BASIC CREDIT RISK ANALYSIS WITH LEVY PROCESSES AND NUMERICAL FFT METHOD

By

BIN XIE

Bachelor of Science in Mathematics
Hefei Normal University
Hefei, Anhui/China
2001

Master of Science in Applied Mathematics
Southwest Jiaotong University
Chengdu, Sichuan/China
2004

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Dissertation Approved:

Dr. Weiping Li

Dissertation Advisor

Dr. Jiahong Wu

Dr. Birne Binegar

Dr. Weiwei Hu

Dr. Timothy Krehbiel

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Name: Bin Xie

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Abstract:

Levy processes application is becoming a hot topic in financial modeling and empirical calibration on recent decades. Due to the infinite divisibility, independent and stationary increments properties, Levy processes match the market price dynamics intuitively.

In this thesis, some properties of Levy processes which outbreak the restricts of classic continuous Black-Scholes model with jumps are explored. Moreover, the explicit sensitivities for the bond price according to the log-normal distributed compound Poisson processes are deduced strictly. Meanwhile, the analytic illustrations are provided.

To find the inherent Levy processes evidences of the market, the S&P 500 index option prices are studied since those are the easiest and representative data source. Besides the classic Black-Scholes model, the Heston model is considered since its stochastic volatility embedding. Then non-iid models which violate the assumption of identically and independently distributed jumps are checked for next. Furthermore, Levy processes are discussed for the Partial Integro-Differential Equation (PIDE) question and numerical estimation by applying Fast Fourier Transform (FFT) algorithm. By investigating the typical three Levy processes: General Hyperbolic model (GH), Normal Inverse Gaussian model (NIG), Carr-Geman-Madan-Yor model (CGMY), numerical signs about the parameters sensitivities show up. The empirical indications and comparisons which reveal the more stable prediction of Levy processes are observed as well.

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CHAPTER 1

BLACK-SCHOLES-MERTON MODEL WITH POISSON JUMPS

In a financial market, the price variation over time of financial instruments can be studied by Black-Scholes-Merton(BSM) model basically. The BSM model is the fundamental tool to discover the mechanism of price movement.

1.1 Black-Scholes-Merton Model

The Black-Scholes-Merton model is a mathematical method in using Brownian motion and trying to describe the continuous case of price changing over time.

1.1.1 The BSM Model

Assume that we are in the setting of the standard Black-Scholes model like a perfect market. Stock price is assumed to follow a geometric Brownian motion:

$$S_t = S_0 exp((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t).$$

Here, W is a standard Brownian motion under the probability measure $\mathbf{P}$.

Let the starting value of assets is S_0. Then by Ito-Doeblin formula, we have:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t.$$

For the discount price $\tilde{S}(t) = e^{-rt}S(t)$, there is

$$\begin{aligned} d(\tilde{S}(t)) &= d(e^{-rt}S(t)) \\ &= -re^{-rt}S(t)dt + e^{-rt}dS(t) \\ &= \tilde{S}(t)((\mu - r)dt + \sigma dW_t. \end{aligned}$$

That means

$$\frac{d\tilde{S}(t)}{\tilde{S}(t)} = (\mu - r)dt + \sigma dW_t = \sigma(\frac{\mu - r}{\sigma}dt + dW_t).$$

To make sure we have a no arbitrage market, the return of investment should be the interest rate. There are two theorem we needed as Radon-Nikodym and Girsanov to change the measure .

Theorem 1.1.1 *[1] (Radon-Nikodym)* $\mathbf{Q}$ *is absolutely continuous with respect to* $\mathbf{P}$ *if and only if there exists a random variable* $Z \geq 0$ *such that for any subset* $A \in \mathcal{A}$*,*

$$\mathbf{Q}(A) = \int_A Z(\omega) d\mathbf{P}(\omega).$$

Z *is called density of* $\mathbf{Q}$ *with respect to* P*, and* $Z = \frac{d\mathbf{Q}}{d\mathbf{P}}$.

Theorem 1.1.2 *[1] (Girsanov) If* $\int_0^t \theta_s^2 ds < \infty$ *a.s. and* $Z_t = exp(-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds) > 0$ *is a martingale. Under* $\mathbf{Q}$ *with density* Z_T *with respect to* P, $\tilde{W}_t = W_t + \int_0^t \theta_s ds$ *is a standard Brownian motion.*

Let $\frac{\mu - r}{\sigma} = \theta$, due to the fact $\int_0^t \theta_s^2 ds = \theta^2 t < \infty$ and $L_t = exp(-\frac{1}{2}\theta^2 t - \theta W_t)$ is a martingale, then by theorem 1.1.2,

$$\frac{d\tilde{S}(t)}{\tilde{S}(t)} = \sigma(\theta dt + dW_t) = \sigma d\tilde{W}_t.$$

Where $\tilde{W}_t = \theta t + W_t$ is a martingale in $(\Omega, \mathbf{Q})$ with $\mathbf{Q}$ is the risk-neutral measure.

Then we have,

$$\begin{aligned}\frac{dS(t)}{S(t)} &= \mu dt + \sigma dW_t \\ &= \mu dt + \sigma(d\tilde{W}_t - \frac{\mu - r}{\sigma} dt) \\ &= r dt + \sigma d\tilde{W}_t.\end{aligned}$$

For a perfect, no arbitrage market, the discount portfolio value $\tilde{V}(t, S_t) = e^{-rt}V(t, S_t)$ should be a martingale. In other words, $E^Q[\tilde{V}(T)|\mathcal{F}_t] = \tilde{V}(t)$ for $0 < t < T$.

Setting $S(t) = x$,then by Ito-Doeblin formula:

$$\begin{aligned}d(\tilde{V}(t, S(t))) &= d(e^{-rt}V(t, x)) \\ &= -re^{-rt}V dt + e^{-rt}V_t dt + e^{-rt}V_x dx + \frac{1}{2}e^{-rt}V_{xx} dx dx \\ &= e^{-rt}[-rV dt + V_t dt + V_x x(r dt + \sigma d\tilde{W}_t) + \frac{1}{2}V_{xx}x^2\sigma^2 dt] \\ &= e^{-rt}[(-rV + V_t + rxV_x + \frac{1}{2}\sigma^2 x^2 V_{xx})dt + \sigma x V_x d\tilde{W}_t]\end{aligned}$$

The last result should have no dt term since $\tilde{V}$ is a martingale, then we have the Black-Scholes partial differential equation:

$$-rV + V_t + rxV_x + \frac{1}{2}\sigma^2 x^2 V_{xx} = 0 \tag{1.1}$$

1.1.2 The Solution of BSM Model for European Call Price

To solve the equation (1.1), we need to use the normal distribution property of Brownian motion and the definition of martingale.

For a European call price $c(t, S(t))$ with the payoff as $V(T) = (S(T) - K)^+$. Then under the risk-neutral measure,

$$c(t, S(t)) = \tilde{E}[e^{-r(T-t)}(S(T) - K)^+ | \mathcal{F}_t].$$

With constant σ and r, stock price becomes:

$$S(t) = S(0) exp(\sigma \tilde{W}(t) + (r - \frac{1}{2}\sigma^2)t),$$

and then,

$$\begin{aligned} S(T) &= S(t) exp(\sigma(\tilde{W}(T) - \tilde{W}(t)) + (r - \frac{1}{2}\sigma^2)\tau) \\ &= S(t) exp(-\sigma\sqrt{\tau}Y + (r - \frac{1}{2}\sigma^2)\tau). \end{aligned}$$

where $\tau = T - t$ and Y is the standard normal random variable

$$Y = -\frac{\tilde{W}(T) - \tilde{W}(t)}{\sqrt{T - t}}.$$

and now the exponential part of $S(T)$ is independent to $\mathcal{F}_t$.

Therefore,

$$\begin{aligned} c(t, S(t)) &= \tilde{E}[e^{-r(T-t)}(S(T) - K)^+ | \mathcal{F}_t] \\ &= \tilde{E}[e^{-r\tau}(S(t) exp(-\sigma\sqrt{\tau}Y + (r - \frac{1}{2}\sigma^2)\tau) - K)^+] \\ &= \frac{1}{\sqrt{2\pi}} \int_R e^{-r\tau}(x exp(-\sigma\sqrt{\tau}y + (r - \frac{1}{2}\sigma^2)\tau) - K)^+ e^{-\frac{1}{2}y^2} dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\tau,x)} e^{-r\tau}(x exp(-\sigma\sqrt{\tau}y + (r - \frac{1}{2}\sigma^2)\tau) - K) e^{-\frac{1}{2}y^2} dy \\ &\{x exp(-\sigma\sqrt{\tau}y + (r - \frac{1}{2}\sigma^2)\tau) > K \Leftrightarrow y < d_-(\tau, x) = \frac{1}{\sigma\sqrt{\tau}}[log\frac{x}{K} + (r - \frac{1}{2}\sigma^2)\tau]\} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\tau,x)} x exp(-\sigma\sqrt{\tau}y - \frac{1}{2}\sigma^2\tau - \frac{1}{2}y^2) dy - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\tau,x)} e^{-r\tau} K e^{-\frac{1}{2}y^2} dy \\ &= \frac{x}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\tau,x)} exp(-\frac{1}{2}(y + \sigma\sqrt{\tau})^2) dy - e^{-r\tau} K N(d_-(\tau, x)) \\ &= \frac{x}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\tau,x)+\sigma\sqrt{\tau}} exp(-\frac{1}{2}z^2) dz - e^{-r\tau} K N(d_-(\tau, x)) \\ &= x N(d_+(\tau, x)) - e^{-r\tau} K N(d_-(\tau, x)). \end{aligned}$$

where

$$d_+(\tau, x) = d_-(\tau, x) + \sigma\sqrt{\tau} = \frac{1}{\sigma\sqrt{\tau}}[log\frac{x}{K} + (r + \frac{1}{2}\sigma^2)\tau].$$

With this solution, the BSM pde equation (1.1) can be verified explicitly.

1.2 BSM Model with Poisson Jumps

For the real world, financial market always has the interruption events which is called jumps occur and make the price changing be discontinuous. The simple improvement of continuous BSM model is considering the Poisson jumps.

In this section, we will discuss the fundamental jump process as Poisson process and the call pricing for asset driven by a Brownian motion and a compound Poisson process further. (See details in [1])

1.2.1 Poisson Process

Definition 1.2.1 *[1] Let τ be a random variable with density*

$$f(t) = \begin{cases} \lambda e^{-\lambda t}, & t \leq 0 \\ 0, & t < 0 \end{cases}$$

*where λ is a positive constant. We say τ is an **exponential random variable**.*

The expectation can be computed by an integral by parts:

$$E\tau = \int_0^\infty t f(t) dt = \frac{1}{\lambda}$$

.

The cumulative distribution function is:

$$F(t) = P(\tau < t) = \int_0^t \lambda e^{-\lambda u} du = 1 - e^{-\lambda t}, t \geq 0,$$

and hence

$$P(\tau > t) = e^{-\lambda t}, t \geq 0.$$

Then

$$\begin{aligned} P(\tau > t + s | \tau > s) &= \frac{P(\tau > t + s \text{ and } \tau > s)}{P(\tau > s)} = \frac{P(\tau > t + s)}{P(\tau > s)} \\ &= \frac{e^{-\lambda(t+s)}}{e^{\lambda s}} = e^{-\lambda t}. \end{aligned}$$

In other words, after waiting s time units, the probability that we will have to wait an additional t time units is the same as the probability of having to wait t time units when we starting at time 0. This property for the exponential distribution is called *memorylessness* [1].

Definition 1.2.2 *[1] There is a sequence τ_1, τ_2,...of independent exponential random variables, all with the same mean $\frac{1}{\lambda}$. The first event we called "jump" occurs at time τ_1, the second occurs at time τ_2 after the first, the third occurs at τ_3 ,etc. The **Poisson process** N(t) counts the number of jumps that occur at or before time t.*

The τ_k random variables are called the *interarrival times*. The *arrival times* are

$$S_n = \sum_{k=1}^{n} \tau_k.$$

Note that at the jump times $N(t)$ is defined so that it is $right - continuous$ (i.e., $N(t) = \lim_{s \downarrow t} N(s)$). We denote by $\mathcal{F}(t)$ the σ-algebra of information acquired by observing $N(s)$ for $0 \leq s < t$.

The jumps are arriving at an average rate of λ per unit time since the expected time between jumps is $\frac{1}{\lambda}$. We say the Poisson process $N(t)$ has *intensity* λ.

Now we can see some lemma and theorems for Poisson process without proof. (See details in [1])

Lemma 1.2.1 *[1] For $n \geq 1$, the random variable arrival times S_n has the gamma density*

$$g_n(s) = \frac{(\lambda s)^{n-1}}{(n-1)!} \lambda e^{-\lambda s}, s \geq 0.$$

Lemma 1.2.2 *[1] The Poisson process $N(t)$ with intensity λ has the distribution*

$$P(N(t) = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, k = 0, 1, ...$$

Theorem 1.2.1 *[1] Let $N(t)$ be a Poisson process with intensity $\lambda > 0$, and let $0 = t_0 < t_1 < ... < t_n$ be given. Then the increments*

$$N(t_1) - N(t_0), N(t_2) - N(t_1), ..., N(t_n) - N(t_{n-1})$$

are stationary and independent, and

$$P(N(t_j) - N(t_{j-1}) = k) = \frac{\lambda^k (t_j - t_{j-1})^k}{k!} e^{\lambda(t_j - t_{j-1})}, k = 0, 1, ...$$

Base on this probability, we can calculate the expectation and variance of Poisson increment $N(t) - N(s)$ as

$$E[N(t) - N(s)] = \lambda(t - s)$$

$$Var(N(t) - N(s)) = \lambda(t - s)$$

Theorem 1.2.2 *[1] Let $N(t)$ be a Poisson process with intensity λ. We define the **compensated Poisson process***

$$M(t) = N(t) - \lambda t.$$

Then $M(t)$ is a martingale.

1.2.2 Compound Poisson Process

The Poisson jump has the step size of one only, we need to construct a new jump process called compound Poisson process with random step size for the financial market.

Definition 1.2.3 *[1] Let $N(t)$ be a Poisson process with intensity λ, and let $Y_1, Y_2...$ be a sequence of independent identically distributed random variables with mean $\beta = EY_i$. We assume the random variable Y_i are also independent of the Poisson process $N(t)$. Then the **compound Poisson process** is*

$$Q(t) = \sum_{i=1}^{N(t)} Y_i, t \geq 0. \tag{1.2}$$

The jumps in $Q(t)$ occur at the same times as the jumps in $N(t)$, but whereas the jumps in $N(t)$ are always of size 1, the jumps in $Q(t)$ are of random size as Y_i.

Like the simple Poisson process $N(t)$, the increment of the compound Poisson process $Q(t)$,

$$Q(t) - Q(s) = \sum_{i=N(s)-1}^{N(t)} Y_i, 0 \leq s < t$$

are independent. Moreover, $Q(t) - Q(s)$ has the same distribution as $Q(t-s)$ because $N(t) - N(s)$ has the same distribution as $N(t-s)$.

The mean of compound Poisson process is $\beta\lambda t$. On average, there are λt jumps in the time interval $[0, t]$, the average jump size is β, and the number of jumps is independent of the size of the jumps. Hence $EQ(t)$ is the product $\beta\lambda t$.

Theorem 1.2.3 *[1] Let $Q(t)$ be the compound Poisson process defined above. Then the **compensated compound Poisson process***

$$Q(t) - \beta\lambda t$$

is a martingale.

Though the compound Poisson process has the random step size, alternatively we can present the same process with compound Poisson of fixed step size.

Theorem 1.2.4 *[1] (Decomposition of a compound Poisson process). Let $y_1, ...y_M$ be a finite set of nonzero numbers, and let $p(y_1), ..., p(y_M)$ be positive numbers that sum to 1. Let $Y_1, Y_2, ...$ be a sequence of independent, identically distributed random variables with $P(Y_i = y_m) = p(y_m), m = 1, ..., M$. Let $N(t)$ be a Poisson process and define the compound Poisson process*

$$Q(t) = \sum_{i=1}^{N(t)} Y_i$$

For $m = 1, ..., M$, let $N_m(t)$ denote the number of jumps in Q of size y_m up to and including time t. Then

$$N(t) = \sum_{m=1}^{M} N_m(t) \text{ and } Q(t) = \sum_{m=1}^{M} y_m N_m(t).$$

The processes $N_1, ...N_M$ defined this way are independent Poisson process, and each N_m has intensity $\lambda p(y_m)$.

1.2.3 Jump Process and Their Integrals

For then, we need to derive the stochastic integral for the process with jumps, either the Poisson process or compound Poisson process.

Definition 1.2.4 *[1] Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $\mathcal{F}, t \geq 0$, be a filtration on this space. We say that a Brownian motion W is a Brownian motion relative to this filtration if $W(t)$ is $\mathcal{F}(t)$ -measurable for every t and for every $u > t$ the increment $W(u) - W(t)$ is independent of $\mathcal{F}(t)$. Similarly, we say that a Poisson process N is a Poisson process relative to this filtration if $N(t)$ is $\mathcal{F}(t)$-measurable for every t and for every $u > t$ the increment $N(u) - N(t)$ is independent of $\mathcal{F}(t)$. Finally, we say that a compound Poisson process is a compound Poisson process relative to this filtration if $Q(t)$ is $\mathcal{F}(t)$-measurable for every t and for every $u > t$ the increment $Q(u) - Q(t)$ is independent of $\mathcal{F}(t)$.*

We wish to define the stochastic integral [1]

$$\int_0^t \Phi(s) dX(s)$$

where the integrator X can have jumps. We consider in this section will be right-continuous and of the form

$$X(t) = X^c(t) + J(t) = X(0) + I(t) + R(t) + J(t) \tag{1.3}$$

$$= X(0) + \int_0^t \Gamma(s) dW(s) + \int_0^t \Theta(s) ds + J(t) \tag{1.4}$$

where $X(0)$ is the nonrandom initial condition, $I(t)$ is the Ito-integral part and $R(t)$ is the Riemann integral part of X.

So, the quadratic variation of this continuous part $X^c(t)$ is [1]

$$[X^c, X^c](t) = \int_0^t \Gamma^2(s) ds,$$

or

$$dX^c(t) dX^c(t) = \Gamma^2(t) dt.$$

We assume that J does not jump at time zero, has only finitely many jumps on each finite time interval $(0, T]$, and is constant between jumps, which is called *pure* jump process. [1]

Definition 1.2.5 *[1] A process $X(t)$ of the form (1.3), with Ito integral part, Riemann integral part, and pure jump part will be called a **jump process**.*

A jump process $X(t)$ is right-continuous and adapted. Because both $I(t)$ and $R(t)$ are continuous, the left continuous version of $X(t)$ is [1]

$$X(t-) = X(0) + I(t) + R(t) + J(t-).$$

The jump size of X at time t is

$$\Delta X(t) = X(t) - X(t-) = \Delta J(t) = J(t) - J(t-).$$

Definition 1.2.6 *[1] Let $X(t)$ be a jump process of the form (1.3) and let $\Phi(s)$ be an adapted process. The **stochastic integral** of Φ with respect to X is defined to be*

$$\int_0^t \Phi(s)dX(s) = \int_0^t \Phi(s)\Gamma(s)dW(s) + \int_0^t \Phi(s)\Theta(s)ds + \sum_{0<s\leq t} \Phi(s)\Delta J(s).$$

Or

$$\Phi(t)dX(t) = \Phi(t)dI(t) + \Phi(t)dR(t) + \Phi(t)dJ(t) = \Phi(t)dX^c(t) + \Phi(t)dJ(t).$$

Theorem 1.2.5 *[1] Assume that the jump process $X(s)$ of form (1.3) is a martingale, the integrand $\Phi(s)$ is left-continuous and adapted, and*

$$E\int_0^t \Gamma^2(s)\Phi^2(s)ds < \infty \text{ for all } t \geq 0.$$

Then the stochastic integral $\int_0^t \Phi(s)dX(s)$ is also a martingale.

Theorem 1.2.6 *[1] Let $X_1(t)$ and $X_2(t)$ be two jump processes defined as 1.3, then*

$$[X_1, X_1](t) = [X_1^c, X_1^c](t) + [J_1, J_1](t) = \int_0^t \Gamma_1^2(s)ds + \sum_{0<s\leq t} (\Delta J_1(s))^2.$$

and

$$[X_1, X_2](t) = [X_1^c, X_2^c](t) + [J_1, J_2](t) = \int_0^t \Gamma_1(s)\Gamma_2(s)ds + \sum_{0<s\leq t} \Delta J_1(s)\Delta J_2(s).$$

Remark 1.2.1 *[1] Generally, the cross variation between two processes is zero if one of them is continuous and the other has no Ito integral part.*

Specially, let $W(t)$ be a Brownian motion and $M(t) = N(t) - \lambda t$ be a compensated Poisson process relative to the same filtration $\mathcal{F}(t)$. Then

$$[W, M](t) = 0, t \geq 0.$$

And hence, $W(t)$ and $N(t)$ are independent.

1.2.4 Stochastic Calculus for Jump Processes

As the same as continuous case, we now introduce the Ito formulas for jump processes.

Theorem 1.2.7 *[1] (Ito-Doeblin formula for on jump process) Let $X(t)$ be a jump process and $f(x)$ a function for which $f'(x)$ and $f''(x)$ are defined and continuous. Then*

$$
\begin{aligned}
f(X(t)) = f(X(0)) + \int_0^t f'(X(s))dX^c(s) + \frac{1}{2}\int_0^t f''(X(s))dX^c(s)dX^c(s) \\
+ \sum_{0<s\leq t} [f(X(s)) - f(x(s-))].
\end{aligned}
$$

Theorem 1.2.8 *[1] (Two-dimensional Ito-Doeblin formula for process with jumps) Let X_1 and X_2 be jump processes, and let $f(t, x_1, x_2)$ be a funtion whose first and second partial derivatives appearing in the following formula are defined and are continuous. Then*

$$
\begin{aligned}
f(t, X_1(t), X_2(t)) = f(0, X_1(0), X_2(0)) &+ \int_0^t f_t(s, X_1(s), X_2(s))d(s) \\
&+ \int_0^t f_{x_1}(s, X_1(s), X_2(s))dX_1^c(s) + \int_0^t f_{x_2}(s, X_1(s), X_2(s))dX_2^c(s) \\
&+ \frac{1}{2}\int_0^t f_{x_1,x_1}(s, X_1(s), X_2(s))dX_1^c(s)dX_1^c(s) \\
&+ \int_0^t f_{x_1,x_2}(s, X_1(s), X_2(s))dX_1^c(s)dX_2^c(s) \\
&+ \frac{1}{2}\int_0^t f_{x_2,x_2}(s, X_1(s), X_2(s))dX_2^c(s)dX_2^c(s) \\
&+ \sum_{0<s\leq t} [f(s, X_1(s), X_2(s)) - f(s, X_1(s-), X_2(s-))]
\end{aligned}
$$

Corollary 1.2.1 *[1] (Ito's product rule for jump processes). Let $X_1(t)$ and $X_2(t)$ be jump processes. Then*

$$
\begin{aligned}
X_1(t)X_2(t) =& X_1(0)X_2(0) + \int_0^t X_2(s)dX_1^c(s) + \int_0^t X_1(s)dX_2^c(s) \\
&+ [X_1^c, X_2^c](t) + \sum_{0<s\leq t} [X_1(s)X_2(s) - X_1(s-)X_2(s-)] \\
=& X_1(0)X_2(0) + \int_0^t X_2(s-)dX_1(s) + \int_0^t X_1(s-)dX_2(s) + [X_1, X_2](t).
\end{aligned}
$$

1.2.5 Change of Measure for Compound Poisson Process

Just as we use Girsanov's Theorem to change the measure so that a Brownian motion with drift becomes a Brownian motion without drift, we can change the measure for Poisson processed and compound Poisson process.For a Poisson process, the change of measure affects the intensity. For a compound Poisson process, the change of measure can affect both the intensity and the distribution of the jump sizes. We also include a Brownian motion component in the process under consideration. [1]

We define [1]

$$Z(t) = e^{(\lambda - \tilde{\lambda})t} (\frac{\tilde{\lambda}}{\lambda})^{N(t)}.$$

then the process $Z(t)$ satisfies

$$dZ(t) = \frac{\tilde{\lambda} - \lambda}{\lambda} Z(t-) dM(t).$$

In particular, $Z(t)$ is a martingale under $\mathbf{P}$ and $EZ(t) = 1$ for all t.

We may now fix a positive time T and use $Z(T)$ to change the measure by defining [1]

$$\mathbf{Q}(A) = \int_A Z(T) dP \text{ for all } A \in \mathcal{F}.$$

Theorem 1.2.9 *[1] (Change of Poisson intensity). Under the probability measure* $\mathbf{Q}$*, the process* $N(t), 0 \leq t \leq T$ *is Poisson process with intensity* $\tilde{\lambda}$*.*

For compound Poisson process, let $\tilde{\lambda_1}, ..., \tilde{\lambda}_M$ be given positive numbers, and set

$$Z_m(t) = e^{(\lambda_m - \tilde{\lambda}_m)t} (\frac{\tilde{\lambda}_m}{\lambda_m}) \text{ and } Z(t) = \prod_{m=1}^{M} Z_m(t)$$

Then the process $Z(t)$ is a martingale. In particular, $EZ(t) = 1$ for all t.

Fix $T > 0$, because $Z(t) > 0$ almost surely and $EZ(t) = 1$, we can use $Z(t)$ to change the measure, defining [1]

$$\mathbf{Q}(A) = \int_A Z(T) dP \text{ for all } A \in \mathcal{F}.$$

Theorem 1.2.10 *[1] (Change of compound Poisson intensity and jump distribution for finitely many jump sizes). Under* $\mathbf{Q}$*,* $Q(t)$ *is a compound Poisson process with intensity* $\tilde{\lambda} = \sum_{m=1}^{M} \tilde{\lambda}$*, and* $Y_1, Y_2, ...$ *are independent,*

$$\mathbf{Q}\{Y_i = y_m\} = \tilde{p}(y_m) = \frac{\tilde{\lambda}_m}{\tilde{\lambda}}.$$

Using the same technique, we can get combined case as

Theorem 1.2.11 *[1] (Change of compound Poisson intensity and jump distribution for continuum of jump sizes). Under the probability measure* $\mathbf{Q}$*, the compound Poisson process* $Q(t), 0 \leq t \leq T$ *as (1.2), is a compound Poisson process with Intensity* $\tilde{\lambda}$*.Furthermore, the jumps in* $Q(t)$ *are independent and identically distributed with density* $\tilde{f}(y)$*.*

Furthermore, we also can change the measure for a compound Poisson process and a Brownian motion as

Theorem 1.2.12 *[1] Under the probability measure* $\mathbf{Q}$*, the process*

$$\tilde{W}(t) = W(t) + \int_0^t \Theta(s)ds$$

is a Brownian motion, $Q(t)$ *is a compound Poisson process with intensity* $\tilde{\lambda}$ *and independent, identically distributed jump sizes having density* $\tilde{f}(y)$*, and the processes* $\tilde{W}(t)$ *and* $Q(t)$ *are independent.*

1.2.6 Underlying Asset Driven by a Brownian Motion and a Compound Poisson Jump Process

For the price driven by a Brownian motion and a compound Poisson process, we discuss the risk-neutral measure change which is suitable for no arbitrage market.

Consider the compound Poisson process $Q(t)$ as (1.2), we decompose it due to theorem 1.2.4. And

$$Q(t) - \beta\lambda t = Q(t) - t\sum_{m=1}^{M} \lambda_m y_m$$

is a martingale. [1]

For the underlying asset price driven by [1]

$$S(t) = S(0)exp\{\sigma W(t) + (\alpha - \beta\lambda - \frac{1}{2}\sigma^2)t\} \prod_{i=1}^{N(t)} (Y_i + 1).$$

Define the continuous stochastic process

$$X(t) = S(0)exp\{\sigma W(t) + (\alpha - \beta\lambda - \frac{1}{2}\sigma^2)t\},$$

and the jump process

$$J(t) = \prod_{i=1}^{N(t)} (Y_i + 1).$$

Then $S(t) = X(t)J(t)$.

The Ito-Doeblin formula for a continuous process says that

$$dX(t) = (\alpha - \beta\lambda)X(t)dt + \sigma X(t)dW(t).$$

At the time of the ith jump, $J(t) = J(t-)(Y_i + 1)$ and hence

$$\Delta J(t) = J(t) - J(t-) = J(t-)Y_i = J(t-)\Delta Q.$$

That means $dJ(t) = J(t-)\Delta Q$ and it also holds at nonjump times, with both sides equal to zero. [1]

Therefore, Ito's product rule for jump process implies [1]

$$\begin{aligned} dS(t) &= dX(t)J(t) \\ &= X(t-)dJ(t) + J(t-)dX(t) + [X,J](t) \\ &= X(t-)J(t-)dQ(t) + J(t-)(\alpha - \beta\lambda)X(t)dt + J(t-)\sigma X(t)dW(t) \\ &= S(t-)dQ(t) + (\alpha - \beta\lambda)S(t)dt + \sigma S(t)dW(t) \end{aligned}$$

since X and J are independent, and $J(t) = J(t-)$ for continuous part.

We now undertake to construct a risk-neutral measure. Let θ be a constant and let $\tilde{\lambda}_1, ...\tilde{\lambda}_M$ be positive constants. Define [1]

$$Z_0 = exp\{-\theta W(t) - \frac{1}{2}\theta^2 t\}$$

$$Z_m(t) = e^{(\lambda_m - \tilde{\lambda}_m)t}(\frac{\tilde{\lambda}_m}{\lambda_m})^{N_m(t)}$$

$$Z(t) = Z_0(t)\prod_{m=1}^{M} Z_m(t)$$

$$\mathbf{Q}(A) = \int_A Z(T)d\mathbf{P} \text{ for all } A \in \mathcal{F}.$$

Then [1]

(1) the process $\tilde{W}(t) = W(t) + \theta t$ is a Brownian motion.

(2) each N_m is a Poisson process with intensity $\tilde{\lambda}_m$, and

(3) $\tilde{W}$ and $N_1, ..., N_m$ are independent of one another.

Define [1]

$$\tilde{\lambda} = \sum_{m=1}^{M} \tilde{\lambda}_m, \tilde{p}(y_m) = \frac{\tilde{\lambda}_m}{\tilde{\lambda}}.$$

Under $\mathbf{Q}$, the process $N(t) = \sum_{m=1}^{M} N_m(t)$ is Poisson with $\tilde{\lambda}$, the jump-size random variables $Y_1, Y_2, ...$ are independent and identically distributed with $\mathbf{Q}\{Y_i = y_m\} = \tilde{p}(y_m)$, and $Q(t) - \tilde{\beta}\tilde{\lambda}t$ is a martingale, where

$$\tilde{\beta} = \tilde{E}Y_i = \sum_{m=1}^{M} y_m \tilde{p}(y_m) = \frac{1}{\tilde{\lambda}}\sum_{m=1}^{M} \tilde{\lambda}_m y_m.$$

The probability measure $\mathbf{Q}$ is risk-neutral if and only if the mean rate of return of the stock under $\mathbf{Q}$ is the interest rate r. i.e. [1]

$$\begin{aligned} dS(t) &= (\alpha - \beta\lambda)S(t)dt + \sigma S(t)dW(t) + S(t-)dQ(t) \\ &= rS(t)dt + \sigma S(t)d\tilde{W}(t) + S(t-)d(Q(t) - \tilde{\beta}\tilde{\lambda}t). \end{aligned}$$

This is equivalent to the equation [1]

$$\alpha - \beta\lambda = r + \sigma\theta - \tilde{\beta}\tilde{\lambda},$$

which is called *market price of risk equation* for this model. Or,

$$\alpha - r = \sigma\theta + \sum_{m=1}^{M}(\lambda_m - \tilde{\lambda}_m)y_m.$$

Let us choose some θ and $\tilde{\lambda}_1, ...\tilde{\lambda}_M$ satisfying the market price of risk equation. Then, we have [1]

$$\begin{aligned} dS(t) &= rS(t)dt + \sigma S(t)d\tilde{W}(t) + S(t-)d(Q(t) - \tilde{\beta}\tilde{\lambda}t) \\ &= (r - \tilde{\beta}\tilde{\lambda})S(t)dt + \sigma S(t)d\tilde{W}(t) + S(t-)dQ(t). \end{aligned} \tag{1.5}$$

with solution

$$S(t) = S(0)exp\{\sigma\tilde{W}(t) + (r - \tilde{\beta}\tilde{\lambda} - \frac{1}{2}\sigma^2)t\}\prod_{i=1}^{N(t)}(Y_i + 1). \tag{1.6}$$

In this equation (1.6), the drifting term α is replaced by interest rate r. That means the market has no arbitrage due to the asset return value of r.

1.2.7 Call Pricing for Jump Process

In this section, the call price formula is obtained due to the properties of Brownian motion and compound Poisson process.

For the next, we use $C(\tau, x)$ to denote the standard Black-Scholes-Merton call price on a geometric Brownian motion with volatility σ when the current stock price is x, the time to maturity is τ, the interest rate r, and the strike price is K. [1]

$$C(\tau, x) = xN(d_1(\tau, x)) - Ke^{-r\tau}N(d_2(\tau, x))$$

Where N is the standard normal distribution function and

$$d_{1,2} = \frac{1}{\sigma\sqrt{\tau}}[log(x/K) + (r \pm \frac{1}{2}\sigma^2)\tau].$$

We have

$$C(\tau, x) = \tilde{E}[e^{-r\tau}(xexp\{-\sigma\sqrt{\tau}Y + (r - \frac{1}{2}\sigma^2)\tau\} - K)^+]$$

where Y is a standard normal random variable under $\mathbf{Q}$.

Theorem 1.2.13 *[1] For $0 \le t < T$, the risk-neutral price of a call,*

$$V(t) = \tilde{E}[e^{-r\tau}(S(t) - K)^+|\mathcal{F}(t)]$$

is given by $V(t) = c(t, S(t))$, *where*

$$c(t,x) = \sum_{j=1}^{\infty} e^{-\tilde{\lambda}\tau} \frac{(\tilde{\lambda}\tau)^j}{j!} \tilde{E}C(\tau, xe^{-\tilde{\beta}\tilde{\lambda}\tau} \prod_{i=1}^{j} (Y_i + 1)) \tag{1.7}$$

Proof. [1] Let $t \in [0, T)$ be given and define $\tau = T - t$. Then from (1.6),

$$S(T) = S(t)exp\{\sigma(\tilde{W}(T) - \tilde{W}(t)) + (r - \tilde{\beta}\tilde{\lambda} - \frac{1}{2}\sigma^2)t\} \prod_{i=N(t)+1}^{N(T)} (Y_i + 1).$$

$S(t)$ is $\mathcal{F}(t)$-measurable and then from independence,

$$V(t) = \tilde{E}[e^{-r\tau}(S(T) - K)^+ | \mathcal{F}(t)] = c(t, S(t)),$$

where

$$\begin{aligned} c(t,x) =& \tilde{E}[e^{-r\tau}(xe^{\sigma(\tilde{W}(T)-\tilde{W}(t))+(r-\tilde{\beta}\tilde{\lambda}-\frac{1}{2}\sigma^2)t} \prod_{i=N(t)+1}^{N(T)} (Y_i + 1) - K)^+] \\ =& \tilde{E}[\tilde{E}[e^{-r\tau}(xe^{\sigma(\tilde{W}(T)-\tilde{W}(t))+(r-\tilde{\beta}\tilde{\lambda}-\frac{1}{2}\sigma^2)t} \\ & \times \prod_{i=N(t)+1}^{N(T)} (Y_i + 1) - K)^+ | \sigma(\prod_{i=N(t)+1}^{N(T)} (Y_i + 1))]] \\ =& \tilde{E}[\tilde{E}[e^{-r\tau}(xe^{-\tilde{\beta}\tilde{\lambda}\tau} exp\{-\sigma\sqrt{\tau}Y + (r - \frac{1}{2}\sigma^2)t\} \\ & \times \prod_{i=N(t)+1}^{N(T)} (Y_i + 1) - K)^+ | \sigma(\prod_{i=N(t)+1}^{N(T)} (Y_i + 1))]] \end{aligned}$$

where $Y = -\frac{\tilde{W}(T)-\tilde{W}(t)}{\sqrt{\tau}} \sim N(0,1)$ under $\mathbf{Q}$. And since $\prod_{i=N(t)+1}^{N(T)}(Y_i + 1)$ is $\sigma(\prod_{i=N(t)+1}^{N(T)}(Y_i + 1))$-measurable and Y is independent of $\sigma(\prod_{i=N(t)+1}^{N(T)}(Y_i + 1))$, we may get

$$\begin{aligned} &\tilde{E}[e^{-r\tau}(xe^{-\tilde{\beta}\tilde{\lambda}\tau} exp\{-\sigma\sqrt{\tau}Y + (r - \frac{1}{2}\sigma^2)t\} \\ &\quad \times \prod_{i=N(t)+1}^{N(T)} (Y_i + 1) - K)^+ | \sigma(\prod_{i=N(t)+1}^{N(T)} (Y_i + 1))] \\ &\quad = C(\tau, xe^{-\tilde{\beta}\tilde{\lambda}\tau} \prod_{i=N(t)+1}^{N(T)} (Y_i + 1)). \end{aligned}$$

It follows

$$c(t,x) = E[C(\tau, xe^{-\tilde{\beta}\tilde{\lambda}\tau} \prod_{i=N(t)+1}^{N(T)} (Y_i + 1))].$$

On the other hand, we note that conditioned on $N(T) - N(t) = j$, the random variable $\prod_{i=N(t)+1}^{N(T)}(Y_i + 1)$ has the same distribution as $\prod_{i=1}^{j}(Y_i + 1)$. And

$$\mathbf{P}\{N(T) - N(t) = j\} = e^{-\tilde{\lambda}\tau} \frac{(\tilde{\lambda}\tau)^j}{j!}.$$

□

Remark 1.2.2 *[1] (Continuous jump distribution). Suppose the jump sizes Y_i have a density $f(y)$ rather than a probability mass function $p(y_1), ..., p(y_m)$, and this density is strictly positive on a set $B \subset (-1, \infty)$ and zero elsewhere. In this case, we can use*

$$\beta = EY_i = \int_{-1}^{\infty} yf(y)dy.$$

For a risk-neutral measure, we can use

$$\tilde{\beta} = \tilde{E}Y_i = \int_{-1}^{\infty} y\tilde{f}(y)dy.$$

Theorem 1.2.13 still holds.

1.2.8 Partial Integro-Differential Equation

In the calculation of previous section, we can reach the partial integro-differential equation (PIDE) since the discount call price should be a martingale.

Theorem 1.2.14 *[1] The call price $c(t, x)$ of (1.7) satisfies the equation*

$$\begin{aligned} &-rc(t,x) + c_t(t,x) + (r - \tilde{\beta} - \tilde{\lambda})xc_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x) \\ &\quad + \tilde{\lambda}[\sum_{m=1}^{M} c(t,(y_m+1)x) - c(t,x)] = 0, 0 \le t < T, x \ge 0. \end{aligned} \tag{1.8}$$

and the terminal condition

$$c(T, x) = (x - K)^+, x \ge 0.$$

Proof. [1] From 1.5, the continuous part of the stock price satisfies $dS^c = (r - \tilde{\beta}\tilde{\lambda}S(t)dt + \sigma S(t)d\tilde{W}(t)$. Therefore, the Ito-Doeblin formula implies

$$\begin{aligned} &e^{-rt}c(t,S(t)) - c(0,S(0)) \\ &= \int_0^t e^{-rt}[-rc + c_t + (r - \tilde{\beta}\tilde{\lambda})S(u)c_x + \frac{1}{2}\sigma^2 S(u)c_{xx}]du \\ &\quad + \int_0^t e^{-ru}\sigma S(u)c_x d\tilde{W}(u) + \sum_{0<u\le u} e^{-rt}[c(u,S(u) - c(u,S(u-))] \end{aligned}$$

For the last term, if u is a jump time of the mth Poisson process N_m, the stock price satisfies

$S(u) = (y_m + 1)S(u-)$. then

$$
\begin{aligned}
&\sum_{0<u\leq u} e^{-rt}[c(u, S(u) - c(u, S(u-))] \\
=&\sum_{m=1}^{M} \sum_{0<t\leq u} e^{-rt}[c(u, (y_m + 1)S(u-) - c(u, S(u-))]\Delta N_m(u) \\
=&\sum_{m=1}^{M} \int_0^t e^{-rt}[c(u, (y_m + 1)S(u-) - c(u, S(u-))]d(N_m(u) - \tilde{\lambda}_m u) \\
&+ \int_0^t e^{-rt}[\sum_{m=1}^{M} \frac{\tilde{\lambda}_m}{\tilde{\lambda}} c(u, (y_m + 1)S(u-) - c(u, S(u-))]\tilde{\lambda} du \\
=&\sum_{m=1}^{M} \int_0^t e^{-rt}[c(u, (y_m + 1)S(u-) - c(u, S(u-))]d(N_m(u) - \tilde{\lambda}_m u) \\
&+ \int_0^t e^{-rt}[\sum_{m=1}^{M} \tilde{p}(y_m) c(u, (y_m + 1)S(u-) - c(u, S(u-))]\tilde{\lambda} du
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
&d(e^{-rt}c(t, S(t)) \\
=&e^{-rt}\{-rc + c_t + (r - \tilde{\beta}\tilde{\lambda})S(u)c_x + \frac{1}{2}\sigma^2 S(u)c_{xx} \\
&+ \tilde{\lambda}[\sum_{m=1}^{M} \tilde{p}(y_m)c(u, (y_m + 1)S(u-) - c(u, S(u-))]\}dt \\
&+ e^{-rt}\sigma S(t)c_x d\tilde{W}(t) \\
&+ \sum_{m=1}^{M} e^{-rt}[c(t, (y_m + 1)S(t-) - c(t, S(t-))]d(N_m(t) - \tilde{\lambda}_m t)
\end{aligned}
$$

The integrators $N_m(t) - \tilde{\lambda}t$ in the last term are martingales under $\mathbf{Q}$, and the integrands $e^{-rt}[c(t, (y_m + 1)S(t-) - c(t, S(t-))]$ are left-continuous. Therefore, the integral of this term is a martingale. Likewise, the integral of the term $e^{-rt}\sigma S(t)c_x d\tilde{W}(t)$ is a martingale. Since the discounted option price appearing on the left-hand side is also a martingale, the remaining term is a martingale as well. i.e. by replacing $S(t)$ as x, the *Partial Integro-Differential equation* (1.8) holds. □

Corollary 1.2.2 *[1] The call price $c(t,x)$ (1.7) satisfies*

$$\begin{aligned}
&d(e^{-rt}c(t,S(t))\\
=&e^{-rt}\sigma S(t)c_x d\tilde{W}(t)\\
&+\sum_{m=1}^{M} e^{-rt}[c(t,(y_m+1)S(t-)-c(t,S(t-))]d(N_m(t)-\tilde{\lambda}_m t)\\
=&e^{-rt}\sigma S(t)c_x d\tilde{W}(t)+e^{-rt}[c(t,S(t))-c(t,S(t-))]dN(t)\\
&+e^{-rt}\tilde{\lambda}[\sum_{m=1}^{M}\tilde{p}(y_m)c(t,(y_m+1)S(t-)-c(t,S(t-))]dt\\
=&e^{-rt}\sigma S(t)c_x d\tilde{W}(t)+e^{-rt}[c(t,S(t))-c(t,S(t-))]dN(t)\\
&+e^{-rt}\tilde{\lambda}E[\Delta c(t,S(t))]dt
\end{aligned} \tag{1.9}$$

Remark 1.2.3 *[1] (Continuous jump distribution). There are modification of equation (1.8) and (1.9) for the case when the jump sizes Y_i have a density $\tilde{f}(y)$ under the risk-neutral measure $\mathbf{Q}$. The term $\sum_{m=1}^{M}\tilde{p}(y_m)c(t,(y_m+1)S(t-)$ would be replaced by $\int_{-1}^{\infty}c(t,(y+1)S(t-))\tilde{f}(y)dy$.*

1.2.9 Underlying Asset Dynamics Associated with Log-normal Jumps

For the call price formula (1.7), there is a special case of log-normal jumps which can be presented explicitly for further discussion.

Suppose V is the total market value of the assets of the firm. The dynamics of V are given by the following jump-diffusion process under risk-neutral measure [5]

$$V(t) = V(0)exp\{\sigma W(t) + (r-\beta\lambda-\frac{1}{2}\sigma^2)t\}\prod_{i=1}^{N(t)}(Y_i+1)$$

where μ,λ,β and σ are positive constants. $N(t)$ is a Poisson process with intensity λ. Y_i is the jump size with expectation β. And Brownian motion $W(t)$, Poisson process $N(t)$, jump Y_i are mutually independent. We assume that Y_i is identically independent distributed and Y_i+1 is log-normal random variables, such that

$$ln(Y_i+1)\sim N(\mu_0,\sigma_0^2)$$

That implies $\beta = EY_i = e^{\mu_0+\frac{1}{2}\sigma_0^2}-1$. Then, we get continuous case as

$$V(t) = V(0)exp\{\sigma W(t) + (r-\beta\lambda-\frac{1}{2}\sigma^2)t\}(Y_i+1)^{N(t)}$$

Let $V(t) = f(X(t))$ where $f(x) = V(0)e^x$ with

$$X(t) = \sigma W(t) + (r-\beta\lambda-\frac{1}{2}\sigma^2)t + N(t)ln(Y_i+1)$$

is a jump process. Then the continuous part is

$$X^c(t) = \sigma W(t) + (r - \beta\lambda - \frac{1}{2}\sigma^2)t,$$

and jump part is

$$J(t) = N(t)ln(Y_i + 1).$$

By Ito-Doeblin formula for one jump process [5]

$$\begin{aligned} dV(t) &= df(X(t)) \\ &= f'(x)dX^c(t) + \frac{1}{2}f''(x)dX^c(t)dX^c(t) + [f(X(t) - f(X(t-))]dN(t) \\ &= V(t)(r - \beta\lambda - \frac{1}{2}\sigma^2)dt + \frac{1}{2}V(t)\sigma^2 dt + (V(t) - V(t-))dN(t) \\ &= V(t)[(r - \beta\lambda)dt + \sigma dW(t)] + V(t-)Y_i dN(t) \end{aligned} \tag{1.10}$$

Theorem 1.2.15 *[5] Let H be the price of any derivative security with payoff at time T contingent on the firm's V. Using Merton's result, we know that the assumption that the jump risk is not systematic and that arbitrage opportunities are excluded, the derivative price H must satisfy the following equation*

$$-rH + H_t + V(r - \beta\lambda)H_V + \frac{1}{2}\sigma^2 V^2 H_{VV} + \lambda E[\Delta H(t, V)] = 0$$

Proof. Denote $Y_i{+}1 = \nu$ follows log-normal distribution with density function $p(\nu)$. Since $e^{-rt}H(t, V(t))$ is a martingale. For

$$dV(t) = V(t)[(r - \beta\lambda)dt + \sigma dW(t)] + V(t-)Y_i dN(t) = dV^c(t) + dJ(t).$$

Then

$$\begin{aligned} & d(e^{-rt}H(t, V(t))) \\ =& e^{-rt}\{-rHdt + H_t dt + H_V dV^c + \frac{1}{2}H_{VV}dV^c dV^c \\ & + \int_0^\infty (H(t, \nu V) - H(t, V))p(\nu)d\nu dN(t)\} \\ =& e^{-rt}\{-rH + H_t + V(r - \beta\lambda)H_V + \frac{1}{2}\sigma^2 V^2 H_{VV} \\ & + \int_0^\infty (H(t, \nu V) - H(t, V))p(\nu)d\nu\lambda\}dt \\ & + H_V V\sigma dW(t) + \int_0^\infty (H(t, \nu V) - H(t, V))p(\nu)d\nu d(N(t) - \lambda t) \end{aligned}$$

should has no dt term since $W(t)$ and $N(t) - \lambda t$ are martingales. Hence

$$\begin{aligned}
-rH + H_t + V(r - \beta\lambda)H_V + \frac{1}{2}\sigma^2 V^2 H_{VV} + \int_0^\infty (H(t, \nu V) - H(t, V))p(\nu)d\nu\lambda \\
= -rH + H_t + V(r - \beta\lambda)H_V + \frac{1}{2}\sigma^2 V^2 H_{VV} + \lambda E[H(t, \nu V) - H(t, V)] \\
= -rH + H_t + V(r - \beta\lambda)H_V + \frac{1}{2}\sigma^2 V^2 H_{VV} + \lambda E[\Delta H(t, V)] \\
= 0
\end{aligned}$$

□

1.2.10 Call Pricing for Log-normal Jumps

Due to the log-normal distribution of jump size Y_i, the call price can be calculated explicitly with some new defined parameters.

Lemma 1.2.3 *For call option price of Black-Scholes, $xN'(d_1) = Ke^{-r\tau}N'(d_2)$. Let $S_t = x$, $T - t = \tau$, $K_t = Ke^{-r(T-t)}$. Since*

$$N'(y) = (\int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx)' = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

and also,

$$\begin{aligned}
d_1^2 - d_2^2 &= (d_1 + d_2)(d_1 - d_2) \\
&= \frac{2}{\sigma\sqrt{\tau}}(log(x/K) + r\tau)\sigma\sqrt{\tau} \\
&= 2(log(x/K) + r_n\tau)
\end{aligned}$$

Then,

$$\begin{aligned}
K_t N'(d_2) &= K_t \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} = K_t \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2} + \frac{d_1^2 - d_2^2}{2}} \\
&= K_t \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} e^{ln(x/K) + r\tau} = x \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \\
&= xN'(d_1)
\end{aligned}$$

Utilizing this Lemma, the call price is achieved as below with some parameter modifications.

Theorem 1.2.16 *[4] For log-normal case of jump's size distribution, the price of call option C^J is*

$$C^J(V, \tau, r, \sigma) = \sum_{n=0}^{\infty} \frac{(\lambda'\tau)^n}{n!} e^{-\lambda'\tau} C(V, \tau, r_n, \sigma_n) \tag{1.11}$$

where $C(V, \tau, r_n, \sigma_n)$ is the standard Black-Scholes formula for a call and

$$\lambda' = \lambda(1 + \beta),$$

$$r_n = r + n\gamma/\tau - \lambda\beta,$$

$$\sigma_n^2 = \sigma^2 + n\sigma_0^2/\tau,$$

$$\gamma = ln(1+\beta) = \mu_0 + \frac{1}{2}\sigma_0^2.$$

Proof. Since $ln(Y_i + 1) \sim N(\mu_0, \sigma_0^2)$, and $Y_1, Y_2, ...$ are i.i.d.

$$E[\sum_{i=1}^{n} ln(Y_i + 1)] = n\mu_0$$

$$Var(\sum_{i=1}^{n} ln(Y_i + 1)) = n\sigma_0^2$$

And

$$\sum_{i=1}^{n} ln(Y_i + 1) \sim N(n\mu_0, n\sigma_0^2).$$

Then there is the normal distribution variable Y,

$$Y = \sigma(W(T) - W(t)) + nln(Y_i + 1) \sim N(n\mu_0, \sigma^2\tau + n\sigma_0^2) = N(n\mu_0, \sigma_n^2\tau),$$

we have

$$\xi = \frac{Y - n\mu_0}{\sqrt{\sigma_n^2\tau}} \sim N(0,1).$$

On the other hand, denote

$$d_{1n} = \frac{1}{\sigma_n\sqrt{\tau}}[ln\frac{x}{K} + (r_n + \frac{1}{2}\sigma_n^2)\tau] = d_{2n} + \sigma_n\sqrt{\tau}.$$

For call option, we get

$$S(T) = S(t)e^{(r - \frac{1}{2}\sigma^2 - \beta\lambda)\tau + \sigma(W(T) - W(t))} \times \prod_{i=1}^{n}(Y_i + 1) > K.$$

Since independence of Y_i's,i.e.

$$S(T) = S(t)e^{(r - \frac{1}{2}\sigma^2 - \beta\lambda)\tau + \sigma(W(T) - W(t)) + nln(Y_i + 1)} > K.$$

Therefore, we have

$$(r - \frac{1}{2}\sigma^2 - \beta\lambda)\tau + Y > ln(K/x)$$

$$(r - \frac{1}{2}\sigma^2 - \beta\lambda)\tau + \sqrt{\sigma_n^2\tau}\xi + n\mu_0 > ln(K/x)$$

$$\sqrt{\sigma_n^2\tau}\xi - \frac{1}{2}\sigma^2\tau - \frac{1}{2}n\sigma_0^2 > ln(K/x) - r_n\tau$$

$$\sqrt{\sigma_n^2\tau}\xi - \frac{1}{2}\sigma_n^2\tau > ln(K/x) - r_n\tau$$

Which is $\xi > -d_{2n}$.

Base on (1.7), we can get

$$
\begin{aligned}
C^J(V,\tau,r,\sigma) &= E[e^{-r\tau}(xe^{-\beta\lambda\tau}e^{\sigma(W(T)-W(t))+(r-\frac{1}{2}\sigma^2)\tau} \times \prod_{i=1}^{n}(Y_i+1) - K)^+] \\
&= E[e^{-r\tau}(xe^{-\beta\lambda\tau}e^{\sigma(W(T)-W(t))+(r-\frac{1}{2}\sigma^2)\tau+\sum_{i=1}^{n} ln(Y_i+1)} - K)^+] \\
&= E[e^{-\beta\lambda\tau+n\gamma}(xe^{-r\tau+Y-n\gamma+(r-\frac{1}{2}\sigma^2)\tau} - e^{-r\tau+\beta\lambda\tau-n\gamma}K)^+] \\
&= e^{-\beta\lambda\tau+n\gamma}E[(xe^{\sqrt{\sigma_n^2\tau}\xi+n\mu_0-n\gamma-\frac{1}{2}\sigma^2\tau} - e^{-r_n\tau}K)^+] \\
&= e^{-\beta\lambda\tau+n\gamma}(\int_{-d_{2n}}^{\infty}(xe^{\sqrt{\sigma_n^2\tau}y-\frac{1}{2}\sigma_n^2\tau} - e^{-r_n\tau}K)e^{-\frac{y^2}{2}}dy) \\
&= e^{-\beta\lambda\tau+n\gamma}(\int_{-d_{2n}}^{\infty} xe^{-\frac{(y-\sigma_n\sqrt{\tau})^2}{2}}dy - e^{-r_n\tau}KN(d_{2n})) \\
&= e^{-\beta\lambda\tau+n\gamma}(\int_{-d_{1n}}^{\infty} xe^{-\frac{z^2}{2}}dz - e^{-r_n\tau}KN(d_{2n})) \\
&= e^{-\beta\lambda\tau+n\gamma}(xN(d_{1n}) - e^{-r_n\tau}KN(d_{2n})) \\
&= e^{-\beta\lambda\tau+n\gamma}C(V,\tau,r_n,\sigma_n).
\end{aligned}
$$

Combining with the summation term, we get

$$
\begin{aligned}
C^J(V,\tau,r,\sigma) &= \sum_{n=0}^{\infty}\frac{(\lambda\tau)^n}{n!}e^{-\lambda\tau}e^{-\beta\lambda\tau+n\gamma}C(V,\tau,r_n,\sigma_n) \\
&= \sum_{n=0}^{\infty}\frac{(\lambda\tau)^n}{n!}e^{-\lambda(\beta+1)\tau}e^{nln(\beta+1)}C(V,\tau,r_n,\sigma_n) \\
&= \sum_{n=0}^{\infty}\frac{(\lambda'\tau)^n}{n!}e^{-\lambda'\tau}C(V,\tau,r_n,\sigma_n)
\end{aligned}
$$

□

1.2.11 The Influence of Jump Intensity in the Log-normal Jump Process

Beyond the previous analysis, we want to know what is the contribution of jump in the call price.

Proposition 1.2.1 *The call price is increasing in λ.*

Take partial derivative with respect to λ,

$$\frac{\partial C^J(V,\tau,r,\sigma)}{\partial\lambda} = \sum_{n=1}^{\infty}[n(\lambda'\tau)^{n-1}(1+\beta)\frac{\tau}{n!}e^{-\lambda'\tau}C(V,\tau,r_n,\sigma_n)]$$

$$+\sum_{n=0}^{\infty}[\frac{(\lambda'\tau)^n}{n!}(-\tau)(1+\beta)e^{-\lambda'\tau}C(V,\tau,r_n,\sigma_n)] + \sum_{n=0}^{\infty}[\frac{(\lambda'\tau)^n}{n!}e^{-\lambda'\tau}\frac{\partial C(V,\tau,r_n,\sigma_n)}{\partial\lambda}]$$

$$=\sum_{n=1}^{\infty}[\frac{n}{\lambda}\frac{(\lambda'\tau)^n}{(n)!}e^{-\lambda'\tau)}C(V,\tau,r_n,\sigma_n)] + \sum_{n=0}^{\infty}[\frac{(\lambda'\tau)^n}{n!}(-\tau)(1+\beta)e^{-\lambda'\tau}C(V,\tau,r_n,\sigma_n)]$$

$$+\sum_{n=0}^{\infty}[\frac{(\lambda'\tau)^n}{n!}e^{-\lambda'\tau}\frac{\partial C(V,\tau,r_n,\sigma_n)}{\partial\lambda}]$$

$$=\sum_{n=1}^{\infty}[\frac{(\lambda'\tau)^n}{n!}e^{-\lambda'\tau}C(V,\tau,r_n,\sigma_n)(\frac{n}{\lambda}-(1+\beta)\tau)] + (-\tau)(1+\beta)e^{-\lambda'\tau}C_0(V,\tau,r_n,\sigma_n)$$

$$+\sum_{n=0}^{\infty}[\frac{(\lambda'\tau)^n}{n!}e^{-\lambda'\tau}(xN'(d_{1n})\frac{\partial d_{1n}}{\partial r_n}(-\beta) - Ke^{-r_n\tau}(-\tau)(-\beta)N(d_{2n})$$

$$-Ke^{-r_n\tau}N'(d_{2n})\frac{\partial d_{2n}}{\partial r_n}(-\beta))]$$

$$=\sum_{n=1}^{\infty}[\frac{(\lambda'\tau)^n}{n!}e^{-\lambda'\tau}C(V,\tau,r_n,\sigma_n)_n(\frac{n}{\lambda}-(1+\beta)\tau)] + (-\tau)(1+\beta)e^{-\lambda'\tau}C_0(V,\tau,r_n,\sigma_n)$$

$$+\sum_{n=0}^{\infty}[\frac{(\lambda'\tau)^n}{n!}e^{-\lambda'\tau}(-Ke^{-r_n\tau}\tau\beta N(d_{2n}))]$$

$$=\sum_{n=1}^{\infty}[\frac{(\lambda'\tau)^n}{n!}e^{-\lambda'\tau}C_n(V,\tau,r_n,\sigma_n)(\frac{n}{\lambda}-\tau-\beta\tau+\beta\tau\frac{-Ke^{-r_n\tau}N(d_{2n})}{C_n(V,\tau,r_n,\sigma_n)})]$$

$$-\tau(1+\beta)e^{-\lambda'\tau}C_0(V,\tau,r_n,\sigma_n) - e^{-\lambda'\tau}Ke^{-r_0\tau}\tau\beta N(d_{20})$$

$$=\sum_{n=1}^{\infty}[\frac{(\lambda'\tau)^n}{n!}e^{-\lambda'\tau}C_n(V,\tau,r_n,\sigma_n)(\frac{n}{\lambda}-\tau-\beta\tau\frac{xN(d_{1n})}{C_n(V,\tau,r_n,\sigma_n)})]$$

$$-e^{-\lambda'\tau}\tau((1+\beta)C_0(V,\tau,r_n,\sigma_n) + Ke^{-r_0\tau}\beta N(d_{20}))$$

$$=\sum_{n=1}^{\infty}[\frac{(\lambda'\tau)^n}{n!}e^{-\lambda'\tau}C_n(V,\tau,r_n,\sigma_n)(\frac{n}{\lambda}-\tau-\beta\tau\frac{xN(d_{1n})}{C_n(V,\tau,r_n,\sigma_n)})]$$

$$-e^{-\lambda'\tau}\tau(C(V,\tau,r_n,\sigma_n)_0 + \beta xN(d_{10}))$$

$$=\sum_{n=0}^{\infty}[\frac{(\lambda'\tau)^n}{n!}e^{-\lambda'\tau}C_n(V,\tau,r_n,\sigma_n)(\frac{n}{\lambda}-\tau-\beta\tau\frac{xN(d_{1n})}{C_n(V,\tau,r_n,\sigma_n)})]$$

where we used the fact that

$$\frac{\partial d_1}{\partial r_n} = \frac{\partial d_2}{\partial r_n},$$

$$xN'(d_{1n}) = Ke^{-r_n\tau}N'(d_{2n}),$$

$$C_n(V,\tau,r_n,\sigma_n) = xN(d_{1n}) - Ke^{-r_n\tau}N(d_{2n}).$$

Since n can be a very large number in the sum, then the term $\frac{n}{\lambda}-\tau-\beta\tau\frac{xN(d_{1n})}{C_n}$ is positive for most cases. That means the partial derivative w.r.t λ is positive. So the call price is increasing against λ.

In fact, since λ is the intensity of Poisson process, λ increasing means the jump rate is bigger, the fluctuation will make the call price go up.(See Figure 1.1.)

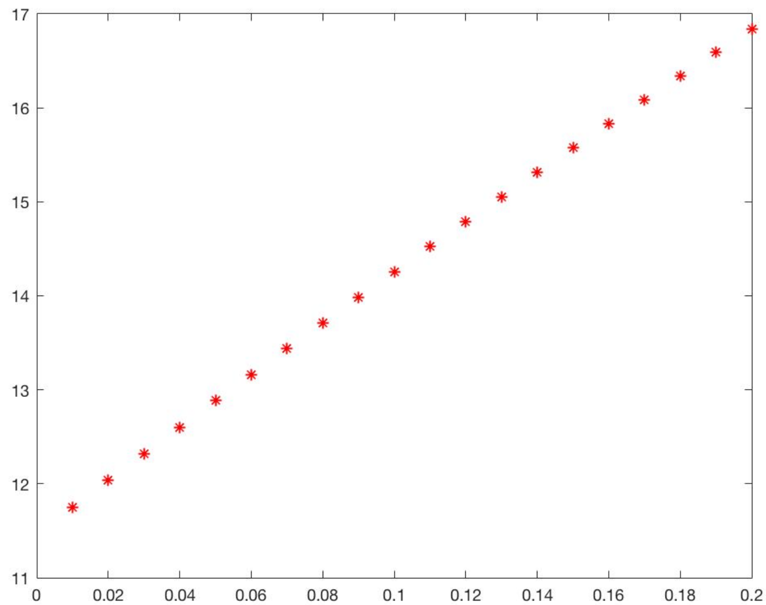

Figure 1.1: Call price - λ (λ is from 0.01 to 0.2 with step size 0.01, $V = 100$, $K = 110$, $\tau = 2$, $\sigma = 0.2$, $r = 0.05$, $\mu_0 = -0.2$, $\sigma_0 = 0.6$, upbound of summation $n = 50$.)

CHAPTER 2

EMPIRICAL STUDY

Empirical study is an important part of financial research by using empirical data to verify model validity and try to make a prediction. In financial mathematics, the major method is quantitative analysis.

2.1 Heston Model

In the last four decades, many modifications were made to relax the restrictions of Black-Scholes model of 1973. Heston model is a solid way by setting the volatility as a stochastic process as well.

2.1.1 Model Dynamics

The Heston model assumes that the underlying stock price S_t follows a Black-Scholes stochastic process.And the stochastic variance v_t follows a CIR process. Then the Heston model has the bi-variate system of SDEs [7]:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_1(t) \tag{2.1}$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_2(t) \tag{2.2}$$

Where μ is the drift, κ is the mean reversion speed for the variance, θ is the long-run mean, σ is the volatility of the variance, dW_1 and dW_2 are two Brownian motions with correlation coefficient ρ. We denote v_0 as the initial value of the variance.

Indeed, replacing $\sigma = 0$ and $\theta = v_t$ and letting $\sigma_{BS} = \sqrt{v_t}$, the Heston model is reduced to general Black-Scholes equation.

By Girsanov's theorem, under the risk-neutral measure $\mathbb{Q}$, the new process SDEs are:

$$dS_t = rS_t dt + \sqrt{v_t} S_t d\tilde{W}_1(t) \tag{2.3}$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} d\tilde{W}_2(t) \tag{2.4}$$

If the stock pays a continuous dividend yield q, we may replace r in the equation (2.3) as $r - q$.

2.1.2 Solution of Heston Model

Like the result of BSM model, the Heston equations also have the solution of a difference including two expressions of probability.

For a European call option written on a stock with strike price K and time-to-maturity τ, has price $C(t,\tau)$ subject to $C(t,\tau) = max\{S(t+\tau, 0) - K, 0\}$ with solution:

$$C(t,\tau) = S(t)\Pi_1(t,\tau; S, R, V) - KB(t,\tau)\Pi_2(t,\tau; S, R, V) \tag{2.5}$$

Where the risk-neutral probability, Π_1 and Π_2, are represented by the respective characteristic functions f_j's for the SVJ model(see Bates ([17]), Heston([18]), Scott([13]) and Bakshi et al ([22])):

$$\Pi_j(t,\tau; S_t, R_t, V_t)) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re[\frac{e^{-i\phi lnK} f_j(t,\tau,S_t,R_t,V_t;\phi)}{i\phi}]d\phi, j = 1,2 \tag{2.6}$$

$$\begin{aligned} f_1 =& exp\{-i\phi lnB(t,\tau) - \frac{\theta_v}{\sigma_v^2}[2ln(1 - \frac{[\xi_v - \kappa_v + (1+i\phi)\rho\sigma_v](1-e^{-\xi_v\tau})}{2\xi_v})] \\ &- \frac{\theta_v}{\sigma_v^2}[\xi_v - \kappa_v + (1+i\phi)\rho\sigma_v]\tau + i\phi lnS(t)] \\ &+ \lambda(1+\mu_J)\tau[(1+\mu_J)^{i\phi}e^{(i\phi/2)(1+i\phi)\sigma_J^2} - 1] - \lambda i\phi\mu_J\tau \\ &+ \frac{i\phi(i\phi+1)(1-e^{-\xi_v\tau})V_t}{2\xi_v - [\xi_v - \kappa_v + (1+i\phi)\rho\sigma_v](1-e^{-\xi_v\tau})}\}, \\ f_2 =& exp\{-i\phi lnB(t,\tau) - \frac{\theta_v}{\sigma_v^2}[2ln(1 - \frac{[\xi_v^* - \kappa_v + i\phi\rho\sigma_v](1-e^{-\xi_v^*\tau})}{2\xi_v^*})] \\ &- \frac{\theta_v}{\sigma_v^2}[\xi_v^* - \kappa_v + i\phi\rho\sigma_v]\tau + i\phi lnS(t)] \\ &+ \lambda(1+\mu_J)\tau[(1+\mu_J)^{i\phi}e^{(i\phi/2)(i\phi-1)\sigma_J^2} - 1] - \lambda i\phi\mu_J\tau \\ &+ \frac{i\phi(i\phi-1)(1-e^{-\xi_v^*\tau})V_t}{2\xi_v^* - [\xi_v^* - \kappa_v + i\phi\rho\sigma_v](1-e^{-\xi_v^*\tau})}\}, \end{aligned}$$

where setting $R(t) = R$ as a constant and $B(t,\tau) = e^{-r\tau}$ in the model, and

$$\xi_v = \sqrt{[\kappa_v - (1+i\phi)\rho\sigma_v]^2 - i\phi(i\phi+1)\sigma_v^2}$$
$$\xi_v^* = \sqrt{[\kappa_v - i\phi\rho\sigma_v]^2 - i\phi(i\phi-1)\sigma_v^2}$$

The SV model can be obtained by setting $\lambda = 0$.

2.1.3 Empirical Test with SSE Results for S&P 500 Index

The Standard & Poor (S&P) 500 option index is a kind of American stock market index, which synthesize the market values of 500 largest companies who have the common stock in NYSE or

NASDAQ. It becomes the most popular subject in financial quantitative study since it is representative and easy to achieve.

Collect N option prices on S&P 500 option and taken from the same period. For each $n = 1, ..., N$, setting τ_n and K_n be the time-to-maturity and the strike prices of the n-th option respectively. Let $\hat{C}_n(t, \tau_n, K_n)$ be the observed price and $C_n(t, \tau_n, K_n)$ the model price as determined by equation (2.5). The difference between $\hat{C}_n$ and C_n is a function of $V(t)$ and parameters. Define

$$\epsilon = \hat{C}_n(t, \tau_n, K_n) - C_n(t, \tau_n, K_n)$$

Then sum of squared dollar pricing errors is defined by

$$SSE(t) = min \sum_{n=1}^{N} |\frac{\epsilon}{BSVega}|^2. \tag{2.7}$$

where $BSVega$ is the Black-Scholes sensitivity of the option price with respect to the market implied volatility V_t:

$$BSVega = Se^{(-q\tau_n)}n(d_n)\sqrt{\tau_n}$$

with

$$d_n = \frac{ln(S/K_n) + (r - q + V_n^2/2)\tau_n}{V_n^2\sqrt{\tau_n}}$$

and $n(x) = e^{-x^2/2}/\sqrt{2\pi}$ is the standard normal density(see Lewis([29]) or Christoffersen et al.([30])).

Considering the S&P 500 index option during September 2012 to August 2013, the previous half year's data, from September 4th 2012 to February 28th 2013, is treated as in-the-sample and the latter period, from March 1st 2013 to August 30th 2013, as out-of-sample.

Before the data analysis, we need to exclude some prices. The filtration includes three criteria[22]. First, the options expire within six days are eliminated since they could induce liquidity related biases. Second, the option price which is low than $\$\frac{3}{8}$ is excluded. Finally, quotes should satisfy the arbitrage restriction:

$$C(t, \tau) \geq max(0, S_0 - K, S_0e^{-qt} - Ke^{-rt}),$$

where the q is the treasure bond rate, which is taken as the dividend yield in the study.

The data is download from the website of Warton Research Data Services (WRDS). After the filtration, 27,363 prices are extracted from the in-the-sample 176,105 entries. Meanwhile, 34,468 observations are left from the out-of-sample 203,709 prices.

The parameters are estimated by $fmincon$ function in Matlab to find the group of parameters which make the SSE value be the minimum.

For in-the-sample, total is $27,363$ prices. The parameters are listed in Table 1 and 2.

κ	θ	σ	ρ	V_t	SSE
20.0000	0.0112	0.7329	0.1954	$imp.V_t^2$	5.12
12.8826	0.0125	0.1399	0.7938	0.0089	4.13

Table 2.1: Heston Model Parameters Optimal Estimation for No Jump Case.

κ	θ	σ	ρ	V_t	λ	μ_j	σ_j	SSE
19.9994	0.0109	0.7244	0.1921	$imp.V_t^2$	0.5207	0.0001	0.0002	5.10
13.2477	0.0131	0.1551	0.6891	0.0090	0.0530	-0.0001	0.0009	4.09

Table 2.2: Heston Model Parameters Optimal Estimation for With Jump Case

	SSE	Average Error	Average Relative Error	
BSM	96.35	15.81	4.4620	
No jump	5.1238	1.9272	0.6035	($V_t = imp.V_t^2$)
	4.1293	1.8763	0.5400	(estimate V_t)
With jump	5.1045	1.8982	0.5770	($V_t = imp.V_t^2$)
	4.0942	1.8305	0.5082	(estimate V_t)

Table 2.3: Comparison with BSM and Heston models for in-the-sample Data

The comparison with BSM model listed in Table 3.

For out-of-sample, total is $34,468$ prices, use the parameters of in-the-sample estimation, results as in Table 4.

	SSE	Average Error	Average Relative Error	
BSM	104.62	17.23	2.4976	
No jump	6.6083	2.9657	0.5995	$V_t = imp.V_t^2$
	5.3216	2.6315	0.4717	(estimate V_t)
With jump	6.5496	2.9459	0.5856	$V_t = imp.V_t^2$
	5.2743	2.5960	0.4561	(estimate V_t)

Table 2.4: Comparison with BSM and Heston models for out-of-sample Data

Heston model has the sum of squared error dramatically less than BSM model by comparing the consequence in the tables above. We can also see for in-the-sample, the SSE values,averages of $|C_{Heston} - C_{market}|$ and averages of relative error $|\frac{C_{Heston} - C_{market}}{C_{market}}|$ improved 1%, 2% and 6% respectively when jump was considered. And for out-of-sample, 1%, 1%, and 3% respectively.

2.2 Option Pricing with Non-IID Jumps

There is an important assumption of jumps diffusion model as we discussed before is that jumps should be identically and independently distribution (IID). In Camara and Li([16]), the non-iid case was considered and the formulas can be modified with dividend yield q.

2.2.1 Jumps with Time-varying Means

The first non-iid case can happen only on the means of the price jumps.

Corollary 2.2.1 *[16] **(Jumps with time-varying means)***

Let $Y_i \sim N(\alpha_i, \gamma^2)$ *and* $Cov(Y_i, Y_j) = 0$. *Then*

$$P_c = \frac{e^{-\lambda T}}{K_1} \sum_{n=0}^{\infty} \frac{(\lambda_n' T)^n}{n!} (S_0 e^{-qT} N(d_{1,n}) - K e^{-r_n T} N(d_{2,n})) \tag{2.8}$$

where

$$d_{1,n} = \frac{ln(\frac{S_0}{K}) + (r_n - q + \frac{\sigma_n^2}{2})T}{\sigma_n \sqrt{T}}, \ d_{2,n} = d_{1,n} - \sigma_n \sqrt{T},$$

$$\lambda_n' = \lambda e^{\bar{\alpha}_n + \gamma^2/2}, \ \sigma_n^2 = \frac{n}{T}\gamma^2 + \sigma^2,$$

$$r_n = r - \frac{ln(K_1)}{T} + \frac{n}{T}(\bar{\alpha}_n + \frac{\gamma^2}{2}), \ \bar{\alpha}_n = \sum_{i=0}^{n} \frac{\alpha_i}{n}, \bar{\alpha}_0 = 0.$$

Where

$$K_1 = \frac{e^{-\lambda T}}{K_1} \sum_{n=0}^{\infty} \frac{(\lambda_n' T)^n}{n!}, \lambda_n' = \lambda e^{\bar{\alpha}_n} + n\bar{\gamma}_n/2,$$

$$\bar{\alpha}_n = \sum_{i=0}^{n} \frac{\alpha_i}{n}, \bar{\gamma}_n = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\gamma_{ij}}{n^2}, \bar{\alpha}_0 = 0, \bar{\gamma}_0 = 0.$$

For the case with jumps, the parameters are using $V_t = 0.009$, $\lambda = 0.053$, $\alpha = \mu_J = -0.0001$, $\gamma = \sigma_J = 0.0009$ consistently.

For corollary 2.2.1, take $n = 50$, $\mu_1 = \mu_2 = \mu_3 = \mu_J = -0.0001$, $\mu_4 = -0.03$, $\mu_5 = -0.01$, μ_6 and the others μ_i are zeroes; $\sigma_i = \sigma_J$. We got the results as $SSE = 96.3076$ for in-the-sample, $SSE = 104.5680$ for out-of-sample. They are very close to BSM model results.

2.2.2 Jumps with Time-varying Variances

The second non-iid case may occur on the variances of the price jumps only.

Corollary 2.2.2 *[16] **(Jumps with time-varying variances)***

Let $Y_i \sim N(\alpha, \gamma_i^2 = \gamma_{ii})$ *and* $Cov(Y_i, Y_j) = 0$. *Then*

$$P_c = \frac{e^{-\lambda T}}{K_1} \sum_{n=0}^{\infty} \frac{(\lambda_n' T)^n}{n!} (S_0 e^{-qT} N(d_{1,n}) - K e^{-r_n T} N(d_{2,n})) \tag{2.9}$$

where

$$d_{1,n} = \frac{ln(\frac{S_0}{K}) + (r_n - q + \frac{\sigma_n^2}{2})T}{\sigma_n \sqrt{T}},\ d_{2,n} = d_{1,n} - \sigma_n \sqrt{T},$$

$$\lambda_n' = \lambda e^{\alpha + \bar{\gamma}_n^2/2},\ \sigma_n^2 = \frac{n}{T}\bar{\gamma}_n^2 + \sigma^2,$$

$$r_n = r - \frac{ln(K_1)}{T} + \frac{n}{T}(\alpha_n + \frac{\bar{\gamma}_n^2}{2}),\ \bar{\gamma}_n^2 = \sum_{i=0}^{n} \frac{\gamma_i^2}{n}, \bar{\gamma}_0^2 = 0.$$

For corollary 2.2.2, take $n = 50$, $\mu_i = \mu_J = -0.0001$; $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_J = 0.0009$, $\sigma_4 = 0.05$, $\sigma_5 = 0.03$, $\sigma_6 = 0.01$ and the others are zeroes. The results are $SSE = 96.3075$ for in-the-sample, $SSE = 104.5680$ for out-of-sample.

2.2.3 Autocorrelated Jumps

The third case of non-iid is only on the autocorrelations of price jumps.

Corollary 2.2.3 *[16]* ***(Autocorrelated Jumps)***

Let $Y_i \sim N(\alpha, \gamma^2)$ *and* $Cov(Y_i, Y_l) = \gamma^2 \rho_{il}$. *Then*

$$P_c = \frac{e^{-\lambda T}}{K_1} \sum_{n=0}^{\infty} \frac{(\lambda_n' T)^n}{n!} (S_0 e^{-qT} N(d_{1,n}) - K e^{-r_n T} N(d_{2,n})) \tag{2.10}$$

where

$$d_{1,n} = \frac{ln(\frac{S_0}{K}) + (r_n - q + \frac{\sigma_n^2}{2})T}{\sigma_n \sqrt{T}},\ d_{2,n} = d_{1,n} - \sigma_n \sqrt{T},$$

$$\bar{\rho}_n = \sum_{i=1, i\neq l}^{n} \sum_{l=1}^{n} \frac{\rho_{il}}{n(n-1)}, \bar{\rho}_0 = 0, \bar{\rho}_1 = 0$$

$$\lambda_n' = \lambda e^{\alpha + \frac{\gamma^2}{2}[1+(n-1)\bar{\rho}_n]},\ \sigma_n^2 = \frac{n}{T}\gamma^2[1 + (n-1)\bar{\rho}_n] + \sigma^2,$$

$$r_n = r - \frac{ln(K_1)}{T} + \frac{n}{T}(\alpha + \frac{\gamma^2}{2}[1 + (n-1)\bar{\rho}_n]).$$

As using the parameters from Heston model, $V_t = 0.009$, $\lambda = 0.053$, $\alpha = \mu_J = -0.0001$, $\gamma = \sigma_J = 0.0009$, $\rho_{il, i\neq l} = 0.6891$, $n = 50$, the estimation results are: $SSE = 93.1146$, average of absolute error is 15.4337,and the average of relative error is 4.3081 for in-the-sample. Respectively, results are $101.0453, 16.8378, 2.4303$ for out-of-sample.

Optionally, suppose autocorrelation is 0.95 between two consecutive jumps and decreases for 5 percent for every next period. Choose $n = 15$, then the farthest autocorrelation is 0.3. The same outcomes are obtained for either in-the-sample or out-of-sample.

2.3 Comparison of BSM Model and Non-iid Cases.

With the models of three cases of non-iid price jumps distribution, we may compare with classic BSM model together.

Firstly, the half year in-the-sample and out-of-sample data are compared within BSM model and non-iid cases.

In-the-Sample	SSE	Average Error	Average Relative Error		improvement percent		
BSM	96.35	15.81	4.462				
Cor 1	96.3076	15.8093	4.4606	$(V_t = imp.V_t^2)$	0.044%	0.004%	0.031%
	93.1146	15.4337	4.3081	$(V_t = 0.009)$	3.358%	2.380%	3.449%
Cor 2	96.3075	15.8093	4.4606	$(V_t = imp.V_t^2)$	0.044%	0.004%	0.031%
	93.1146	15.4337	4.3081	$(V_t = 0.009)$	3.358%	2.380%	3.449%
Cor 3 (n=50)	96.3076	15.8093	4.4606	$(V_t = imp.V_t^2)$	0.044%	0.004%	0.031%
	93.1145	15.4337	4.3081	$(V_t = 0.009)$	3.358%	2.380%	3.449%
(n=15)	96.3076	15.8093	4.4606	$(V_t = imp.V_t^2)$	0.044%	0.004%	0.031%
	93.1146	15.4337	4.3081	$(V_t = 0.009)$	3.358%	2.380%	3.449%
Out-of-Sample							
BSM	104.62	17.23	2.4976				
Cor 1	104.568	17.2294	2.4967	$(V_t = imp.V_t^2)$	0.050%	0.003%	0.036%
	101.0453	16.8378	2.4303	$(V_t = 0.009)$	3.417%	2.276%	2.695%
Cor 2	104.568	17.2294	2.4967	$(V_t = imp.V_t^2)$	0.050%	0.003%	0.036%
	101.0452	16.8378	2.4303	$(V_t = 0.009)$	3.417%	2.276%	2.695%
Cor 3 (n=50)	104.568	17.2294	2.4967	$(V_t = imp.V_t^2)$	0.050%	0.003%	0.036%
	101.0453	16.8378	2.4303	$(V_t = 0.009)$	3.417%	2.276%	2.695%
(n=15)	104.568	17.2294	2.4967	$(V_t = imp.V_t^2)$	0.050%	0.003%	0.036%
	101.0453	16.8378	2.4303	$(V_t = 0.009)$	3.417%	2.276%	2.695%

Table 2.5: Non-iid Cases Analysis for Two Half Years Data

Table 5 shows the comparison with BSM model and three non-iid cases for half-year samples.

Checking the results, the improvement percentages are around 3% for the three noniid corollaries when V_t is fixed as 0.009.

Meanwhile, we want to know the sensitivities for parameters α and γ. Furthermore, the influence of combined movements in both is tested. We checked one day data for simplicity.

For one day data, we analysis 3 cases: First is let α increase from -0.03, -0.01 to -0.0001 then stay. γ is always 0.0009; Second is let α be fixed in -0.0001, γ decrease from 0.05, 0.03 to 0.01, then stay in 0.0009; Third case is α increase from -0.03, -0.01 to -0.0001 then stays. Meanwhile γ decreases from 0.05, 0.03 to 0.01, then stays in 0.0009. Choosing the two first days of the half-year samples then the results are listed in Table 6.

The improvement percentages show that both case 2 and 3 are better than the setup of table 5 but the case 1 is worse. And the case 2 has the best improvement. In-the-sample data has more upgrading percentage than out-of-sample. Corollary 1 and corollary 2 have almost the same improvement effects considering 0.03% fluctuation.

In summary, the non-iid cases model don't have the dramatic different result with BSM model since the non-iid formulas are derived from the classic BSM model without uniform distributions. The core dynamics doesn't change a lot.

In-the-sample	(9/4/2012)	(272 prices)				
	SSE	Average Error	Average Relative Error	Improvement Percent		
BSM	1.0997	15.7474	2.8612			
Cor 1	1.0519	15.2809	2.7669	4.35%	2.96%	3.30%
Case 1	1.0614	15.3664	2.7961	3.48%	2.42%	2.28%
Case 2	1.0423	15.1907	2.7537	5.22%	3.54%	3.76%
Case 3	1.049	15.2544	2.7722	4.61%	3.13%	3.11%
Cor 2	1.0519	15.2809	2.7669	4.35%	2.96%	3.30%
Case 1	1.0614	15.3664	2.7961	3.48%	2.42%	2.28%
Case 2	1.0426	15.1929	2.7541	5.19%	3.52%	3.74%
Case 3	1.0492	15.2561	2.7724	4.59%	3.12%	3.10%
Out-of–sample	(3/1/2013)	(187 prices)				
	SSE	Average Error	Average Relative Error	Improvement Percent		
BSM	0.5747	16.2018	4.5028			
Cor 1	0.5572	15.8389	4.3612	3.05%	2.24%	3.14%
Case 1	0.5623	15.9257	4.4041	2.16%	1.70%	2.19%
Case 2	0.5521	15.7393	4.345	3.93%	2.85%	3.50%
Case 3	0.5551	15.7959	4.3655	3.41%	2.51%	3.05%
Cor 2	0.5572	15.8389	4.3612	3.05%	2.24%	3.14%
Case 1	0.5623	15.9257	4.4041	2.16%	1.70%	2.19%
Case 2	0.5523	15.7425	4.3461	3.90%	2.83%	3.48%
Case 3	0.5553	15.7984	4.3663	3.38%	2.49%	3.03%

Table 2.6: Noniid Cases Analysis of one Day Data for In-the-sample and Out-of-sample.

CHAPTER 3

OPTION PRICING WITH LEVY JUMPS

Levy process is named after French mathematician Paul Levy. In recent two decades, Levy processes are studied in financial quantitative analysis in lots of literature. Levy process is a category of stochastic processes which have independent and stationary increments. Levy process is thus an analog of random walk which is the basic simulation of dynamics of market price over time.

3.1 Levy Processes and Levy Jumps

The classic definition of Levy process is:

Definition 3.1.1 *[2] (Levy process) A cadlag (right continuous and left limit exists) random process is a Levy process if it has $X_0 = 0$ and:*

1. *independent increments;*
2. *stationary increments;*
3. *stochastic continuity.*

The measure ν on $\mathbf{R}^d$ is called Levy measure:

$$\nu(A) = E[\#\{t \in [0,1] : \Delta X_t \neq 0, \Delta X_t \in A\}], A \in \mathcal{B}(\mathbf{R}^d).$$

Any Levy process may be decomposed into the sum of a liner drifting term, a Brownian motion and pure jumps. This is called Levy-Ito decomposition theorem.

Theorem 3.1.1 *(Levy-Ito Decomposition) [2] Let X_t be a Levy process on $\mathbf{R}^d$ and ν its Levy measure.*

1. ν is a Radon measure on $\mathbf{R}^d \backslash \{0\}$ and verifies:

$$\int_{|x|\leq 1} |x|^2 \nu(dx) < \infty, \int_{|x|>1} \nu(dx) < \infty.$$

2. The jump measure of X, denoted by J_X, is a Poisson random measure on $[0,\infty) \times \mathbf{R}^d$ with intensity measure $\nu(dx)dt$.

3. There exist a vector γ and a d-dimensional Brownian motion B_t with covariance matrix A such that

$$X_t = \gamma t + B_t + X_t^l + \lim_{\epsilon \downarrow 0} \tilde{X}_t^\epsilon. \tag{3.1}$$

where

$$\begin{aligned} X_t^l &= \int_{|x|\geq 1, s\in[0,1]} x J_X(ds \times ds) \\ \tilde{X}_t^\epsilon &= \int_{\epsilon\leq|x|<1, s\in[0,1]} x(J_X(ds\times dx) - \nu(dx)ds) \\ &= \int_{\epsilon\leq|x|<1, s\in[0,1]} x\tilde{J}_X(ds\times dx). \end{aligned}$$

The terms in equation (3.1) are independent and the convergence in the last term is a.s. and uniform in t on [0,T].

Any Levy process can be characterized by its characteristic function, which is said Levy-Khinchin representation theorem.

Theorem 3.1.2 *(Levy-Khinchin representation) [2] Let X_t be a Levy process on $\mathbf{R}^d$ with characteristic triplet (A, γ, ν). Then*

$$E[e^{iz\cdot X_t}] = e^{t\psi(z)}, z \in \mathbf{R}^d.$$

where

$$\psi(z) = -\frac{1}{2} z\cdot Az + i\gamma\cdot z + \int_{\mathbf{R}^d} (e^{iz\cdot x} - 1 - iz\cdot x 1_{|x|\leq 1})\nu(dx).$$

Three simple examples are listed afterward to show how to decide the Levy triplets:

Example 1. For a 1-dimensional Brownian motion $W_t \sim N(0,t)$. Then the characteristic function is:

$$E[e^{izW_t}] = \frac{1}{\sqrt{2\pi t}}\int_{\mathbf{R}} e^{izx}e^{-\frac{x^2}{2t}}dx = \frac{1}{\sqrt{2\pi t}}\int_{\mathbf{R}} e^{-\frac{(x-itz)^2-(itz)^2}{2t}}dx = e^{t(\frac{-z^2}{2})}.$$

which means it has the Levy triplet as (1,0,0).

Example 2. For a Poisson process N_t with density λ. The characteristic function is:

$$E[e^{izN_t}] = \sum_{x=0}^{\infty} e^{izx}\frac{e^{-\lambda t}(\lambda t)^x}{x!} = e^{-\lambda t}e^{\lambda t e^{iz}} = e^{\lambda t(e^{iz}-1)}.$$

thereafter,

$$\psi(z) = \lambda(e^{iz}-1).$$

Denote δ_1 as the Dirac function at single point $x = 1$.

Then the Levy triplet is $(0, 0, \lambda\delta_1)$.

Example 3. For a compound Poisson process $Q_t = \sum_{i=1}^{N_t} X_t$, where X_t's are iid with density $f(x)$. The characteristic function is:

$$\begin{aligned}E[e^{izQ_t}] =& E[E[e^{iz\sum_{i=1}^{N_t} X_t}|N_t = n]] = E[E[e^{izX_1}]^n|N_t = n]\\ =& \sum_{n=0}^{\infty}(\phi_{X_1}(z))^n P(N_t = n) = e^{\lambda t(\phi_{X_1}-1)}.\end{aligned}$$

where $\phi_{X_1} = E[e^{izX_1}] = \int_{\mathbf{R}} e^{izx} f(x)dx$ is the characteristic function if X_1. Therefore,

$$\psi(z) = \lambda(\phi_{X_1} - 1) = \int_{\mathbf{R}} \lambda(e^{izx} - 1)f(x)dx$$

since $\int_{\mathbf{R}} f(x) = 1$.

So the Levy triplet is $(\int_{-1}^{1} \lambda x f(dx), 0, \lambda f(x))$.

Theorem 3.1.3 *[2] Let (X_t) be a Levy process on $\mathbf{R}$ with characteristic triplet (A, ν, γ).*

1. (X_t) is a martingale if and only if $\int_{|x|\geq 1} |x|\nu(dx) < \infty$ and $\gamma + \int_{|x|\geq 1} x\nu(dx) = 0$.

2. $exp(X_t)$ is a martingale if and only if $\int_{|x|\geq 1} e^x \nu(dx) < \infty$ and $\frac{A}{2} + \gamma + \int_{\mathbf{R}}(e^x - 1 - x1_{|x|\leq 1})\nu(dx) = 0$.

For a jump-diffusion process,

$$X_t = \sigma W_t + \mu t + J_t = X^c(t) + J_t.$$

The Ito formula is

$$\begin{aligned}f(X_t) - f(X_0) =& \int_0^t f'(X_s)dX_s^c + \int_0^t \frac{\sigma}{2} f''(X_s)ds\\ &+ \sum_{0\leq s\leq t, \Delta X_s\neq 0} [f(X_{s-} + \Delta X_s) - f(X_{s-})]\\ =& \int_0^t f'(X_s)dX_s + \int_0^t \frac{\sigma}{2} f''(X_s)ds\\ &+ \sum_{0\leq s\leq t, \Delta X_s\neq 0} [f(X_{s-} + \Delta X_s) - f(X_{s-}) - \Delta X_s f'(X_{s-})].\end{aligned}$$

since $dX_s^c = dX_s - \Delta X_s$.

This is also true for Levy process. Due to the Ito formula and Levy-Khinchin decomposition, we have

Theorem 3.1.4 *(Martingale-drift decomposition of functions of a Levy process) [2] Let (X_t) be a Levy process with Levy triplet (σ^2, γ, ν) and $f : \mathbf{R} \to \mathbf{R}$ is a C^2 function. Then $Y_t = f(X_t) = M_t + V_t$ where M is the martingale part as*

$$M_t = f(X_0) + \int_0^t f'(X_s)\sigma dW_s + \int_{[0,t]\times\mathbf{R}^d} \tilde{J}_X(ds \times dy)[f(X_{s-} + y) - f(X_{s-})],$$

and V_t is a continuous finite variation process:

$$V_t = \int_0^t \frac{\sigma^2}{2} f''(X_s)ds + \int_0^t \gamma f'(X_s)ds$$
$$+ \int_{[0,t]\times\mathbf{R}} ds\nu(dy)[f(X_{s-}+y) - f(X_{s-}) - yf'(X_{s-})1_{|y|\leq 1}].$$

3.2 Change Measure

As what we discussed before, the utilization of Levy process in financial market also needs the risk-neutral measure change. There is the general theorem about the equivalence of Levy processes, proof is referred to [3] for more details.

Theorem 3.2.1 *Let $(X_t, \mathbf{P})$ and $(X_t, \mathbf{Q})$ be two Levy processes on $\mathbf{R}$ with characteristic triplets (σ^2, γ, ν) and $(\sigma'^2, \gamma', \nu')$. Then $\mathbf{P}|_{\mathcal{F}_t}$ and $\mathbf{Q}|_{\mathcal{F}_t}$ are equivalent for all t if and only if three following conditions are satisfied*

1. *$\sigma = \sigma'$;*
2. *The Levy measures are equivalent with*

$$\int_{\mathbf{R}} (e^{\phi(x)/2} - 1)^2 \nu(dx) < \infty.$$

When $\mathbf{P}$ and $\mathbf{Q}$ are equivalent, the Radon-Nikodym derivative is

$$\frac{d\mathbf{P}|_{\mathcal{F}_t}}{d\mathbf{Q}|_{\mathcal{F}_t}} = e^{U_t}.$$

with

$$U_t = \eta X_t^c - \frac{\eta^2\sigma^2 t}{2} - \eta\gamma t + \lim_{\epsilon\downarrow 0}(\sum_{s\leq t, |\Delta X_s|>\epsilon} \phi(\Delta X_s) - t\int_{|x|>\epsilon} (e^{\phi(x)} - 1)\nu(dx)).$$

Here (X_t^c) is the continuous part and η is such that

$$\gamma' - \gamma - \int_{-1}^{1} x(\nu' - \nu)(dx) = \sigma^2\eta$$

if $\sigma > 0$ or zero if $\sigma = 0$.

U_t is a Levy process with characteristic triplet (a_U, ν_U, γ_U) given as

$$a_U = \sigma^2\eta^2,$$

$$\nu_u = \nu\phi_{-1},$$

$$\gamma_U = -\frac{1}{2}\sigma^2\eta^2 - \int_{\mathbf{R}} (e^y - 1 - y1_{|y|\leq 1})(\nu\phi^{-1})(dy).$$

3.3 Option Pricing with Levy Jumps

In the BSM model, the dynamics of asset price is described as an exponential of Brownian motion with drifting term like

$$S_t = S_0 exp(\mu t + \sigma W_t) = S_0 exp(B_t),$$

where $B_t = \mu t + \sigma W_t$.

Replace B_t by a Levy process, we get $S_t = S_0 exp X_t$, where X_t is a Levy process. We call it as exponential-Levy model.

3.3.1 Exponential of Levy Process

For the exponential of Levy process, the similar result is given as theorem 3.1.4:

Theorem 3.3.1 *[2] Let (X_t) be a Levy process with Levy triplet (σ^2, γ, ν) satisfying*

$$\int_{|y|\geq 1} e^y \nu(dy) < \infty.$$

Then $Y_t = exp(X_t)$ is a semimartingale with decomposition $Y_t = M_t + A_t$, M_t is the martingale part

$$M_t = 1 + \int_0^t Y_{s-}\sigma dw_t + \int_{[0,t]\times\mathbf{R}} Y_{s-}(e^z - 1)\tilde{J}_X(dsdz),$$

and the continuous finite variation drift part is

$$A_t = \int_0^t Y_{s-}[\gamma + \frac{\sigma^2}{2} + \int_{\mathbf{R}} (e^z - 1 - z1_{|z|\leq 1})\nu(dz)]ds.$$

Therefore, (Y_t) is a martingale if and only if

$$\gamma + \frac{\sigma^2}{2} + \int_{\mathbf{R}} (e^z - 1 - z1_{|z|\leq 1})\nu(dz) = 0.$$

3.3.2 European Option with Levy Process and PIDE

With exponential-Levy model, the European option price can be studied as well and we have the PIDE result since the martingale property.

Consider the European option with maturity T and payoff $H(S_T)$ which satisfies Lipschitz condition $|H(x) - H(y)| \leq c|x - y|$ for some $c > 0$. The option value is $C(t, s) = e^{-r\tau} E[H(S_T)|\mathcal{F}_t] = e^{-r\tau} E[H(S_t e^{r\tau + X_\tau})]$. The risk-neutral dynamic of S_t is then given by [2]

$$S_t = S_0 + \int_0^t rS_{u-}du + \int_0^t S_{u-}\sigma dW_u + \int_0^t \int_{\mathbf{R}} (e^x - 1)S_{u-}\hat{J}_X(dudx),$$

where $\hat{J}_X$ is the compensated jump measure of the Levy process X and $\hat{S}_t = e^{X_t}$ is a martingale:

$$\frac{d\hat{S}_t}{\hat{S}_{t-}} = \sigma dW_t + \int_{\mathbf{R}} (e^x - 1)\hat{J}_X(dtdx),$$

verifies $\sup_{t\in[0,T]} E[\hat{S}^2] < \infty$.

Proposition 3.3.1 *(Backward PIDE for European option with Levy process)[2] Consider a market with the risk-neutral dynamics of asset given by an exponential Levy process* $S_t = S_0 exp(rt + X_t)$, *where* (X_t) *is a Levy process with Levy triplet* (σ^2, γ, ν) *under* $\mathbf{Q}$ *such that* $\tilde{S}_t = e^{-rt}S_t = e^{X_t}$ *is a martingale.*

Suppose $\int_{|y|\geq 1} e^{2y}\nu(dy) < \infty$. *If either* $\sigma > 0$ *or*

$$\exists\beta \in [0, 2], \lim_{\epsilon\downarrow 0} \epsilon^{-\beta} \int_{-\epsilon}^{\epsilon} |x|^2 d\nu(x) > 0,$$

then the value of a European option with terminal payoff $H(S_T)$ *is given by* $C(t, s) : [0, T] \times (0, \infty) \to \mathbf{R}$, *which is continuous and verifies the partial integro-differential equation:*

$$\begin{aligned} &\frac{\partial C}{\partial t}(t, S) + r\frac{\partial C}{\partial S}(t, S) + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial^2 S}(t, S) - rC(t, S) \\ &\quad + \int \nu(dy)[C(t, Se^y) - C(t, S) - S(e^y - 1)\frac{\partial C}{\partial S}(t, S)] = 0 \end{aligned} \tag{3.2}$$

on $[0, T] \times (0, \infty)$ *with the terminal condition:*

$$C(T, S) = H(S).$$

Proof. [2] Applying the martingale $\hat{C}(t, S_t) = e^{-rt}C(t, S_t)$,

$$\begin{aligned} d\hat{C}(t, S_t) =& e^{-rt}[-rC + \frac{\partial C}{\partial t}(t, S_{t-}) + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 C}{\partial S^2}(t, S_{t-})dt + \frac{\partial C}{\partial S}(t, S_{t-})dS_t] \\ &+ e^{-rt}[C(t, S_{t-}e^{\Delta X_t)}) - C(t, S_{t-}) - S_{t-}(e^{\Delta X_t} - 1)\frac{\partial C}{\partial S}(t, S_{t-})] \\ =& a(t)dt + dM_t, where \\ a(t) =& e^{-rt}[-rC + \frac{\partial C}{\partial t} + \frac{\sigma^2 S_{t-}^2}{2}\frac{\partial^2 C}{\partial S^2}](t, S_{t-}) \\ &+ \int_{\mathbf{R}} e^{-rt}[C(t, S_{t-}e^x) - C(t, S_{t-}) - S_{t-}(e^x - 1)\frac{\partial C}{\partial S}(t, S_{t-})]\nu(dx), \\ dM_t =& e^{-rt}[\frac{\partial C}{\partial t}(t, S_{t-})\sigma S_{t-}dW_t + \int_{\mathbf{R}}[C(t, S_{t-}e^x) - C(t, S_{t-})]\hat{J}_X(dtdx)]. \end{aligned}$$

Since the payoff function H is Lipschitz, C is Lipschitz w.r.t x as well:

$$\begin{aligned} C(t, x) - C(t, y) &= e^{-t\tau}[E[H(xe^{r\tau + X_T - X_t}] - E[H(ye^{r\tau + X_T - X_t}]] \\ &\leq c|x - y|E[e^{X_\tau}] = c|x - y|, \end{aligned}$$

since e^{X_t} is a martingale with expectation of 1.

Then the predictable random function $C(t, S_{t-}e^x) - C(t, S_{t-})$ satisfies:

$$\begin{aligned} E[\int_0^T dt \int_{\mathbf{R}} |C(t, S_{t-}e^x) - C(t, S_{t-})^2|]\nu(dx) &\leq E[\int_0^T dt \int_{\mathbf{R}} c^2(e^{2x} + 1)S_{t-}^2\nu(dx) \\ &\leq c^2 \int_{\mathbf{R}} (e^{2x} + 1)\nu(dx)E[\int_0^T S_{t-}^2 dt] < \infty. \end{aligned}$$

so by the isometric property, the compensated integral

$$\int_0^t \int_{\mathbf{R}} e^{-rt}[C(t, S_{t-}e^x) - C(t, S_{t-})]\hat{J}_X(dtdx)$$

is a martingale.

On the other hand, C is Lipschitz, $sup\frac{\partial C}{\partial S}(t, .) \leq c$ insults

$$E[\int_0^T S_{t-}^2|\frac{\partial C}{\partial S}(t, S_{t-})dt| \leq c^2 E[\int_0^T S_{t-}^2 dt] < \infty.$$

Then $\int_0^T \sigma S_{t-}\frac{\partial C}{\partial S}(t, S_{t-})dW_t$ is also a martingale by isometry.

Thereafter, $\hat{C}$ is a martingale makes the dt term is zero, that means $a(t) = 0$. We proved PIDE (3.2). □

CHAPTER 4

BASIC CREDIT RISK ANALYSIS WITH JUMPS

The financial market has a big part which is called credit market or bond market. The participants can issue new debts or securities on the credit market. Credit risk is the crucial problem for the loss risk of borrower's failure to meet the obligations.

4.1 Basic Credit Risk Concepts

Assume that we are in the setting of the standard Black-Scholes model, i.e. we analyze a market with continuous trading which is frictionless and competitive with assumptions. (See [4])

1. agents are price takers.

2. there are no transaction costs.

3. there is unlimited access to short selling and no indivisibilities of assets.

4. borrowing and lending through a money-market account can be done at some riskless, continuously compounded rate r.

We want to price bonds issued by a firm whose assets are assumed to follow a geometric Brownian motion:

$$dV_t = \mu V_t dt + \sigma V_t dW_t$$

Here, W is a standard Brownian motion under the probability measure $\mathbf{P}$.

Let the starting value of assets is V_0. Then by Ito-Doeblin formula:

$$V_t = V_0 exp((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t)$$

We take it to be well known that in an economy consisting of these two assets, the price C_0 at time 0 of a contingent claim paying $C(V_T)$ at time T is equal to

$$C_0 = E^Q[e^{-rt}C_T]$$

where $\mathbf{Q}$ is the equivalent martingale measure under which the dynamics of V are given as

$$V_t = V_0 exp((r - \frac{1}{2}\sigma^2)t + \sigma W_t^Q)$$

Here, W_t^Q is a Brownian motion and we can see that the drift term μ has been replaced by r.[4]

Now, assume that the firm at time 0 has issued two types of claims: debt and equity. In the simple model, debt is a zero-coupon bond with a face value of D and maturity date T. We think of the firm run by the equity owners. At maturity of bond, equity holder pay the face value of debt precisely when the assets value is higher than the face value of the bond. On the other hand, if assets are worth less than D, equity owners do not want to pay D. And since they have limited liability they don't have to do that. Bond holders then take over the remaining assets of V_T instead of the promised payment D. With this assumption, the payoffs to debt, B_T, and equity, S_T, at date T are given as:

$$B_T = min(D, V_T) = D - max(D - V_T, 0)$$

$$S_T = max(V_T - D, 0)$$

From the structure, debt can be viewed as the difference between a riskless bond and a put option, and equity can be viewed as a call option on the firm's assets. [4]

We assumed there are no transaction costs, bankruptcy costs, taxes and so on for simpleness. We then get $V_T = B_T + S_T$. Given the current level V and volatility σ of assets, and the riskless rate r, we denote the Black-Scholes model of European call as $C(V_t, D, \sigma, r, T - t)$ with strike price D and maturity time T, [4] i.e.

$$C(V_t, D, \sigma, r, T - t) = V_t N(d_1) - De^{-r(T-t)} N(d_2)$$

Where N is the standard normal distribution function and

$$d_{1,2} = \frac{ln(V_t/D) + (r \pm \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}},$$

$$d_1 - d_2 = \sigma\sqrt{T - t}.$$

Applying the Black-Scholes formula to price these options, we obtain the Merton model for values of debt and equity at time t as:

$$S_t = C(V_t, D, \sigma, r, T - t)$$

$$B_t = De^{(-r(T-t))} - P(V_t, D, \sigma, r, T - t)$$

From the put-call parity for European options on non-dividend paying stocks

$$C(V_t) - P(V_t) = V_t - De^{-r(T-t)}$$

We get

$$
\begin{aligned}
B_t &= De^{(-r(T-t))} - P(V_t) \\
&= De^{(-r(T-t))} + V_t - De^{(-r(T-t))} - C(V_t) \\
&= V_t - C(V_t) \\
&= V_t - (V_t N(d_1) - De^{-r(T-t)} N(d_2)) \\
&= V_t(1 - N(d_1)) + De^{-r(T-t)} N(d_2).
\end{aligned}
$$

4.2 Basic Credit Risk Analysis with Compound Poisson Jumps

For the BSM model with Levy jumps, we may consider the case of compound Poisson jumps which has the explicit formula for the the call price. Therefore, the bond price is obvious from the call-put parity and equality $V_T = B_T + S_T$.

Suppose asset value V_t has dynamics of jumps, then the equity value S_t is priced as a call option C^J with jumps. First, we focus the compound Poisson jumps with i.i.d.log-normal distributed $Y_i + 1$ (ie, $ln(Y_i + 1) \sim N(\mu, \delta^2)$) which has the explicit formula, the price of call option C^J is (1.11)([6]):

$$
C^J(V_t, D, \tau, \sigma^2, r, \delta^2, \lambda, k) = \sum_{n=0}^{\infty} \frac{(\lambda' \tau)^n}{n!} e^{-\lambda' \tau} C(V_t, D, \tau, \sigma_n, r_n)
$$

where $C(V_t, D, \tau, \sigma_n, r_n)$ is the standard Black-Scholes formula for a call and

$$
k = E(Y_i),
$$

$$
\lambda' = \lambda(1 + k),
$$

$$
r_n = r + n\gamma/\tau - \lambda k,
$$

$$
\sigma_n^2 = \sigma^2 + n\delta^2/\tau,
$$

$$
\gamma = ln(1 + k) = \mu + \frac{1}{2}\delta^2.
$$

In advance, some facts of general Black-Scholes call price are listed:

$$
C_x = N(d_1) = \Delta > 0,
$$

$$
C_\tau = \frac{S_t \sigma}{2\sqrt{\tau}} n(d_1) + Kre^{-r\tau} N(d_2) = \Theta > 0,
$$

$$
C_\sigma = S_t \sqrt{\tau} n(d_1) = Vega > 0,
$$

$$
C_r = \tau K e^{-r\tau} N(d_2) = Rho > 0,
$$

$$
C_K = -e^{-r\tau} N(d_2) < 0.
$$

4.2.1 Sensitivities of Bond Pricing for Log-normal Jumps Process

Since the explicit formula, the derivatives with respect to all parameters are disclosed as the sensitivities of bond price.

Proposition 4.2.1 *(i) The bond price is increasing in V_t for log-normal jumps process.*
(ii) $\frac{\partial B_t}{\partial x} \in (0, 1)$.

Let $V_t = x$, $T - t = \tau$. Then we check the partial derivative of B_t with respect to x.

Hence,

$$\begin{aligned}\frac{\partial B_t}{\partial x} &= \frac{\partial}{\partial x}(V_t - C^J(V_t, D, \tau, \sigma^2, r, \delta^2, \lambda, k)) \\ &= 1 - \frac{\partial}{\partial x}(\sum_{n=0}^{\infty} \frac{(\lambda'\tau)^n}{n!} e^{-\lambda'\tau} C(V_t, D, \tau, \sigma_n, r_n)) \\ &= 1 - \sum_{n=0}^{\infty} \frac{(\lambda'\tau)^n}{n!} e^{-\lambda'\tau} \frac{\partial C(V_t, D, \tau, \sigma_n, r_n)}{\partial x}) \\ &= 1 - \sum_{n=0}^{\infty} \frac{(\lambda'\tau)^n}{n!} e^{-\lambda'\tau} N(d_{1n}) \\ &= \sum_{n=0}^{\infty} \frac{(\lambda'\tau)^n}{n!} e^{-\lambda'\tau}(1 - N(d_{1n})) \in (0, 1).\end{aligned}$$

Where $\sum_{n=0}^{\infty} \frac{(\lambda'\tau)^n}{n!} e^{-\lambda'\tau} = 1$ is convergent.

It is clear that the bond price goes up as V_t increases.(See Figure **??**).

Proposition 4.2.2 *(i) The bond price is increasing in face value D for log-normal jumps process.*
(ii) $\frac{\partial B_t}{\partial D} \in (0, e^{-(r-\lambda k)\tau})$.

In fact,

$$\begin{aligned}\frac{\partial C_J(V_t, D, \tau, \sigma^2, r, \delta^2, \lambda, k)}{\partial D} &= \frac{\partial}{\partial D}(\sum_{n=0}^{\infty} \frac{(\lambda'\tau)^n}{n!} e^{-\lambda'\tau} C(V_t, D, \tau, \sigma_n, r_n)) \\ &= \sum_{n=0}^{\infty} \frac{(\lambda'\tau)^n}{n!} e^{-\lambda'\tau} \frac{\partial C(V_t, D, \tau, \sigma_n, r_n)}{\partial D} \\ &= -\sum_{n=0}^{\infty} \frac{(\lambda'\tau)^n}{n!} e^{-\lambda'\tau} e^{-r_n\tau} N(d_{2n}) \\ &= -\sum_{n=0}^{\infty} \frac{(\lambda'\tau)^n}{n!} e^{-\lambda'\tau} e^{-(r\tau + n\gamma - \lambda k\tau)} N(d_{2n}) \\ &= -e^{-(r-\lambda k)\tau} \sum_{n=0}^{\infty} \frac{(\lambda'\tau)^n}{n!} e^{-\lambda'\tau} e^{-n\gamma} N(d_{2n}) \\ &\in (-e^{-(r-\lambda k)\tau}, 0).\end{aligned}$$

Then, $\frac{\partial B_t}{\partial D} = -\frac{\partial C_J(V_t, D, \tau, \sigma^2, r, \delta^2, \lambda, k)}{\partial D} \in (0, e^{-(r-\lambda k)\tau})$.

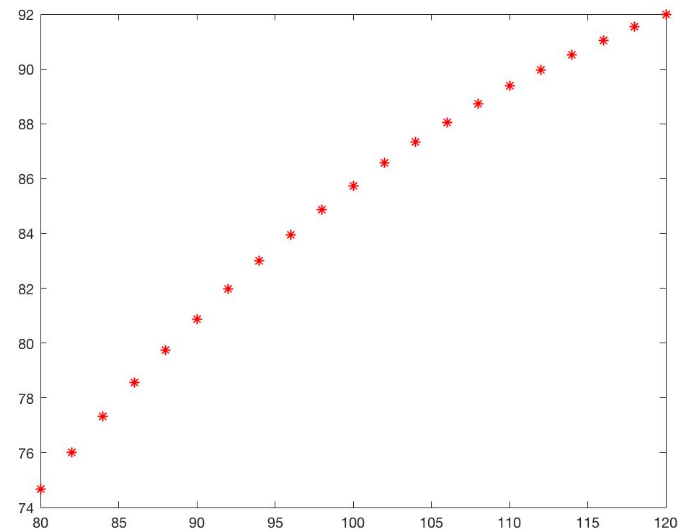

Figure 4.1: Bond price - V_t. (V_t is from 80 to 120 with step size 2, $D = 110$, $\tau = 2$, $\sigma = 0.2$, $r = 0.05$, $\lambda = 0.1$, $\mu = -0.2$, $\delta = 0.6$, upbound of summation $n = 50$.)

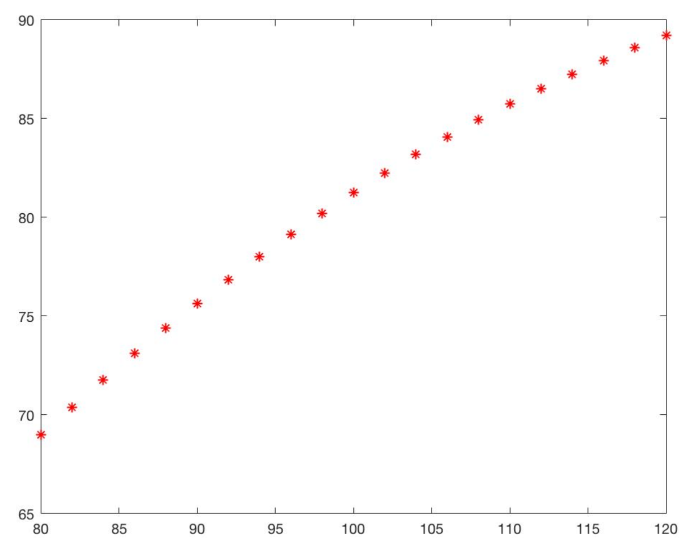

Figure 4.2: Bond price - D. (D is from 80 to 120 with step size 2, $V_t = 100$, $\tau = 2$, $\sigma = 0.2$, $r = 0.05$, $\lambda = 0.1$, $\mu = -0.2$, $\delta = 0.6$, upbound of summation $n = 50$.)

Definitely, increasing the face value typically will produce a larger payoff. (See Figure 4.2).

Proposition 4.2.3 *The bond price is decreasing in volatility, σ, for log-normal jumps process.*

Mathematically,

$$\begin{aligned}\frac{\partial C_J(V_t, D, \tau, \sigma^2, r, \delta^2, \lambda, k)}{\partial \sigma} &= \frac{\partial}{\partial \sigma}(\sum_{n=0}^{\infty} \frac{(\lambda' \tau)^n}{n!} e^{-\lambda' \tau} C(V_t, D, \tau, \sigma_n, r_n)) \\ &= \sum_{n=0}^{\infty} \frac{(\lambda' \tau)^n}{n!} e^{-\lambda' \tau} \frac{\partial C(V_t, D, \tau, \sigma_n, r_n)}{\partial \sigma_n} \\ &= \sum_{n=0}^{\infty} \frac{(\lambda' \tau)^n}{n!} e^{-\lambda' \tau} x \sqrt{\tau} n(d_{1n}) \\ &= x\sqrt{\tau} \sum_{n=0}^{\infty} \frac{(\lambda' \tau)^n}{n!} e^{-\lambda' \tau} n(d_{1n}) > 0.\end{aligned}$$

Then, $\dfrac{\partial B_t}{\partial \sigma} = -\dfrac{\partial C_J(V_t, D, \tau, \sigma^2, r, \delta^2, \lambda, k)}{\partial \sigma} < 0.$

When the volatility goes up, B_t must decrease because the sum of S_t and B_t remains unchanged, call price increases as V_t is more fluctuable. (See Figure 4.3).

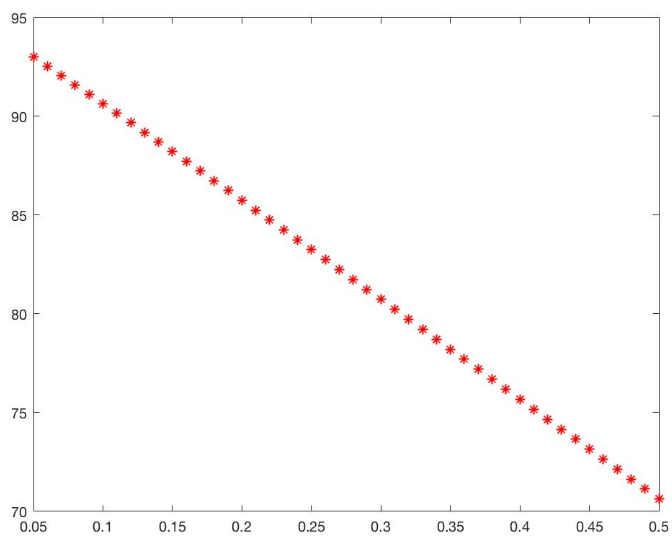

Figure 4.3: Bond price - σ. (σ is from 0.05 to 0.5 with step size 0.01, $V_t = 100$, $D = 110$, $\tau = 2$, $r = 0.05$, $\lambda = 0.1$, $\mu = -0.2$, $\delta = 0.6$, upbound of summation $n = 50$.)

Proposition 4.2.4 *(i) The bond price is decreasing in risk-free interest rate r for log-normal jumps process.*

(ii) $\dfrac{\partial B_t}{\partial r} \in (-\tau D e^{-(r-\lambda k)\tau}, 0).$

Actually,

$$
\begin{aligned}
\frac{\partial C_J(V_t, D, \tau, \sigma^2, r, \delta^2, \lambda, k)}{\partial r} &= \frac{\partial}{\partial r}\left(\sum_{n=0}^{\infty} \frac{(\lambda'\tau)^n}{n!} e^{-\lambda'\tau} C(V_t, D, \tau, \sigma_n, r_n)\right) \\
&= \sum_{n=0}^{\infty} \frac{(\lambda'\tau)^n}{n!} e^{-\lambda'\tau} \frac{\partial C(V_t, D, \tau, \sigma_n, r_n)}{\partial r_n} \\
&= \sum_{n=0}^{\infty} \frac{(\lambda'\tau)^n}{n!} e^{-\lambda'\tau} \tau D e^{-r_n\tau} N(d_{2n}) \\
&= \tau D e^{-(r-\lambda k)\tau} \sum_{n=0}^{\infty} \frac{(\lambda'\tau)^n}{n!} e^{-n\gamma} N(d_{2n}) \\
&\in (0, \tau D e^{-(r-\lambda k)\tau}).
\end{aligned}
$$

Then, $\frac{\partial B_t}{\partial r} = -\frac{\partial C_J(V_t, D, \tau, \sigma^2, r, \delta^2, \lambda, k)}{\partial r} \in (-\tau D e^{-(r-\lambda k)\tau}, 0)$.

Since the call option increases as r goes up, B_t must decrease the money market looks more attractive.(See Figure 4.4).

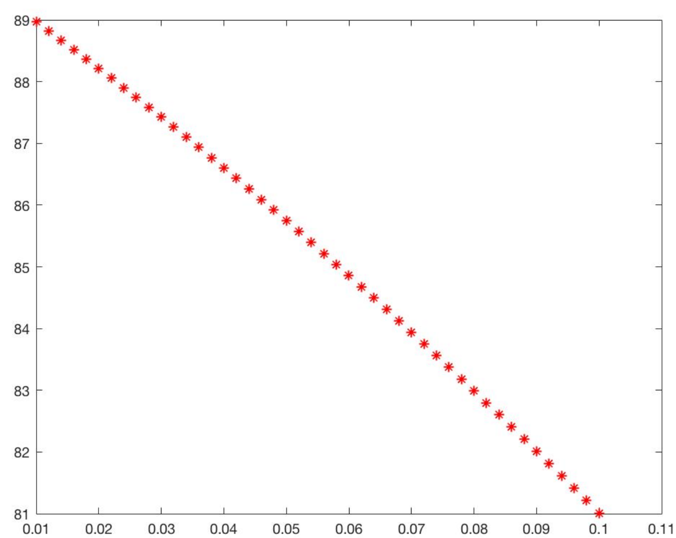

Figure 4.4: Bond price - r. (r is from 0.01 to 0.1 with step size 0.002, $V_t = 100$, $D = 110$, $\tau = 2$, $\sigma = 0.2$, $\lambda = 0.1$, $\mu = -0.2$, $\delta = 0.6$, upbound of summation $n = 50$.)

Proposition 4.2.5 *(i) The call price is increasing in time-to-maturity, τ, for the log-normal jumps process if $r - \lambda k \geq 0$.*

(ii) The bond price is decreasing in time-to-maturity, τ, for the log-normal jumps process $r - \lambda k \geq 0$.

$$
\begin{aligned}
&\frac{\partial C^J(V_t, D, \tau, \sigma^2, r, \delta^2, \lambda, k)}{\partial \tau} = \frac{\partial}{\partial \tau}(\sum_{n=0}^{\infty} \frac{(\lambda' \tau)^n}{n!} e^{-\lambda' \tau} C(V_t, D, \tau, \sigma_n, r_n)) \\
=&\sum_{n=1}^{\infty}[n(\lambda'\tau)^{n-1}\frac{\lambda'}{n!}e^{-\lambda'\tau}C_n(V_t, D, \tau, \sigma_n, r_n)] + \sum_{n=0}^{\infty}[\frac{(\lambda'\tau)^n}{n!}(-\lambda')e^{-\lambda'\tau}C_n(V_t, D, \tau, \sigma_n, r_n)] \\
&+ \sum_{n=0}^{\infty}[\frac{(\lambda'\tau)^n}{n!}e^{-\lambda'\tau}(\frac{\partial C(V_t, D, \tau, \sigma_n, r_n)}{\partial \tau} + \frac{\partial C(V_t, D, \tau, \sigma_n, r_n)}{\partial r_n}\frac{\partial r_n}{\partial \tau} \\
&+ \frac{\partial C(V_t, D, \tau, \sigma_n, r_n)}{\partial \tau}\frac{\partial \sigma_n}{\partial \tau})] \\
=&\sum_{n=1}^{\infty}[\frac{(\lambda'\tau)^{n-1}}{(n-1)!}\lambda' e^{-\lambda'\tau}C_n(V_t, D, \tau, \sigma_n, r_n)] + \sum_{n=0}^{\infty}[\frac{(\lambda'\tau)^n}{n!}(-\lambda')e^{-\lambda'\tau}C_n(V_t, D, \tau, \sigma_n, r_n)] \\
&+ \sum_{n=0}^{\infty}[\frac{(\lambda'\tau)^n}{n!}e^{-\lambda'\tau}(\frac{\partial C(V_t, D, \tau, \sigma_n, r_n)}{\partial \tau} + \frac{\partial C(V_t, D, \tau, \sigma_n, r_n)}{\partial r_n}\frac{\partial r_n}{\partial \tau} \\
&+ \frac{\partial C(V_t, D, \tau, \sigma_n, r_n)}{\partial \tau}\frac{\partial \sigma_n}{\partial \tau})] \\
=&\sum_{n=0}^{\infty}[\frac{(\lambda'\tau)^n}{n!}\lambda' e^{-\lambda'\tau}(C_{n+1}(V_t, D, \tau, \sigma_n, r_n) - C_n(V_t, D, \tau, \sigma_n, r_n))] \\
&+ \sum_{n=0}^{\infty}[\frac{(\lambda'\tau)^n}{n!}e^{-\lambda'\tau}(\frac{\partial C(V_t, D, \tau, \sigma_n, r_n)}{\partial \tau} + \frac{\partial C(V_t, D, \tau, \sigma_n, r_n)}{\partial r_n}\frac{\partial r_n}{\partial \tau} \\
&+ \frac{\partial C(V_t, D, \tau, \sigma_n, r_n)}{\partial \tau}\frac{\partial \sigma_n}{\partial \tau})] \\
=&S_1 + S_2,
\end{aligned}
$$

Separately, we consider the 1st part S_1 and 2nd part S_2, Since

$$
\begin{aligned}
\frac{\partial C(V_t, D, \tau, \sigma_n, r_n)}{\partial n} &= xN'(d_{1n})\frac{\partial d_{1n}}{\partial n} - De^{-r_n\tau}(-\frac{\partial r_n}{\partial n})\tau N(d_{2n}) + De^{-r_n\tau}N'(d_{2n})\frac{\partial d_{2n}}{\partial n} \\
&= xN'(d_{1n})(\frac{\partial d_{1n}}{\partial n} - \frac{\partial d_{2n}}{\partial n}) + De^{-r_n\tau}\gamma\tau N(d_{2n}) > 0
\end{aligned}
$$

Where we used the facts:

$$
xN'(d_{1n}) = Ke^{-r_n\tau}N'(d_{2n}),
$$

$$
d_{1n} - d_{2n} = \sigma_n\sqrt{\tau} \Rightarrow \frac{\partial d_{1n}}{\partial n} - \frac{\partial d_{2n}}{\partial n} = \frac{1}{2\sqrt{\sigma_n}}\frac{\delta^2}{\tau}\sqrt{\tau} > 0.
$$

That means the function $C(V_t, D, \tau, \sigma_n, r_n)$ is increasing w.r.t n, such that S_1 part is positive.

For S_2, we taking partial derivative of $C(V_t, D, \tau, \sigma_n, r_n)$ w.r.t τ,

$$
\begin{aligned}
S_2 =& \frac{\partial C(V_t, D, \tau, \sigma_n, r_n)}{\partial \tau} + \frac{\partial C(V_t, D, \tau, \sigma_n, r_n)}{\partial r_n}\frac{\partial r_n}{\partial \tau} + \frac{\partial C(V_t, D, \tau, \sigma_n, r_n)}{\partial \tau}\frac{\partial \sigma_n}{\partial \tau} \\
=& \frac{x\sigma_n}{2\sqrt{\tau}} n(d_{1n}) + Dr_n e^{-r_n\tau} N(d_{2n}) + \tau D e^{-r_n\tau} N(d_{2n})[n\gamma(-\tau^{-2})] \\
& + x\sqrt{\tau} n(d_{1n})[\frac{1}{2}(\sigma_n^2)^{-\frac{1}{2}} n\delta^2(-\tau^{-2})] \\
=& \frac{1}{2\sqrt{\tau}} x n(d_{1n}) \frac{\sigma_n^2\tau - n\delta^2}{\sigma_n \tau} + De^{-r_n\tau} N(d_{2n}) \frac{r_n\tau - n\gamma}{\tau} \\
=& \frac{1}{2\sqrt{\tau}} x n(d_{1n}) \frac{\sigma^2}{\sigma_n} + De^{-r_n\tau} N(d_{2n})(r - \lambda k).
\end{aligned}
$$

here $r - \lambda k$ is non-negative as the condition, then S_2 is also positive, such that the initial $\dfrac{\partial C_J(V_t, D, \tau, \sigma^2, r, \delta^2, \lambda, k)}{\partial \tau}$ is positive.

Hence $\dfrac{\partial B_t}{\partial k} = -\dfrac{\partial C_J(V_t, D, \tau, \sigma^2, r, \delta^2, \lambda, k)}{\partial k} < 0$. B_t is decreasing due to the value of call increases when time-to-maturity is bigger. (See Figure 4.5).

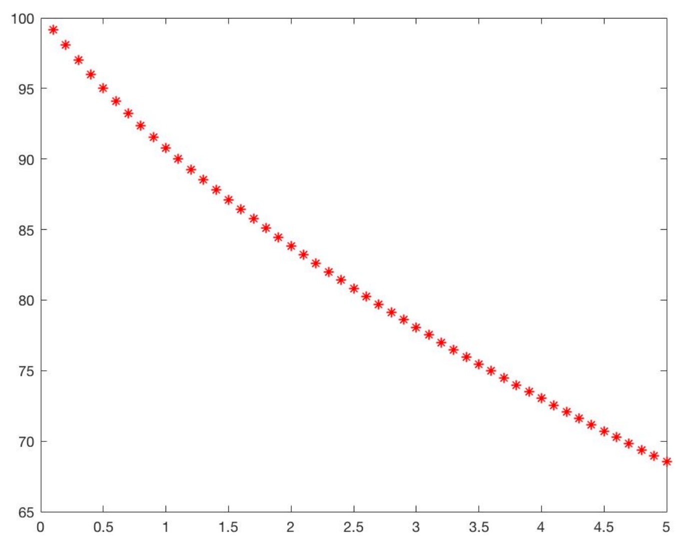

Figure 4.5: Bond price - τ with condition satisfied. (τ is from 0.1 to 5 with step size 0.1, $V_t = 100$, $D = 110$, $\sigma = 0.2$, $r = 0.05$, $\lambda = 0.1$, $\mu = -0.2$, $\delta = 0.6$, upbound of summation $n = 50$.)

In some extreme case, when S_2 is very small as a negative number, the sum of S_1 and S_2 can be negative which causes the tendency of B_t w.r.t τ is not decreasing. (See figure 4.6.)

Proposition 4.2.6 *The bond price is decreasing in δ for the log-normal jumps process.*

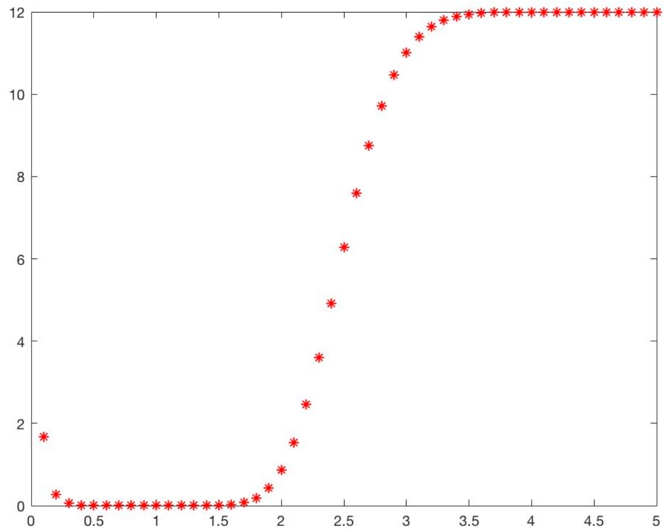

Figure 4.6: Bond price - τ trend is somehow increasing when condition $r - \lambda k \geq 0$ is not satisfied at extreme case. (Here $k = 199.34$ under $\mu = 0.8, \delta = 3$. $\lambda = 0.1, r = 0.05$. $r - \lambda k = -19.88$, $V_t = 12, D = 10$. τ is from 0.1 to 5 with step pace 0.1.)

We have

$$\begin{aligned}\frac{\partial C_J(V_t, D, \tau, \sigma^2, r, \delta^2, \lambda, k)}{\partial \delta} &= \frac{\partial}{\partial \delta}(\sum_{n=0}^{\infty} \frac{(\lambda' \tau)^n}{n!} e^{-\lambda' \tau} C(V_t, D, \tau, \sigma_n, r_n)) \\ &= \sum_{n=0}^{\infty} \frac{(\lambda' \tau)^n}{n!} e^{-\lambda' \tau} \frac{\partial C(V_t, D, \tau, \sigma_n, r_n))}{\partial \sigma_n} \frac{\partial \sigma_n}{\partial \delta} \\ &= \sum_{n=0}^{\infty} \frac{(\lambda' \tau)^n}{n!} e^{-\lambda' \tau} x \sqrt{\tau} n(d_{1n}) \frac{2n\delta}{\tau} \\ &= \frac{2x\delta}{\sqrt{\tau}} \sum_{n=0}^{\infty} \frac{(\lambda' \tau)^n}{n!} e^{-\lambda' \tau} n(d_{1n}) n > 0.\end{aligned}$$

Then, $\dfrac{\partial B_t}{\partial \delta} = -\dfrac{\partial C_J(V_t, D, \tau, \sigma^2, r, \delta^2, \lambda, k)}{\partial \delta} < 0$. (See Figure 4.7).

Proposition 4.2.7 *The bond price is decreasing in λ for the log-normal jumps process.*

From property (1.2.1), we know $\dfrac{\partial C_J(V_t, D, \tau, \sigma^2, r, \delta^2, \lambda, k)}{\partial \lambda} > 0$. Therefore, $\dfrac{\partial B_t}{\partial \lambda} = -\dfrac{\partial C_J(V_t, D, \tau, \sigma^2, r, \delta^2, \lambda, k)}{\partial \lambda} < 0$ holds. (See Figure 4.8).

Proposition 4.2.8 *The bond price is decreasing in k for the log-normal jumps process.*

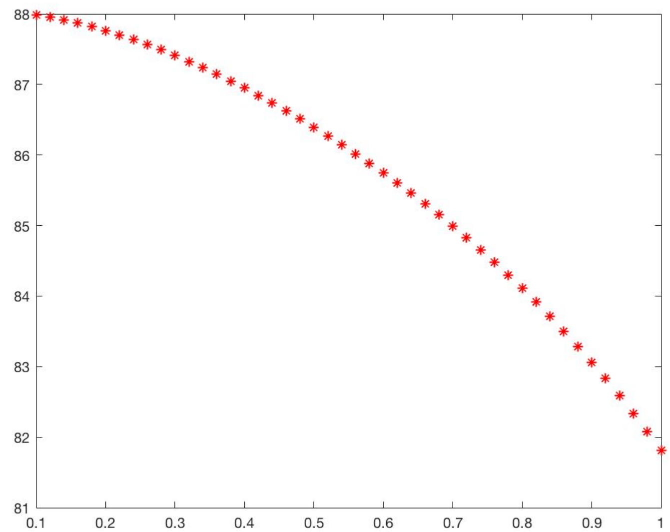

Figure 4.7: Bond price - δ. (δ is from 0.01 to 1 with step size 0.02, $V_t = 100$, $D = 110$, $\tau = 2$, $\sigma = 0.2$, $r = 0.05$, $\lambda = 0.1$, $\mu = -0.2$, upbound of summation $n = 50$.)

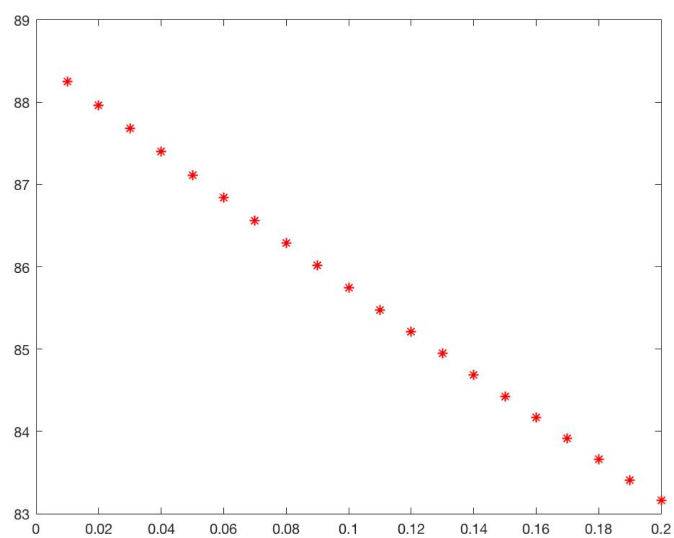

Figure 4.8: Bond price - λ. (λ is from 0.01 to 0.2 with step size 0.01, $V_t = 100$, $D = 110$, $\tau = 2$, $\sigma = 0.2$, $r = 0.05$, $\mu = -0.2$, $\delta = 0.6$, upbound of summation $n = 50$.)

Since

$$\frac{\partial C_J(V_t, D, \tau, \sigma^2, r, \delta^2, \lambda, k)}{\partial k} = \frac{\partial}{\partial k}(\sum_{n=0}^{\infty} \frac{(\lambda'\tau)^n}{n!} e^{-\lambda'\tau} C_n(V_t, D, \tau, \sigma_n, r_n))$$

$$= \sum_{n=1}^{\infty}[n(\lambda'\tau)^{n-1}\frac{\lambda\tau}{n!}e^{-\lambda'\tau}C_n(V_t, D, \tau, \sigma_n, r_n)] + \sum_{n=0}^{\infty}[\frac{(\lambda'\tau)^n}{n!}(-\tau\lambda)e^{-\lambda'\tau}C_n(V_t, D, \tau, \sigma_n, r_n)]$$

$$+ \sum_{n=0}^{\infty}[\frac{(\lambda'\tau)^n}{n!}e^{-\lambda'\tau}\frac{\partial C_n(V_t, D, \tau, \sigma_n, r_n)}{\partial k}]$$

$$= \sum_{n=1}^{\infty}[\frac{n}{1+k}\frac{(\lambda'\tau)^n}{(n)!}e^{-\lambda'\tau)}C_n(V_t, D, \tau, \sigma_n, r_n)] + \sum_{n=0}^{\infty}[\frac{(\lambda'\tau)^n}{n!}(-\lambda\tau)e^{-\lambda'\tau}C_n(V_t, D, \tau, \sigma_n, r_n)]$$

$$+ \sum_{n=0}^{\infty}[\frac{(\lambda'\tau)^n}{n!}e^{-\lambda'\tau}\frac{\partial C_n(V_t, D, \tau, \sigma_n, r_n)}{\partial k}]$$

$$= \sum_{n=1}^{\infty}[\frac{(\lambda'\tau)^n}{n!}e^{-\lambda'\tau}C_n(V_t, D, \tau, \sigma_n, r_n)(\frac{n}{1+k} - \lambda\tau)] + (-\lambda\tau)e^{-\lambda'\tau}C_0(V_t, D, \tau, \sigma_n, r_n)$$

$$+ \sum_{n=0}^{\infty}[\frac{(\lambda'\tau)^n}{n!}e^{-\lambda'\tau}(xN'(d_{1n})\frac{\partial d_{1n}}{\partial r_n}(\frac{n}{(1+k)\tau} - \lambda)$$

$$- De^{-r_n\tau}(-\tau)(\frac{n}{(1+k)\tau} - \lambda)N(d_{2n}) - De^{-r_n\tau}N'(d_{2n})\frac{\partial d_{2n}}{\partial r_n}(\frac{n}{(1+k)\tau} - \lambda))]$$

$$= \sum_{n=1}^{\infty}[\frac{(\lambda'\tau)^n}{n!}e^{-\lambda'\tau}C_n(V_t, D, \tau, \sigma_n, r_n)(\frac{n}{1+k} - \lambda\tau)] + (-\lambda\tau)e^{-\lambda'\tau}C_0(V_t, D, \tau, \sigma_n, r_n)$$

$$+ \sum_{n=0}^{\infty}[\frac{(\lambda'\tau)^n}{n!}e^{-\lambda'\tau}(De^{-r_n\tau}(\frac{n}{1+k} - \lambda\tau)N(d_{2n}))]$$

$$= \sum_{n=1}^{\infty}[\frac{(\lambda'\tau)^n}{n!}e^{-\lambda'\tau}C_n(V_t, D, \tau, \sigma_n, r_n)(\frac{n-\lambda'\tau}{1+k})\frac{C_n(V_t, D, \tau, \sigma_n, r_n) + De^{-r_n\tau}N(d_{2n})}{C_n(V_t, D, \tau, \sigma_n, r_n)})]$$

$$- \lambda\tau e^{-\lambda'\tau}C_0(V_t, D, \tau, \sigma_n, r_n) - e^{-\lambda'\tau}De^{-r_0\tau}\lambda\tau N(d_{20})$$

$$= \sum_{n=1}^{\infty}[\frac{(\lambda'\tau)^n}{n!}e^{-\lambda'\tau}C_n(V_t, D, \tau, \sigma_n, r_n)(\frac{n-\lambda'\tau}{1+k})\frac{xN(d_{1n})}{C_n(V_t, D, \tau, \sigma_n, r_n)})]$$

$$- e^{-\lambda'\tau}\lambda\tau(C_0(V_t, D, \tau, \sigma_n, r_n) + De^{-r_0\tau}N(d_{20}))$$

$$= \sum_{n=1}^{\infty}[\frac{(\lambda'\tau)^n}{n!}e^{-\lambda'\tau}(\frac{n-\lambda'\tau}{1+k})xN(d_{1n})] - e^{-\lambda'\tau}\lambda\tau xN(d_{10})$$

$$= \sum_{n=0}^{\infty}[\frac{(\lambda'\tau)^n}{n!}e^{-\lambda'\tau}(\frac{n-\lambda'\tau}{1+k})xN(d_{1n})].$$

where we used the fact that

$$\frac{\partial d_1}{\partial r_n} = \frac{\partial d_2}{\partial r_n},$$

$$xN'(d_{1n}) = De^{-r_n\tau}N'(d_{2n}),$$

$$C_n(V_t, D, \tau, \sigma_n, r_n) = xN(d_{1n}) - Ke^{-r_n\tau}N(d_{2n}).$$

Since n goes to larger and large in the sum, then the term $\frac{n-\lambda'\tau}{1+k}$ is positive for most cases. That means the partial derivative w.r.t k is positive. Then, $\frac{\partial B_t}{\partial k} = -\frac{\partial C_J}{\partial k} < 0$.

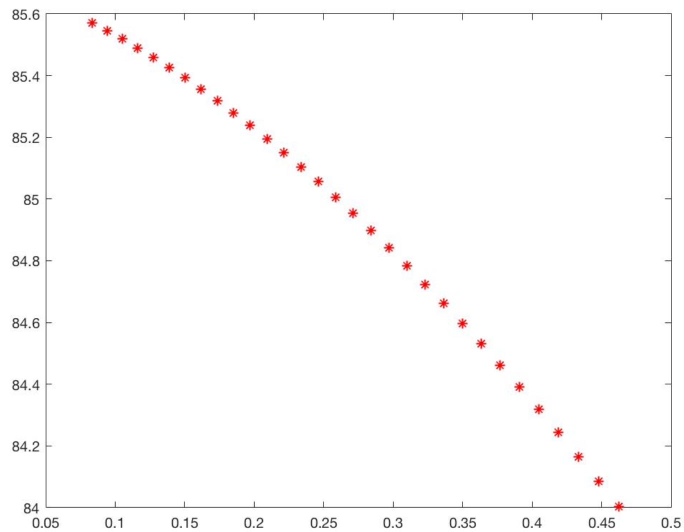

Figure 4.9: Bond price - k. ($k = exp(\mu + \frac{1}{2}\sigma^2) - 1$, where μ is from -0.1 to 0.2 with step size 0.01, $V_t = 100$, $D = 110$, $\tau = 2$, $\sigma = 0.2$, $r = 0.05$, $\lambda = 0.1$, $\delta = 0.6$, upbound of summation $n = 50$.)

4.2.2 Summary

In total, the bond price with compound Poisson jumps has propositions listed as:

1.(i) The bond price is increasing in V_t for log-normal jumps process.

(ii) $\dfrac{\partial B_t}{\partial x} \in (0, 1)$.

2.(i) The bond price is increasing in face value D for log-normal jumps process.

(ii)$\dfrac{\partial B_t}{\partial D} \in (0, e^{-(r-\lambda k)\tau})$.

3.The bond price is decreasing in volatility, σ, for log-normal jumps process.

4.(i) The bond price is decreasing in risk-free interest rate r for log-normal jumps process.

(ii) $\dfrac{\partial B_t}{\partial r} \in (-\tau D e^{-(r-\lambda k)\tau}, 0)$.

5.(i) The call price is increasing in time-to-maturity, τ, for the log-normal jumps process if $r - \lambda k \geq 0$.

(ii) The bond price is decreasing in time-to-maturity, τ, for the log-normal jumps process $r-\lambda k \geq 0$.

6.The bond price is decreasing in δ for the log-normal jumps process.

7.The bond price is decreasing in λ for the log-normal jumps process.

8.The bond price is decreasing in k for the log-normal jumps process.

CHAPTER 5

NUMERICAL ANALYSIS PRICING OF LEVY PROCESS

By analyzing how to solve the PIDE (3.2) which is derived in exponential Levy processes dynamics. We found it is extremely difficult to get the theoretical solution since two facts. Firstly, it is a backwards partial differential equation with the boundary condition of maturity time. Secondly, the integral part in the equation is very tough to handle. Fortunately, we can use Fourier transform to estimate the solution numerically.

5.1 Fast Fourier Transform Method to Price Levy Process Dynamics

The Levy-Khinchin representation theorem reveals the characteristic function of Levy process. Which is the key to connect Fourier transform due the the similar mathematical form.

5.1.1 Basic Concepts

We consider the integrand as a Fourier transform form:

$$\phi(u) = \int_{-\infty}^{\infty} e^{iux} f(x) dx$$

It can be recovered by using inverse Fourier transform to get function $f(x)$ as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \phi(u) du.$$

Recall the European call pricing for asset price S_t. The characteristic function of logarithm value $X = lnS_t$ is defined as

$$\phi_X(u) = E[e^{iuX}] = \int_{-\infty}^{\infty} e^{iux} p_X(x) dx. \tag{5.1}$$

where p_X is the risk neutral density for the random variable X.

The characteristic functions are analytically known in many financial cases in lots of literature. Which makes the pricing models become very easy to utilize the Fourier transform to the dynamics of log price coupled with divisible process of independent increments. When the characteristic function is given naturally by Levy-Khinchin representation theorem.

In those processes, we chose three typical models to discuss. First is general hyperbolic process introduces by Barndorff-Nielsen (1977) [10]. Second is normal inverse Gausian derived by Barndorff-Nielsen (1997) [11] as a subclass of general hyperbolic processes. Third is CGMY model which contains the variance gamma (Madan and Senata (1987)[19]) as a subclass proposed by Carr, Geman, Madan, and Yor (1999) [20].

With the known characteristic functions, the numerical results of risk neutral probability was obtained in many literature as Bakshi and Madan [12] and Scott [13]:

$$P(S_T > K) = \Pi_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re[\frac{e^{-iulnK}\phi_T(u)}{iu}]du.$$

The delta of the option Π_1 is given numerically as:

$$\Pi_1 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re[\frac{e^{-iulnK}\phi_T(u-i)}{iu\phi_T(-i)}]du.$$

Therefore, assuming constant interest rate r and zero dividends, the call option price is:

$$C = S\Pi_1 - Ke^{-rT}\Pi_2.$$

5.1.2 Fourier Transform for Option Price

Suppose $f(X_T)$ is the risk neutral probability density function of underlying value X_T. Denote $q_T(x)$ as the risk neutral density of log price $x_T = lnX_T$. $k = lnK$ is the log value of strike price K. Then the characteristic function of x_T is

$$\phi_x(z) = \int_{-\infty}^\infty e^{izx_T} q(x_T)dx_T.$$

Therefore the European call option price $C_T(k)$ has the form [23]:

$$\begin{aligned} E[(X_T - K)^+] &= \int_K^\infty (X_T - K)f(X_T)dX_T = \int_k^\infty (e^{x_T} - e^k)q(x_T)dx_T \\ &= \int_k^\infty (e^x - e^k)q(x)dx = C_T(k). \end{aligned}$$

where we get rid of the subscript T in the last step for simplicity. Then the Fourier transform of $C_T(k)$ is

$$\begin{aligned} \Phi_T(z) &= \int_{-\infty}^\infty e^{izk}C_T(k)dk = \int_{-\infty}^\infty (\int_k^\infty (e^x - e^k)q(x)dx)dk \\ &= \int_{-\infty}^\infty \int_{-\infty}^x e^{izk}(e^x - e^k)q(x)dkdx \\ &= \int_{-\infty}^\infty q(x)(\int_{-\infty}^x e^{izk}(e^x - e^k)dk)dx. \end{aligned}$$

We changed the integral order due to Fubini's theorem in the last step. Therefore,

$$\int_{-\infty}^{x} e^{izk}(e^x - e^k)q(x)dk = \int_{-\infty}^{x} e^{izk}(e^x)q(x)dk - \int_{-\infty}^{x} e^{izk}(e^k)q(x)dk$$
$$= e^x \frac{e^{izk}}{iz}|_{-\infty}^{x} - \frac{e^{(iz+1)k}}{iz+1}|_{-\infty}^{x}.$$

Then the Fourier transform has a singularity point at zero for the first term. In Carr and Madan (1999) [21], we use the damping function $e^{\alpha k}$ to solve the issue of convergence. Define:

$$c_T(k) = e^{\alpha k} C_T(k).$$

Hence,

$$\Phi_T(z) = \int_{-\infty}^{\infty} e^{izk} e^{\alpha k} C_T(k)dk = \int_{-\infty}^{\infty} (e^{\alpha k} \int_{k}^{\infty} (e^x - e^k)q(x)dx)dk$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{x} e^{(\alpha+iz)k}(e^x - e^k)q(x)dkdx$$
$$= \int_{-\infty}^{\infty} q(x)(\int_{-\infty}^{x} e^{(\alpha+iz)k}(e^x - e^k)dk)dx.$$

Check the integral again, we can get:

$$\int_{-\infty}^{x} e^{(\alpha+iz)k}(e^x - e^k)q(x)dk = \int_{-\infty}^{x} e^{izk}(e^x)q(x)dk - \int_{-\infty}^{x} e^{izk}(e^k)q(x)dk$$
$$= e^x \frac{e^{(\alpha+iz)k)}}{\alpha+iz}|_{-\infty}^{x} - \frac{e^{(\alpha+iz+1)k}}{\alpha+iz+1}|_{-\infty}^{x} = e^x \frac{e^{(\alpha+iz)x)}}{\alpha+iz} - \frac{e^{(\alpha+iz+1)x}}{\alpha+iz+1}$$
$$= \frac{e^{(\alpha+iz+1)x}}{(\alpha+iz)(\alpha+iz+1)}.$$

Now, the Fourier transform results of dampening call price is [23]

$$\Phi_T(z) = \int_{-\infty}^{\infty} q(x) \frac{e^{(\alpha+iz+1)x}}{(\alpha+iz)(\alpha+iz+1)} dx$$
$$= \frac{1}{(\alpha+iz)(\alpha+iz+1)} \int_{-\infty}^{\infty} q(x) e^{i(z-(\alpha+i)i)x} dx$$
$$= \frac{\phi(z-(\alpha+1)i)}{(\alpha+iz)(\alpha+iz+1)}.$$

On the other hand, the modified call price characteristic function is

$$\Phi_T(z) = \int_{-\infty}^{\infty} e^{izk} e^{\alpha k} C_T(k)dk$$

Therefore,we can solve the $C_T(k)$ by the inverse Fourier transform,

$$C_T(k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-izk} \Phi_T(z)dz = \frac{e^{-\alpha k}}{\pi} \int_{0}^{\infty} e^{-izk} \Phi_T(z)dz,$$

where the characteristic function $\Phi_T(z)$ can be determined according to particular Levy processes.

The last result is true since the call price ought to be a real number, the imaginary part of integral should be odd and the real part is even.

The put price formula is in the same mode since the put value $P_T(x)$ is the expectation of $(K - X_T)^+$.(See Hirsa 2011 [23])

5.1.3 Price Approximation with FFT Algorithm

Fast Fourier transforms are used in modern applications in engineering, science, and mathematics widely. The Cooley-Turkey FFT algorithm can reduce the N^2 multiplication of discrete Fourier transform to $NlnN$ by using a divide. Gilbert Strang described the FFT in 1994 as "the most important numerical algorithm of our lifetime" and it was included in Top 10 Algorithms of 20th Century by the IEEE journal.

The Fourier transform allow us to compute the option price when the PIDE can't be solved but the characteristic function is known. Due to the derived formula of last section, set up the upper bound of integral is a and then the equidistant interval length is $\eta = a/N$. And the integral interval endpoints are $z_j = (j-1)\eta$ for $j = 1, 2, ..., N+1$. Then the discretized sum applying trapezoidal rule is:

$$\begin{aligned}
C_T(k) &= \frac{e^{-\alpha k}}{\pi} \int_0^\infty e^{-izk} \Phi_T(z) dz \approx \frac{e^{-\alpha k}}{\pi} \int_0^a e^{-izk} \Phi_T(z) dz \\
&\approx \frac{e^{-\alpha k}}{\pi} (e^{-iz_1 k} \Phi_T(z_1) + 2e^{-iz_2 k} \Phi_T(z_2) + ... + 2e^{-iz_N k} \Phi_T(z_N) \\
&+ e^{-iz_{N+1} k} \Phi_T(z_{N+1})) \frac{\eta}{2} \\
&= \frac{e^{-\alpha k}}{\pi} \sum_{j=1}^{N} e^{-iz_j k} \Phi_T(z_j) w_j.
\end{aligned}$$

where $w_j = \frac{\eta}{2}(2 - \delta_{j-1})$.

Furthermore, we can use Simpson rule such that $w_j = \frac{\eta}{2}(3 + (-1)^j - \delta_{j-N})$. Where δ_n is the Dirac function which is 1 when $n = 0$ and 0 otherwise.

Since FFT algorithm is computing such summation:

$$\omega(k) = \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(k-1)} x(j), \text{ for } k = 1, 2, ..., N. \tag{5.2}$$

where $N = 2^n$ have to be the power of 2 as the restrict of FFT algorithm.

We now need to change the form of $C_T(k)$ to formula (5.2). Setting $b = \frac{N\lambda}{2}$, define the strike logarithm value as $k_u = -b + \lambda(u-1),$ for $u = 1, 2, ..., N$ which changes for $-b$ to b. That means

$$C_T(k_u) \approx \frac{e^{-\alpha k_u}}{\pi} \sum_{j=1}^{N} e^{-iz_j(-b+\lambda(u-1))} \Phi_T(z_j) w_j$$
$$= \frac{e^{-\alpha k_u}}{\pi} \sum_{j=1}^{N} e^{-i\lambda\eta(j-1)(u-1)} e^{ibz_j} \Phi_T(z_j) w_j.$$

Therefore we can see $\lambda\eta = \frac{2\pi}{N}$ satisfies the form of FFT algorithm and $x_j = e^{ibz_j}\Phi_T(z_j)w_j$.

For the last step, the interpolation is utilized to find the call price value of lnK according to the scheme of $C_T(k_u)$ values.

Considering discount factor of constant rate r, apply the discount factor to call price value:

$$C_T(k) = \frac{e^{-rT}e^{-\alpha k}}{\pi} \int_0^\infty e^{-izk} \Phi_T(z) dz.$$

The same modification applies to FFT results.[21][24]

For an example, FFT algorithm [24] for simple Black-Scholes model is attached as figure (5.1).

5.2 Several Typical Levy Processes in Finance and Numerical Sensitivity Studies

In this section, the brief descriptions of some kinds of typical Levy processes are given and the explicit density function and characteristic functions are listed. Therefore the explicit formulas of FFT algorithm are decided and we may discuss the sensitivities again the parameters.

5.2.1 General Hyperbolic (GH) Distribution

General Hyperbolic distribution is a class of Lebesgue continuous infinitely divisible distribution of 5 parameters. The Lebesgue density is [10][25]

$$\rho_{GH}(x+\mu) = \frac{e^{\beta x}}{\sqrt{2\pi}\alpha^{2\lambda-1}\delta^{2\lambda}} \frac{\delta\sqrt{\alpha^2-\beta^2}}{K_\lambda(\delta\sqrt{\alpha^2-\beta^2})} (\alpha\sqrt{\delta^2+x^2})^{\lambda-1/2} K_{\lambda-1/2}(\alpha\sqrt{\delta^2+x^2}).$$

The domain of parameters is: $\lambda \in \mathbf{R}, \alpha > 0, \beta \in (-\alpha, \alpha), \delta > 0, \mu \in \mathbf{R}^2$. Where K_λ and $K_{lm-1/2}$ are the modified third kind of Bessel functions with the order as subscripts.

The characteristic function was given by Prause (1999)([26]):

$$\phi(u) = e^{i\mu u} \frac{\delta\sqrt{\alpha^2-\beta^2}}{K_\lambda(\delta\sqrt{\alpha^2-\beta^2})} \frac{K_\lambda(\delta\sqrt{\alpha^2-\beta^2+iu)^2})}{\delta\sqrt{\alpha^2-\beta^2+iu)^2}}.$$

Setting parameter $\lambda = 1$ yields the class of hyperbolic distribution with 4 parameters.Applying $K_{1/2}(z) = \sqrt{\pi/(2z)}d^{-z}$, the density is

$$\rho_{(\alpha,\beta,\delta,\mu)}(x+\mu) = \frac{e^{\beta x}}{2\alpha\delta^2} \frac{\delta\sqrt{\alpha^2-\beta^2}}{K_1(\delta\sqrt{\alpha^2-\beta^2})} e^{\alpha\sqrt{\delta^2+x^2}}.$$

```
function CallFFTBS = CallFFTBS(n,S,K,T,r,d,sigma)

s = log(S);
k = log(K);

Discount = exp(-r*T);
alpha = .75;

FFT_N = 2^n;
FFT_eta = 0.05;

FFT_lambda = (2 * pi) / (FFT_N * FFT_eta);
FFT_b = (FFT_N * FFT_lambda) / 2;

uvec = 1:FFT_N;
ku = - FFT_b + FFT_lambda * (uvec - 1);

jvec = 1:FFT_N;
zj = (jvec-1) * FFT_eta;

w = Discount .* exp(1i .* zj .* (FFT_b)).* psi(zj,alpha,s,T,r,d,sigma).* FFT_eta;
w = (w / 3) .* (3 + (-1).^jvec - ((jvec - 1) == 0) );   % simpson's rule
cpvec = exp(-alpha .* ku)./ pi .* real(fft(w));

strikenum = round((k + FFT_b)/FFT_lambda + 1) ;
index = min(strikenum)-1:1:max(strikenum)+1;
xp = ku(index);
yp = cpvec(index);
CallFFTBS = real(interp1(xp,yp,k));

end

function y = char_BS(u,s,T,r,d,sigma)
    y = 1i.*u.*(s+(r-d-0.5.*sigma.^2).*T) - 0.5.*sigma.^2.*u.^2.*T;
end

function Phi = psi(v,alpha,s,T,r,d,sigma)
  Phi = exp (char_BS(v - (alpha + 1) .* 1i,s,T,r,d,sigma)) ...
      ./ (alpha.^2 + alpha - v.^2 + 1i .* (2 .* alpha + 1) .* v);
end
```

Figure 5.1: Black-Scholes code FFT algorithm

At the very beginning, we borrowed some calibration parameters of call option prices of S&P 500 index from Wim [9]. Which estimates the model parameters by minimizing the root-mean-square error for the price differences of market and models.

For GH model, we have $\alpha = 3.8288, \beta = -3.8286, \delta = 0.2375, \nu = -1, 7555$. Base on this group data, we computed several call prices around the existing parameters. See table (5.1).

GH							
	Call price		Call price		Call price		Call price
S_t=9.5	0.4975	K=10.5	1.3978	T=1	0.2087	r=0.01	0.4296
10	0.7224	11	1.1407	1.25	0.3274	0.02	0.4938
10.5	0.9929	11.5	0.9154	1.5	0.4558	0.03	0.5641
11	1.3037	12	0.7224	1.75	0.5884	0.04	0.6403
11.5	1.6486	12.5	0.5611	2	0.7224	0.05	0.7224

q=0	0.7224	$\alpha = 0$	0.0376	$\beta = -8$	-0.2219	Cont. β=1	0.5801
0.005	0.6738	2	0.0873	-6	0.2093	3	NaN
0.01	0.6276	3.8288	0.7224	-3	0.5936	4	-0.9794
0.015	0.5837	6	0.2116	-3.8286	0.7224	5	-0.0523
0.02	0.5419	8	0.1634	0	0.5375	8	1.095

δ=-0.5	NaN	ν=-3	0.3328
0.1	0.1278	-1	1.8584
0.2375	0.7224	0	0.7224
0.5	2.2071	1	9.9999
1	4.8908	2	10

Table 5.1: Prices trend of GH model parameters. (The price is increasing w.r.t to S_t, T, r, δ, is decreasing w.r.t to K, q. The price is a sort of symmetric w.r.t. α. And there are no tendency about β and ν. The β part is extremely wired since it is the asymmetry parameter of general hyperbolic distribution.)

By checking the data, we can judge that the price is increasing w.r.t to S_t, T, r, δ, is decreasing w.r.t to K, q. The price is a sort of symmetric w.r.t. α. And there are no tendency about β and ν. The β part is extremely wired since it is the asymmetry parameter of general hyperbolic distribution.

Visualize the trends, we get some figures list below from figure (5.2) to (5.10).

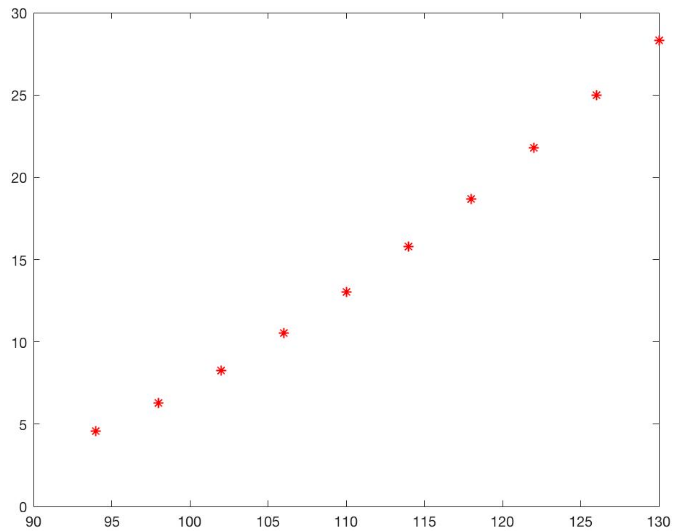

Figure 5.2: GH model for call prices - S. (n=16, S=90 with step size 3, K=120, T=2, r=0.05, q=0,$\alpha = 3.8288, \beta = -3.8286, \delta = 0.2375, \nu = -1.7555.$)

5.2.2 Normal Inverse Gaussian (NIG) Distribution

Setting $\lambda = -1/2$, we have Normal Inverse Gaussian distribution from hyperbolic. The characteristic function is given as

$$\phi_{\alpha,\beta,\delta,\mu} = e^{i\mu u} e^{\delta\sqrt{\alpha^2-\beta^2} - \delta\sqrt{\alpha^2-(\beta^2+iu)^2}}.$$

For NIG model, we have $\alpha = 6.1882, \beta = -3.8941, \delta = 0.1622$ [9]. There is no μ given, we can set it as 0 as the initial value since it is the drifting term of process.

Therefore, we have the table (5.2).

For NIG model, call price increases w.r.t. S_t, T, r, positive δ and μ, decreases w.r.t. K, q, α. For the parameter β, the behavior is undetermined since NIG is a special case of GH model.

The figures are omitted since they are almost the same with GH process.

5.2.3 The Carr-Geman-Madan-Yor (CGMY) Class of Distribution

The CGMY distribution class is defined by Carr, Geman, Madan and Yor (1999) [20]. The Levy density is

$$K_{CGMY} = \frac{C}{|x|^{1+Y}} exp(\frac{G-M}{2}x - \frac{G+M}{x}|x|).$$

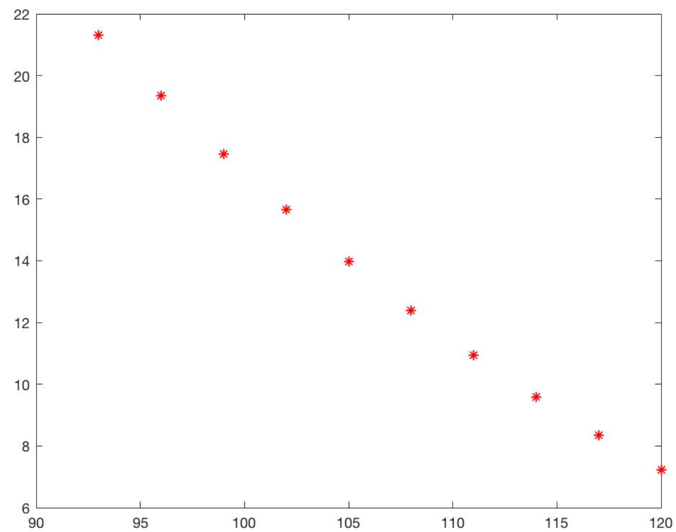

Figure 5.3: GH model for call prices - K. (n=16, S=100,K=90 with step size 3,T=2,r=0.05,q=0,$\alpha = 3.8288, \beta = -3.8286, \delta = 0.2375, \nu = -1.7555.$)

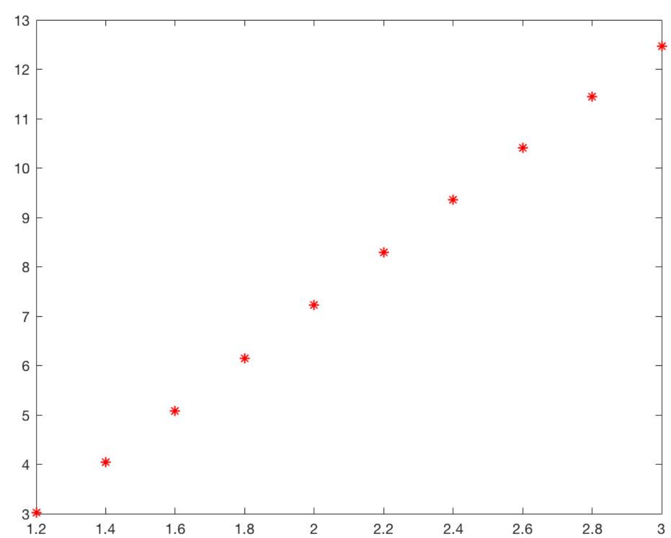

Figure 5.4: GH model for call prices - T. (n=16, S=100,K=120,T=1 with step size 0.2, r=0.05, q=0, $\alpha = 3.8288, \beta = -3.8286, \delta = 0.2375, \nu = -1.7555.$)

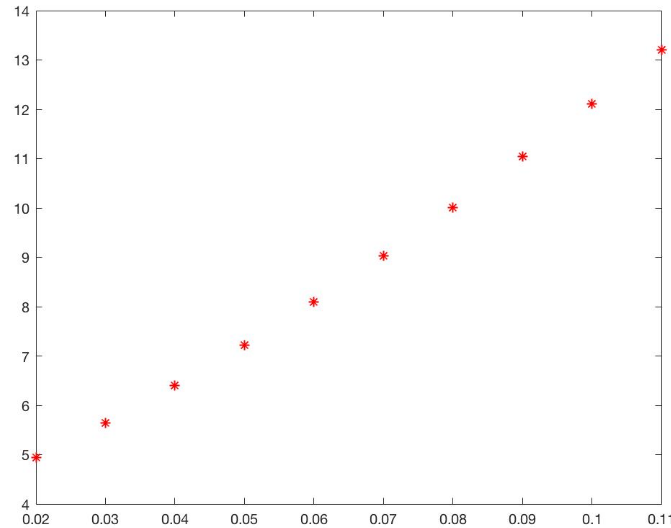

Figure 5.5: GH model for call prices - r. (n=16, S=100, K=120, T=2, r=0.01 with step size 0.01, q=0, $\alpha = 3.8288, \beta = -3.8286, \delta = 0.2375, \nu = -1.7555$.)

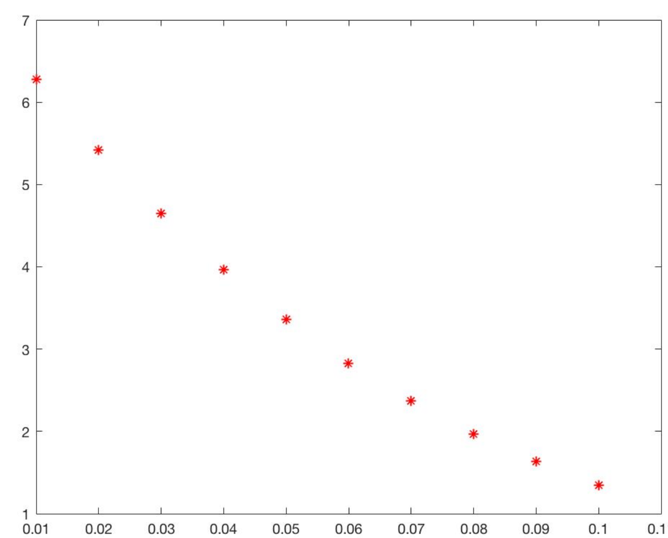

Figure 5.6: GH model for call prices - q. (n=16, S=100, K=120, T=2, r=0.05 q=0 with step size 0.01, $\alpha = 3.8288, \beta = -3.8286, \delta = 0.2375, \nu = -1.7555$.)

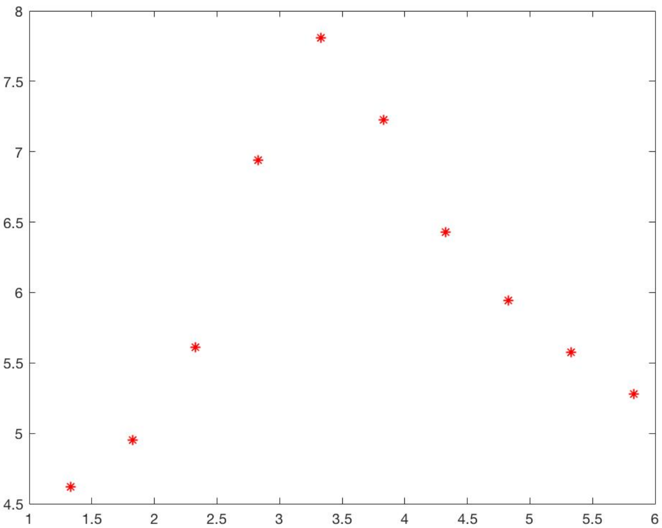

Figure 5.7: GH model for call prices - α. (n=16, S=100, K=120, T=2, r=0.05 q=0, $\alpha = 0.8288$ with step size 0.5, $\beta = -3.8286, \delta = 0.2375, \nu = -1.7555$.)

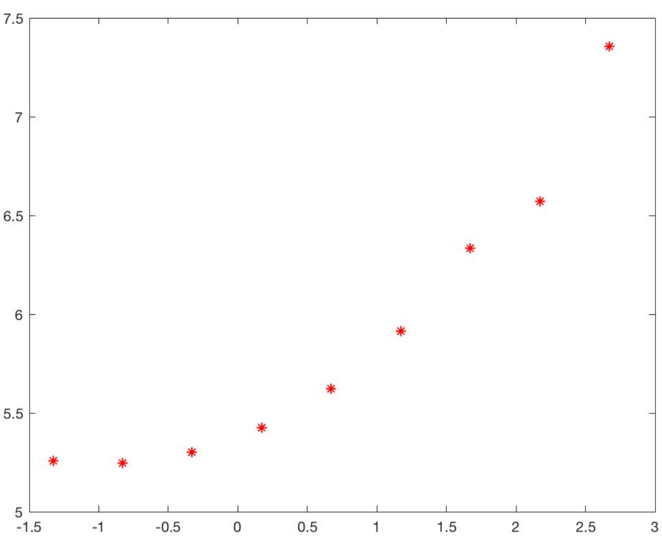

Figure 5.8: GH model for call prices - β. (n=16, S=100, K=120, T=2, r=0.05 q=0 , $\alpha = 3.8288$,$\beta = -1.8286$ with step size 0.5,$\delta = 0.2375, \nu = -1.7555$.)

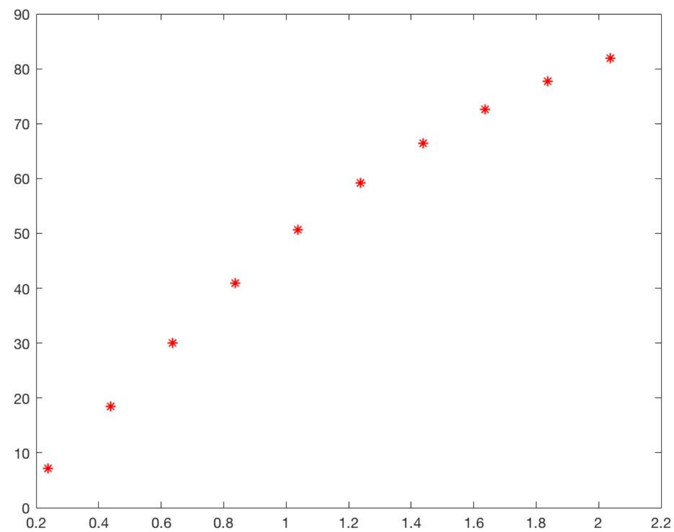

Figure 5.9: GH model for call prices - δ. (n=16, S=100, K=120, T=2, r=0.05 q=0, $\alpha = 3.8288$, $\beta = -1.8286$, $\delta = 0.0375$ with step size 0.2,$\nu = -1.7555$.)

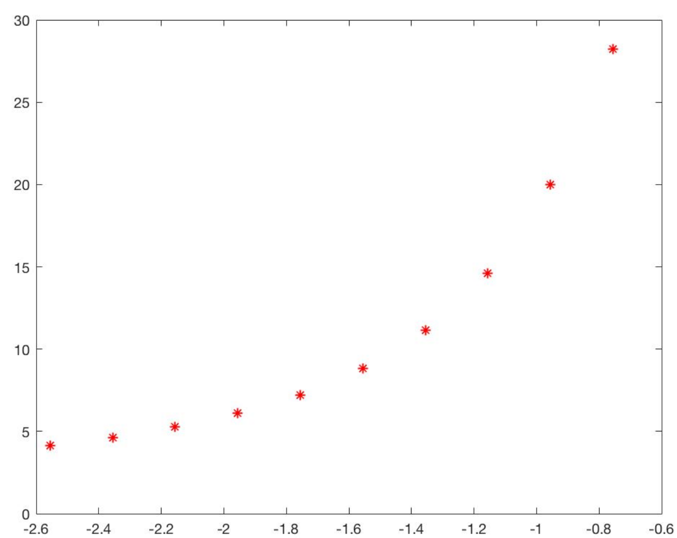

Figure 5.10: GH model for call prices - ν. (n=16, S=100, K=120, T=2, r=0.05 q=0 , $\alpha = 3.8288$, $\beta = -1.8286$, $\delta = 0.0375$, $\nu = -2.7555$ with step size 0.2.)

NIG					
	Call price		Call price		Call price
S_t=9.5	0.4979	K=10.5	1.3851	T=1	0.2091
10	0.721	11	1.133	1.25	0.3286
10.5	0.9882	11.5	0.9114	1.5	0.4573
11	1.2943	12	0.721	1.75	0.5891
11.5	1.6331	12.5	0.5612	2	0.721
r=0.01	0.4307	q=0	0.721	α=5.8	0.7954
0.02	0.4945	0.005	0.6729	6	0.7547
0.03	0.5642	0.01	0.6271	6.1882	0.721
0.04	0.6398	0.015	0.5835	6.4	0.719
0.05	0.721	0.02	0.5421	6.6	0.6874
β=-9	-5.43E+208	δ=-2	NaN	μ=-0.01	0.6271
-6	2.3036	0	-1.04E-07	-0.005	0.6729
-3.8941	0.721	0.1622	0.721	0	0.721
0	0.556	2	3.7332	0.005	0.7713
5	0.9341	8	6.9534	0.01	0.8239

Table 5.2: Prices trend of NIG model parameters. (Call price increases w.r.t. S_t, T, r, positive δ and μ, decreases w.r.t. K, q, α. For the parameter β, the behavior is undetermined since NIG is a special case of GH model.)

The characteristic function is

$$\phi_{CGMY} = exp\{C\Gamma(-Y)[(M - iu)^Y - M^Y + (G + iu)^Y - G^Y]\}.$$

For the CGMY model, we have $C = 0.0244, G = 0.0765, M = 7.5515, Y = 1.2945$ [9]. Check the table(5.3).

Call price is increasing w.r.t. S_t, T, r, C, G, Y, and decreasing w.r.t. K, q, G, M.

The figures are omitted since the same mode.

5.3 Empirical Test for S&P 500 Call Option With Levy Processes

As the last part of empirical test, those three types of Levy Processes are checked by optimal parameters searching. The comparison with previous Heston model is considered as well.

For the data of S&P 500 option prices, we may check the SSE values for these three Levy processes. The first step is to estimate the parameters for in-the-sample data, which is from September 4th 2012 to February 28th 2013. The task is to make the SSE value be the minimum by using Matlab optimal function "$fmincon$". And then the SSE value is calculated by using estimated parameters from the first step for out-of-sample data, which is from March 1st 2013 to August 30th 2013.

As the initial parameter values for the optimal searching, results of Wim [9], which calibrated the S&P 500 option index as well are referred. For the GH model, the initial values are $\alpha = 3.8$, $\beta = -3$, $\delta = 1$, $\nu = 2$. The CGMY model use $C = 0.02$, $G = 0.08$, $M = 7.55$, $Y = 1.3$. The NIG model utilize $\alpha = 6$, $\beta = -3$, $\mu = 0.01$, $\delta = 1$. And all the estimations have upper bound of 20 and lower bound of -20.

Table (5.4) below shows the parameters estimation and the corresponding SSE values.

From the final results, we can check that the Levy processes have the equivalent order of magnitude of SSE values with Heston model but slightly larger. The greatest in-the-sample SSE result is 7.57 for GH model and the lowest is 4.13 for NIG model, the average is 6.21. The greatest out-of-sample SSE estimation is 8.15 for CGMY model and the lowest is 5.81 for NIG model, the average is 7.13. According to Heston model, the greatest is 5.12 and lowest is 4.10 for in-the-sample with average of 4.61, the greatest is 6.61 and lowest is 5.27 for out-of-sample with average of 5.94. By comparison, and the NIG model has the both lowest answers in the three kinds of Levy processes and very close to Heston model.

CGMY					
	Call price		Call price		Call price
S_t=9.5	0.4969	K=10.5	1.4056	T=1	0.2077
10	0.7231	11	1.1456	1.25	0.326
10.5	0.9957	11.5	0.9178	1.5	0.4544
11	1.3095	12	0.7231	1.75	0.5878
11.5	1.6581	12.5	0.5607	2	0.7231
r=0.01	0.4286	q=0	0.7231	C=0.01	0.2832
0.02	0.493	0.005	0.6742	0.02	0.6004
0.03	0.5637	0.01	0.6277	0.0244	0.7231
0.04	0.6404	0.015	0.5835	0.03	0.8676
0.05	0.7231	0.02	0.5416	0.035	0.9876
G=0.065	0.7276	M=7.45	0.7244	Y=1.2	0.623
0.07	0.7256	7.5	0.7237	1.25	0.6736
0.0765	0.7231	7.5515	0.7231	1.2945	0.7231
0.08	0.7217	8	0.7176	1.3	0.7295
0.085	0.7199	8.5	0.712	1.35	0.7918

Table 5.3: Prices trend of NIG model parameters. (Call price is increasing w.r.t. S_t, T, r, C, G, Y, and decreasing w.r.t. K, q, G, M.)

GH model	α	β	δ	ν	SSE
in-the-sample	-17.3388	15.6454	0.3554	-17.6347	6.92
out-of-sample	-17.3388	15.6454	0.3554	-17.6347	7.43
CGMY model	C	G	M	Y	SSE
in-the-sample	4.47E-06	4.5725	6.8383	1.9975	7.57
out-of-sample	4.47E-06	4.5725	6.8383	1.9975	8.15
NIG model	α	β	μ	δ	SSE
in-the-sample	19.999	-15.421	-0.1667	0.2295	4.13
out-of-sample	19.999	-15.421	-0.1667	0.2295	5.81
Heston model	Cases				SSE
in-the-sample	No jump and $V_t = imp.V_t^2$				5.12
	No jump and estimate V_t				4.13
	With jump and $V_t = imp.V_t^2$				5.10
	With jump and estimate V_t				4.10
out-of-sample	No jump and $V_t = imp.V_t^2$				6.61
	No jump and estimate V_t				5.32
	With jump and $V_t = imp.V_t^2$				6.55
	With jump and estimate V_t				5.27

Table 5.4: SSE estimations according to 3 types of Levy Processes comparing previous Heston models. (The in-the-sample data comes from September 4th 2012 to February 28th 2013 and the out-of-sample data comes from March 1st 2013 to August 30th 2013. These three Levy processes have the equivalent order of magnitudes with Heston model for the final SSE. The greatest in-the-sample SSE result is 7.57 for GH model and the lowest is 4.13 for NIG model, the average is 6.21. The greatest out-of-sample SSE estimation is 8.15 for CGMY model and the lowest is 5.81 for NIG model, the average is 7.13. According to Heston model, the greatest is 5.12 and lowest is 4.10 for in-the-sample with average 4.61, the greatest is 6.61 and lowest is 5.27 for out-of-sample with average 5.94. By comparison, and the NIG model has the both lowest answers in the 3 kinds of Levy processes and very close to Heston model.)

GH model	α	β	δ	ν	SSE
Sept. 4th 2012	3.1068	2.0105	0.4748	-18.3818	0.0415
CGMY model	C	G	M	Y	SSE
Sept. 4th 2012	4.72E-06	1.4314	7.2307	1.9984	0.0453
NIG model	α	β	μ	δ	SSE
Sept. 4th 2012	9.2051	-9.2051	-1.1726	0.3160	0.0530

Table 5.5: Parameters calibration of three Levy processes for the first day, Sept. 4th 2012.

5.3.1 Two Scenario Calibrations for One Day Prediction.

To detect the prediction of Levy processes, another test about daily SSE computation is undertaken. We use 273 option prices of September 4th 2012 as in-the-sample data and 187 prices of the first day of second half year, March 1st 2013, as out-of-sample data.

Table (5.5) shows the parameter results of three Levy processes for in-the-sample case.

With the first day parameters, the out-of-sample results are listed in the table (5.6) as below. The SSE values are computed under two scenarios: using the first day parameters and using the first half year parameters.

In table (5.6), for example, the SSE value in the first row is 0.0415, and the SSE value for second row is 0.0845. The results of out-of-sample by using one day calibration parameters are more than double of in-the-sample SSE values. On the contrary, the SSE value on third row is 0.02774, which means by using the 1st year's optimal parameters, the value becomes almost the half. The effect of NIG model is as dramatic as around fourfold.

The SSE values change declares the fact that the prediction error should be narrowed by using previous long period calibration parameters, not the one day optimal parameters far away from the history.

5.3.2 Daily Prediction Results Comparison with Previous Heston Models

For the daily calibration, the parameter of Heston models are estimated by Matlab optimal function and listed in table (5.7).

With Heston models parameters, table (5.8) shows the results and comparison of Levy processes with previous Heston model.

Comparing the four cases of Heston models, the SSE values show some facts of oscillation. For the first case, the estimation of out-of-sample by using 1st day parameters has SSE value as 0.0271.

GH model	α	β	δ	ν	SSE
Sept. 4th 2012	3.1068	2.0105	0.4748	-18.3818	0.0415
March 1st 2013(using 1st day Para's)	3.1068	2.0105	0.4748	-18.3818	0.084492
March 1st 2013(using 1st half year Para's)	-17.3388	15.6454	0.3554	-17.6347	0.027741
CGMY model	C	G	M	Y	SSE
Sept. 4th 2012	4.72E-06	1.4314	7.2307	1.9984	0.0453
March 1st 2013(using 1st day Para's)	4.72E-06	1.4314	7.2307	1.9984	0.085249
March 1st 2013(using 1st half year Para's)	4.47E-06	4.5725	6.8383	1.9975	0.02879
NIG model	α	β	μ	δ	SSE
Sept. 4th 2012	9.2051	-9.2051	-1.1726	0.3160	0.053
March 1st 2013(using 1st day Para's)	9.2051	-9.2051	-1.1726	0.3160	0.20194
March 1st 2013(using 1st half year Para's)	19.999	-15.421	-0.1667	0.2295	0.013898

Table 5.6: Estimation about one day of out-of-sample, March 1st 2013, in three Levy processes. (The SSE value in the first row is 0.0415, and the SSE value for second row is 0.0845. The results of out-of-sample by using one day calibration parameters are more than double of in-the-sample SSE values. On the contrary, the SSE value on third row is 0.02774, which means by using the 1st year's optimal parameters, the value becomes almost the half. The effect of NIG model is as dramatic as around fourfold.)

Heston para's for Sept. 4th 2012									
no jump and	κ	θ	σ	ρ					SSE
$V_t = imp.V_t^2$	19.9998	0.0546	2.0127	-0.0172					4.31E-02
no jump and	κ	θ	σ	ρ	V_t				SSE
estimate V_t	4.7985	0.0062	0.1664	0.9984	0.009				5.87E-03
with jump and	κ	θ	σ	ρ	λ	μ_j	σ_j		SSE
$V_t = imp.V_t^2$	4.7747	0.322	2.4965	0.4356	0.0034	0.5415	1.2229		3.66E-02
with jump and	κ	θ	σ	ρ	V_t	λ	μ_j	σ_j	SSE
estimate V_t	5.9353	0.0073	0.1242	0.9984	0.0098	1.59E-04	-0.0518	1.0009	6.75E-03

Table 5.7: Optimal parameters for four cases of Heston models according to Sept. 4th 2012.

Levy Processes	Cases	SSE
in-the-sample	GH model	0.0415
Sept. 4th 2012	CGMY model	0.0453
	NIG model	0.0530
out-of-sample	GH model	0.0845
March 1st 2013	CGMY model	0.0852
(using 1st day Para's)	NIG model	0.2019
out-of-sample	GH model	0.0277
March 1st 2013	CGMY model	0.0288
(using 1st half year Para's)	NIG model	0.0139
Heston model	Cases	SSE
in-the-sample	no jump and $V_t = imp.V_t^2$	0.0431
Sept. 4th 2012	no jump and estimate V_t	0.0059
	with jump and $V_t = imp.V_t^2$	0.0366
	with jump and estimate V_t	0.0068
out-of-sample	no jump and $V_t = imp.V_t^2$	0.0271
March 1st 2013	no jump and estimate V_t	0.0857
(using 1st day Para's)	with jump and $V_t = imp.V_t^2$	7.4027
	with jump and estimate V_t	0.1025
out-of-sample	no jump and $V_t = imp.V_t^2$	0.0278
March 1st 2013	no jump and estimate V_t	0.0202
(using 1st half year Para's)	with jump and $V_t = imp.V_t^2$	0.0484
	with jump and estimate V_t	0.0807

Table 5.8: Comparison with three types of Levy processes and Heston models. (Comparing the four cases of Heston models, the SSE values show some facts of oscillation. Extraordinarily, the third case of SSE value is 7.4027 for using first day estimation parameters, which looks protruding large. However, the three Levy processes have the same shrink trend when we use the long period data calibration parameters. For example, the NIG model has the change from 0.2019 to 0.0139 and lower than any Heston model. Which means The Levy processes have the more stable prediction which is expectable with less error.)

And SSE value becomes 0.0278, a little bit bigger, for using first half year parameters. But for the second case, the SSE value becomes from 0.0857 to 0.0202 which means lower. Extraordinarily, the third case of SSE value is 7.4027 for using first day estimation parameters, which looks protruding large. However, the three Levy processes have the same shrink trend when we use the long period data calibration parameters. For example, the NIG model has the change from 0.2019 to 0.0139 and lower than any Heston model. Which means The Levy processes have the more stable prediction which is expectable with less error.

In summary, the SSE values of Levy processes have the same order of magnitude with Heston models we did before and even slightly greater. However, when we check the one day prediction for out-of-sample data, we can see that the Levy processes have the shrink effect and more stable than Heston models. And also, the NIG model has the lowest SSE value prediction within all the models.

CHAPTER 6

CONCLUSIONS

In financial term structure study, Brownian motion is the fundamental method to simulate the dynamics of asset price movement over time. Black-Scholes-Merton model studied the continuous case that asset price follows geometric Brownian motion.

Generally, the price activity has jumps which can be observed in the real financial market. For this case of discontinuity, the simple scenario of one kind of Levy processes, compound Poisson process, are considered. And the explicit call/put price formulas can also be derived when the jump size follows log-normal distribution base on the fundamental BSM model. Therefore, the sensitivities of parameters for bond pricing are arising after the complicated mathematical deduce.

Empirical study reveals the knowledge of history data from the market. In the period we selected, the sum of squared error (SSE) of BSM model can be around 100 for half year S&P 500 option index. When loosening the restriction of iid jumps condition, we have the SSE results of non-iid cases are less than the BSM model a little. The advanced Heston model which considers the volatility is also a stochastic process has the SSE value less than 10. The Heston model provide a excellent approach to remove the limitation of BSM model which fixes volatility constant.

Levy processes are outstanding methods in term structure research for financial mathematics since their infinitely divisible, independent and stationary increments properties match financial market intuitively. For the exponential of Levy process, the PIDE can be derived but it is very hard to solve. Fourier transform method can be used to solve the question numerically due to the analogy form of Levy process' characteristic function derived from Levy-Khinchin theorem. Three typical cases of Levy processes, GH model, NIG model, CGMY model are calibrated by using fast Fourier transform (FFT) method numerically. Not only the parameter sensitivities of these Levy processes are checked. But also we can see all of them have the SSE result of half year data below 10 as Heston model does. Furthermore, the Levy processes have the shrink effect and more stable prediction for chosen data.

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VITA

Bin Xie

Candidate for the Degree of

Doctor of Philosophy

Dissertation: BASIC CREDIT RISK ANALYSIS WITH LEVY PROCESSES AND NUMERICAL FFT METHOD

Major Field: Financial Mathematics

Biographical:

Education:

Completed the requirements for the Doctor of Philosophy in Mathematics at Oklahoma State University, Stillwater, Oklahoma in July, 2018.

Completed the requirements for the Master of Science degree in Applied Mathematics at Southwest Jiaotong University of China in July, 2004.

Completed the requirements for the Bachelor of Science degree in Mathematics at Hefei Normal University of China in July, 2001.

Experience:

Employed by Beijing Wuzi University of China, as a mathematics lecturer from June 2004 to August 2012.

Employed by Department of Mathematics, Oklahoma State University, USA, as a teaching assistant from August 2012 to July 2018.

Professional Memberships:

American Mathematical Society.