) - l| < \epsilon$$

for all $\mathbf{x} \in X$ satisfying:

$$|\mathbf{x} - \mathbf{a}| < \delta$$

If X is unbounded, then $f(\mathbf{x})$ is said to approach the limit, l, as $\mathbf{x}$ approaches infinity, denoted as $\lim_{\mathbf{x} \to \infty} f(\mathbf{x}) = l$, if for every $\epsilon > 0$ there is a $\delta > 0$ such that:

$$|f(\mathbf{x}) - l| < \epsilon$$

for all $\mathbf{x} \in X$ satisfying:

$|\mathbf{x}| > \delta$

Definition (Limit of a Sequence): A sequence, $\{\mathbf{x}_k\}$, is said to converge to the limit $\bar{\mathbf{x}}$, denoted as $\lim_{k\to\infty} \mathbf{x}_k = \bar{\mathbf{x}}$, if for every $\epsilon > 0$ there is an integer, K, such that:

$$|\mathbf{x}_k - \bar{\mathbf{x}}| < \epsilon; \qquad \forall\ k \geq K$$

Definition (Continuity): Let X be a nonempty subset of R^n, and let $f(\mathbf{x}):X \to R^1$. $f(\mathbf{x})$ is said to be "continuous" at $\mathbf{a} \in X$ if:

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$$

$f(\mathbf{x})$ is said to be continuous on X if it is continuous at every $\mathbf{a} \in X$.

Definition (Differentiability): Let X be a nonempty open subset of R^n, and let $f(\mathbf{x}):X \to R^1$. Denote the gradient of $f(\mathbf{x})$ by $\nabla \mathbf{f}(\mathbf{x})$. $f(\mathbf{x})$ is said to be "differentiable" at $\mathbf{a} \in X$ if its gradient exists at $\mathbf{a}$, and if there exists a function, $\omega(\mathbf{a},\mathbf{x}-\mathbf{a})$, satisfying

$$\lim_{\mathbf{x}\to\mathbf{a}} \omega(\mathbf{a},\mathbf{x}-\mathbf{a}) = 0$$

and:

$$f(\mathbf{x}) = f(\mathbf{a}) + \nabla \mathbf{f}'(\mathbf{a})(\mathbf{x} - \mathbf{a}) + |\mathbf{x} - \mathbf{a}|\omega(\mathbf{a},\mathbf{x}-\mathbf{a}); \qquad \forall\ \mathbf{x} \in X$$

Let $\mathbf{H}(\mathbf{x})$ denote the Hessian matrix of $f(\mathbf{x})$. $f(\mathbf{x})$ is said to be "twice differentiable" at $\mathbf{a} \in X$ if its gradient and Hessian matrix exist at $\mathbf{a}$, and if there exists a function, $\omega(\mathbf{a},\mathbf{x}-\mathbf{a})$, satisfying

$$\lim_{\mathbf{x}\to\mathbf{a}} \omega(\mathbf{a},\mathbf{x}-\mathbf{a}) = 0$$

and:

$$f(\mathbf{x}) = f(\mathbf{a}) + \nabla f'(\mathbf{a})(\mathbf{x} - \mathbf{a}) + (1/2)(\mathbf{x} - \mathbf{a})'H(\mathbf{a})(\mathbf{x} - \mathbf{a}) + |\mathbf{x} - \mathbf{a}|^2\omega(\mathbf{a},\mathbf{x}-\mathbf{a}); \qquad \forall\ \mathbf{x} \in X$$

$f(\mathbf{x})$ is said to be differentiable (twice differentiable) on X if it is differentiable (twice differentiable) at every $\mathbf{a} \in X$.

Theorem (Taylor's Theorem, Second Order): Let X be a nonempty open convex set in R^n, and let $f(\mathbf{x}):X \to R^1$ be twice differentiable on X. For every $\mathbf{x}_1,\mathbf{x}_2 \in X$:

$$f(\mathbf{x}_2) = f(\mathbf{x}_1) + \nabla f'(\mathbf{x}_1)(\mathbf{x}_2 - \mathbf{x}_1) + (1/2)(\mathbf{x}_2 - \mathbf{x}_1)'H(\bar{\mathbf{x}})(\mathbf{x}_2 - \mathbf{x}_1)$$

where $\bar{\mathbf{x}} = \alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2$ for some $\alpha \in (0,1)$.

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VITA

David Allan Pyles

Candidate for the Degree of

Doctor of Philosophy

Thesis: SPATIAL EQUILIBRIUM THEORY AND PRICE DISCRIMINATION IN THE SPATIAL MARKET

Major Field: Agricultural Economics

Biographical:

Personal Data: Born in Memphis, Tennessee; August 6, 1958; the son of Mr. and Mrs. W.A. Pyles.

Education: Graduated from Graham High School; Graham, Texas; May 1976. Received Bachelor of Science degree in Agricultural Economics at Texas Tech University in December 1980. Received Master of Science degree in Agricultural Economics from Oklahoma State University in December 1983. Completed requirements for Doctor of Philosophy degree in Agricultural Economics at Oklahoma State University in December 1986.

Professional Experience: Research Assistant and Associate at Texas Tech University, Department of Agricultural Economics; January 1979 through December 1980. Research Assistant at Oklahoma State University, Department of Agricultural Economics; January 1981 through August 1986.