FREE EXTENSIONS OF PARTIAL ℓ–GROUPS

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in partial fulfillment of
the requirements for
the Degree of
DOCTOR OF PHILOSOPHY
December, 1994

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ACKNOWLEDGMENTS

This is very difficult for me to write. There are so many people to thank, I am sure to forget someone and the acknowledgment section probably shouldn't be longer than the thesis. Let me begin by thanking Northwest Nazarene College, who provided much of the financial support and time off to pursue this degree. I cannot forget my colleagues in the science division, especially my good friend Dr. Gary Ganske, who called me every month to keep me informed as to the goings on back home as well as offer encouragement and words of wisdom when I was down. He'll never know how much those calls meant to me. I've made many new friends while here at OSU. One of my office mates, John Lorch, is a good listener as well as a good mathematician and helped me with some ideas and problems early on. There are many others who showed hospitality, gave encouragement, and offered prayers, to name but a few: Crystal (Clough) Lorch, Alan and Elizabeth Noell, Jon and Elizabeth Beal, and Stillwater First Church of the Nazarene.

When it comes to my family, I don't think I'll forget anyone, but it is difficult to express in words my gratitude for all their sacrifices. They were supportive the entire time, offered up countless prayers, without which I'm sure I would not have made it this far, and demonstrated their love in action as well as with words. In spite of the difficulties, Debra and Daniel found new friends, and Anne received her Masters Degree in Speech Communication here at OSU. My mother-in-law, Jerri Mangum, was one of my strongest supporters, as she and her husband Rev. Paul Mangum would call every Saturday to find out how we were doing. Jerri lost a long hard battle to cancer just months before I finished and was unable to see me get hooded, a dream she had had for

sometime. She will be sorely missed and it is in honor to her memory that I dedicate my efforts and this dissertation.

Finally, let me turn my attention to the OSU mathematics department, my committee and advisor. My stay here has been a very pleasant one. This is largely due to the fact that, on the whole, the math faculty are interested in their students. They make themselves available and offer support and encouragement. I want to thank my committee, Dr. Alan Noell, Dr. Dennis Bertholff, Dr. Mark McConnell, and Dr. George Hedrick for their support and encouragement and for attending meetings on short notice! I want to express my sincere thanks and appreciation to my advisor, Dr. Wayne Powell, for his excellent supervision, brilliant insights, constructive suggestions, and incredible patience! He and his wife, Joyce, went above and beyond their normal duties by opening their home to me and my family on several occasions and extending genuine hospitality and friendship. I am indeed the fortunate one to count him among my friends.

TABLE OF CONTENTS

Chapter Page

LIST OF FIGURES

Figure Page

Chapter 1

Partially Ordered Algebraic Systems

Introduction and History

Lattices, and in particular lattice ordered groups (ℓ-groups), have been studied for almost 100 years. Dedekind [7 and 8] is credited for having started the work on lattices by the publication of his papers around the turn of the century. However, it was G. Birkhoff [3] who really was the driving force behind the development and promotion of lattice theory. And as he points out in [4], "... lattices give important results concerning classical analysis, ... measure theory, general topology, and other aspects of modern functional analysis." He closes his talk by stating that "... lattices can do things for you, no matter what kind of mathematician you are!" Throughout the 1930's, 40's, and 50's major contributions were made in the area of general lattice theory. During the 40's and 50's ℓ-groups were developed as applications to functional analysis, these were purely group theoretic questions about the orderability of groups. During the 60's and 70's attention was on partially ordered groups and equational classes of ℓ-groups, also known as varieties. The 80's seemed to be a time of development of structure properties within varieties of ℓ-groups, including an analysis of free products and free ℓ-groups. During the last 20 years, emphasis has been in the area of universal algebra and structure and existence theorems for free objects in different varieties.

In this chapter we review some of the basic definitions and the notation necessary for our study of free extensions of partial ℓ-groups. A more complete discussion can be found in Fuchs [9 and 10] and Birkhoff [3].

Background and Notation

If a binary relation $\leq$ is defined on a set G such that for all $a, b, c \in G$ the following hold:

1). Reflexive: $a \leq a$

2). Antisymmetric: $a \leq b,\ b \leq a \Rightarrow a = b$

3). Transitive: $a \leq b,\ b \leq c \Rightarrow a \leq c$

then $(G, \leq)$ is called a *partially ordered set* (abbreviated: poset) and $\leq$ is called a *partial order*. Throughout this document it will be clear what symbol designates the partial order and we will denote by G the poset $(G, \leq)$.

If G and G' are posets, a mapping $\varphi : G \to G'$ is called *isotone* if it is single valued and preserves order, that is $a \underset{G}{\leq} b \Rightarrow a' \underset{G'}{\leq} b'$, where $\underset{G}{\leq}$ and $\underset{G'}{\leq}$ are the partial orders of G and G', respectively.

G is *trivially ordered* if, for all $a, b \in G,\ a \leq b \Rightarrow a = b$. If $a \nleq b$ and $b \nleq a$, that is a and b are *not comparable*, we denote this by $a \| b$. The order $\leq$ is called *total* if, in addition to G being a poset, for all $a, b \in G$, either $a \leq b$ or $b \leq a$. If B is a subset of G, we say that B has an *upper bound (lower bound)* if and only if there exists an element $x \in G$ such that $b \leq x$ $(x \leq b)$ for every element $b \in B$. We say x is the *least upper bound (greatest lower bound)* for B if x is an upper bound (lower bound) and if y is any other upper bound (lower bound), then $x \leq y$ $(y \leq x)$. We denote the least upper bound (greatest lower bound) of $\{a, b\}$ by $\sup\{a, b\}$ $(\inf\{a, b\})$. A poset G is called a *lattice* if for all $a, b \in G$, both $\sup\{a, b\}$ and $\inf\{a, b\}$ exist in G. If these elements exist, they are unique and are denoted by $a \vee b$ and $a \wedge b$, and are called the *join* and *meet*, respectively.

Alternatively, a *lattice* may be defined as an algebraic system in which two operations, $\vee$ and $\wedge$, are defined such that for all $a, b, c \in G$ the following laws hold:

1). Idempotent $a \vee a = a$ and $a \wedge a = a$

2). Commutative $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$

3). Associative $(a \vee b) \vee c = a \vee (b \vee c)$ and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$

4). Absorption $(a \vee b) \wedge a = a$ and $(a \wedge b) \vee a = a$

If $(G, \wedge, \vee)$ is a lattice and we define $a \leq b$ if and only if $a \wedge b = a$, or $a \vee b = b$, then it can be easily shown that the above four properties show $(G, \leq)$ is a poset, such that $\inf\{a, b\}$ exists for all $a, b \in G$. Since this is such an important concept and an idea that is used constantly, we prove

Theorem 1.1: *Under the conditions discussed in the previous paragraph,* $a \wedge b = c$ *if and only if* $\inf\{a, b\} = c$.

Proof: $(\rightarrow)$ Suppose $a \wedge b = c$, then $a = a \vee (a \wedge b) = a \vee c$, so $a \geq c$, similarly, $b \geq c$, hence c is a lower bound. Now suppose d is another lower bound of a and b, then $c \wedge d = (a \wedge b) \wedge d = a \wedge (b \wedge d) = a \wedge d = d$, therefore $\inf\{a, b\} = c$.

$(\leftarrow)$ Suppose $\inf\{a, b\} = c$, then $c \leq a, b$, so $a \wedge c = c$ and $b \wedge c = c$. But then $(a \wedge b) \wedge c = a \wedge (b \wedge c) = a \wedge c = c$, so that $c \leq a \wedge b$. On the other hand $(a \wedge b) \vee b = b$, so $a \wedge b \leq b$, similarly $a \wedge b \leq a$. Hence $a \wedge b$ is a lower bound for a, b, hence $a \wedge b \leq c$. Therefore $a \wedge b = c$. □

Throughout this paper we will use $+$ to denote the group operation of any group. A *partially ordered group* (po group) is a set G such that the following hold:

1). $(G, +)$ is a group,

2). $(G, \leq)$ is a poset, and

3). $a \leq b \Rightarrow c + a + d \leq c + b + d$, for all $c, d \in G$.

Property 3, which ties the group operation to the partial order, is called by various names, *homogeneity law, isotone property of* $\leq$, and an amusing one, *the monotony law*. This last one probably lost something in the translation from German to English! A set G is called a *lattice ordered group* (ℓ-group) if G is a po group, such that for all $a, b \in G$, $a \wedge b$ and $a \vee b$ both exist in G. The class of ℓ-groups is equationally definable.

We now collect a brief list of useful properties, definitions, and examples of po groups and ℓ-groups.

In a po group, G, we say an element a is *positive* if $a \geq 0$ and *negative* if $a \leq 0$. The collection of positive elements, denoted by $P = \{a \in G : a \geq 0\}$, is called *the positive cone* of G, sometimes denoted by G^+, while the negative cone is denoted by $-P$ and G^- (or even $-G^+$). In this manner the partial order $\leq$ is uniquely determined by the corresponding positive cone, that is

$$(*) \qquad a \leq b \text{ is equivalent to } (b - a) \in P, \text{ and to } (-a + b) \in P.$$

As a result, we will adopt the common practice and slightly abuse the language and say "the partial order P" when we mean "the partial order with the positive cone P." In addition there are some conditions on a subset P of G that determine a partial order on G. These are given below.

Theorem 1.2: *A subset P of a group G is the positive cone of some partial order of G, if and only if the following three conditions are satisfied:*

1). $P + P \subseteq P$

2). $g + P - g \subseteq P$, *for all* $g \in G$.

3). $P \cap -P = \{0\}$.

In other words, P is a normal subsemigroup of G containing no other element along with its inverse except 0. *Furthermore, G is totally ordered if, in addition, G satisfies*

4). $P \cup -P = G$.

Sketch of Proof: By using $(*)$ above to define $\leq$ from P, if P is given. Then we notice that *reflexivity* of $\leq$ is equivalent to $0 \in P$, *antisymmetry* is equivalent to *3),* *transitivity* is equivalent to *1),* and the *isotonicity* of $\leq$ is equivalent to *2).* The *total order* (i.e. for all $a \in G$, $a \geq 0$ or $a \leq 0$), is equivalent to *4).* $\square$

A partial order P is said to be *semi-closed* if whenever $na \in P$, for $n \in \mathbb{Z}^+$, we have $a \in P$.

A po group G, is an ℓ-group if and only if for all $a \in G$, $a \vee 0$ exists in G (or dually $a \wedge 0$ exists in G).

Theorem 1.3: *The following hold for any ℓ-group G:*

1). $c - (a \wedge b) + d = (c - a + d) \vee (c - b + d)$, *and dually.*

2). $a - (a \wedge b) + b = a \vee b$, *(special case of 1).*

3). $a + b = (a \wedge b) + (a \vee b)$, *if G is abelian.*

4). $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, *and dually--the distributive property.*

5). $na \geq 0,\ for\, n \geq 0 \Rightarrow a \geq 0,\ i.e.$ *G is semi-closed.*

6). $na = 0,\ for\, n \neq 0 \Rightarrow a = 0,\ i.e.$ *G is torsion-free.*

7). $a \wedge b = 0$ *and* $c \geq 0 \Rightarrow a \wedge (b + c) = a \wedge c$.

8). $a \wedge b = 0$ *and* $a \wedge c = 0 \Rightarrow a \wedge (b + c) = 0$, *(special case of 7).*

Let $\varphi : G \to H$, be a map between two ℓ-groups. Then we say φ is an

1). *ℓ-homomorphism,* if φ is a group homomorphism and

$\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$ and $\varphi(a \vee b) = \varphi(a) \vee \varphi(b)$.

2). *ℓ-monomorphism,* if φ is an injective ℓ-homomorphism.

3). *ℓ-epimorphism,* if φ is an onto ℓ-homomorphism, such that $\varphi(G^+) = H^+$.

4). *ℓ-isomorphism,* if φ is a bijection such that φ and φ^{-1} are ℓ-epimorphisms.

In an analogous way we define an order preserving homomorphism: *o-homomorphism,* if $a \leq b \Rightarrow \varphi(a) \leq \varphi(b)$, an *o-monomorphism, o-epimorphism,* and *o-isomorphism* are defined similarly.

A partial order P on a po group G, *induces a partial order* on a subgroup H of G under which H is again a po group, in this case we have $H^+ = H \cap G^+$. If $A \subseteq G$ a poset, is such that $x \in A$, whenever $a, b \in A$, $x \in G$ and $a \leq x \leq b$, we say A is a *convex* subset of G. A *convex subgroup* of a po group G, is a subgroup of G which is a convex subset of G. A subgroup B of a po group G, is a convex subgroup of G if and only if B^+ is a convex subset of G^+. The intersection of convex subgroups is again convex, so we denote by $\{X\}$, to mean the *convex subgroup generated by X.* We call a normal convex subgroup of a po group G, an *o-ideal*, and a normal convex subgroup of an ℓ-group G, which is also a *sublattice*, we call an *ℓ-ideal.*

If G is an ℓ-group, and N an ℓ-ideal of G, we can make G/N into an ℓ-group by defining $g + N \geq h + N$ if and only if there exists $k \in N$ such that $k + g \geq h$. An equivalent definition is $(G/N)^+$is the image of G^+ under the natural homomorphism of G onto G/N. This partial order may again be called induced. The next theorem should look familiar.

Theorem 1.4: *Let $\varphi : G \to H$, be an ℓ-homomorphism onto G and H.*

1). A normal subgroup N of G is the kernel of an ℓ-homomorphism if and only if it is convex.

2). *The kernel,* ker φ, *is an ℓ-ideal.*

3). *If N is an ℓ-ideal of G, then the set of right cosets G/N can be provided with an order which makes it an ℓ-group, so the natural map $G \to G/N$ is an ℓ-homomorphism.*

4). *$G/\ker\varphi$ is ℓ-isomorphic to H.*

Examples

We close this chapter by giving several examples of po groups and ℓ-groups, some of which we will refer to throughout this paper.

Example 1.5: Let G be $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$ where $\leq$ has the usual meaning. These are all po groups, ℓ-groups, and in fact totally ordered groups. Note that there are only two orders on $\mathbb{Z}$ and $\mathbb{Q}$, both of which are total, however, this is not true of $\mathbb{R}$.

Definition 1.6: Let $\{G_i : i \in I\}$ be a collection of ℓ-groups. We define the *cardinal product*, $\prod_{i \in I} G_i$, to be the usual Cartesian product of the G_i, with all operations being performed componentwise. The order inherited from the lattice operations is determined componentwise by the orders of the cardinal factors, i.e. the G_i. When only two ℓ-groups, G_1 and G_2, are involved in a cardinal product we refer to the *cardinal sum* and write $G_1 \boxplus G_2$. The symbol $\boxplus$ is intended to distinguish this cardinal sum from the frequently occurring group direct sum $G_1 \oplus G_2$ where no order is involved. In particular, consider the following example.

Example 1.7: Let G be $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Q} \times \mathbb{Q}$, or $\mathbb{R} \times \mathbb{R}$ and define $(x, y) \in G^+$ if $x \geq 0$ and $y \geq 0$. These are all po groups and ℓ-groups, but they are not totally ordered. Based on the previous definition, these are called the cardinal sum, denoted by: $\mathbb{Z} \boxplus \mathbb{Z}$,

$\mathbb{Q} \boxplus \mathbb{Q}$, and $\mathbb{R} \boxplus \mathbb{R}$, respectively. The positive cone in all cases is the first quadrant, including the axes, as illustrated by the following diagram

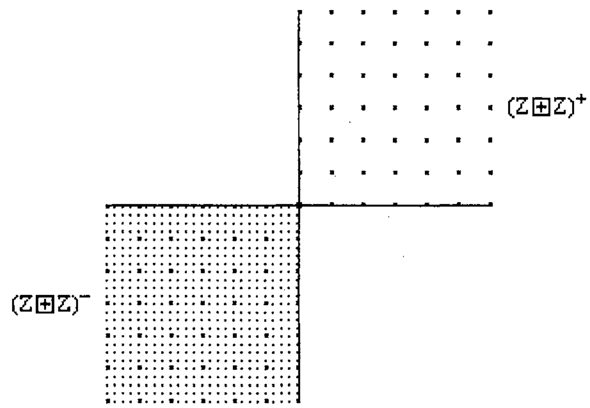
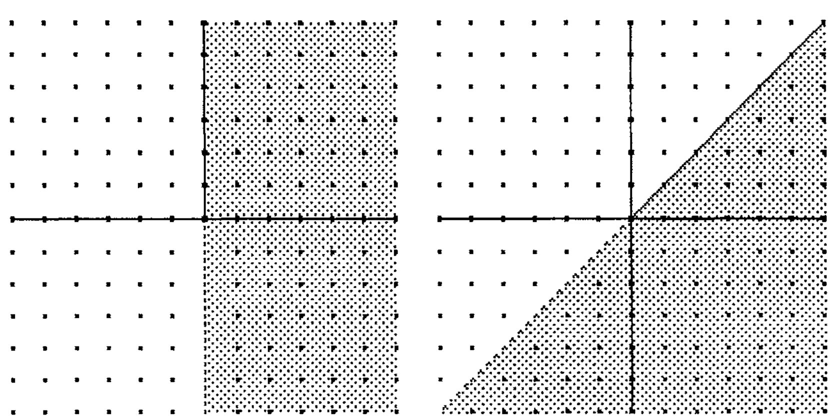


Figure 1 -- $\mathbb{Z} \boxplus \mathbb{Z}$

Example 1.8: Let G be $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Q} \times \mathbb{Q}$, or $\mathbb{R} \times \mathbb{R}$, only define $(x, y) \in G^+$ if $x > 0$, or $x = 0$ and $y \geq 0$. This is the standard lexicographic order, denoted by $\mathbb{Z} \circ \mathbb{Z}$, $\mathbb{Q} \circ \mathbb{Q}$, or $\mathbb{R} \circ \mathbb{R}$ respectively. All of which are totally ordered. A variation of this total order is to take any line $y = mx$ with $(x, y) \in G^+$ if $y \leq mx$ when $x \geq 0$, or $y < mx$ when $x < 0$. Thus there are uncountably many total orders on $\mathbb{Q} \times \mathbb{Q}$, for example. The shaded areas in the diagrams below represent the positive cones for the lexicographic order on $\mathbb{Q} \circ \mathbb{Q}$ and $\mathbb{Q} \times \mathbb{Q}$. Each of the respective orders are total orders.



$(\mathbb{Q} \circ \mathbb{Q})^+$ G^+

Figure 2 -- Lexicographic Order

Example 1.9: Let G be the additive group of continuous functions on $[0, 1]$, and let $f \in G^+$ if $f(x) \geq 0$ for all $x \in [0, 1]$. This is a po group and an ℓ-group. Notice in figure 3a, h is positive, f and g are neither positive nor negative, but $f \vee g$ exists and is positive, where $f \vee g = \max\{f(x), g(x)\}$ for all $x \in [0, 1]$.

Example 1.10: Let H be all polynomials on $[0, 1]$, and as in example 1.9, $f \in H^+$ if $f(x) \geq 0$ for all $x \in [0, 1]$, then H is a po subgroup of G, but not an ℓ-subgroup of G. H is not even an ℓ-group, since there may not be a "smallest" polynomial larger than both making up the join. For example, in figure 3a, even though f and g are polynomials, there does not exist a smallest polynomial larger than both f and g. So $f \vee g$ does not exist in H.

Example 1.11: Let K be the additive group of linear functions on $[0, 1]$, and let $f \in K^+$ if $f(x) \geq 0$ for all $x \in [0, 1]$. Then K is a subgroup of the G described in example 1.9. K has the same partial order as G and is indeed a po subgroup of G. K is an ℓ-group, but K is not a sublattice of G. This is because there is a "smallest" linear function bigger than both in the functions in the join. It is merely the straight line connecting the maximums on $[0, 1]$. If we look at figure 3b we see that again both f and g are neither positive nor negative, but that $f \vee g$ is positive and is the smallest straight line larger than both f and g.

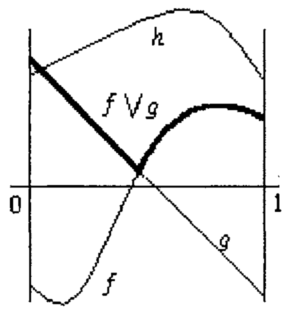


Figure 3a -- $C[0, 1]$

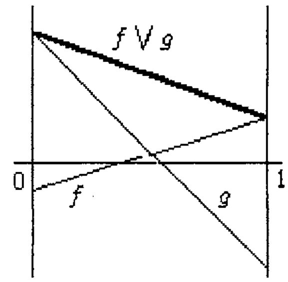


Figure 3b -- Linear functions on $[0, 1]$

Chapter 2

Partial ℓ-Groups

Introduction

Prior to now, partial ℓ-groups, have not been considered in the most general sense. All work previously done has been when the underlying group was a partially ordered group. In a restricted way, a partially ordered group is a partial ℓ-group, where the only lattice operations considered are among comparable elements. That is, $a \wedge b$ is in the partial ℓ-group if and only if $a \geq b$ or $b \geq a$, in other words when $a \wedge b = b$ or $a \wedge b = a$, respectively. In the context of free extensions, the only mappings considered were o-homomorphisms (order preserving group homomorphisms) rather than partial ℓ-homomorphisms (lattice preserving group homomorphisms). In this chapter we define and clarify the concept of partial ℓ-groups and look at some of its properties as well as some examples.

Notation and underlying assumptions

We will work only in $\mathcal{A}$, the variety of abelian ℓ-groups. G will denote a torsion free, abelian group. Additive notation ($+$) will be used for the group operation. P will denote the positive cone of a partial order on G, so (G, P) will mean G is a partially ordered group with the order determined by the positive cone P.

Definition 2.1: Let G be a torsion free abelian group. By a *partial operation* in G we mean an equality formed by applying lattice and group operations to some specific elements of G. If Γ is a collection of partial operations in G that are consistent with all ℓ-group laws, then (G, Γ) is called a *partial ℓ-group.* Note that if (G, Γ) is a partial

ℓ-group and both $a \wedge b = c$ and $a \vee b = d$ are in Γ for all $a, b \in G$ and some $c, d \in G$, then (G, Γ) is an ℓ-group.

We look for conditions on Γ that enable us to embed (G, Γ) into a partial order on G so that all partial operations in Γ are preserved. Consider the following examples:

Example 2.2: Let $G = \mathbb{Z} \times \mathbb{Z}$, and $\Gamma = \{(1,0) \wedge (0,1) = (0,0)\}$, then (G, Γ) is the partial ℓ-group with just one partial lattice operation. In this example, we can embed (G, Γ) in a partial order on G, by defining $P_\Gamma = \{n(1,0) + m(0,1) : n, m \geq 0\}$. See the diagram below

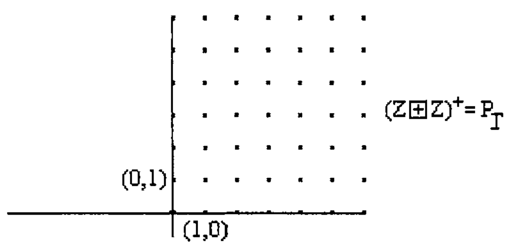


Figure 4 -- Partial ℓ-Group

Later we will show that (G, P_Γ) is an ℓ-group. However consider the following example.

Example 2.3: Let $G = \mathbb{Z}$, and $\Gamma = \{1 \wedge 0 = 0 \text{ and } 1 \vee 0 = 2\}$, then (G, Γ) can never be embedded in (G, P) where P is a partial order on G. This is because, in any partial order, $a \wedge b = a$ is equivalent to $a \leq b$, which is also equivalent to $a \vee b = b$. In this example, we would need for $1 \vee 0 = 1$, for us to be able to embed (G, Γ) into a partially ordered group. Also in a partially ordered group, if $a \vee b = c$, then $c = \sup\{a, b\}$, if and only if $-c = \inf\{-a, -b\}$. In our example this leads to the following: $1 \vee 0 = 2 \Leftrightarrow -1 \vee -2 = 0 \Leftrightarrow -(1 \wedge 2) = 0 \Leftrightarrow 1 \wedge 2 = 0$. It is important to note that in this example with $G = \mathbb{Z}$, that 1 and 2 are <u>not</u> linearly independent.

We now look at some of the ℓ-group properties that are satisfied when the partial operations exist in Γ. When we say "a dual statement holds", we mean an equivalent statement holds if $\vee$'s are replaced by $\wedge$'s and vice versa.

A partial order on an ℓ-group is uniquely determined by defining $a \wedge b = b$ if and only if $a \geq b$. We define the same order in a partial ℓ-group, by saying if $a \wedge b = b$ exists in Γ, then $a \geq b$. This does not imply there is a partial order on G, only that there is part of an order on G and if G could be extended to a partially ordered group, the "order" determined by those operations in Γ would still have to hold.

We list some of the algebraic rules that hold for partial ℓ-groups if and only if all meets and joins listed in the statement exist in Γ. All of these have a dual statement.

1). $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

2). $c + (a \wedge b) + d = (c + a + d) \wedge (c + b + d)$.

3). $c - (a \wedge b) + d = (c - a + d) \vee (c - b + d)$.

4). $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

5). If $a \wedge b = 0$ and $c \wedge 0 = 0$, then $a \wedge (b + c) = a \wedge c$.

6). If $a \wedge b = 0$ then $a \vee b = a + b$.

Now we look at conditions on Γ when (G, Γ) can be extended to a partial order.

Theorem 2.4: *Let M be an index set and let*

$$\Gamma = \{a_i \wedge a_j = 0 \textit{ for } i \neq j \textit{ and } i, j \in M\}$$

and suppose that all the a_i's are linearly independent with respect to integers. That is, the elements making up the partial equations in Γ form a collection of pairwise disjoint, *integer linearly independent elements from G. Define*

$$P_\Gamma = \left\{ \sum_{i \in I} m_i a_i : m_i \geq 0 \textit{ for } I \textit{ a finite subset of } M \right\}.$$

then P_Γ is a positive cone of some partial order on G.

Proof: We recall from Theorem 1.2 that we need to show that P_Γ satisfies the following three properties:

1). $P_\Gamma + P_\Gamma \subseteq P_\Gamma$

2). $g + P_\Gamma - g \subseteq P_\Gamma$ for all $g \in G$

3). $P_\Gamma \cap -P_\Gamma = \{0\}$

By the definition of P_Γ and the fact that G is abelian, $P_\Gamma + P_\Gamma \subseteq P_\Gamma$ and $g + P_\Gamma - g \subseteq P_\Gamma$ is clear. So we must only show that $P_\Gamma \cap -P_\Gamma = \{0\}$. To this end, suppose $x \in P_\Gamma \cap -P_\Gamma$, so $x \in P_\Gamma$ and $-x \in P_\Gamma$, thus we have

$$x = \sum_{i \in I} m_i a_i \text{ and}$$

$$-x = \sum_{j \in J} r_j a_j$$

with m_i , $r_j \geq 0$ for all i and j. Now let K be a common refinement of the finite index sets (i.e. $K = I \cup J$), then we have

$0 = \sum_{k \in K} (m_k + r_k) a_k$ and by linear independence we have

$0 = m_k + r_k$, for all k, but $m_k, r_k \geq 0$, thus

$0 = m_k = r_k$ for all $k \in K$, therefore

$0 = x.$ □

Later we will show that the free extension of (G, Γ), denoted by $\mathcal{F}(G, \Gamma)$ exists if and only if Γ is a collection of partial operations all of whose elements that satisfy the equations in Γ are pairwise disjoint integer linearly independent. Next we show that all the operations in Γ are preserved in the partial order defined by Γ.

Theorem 2.5: Γ *is preserved in* P_Γ, *that is if* $a_i \wedge a_j = 0$ *in* Γ *then* $\inf\{a_i, a_j\} = 0$ *in* P_Γ.

Proof: We need to show that $0 = \inf\{a_i, a_j\}$ in P_Γ. Clearly $a_i, a_j \geq 0$, since they are in P_Γ. Hence 0 is a lower bound. Now suppose $f \leq a_i, a_j$. We need to show $f \leq 0$. Since $a_i - f \geq 0$ and $a_j - f \geq 0$ then after a common refinement of the finite index sets (i.e. $K = I \cup J$, with $|K| = n$) we have

$a_i - f = \sum m_k a_k$

$a_j - f = \sum r_k a_k$, so these lead to

$(*) \qquad -f = m_1 a_1 + m_2 a_2 + \cdots + (m_i - 1)a_i + \cdots + m_n a_n$

$-f = r_1 a_1 + r_2 a_2 + \cdots + (r_j - 1)a_j + \cdots + r_n a_n$, subtracting from $(*)$

$0 = (m_1 - r_1)a_1 + \cdots + (m_i - 1 - r_i)a_i + \cdots$

$+ (m_j - r_j + 1)a_j + \cdots + (m_n - r_n)a_n$, so by independence

$0 = m_1 - r_1 = \cdots = m_i - 1 - r_i = \cdots = m_j - r_j + 1 = \cdots = m_n - r_n$

$0 \leq r_i = m_i - 1$, so by $(*)$, therefore

$f \leq 0$. $\square$

Before we proceed we notice some results from ℓ-group theory that we need to be true in P_Γ.

Theorem 2.6: *If* $a \wedge b = 0$ *and* $c \geq 0$ *then* $a \wedge c = a \wedge (c + b)$.

Proof:

$$
\begin{aligned}
a \wedge c &= a \wedge (0 + c) \\
(1) \qquad &= a \wedge ((a \wedge b) + c) \\
(2) \qquad &= a \wedge (a + c) \wedge (b + c) \\
(3) \qquad &= a \wedge (b + c).
\end{aligned}
$$

The reason (2) follows from (1) is that in an ℓ-group $+$ distributes over $\wedge$'s and $\vee$'s.

(3) follows from (2) because $\geq$ is isotone and since $c \geq 0$, we have $a + c \geq a$, hence $(a + c) \wedge a = a$. $\square$

An immediate corollary is the following:

Corollary 2.7: *If* $a \wedge b = 0$ *then* $na \wedge mb = 0$ *for* $n, m \in \mathbb{N}$.

Proof: From Theorem 2.6, let $c = a$ or b above and use induction twice. $\square$

Now we show that this same property still holds in our particular P_Γ.

Theorem 2.8: *Let* P_Γ *be as defined in Theorem 2.4, then for all* $n, m \in \mathbb{Z}^+$, $na_i \wedge ma_j = 0$, *whenever* $i \neq j$.

Proof: Since by Theorem 2.5, $a_i \wedge a_j = 0$, so $a_i, a_j \geq 0$ in P_Γ. Hence for any $n, m \geq 0$, we have that $na_i, ma_j \geq 0$, thus 0 is a lower bound. So suppose $e \leq na_i, ma_j$ in P_Γ, then $na_i - e$ and $ma_j - e \in P_\Gamma$. Then we have

$$na_i - e = \sum r_k a_k$$

$$ma_j - e = \sum s_k a_k$$

$$(*) \qquad -e = r_1 a_1 + \cdots + (r_i - n)a_i + \cdots + r_n a_n$$

$$-e = s_1 a_1 + \cdots + s_i a_i + \cdots + (s_j - m)a_j + \cdots + s_n a_n$$

$0 \leq s_i = (r_i - n)$, this follows from linear independence.

Now since $(r_i - n) \geq 0$, it follows by $(*)$ that $-e \in P_\Gamma$, in other words $e \leq 0$. Therefore $0 = \inf\{na_i, ma_j\}$ or $na_i \wedge ma_j = 0$. $\square$

Recapping what we have done up to now, we started with a set Γ of partial operations and extended (G, Γ) to a partially ordered group (G, P_Γ). We now turn the question around and ask, if we start with a partial order P, and collect all the disjoint elements that exist because of P, can we form a new partial order. The next theorem answers this question affirmatively.

Theorem 2.9: *Let G be a torsion free abelian group with partial order P. Define $\Gamma_P = \{a_i \wedge a_j = 0$ that exist because of $P\}$, and form P_{Γ_P} as in Theorem 2.4. Then $P_{\Gamma_P} = P$.*

Proof: First notice that $P \subseteq P_{\Gamma_P}$, since all $a_i \wedge a_j = 0$ in P, by definition hold in P_{Γ_P}. Now suppose $x \in P_{\Gamma_P}$ then $x = \sum\limits_{j=1}^{k}(n_{i_j}a_{i_j} + m_{i_j}b_{i_j})$, where a_{i_j}, and b_{i_j} are in Γ_P. But again by the definition of Γ_P, since $a_{i_j} \wedge b_{i_j} = 0$, then $a_{i_j}, b_{i_j} \in P$. But P is a partial order, hence closed under $+$, therefore we have $x \in P$. □

Before we continue we need another definition and some properties relating to P_Γ.

Definition 2.10: A partial order, P, on a group G, is said to be *semi-closed* if whenever $na \in P$ for $n \geq 0$ we have that $a \in P$.

We do not require (G, P_Γ) to be semi-closed, but we can embed P_Γ in a partial order that is semi-closed. We do this as follows. Define

$$\overline{P_\Gamma} = \{x \in G : nx \in P_\Gamma \text{ for some } n \geq 0\}.$$

We prove several important properties about $\overline{P}$ that we will make use of later.

Theorem 2.11 *Assume G is a torsion-free abelian group and P_Γ is the partial order generated by Γ as defined in Theorem 2.4. Then*

A). $P_\Gamma \subseteq \overline{P_\Gamma}$.

B). $\overline{P_\Gamma}$ *is semi-closed.*

C). $\overline{P_\Gamma}$ *is a positive cone for G.*

D). *If P_Γ is not semi-closed, there does not exist a total order with positive cone T such that $P_\Gamma \subseteq T$ but $\overline{P_\Gamma} \not\subseteq T$.*

Proof of A): If $y \in P_\Gamma$, then $ny \in P_\Gamma$, so $y \in \overline{P_\Gamma}$.

Proof of B): If $mx \in \overline{P_\Gamma}$, then $(nm)x \in P_\Gamma$, so $x \in \overline{P_\Gamma}$.

Proof of C): *i).* Suppose $x, y \in \overline{P_\Gamma}$ then nx, $my \in P_\Gamma$, so $m(nx), n(my) \in P_\Gamma$, so $mnx + mny \in P_\Gamma$, so $mn(x+y) \in P_\Gamma$, thus $x + y \in \overline{P_\Gamma}$. *ii).* Clearly $g + \overline{P_\Gamma} - g \subseteq \overline{P_\Gamma}$ since G is abelian. *iii).* Suppose $x \in \overline{P_\Gamma} \cap -\overline{P_\Gamma}$, then $x \in \overline{P_\Gamma}$ and $x \in -\overline{P_\Gamma}$ or $-x \in \overline{P_\Gamma}$. Thus $nx \in P_\Gamma$ and $m(-x) \in P_\Gamma$. So $nmx, -nmx \in P_\Gamma$, hence $nmx = 0$, but G is torsion free, so $x = 0$.

Proof of D): Suppose there exists a total order with positive cone T such that $P_\Gamma \subseteq T$ and yet $\overline{P_\Gamma} \not\subseteq T$. Then there exists $x \in \overline{P_\Gamma}$ and $x \notin T$. So $nx \in P_\Gamma \subseteq T$ for some $n > 0$. But total orders are semi-closed, so $x \in T$, a contradiction. □

There is an interesting claim made by Weinberg [24]. He states in his paper (without proof), Theorem 1.3, pg188, the following: Let P be a semi-closed partial order of G. Then if $x \notin -P$ then there exists a total order of G that contains both P and x.

Here we prove this theorem by removing the semi-closed requirement on P and use instead $\overline{P}$. First we need a lemma.

Lemma 2.12: *Suppose $x \notin -\overline{P}$ and define*

$$Q = \{y \in G : y = nx + mp, \text{ where } p \in P,\ n, m \geq 0\}.$$

Then Q is a positive cone of some partial order on G and $Q \supseteq \{P,x\}$.

Proof: Clearly $Q \supseteq \{P,x\}$ since $x = 1x + 0 \in Q$, and $p = 0 + 1p \in Q$. Also

$g + Q - g \subseteq Q$ since G is abelian. If a, $b \in Q$ then $a = nx + mp_1$ and $b = rx + sp_2$, for some $n, m, r, s \geq 0$. So $a + b = (n + r)x + 1p_3 \in Q$, where $p_3 = mp_1 + sp_2$. Therefore $Q + Q \subseteq Q$. Finally, if $a \in Q \cap -Q$, then $a \in Q$ and $-a \in Q$. Hence we have

$a = nx + mp_1$ and

$-a = rx + sp_2$, but Q is closed, so

$0 = a - a = (n + r)x + p_3$where $p_3 = mp_1 + sp_2$, hence

$(n + r)(-x) = -(n + r)x = p_3 \in P$, so

$-x \in \overline{P}$, or

$x \in -\overline{P}$, a contradiction unless $n + r = 0$, but $n, r \geq 0$, so

$n = r = 0$, hence

$a = mp_1 = -sp_2$, which implies that $a, -a \in P$, and

$a = 0$.

Therefore $Q \cap -Q = \{0\}$, and Q is a positive cone for G. □

Theorem 2.13: *Let P be a partial order, which is not necessarily semi-closed, of a torsion-free abelian group G. If $x \notin -\overline{P}$, then there exists a total order with positive cone $T \supseteq \{P,x\}$.*

Proof: Since G is a torsion-free abelian group, it is $\mathcal{O}^*$. That is every partial order can be extended to a total order (see Theorem 3.21). By lemma 2.12 Q is a partial order on G and thus can be extended to a total order with positive cone T and we have $T \supseteq Q \supseteq \{P,x\}$. □

Having proved some results regarding $\overline{P}$, we return to a theorem to show that the partial operations determined by Γ are preserved in $\overline{P_\Gamma}$. This is very important, when later we need to embed (P, Γ) in an ℓ-group to show the existence of free extensions.

Theorem 2.14: $a_i \wedge a_j = 0$, *for* $i \neq j$, *still holds in* $\overline{P_\Gamma}$.

Proof: Since $a_i, a_j \in P_\Gamma$ by the definition of P_Γ and from Theorem 2.11 $P_\Gamma \subseteq \overline{P_\Gamma}$. Then $a_i, a_j \geq 0$ in $\overline{P_\Gamma}$. Now suppose $e \leq a_i, a_j$ in $\overline{P_\Gamma}$. Then $a_i - e$ and $a_j - e \in \overline{P_\Gamma}$. Thus $l(a_i - e)$ and $m(a_j - e) \in P_\Gamma$, for some $l, m \in \mathbb{N}$. Hence we have

$$l(a_i - e) = \sum r_k a_k$$

$$m(a_j - e) = \sum s_k a_k$$

$$(*) \qquad -le = r_1 a_1 + \cdots + (r_i - l)a_i + \cdots + r_n a_n$$

$$-me = s_1 a_1 + \cdots + (s_j - m)a_j + \cdots + s_n a_n$$

$$-mle = mr_1 a_1 + \cdots + m(r_i - l)a_i + \cdots + mr_n a_n$$

$$-mle = ls_1 a_1 + \cdots + ls_i a_i + \cdots + l(s_j - m)a_j + \cdots + ls_n a_n$$

$$0 \leq ls_i = m(r_i - l).$$

But $m \geq 0$, hence $r_i - l \geq 0$. Thus by $(*)$ $-le \in P_\Gamma$. Thus $l(-e) \in P_\Gamma$, so $-e \in \overline{P_\Gamma}$, and hence $-e \geq 0$, so that $e \leq 0$ in $\overline{P_\Gamma}$. Therefore $0 = \inf\{a_i, a_j\}$ in $\overline{P_\Gamma}$, or $a_i \wedge a_j = 0$ in $\overline{P_\Gamma}$ $\square$

The next theorem demonstrates that in order for us to construct a partial order on G, then the only Γ we need to consider is the collection of partial operations consisting of disjoint meets.

Theorem 2.15: *Let*

$$\Gamma = \{a_\delta \wedge a_\gamma = c_\alpha : a_\delta, a_\gamma, c_\alpha \in G\} \text{ and}$$

$$\overline{\Gamma} = \{(a_\delta - c_\alpha) \wedge (a_\gamma - c_\alpha) = 0 : a_\delta, a_\gamma, c_\alpha \in G\}$$

If there exists a partial order with positive cone P, *such that* Γ *is preserved, then* $\overline{\Gamma}$ *is also preserved in* P, *and conversely.*

Proof: Suppose Γ is preserved in some partial order on G with positive cone P.

Then since $a_\delta \wedge a_\gamma = c_\alpha$ in Γ, by definition then $c_\alpha = \inf\{a_\delta, a_\gamma\}$, thus $a_\delta, a_\gamma \geq c_\alpha$ in P and so $(a_\delta - c_\alpha), (a_\gamma - c_\alpha) \geq 0$. Now suppose that d is any other lower bound for $(a_\delta - c_\alpha), (a_\gamma - c_\alpha)$, that is $d \leq (a_\delta - c_\alpha), (a_\gamma - c_\alpha)$. Then $(d + c_\alpha) \leq a_\delta, a_\gamma$, hence $(d + c_\alpha) \leq c_\alpha$, and therefore $d \leq 0$, so that $0 = \inf\{(a_\delta - c_\alpha), (a_\gamma - c_\alpha)\}$ in P. Thus $(a_\delta - c_\alpha) \wedge (a_\gamma - c_\alpha) = 0$, and $\overline{\Gamma}$ is preserved. An entirely similar argument shows the converse is also true. □

Theorem 2.16: *If $(a \wedge b = c) \in \Gamma$ is preserved in some partial order on G with positive cone P, then $-c = -a \vee -b$ in P.*

Proof: By definition of $a \wedge b = c$, we have $c \leq a, b$, thus $-a, -b \leq -c$, so that $-c$ is an upper bound of $-a$ and $-b$. Further suppose that $d \geq -a, -b$. Then $a, b \geq -d$ and therefore, $c \geq -d$. Then $d \geq -c$, and thus $-c = \sup\{-a, -b\}$. In other words, $-c = -a \vee -b$. □

In light of the previous two theorems, we can without loss of generality, assume that Γ contains only $\wedge$'s (meets) and that all $\wedge$'s in Γ are equal to 0, that is Γ is a collection of partial operations representing pairwise disjoint elements.

We now list other results about the relationships of Γ, P_Γ, and $\overline{P_\Gamma}$.

So that our examples are a little easier to visualize we consider first a Γ with only one lattice operation. Let $\Gamma = \{a \wedge b = 0\}$, $P_\Gamma = \{na + mb : n, m \geq 0\}$, and $\overline{P_\Gamma}$ be the semi-closure of P_Γ. After our examples we will generalize to an arbitrarily large Γ.

Theorem 2.17: *If $x, y \in P_\Gamma$, say $x = n_1 a + m_1 b$ and $y = n_2 a + m_2 b$, with $n_1, n_2, m_1, m_2 \in \mathbb{Z}^+$, then $x \wedge y$ exists and is determined by the minimum of each of the coefficients of x and y. That is, $x \wedge y = \min\{n_1, n_2\}a + \min\{m_1, m_2\}b$.*

Proof: Clearly $x, y \geq \min\{n_1, n_2\}a + \min\{m_1, m_2\}b$ in P_Γ. So we need only show this sum is the greatest lower bound of x and y. To that end consider the following cases:

Case 1: $n_1 \leq n_2$ and $m_1 \leq m_2$. But then $x \leq y$ and thus $x \wedge y = x$.

Case 2: $n_2 \leq n_1$ and $m_2 \leq m_1$. But then $y \leq x$ and thus $x \wedge y = y$.

Case 3: $n_1 \leq n_2$ and $m_2 \leq m_1$. Let $c = n_1 a + m_2 b$. We'll show $c = \inf\{x, y\}$. Now suppose $d \leq x, y$. Then $x - d$ and $y - d \in P_\Gamma$. So

$$
\begin{aligned}
x - d &= n_3 a + m_3 b \\
y - d &= n_4 a + m_4 b, \text{ so} \\
-d &= (n_3 - n_1)a + (m_3 - m_1)b, \text{ and} \\
-d &= (n_4 - n_2)a + (m_4 - m_2)b, \text{ so} \\
c - d &= (n_1 + n_3 - n_1)a + (m_2 + m_4 - m_2)b \\
&\qquad \text{above is true since } m_3 - m_1 = m_4 - m_2, \text{ so} \\
c - d &= n_3 a + m_4 b \geq 0, \text{ therefore} \\
c &\geq d, \text{ in other words} \\
c &= \inf\{x, y\}.
\end{aligned}
$$

Case 4: Similar to case 3 only $n_2 \leq n_1$ and $m_1 \leq m_2$.

Hence $x \wedge y$ exists when $x, y \geq 0$, and $x \wedge y = \min\{n_1, n_2\}a + \min\{m_1, m_2\}b$. □

An immediate corollary, at least with this simple Γ, indicates that the only disjoint elements occur along the "lines" from the origin generated by a and b. This is particularly useful in our examples of $\mathbb{Z} \times \mathbb{Z}$ that follow.

Corollary 2.18: *If $x \wedge y = 0$ in P_Γ and neither x nor y is 0, then $x = na$ and $y = mb$.*

Proof: By Theorem 2.17 above, $0 = x \wedge y = \min\{n_1, n_2\}a + \min\{m_1, m_2\}b$. So by independence, $\min\{n_1, n_2\} = \min\{m_1, m_2\} = 0$, but neither x nor y are 0. So if $n_1 = 0$, then $n_2 \neq 0, m_1 \neq 0$, and $m_2 = 0$ or $x = m_1 b$ and $y = n_2 a$. On the other hand if $n_2 = 0$, then $n_1 \neq 0, m_1 = 0$, and $m_2 \neq 0$ or $x = n_1 a$ and $y = m_2 b$. □

Theorem 2.19: *If $x \wedge y = c$ in $\overline{P_\Gamma}$, and $x, y \in \overline{P_\Gamma}$, then there exists a $k \in \mathbb{Z}^+$ such that $kx \wedge ky = kc$ in P_Γ.*

Proof: By hypothesis we have $x, y, c, x - c, y - c \in \overline{P_\Gamma}$. Thus $nx, my, rc, s(x - c), t(y - c) \in P_\Gamma$, for $n, m, r, s, t \in \mathbb{Z}^+$. Let $k = nmrst$. Then by Theorem 2.17, $kx \wedge ky$ exists in P_Γ, say $kx \wedge ky = t$. Hence $kx \wedge ky = t$ in $\overline{P_\Gamma}$, since $P_\Gamma \subseteq \overline{P_\Gamma}$ and $\overline{P_\Gamma}$ preserves the order from P_Γ. But $kx \wedge ky = k(x \wedge y) = kc$ in $\overline{P_\Gamma}$. The last statement holds because $\overline{P_\Gamma}$ is semi-closed and both $kx \wedge ky$ and $k(x \wedge y)$ exist in $\overline{P_\Gamma}$, and in an abelian ℓ-group $n(x \wedge y) = ny \wedge ny$. Hence $t = kc$ in $\overline{P_\Gamma}$ so $j(t - kc) = 0$ in P_Γ, but G is torsion-free, so $t = kc$ in P_Γ. □

Examples

Before we generalize these results to an arbitrarily large Γ, we first consider some examples of positive cones, P_Γ and $\overline{P_\Gamma}$, generated by a simple Γ, and look at when Γ forces (G, P_Γ) and $(G, \overline{P_\Gamma})$ into being an ℓ-group. These examples are interesting in their own right.

Example 2.20: Let $G = \mathbb{Z} \times \mathbb{Z}$, and let $a = (a_1, a_2)$, $b = (b_1, b_2)$ be two linearly independent points in G. That is assume $\Gamma = \{a \wedge b = 0\}$, and let $P_\Gamma = \{na + mb : n, m \geq 0\}$, which is a positive cone for G, and let $\overline{P_\Gamma}$ be the semi-closure of P_Γ. Let

$$A = \begin{pmatrix} a_2 & b_2 \\ a_1 & b_1 \end{pmatrix}, \text{ so that}$$

$$\det A = a_2 b_1 - b_2 a_1.$$

Theorem 2.21: *If* $\det A | (a_1, a_2, b_1, b_2)$ *then* $(\mathbb{Z} \times \mathbb{Z}, \overline{P_\Gamma})$ *is an* ℓ*-group. Furthermore, if* $\det A = \pm 1$*, then* $P_\Gamma = \overline{P_\Gamma}$.

Proof: From ℓ-group theory, we need only show that for all $z \in \mathbb{Z} \times \mathbb{Z}$, $z \vee 0$ exists. Let $z \in \mathbb{Z} \times \mathbb{Z}$, say $z = (z_1, z_2)$. Now the $\overline{P_\Gamma}$ boundary lines are

$$l_1 : y = \left(\frac{a_2}{a_1}\right) x \text{ and}$$

$$l_2 : y = \left(\frac{b_2}{b_1}\right) x, \text{ so that}$$

$$\overline{P_\Gamma} = \{(u_1, u_2) \in \mathbb{Z} \times \mathbb{Z} : (u_1, u_2) \text{ falls "between" } l_1 \text{ and } l_2\}.$$

Now consider the lines through z parallel to l_1, l_2, say l_1' and l_2'. Since l_1 and l_2 are not parallel, one of l_1' or l_2'must intersect l_1 or l_2. Without loss of generality, suppose l_2' intersects l_1. See figure 2.5, below.

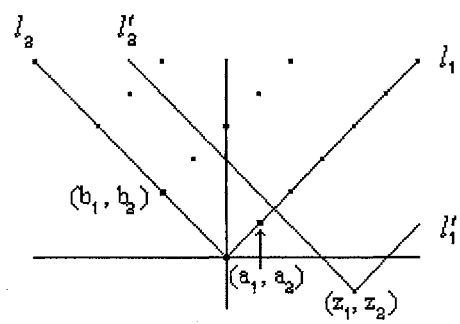


Figure 2.5 -- $(\mathbb{Z} \times \mathbb{Z}, \overline{P_\Gamma})$ is an ℓ-group

Then we have the following:

$$l_2' : y - z_2 = \frac{b_2}{b_1}(x - z_1)$$

$$(1) \qquad l_1 : y = \frac{a_2}{a_1} x, \text{ hence}$$

$$\frac{a_2}{a_1} x = \frac{b_2}{b_1}(x - z_1) + z_2, \text{ or}$$

$$\left(\frac{a_2b_1 - a_1b_2}{a_1b_1}\right)x = \frac{1}{b_1}(b_1z_2 - b_2z_1), \text{ or}$$

$$(2) \qquad x = \frac{a_1}{a_2b_1 - a_1b_2}(b_1z_2 - b_2z_1), \text{ so by (1)}$$

$$(3) \qquad y = \frac{a_2}{a_2b_1 - a_1b_2}(b_1z_2 - b_2z_1)$$

Hence $x, y \in \mathbb{Z}$, since in (2) and (3) $\det A | (a_1, a_2)$

Thus the lines l_1 and l_2' intersect at integer points, hence $z \vee 0 \in \mathbb{Z} \times \mathbb{Z}$. To finish the proof, suppose $\det A = 1$, then since $P_\Gamma \subseteq \overline{P_\Gamma}$, we need only show $\overline{P_\Gamma} \subseteq P_\Gamma$. For simplicity, assume the boundary lines have positive slopes and that $0 \leq \frac{b_2}{b_1} \leq \frac{a_2}{a_1}$. Now suppose $(u, v) \in \overline{P_\Gamma}$ and thus (u, v) is "between" l_1 and l_2. Then there exists $n, m, r \geq 0$ such that $n(a_1, a_2) + m(b_1, b_2) = r(u, v)$. Note that if (u, v) are between l_1 and l_2 then in this case $v \geq \frac{b_2}{b_1}u$ and $v \leq \frac{a_2}{a_1}u$, so that $b_1v - b_2u \geq 0$ and $a_2u - a_1v \geq 0$. So consider

$$\begin{aligned}
r(u, v) &= n(a_1, a_2) + m(b_1, b_2), \text{ then}\\
ru &= na_1 + mb_1 \text{ and}\\
(*) \qquad rv &= na_2 + mb_2, \text{ so we have}\\
rua_2 &= na_1a_2 + mb_1a_2 \text{ and}\\
rva_1 &= na_1a_2 + mb_2a_1, \text{ hence}\\
0 \leq r(ua_2 - va_1) &= m(a_2b_1 - a_1b_2) = m, \text{ since } \det A = 1, \text{ thus}\\
m &\in \mathbb{Z}, \text{ and substituting into } (*) \text{ we have}\\
rv &= na_2 + r(a_2u - a_1v)b_2, \text{ or}\\
na_2 &= r(v + a_1b_2v - a_2b_2u)\\
&= r((1 + a_1b_2)v - a_2b_2u)\\
&= r(a_2b_1v - a_2b_2u), \text{ so that}\\
n &= r(b_1v - b_2u) \geq 0, \text{ and thus } n \in \mathbb{Z}
\end{aligned}$$

Therefore $(u, v) \in P_\Gamma$ □

Can there be any points missing between P_Γ and $\overline{P_\Gamma}$? By that we mean can there be a point between the boundary lines, not in P_Γ and also not in $\overline{P_\Gamma}$? In other words is $\overline{P_\Gamma}$ what we think it is? That is, if $(u,v) \notin P_\Gamma$ but (u,v) is "between" the boundary lines of l_1 and l_2 does there exist $k \in \mathbb{Z}^+$ such that $k(u,v) \in P_\Gamma$, i.e. $(u,v) \in \overline{P_\Gamma}$? As in the previous example we will assume $0 \leq \frac{b_2}{b_1} \leq \frac{a_2}{a_1}$, so det$A > 0$. Now consider the following

$$k(u,v) = n(a_1,a_2) + m(b_1,b_2), \text{ hence}$$

$$ku = na_1 + mb_1, \text{ and} \tag{1}$$

$$kv = na_2 + mb_2, \text{ so } (1) \times a_2 - (2) \times a_1 \text{ gives} \tag{2}$$

$$m(a_2b_1 - a_1b_2) = k(a_2u - a_1v), \text{ note } a_2u - a_1v \geq 0$$

$$m = \frac{k(a_2u - a_1v)}{\det A}, \text{ so from (1) we get}$$

$$n = \frac{k(b_1v - b_2u)}{\det A}, \text{ also note that } b_1v - b_2u \geq 0, \text{ thus}$$

$$n, m \geq 0.$$

So pick k to be any multiple of det$A$, then $k(u,v) \in P_\Gamma$.

There actually is a slightly weaker condition for $\overline{P_\Gamma}$ to be an ℓ-group that we mention in the next corollary, however, the determinant condition seems to be the most useful.

Corollary 2.22: *If* det$A$ *divides all row and diagonal products, then* $(\mathbb{Z} \times \mathbb{Z}, \overline{P_\Gamma})$ *is an ℓ-group.*

Proof: In the proof of Theorem 2.21 note that in equations 2 and 3, if det$A | (a_1b_1,\ a_1b_2,\ a_2b_1,\ a_2b_2)$, then x and y are integers. These are merely the row and diagonal products of A. □

Notice in the following example, the positive cone is generated by (2,0) and (2,2), which is not semi-closed, i.e. $(1,1) \notin P_\Gamma$, so $P_\Gamma \neq \overline{P_\Gamma}$. Note that $\begin{vmatrix} 2 & 0 \\ 2 & 2 \end{vmatrix} = 4$ and $4 \nmid (2,2,2)$, but 4 does divide all row and diagonal products, so $\overline{P_\Gamma}$ is an ℓ-group. A sample of P_Γ is given in the diagram below. In the diagram, notice that $(6,7) \wedge (7,5) = (4,5)$. None of those points are in (G, P_Γ), but they are all in $(G, \overline{P_\Gamma})$.

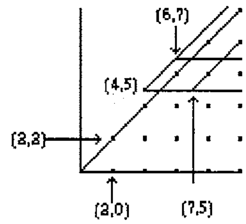


Figure 6 --A $\overline{P_\Gamma}$ that is an ℓ-group

We consider one final example here.

Example 2.23: We demonstrate where given a certain Γ, the generated positive cone, P_Γ is such that all $\wedge$'s in P_Γ exist, but not all $\wedge$'s in $\overline{P_\Gamma}$ exist. Let $G = \mathbb{Z} \times \mathbb{Z}$, $a = (1,1)$, and $b = (-1,1)$, so

$$P_\Gamma = \{n(1,1) + m(-1,1) : n, m \in \mathbb{Z}^+\}.$$

Then $\overline{P_\Gamma}$ is all integer ordered pairs "between" the lines $y = x$ and $y = -x$ (i.e. $y \geq |x|$). Now by what was done in Theorem 2.17, $x \wedge y$ exists for all $x, y \in P_\Gamma$. But we show that $(0,6)$, $(5,6) \in \overline{P_\Gamma}$, yet $(0,6) \wedge (5,6)$ does not exist in $\overline{P_\Gamma}$.

To this end note that $(0,6) = 3(1,1) + 3(-1,1) \in P_\Gamma \subseteq \overline{P_\Gamma}$ and $(5,6) \in \overline{P_\Gamma}$ since $2(5,6) = 11(1,1) + 1(-1,1) \in P_\Gamma$. Also note that both $(2,3)$ and (3,3) are smaller than both $(0,6)$ and $(5,6)$. This follows from $(0,6) - (2,3) = (-2,3) \in \overline{P_\Gamma}$, since $2(-2,3) = 1(1,1) + 5(-1,1) \in P_\Gamma$ and $(5,6) - (2,3) = (3,3) \in P_\Gamma$. Also $(0,6) - (3,3) = (-3,3) \in P_\Gamma \subseteq \overline{P_\Gamma}$ and $(5,6) - (3,3) = (2,3) \in \overline{P_\Gamma}$ since $2(2,3) = 5(1,1) + 1(-1,1) \in P_\Gamma$. But $(2,3) \| (3,3)$ (i.e. not comparable) since $(2,3) - (3,3) = (-1,0) \notin \pm P_\Gamma$ or $\pm\overline{P_\Gamma}$. So neither $(2,3)$ nor $(3,3)$ can be the inf$\{(0,6), (5,6)\}$.

Now suppose $(0,6) \wedge (5,6) = (u,v)$ in $\overline{P_\Gamma}$. This implies that $(-u, 6-v)$, $(5-u, 6-v)$, $(u-2, v-3)$, and $(u-3, v-3)$ are all in $\overline{P_\Gamma}$. Note the last two come from the fact that if $(u,v) = \inf\{(0,6),(5,6)\}$. Then $(u,v) \geq (2,3)$ and $(3,3)$. Because all these are in $\overline{P_\Gamma}$ (that is $y \geq x$ and $y \geq -x$) we have the following inequalities :

$$\begin{array}{lcl} 6-v \geq -u & \text{and} & 6-v \geq u \\ 6-v \geq 5-u & \text{and} & 6-v \geq u-5 \\ v-3 \geq u-2 & \text{and} & v-3 \geq 2-u \\ v-3 \geq u-3 & \text{and} & v-3 \geq 3-u \end{array}$$

these lead to the following

$$\begin{array}{llcl} (1) & 6 \geq v-u & \text{and} & 6 \geq v+u \\ (2) & 1 \geq v-u & \text{and} & 11 \geq v+u \\ (3) & v-v \geq 1 & \text{and} & v+u \geq 5 \\ (4) & v-u \geq 0 & \text{and} & v+u \geq 6 \end{array}$$

Now (2) and (3) imply that $1 \geq v-u \geq 1$, or $v-u = 1$, while (1) and (4) imply that $6 \geq v+u \geq 6$, or $v+u = 6$. Thus $2v = 7$ or $v = 7/2$, and $u = 5/2$, but $(5/2, 7/2) \notin \mathbb{Z} \times \mathbb{Z}$. Therefore $(0,6) \wedge (5,6)$ does not exist in $\overline{P_\Gamma}$. Notice in the figure below, that the potential meet misses all integer coordinates, indicated by the "white space" and ends up on the cell $(5/2,\ 7/2)$.

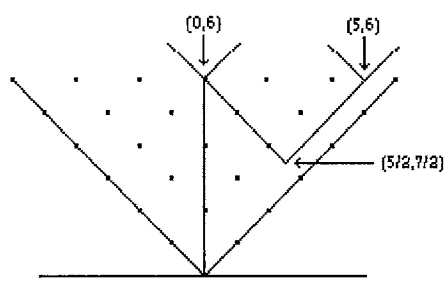


Figure 7 -- A $\overline{P_\Gamma}$ that is not an ℓ-group

Generalized Γ

We now generalize the above results for an arbitrarily large pairwise disjoint Γ, namely

$$\Gamma = \{a_i \wedge a_j = 0 : i \neq j \text{ and } i, j \in M, \text{ some index set}\}.$$

This leads to

$$P_\Gamma = \left\{ \sum_{i \in I} n_i a_i : n_i \geq 0, I \text{ a finite subset of } M \right\}.$$

Theorem 2.24: *If $x, y \in P_\Gamma$ then $x \wedge y$ exists and is the minimum value of the coefficients of x and y, as described below:*

$$x = \sum_{i \in I} {}_x n_i a_i, \textit{ for finite } I$$

$$y = \sum_{j \in J} {}_y n_j a_j, \textit{ for finite } J, \textit{ then}$$

$$x \wedge y = \sum_{k \in K} n'_k a_k, \textit{ for } K = I \cup J, \textit{ where}$$

$$n'_k = \min\{{}_x n_k, {}_y n_k\}.$$

Proof: From now on we assume that K is a common refinement of the finite subsets I, J. For notational benefit the x and y "pre-subscript" on the n_i and n_j are to identify that these integers go with the x and y variables. Also for ease of use, let $c = x \wedge y$. Clearly $x, y \geq c$.

Cases 1 and 2: $x \geq y$ or $y \geq x$. These work just as in Theorem 2.17.

Case 3: Suppose there exists some i's and j's such that ${}_x n_{k_i} \leq {}_y n_{k_i}$ while ${}_x n_{k_j} \geq {}_y n_{k_j}$.

We show that $c = \inf\{x, y\}$. Suppose $d \leq x, y$, then $x - d$ and $y - d \in P_\Gamma$. So

$$x - d = \sum_{k \in K} {}_{x'} n_k a_k$$

$$y - d = \sum_{k \in K} {}_{y'} n_k a_k, \text{ hence}$$

$$- d = \sum_{k \in K} ({}_{x'}n_k - {}_{x}n_k)a_k$$

$$- d = \sum_{k \in K} ({}_{y'}n_k - {}_{y}n_k)a_k, \text{ so for all } k, \text{ by independence}$$

(1) $({}_{x'}n_k - {}_{x}n_k) = ({}_{y'}n_k - {}_{y}n_k)$, hence

(2) $$c - d = \sum_{k \in K} (n'_k + {}_{x'}n_k - {}_{x}n_k)a_k, \text{ which simplifies to}$$

(3) $$c - d = \sum_{k \in K} {}_{i}n_k a_k \geq 0, \text{ where } i = x' \text{ or } y', \text{ therefore}$$

$$c \geq d, \text{ and thus we have}$$

$$c = \inf\{x, y\}.$$

The reason (3) follows from (2) is that in (2), for all $k \in K$, $n'_k = {}_{x}n_k$, or ${}_{y}n_k$. So if $n'_k = {}_{x}n_k$ then we are left with ${}_{x'}n_k$ while if $n'_k = {}_{y}n_k$ then we can substitute by (1), and we are left with ${}_{y'}n_k$. □

Theorem 2.25: *If $x \wedge y = 0$ in P_Γ and neither x nor y is 0, say*

$$x = \sum_{i \in I} {}_{x}n_i a_i \textit{ and } y = \sum_{j \in J} {}_{y}n_j a_j$$

*and if $I_x = \{a_i\text{'s that make up } x\}$ and $J_y = \{a_j\text{'s that make up } y\}$, then $I_x \cap J_y = \emptyset$. That is, if a_i is in the sum that makes up x, then it is **not** in the sum that makes up y.*

Proof: If $x \wedge y = 0$, then by Theorem 2.24 above, $0 = \sum_{k \in K} n'_k a_k$, where $n'_k = \min\{{}_{x}n_k, {}_{y}n_k\}$, but by linear independence, $n'_k = 0$, for all k. Thus for each k, the corresponding coefficient for a_k must be 0 in either the x variable or the y variable. So for example, if $n_i \neq 0$ in x, so that a_i is in I_x, then the corresponding n_i must be 0 in y, so that a_i would not be in J_y. □

Theorem 2.26: *If $x \wedge y = c$ in $\overline{P_\Gamma}$, and $x, y \in \overline{P_\Gamma}$, then there exists $k \in \mathbb{Z}^+$ such that $kx \wedge ky = kc$ in P_Γ.*

Proof: This follows exactly as in Theorem 2.19, since it only deals with P_Γ and $\overline{P_\Gamma}$ and not the construction of x or y. □

Now consider the following situation, suppose Γ' has only two partial operations in $\mathbb{Z} \times \mathbb{Z}$, that is let $\Gamma' = \{e_1 \wedge e_2 = e_2 \wedge e_3 = 0\}$. Then Γ' is not pairwise disjoint, yet still generates the same partial order as $\Gamma = \{e_1 \wedge e_2 = e_2 \wedge e_3 = e_1 \wedge e_3 = 0\}$, as shown in the figure below.

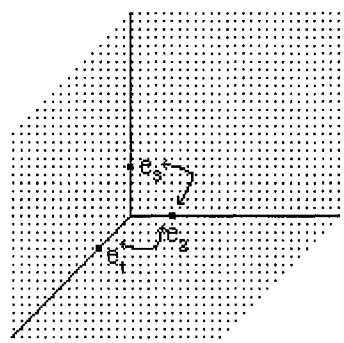


Figure 8 -- Minimal Γ

This leads us to consider if there is a minimal Γ' that generates the same partial order. We use Zorn's lemma to prove the following.

Theorem 2.27: *If G is a po group with positive cone P, then there exists a minimal Γ such that $P_\Gamma = P$.*

Proof: Let $\mathcal{X}$ = Set of all Γ's, where Γ is a collection of partial operations of the form $a \wedge b = c$ such that $P_\Gamma = P$. Where

$$P_\Gamma = \left\{ \sum_{i \in I} n_i a_i : n_i \in \mathbb{Z}^+, I \text{ a finite subset of } M \right\}.$$

Since in a po group, $a \wedge b = c$ if and only if $(a - c) \wedge (b - c) = 0$, we may as well assume all partial operations in Γ have the form $a \wedge b = 0$. First $\mathcal{X} \neq \emptyset$, since if $\Gamma = \{x \wedge 0 = 0 : x \in P\}$, then clearly $P_\Gamma = P$. Next, let $\mathcal{C}$ be a chain in $\mathcal{X}$, and let $\overline{\Gamma} = \cap\{\Gamma : \Gamma \in \mathcal{C}\}$. Then if $p \in P_{\overline{\Gamma}}$ then p is the linear combination of $\wedge$'s in $\overline{\Gamma}$, which are contained in all Γ's, so p is a linear combination of $\wedge$'s in $\Gamma \in \mathcal{C}$. Hence $p \in P_\Gamma = P$ and $P_\Gamma \subseteq P$. On the other hand, if $p \in P$ then since $P_\Gamma = P$, for all $\Gamma \in \mathcal{C}$, then $p \in P_\Gamma$, for all $\Gamma \in \mathcal{C}$, thus $p \in P_{\overline{\Gamma}}$, so $P \subseteq P_{\overline{\Gamma}}$. Therefore $P = P_{\overline{\Gamma}}$, and by Zorn's Lemma there exists a minimal element of $\mathcal{X}$, say Γ'. □

Theorem 2.28: *Let (G, P) be an abelian po group such that $x \wedge y$ exists for all $x, y \in P$. If $H = P - P = \{x - y : x, y \in P\}$, then (H, P) is an ℓ-group.*

Proof: First we show that H is a group. H is closed since if $a, b \in H$, then $a = x_1 - y_1$ and $b = x_2 - y_2$, where $x_1, x_2, y_1, y_2 \in P$. Since G is abelian, then $a + b = (x_1 + x_2) - (y_1 + y_2) \in H$. That $(H, +)$ is associative follows since $(G, +)$ is, and clearly $0 \in H$, since $0 \in P$. Finally, if $a \in H$, then $a = x - y$ and thus $-a = y - x \in H$. To show (H, P) is an ℓ-group we need only show that $a \wedge 0$ exists for all $a \in H$. To that end, let $a \in H$, so that $a = x - y$, for some $x, y \in P$. So we need only show that $(x - y) \wedge 0 = (x \wedge y) - y$. First $(x \wedge y) - y$ is a lower bound for $x - y$ and 0. Since $x \wedge y$ exists by assumption, then $x, y \geq x \wedge y$, so that $x - y \geq (x \wedge y) - y$ and $0 \geq (x \wedge y) - y$. Finally suppose $d \leq x - y, 0$, so that $d + y \leq x$ and $d + y \leq y$, thus $d + y \leq x \wedge y$. So $d \leq (x \wedge y) - y$. Therefore $(x \wedge y) - y = \inf\{a, 0\}$, and thus (H, P) is an ℓ-group. □

Chapter 3

Existence of Free Extensions

Introduction

The concept of a free algebra has a rich and interesting history. With the tools now available, the study of free algebras has shifted from creating all possible "words" using "letters" from a set and operations from the algebra to using universal mapping properties. This perhaps has the disadvantage of not "seeing" what a particular free object looks like, but is overshadowed by the advantage of discovering the similarities within a class of algebras and the properties shared by subalgebras within that particular class, not to mention the fact that some of the proofs are very elegant. Intuitively a free algebra can be thought of as the loosest way possible of constructing an algebra in a particular class from a set and the operations defined by the class of algebras.

There has been quite a bit of work done looking at free ℓ-groups in various varieties as well as free products, but there has been precious little done when it comes to partial ℓ-groups. Bernau [2], Conrad [6], and Weinberg [24] were among the first to look at free ℓ-groups. Many interesting results were obtained, even though in most cases the underlying group was equipped only with the trivial order.

In this chapter we define what a free extension of a partial ℓ-group is, determine when they exist and explore some of their properties. We begin with some definitions.

Free ℓ-groups

Definition 3.1: Let $\mathcal{U}$ be any variety of ℓ-groups and let X be a nonempty set.

The algebra $\mathcal{F}_{\mathcal{U}}(X)$ is called the *$\mathcal{U}$-free ℓ-group* if $\mathcal{F}_{\mathcal{U}}(X) \in \mathcal{U}$, X generates $\mathcal{F}_{\mathcal{U}}(X)$ as an ℓ-group, and whenever $H \in \mathcal{U}$ and $\beta : X \to H$ is a map, then there exists an ℓ-homomorphism $\lambda : \mathcal{F}_{\mathcal{U}}(X) \to H$, such that $\beta = \lambda i$. That is, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "\beta"'] & \mathcal{F}_{\mathcal{U}}(X) \arrow[d, "\lambda"] \\ & H \end{tikzcd}$$

Figure 9 -- $\mathcal{U}$-free ℓ-group

This is the standard definition of a free ℓ-group *over a set* (i.e. no structure). It has been known for sometime that $\mathcal{U}$-free ℓ-groups in any variety exist for any nonempty set X (see Powell [22] and Birkhoff [3]). Weinberg [24] defined the free ℓ-group *over a partially ordered group* as follows:

Definition 3.2: Let (G, P) be a torsion-free abelian group with semi-closed partial order P. An ℓ-group $\mathcal{F}_{\mathcal{W}}(G, P)$ is a *free ℓ-group over* (G, P) if

i). there exists an *o-isomorphism* $\psi : (G, P) \to \mathcal{F}_{\mathcal{W}}(G, P)$,

ii). $\psi(G, P)$ generates $\mathcal{F}_{\mathcal{W}}(G, P)$, and

iii). if $\beta : (G, P) \to H$ is an o-homomorphism into an ℓ-group, then there exists an ℓ-homomorphism $\lambda : \mathcal{F}_{\mathcal{W}}(G, P) \to H$, such that $\beta = \lambda\psi$. That is, the following diagram commutes:

$$\begin{tikzcd}(G,P) \arrow[r, "\psi"] \arrow[dr, "\beta"'] & \mathcal{F}_W(G,P) \arrow[d, "\lambda"] \\ & H\end{tikzcd}$$

Figure 10 -- Weinberg free ℓ-group

We now define what we mean by a free extension of a partial ℓ-group and then we compare the differences in the definitions.

Definition 3.3: Let (G,Γ) be a partial ℓ-group. G is assumed to be a torsion-free abelian group with partial lattice operations defined by Γ. An ℓ-group $\mathcal{F}(G,\Gamma)$ is the *free extension of the partial ℓ-group* (G, Γ) if

i). there exists a *partial ℓ-monomorphism* $i : (G, \Gamma) \to \mathcal{F}(G,\Gamma)$,

ii). $i(G,\Gamma)$ generates $\mathcal{F}(G,\Gamma)$ as an ℓ-group, and

iii). if $\beta : (G,\Gamma) \to H$ is a partial ℓ-homomorphism into an ℓ-group, then there exists an ℓ-homomorphism $\lambda : (G,\Gamma) \to H$ such that $\lambda i = \beta$. That is, the following diagram commutes:

$$\begin{tikzcd}(G,\Gamma) \arrow[r, "i"] \arrow[dr, "\beta"'] & \mathcal{F}(G,\Gamma) \arrow[d, "\lambda"] \\ & H\end{tikzcd}$$

Figure 11 -- Free extension

The difference between Weinberg's definition and ours rests solely in item *i).* Weinberg requires his embedding to be an order preserving isomorphism. This means $a \leq b$ if and only if $\psi(a) \leq \psi(b)$. Since the image of ψ is an ℓ-group, this requires that the partial order of the underlying group be semi-closed. This follows from the fact that an ℓ-group is semi-closed or isolated as some authors call it (see Fuchs [9]). Thus

$$\begin{aligned} &na \geq 0 \\ &\Leftrightarrow \psi(na) \geq 0 \\ &\Leftrightarrow n\psi(a) \geq 0 \\ &\Leftrightarrow \psi(a) \geq 0 \\ &\Leftrightarrow a \geq 0 \end{aligned}$$

On the other hand, our definition only requires that the embedding be a *monomorphism* that preserves the partial lattice operations. For example if $a \wedge b = 0$ in Γ then $i(a) \wedge i(b) = 0$. As a result, we do not need to require that any partial order extending the partial lattice order on the underlying group be semi-closed.

Now because ℓ-groups are distributive lattices, $+$ distributes over $\wedge$ and $\vee$, and the definition requires that (G, Γ) generate $\mathcal{F}(G, \Gamma)$, we see that elements of $\mathcal{F}(G, \Gamma)$ can be represented in the form

$$\bigvee_{i \in I} \bigwedge_{j \in J} i(g_{ij}), \text{ where } I \text{ and } J \text{ are finite and } g_{ij} \in G$$

Before we continue we show that if free extensions exist, they are unique.

Uniqueness

Theorem 3.4: *If $\mathcal{F}_1(G, \Gamma)$ and $\mathcal{F}_2(G, \Gamma)$ are free extensions of the partial ℓ-group (G, Γ), with λ_1, λ_2 as the respective ℓ-homomorphisms, then there exists a unique ℓ-isomorphism $\varphi : \mathcal{F}_1(G, \Gamma) \to \mathcal{F}_2(G, \Gamma)$, such that the following diagram commutes:*

$$\begin{tikzcd}
(G,\Gamma) \arrow[r, "\lambda_1"] \arrow[dr, "\lambda_2"'] & \mathcal{F}_1(G,\Gamma) \arrow[d, "\varphi"] \\
& \mathcal{F}_2(G,\Gamma)
\end{tikzcd}$$

Figure 12a -- Uniqueness of Free Extensions

Proof: By definition there exists ℓ-homomorphisms φ_1 and φ_2 so that the following diagram commutes:

$$\begin{tikzcd}
(G,\Gamma) \arrow[r, "\lambda_1"] \arrow[dr, "\lambda_2"'] & \mathcal{F}_1(G,\Gamma) \arrow[d, shift right, "\varphi_1"'] \\
& \mathcal{F}_2(G,\Gamma) \arrow[u, shift right, "\varphi_2"']
\end{tikzcd}$$

Figure 12b -- Uniqueness of Free Extensions

Thus if $x \in \mathcal{F}_1(G,\Gamma)$, then

$$x = \bigvee_I \bigwedge_J \lambda_1(g_{ij}), \text{ so that}$$

$$\begin{aligned}
\varphi_2\varphi_1(x) &= \varphi_2\varphi_1(\bigvee_I \bigwedge_J \lambda_1(g_{ij})) \\
&= \varphi_2(\bigvee_I \bigwedge_J \varphi_1\lambda_1(g_{ij})) \\
&= \varphi_2(\bigvee_I \bigwedge_J \lambda_2(g_{ij})) \\
&= \bigvee_I \bigwedge_J \varphi_2\lambda_2(g_{ij}) \\
&= \bigvee_I \bigwedge_J \lambda_1(g_{ij}) \\
&= x
\end{aligned}$$

Thus $\varphi_2\varphi_1$ is the identity on $\mathcal{F}_1(G,\Gamma)$, and similarly $\varphi_1\varphi_2$ is the identity on $\mathcal{F}_2(G,\Gamma)$. Therefore φ_1 is an ℓ-isomorphism. □

Since we are no longer looking at the free ℓ-group over a set, we are now faced with another problem. Do any free extensions of partial ℓ-groups exist and if they do, under what circumstances? We answer this question by invoking a theorem of R. S. Pierce [15, see page 101], which we state below for completeness.

Existence

Theorem 3.5: *Let $\mathcal{U}$ be a class of partial algebras of type τ which is closed under the formation of direct products and subalgebras. Let* **A** *be a partial algebra of type τ such that there is a partial monomorphism of* **A** *to some partial algebra belonging to $\mathcal{U}$. Then there exists a $\mathcal{U}$-free extension of* **A**.

We restate this theorem in terms of partial ℓ-groups and varieties.

Theorem 3.6: *If (G,Γ) is a partial ℓ-group in a variety $\mathcal{U}$, then the $\mathcal{U}$-free extension, $\mathcal{F}_{\mathcal{U}}(G,\Gamma)$, exists if and only if (G,Γ) can be embedded in an ℓ-group in the variety $\mathcal{U}$, using a partial ℓ-homomorphism.*

Because we are only considering torsion-free abelian groups in the abelian variety $\mathcal{A}$, we will refer to the $\mathcal{A}$-free extension of (G,Γ) as the free extension of (G,Γ), and will denote it by $\mathcal{F}(G,\Gamma)$. We now are in a position to show some properties and relationships among free extensions of partial ℓ-groups and partially ordered groups.

Properties

Theorem 3.7: *Let*

$$\Gamma = \{a_\delta \wedge a_\gamma = c_\alpha : a_\delta, a_\gamma, c_\alpha \in G\} \text{ and}$$
$$\Gamma' = \{(a_\delta - c_\alpha) \wedge (a_\gamma - c_\alpha) = 0 : a_\delta, a_\gamma, c_\alpha \in G\}$$

Then $\mathcal{F}(G, \Gamma)$ exists if and only if $\mathcal{F}(G, \Gamma')$ exists.

Proof: ($\rightarrow$) Suppose $\mathcal{F}(G, \Gamma)$ exists. If we can embed (G, Γ') in an ℓ-group, then by Theorem 3.6, $\mathcal{F}(G, \Gamma')$ exists. To this end, let $(G, \Gamma') \xrightarrow{i} (G, \Gamma) \xrightarrow{\gamma} \mathcal{F}(G, \Gamma)$, where i is the inclusion map, and γ is an embedding. Define $\beta = \gamma i$. Now β is clearly a group monomorphism, because both γ and i are. We only need to show that β preserves all the $\wedge$'s (meets) of Γ' in $\mathcal{F}(G, \Gamma)$. But in $\mathcal{F}(G, \Gamma)$,

$$\beta(c_\alpha) = \gamma(c_\alpha) = \gamma(a_\delta \wedge a_\gamma) = \gamma(a_\delta) \wedge \gamma(a_\gamma), \text{ so}$$
$$0 = (\gamma(a_\delta) \wedge \gamma(a_\gamma)) - \gamma(c_\alpha), \text{ hence}$$
$$0 = (\gamma(a_\delta) - \gamma(c_\alpha)) \wedge (\gamma(a_\gamma) - \gamma(c_\alpha)), \text{ thus}$$
$$0 = \gamma(a_\delta - c_\alpha) \wedge \gamma(a_\gamma - c_\alpha), \text{ hence}$$
$$0 = \beta(a_\delta - c_\alpha) \wedge \beta(a_\gamma - c_\alpha)$$

So β is a partial ℓ–monomorphism, hence $\mathcal{F}(G, \Gamma')$ exists.

($\leftarrow$) An entirely similar argument works the other way. □

Under these conditions, an even stronger statement can be made.

Theorem 3.8: *Under the hypotheses of Theorem 3.7,*

$$\mathcal{F}(G, \Gamma) \simeq \mathcal{F}(G, \Gamma').$$

Proof: Let $(G,\Gamma') \xrightarrow{i} (G, \Gamma) \xrightarrow{\gamma} \mathcal{F}(G, \Gamma)$ as above. Let β' be a partial ℓ-homomorphism from (G, Γ') to any $H \in \mathcal{A}$. Define $\beta : (G, \Gamma) \rightarrow H$ by $\beta i = \beta'$. Now suppose that $a_\delta \wedge a_\gamma = c_\alpha$ in (G, Γ). We need to show that $\beta(a_\delta) \wedge \beta(a_\gamma) = \beta(c_\alpha)$ in H. This follows from:

$$
\begin{aligned}
0 &= \beta'(a_\delta - c_\alpha) \wedge \beta'(a_\gamma - c_\alpha) \\
&= (\beta'(a_\delta) - \beta'(c_\alpha)) \wedge (\beta'(a_\gamma) - \beta'(c_\alpha)) \\
&= (\beta'(a_\delta) \wedge \beta'(a_\gamma)) - \beta'(c_\alpha) \\
&= \beta(a_\delta) \wedge \beta(a_\gamma) - \beta(c_\alpha).
\end{aligned}
$$

Thus β is a partial ℓ–homomorphism, so there exists a unique $\lambda : \mathcal{F}(G,\Gamma) \to H$, an ℓ-homomorphism, so that $\lambda\gamma = \beta$. Therefore, $\lambda\gamma i = \beta'$, and the following diagram commutes.

$$
\begin{tikzcd}
(G,\Gamma') \arrow[r, "i"] \arrow[dr, "\beta'"'] & (G,\Gamma) \arrow[r, "\gamma"] \arrow[d, "\beta"] & \mathcal{F}(G,\Gamma) \arrow[dl, "\lambda"] \\
& H &
\end{tikzcd}
$$

Figure 13 -- $\mathcal{F}(G,\Gamma) \simeq \mathcal{F}(G,\Gamma')$

Thus by uniqueness of free extensions, $\mathcal{F}(G,\Gamma) \simeq \mathcal{F}(G,\Gamma')$. $\square$

We now turn our attention to comparing the free extension of (G,Γ) with the free extensions of (G, P_Γ) and $(G, \overline{P_\Gamma})$. Recall that

$$
\Gamma = \{a_i \wedge a_j = 0 : i \neq j \text{ and } i,j \in M, \text{ some index set}\}
$$

$$
P_\Gamma = \left\{ \sum_{i\in I} n_i a_i : n_i \geq 0, I \text{ a finite subset of } M \right\}.
$$

$$
\overline{P_\Gamma} = \{x \in G : nx \in P_\Gamma \text{ for some } n \geq 0\}.
$$

Before the next theorem, we recall a fact from ℓ-group theory that we will need.

Lemma 3.9: If $a \wedge b = 0$ then $a + b = a \vee b$, and in general if $a_i \wedge a_j = 0$ for

$i \neq j$ then $\sum\limits_{i=1}^{n} a_i = \bigvee\limits_{i=1}^{n} a_i$. This follows from:

$$\begin{aligned} a+b &= a-(a \wedge b)+b \\ &= a+(-a \vee -b)+b \\ &= (0+b) \vee (a+0) \\ &= b \vee a \end{aligned}$$

A straight forward application of induction completes the proof. $\square$

Theorem 3.10: *If $\mathcal{F}(G,\Gamma)$ and $\mathcal{F}(G,P_\Gamma)$ exist, then they are the same.*

Proof: Let $(G,\Gamma) \xrightarrow{i} (G,P_\Gamma) \xrightarrow{\gamma} \mathcal{F}(G,P_\Gamma)$ be embeddings. Let $H \in \mathcal{A}$, and $(G,\Gamma) \xrightarrow{\beta'} H$ be a partial ℓ-homomorphism. Define $(G,P_\Gamma) \xrightarrow{\beta} H$ by $\beta i = \beta'$. Clearly, β is a group homomorphism, since both β' and i are. Next suppose $x \wedge y = 0$ in P_Γ. Then by Theorem 2.25, the collection of a_i's that make up x are disjoint from the collection of a_j's that make up y. Thus

$$\begin{aligned} \beta(x) \wedge \beta(y) &= \beta'(x) \wedge \beta'(y) \\ &= \beta'\Big(\sum_{i\in I} {}_x n_i a_i\Big) \wedge \beta'\Big(\sum_{j\in J} {}_y n_j a_j\Big) \\ &= \Big[\sum_{i\in I} {}_x n_i \beta'(a_i)\Big] \bigwedge \Big[\sum_{j\in J} {}_y n_j \beta'(a_j)\Big] \\ (1) \qquad &= \Big[\bigvee_{i\in I} {}_x n_i \beta'(a_i)\Big] \bigwedge \Big[\bigvee_{j\in J} {}_y n_j \beta'(a_j)\Big] \\ (2) \qquad &= \bigvee_{\substack{j\in J \\ i\in I}} \Big[{}_x n_i \beta'(a_i) \bigwedge {}_y n_j \beta'(a_j)\Big] \\ (3) \qquad &= 0. \end{aligned}$$

(1) follows from the previous lemma, the fact that β' preserves all defined lattice operations, and because all elements in Γ are pairwise disjoint. (2) follows from the distributive property of ℓ-groups. (3) follows from Theorem 2.25 and pairwise disjointness being preserved by β'.

Therefore, β is a partial l-homomorphism and thus there exists a unique ℓ-homomorphism, $\lambda : \mathcal{F}(G, P_\Gamma) \to H$ such that $\lambda\gamma i = \beta'$ so that the following diagram commutes.

$$\begin{tikzcd}
(G,\Gamma) \arrow[r, "i"] \arrow[dr, "\beta'"'] & (G,P_\Gamma) \arrow[r, "\gamma"] \arrow[d, "\beta"] & \mathcal{F}(G,P_\Gamma) \arrow[dl, "\lambda"] \\
& H &
\end{tikzcd}$$

Figure 14 -- $\mathcal{F}(G,\Gamma) \simeq \mathcal{F}(G, P_\Gamma)$

Therefore, by uniqueness of free extensions, $\mathcal{F}(G,\Gamma) \simeq \mathcal{F}(G, P_\Gamma)$. $\square$

The next theorem ties the free extension of (G, P_Γ) to the free extension of $(G, \overline{P_\Gamma})$.

Theorem 3.11: *If $\mathcal{F}(G, P_\Gamma)$ and $\mathcal{F}(G, \overline{P_\Gamma})$ exist, then they are the same.*

Proof: Let $(G, P_\Gamma) \xrightarrow{i} (G, \overline{P_\Gamma}) \xrightarrow{\gamma} \mathcal{F}(G, \overline{P_\Gamma})$ be embeddings. Let $H \in \mathcal{A}$, and $(G, P_\Gamma) \xrightarrow{\beta'} H$ be a partial ℓ-homomorphism. Define $(G, \overline{P_\Gamma}) \xrightarrow{\beta} H$ by $\beta i = \beta'$. Clearly, β is a group homomorphism, since both β' and i are. Next suppose $x \wedge y = 0$ in $\overline{P_\Gamma}$, so that $x, y \in \overline{P_\Gamma}$. Hence by Theorem 2.26, there exists $k \in \mathbb{Z}^+$ such that $kx, ky \in P_\Gamma$ and $kx \wedge ky = 0$ in P_Γ. Then we have

$$\begin{aligned}
k(\beta(x) \wedge \beta(y)) &= k(\beta'(x) \wedge \beta'(y)) \\
&= k\beta'(x) \wedge k\beta'(y) \\
&= \beta'(kx) \wedge \beta'(ky) \\
&= \beta'(kx \wedge ky) \\
&= \beta'(0) \\
&= 0
\end{aligned}$$

but $H \in \mathcal{A}$ and all ℓ-groups are torsion-free, so $0 = \beta(x) \wedge \beta(y)$. Therefore, β is a partial l-homomorphism and hence there exists a unique ℓ-homomorphism, $\lambda : \mathcal{F}(G, \overline{P_\Gamma}) \to H$ such that $\lambda\gamma i = \beta'$ so that the following diagram commutes.

$$\begin{tikzcd}
(G, P_\Gamma) \arrow[r, "i"] \arrow[dr, "\beta'"'] & (G, \overline{P_\Gamma}) \arrow[r, "\gamma"] \arrow[d, "\beta"] & \mathcal{F}(G, \overline{P_\Gamma}) \arrow[dl, "\lambda"] \\
& H &
\end{tikzcd}$$

Figure 15 --$\mathcal{F}(G, P_\Gamma) \simeq \mathcal{F}(G, \overline{P_\Gamma})$

Therefore, by uniqueness of free extensions, $\mathcal{F}(G, P_\Gamma) \simeq \mathcal{F}(G, \overline{P_\Gamma})$. □

Therefore, if the free extensions exist, then $\mathcal{F}(G, \Gamma) \simeq \mathcal{F}(G, P_\Gamma) \simeq \mathcal{F}(G, \overline{P_\Gamma})$.

We have looked at several properties pertaining to the partial ℓ-group (G, Γ), the partial orders (G, P_Γ) and $(G, \overline{P_\Gamma})$ generated by Γ, and the associated free extensions. We need to ask when a free extension does not exist. The next theorem answers this question and in the next chapter we construct free extensions and show when they do exist .

Theorem 3.12: *Let M be some index set. If $\Gamma = \{a_i \wedge a_j = 0 : \ i, j \in M\}$ and the collection of a_i's that make up the partial operations in Γ are linearly dependent then the free extension $\mathcal{F}(G, \Gamma)$ does not exist.*

Proof: Suppose the free extension did exist and $\sum_{i \in I} n_i a_i = 0$, for some $n_i \neq 0$.

Since this is a finite list, regroup and put all positive coefficients on one side of the equality and the negative coefficients on the other. After renumbering and relabeling, if necessary, we have

$$\sum_{j=1}^{k} n_{i_j} a_{i_j} = \sum_{l=1}^{m} n_{s_l} a_{s_l}, \text{ with } n_{i_j}, n_{s_l} \geq 0, \text{ for all } j,\ l, \text{ so}$$

$$\sum_{j=1}^{k} n_{i_j} a_{i_j} \geq n_{s_1} a_{s_1}, \text{ hence}$$

$$\left(\sum_{j=1}^{k} n_{i_j} a_{i_j}\right) \wedge n_{s_1} a_{s_1} = n_{s_1} a_{s_1} \geq 0, \text{ but}$$

$$\left(\sum_{j=1}^{k} n_{i_j} a_{i_j}\right) \wedge n_{s_1} a_{s_1} \leq \sum_{j=1}^{k} \left(n_{i_j} a_{i_j} \wedge n_{s_1} a_{s_1}\right)$$

$$\text{but } \sum_{j=1}^{k} \left(n_{i_j} a_{i_j} \wedge n_{s_1} a_{s_1}\right) = 0, \text{ since } a_i\text{'s and } a_s'\text{s are disjoint, hence}$$

$$n_{s_1} a_{s_1} = 0, \text{ but } \ell\text{-groups are torsion-free, so}$$

$$a_{s_1} = 0, \text{ which is a contradiction.}$$

□

Embeddings

We have already shown we can embed (G, Γ) into (G, P_Γ) and thus into $(G, \overline{P_\Gamma})$. In this section we now show how to embed $(G, \overline{P_\Gamma})$ into an ℓ-group so that we can invoke Theorem 3.6 to ensure the existence of the free extension, $\mathcal{F}(G, \Gamma)$. The approach we take here is to embed $(G, \overline{P_\Gamma})$ into appropriate factor groups, that inherit the order from $(G, \overline{P_\Gamma})$ and in such a way that the partial order of the factor groups can be extended to a total order. Finally, we will form the cardinal sum of these total orders, which is an ℓ-group, thus completing our embeddings, such that the partial orders in Γ still hold.

We first need to examine some properties of the convex, normal subgroups $\langle a_i, a_j \rangle$ generated by $a_i, a_j \in \Gamma$, for all $i \neq j$. These are defined as

$$\langle a_i, a_j \rangle = \{x \in G \mid na_i + ma_j \leq x \leq n'a_i + m'a_j, \text{ where } n,\ m,\ n',\ m' \in \mathbb{Z}\}.$$

Theorem 3.13: $\langle a_i, a_j \rangle = \{x \in G \mid x = na_i + ma_j, \text{ with } n, m \in \mathbb{Z}\}$.

Proof: Let $x \in \langle a_i, a_j \rangle$ then $na_i + ma_j \leq x \leq n'a_i + m'a_j$, so $x - (na_i + ma_j)$ and $n'a_i + m'a_j - x$ are both in P_Γ. Thus we have, after a common refinement of the finite index sets ($K = I \cup J, |K| = k$).

$$x - (na_i + ma_j) = \sum_{t \in K} r_t a_t$$

$$n'a_i + m'a_j - x = \sum_{t \in K} s_t a_t, \text{ so}$$

$$x = r_1 a_1 + \cdots + (n + r_i)a_i + \cdots + (m + r_j)a_j + \cdots + r_k a_k$$

$$-x = s_1 a_1 + \cdots + (s_i - n')a_i + \cdots + (s_j - m')a_j + \cdots + s_k a_k$$

$$0 = (r_1 + s_1)a_1 + \cdots + (n + r_i + s_i - n')a_i + \cdots$$
$$+ (m + r_j + s_j - m')a_j + \cdots + (r_k + s_k)a_k.$$

Therefore by linear independence and since $r_t, s_t \geq 0$ we have,

$0 = r_t = s_t$, for all $t \neq i, j$, hence

$x = ra_i + sa_j$. □

Next we show that these convex normal subgroups have nothing in common, except 0.

Theorem 3.14: *Let $\langle a_i, a_j \rangle$ be the convex normal subgroup generated by $a_i, a_j \in \Gamma$, then*

$$\bigcap_{\substack{i \neq j \\ i,j \in M}} \langle a_i, a_j \rangle = \{0\}$$

Proof: We break this down into two cases, one where the cardinality of Γ is 3 and the other where it is bigger.

Case 1: If $|\Gamma| = 3$, then we have $\langle a, b \rangle \cap \langle a, c \rangle \cap \langle b, c \rangle$ and suppose x is in the intersection, then by Theorem 3.13 we have

$x = na + mb = ra + sc = qb + tc$, thus we get

$0 = (n - r)a + mb - sc$, and

$0 = ra - qb + (s - t)c$, and by independence we have

$0 = n - r = m = s = q = r = s - t$, hence

$x = 0$

Case 2: If $|\Gamma| \geq 4$, then there exist $a_i,\ a_j,\ a_k,\ a_l$, all distinct. Hence if $x \in \bigcap_{i \neq j} \langle a_i, a_j \rangle$, then $x \in \langle a_i, a_j \rangle \cap \{a_k, a_l\}$ and by Theorem 3.13 we have

$x = na_i + ma_j = ra_k + sa_l$, thus

$0 = na_i + ma_j - ra_k - sa_l$, hence by independence,

$0 = n = m = r = s$, hence $x = 0$. □

Recall the definition of the semi-closure of these convex normal subgroups, $\langle a_i, a_j \rangle$, denoted by $\overline{\langle a_i, a_j \rangle} = \{x \in G : nx \in \langle a_i, a_j \rangle$, for some $n \in \mathbb{Z}\}$. So as a Corollary of the previous Theorem, we have

Corollary 3.15: *Let $\overline{\langle a_i, a_j \rangle}$ be the pure, convex normal subgroup generated by a_i, a_j, then*

$$\bigcap_{\substack{i \neq j \\ i,j \in M}} \overline{\langle a_i, a_j \rangle} = \{0\}.$$

Proof: Let $x \in \bigcap_{i \neq j} \overline{\langle a_i, a_j \rangle}$, so $x \in \overline{\langle a_i, a_j \rangle}$ for all $i \neq j$. Thus $m_{ij}x \in \langle a_i, a_j \rangle$ so $(\prod_{i \neq j} m_{ij})x \in \bigcap_{i \neq j} \langle a_i, a_j \rangle = \{0\}$. But G is torsion free, so $x = 0$. □

Before we continue we need to recall some group theoretic properties and definitions, as well as some additional partial order concepts..

Definition 3.16: For $x \in G$, $n \in \mathbb{N}$, *x is divisible* by n, if there exists $y \in G$ such that $x = ny$.

Definition 3.17: For some fixed $n \in \mathbb{N}$, define

$$nG = \{\mathrm{g} \in G : \mathrm{g} = nx, \text{ for some } x \in G\}.$$

H is a *pure subgroup* of G if $H \cap nG = nH$ for all $n \in \mathbb{N}$. Since $H \cap nG \supseteq nH$ is always true, we need only show $nH \supseteq H \cap nG$ for H to be pure. That is, if $\mathrm{h} = n\mathrm{g}$, for $\mathrm{h} \in H$ and $\mathrm{g} \in G$ then there exists an $\mathrm{h}' \in H$ such that $\mathrm{h} = n\mathrm{h}'$. In other words, if an element of H is divisible by n in G then it is also divisible by n in H.

Theorem 3.18: *Let X be any subset of G and $\langle X \rangle$ the convex normal subgroup generated by X, then $\overline{\langle X \rangle}$ is a convex, normal, pure subgroup of G.*

Proof: Normality: This follows from G being abelian.

Convexity: Suppose $x \leq c \leq y$, and $x, y \in \overline{\langle X \rangle}$, then $nx \in \langle X \rangle$ and $mx \in \langle X \rangle$ for some $n, m \in \mathbb{Z}$. So $nmx \leq nmc \leq nmy$ and $nmx, nmy \in \langle X \rangle$ which is convex so $nmc \in \langle X \rangle$, thus $c \in \overline{\langle X \rangle}$. Therefore $\overline{\langle X \rangle}$ is convex.

Purity: Suppose $x \in \overline{\langle X \rangle} \cap nG$, say $x = ng$ and $x \in \overline{\langle X \rangle}$. Hence $mx \in \langle X \rangle$ hence $m(ng) \in \langle X \rangle$ or $(nm)g \in \langle X \rangle$, so $g \in \overline{\langle X \rangle}$, thus $x \in n\overline{\langle X \rangle}$, hence $\overline{\langle X \rangle}$ is pure. □

Theorem 3.19: *Suppose G is torsion free, then H is pure if and only if G/H is torsion free.*

Proof: ($\rightarrow$) Suppose H is pure. Let $\mathrm{k} \in G/H$, say $\mathrm{k} = \mathrm{g} + H$ and suppose $n\mathrm{k} = 0$, that is suppose $n\mathrm{g} \in H$, say $n\mathrm{g} = \mathrm{h}$. But H is pure so there exists $\mathrm{h}' \in H$ such that $n\mathrm{h}' = \mathrm{h}$, but $n\mathrm{h}' = \mathrm{h} = n\mathrm{g}$, so $n(\mathrm{h}' - \mathrm{g}) = 0$. But G is torsion free so $\mathrm{h}' - \mathrm{g} = 0$, hence $\mathrm{h}' =\ \mathrm{g}$, thus $\mathrm{g} \in H$, therefore G/H is torsion free.

($\leftarrow$) Suppose G/H is torsion free. Let $x \in nG \cap H$, say $x \in H$ and $x = n\mathrm{g}$. But $n\mathrm{g} = x \in H$ and G/H is torsion free so $\mathrm{g} \in H$, thus $x \in nH$, hence H is pure. □

Definition 3.20: A group G is called an *$\mathcal{O}^*$-group* if every partial order of G can be extended to a total order of G.

Theorem 3.21: [Fuchs 9] *An abelian group is an $\mathcal{O}^*$-group if and only if it is torsion-free.*

Now since quotient groups inherit the partial order from the underlying group, that is, using the natural homomorphism, the positive cone of $(G, \overline{P_\Gamma})/\overline{\langle a_i, a_j\rangle}$ is the image of the positive cone of $(G, \overline{P_\Gamma})$. So in light of the above definitions and theorems, since $\overline{\langle a_i, a_j\rangle}$ is pure, then $(G, \overline{P_\Gamma})/\overline{\langle a_i, a_j\rangle}$ is torsion-free, hence an $\mathcal{O}^*$-group, and therefore has a total order T_{ij}, so that $T_{ij} \supseteq ((G, \overline{P_\Gamma})/\overline{\langle a_i, a_j\rangle})^+$. To complete our embeddings we need to embed $(G, \overline{P_\Gamma})$ into the cardinal sum of all quotient groups $(G, \overline{P_\Gamma})/\overline{\langle a_i, a_j\rangle}$, for all $i \neq j$ so that the partial lattice operations are preserved. We do this by defining the map:

$$\alpha : (G,\overline{P_\Gamma}) \to \underset{\substack{i \neq j \\ i,j \in M}}{\boxplus} \left(G/\overline{\langle a_i, a_j\rangle}\right)$$

by the following, for all $g \in G$:

$$\alpha(g) = \prod_{\substack{i \neq j \\ i,j \in M}} \left(g + \overline{\langle a_i, a_j\rangle}\right).$$

We need to show the following:

Theorem 3.22 *The map, α, as defined above is an order preserving partial ℓ-monomorphism.*

Proof: 1). α is a group homomorphism.

$$\begin{aligned}\alpha(g+h) &= \prod_{\substack{i\neq j\\ i,j\in M}} \left((g+h)+\overline{\langle a_i,a_j\rangle}\right)\\ &= \prod_{\substack{i\neq j\\ i,j\in M}} \left((g+\overline{\langle a_i,a_j\rangle})+(h+\overline{\langle a_i,a_j\rangle})\right)\\ &= \prod_{\substack{i\neq j\\ i,j\in M}} (g+\overline{\langle a_i,a_j\rangle})+\prod_{\substack{i\neq j\\ i,j\in M}} (h+\overline{\langle a_i,a_j\rangle})\\ &= \alpha(g)+\alpha(h)\end{aligned}$$

2). α preserves order since on each component of the product $(G/\overline{\langle a_i,a_j\rangle})^+$ is the image of $(G,\overline{P_\Gamma})^+$ under the natural homomorphism of G onto $G/\overline{\langle a_i,a_j\rangle}$.

3). α is a monomorphism, for suppose $\alpha(g)=\alpha(h)$, then

$$\alpha(g-h)=0=\prod_{\substack{i\neq j\\ i,j\in M}} (\overline{\langle a_i,a_j\rangle}), \text{ thus}$$

$$g-h\in\bigcap_{\substack{i\neq j\\ i,j\in M}} \overline{\langle a_i,a_j\rangle}=\{0\}, \text{ so}$$

$$g=h.$$

4). Finally, we show α preserves the lattice operations in Γ, that is

$$0=\alpha(a_i\wedge a_j)=\alpha(a_i)\wedge\alpha(a_j).$$

Now since we are in a partial order, this is equivalent to showing that

$$\prod_{\substack{i\neq j\\ i,j\in M}} (\overline{\langle a_i,a_j\rangle})=\inf\{\alpha(a_i),\alpha(a_j)\}.$$

Now $\alpha(a_i)\geq 0$, since $a_i\geq 0$ and

$$\alpha(a_i) = \left(\cdots,\ \overline{\langle a_i, a_j\rangle},\ \cdots,\ \overline{\langle a_i, a_k\rangle},\ \cdots,\ a_i + \overline{\langle a_j, a_k\rangle},\ \cdots\right) \geq 0.$$

So 0 is a lower bound for $\alpha(a_i)$. Now pick $k \in \underset{i \neq j}{\boxplus}\left(G/\overline{\langle a_i, a_j\rangle}\right)$, such that $k \leq \alpha(a_i), \alpha(a_j)$, say $k = \prod_{i \neq j}\left(g_{ij} + \overline{\langle a_i, a_j\rangle}\right)$. We need to show $k \leq 0$. But

$$\prod_{i \neq j}\left(g_{ij} + \overline{\langle a_i, a_j\rangle}\right) \leq \left(\cdots,\ \overline{\langle a_i, a_j\rangle},\ \cdots,\ \overline{\langle a_i, a_k\rangle}\ ,\ \cdots,\ a_i + \overline{\langle a_j, a_k\rangle}\ ,\ \cdots\right) \text{ and}$$

$$\leq \left(\cdots,\ \overline{\langle a_i, a_j\rangle},\ \cdots,\ a_j + \overline{\langle a_i, a_k\rangle}\ ,\ \cdots,\ \overline{\langle a_j, a_k\rangle}\ ,\ \cdots\right), \text{ so}$$

$$g_{ij} + \overline{\langle a_i, a_j\rangle} \leq \overline{\langle a_i, a_j\rangle}, \text{ for all } i \neq j, \text{ therefore}$$

$$k \leq 0, \text{ therefore}$$

$$\alpha(a_i) \wedge \alpha(a_j) = 0$$

Thus we have that α is an order preserving partial ℓ-monomorphism. □

Now since the cardinal product of a collection ot total orders is an ℓ-group, this completes all our embeddings, hence by Theorem 3.6, $\mathcal{F}(G, \Gamma)$ exists. We have established the following embeddings:

$$(G, \Gamma) \to (G, P_\Gamma) \to (G, \overline{P_\Gamma}) \to \underset{i \neq j}{\boxplus}\left(G/\overline{\langle a_i, a_j\rangle}\right) \to \underset{i \neq j}{\boxplus}\left(G/\overline{\langle a_i, a_j\rangle}\right)_{T_{ij}}$$

Before we close with an interesting comparison of existence theorems, we show that we can relax the requirement that Γ be pairwise disjoint. All we require is for the elements of Γ to be disjoint and linearly independent.

Definition: 3.23: If Γ' is a collection of partial operations on the partial ℓ-group (G, Γ) whose equations are disjoint meets and whose elements are linearly independent, we say that Γ is the *pairwise disjoint completion of* Γ' if $\Gamma \supseteq \Gamma'$ and all equations of meets in Γ are pairwise disjoint, that is $\Gamma = \{a_i \wedge a_j = 0 : i \neq j\}$.

Theorem 3.24: *Let (G, Γ') be a torsion-free abelian partial ℓ-group whose* *partial lattice operations are disjoint and whose elements are linearly independent, then* $\mathcal{F}(G,\Gamma')$ *exists. Furthermore, there is an* ℓ*-epimorphism from* $\mathcal{F}(G,\Gamma')$ *onto* $\mathcal{F}(G,\Gamma)$*, where* Γ *is the pairwise disjoint completion of* Γ'.

Proof: Let Γ be the pairwise disjoint completion of Γ' by adding enough lattice operations to make Γ pairwise disjoint. Thus by the above embeddings and Theorem 3.6, $\mathcal{F}(G,\Gamma')$ exists. Now embed (G,Γ') into (G,Γ) by the inclusion map i. Let γ' and γ be the embeddings into the corresponding free extensions. That is

$$\gamma' : (G,\Gamma') \to \mathcal{F}(G,\Gamma')$$
$$\gamma : (G,\Gamma) \to \mathcal{F}(G,\Gamma)$$

By existence of free extensions, there exists an ℓ-homomorphism, $\lambda : \mathcal{F}(G,\Gamma') \to \mathcal{F}(G,\Gamma)$ such that $\lambda\gamma'(x) = \gamma i(x) = \gamma(x)$ for all $x \in G$. That is the following diagram commutes.

$$\begin{tikzcd}
(G,\Gamma') \arrow[rr, "\gamma'"] \arrow[dr, "i"'] & & \mathcal{F}(G,\Gamma') \arrow[dd, "\lambda"] \\
& (G,\Gamma) \arrow[dr, "\gamma"'] & \\
& & \mathcal{F}(G,\Gamma)
\end{tikzcd}$$

Figure 16 -- Disjoint vs Pairwise Disjoint

Now let $y \in \mathcal{F}(G,\Gamma)$, say $y = \bigvee_R \bigwedge_S \gamma(x_{ij})$, where $x_{ij} \in G$, and R and S are finite index sets. Then we have

$$\begin{aligned}
y &= \bigvee_R \bigwedge_S \gamma(x_{ij}) \\
&= \bigvee_R \bigwedge_S \lambda\gamma'(x_{ij}) \\
&= \lambda(\bigvee_R \bigwedge_S \gamma'(x_{ij})) \\
&= \lambda(x)
\end{aligned}$$

where $x = \bigvee_R \bigwedge_S \gamma'(x_{ij}) \in \mathcal{F}(G, \Gamma')$, thus λ is an ℓ-epimorphism. Therefore

$\mathcal{F}(G, \Gamma')/\ker \lambda \simeq \mathcal{F}(G, \Gamma)$. □

We now compare an existence theorem used by Weinberg [24], Conrad [6], and Bernau [2] to our existence theorem. The theorem below can be found in Conrad [see 5, page 6.8, Corollary II]. We then close this chapter with a discussion of the differences and similarities of these two theorems, along with some examples.

Theorem 3.25: *[Conrad] For an abelian po-group G, the following are equivalent.*

1). There exists a free ℓ-group over G.

2). There exists an o-isomorphism of G into an ℓ-group.

3). G^+ is the intersection of total orders.

4). The partial order of G is semi-closed.

Theorem 3.26: *For an abelian partial ℓ-group (G, Γ), the following are equivalent.*

1'). The free-extension of the partial ℓ-group (G, Γ) exists.

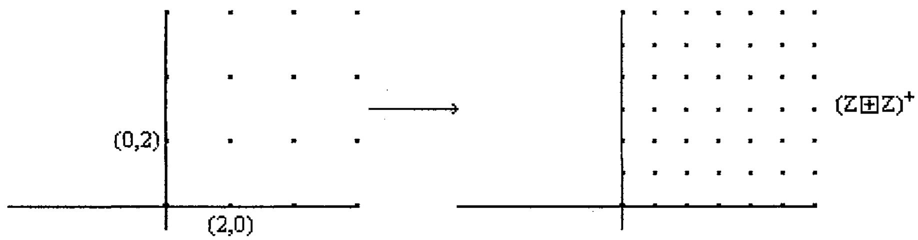
2'). There exists an embedding of (G, Γ) into an ℓ-group.

3'/4'). The partial lattice operations of Γ are disjoint, and the elements of these operations are (integer) linearly independent.

Proof: That 1' and 2' are equivalent follows from Theorem 3.6. That 2' and 3'/4' are equivalent follows from Theorems 2.4, 3.12, and 3.24. □

Example 3.27: Let $G = (\mathbb{Z} \times \mathbb{Z})$, with $\Gamma = \{(2,0) \wedge (0,2) = (0,0)\}$. Then $P_\Gamma = \{n(2,0) + m(0,2) : n, m \geq 0\}$ and $\overline{P_\Gamma} = (\mathbb{Z} \times \mathbb{Z})^+$. Therefore by Theorem 3.26,

$\mathcal{F}(G,\Gamma)$ exists, and is in fact $\mathbb{Z} \boxplus \mathbb{Z}$. Also note by Theorem 3.10, that $\mathcal{F}(G,\Gamma) = \mathcal{F}(G,P_\Gamma)$. See diagram below



(G,P_Γ) $\qquad$ $(G,\overline{P_\Gamma}) = \mathcal{F}(G,\Gamma) = \mathbb{Z} \boxplus \mathbb{Z}$

Figure 17 -- P_Γ not semi-closed, $\mathcal{F}(G,\Gamma)$ still exists

On the other hand, (G,P_Γ) is not semi-closed, since $2(1,1) = (2,2) \in P_\Gamma$, but $(1,1) \notin P_\Gamma$. Therefore by Theorem 3.25 $\mathcal{F}_W(G,P_\Gamma)$ does not exist.

However, as in Weinberg's case, if we assume G is a semi-closed partially ordered group with positive cone P, and we let $\Gamma = \{a \wedge 0 = 0 : a \in P\}$, then our (G,Γ) is a partial ℓ-group whose only elements are comparable ones. Recall, $a \geq b$ if and only if $(a-b) \geq 0$ if and only if $(a-b) \wedge 0 = 0$. Now by Theorem 3.25, $\mathcal{F}_W(G,P)$ does exist. But by Theorems 3.24 and 3.26, $\mathcal{F}(G,\Gamma)$ also exists. If we further assume that the specific elements that satisfy the partial operations of Γ are linearly independent, then $\mathcal{F}_W(G,P) \simeq \mathcal{F}(G,\Gamma)$.

Theorem 3.28: *Under the conditions discussed above,*

$$\mathcal{F}_W(G,P) \simeq \mathcal{F}(G,\Gamma).$$

Proof: By Theorem 2.9 $P_{\Gamma_P} = P = \overline{P}$, since P is semi-closed. Let $H \in \mathcal{A}$ and let $i : (G,\Gamma) \to (G,P)$ be the inclusion map. Let $\beta' : (G,\Gamma) \to H$ be a partial ℓ-homomorphism. Define $\beta : (G,P) \to H$ by $\beta(i(x) = \beta'(x)$. Clearly β is well-defined. It is also clear that β is a group homomorphism since β' and i both are. It remains to show that β preserves order. To this end, let $x \in P$. We need to show $\beta(x) \geq 0$. So we have

$$\begin{aligned} \beta(x) &= \beta i(x) \\ &= \beta'(x), \text{ but } x \in P \text{ so } x \wedge 0 = 0, \text{ but } \beta' \text{ is a partial } \ell\text{-homomorphism, so} \\ &\geq 0. \end{aligned}$$

Therefore, β is an o-homomorphism. Therefore, $\lambda\gamma i = \beta'$. Hence by uniqueness of the free extension, $\mathcal{F}(G,\Gamma) \simeq \mathcal{F}_W(G,P)$. That is, the following diagram commutes.

$$\begin{tikzcd}
(G,\Gamma) \arrow[r, "i"] \arrow[dr, "\beta'"'] & (G,P) \arrow[r, "\gamma"] \arrow[d, "\beta"] & \mathcal{F}_W(G,P) \arrow[dl, "\lambda"] \\
& H &
\end{tikzcd}$$ $\square$

Figure 18 -- $\mathcal{F}(G,\Gamma) \simeq \mathcal{F}_W(G,P)$

Thus when our Γ contains the comparable elements, and no others, our free-extension coincides with Weinberg's. On the other hand, we have partial ℓ-groups whose free-extensions do exist, that do not exist under Weinberg's existence conditions. We turn our attention to the actual construction of the free-extensions in the next chapter.

Chapter 4

Construction of Free Extensions

Introduction

With the foundation we now have, we are in a position to construct free extensions of partial ℓ-groups. Showing they exist actually gives us insight into how the construction should go. The goal seems straightforward enough, generate an ℓ-group from G, in such a way that all the partial lattice operations still hold, while at the same time, for any ℓ-group, H, we have an ℓ-homomorphism, extending G. In other words, our object satisfies the universal mapping property. Generating the ℓ-group is easy enough. It is ensuring that we have an ℓ-homomorphism that provides us the challenge. We try to use some techniques and ideas of those who have gone before, Weinberg [24 and 25], Bernau [2], Conrad [6], and Powell and Tsinakis [17, 18, and 20], to name but a few. The approach they have used is both simple and brilliant. Form the direct product of all total orders that preserve the partial lattice operations, then generate a sublattice so that the mapping property is satisfied. To do this we need some additional notions.

Positively Independent

Definition 4.1: A nonempty set A, of a partially ordered abelian group G, is said to be *positively independent*, if for any finite subsets $\{a_1, \ldots, a_n\} \subseteq A\backslash\{0\}$ and $\{k_1, \ldots, k_n\} \subseteq \mathbb{Z}^+$, we have $k_i = 0$ for each $i = 1, \ldots, n$, whenever $\sum_{i=1}^{n} k_i a_i \in G^-$. An important result from this definition, which we will make repeated use of, follows.

Theorem 4.2: *Let A be a nonempty subset of a partially ordered torsion-free abelian group G. There exists a total order on G with positive cone $T \supseteq (G^+ \cup A)$ if* *and only if A is positively independent.*

Proof: $(\rightarrow)$ Suppose there exists a total order on G with positive cone $T \supseteq (G^+ \cup A)$. Let $\{a_1, \ldots, a_n\} \subseteq A\backslash\{0\}$ and $\{k_1, \ldots, k_n\} \subseteq \mathbb{Z}^+$ and suppose $\sum_{i=1}^n k_i a_i \in G^-$. Now $\sum_{i=1}^n k_i a_i \in T$ since $k_i \geq 0$ for all i, and $\{a_1, \ldots, a_n\} \subseteq A \subseteq T$. On the other hand $\sum_{i=1}^n k_i a_i \in -T$, since $\sum_{i=1}^n k_i a_i \in -G \subseteq -T$. Therefore $\sum_{i=1}^n k_i a_i = 0$. Now if at least one $k_i \neq 0$, then $k_i a_i$ can be written as a linear combination of the others, so after renumbering, if necessary, $k_1 a_1 = \sum_{i=2}^n k_i a_i \in -T$, so that $k_1 a_1 \in T \cap -T = 0$. But G is torsion-free, so $a_i = 0$, a contradiction. Therefore $k_i = 0$, for all $i = 1, \ldots, n$.

$(\leftarrow)$ Suppose A is positively independent. Define

$$P = \left\{\sum n_i a_i + g : n_i \in \mathbb{Z}^+,\ a_i \in A,\ g \in G^+\right\}.$$

Clearly, $P \supseteq (G^+ \cup A)$. We now show that P is a positive cone, of some partial order on G. That $P + P \subseteq P$ and $g + P - g \subseteq P$ both hold, follows easily since G is an abelian partially ordered group, and sums of positive integers are positive. Finally, we show that $P \cap -P = \{0\}$. To this end, suppose $x \in P \cap -P$, so $x \in P$ and $-x \in P$, that is

$$x = \sum n_i a_i + g_1, \text{ and}$$

$$-x = \sum m_i a_i + g_2, \text{ so that}$$

$$0 = \sum (n_i + m_i) a_i + (g_1 + g_2), \text{ thus}$$

$$\sum (n_i + m_i) a_i = -(g_1 + g_2) \in G^-, \text{ so}$$

$$0 = n_i + m_i, \text{ since } A \text{ is positively independent}$$

$$0 = n_i = m_i, \text{ because } n_i, m_i \geq 0, \text{ so we have}$$

$$x = g_1$$

$$-x = g_2, \text{ which implies that}$$

$$x \in G^+ \cap G^-, \text{ therefore}$$

$$x = 0.$$

Therefore, P is a positive cone of some partial order on G which is an $\mathcal{O}^*$-group. So there exists a total order with positive cone $T \supseteq P \supseteq (G^+ \cup A)$. □

Intuitively, this means that if there exists a positive integer linear combination of elements of A that end up in the negative cone, then there is no way to surround both A and G^+ by a total order. For example, in the following diagram, there is no way to draw a straight line, in $\mathbb{Z} \times \mathbb{Z}$, that keeps A and G^+ on the same side of the line.

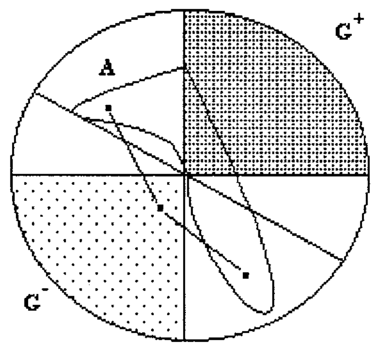


Figure 19 -- Positively Independent

Generated Sublattice

We now have all the necessary machinery to finish our construction which culminates in the following theorem.

Theorem 4.3 *Let G be a torsion-free abelian group with partial lattice operations defined by*

$$\Gamma = \{a_i \wedge a_j = 0 \textit{ for } i \neq j \textit{ and } i, j \in M \textit{ some index set}\}.$$

Then the free extension of (G, Γ) *is the sublattice generated by the diagonal map,*

$$\gamma : (G, \Gamma) \to \prod_{\Lambda} \left(\boxplus_{i \neq j} \left(\left(G / \overline{\langle a_i, a_j \rangle} \right),\ T_{ij} \right) \right)$$

defined by

$$\gamma(g) = \left(\cdots, \left(\boxplus_{i \neq j}\left(g + \overline{\langle a_i, a_j \rangle}\right)_{T_{ij}}\right), \cdots\right)$$

where Λ *is the collection of all total orders extending the partial orders of* $G/\overline{\langle a_i, a_j \rangle}$, and T_{ij} *are the positive cones of total orders such that* $T_{ij} \supseteq (G/\overline{\langle a_i, a_j \rangle})^+$ *for all* $T_{ij} \in \Lambda$. *In other words*

$$\mathcal{F}(G, \Gamma) = \left\{ \bigvee_R \bigwedge_S \gamma(g_{rs}) : R,\ S \text{ are finite index sets} \right\}.$$

Proof: First, we show γ is an embedding. Clearly, γ is a group homomorphism. Next suppose $g \in \ker \gamma$, then $\gamma(g) = 0$, in other words, $g \in \bigcap_{i \neq j} \overline{\langle a_i, a_j \rangle} = \{0\}$, so $g = 0$, hence γ is a monomorphism. Now because of the natural epimorphism from $(G, P_\Gamma) \to \left(G/\overline{\langle a_i, a_j \rangle}, P_\Gamma/\overline{\langle a_i, a_j \rangle}\right)$, every component extends the inherited order from (G, P_Γ). That is, $0 = \gamma(0) = \gamma(a_i \wedge a_j) = \gamma(a_i) \wedge \gamma(a_j)$, for all $i \neq j$. Therefore, γ is a partial ℓ-monomorphism.

Now let $\beta : (G, \Gamma) \to H \in \mathcal{A}$, be a partial ℓ-homomorphism. Without loss of generality, we can assume H is totally ordered. This follows because $H \in \mathcal{A} \subseteq \mathcal{R}$, the variety of representable ℓ-groups. That is, H is contained in a product of totally ordered groups, each of whose projection maps are onto. Since the free extension, $\mathcal{F}(G, \Gamma)$, exists there is an ℓ-homomorphism $\lambda : \mathcal{F}(G, \Gamma) \to H$ so that $\lambda\gamma = \beta$. Now $H = \prod H_i$ where each H_i is totally ordered and each projection map, $\pi_i : H \to H_i$, is onto. Hence for each i there exists a unique λ_i such that $\lambda_i \gamma = \pi_i \beta$. Thus

$$\begin{aligned} \lambda_i(\gamma(g)) &= \pi_i(\beta(g)) \\ &= \pi_i(\lambda(\gamma(g))), \text{ so} \\ \lambda_i &= \pi_i \lambda, \text{ for all } i. \end{aligned}$$

This is illustrated in the following commutative diagram.

$$
\begin{tikzcd}
(G,\Gamma) \arrow[rr, "\gamma"] \arrow[dr, "\beta"'] & & \mathcal{F}(G,\Gamma) \arrow[dl, "\lambda"] \arrow[dd, "\lambda_i"] \\
& H \arrow[dr, "\pi_i"'] & \\
& & H_i
\end{tikzcd}
$$

Figure 20 -- $H \in \mathcal{A} \subseteq \mathcal{R}$, Variety of representable ℓ-groups

So if $x \in \mathcal{F}(G,\Gamma)$ and $\lambda(x) \neq 0$ then $\lambda_i(x) = \pi_i\lambda(x)$ for all i and since π_i are projection maps, there exists some i such that $\pi_i\lambda(x) \neq 0$ hence $\lambda_i(x) \neq 0$. But λ_i is well-defined so $x \neq 0$. So whatever happens in H because of λ, a similar thing happens in some totally ordered group H_i because of the projection map.

Now if we can exhibit an ℓ-homomorphism,

$$\varphi : \prod_{\Lambda}\left(\boxplus_{i\neq j}\left((G/\overline{\langle a_i, a_j\rangle}),\ T_{ij}\right)\right) \to H$$

such that $\varphi\gamma = \beta$, then by uniqueness of free extensions, $\mathcal{F}(G,\Gamma)$ will be the sublattice generated by γ, as described in the theorem. To this end define φ, by

$$\varphi(\bigvee_R \bigwedge_S \gamma(g_{rs})) = \bigvee_R \bigwedge_S \beta(g_{rs})$$

where R and S are finite index sets. Clearly, φ is the right map, provided it is well-defined! Hence we need only show that

$$\bigvee_R \bigwedge_S \beta(g_{rs}) \neq 0 \Rightarrow \bigvee_R \bigwedge_S \gamma(g_{rs}) \neq 0.$$

Since, by assumption, H is totally ordered we can break this down into two cases.

Case 1: $\bigvee_R \bigwedge_S \beta(g_{rs}) > 0.$

Then there exists r_0 such that $\bigwedge_S \beta(g_{r_0 s}) > 0$ for all s. Hence $\beta(g_{r_0 s}) > 0$, for all s. So $g_{r_0 s} \notin \ker\beta$, for all s. Now $\ker\beta$ is an o-ideal, so $G/\ker\beta$ is isomorphic to a subgroup of H. Since subgroups of totally ordered groups are totally ordered, $G/\ker\beta$ is totally ordered, so $(g_{r_0 s} + \ker\beta) > 0$, for all s in $G/\ker\beta$.

Claim 1: $\{g_{r_0 s} : s \in S\}$ is positively independent in G.

Proof: Suppose $\sum_{s=1}^{n} k_s g_{r_0 s} \in G^-$, $k_s \geq 0$. We need to show that $k_s = 0$, for all s. But $\beta(\sum_{s=1}^{n} k_s g_{r_0 s}) \in H^-$ since β is an o-homomorphism and H is totally ordered, hence $\sum_{s=1}^{n} k_s \beta(g_{r_0 s}) \in H^-$, but $\beta(g_{r_0 s}) \in H^+$, for all s, so $k_s\beta(g_{r_0 s}) \in H^+$, for all s, so $\sum_{s=1}^{n} k_s\beta(g_{r_0 s}) \in H^+$. Hence $\sum_{s=1}^{n} k_s\beta(g_{r_0 s}) = 0$, thus $k_s\beta(g_{r_0 s}) = 0$, for all s. But H is torsion-free, so $k_s = 0$, for all s.

Claim 2: $\{g_{r_0 s} + \overline{\langle a_i, a_j \rangle}\}$ is positively independent in $G/\overline{\langle a_i, a_j\rangle}$.

Proof: Suppose $\sum_{s=1}^{n} k_s(g_{r_0 s} + \overline{\langle a_i, a_j\rangle}) \in (G/\overline{\langle a_i, a_j\rangle})^-$, then $\sum_{s=1}^{n} k_s g_{r_0 s} + \overline{\langle a_i, a_j\rangle} \in G^-/\overline{\langle a_i, a_j\rangle}$, hence $\sum_{s=1}^{n} k_s g_{r_0 s} \in G^-$, so by the previous claim, $k_s = 0$, for all s.

Note that in Claim 2 above, we use the fact that $(G/H)^- = G^-/H$. This follows since, if $x \in (G/H)^-$, then $x = g + H$, with $g + H \leq H$. So $g \leq h'$, and $g - h' \leq 0$. Thus $g - h' \in G^-$, and hence $g - h' + H \in G^-/H$. But $g - h' + H = g + H = x$, so $x \in G^-/H$. On the other hand, suppose $x \in G^-/H$, say $x = g + H$ with $g \in G^-$. That is, $g \leq 0$, thus $g + H \leq H$, so $x \in (G/H)^-$.

Therefore, by Theorem 4.2 there exists a total order with positive cone T_{ij} such that the following is true

$$\left((G/\overline{\langle a_i, a_j\rangle})^+, \{g_{r_0 s} + \overline{\langle a_i, a_j\rangle}\}\right) \subset T_{ij}, \text{ thus}$$

$$\left(g_{r_0 s} + \overline{\langle a_i, a_j\rangle}\right)_{T_{ij}} > 0, \text{ for all } s, \text{ thus}$$

$$\bigwedge_S \left(g_{r_0 s} + \overline{\langle a_i, a_j\rangle}\right)_{T_{ij}} > 0, \text{ so that}$$

$$\bigvee_R \bigwedge_S \left(g_{rs} + \overline{\langle a_i, a_j\rangle}\right)_{T_{ij}} > 0, \text{ therefore}$$

$$\boxplus_{i \neq j}\left(\bigvee_R \bigwedge_S \left(g_{rs} + \overline{\langle a_i, a_j\rangle}\right)_{T_{ij}} \right) \neq 0, \text{ thus}$$

$$\prod_{T_{ij} \in \Lambda} \left(\boxplus_{i \neq j}\left(\bigvee_R \bigwedge_S \left(g_{rs} + \overline{\langle a_i, a_j\rangle}\right)_{T_{ij}} \right)\right) \neq 0, \text{ therefore}$$

$$\bigvee_R \bigwedge_S \gamma(g_{rs}) \neq 0.$$

Case 2: $\bigvee_R \bigwedge_S \beta(g_{rs}) < 0$.

Because ℓ-groups are distributive lattices, we'll use the following notation from Anderson and Feil [1] and Conrad [5]:

$$\bigwedge_R \bigvee_S a_{rs} = \bigvee_{S^R} \bigwedge_R a_{rf(r)}, \text{ where } S^R \text{ is all permutations on } R.$$

So using this and the inverse properties of the lattice operations of ℓ-groups we have

$$0 > \bigvee_R \bigwedge_S \beta(g_{rs}), \text{ so}$$

$$0 < -(\bigvee_R \bigwedge_S \beta(g_{rs})), \text{ which is, by lattice inverse operations}$$

$$= \bigwedge_R \bigvee_S \beta(-g_{rs}), \text{ and because } \ell\text{-groups are distributive, this is}$$

$$= \bigvee_{S^R} \bigwedge_R \beta(-g_{rf(r)}), \text{ so there exists an } f \in S^R \text{ such that}$$

$$0 < \bigwedge_R \beta(-g_{rf(r)}), \text{ for all } r \in R, \text{ therefore}$$

$$0 < \beta(-g_{rf(r)}), \text{ for all } r.$$

Hence $-g_{rf(r)} \notin \ker \beta$, so as with Case 1, $\{-g_{rf(r)}\}$ is positively independent in G. Thus, $\{-g_{rf(r)} + \overline{\langle a_i, a_j \rangle}\}$ is positively independent in $G/\overline{\langle a_i, a_j \rangle}$. Therefore there exists a total order with positive cone T_{ij} such that the following is true

$$\left((G/\overline{\langle a_i, a_j\rangle})^+, \{-g_{rf(r)} + \overline{\langle a_i, a_j\rangle}\}\right) \subset T_{ij}, \text{ thus}$$

$$\left(-g_{rf(r)} + \overline{\langle a_i, a_j\rangle}\right)_{T_{ij}} > 0, \text{ for all } r, \text{ thus}$$

$$\bigwedge_R \left(-g_{rf(r)} + \overline{\langle a_i, a_j\rangle}\right)_{T_{ij}} > 0, \text{ so that}$$

$$\bigvee_{S^R} \bigwedge_R \left(-g_{rf(r)} + \overline{\langle a_i, a_j\rangle}\right)_{T_{ij}} > 0, \text{ therefore}$$

$$\bigwedge_R \bigvee_S \left(-g_{rs} + \overline{\langle a_i, a_j\rangle}\right)_{T_{ij}} > 0, \text{ or equivalently}$$

$$\bigvee_R \bigwedge_S \left(g_{rs} + \overline{\langle a_i, a_j\rangle}\right) < 0, \text{ hence}$$

$$\boxplus_{i\neq j}\left(\bigvee_R \bigwedge_S \left(g_{rs} + \overline{\langle a_i, a_j\rangle}\right)_{T_{ij}}\right) \neq 0, \text{ thus}$$

$$\prod_{T_{ij}\in\Lambda}\left(\boxplus_{i\neq j}\left(\bigvee_R \bigwedge_S \left(g_{rs} + \overline{\langle a_i, a_j\rangle}\right)_{T_{ij}}\right)\right) \neq 0, \text{ therefore}$$

$$\bigvee_R \bigwedge_S \gamma(g_{rs}) \neq 0.$$

So φ is well-defined, and by its definition it is an ℓ-homomorphism. So by uniqueness of free extensions we have

$$\mathcal{F}(G,\Gamma) = \left\{ \bigvee_R \bigwedge_S \gamma(g_{rs}) : R,\ S \textit{ are finite index sets}\right\}. \qquad \square$$

Chapter 5

Structure of Free Extensions

Introduction

Prior to this we have been concerned with the existence of free extensions of partial ℓ-groups and their construction. We now turn our attention to some of the characteristics and attributes contained in the structure of these free extensions. The ideas we discuss in this chapter are the subalgebra property, disjoint sets, and cardinal decomposition.

In universal algebra, the subalgebra property is an important idea that enables one to look at smaller objects and still retain the essence of a larger free object. Since we are dealing with partial ℓ-groups, which are partial algebras, we have to define in our context what a reasonable subalgebra property is.

In the study of ℓ-groups, disjoint sets or orthogonal elements have many interesting properties. They can be put into equivalence classes called filets or carriers. The collection of these carriers forms a lattice. Also each equivalence class, consists of those elements that are orthogonal to the same set and forms a convex subsemigroup. Since all ℓ-groups are infinite, and often uncountably infinite, it is interesting that in some cases the size of these pairwise disjoint sets cannot be uncountable.

Finally, an algebra, in a split exact sequence tells you more about the structure of the object in question. In ℓ-groups an analogous characteristic is whether or not the object has a nontrivial cardinal decomposition.

We examine each of these ideas briefly.

Subalgebra Properties

In universal algebra, there is really only one reasonable way to define the subalgebra of an algebra. However, in our setting we are dealing with partial algebras and as Grätzer [14, page 80] points out there are three ways to define a subalgebra of a partial algebra. Using his terminology, these are *subalgebra, relative subalgebra,* and *weak subalgebra.* Pierce [15, page 28] also mentions that there is "... no obvious "right way" to define subalgebras of relational systems." Although there is not universal agreement on what is meant by a subalgebra of a partial algebra or rather what is the most useful definition, we get the distinct feeling that Gräzter and Pierce lean toward the relative subalgebra concept. We state this below in universal algebra notation and then in the context of partial ℓ-groups. See Grätzer [14] and Pierce [15] for a more complete discussion of subalgebras.

Definition 5.1: Let $\mathbf{A} = \langle A; F_\xi \rangle_{\xi<\rho}$ be a partial algebra of similarity type τ. A subset B of A *determines a subalgebra of* $\mathbf{A}$ if the condition

$$\mathbf{b} \in \mathfrak{D}(F_\xi) \cap B^{\tau(\xi)} \text{implies } F_\xi(\mathbf{b}) \in B$$

is satisfied. An algebra $\mathbf{B}$ of type τ is called a *subalgebra of* $\mathbf{A}$ if $B \subseteq A$, B determines a subalgebra of $\mathbf{A}$, and $\mathbf{B} = \mathbf{A}|B$ ($= \langle B; F_\xi \cap B^{\tau(\xi)} \rangle_{\xi<\rho}$ the restriction of $\mathbf{A}$ to B).

For our purposes we will say the subalgebra property holds for free extensions in the variety of ℓ-groups $\mathcal{U}$, if whenever H and G are partial ℓ-groups with $H \subseteq G$, there is an ℓ-monomorphism $\varphi : \mathcal{F}_\mathcal{U}(H) \to \mathcal{F}_\mathcal{U}(G)$, where $\mathcal{F}_\mathcal{U}(H)$, $\mathcal{F}_\mathcal{U}(G)$ are the $\mathcal{U}$-free extensions of H and G, respectively. That is, $H \subseteq G \Rightarrow \mathcal{F}_\mathcal{U}(H) \subseteq \mathcal{F}_\mathcal{U}(G)$.

Powell and Tsinakis [17 and 18] have shown that the corresponding subalgebra property holds for free products in $\mathcal{A}$, the variety of abelian ℓ-groups. It is interesting to note that there are uncountably many other varieties for which the subalgebra property fails for free products (see Powell and Tsinakis [17, 21, and 22]).

In our setting, however, H and G discussed above are partial ℓ-groups. Therefore we need to modify the definition of the subalgebra property in applying it to free extensions to what appears to be the most natural way. Unfortunately this is not as easy as it seems. The next theorem shows that the "subalgebra property" holds for a fixed Γ over two different abelian torsion-free groups.

Theorem 5.2: *For a fixed* Γ, *if* $(G_1, \Gamma) \subseteq (G_2, \Gamma)$ *then* $\mathcal{F}(G_1,\Gamma) \subseteq \mathcal{F}(G_2,\Gamma)$.

Proof: Embed (G_1, Γ) into (G_2, Γ) by the inclusion map, and let γ_k be embeddings into the corresponding free extensions. That is,

$$\gamma_1 : (G_1,\Gamma) \to \mathcal{F}(G_1,\Gamma)$$
$$\gamma_2 : (G_2,\Gamma) \to \mathcal{F}(G_2,\Gamma).$$

Then $\gamma_2 i : (G_1,\Gamma) \to \mathcal{F}(G_2,\Gamma)$ is a partial ℓ-homomorphism. Hence there exists an ℓ-homomorphism λ_1: $\mathcal{F}(G_1,\Gamma) \to \mathcal{F}(G_2,\Gamma)$ such that $\lambda_1\gamma_1 = \gamma_2 i$. That is, the following diagram commutes.

$$\begin{tikzcd}
(G_1,\Gamma) \arrow[rr, "\gamma_1"] \arrow[dr, "i"'] & & \mathcal{F}(G_1,\Gamma) \arrow[dd, "\lambda_1"] \\
& (G_2,\Gamma) \arrow[dr, "\gamma_2"'] & \\
& & \mathcal{F}(G_2,\Gamma)
\end{tikzcd}$$

Figure 21 -- Subalgebra Property

But γ_k (for $k = 1, 2$) are the diagonal maps from

$$(G_k, \Gamma) \to \prod_{T_{ij} \in \Lambda_k} \left(\boxplus_{i \neq j} (G_k / \overline{\langle a_i, a_j \rangle})_{T_{ij}} \right)$$

defined by

$$\gamma_k(g) = \left(\cdots, \left(\boxplus_{i \neq j} (g + \overline{\langle a_i, a_j \rangle}) \right), \cdots \right)$$

where $a_i, a_j \in \Gamma$. Now since Γ is fixed for both G_1 and G_2 we have that the positive cones, P_1 and P_2, induced by Γ are the same, say P. But since $G_1 \subseteq G_2$, all total orders that extend (G_1, P) also extend (G_2, P). Thus if Λ_1 is the collection of total orders on G_1 with positive cone $T \supseteq P$ and Λ_2 is a similar collection for G_2, then $\Lambda_2 \supseteq \Lambda_1$. Hence on G_1, $\gamma_1 = \gamma_2$ on all total orders in Λ_1. Therefore if $x \in \ker \lambda_1$, say $x = \bigvee_R \bigwedge_S \gamma_1(x_{ij})$, where R and S are finite index sets and all $x_{ij} \in G_1$, then

$$\begin{aligned} 0 &= \lambda_1(x) \\ &= \lambda_1(\bigvee_R \bigwedge_S \gamma_1(x_{ij})) \\ &= \bigvee_R \bigwedge_S \lambda_1 \gamma_1(x_{ij}) \\ &= \bigvee_R \bigwedge_S \gamma_2(x_{ij}) \\ &= \bigvee_R \bigwedge_S \gamma_1(x_{ij}) \\ &= x. \end{aligned}$$

Thus λ_1 is an ℓ-monomorphism and hence $\mathcal{F}(G_1, \Gamma) \subseteq \mathcal{F}(G_2, \Gamma)$. □

The next result, using a fixed group G, but different Γ's, leads us to believe, that the subalgebra property fails in the $\mathcal{A}$ variety, which we do verify by an example.

Theorem 5.3: *For a fixed group G, if $\Gamma_1 \subseteq \Gamma_2$, then there exists an*

ℓ-epimorphism $\lambda_1 : \mathcal{F}(G, \Gamma_1) \to \mathcal{F}(G, \Gamma_2)$.

Proof: Embed (G, Γ_1) into (G, Γ_2) by the inclusion map i. Let γ_i be embeddings into the corresponding free extensions. That is

$$\gamma_1 : (G, \Gamma_1) \to \mathcal{F}(G, \Gamma_1)$$
$$\gamma_2 : (G, \Gamma_2) \to \mathcal{F}(G, \Gamma_2)$$

By the existence of free extensions, there exists an ℓ-homomorphism,

$\lambda_1 : \mathcal{F}(G, \Gamma_1) \to \mathcal{F}(G, \Gamma_2)$ such that $\lambda_1\gamma_1(x) = \gamma_2 i(x) = \gamma_2(x)$ for all $x \in G$. That is, the following diagram commutes.

$$\begin{tikzcd}
(G, \Gamma_1) \arrow[rr, "\gamma_1"] \arrow[dr, "i"'] & & \mathcal{F}(G,\Gamma_1) \arrow[dd, "\lambda_1"] \\
& (G, \Gamma_2) \arrow[dr, "\gamma_2"'] & \\
& & \mathcal{F}(G, \Gamma_2)
\end{tikzcd}$$

Figure 22 -- Fixed G, different Γ's

Now let $y \in \mathcal{F}(G, \Gamma_2)$, say $y = \bigvee_R \bigwedge_S \gamma_2(x_{ij})$, where $x_{ij} \in G$, and R and S are finite index sets. Then we have

$$\begin{aligned}
y &= \bigvee_R \bigwedge_S \gamma_2(x_{ij}) \\
&= \bigvee_R \bigwedge_S \lambda_1\gamma_1(x_{ij}) \\
&= \lambda_1(\bigvee_R \bigwedge_S \gamma_1(x_{ij})) \\
&= \lambda_1(x)
\end{aligned}$$

where $x = \bigvee_R \bigwedge_S \gamma_1(x_{ij}) \in \mathcal{F}(G, \Gamma_1)$. Thus λ_1 is an ℓ-epimorphism. Therefore

$\mathcal{F}(G, \Gamma_1)/\ker \lambda_1 \simeq \mathcal{F}(G, \Gamma_2)$. $\square$

The following example verifies the above theorem and shows that in some cases the subalgebra property does not hold.

Example 5.4:

$$\begin{aligned} &\text{Let } G = \mathbb{Z}, \\ &\quad \Gamma_1 = \{0 \wedge 0 = 0\}, \text{ and} \\ &\quad \Gamma_2 = \{1 \wedge 0 = \ 0 \wedge 0 = 0\}, \text{ so that} \\ &(G, \Gamma_1) \subseteq (G, \Gamma_2) \end{aligned}$$

We show that $\mathcal{F}(G, \Gamma_1) \not\subseteq \mathcal{F}(G, \Gamma_2)$. In particular, $\mathcal{F}(G, \Gamma_2) = (\mathbb{Z}, \mathbb{Z}^+)$, since (G, Γ_2) embeds into $(\mathbb{Z}, \mathbb{Z}^+)$ which is an ℓ-group. On the other hand $(\mathbb{Z}, \Gamma_1)$ embeds into $(\mathbb{Z}, \{0\})$, that is the partially ordered group $\mathbb{Z}$, with the trivial order. But this is the free abelian group on one generator and by Birkhoff [3] the free ℓ-group on one generator is $\mathbb{Z} \boxplus \mathbb{Z}$, which is not totally ordered. Now since $(\mathbb{Z}, \mathbb{Z}^+)$ is totally ordered and all ℓ-subgroups of totally ordered groups are totally ordered we have that $\mathbb{Z} \boxplus \mathbb{Z} \not\subseteq \mathbb{Z}$.

Disjoint Sets

Let m be an infinite cardinal. If G is an ℓ-group, then G is said to satisfy the *m-disjointness condition* if every set S of pairwise disjoint elements has cardinality less than m. Based on the results above and the results of Powell and Tsinakis [19] we show that free extensions of partial ℓ-groups can have uncountably large disjoint sets.

Weinberg [25] has shown that free extensions of partially ordered groups with the trivial order satisfies the $\aleph_1$-disjointness condition. This is extension of the fact that free abelian ℓ-groups satisfy the $\aleph_1$-disjointness condition. In general, Powell and Tsinakis [19] have shown that free objects in any variety (in a countable language) satisfy the $\aleph_1$-disjointness condition. Hence free ℓ-groups in any variety of ℓ-groups have no uncountable disjoint sets. However free extensions of partial ℓ-groups is another matter. In the following examples we use the result that for G_i, a collection of totally ordered groups,

$$\mathcal{F}(\boxplus_{i\in I} G_i) = \bigsqcup_{i\in I} G_i.$$

Example 5.5: Let $G = \mathbb{Z} \boxplus \mathbb{Z}$ and $\Gamma = \{a \wedge 0 = 0 : a \geq 0\}$. The positive cone generated by Γ is $P = (\mathbb{Z} \boxplus \mathbb{Z})^+$. Thus (G, P) is an ℓ-group so that $(G, P) = \mathcal{F}(G, \Gamma)$. But

$$\begin{aligned}\mathcal{F}(\mathbb{Z} \boxplus \mathbb{Z}, \Gamma) &\simeq \mathbb{Z} \sqcup \mathbb{Z} \\ &\subseteq (\mathbb{Z} \boxplus \mathbb{Z}) \sqcup (\mathbb{Z} \boxplus \mathbb{Z}) \\ &\simeq \mathcal{F}(\{x_1, x_2\})\end{aligned}$$

and since $\mathcal{F}(\{x_1, x_2\})$ satisfies the $\aleph_1$-disjointness condition (Powell [19] and Weinberg [25]), so does $\mathcal{F}(\mathbb{Z} \boxplus \mathbb{Z}, \Gamma)$.

Example 5.6: Let $G = \mathbb{R} \boxplus \mathbb{R}$, with Γ as in the previous example, so that the positive cone generated by Γ is $P = (\mathbb{R} \boxplus \mathbb{R})^+$. Powell and Tsinakis [19] have shown that $\mathbb{R} \sqcup \mathbb{R}$ has an uncountably large disjoint set and since $\mathcal{F}(\mathbb{R} \boxplus \mathbb{R}, \Gamma) \simeq \mathbb{R} \sqcup \mathbb{R}$, then $\mathcal{F}(\mathbb{R} \boxplus \mathbb{R}, \Gamma)$ also has an uncountably large disjoint set.

In light of the subalgebra property discussed above for a fixed Γ and $G_1 \subseteq G_2$, if A is a disjoint set of $\mathcal{F}(G_1, \Gamma)$ then A will also be a disjoint set of $\mathcal{F}(G_2, \Gamma)$. So we only need look at disjoint sets in "small" groups for a fixed Γ.

Cardinal Decomposition

For an ℓ-group, G, to be cardinally indecomposable means that G cannot be written as a nontrivial cardinal sum, that is if $G = H \boxplus K$, then either H or K is 0. Unlike the results of Bernau [2] and Powell and Tsinakis [16], free extensions of partial ℓ-groups can be cardinally decomposed.

Example 5.7: Let $G = \boxplus \mathbb{Z}^n$ and $\Gamma = \{e_i \wedge e_j = 0 : i \neq j$, where e_i is the standard basis vector for $\mathbb{Z}^n\}$. Then the positive cone generated by Γ is $P_\Gamma = (\boxplus \mathbb{Z}^n)^+$ and hence (G, P_Γ) is an ℓ-group. Thus $\mathcal{F}(G, P_\Gamma) = \boxplus \mathbb{Z}^n$.

Chapter 6

Construction from Simple Γ

Introduction

While the construction of free extensions of a partial ℓ-group (G, Γ) are of interest for arbitrarily large Γ, it is frequently the case that the free extension of an abelian group with only a very small Γ may be equally important. In this chapter we look at some special sets Γ of partial operations and determine corresponding free extensions.

One Comparable Lattice Operation

Let $\Gamma = \{a \wedge 0 = 0\}$. For (G, Γ) to be embedded in an ℓ-group we need a positive cone P on G with $a \in P$.

Theorem 6.1: $P = \{na : n \geq 0\}$ *is a positive cone of* G *in which* Γ *is preserved.*

Proof: Recall from Theorem 1.2, for P to be a positive cone it must satisfy the following three properties:

(1) $P + P \subseteq P$;

(2) $g + P - g \subseteq P$; and

(3) $P \cap -P = \{0\}$.

(1) and (2) are trivial since the sum of positive integers is positive and G is abelian. To prove (3), let $x \in P \cap -P$, then $x \in P$ and $-x \in P$. So $x = na$ and $-x = ma$ for some $n, m \geq 0$. Hence $(n+m)a = 0$, but G is torsion-free and $a \neq 0$, so $n + m = 0$. But $n, m \geq 0$, so $n = m = 0$, therefore $x = 0$. That Γ is preserved in P follows easily from the fact that 0 is a lower bound of 0 and a, and if d is any other lower bound, then $d \leq 0$, therefore $0 = \inf\{a, 0\}$ in P. $\square$

We can easily generalize the above in the case where $\Gamma = \{a \wedge b = a\}$, just define $P = \{n(b-a) : n \geq 0\}$.

Since P only contains comparable elements we can apply Weinberg's construction to obtain $\mathcal{F}(G, P)$. However, Weinberg required that P be semi-closed. He needs this because his construction is based upon $P = \cap T$, where the intersection is over all total orders that contain P. This is true if and only if P is semi-closed (see Conrad [5]). Since $P \subseteq \overline{P}$, we can embed P into $\overline{P}$ via the inclusion map and apply Weinberg's construction to $\overline{P}$. This is the sublattice generated by the diagonal map of the direct product over Λ, the collection of all total orders with positive cone T, containing $\overline{P}$. That is, the natural sublattice of $\prod(G, T \in \Gamma)$. So if $\gamma\colon (G, \overline{P}) \to \mathcal{F}(G, \overline{P})$ is an embedding and $H \in \mathcal{A}$ and $\beta\colon (G, \overline{P}) \to H$ is a partial o-homomorphism, then there exists a unique o-homomorphism $\lambda\colon \mathcal{F}(G, \overline{P}) \to H$ such that $\lambda\gamma = \beta$. That is, the diagram below commutes.

$$\begin{tikzcd}
(G, P) \arrow[rr, "\gamma"] \arrow[dr, "\beta"'] & & \mathcal{F}(G, \overline{P}) \arrow[dl, "\lambda"] \\
& H &
\end{tikzcd}$$

Figure 23 -- Simple Γ - Weinberg case

Two questions come to mind: Does $\mathcal{F}(G, \Gamma)$ exist? If so, what does it look like?

Theorem 6.2: *$\mathcal{F}(G, \Gamma)$ exists and $\mathcal{F}(G, \Gamma) \simeq \mathcal{F}(G, \overline{P})$, where $\overline{P} = \{x \in G : nx \in P, n \geq 0\}$ and $P = \{na : n \geq 0\}$, as defined earlier.*

Proof: This follows as an easy application of Theorems 3.6 and 3.10. □

One Non-comparable Lattice Operation

Let $\Gamma = \{a \wedge b = 0 : a, b, 0 \text{ all distinct}\}$. Does $\mathcal{F}(G, \Gamma)$ exist, and if so, what does it look like? According to Theorem 3.6, $\mathcal{F}(G, \Gamma)$ exists if and only if (G, Γ) can be embedded in an ℓ-group by some partial ℓ-homomorphism. We approach the problem like we did in the earlier chapters. First embed (G,Γ) into a po group, i.e. (G,P), such that the lattice operations in Γ are preserved in P. Then embed (G,P) in an ℓ-group, so we can apply Theorem 3.6 to ensure the existence of $\mathcal{F}(G,\Gamma)$. By then, it will be more clear what the free extension should look like. First notice when the free extension does not exist.

Theorem 6.3: *If a,b are integer dependent (i.e. there exist integers $n,m \neq 0$ yet $na = mb$), then the free extension does not exist.*

Proof: Suppose the free extension does exist. Since it is an ℓ-group we must have $a \wedge b = 0$, but $na = mb$, and neither n nor m is 0. Without loss of generality, suppose $n \geq m$, then

$$\begin{aligned} n - m &\geq 0, \text{ so} \\ (n-m+m)a &= mb, \text{ hence} \\ (n-m)a + ma &= mb, \text{ so that} \\ ma &\leq mb, \text{ thus} \\ m(b-a) &\geq 0, \text{ but } \ell\text{–groups are semi-closed, so} \\ b - a &\geq 0, \text{ hence} \\ b &\geq a, \text{ which finally leads to} \\ a &= b \wedge a = 0, \text{ a contradiction.} \end{aligned}$$

□

Theorem 6.4: *If a and b are linearly independent with respect to integers, then there exists a positive cone P, such that Γ is preserved.*

Proof: Suppose a, b are integer linearly independent, i.e. $na + mb = 0 \Rightarrow n = m = 0$. Define $P = \{na + mb : n, m \geq 0\}$. As before, for P to be a positive cone it must satisfy the following three properties:

(1) $P + P \subseteq P$;

(2) $g + P - g \subseteq P$; and

(3) $P \cap -P = \{0\}$.

(1) and (2) are clear since the sum of positive integers is positive and G is abelian.

(3) Let $x \in P \cap -P$. So we have $x \in P$ and $-x \in P$, thus

$$
\begin{aligned}
x &= na + mb, \text{ and} \\
-x &= ra + sb, \text{ for some } n,\ m,\ r,\ s \geq 0, \text{ thus} \\
0 &= (n + r)a + (m + s)b, \text{ so by linear independence,} \\
0 &= n + r = m + s, \text{ and since } n,\ m,\ r,\ s \geq 0, \\
0 &= n = r = m = s, \text{ therefore} \\
x &= 0.
\end{aligned}
$$

Finally, we must show Γ is preserved in P, that is $0 = a \wedge b$ in P or equivalently, $0 = \inf\langle a, b\rangle$ in P. Clearly $a, b \geq 0$, so suppose $a, b \geq d$, hence $a - d$, and $b - d \in P$. Hence

$$
\begin{aligned}
a - d &= n_1 a + m_1 b \text{ and} \\
b - d &= n_2 a + m_2 b \text{ for } n_1, m_1, n_2, m_2 \geq 0, \text{ these imply} \\
(*) \qquad - d &= (n_1 - 1)a + m_1 b \text{ and} \\
- d &= n_2 a + (m_2 - 1)b, \text{ subtracting this from } (*) \text{ yields} \\
0 &= (n_1 - 1 - n_2)a + (1 + m_1 - m_2)b, \text{ which implies} \\
0 &= (n_1 - 1 - n_2) = (1 + m_1 - m_2), \text{ hence} \\
n_1 - 1 &= n_2 \geq 0, \text{ thus from } (*) \text{ we have } - d \in P \text{ or} \\
d &\leq 0, \text{ therefore} \\
0 &= \inf\{a, b\}.
\end{aligned}
$$

□

The above work can be generalized as follows. Suppose $\Gamma = \{a \wedge b = c\}$ and that $(a - c)$ and $(b - c)$ are linearly independent. We then define $P = \{n(a - c) + m(b - c) : n, m \geq 0\}$. By an argument similar to the one above we can show P is a positive cone for a partial order on G. We can further claim that not only is $a \wedge b = c$ in P, but $(a-c) \wedge (b-c) = 0$ in P.

First we show $c = \inf\langle a, b\rangle$ in P. Clearly $c \leq a, b$ since $(a-c)$ and $(b-c) \in P$. So suppose $d \leq a, b$, then $(a-d)$ and $(b-d) \in P$, so we have

$a - d = n_1(a-c) + m_1(b-c)$

$b - d = n_2(a-c) + m_2(b-c)$, and these lead to

$-d = (n_1 - 1)a - n_1 c + m_1(b-c)$, and

$-d = n_2(a-c) + (m_2 - 1)b - m_2 c$, which lead to

$(*)$ $c - d = (n_1 - 1)(a-c) + m_1(b-c)$, and

$c - d = n_2(a-c) + (m_2 - 1)(b-c)$, subtracting these two give

$0 = (n_1 - 1 - n_2)(a-c) + (m_1 - m_2 + 1)(b-c)$, hence

$0 = n_1 - 1 - n_2 = m_1 - m_2 + 1$, which follows from independence, so

$n_1 - 1 = n_2 \geq 0$, so from $(*)$ we have that

$c - d \geq 0$, so $c \geq d$, therefore

$c = \inf\{a, b\}$ in P.

Finally we show $0 = \inf\{(a-c), (b-c)\}$. Clearly 0 is a lower bound since $(a-c)$ and $(b-c) \in P$. Next suppose $d \leq a-c$ and $b-c$, hence $d + c \leq a, b$, so $d + c$ is a lower bound for a, b, hence $d + c \leq c = \inf\langle a, b\rangle$. Therefore $d \leq 0$, so
$0 = \inf\{(a-c), (b-c)\}$. Hence $(a-c) \wedge (b-c) = 0$ in P.

We wish to embed (G, P) into an ℓ–group so we can invoke Theorem 3.6 to prove $\mathcal{F}(G, P)$ exists. Here is a brief plan of attack:

1). Look at some of the properties of the normal convex subgroup generated by a, denoted by $\langle a\rangle = \{x \in G : na \leq x \leq ma$, for some $n, m \in \mathbb{Z}\}$

2). Look at some of the properties of the pure closure of $\langle a\rangle$, denoted by

$\overline{\langle a\rangle} = \{x \in G : nx \in \langle a\rangle$ for some $n \in \mathbb{Z}\}$. This is the smallest pure subgroup containing $\langle a\rangle$.

3). Look at some of the properties of the semi-closure of P, denoted by

$\overline{P} = \{x \in G : nx \in P$, for some $n \in \mathbb{N}\}$.

4). Embed $(G, P) \to (G, \overline{P})$.

5). Since G is torsion free and $\overline{\langle a\rangle}$ is pure, then $G/\overline{\langle a\rangle}$ is torsion free and hence $G/\overline{\langle a\rangle}$ is an $\mathcal{O}^*$-group, so there exists a total order with positive cone T_1, extending $G/\overline{\langle a\rangle}$.

6). Since the above works equally well for $G/\overline{\langle b\rangle}$, we will have the following embeddings:

$$(G,\Gamma) \to (G,P) \to (G,\overline{P}) \to (G/\overline{\langle a\rangle} \boxplus G/\overline{\langle b\rangle}) \to ((G/\overline{\langle a\rangle}, T_1) \boxplus (G/\overline{\langle b\rangle}, T_2)).$$

Therefore by Theorem 3.6, $\mathcal{F}(G,\Gamma)$ exists.

Now for some of the details. First, some properties of $\langle a\rangle$ and $\langle b\rangle$.

Theorem 6.5: *$\langle a\rangle$ is the infinite cyclic group $[\,a\,]$ generated by a.*

Proof: Clearly $[\,a\,] \subseteq \langle a\rangle$, since $na \leq na \leq na$. Now let $\mathrm{g} \in \langle a\rangle$, so $ra \leq \mathrm{g} \leq sa$, for some $r, s \in \mathbb{Z}$. Hence $(\mathrm{g} - ra)$ and $(sa - \mathrm{g}) \in P$. So we have the following:

$\mathrm{g} - ra = n_1a + m_1b$ and

$sa - \mathrm{g} = n_2a + m_2b$, now these lead to

(1) $\mathrm{g} = (r + n_1)a + m_1b$, and

(2) $-\mathrm{g} = (n_2 - s)a + m_2b$, adding (1) and (2) yield

$0 = (r + n_1 - s + n_2)a + (m_2 + m_1)b$, so by linear independence

$0 = m_2 + m_1$, but $m_1, m_2 \geq 0$, so

$0 = m_1 = m_2$. Thus by (1)

$\mathrm{g} = (r + n_1)a \in [\,a\,]$. So

$\langle a\rangle \subseteq [\,a\,]$, therefore

$\langle a\rangle = [\,a\,]$. □

Theorem 6.6: $\langle a\rangle \cap \langle b\rangle = \{0\}$.

Proof: Let $x \in \langle a\rangle \cap \langle b\rangle$, hence by the previous theorem, $x \in [\,a\,] \cap [\,b\,]$. So $x = na$ and $x = mb$, thus $na = mb$, and by independence, $n = m = 0$, so $x = 0$. □

Theorem 6.7: $\overline{\langle a\rangle} \cap \overline{\langle b\rangle} = \{0\}$.

Proof: Let $x \in \overline{\langle a\rangle} \cap \overline{\langle b\rangle}$, so $nx \in \langle a\rangle$ and $mx \in \langle b\rangle$, for some $n, m \in \mathbb{Z}$. Hence $nmx \in \langle a\rangle \cap \langle b\rangle = \{0\}$, but G is torsion free, so $x = 0$. □

Since $\overline{\langle a\rangle}$ is a normal convex pure subgroup, $G/\overline{\langle a\rangle}$ makes sense and inherits the order from G. This holds equally well for $G/\overline{\langle b\rangle}$, so define α as follows:

$$\alpha\colon (G, \overline{P}) \to (G/\overline{\langle a\rangle} \boxplus G/\overline{\langle b\rangle}) \text{ by } \alpha(\mathrm{g}) = (\mathrm{g} + \overline{\langle a\rangle}, \mathrm{g} + \overline{\langle b\rangle}).$$

We show the following:

Theorem 6.8: *α is a partial ℓ-monomorphism.*

Proof: α is a group homomorphism:

$$\begin{aligned}\alpha(\mathrm{g}+\mathrm{h}) &= (\mathrm{g}+\mathrm{h}+\overline{\langle a\rangle}, \mathrm{g}+\mathrm{h}+\overline{\langle b\rangle})\\ &= (\mathrm{g}+\overline{\langle a\rangle}+\mathrm{h}+\overline{\langle a\rangle}, \mathrm{g}+\overline{\langle b\rangle}+\mathrm{h}+\overline{\langle b\rangle})\\ &= (\mathrm{g}+\overline{\langle a\rangle}, \mathrm{g}+\overline{\langle b\rangle}) + (\mathrm{h}+\overline{\langle a\rangle}, \mathrm{h}+\overline{\langle b\rangle})\\ &= \alpha(\mathrm{g}) + \alpha(\mathrm{h}).\end{aligned}$$

α is $1-1$: Suppose $\alpha(\mathrm{g}) = \alpha(\mathrm{h})$, then

$$0 = \alpha(\mathrm{g} - \mathrm{h}), \text{ so}$$
$$= (\mathrm{g} - \mathrm{h} + \overline{\langle a \rangle}, \mathrm{g} - \mathrm{h} + \overline{\langle b \rangle}), \text{ thus}$$
$$\mathrm{g} - \mathrm{h} \in \overline{\langle a \rangle}, \text{ and}$$
$$\mathrm{g} - \mathrm{h} \in \overline{\langle b \rangle}, \text{ hence}$$
$$\mathrm{g} - \mathrm{h} \in \overline{\langle a \rangle} \cap \overline{\langle b \rangle} = 0, \text{therefore}$$
$$\mathrm{g} = \mathrm{h}.$$

α preserves order: This follows since $(G/\overline{\langle a \rangle})^+$ inherits the order from $\overline{P}$ via the natural map. Therefore, α is an o-monomorphism.

Finally, α is a partial ℓ-monomorphism. Since our only partial lattice operation in Γ is $a \wedge b = 0$, we only need to show that $\alpha(a \wedge b) = \alpha(a) \wedge \alpha(b)$. And since $a \wedge b = 0$, we only need show $\alpha(a) \wedge \alpha(b) = \alpha(0) = (\overline{\langle a \rangle}, \overline{\langle b \rangle})$. We'll do this by showing: $\inf\{\alpha(a), \alpha(b)\} = (\overline{\langle a \rangle}, \overline{\langle b \rangle})$. First $(\overline{\langle a \rangle}, \overline{\langle b \rangle})$ is a lower bound since $a, b \geq 0$, hence $\alpha(a) = (\overline{\langle a \rangle}, a + \overline{\langle b \rangle}) \geq (\overline{\langle a \rangle}, \overline{\langle b \rangle})$, and similarly $\alpha(b) = (b + \overline{\langle a \rangle}, \overline{\langle b \rangle}) \geq (\overline{\langle a \rangle}, \overline{\langle b \rangle})$. Now suppose $\mathrm{k} \in (G/\overline{\langle a \rangle} \boxplus G/\overline{\langle b \rangle})$ is a lower bound for $\{\alpha(a), \alpha(b)\}$. So $\mathrm{k} = (\mathrm{g} + \overline{\langle a \rangle}, \mathrm{h} + \overline{\langle b \rangle}) \leq \alpha(a), \alpha(b)$. Hence $(\mathrm{g} + \overline{\langle a \rangle}, \mathrm{h} + \overline{\langle b \rangle}) \leq (\overline{\langle a \rangle}, \overline{\langle b \rangle})$. Therefore $(\overline{\langle a \rangle}, \overline{\langle b \rangle}) = \inf\{\alpha(a), \alpha(b)\} = \alpha(a) \wedge \alpha(b)$. □

Now since $\overline{\langle a \rangle}$ and $\overline{\langle b \rangle}$ are pure subgroups, by an earlier property $G/\overline{\langle a \rangle}$ and $G/\overline{\langle b \rangle}$ are torsion free. So $G/\overline{\langle a \rangle}$ and $G/\overline{\langle b \rangle}$ are $\mathcal{O}^*$-groups, hence there exist total orders with positive cones T_1 and T_2, respectively, extending the orders of $G/\overline{\langle a \rangle}$ and $G/\overline{\langle b \rangle}$. This completes all our embeddings stated earlier, so by Theorem 3.6, $\mathcal{F}(G, \Gamma)$ exists. To recap, we have the following embeddings:

$$(G, \Gamma) \to (G, P) \to (G, \overline{P}) \to (G/\overline{\langle a \rangle} \boxplus G/\overline{\langle b \rangle}) \to ((G/\overline{\langle a \rangle}, T_1) \boxplus (G/\overline{\langle b \rangle}, T_2)).$$

Now lets see what $\mathcal{F}(G, \Gamma)$ looks like.

Theorem 6.9: *$\mathcal{F}(G,\Gamma) = \{ \bigvee_J \bigwedge_K \gamma(g_{jk}) : J, K$ are finite index sets$\}$. This is the sublattice generated by the diagonal map*

$$\gamma : (G,\Gamma) \to \prod_{\Lambda} \left((G/\overline{\langle a \rangle}, T_a) \boxplus (G/\overline{\langle b \rangle}, T_b) \right)$$

defined by

$$\gamma(g) = (\cdots, (g + \overline{\langle a \rangle}, g + \overline{\langle b \rangle}), \cdots)$$

where Λ is an index set over total orders with positive cones T_a and T_b such that

$$T_a, T_b \supseteq (G/\overline{\langle a \rangle})^+, (G/\overline{\langle b \rangle})^+, \text{ respectively.}$$

Proof: Now if $\gamma(g) = 0$ then $g \in \overline{\langle a \rangle} \cap \overline{\langle b \rangle}$ hence $g = 0$, so γ is $1-1$. And because of the natural epimorphism, $(G/\overline{\langle a \rangle}, P/\overline{\langle a \rangle})$ inherits the order from (G, P), thus every component of the direct product maintains the meet (i.e. $a \wedge b = 0$), hence $\gamma(a \wedge b) = \gamma(a) \wedge \gamma(b)$. Therefore γ is a partial ℓ-monomorphism.

Now let $\beta : (G,\Gamma) \to H \in \mathcal{A}$, be a partial ℓ-homomorphism. Without loss of generality, we can assume H is totally ordered. If we can exhibit an ℓ-homomorphism,

$$\varphi: \prod_{\Lambda} ((G/\overline{\langle a \rangle}, T_a) \boxplus (G/\overline{\langle b \rangle}, T_b)) \to H$$

such that $\varphi\gamma = \beta$, then by uniqueness of free extensions, $\mathcal{F}(G,\Gamma)$ will be the sublattice of $\prod_{\Lambda}((G/\overline{\langle a \rangle}, T_a) \boxplus (G/\overline{\langle b \rangle}, T_b))$ generated by γ. To this end define φ, by $\varphi(\bigvee_R \bigwedge_S \gamma(g_{rs})) = \bigvee_R \bigwedge_S \beta(g_{rs})$, where R, S are finite index sets. Clearly, φ is the right map, provided it is well-defined! Hence we need only show that $\bigvee_R \bigwedge_S \beta(g_{rs}) \neq 0 \Rightarrow \bigvee_R \bigwedge_S \gamma(g_{rs}) \neq 0$. Since H is totally ordered we can break this down into two cases.

Case 1: $\bigvee_R \bigwedge_S \beta(g_{rs}) > 0$.

Then there exists r_0 such that $\bigwedge_S \beta(g_{r_0 s}) > 0$ for all s, hence $\beta(g_{r_0 s}) > 0$, for all s. So $g_{r_0 s} \notin \ker\beta$, for all s. Now $\ker\beta$ is an o-ideal, so $G/\ker\beta$ is isomorphic to a subgroup of H. Since subgroups of totally ordered groups are totally ordered, $G/\ker\beta$ is totally ordered, so $(g_{r_0 s} + \ker\beta) > 0$, for all s in $G/\ker\beta$.

Claim 1: $\{g_{r_0 s} : s \in S\}$ is positively independent in G.

Proof: Suppose $\sum_{s=1}^{n} k_s g_{r_0 s} \in G^-$, $k_s \geq 0$. We need to show that $k_s = 0$, for all s. But $\beta(\sum_{s=1}^{n} k_s g_{r_0 s}) \in H^-$ since β is o-homomorphism and H is totally ordered, hence $\sum_{s=1}^{n} k_s \beta(g_{r_0 s}) \in H^-$, but $\beta(g_{r_0 s}) \in H^+$,for all s, so $k_s\beta(g_{r_0 s}) \in H^+$, for all s, so $\sum_{s=1}^{n} k_s \beta(g_{r_0 s}) \in H^+$. Hence $\sum_{s=1}^{n} k_s \beta(g_{r_0 s}) = 0$, thus $k_s\beta(g_{r_0 s}) = 0$, for all s. But H is torsion-free, so $k_s = 0$, for all s.

Claim 2: $\{g_{r_0 s} + \overline{\langle a \rangle}\}$ is positively independent in $G/\overline{\langle a \rangle}$.

Proof: Suppose $\sum_{s=1}^{n} k_s (g_{r_0 s} + \overline{\langle a \rangle}) \in (G/\overline{\langle a \rangle})^-$, then $\sum_{s=1}^{n} k_s g_{r_0 s} + \overline{\langle a \rangle} \in G^-/\overline{\langle a \rangle}$, hence $\sum_{s=1}^{n} k_s g_{r_0 s} \in G^-$, so by the previous claim $k_s = 0$, for all s.

Therefore there exists a total order with positive cone T'_a such that the following hold:

$$T'_a \supset ((G/\overline{\langle a \rangle})^+, \{g_{r_0 s} + \overline{\langle a \rangle}\})$$

$$(g_{r_0 s} + \overline{\langle a \rangle}) > 0, \text{ for all } s, \text{ therefore}$$

$$\bigwedge_S (g_{r_0 s} + \overline{\langle a \rangle}) > 0, \text{ thus}$$

$$\bigvee_R \bigwedge_S (g_{rs} + \overline{\langle a \rangle}) > 0, \text{ therefore}$$

$$\bigvee_R \bigwedge_S \gamma(g_{rs}) \neq 0 \text{ in } \prod_{\Lambda}((G/\overline{\langle a \rangle}, T_a) \boxplus (G/\overline{\langle b \rangle}, T_b)).$$

Case 2: $\bigvee_R \bigwedge_S \beta(g_{rs}) < 0$.

Because ℓ-groups are distributive lattices, we'll use the following notation from Anderson and Feil (and Conrad):

$$\bigwedge_R \bigvee_S a_{is} = \bigvee_{S^R} \bigwedge_R a_{if(i)}, \text{ where } S^R \text{ is all functions } f : R \to S.$$

So we have

$$0 > \bigvee_R \bigwedge_S \beta(g_{rs}), \text{ so}$$

$$0 < -(\bigvee_R \bigwedge_S \beta(g_{rs})) = \bigwedge_R \bigvee_S \beta(-g_{rs})$$

$$= \bigvee_{S^R} \bigwedge_R \beta(-g_{rf(r)}), \text{ so there exists an } f \in S^R \text{ so that}$$

$$0 < \bigwedge_R \beta(-g_{rf(r)}), \text{ for all } r \in R, \text{ therefore}$$

$$0 < \beta(-g_{rf(r)}), \text{ for all } r.$$

Hence $-g_{rf(r)} \notin \ker \beta$, so as with the first case, $\{-g_{rf(r)}\}$ is positively independent in G. Thus, $\{-g_{rf(r)} + \overline{\langle a \rangle}\}$ is positively independent in $G/\overline{\langle a \rangle}$. Therefore there exists a total order with positive cone T_a' such that

$$T_a' \supset ((G/\overline{\langle a \rangle})^+, \{-g_{rf(r)} + \overline{\langle a \rangle}\}) \text{ so that}$$

$$0 < (-g_{rf(r)} + \overline{\langle a \rangle}), \text{ for all } r \in R, \text{ hence}$$

$$0 < \bigwedge_R (-g_{rf(r)} + \overline{\langle a \rangle}), \text{ so}$$

$$0 < \bigvee_{S^R} \bigwedge_R (-g_{rf(r)} + \overline{\langle a \rangle}), \text{ which is}$$

$$= \bigwedge_R \bigvee_S (-g_{rs} + \overline{\langle a \rangle})$$

$$= -\bigvee_R \bigwedge_S (g_{rs} + \overline{\langle a \rangle}).$$

$$\text{Therefore } 0 > \bigvee_R \bigwedge_S (g_{rs} + \overline{\langle a \rangle}), \text{ thus}$$

$$\bigvee_R \bigwedge_S \gamma(g_{rs}) \neq 0 \text{ in } \prod_\Lambda ((G/\overline{\langle a \rangle}, T_a) \boxplus (G/\overline{\langle b \rangle}, T_b)).$$

So γ is well-defined and we are done! Note that throughout all this we could have generalized to $\Gamma = \{a \wedge b = c : a, b, c \text{ all distinct}\}$. We would have then defined $P = \{n(a-c) + m(b-c) : n, m \geq 0\}$.

Three Pairwise Disjoint Partial Lattice Operations

Theorem 6.10: *Let* $\Gamma = \{a \wedge b = b \wedge c = a \wedge c = 0\}$, *with* $a, b, c, 0$ *all distinct, further suppose* a, b, c *are linearly independent with respect to integers. Define* $P = \{na + mb + rc : n, m, r \geq 0\}$, *then* P *is a positive cone of* G.

Proof: Clearly $P + P \subseteq P$ and $\mathrm{g} + P - \mathrm{g} \subseteq P$, since the sum of positive integers is still positive and G is abelian. So we need to show: $P \cap -P = \{0\}$. To this end, let $\mathrm{x} \in P \cap -P$, then $\mathrm{x} \in P$ and $-\mathrm{x} \in P$. So we now have

$\mathrm{x} = n_1 a + m_1 b + r_1 c$ and

$-\mathrm{x} = n_2 a + m_2 b + r_2 c$ for some $n_i, m_i, r_i \geq 0$, $i = 1, 2$. Hence

$0 = (n_1 + n_2)a + (m_1 + m_2)b + (r_1 + r_2)c$.

But a, b, c are linearly independent, so $n_1 + n_2 = m_1 + m_2 = r_1 + r_2 = 0$, but $n_i, m_i, r_i \geq 0$, so $n_1 = n_2 = m_1 = m_2 = r_1 = r_2 = 0$. Therefore $\mathrm{x} = 0$.

Theorem 6.11: *If* $\Gamma = \{a \wedge b = b \wedge c = a \wedge c = 0\}$ *and* a, b, c *are linearly dependent, then the free extension does not exist.*

Proof: For suppose the free extension does exist, and $na + mb + rc = 0$ where not all n, m, r are 0. Then consider the following cases.

Case 1: $n, m, r \geq 0$. Then $-na = mb + rc \in P$, so $na, -na \in P$, hence $na = 0$. But G is torsion free so $a = 0$, a contradiction.

Case 2: $n, m, r \leq 0$. This is similar to Case 1.

Case 3: Some coefficients are positive and some are negative. Leave all positive coefficients alone and move all negative coefficients to the other side. Hence both the left hand side and right hand side are positive, say $na = -mb - rc$. So that $na \geq -mb$ thus

$na \wedge -mb = -mb$, but $a \wedge b = 0$, so $na \wedge -mb = 0$. Hence $-mb = 0$ and since G is torsion free, $b = 0$, a contradiction. $\square$

Theorem 6.12: *Γ is preserved in P. That is,*

$$0 = \inf\{a, b\} = \inf\{a, c\} = \inf\{b, c\} \text{ in } P.$$

Proof: We'll show $0 = \inf\{a, b\}$, since the others follow analogously. Clearly $a, b \geq 0$ since $a, b \in P$, hence 0 is a lower bound. So suppose $f \leq a, b$. We need to show $f \leq 0$. But since $f \leq a, b$, then $a - f$, $b - f \in P$. Hence we have

$a - f = n_1 a + m_1 b + r_1 c$ and

$b - f = n_2 a + m_2 b + r_2 c$, these imply

$(*)$ $\quad -f = (n_1 - 1)a + m_1 b + r_1 c$, and

$-f = n_2 a + (m_2 - 1)b + r_2 c$, subtracting from $(*)$ we have

$0 = (n_1 - 1 - n_2)a + (m_1 + 1 - m_2)b + (r_1 - r_2)c$,

$0 = n_1 - 1 - n_2 = m_1 + 1 - m_2 = r_1 - r_2$, from independence, so

$0 \leq n_2 = n_1 - 1$, so by $(*)$,

$-f \in P$, therefore

$f \leq 0$. $\square$

Now consider $\langle a, b \rangle, \langle a, c \rangle, \langle b, c \rangle$, the convex normal subgroups generated by $\{a, b\}$, $\{a, c\}$, $\{b, c\}$ respectively. So in particular

$$\langle a, b \rangle = \{x \in G \mid n_1 a + m_1 b \leq x \leq n_2 a + m_2 b, \text{ where } n_i, m_i \in \mathbb{Z}\}.$$

Theorem 6.13: $\langle a, b \rangle = \{x \in G \mid x = na + mb,$ *with* $n, m \in \mathbb{Z}\}$.

Proof: Let $x \in \langle a, b \rangle$ then $n_1 a + m_1 b \leq x \leq n_2 a + m_2 b$, so $x - (n_1 a + m_1 b)$ and $n_2 a + m_2 b - x$ are both in P, thus we have

$x - (n_1 a + m_1 b) = n_3 a + m_3 b + r_3 c$

$n_2a + m_2b - x = n_4a + m_4b + r_4c$, so

$$(1) \qquad x = (n_1 + n_3)a + (m_1 + m_3)b + r_3c$$

$$-x = (n_4 - n_2)a + (m_4 - m_2)b + r_4c$$

$$0 = (n_1 + n_3 + n_4 - n_2)a$$

$$+ (m_1 + m_3 + m_4 - m_2)b + (r_3 + r_4)c, \text{ so}$$

$r_3 = r_4 = 0$, from independence and since $r_3, r_4 \geq 0$, so by (1)

$$x = na + mb. \qquad \square$$

Theorem 6.14: $\langle a, b\rangle \cap \langle a, c\rangle \cap \langle b, c\rangle = \{0\}$.

Proof: Suppose x is in the intersection, then we have

$x = na + mb = ra + sc = qb + tc$, thus we get

$0 = (n - r)a + mb - sc$, and

$0 = ra - qb + (s - t)c$, and by independence we have

$0 = n - r = m = s = q = r = s - t$, hence

$x = 0$. $\square$

Theorem 6.15: $\overline{\langle a, b\rangle} \cap \overline{\langle a, c\rangle} \cap \overline{\langle b, c\rangle} = \{0\}$.

Proof: This follows from theorem above. Suppose x is in the intersection. Then $nx \in \langle a, b\rangle$, $mx \in \langle a, c\rangle$, and $rx \in \langle b, c\rangle$, thus $nmrx \in \langle a, b\rangle \cap \langle a, c\rangle \cap \langle b, c\rangle$. Hence $nmrx = 0$. But G is torsion free, so $x = 0$. $\square$

Theorem 6.16: $a \wedge b = a \wedge c = b \wedge c = 0$ still holds in $\overline{P}$.

Proof: Recall $\overline{P} = \{x \in G : nx \in P\}$.

Since $a, b \in P \subseteq \overline{P}$, then $a, b \geq 0$ in $\overline{P}$. So suppose $e \leq a, b$ in $\overline{P}$, then $a - e$ and $b - e \in \overline{P}$. Thus $n(a - e)$ and $m(b - e) \in P$, for some $n, m \in \mathbb{N}$. Hence

(1) $\quad n(a - e) = n_1a + m_1b + r_1c$

(2) $\quad m(b - e) = n_2a + m_2b + r_2c$, now (1) and (2) lead to

(3) $\quad -ne = (n_1 - n)a + m_1 b + r_1 c$

(4) $\quad -me = n_2 a + (m_2 - m)b + r_2 c$, multiply (3) by m and (4) by n

(5) $\quad -mne = m(n_1 - n)a + mm_1 b + mr_1 c$

(6) $\quad -nme = nn_2 a + n(m_2 - m)b + nr_2 c$, by independence we have

$m(n_1 - n) = nn_2 \geq 0$, and $m \geq 0$, hence $n_1 - n \geq 0$, thus by (3) $-ne \in P$.

Thus $n(-e) \in P$, so $-e \in \overline{P}$, hence $-e \geq 0$, so that $e \leq 0$. Therefore $0 = \inf\{a, b\}$ in $\overline{P}$, or $a \wedge b = 0$ in $\overline{P}$. A similar argument works for $a \wedge c$ and $b \wedge c$ in $\overline{P}$. $\square$

Following the earlier plan of attack we now wish to embed $(G, \overline{P})$ into some $\mathcal{O}^*$-group. Define $\alpha : (G, \overline{P}) \to (G/\overline{\langle a,b\rangle} \boxplus G/\overline{\langle a,c\rangle} \boxplus G/\overline{\langle b,c\rangle})$ by the following $\alpha(\mathrm{g}) = (\mathrm{g} + \overline{\langle a,b\rangle}, \mathrm{g} + \overline{\langle a,c\rangle}, \mathrm{g} + \overline{\langle b,c\rangle}\ )$. We need to show α is an ℓ-monomorphism: Clearly α is a o-homomorphism, since $(G/\overline{\langle a,b\rangle})^+$ is the image of $\overline{P}$ under the natural homomorphism of G onto $G/\overline{\langle a,b\rangle}$. That α is $1-1$ follows from theorem 6.15 above. Suppose $\alpha(\mathrm{g}) = \alpha(\mathrm{h})$. Then $\alpha(\mathrm{g}-\mathrm{h}) = 0 = (\overline{\langle a,b\rangle}, \overline{\langle a,c\rangle}, \overline{\langle b,c\rangle})$. Thus $\mathrm{g} - \mathrm{h} \in \overline{\langle a,b\rangle} \cap \overline{\langle a,c\rangle} \cap \overline{\langle b,c\rangle} = 0$. So, $\mathrm{g} = \mathrm{h}$.

Finally we need to show α preserves the lattice operations: $0 = \alpha(a \wedge b) = \alpha(a) \wedge \alpha(b)$. So we must show that $(\overline{\langle a,b\rangle}, \overline{\langle a,c\rangle}, \overline{\langle b,c\rangle}) = \inf\{\alpha(a), \alpha(b)\}$. Now $\alpha(a) \geq 0$, since $a \geq 0$ and $\alpha(a) = (\overline{\langle a,b\rangle}, \overline{\langle a,c\rangle}\ ,\ a + \overline{\langle b,c\rangle}) \geq (\langle a,b\rangle, \langle a,c\rangle\ , \langle b,c\rangle) = 0$. So pick $k \in (G/\overline{\langle a,b\rangle} \boxplus G/\overline{\langle a,c\rangle} \boxplus G/\overline{\langle b,c\rangle})$, such that $k \leq \alpha(a), \alpha(b)$. We need to show $k \leq 0$. Say $k = (\mathrm{g}_1 + \overline{\langle a,b\rangle},\ \mathrm{g}_2 + \overline{\langle a,c\rangle},\ \mathrm{g}_3 + \overline{\langle b,c\rangle})$, then we have

$$k \leq (\overline{\langle a,b\rangle}, \overline{\langle a,c\rangle}\ ,\ a + \overline{\langle b,c\rangle}) \text{ and}$$

$$\leq (\overline{\langle a,b\rangle},\ b + \overline{\langle a,c\rangle}\ ,\ \overline{\langle b,c\rangle}), \text{ so}$$

$$g_1 + \overline{\langle a,b\rangle} \le \overline{\langle a,b\rangle}$$
$$g_2 + \overline{\langle a,c\rangle} \le \overline{\langle a,c\rangle}$$
$$g_3 + \overline{\langle b,c\rangle} \le \overline{\langle b,c\rangle}, \text{ therefore}$$
$$k \le 0.$$

Therefore, $\alpha(a) \wedge \alpha(b) = 0$. The other lattice operations follow analogously. Hence we have that α is an ℓ-monomorphism.

Now since $\overline{\langle a,b\rangle}$, $\overline{\langle a,c\rangle}$, $\overline{\langle b,c\rangle}$ are pure then $G/\overline{\langle a,b\rangle}$, $G/\overline{\langle a,c\rangle}$, and $G/\overline{\langle b,c\rangle}$ are torsion free, hence $\mathcal{O}^*$-groups. So there exist total orders, hence forming ℓ-groups, with positive cones T_1, T_2, and T_3, extending $G/\overline{\langle a,b\rangle}$, $G/\overline{\langle a,c\rangle}$, and $G/\overline{\langle b,c\rangle}$, respectively. This completes all our embeddings (shown below), so by Theorem 3.6, $\mathcal{F}(G, \Gamma)$ exists. We have established the following embeddings:

$$(G,\Gamma) \to (G,P) \to (G,\overline{P})$$
$$\to (G/\overline{\langle a,b\rangle} \boxplus G/\overline{\langle a,c\rangle} \boxplus G/\overline{\langle b,c\rangle})$$
$$\to ((G/\overline{\langle a,b\rangle}, T_1) \boxplus (G/\overline{\langle a,c\rangle}, T_2) \boxplus (G/\overline{\langle b,c\rangle}, T_3)).$$

Finally, we have

Theorem 6.17: $\mathcal{F}(G,\Gamma) = \{ \bigvee_J \bigwedge_K \gamma(g_{sk}) : J, K$ *are finite index sets*$\}$. *That is,* $\mathcal{F}(G,\Gamma)$ *is the sublattice generated by the diagonal map*

$$\gamma : (G,\Gamma) \to \prod_{\Lambda} \left((G/\overline{\langle a,b\rangle}, T_1) \boxplus (G/\overline{\langle a,c\rangle}, T_2) \boxplus (G/\overline{\langle b,c\rangle}, T_3)\right)$$

where γ *is defined by* $\gamma(g) = (\cdots, (g + \overline{\langle a,b\rangle}, g + \overline{\langle a,c\rangle}, g + \overline{\langle b,c\rangle}), \cdots)$, *and* Λ *is an index set over total orders with positive cones as discussed above.*

Proof: The proof of this claim is identical to the one given earlier.

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NOTE

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