APPROXIMATE CONFIDENCE INTERVALS AND
APPROXIMATE CONFIDENCE BANDS
FOR LOGISTIC MODELS

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CHAPTER I

INTRODUCTION

The logistic model is a commonly used technique for the analysis of epidemiological and clinical studies with binary data or growth data. The use of the logistic model in the biological sciences dates back over 50 years (Berkson, 1944), and today it enjoys a wide variety of applications.

Estimation of the parameters of a logistic model can be accomplished by maximum likelihood or least squares, although these require iterative computational methods. Packaged computer programs simplify this. We can use SAS/STAT (1990) which contains the procedures CATMOD, NLIN and LOGIST. SAS/IML (1990) also can be implemented in one of the statistical programming languages for other more sophisticated approximation methods.

Although point estimates of model parameters usually play a central role, descriptive presentation that goes beyond direct focus on the parameter estimates themselves is often desirable. Such a case occurs in connection with the logistic models.

We will consider the problem of constructing approximate confidence intervals and approximate confidence bands for several logistic models.

Since we do not know the distribution of maximum likelihood estimates of parameters in logistic models, we will use large-sample methods which lead directly to an approximate confidence set for parameters in logistic models.

We use the asymptotic properties of maximum likelihood estimates. Maximum likelihood estimates of parameters have a large-sample normal distribution with covariance matrix equal to the inverse of the information matrix. The information matrix is the negative expected value of the matrix of second partial derivative of log likelihood functions. We estimate the covariance matrix by substituting maximum likelihood estimates into the information matrix and inverting it.

Logistic regression models have been increasingly employed as a technique for analyzing binary response data. In chapter II, we consider the logistic regression model for binary response data.

What distinguishes logistic regression models from linear regression models is that the response variable is binary. This difference between logistic regression and linear regression is reflected both in the choice of a parametric model and in the assumptions. Cleary and Angel (1984) discussed the difference between the general linear model and logistic regression model.

In a regression analysis when the response variable is binary:

(1) The conditional mean of the regression equation must be formulated to be bounded between 0 and 1.

(2) The binomial distribution describes the distribution of the errors and will be the statistical distribution upon which the analysis is based.

We will study approximate confidence intervals and approximate confidence bands for the logistic regression model. There are two different methods, the inverse logit transformation method and the delta method, to obtain the approximate confidence intervals for the logistic regression model.

Brand, Pinnock and Jackson (1973) discussed approximate confidence bands for the logistic regression curve with one explanatory variable using an approximate elliptical confidence set of parameters in the logistic regression model. Hauck (1983) extended this to multiple explanatory variables.

We will consider approximate confidence bands for the logistic regression curve using an approximate rectangular confidence set of parameters in the logistic regression model.

In chapter III, we consider the logistic model for categorical binary response data. Explanatory variables in logistic models can be categorical. When they are categorical, models with logit link are equivalent to log-linear models. When modeling categorical explanatory variables, there is an alternative technique that can be used to estimate a logistic model. Grizzle, Starmer and Koch (1969) discussed GSK weighted least squares logistic models and log-linear models.

We will obtain approximate confidence regions for the logistic model with IxJx2 tables.

In chapter IV, we will study logistic regression models for binary bivariate data. Bonney (1987) transformed the multivariate problem into one of univariate logistic regression for n independent observations in which the response is binary and the same set of explanatory variables are associated with all responses.

We will obtain approximate confidence bands for the logistic regression model for binary bivariate data.

Finally, in chapter V, we consider the logistic model for growth curves. Processes producing S-shaped growth curves are widespread in biology, agriculture, engineering, and economics. Such curves start at some fixed point and increase their growth rate monotonically to reach an inflection point; after this the growth rate decreases to approach asymptotically some final value.

Numerous mathematical functions have been proposed for modeling S-shaped curves, many of which are claimed to have some underlying theoretical basis. The logistic growth curve is perhaps more widely used than any other mathematical curve in biological investigations of growth. The interrelations between growth curves and their rate equations are given by Turner, Bradley, Kirk, and Pruitt (1976).

We have two methods, the linearization method and the maximum likelihood method, to obtain approximate confidence intervals and approximate prediction intervals for the logistic model for growth curve. We consider a polynomial approximation to obtain approximate confidence bands on the logistic model for growth curve.

CHAPTER II

LOGISTIC REGRESSION MODEL FOR BINARY DATA

Let Y denote a binary response variable. For instance Y might indicate diagnosis of breast cancer (present, absent), vote in an election (Democrat, Republican) or choice of automobile (domestic, foreign import). Denote the two responses by 0 and 1. In those cases where the binary response, Y, is affected by a predictor variables, x, logistic regression is often employed. Let the probability of response given x be $\Pr(Y|x) = \pi(x)$ to simplify notation. The logistic regression specifies the probability of response as

$$\pi(x) = \frac{\exp(\beta_0 + \beta_1 x)}{1 + \exp(\beta_0 + \beta_1 x)}, \quad -\infty < x < +\infty. \tag{2.1}$$

Properties of $\pi(x)$

(1) Since $0 \le \exp(\beta_0 + \beta_1 x) \le 1 + \exp(\beta_0 + \beta_1 x)$, $0 \le \pi(x) \le 1$.

(2) Let $\beta_1 > 0$, then

$\lim_{x \to \infty} \pi(x) \to 1$: upper asymptote.

$\lim_{x \to -\infty} \pi(x) \to 0$: lower asymptote.

(3) Let $\beta_1 < 0$, then

$\lim_{x \to \infty} \pi(x) \to 0$: lower asymptote.

$\lim_{x \to -\infty} \pi(x) \to 1$: upper asymptote.

(4) Let $\beta_1 = 0$, then

The binary response is independent of the predictor variable.

(5) The logistic regression (2.1) has

$$\frac{\partial \pi(x)}{\partial x} = \frac{\beta_1 \exp(\beta_0 + \beta_1 x)}{(1 + \exp(\beta_0 + \beta_1 x))^2}$$

and

$$\frac{\partial^2 \pi(x)}{\partial x^2} = \frac{\beta_1^2 \exp(\beta_0 + \beta_1 x)(1 - \exp(\beta_0 + \beta_1 x))}{(1 + \exp(\beta_0 + \beta_1 x))^4}.$$

Let $\frac{\partial^2 \pi(x)}{\partial x^2} = 0$ then $x = \frac{-\beta_0}{\beta_1}$.

Thus, the point of inflection of $\pi(x)$ is $x = \frac{-\beta_0}{\beta_1}$.

(6) Since $\pi\left(\frac{-\beta_0}{\beta_1}\right) = \frac{1}{2}$, the inflection point, $x = \frac{-\beta_0}{\beta_1}$, has reached one half of its maximum.

(7) The log odds has the linear relationship:

$$\ln\left(\frac{\pi(x)}{1 - \pi(x)}\right) = \beta_0 + \beta_1 x.$$

The log odds is also referred to as the logit model.

Maximum Likelihood Estimates in Logistic Regression Model

Cox (1970) discussed the maximum likelihood estimates in logistic regression models. Silvapulle (1981) made necessary and sufficient conditions for the existence of the maximum likelihood estimators for logit models. Albert and Anderson (1984) proved existence theorems by considering the possible patterns of data points.

We study the mechanics of maximum likelihood estimation for logistic regression. It is assumed that *I* binary responses are independent Bernoulli random variables. Designate the vector of independent variables for the *i*th individual in the sample as $\mathbf{x}_i = (x_{i0}\ x_{i1}\ x_{i2} \cdots x_{is})'$, where $x_{i0} = 1$. We also define the *I*x(*s*+1) matrix of independent variables for the sample as

$$\mathbf{X} = \begin{bmatrix} 1 & x_{11} & \cdots & x_{1s} \\ 1 & x_{21} & \cdots & x_{2s} \\ \vdots & \vdots & & \vdots \\ 1 & x_{I1} & \cdots & x_{Is} \end{bmatrix}$$

and $\boldsymbol{\beta} = (\beta_0\ \beta_1 \cdots \beta_s)'$ is the unknown parameters vector to be estimated.

We express the logistic regression model (2.1) as

$$\pi(\mathbf{x}_i) = \frac{\exp(\beta_0 + \beta_1 x_{1i} + \cdots + \beta_s x_{is})}{1 + \exp(\beta_0 + \beta_1 x_{1i} + \cdots + \beta_s x_{is})}$$

$$= \frac{\exp(\mathbf{x}_i'\boldsymbol{\beta})}{1 + \exp(\mathbf{x}_i'\boldsymbol{\beta})}. \tag{2.2}$$

When more than one observation on Y occurs at a fixed x_i value, it is sufficient to record the number of observations n_i and the number of "1" outcomes.

The $\{Y_i,\ i = 1,\cdots,I\}$ are independent binomial random variables with $E(Y_i) = n_i\pi(x_i)$, where $n_1 + \cdots + n_I = N$. The likelihood function is

$$l(\boldsymbol{\beta}) = \prod_{i=1}^{I} \pi(\mathbf{x}_i)^{y_i} (1 - \pi(\mathbf{x}_i))^{n_i - y_i}$$

$$= \prod_{i=1}^{I} \left[\frac{\exp\left(\sum_{j=0}^{s} \beta_j x_{ij} y_i\right)}{\left(1 + \exp\left(\sum_{j=0}^{s} \beta_j x_{ij}\right)\right)^{n_i}} \right].$$

The log likelihood equals

$$L(\boldsymbol{\beta}) = \ln[l(\boldsymbol{\beta})] = \sum_{i=1}^{I} \left[\sum_{j=0}^{s} \beta_j x_{ij} y_i - n_i \ln\left(1 + \exp\left(\sum_{j=0}^{s} \beta_j x_{ij}\right)\right) \right]. \tag{2.3}$$

To find the value of $\boldsymbol{\beta}$ that maximizes $L(\boldsymbol{\beta})$ we differentiate $L(\boldsymbol{\beta})$ with respect to the elements of $\boldsymbol{\beta}$ and set the resulting expressions equal to zero. Since

$$\frac{\partial L(\boldsymbol{\beta})}{\partial \beta_j} = \sum_{i=1}^{I} y_i x_{ij} - \sum_{i=1}^{I} n_i x_{ij} \left[\frac{\exp\left(\sum_{j=0}^{s} \beta_j x_{ij}\right)}{1 + \exp\left(\sum_{j=0}^{s} \beta_j x_{ij}\right)} \right],$$

the likelihood equations are

$$\sum_{i=1}^{I} y_i x_{ij} - \sum_{i=1}^{I} n_i x_{ij} \hat{\pi}(\mathbf{x}_i) = 0, \quad j = 0, \ldots, s, \tag{2.4}$$

where $\hat{\pi}(\mathbf{x}_i) = \dfrac{\exp\left(\sum_{j=0}^{s} \hat{\beta}_j x_{ij}\right)}{1 + \exp\left(\sum_{j=0}^{s} \hat{\beta}_j x_{ij}\right)}$ denotes the maximum likelihood estimate of $\pi(\mathbf{x}_i)$.

Since the likelihood equations (2.4) are nonlinear functions of the maximum likelihood estimates $\hat{\boldsymbol{\beta}}$, they require an iterative solution.

Bradley and Gart (1962) and Cox (1970) discussed the asymptotic properties of maximum likelihood estimators.

The information matrix is the negative expected value of the matrix of second partial derivative of the log likelihood. Under regularity conditions, maximum likelihood estimators of parameters have a large-sample normal distribution with a covariance matrix equal to the inverse of the information matrix. We have, asymptotically,

$$\hat{\boldsymbol{\beta}} \sim N_{s+1}(\boldsymbol{\beta}, \quad \boldsymbol{\Sigma}^{-1}), \tag{2.5}$$

where the $(h, \ l)$ element of matrix $\boldsymbol{\Sigma}$ is $\Sigma_{hl} = -E\left(\frac{\partial^2 L(\boldsymbol{\beta})}{\partial\beta_h \partial\beta_l}\right)$, $h, l = 0, 1, \cdots, s$.

For logistic regression model

$$\frac{\partial^2 L(\boldsymbol{\beta})}{\partial\beta_h \partial\beta_l} = -\sum_{i=1}^{I} \frac{n_i x_{ih} x_{il} \exp\left(\sum_{j=0}^{s} \beta_j x_{ij}\right)}{\left[1+\exp\left(\sum_{j=0}^{s} \beta_{ij} x_{ij}\right)\right]^2}$$

$$= -\sum_{i=1}^{I} n_i x_{ih} x_{il} \pi(\mathbf{x}_i)(1-\pi(\mathbf{x}_i)). \tag{2.6}$$

Since second partial derivatives of the log likelihood (2.6) are not a function of $\{y_i\}$, the observed and expected second derivative matrix are identical. We estimate the covariance matrix by substituting $\hat{\beta}$ into the matrix having elements equal to the negative of (2.6) and inverting.

The asymptotic estimated covariance matrix takes the form

$$\hat{\mathbf{\Sigma}}^{-1} = \left\{\mathbf{X}' diag[n_i \hat{\pi}(\mathbf{x}_i)(1-\hat{\pi}(\mathbf{x}_i))]\mathbf{X}\right\}^{-1}, \tag{2.7}$$

where $diag[n_i \hat{\pi}(\mathbf{x}_i)(1-\hat{\pi}(\mathbf{x}_i))]$ denotes an IxI diagonal matrix.

Newton-Raphson Method

The Newton-Raphson method is a method for solving nonlinear equations.

Let $S(\boldsymbol{\beta})$ be a function of $\boldsymbol{\beta}$. Let $\mathbf{g}'(\boldsymbol{\beta}) = \left(\frac{\partial S(\boldsymbol{\beta})}{\partial \beta_0} \; \frac{\partial S(\boldsymbol{\beta})}{\partial \beta_1} \; \cdots \; \frac{\partial S(\boldsymbol{\beta})}{\partial \beta_s}\right)$ and $\mathbf{H}$ the matrix

having entries $h_{hl} = \frac{\partial S(\boldsymbol{\beta})}{\partial \beta_h \partial \beta_l}$, denote, respectively, the so-called gradient vector and

Hessian matrix of $S(\boldsymbol{\beta})$.

Then $S(\boldsymbol{\beta})$ is approximated near $\boldsymbol{\beta}^{(t)}$ by terms up to second order in its Taylor

series expansion,

$$S(\boldsymbol{\beta}) \approx Q^{(t)}(\boldsymbol{\beta}) = S(\boldsymbol{\beta}^{(t)}) + \mathbf{g}'(\boldsymbol{\beta}^{(t)})(\boldsymbol{\beta}-\boldsymbol{\beta}^{(t)}) + \frac{1}{2}(\boldsymbol{\beta}-\boldsymbol{\beta}^{(t)})'\mathbf{H}(\boldsymbol{\beta}^{(t)})(\boldsymbol{\beta}-\boldsymbol{\beta}^{(t)}). \tag{2.8}$$

Solving $\frac{\partial Q^{(t)}(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} = \mathbf{g}(\boldsymbol{\beta}^{(t)}) - \mathbf{H}(\boldsymbol{\beta}^{(t)})(\boldsymbol{\beta}-\boldsymbol{\beta}^{(t)}) = \mathbf{0}$, for $\boldsymbol{\beta}$ yields the next guess, assuming

$\mathbf{H}^{(t)}$ is nonsingular,

$$\boldsymbol{\beta}^{(t+1)} = \boldsymbol{\beta}^{(t)} - (\mathbf{H}^{(t)})^{-1}\mathbf{g}(\boldsymbol{\beta}^{(t)}). \tag{2.9}$$

Let $S(\boldsymbol{\beta})$ be the log likelihood (2.3) for the logistic regression model. From the likelihood equations (2.4) and the second partial derivatives of the log likelihood (2.6) for the logistic model, we have

$$g_j^{(t)} = \left.\frac{\partial L(\boldsymbol{\beta})}{\partial \beta_j}\right|_{\boldsymbol{\beta}^{(t)}} = \sum_{i=1}^{I}(y_i - n_i\pi^{(t)}(\mathbf{x}_i))x_{ij}$$

and

$$h_{hl}^{(t)} = \left.\frac{\partial^2 L(\boldsymbol{\beta})}{\partial \beta_h \partial \beta_l}\right|_{\boldsymbol{\beta}^{(t)}} = -\sum_{i=1}^{I} x_{ih}x_{il}n_i\pi^{(t)}(\mathbf{x}_i)(1-\pi^{(t)}(\mathbf{x}_i)).$$

Here $\pi^{(t)}(\mathbf{x}_i)$, the tth approximation for $\hat{\pi}(\mathbf{x}_i)$, is obtained from $\boldsymbol{\beta}^{(t)}$ through

$$\pi^{(t)}(\mathbf{x}_i) = \frac{\exp\left(\sum_{j=o}^{k}\beta_j^{(t)}x_{ij}\right)}{\left[1+\exp\left(\sum_{j=0}^{k}\beta_j^{(t)}x_{ij}\right)\right]}. \tag{2.10}$$

We use $\mathbf{g}^{(t)}$ and $\mathbf{H}^{(t)}$ with formula (2.8) to obtain the next value $\boldsymbol{\beta}^{(t+1)}$, which in this context is

$$\boldsymbol{\beta}^{(t+1)} = \boldsymbol{\beta}^{(t)} + \{\mathbf{X}'diag[n_i\pi^{(t)}(\mathbf{x}_i)(1-\pi^{(t)}(\mathbf{x}_i))]\mathbf{X}\}^{-1}\mathbf{X}'(\mathbf{y}-\mathbf{m}^{(t)}), \tag{2.11}$$

where $\mathbf{m}^{(t)} = \left[n_1\pi^{(t)}(\mathbf{x}_1) \quad \cdots \quad n_I\pi^{(t)}(\mathbf{x}_I)\right]'$.

Approximate Confidence Intervals for Logistic Regression Model

We have approximate confidence intervals for $\pi(\mathbf{x}_i)$ at a fixed $\mathbf{x}_i$, using the inverse logit transformation method. We have asymptotic normal distribution , from (2.5), such as

$$\mathbf{x}_i'\hat{\boldsymbol{\beta}} \sim N(\mathbf{x}_i'\boldsymbol{\beta}, \quad \mathbf{x}_i'\boldsymbol{\Sigma}^{-1}\mathbf{x}_i),$$

and we estimate the asymptotic variance of $\mathbf{x}_i'\hat{\boldsymbol{\beta}}$ as

$$\hat{V}(\mathbf{x}_i'\hat{\boldsymbol{\beta}}) = \mathbf{x}_i'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}_i,$$

where $\hat{\boldsymbol{\Sigma}}^{-1}$ is given by (2.7).

We have

$$\Pr\left\{-z_{\alpha/2} \le \frac{\mathbf{x}_i'\hat{\boldsymbol{\beta}} - \mathbf{x}_i'\boldsymbol{\beta}}{\sqrt{\mathbf{x}_i'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}_i}} \le z_{\alpha/2}\right\}$$

$$= \Pr\left\{\mathbf{x}_i'\hat{\boldsymbol{\beta}} - z_{\alpha/2}\sqrt{\mathbf{x}_i'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}_i} \le \mathbf{x}_i'\boldsymbol{\beta} \le \mathbf{x}_i\hat{\boldsymbol{\beta}} + z_{\alpha/2}\sqrt{\mathbf{x}_i'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}_i}\right\}$$

$$= 1 - \alpha,$$

where $z_{\alpha/2}$ is the 100(1−α/2) percentile point of a standard normal distribution.

The asymptotic 100(1−α)% confidence interval for $\mathbf{x}_i'\boldsymbol{\beta}$ is given by

$$\left[\mathbf{x}_i'\hat{\boldsymbol{\beta}} - z_{\alpha/2}\sqrt{\mathbf{x}_i'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}_i},\quad \mathbf{x}_i'\hat{\boldsymbol{\beta}} + z_{\alpha/2}\sqrt{\mathbf{x}_i'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}_i}\right]. \tag{2.12}$$

The corresponding 100(1−α)% approximate confidence interval for $\pi(\mathbf{x}_i)$ at a fixed $\mathbf{x}_i$ is given by taking the inverse logit transform of (2.12):

$$\frac{\exp\left(\mathbf{x}_i'\hat{\boldsymbol{\beta}} - z_{\alpha/2}\sqrt{\mathbf{x}_i'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}_i}\right)}{1+\exp\left(\mathbf{x}_i'\hat{\boldsymbol{\beta}} - z_{\alpha/2}\sqrt{\mathbf{x}_i'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}_i}\right)} \le \pi(\mathbf{x}_i) \le \frac{\exp\left(\mathbf{x}_i'\hat{\boldsymbol{\beta}} + z_{\alpha/2}\sqrt{\mathbf{x}_i'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}_i}\right)}{1+\exp\left(\mathbf{x}_i'\hat{\boldsymbol{\beta}} + z_{\alpha/2}\sqrt{\mathbf{x}_i'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}_i}\right)}. \tag{2.13}$$

We can have approximate confidence intervals for $\pi(\mathbf{x}_i)$ at a fixed $\mathbf{x}_i$ using the delta method (Appendix A).

Let $h(\mathbf{z}) = \dfrac{\exp(\mathbf{x}_i'\mathbf{z})}{1+\exp(\mathbf{x}_i'\mathbf{z})}$, for fixed $\mathbf{x}_i$,

where $\mathbf{z} = (z_0\ z_1\ \cdots\ z_s)'$.

Then $h(\mathbf{z})$ has nonzero first derivative at $\mathbf{z} = \boldsymbol{\beta}$, where $\boldsymbol{\beta} = (\beta_0\ \beta_1\ \cdots\ \beta_s)'$. From (2.5) and the delta method, the asymptotic normal distribution given by

$$h(\hat{\boldsymbol{\beta}}) \sim N(h(\boldsymbol{\beta}),\ \mathbf{d}'\boldsymbol{\Sigma}^{-1}\mathbf{d}),$$

where $\mathbf{d}' = \left(\frac{\partial h(\boldsymbol{\beta})}{\partial \beta_0} \frac{\partial h(\boldsymbol{\beta})}{\partial \beta_1} \cdots \frac{\partial h(\boldsymbol{\beta})}{\partial \beta_s} \right)$.

The estimated asymptotic variance of $h(\hat{\boldsymbol{\beta}})$ is given by

$$\hat{\mathrm{V}}(h(\hat{\boldsymbol{\beta}})) = \hat{\mathbf{d}}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\mathbf{d}},$$

where $\hat{\mathbf{d}}' = \mathbf{d}'|_{\boldsymbol{\beta}=\hat{\boldsymbol{\beta}}}$ and $\hat{\boldsymbol{\Sigma}}^{-1}$ is given in (2.7).

We have, asymptotically,

$$\Pr\left\{ -z_{\alpha/2} \le \frac{h(\hat{\boldsymbol{\beta}}) - h(\boldsymbol{\beta})}{\sqrt{\hat{\mathbf{d}}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\mathbf{d}}}} \le z_{\alpha/2} \right\} = 1 - \alpha,$$

where $z_{\alpha/2}$ is the $100(1-\alpha/2)$ percentile point of a standard normal distribution.

Since $h(\boldsymbol{\beta}) = \pi(\mathbf{x}_i)$, we have $100(1-\alpha)\%$ approximate confidence intervals for $\pi(\mathbf{x}_i)$ at a fixed $\mathbf{x}_i$ given by

$$\hat{\pi}(\mathbf{x}_i) - z_{\alpha/2}\sqrt{\hat{\mathbf{d}}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{d}} \le \pi(\mathbf{x}_i) \le \hat{\pi}(\mathbf{x}_i) + z_{\alpha/2}\sqrt{\mathbf{d}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{d}}, \qquad (2.14)$$

where $\hat{\pi}(\mathbf{x}_i) = \frac{\exp(\mathbf{x}_i'\hat{\boldsymbol{\beta}})}{1 + \exp(\mathbf{x}_i'\hat{\boldsymbol{\beta}})}$.

A problem with the approximate confidence intervals in (2.14) is that a lower bound may be less than zero or a upper bound may be greater than one.

Approximate Confidence Bands for Logistic Regression Model

There have been lots of papers about confidence bands for linear regression models. But not much has been published on confidence bands for logistic regression models. We consider the problem of constructing approximate confidence bands for logistic regression models.

Approximate Confidence Bands for Logistic Regression using Elliptical Confidence Set of Parameters

Hauck (1983) obtained approximate confidence bands for the logistic regression curve using elliptical confidence set of parameters.

From (2.5) and the asymptotic estimated covariance matrix (2.7),

$$(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta}) \sim \chi^2_{s+1},$$

where χ^2_{s+1} is the chi-square distribution with s+1 degrees of freedom.

That is,

$$\Pr\{(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta}) \leq \chi^2_{s+1,\alpha}\} = 1-\alpha, \tag{2.15}$$

where $\chi^2_{s+1,\alpha}$ is the upper a percentage point of the chi-square distribution with s+1 degrees of freedom.

Probability (2.15) gives an approximate elliptical confidence set on $\boldsymbol{\beta}$ with confidence coefficient of 1−α such as

$$(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta}) \le \chi^2_{s+1,\alpha}.$$

Applying a form of the Cauchy-Schwartz inequality (Appendix B), Hauck obtained, for all $\mathbf{x}_i \neq \mathbf{0}$,

$$\frac{[\mathbf{x}_i'(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})]^2}{\mathbf{x}_i'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}_i} \le (\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta}). \tag{2.16}$$

Combining (2.15) and (2.16), he concluded that

$$\Pr\left\{\frac{[\mathbf{x}_i'(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})]^2}{\mathbf{x}_i'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}_i} \le \chi^2_{s+1,\alpha} \quad for \;\; all \;\; \mathbf{x}_i\right\}$$

$$= \Pr\left\{\left|\mathbf{x}_i'(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})\right| \le \sqrt{\chi^2_{s+1,\alpha}\mathbf{x}_i'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}} \quad for \;\; all \;\; \mathbf{x}_i\right\}$$

$$= \Pr\left\{\mathbf{x}_i'\hat{\boldsymbol{\beta}} - \sqrt{\chi^2_{s+1,\alpha}\mathbf{x}_i'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}_i} \le \mathbf{x}_i'\boldsymbol{\beta} \le \mathbf{x}_i'\hat{\boldsymbol{\beta}} + \sqrt{\chi^2_{s+1,\alpha}\mathbf{x}_i'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}_i} \quad for \;\; all \;\; \mathbf{x}_i\right\}$$

$$\ge 1-\alpha. \tag{2.17}$$

So, a 100(1−α)% approximate confidence band for $\mathbf{x}_i'\boldsymbol{\beta}$ over all $\mathbf{x}_i$ is given by

$$\left[\mathbf{x}_i'\hat{\boldsymbol{\beta}} - \sqrt{\chi^2_{s+1,\alpha}\mathbf{x}_i'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}_i}\,,\;\; \mathbf{x}_i'\hat{\boldsymbol{\beta}} + \sqrt{\chi^2_{s+1,\alpha}\mathbf{x}_i'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}_i}\right] \tag{2.18}$$

The corresponding 100(1−α)% approximate confidence bands on $\pi(\mathbf{x}_i)$ over all $\mathbf{x}_i$ are given by taking the inverse logit transform of (2.18):

$$\frac{\exp\left(\mathbf{x}_i'\hat{\boldsymbol{\beta}} - \sqrt{\chi_{s+1,\alpha}\mathbf{x}_i'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}_i}\right)}{1+\exp\left(\mathbf{x}_i'\hat{\boldsymbol{\beta}} - \sqrt{\chi_{s+1,\alpha}\mathbf{x}_i'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}_i}\right)} \le \pi(\mathbf{x}_i) \le \frac{\exp\left(\mathbf{x}_i'\hat{\boldsymbol{\beta}} + \sqrt{\chi_{s+1,\alpha}\mathbf{x}_i'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}_i}\right)}{1+\exp\left(\mathbf{x}_i'\hat{\boldsymbol{\beta}} + \sqrt{\chi_{s+1,\alpha}\mathbf{x}_i'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}_i}\right)} \tag{2.19}$$

Approximate Confidence Bands for Logistic

Regression using Rectangular Confidence

Set of Parameters

We obtain approximate confidence bands on $\pi(\mathbf{x}_i)$ for all $\mathbf{x}_i$ using a rectangular confidence set of parameters in the logistic regression model.

From (2.5), we know that, asymptotically,

$$\hat{\boldsymbol{\beta}} \sim N_{s+1}(\boldsymbol{\beta},\ \boldsymbol{\Sigma}^{-1}).$$

Let $\mathbf{D} = diag\{\lambda_k\}$ be the diagonal matrix of the eigenvalues of $\boldsymbol{\Sigma}$ and let $\mathbf{U}$ be the matrix of corresponding orthonormal eigenvectors. Then

$$\boldsymbol{\Sigma} = \mathbf{UDU}' \text{ and } \boldsymbol{\Sigma}^{-1} = (\mathbf{UDU}')^{-1} = \mathbf{UD}^{-1}\mathbf{U}'.$$

Let $\mathbf{d}_i = \mathbf{D}^{\frac{-1}{2}}\mathbf{U}'\mathbf{x}_i$ and $\boldsymbol{\eta} = \mathbf{D}^{\frac{1}{2}}\mathbf{U}'\boldsymbol{\beta}$,

where

$$\mathbf{d}_i = (\mathrm{d}_{i0}\ \mathrm{d}_{i1}\ \ldots\ \mathrm{d}_{is})',$$

$$\boldsymbol{\eta} = (\eta_0\ \eta_1\ \ldots\ \eta_s)',$$

$\mathbf{D}^{\frac{1}{2}}$ is defined as $diag\{\lambda_k^{\frac{1}{2}}\}$.

Let $\hat{\boldsymbol{\eta}}$ be maximum likelihood estimator of $\boldsymbol{\eta}$. Then we have

$$E(\hat{\boldsymbol{\eta}}) = \mathbf{D}^{\frac{1}{2}}\mathbf{U}E(\hat{\boldsymbol{\beta}}) = \mathbf{D}^{\frac{1}{2}}\mathbf{U}\boldsymbol{\beta} = \boldsymbol{\eta}.$$

and $$\mathrm{V}(\hat{\boldsymbol{\eta}}) = \mathbf{D}^{\frac{1}{2}}\mathbf{U}'\mathrm{V}(\hat{\boldsymbol{\beta}})\mathbf{UD}^{\frac{1}{2}} = \mathbf{D}^{\frac{1}{2}}\mathbf{U}'\boldsymbol{\Sigma}^{-1}\mathbf{UD}^{\frac{1}{2}} = \mathbf{I},$$

where $\mathbf{I}$ is an $(s+1)\mathrm{x}(s+1)$ identity matrix.

Thus, asymptotically,

$$\hat{\boldsymbol{\eta}} \sim N_{s+1}(\boldsymbol{\eta},\ \mathbf{I}).$$

Let $\theta_j = \hat{\eta}_j - \eta_j, \quad j = 0, 1, \dots, s$, and $\boldsymbol{\theta} = (\theta_0\ \theta_1\ \dots\ \theta_s)'$. Then we have s+1 variates asymptotic standard normal distribution,

$$\boldsymbol{\theta} \sim N_{s+1}(\mathbf{0},\ \mathbf{I}).$$

Since θ_j, $j = 0, 1, \dots, s$, are independent, we have

$$\Pr\left\{-c_{\alpha/2} \le \theta_0 \le c_{\alpha/2},\ -c_{\alpha/2} \le \theta_1 \le c_{\alpha/2}, \dots,\ -c_{\alpha/2} \le \theta_s \le c_{\alpha/2}\right\}$$

$$= \Pr\left\{-c_{\alpha/2} \le \theta_0 \le c_{\alpha/2}\right\}\Pr\left\{-c_{\alpha/2} \le \theta_1 \le c_{\alpha/2}\right\}\dots \Pr\left\{-c_{\alpha/2} \le \theta_s \le c_{\alpha/2}\right\}$$

$$= 1 - \alpha, \tag{2.20}$$

where $c_{\alpha/2}$ is a number such that $\int_{c_{\frac{\alpha}{2}}}^{c_{\frac{\alpha}{2}}} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{1}{2}\theta_j^2\right) d\theta_j = (1-\alpha)^{\frac{1}{s+1}}$.

Substituting $\theta_j = \hat{\eta}_j - \eta_j, \quad j = 0, 1, \dots, s$, into (2.20), we have

$$\Pr\left\{\hat{\eta}_0 - c_{\alpha/2} \le \eta_0 \le \hat{\eta}_0 + c_{\alpha/2}\right\} \Pr\left\{\hat{\eta}_1 - c_{\alpha/2} \le \eta_1 \le \hat{\eta}_1 + c_{\alpha/2}\right\} \cdots \Pr\left\{\hat{\eta}_s - c_{\alpha/2} \le \eta_s \le \hat{\eta}_s + c_{\alpha/2}\right\}$$

$$= 1 - \alpha. \tag{2.21}$$

Probability (2.21) gives the following an approximate rectangular confidence set on $\boldsymbol{\eta}$ with confidence coefficient of 1–α:

$$\hat{\eta}_0 - c_{\alpha/2} \le \eta_0 \le \hat{\eta}_0 + c_{\alpha/2}. \tag{2.22}$$

$$\vdots$$

$$\hat{\eta}_s - c_{\alpha/2} \le \eta_s \le \hat{\eta}_s + c_{\alpha/2}. \tag{2.23}$$

When $d_{i0} \ge 0$, we have an inequality, from (2.22),

$$\hat{\eta}_0 d_{i0} - c_{\alpha/2} d_{i0} \le \eta_0 d_{i0} \le \hat{\eta}_0 d_{i0} + c_{\alpha/2} d_{i0},$$

When $d_{i0} \leq 0$, we have an inequality, from (2.22),

$$\hat{\eta}_0 d_{i0} + c_{\alpha/2} d_{i0} \leq \eta_0 d_{i0} \leq \hat{\eta}_0 d_{i0} - c_{\alpha/2} d_{i0}.$$

Thus, for all d_{i0},

$$\hat{\eta}_0 d_{i0} - c_{\alpha/2} \left| d_{i0} \right| \leq \eta_0 d_{i0} \leq \hat{\eta}_0 d_{i0} + c_{\alpha/2} \left| d_{i0} \right|. \tag{2.24}$$

Similarly, from (2.23), we have

$$\hat{\eta}_s d_{is} - c_{\alpha/2} \left| d_{is} \right| \leq \eta_s d_{is} \leq \hat{\eta}_s d_{is} + c_{\alpha/2} \left| d_{is} \right|, \text{ for all } d_{is}. \tag{2.25}$$

Therefore, we have an inequality

$$\hat{\eta}_0 d_{i0} + \cdots + \hat{\eta}_s d_{is} - c_{\alpha/2}\left(\left| d_{i0} \right| + \cdots + \left| d_{is} \right| \right) \leq \eta_0 d_{i0} + \cdots + \eta_s d_{is} \leq \hat{\eta}_0 d_{i0} + \cdots + \hat{\eta}_s d_{is} - c_{\alpha/2}\left(\left| d_{i0} \right| + \cdots + \left| d_{is} \right| \right)$$

for all $d_{i0}, d_{i1}, \ldots, d_{is}$.

This is equivalent to writing in matrix form

$$\mathbf{d}_i{}'\hat{\boldsymbol{\eta}} - c_{\alpha/2}\left(\sum_{j=0}^{s} \left| d_{ij} \right| \right) \leq \mathbf{d}_i{}'\boldsymbol{\eta} \leq \mathbf{d}_i{}'\hat{\boldsymbol{\eta}} + c_{\alpha/2}\left(\sum_{j=0}^{s} \left| d_{ij} \right| \right) \text{ for all } \mathbf{d}_i. \tag{2.26}$$

Substituting $\mathbf{d}_i = \mathbf{D}^{\frac{-1}{2}}\mathbf{U}'\mathbf{x}_i$ and $\boldsymbol{\eta} = \mathbf{D}^{\frac{1}{2}}\mathbf{U}'\boldsymbol{\beta}$ into (2.26), we have

$$\mathbf{x}_i{}'\hat{\boldsymbol{\beta}} - c_{\alpha/2}\left(\sum_{j=0}^{s} \left| d_{ij} \right| \right) \leq \mathbf{x}_i{}'\boldsymbol{\beta} \leq \mathbf{x}_i{}'\hat{\boldsymbol{\beta}} + c_{\alpha/2}\left(\sum_{j=0}^{s} \left| d_{ij} \right| \right) \text{ for all } \mathbf{x}_i. \tag{2.27}$$

The $100(1-\alpha)\%$ approximate confidence bands for $\mathbf{x}_i{}'\boldsymbol{\beta}$ over all $\mathbf{x}_i$ are given by

$$\left[\mathbf{x}_i{}'\hat{\boldsymbol{\beta}} - c_{\alpha/2}\left(\sum_{j=0}^{s} \left| d_{ij} \right| \right), \quad \mathbf{x}_i{}'\hat{\boldsymbol{\beta}} + c_{\alpha/2}\left(\sum_{j=0}^{s} \left| d_{ij} \right| \right) \right]. \tag{2.28}$$

The corresponding 100(1−α)% approximate confidence bands on $\pi(\mathbf{x}_i)$ over all $\mathbf{x}_i$ are given by taking the inverse logit transform of (2.27):

$$\frac{\exp\left[\mathbf{x}_i'\hat{\boldsymbol{\beta}} - c_{\alpha/2}\left(\sum_{j=0}^{s}\left|d_{ij}\right|\right)\right]}{1+\exp\left[\mathbf{x}_i'\hat{\boldsymbol{\beta}} - c_{\alpha/2}\left(\sum_{j=0}^{s}\left|d_{ij}\right|\right)\right]} \le \pi(\mathbf{x}_i) \le \frac{\exp\left[\mathbf{x}_i'\hat{\boldsymbol{\beta}} + c_{\alpha/2}\left(\sum_{j=0}^{s}\left|d_{ij}\right|\right)\right]}{1+\exp\left[\mathbf{x}_i'\hat{\boldsymbol{\beta}} + c_{\alpha/2}\left(\sum_{j=0}^{s}\left|d_{ij}\right|\right)\right]}. \tag{2.29}$$

Comparison of Two Methods for Approximate Confidence Bands

Since the logit transform is a monotone function of $\pi(\mathbf{x}_i)$, to compare widths of approximate confidence bands given in (2.19) and (2.29) is the same as comparing widths of approximate confidence bands given in (2.18) and (2.28).

We compare widths of bands given in (2.18) and (2.28) when $s = 1$. The width of bands in (2.18) is

$$W_e = 2\sqrt{\chi^2_{2,\alpha}}\sqrt{\mathbf{x}_i'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}_i}\,. \tag{2.30}$$

and the width of bands in (2.28) is

$$W_r = 2c_{\alpha/2}\left(\left|d_{i0}\right| + \left|d_{i1}\right|\right). \tag{2.31}$$

Since $\hat{\boldsymbol{\Sigma}}^{-1} = \mathbf{U}\mathbf{D}^{-1}\mathbf{U}'$ and $\mathbf{x}_i'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}_i = \left(\mathbf{D}^{-1/2}\mathbf{U}'\mathbf{x}_i\right)'\left(\mathbf{D}^{-1/2}\mathbf{U}'\mathbf{x}_i\right) = \mathbf{d}_i'\mathbf{d}_i = d_{i0}^2 + d_{i1}^2$, Equation (2.30) is

$$W_e = 2\sqrt{\chi^2_{2,\alpha}}\sqrt{d_{i0}^2 + d_{i1}^2}\,. \tag{2.32}$$

Let R be the ratio of the two widths.

$$R = \frac{W_e}{W_r} = \frac{k\sqrt{d_{i0}^2 + d_{i1}^2}}{|d_{i0}| + |d_{i1}|}, \tag{2.33}$$

where $k = \dfrac{\sqrt{\chi^2_{2,\alpha}}}{c_{\alpha/2}}$.

Now $c_{\alpha/2}$ is the number such that

$$\Pr\{\theta_0{}^2 \le (c_{\alpha/2})^2,\ \theta_1{}^2 \le (c_{\alpha/2})^2\} = 1-\alpha,$$

or

$$\Pr\{\max \theta_j{}^2 \le (c_{\alpha/2})^2\} = 1-\alpha, \quad j = 0,\ 1.$$

But

$$\Pr\{\max \theta_j{}^2 \le (c_{\alpha/2})^2\} = \Pr\{\theta_0{}^2 + \theta_1{}^2 \le \chi^2_{2,\alpha}\}$$

and

$$\Pr\{\theta_0{}^2 + \theta_1{}^2 \le \chi^2_{2,\alpha}\} < \Pr\{\max \theta_j{}^2 \le \chi^2_{2,\alpha}\}.$$

Hence $(c_{\alpha/2})^2 < \chi^2_{2,\alpha}$. Thus it is that $1 < k^2$.

We shall find the values of x for which R=1. When R=1, from (2.33), we have

$$k\sqrt{d_0^2 + d_1^2} = \sqrt{d_0^2} + \sqrt{d_1^2} \tag{2.34}$$

We square both sides of (2.34), then we have

$$k^2(d_0^2 + d_1^2) = d_0^2 + d_1^2 + 2\sqrt{d_0^2 d_1^2}$$

or

$$(k^2 - 1)(d_0^2 + d_1^2) = 2\sqrt{d_0^2 d_1^2} \tag{2.35}$$

We square both sides of (2.35) again, then we have

$$(k^2-1)^2(d_0^4+d_1^4+2d_0^2d_1^2)=4d_0^2d_1^2$$

or

$$d_0^4+d_1^4+\left(2-\frac{4}{(k^2-1)^2}\right)d_0^2d_1^2=0. \tag{2.36}$$

Let $c=2-\dfrac{4}{(k^2-1)^2}$ and $t=d_0^2$. Then we have, from (2.36),

$$t^2+cd_1^2t+d_1^4=0 \tag{2.37}$$

and the roots are

$$\begin{aligned} t &= \frac{-cd_1^2\pm\sqrt{c^2d_1^4-4d_1^4}}{2} \\ &= \frac{d_1^2(-c\pm\sqrt{c^2-4})}{2}. \end{aligned} \tag{2.38}$$

Substituting $t=d_0^2$ into (2.38), we have

$$d_0^2=\frac{d_1^2(-c+\sqrt{c^2-4})}{2}, \tag{2.39}$$

and

$$d_0^2=\frac{d_1^2(-c-\sqrt{c^2-4})}{2}. \tag{2.40}$$

Let $a_1=\dfrac{-c+\sqrt{c^2-4}}{2}$ and $a_2=\dfrac{-c-\sqrt{c^2-4}}{2}$. Then equations (2.39) and (2.40) can be simplified to

$$d_0^2=a_1d_1^2 \tag{2.41}$$

and

$$d_0^2=a_2d_1^2. \tag{2.42}$$

Let $\mathbf{D}^{\frac{-1}{2}}\mathbf{U}'=\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$. Since $\begin{bmatrix} d_0 \\ d_1 \end{bmatrix}=\mathbf{D}^{-\frac{1}{2}}\mathbf{U}'\begin{bmatrix} 1 \\ x \end{bmatrix}$, from (2.41), we have the equation in x,

$$(r_{12}^2 - a_1 r_{22}^2)x^2 + 2(r_{11}r_{12} - a_1 r_{21} r_{22})x + (r_{11}^2 - a_1 r_{21}^2) = 0. \tag{2.43}$$

From (2.43), we have the roots such as

$$x = \frac{-(r_{11}r_{12} - a_1 r_{21} r_{22}) \pm \sqrt{(r_{11}r_{12} - a_1 r_{21} r_{22})^2 - (r_{12}^2 - a_1 r_{22}^2)(r_{11}^2 - a_1 r_{21}^2)}}{(r_{12}^2 - a_1 r_{22}^2)}$$

$$= \frac{-(r_{11}r_{12} - a_1 r_{21} r_{22}) \pm \sqrt{a_1 (r_{12}r_{21} - r_{11}r_{22})^2}}{(r_{12}^2 - a_1 r_{22}^2)}.$$

Similarly, from equation (2.42), we have the roots such as

$$x = \frac{-(r_{11}r_{12} - a_2 r_{21} r_{22}) \pm \sqrt{(r_{11}r_{12} - a_2 r_{21} r_{22})^2 - (r_{12}^2 - a_2 r_{22}^2)(r_{11}^2 - a_2 r_{21}^2)}}{(r_{12}^2 - a_2 r_{22}^2)}$$

$$= \frac{-(r_{11}r_{12} - a_2 r_{21} r_{22}) \pm \sqrt{a_2 (r_{12}r_{21} - r_{11}r_{22})^2}}{(r_{12}^2 - a_2 r_{22}^2)}.$$

Thus approximate confidence bands given in (2.19) intersect approximate confidence bands given in (2.29) in four places.

The set of values of x for which R is greater than one occurs when the width of bands given in (2.29) is smaller than the width of bands given in (2.19). If R is less than one then the contrary is true.

We shall find the values of x for which $R > 1$.

When $R > 1$, we have the inequalities

$$d_0^2 > a_1 d_1^2 \tag{2.44}$$

and

$$d_0^2 < a_2 d_1^2. \tag{2.45}$$

From (2.44), we have the inequality, for x,

$$(r_{12}^2 - a_1 r_{22}^2)x^2 + 2(r_{11}r_{12} - a_1 r_{21}r_{22})x + (r_{11}^2 - a_1 r_{21}^2) > 0. \tag{2.46}$$

When $r_{12}^2 - a_1 r_{22}^2 > 0$, the solutions of inequality (2.46) are

$$x > \frac{-(r_{11}r_{12} - a_1 r_{21}r_{22}) + \sqrt{a_1 (r_{12}r_{21} - r_{11}r_{22})^2}}{(r_{12}^2 - a_1 r_{22}^2)}$$

and

$$x < \frac{-(r_{11}r_{12} - a_1 r_{21}r_{22}) - \sqrt{a_1 (r_{12}r_{21} - r_{11}r_{22})^2}}{(r_{12}^2 - a_1 r_{22}^2)}.$$

When $r_{12}^2 - a_1 r_{22}^2 < 0$, the solution of inequality (2.46) is

$$\frac{-(r_{11}r_{12} - a_1 r_{21}r_{22}) + \sqrt{a_1 (r_{12}r_{21} - r_{11}r_{22})^2}}{(r_{12}^2 - a_1 r_{22}^2)} < x < \frac{-(r_{11}r_{12} - a_1 r_{21}r_{22}) - \sqrt{a_1 (r_{12}r_{21} - r_{11}r_{22})^2}}{(r_{12}^2 - a_1 r_{22}^2)}.$$

Similarly, from (2.45), we have the inequality

$$(r_{12}^2 - a_2 r_{22}^2)x^2 + 2(r_{11}r_{12} - a_2 r_{21}r_{22})x + (r_{11}^2 - a_2 r_{21}^2) < 0. \tag{2.47}$$

When $r_{12}^2 - a_2 r_{22}^2 > 0$, the solution of inequality (2.47) is given by

$$\frac{-(r_{11}r_{12} - a_2 r_{21}r_{22}) - \sqrt{a_2 (r_{12}r_{21} - r_{11}r_{22})^2}}{(r_{12}^2 - a_2 r_{22}^2)} < x < \frac{-(r_{11}r_{12} - a_2 r_{21}r_{22}) + \sqrt{a_2 (r_{12}r_{21} - r_{11}r_{22})^2}}{(r_{12}^2 - a_2 r_{22}^2)}.$$

When $r_{12}^2 - a_2 r_{22}^2 < 0$, the solution of inequality (2.47) is given by

$$x > \frac{-(r_{11}r_{12} - a_2 r_{21}r_{22}) + \sqrt{a_2 (r_{12}r_{21} - r_{11}r_{22})^2}}{(r_{12}^2 - a_2 r_{22}^2)}$$

and

$$x < \frac{-(r_{11}r_{12} - a_2 r_{21}r_{22}) - \sqrt{a_2 (r_{12}r_{21} - r_{11}r_{22})^2}}{(r_{12}^2 - a_2 r_{22}^2)}.$$

If x does not satisfy the above inequalities, then R is less than one.

In the simulation study, we examine the probabilities that false probabilities are contained in the bands. Let

$$P= \Pr(Y|x_i) = \frac{\exp(-3+0.07x_i)}{1+\exp(-3+0.07x_i)}, \ -\infty < x_i < \infty, \ i= 1, 2, ..., n,$$

be the true probability.

The zero-one values of Y are generated using the true probabilities.

We restrict the regions for x_i. These restricted regions are as follows:

Region 1: R1=$\{x_i | 10.0 \le x_i \le 34.5\}$

Region 2: R2=$\{x_i | 35.0 \le x_i \le 59.5\}$

Region 3: R3=$\{x_i | 60.0 \le x_i \le 84.5\}$.

Let method 1 be approximate confidence bands with the elliptical confidence set of parameters in logistic regression and method 2 be approximate confidence bands with the rectangular confidence set of parameters in logistic regression.

We selected sample sizes n=10, 20, 30, 40, and 50, from each region. We add small numbers, 0.001, 0.005, 0.02, 0.06 and 0.1 to the true probabilities. That is, P+0.001, P+0.005, P+0.02, P+0.06 and P+0.1 are called false probabilities.

Finally, 1000 separate runs are made for each combination of sample size, the false probabilities and methods.

In any case, the probabilities that false probabilities are contained in the bands for method 1 are less than for method 2 at some middle points of each regions. Tables provide the probabilities that false probabilities are contained in the bands for the combinations of sample sizes, false probabilities and methods. Tables are given in Appendix C.

Numerical Example

Table I lists in years (Age), and presence or absence of evidence of significant coronary heart disease (CHD). The outcome variable is CHD, which is coded with a value of zero to indicate CHD is absent, or 1 to indicate that it is present in the individual.

Let x_i be Age and y_i be the outcome variable in the ith individual. Let $\Pr(Y_i = 1|x_i) = \pi(\mathbf{x}_i)$, where $\mathbf{x}_i = (1 \;\; x_i)'$. We express the logistic model as

$$\pi(\mathbf{x}_i) = \frac{\exp(\beta_0 + \beta_1 x_i)}{1 + \exp(\beta_0 + \beta_1 x_i)}$$

$$= \frac{\exp(\mathbf{x}_i'\boldsymbol{\beta})}{1 + \exp(\mathbf{x}_i'\boldsymbol{\beta})} \tag{2.48}$$

where $\mathbf{x}_i = (1 \;\; x_i)'$ and $\boldsymbol{\beta} = (\beta_0 \;\; \beta_1)'$.

Using SAS/IML we have the maximum likelihood estimates of β_0 and β_1 which are $\hat{\beta}_0 = -5.310$ and $\hat{\beta}_1 = 0.110$. The estimated asymptotic covariance matrix of maximum likelihood estimates is

$$\hat{\boldsymbol{\Sigma}}^{-1} = \begin{bmatrix} 1.2851728 & -0.026677 \\ -0.026677 & 0.0005789 \end{bmatrix}.$$

Figure 1 shows the fitted logistic regression model given in (2.48).

We have two approximate confidence intervals from the inverse logit transformation method and the delta method respectively in Table II, III and IV. The solid line denotes approximate confidence intervals from the inverse logit transformation method and the dotted line denotes approximate confidence intervals from the delta method in Figure 2, 3 and 4.

We have two approximate confidence bands on the entire curve of the logistic regression model given in (2.19) and (2.29) respectively in Table V, VI and VII. The solid line denotes bands given in (2.19) and the dotted line denotes bands given in (2.29) in Figure 5, 6 and 7.

TABLE I

AGE AND CORONARY HEART DISEASE STATUS (CHD) OF 100 SUBJECTS

AGE	CHD	AGE	CHD	AGE	CHD	AGE	CHD	AGE	CHD
20	0	34	0	41	0	48	1	57	0
23	0	34	0	42	0	48	1	57	1
24	0	34	1	42	0	49	0	57	1
25	0	34	0	42	0	49	0	57	1
25	1	34	0	42	1	49	1	57	1
26	0	35	0	43	0	50	0	58	0
26	0	35	0	43	0	50	1	58	1
28	0	36	0	43	1	51	0	58	1
28	0	36	1	44	0	52	0	59	1
29	0	36	0	44	0	52	1	59	1
30	0	37	0	44	1	53	1	60	0
30	0	37	1	44	1	53	1	69	1
30	0	37	0	45	0	54	1	61	1
30	0	38	0	45	1	55	0	62	1
30	0	38	0	46	0	55	1	62	1
30	1	39	0	46	1	55	1	63	1
32	0	39	1	47	0	56	1	64	0
32	0	40	0	47	0	56	1	64	1
33	0	40	1	47	1	56	1	65	1
33	1	41	0	48	0	57	0	69	1

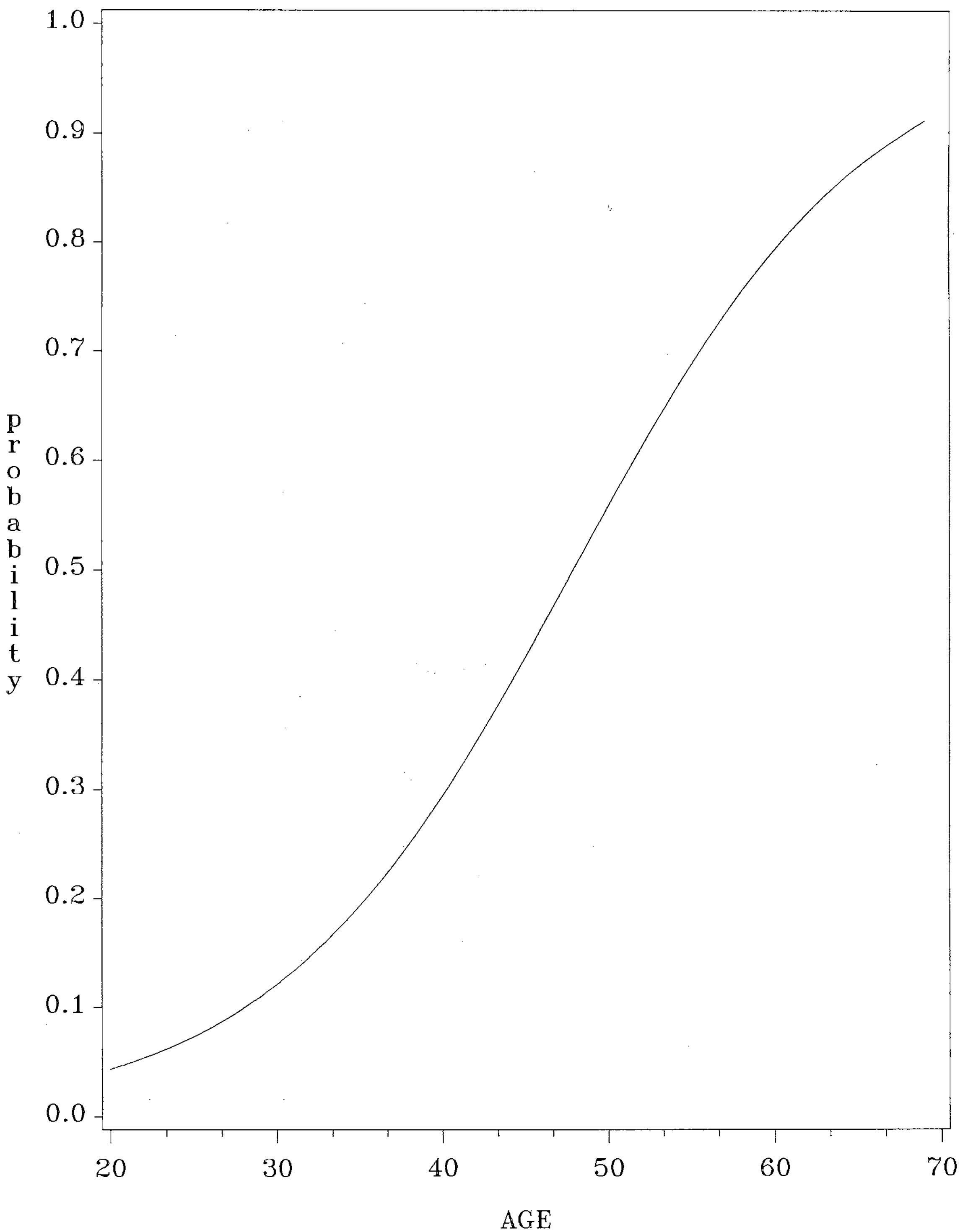


Figure 1. Fitted Logistic Regression

TABLE II

95% APPROXIMATE CONFIDECE INTERVALS
FOR LOGISTIC REGRESSION

	Intervals in (2.13)		Intervals in (2.14)	
Age	Lower	Upper	Lower	Upper
20	0.01206	0.14471	-0.0112	0.09814
23	0.01905	0.17146	-0.0067	0.12594
24	0.02216	0.18129	-0.0043	0.13656
25	0.02575	0.19160	-0.0012	0.14788
26	0.02990	0.20241	0.00256	0.15994
28	0.04016	0.22560	0.01254	0.18630
29	0.04645	0.23802	0.01898	0.20063
30	0.05364	0.25101	0.02653	0.21572
32	0.07112	0.27881	0.04540	0.24819
33	0.08163	0.29366	0.05692	0.26556
34	0.09344	0.30918	0.06995	0.28367
35	0.10665	0.32542	0.08456	0.30251
36	0.12132	0.34240	0.10078	0.32209
37	0.13751	0.36017	0.11864	0.34240
38	0.15523	0.37877	0.13810	0.36346
39	0.17445	0.39828	0.15910	0.38529
40	0.19510	0.41873	0.18151	0.40792
41	0.21704	0.44020	0.20516	0.43140
42	0.24010	0.46273	0.22986	0.45578
43	0.26404	0.48635	0.25534	0.48111
44	0.28861	0.51108	0.28135	0.50742
45	0.31351	0.53687	0.30762	0.53470
46	0.33849	0.56362	0.33391	0.56292
47	0.36329	0.59116	0.36001	0.59194
48	0.38772	0.61926	0.38578	0.62160
49	0.41163	0.64762	0.41113	0.65163
50	0.43492	0.67590	0.43601	0.68175
51	0.45755	0.70376	0.46042	0.71162
52	0.47950	0.73085	0.48438	0.74091
53	0.50077	0.75688	0.50793	0.76930
54	0.52139	0.78159	0.53109	0.79652
55	0.54137	0.80480	0.55388	0.82231
56	0.56073	0.82637	0.57629	0.84648
57	0.57951	0.84623	0.59833	0.86891
58	0.59771	0.86436	0.61995	0.88950
59	0.61535	0.88079	0.64113	0.90821
60	0.63245	0.89557	0.66183	0.92506
61	0.64901	0.90878	0.68198	0.94008
62	0.66504	0.92052	0.70154	0.95335
63	0.68055	0.93092	0.72047	0.96496
64	0.69554	0.94008	0.73871	0.97502
65	0.71001	0.94811	0.75623	0.98365
69	0.76287	0.97124	0.81856	1.00637

TABLE III

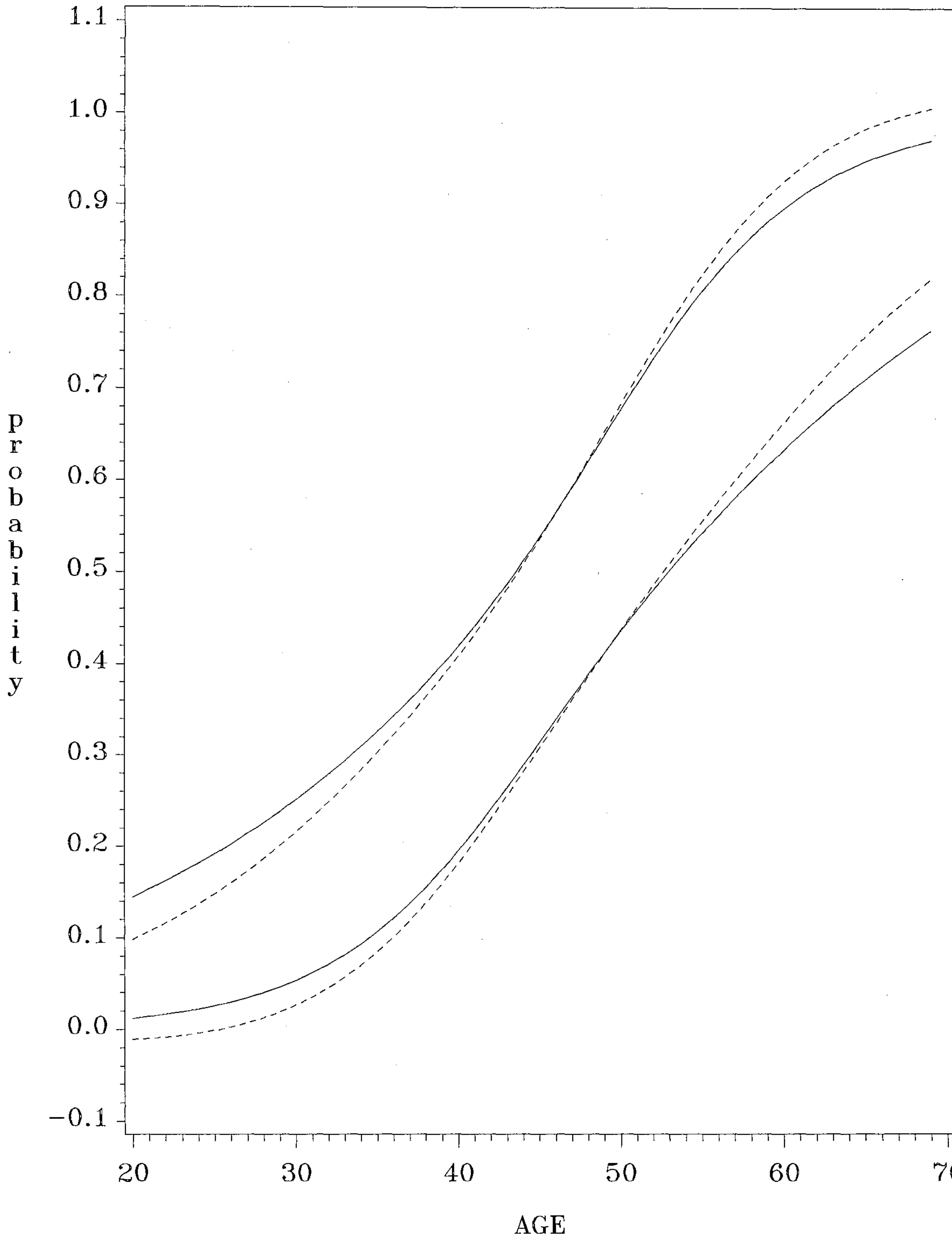
90% APPROXIMATE CONFIDENCE INTERVALS
FOR LOGISTIC REGRESSION

	Intervals in (2.13)		Intervals in (2.14)	
Age	Lower	Upper	Lower	Upper
20	0.01491	0.12012	-0.0023	0.08921
23	0.02302	0.14573	0.00413	0.11512
24	0.02657	0.15529	0.00724	0.12506
25	0.03065	0.16538	0.01097	0.13571
26	0.03531	0.17604	0.01540	0.14709
28	0.04673	0.19913	0.02672	0.17212
29	0.05365	0.21161	0.03381	0.18580
30	0.06149	0.22473	0.04197	0.20028
32	0.08037	0.25302	0.06195	0.23163
33	0.09158	0.26823	0.07395	0.24852
34	0.10410	0.28418	0.08739	0.26622
35	0.11801	0.30091	0.10235	0.28472
36	0.13336	0.31843	0.11885	0.30402
37	0.15019	0.33679	0.13691	0.32413
38	0.16851	0.35602	0.15650	0.34506
39	0.18828	0.37617	0.17756	0.36682
40	0.20944	0.39726	0.19999	0.38943
41	0.23185	0.41934	0.22363	0.41293
42	0.25535	0.44245	0.24830	0.43733
43	0.27972	0.46659	0.27377	0.46268
44	0.30472	0.49176	0.29981	0.48896
45	0.33011	0.51792	0.32616	0.51617
46	0.35562	0.54495	0.35260	0.54422
47	0.38103	0.57270	0.37895	0.57301
48	0.40615	0.60095	0.40503	0.60235
49	0.43085	0.62943	0.43076	0.63200
50	0.45502	0.65783	0.45607	0.66169
51	0.47859	0.68584	0.48092	0.69111
52	0.50155	0.71314	0.50532	0.71997
53	0.52387	0.73946	0.52927	0.74797
54	0.54555	0.76457	0.55276	0.77485
55	0.56659	0.78826	0.57579	0.80039
56	0.58700	0.81041	0.59835	0.82443
57	0.60678	0.83094	0.62041	0.84682
58	0.62593	0.84982	0.64196	0.86749
59	0.64446	0.86704	0.66294	0.88641
60	0.66236	0.88265	0.68332	0.90357
61	0.67965	0.89672	0.70305	0.91902
62	0.69631	0.90933	0.72210	0.93280
63	0.71236	0.92058	0.74043	0.94500
64	0.72778	0.93058	0.75800	0.95573
65	0.74260	0.93942	0.77479	0.96508
69	0.79583	0.96537	0.83389	0.99103

TABLE IV

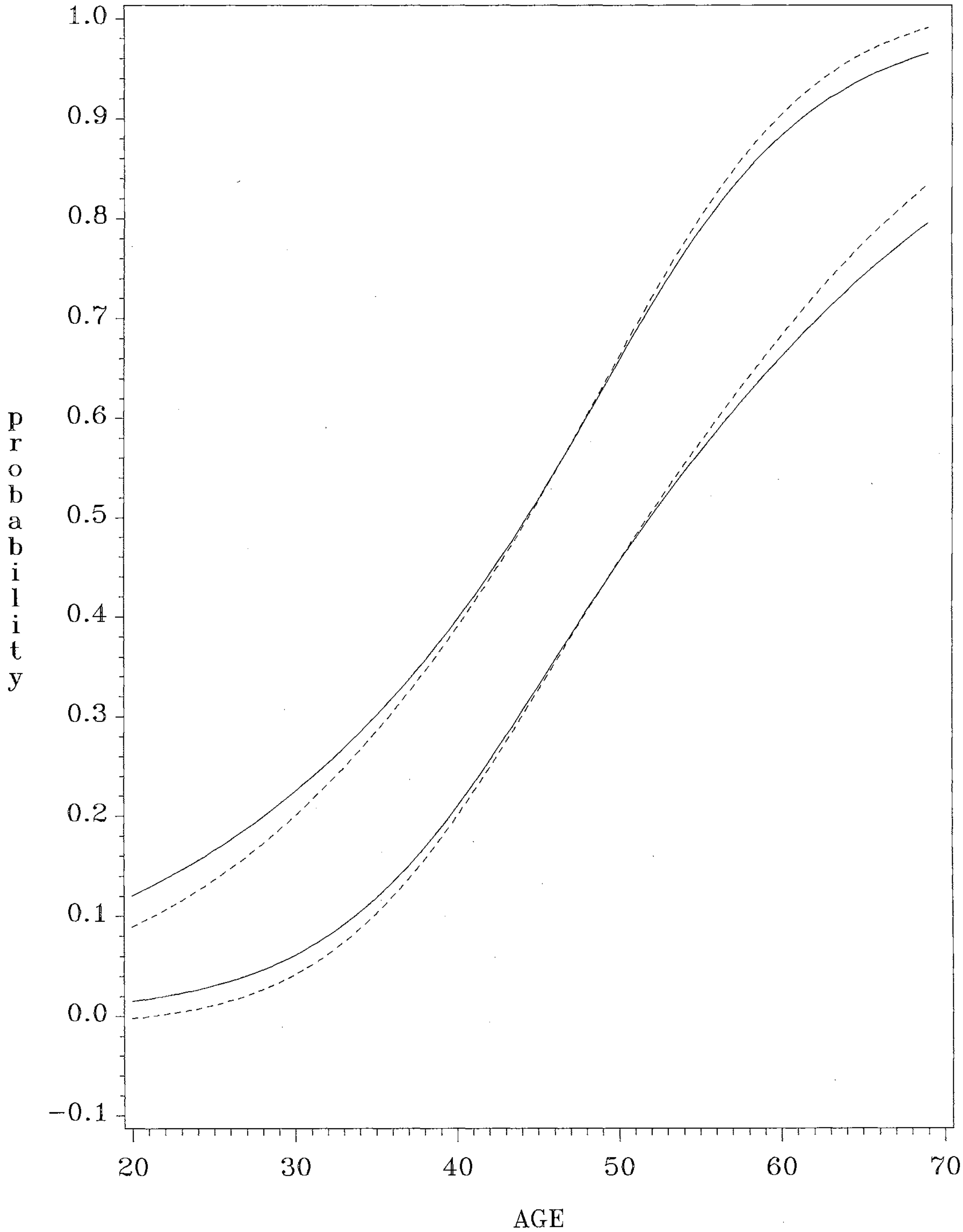
75% APPROXIMATE CONFIDENCE INTERVALS
FOR LOGISTIC REGRESSION

	Intervals in (2.13)		Intervals in (2.14)	
Age	Lower	Upper	Lower	Upper
20	0.02059	0.08949	0.01141	0.07555
23	0.03070	0.11263	0.02071	0.09854
24	0.03503	0.12146	0.02484	0.10746
25	0.03993	0.13091	0.02961	0.11708
26	0.04548	0.14100	0.03508	0.12742
28	0.05880	0.16324	0.04845	0.15040
29	0.06674	0.17544	0.05651	0.16309
30	0.07563	0.18840	0.06562	0.17663
32	0.09665	0.21671	0.08730	0.20629
33	0.10894	0.23210	0.10003	0.22244
34	0.12251	0.24836	0.11411	0.23950
35	0.13742	0.26551	0.12959	0.25747
36	0.15372	0.28355	0.14651	0.27636
37	0.17144	0.30252	0.16488	0.29616
38	0.19057	0.32244	0.18467	0.31689
39	0.21107	0.34331	0.20584	0.33855
40	0.23287	0.36516	0.22829	0.36113
41	0.25587	0.38798	0.25191	0.38465
42	0.27990	0.41179	0.27654	0.40909
43	0.30480	0.43656	0.30199	0.43446
44	0.33035	0.46226	0.32806	0.46070
45	0.35634	0.48882	0.35454	0.48778
46	0.38256	0.51613	0.38123	0.51560
47	0.40879	0.54404	0.40794	0.54402
48	0.43488	0.57236	0.43451	0.57287
49	0.46067	0.60085	0.46082	0.60194
50	0.48605	0.62926	0.48678	0.63097
51	0.51094	0.65730	0.51232	0.65971
52	0.53527	0.68472	0.53739	0.68790
53	0.55900	0.71128	0.56194	0.71530
54	0.58209	0.73676	0.58594	0.74167
55	0.60451	0.76099	0.60934	0.76684
56	0.62623	0.78384	0.63212	0.79065
57	0.64722	0.80522	0.65424	0.81300
58	0.66748	0.82508	0.67565	0.83380
59	0.68698	0.84340	0.69632	0.85303
60	0.70571	0.86021	0.71622	0.87067
61	0.72366	0.87553	0.73531	0.88675
62	0.74082	0.88944	0.75358	0.90132
63	0.75721	0.90201	0.77099	0.91444
64	0.77281	0.91331	0.78754	0.92619
65	0.78764	0.92344	0.80322	0.93666
69	0.83947	0.95408	0.85737	0.96756



______: Intervals in (2.13)
_ _ _ _ _: Intervals in (2.14)

Figure 2. 95% Approximate Confidence Intervals
for Logistic Regression



______: Intervals in (2.13)
_ _ _ _ _: Intervals in (2.14)

Figure 3. 90% Approximate Confidence Intervals for Logistic Regression

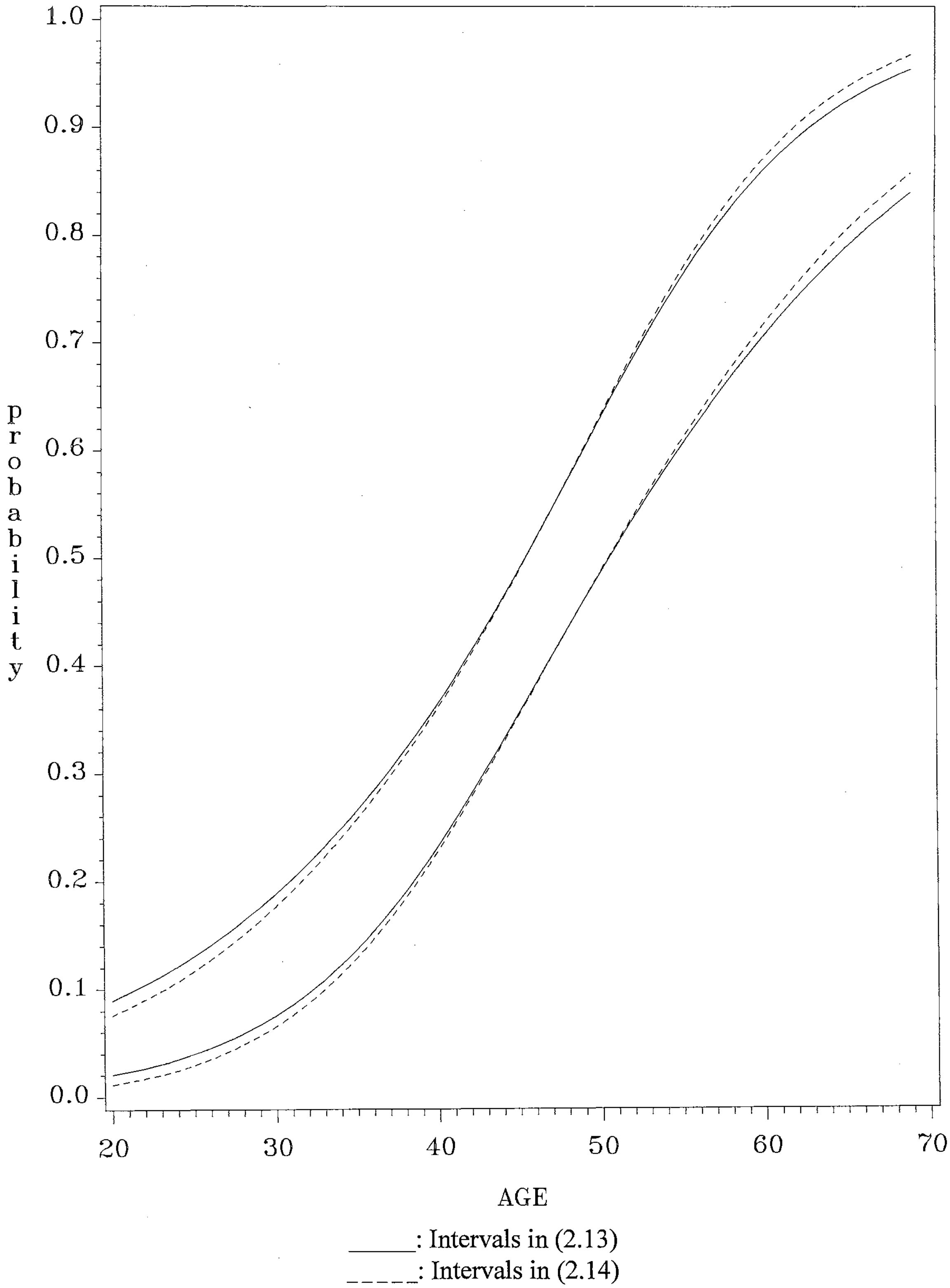


Figure 4. 75% Approximate Confidence Intervals
for Logistic Regression

TABLE V

95% APPORXIMATE CONFIDENCE BANDS
FOR LOGISTIC REGRESSION

	Bands in (2.19)		Bands in (2.29)	
Age	Lower	Upper	Lower	Upper
20	0.00873	0.19002	0.00816	0.20082
23	0.01427	0.21736	0.01282	0.23633
24	0.01678	0.22719	0.01490	0.24909
25	0.01973	0.23741	0.01731	0.26230
26	0.02316	0.24803	0.02011	0.27596
28	0.03182	0.27052	0.02708	0.30455
29	0.03722	0.28242	0.03140	0.31945
30	0.04346	0.29479	0.03638	0.33473
32	0.05892	0.32103	0.04871	0.36633
33	0.06835	0.33496	0.05628	0.38259
34	0.07908	0.34947	0.06494	0.39912
35	0.09120	0.36463	0.07483	0.41588
36	0.10480	0.38046	0.08609	0.43284
37	0.11994	0.39705	0.09887	0.44996
38	0.13666	0.41445	0.11330	0.46719
39	0.15494	0.43274	0.12953	0.48451
40	0.17470	0.45201	0.14771	0.50186
41	0.19581	0.47235	0.16795	0.51921
42	0.21809	0.49383	0.19033	0.53651
43	0.24127	0.51651	0.21493	0.55373
44	0.26505	0.54042	0.24176	0.57082
45	0.28914	0.56551	0.27079	0.58774
46	0.31321	0.59170	0.30191	0.60445
47	0.33699	0.61879	0.33497	0.62093
48	0.36027	0.64651	0.36973	0.63713
49	0.38288	0.67452	0.38508	0.67247
50	0.40474	0.70244	0.39627	0.70977
51	0.42581	0.72988	0.40756	0.74443
52	0.44610	0.75646	0.41895	0.77625
53	0.46562	0.78185	0.43043	0.80515
54	0.48443	0.80579	0.44199	0.83113
55	0.50258	0.82808	0.45361	0.85427
56	0.52011	0.84861	0.46527	0.87472
57	0.53708	0.86733	0.47698	0.89266
58	0.55351	0.88423	0.48871	0.90830
59	0.56945	0.89936	0.50046	0.92186
60	0.58492	0.91282	0.51220	0.93357
61	0.59995	0.92472	0.52393	0.94362
62	0.61456	0.93516	0.53564	0.95224
63	0.62876	0.94429	0.54730	0.95959
64	0.64257	0.95223	0.55891	0.96585
65	0.65600	0.95912	0.57046	0.97117
69	0.70601	0.97838	0.61578	0.98546

TABLE VI

90% APPROXIMATE CONFIDENCE BANDS
FOR LOGISTIC REGRESSION

	Bands in (2.19)		Bands in (2.29)	
Age	Lower	Upper	Lower	Upper
20	0.01066	0.16093	0.01016	0.16763
23	0.01705	0.18810	0.01570	0.20131
24	0.01992	0.19799	0.01814	0.21360
25	0.02325	0.20834	0.02095	0.22644
26	0.02711	0.21914	0.02419	0.23981
28	0.03673	0.24219	0.03220	0.26813
29	0.04267	0.25448	0.03711	0.28306
30	0.04949	0.26729	0.04273	0.29849
32	0.06618	0.29461	0.05652	0.33072
33	0.07628	0.30916	0.06489	0.34749
34	0.08767	0.32435	0.07440	0.36463
35	0.10046	0.34021	0.08518	0.38213
36	0.11473	0.35680	0.09736	0.39994
37	0.13053	0.37415	0.11107	0.41802
38	0.14787	0.39233	0.12644	0.43633
39	0.16674	0.41141	0.14359	0.45480
40	0.18706	0.43144	0.16264	0.47340
41	0.20870	0.45249	0.18367	0.49208
42	0.23147	0.47464	0.20675	0.51078
43	0.25514	0.49793	0.23190	0.52945
44	0.27942	0.52236	0.25911	0.54803
45	0.30403	0.54790	0.28832	0.56649
46	0.32867	0.57446	0.31941	0.58476
47	0.35309	0.60184	0.35218	0.60280
48	0.37709	0.62982	0.38641	0.62056
49	0.40051	0.65807	0.40344	0.65532
50	0.42327	0.68623	0.41681	0.69192
51	0.44531	0.71395	0.43030	0.72625
52	0.46664	0.74088	0.44389	0.75809
53	0.48726	0.76669	0.45757	0.78732
54	0.50721	0.79113	0.47132	0.81388
55	0.52651	0.81401	0.48511	0.83781
56	0.54521	0.83520	0.49892	0.85919
57	0.56333	0.85464	0.51273	0.87816
58	0.58089	0.87231	0.52653	0.89489
59	0.59793	0.88825	0.54028	0.90956
60	0.61445	0.90252	0.55397	0.92236
61	0.63049	0.91522	0.56758	0.93348
62	0.64604	0.92647	0.58109	0.94311
63	0.66111	0.93637	0.59448	0.95142
64	0.67572	0.94505	0.60773	0.95856
65	0.68987	0.95263	0.62082	0.96470
69	0.74198	0.97422	0.67131	0.98155

TABLE VII

75% APPROXIMATE CONFIDENCE BANDS
FOR LOGISTIC REGRESSION

	Bands in (2.19)		Bands in (2.29)	
Age	Lower	Upper	Lower	Upper
20	0.01467	0.12185	0.01426	0.12498
23	0.02269	0.14757	0.02146	0.15491
24	0.02621	0.15715	0.02457	0.16610
25	0.03024	0.16727	0.02813	0.17794
26	0.03487	0.17794	0.03218	0.19043
28	0.04619	0.20106	0.04203	0.21739
29	0.05306	0.21354	0.04799	0.23187
30	0.06086	0.22666	0.05473	0.24700
32	0.07963	0.25492	0.07099	0.27921
33	0.09079	0.27011	0.08070	0.29624
34	0.10325	0.28604	0.09161	0.31387
35	0.11711	0.30273	0.10382	0.33204
36	0.13241	0.32022	0.11746	0.35073
37	0.14919	0.33854	0.13261	0.36989
38	0.16747	0.35773	0.14940	0.38947
39	0.18720	0.37783	0.16789	0.40941
40	0.20832	0.39888	0.18817	0.42965
41	0.23070	0.42092	0.21028	0.45013
42	0.25416	0.44398	0.23424	0.47078
43	0.27851	0.46809	0.26003	0.49153
44	0.30348	0.49323	0.28759	0.51232
45	0.32883	0.51936	0.31682	0.53305
46	0.35430	0.54637	0.34758	0.55368
47	0.37967	0.57411	0.37966	0.57412
48	0.40474	0.60235	0.41282	0.59431
49	0.42938	0.63082	0.43244	0.62791
50	0.45348	0.65922	0.44923	0.66307
51	0.47699	0.68722	0.46613	0.69651
52	0.49987	0.71451	0.48311	0.72799
53	0.52212	0.74082	0.50013	0.75735
54	0.54372	0.76589	0.51715	0.78447
55	0.56468	0.78955	0.53414	0.80933
56	0.58502	0.81166	0.55104	0.83193
57	0.60472	0.83215	0.56782	0.85235
58	0.62381	0.85097	0.58445	0.87067
59	0.64228	0.86813	0.60089	0.88702
60	0.66013	0.88368	0.61711	0.90153
61	0.67737	0.89769	0.63307	0.91436
62	0.69399	0.91023	0.64874	0.92566
63	0.71001	0.92142	0.66410	0.93557
64	0.72541	0.93135	0.67912	0.94424
65	0.74021	0.94013	0.69377	0.95180
69	0.79345	0.96585	0.74843	0.97335

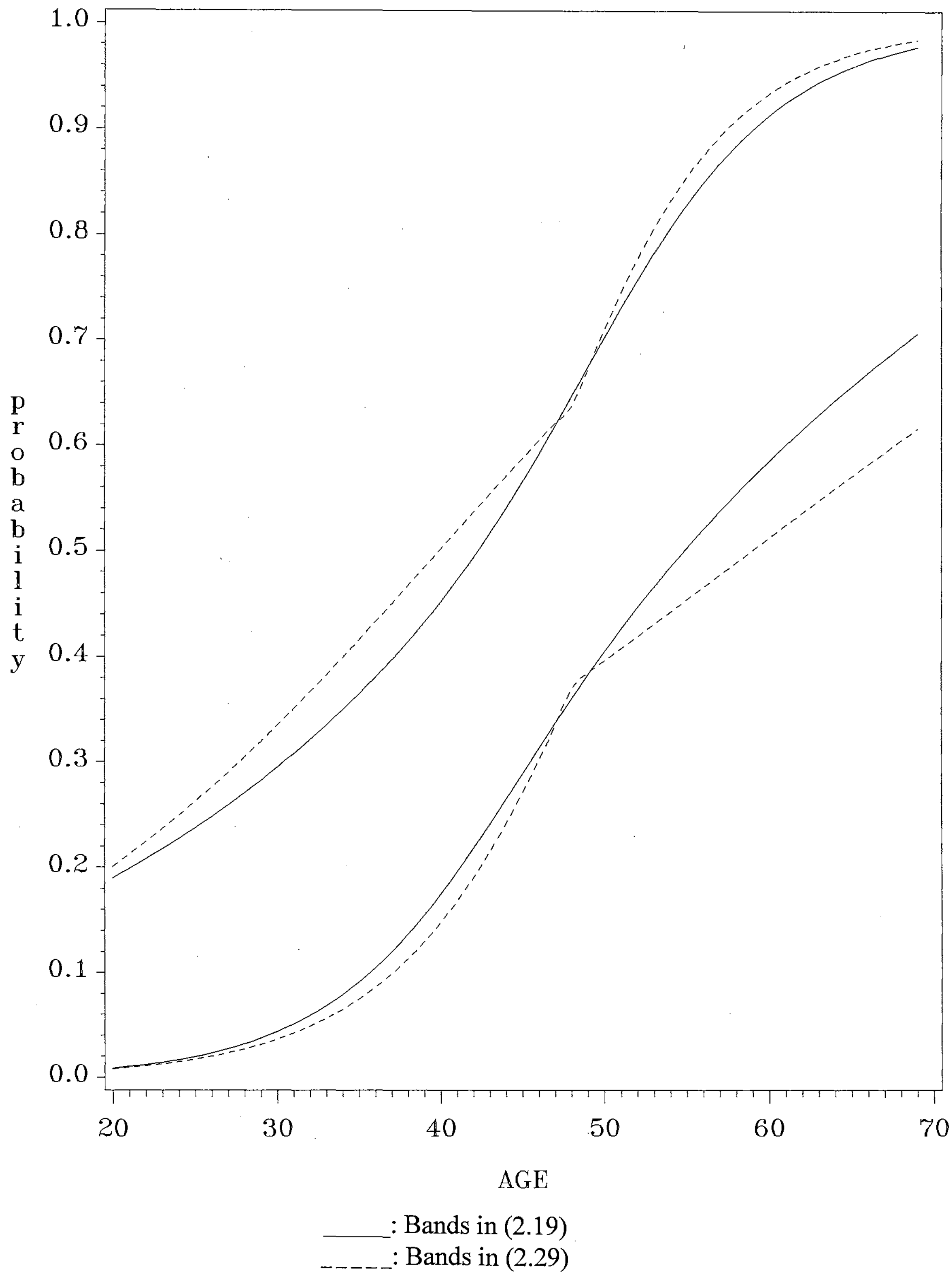


Figure 5. 95% Approximate Confidence Bands
for Logistic Regression

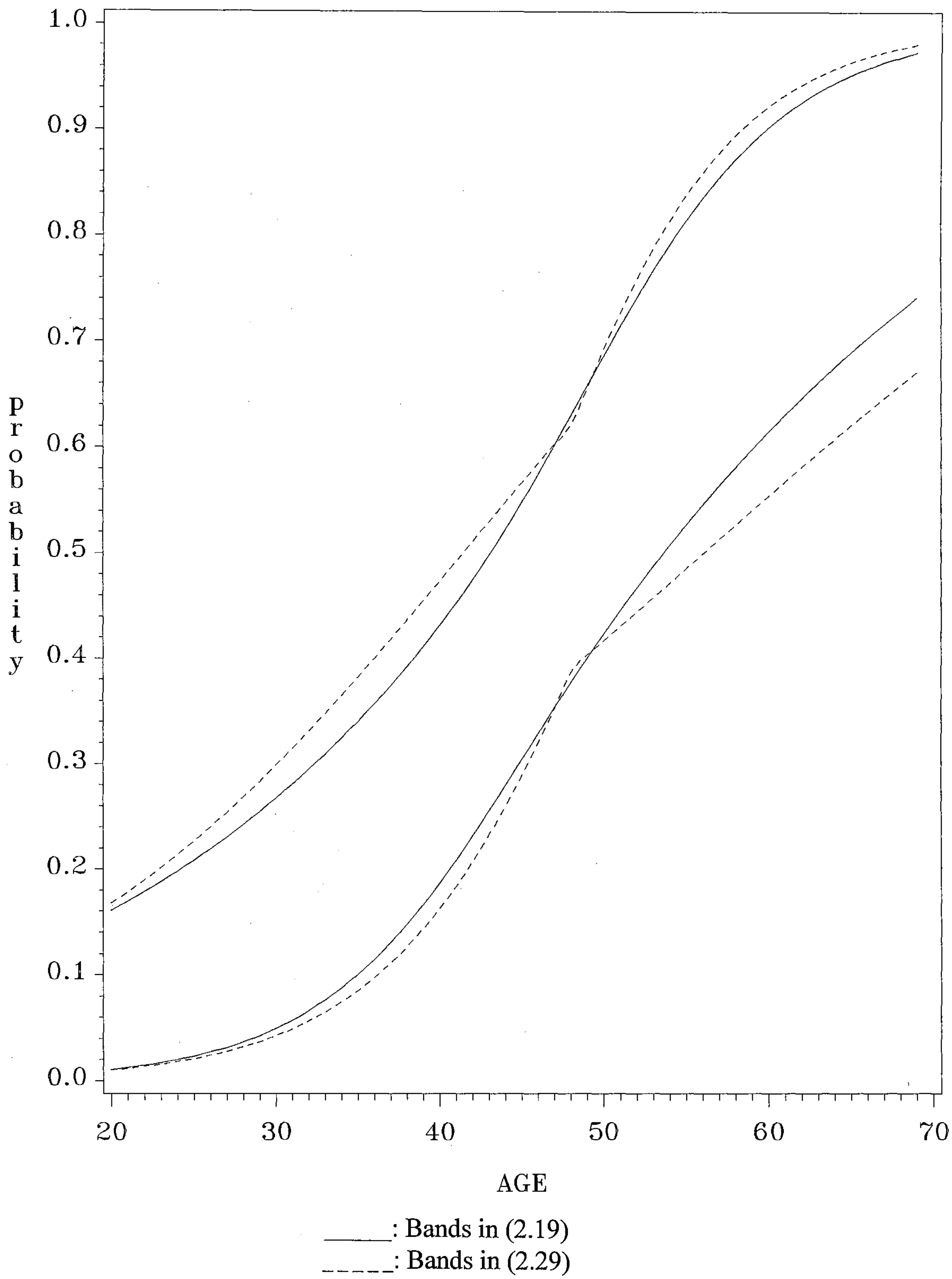


Figure 6. 90% Approximate Confidence Bands for Logistic Regression

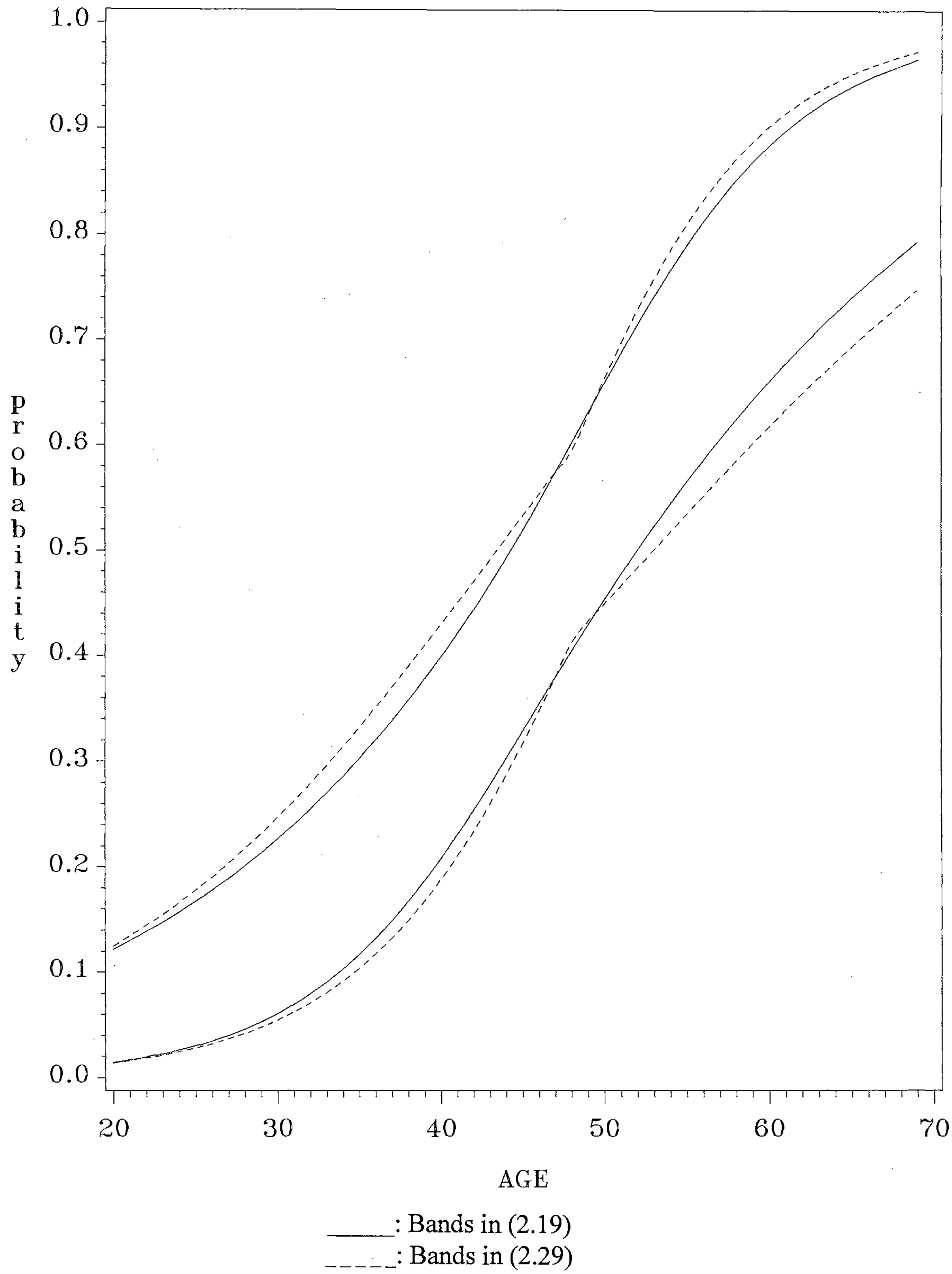


Figure 7. 75% Approximate Confidence Bands
for Logistic Regression

CHAPTER III

LOGISTIC MODELS FOR CATEGORICAL DATA

Logistic Model for *I*x*J*x2 Tables

Suppose there are two factors, A and B, for the binary response. Let the response variable, y_{ijk}, $i = 1, 2, \cdots, I$, $j = 1, 2, \cdots, J$ and $k = 1, 2, \cdots, m_{ij}$, be values 0 and 1.

Let μ denote the effect of the general mean. Let α_i be the effect of factor A and β_j be the effect of factor B. We assume that $\sum_{i=1}^{I} \alpha_i = 0$ and $\sum_{j=1}^{J} \beta_j = 0$.

Denote by $\pi_{1|ij} = \dfrac{\exp(\mu + \alpha_i + \beta_j)}{1 + \exp(\mu + \alpha_i + \beta_j)}$ the probability of response 1, when factor A is at level *i* and factor B is at level *j*, so $\pi_{1|ij} + \pi_{0|ij} = 1$. The logit model is

$$\ln\left(\frac{\pi_{1|ij}}{1-\pi_{1|ij}}\right) = \ln\left(\frac{\pi_{1|ij}}{\pi_{0|ij}}\right) = \mu + \alpha_i + \beta_j. \tag{3.1}$$

Let $\{n_{1|ij}\}$ denote the number of times response 1 occurs when the factor A is at level *i* and the factor B at level *j*.

Assume that $n_{1|ij} \neq 0$ for all *i* and *j*. Then $\{n_{1|ij}\}$ are independent binomial random variables with parameters $\{\pi_{1|ij}\}$.

To fix ideas, consider the frequency distribution shown in Table VIII and the expected cell probabilities shown in Table IX.

TABLE VIII

FREQUENCY DISTRIBUTION

Level of Factor A	Level of Factor B 1	2	....	J
1	$n_{1\|11}$	$n_{1\|12}$	....	$n_{1\|1J}$
2	$n_{1\|21}$	$n_{1\|22}$	....	$n_{1\|2J}$
.	.	.		.
.	.	.		.
I	$n_{1\|I1}$	$n_{1\|I2}$	....	$n_{1\|IJ}$

TABLE IX

EXPECTED CELL PROBABILITIES

Level of Factor A	Level of Factor B 1	2	....	J
1	$\pi_{1\|11}$	$\pi_{1\|12}$	....	$\pi_{1\|1J}$
2	$\pi_{1\|21}$	$\pi_{1\|22}$	....	$\pi_{1\|2J}$
.	.	.		.
.	.	.		.
I	$\pi_{1\|I1}$	$\pi_{1\|I2}$	....	$\pi_{1\|IJ}$

Maximum Likelihood Estimates in

Logistic Model for *I*x*J*x2 Table

The likelihood function is

$$l(\mu, \alpha_1, \cdots, \alpha_I, \beta_1, \cdots, \beta_J) = \prod_{i=1}^{I} \prod_{j=1}^{J} \binom{m_{ij}}{n_{1|ij}} \left(\pi_{1|ij}\right)^{n_{1|ij}} \left(1 - \pi_{1|ij}\right)^{m_{ij} - n_{1|ij}}. \tag{3.2}$$

The log likelihood function is defined as

$$L = \ln\left[l(\mu, \alpha_1, \cdots, \alpha_I, \beta_1, \cdots, \beta_J)\right]$$

$$= \sum_{i=1}^{I} \sum_{j=1}^{J} \left[\ln\binom{m_{ij}}{n_{1|ij}} + n_{1|ij} \ln\left(\frac{\pi_{1|ij}}{1 - \pi_{1|ij}}\right) + m_{ij} \ln\left(1 - \pi_{1|ij}\right) \right]$$

$$= \sum_{i=1}^{I} \sum_{j=1}^{J} \left[\ln\binom{m_{ij}}{n_{1|ij}} + n_{1|ij}\left(\mu + \alpha_i + \beta_j\right) - m_{ij} \ln\left(1 + \exp\left(\mu + \alpha_i + \beta_j\right)\right) \right]. \tag{3.3}$$

To find the values of $\mu, \alpha_1, \cdots, \alpha_I, \beta_1, \cdots, \beta_J$ that maximize L we differentiate L with respect to $\mu, \alpha_1, \cdots, \alpha_I, \beta_1, \cdots, \beta_J$, respectively, and set the resulting expression equal to zero. The likelihood equations are as follows:

$$\frac{\partial L}{\partial \mu} = \sum_{i=1}^{I} \sum_{j=1}^{J} \left(n_{1|ij} - m_{ij}\, \hat{\pi}_{1|ij}\right) = 0,$$

$$\frac{\partial L}{\partial \alpha_i} = \sum_{j=1}^{J} \left(n_{1|ij} - m_{ij}\, \hat{\pi}_{1|ij}\right) = 0, \text{ for } i = 1, \quad 2, \cdots, I,$$

$$\frac{\partial L}{\partial \beta_j} = \sum_{i=1}^{I} \left(n_{1|ij} - m_{ij}\, \hat{\pi}_{1|ij}\right) = 0, \text{ for } j = 1, \quad 2, \cdots, J,$$

where $\hat{\pi}_{1|ij} = \dfrac{\exp\left(\hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j\right)}{1 + \exp\left(\hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j\right)}$ denotes the maximum likelihood estimate of $\pi_{1|ij}$.

The likelihood equations are nonlinear functions of maximum likelihood estimates $\hat{\mu}, \hat{\alpha}_1, \cdots, \hat{\alpha}_I, \hat{\beta}_1, \cdots, \hat{\beta}_J$, and they require an iterative solution. We can use the Newton-Raphson method to solve the likelihood equations.

Approximate Confidence Region on $\pi_{1|ij}$ for all i and j

Let $\mathbf{F}(\boldsymbol{\pi}) = \left(f(\pi_{1|11})\ f(\pi_{1|12}) \cdots f(\pi_{1|1J}) \cdots\cdots f(\pi_{1|I1})\ f(\pi_{1|I2}) \cdots f(\pi_{1|IJ}) \right)$,

where $f(\pi_{1|ij}) = \ln\left(\dfrac{\pi_{1|ij}}{1-\pi_{1|ij}} \right) = \mu + \alpha_i + \beta_j$ for $i = 1, 2, \cdots, I, \quad j = 1, 2, \cdots, J$.

Let $\boldsymbol{\lambda} = \left(\mu\ \alpha_1 \cdots \alpha_{I-1}\ \beta_1 \cdots \beta_{J-1} \right)$. We assume that $\mathbf{F}(\boldsymbol{\pi}) = \mathbf{X}\boldsymbol{\lambda}$. Then we have a design matrix such as

$$\mathbf{X} = \begin{bmatrix} 1 & 1 & 0 & \cdot & \cdots & 0 & 1 & 0 & \cdot & \cdots & 0 \\ 1 & 1 & 0 & \cdot & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 0 & \cdot & \cdots & 0 & -1 & \cdot & \cdot & \cdots & -1 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdot & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 1 & 0 & \cdots & 0 & -1 & \cdot & \cdot & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & -1 & \cdot & \cdot & \cdots & -1 & 1 & 0 & \cdot & \cdots & 0 \\ 1 & -1 & \cdot & \cdot & \cdots & -1 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdot & \cdot & \cdots & \vdots & \vdots & \cdot & \cdot & \cdots & \vdots \\ 1 & -1 & \cdot & \cdot & \cdots & -1 & -1 & \cdot & \cdot & \cdots & -1 \end{bmatrix}.$$

Denote the maximum likelihood estimator of $\boldsymbol{\lambda}$ by $\hat{\boldsymbol{\lambda}}$. The estimate of α_I can be calculated by $\hat{\alpha}_I = -\hat{\alpha}_1 - \hat{\alpha}_2 - \cdots - \hat{\alpha}_{I-1}$ and similarly $\hat{\beta}_J = -\hat{\beta}_1 - \hat{\beta}_2 - \cdots - \hat{\beta}_{J-1}$. We have, asymptotically,

$$\hat{\boldsymbol{\lambda}} \sim N_{I+J-1}\left(\boldsymbol{\lambda},\ \ \boldsymbol{\Sigma}^{-1}\right).$$

The second partial derivatives of the log likelihood functions are as follows:

$$\frac{\partial^2 L}{\partial \mu^2} = -\sum_{j=1}^{J}\sum_{i=1}^{I}\left(m_{ij}\pi_{1|ij}\pi_{0|ij}\right),$$

$$\frac{\partial^2 L}{\partial \mu \partial \alpha_i} = \frac{\partial^2 L}{\partial \alpha_i^2} = -\sum_{j=1}^{J}\left(m_{ij}\pi_{1|ij}\pi_{0|ij}\right), \text{ for } i = 1, 2, \cdots, I-1,$$

$$\frac{\partial^2 L}{\partial \mu \partial \beta_j} = \frac{\partial^2 L}{\partial \beta_j^2} = -\sum_{i=1}^{I}\left(m_{ij}\pi_{1|ij}\pi_{0|ij}\right), \text{ for } j = 1, 2, \cdots, J-1,$$

$$\frac{\partial^2 L}{\partial \alpha_i \partial \alpha_h} = \frac{\partial^2 L}{\partial \beta_j \partial \beta_k} = 0, \text{ for } i \neq h \text{ and } j \neq k,$$

$$\frac{\partial^2 L}{\partial \alpha_i \partial \beta_j} = -\left(m_{ij}\pi_{1|ij}\pi_{0|ij}\right),\ \ i = 1, 2, \cdots, I-1, j = 1, 2, \cdots, J-1.$$

Since the second partial derivatives of the log likelihood functions are not a function of $\left\{n_{1|ij}\right\}$, the observed and expected second derivative matrix are identical. Thus, the asymptotic estimated covariance matrix is given by

$$\hat{\boldsymbol{\Sigma}}^{-1} = \left\{\mathbf{X}'\, diag\left[m_{ij}\hat{\pi}_{1|ij}\left(1 - \hat{\pi}_{1|ij}\right)\right]\mathbf{X}\right\}^{-1}. \tag{3.4}$$

For large-samples,

$$\left(\hat{\boldsymbol{\lambda}} - \boldsymbol{\lambda}\right)'\hat{\boldsymbol{\Sigma}}\left(\hat{\boldsymbol{\lambda}} - \boldsymbol{\lambda}\right) \sim \chi^2_{I+J-1}.$$

We have

$$\Pr\left\{\left(\hat{\boldsymbol{\lambda}}-\boldsymbol{\lambda}\right)'\hat{\boldsymbol{\Sigma}}\left(\hat{\boldsymbol{\lambda}}-\boldsymbol{\lambda}\right) \le \chi^2_{I+J-1,\alpha}\right\} = 1-\alpha, \tag{3.5}$$

where $\chi^2_{I+J-1,\alpha}$ is the upper a percentage point of the chi-square distribution with I+J - 1 degrees of freedom.

Let $\mathbf{x}'$ be any row vector of a design matrix, $\mathbf{X}$. Applying the Cauchy-Schwartz inequality (Appendix B), we have

$$\frac{\left[\mathbf{x}'\left(\hat{\boldsymbol{\lambda}}-\boldsymbol{\lambda}\right)\right]^2}{\mathbf{x}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}} \le \left(\hat{\boldsymbol{\lambda}}-\boldsymbol{\lambda}\right)'\hat{\boldsymbol{\Sigma}}\left(\hat{\boldsymbol{\lambda}}-\boldsymbol{\lambda}\right) \text{ for all } \mathbf{x}. \tag{3.6}$$

Combining (3.6) and (3.7), we have

$$\Pr\left\{\frac{\left[\mathbf{x}'\left(\hat{\boldsymbol{\lambda}}-\boldsymbol{\lambda}\right)\right]^2}{\mathbf{x}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}} \le \chi^2_{I+J-1,\alpha} \quad \textit{for} \quad \textit{all} \quad \mathbf{x}\right\}$$

$$= \Pr\left\{\left|\mathbf{x}'\left(\hat{\boldsymbol{\lambda}}-\boldsymbol{\lambda}\right)\right| \le \sqrt{\chi^2_{I+J-1,\alpha}\mathbf{x}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}} \quad \textit{for} \quad \textit{all} \quad \mathbf{x}\right\}$$

$$= \Pr\left\{\mathbf{x}'\hat{\boldsymbol{\lambda}} - \sqrt{\chi^2_{I+J-1,\alpha}\mathbf{x}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}} \le \mathbf{x}'\boldsymbol{\lambda} \le \mathbf{x}'\hat{\boldsymbol{\lambda}} + \sqrt{\chi^2_{I+J-1}\mathbf{x}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}} \quad \textit{for} \quad \textit{all} \quad \mathbf{x}\right\}$$

$$= \Pr\left\{\left(\hat{\mu}+\hat{\alpha}_i+\hat{\beta}_j\right) - \sqrt{\chi^2_{(I+J-1),\alpha}\mathbf{x}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}} \le \mu+\alpha_i+\beta_j \le \left(\hat{\mu}+\hat{\alpha}_i+\hat{\beta}_j\right)\right.$$

$$\left. + \sqrt{\chi^2_{(I+J-1),\alpha}\mathbf{x}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}}, \quad \textit{for} \quad \textit{all} \quad \mathbf{x}\right\}$$

$$\ge 1-\alpha.$$

So, 100(1−α)% approximate confidence bands for $\mu+\alpha_i+\beta_j$ for all i and j are

$$[l, \ u] = \left[\mathbf{x}'\hat{\boldsymbol{\lambda}} - \sqrt{\chi^2_{I+J-1}\mathbf{x}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}}, \quad \mathbf{x}'\hat{\boldsymbol{\lambda}} + \sqrt{\chi^2_{I+J-1}\mathbf{x}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}}\right]. \tag{3.7}$$

By taking the inverse logit transform, 100(1−α)% approximate confidence region on $\pi_{1|ij}$ for all i and j is given by

$$\frac{\exp(l)}{1+\exp(l)} \leq \pi_{1|ij} \leq \frac{\exp(u)}{1+\exp(u)}. \tag{3.8}$$

Since $\pi_{0|ij} = 1-\pi_{1|ij}$, the 100(1−α)% approximate confidence region on $\pi_{0|ij}$ for all i and j is given by

$$1-\frac{\exp(u)}{1+\exp(u)} \leq \pi_{0|ij} \leq 1-\frac{\exp(l)}{1+\exp(l)}. \tag{3.9}$$

Numerical Example

A 2x3x2 tables is examined. The data in Table X for illustrating these confidence region calculations came from National Opinion Research Center, 1975 General Social Survey.

Denote by $\pi_{1|ij} = \dfrac{\exp(\mu+\alpha_i+\beta_j)}{1+\exp(\mu+\alpha_i+\beta_j)}$, $i=1,2,\quad j=1,2,3$, the probability of response Agree, when factor sex is at level i and factor education is at level j.

I used SAS/IML (1990) for a computation. The maximum likelihood estimates of μ, α_i and β_j are $\hat{\mu}$= −0.511551, $\hat{\alpha}_1$= −0.01172, $\hat{\alpha}_2$= 0.01172, $\hat{\beta}_1$=1.1312745, $\hat{\beta}_2$= −0.017027 and $\hat{\beta}_3$= −1.1295718.

The estimated asymptotic variance-covariance matrix of maximum likelihood estimators is given by

$$\mathbf{\Sigma}^{-1} = \begin{bmatrix} 0.0201886 & -0.007061 & -0.016718 & -0.017363 \\ -0.007061 & 0.0139763 & 0.0001919 & 0.0014686 \\ -0.016718 & 0.0001919 & 0.0347185 & 0.0166416 \\ -0.017363 & 0.0014686 & 0.0166416 & 0.0223972 \end{bmatrix}.$$

The estimated number of response and the estimated probability of response are in Table XI, and 95%, 90% and 75% approximate confidence regions on $\pi_{1|ij}$ for all i and j are in Table XII.

TABLE X

SUBJECT IN 1975 GENERAL SOCIAL SURVEY, CROSS-CLASSIFIED BY ATTITUDE TOWARD WOMEN STAYING AT HOME, SEX OF RESPONDENT, AND EDUCATION OF RESPONDENT

Sex of respondent	Education of respondent, yrs	Response: Agree No.	Response: Agree Prob.	Response: Disagree No.	Response: Disagree Prob.	Total No.
Male	≤8	72	0.605	47	0.395	119
	9-12	110	0.359	196	0.641	306
	≥13	44	0.197	179	0.803	223
Female	≤8	86	0.694	38	0.306	124
	9-12	173	0.379	283	0.621	456
	≥13	28	0.130	187	0.870	215

TABLE XI

THE ESTIMATED NUMBER OF RESPONSE AND THE ESTIMATED PERCENTAGE OF RESPONSE

Sex of respondent	Education of respondent, yrs	Response: Agree No.	Response: Agree Prob.	Response: Disagree No.	Response: Disagree Prob.	Total No.
Male	≤8	77.05	0.647	41.95	0.353	119
	9-12	112.64	0.368	193.36	0.632	306
	≥13	36.31	0.163	186.69	0.837	223
Female	≤8	80.95	0.653	43.05	0.347	124
	9-12	170.36	0.374	285.64	0.626	456
	≥13	35.69	0.166	179.31	0.834	215

TABLE XII

CONFIDENCE REGIONS ON $\pi_{1|ij}$ FOR ALL i AND j

			Response			
	Sex of	Education of	Agree		Disagree	
1−α	respondent	respondent, yrs	Lower	Upper	Lower	Upper
0.95	Male	≤ 8	0.538	0.743	0.257	0.462
		9–12	0.298	0.445	0.555	0.702
		≥13	0.112	0.231	0.769	0.888
	Female	≤ 8	0.545	0.747	0.253	0.455
		9–12	0.312	0.439	0.561	0.688
		≥13	0.114	0.236	0.764	0.886
0.90	Male	≤ 8	0.549	0.735	0.265	0.451
		9–12	0.304	0.437	0.563	0.696
		≥13	0.116	0.224	0.776	0.884
	Female	≤ 8	0.555	0.739	0.261	0.444
		9–12	0.318	0.433	0.567	0.682
		≥13	0.188	0.228	0.771	0.881
0.75	Male	≤ 8	0.566	0.721	0.279	0.434
		9–12	0.314	0.425	0.575	0.686
		≥13	0.123	0.213	0.787	0.877
	Female	≤ 8	0.572	0.725	0.274	0.428
		9–12	0.327	0.423	0.577	0.673
		≥13	0.125	0.216	0.783	0.875

CHAPTER IV

LOGISTIC REGRESSION MODEL FOR BIVARIATE BINARY DATA

Consider n bivariate binary observations $\mathbf{Y}_1, \mathbf{Y}_2, \cdots, \mathbf{Y}_n$, where $\mathbf{Y}_i = (Y_{i1}, Y_{i2})$ and Y_{ij}, j=1, 2, are coded 1 and 0 for $i = 1, 2, \cdots, n$. Let $\mathbf{Y}_i$ have associated with it measurement on an explanatory variable x_i. It is assumed that the paired marginal observations Y_{i1} and Y_{i2} may be correlated, but that the vectors $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n$ are independent.

Examples where an analysis of this type may be required include twin studies and ophthalmological data.

The joint probability is decomposed into a product of successive conditional probabilities each of which is assumed to be a univariate logistic regression. Decompose the probability of $\mathbf{Y}_i = (Y_{i1}, \; Y_{i2})$ given x_i into a product of two probabilities:

$$\Pr(\mathbf{Y}_i | x_i) = \Pr(Y_{i1}, Y_{i2} | x_i) = \Pr(Y_{i1} | x_i) \Pr(Y_{i2} | Y_{i1}, x_i) \; . \tag{4.1}$$

The decomposition (4.1) is always valid. It is however important that only one order is assumed for all independent subsets of data because a different order generally implies a different model and the joint probabilities are not necessarily the same for different orders. The positions of Y_{i1} and Y_{i2} can be interchanged without changing their joint probability if, and only if, $\pi_{i10} = \pi_{i01}$.

We define that

$$\Pr(Y_{i1} = 1|x_i) = \frac{\exp(\beta_0 + \beta_1 x_i)}{1 + \exp(\beta_0 + \beta_1 x_i)} \tag{4.2}$$

and

$$\Pr(Y_{i2} = 1|Y_{i1}, x_i) = \frac{\exp(\beta_0 + \gamma Z + \beta_1 x_i)}{1 + \exp(\beta_0 + \gamma Z + \beta_1 x_i)} \tag{4.3}$$

where Z is a known linear function of Y_{i1} and although other definitions are possible, we shall let $Z = 2Y_{i1} - 1$.

Let, for s =0,1, and t =0,1,

$$\pi_{ist}(x_i) = \Pr(Y_{i1} = s, Y_{i1} = t|x_i). \tag{4.4}$$

Then combining (4.1) and (4.4), we have four probabilities as follows:

$$\begin{aligned}\pi_{i11}(x_i) &= \Pr(Y_{i1} = 1, Y_{i2} = 1|x_i) \\ &= \Pr(Y_{i1} = 1|x_i)\Pr(Y_{i2} = 1|Y_{i1} = 1, x_i) \\ &= \frac{\exp(\beta_0 + \beta_1 x_i)}{1 + \exp(\beta_0 + \beta_1 x_i)} \frac{\exp(\beta_0 + \gamma + \beta_1 x_i)}{1 + \exp(\beta_0 + \gamma + \beta_1 x_i)}\end{aligned}$$

$$\begin{aligned}\pi_{i10}(x_i) &= \Pr(Y_{i1} = 1, Y_{i2} = 0|x_i) \\ &= \Pr(Y_{i1} = 1|x_i)\Pr(Y_{i2} = 0|Y_{i1} = 1, x_i) \\ &= \frac{\exp(\beta_0 + \beta_1 x_i)}{1 + \exp(\beta_0 + \beta_1 x_i)} \frac{1}{1 + \exp(\beta_0 + \gamma + \beta_1 x_i)}\end{aligned}$$

$$\pi_{i01}(x_i) = \Pr(Y_{i1} = 0, Y_{i2} = 1|x_i)$$

$$= \Pr(Y_{i1} = 0|x_i)\Pr(Y_{i2} = 1|Y_{i1} = 0, x_i)$$

$$= \frac{1}{\exp(\beta_0 + \beta_1 x_i)} \frac{\exp(\beta_0 - \gamma + \beta_1 x_i)}{1 + \exp(\beta_0 - \gamma + \beta_1 x_i)}$$

$$\pi_{i00}(x_i) = \Pr(Y_{i1} = 0, Y_{i2} = 0|x_i)$$

$$= \Pr(Y_{i1} = 0|x_i)\Pr(Y_{i2} = 0|Y_{i1} = 0, x_i)$$

$$= \frac{1}{1 + \exp(\beta_0 + \beta_1 x_i)} \frac{1}{1 + \exp(\beta_0 - \gamma + \beta_1 x_i)}.$$

Thus, we have

$$\pi_{i11}(x_i) + \pi_{i10}(x_i) + \pi_{i01}(x_i) + \pi_{i00}(x_i) = 1.$$

Correspondingly, we can define the i th logit as follows:

$$\lambda_{1i} = \ln\left[\frac{\Pr(Y_{i1} = 1|x_i)}{\Pr(Y_{i1} = 0|x_i)}\right] = \beta_0 + \beta_1 x_i$$

$$\lambda_{2i} = \ln\left[\frac{\Pr(Y_{i2} = 1|Y_{i1} = 1, x_i)}{\Pr(Y_{i2} = 0|Y_{i1} = 1, x_i)}\right] = \beta_0 + \gamma + \beta_1 x_i$$

$$\lambda_{3i} = \ln\left[\frac{\Pr(Y_{i2} = 1|Y_{i1} = 0, x_i)}{\Pr(Y_{i2} = 0|Y_{i1} = 0, x_i)}\right] = \beta_0 - \gamma + \beta_1 x_i.$$

We have therefore transformed the bivariate problem into one of univariate logistic regression.

The logit model can be written in matrix form by setting

$$\boldsymbol{\lambda}' = (\lambda_{11} \quad \lambda_{12} \quad \cdots \quad \lambda_{1n} \quad \lambda_{21} \quad \lambda_{22} \quad \cdots \quad \lambda_{2n} \quad \lambda_{31} \quad \lambda_{32} \quad \cdots \quad \lambda_{3n}),$$

$$\mathbf{X}_1 = \begin{bmatrix} 1 & 0 & x_1 \\ 1 & 0 & x_2 \\ \vdots & \vdots & \vdots \\ 1 & 0 & x_n \end{bmatrix}, \quad \mathbf{X}_2 = \begin{bmatrix} 1 & 1 & x_1 \\ 1 & 1 & x_2 \\ \vdots & \vdots & \vdots \\ 1 & 1 & x_n \end{bmatrix}, \quad \mathbf{X}_3 = \begin{bmatrix} 1 & -1 & x_1 \\ 1 & -1 & x_2 \\ \vdots & \vdots & \vdots \\ 1 & -1 & x_n \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \mathbf{X}_3 \end{bmatrix}, \quad \boldsymbol{\rho} = \begin{bmatrix} \beta_0 \\ \gamma \\ \beta_1 \end{bmatrix}.$$

Then $\boldsymbol{\lambda} = \mathbf{X}\boldsymbol{\rho}$.

Maximum Likelihood Estimates in Logistic Regression Model for Bivariate Binary Data

Let $\{n_{ist}\}$ denote the number of times response $\mathbf{Y}_i = (Y_{i1} = s,\ Y_{i2} = t)$. Then the likelihood function is

$$l(\boldsymbol{\rho}) = \prod_{i=1}^{n} \left[\frac{\exp(\beta_0 + \beta_1 x_i)s}{1 + \exp(\beta_0 + \beta_1 x_i)} \; \frac{\exp(\beta_0 + \gamma Z + \beta_1 x_i)t}{1 + \exp(\beta_0 + \gamma Z + \beta_1 x_i)} \right]^{n_{ist}}$$

where $Z = 2s - 1$.

The log likelihood is

$$L(\boldsymbol{\rho}) = \sum_{i=1}^{n} n_{ist} \{ (\beta_0 + \beta_1 x_i)s - \ln(1 + \exp(\beta_0 + \beta_1 x_i)) + (\beta_0 + \gamma Z + \beta_1 x_i)t$$

$$- \ln(1 + (\beta_0 + \gamma Z + \beta_1 x_i)) \} \tag{4.5}$$

To find the values of $\boldsymbol{\rho} = (\beta_0 \quad \gamma \quad \beta_1)$ that maximize $L(\boldsymbol{\rho})$ we differentiate $L(\boldsymbol{\rho})$ with respect to β_0, γ and β_1 respectively and set the resulting expression equal to zero. These equations are as follows:

$$\frac{\partial L(\boldsymbol{\rho})}{\partial \beta_0} = \sum_{i=1}^{n} n_{ist} \left[s - \frac{\exp(\beta_0 + \beta_1 x_i)}{1 + \exp(\beta_0 + \beta_1 x_i)} + t - \frac{\exp(\beta_0 + \gamma Z + \beta_1 x_i)}{1 + \exp(\beta_0 + \gamma Z + \beta_1 x_i)} \right] = 0$$

$$\frac{\partial L(\boldsymbol{\rho})}{\partial \beta_1} = \sum_{i=1}^{n} n_{ist}\left[x_i s - \frac{x_i \exp(\beta_0 + \beta_1 x_i)}{1+\exp(\beta_0 + \beta_1 x_i)} + x_i t - \frac{x_i \exp(\beta_0 + \gamma Z + \beta_1 x_i)}{1+\exp(\beta_0 + \gamma Z + \beta_1 x_i)} \right] = 0$$

$$\frac{\partial L(\boldsymbol{\rho})}{\partial \gamma} = \sum_{i=1}^{n} n_{ist}\left[Zt - \frac{Z \exp(\beta_0 + \gamma Z + \beta_1 x_i)}{1+\exp(\beta_0 + \gamma Z + \beta_1 x_i)} \right] = 0.$$

Let $\hat{\boldsymbol{\rho}}$ be the maximum likelihood estimator of $\boldsymbol{\rho}$. We have, asymptotically,

$$\hat{\boldsymbol{\rho}} \sim N_3(\boldsymbol{\rho},\ \boldsymbol{\Sigma}^{-1}). \tag{4.6}$$

The estimated asymptotic variance-covariance matrix of $\hat{\boldsymbol{\rho}}$ is given by

$$\hat{\boldsymbol{\Sigma}}^{-1} = \begin{bmatrix} \hat{\Sigma}_{11} & \hat{\Sigma}_{12} & \hat{\Sigma}_{13} \\ \hat{\Sigma}_{21} & \hat{\Sigma}_{22} & \hat{\Sigma}_{23} \\ \hat{\Sigma}_{31} & \hat{\Sigma}_{32} & \hat{\Sigma}_{33} \end{bmatrix}^{-1}$$

where

$$\hat{\Sigma}_{11} = \sum_{i=1}^{n} n_{ist}\left[\frac{\exp(\hat{\beta}_0 + \hat{\beta}_1 x_i)}{[1+\exp(\hat{\beta}_0 + \hat{\beta}_1 x_i)]^2} + \frac{\exp(\hat{\beta}_0 + \hat{\gamma} Z + \hat{\beta}_1 x_i)}{[1+\exp(\hat{\beta}_0 + \hat{\gamma} Z + \hat{\beta}_1 x_i)]^2} \right]$$

$$\hat{\Sigma}_{12} = \hat{\Sigma}_{21} = \sum_{i=1}^{n} n_{ist}\left[\frac{x_i \exp(\hat{\beta}_0 + \hat{\beta}_1 x_i)}{[1+\exp(\hat{\beta}_0 + \hat{\beta}_1 x_i)]^2} + \frac{x_i \exp(\beta_0 + \hat{\gamma} Z + \hat{\beta}_1 x_i)}{[1+\exp(\hat{\beta}_0 + \hat{\gamma} Z + \hat{\beta}_1 x_i)]^2} \right]$$

$$\hat{\Sigma}_{13} = \hat{\Sigma}_{31} = \sum_{i=1}^{n} n_{ist}\left[\frac{Z \exp(\hat{\beta}_0 + \hat{\gamma} Z + \hat{\beta}_1 x_i)}{[1+\exp(\hat{\beta}_0 + \hat{\gamma} Z + \hat{\beta}_1 x_i)]^2} \right]$$

$$\hat{\Sigma}_{22} = \sum_{i=1}^{n} n_{ist}\left[\frac{x_i^2 \exp(\hat{\beta}_0 + \hat{\beta}_1 x_i)}{[1+\exp(\hat{\beta}_0 + \hat{\beta}_1 x_i)]^2} + \frac{x_i^2 \exp(\hat{\beta}_0 + \hat{\gamma} Z + \hat{\beta}_1 x_i)}{[1+\exp(\hat{\beta}_0 + \hat{\gamma} Z + \hat{\beta}_1 x_i)]^2} \right]$$

$$\hat{\Sigma}_{23} = \hat{\Sigma}_{32} = \sum_{i=1}^{n} n_{ist}\left[\frac{x_i Z \exp(\hat{\beta}_0 + \hat{\gamma} Z + \hat{\beta}_1 x_i)}{[1+\exp(\hat{\beta}_0 + \hat{\gamma} Z + \hat{\beta}_1 x_i)]^2} \right]$$

$$\hat{\Sigma}_{33} = \sum_{i=1}^{n} n_{ist}\left[\frac{Z^2 \exp(\hat{\beta}_0 + \hat{\gamma} Z + \hat{\beta}_1 x_i)}{[1+\exp(\hat{\beta}_0 + \hat{\gamma} Z + \hat{\beta}_1 x_i)]^2} \right].$$

Approximate Confidence Bands on $\pi_{ist}(x_i)$

From (4.6) and (4.7), we have an asymptotical chi-square distribution with 3 degrees of freedom such as

$$(\hat{\boldsymbol{\rho}}-\boldsymbol{\rho})'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\rho}}-\boldsymbol{\rho}) \sim \chi_3^2.$$

We have

$$\Pr\left\{(\hat{\boldsymbol{\rho}}-\boldsymbol{\rho})'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\rho}}-\boldsymbol{\rho}) \le \chi_{3,\alpha}^2\right\} = 1-\frac{\alpha}{2}, \tag{4.8}$$

where $\chi_{3,\alpha}^2$ is a number such that $\int_0^{\chi_{3,\alpha}^2} \frac{1}{2^{3/2}\Gamma(3/2)} \upsilon^{(3/2)-1} e^{-\upsilon/2} d\upsilon = 1-\frac{\alpha}{2}$.

Applying the Cauchy-Schwartz inequality (Appendix B), we have, for any row vector $\mathbf{x}_1{}'$ of $\mathbf{X}_1$,

$$\frac{[\mathbf{x}_1{}'(\hat{\boldsymbol{\rho}}-\boldsymbol{\rho})]^2}{\mathbf{x}_1{}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}_1} \le (\hat{\boldsymbol{\rho}}-\boldsymbol{\rho})'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\rho}}-\boldsymbol{\rho}), \tag{4.9}$$

Combining (4.8) and (4.9), we have

$$1-\frac{\alpha}{2} \le \Pr\left\{\frac{[\mathbf{x}_1{}'(\hat{\boldsymbol{\rho}}-\boldsymbol{\rho})]^2}{\mathbf{x}_1{}'\hat{\boldsymbol{\Sigma}}\mathbf{x}_1} \le \chi_{3,\alpha}^2 \quad \textit{for all } \mathbf{x}_1\right\}$$

$$= \Pr\left\{\left|\mathbf{x}_1{}'(\hat{\boldsymbol{\rho}}-\boldsymbol{\rho})\right| \le \sqrt{\chi_{3,\alpha}^2 \mathbf{x}_1{}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}_1} \quad \textit{for all } \mathbf{x}_1\right\}$$

$$= \Pr\left\{\mathbf{x}_1{}'\hat{\boldsymbol{\rho}} - \sqrt{\chi_{3,\alpha}^2 \mathbf{x}_1{}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}_1} \le \mathbf{x}_1{}'\boldsymbol{\rho} \le \mathbf{x}_1{}'\hat{\boldsymbol{\rho}} + \sqrt{\chi_{3,\alpha}^2 \mathbf{x}_1{}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}_1} \quad \textit{for all } \mathbf{x}_1\right\}$$

$$= \Pr\left\{l_{1i} \le \beta_0 + \beta_1 x_i \le u_{1i}, \ \textit{for all } x_i\right\}, \tag{4.10}$$

where $l_{1i} = \mathbf{x}_1{}'\hat{\boldsymbol{\rho}} - \sqrt{\chi_{3,\alpha}^2 \mathbf{x}_1{}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}_1}$

$$u_{1i} = \mathbf{x}_1{}'\hat{\boldsymbol{\rho}} + \sqrt{\chi_{3,\alpha}^2 \mathbf{x}_1{}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}_1}\,.$$

Similar to (4.10), we have the probabilities, for any row vector $\mathbf{x}_2{}'$ of $\mathbf{X}_2$,

$$1-\frac{\alpha}{2} \le \Pr\{l_{2i} \le \beta_0 + \gamma + \beta_1 x_i \le u_{2i} \quad for \quad all \quad x_i\}, \tag{4.11}$$

where $$l_{2i} = \mathbf{x}_2{}'\hat{\boldsymbol{\rho}} - \sqrt{\chi^2_{3,\alpha}\mathbf{x}_2{}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}_2}$$

$$u_{2i} = \mathbf{x}_2{}'\hat{\boldsymbol{\rho}} + \sqrt{\chi^2_{3,\alpha}\mathbf{x}_2{}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}_2}$$

and, for any row vector $\mathbf{x}_3{}'$ of $\mathbf{X}_3$,

$$1-\frac{\alpha}{2} \le \Pr\{l_{3i} \le \beta_0 - \gamma + \beta_1 x_1 \le u_{3i} \quad for \quad all \quad x_i\}, \tag{4.12}$$

where $$l_{3i} = \mathbf{x}_3{}'\hat{\boldsymbol{\rho}} - \sqrt{\chi^2_{3,\alpha}\mathbf{x}_3{}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}_3}$$

$$u_{3i} = \mathbf{x}_3{}'\hat{\boldsymbol{\rho}} + \sqrt{\chi^2_{3,\alpha}\mathbf{x}_3{}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}_3}.$$

By taking the inverse logit transform of (4.10), (4.11) and (4.12), we have approximate confidence bands over all x_i as follows:

$$\frac{\exp(l_{1i})}{1+\exp(l_{1i})} \le \frac{\exp(\beta_0+\beta_1 x_i)}{1+\exp(\beta_0+\beta_1 x_i)} \le \frac{\exp(u_{1i})}{1+\exp(u_{1i})}$$

$$\frac{\exp(l_{2i})}{1+\exp(l_{2i})} \le \frac{\exp(\beta_0+\gamma+\beta_1 x_i)}{1+\exp(\beta_0+\gamma+\beta_1 x_i)} \le \frac{\exp(u_{2i})}{1+\exp(u_{2i})}$$

$$\frac{\exp(l_{3i})}{1+\exp(l_{3i})} \le \frac{\exp(\beta_0-\gamma+\beta_1 x_i)}{1+\exp(\beta_0-\gamma+\beta_1 x_i)} \le \frac{\exp(u_{3i})}{1+\exp(u_{3i})}.$$

Thus we have 100(1–α)% approximate confidence bands on $\pi_{ist}(x_i)$ over all x_i given by

$$\frac{\exp(l_{1i})}{1+\exp(l_{1i})}\,\frac{\exp(l_{2i})}{1+\exp(l_{2i})} \le \pi_{i11}(x_i) \le \frac{\exp(u_{1i})}{1+\exp(u_{1i})}\,\frac{\exp(u_{2i})}{1+\exp(u_{2i})}$$

$$\frac{\exp(l_{1i})}{1+\exp(l_{1i})}\,\frac{1}{1+\exp(u_{2i})} \le \pi_{i10}(x_i) \le \frac{\exp(u_{1i})}{1+\exp(u_{1i})}\,\frac{1}{1+\exp(l_{2i})}$$

$$\frac{1}{1+\exp(u_{1i})}\frac{\exp(l_{3i})}{1+\exp(l_{3i})} \le \pi_{i01}(x_i) \le \frac{1}{1+\exp(l_{1i})}\frac{\exp(u_{3i})}{1+\exp(u_{3i})}$$

$$\frac{1}{1+\exp(u_{1i})}\frac{1}{1+\exp(u_{3i})} \le \pi_{i00}(x_i) \le \frac{1}{1+\exp(l_{1i})}\frac{1}{1+\exp(l_{3i})}.$$

CHAPTER V

LOGISTIC MODEL FOR GROWTH CURVE

One of the most versatile and useful models for fitting an S-shape response having a lower asymptote of zero and a finite upper asymptote is the logistic model in the form

$$f(x) = \frac{\beta_0}{1+\exp(\beta_1 - \beta_2 x)}, \quad -\infty < x < +\infty, \quad \beta_0 > 0 \text{ and } \beta_2 > 0 \qquad (5.1)$$

Properties of $f(x)$

(1) $\lim_{x \to +\infty} f(x) = \beta_0$: the upper asymptote.

(2) $\lim_{x \to -\infty} f(x) = 0$: the lower asymptote.

(3) From (1) and (2), $0 \le f(x) \le \beta_0$.

(3) The logistic model (5.1) has

$$\frac{\partial f(x)}{\partial x} = \frac{\beta_0 \beta_1 \exp(\beta_1 - \beta_2 x)}{[1+\exp(\beta_1 - \beta_2 x)]^2}$$

and

$$\frac{\partial^2 f(x)}{\partial x^2} = \frac{\beta_0 \beta_2^2 \exp(\beta_1 - \beta_2 x)[1+\exp(\beta_1 - \beta_2 x)][-1+\exp(\beta_1 - \beta_2 x)]}{[1+\exp(\beta_1 - \beta_2 x)]^4}.$$

Let $\frac{\partial^2 f(x)}{\partial x^2} = 0$. Then $x = \frac{\beta_1}{\beta_2}$.

Thus, the point of inflection of $f(x)$ is $x = \dfrac{\beta_1}{\beta_2}$.

(4) Since $f\left(\dfrac{\beta_1}{\beta_2}\right) = \dfrac{\beta_0}{2}$, the inflection point, $x = \dfrac{\beta_1}{\beta_2}$, has reached one half of its maximum.

Least-Squares Estimates in Logistic Model

for Growth Curve

Suppose that we have n observations $(x_i,\ y_i), i = 1, 2, \cdots, n$. Let

$$y_i = \frac{\beta_0}{1+\exp(\beta_1 - \beta_2 x_i)} + \varepsilon_i. \tag{5.2}$$

Let β_0, β_1 and β_2 be unknown parameters to be estimated. Assuming the ε_i, i = 1, 2, ..., n, to be independently and identically normal distribution with mean zero and unknown variance σ^2.

We shall use the notations:

$\boldsymbol{\beta} = (\beta_0\ \ \beta_1\ \ \beta_2)'$: a 3x1 vector.

$f_i(\boldsymbol{\beta}) = \dfrac{\beta_0}{1+\exp(\beta_1 - \beta_2 x_i)}$ for $i = 1, 2, \cdots, n$.

$\mathbf{f}(\boldsymbol{\beta}) = (\mathrm{f}_1(\boldsymbol{\beta})\ \ \mathrm{f}_2(\boldsymbol{\beta})\ \ \cdots\ \ \mathrm{f}_n(\boldsymbol{\beta}))'$: an n x 1 vector. (5.3)

$\mathbf{y} = (\mathrm{y}_1\ \ \mathrm{y}_2\ \ \cdots\ \ \mathrm{y}_n)'$: an n x 1 vector. (5.4)

$\boldsymbol{\varepsilon} = (\varepsilon_1\ \ \varepsilon_2\ \ \cdots\ \ \varepsilon_n)'$: an n x 1 vector.

J(**b**) is a 3xn matrix such that

$$\mathbf{J}(\boldsymbol{\beta}) = \begin{bmatrix} \frac{1}{1+\exp(\beta_1 - \beta_2 x_1)} & \frac{-\beta_0 \exp(\beta_1 - \beta_2 x_i)}{[1+\exp(\beta_1 - \beta_2 x_1)]^2} & \frac{\beta_0 x_1 \exp(\beta_1 - \beta_2 x_1)}{[1+\exp(\beta_1 - \beta_2 x_1)]^2} \\ \vdots & \vdots & \vdots \\ \frac{1}{1+\exp(\beta_1 - \beta_2 x_n)} & \frac{-\beta_0 \exp(\beta_1 - \beta_2 x_n)}{[1+\exp(\beta_1 - \beta_2 x_n)]^2} & \frac{\beta_0 x_n \exp(\beta_1 - \beta_2 x_n)}{[1+\exp(\beta_1 - \beta_2 x_n)]^2} \end{bmatrix}. \tag{5.5}$$

The least-squares estimate of $\boldsymbol{\beta}$, denote by $\hat{\boldsymbol{\beta}}$, minimizes the error sum of squares

$$S(\boldsymbol{\beta}) = \sum_{i=1}^{n} [y_i - f_i(\boldsymbol{\beta})]^2 = [\mathbf{y} - \mathbf{f}(\boldsymbol{\beta})]'[\mathbf{y} - \mathbf{f}(\boldsymbol{\beta})]. \tag{5.6}$$

When each $f_i(\boldsymbol{\beta})$ is differentiable with respect to $\boldsymbol{\beta}$, $\hat{\boldsymbol{\beta}}$ will satisfy normalequations,

$$\left.\frac{\partial S(\boldsymbol{\beta})}{\partial \beta_i}\right|_{\boldsymbol{\beta}=\hat{\boldsymbol{\beta}}} = 0, \quad i = 0, 1, 2. \tag{5.7}$$

The normal equations are

$$\sum_{i=1}^{n} \frac{1}{1+\exp(\hat{\beta}_1 - \hat{\beta}_2 x_i)} = 0,$$

$$\sum_{i=1}^{n} \frac{-\hat{\beta}_0 \exp(\hat{\beta}_1 - \hat{\beta}_2 x_i)}{1+\exp(\hat{\beta}_1 - \hat{\beta}_2 x_i)} = 0,$$

$$\sum_{i=1}^{n} \frac{\hat{\beta}_0 x_i \exp(\hat{\beta}_1 - \hat{\beta}_2 x_i)}{[1+\exp(\hat{\beta}_1 - \hat{\beta}_2 x_i)]^2} = 0.$$

Since the normal equations are nonlinear functions of the least-squares estimates $\hat{\boldsymbol{\beta}}$, these equations need an iterative method to be solved.

Gauss-Newton Method

The Gauss-Newton method is a method for solving nonlinear equations. We may expand $f_i(\boldsymbol{\beta})$ in a Taylor series about $\boldsymbol{\beta}^{(t)}$ by terms up to first order,

$$\mathbf{f}(\boldsymbol{\beta}) \approx \mathbf{f}(\boldsymbol{\beta}^{(t)}) + \mathbf{J}(\boldsymbol{\beta}^{(t)})(\boldsymbol{\beta} - \boldsymbol{\beta}^{(t)}).$$

From (5.3), it follows that

$$\begin{aligned}[\mathbf{y} - \mathbf{f}(\boldsymbol{\beta})]'[\mathbf{y} - \mathbf{f}(\boldsymbol{\beta})] &\cong Q^{(t)}(\boldsymbol{\beta}) \\ &= [\mathbf{y} - \mathbf{f}(\boldsymbol{\beta}^{(t)})]'[\mathbf{y} - \mathbf{f}(\boldsymbol{\beta}^{(t)})] - 2[\mathbf{y} - \mathbf{f}(\boldsymbol{\beta})]'\mathbf{J}(\boldsymbol{\beta}^{(t)})(\boldsymbol{\beta} - \boldsymbol{\beta}^{(t)}) \\ &\quad + (\boldsymbol{\beta} - \boldsymbol{\beta}^{(t)})\mathbf{J}'(\boldsymbol{\beta}^{(t)})\mathbf{J}(\boldsymbol{\beta}^{(t)})(\boldsymbol{\beta} - \boldsymbol{\beta}^{(t)}).\end{aligned}$$

Solving

$$\frac{\partial Q^{(t)}(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} = -2\mathbf{J}'(\boldsymbol{\beta}^{(t)})[\mathbf{y} - \mathbf{f}(\boldsymbol{\beta}^{(t)})]' + 2\mathbf{J}'(\boldsymbol{\beta}^{(t)})\mathbf{J}(\boldsymbol{\beta}^{(t)})(\boldsymbol{\beta} - \boldsymbol{\beta}^{(t)}) = \mathbf{0}$$

for $\boldsymbol{\beta}$ yields the next guess,

$$\boldsymbol{\beta}^{(t+1)} = \boldsymbol{\beta}^{(t)} + [\mathbf{J}'(\boldsymbol{\beta}^{(t)})\mathbf{J}(\boldsymbol{\beta}^{(t)})]^{-1}\mathbf{J}'(\boldsymbol{\beta}^{(t)})[\mathbf{y} - \mathbf{f}(\boldsymbol{\beta}^{(t)})]. \tag{5.8}$$

Approximate Confidence Intervals for the Logistic Model for Growth Curve

The linearization method and the maximum likelihood method are used for finding confidence intervals for the logistic model.

Linearization Method

The least squares estimator is asymptotically normally distributed with mean vector $\boldsymbol{\beta}$ and covariance matrix $\sigma^2[\mathbf{J}'(\boldsymbol{\beta})\mathbf{J}(\boldsymbol{\beta})]^{-1}$.

That is,

$$\hat{\boldsymbol{\beta}} \sim N_3(\boldsymbol{\beta},\ \sigma^2[\mathbf{J}'(\boldsymbol{\beta})\mathbf{J}(\boldsymbol{\beta})]^{-1}). \tag{5.9}$$

The estimated asymptotic covariance matrix of $\hat{\boldsymbol{\beta}}$ is

$$\hat{\mathrm{V}}(\hat{\boldsymbol{\beta}}) = \mathrm{s}^2[\mathbf{J}'(\hat{\boldsymbol{\beta}})\mathbf{J}(\hat{\boldsymbol{\beta}})]^{-1},$$

where $s^2 = \dfrac{S(\hat{\boldsymbol{\beta}})}{n-3}$.

Let $g(\boldsymbol{\beta}) = \dfrac{\beta_0}{1+\exp(\beta_1 - \beta_2 x_i)}$ and x_i be fixed. Since, for large n, $\hat{\boldsymbol{\beta}}$ is close to the true $\boldsymbol{\beta}$, the linear Taylor expansion gives

$$g(\hat{\boldsymbol{\beta}}) \approx g(\boldsymbol{\beta}) + \mathbf{d}'(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}),$$

where $\mathbf{d}' = \left(\dfrac{1}{1+\exp(\beta_1 - \beta_2 x_i)} \quad \dfrac{-\beta_0 \exp(\beta_1 - \beta_2 x_i)}{[1+\exp(\beta_1 - \beta_2 x_i)]^2} \quad \dfrac{\beta_0 x_i \exp(\beta_1 - \beta_2 x_i)}{[1+\exp(\beta_1 - \beta_2 x_i)]^2} \right)$.

From (5.9), we have

$$E\left[g(\hat{\boldsymbol{\beta}})\right] \approx g(\boldsymbol{\beta}) + \mathbf{d}' E(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) = g(\boldsymbol{\beta}),$$

and

$$\mathrm{V}\left[\mathrm{g}(\hat{\boldsymbol{\beta}})\right] \approx \mathbf{d}' \mathrm{V}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})\mathbf{d} = \sigma^2 \mathbf{d}'[\mathbf{J}'(\boldsymbol{\beta})\mathbf{J}(\boldsymbol{\beta})]^{-1}\mathbf{d}.$$

Hence,

$$g(\hat{\boldsymbol{\beta}}) \sim N(g(\boldsymbol{\beta}),\ \sigma^2 \mathbf{d}'[\mathbf{J}'(\boldsymbol{\beta})\mathbf{J}(\boldsymbol{\beta})]^{-1}\mathbf{d}). \tag{5.10}$$

The asymptotic estimated covariance matrix of $g(\hat{\boldsymbol{\beta}})$ is

$$\hat{\mathrm{V}}\left[\mathrm{g}(\hat{\boldsymbol{\beta}})\right] = \mathrm{s}^2\hat{\mathbf{d}}'\left[\mathbf{J}'(\hat{\boldsymbol{\beta}})\mathbf{J}(\hat{\boldsymbol{\beta}})\right]^{-1}\hat{\mathbf{d}}, \tag{5.11}$$

where $\hat{\mathbf{d}}' = \mathbf{d}'|_{\boldsymbol{\beta}=\hat{\boldsymbol{\beta}}}$.

We have, from (5.10) and (5.11),

$$\Pr\left\{-z_{\alpha/2} \le \frac{g(\hat{\boldsymbol{\beta}}) - g(\boldsymbol{\beta})}{s\sqrt{\hat{\mathbf{d}}'[\mathbf{J}'(\hat{\boldsymbol{\beta}})\mathbf{J}(\hat{\boldsymbol{\beta}})]^{-1}\hat{\mathbf{d}}}} \le z_{\alpha/2}\right\}$$

$$= \Pr\left\{g(\hat{\boldsymbol{\beta}}) - z_{\alpha/2}s\sqrt{\hat{\mathbf{d}}'[\mathbf{J}'(\hat{\boldsymbol{\beta}})\mathbf{J}(\hat{\boldsymbol{\beta}})]^{-1}\hat{\mathbf{d}}} \le g(\boldsymbol{\beta}) \le g(\hat{\boldsymbol{\beta}}) + z_{\alpha/2}s\sqrt{\hat{\mathbf{d}}'[\mathbf{J}'(\hat{\boldsymbol{\beta}})\mathbf{J}(\hat{\boldsymbol{\beta}})]^{-1}\hat{\mathbf{d}}}\right\}$$

$$= 1 - \alpha,$$

where $z_{\alpha/2}$ is the $100(1-\alpha/2)$ percentile point of a standard normal distribution.

The approximate $100(1-\alpha)\%$ confidence interval for

$E(Y_i|x_i) = \dfrac{\beta_0}{1+\exp(\beta_1 - \beta_2 x_i)}$ at a fixed x_i is given by

$$\left[\frac{\hat{\beta}_0}{1+\exp(\hat{\beta}_1 - \hat{\beta}_2 x_i)} - z_{\alpha/2}s\sqrt{\hat{\mathbf{d}}'[\mathbf{J}'(\hat{\boldsymbol{\beta}})\mathbf{J}(\hat{\boldsymbol{\beta}})]^{-1}\hat{\mathbf{d}}},\ \ \frac{\hat{\beta}_0}{1+\exp(\hat{\beta}_1 - \hat{\beta}_2 x_i)} + z_{\alpha/2}s\sqrt{\hat{\mathbf{d}}'[\mathbf{J}'(\hat{\boldsymbol{\beta}})\mathbf{J}(\hat{\boldsymbol{\beta}})]^{-1}\hat{\mathbf{d}}}\right] \tag{5.12}$$

<u>Maximum Likelihood Method</u>

Let the maximum likelihood estimator of $\boldsymbol{\beta}$ be denoted by $\hat{\boldsymbol{\beta}}$ that maximizes the likelihood function. From (5.2), we have a likelihood function such as

$$l(\boldsymbol{\beta}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\sigma^2}} \exp\left[\frac{-1}{2\sigma^2} \left(y_i - \frac{\beta_0}{1+\exp(\beta_1 - \beta_2 x_i)} \right)^2 \right].$$

The log likelihood equals

$$L(\boldsymbol{\beta}) = \ln[l(\boldsymbol{\beta})] = \frac{-n}{2}\ln(2\pi) - \frac{n}{2}\ln\left(\sigma^2\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(y_i - \frac{\beta_0}{1+\exp(\beta_1 - \beta_2 x_i)} \right)^2 .$$

The maximum likelihood estimator is asymptotically normally distributed with mean vector $\boldsymbol{\beta}$ and covariance matrix by $\boldsymbol{\Sigma}^{-1}$.

That is,

$$\hat{\boldsymbol{\beta}} \sim N_3(\boldsymbol{\beta},\ \boldsymbol{\Sigma}^{-1}). \tag{5.13}$$

The estimated asymptotic covariance matrix of $\hat{\boldsymbol{\beta}}$ is

$$\hat{\boldsymbol{\Sigma}}^{-1} = \begin{bmatrix} \hat{\Sigma}_{00} & \hat{\Sigma}_{01} & \hat{\Sigma}_{02} \\ \hat{\Sigma}_{10} & \hat{\Sigma}_{11} & \hat{\Sigma}_{12} \\ \hat{\Sigma}_{20} & \hat{\Sigma}_{21} & \hat{\Sigma}_{22} \end{bmatrix}^{-1} \tag{5.14}$$

The element of $\hat{\boldsymbol{\Sigma}}$ is given by

$$\hat{\Sigma}_{00} = \frac{1}{s^2}\sum_{i=1}^{n}\left(\frac{1}{(1+\varphi)^2} \right)$$

$$\hat{\Sigma}_{01} = \hat{\Sigma}_{10} = \frac{-1}{s^2}\sum_{i=1}^{n}\left(\frac{\hat{\beta}_0 \varphi}{(1+\varphi)^3} \right)$$

$$\hat{\Sigma}_{02} = \hat{\Sigma}_{20} = \frac{1}{s^2}\sum_{i=1}^{n}\left(\frac{\hat{\beta}_0 x_i \varphi}{(1+\varphi)^3} \right)$$

$$\hat{\Sigma}_{11} = \frac{1}{s^2}\sum_{i=1}^{n}\left(\frac{\hat{\beta}_0 x_i \varphi(1-\varphi) + \hat{\beta}_0^2 \varphi(1-2\varphi)}{(1+\varphi)^4} \right)$$

$$\hat{\Sigma}_{12} = \hat{\Sigma}_{21} = \frac{-1}{s^2}\sum_{i=1}^{n}\left(\frac{\hat{\beta}_0^2 x_i \varphi^2}{(1+\varphi)^4} \right)$$

$$\hat{\Sigma}_{22} = \frac{1}{s^2}\sum_{i=1}^{n}\left(\frac{\hat{\beta}_0^2 x_i \varphi(1-\varphi) + \hat{\beta}_0^2 x_i^2 \varphi(1-2\varphi)}{(1+\varphi)^4}\right)$$

where $\varphi = \exp(\hat{\beta}_1 - \hat{\beta}_2 x_i)$ and $s^2 = \frac{S(\hat{\boldsymbol{\beta}})}{n-3}$.

Let $g(\boldsymbol{\beta}) = \frac{\beta_0}{1+\exp(\beta_1 - \beta_2 x_i)}$ and x_i be fixed. Then similar to linearization method, we have, from (5.12),

$$g(\hat{\boldsymbol{\beta}}) \sim N_3\left(g(\boldsymbol{\beta}),\ \mathbf{d}'\boldsymbol{\Sigma}^{-1}\mathbf{d}\right), \tag{5.15}$$

where $\mathbf{d}' = \left(\frac{1}{1+\exp(\beta_1 - \beta_2 x_i)} \quad \frac{-\beta_0 \exp(\beta_1 - \beta_2 x_i)}{[1+\exp(\beta_1 - \beta_2 x_i)]^2} \quad \frac{\beta_0 x_i \exp(\beta_1 - \beta_2 x_i)}{[1+\exp(\beta_1 - \beta_2 x_i)]^2}\right)$.

The estimated asymptotic variance-covariance matrix of $g(\hat{\boldsymbol{\beta}})$ is

$$\hat{\mathrm{V}}\left(g(\hat{\boldsymbol{\beta}})\right) = \hat{\mathbf{d}}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\mathbf{d}}, \tag{5.16}$$

where $\hat{\mathbf{d}} = \mathbf{d}|_{\boldsymbol{\beta}=\hat{\boldsymbol{\beta}}}$.

From (5.15) and (5.16), we have

$$\Pr\left\{-z_{\alpha/2} \le \frac{g(\hat{\boldsymbol{\beta}}) - g(\boldsymbol{\beta})}{\sqrt{\hat{\mathbf{d}}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\mathbf{d}}}} \le z_{\alpha/2}\right\}$$

$$= \Pr\left\{g(\hat{\boldsymbol{\beta}}) - z_{\alpha/2}\sqrt{\hat{\mathbf{d}}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\mathbf{d}}} \le g(\boldsymbol{\beta}) \le g(\hat{\boldsymbol{\beta}}) + z_{\alpha/2}\sqrt{\hat{\mathbf{d}}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\mathbf{d}}}\right\}$$

$$= 1-\alpha,$$

where $z_{\alpha/2}$ is the $100(1-\alpha/2)$ percentile point of a standard normal distribution.

Therefore, we have an approximate $100(1-\alpha)\%$ confidence interval for $E\left(Y_i \middle| x_i\right) = \frac{\beta_0}{1+\exp(\beta_1 - \beta_2 x_i)}$ at a fixed x_i given by

$$\left[\frac{\hat{\beta}_0}{1+\exp(\hat{\beta}_1-\hat{\beta}_2 x_i)} - z_{\alpha/2}\sqrt{\hat{\mathbf{d}}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\mathbf{d}}},\ \frac{\hat{\beta}_0}{1+\exp(\hat{\beta}_1-\hat{\beta}_2 x_i)} + z_{\alpha/2}\sqrt{\hat{\mathbf{d}}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\mathbf{d}}}\right]. \tag{5.17}$$

Approximate Prediction Intervals for

Logistic Model for Growth Curve

We can apply existing linear method to finding a prediction interval for y_{n+1} at fixed $x = x_0$ using the asymptotic results (5.9). Let

$$y_{n+1} = \frac{\beta_0}{1+\exp(\beta_1-\beta_2 x_0)} + \varepsilon_{n+1}\ .$$

We assume that $\varepsilon_{n+1} \sim N(0,\ \sigma^2)$ and independent of ε_i , $i = 1, 2, \cdots, n$.

Let $g(x_0;\boldsymbol{\beta}) = \dfrac{\beta_0}{1+\exp(\beta_1-\beta_2 x_0)}$. Since, for large n, $\hat{\boldsymbol{\beta}}$ is close to the true value $\boldsymbol{\beta}$, we have the Taylor expansion

$$g(x_0;\hat{\boldsymbol{\beta}}) \approx g(x_0;\boldsymbol{\beta}) + \mathbf{d}_0'(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta}),$$

where $\mathbf{d}_0' = \left(\dfrac{1}{1+\exp(\beta_1-\beta_2 x_0)}\quad \dfrac{-\beta_0\exp(\beta_1-\beta_2 x_0)}{[1+\exp(\beta_1-\beta_2 x_0)]^2}\quad \dfrac{\beta_0 x_0\exp(\beta_1-\beta_2 x_0)}{[1+\exp(\beta_1-\beta_2 x_0)]^2}\right).$

Hence,

$$y_{n+1} - \hat{y}_{n+1} \approx y_{n+1} - g(x_0;\boldsymbol{\beta}) - \mathbf{d}_0'(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta}) = \varepsilon_0 - \mathbf{d}_0'(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta}). \tag{5.18}$$

From asymptotic result (5.9) and (5.18), we have

$$E(Y_{n+1} - \hat{Y}_{n+1}) \approx E(\varepsilon_{n+1}) - \mathbf{d}_0' E(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta}) = 0,$$

and

$$\mathrm{V}(Y_{n+1} - \hat{Y}_{n+1}) \approx \sigma^2[1 + \mathbf{d}_0'(\mathbf{J}'(\boldsymbol{\beta})\mathbf{J}(\boldsymbol{\beta}))^{-1}\mathbf{d}_0].$$

Thus, asymptotically,

$$y_{n+1} - \hat{y}_{n+1} \sim N(0,\ \sigma^2[1 + \mathbf{d}_0'(\mathbf{J}'(\boldsymbol{\beta})\mathbf{J}(\boldsymbol{\beta}))^{-1}\mathbf{d}_0]). \tag{5.19}$$

The estimated asymptotic covariance matrix of $y_{n+1} - \hat{y}_{n+1}$ is

$$\hat{\mathrm{V}}(Y_{n+1} - \hat{Y}_{n+1}) = s^2[1 + \hat{\mathbf{d}}_0'(\mathbf{J}'(\hat{\boldsymbol{\beta}})\mathbf{J}(\hat{\boldsymbol{\beta}}))^{-1}\hat{\mathbf{d}}_0], \tag{5.20}$$

where $s^2 = \dfrac{S(\hat{\boldsymbol{\beta}})}{n-3}$ and $\hat{\mathbf{d}}_0 = \mathbf{d}_0\big|_{\boldsymbol{\beta}=\hat{\boldsymbol{\beta}}}$.

We have, from (5.19) and (5.20),

$$\Pr\left\{-z_{\alpha/2} \le \frac{y_{n+1} - \hat{y}_{n+1}}{s\sqrt{1 + \hat{\mathbf{d}}_0'(\mathbf{J}'(\hat{\boldsymbol{\beta}})\mathbf{J}(\hat{\boldsymbol{\beta}}))^{-1}\hat{\mathbf{d}}_0}} \le z_{\alpha/2}\right\}$$

$$= \Pr\left\{\hat{y}_{n+1} - z_{\alpha/2}s\sqrt{1 + \hat{\mathbf{d}}_0'(\mathbf{J}'(\hat{\boldsymbol{\beta}})\mathbf{J}(\hat{\boldsymbol{\beta}}))^{-1}\hat{\mathbf{d}}_0} \le y_{n+1} \le \hat{y}_{n+1} + z_{\alpha/2}s\sqrt{1 + \hat{\mathbf{d}}_0'(\mathbf{J}'(\hat{\boldsymbol{\beta}})\mathbf{J}(\hat{\boldsymbol{\beta}}))^{-1}\hat{\mathbf{d}}_0}\right\}$$

$$= 1 - \alpha,$$

where $z_{\alpha/2}$ is the $100(1-\alpha/2)$ percentile point of a standard normal distribution.

The $100(1-\alpha)\%$ approximate confidence interval for y_{n+1} is

$$\left[\hat{y}_{n+1} - z_{\alpha/2}s\sqrt{1 + \hat{\mathbf{d}}_0'(\mathbf{J}'(\hat{\boldsymbol{\beta}})\mathbf{J}(\hat{\boldsymbol{\beta}}))^{-1}\hat{\mathbf{d}}_0},\ \ \hat{y}_{n+1} + z_{\alpha/2}s\sqrt{1 + \hat{\mathbf{d}}_0'(\mathbf{J}'(\hat{\boldsymbol{\beta}})\mathbf{J}(\hat{\boldsymbol{\beta}}))^{-1}\hat{\mathbf{d}}_0}\right]. \tag{5.21}$$

We can also use another asymptotic result (5.13) to find a prediction interval for y_{n+1} at fixed $x = x_0$. Similar to the above procedure, we have

$$y_{n+1} - \hat{y}_{n+1} \sim N(0,\ \sigma^2(1 + \mathbf{d}_0'\mathbf{H}^{-1}\mathbf{d}_0)), \tag{5.22}$$

where $\mathbf{H}^{-1} = \dfrac{1}{\sigma^2}\boldsymbol{\Sigma}^{-1}$.

The estimated asymptotic covariance matrix of $y_{n+1} - \hat{y}_{n+1}$ is

$$\hat{\mathrm{V}}(Y_{n+1} - \hat{Y}_{n+1}) = s^2(1 + \hat{\mathbf{d}}_0'\hat{\mathbf{H}}^{-1}\hat{\mathbf{d}}_0), \tag{5.23}$$

where $s^2 = \dfrac{S(\hat{\boldsymbol{\beta}})}{n-3}$, $\hat{\mathbf{d}}_0 = \mathbf{d}_0\big|_{\boldsymbol{\beta}=\hat{\boldsymbol{\beta}}}$ and $\hat{\mathbf{H}}^{-1} = \dfrac{1}{s^2}\hat{\boldsymbol{\Sigma}}^{-1}$.

From (5.22) and (5.23), the 100(1–α)% approximate confidence interval for y_{n+1} is given by

$$\left[\hat{y}_{n+1} - z_{\alpha/2}\, s\sqrt{1+\hat{\mathbf{d}}_0'\hat{\mathbf{H}}^{-1}\hat{\mathbf{d}}_0},\ \ \hat{y}_{n+1} + z_{\alpha/2}\, s\sqrt{1+\hat{\mathbf{d}}_0'\hat{\mathbf{H}}^{-1}\hat{\mathbf{d}}_0}\right]. \tag{5.24}$$

Approximate Confidence Bands in Logistic Model for Growth Curve

We consider polynomial approximation to obtain approximate confidence bands in logistic model for growth curve.

Use the logistic model for growth curve given in (5.2). Let the polynomial approximation be

$$y_i \approx \alpha_0 + \alpha_1 x_i + \alpha_2 x_i^2 + \alpha_3 x_i^3 + \varepsilon_i, \tag{5.25}$$

where α_0, α_1, α_2 and α_3 are unknown parameters to be estimated. Let

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix},\ \mathbf{X} = \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & x_n^3 \end{bmatrix},\ \boldsymbol{\alpha} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix},\ \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}.$$

Then model (5.25) can be written in matrix form such as

$$\mathbf{y} \approx \mathbf{X}\boldsymbol{\alpha} + \boldsymbol{\varepsilon}. \tag{5.26}$$

We can use the general multiple linear regression method to obtain approximate confidence bands for the model in (5.26).

Let $\hat{\boldsymbol{\alpha}}$ denote the maximum likelihood estimator of $\boldsymbol{\alpha}$.

We have

$$\hat{\boldsymbol{\alpha}} \sim N_4(\boldsymbol{\alpha},\ \sigma^2(\mathbf{X}'\mathbf{X})^{-1}), \tag{5.27}$$

and

$$\frac{(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha})'(\mathbf{X}'\mathbf{X})(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha})}{4s^2} \sim F_{4,n-4}, \tag{5.28}$$

where $$s^2 = \frac{(\mathbf{y}-\mathbf{X}\hat{\boldsymbol{\alpha}})'(\mathbf{y}-\mathbf{X}\hat{\boldsymbol{\alpha}})}{n-4},$$

$F_{4,n-4}$ is the F-distribution with 4 and n-4 degrees of freedom.

We have two different confidence set of parameters in model (5.26) to obtain approximate confidence bands.

Approximate Confidence Bands using

Elliptical Confidence Set

of Parameters

From (5.28) we have

$$\Pr\left\{(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha})'(\mathbf{X}'\mathbf{X})(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}) \le 4s^2 F^{\alpha}_{4,n-4}\right\} = 1-\alpha, \tag{5.29}$$

where $F^{\alpha}_{4,n-4}$ is the upper 100α percent point of F-distribution with 4 and n-4 degrees of freedom.

Applying the Cauchy-Schwartz inequality we have, for any 4x1 vector $\mathbf{x}\neq\mathbf{0}$,

$$\frac{[\mathbf{x}'(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha})]^2}{\mathbf{x}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}} \le (\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha})'(\mathbf{X}'\mathbf{X})(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}). \tag{5.30}$$

From (5.29) and (5.30), we have

$$\Pr\left\{\left|\mathbf{x}'(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha})\right| \le 2s\sqrt{F_{4,n-4}^{\alpha}\,\mathbf{x}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}} \quad \textit{for all } \mathbf{x}\right\}$$

$$= \Pr\left\{\mathbf{x}'\hat{\boldsymbol{\alpha}} - 2s\sqrt{F_{4,n-4}^{\alpha}\mathbf{x}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}} \le \mathbf{x}'\boldsymbol{\alpha} \le \mathbf{x}'\hat{\boldsymbol{\alpha}} + 2s\sqrt{F_{4,n-4}^{\alpha}\mathbf{x}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}}\right\}$$

$$\ge 1-\alpha.$$

Therefore, 100(1−α)% approximate confidence bands in logistic model for growth curve are given by

$$\left[\mathbf{x}'\hat{\boldsymbol{\alpha}} - 2s\sqrt{F_{4,n-4}^{\alpha}\mathbf{x}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}},\quad \mathbf{x}'\hat{\boldsymbol{\alpha}} + 2s\sqrt{F_{4,n-4}^{\alpha}\mathbf{x}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}}\right]. \tag{5.31}$$

Approximate Confidence Bands using

Rectangular Confidence Set

of Parameters

Let $\boldsymbol{\Lambda} = diag\{\lambda_k\}$ be the diagonal matrix of the eigenvalues of $\mathbf{X}'\mathbf{X}$ and $\mathbf{T}$ be the matrix of corresponding orthonomal eigenvectors. Then $\mathbf{X}'\mathbf{X} = \boldsymbol{\Lambda}\mathbf{T}\boldsymbol{\Lambda}$.

Let $\mathbf{z}_i = \boldsymbol{\Lambda}^{\frac{-1}{2}}\mathbf{T}'\mathbf{x}_i$ and $\boldsymbol{\gamma} = \boldsymbol{\Lambda}^{\frac{1}{2}}\mathbf{T}'\boldsymbol{\alpha}$,

where

$$\mathbf{z}_i = \begin{pmatrix} z_{i0} & z_{i1} & z_{i2} & z_{i3} \end{pmatrix}',$$

$$\mathbf{x}_i = \begin{pmatrix} 1 & x_i & x_i^2 & x_i^3 \end{pmatrix}',$$

$$\boldsymbol{\gamma} = \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}',$$

$\boldsymbol{\Lambda}^{\frac{1}{2}}$ is defined as $diag\{\lambda_k^{\frac{1}{2}}\}$.

Let $\hat{\boldsymbol{\gamma}}$ be the maximum likelihood estimator of $\boldsymbol{\gamma}$. We have

$$E(\hat{\boldsymbol{\gamma}}) = \boldsymbol{\Lambda}^{\frac{1}{2}}\mathbf{T}'\boldsymbol{\alpha} = \boldsymbol{\gamma}$$

and $$\mathrm{V}(\hat{\boldsymbol{\gamma}}) = \boldsymbol{\Lambda}^{\frac{1}{2}}\mathbf{T}'\mathrm{V}(\hat{\boldsymbol{\alpha}})\mathbf{T}\boldsymbol{\Lambda}^{\frac{1}{2}} = \sigma^2\mathbf{I},$$

where $\mathbf{I}$ is an 4x4 identity matrix.

Therefore,

$$\hat{\boldsymbol{\gamma}} \sim N_4\left(\boldsymbol{\gamma},\ \sigma^2\mathbf{I}\right). \tag{5.32}$$

Let $\boldsymbol{\theta} = \begin{pmatrix}\theta_0 & \theta_1 & \theta_2 & \theta_3\end{pmatrix}$, where $\theta_k = \dfrac{\hat{\gamma}_k - \gamma_k}{\sigma}$, k=0, 1, 2, 3.

Then

$$\boldsymbol{\theta} \sim N_4(\mathbf{0},\ \mathbf{I}). \tag{5.33}$$

Let $t_k = \dfrac{\hat{\gamma}_k - \gamma_k}{s}$, k=0, 1, 2, 3. Since $\dfrac{(n-4)s^2}{\sigma^2}$ is distributed as a chi-square variable with n-4 degrees of freedom, $\mathbf{t} = \begin{pmatrix}t_0 & t_1 & t_2 & t_3\end{pmatrix}$ is distributed as the 4-variate Student t with n-4 degrees of freedom and its density is given by

$$f\left(t_0,\ t_1,\ t_2,\ t_3\right) = \frac{n-2}{(n-4)(2\pi)^2}\left(1 + \frac{t_0^2 + t_1^2 + t_2^2 + t_3^2}{n-4}\right)^{\frac{-n}{2}},\ -\infty < t_k < +\infty,$$

for k=0, 1, 2, 3.

We have

$$\Pr\left\{-d_\alpha \le t_0 \le d_\alpha, -d_\alpha \le t_1 \le d_\alpha, -d_\alpha \le t_2 \le d_\alpha, -d_\alpha \le t_3 \le d_\alpha\right\} = 1-\alpha, \tag{5.34}$$

where d_α is a number such that

$$\int_{-d_\alpha}^{d_\alpha}\int_{-d_\alpha}^{d_\alpha}\int_{-d_\alpha}^{d_\alpha}\int_{-d_\alpha}^{d_\alpha} f\left(t_0,t_1,t_2,t_3\right)dt_0dt_1dt_2dt_3 = 1-\alpha.$$

The values of d_α are given by Hahn and Hendrickson (1971).

Substituting $t_k = \dfrac{\hat{\gamma}_k - \gamma_k}{s}$, k=0, 1, 2, 3, into (5.34), we have

$$\Pr\{\hat{\gamma}_0 - sd_\alpha \le \gamma_0 \le \hat{\gamma}_0 + sd_\alpha, \hat{\gamma}_1 - sd_\alpha \le \gamma_1 \le \gamma_1 + sd_\alpha,$$

$$\hat{\gamma}_2 - sd_\alpha \le \gamma_2 \le \hat{\gamma}_2 + sd_\alpha, \hat{\gamma}_3 - sd_\alpha \le \gamma_3 \le \hat{\gamma}_3 + sd_\alpha\} = 1-\alpha. \quad (5.35)$$

Equation (5.35) gives the following simultaneous rectangular confidence region on $\boldsymbol{\gamma}$ with confidence coefficient of $1-\alpha$:

$$\hat{\gamma}_0 - sd_\alpha \le \gamma_0 \le \hat{\gamma}_0 + sd_\alpha \quad (5.36)$$

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$$\hat{\gamma}_3 - sd_\alpha \le \gamma_3 \le \hat{\gamma}_3 + sd_\alpha \quad (5.37)$$

When $z_{i0} \ge 0$, from (5.36) we have

$$\hat{\gamma}_0 z_{i0} - sd_\alpha z_{i0} \le \gamma_0 z_{i0} \le \hat{\gamma}_0 z_{i0} + sd_\alpha z_{i0}.$$

When $z_{i0} \le 0$, from (5.36) we have

$$\hat{\gamma}_0 z_{i0} + sd_\alpha z_{i0} \le \gamma_0 z_{i0} \le \hat{\gamma}_0 z_{i0} - sd_\alpha z_{i0}.$$

Thus,

$$\hat{\gamma}_0 z_{i0} - sd_\alpha |z_{i0}| \le \gamma_0 z_{i0} \le \hat{\gamma}_0 z_{i0} + sd_\alpha |z_{i0}|, \text{ for all } z_{i0}. \quad (5.38)$$

Similarly from (5.37) we have

$$\hat{\gamma}_3 z_{i3} - sd_\alpha |z_{i3}| \le \gamma_3 z_{i3} \le \hat{\gamma}_3 z_{i3} + sd_\alpha |z_{i3}|, \text{ for all } z_{i3.} \quad (5.39)$$

Thus we have

$$\hat{\gamma}_0 z_{i0} + \cdots + \hat{\gamma}_3 z_{i3} - sd_\alpha(|z_{i0}| + \cdots + |z_{i3}|) \le \gamma_0 z_{i0} + \cdots + \gamma_3 z_{i3} \le \hat{\gamma}_0 z_{i0} + \cdots \hat{\gamma}_3 z_{i3} + sd_\alpha(|z_{i0}| + \cdots + |z_{i3}|)$$

for all $z_{i0}, \ldots, z_{i1}$.

This is equivalent to writing

$$\mathbf{x}_i'\hat{\boldsymbol{\alpha}} - sd_\alpha(|z_{i0}|+\cdots+|z_{i3}|) \le \mathbf{x}_i'\boldsymbol{\alpha} \le \mathbf{x}_i'\hat{\boldsymbol{\alpha}} + sd_\alpha(|z_{i0}|+\cdots+|z_{i3}|) \text{ for all } \mathbf{x}_i.$$

Thus, $100(1-\alpha)\%$ approximate confidence bands in logistic model for growth curve are given by

$$\left[\mathbf{x}_i'\hat{\boldsymbol{\alpha}} - sd_\alpha(|z_{i0}|+\cdots+|z_{i3}|),\quad \mathbf{x}_i'\hat{\boldsymbol{\alpha}} + sd_\alpha(|z_{i0}|+\cdots+|z_{i3}|)\right]. \tag{5.40}$$

NUMERICAL EXAMPLE

The data in table XIII for illustrating these confidence intervals came from Heyes and Brown (1959). Let y be water content of bean root cell and x be distance from tip. Let us have the logistic model given in (5.2).

Under normality, the least square estimators are the same as the maximum likelihood estimators. Using SAS/IML (1990), we have $\hat{\beta}_0 = 21.509$, $\hat{\beta}_1 = 3.957$ and $\hat{\beta}_2 = 0.622$. Figure 8 shows the fitted logistic curve.

The estimated asymptotic covariance for linearization method is

$$s^2[\mathbf{J}'(\hat{\boldsymbol{\beta}})\mathbf{J}(\hat{\boldsymbol{\beta}})]^{-1} = \begin{bmatrix} 0.1725246 & -0.046794 & -0.011097 \\ -0.046794 & 0.0685794 & 0.0111312 \\ -0.011097 & 0.0111312 & 0.0019894 \end{bmatrix}.$$

The estimated asymptotic covariance matrix for the maximum likelihood method is

$$\boldsymbol{\Sigma}^{-1} = \begin{bmatrix} 0.115423 & 0.0143732 & -0.000166 \\ 0.0143732 & 0.0072261 & 0.0001661 \\ -0.000166 & 0.0001661 & 0.0000297 \end{bmatrix}.$$

The approximate confidence intervals for the logistic model using two different asymptotic covariance matrix are in Table XIV and these confidence intervals are compared using graphs in Figures 9, 10 and 11.

Table XV presents the approximate prediction intervals and Figures 12, 13 and 14 shows these prediction intervals.

The dot line and the solid line in Figures 9 to 14 denote the linearization method and the maximum likelihood method respectively.

Approximate confidence bands for logistic model are in table XVI. The solid line denotes bands given in (5.31) and the dot line denotes bands given in (5.40) in Figures 15 and 16.

TABLE XIII

THE GROWTH OF LEAVES

Distance x_i	Water Content y_i
0.5	1.3
1.5	1.3
2.5	1.9
3.5	3.4
4.5	5.3
5.5	7.1
6.5	10.6
7.5	16.0
8.5	16.4
9.5	18.3
10.5	20.9
11.5	20.5
12.5	21.3
13.5	21.2
14.5	20.9

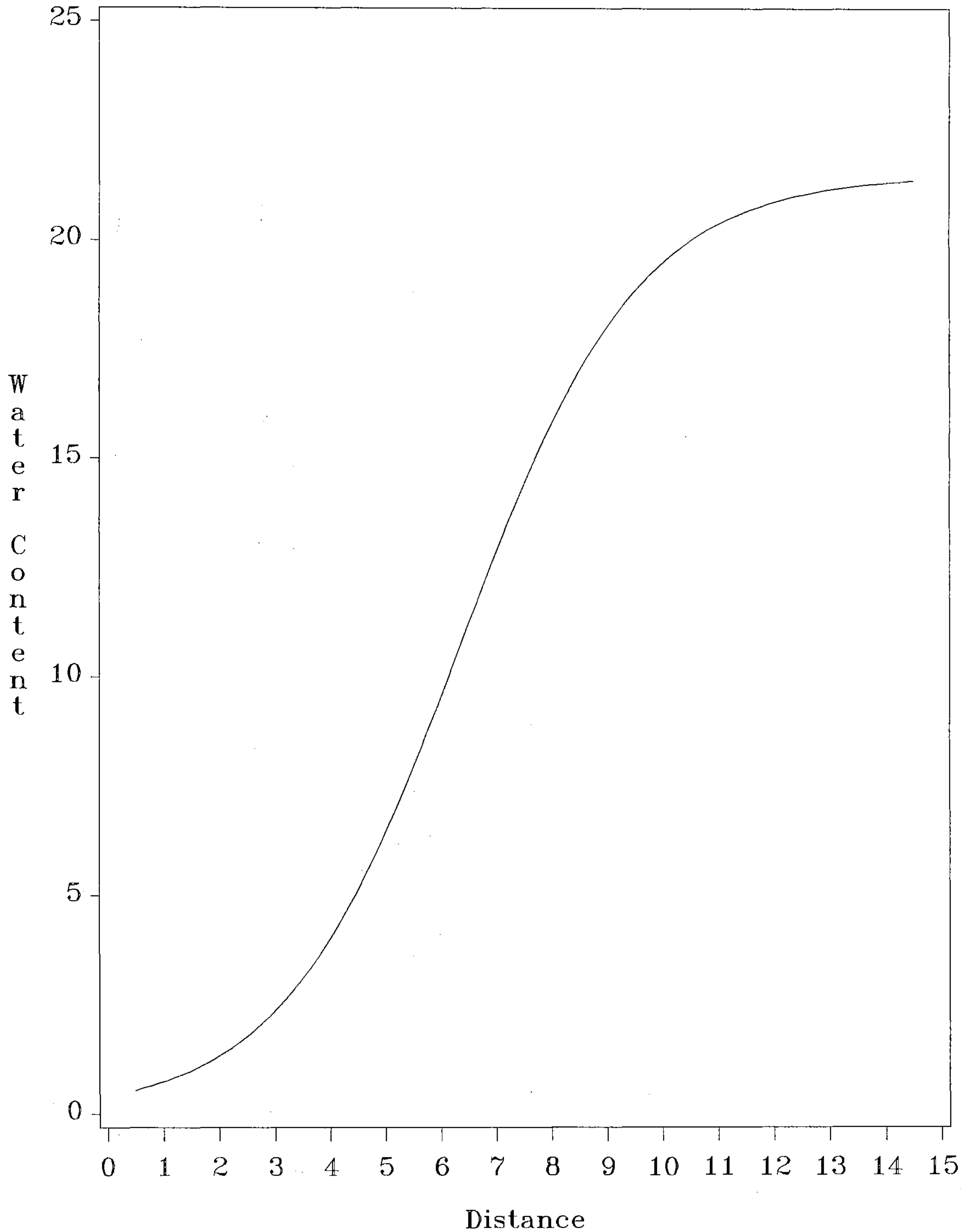


Figure 8. Fitted Logistic Model for Growth Curve

TABLE XIV

APPROXIMATE CONFIDENCE INTERVALS IN LOGISTIC MODEL FOR GROWTH CURVE

1-α	Distance	Intervals in (5.12) Lower	Intervals in (5.12) Upper	Intervals in (5.17) Lower	Intervals in (5.17) Upper
0.95	0.5	0.466	0.627	0.286	0.808
	1.5	0.857	1.137	0.611	1.383
	2.5	1.551	2.021	1.251	2.320
	3.5	2.732	3.478	2.438	3.771
	4.5	4.606	5.680	4.418	5.869
	5.5	7.273	8.612	7.238	8.648
	6.5	10.526	11.917	10.506	11.937
	7.5	13.810	15.020	13.654	15.176
	8.5	16.520	17.510	16.302	17.728
	9.5	18.369	19.307	18.242	19.434
	10.5	19.477	20.499	19.455	20.521
	11.5	20.102	21.227	20.100	21.230
	12.5	20.447	21.648	20.414	21.681
	13.5	20.635	21.883	20.563	21.955
	14.5	20.736	22.736	20.633	22.115
0.90	0.5	0.480	0.614	0.329	0.765
	1.5	0.880	1.114	0.674	1.320
	2.5	1.589	1.983	1.339	2.233
	3.5	2.793	3.417	2.547	3.663
	4.5	4.694	5.593	4.536	5.750
	5.5	7.383	8.503	7.353	8.533
	6.5	10.639	11.803	10.623	11.820
	7.5	13.909	14.922	13.779	15.052
	8.5	16.600	17.429	16.418	17.611
	9.5	18.446	19.230	18.339	19.337
	10.5	19.560	20.415	19.542	20.434
	11.5	20.194	21.136	20.192	21.137
	12.5	20.545	21.550	20.517	21.578
	13.5	20.737	21.781	20.676	21.841
	14.5	20.840	21.907	20.754	21.994
0.75	0.5	0.500	0.594	0.394	0.700
	1.5	0.915	1.079	0.770	1.224
	2.5	1.648	1.924	1.472	2.100
	3.5	2.886	3.324	2.713	3.496
	4.5	4.828	5.458	4.718	5.569
	5.5	7.550	8.336	7.529	8.356
	6.5	10.813	11.629	10.801	11.641
	7.5	14.060	14.770	13.969	14.862
	8.5	16.724	17.305	16.596	17.433
	9.5	18.563	19.113	18.488	19.188
	10.5	19.688	20.288	19.675	20.301
	11.5	20.335	20.995	20.333	20.996
	12.5	20.695	21.400	20.676	21.419
	13.5	20.893	21.625	20.850	21.667
	14.5	20.100	21.748	20.939	21.809

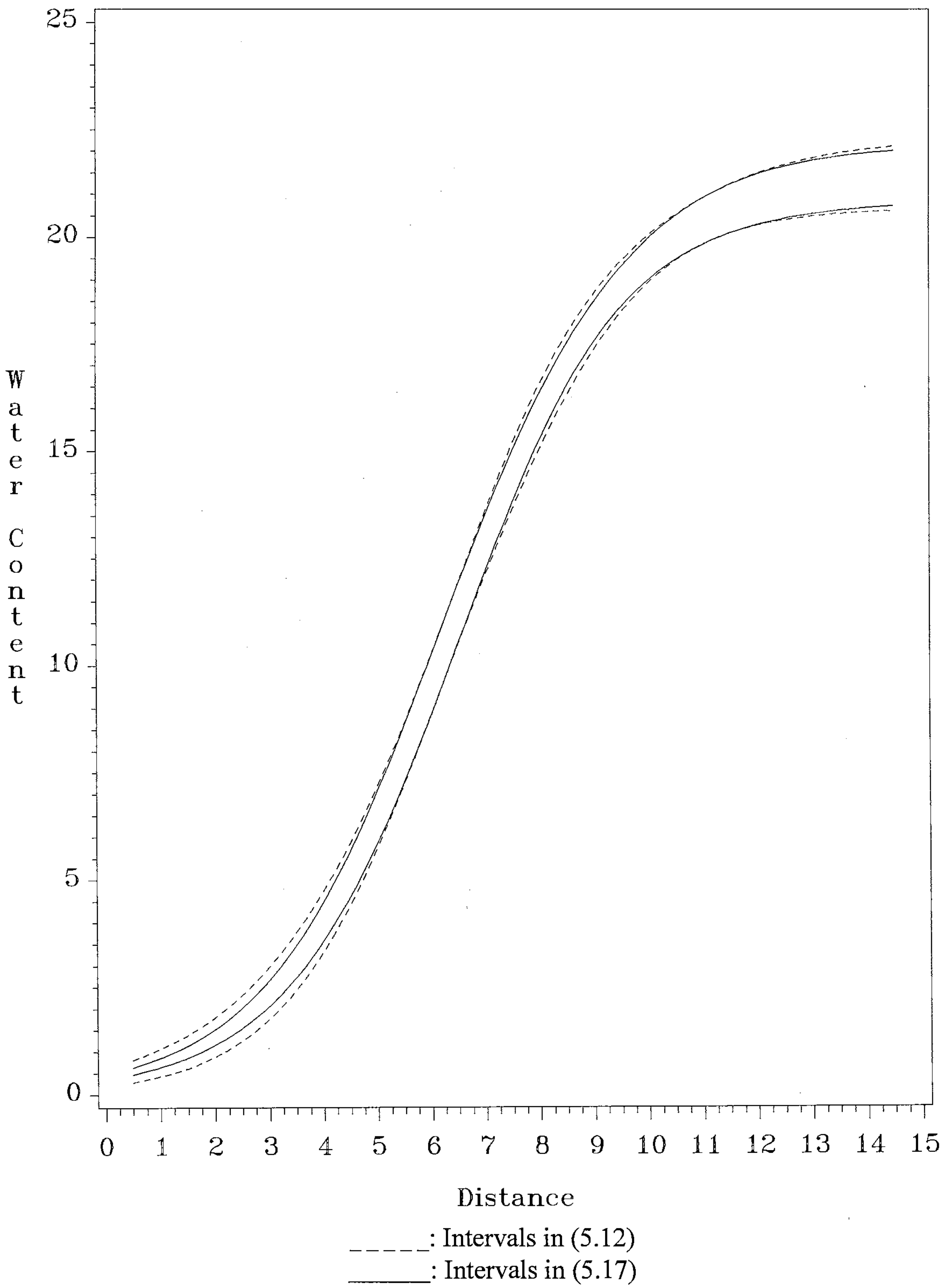
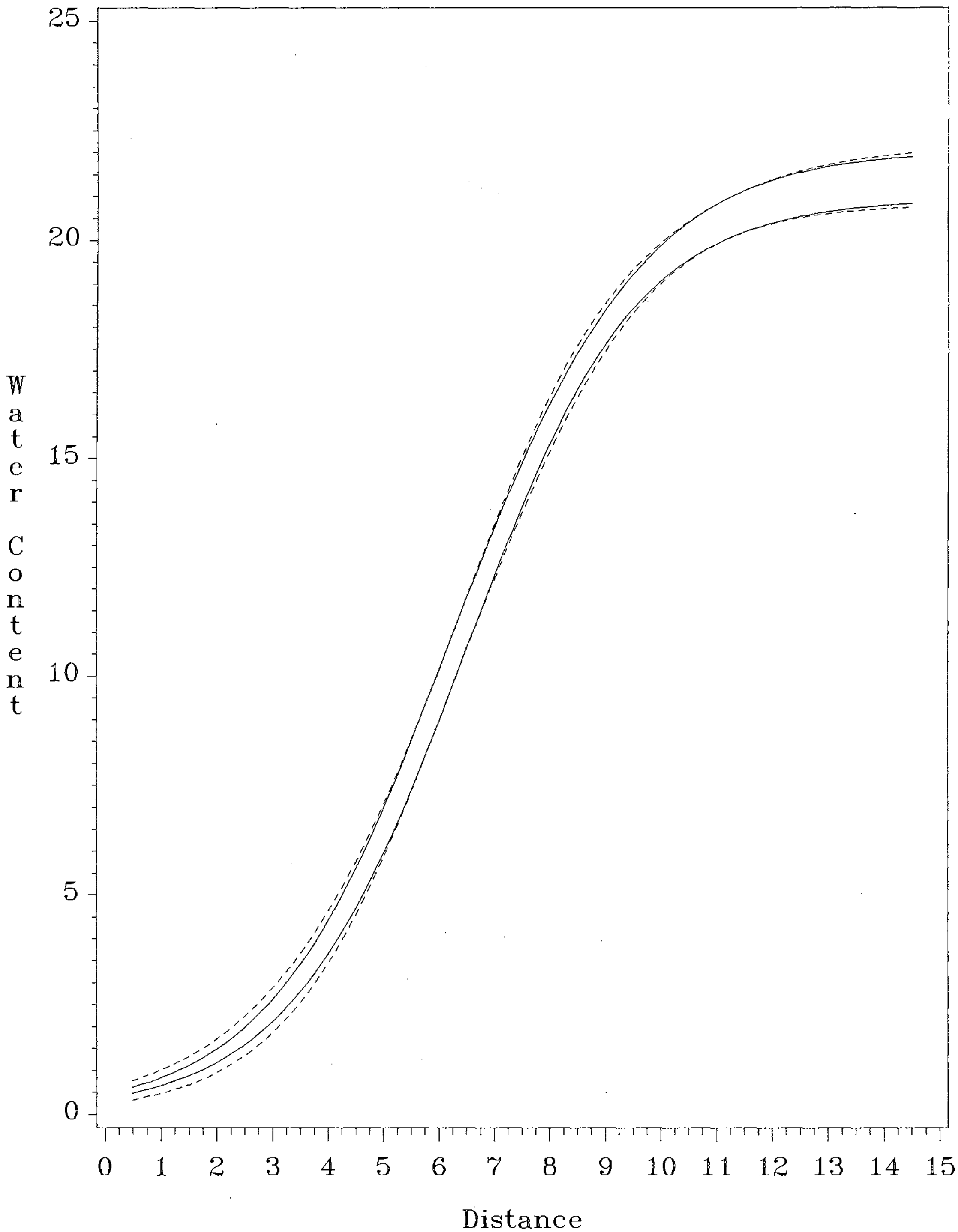


Figure 9. 95% Approximate Confidence Intervals in the Logistic Model for Growth Curve



_ _ _ _ _: Intervals in (5.12)
_______: Intervals in (5.17)

Figure 10. 90% Approximate Confidence Intervals in the Logistic Model for Growth Curve

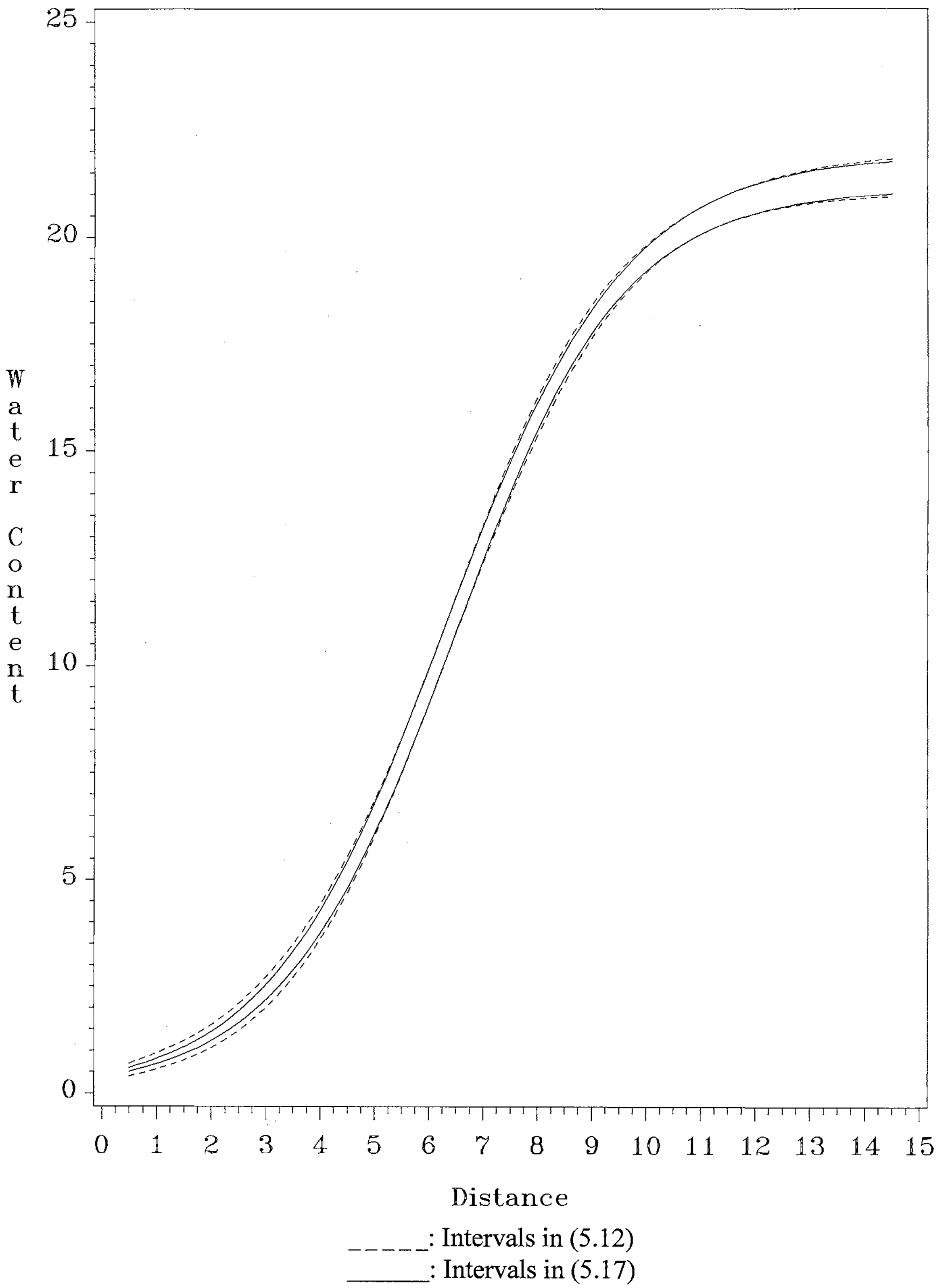


Figure 11. 75% Approximate Confidence Intervals in the Logistic Model for Growth Curve

TABLE XV

APPROXIMATE PREDICTION INTERVALS IN LOGISTIC MODEL FOR GROWTH CURVE

1-α	Distance	Intervals in (5.21)		Intervals in (5.24)	
		Lower	Upper	Lower	Upper
0.95	0.5	-0.944	2.037	-1.124	2.218
	1.5	-0.553	2.547	-0.800	2.793
	2.5	0.141	3.431	-0.159	3.730
	3.5	1.322	4.888	1.028	5.182
	4.5	3.196	7.090	3.008	7.279
	5.5	5.863	10.022	5.828	10.058
	6.5	9.116	13.327	9.096	13.347
	7.5	12.400	16.430	12.244	16.586
	8.5	15.110	18.920	14.892	19.138
	9.5	16.959	20.717	16.832	20.844
	10.5	18.067	21.909	18.045	21.931
	11.5	18.692	22.637	18.690	22.640
	12.5	19.037	23.058	19.004	23.091
	13.5	19.225	23.293	19.153	23.365
	14.5	19.326	23.422	19.223	23.525
0.90	0.5	-0.700	1.794	-0.851	1.945
	1.5	-0.300	2.294	-0.506	2.500
	2.5	0.409	3.163	0.159	3.413
	3.5	1.623	4.597	1.367	4.843
	4.5	3.514	6.772	3.357	6.930
	5.5	6.203	9.683	6.173	9.712
	6.5	9.460	12.983	9.443	13.000
	7.5	12.729	16.101	12.599	16.232
	8.5	15.421	18.601	15.238	18.791
	9.5	17.266	20.410	17.159	20.517
	10.5	18.380	21.595	18.362	21.517
	11.5	19.014	22.315	19.012	22.317
	12.5	19.365	22.730	19.337	22.758
	13.5	19.557	22.961	19.496	23.021
	14.5	19.661	23.087	19.574	23.174
0.75	0.5	-0.328	1.421	-0.433	1.527
	1.5	0.088	1.906	-0.057	2.051
	2.5	0.821	2.751	0.645	2.927
	3.5	2.059	4.151	1.886	4.323
	4.5	4.001	6.286	3.890	6.396
	5.5	6.723	9.163	6.702	9.184
	6.5	9.986	12.457	9.974	12.469
	7.5	13.233	15.598	13.141	15.689
	8.5	15.900	18.132	15.769	18.260
	9.5	17.736	19.940	17.661	20.015
	10.5	18.861	21.115	18.848	21.128
	11.5	19.507	21.822	19.506	21.823
	12.5	19.868	22.227	19.848	22.247
	13.5	20.065	22.452	20.023	22.494
	14.5	20.173	22.575	20.112	22.636

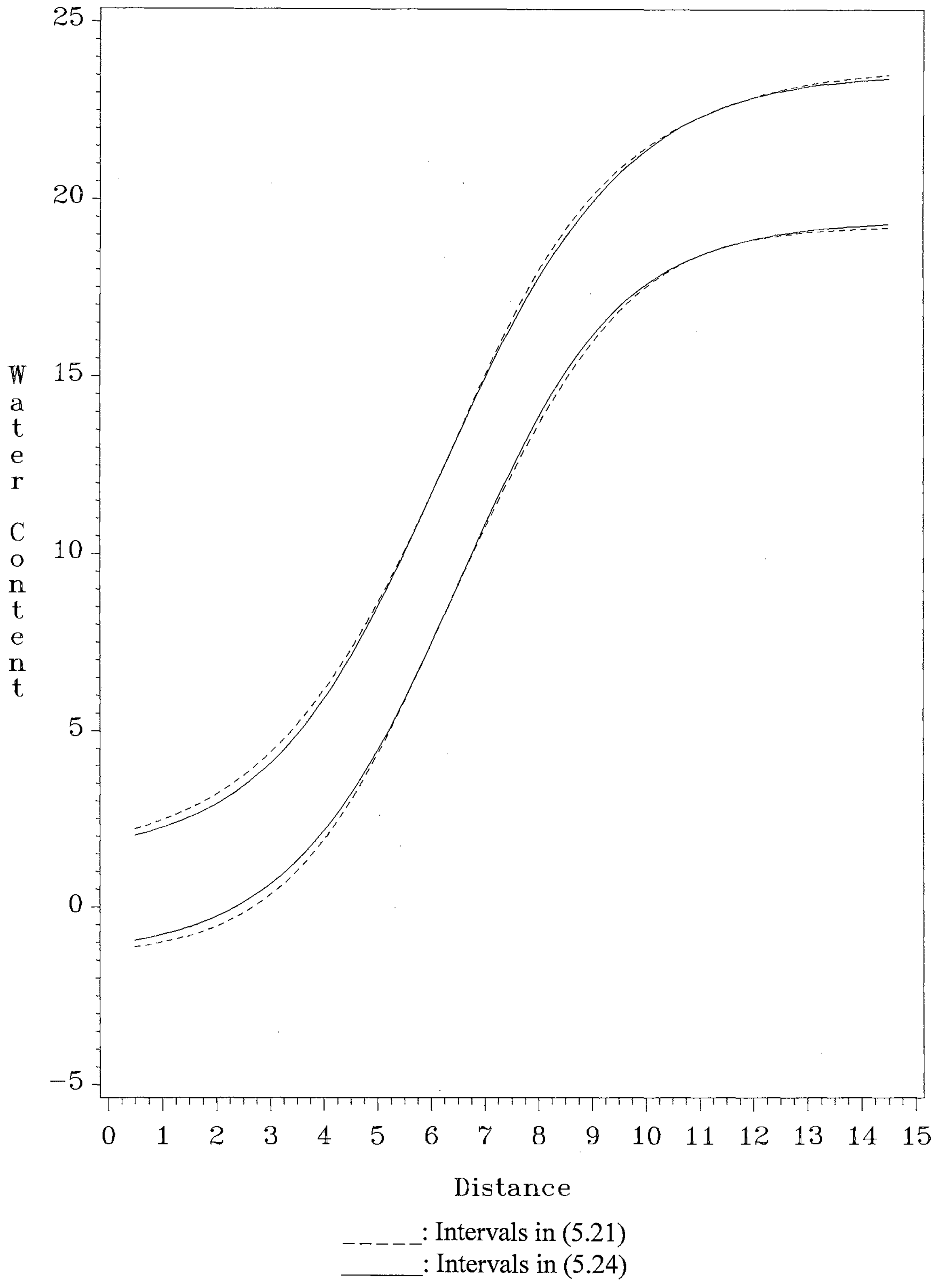


Figure 12. 95% Approximate Prediction Intervals in the Logistic Model for Growth Curve

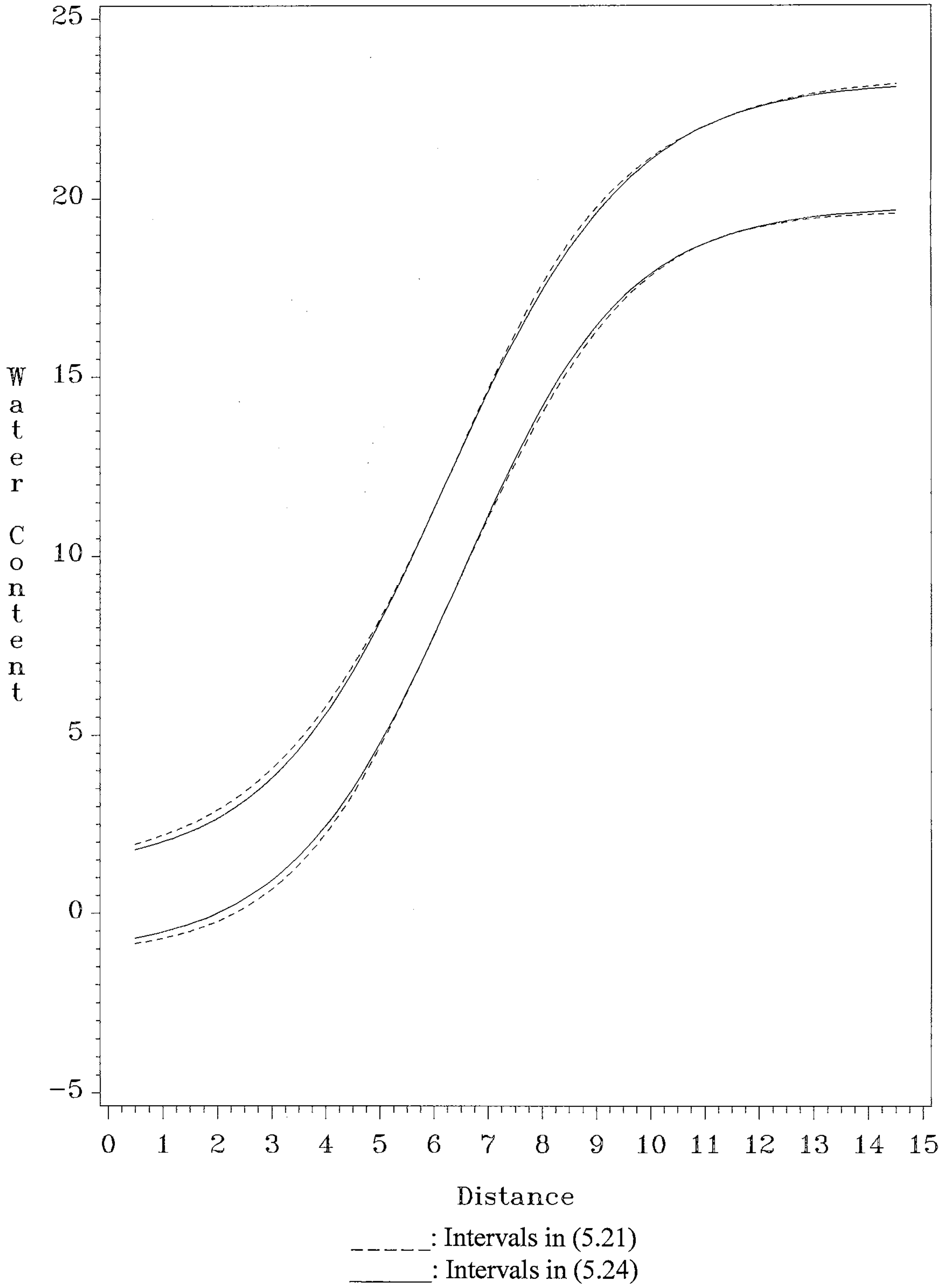
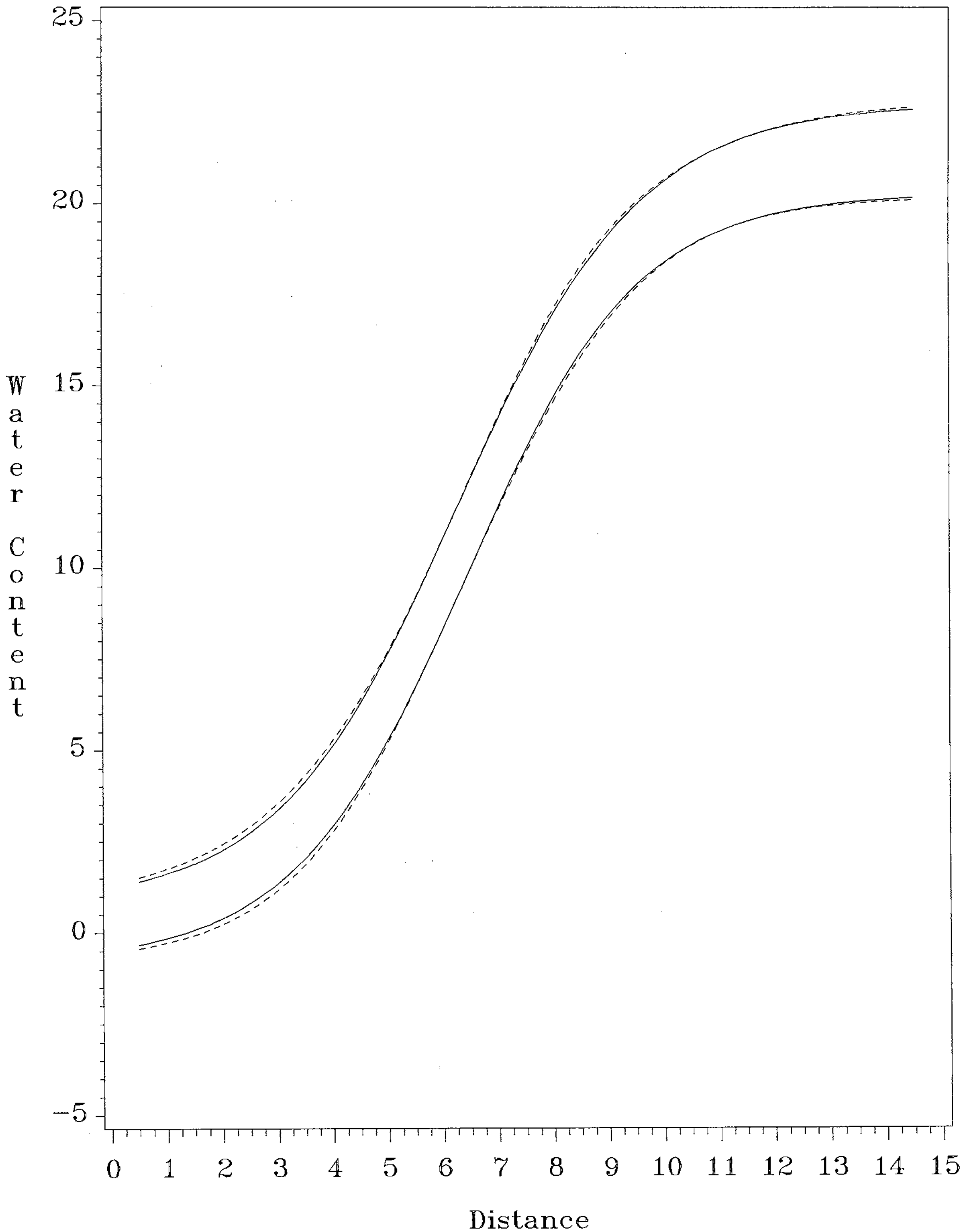


Figure 13. 90% Approximate Prediction Intervals in the Logistic Model for Growth Curve



_ _ _ _ _: Intervals in (5.21)
_______: Intervals in (5.24)

Figure 14. 75% Approximate Prediction Intervals in the Logistic Model for Growth Curve

TABLE XVI

APPROXIMATE CONFIDENCE BANDS IN LOGISTIC MODEL FOR GROWTH CURVE

1-α	Distance	Bands in (5.31) Lower	Bands in (5.31) Upper	Bands in (5.40) Lower	Bands in (5.40) Upper
0.95	0.5	-1.8652	4.0018	-1.8652	4.0019
	1.5	-0.7403	2.9921	-1.0818	3.3336
	2.5	0.4442	3.6145	0.3458	3.7129
	3.5	1.9779	5.2563	1.6672	5.5671
	4.5	4.0694	7.3859	3.4731	7.9822
	5.5	6.6264	9.7723	5.9412	10.4575
	6.5	9.4214	12.3198	8.8302	12.9110
	7.5	12.1896	14.9702	11.8989	15.2610
	8.5	14.7164	17.6149	14.4699	17.8614
	9.5	16.8933	20.0392	16.2544	20.6781
	10.5	18.6619	21.9783	17.7595	22.8807
	11.5	19.9265	23.2049	19.0155	24.1159
	12.5	20.4563	23.6266	19.9742	24.1087
	13.5	19.7195	23.4519	19.6421	23.5293
	14.5	17.1035	22.9705	15.7254	24.3486
0.90	0.5	-1.4822	3.6189	-1.4612	3.5978
	1.5	-0.4967	2.7485	-0.7777	3.0295
	2.5	0.6511	3.4076	0.5777	3.4810
	3.5	2.1919	5.0423	1.9358	5.2985
	4.5	4.2859	7.1694	3.7837	7.6716
	5.5	6.8318	9.5669	6.2523	10.1464
	6.5	9.6106	12.1306	9.1113	12.6300
	7.5	12.3711	14.7887	12.1304	15.0294
	8.5	14.9056	17.4257	14.7035	17.6278
	9.5	17.0986	19.8338	16.5591	20.3734
	10.5	18.8784	21.7618	18.1122	22.5280
	11.5	20.1405	22.9909	19.3668	23.7646
	12.5	20.6632	23.4196	20.2589	23.8239
	13.5	19.9631	23.2083	19.9098	23.2616
	14.5	17.4864	22.5875	16.3193	23.7547

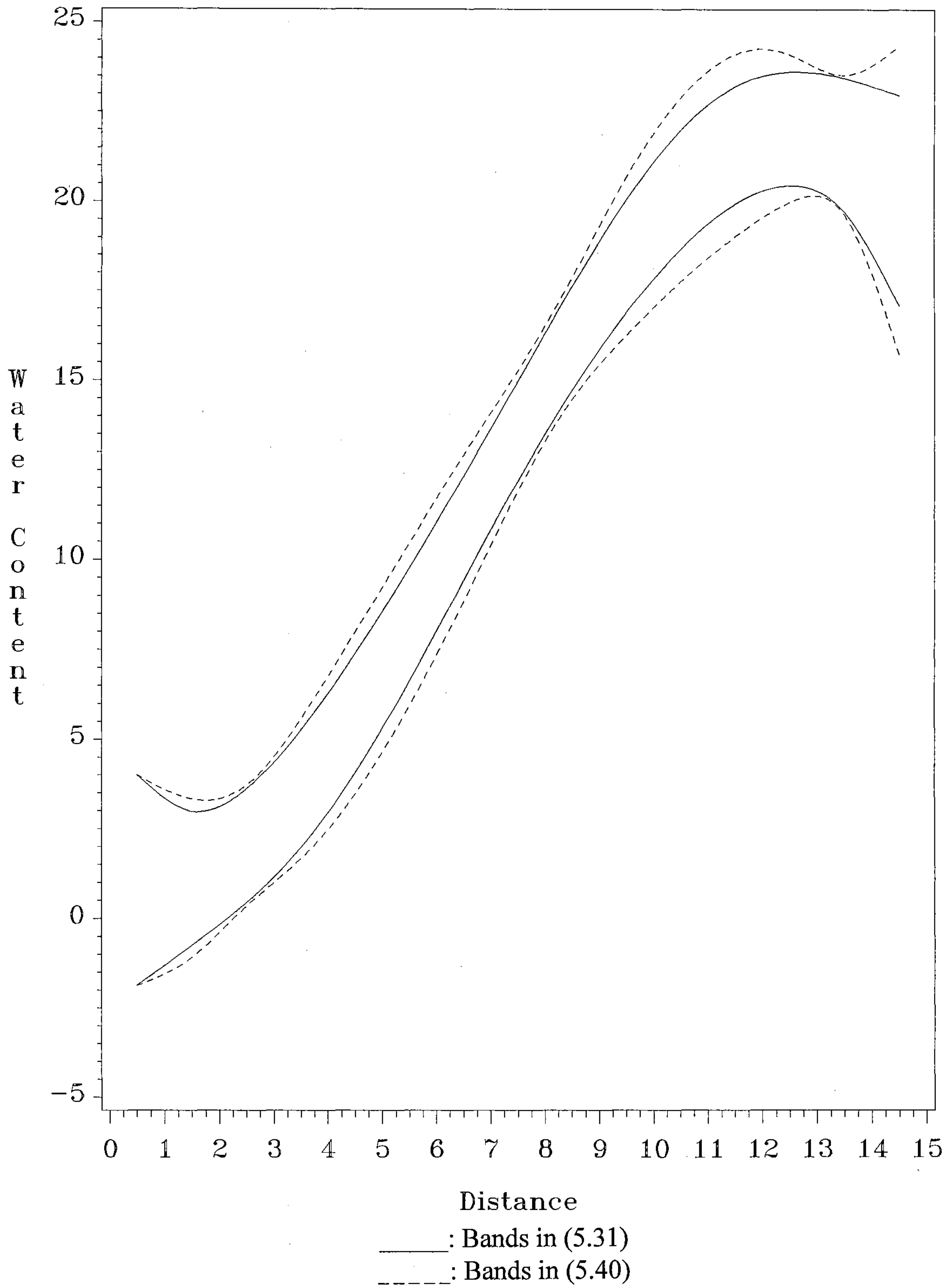
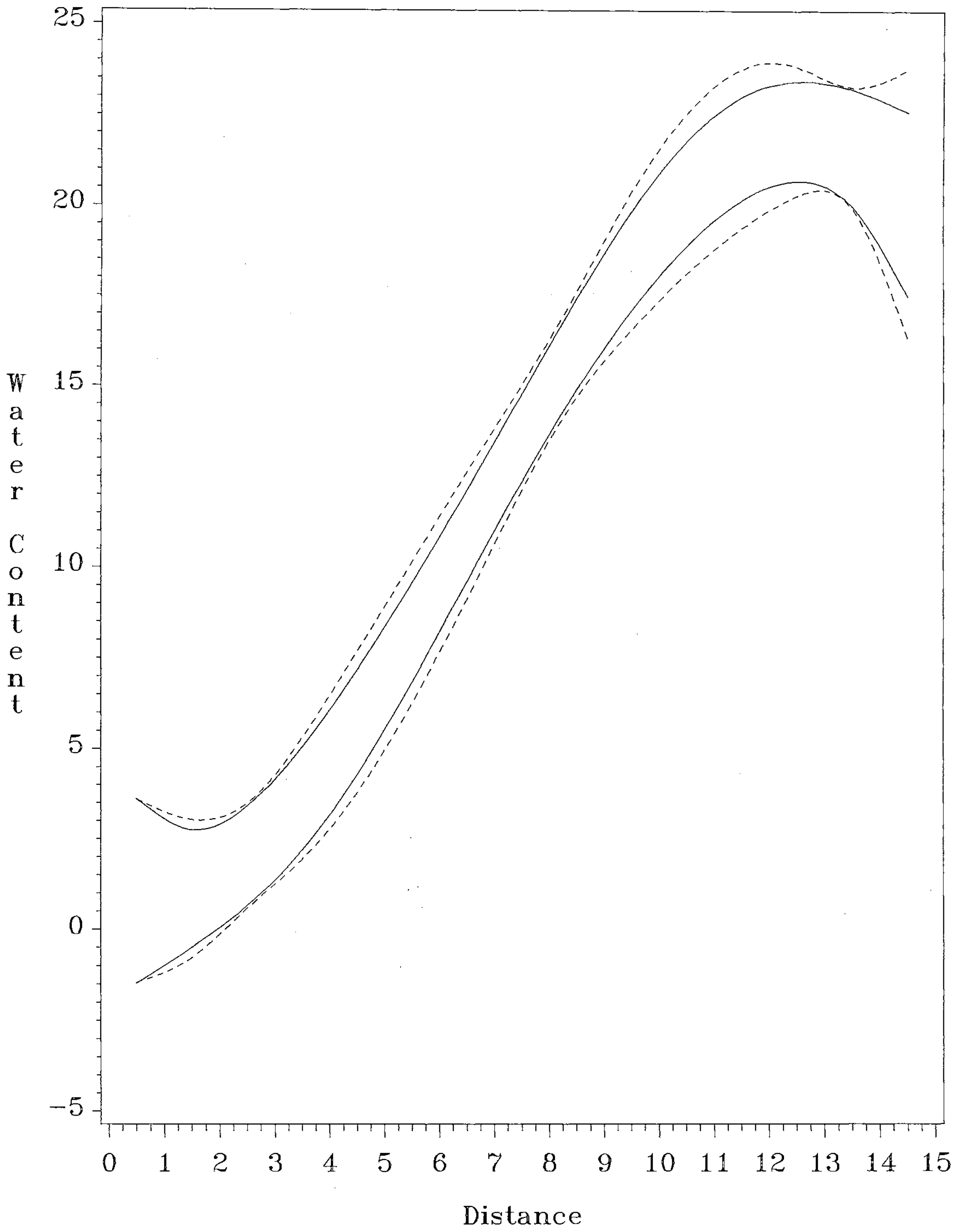


Figure 15. 95% Approximate Confidence Bands in the Logistic Model for Growth Curve



______: Bands in (5.31)
_ _ _ _ _: Bands in (5.40)

Figure 16. 90% Approximate Confidence Bands in the Logistic Model for Growth Curve

CHAPTER VI

SUMMARY AND CONCLUSIONS

The purpose of this study was to find approximate confidence intervals and approximate confidence bands for logistic models. We used large-sample methods to obtain an approximate confidence set for the parameters of logistic models.

Approximate confidence intervals for the logistic regression model using the delta method had a problem that the lower bound may be less than zero or the upper bound may be greater than one. Approximate confidence intervals in the logistic model for a growth curve using the linearization method had the advantage that their resulting confidence intervals were simpler to construct.

We have derived an approximate confidence band for the logistic regression model using a rectangular confidence set for parameters. The widths of this band were compared to the widths of the band obtained from an elliptical confidence set for parameters. We have found that the elliptical set gives a narrower band for most values of the independent variables. However the band derived from the rectangular set is narrower at some intermediate values of the independent variables.

The methods of obtaining approximate confidence bands for the logistic model of a IxJx2 table and in logistic regression for a bivariate binary response were derived.

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APPENDIX A

DELTA METHOD

Theorem:(Serfling, 1980) suppose that $\mathbf{X}_n = (X_{n1}, \ldots, X_{nk})$ is $AN(\boldsymbol{\mu}, b_n^2 \boldsymbol{\Sigma})$, with $\boldsymbol{\Sigma}$ a covariance matrix and $b_n \to 0$. Let $g(\mathbf{x}) = (g_1(\mathbf{x}), \ldots, g_m(\mathbf{x}))$, $\mathbf{x} = (x_1, \ldots, x_k)$, be a vector-valued function for which each component function $g_i(\mathbf{x})$ is real-valued and has a nonzero differential $g_i(\boldsymbol{\mu}\,;\mathbf{t})$, $\mathbf{t} = (t_1, \ldots, t_k)$, at $\mathbf{x} = \boldsymbol{\mu}$. Put

$$\mathbf{D} = \left[\left. \frac{\partial g_i}{\partial x_j} \right|_{\mathbf{x}=\boldsymbol{\mu}} \right]_{mxk}.$$

Then

$$g(\mathbf{X}_n) \text{ is } AN(g(\boldsymbol{\mu}), b_n^2 \mathbf{D}\boldsymbol{\Sigma}\mathbf{D}').$$

Proof: Put $\boldsymbol{\Sigma}_n = b_n^2 \boldsymbol{\Sigma}$. By the definition of asymptotic multivariate normality, we need to show that for every vector $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_m)$ such that $\boldsymbol{\lambda}\mathbf{D}\boldsymbol{\Sigma}_n\mathbf{D}'\boldsymbol{\lambda}' > 0$ for all sufficiently large n, we have

$$\frac{\boldsymbol{\lambda}[g(\mathbf{X}_n) - g(\boldsymbol{\mu})]'}{(\boldsymbol{\lambda}\mathbf{D}\boldsymbol{\Sigma}_n\mathbf{D}'\boldsymbol{\lambda}')^{1/2}} \xrightarrow{d} N(0, 1). \tag{1}$$

Let $\boldsymbol{\lambda}$ satisfy the required condition and suppose that n is already sufficiently large, and put $b_{\lambda n} = (\boldsymbol{\lambda}\mathbf{D}\boldsymbol{\Sigma}_n\mathbf{D}'\boldsymbol{\lambda}')^{1/2}$. Define functions h_i, $1 \le i \le m$, by $h_i(\boldsymbol{\mu}) = 0$ and

$$h_i(\mathbf{x}) = \frac{g_i(\mathbf{x}) - g_i(\boldsymbol{\mu}) - g_i(\boldsymbol{\mu}; \mathbf{x} - \boldsymbol{\mu})}{\|\mathbf{x} - \boldsymbol{\mu}\|}, \quad \mathbf{x} \neq \boldsymbol{\mu}$$

By the definition of g_i having a differential at $\boldsymbol{\mu}$, $h_i(\mathbf{x})$ is continuous at $\boldsymbol{\mu}$.

Now

$$\boldsymbol{\lambda}[g(\mathbf{X}_n) - g(\boldsymbol{\mu})]' b_{\lambda n}^{-1} = \sum_{i=1}^{m} \lambda_i b_{\lambda n}^{-1} [g_i(\mathbf{X}_n) - g_i(\boldsymbol{\mu})]$$

$$= \sum_{i=1}^{m} \lambda_i b_{\lambda n}^{-1} h_i(\mathbf{X}_n) \|\mathbf{X}_n - \boldsymbol{\mu}\| + \sum_{i=1}^{m} \lambda_i b_{\lambda n}^{-1} g_i(\boldsymbol{\mu}; \mathbf{X}_n - \boldsymbol{\mu}). \quad (2)$$

By the linear form of the differential, we have

$$\sum_{i=1}^{m} \lambda_i b_{\lambda n}^{-1} g_i(\boldsymbol{\mu}; \mathbf{X}_n - \boldsymbol{\mu}) = \sum_{i=1}^{m} \lambda_i b_{\lambda n}^{-1} \sum_{j=1}^{k} (X_{nj} - \mu_j) \frac{\partial g_i}{\partial x_j}\Big|_{\mathbf{x}=\boldsymbol{\mu}}$$

$$= b_{\lambda n}^{-1} \sum_{j=1}^{k} (X_{nj} - \mu_j) \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_i}\Big|_{\mathbf{x}=\boldsymbol{\mu}}$$

$$= \frac{(\boldsymbol{\lambda}\mathbf{D})(\mathbf{X}_n - \boldsymbol{\mu})'}{[(\boldsymbol{\lambda}\mathbf{D})\boldsymbol{\Sigma}_n(\boldsymbol{\lambda}\mathbf{D})']^{1/2}}. \quad (3)$$

By the assumption on $\boldsymbol{\lambda}$, and by the definition of asymptotic multivariate normality, the right-hand side of (3) converges in distribution to $N(0,\ 1)$. Thus

$$\sum_{i=1}^{m} \lambda_i b_{\lambda n}^{-1} g_i(\boldsymbol{\mu}; \mathbf{X}_n - \boldsymbol{\mu}) \xrightarrow{d} N(0,\ 1). \quad (4)$$

Now write

$$\sum_{i=1}^{m} \lambda_i b_{\lambda n}^{-1} h_i(\mathbf{X}_n) \|\mathbf{X}_n - \boldsymbol{\mu}\| = b_{\lambda n}^{-1} \|\mathbf{X}_n - \boldsymbol{\mu}\| \sum_{i=1}^{m} \lambda_i h_i(\mathbf{X}_n). \quad (5)$$

Since $\boldsymbol{\Sigma}_n \to \mathbf{0}$, we have $\mathbf{X}_n \xrightarrow{p} \boldsymbol{\mu}$. Therefore, since each h_i is continuous at $\boldsymbol{\mu}$,

$$\sum_{i=1}^{m} \lambda_i h_i(\mathbf{X}_n) \xrightarrow{p} \sum_{i=1}^{m} \lambda_i h_i(\boldsymbol{\mu}) = 0.$$

Also, now utilizing the fact that $\mathbf{S}_n$ is of the form $b_n^2 \boldsymbol{\Sigma}$, we have

$$b_{\lambda n}^{-1} \|\mathbf{X}_n - \boldsymbol{\mu}\| = (\boldsymbol{\lambda}\mathbf{D}\boldsymbol{\Sigma}\mathbf{D}'\boldsymbol{\lambda}')^{-1/2} b_n^{-1} \|\mathbf{X}_n - \boldsymbol{\mu}\| \xrightarrow{d} (\cdot).$$

It follows by Slutsky's Theorem that the right-hand side of (5) converges in probability to 0. Combining this result with (4) and (2), we have (1).

APPENDIX B

INEQUALITY

Let $(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})$ and $\mathbf{x}$ be an (s+1)x1 vectors. Let $\hat{\boldsymbol{\Sigma}}$ be an (s+1)x(s+1) positive definite matrix. Then

$$\frac{[\mathbf{x}'(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})]^2}{\mathbf{x}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}} \leq (\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta}). \tag{1}$$

Proof: Since $\hat{\boldsymbol{\Sigma}}$ is a positive definite matrix, there exits an (s+1)x(s+1) matrix $\mathbf{A}$ such that $\hat{\boldsymbol{\Sigma}}=\mathbf{A}'\mathbf{A}$ and rank(A)=s+1. Cauchy-Schwartz inquality is

$$(\mathbf{u}'\mathbf{v})^2 \leq (\mathbf{u}'\mathbf{u})(\mathbf{v}'\mathbf{v}), \tag{2}$$

where $\mathbf{u}$ and $\mathbf{v}$ are any column vectors.

Let $\mathbf{u}=(\mathbf{A}^{-1})'\mathbf{x}$ and $\mathbf{v}=\mathbf{A}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})$. Then, by (2), we have

$$\{[(\mathbf{A}^{-1})'\mathbf{x}]'[\mathbf{A}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})]\}^2 \leq \{[(\mathbf{A}^{-1})'\mathbf{x}]'[(\mathbf{A}^{-1})'\mathbf{x}]\}\{[\mathbf{A}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})]'[\mathbf{A}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})]\}$$

or $$[\mathbf{x}'\mathbf{A}^{-1}\mathbf{A}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})]^2 \leq [\mathbf{x}'\mathbf{A}^{-1}(\mathbf{A}^{-1})'\mathbf{x}][(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})'\mathbf{A}'\mathbf{A}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})]$$

or $$[\mathbf{x}'(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})]^2 \leq [\mathbf{x}'(\mathbf{A}'\mathbf{A})^{-1}\mathbf{x}][(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})]$$

or $$[\mathbf{x}'(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})]^2 \leq (\mathbf{x}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x})[(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})].$$

Since $\mathbf{x}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}>0$, we have

$$\frac{[\mathbf{x}'(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})]^2}{\mathbf{x}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{x}} \leq (\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta}).$$

APPENDIX C

SIMULATION RESULTS

APPENDIX C-1

PROBABILITIES THAT FALSE PROBABILITIES ARE CONTAINED IN THE BANDS (n=50, R1, method 1)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
10.0	0.989	0.992	0.994	0.998	0.999
10.5	0.990	0.992	0.995	0.998	0.999
11.0	0.990	0.992	0.995	0.999	0.999
11.5	0.991	0.993	0.995	0.999	0.999
12.0	0.991	0.993	0.995	0.999	0.999
12.5	0.991	0.992	0.995	0.999	0.999
13.0	0.991	0.992	0.995	0.999	0.999
13.5	0.992	0.992	0.994	0.999	0.999
14.0	0.992	0.992	0.994	0.999	0.999
14.5	0.992	0.992	0.995	0.999	1.000
15.0	0.992	0.992	0.996	0.999	1.000
15.5	0.993	0.992	0.997	0.999	1.000
16.0	0.992	0.991	0.996	1.000	1.000
16.5	0.991	0.992	0.995	1.000	1.000
17.0	0.990	0.993	0.994	1.000	1.000
17.5	0.990	0.993	0.994	1.000	1.000
18.0	0.989	0.993	0.994	1.000	0.999
18.5	0.991	0.994	0.994	1.000	0.997
19.0	0.992	0.994	0.993	1.000	0.995
19.5	0.992	0.994	0.994	1.000	0.987
20.0	0.994	0.994	0.994	0.998	0.985
20.5	0.996	0.992	0.994	0.995	0.961
21.0	0.996	0.992	0.993	0.995	0.957
21.5	0.994	0.993	0.993	0.989	0.925
22.0	0.994	0.992	0.992	0.987	0.905
22.5	0.993	0.994	0.992	0.986	0.902
23.0	0.994	0.994	0.993	0.985	0.901
23.5	0.992	0.993	0.993	0.975	0.897
24.0	0.992	0.993	0.993	0.969	0.896
24.5	0.992	0.994	0.994	0.966	0.896
25.0	0.989	0.995	0.995	0.967	0.896
25.5	0.989	0.993	0.995	0.968	0.898
26.0	0.990	0.992	0.995	0.971	0.899
26.5	0.990	0.994	0.995	0.977	0.907
27.0	0.987	0.994	0.995	0.977	0.917
27.5	0.988	0.995	0.997	0.976	0.931
28.0	0.986	0.994	0.996	0.978	0.941
28.5	0.984	0.993	0.997	0.980	0.954

APPENDIX C-1(Continued)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
29.0	0.983	0.993	0.998	0.982	0.960
29.5	0.982	0.992	0.998	0.984	0.966
30.0	0.983	0.991	0.998	0.988	0.971
30.5	0.984	0.991	0.998	0.988	0.975
31.0	0.983	0.992	0.998	0.990	0.980
31.5	0.983	0.992	0.998	0.989	0.980
32.0	0.983	0.991	0.998	0.989	0.983
32.5	0.983	0.991	0.998	0.989	0.984
33.0	0.983	0.991	0.998	0.990	0.985
33.5	0.983	0.991	0.998	0.990	0.985
34.0	0.983	0.991	0.997	0.990	0.986
34.5	0.987	0.991	0.997	0.988	0.987

APPENDIX C-2

PROBABILITIES THAT FALSE PROBABILITIES
ARE CONTAINED IN BANDS
(n=50, R2, method 1)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
35.0	0.995	0.989	0.989	0.987	0.970
35.5	0.995	0.989	0.989	0.986	0.969
36.0	0.995	0.989	0.988	0.986	0.967
36.5	0.996	0.989	0.988	0.985	0.964
37.0	0.996	0.989	0.989	0.984	0.962
37.5	0.996	0.989	0.989	0.984	0.959
38.0	0.996	0.989	0.987	0.984	0.951
38.5	0.996	0.989	0.986	0.984	0.947
39.0	0.996	0.989	0.986	0.984	0.944
39.5	0.996	0.989	0.986	0.983	0.943
40.0	0.996	0.989	0.986	0.984	0.937
40.5	0.994	0.990	0.987	0.982	0.930
41.0	0.993	0.990	0.989	0.980	0.925
41.5	0.993	0.990	0.990	0.977	0.917
42.0	0.992	0.991	0.989	0.971	0.909
42.5	0.991	0.991	0.989	0.971	0.902
43.0	0.990	0.989	0.987	0.965	0.892
43.5	0.990	0.989	0.986	0.963	0.887
44.0	0.989	0.988	0.986	0.963	0.878
44.5	0.989	0.988	0.985	0.962	0.869
45.0	0.987	0.988	0.986	0.955	0.864
45.5	0.986	0.992	0.986	0.952	0.856
46.0	0.985	0.992	0.988	0.944	0.843
46.5	0.986	0.993	0.988	0.938	0.827
47.0	0.984	0.991	0.981	0.926	0.826
47.5	0.983	0.991	0.979	0.929	0.835
48.0	0.985	0.991	0.980	0.932	0.832
48.5	0.986	0.992	0.980	0.936	0.831
49.0	0.987	0.993	0.983	0.935	0.835
49.5	0.987	0.994	0.986	0.934	0.837
50.0	0.990	0.995	0.986	0.939	0.841
50.5	0.991	0.995	0.984	0.938	0.843
51.0	0.991	0.995	0.985	0.942	0.843
51.5	0.992	0.995	0.984	0.939	0.848
52.0	0.992	0.995	0.983	0.943	0.853
52.5	0.993	0.995	0.980	0.943	0.856
53.0	0.992	0.995	0.982	0.945	0.859

APPENDIX C-2 (Continued)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
53.5	0.994	0.995	0.982	0.946	0.858
54.0	0.993	0.996	0.983	0.949	0.862
54.5	0.993	0.995	0.984	0.949	0.865
55.0	0.993	0.995	0.984	0.953	0.874
55.5	0.992	0.996	0.983	0.955	0.875
56.0	0.992	0.996	0.985	0.954	0.880
56.5	0.992	0.996	0.985	0.953	0.883
57.0	0.992	0.996	0.985	0.956	0.889
57.5	0.992	0.994	0.986	0.957	0.894
58.0	0.992	0.994	0.986	0.955	0.898
58.5	0.992	0.994	0.985	0.955	0.899
59.0	0.992	0.994	0.986	0.956	0.902
59.5	0.992	0.993	0.986	0.957	0.905

APPENDIX C-3

PROBABILITIES THAT FALSE PROBABILITIES ARE CONTAINED IN BANDS (n=50, R3, method 1)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
60.0	0.988	0.988	0.981	0.966	0.897
60.5	0.987	0.986	0.981	0.963	0.884
61.0	0.987	0.985	0.980	0.957	0.874
61.5	0.987	0.984	0.979	0.953	0.854
62.0	0.987	0.982	0.980	0.953	0.844
62.5	0.987	0.983	0.981	0.949	0.826
63.0	0.990	0.984	0.981	0.946	0.813
63.5	0.991	0.986	0.981	0.939	0.793
64.0	0.993	0.984	0.978	0.934	0.767
64.5	0.994	0.985	0.977	0.930	0.747
65.0	0.994	0.984	0.978	0.923	0.715
65.5	0.994	0.984	0.977	0.917	0.693
66.0	0.994	0.983	0.976	0.909	0.654
66.5	0.993	0.983	0.977	0.902	0.625
67.0	0.995	0.983	0.980	0.903	0.577
67.5	0.993	0.984	0.979	0.894	0.535
68.0	0.993	0.984	0.977	0.877	0.497
68.5	0.993	0.982	0.979	0.873	0.472
69.0	0.994	0.981	0.980	0.869	0.454
69.5	0.993	0.981	0.977	0.833	0.367
70.0	0.993	0.988	0.972	0.820	0.334
70.5	0.993	0.986	0.970	0.808	0.314
71.0	0.994	0.988	0.971	0.806	0.267
71.5	0.995	0.988	0.972	0.807	0.235
72.0	0.995	0.990	0.973	0.789	0.219
72.5	0.995	0.988	0.968	0.776	0.179
73.0	0.996	0.987	0.966	0.775	0.150
73.5	0.997	0.987	0.966	0.754	0.122
74.0	0.998	0.987	0.963	0.740	0.079
74.5	0.998	0.987	0.964	0.732	0.000
75.0	0.997	0.985	0.964	0.711	0.000
75.5	0.997	0.984	0.961	0.690	0.000
76.0	0.997	0.983	0.962	0.684	0.000
76.5	0.996	0.984	0.963	0.675	0.000
77.0	0.996	0.983	0.960	0.652	0.000
77.5	0.996	0.983	0.960	0.635	0.000
78.0	0.996	0.982	0.959	0.614	0.000

APPENDIX C-3 (Continued)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
78.5	0.995	0.982	0.958	0.593	0.000
79.0	0.994	0.981	0.956	0.573	0.000
79.5	0.993	0.980	0.956	0.548	0.000
80.0	0.993	0.981	0.955	0.515	0.000
80.5	0.992	0.981	0.955	0.469	0.000
81.0	0.993	0.980	0.955	0.417	0.000
81.5	0.992	0.980	0.954	0.341	0.000
82.0	0.992	0.981	0.956	0.183	0.000
82.5	0.992	0.980	0.953	0.000	0.000
83.0	0.991	0.981	0.953	0.000	0.000
83.5	0.991	0.982	0.953	0.000	0.000
84.0	0.991	0.984	0.953	0.000	0.000
84.5	0.991	0.984	0.952	0.000	0.000

APPENDIX C-4

PROBABILITIES THAT FALSE PROBABILITIES ARE CONTAINED IN BANDS (n=50, R1, method 2)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
10.0	0.993	0.996	0.999	0.998	0.998
10.5	0.994	0.996	0.999	0.999	0.998
11.0	0.994	0.998	0.999	0.999	0.998
11.5	0.994	0.998	0.999	0.999	0.998
12.0	0.994	0.999	0.999	0.999	0.999
12.5	0.995	0.999	1.000	0.999	1.000
13.0	0.996	0.999	1.000	0.999	1.000
13.5	0.996	0.999	1.000	1.000	1.000
14.0	0.996	0.999	1.000	1.000	1.000
14.5	0.996	0.999	1.000	1.000	1.000
15.0	0.996	0.999	1.000	1.000	1.000
15.5	0.997	0.999	1.000	1.000	1.000
16.0	0.997	1.000	1.000	1.000	1.000
16.5	0.998	1.000	1.000	1.000	1.000
17.0	0.998	1.000	1.000	1.000	1.000
17.5	0.999	1.000	1.000	1.000	1.000
18.0	0.998	1.000	1.000	1.000	1.000
18.5	0.997	1.000	1.000	1.000	1.000
19.0	0.997	1.000	1.000	1.000	1.000
19.5	0.997	1.000	1.000	1.000	1.000
20.0	0.998	1.000	1.000	1.000	1.000
20.5	0.998	1.000	1.000	1.000	1.000
21.0	0.998	1.000	1.000	1.000	0.999
21.5	0.998	1.000	1.000	1.000	0.995
22.0	0.998	1.000	1.000	0.999	0.992
22.5	0.995	1.000	1.000	0.997	0.990
23.0	0.995	1.000	0.998	0.995	0.986
23.5	0.993	1.000	0.997	0.991	0.972
24.0	0.993	0.996	0.995	0.991	0.956
24.5	0.990	0.993	0.995	0.988	0.943
25.0	0.990	0.994	0.994	0.987	0.934
25.5	0.990	0.992	0.992	0.989	0.926
26.0	0.992	0.992	0.988	0.982	0.932
26.5	0.994	0.995	0.992	0.983	0.942
27.0	0.996	0.996	0.992	0.987	0.958
27.5	0.996	0.996	0.995	0.995	0.975
28.0	0.999	0.997	0.998	0.996	0.984

APPENDIX C-4 (Continued)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
28.5	0.998	0.999	0.999	0.998	0.988
29.0	0.999	0.999	0.998	1.000	0.996
29.5	0.999	0.999	0.998	1.000	0.999
30.0	0.999	1.000	0.999	1.000	0.999
30.5	0.999	1.000	0.999	1.000	1.000
31.0	0.999	1.000	0.999	1.000	1.000
31.5	0.999	1.000	1.000	1.000	1.000
32.0	1.000	1.000	1.000	1.000	1.000
32.5	1.000	1.000	1.000	1.000	1.000
33.0	1.000	1.000	1.000	1.000	1.000
33.5	1.000	1.000	1.000	1.000	1.000
34.0	1.000	1.000	1.000	1.000	1.000
34.5	1.000	1.000	1.000	1.000	1.000

APPENDIX C-5

PROBABILITIES THAT FALSE PROBABILITIES
ARE CONTAINED IN BANDS
(n=50, R2, method 2)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
35.0	0.996	1.000	0.999	0.998	0.997
35.5	0.996	1.000	0.999	0.999	0.997
36.0	0.996	1.000	0.999	0.999	0.996
36.5	0.996	1.000	0.999	0.999	0.996
37.0	0.998	1.000	0.999	0.999	0.996
37.5	0.998	1.000	0.999	0.999	0.995
38.0	0.998	1.000	0.999	0.999	0.995
38.5	0.998	1.000	0.999	1.000	0.995
39.0	0.998	1.000	0.999	1.000	0.995
39.5	0.998	1.000	1.000	1.000	0.995
40.0	0.998	1.000	1.000	1.000	0.995
40.5	0.998	1.000	1.000	1.000	0.994
41.0	0.998	1.000	1.000	1.000	0.994
41.5	0.998	1.000	1.000	1.000	0.992
42.0	0.998	1.000	1.000	1.000	0.990
42.5	0.998	1.000	1.000	1.000	0.989
43.0	0.999	1.000	1.000	1.000	0.983
43.5	0.999	1.000	1.000	0.999	0.979
44.0	0.999	1.000	1.000	0.997	0.975
44.5	0.999	1.000	1.000	0.993	0.974
45.0	0.999	1.000	1.000	0.991	0.966
45.5	0.998	1.000	1.000	0.988	0.954
46.0	0.998	1.000	0.999	0.985	0.945
46.5	0.997	0.997	0.998	0.981	0.907
47.0	0.996	0.998	0.997	0.972	0.898
47.5	0.996	0.997	0.994	0.959	0.857
48.0	0.995	0.993	0.987	0.934	0.807
48.5	0.992	0.988	0.978	0.920	0.810
49.0	0.991	0.987	0.978	0.929	0.852
49.5	0.995	0.990	0.984	0.945	0.879
50.0	0.997	0.991	0.990	0.961	0.904
50.5	0.999	0.992	0.995	0.969	0.920
51.0	1.000	0.994	0.999	0.974	0.937
51.5	1.000	0.995	0.999	0.977	0.946
52.0	1.000	0.996	1.000	0.978	0.953
52.5	1.000	0.996	1.000	0.981	0.961
53.0	1.000	0.996	1.000	0.982	0.967

APPENDIX C-5 (Continued)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
53.5	1.000	0.997	1.000	0.986	0.969
54.0	1.000	0.997	1.000	0.990	0.976
54.5	1.000	0.998	1.000	0.991	0.976
55.0	1.000	0.998	1.000	0.991	0.978
55.5	1.000	0.998	1.000	0.991	0.978
56.0	1.000	0.998	1.000	0.991	0.979
56.5	1.000	0.998	1.000	0.991	0.980
57.0	1.000	0.998	1.000	0.991	0.979
57.5	1.000	0.998	1.000	0.992	0.979
58.0	1.000	0.998	1.000	0.992	0.979
58.5	1.000	0.998	1.000	0.992	0.980
59.0	1.000	0.998	1.000	0.992	0.980
59.5	0.999	0.998	1.000	0.993	0.980

APPENDIX C-6

PROBABILITIES THAT FALSE PROBABILITIES ARE CONTAINED IN BANDS (n=50, R3, method 2)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
60.0	1.000	1.000	0.998	0.996	0.974
60.5	0.999	1.000	0.998	0.996	0.971
61.0	0.999	1.000	0.998	0.996	0.964
61.5	0.999	1.000	0.998	0.995	0.959
62.0	0.999	1.000	0.998	0.993	0.950
62.5	0.999	1.000	0.998	0.990	0.943
63.0	0.999	1.000	0.998	0.989	0.932
63.5	0.999	1.000	0.998	0.987	0.915
64.0	0.999	1.000	0.998	0.981	0.896
64.5	0.999	1.000	0.998	0.979	0.867
65.0	0.999	0.999	0.998	0.977	0.836
65.5	0.999	0.999	0.997	0.972	0.809
66.0	0.999	0.999	0.996	0.965	0.762
66.5	0.999	0.999	0.996	0.958	0.733
67.0	0.999	0.999	0.994	0.951	0.680
67.5	0.998	0.995	0.992	0.928	0.632
68.0	0.997	0.995	0.988	0.909	0.587
68.5	0.996	0.995	0.984	0.888	0.549
69.0	0.996	0.994	0.980	0.865	0.494
69.5	0.993	0.991	0.977	0.840	0.455
70.0	0.993	0.990	0.978	0.835	0.406
70.5	0.991	0.988	0.981	0.827	0.380
71.0	0.992	0.988	0.981	0.821	0.345
71.5	0.994	0.983	0.975	0.816	0.314
72.0	0.995	0.982	0.975	0.800	0.282
72.5	0.993	0.983	0.977	0.797	0.250
73.0	0.993	0.988	0.982	0.795	0.208
73.5	0.992	0.989	0.977	0.778	0.157
74.0	0.990	0.989	0.977	0.775	0.098
74.5	0.993	0.990	0.977	0.775	0.000
75.0	0.992	0.992	0.977	0.774	0.000
75.5	0.992	0.992	0.977	0.775	0.000
76.0	0.992	0.992	0.976	0.768	0.000
76.5	0.992	0.992	0.976	0.768	0.000
77.0	0.992	0.993	0.980	0.762	0.000
77.5	0.992	0.993	0.981	0.752	0.000
78.0	0.993	0.993	0.981	0.736	0.000

APPENDIX C-6 (Continued)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
78.5	0.994	0.993	0.982	0.724	0.000
79.0	0.994	0.993	0.983	0.705	0.000
79.5	0.994	0.993	0.983	0.683	0.000
80.0	0.994	0.994	0.983	0.637	0.000
80.5	0.994	0.994	0.982	0.580	0.000
81.0	0.994	0.995	0.983	0.514	0.000
81.5	0.994	0.995	0.983	0.402	0.000
82.0	0.994	0.994	0.984	0.226	0.000
82.5	0.994	0.994	0.983	0.000	0.000
83.0	0.994	0.993	0.982	0.000	0.000
83.5	0.994	0.993	0.981	0.000	0.000
84.0	0.994	0.993	0.981	0.000	0.000
84.5	0.994	0.993	0.981	0.000	0.000

APPENDIX C-7

PROBABILITIES THAT FALSE PROBABILITIES ARE CONTAINED IN BANDS (n=40, R1, method 1)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
10.0	0.990	0.987	0.998	0.999	1.000
10.5	0.990	0.987	0.998	0.999	1.000
11.0	0.990	0.987	0.997	0.999	1.000
11.5	0.992	0.987	0.998	0.999	1.000
12.5	0.992	0.986	0.997	0.999	1.000
13.0	0.993	0.986	0.997	0.999	1.000
13.5	0.993	0.986	0.997	0.999	1.000
14.0	0.992	0.986	0.997	0.999	1.000
15.0	0.993	0.987	0.997	0.999	1.000
15.5	0.993	0.987	0.997	0.999	1.000
16.0	0.992	0.988	0.997	0.999	1.000
16.5	0.992	0.988	0.997	0.999	1.000
17.5	0.992	0.990	0.996	0.999	1.000
18.0	0.992	0.990	0.996	0.999	1.000
18.5	0.990	0.990	0.996	0.999	1.000
19.0	0.993	0.993	0.995	0.999	1.000
20.0	0.991	0.994	0.995	1.000	0.998
20.5	0.991	0.994	0.995	1.000	0.993
21.0	0.992	0.995	0.994	1.000	0.989
21.5	0.992	0.996	0.993	0.999	0.987
22.5	0.997	0.997	0.996	0.996	0.957
23.0	0.997	0.997	0.995	0.996	0.946
23.5	0.997	0.996	0.995	0.996	0.944
24.0	0.997	0.996	0.995	0.993	0.940
25.0	0.997	0.998	0.993	0.998	0.939
25.5	0.993	0.998	0.993	0.998	0.940
26.0	0.995	0.998	0.995	0.998	0.949
26.5	0.996	0.998	0.995	0.998	0.960
27.5	0.995	0.999	0.996	1.000	0.970
28.0	0.994	0.999	0.997	1.000	0.979
28.5	0.994	0.999	0.997	1.000	0.990
29.0	0.994	0.998	0.997	1.000	0.993
30.0	0.995	0.996	0.996	1.000	0.996
30.5	0.995	0.995	0.996	1.000	0.996
31.0	0.993	0.995	0.996	1.000	0.995
31.5	0.994	0.995	0.996	0.999	0.995
32.5	0.993	0.995	0.997	0.999	0.995
33.0	0.994	0.995	0.997	0.999	0.994
33.5	0.994	0.995	0.997	0.998	0.994
34.0	0.994	0.996	0.997	0.998	0.993

APPENDIX C-8

PROBABILITIES THAT FALSE PROBABILITIES ARE CONTAINED IN BANDS (n=40, R2, method 1)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
35.0	0.996	0.995	0.992	0.984	0.973
35.5	0.996	0.995	0.993	0.984	0.974
36.0	0.996	0.993	0.992	0.984	0.972
36.5	0.996	0.993	0.993	0.983	0.968
37.5	0.996	0.993	0.994	0.982	0.965
38.0	0.996	0.993	0.994	0.981	0.964
38.5	0.996	0.993	0.994	0.982	0.962
39.0	0.996	0.991	0.994	0.981	0.960
40.0	0.996	0.991	0.994	0.981	0.951
40.5	0.995	0.990	0.994	0.979	0.948
41.0	0.995	0.991	0.993	0.979	0.945
41.5	0.995	0.991	0.992	0.978	0.943
42.5	0.994	0.992	0.993	0.971	0.937
43.0	0.994	0.991	0.992	0.971	0.932
43.5	0.995	0.992	0.992	0.970	0.930
44.0	0.989	0.991	0.992	0.969	0.922
45.0	0.988	0.987	0.990	0.963	0.911
45.5	0.989	0.987	0.991	0.961	0.899
46.0	0.990	0.989	0.990	0.955	0.884
46.5	0.988	0.986	0.990	0.951	0.875
47.5	0.985	0.987	0.988	0.949	0.887
48.0	0.984	0.988	0.985	0.946	0.888
48.5	0.985	0.988	0.986	0.945	0.888
49.0	0.985	0.989	0.984	0.948	0.892
50.0	0.987	0.988	0.985	0.951	0.896
50.5	0.989	0.988	0.986	0.951	0.897
51.0	0.988	0.991	0.986	0.948	0.896
51.5	0.988	0.990	0.987	0.952	0.898
52.5	0.988	0.992	0.987	0.953	0.905
53.0	0.988	0.993	0.987	0.954	0.907
53.5	0.989	0.993	0.990	0.955	0.910
54.0	0.990	0.993	0.992	0.955	0.913
55.0	0.990	0.994	0.992	0.959	0.918
55.5	0.991	0.994	0.992	0.959	0.919
56.0	0.992	0.995	0.991	0.959	0.920
56.5	0.992	0.995	0.992	0.958	0.919
57.5	0.993	0.996	0.993	0.959	0.923
58.0	0.994	0.996	0.993	0.961	0.921
58.5	0.995	0.996	0.993	0.964	0.920
59.0	0.996	0.995	0.992	0.964	0.919

APPENDIX C-9

PROBABILITIES THAT FALSE PROBABILITIES ARE CONTAINED IN BANDS (n=40, R3, method 1)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
60.0	0.994	0.994	0.989	0.970	0.920
60.5	0.994	0.994	0.989	0.967	0.908
61.0	0.993	0.994	0.989	0.966	0.899
61.5	0.993	0.993	0.989	0.963	0.884
62.5	0.993	0.993	0.988	0.955	0.863
63.0	0.993	0.993	0.989	0.950	0.848
63.5	0.992	0.993	0.986	0.944	0.829
64.0	0.992	0.991	0.987	0.942	0.809
65.0	0.992	0.992	0.986	0.922	0.771
65.5	0.990	0.992	0.984	0.918	0.744
66.0	0.991	0.992	0.984	0.911	0.716
66.5	0.991	0.992	0.982	0.904	0.695
67.5	0.990	0.990	0.980	0.900	0.607
68.0	0.990	0.991	0.985	0.895	0.563
68.5	0.992	0.991	0.982	0.884	0.519
69.0	0.992	0.991	0.982	0.863	0.502
70.0	0.994	0.991	0.982	0.815	0.427
70.5	0.994	0.985	0.982	0.810	0.389
71.0	0.993	0.984	0.972	0.808	0.364
71.5	0.989	0.984	0.970	0.810	0.351
72.5	0.989	0.983	0.977	0.792	0.262
73.0	0.991	0.980	0.978	0.782	0.228
73.5	0.987	0.979	0.974	0.779	0.168
74.0	0.986	0.979	0.976	0.750	0.103
75.0	0.986	0.984	0.973	0.739	0.000
75.5	0.987	0.983	0.975	0.723	0.000
76.0	0.987	0.983	0.976	0.707	0.000
76.5	0.986	0.983	0.968	0.695	0.000
77.5	0.987	0.981	0.963	0.664	0.000
78.0	0.985	0.980	0.963	0.649	0.000
78.5	0.985	0.980	0.963	0.634	0.000
79.0	0.985	0.980	0.959	0.616	0.000
80.0	0.984	0.980	0.952	0.554	0.000
80.5	0.984	0.982	0.951	0.514	0.000
81.0	0.983	0.984	0.950	0.459	0.000
81.5	0.983	0.985	0.950	0.370	0.000
82.5	0.981	0.985	0.948	0.000	0.000
83.0	0.980	0.985	0.948	0.000	0.000
83.5	0.981	0.984	0.948	0.000	0.000
84.0	0.981	0.984	0.946	0.000	0.000

APPENDIX C-10

PROBABILITIES THAT FALSE PROBABILITIES ARE CONTAINED IN BANDS (n=40, R1, method 2)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
10.0	0.988	0.994	0.999	0.999	1.000
10.5	0.989	0.994	0.999	0.999	1.000
11.0	0.991	0.994	0.999	0.999	1.000
11.5	0.991	0.995	0.999	0.999	1.000
12.5	0.993	0.998	0.999	0.999	1.000
13.0	0.995	0.998	0.999	0.999	1.000
13.5	0.995	0.998	0.999	0.999	1.000
14.0	0.996	0.999	0.999	1.000	1.000
15.0	0.997	0.999	1.000	1.000	1.000
15.5	0.997	0.999	1.000	1.000	1.000
16.0	0.997	0.999	1.000	1.000	1.000
16.5	0.997	0.999	1.000	1.000	1.000
17.5	0.997	0.998	1.000	1.000	1.000
18.0	0.997	0.999	1.000	1.000	1.000
18.5	0.997	0.999	1.000	1.000	1.000
19.0	0.997	0.999	1.000	1.000	1.000
20.0	0.998	0.999	1.000	1.000	0.998
20.5	0.997	0.999	1.000	1.000	0.998
21.0	0.997	0.999	1.000	1.000	0.997
21.5	0.997	0.999	1.000	0.999	0.994
22.5	0.997	0.998	1.000	0.997	0.995
23.0	0.997	0.998	1.000	0.996	0.989
23.5	0.995	0.996	0.999	0.995	0.983
24.0	0.993	0.994	0.999	0.994	0.979
25.0	0.991	0.994	0.994	0.995	0.977
25.5	0.989	0.992	0.997	0.996	0.981
26.0	0.991	0.993	0.998	0.994	0.981
26.5	0.995	0.996	0.998	0.992	0.983
27.5	0.997	0.999	0.999	1.000	0.998
28.0	0.998	0.999	0.999	0.999	0.999
28.5	0.999	0.999	0.999	0.999	0.999
29.0	0.999	0.999	0.999	1.000	0.999
30.0	0.999	0.999	0.999	1.000	1.000
30.5	1.000	0.999	0.999	1.000	1.000
31.0	1.000	0.999	1.000	1.000	1.000
31.5	1.000	0.999	1.000	1.000	1.000
32.5	1.000	0.999	1.000	1.000	1.000
33.0	1.000	0.999	1.000	1.000	1.000
33.5	1.000	0.999	1.000	1.000	1.000
34.0	1.000	0.999	1.000	1.000	1.000

APPENDIX C-11

PROBABILITIES THAT FALSE PROBABILITIES ARE CONTAINED IN BANDS (n=40, R2, method 2)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
35.0	1.000	0.996	1.000	1.000	0.994
35.5	1.000	0.997	1.000	1.000	0.994
36.0	1.000	0.997	1.000	1.000	0.995
36.5	1.000	0.997	1.000	1.000	0.995
37.5	1.000	0.999	1.000	1.000	0.997
38.0	1.000	0.999	1.000	1.000	0.997
38.5	1.000	0.999	1.000	1.000	0.997
39.0	1.000	1.000	1.000	1.000	0.997
40.0	1.000	1.000	1.000	1.000	0.996
40.5	1.000	1.000	1.000	1.000	0.996
41.0	1.000	1.000	1.000	1.000	0.995
41.5	1.000	1.000	1.000	1.000	0.995
42.5	1.000	1.000	1.000	1.000	0.994
43.0	1.000	0.999	0.999	1.000	0.991
43.5	1.000	0.999	0.999	0.999	0.989
44.0	1.000	0.998	0.999	0.999	0.985
45.0	0.999	0.998	0.997	0.997	0.975
45.5	0.996	0.995	0.994	0.993	0.957
46.0	0.996	0.996	0.995	0.992	0.946
46.5	0.995	0.995	0.993	0.978	0.925
47.5	0.992	0.994	0.987	0.957	0.873
48.0	0.995	0.994	0.977	0.945	0.830
48.5	0.993	0.990	0.969	0.936	0.847
49.0	0.994	0.994	0.971	0.946	0.867
50.0	0.995	0.996	0.988	0.964	0.911
50.5	0.997	0.998	0.990	0.973	0.926
51.0	0.998	0.998	0.991	0.980	0.938
51.5	0.999	0.999	0.991	0.984	0.950
52.5	0.999	0.999	0.995	0.988	0.956
53.0	0.999	0.999	0.996	0.990	0.963
53.5	0.998	0.998	0.996	0.992	0.965
54.0	0.998	0.998	0.996	0.992	0.969
55.0	0.999	0.998	0.997	0.992	0.972
55.5	0.998	0.998	0.997	0.992	0.973
56.0	0.998	0.998	0.998	0.992	0.973
56.5	0.998	0.998	0.998	0.994	0.974
57.5	0.998	0.998	0.998	0.994	0.973
58.0	0.998	0.998	0.998	0.994	0.974
58.5	0.998	0.998	0.998	0.995	0.974
59.0	0.998	0.998	0.999	0.996	0.973

APPENDIX C-12

PROBABILITIES THAT FALSE PROBABILITIES
ARE CONTAINED IN BANDS
(n=40, R3, method 2)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
60.0	0.999	0.999	0.999	0.996	0.981
60.5	0.999	0.999	0.999	0.996	0.980
61.0	0.999	0.999	0.999	0.994	0.976
61.5	0.999	0.999	0.999	0.992	0.970
62.5	0.999	0.999	0.999	0.988	0.958
63.0	0.999	0.999	0.999	0.987	0.942
63.5	0.999	0.999	0.999	0.981	0.928
64.0	0.999	0.999	0.999	0.978	0.906
65.0	0.998	0.999	0.996	0.969	0.861
65.5	0.998	0.999	0.995	0.967	0.831
66.0	0.998	0.999	0.992	0.962	0.806
66.5	0.997	0.999	0.991	0.956	0.774
67.5	0.997	0.998	0.988	0.943	0.689
68.0	0.995	0.998	0.988	0.936	0.650
68.5	0.994	0.998	0.984	0.923	0.607
69.0	0.996	0.994	0.981	0.906	0.570
70.0	0.993	0.995	0.972	0.881	0.505
70.5	0.991	0.992	0.970	0.864	0.465
71.0	0.987	0.987	0.975	0.867	0.437
71.5	0.988	0.985	0.974	0.850	0.411
72.5	0.989	0.982	0.981	0.841	0.317
73.0	0.991	0.987	0.975	0.841	0.269
73.5	0.989	0.990	0.973	0.839	0.211
74.0	0.991	0.991	0.976	0.837	0.145
75.0	0.993	0.995	0.979	0.836	0.000
75.5	0.993	0.995	0.979	0.828	0.000
76.0	0.994	0.995	0.978	0.825	0.000
76.5	0.994	0.997	0.980	0.822	0.000
77.5	0.995	0.998	0.983	0.805	0.000
78.0	0.995	0.998	0.984	0.797	0.000
78.5	0.996	1.000	0.984	0.780	0.000
79.0	0.995	0.999	0.985	0.767	0.000
80.0	0.996	0.999	0.985	0.718	0.000
80.5	0.996	0.999	0.986	0.682	0.000
81.0	0.996	0.999	0.986	0.617	0.000
81.5	0.996	0.999	0.987	0.507	0.000
82.5	0.996	0.999	0.987	0.000	0.000
83.0	0.996	0.999	0.987	0.000	0.000
83.5	0.996	0.999	0.987	0.000	0.000
84.0	0.996	0.999	0.986	0.000	0.000

APPENDIX C-13

PROBABILITIES THAT FALSE PROBABILITIES
ARE CONTAINED IN BANDS
(n=30, R1, method 1)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
10.5	0.995	0.991	0.997	0.999	0.999
11.0	0.995	0.991	0.997	0.999	0.999
12.0	0.996	0.991	0.997	0.999	0.999
12.5	0.996	0.991	0.998	0.999	0.999
13.5	0.996	0.991	0.998	0.999	0.999
14.0	0.995	0.991	0.998	0.999	0.999
15.5	0.996	0.990	0.998	0.999	1.000
16.0	0.996	0.990	0.998	0.999	1.000
17.0	0.996	0.989	0.998	0.999	1.000
17.5	0.995	0.990	0.998	0.999	1.000
18.5	0.994	0.990	0.997	0.999	1.000
19.0	0.994	0.990	0.997	0.999	1.000
20.5	0.993	0.990	0.997	0.999	0.999
21.0	0.993	0.990	0.997	0.999	0.999
22.0	0.994	0.992	0.997	0.998	0.999
22.5	0.994	0.992	0.998	0.998	0.999
23.5	0.993	0.991	0.998	0.999	0.998
24.0	0.993	0.991	0.998	0.999	0.994
25.5	0.996	0.994	0.996	0.999	0.994
26.0	0.996	0.995	0.997	0.999	0.995
27.0	0.995	0.995	0.997	0.999	0.999
27.5	0.995	0.996	0.997	0.999	0.999
28.5	0.995	0.997	0.997	0.999	1.000
29.0	0.996	0.998	0.997	0.999	1.000
30.5	0.995	0.999	0.999	0.999	1.000
31.0	0.995	0.999	0.999	0.999	1.000
32.0	0.996	0.999	0.998	0.999	1.000
32.5	0.996	0.999	0.998	0.999	1.000
33.5	0.996	0.999	0.998	1.000	1.000
34.0	0.997	0.999	0.998	1.000	1.000

APPENDIX C-14

PROBABILITIES THAT FALSE PROBABILITIES
ARE CONTAINED IN BANDS
(n=30, R2, method 1)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
35.5	0.999	0.997	0.996	0.998	0.990
36.0	0.999	0.997	0.996	0.997	0.988
37.0	0.999	0.997	0.996	0.996	0.988
37.5	0.999	0.997	0.995	0.997	0.987
38.5	0.999	0.996	0.996	0.997	0.983
39.0	0.999	0.996	0.996	0.997	0.982
40.5	0.999	0.997	0.995	0.996	0.976
41.0	0.999	0.996	0.995	0.996	0.974
42.0	0.997	0.996	0.993	0.994	0.965
42.5	0.996	0.996	0.992	0.993	0.962
43.5	0.995	0.995	0.990	0.985	0.949
44.0	0.995	0.995	0.989	0.981	0.943
45.5	0.995	0.993	0.985	0.976	0.933
46.0	0.994	0.997	0.985	0.972	0.924
47.0	0.994	0.998	0.984	0.964	0.916
47.5	0.995	0.998	0.986	0.965	0.916
48.5	0.995	0.997	0.987	0.968	0.900
49.0	0.996	0.996	0.987	0.964	0.899
50.5	0.997	0.997	0.985	0.961	0.904
51.0	0.996	0.997	0.989	0.964	0.904
52.0	0.996	0.996	0.990	0.964	0.910
52.5	0.996	0.997	0.990	0.966	0.912
53.5	0.996	0.996	0.989	0.967	0.921
54.0	0.997	0.995	0.989	0.968	0.926
55.5	0.996	0.996	0.991	0.970	0.931
56.0	0.996	0.996	0.992	0.969	0.936
57.0	0.996	0.996	0.991	0.969	0.937
57.5	0.995	0.996	0.991	0.972	0.937
58.5	0.996	0.996	0.992	0.973	0.940
59.0	0.996	0.996	0.992	0.972	0.940

APPENDIX C-15

PROBABILITIES THAT FALSE PROBABILITIES ARE CONTAINED IN BANDS (n=30, R3, method 1)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
60.5	0.996	0.992	0.985	0.966	0.937
61.0	0.995	0.991	0.985	0.963	0.933
62.0	0.995	0.991	0.984	0.951	0.911
62.5	0.993	0.989	0.983	0.946	0.902
63.5	0.994	0.988	0.982	0.937	0.876
64.0	0.992	0.988	0.982	0.932	0.851
65.5	0.991	0.989	0.986	0.928	0.809
66.0	0.991	0.990	0.986	0.919	0.790
67.0	0.990	0.988	0.987	0.907	0.756
67.5	0.988	0.986	0.986	0.904	0.735
68.5	0.987	0.984	0.984	0.890	0.636
69.0	0.986	0.988	0.983	0.891	0.598
70.5	0.982	0.993	0.983	0.878	0.545
71.0	0.982	0.989	0.982	0.883	0.458
72.0	0.980	0.990	0.975	0.857	0.389
72.5	0.981	0.990	0.972	0.843	0.360
73.5	0.982	0.990	0.974	0.837	0.254
74.0	0.983	0.989	0.976	0.837	0.169
75.5	0.981	0.988	0.967	0.791	0.000
76.0	0.982	0.986	0.967	0.783	0.000
77.0	0.980	0.989	0.962	0.757	0.000
77.5	0.980	0.987	0.961	0.747	0.000
78.5	0.979	0.982	0.963	0.709	0.000
79.0	0.978	0.982	0.962	0.680	0.000
80.5	0.977	0.983	0.961	0.616	0.000
81.0	0.977	0.982	0.961	0.561	0.000
82.0	0.977	0.983	0.962	0.321	0.000
82.5	0.978	0.983	0.961	0.000	0.000
83.5	0.980	0.983	0.961	0.000	0.000
84.0	0.981	0.984	0.961	0.000	0.000

APPENDIX C-16

PROBABILITIES THAT FALSE PROBABILITIES ARE CONTAINED IN BANDS (n=30, R1, method 2)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
10.5	0.996	1.000	0.999	1.000	1.000
11.0	0.997	1.000	0.999	1.000	1.000
12.0	0.997	1.000	0.999	1.000	1.000
12.5	0.997	1.000	0.999	1.000	1.000
13.5	0.998	1.000	1.000	1.000	1.000
14.0	0.998	1.000	1.000	1.000	1.000
15.5	0.998	1.000	1.000	1.000	1.000
16.0	0.998	1.000	1.000	1.000	1.000
17.0	0.998	1.000	1.000	1.000	1.000
17.5	0.998	0.999	1.000	1.000	1.000
18.5	0.997	0.999	1.000	1.000	1.000
19.0	0.997	0.999	1.000	1.000	1.000
20.5	0.997	0.999	1.000	1.000	1.000
21.0	0.997	0.999	1.000	1.000	1.000
22.0	0.997	0.999	1.000	0.999	0.999
22.5	0.997	0.999	1.000	0.999	0.998
23.5	0.997	0.998	0.999	0.999	0.995
24.0	0.995	0.996	0.999	0.999	0.995
25.5	0.997	0.991	0.998	0.997	0.993
26.0	0.994	0.990	0.999	1.000	0.995
27.0	0.998	0.994	0.999	1.000	0.999
27.5	0.998	0.995	0.999	1.000	0.997
28.5	0.999	0.999	0.999	1.000	1.000
29.0	0.999	0.999	0.999	1.000	1.000
30.5	1.000	0.999	1.000	1.000	1.000
31.0	1.000	0.999	1.000	1.000	1.000
32.0	1.000	1.000	1.000	1.000	1.000
32.5	1.000	1.000	1.000	1.000	1.000
33.5	1.000	1.000	1.000	1.000	1.000
34.0	1.000	1.000	1.000	1.000	1.000

APPENDIX C-17

PROBABILITIES THAT FALSE PROBABILITIES ARE CONTAINED IN BANDS (n=30, R2, method 2)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
35.5	1.000	1.000	1.000	1.000	1.000
36.0	1.000	1.000	1.000	1.000	1.000
37.0	1.000	1.000	1.000	1.000	1.000
37.5	1.000	1.000	1.000	1.000	1.000
38.5	1.000	1.000	1.000	1.000	1.000
39.0	1.000	1.000	1.000	1.000	1.000
40.5	1.000	1.000	1.000	1.000	1.000
41.0	1.000	1.000	1.000	1.000	1.000
42.0	1.000	1.000	1.000	1.000	1.000
42.5	0.999	1.000	1.000	1.000	1.000
43.5	0.999	1.000	1.000	0.999	0.999
44.0	0.997	1.000	1.000	0.999	0.998
45.5	0.999	0.998	0.999	0.996	0.991
46.0	1.000	0.998	0.998	0.993	0.991
47.0	0.996	0.998	0.998	0.986	0.953
47.5	0.997	0.996	0.996	0.980	0.927
48.5	0.996	0.993	0.984	0.954	0.876
49.0	0.995	0.991	0.986	0.958	0.892
50.5	0.998	0.995	0.995	0.980	0.939
51.0	0.998	0.995	0.997	0.987	0.953
52.0	0.999	0.998	0.999	0.993	0.964
52.5	1.000	0.998	0.999	0.994	0.972
53.5	1.000	1.000	0.999	0.995	0.976
54.0	1.000	1.000	0.999	0.995	0.977
55.5	1.000	0.998	1.000	0.997	0.979
56.0	1.000	0.998	1.000	0.998	0.982
57.0	1.000	0.998	1.000	0.998	0.982
57.5	1.000	0.998	1.000	0.998	0.983
58.5	1.000	0.998	1.000	0.998	0.983
59.0	1.000	0.998	1.000	0.998	0.984

APPENDIX C-18

PROBABILITIES THAT FALSE PROBABILITIES ARE CONTAINED IN BANDS (n=30, R3, method 2)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
60.5	0.998	1.000	1.000	0.997	0.980
61.0	0.998	1.000	1.000	0.997	0.978
62.0	0.998	1.000	1.000	0.997	0.967
62.5	0.998	1.000	1.000	0.997	0.961
63.5	0.998	1.000	1.000	0.996	0.952
64.0	0.998	1.000	0.999	0.995	0.939
65.5	0.996	0.999	0.999	0.978	0.861
66.0	0.995	0.998	0.999	0.974	0.854
67.0	0.995	0.997	0.998	0.968	0.797
67.5	0.995	0.994	0.996	0.954	0.774
68.5	0.994	0.996	0.992	0.922	0.700
69.0	0.993	0.994	0.991	0.911	0.661
70.5	0.993	0.992	0.987	0.881	0.564
71.0	0.991	0.991	0.987	0.884	0.518
72.0	0.984	0.989	0.977	0.853	0.443
72.5	0.986	0.990	0.978	0.855	0.406
73.5	0.991	0.993	0.985	0.861	0.299
74.0	0.990	0.995	0.979	0.852	0.215
75.5	0.995	0.994	0.986	0.848	0.000
76.0	0.995	0.993	0.988	0.841	0.000
77.0	0.996	0.995	0.990	0.835	0.000
77.5	0.996	0.994	0.990	0.830	0.000
78.5	0.996	0.993	0.992	0.805	0.000
79.0	0.995	0.993	0.995	0.787	0.000
80.5	0.997	0.993	0.996	0.709	0.000
81.0	0.997	0.993	0.996	0.647	0.000
82.0	0.997	0.993	0.996	0.365	0.000
82.5	0.997	0.993	0.996	0.000	0.000
83.5	0.997	0.993	0.996	0.000	0.000
84.0	0.997	0.994	0.995	0.000	0.000

APPENDIX C-19

PROBABILITIES THAT FALSE PROBABILITIES ARE CONTAINED IN BANDS (n=20, R1, method 1)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
10.0	0.989	0.993	0.995	0.997	1.000
11.5	0.989	0.993	0.996	0.997	1.000
13.0	0.986	0.994	0.996	0.997	1.000
14.5	0.986	0.994	0.996	0.997	1.000
15.0	0.986	0.994	0.997	0.999	1.000
16.5	0.986	0.994	0.999	1.000	1.000
18.0	0.984	0.993	0.999	1.000	1.000
19.5	0.985	0.996	0.997	1.000	1.000
20.0	0.985	0.996	0.997	1.000	1.000
21.5	0.987	0.996	0.996	1.000	1.000
23.0	0.987	0.995	0.996	1.000	1.000
24.5	0.991	0.996	0.996	1.000	1.000
25.0	0.992	0.995	0.996	1.000	1.000
26.5	0.993	0.997	0.997	1.000	1.000
28.0	0.995	0.999	0.998	1.000	1.000
29.5	0.995	0.999	0.999	1.000	1.000
30.0	0.995	0.999	0.999	1.000	1.000
31.5	0.996	0.999	0.999	1.000	1.000
33.0	0.997	0.999	1.000	1.000	1.000
34.5	0.999	0.999	1.000	1.000	1.000

APPENDIX C-20

PROBABILITIES THAT FALSE PROBABILITIES ARE CONTAINED IN BANDS (n=20, R2, method 1)

x	P+0.001	P+0.005	P+0.02	P+0.05	P+0.1
35.0	0.999	1.000	0.995	0.997	1.000
36.5	1.000	1.000	0.996	0.997	1.000
38.0	1.000	1.000	0.996	0.997	1.000
39.5	1.000	1.000	0.996	0.997	1.000
40.0	1.000	1.000	0.997	0.999	1.000
41.5	1.000	1.000	0.999	1.000	1.000
43.0	1.000	1.000	0.999	1.000	1.000
44.5	1.000	1.000	0.997	1.000	1.000
45.0	1.000	1.000	0.997	1.000	1.000
46.5	0.999	0.996	0.996	1.000	1.000
48.0	0.997	0.992	0.996	1.000	1.000
49.5	0.997	0.991	0.996	1.000	1.000
50.0	0.996	0.992	0.996	1.000	1.000
51.5	0.996	0.992	0.997	1.000	1.000
53.0	0.997	0.993	0.998	1.000	1.000
54.5	0.995	0.996	0.999	1.000	1.000
55.0	0.993	0.996	0.999	1.000	1.000
56.5	0.993	0.995	0.999	1.000	1.000
58.0	0.993	0.993	1.000	1.000	1.000
59.5	0.992	0.993	1.000	1.000	1.000

APPENDIX C-21

PROBABILITIES THAT INCLUDED PROBABILITIES ARE CONTAINED IN BANDS (n=20, R3, method 1)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
60.0	0.999	0.999	0.996	0.992	0.960
61.5	0.998	0.993	0.995	0.985	0.942
63.0	0.993	0.989	0.988	0.975	0.905
64.5	0.991	0.992	0.985	0.969	0.880
65.0	0.990	0.991	0.984	0.969	0.868
66.5	0.987	0.991	0.982	0.954	0.823
68.0	0.993	0.991	0.977	0.939	0.791
69.5	0.992	0.994	0.972	0.927	0.670
70.0	0.992	0.994	0.972	0.926	0.634
71.5	0.991	0.989	0.973	0.898	0.608
73.0	0.983	0.992	0.982	0.870	0.421
74.5	0.985	0.992	0.976	0.860	0.000
75.0	0.987	0.991	0.976	0.839	0.000
76.5	0.989	0.987	0.974	0.819	0.000
78.0	0.984	0.985	0.974	0.778	0.000
79.5	0.983	0.986	0.967	0.732	0.000
80.0	0.984	0.982	0.966	0.701	0.000
81.5	0.983	0.977	0.962	0.550	0.000
83.0	0.980	0.975	0.957	0.000	0.000
84.5	0.982	0.969	0.958	0.000	0.000

APPENDIX C-22

PROBABILITIES THAT FALSE PROBABILITIES ARE CONTAINED IN BANDS (n=20, R1, method 2)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
10.0	0.994	0.995	0.999	1.000	1.000
11.5	0.997	0.996	0.999	1.000	1.000
13.0	0.998	0.997	1.000	1.000	1.000
14.5	0.998	0.997	1.000	1.000	1.000
15.0	0.998	0.998	1.000	1.000	1.000
16.5	0.999	0.999	1.000	1.000	1.000
18.0	1.000	0.999	0.999	1.000	1.000
19.5	0.999	0.999	0.998	1.000	1.000
20.0	0.998	0.999	0.998	1.000	1.000
21.5	0.996	0.997	0.997	1.000	1.000
23.0	0.994	0.996	0.995	1.000	1.000
24.5	0.989	0.990	0.994	0.999	1.000
25.0	0.989	0.991	0.994	1.000	1.000
26.5	0.996	0.999	0.999	1.000	1.000
28.0	1.000	1.000	1.000	1.000	1.000
29.5	1.000	1.000	1.000	1.000	1.000
30.0	1.000	1.000	1.000	1.000	1.000
31.5	1.000	1.000	1.000	1.000	1.000
33.0	1.000	1.000	1.000	1.000	1.000
34.5	1.000	1.000	1.000	1.000	1.000

APPENDIX C-23

PROBABILITIES THAT FALSE PROBABILITIES ARE CONTAINED IN BANDS (n=20, R2, method 2)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
35.0	1.000	1.000	0.999	1.000	1.000
36.5	1.000	1.000	0.999	1.000	1.000
38.0	1.000	1.000	1.000	1.000	1.000
39.5	1.000	1.000	1.000	1.000	1.000
40.0	1.000	1.000	1.000	1.000	1.000
41.5	1.000	1.000	1.000	1.000	1.000
43.0	1.000	1.000	0.999	1.000	1.000
44.5	1.000	0.998	0.998	1.000	1.000
45.0	1.000	0.996	0.998	1.000	1.000
46.5	0.999	0.998	0.997	1.000	1.000
48.0	1.000	0.995	0.995	1.000	1.000
49.5	0.998	0.995	0.994	0.999	1.000
50.0	0.998	0.994	0.994	1.000	1.000
51.5	0.999	0.999	0.999	1.000	1.000
53.0	1.000	0.998	1.000	1.000	1.000
54.5	1.000	1.000	1.000	1.000	1.000
55.0	1.000	1.000	1.000	1.000	1.000
56.5	1.000	1.000	1.000	1.000	1.000
58.0	1.000	1.000	1.000	1.000	1.000
59.5	1.000	1.000	1.000	1.000	1.000

APPENDIX C-24

PROBABILITIES THAT FALSE PROBABILITIES
ARE CONTAINED IN BANDS
(n=20, R3, method 2)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
60.0	1.000	1.000	1.000	1.000	0.997
61.5	1.000	1.000	1.000	1.000	0.993
63.0	1.000	1.000	1.000	0.998	0.981
64.5	0.999	1.000	0.999	0.997	0.959
65.0	0.999	1.000	0.999	0.995	0.946
66.5	0.999	1.000	0.997	0.978	0.894
68.0	0.995	1.000	0.993	0.954	0.809
69.5	0.991	0.999	0.985	0.940	0.706
70.0	0.990	0.997	0.984	0.933	0.679
71.5	0.988	0.987	0.985	0.893	0.610
73.0	0.988	0.987	0.974	0.884	0.491
74.5	0.991	0.987	0.978	0.887	0.000
75.0	0.992	0.989	0.979	0.883	0.000
76.5	0.994	0.993	0.981	0.880	0.000
78.0	0.993	0.994	0.983	0.859	0.000
79.5	0.997	0.994	0.986	0.830	0.000
80.0	0.997	0.992	0.988	0.811	0.000
81.5	0.996	0.992	0.987	0.673	0.000
83.0	0.997	0.993	0.989	0.000	0.000
84.5	0.997	0.997	0.989	0.000	0.000

APPENDIX C-25

PROBABILITIES THAT FALSE PROBABILITIES
ARE CONTAINED IN BANDS
(n=10, R1, method 1)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
12.0	0.993	0.998	0.999	1.000	1.000
14.5	0.993	0.996	0.999	1.000	1.000
17.0	0.991	0.994	0.997	1.000	1.000
19.5	0.990	0.992	0.997	0.999	1.000
22.0	0.995	0.994	0.996	1.000	0.999
24.5	0.996	0.994	0.999	1.000	1.000
27.0	1.000	1.000	1.000	1.000	1.000
29.5	1.000	1.000	1.000	1.000	1.000
32.0	1.000	1.000	1.000	1.000	1.000
34.5	1.000	1.000	1.000	1.000	1.000

APPENDIX C-26

PROBABILITIES THAT FALSE PROBABILITIES
ARE CONTAINED IN BANDS
(n=10, R2, method 1)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
37.0	1.000	1.000	1.000	1.000	1.000
39.5	1.000	1.000	1.000	1.000	1.000
42.0	0.998	0.993	1.000	1.000	1.000
44.5	0.986	0.981	1.000	1.000	1.000
47.0	0.983	0.981	1.000	1.000	0.997
49.5	0.984	0.979	0.999	0.996	0.974
52.0	0.988	0.976	0.999	0.996	0.977
54.5	0.984	0.966	1.000	0.998	0.991
57.0	0.979	0.976	1.000	1.000	0.996
59.5	0.986	0.980	1.000	1.000	0.995

APPENDIX C-27

PROBABILITIES THAT FALSE PROBABILITIES ARE CONTAINED IN BANDS (n=10, R3, method 1)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
62.0	1.000	1.000	1.000	1.000	0.989
64.5	1.000	1.000	1.000	0.993	0.931
67.0	1.000	1.000	0.984	0.953	0.871
69.5	1.000	1.000	0.979	0.899	0.725
72.0	1.000	1.000	0.975	0.874	0.605
74.5	1.000	1.000	0.984	0.877	0.000
77.0	1.000	1.000	0.971	0.790	0.000
79.5	1.000	1.000	0.972	0.696	0.000
82.0	1.000	1.000	0.947	0.433	0.000
84.5	1.000	1.000	0.952	0.000	0.000

APPENDIX C-28

PROBABILITIES THAT FALSE PROBABILITIES ARE CONTAINED IN BANDS (n=10, R1, method 2)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
12.0	1.000	1.000	1.000	1.000	1.000
14.5	1.000	1.000	1.000	1.000	1.000
17.0	1.000	1.000	1.000	1.000	1.000
19.5	1.000	1.000	1.000	1.000	1.000
22.0	1.000	0.999	0.999	1.000	1.000
24.5	1.000	0.994	0.998	1.000	1.000
27.0	1.000	1.000	1.000	1.000	1.000
29.5	1.000	1.000	1.000	1.000	1.000
32.0	1.000	1.000	1.000	1.000	1.000
34.5	1.000	1.000	1.000	1.000	1.000

APPENDIX C-29

PROBABILITIES THAT FALSE PROBABILITIES
ARE CONTAINED IN BANDS
(n=10, R2, method 2)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
37.0	1.000	1.000	1.000	1.000	1.000
39.5	1.000	1.000	1.000	1.000	1.000
42.0	1.000	1.000	1.000	1.000	1.000
44.5	1.000	0.999	1.000	1.000	1.000
47.0	0.989	0.978	1.000	1.000	1.000
49.5	0.984	0.988	0.999	0.998	0.985
52.0	0.993	0.986	0.998	0.998	0.995
54.5	0.995	0.997	1.000	1.000	0.999
57.0	0.997	1.000	1.000	1.000	1.000
59.5	1.000	1.000	1.000	1.000	1.000

APPENDIX C-30

PROBABILITIES THAT FALSE PROBABILITIES
ARE CONTAINED IN BANDS
(n=10, R3, method 2)

x	P+0.001	P+0.005	P+0.02	P+0.06	P+0.1
62.0	1.000	1.000	1.000	1.000	1.000
64.5	1.000	1.000	1.000	1.000	0.997
67.0	1.000	1.000	1.000	0.995	0.960
69.5	1.000	1.000	0.999	0.970	0.830
72.0	1.000	1.000	0.986	0.885	0.652
74.5	0.998	0.998	0.978	0.891	0.000
77.0	1.000	0.999	0.988	0.863	0.000
79.5	1.000	1.000	0.994	0.828	0.000
82.0	1.000	1.000	0.998	0.532	0.000
84.5	1.000	1.000	0.998	0.000	0.000

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VITA

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