INTERSECTION NUMBERS: A DEVELOPMENT OF FORMULAS FOR DEGREE AND GENUS RELEVANT TO COMPUTER AIDED GEOMETRIC DESIGN

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Submitted to the Faculty of the
Graduate College of the
Oklahoma State University
in partial fulfillment of
the requirements for
the Degree of
DOCTOR OF EDUCATION
May, 1994

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ACKNOWLEGDMENTS

This paper is a culmination of many years work on my part, but mostly, it is a culmination of the influence upon me by many people. I can only acknowledge a few of them here.

The style of exposition in this paper has been influenced by many people. It all began at the University of Mississippi where the Drs. Stokes and Cook not only required me to write proofs but to present them to the critical eye of 27 other students. Ray Hamlett provided helpful comments about my writing during my years at Arkansas Tech. John Wolfe taught me that the flow of the writing was at least as important as the mathematics. I am sure that bits of all of these people are somewhere in this paper and I thank them all for the guidance they provided.

The TEX file used to create this document would have taken much longer to produce had it not been for the help of David Farmer and Greg Force. David gave me the files he used for his dissertation so that I could see how he formatted it to meet the Graduate College requirements. In an amazingly short amount of time, Greg became an expert with TEX and as a result became my guru. He is responsible for the macros which created the page numbers and helped me with the formats of the table of contents, list of figures, and list of tables.

I had originally wanted to hand draw the figures used in this paper, but then I thought it would be ironic to not use graphics to create the figures for a paper designed for people in CAGD. The main reason I wanted to hand draw the figures is that I thought that would be quicker than learning how to use a graphing package. I was also afraid that I would not be able to find a graphing package that could create the type of figures I wanted. Anthony Kable came to my rescue and in a short time taught me how to use Maple and xfig and how to imbed postscript files in a TeX document. He also created the graph used for Figure 3.2.

I want to thank all the instructors that I have had over the last ten years at OSU. Learning, however, went on in many other places than just in the classroom. In various conversations, Charles Vuono, Paulo Aluffi, Anthony Kable, and Tom Zerger helped me "see" the mathematics that I wanted to write about. I thank Dennis Bertholf, Joel Haack, and Sheldon Katz for writing the letters of recommendation that helped get the job at Simpson College. Of course, I thank Mark McConnell, Jim Cogdell, Alan Noell, David Webster, and Sheldon Katz for serving on my advisory committee. Sheldon, in particular, deserves special thanks. Not only did he spend a great amount of time helping me and reading this paper, he also suggested the topic of this paper. As much as I have complained about having to learn some bits of algebraic geometry, it is still a excellent topic for an Ed. D. dissertation.

A little over a decade ago I got a call from John Jobe asking if I wanted to go to graduate school. Since then, I have finished a Master's degree, worked for four years full time, lived in 3 different states and almost finished an Ed.D. Without the support of my family, friends, and colleagues, I would not have made it through this decade. They provided help to me in various ways: housing when I was homeless, boarding my cats, lending books, giving rides to the emergency room, lending luggage, taking me to movies, giving impromptu lectures, letting me bum cigarettes, discussing the merits of TeX over LaTeX, and demonstrating software. To all of those people who have been a part of my life during these years I owe much appreciation especially: Narendhar Jambunathan, who housed me, fed me, and never said my work was unimportant; Saigheetha Jambunathan, who also housed and fed me;

Yiannis Vlassapolous, who can shoot pool continuously for 6 hours and still want more; my students who never complained all the times I lost my voice; Colin Burvill, who I have never met but knows me better than anyone else does; and that group of adult students at OSU who know how to have a good time.

I especially want to thank those people who are closest to me. My sister and her husband, B.J. and David Abbott, provided both housing and much needed recreation. David also read and provided helpful comments on part of the paper. Mike and Jennifer Shallop not only took me into their home but reminded me that teaching and learning are one and the same. Linda Hand Noell gave me a number of textbooks that I could not have afforded to buy. I greatly appreciate my parents, Rose Mary and Alfred Waggoner, not only for financial support but also for never telling me that I was a girl. I never thought there was anything I could not or should not do because I was female. And finally, Werner Kolln deserves special thanks for driving from Indianola to Stillwater and back so I would not have to and for providing just the right amount of support when I needed it most.

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$\mathbb{C}^n$	n-dimensional complex space
$\mathbb{P}^n$	n-dimensional projective space
$f : X \dashrightarrow Y$	rational map from X to Y
$(x_1, \ldots, x_n)$	affine coordinates
$(x_0 : \ldots : x_n)$	homogeneous coordinates
$k[x_1, \ldots, x_n]$	set of all polynomials in variables $x_1, \ldots, x_n$ over the field k
$m_{\mathbf{p}}(f)$	multiplicity of f at $\mathbf{p}$
$i(\mathbf{p}, f \cap g)$	intersection multiplicity of f and g at $\mathbf{p}$
$\mathcal{O}_{\mathbf{p}}$	vector space of rational functions defined at $\mathbf{p}$
V/W	quotient vector space of V over W
$\tilde{X}$	blowup of the manifold X
$\tilde{C}$	strict transform of the curve C
$\pi : \tilde{X} \longrightarrow X$	projection map of the blowup of X
$\mathrm{Div}\,(X)$	set of all divisors on X
$\mathrm{Pic}\,(X)$	Picard group on X
(f)	divisor of the function or curve f
π^*	pullback map
$[D]$	class of divisors linearly equivalent to D or the line bundle associated with the divisor D
$\mathcal{J}f(x,y)$	Jacobian of f
$\|D\|$	complete linear system of divisors linearly equivalent to D
$D\|_C$	restriction of the divisor D to the curve C
$C \cdot D$	intersection number of the divisors C and D
D^2	self intersection number of the divisor D
$p_g(C)$	geometric genus of C
$p_a(C)$	arithmetic genus of C
K_X	canonical divisor on the manifold X

CHAPTER ONE

INTRODUCTION AND BACKGROUND MATERIAL

1.1. Background and Calculations to Be Described

It is well accepted that algebraic geometry is a difficult subject. "Algebraic geometry is among the oldest and most highly developed subjects in mathematics. ... Moreover, in recent years algebraic geometry has undergone vast changes in style and language. For these reasons there has arisen about the subject a reputation of inaccessibility [GH p. v]." However, there are people in other fields, such as computer aided geometric design (CAGD), who need some of the results from algebraic geometry. The inaccessibility of the algebraic geometry literature makes it difficult for outsiders to use. "It is no secret that Algebraic Geometry has a vast literature, largely indigestible. ... The papers found in modern research journals are so written that the reader will not be able to make headway unless his mind is already well supplied with the notions being employed. ... At most the situation can be occasionally alleviated by expository articles [Sei p. vii]." The purpose of this paper is to make a small part of algebraic geometry accessible to a larger audience.

There are no new ideas in this paper, but this is the first time these ideas have been presented in this context. This paper brings together specifically the topics necessary for understanding the calculations of degree and genus and takes these topics out of the context of general results. It is hoped that this paper will allow those interested, especially the CAGD community, to gain access to this material without having to wade through the "vast literature" of algebraic geometry.

In [KS], published in 1988, there are formulas for calculating the genus of a curve of intersection of two rational surfaces which would in turn allow someone to determine if the curve of intersection could be represented by a rational curve. In the CAGD community, robust surface intersection algorithms are a lively topic of investigation and the underlying theory of degree and genus is crucial to that investigation [S1]. In [KS], the formulas for the implicit degree is given only for surfaces with simple base points and the formulas for genus was given only for the general intersection of two surfaces with simple base points. There is also a small discussion about the genus of surfaces which are tangent.

This paper is an attempt to expand and clarify the information presented in [KS]. Formulas for implicit degree are developed for general triangular and tensor product surfaces with any number and type of base points. This includes the complete development of intersection numbers for divisors on blow-ups of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ which is done using few outside results. The formulas for genus presented here are only for general intersection curves as in [KS]. I hope that these formulas are developed fully enough so that the reader could extend these formulas for more general situations. Instead of developing the formula for genus from the definition, this paper begins with the adjunction formula which gives the genus of a nonsingular curve using intersection numbers. Also, the nature of singularities on a surface is used but not developed here. The formula for the genus of the general intersection of two tensor product surfaces with simple base points presented in this paper is a correction of the one given in [KS].

The proofs in this paper often follow closely the proofs from other sources. In particular, the proof of Bertini's Theorem comes from [GH], the proof of Proposition 1.1 was taken from [W], and the definition of intersection numbers was taken from [Har]. The section on intersection multiplicity follows [Ful1] except that in this paper the intersection multiplicity is defined as the dimension of a quotient vector space instead of the dimension of a quotient ring.

The audience of this paper is assumed to have some background in algebra, topology, and complex analysis. Although the topological terms used in this paper are defined here, the reader is assumed to be familiar with these topics. The implicit function theorem, the open mapping theorem, and holomorphic and meromorphic functions are used from multivariable complex analysis without explanation. Vector spaces, quotient vector spaces, and dimension of vector spaces are used throughout the paper. A thorough knowledge of algebra is not necessary to read this paper, but an understanding of groups would be helpful to read the section on divisors and line bundles.

There are basically two formulas presented in this paper: implicit degree of a rational surface and the genus of the curve of intersection of two rational surfaces. The purpose of calculating the genus of the curve of intesection is to determine if the curve can be represented as a rational curve. A *rational surface* in $\mathbb{C}^3$ is defined as the closure of the set of points $(x_1, x_2, x_3) \in \mathbb{C}^3$ defined by parametric equations

$$x_i = \frac{f_i(x,y)}{f_0(x,y)}$$

for $i = 1, 2, 3$ where each f_i is a polynomial and $(x,y) \in \mathbb{C}^2$. An *implicit surface* in $\mathbb{C}^3$ is the set of points in $\mathbb{C}^3$ satisfying a polynomial equation of the form $F(x_1, x_2, x_3) = 0$. Implicit surfaces are also called *algebraic surfaces.* All rational surfaces are algebraic surfaces and can be expressed by an implicit equation [SAG]. Determining the implicit equation of a rational surface is called *implicitization.* This paper contains an outline of a method of calculating the degree of a rational surface, i.e., the degree of the implicit equation of the surface. This calculation is described in Chapter 5.

A *rational curve* in $\mathbb{C}^3$ is defined as the closure of the set of points $(x_1, x_2, x_3) \in \mathbb{C}^3$

defined by parametric equations

$$x_i = \frac{f_i(s)}{f_0(s)}$$

for $i = 1, 2, 3$ where each f_i is a polynomial and $s \in \mathbb{C}$. The nondegenerate intersection of two rational surfaces results in a curve. The question of whether or not such a curve is rational can be answered by computing its *genus*. It is well known among algebraic geometers that only curves of genus zero are rational[Sa p. 30, W p. 67]. The techniques for calculating the algebraic degree of a surface can be extended to calculate the genus of the curve of intersection of two rational surfaces. The calculation of the genus of some specific curves of intersection is contained in Chapter 6.

The calculation of the degree of a rational surface is complicated by the existence of *base points*, which are parameter pairs (s_j, t_j) for which $f_i(s_j, t_j) = 0$ for all i. The first step in calculating the degree will be to create a new parameter space in which there are no base points. This is accomplished by a process called blowing up which is described in Chapter 3. In a similar way, the calculation of the genus of a curve is complicated by the existence of *singular points*, which are points where the curve does not have a unique tangent direction. The method of blowing up will produce a curve whose genus is the same as the original curve but with no singular points and allows us to calculate the genus of the original curve.

Both calculations involve counting the number of points in the intersection of certain curves. When there are no base points or singularities, these curves are in $\mathbb{P}^2$ and Chapter 2 is devoted to counting the number of points in the intersection of curves in $\mathbb{P}^2$. With the existence of base points or singularities and after blowing up, the curves used will reside in 2-dimensional complex manifolds other than $\mathbb{P}^2$. We will use the power of *divisors* on these manifolds to calculate the number of points in the intersection of certain curves. Divisors and their properties are described in Chapter 4.

The curves and surfaces used in this paper will be subsets of complex projective space and other complex manifolds. The remainder of this chapter is devoted to background material on topological spaces, manifolds, curves, functions, and a more complete discussion of rational surfaces.

1.2. Topological Spaces

The calculations of the degree of a surface and genus of a curve can be approached from either a topological or algebraic point of view. This paper is written from a mostly topological viewpoint. An important tool for the calculations in algebraic geometry is that certain numbers, such as genus, are *topological invariants.* That is, if two curves are topologically equivalent, then their genus will be the same. In this section, we give a brief introduction to the terminology used in topology.

A *topology on a set* X is a collection T of subsets of X such that

1. $\emptyset$ and X are in T;
2. the union of arbitrarily many elements of T is in T; and
3. the intersection of finitely many elements of T is in T.

The elements of T are called the *open sets* of the topology on X. The complements of the elements of T are called the *closed sets.* This rather formal definition is a generalization of the idea of open and closed set in $\mathbb{C}^n$.

Of course, it is not always possible to list all elements of a topology. Instead, we can describe a topology by defining a small collection of subsets of X and a method of determining all other elements of the topology. A collection $\mathcal{B}$ of subsets of X is called a *basis* for a topology on X if

1. for every $x \in X$ there exists a $B \in \mathcal{B}$ such that $x \in B$ and

2. if $x \in B_1 \cap B_2$ and $B_i \in \mathcal{B}$, then there is another element $B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \cap B_2$.

The *topology generated by* $\mathcal{B}$ is defined to be $T = \{U \subseteq X\}$ such that for all $x \in U$ there is an element $B \in \mathcal{B}$ such that $x \in B \subseteq U$. It is easy to check that all such sets indeed form a topology. It is also easy to check that T could also be described as the collection of all arbitrary unions of elements of $\mathcal{B}$.

Example 1.1: Let $\mathcal{B}$ consist of all the subsets of $\mathbb{C}$ of the form

$$\Delta(x,r) = \{y \in \mathbb{C} : |y - x| \leq r\}$$

for each $r \in \mathbb{R}$ such that $r > 0$. Let $\mathcal{B}'$ consist of all subsets of $\mathbb{C}^n$ of the form

$$\Delta(\mathbf{x},\mathbf{r}) = \{\mathbf{y} \in \mathbb{C}^n : |y_i - x_i| \leq r_i \text{ for each } i\}$$

where $\mathbf{r} \in \mathbb{R}^n$ with $r_i > 0$ for each i. Clearly, each of these is a basis for a topology. In fact, these define the usual open neighborhoods in $\mathbb{C}$ and $\mathbb{C}^n$ and the topologies generated are called the *standard topologies* on $\mathbb{C}$ and $\mathbb{C}^n$. Throughout this paper we will use the standard topology on $\mathbb{C}^n$. It is useful to note that a set of points in $\mathbb{C}^2$ defined by a strict inequality is open, such as

$$M = \{(x,y) \in \mathbb{C}^2 : x^3 - 2y^2 \neq 3\}.$$

A *topological space* is a set X with a topology T. The set T will be assumed when we refer to its elements as open sets on X. Let X and Y be two topological spaces and $f : X \longrightarrow Y$ a map. The map f is said to be *continuous* if the inverse image of all open sets in Y are open sets in X. This is exactly the δ-ϵ definition of continuity for functions $f : \mathbb{C} \longrightarrow \mathbb{C}$. Two topological spaces X and Y are equivalent if there is a bijective, bicontinuous map

$f : X \longrightarrow Y$. If such an f exists, the spaces are said to be *homeomorphic* and f is called a *homeomorphism.*

Of course, there are other topologies on $\mathbb{C}^n$. For example, the collection of all one point subsets of $\mathbb{C}^n$ forms a basis for a topology call the *discrete topology.* It is easy to show the discrete topology and the standard topology are not the same. Of more interest to algebraic geometers is a topology on $\mathbb{C}^n$ called the *Zariski topology.* The closed sets of this topology are the sets $\{\mathbf{x} \in \mathbb{C}^n : f(x_1, \cdots, x_n) = 0\}$ where f is a polynomial. The open sets are the complements of all closed sets. At first glance it might appear the Zariski topology is not much different from the standard topology. In fact, the closed sets of the Zariski topology are also closed sets in the standard topology. But these topologies are very different as we shall see.

A topology on X is said to be *Hausdorff* if for each pair of distinct points a and $b \in X$ there are open disjoint nonempty sets U and V such that $a \in U$ and $b \in V$. In other words, we can always separate points with open sets in a Hausdorff topology. Obviously, if $f : X \longrightarrow Y$ is a homeomorphism and X is Hausdorff, then Y is Hausdorff. It is also clear that $\mathbb{C}^n$ with the standard topology is Hausdorff. We will show the Zariski topology is not Hausdorff, and therefore, is not homeomorphic to the standard topology.

Let U and V be any two nonempty open sets in the Zariski topology. The sets $\mathbb{C}^n - U$ and $\mathbb{C}^n - V$ are closed in the Zariski topology and are the zero loci of two polynomials f and g. Let $\mathbf{x} \in \mathbb{C}^n$ such that $f(\mathbf{x}) \neq 0$ and $g(\mathbf{x}) \neq 0$. Thus $\mathbf{x} \in U \cap V$. Therefore, every two open sets in the Zariski topology have nonempty intersection, and the Zariski topology cannot be Hausdorff.

Another way to create a topology is to impose the topological structure of a set X onto any subset of X. Let Y be a subset of X and define the open sets of Y under the *subspace topology* to be all sets U such that $U = Y \cap V$ for some open set V in X. For all sets X and Y and injective maps $f : X \longrightarrow Y$, the set X can be considered as a subspace of Y by identifying it with the subspace $f(X) \subseteq Y$. Let X and Y be topological spaces. If $f : X \longrightarrow f(X) \subseteq Y$ is a homeomorphism, then X can be considered a subset of Y with the subspace topology. The map f is then called an *embedding* of X into Y.

A few more definitions will be useful. A collection of sets $\{U_\alpha\}$ with $\alpha \in A$ is said to *cover* a set X if

$$X \subseteq \bigcup_{\alpha \in A} U_\alpha.$$

If each set U_α is open, the collection is called an *open cover.* The *closure* of a set A is the intersection of all closed sets C containing A. Clearly, the closure of a closed set is the set itself. A set A is *dense in* X if the closure of A is equal to X. For example, if $A = \mathbb{C}^2 - \{\mathbf{p}\}$ in the standard topology, then the closure of A is all of $\mathbb{C}^2$ and so A is dense in $\mathbb{C}^2$. Similarly, the closure of the set M under the standard topology in Example 1.1 is all of $\mathbb{C}^2$. A set A in X is *connected* if there do not exist 2 open nonempty disjoint subsets B and C of X such that $A \subseteq B \cup C$, $A \cap B \neq \emptyset$, and $A \cap C \neq \emptyset$. Finally, a set $A \subseteq X$ is *compact* if for every cover $\{U_\alpha\}$ of A there is a finite subcover $\{U_{\alpha_i}\}_{i=1}^r$ of A.

1.3. Complex Manifolds

1.3.1. *General Complex Manifolds.*

We will begin working in complex projective space, but later it will be necessary to use other topological spaces called n-dimensional complex manifolds. An *n-dimensional complex manifold* X is a Hausdorff topological space which looks locally like $\mathbb{C}^n$, i.e., X is covered by open sets $U_i \subseteq X$ for which there are homeomorphic maps

$$\phi_i : U_i \to V_i$$

onto open sets $V_i \subseteq \mathbb{C}^n$. The U_i are called *coordinate neighborhoods*, and the ϕ_i are called *coordinate charts.* The coordinates $(x_1, \ldots, x_n) = \phi_i(\mathbf{p})$ for $\mathbf{p} \in U_i$ are called the *local* *coordinates of* p *on* U_i. In addition, the maps $\phi_{ij} = \phi_i \circ \phi_j^{-1}$ are required to be holomorphic on $\phi_j(U_i \cap U_j)$. The maps ϕ_{ij} are called *coordinate transformations* or *gluing maps* and give the relationship of different local coordinates of the same point. Note that $\phi_{ij} = \phi_{ji}^{-1}$. Also, if we know X is compact, then $\{U_i\}$ can be assumed to be a finite cover. If the manifold has a coordinate system itself, those coordinates are called the *global coordinates* of the manifold. Not all manifolds have a global coordinate system.

Example 1.2: Let us illustrate this definition with an example of a 2-dimensional real manifold which looks locally like $\mathbb{R}^2$. Consider the surface $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$. We will cover this sphere with open sets which are homeomorphic to subsets of $\mathbb{R}^2$. Let U_0 be the subset of the sphere where $z > 0$ and V_0 be the unit disk D in $\mathbb{R}^2$. The map $\phi_0 : U_0 \longrightarrow V_0$ defined by projection onto the first two coordinates is a homeomorphism. Similarly, the subsets of the surface $U_1 = \{z < 0\}$, $U_2 = \{x > 0\}$, $U_3 = \{x < 0\}$, $U_4 = \{y > 0\}$, and $U_5 = \{y < 0\}$ are homeomorphic to $V_i = D$ and have coordinate charts ϕ_i for i = 1, 2, 3, 4 and 5 defined as projections onto the appropriate coordinates. The open sets U_i cover the unit sphere and any point on the sphere can now be identified by the set U_i to which it belongs and the 2 real coordinates of the point's image in V_i under ϕ_i.

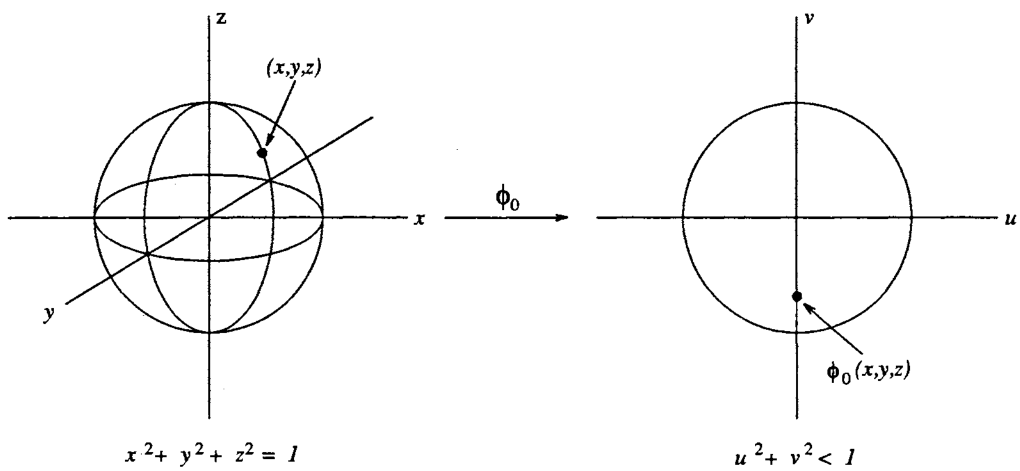


Figure 1.1

Coordinate Map of U_0 onto V_0

Consider the point $\mathbf{p} = \left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right) \in U_0 \cap U_2$ on the sphere. The local coordinates of this point in each of U_0 and U_2 are defined by $\phi_0(\mathbf{p}) = \left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}\right) \in V_0$ and $\phi_2(\mathbf{p}) = \left(\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right) \in V_2$. The gluing map ϕ_{02} is defined by $\phi_{02}(a,b) = \left(\sqrt{1-a^2-b^2}, a\right)$ and allows us to convert from the local coordinates of U_2 to the local coordinates of U_0. Note that ϕ_{02} is differentiable on $U_0 \cap U_2$. We can check $\phi_{02}\left(\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right) = \left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}\right)$. For this real manifold, the gluing maps ϕ_{01}, ϕ_{23}, and ϕ_{45} have empty domains.

Unless otherwise stated, every manifold in this paper is a compact complex manifold and will simply be referred to as a manifold.

1.3.2. *Projective Space as a Complex Manifold.*

The sets $\mathbb{C}^n$ for all n are trivially n-dimensional manifolds. We will work with n-dimensional complex projective space $\mathbb{P}^n$ which is also an n-dimensional manifold. Define *n-dimensional complex projective space* as the quotient space $\mathbb{P}^n = \mathbb{C}^{n+1} - \{\mathbf{0}\}/\sim$ where $\sim$

is the equivalence relation such that $(y_0, \cdots, y_n) \sim (x_0, \cdots, x_n)$ if and only if $(y_0, \cdots, y_n) = (\lambda x_0, \cdots, \lambda x_n)$ for some complex $\lambda \neq 0$. (See Section 2.3 for quotient spaces.) Denote the equivalence class of $(x_0, \cdots, x_n)$ by $(x_0 : \cdots : x_n)$ and call the $(n+1)$-tuple $(x_0 : \cdots : x_n)$ of complex numbers the *homogeneous coordinates* of the point $(x_0, \cdots, x_n)$.

To show that $\mathbb{P}^n$ is a complex n-dimensional manifold we will cover $\mathbb{P}^n$ with the $n+1$ open sets $U_i = \{(x_0 : \cdots : x_n) \in \mathbb{P}^n : x_i \neq 0\}$. The coordinate chart $\phi_i(x_0 : \cdots : x_n) = \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right)$ from U_i to $V_i = \mathbb{C}^n$ is continuous and bijective, and its inverse $\phi_i^{-1}(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$ is also continuous. It is easy to check that the gluing maps ϕ_{ij} are holomorphic for each i, j. Therefore, $\mathbb{P}^n$ is a manifold. The homogeneous coordinates $(x_0 : \ldots : x_n)$ are called the *global coordinates* of $\mathbb{P}^n$. Throughout this paper, the sets U_i will either denote the general open sets in the open cover of a manifold or the specific open sets in the open cover of $\mathbb{P}^n$. The specific use of U_i will be made clear from the context.

The definition of manifold first requires the set $\mathbb{P}^n$ to be a Hausdorff topological space. What, then, is the topology on $\mathbb{P}^n$? The most convenient would be the Zariski topology defined in the same way as it was for $\mathbb{C}^n$. We, however, will use a topology on $\mathbb{P}^n$ which is consistent with the standard topology on each coordinate neighborhood. The basis for this topology is

$$\bigcup_{i=0}^{n} \left\{ \phi_i^{-1} \Delta(\phi_i(\mathbf{x}), \mathbf{r}) : \mathbf{x} \in U_i \text{ and } \mathbf{r} \in \mathbb{R}^n \text{ with each } r_j > 0 \right\}.$$

We will refer to the manifold structure on $\mathbb{P}^2$ frequently and the following notation will be convenient. Let $(x : y : z)$ be the homogeneous coordinates of $\mathbb{P}^2$. To distinguish the local coordinates in each U_i, we will give these coordinates unique names: (y_0, z_0) in U_0, (x_1, z_1) in U_1, and (x_2, y_2) in U_2. The coordinate charts $\phi_i : U_i \longrightarrow V_i$ are defined as above. For example, if $(x : y : z) \in U_0$, then $\phi_0(x : y : z) = (y/x, z/x) = (y_0, z_0)$. The gluing maps ϕ_{ij} for

$\mathbb{P}^2$ are given in Table 1.1 using this notation.

TABLE 1.1

GLUING MAPS ON $\mathbb{P}^2$

j	i	Local coordinates on U_j	Local coordinates on U_i
0	1	(y_0, z_0)	$\phi_{10}(y_0, z_0) = (1/y_0, z_0/y_0) = (x_1, z_1)$
1	0	(x_1, z_1)	$\phi_{01}(x_1, z_1) = (1/x_1, z_1/x_1) = (y_0, z_0)$
0	2	(y_0, z_0)	$\phi_{20}(y_0, z_0) = (1/z_0, y_0/z_0) = (x_2, y_2)$
2	0	(x_2, y_2)	$\phi_{02}(x_2, y_2) = (y_2/x_2, 1/x_2) = (y_0, z_0)$
1	2	(x_1, z_1)	$\phi_{21}(x_1, z_1) = (x_1/z_1, 1/z_1) = (x_2, y_2)$
2	1	(x_2, y_2)	$\phi_{12}(x_2, y_2) = (x_2/y_2, 1/y_2) = (x_1, z_1)$

1.3.3. *Another Complex Manifold:* $\mathbb{P}^1 \times \mathbb{P}^1$.

Another 2-dimensional manifold which we will frequently use is $\mathbb{P}^1 \times \mathbb{P}^1$. Let the coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$ be $(\rho_0 : \rho_1 ; \sigma_0 : \sigma_1)$, where $(\rho_0 : \rho_1)$ and $(\sigma_0 : \sigma_1)$ are the homogeneous coordinates of each $\mathbb{P}^1$. Cover $\mathbb{P}^1 \times \mathbb{P}^1$ with the 4 open sets

$$W_{ij} = \{(\rho_0 : \rho_1 ; \sigma_0 : \sigma_1) : \rho_i \neq 0 \text{ and } \sigma_j \neq 0\}$$

for $i = 0$, 1 and $j = 0$, 1. The coordinate chart $\phi_{(ij)} : W_{ij} \longrightarrow \mathbb{C}^2$ is defined by taking the cartesian product of the coordinate charts for each $\mathbb{P}^1$. For example, the coordinate chart $\phi_{(00)} : W_{00} \longrightarrow \mathbb{C}^2$ is defined by $\phi_{00}(\rho_0 : \rho_1 ; \sigma_0 : \sigma_1) = \left(\frac{\rho_1}{\rho_0}, \frac{\sigma_1}{\sigma_0}\right)$.

For the gluing maps below, name the local coordinates (r_i, s_j) on W_{ij}. Some of the gluing maps $\phi_{kl,ij} : W_{ij} \longrightarrow W_{kl}$ are in the Table 1.2 and it is easy to see they are holomorphic on $\phi_{(ij)}(W_{ij} \cap W_{kl})$.

TABLE 1.2

SOME GLUING MAPS ON $\mathbb{P}^1 \times \mathbb{P}^1$

(k,l)	(i,j)	Local coordinates on W_{kl}	Local coordinates on W_{ij}
(0,0)	(0,1)	(r_0, s_1)	$\phi_{00,01}(r_0, s_1) = (r_0, 1/s_1) = (r_0, s_0)$
(0,0)	(1,0)	(r_1, s_0)	$\phi_{00,11}(r_1, s_0) = (1/r_1, s_1) = (r_0, s_1)$
(0,1)	(1,0)	(r_0, s_1)	$\phi_{10,01}(r_0, s_1) = (1/r_0, 1/s_1) = (r_1, s_0)$
(0,0)	(0,1)	(r_0, s_0)	$\phi_{01,00}(r_0, s_0) = (r_0, 1/s_0) = (r_0, s_1)$

1.4. Depicting Curves in $\mathbb{P}^2$

We do not have a physical model of $\mathbb{C}^2$ (it would require four real dimensions), so to depict curves in $\mathbb{C}^2$ we usually sketch the real part of those curves in $\mathbb{R}^2$. Unfortunately, much important information may be left undiscovered from the real part of the curve. For example, we usually expect to be able to determine the singularities of a curve from its graph. If the graph is smooth and does not intersect itself, we assume there are no singularities. However, the graph of the real part of $x^6 - x^2y^3 - y^5 = 0$ looks very nice, yet it has a multiple point at the origin with one triple tangent line and two simple tangent lines there (see Figure 1.2). Keeping this in mind, the graphs of various curves in $\mathbb{R}^2$ are still quite useful in understanding their geometric properties.

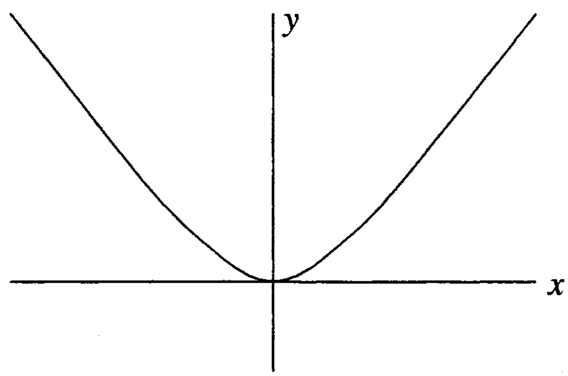


Figure 1.2

$$x^6 - x^2y^3 - y^5 = 0$$

There is no physical model of $\mathbb{P}^2$, but we would also like to depict curves on $\mathbb{P}^2$. Even if we restrict our sketch to the real part of the curve, i.e., the part in two dimensional real projective space $\mathbb{P}^2\mathbb{R}$, we cannot sketch it because there is no physical model of $\mathbb{P}^2\mathbb{R}$ either. At first it might seem we could think of $\mathbb{P}^2\mathbb{R}$ as a surface in $\mathbb{R}^3$ since we used three coordinates to define points in $\mathbb{P}^2\mathbb{R}$. But, in fact, there is no embedding of $\mathbb{P}^2\mathbb{R}$ into $\mathbb{R}^3$[MS p. 120]. However, there are several good, if somewhat imperfect, ways to represent $\mathbb{P}^2\mathbb{R}$ in $\mathbb{R}^3$. One is by using barycentric coordinates.

Let $\mathbf{p}_0$, $\mathbf{p}_1$ and $\mathbf{p}_2$ be the vertices of an equilateral triangle with height 1 in $\mathbb{R}^2$. Let L_0 be the line through $\mathbf{p}_1$ and $\mathbf{p}_2$, L_1 the line through $\mathbf{p}_0$ and $\mathbf{p}_2$, and L_2 the line through $\mathbf{p}_0$ and $\mathbf{p}_1$. Let $\mathbf{p} \in \mathbb{R}^2$. There is a unique set of numbers (a, b, c) such that the coordinates of the points $\mathbf{p}$, $\mathbf{p}_0$, $\mathbf{p}_1$ and $\mathbf{p}_2$ satisfy the equation

$$\mathbf{p} = a\mathbf{p}_0 + b\mathbf{p}_1 + c\mathbf{p}_2$$

and

$$a + b + c = 1.$$

Call (a, b, c) the *barycentric coordinates of* $\mathbf{p}$ *with respect to* $\mathbf{p}_0$, $\mathbf{p}_1$, *and* $\mathbf{p}_2$. The unique point $\mathbf{p}$ represented by barycentric coordinates (a, b, c) lies a distance from L_0, b distance from L_1, and c distance from L_2 (see Figure 1.3). A point with all positive barycentric coordinates lies inside the equilateral triangle; a point lies outside if any barycentric coordinate is negative. In this way, the points $\mathbf{p}_0$, $\mathbf{p}_1$, and $\mathbf{p}_2$ have barycentric coordinates $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, respectively.

Let $\mathbf{p} = (x{:}\,y{:}\,z)$ be any point in $\mathbb{P}^2\mathbb{R}$. It would be nice if we could say we find barycentric coordinates for this point and plot it. Unfortunately, we can only do this when $x+y+z \neq 0$. If $x+y+z \neq 0$, then $\mathbf{p} = (x'{:}\,y'{:}\,z') = \left(\frac{x}{x+y+z}{:}\,\frac{y}{x+y+z}{:}\,\frac{z}{x+y+z}\right) \in \mathbb{P}^2$, $x' + y' + z' = 1$, and we can plot (x', y', z') in $\mathbb{R}^2$ as barycentric coordinates. Using this scheme, we can plot almost all points in $\mathbb{P}^2\mathbb{R}$. We can also plot curves in $\mathbb{P}^2\mathbb{R}$, e.g., the line $x = y$ is the perpendicular bisector of the line segment joining $\mathbf{p}_0$ and $\mathbf{p}_1$ (see Figure 1.4). Keep in mind not all the points of the real part of the curve are plotted. In the example $x = y$, the point $(1{:}\,1{:}\,-2)$ is not plotted.

The manifold structure on $\mathbb{P}^2$ gives us other ways to sketch the real part of curves in $\mathbb{P}^2$. Again, not all points in $\mathbb{P}^2\mathbb{R}$ will be plotted, but by omitting a "nice" set of points we get a picture which is more familiar to us. An algebraic curve in $\mathbb{P}^2$ is defined by a homogeneous polynomial equation $F(x{:}\,y{:}\,z) = 0$. For the equation $F = 0$ to be well defined on $\mathbb{P}^2$, F must be a homogeneous polynomial. A homogeneous polynomial of degree n is called a *form of degree* n. Let $f_i = F \circ \phi_i^{-1}$ where the ϕ_i are the coordinate charts for $\mathbb{P}^2$, i.e., $f_0(y_0, z_0) = F(1{:}\,y_0{:}\,z_0)$, $f_1(x_1, z_1) = F(x_1{:}\,1{:}\,z_1)$, and $f_2(x_2, y_2) = F(x_2{:}\,y_2{:}\,1)$. This is the process of *dehomogenizing* each polynomial with respect to the variables x, y, and z, respectively. Each $f_i = 0$ is defined on $U_i = \mathbb{C}^2$, and we can sketch the real part of each $f_i = 0$ in $\mathbb{R}^2$.

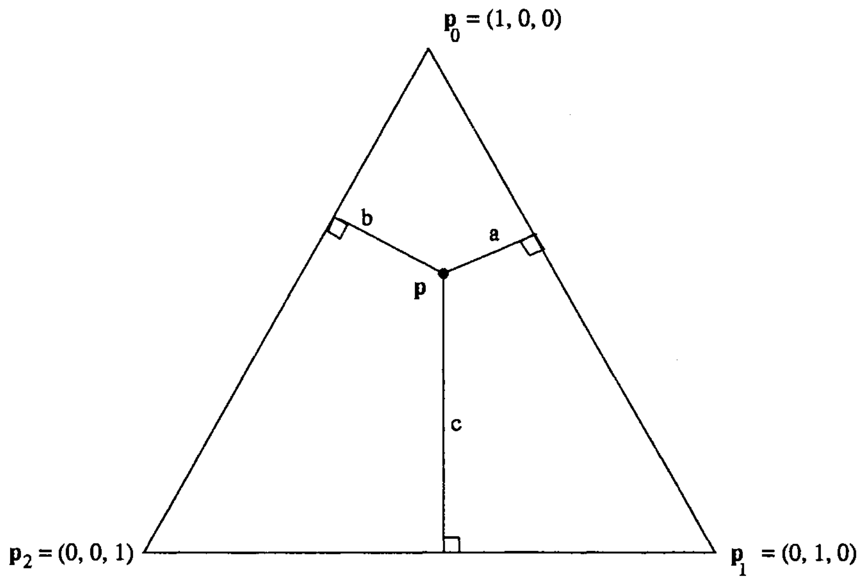


Figure 1.3

Barycentric Coordinates

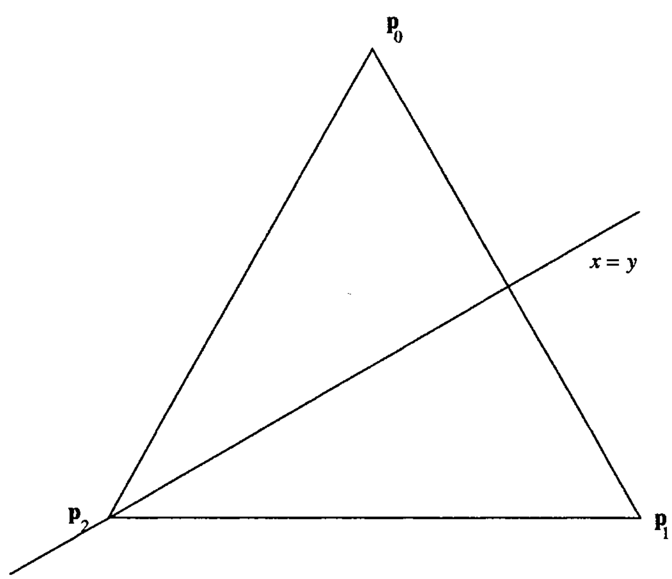


Figure 1.4

Projective Line $x = y$ in Barycentric Coordinates

Example 1.3: The curve $x^2z = y^3$ can be depicted by the real part of each of $z_0 = y_0^3$, $x_1^2z_1 = 1$, $x_2^2 = y_2^3$, or with barycentric coordinates. Figure 1.5 shows the four views of this curve mentioned in this section.

1.5. Curves and Functions on 2-Dimensional Manifolds

Curves, polynomial functions and rational functions are easily defined on $\mathbb{P}^2$ because there is a set of global coordinates. Unfortunately, on a general complex manifold there is not necessarily a set of global coordinates and just as we had to piece together local coordinates to define the manifold, we have to piece together curves and functions on the coordinate neighborhoods to get a definition which works for the entire manifold.

1.5.1. *Curves.*

Let U_i be the coordinate neighborhoods of X. The data

$$\{(U_i, f_i)\}_{i=1}^r$$

defines an *curve* C *on* X if

1. f_i is a holomorphic function on U_i, and
2. $f_i = hf_j \circ \phi_{ji}$ where h is a holomorphic, nowhere vanishing function on $U_i \cap U_j$ for all i, j.

While f_i may be zero somewhere on U_i, condition (2) implies f_i and $f_j \circ \phi_{ji}$ are zero on exactly the same set of points in $U_i \cap U_j$. Thus, the curve defined by the data $\{(U_i, f_i)\}$ is

$$C = \bigcup_i \{\mathrm{p} \in U_i : f_i(\mathrm{p}) = 0\}.$$

Each $f_i = 0$ defines a curve in U_i and condition (2) insures these curves fit together when we take their union. The $f_i = 0$ are called the *local equations* of the curve C. Notice the data $\{(U_i, f_i)\}$ is not unique for a given curve. For example, let g be holomorphic and nowhere

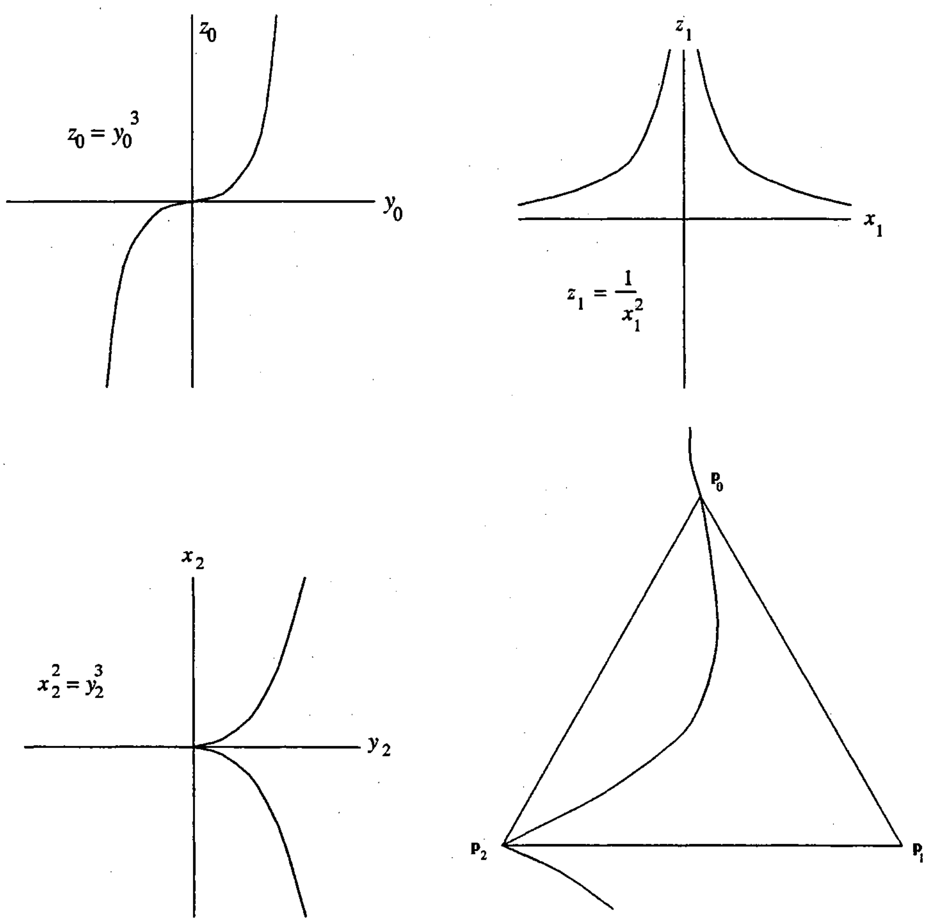


Figure 1.5

Four Representations of the Curve $x^2z = y^3$

vanishing on U_1. Then

$$\{(U_i, f_i)\}_{i=1}^{r}$$

and

$$\{(U_1, f_1 g)\} \cup \{(U_i, f_i)\}_{i=2}^{r}$$

define the same curve C because the set

$$\{\mathbf{p} \in U_1 : f_1(\mathbf{p}) = 0\}$$

is the same as the set

$$\{\mathbf{p} \in U_1 : f_1 g(\mathbf{p}) = 0\}.$$

If, in addition to the definition above for curves, each f_i is also a polynomial, then the curve C is called an *algebraic curve.* All curves referred to in this paper are algebraic curves.

In most written mathematics there is a certain amount of notational abuse and this paper is no different. For instance, the only real effect of ϕ_{ji} is to change the names of the coordinates. Thus, to simplify the notation, condition (2) will be rewritten

2. $f_i = h f_j$ where h is a holomorphic, nowhere vanishing function on $U_i \cap U_j$ for all i, j.

In other words, composition with the gluing maps will be assumed whenever it is needed. Technically, the functions f_i in the notation $\{(U_i, f_i)\}$ are functions from $U_i \subseteq X$ to $\mathbb{C}$. To be able to write the data for a particular curve, however, we will use a function f_i which has $V_i \subseteq \mathbb{C}^2$ as the domain, that is, we will define f_i in terms of its local coordinates on U_i. Thus throughout this paper the data for curves and functions will actually be written as

$$\{(U_i, f_i \circ \phi_i)\}$$

but to simplify the discussion we will always refer to the local equations as f_i instead of $f_i \circ \phi_i$. (See Example 1.4.) Even though f_i may be written in the local coordinates of

U_i, for convenience we will continue to say f_i is holomorphic on U_i. This same convention will apply to the data for holomorphic function, meromorphic functions, and other objects defined using local data. (See definition for Cartier divisors in Section 4.2.)

Is the definition for algebraic curves on a general complex manifold X consistent with the definition for algebraic curves on $\mathbb{P}^2$? It can be shown that any algebraic curve defined by data $\{(U_0, f_0(y_0, z_0)), (U_1, f_1(x_1, z_1)), (U_2, f_2(x_2, y_2))\}$ can also be defined by an implicit equation $F(x{:}y{:}z) = 0$. To calculate F, first homogenize each of $f_0(y, z)$, $f_1(x, z)$, and $f_2(x, y)$ to get F_0, F_1, and F_2, respectively. If this results in polynomials of different degree, let n be the maximum degree of f_0, f_1, and f_2. Now multiply F_0, F_1, and F_2 by powers of z, y, and x, respectively, such that all are polynomials of degree n. Condition (2) insures the three polynomials obtained from this process will be scalar multiples of the same polynomial F.

Example 1.4: The curve $x^2 z = y^3$ in Example 1.3 could have been defined by data

$$F = \{(U_i, f_i)\} = \{(U_0, y_0^3 - z_0), (U_1, x_1^2 z_1 - 1), (U_2, x_2^2 - y_2^3)\}$$

instead. Clearly, there is no advantage here to use this more cumbersome notation. Condition (1) above is easily satisfied since each f_i is a polynomial. One can easily verify condition (2): for instance, $\frac{f_0}{f_1} = \frac{y_0^3 - z_0}{x_1^2 z_1 - 1}$ is holomorphic on $U_0 \cap U_1$ by noticing the zero locus of $f_0(y_0, z_0)$ in $U_0 \cap U_1$ is exactly the zero locus of $f_1 \circ \phi_{01}(y_0, z_0)$ in $U_0 \cap U_1$. (See the comment on notation above.)

A curve in $\mathbb{P}^1 \times \mathbb{P}^1$ is the zero set of a polynomial in indeterminates ρ_0, ρ_1, σ_0, and σ_1 which is homogeneous in the ρ_i and the σ_i separately. The curve $\rho_0^2 \rho_1 \sigma_0^2 + \rho_1^3 \sigma_1^2 = 0$ can also be defined by the data

$$\{(W_{ij}, f_{ij})\} = \{(W_{00}, r_1 + r_1^3 s_1^2), (W_{01}, r_1 s_0^2 + r_1^3), (W_{10}, r_0^2 + s_1^2), (W_{11}, r_0^2 s_0^2 + 1)\}$$

using the local coordinates in Section 1.3.3. Again, it is easy to see $\frac{f_{ij}}{f_{kl}}$ is nonvanishing and homomorphic on $W_{ij} \cap W_{kl}$.

In $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ it is possible to define a curve globally with one polynomial because there is a global set of coordinates. In chapter 3 we will use manifolds which do not have a set of global coordinates and to define a curve there is it necessary for us to define it locally for each element of the cover.

1.5.2. *Holomorphic Functions.*

Holomorphic functions on $\mathbb{C}^2$ are those functions locally defined by a power series. The same is true for holomorphic functions on a manifold X. A *holomorphic function F on X* is defined by data

$$\{(U_i, f_i)\}_{i=1}^r$$

where

1. f_i is a holomorphic function on U_i and
2. $f_i(\mathbf{p}) = f_j(\mathbf{p})$ for all $\mathbf{p} \in U_i \cap U_j$.

In other words, data $\{(U_i, f_i)\}$ defines a holomorphic function F as long as it is well defined on all of X, and thus, $F(\mathbf{p}) = f_i(\mathbf{p})$ for any $\mathbf{p} \in U_i$. Again, the f_i are called the local equations of the function F.

In the following example we find there are no interesting holomorphic functions on $\mathbb{P}^2$.

Example 1.5: Homogeneous polynomials of degree $n \geq 1$ do not define holomorphic functions on $\mathbb{P}^2$. Let $F(x\!:\!y\!:\!z)$ be a nonconstant homogeneous polynomial and $f_0(y_0, x_0)$ and $f_1(x_1, z_1)$ the dehomogenizations of f with respect to x and y respectively. Let $\mathbf{p} = (a\!:\!b\!:\!c) \in U_0 \cap U_1$. Then the local coordinates for $\mathbf{p}$ in U_0 are $\left(\frac{b}{a}, \frac{c}{a}\right)$ and in U_1 are $\left(\frac{a}{b}, \frac{c}{b}\right)$.

Now

$$\begin{aligned} f_0\left(\frac{b}{a},\frac{c}{a}\right) &= F\left(1:\frac{b}{a}:\frac{c}{a}\right) \\ &= \frac{1}{a^n}F\left(a:b:c\right) \\ &= \frac{b^n}{a^n}F\left(\frac{a}{b}:1:\frac{c}{b}\right) \\ &= \frac{b^n}{a^n}f_1\left(\frac{a}{b},\frac{c}{b}\right). \end{aligned}$$

Since a and b are arbitrary, $f_0\left(\frac{b}{a},\frac{c}{a}\right) \neq f_1\left(\frac{a}{b},\frac{c}{b}\right)$. Therefore F does not define a function on $\mathbb{P}^2$ because $f_0(\frac{b}{a},\frac{c}{a})$ must be equal to $f_1(\frac{a}{b},\frac{c}{b})$ for all values of a, b, and c.

If $F(x:y:z)$ is a constant function, then $f_0\left(\frac{b}{a},\frac{c}{a}\right) = f_1\left(\frac{a}{b},\frac{a}{c}\right)$, and in fact, the only holomorphic functions on $\mathbb{P}^2$ are the constant functions. Similarly, the only holomorphic functions on $\mathbb{P}^1 \times \mathbb{P}^1$ are the constant functions.

In fact, the only holomorphic functions on a compact complex manifold are constant on connected components of the manifold. Let F be a nonconstant holomorphic function on a connected compact complex manifold X with connected coordinate neighborhoods $\{U_i\}$. The set $F(X)$ must be closed since X is compact and F is continuous. On the other hand, the open mapping theorem [N p. 6] says the image of a nonconstant holomorphic function on an open subset of $\mathbb{C}^n$ is open. So $F(U_i)$ is open for each i. Now $\cup F(U_i) = F(X)$ is open and closed but cannot be all of $\mathbb{C}$ or the empty set. Therefore, F must be constant on each U_i and, hence, constant on all of X because it is a holomorphic function.

1.5.3. *Meromorphic Functions.*

Meromorphic functions are those functions which can locally be written as the quotient of two holomorphic functions. A *meromorphic function F on X* is also defined by data $\{(U_i, f_i/g_i)\}$ such that

1. f_i and g_i are holomorphic functions on U_i with $g_i(\mathbf{p}) \neq 0$ for some $\mathbf{p} \in U_i$, and
2. $\dfrac{f_i(\mathbf{p})}{g_i(\mathbf{p})} = \dfrac{f_j(\mathbf{p})}{g_j(\mathbf{p})}$ for all $\mathbf{p} \in U_i \cap U_j$ for which $g_i(\mathbf{p}) \neq 0$ and $g_j(\mathbf{p}) \neq 0$.

Meromorphic functions are only defined for part of X: $F(\mathbf{p}) = f_i(\mathbf{p})/g_i(\mathbf{p})$ for all $\mathbf{p} \in U_i$ whenever $g_i(\mathbf{p}) \neq 0$. If all f_i and g_i are polynomials, the data $\{(U_i, f_i/g_i)\}$ defines a *rational function on X*.

The only meromorphic functions on $\mathbb{P}^2$ are of the form

$$\frac{F(x\colon y\colon z)}{G(x\colon y\colon z)}$$

where F and G are homogeneous polynomials of like degree. Similarly, the only meromorphic functions on $\mathbb{P}^1 \times \mathbb{P}^1$ are quotients of polynomials

$$\frac{F(\rho_0\colon \rho_1; \sigma_0\colon \sigma_1)}{G(\rho_0\colon \rho_1; \sigma_0\colon \sigma_1)}$$

of like bidegree. In Section 3.4 there are examples of meromorphic functions on a different manifold, the blow up of $\mathbb{P}^2$.

1.5.4. *Terminology, Notation, and Conventions.*

Every meromorphic function defines 2 curves. If $\{(U_i, f_i/g_i)\}$ defines a meromorphic function F on X, then we may assume each pair (f_i, g_i) is relatively prime in the sense that if h is a nonconstant holomorphic function which divides both f_i and g_i, then h has no zeros on U_i. Under this assumption, the data $\{(U_i, f_i)\}$ and $\{(U_i, g_i)\}$ each define a curve on X because the points where $f_i = 0$ and $g_i = 0$ vanish are well defined on all of X from condition (2) for meromorphic functions. The curve $F_0 = \{(U_i, f_i)\}$ is called the *zeros* of F and the curve $F_\infty = \{(U_i, g_i)\}$ is called the *poles* of F. The converse is not true. If $\{(U_i, f_i)\}$ and $\{(U_i, g_i)\}$ each define a curve on X, the quotients f_i/g_i may not be the local equations for a meromorphic function on X because they may not satisfy condition (2) for meromorphic functions. (See Section 3.4).

It is important to note the different uses of the qualifiers holomorphic and meromorphic. Let $\{(U_i, f_i)\}$ define a curve on the manifold X. Each f_i is holomorphic on U_i and we say each f_i is a *locally holomorphic.* On the other hand, the data does not necessarily define a holomorphic function on X and we cannot say it is globally holomorphic. The data $\{(U_i, f_i)\}$ must satisfy the requirements for a holomorphic function on X to be *globally holomorphic.* Similarly, the meaning of *local meromorphic* and *global meromorphic* functions are very different.

Now, here is some more notational abuse. We will identify the polynomial or data defining a curve with the set of points itself. Say C is the curve defined by the polynomial equation $f = 0$. For convenience we will write $\mathbf{p} \in f$ to mean a point $\mathbf{p}$ is in the zero locus of f. We say a polynomial f *factors* if there are nonconstant polynomials, g and h, with $gh = f$. If $f = f_1 \cdot \cdots \cdot f_n$ and each f_i is irreducible, then the f_i are called the *components of the curve* f. A polynomial f is said to be *irreducible* if any time f can be written as $f = gh$, then either g or h is a constant polynomial. An algebraic curve $f = 0$ in $\mathbb{P}^2$ is *irreducible* if f is irreducible. A curve C defined by data $\{(U_i, f_i)\}$ on a 2-dimensional manifold X is *irreducible* if there is no data $\{(U_i, g_i)\}$ which defines a curve C' which is a nonempty strict subset of C. If $C_1, \ldots, C_n$ are each irreducible curves on X and $\cup C_i = C$, then the C_i are called the *components of* C. Every curve on a 2-dimensional manifold can be written as the unique union of irreducible components. However, it may be true that all the local equations for a curve C defined on a manifold are irreducible, but the curve C is not irreducible.

1.6. Multiplicity of a Point on a Curve

The whole point of this paper depends on being able to count: counting points of intersection, counting curves, etc. We will begin by counting the singularities of a curve at a point. Let X be a 2-dimensional manifold with coordinate neighborhoods $\{U_i\}$. Fix i and let the local coordinates for U_i be (x, y). Since each coordinate neighborhood is topologically equivalent to a subset of $\mathbb{C}^2$, if we want to understand how a curve behaves at a point in X we need only understand how the curve behaves at the point's local coordinates in U_i. Properties which only depend on the behavior of the curve in U_i are called *local properties.* In this paper we will define two local properties of curves: here we introduce the multiplicity of a point on a curve, and Chapter 2 contains the definition of the intersection multiplicity of a point in the intersection of two curves.

1.6.1. *Multiplicity of a Point on a Plane Curve.*

Let us begin by looking at a curve in $\mathbb{C}^2$ defined by the polynomial equation $f(x,y) = 0$, and let $\mathbf{p} \in f$. Write the Taylor expansion of f about $\mathbf{p}$. If $\mathbf{p} = (x_0, y_0)$ and since $f(\mathbf{p}) = 0$,

$$\begin{aligned} f(x,y) =& \frac{\partial f}{\partial x}(\mathbf{p})(x - x_0) + \frac{\partial f}{\partial y}(\mathbf{p})(y - y_0) \\ &+ \frac{\partial^2 f}{\partial x^2}(\mathbf{p})(x - x_0)^2 + \frac{\partial^2 f}{\partial x \partial y}(\mathbf{p})(x - x_0)(y - y_0) + \frac{\partial^2 f}{\partial y^2}(\mathbf{p})(y - y_0)^2 \\ &+ \cdots . \end{aligned}$$

If one of $\partial f/\partial x(\mathbf{p})$ or $\partial f/\partial y(\mathbf{p})$ is not zero, then the curve f has one tangent at $\mathbf{p}$ defined by $\frac{\partial f}{\partial x}(\mathbf{p})(x - x_0) + \frac{\partial f}{\partial y}(\mathbf{p})(y - y_0) = 0$. If all partial derivatives of f up to and including order $r - 1$ vanish at $\mathbf{p}$ but at least one partial derivative of order r does not vanish at $\mathbf{p}$, then the terms of degree r in the Taylor expansion are

$$\frac{\partial^r f}{\partial x^r}(\mathbf{p})(x - x_0)^r + \frac{\partial^r f}{\partial x^{r-1} \partial y}(\mathbf{p})(x - x_0)^{r-1}(y - y_0) + \cdots + \frac{\partial^r f}{\partial y^r}(\mathbf{p})(y - y_0)^r.$$

This polynomial can be factored into r linear terms $[a_1(x - x_0) + b_1(y - y_0)] \cdots [a_r(x - x_0) + b_r(y - y_0)]$, and f has r, not necessarily distinct, tangent directions at the point $\mathbf{p}$ defined by the lines $a_i(x - x_0) + b_i(y - y_0) = 0$.

Define the *multiplicity of* $\mathbf{p}$ *on the curve* f, $m_{\mathbf{p}}(f)$, to be the number of tangent directions of f, counting multiplicities, at the point $\mathbf{p}$. This can be calculated as seen above. Notice $m_{\mathbf{0}}(f)$ is just the degree of the lowest order term of the polynomial f. A point of multiplicity greater than 1 is called *singular.* If there are no points on the curve of multiplicity greater than one then the curve is called *nonsingular.* If all tangent directions at a singular point are distinct, the singularity is called *ordinary.* A *double point* is a point of multiplicity 2, a *triple point* is a point of multiplicity 3, and so on.

1.6.2. *Multiplicity in* $\mathbb{P}^2$*.*

Before we investigate multiplicity of a point of a curve on general 2-dimensional manifolds, let us first look at curves on $\mathbb{P}^2$. A major difference between $\mathbb{P}^2$ and a general manifold is that on $\mathbb{P}^2$ every algebraic curve has a global definition. To show that multiplicity is well defined on $\mathbb{P}^2$, we need only show that the multiplicity of the global homogeneous polynomial is well defined.

Proposition 1.1. *Let* $F(x{:}y{:}z) = 0$ *be a curve on* $\mathbb{P}^2$*. Then* $m_{\mathbf{p}}(F) = r$ *if and only if all* $(r-1)$*-th order partial derivatives of* F *vanish at* $\mathbf{p}$ *but at least one* r*-th order partial derivative of* F *does not vanish at* $\mathbf{p}$*.*

Proof: We may assume, without loss of generality, $\mathbf{p} = (a{:}b{:}1)$. Let $f(x,y) = F(x{:}y{:}1)$ be the dehomogenization of F with respect to z. First, note that $f(a,b) = 0$ if and only $F(\mathbf{p}) = 0$. Also

$$\frac{\partial F}{\partial x}(x{:}y{:}1) = \frac{\partial f}{\partial x}(x,y)$$

and

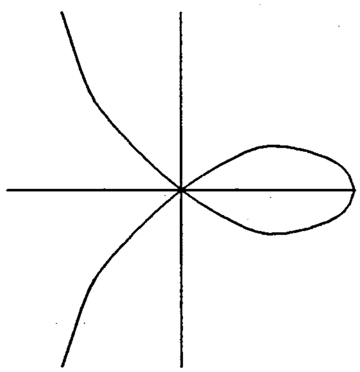
$$\frac{\partial F}{\partial y}(x{:}y{:}1) = \frac{\partial f}{\partial y}(x,y).$$

Thus

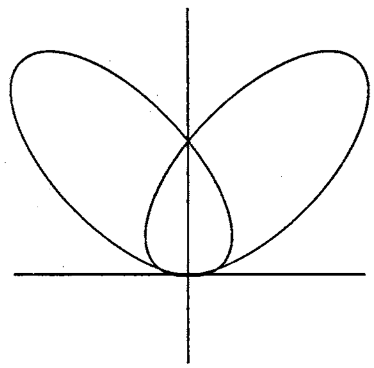
$$\frac{\partial F}{\partial x}(\mathbf{p}) = 0 \text{ and } \frac{\partial F}{\partial y}(\mathbf{p}) = 0$$

if and only if

$$\frac{\partial f}{\partial x}(a,b) = 0 \text{ and } \frac{\partial f}{\partial y}(a,b) = 0.$$



Ordinary double point at the orgin



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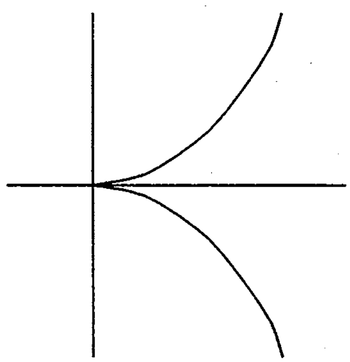
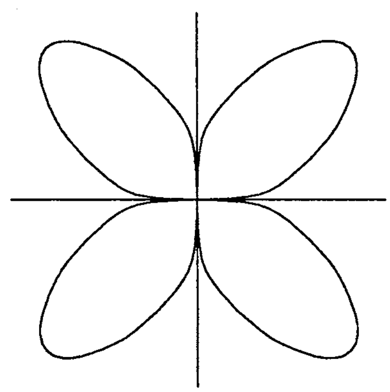


Nonordinary double point at the origin



Point of multiplicity 4 at the origin

Figure 1.6

Examples of Multiple Points

However, it is not at first clear that when $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ vanish at (a,b), $\frac{\partial F}{\partial z}$ will also vanish at $\mathbf{p}$. For each $t \neq 0$, $F(tx\colon ty\colon tz) = t^n F(x\colon y\colon z)$ where n is the homogeneous degree of F. By differentiating both sides of this equation with respect to t we obtain

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z} = nt^{n-1}F.$$

Let $t = 1$, and we see

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z} = nF.$$

Now,

$$\frac{\partial F}{\partial x}(\mathbf{p}) = \frac{\partial F}{\partial y}(\mathbf{p}) = F(\mathbf{p}) = 0$$

if and only if

$$\frac{\partial F}{\partial x}(\mathbf{p}) = \frac{\partial F}{\partial y}(\mathbf{p}) = \frac{\partial F}{\partial z}(\mathbf{p}) = 0.$$

The same argument can be applied to the higher order partial derivatives of f to yield

$$f(a,b) = \frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = \frac{\partial^2 f}{\partial x^2}(a,b) = \cdots = \frac{\partial^{r-1} f}{\partial y^{r-1}}(a,b) = 0$$

if and only if the $(r-1)$-th partials of F vanish at $\mathbf{p}$. This proves the proposition.

1.6.3. *Multiplicity of Points on General Manifolds.*

Let $\{(U_i, f_i)\}$ define a curve C on the 2-dimensional manifold X. We will show $m_{\mathbf{p}}(f_i) = m_{\mathbf{p}}(f_j)$ for all points $\mathbf{p} \in U_i \cap U_j$. There is a holomorphic, nowhere vanishing function h on $U_i \cap U_j$ such that $f_i = h \cdot f_j$ from condition (1) for curves. Let (x,y) be the local coordinates of U_i and (s,t) the local coordinates of U_j. For all i and j and for $l, k \geq 0$,

$$\frac{\partial^{l+k} f_j}{\partial s^l \partial t^k} = \sum_{r=0}^{k} \sum_{q=0}^{l} \binom{k}{r} \binom{l}{q} \frac{\partial^{q+r} h}{\partial s^q \partial t^r} \frac{\partial^{l+k-q-r} f_i}{\partial s^{l-q} \partial t^{k-r}}.$$

Suppose $m_{\mathbf{p}}(f_i) = n+1$. Then all partials of f_i up to and including order n vanish at $\mathbf{p}$. Let $l + k = n$. From the equation above we see that all partials of f_j of order up to and including n vanish at $\mathbf{p}$ also. In fact, $m_{\mathbf{p}}(f_i) = m_{\mathbf{p}}(f_j)$ and multiplicity is well defined for curves on 2-dimensional manifolds.

1.7. Linear Systems

1.7.1. *Definition.*

The base points of a rational surface

$$x_i = \frac{f_i(x,y)}{f_0(x,y)}$$

is the set of all points satisfying $f_i(x,y) = 0$ for all i. Note that this is also the set of all points satisfying the equation $\sum_{i=0}^{3} \alpha_i f_i(x,y) = 0$ for all values $\alpha_i \in \mathbb{C}$. The set of curves

$$\left\{ \sum_{i=0}^{3} \alpha_i f_i(x,y) = 0 \right\} \tag{1.1}$$

is called a linear system and is an essential idea in the remaining chapters. Any set L of curves which can be linearly parameterized by $\mathbb{P}^k$ is called a *linear system* and k is called the *dimension* of the linear system. If $k = 1$ the linear system is called a *pencil*. Let L be a linear system and $B = \{\mathbf{p} \in \mathbb{P}^2 : \mathbf{p} \in F \text{ for all } F \in L\}$. This is the set of all points common to the zero loci of all elements in L and is called the *base locus* of L. The elements of B are called *base points*.

Let L_d be the set of all homogeneous polynomials of degree d in $\mathbb{P}^2$ in indeterminates x_1, x_2, and x_3. We will show L_d can be parameterized by $\mathbb{P}^N$ for some N. Let $A \subset L_d$ be the subset which contains the monomials $x_0^i x_1^j x_2^k$ with $i+j+k = d$. There are $(d+2)(d+1)/2$ such elements. Put $N = (d+2)(d+1)/2 - 1 = d(d+3)/2$ and name these elements $F_0, \ldots, F_N$. Every element F of L_d can be written uniquely as $F = a_0F_0 + \cdots a_NF_N$. Let $\mathbf{a} = (a_0 : \ldots : a_N)$ be the coordinates of the point F in L_d and note that $\mathbf{a} \neq \mathbf{0}$. Equate two points F and F' if their zero loci are the same. Thus $(a_0 : \ldots : a_N)$ and $(\lambda a_0 : \ldots : \lambda a_N)$ are the same when $\lambda \neq 0$ and L_d is parameterized by $\mathbb{P}^N$.

Another way to see how the degree N is calculated is to notice the set of homogeneous polynomials of degree d is a vector space over $\mathbb{C}$ of dimension $n = (d+2)(d+1)/2$, and hence, is isomorphic to the vector space $\mathbb{C}^n$. With the identification of $(a_0:\ldots:a_n)$ and $(\alpha a_0:\ldots:\alpha a_n)$ in $\mathbb{C}^n$ for $\alpha \neq 0$, $\mathbb{C}^n - \{0\}$ is isomorphic to $\mathbb{P}^{n-1}$ (see Section 1.3.2). In fact, any time there is a vector space V over $\mathbb{C}$ of dimension $N+1$, we can impose a condition such that $V - \{0\}$ modulo this condition is isomorphic to $\mathbb{P}^N$.

1.7.2. *Examples.*

There are many ways to define linear systems. First, we will consider subspaces of L_d. For example, when $d = 1$, L_1 is equal to the set of all lines in the projective plane and L_1 is isomorphic to $\mathbb{P}^2$. Fix some point $\mathbf{p} = (p_0:p_1:p_2)$ in the projective plane. Let $L_1^{\mathbf{p}}$ be the set of all elements of L_1 which have $\mathbf{p}$ in their zero locus. The set $L_1^{\mathbf{p}}$ is isomorphic to $\mathbb{P}^1$ as follows. If $(a_0, a_1, a_2) \in L_1^{\mathbf{p}}$ then

$$a_0p_0 + a_1p_1 + a_2p_2 = 0.$$

Assume, without loss of generality, $p_0 = 1$. Now $a_0 = -a_1p_1 - a_2p_2$ and the elements of $L_1^{\mathbf{p}}$ can be identified by (a_1, a_2). One of a_1 or a_2 must be nonzero; otherwise, a_0 would be zero also.

Linear systems can be described in many ways. One way is to impose conditions on the elements of L_d. For example, the subset of all elements of L_d whose zero loci all contain the points $\mathbf{p}_1, \ldots, \mathbf{p}_s$ defines a linear system. We call it a *linear system with s assigned base points.* Each additional assigned base point adds conditions to a linear system and one would expect the dimension of a linear system to drop for each base point added, but this is not always the case[W]. Let us form a linear system in another way. Let $G_1, \ldots, G_r \in L_d$. Then the space of all elements of the form $\alpha_1 G_1 + \cdots \alpha_r G_r$ with $\alpha_i \in \mathbb{C}$ is a linear system. The set $\{G_1, \ldots, G_r\}$ is a basis for the linear system. If the G_i are linearly independent, the dimension of this linear system is r and the set $\{G_1, \ldots, G_r\}$ is a basis for the linear system. In Section 4.12 there are further examples of linear systems.

Example 1.6: Let $f(x,y) = x^2 + 4y^2 - 4$ and $g(x,y) = 4x^2 + y^2 - 4$ and consider the set L of polynomials $\alpha f + \beta g$ where α, $\beta \in \mathbb{R}$ not both zero. An element of L is a polynomial $h = (\alpha + 4\beta)x^2 + (4\alpha + \beta)y^2 - 4(\alpha + \beta)$. The zero locus of h is either an ellipse, a hyperbola, or two lines depending on the values of α and β (see Figure 1.7). The zero loci of f and g intersect at the 4 points $(\pm 2/\sqrt{5}, \pm 2/\sqrt{5})$, and consequently, the zero locus of each h contains these 4 points also. The linear system L is parameterized by $\mathbb{P}^1$ and the base locus is $\{(\pm 2/\sqrt{5}, \pm 2/\sqrt{5})\}$.

1.7.3. *General Elements of Linear Systems.*

A word which shows up often in algebraic geometry is the word *general.* If there is a family of objects parameterized by a complex manifold (such as a linear system) or parameterized by a surface (in the case of line bundles later), the statement "a general member of the family has a certain property" means "the set of objects in the family which do not have the property is contained in a space of strictly smaller dimension." For example, a general line in $\mathbb{R}^2$ intersects a fixed line $L \subseteq \mathbb{R}^2$ since the set of lines which do not intersect L are parameterized by $\mathbb{R} - \{0\}$. Another term which is used almost in the same way in *generic*, although there are some specific uses of generic as an adjective which mean something different.

The following theorem will be useful in Chapter 5.

Proposition 1.2. *(Bertini's Theorem) The general element of a linear system is nonsingular away from the base locus of the system.*

Proof: Consider first a pencil L with basis $\{f, g\}$ defined on a compact complex manifold

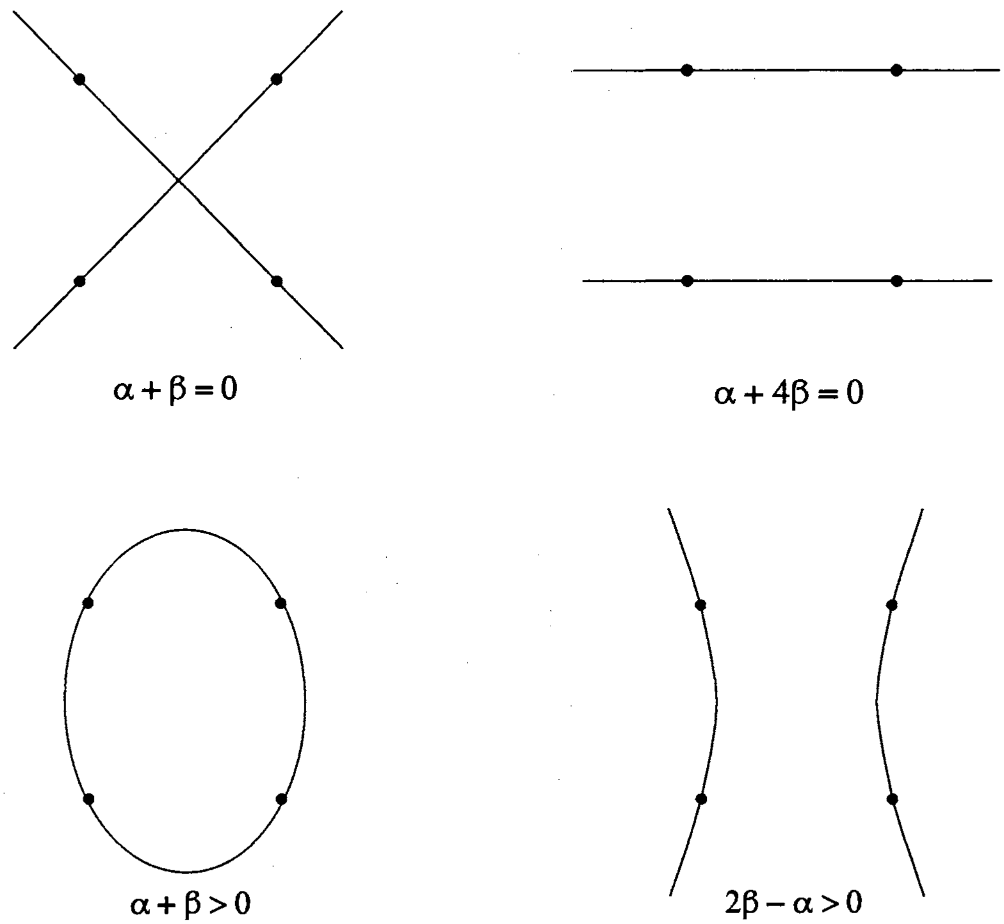


Figure 1.7

Elements of the Linear System in Example 1.6

X. Let $B = \{\mathbf{p} \in X : (\alpha f + \beta g)(\mathbf{p}) = 0 \text{ for all } (\alpha{:}\beta) \in \mathbb{P}^1\}$ be the base locus of L. Define another set V to be all the points $\mathbf{p} \in X$ such that $\alpha f + \beta g$ is singular at $\mathbf{p}$ for some $(\alpha{:}\beta) \in \mathbb{P}^1$.

Let $\mathbf{p} \in V - B$ and U_i an element of the cover of X which contains $\mathbf{p}$. If the local coordinates of U_i are (x, y), then

$$\begin{aligned}(\alpha f + \beta g)(\mathbf{p}) &= 0,\\ \left(\alpha \frac{\partial f}{\partial x} + \beta \frac{\partial g}{\partial x}\right)(\mathbf{p}) &= 0, \text{ and}\\ \left(\alpha \frac{\partial f}{\partial y} + \beta \frac{\partial g}{\partial y}\right)(\mathbf{p}) &= 0\end{aligned}$$

for some $(\alpha{:}\beta) \in \mathbb{P}^1$. One of α or β is not zero, so assume $\alpha \neq 0$. Both f and g cannot vanish at $\mathbf{p}$ since $\mathbf{p} \notin B$ and, in particular, $g(\mathbf{p}) \neq 0$ since $\alpha \neq 0$. Thus,

$$\frac{f(\mathbf{p})}{g(\mathbf{p})} = -\frac{\beta}{\alpha}.$$

Now

$$\frac{\partial}{\partial x}\left(\frac{f}{g}\right)(\mathbf{p}) = \frac{\alpha \frac{\partial f}{\partial x}(\mathbf{p}) + \beta \frac{\partial g}{\partial x}(\mathbf{p})}{\alpha g(\mathbf{p})} = 0.$$

Similarly,

$$\frac{\partial}{\partial y}\left(\frac{f}{g}\right)(\mathbf{p}) = 0.$$

To show that a general element of L is nonsingular away from the base locus, we need to show that the set of elements of L which are singular away from B has dimension less than 1, i.e., there are only finitely many such elements. Consider the set $V - B$. The set V could be all of X, curves in X, points in X, or just empty. In each case, the set $V - B$ has finitely many connected components V_i since X is compact. On each V_i, the function f/g must be constant from the calculations of the partial derivatives above. The element of L which is singular on V_i is exactly $f - (f(\mathbf{p}_i)/g(\mathbf{p}_i))g$ for any $\mathbf{p}_i \in V_i$. Therefore, there are only finitely many elements of L which are singular off of B.

Not only have we shown that the theorem is true for a pencil, but we have also shown that it is true for all pencils of any linear system. We proceed by contradiction. Assume that the general element of a linear system L is singular away from the base locus. That is, the set of elements which are singular away from the base locus is a set of the same dimension as L. Let f be an element of L which is singular at $\mathbf{p}$ with $\mathbf{p}$ not in the base locus of L. Let P be a general pencil in L which contains f and does not have $\mathbf{p}$ in its base locus. By the argument above, f cannot be singular away from the base locus of P. However, f is singular at $\mathbf{p}$ which is not in the base locus of P. This is a contradiction and the proposition is true for all linear systems.

1.8. Rational Surfaces

1.8.1. *Surfaces Parameterized in $\mathbb{C}^2$.*

In the introduction to this chapter a definition of rational surfaces in $\mathbb{C}^3$ with parameter space $\mathbb{C}^2$ was given. Actually, there are two different ways such parameterizations are used in practice. A *tensor product parametric surface* is defined by

$$x_i = \frac{g_i(x,y)}{g_0(x,y)} = \frac{\sum_{j=0}^{m} \sum_{k=0}^{n} c_{jk}^{(i)} x^j y^k}{\sum_{j=0}^{m} \sum_{k=0}^{n} c_{jk}^{(o)} x^j y^k}. \tag{1.2}$$

The surface is called *bilinear*, *biquadratic*, or *bicubic* if $m = n = 1$, 2 or 3 respectively. On the other hand, a *triangular parametric surface* can be defined as

$$x_i = \frac{f_i(x,y)}{f_0(x,y)} = \frac{\sum_{j+k \le n} c_{jk}^{(i)} x^j y^k}{\sum_{j+k \le n} c_{jk}^{(o)} x^j y^k}. \tag{1.3}$$

In the tensor product surface the polynomials g_i have maximum degree m in x and maximum degree n in y. For the triangular surface the polynomials have maximum degree n in x and y.

To use these definitions as is, special cases must be considered. The points where $g_0(x,y) = 0$ or $f_0(x,y) = 0$ are the points on the surface at infinity and have to be considered separately. Also, we could consider the point at infinity in $\mathbb{C}^2$ as a parameter value, but this has to be done as a special case. We can and will avoid these special cases by using projective space for both the parameter space and the image space.

1.8.2. *Surfaces Parameterized in Projective Space.*

Using projective space, the parameter space for tensor product surfaces is $\mathbb{P}^1 \times \mathbb{P}^1$ with coordinates $(x_0{:}x_1; y_0{:}y_1)$ and the image space is $\mathbb{P}^3$ with coordinates $(X_0{:}X_1{:}X_2{:}X_3)$. The parameterization is written

$$X_i = G_i(x_0{:}x_1; y_0{:}y_1) \tag{1.4}$$

for $i = 0, 1, 2,$ and 3 where G_i is homogeneous in x_0 and x_1 with degree n and homogeneous in y_0 and y_1 in degree m. Each G_i is said to be of *bidegree* (n, m). To convert a nonprojective parameterization to one defined in projective space, homogenize the polynomials $g_i(x, y)$ from (1.2) to get the polynomials $G_i(x_0{:}x_1; y_0{:}y_1)$ of (1.4) where x becomes x_0 and y becomes y_0 and then multiply the G_i by the least powers of x_1 and y_1 to insure they are all of the same bidegree. Since there are no quotients in (1.4) the points where $g_0 = 0$ are not treated differently than other points. Also, the points at infinity in $\mathbb{C}^2$ are now represented by the points $(a{:}0; b{:}0) \in \mathbb{P}^1 \times \mathbb{P}^1$ and do not constitute a special case.

Example 1.7: Define a tensor product surface in $\mathbb{P}^3$ by

$$(X_0{:}X_1{:}X_2{:}X_3) = (G_0{:}G_1{:}G_2{:}G_3) = (x_0y_0{:}x_0y_1{:}x_1y_0{:}x_1y_1).$$

The bidegree of each G_i is $(1, 1)$. This surface has no base points because there are no solutions in $\mathbb{P}^1 \times \mathbb{P}^1$ to the simultaneous equations $G_i = 0$. In $\mathbb{P}^3$ this surface is the zero locus of the equation $X_0X_3 - X_2X_1 = 0$. In fact, any quadric surface in $\mathbb{P}^3$ is isomorphic to this surface [GH p. 478] and for this reason it is called the *quadric surface in* $\mathbb{P}^3$.

The parameter space for triangular surfaces using projective space is $\mathbb{P}^2$ with coordinates $(x\colon y\colon z)$ and the image space is $\mathbb{P}^3$ again with coordinates $(X_0\colon X_1\colon X_2\colon X_3)$. The parameterization is written

$$X_i = F_i(x\colon y\colon z) \tag{1.5}$$

for $i = 0, 1, 2$ and 3 where each F_i is a homogeneous polynomial of degree n. Again it is easy to convert the parameterization of (1.3) to one defined on projective space by homogenizing the polynomials f_i of (1.3) to get the polynomials F_i of (1.4) having multiplied by the least power of z to insure all the F_i are of the same homogeneous degree. The point at infinity in $\mathbb{C}^2$ is now represented by the points $(a\colon b\colon 0)$ and is not considered separately.

Example 1.8: Let $\mathbf{p}_1, \ldots, \mathbf{p}_6$ be six points in $\mathbb{P}^2$ such that no 3 lie on the same line and not all 6 line on the same conic. Let L be the linear system of all cubic curves in $\mathbb{P}^3$ and L' the subspace of L of cubics containing the 6 points $\mathbf{p}_i$. The six points impose at most 6 linear conditions on L' and L has projective dimension 9, thus the projective dimension of L' is at least 3. Choose any 4 linearly independent cubics in L' and call them F_0, F_1, F_2 and F_3. Define a triangular surface in $\mathbb{P}^3$ by

$$(X_0\colon X_1\colon X_2\colon X_3) = (F_0\colon F_1\colon F_2\colon F_3).$$

This surface has six base points $\mathbf{p}_i$ and in Chapter 5 it will be shown that this is a cubic surface in $\mathbb{P}^3$. In fact, all smooth cubic surfaces in $\mathbb{P}^3$ can be defined in this way [GH p. 489].

For the most part the discussions in this paper will be restricted to triangular surfaces but will apply to both surface definitions. Sections 3.5 and 3.7 contain examples of triangular surfaces. Appendix C contains an example of a surface defined both with a triangular parameterization and with a tensor product parameterization.

1.8.3. *Rational Maps and One-to-One Parameterization.*

Not all definitions of the form (1.4) and (1.5) actually give a surface. For example, we must assume the F_i's do not have a common factor. If they did have a common factor, say F, then we could multiply $(F_0:F_1:F_2:F_3)$ through by $1/F$. Even then, the set of points $\{(F_0:\ldots:F_3)\}$ may be degenerate, that is, they may define a single point or a curve in $\mathbb{P}^3$ instead of a surface. For example, $\{(x:y:x:y)\}$ defines the line of intersection of the planes $X_0 = X_2$ and $X_1 = X_3$ in $\mathbb{P}^3$. We will assume all parameterizations used in this paper are nondegenerate.

We would like for there to be a function from the parameter space to the image space so we could easily move from one space to the other. Unfortunately, the relation

$$\psi(x:y:z) = (F_0(x:y:z):F_1(x:y:z):F_2(x:y:z):F_3(x:y:z))$$

may not define a map from the parameter space to the image space because there may be points in $\mathbb{P}^2$ for which it is not defined. Curves of the form $F = \alpha_0 F_0 + \cdots + \alpha_3 F_3 = 0$ form a linear system L of dimension 4 or less. If this linear system has any base points, ψ is not a map. Since $F_i \in L$ for each i, any base point $\mathbf{p}$ must be in zero locus of F_i and $(F_0(\mathbf{p}):F_1(\mathbf{p}):F_2(\mathbf{p}):F_3(\mathbf{p})) = (0:0:0:0)$ does not define a point in $\mathbb{P}^3$. The base points are simply the points in the intersection $\bigcap_i \{\mathbf{p} \in \mathbb{P}^2 : F_i(\mathbf{p}) = 0\}$. It is assumed the F_i do not share a common factor, so this set is finite. If F is a general element of L, then $m_{\mathbf{p}}(F) = \min_i\{m_{\mathbf{p}}(F_i)\}$. Define the *multiplicity of the base point* $\mathbf{p}$ to be $m_{\mathbf{p}}(F)$ where F is a general element of L. If there are base points, the relation ψ is said to be a *rational map*, which, for our purposes, is a relation which defines a function when restricted to a dense open subset of $\mathbb{P}^2$ and is written

$$\psi: \mathbb{P}^2 \dashrightarrow \mathbb{P}^3.$$

The closure of the image of this rational map is the rational surface and is denoted $\operatorname{Im}(\psi)$.

We need to make another assumption about our surfaces before we can begin doing the calculations. We will assume all parameterizations are *one-to-one parameterizations.* This does not mean there is a one-to-one correspondence between the elements of the parameter space and the surface. In general, the inverse image of a point in the image space could be empty, finite, or infinite. The set $\psi^{-1}(\mathbf{q})$ is empty for any point $\mathbf{q}$ which is not on the surface. The set $\psi^{-1}(\mathbf{q})$ is infinite if some curve in $\mathbb{P}^2$ maps to $\mathbf{q}$. For example, if we let

$$(F_0\colon F_1\colon F_2\colon F_3) = (xy\colon xz\colon yz\colon x^2)$$

and $\mathbf{p} = (0\colon y\colon z)$ be any point on the line $x = 0$ in $\mathbb{P}^2$, then $\psi(\mathbf{p}) = (0\colon 0\colon yz\colon 0) \in \mathbb{P}^3$ which is a single point. Thus $\psi^{-1}(0\colon 0\colon 1\colon 0)$ contains infinitely many points. There can be only finitely many point points $\mathbf{q} \in \mathbb{P}^3$ with $\psi^{-1}(\mathbf{q})$ infinite. Thus, for all but finitely many points on the surface $\mathrm{Im}\,(\psi)$, the inverse image under ψ is finite. We will assume the inverse image of these points is not only finite but contains exactly one point. This is called a one-to-one parameterization.

1.8.4. *Surfaces Parameterized on Other Manifolds.*

To calculate the degree of a surface we will sometimes replace the parameter space $\mathbb{P}^2$ with some other 2-dimensional manifold. Let X be a 2-dimensional manifold with coordinate neighborhoods U_i. A rational parametric surface using X as the parameter space would be defined by data

$$\{(U_i, (F_{i0}\colon F_{i1}\colon F_{i2}\colon F_{i3}))\}$$

for polynomials F_{ij} where each

$$\{(U_i, F_{ij})\}$$

for each j defines a meromorphic section of the same holomorphic line bundle (see Section 4.8 for the definitions of line bundles and sections.) The rational map

$$\psi : X \dashrightarrow \mathbb{P}^3$$

is defined by $\psi(\mathbf{p}) = (F_{i0}(\mathbf{p}): F_{i1}(\mathbf{p}): F_{i2}(\mathbf{p}): F_{i3}(\mathbf{p}))$ for $\mathbf{p} \in U_i$. For $\alpha_i \in \mathbb{C}$

$$\{(U_i, \alpha_0 F_{i0} + \alpha_1 F_{i1} + \alpha_2 F_{i2} + \alpha_3 F_{i3})\}$$

is a linear system of curves and the base points of this system are the points where ψ is not defined.

1.9. Summary of Conventions

While most of the material in this paper is stated generally for objects defined locally with holomorphic and meromorphic functions, there are a couple of places where the arguments will only work with objects defined locally with polynomial and rational functions and this is indicated in the proof. These statements may also be true in the general case, simply not proven for the general case here.

The name of a polynomial will also be used for the zero locus of the polynomial, e.g., $\mathbf{p} \in f$ means $f(\mathbf{p}) = 0$.

All manifolds here are compact complex manifolds. This is essential for many of the proofs. The standard topology will be used on $\mathbb{C}^n$ and the topology for $\mathbb{P}^n$ is given in Section 1.3.2.

All surfaces are parameterized with polynomial coordinates and all parameterizations are assumed to be one-to-one as described in Section 1.8.3.

CHAPTER TWO

INTERSECTION MULTIPLICITY

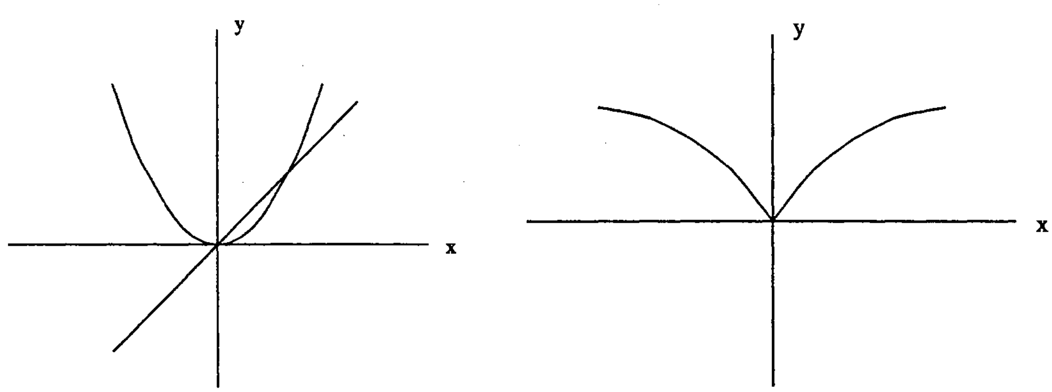
2.1. Introduction

In Section 1.6 we counted the multiplicity of a point on a single curve. Bezout's Theorem for $\mathbb{C}^2$ states that the number of points in the intersection of 2 curves of degree n and m with no common component is no more than nm counting multiplicities. In this chapter we present an algorithm for counting the multiplicity of the point of intersection of 2 curves. This number is called the intersection multiplicity.

The *intersection multiplicity* of a point on the intersection of 2 curves f and g is denoted $i(\mathbf{p}, f \cap g)$ and is the number of times $\mathbf{p}$ is counted in the intersection of $f = 0$ and $g = 0$. We might think of the intersection multiplicity as a way of measuring "how much" two curves intersect at a particular point. There are certain properties the intersection multiplicity should have. For instance, the "smallest" way for 2 curves to intersect at a point $\mathbf{p}$ is for them not to meet at $\mathbf{p}$ at all. Thus, if $\mathbf{p} \notin f \cap g$, we would expect $i(\mathbf{p}, f \cap g)$ to be 0. Two curves f and g meet *transversally* at $\mathbf{p}$ if $\mathbf{p} \in f \cap g$, $\mathbf{p}$ is a simple point of each curve and f and g have distinct tangent directions at $\mathbf{p}$. This would be the "smallest" way to have a nonempty intersection, so we expect $i(\mathbf{p}, f \cap g)$ to be 1, the smallest counting number. The curves f and g are said to meet *properly* at $\mathbf{p}$ if $\mathbf{p} \in f \cap g$ and f and g do not have a common component containing $\mathbf{p}$. The curves meet *improperly* at $\mathbf{p}$ is f and g have a common component containing $\mathbf{p}$. When f and g meet properly but not transversally at $\mathbf{p}$, $i(\mathbf{p}, f \cap g)$ should be an integer greater than or equal to 2. The "largest" way for f and

g to intersect is for the two curves to meet improperly at $\mathbf{p}$. In this case, $i(\mathbf{p}, f \cap g)$ should be larger than any integer, that is, $i(\mathbf{p}, f \cap g)$ should be ∞.

Some intersection multiplicities are illustrated in Figure 2.1. Notice that if the 2 curves share tangent directions, the intersection multiplicity is greater than if they have distinct tangent directions (e.g. $i(\mathbf{0}, x^3 - y^2 \cap x) > i(\mathbf{0}, x^3 - y^2 \cap y)$).



$i((-1,1), x^2 - y \cap y) = 0)$
$i(\mathbf{0}, x^2 - y \cap x - y) = 1)$
$i(\mathbf{0}, x^2 - y \cap y) = 2)$

$i(\mathbf{0}, y^3 - x^2 \cap y) = 2$
$i(\mathbf{0}, y^3 - x^2 \cap x) = 3$
$i(\mathbf{0}, y^3 - x^2 \cap y^3 - x^2) = \infty$

Figure 2.1

Intersection Multiplicities

Intersection multiplicity will be defined in several steps. First, it will be defined for nonsingular intersecting curves in $\mathbb{C}^2$ before it is defined for all curves in $\mathbb{C}^2$. After defining $i(\mathbf{p}, f \cap g)$ for all curves in $\mathbb{C}^2$, we will show that intersection multiplicity is well defined for curves on any 2-dimensional manifold.

2.2. Defining Intersection Multiplicity for a Special Case

Let us first look at another way of calculating the multiplicity of a curve at a point $m_{\mathbf{p}}(g)$. Bezout's Theorem tells us the zero locus of a line $l(x, y) = 0$ and the zero locus of a polynomial $g(x, y) = 0$ of degree n not containing the line l intersect in at most n points counting multiplicities. Let $\mathbf{p} = (a, b) \in l \cap g$ where l is parameterized by $x(t) = t + a$ and $y(t) = t + b$. If we define $g^*(t)$ to be $g(x(t), y(t))$, then g^* is a polynomial in t with $g^*(0) = 0$. The multiplicity of the zero of g^* at 0 can now be calculated and is exactly $m_{\mathbf{p}}(g)$.

Now, do the same thing for a polynomial f instead of a line l. The polynomial g^* was found by first parameterizing $l(x, y) = 0$. This cannot be done globally for every polynomial $f(x, y) = 0$, but by using the implicit function theorem it can be done locally at any nonsingular point of f. Assume for now f and g are polynomials, $f(\mathbf{p}) = 0$ and $f_y(\mathbf{p}) \neq 0$. Using the implicit function theorem, f can be parameterized near $\mathbf{p}$. This parameterization can be used to obtain a power series in a parameter t which contains information from both f and g. This power series will be used to define $i(\mathbf{p}, f \cap g)$ for this special case.

By the implicit function theorem, there is a neighborhood U of a and a unique holomorphic function, $y(x)$ on U such that $y(a) = b$, $f(x, y(x)) = 0$ for all $x \in U$, and there is a neighborhood V of $\mathbf{p}$ such that for all $\mathbf{q} = (q_1, q_2) \in V \cap f$, $y(q_1) = q_2$. Thus the y-coordinates of the set f can be written as an holomorphic function of the x-coordinates, and hence, can be written as a power series, say

$$y(x) = \sum_{i=0}^{\infty} \alpha_i (x - a)^i.$$

Note that this power series is only defined sufficiently near a.

Now, parameterize x by $x(t) = t + a$. Then

$$y(x(t)) = \sum_{i=0}^{\infty} \alpha_i t^i$$

and $(x(t), y(x(t)))$ is a parameterization of the portion of the set f defined in a sufficiently small neighborhood of $t = 0$. Consider the function in t, $g^*(t) = g(x(t), y(x(t)))$. Since $g(x, y)$ is a polynomial, $x(t)$ is a polynomial and $y(x(t))$ is a power series, $g^*(t)$ is a power series defined in a neighborhood of $t = 0$, say

$$g^*(t) = \sum_{i=0}^{\infty} \beta_i t^i.$$

Put $i(\mathbf{p}, f \cap g) = k$ where k is the least integer with $\beta_k \neq 0$, i.e., $i(\mathbf{p}, f \cap g)$ is the degree of the lowest order term of g^*. If $g^* \equiv 0$, put $i(\mathbf{p}, f \cap g) = \infty$. Since $y(x)$ is uniquely determined by f and $\mathbf{p}$, $i(\mathbf{p}, f \cap g)$ is uniquely determined by f, g and $\mathbf{p}$.

Example 2.1: Consider the polynomials $f(x,y) = x^2 - xy + y$ and $f(x,y) = y$. (See Figure 2.2.) By Bezout's Theorem we know f and g intersect in at most two points counting multiplicities. The only point of intersection of the sets f and g, however, is the origin. Therefore, it is reasonable to suspect that $i(\mathbf{0}, f \cap g) = 2$. Since $f(\mathbf{0}) = 0$ and $f_y(\mathbf{0}) \neq 0$, apply the implicit function theorem to yield $y(x) = \sum_{n=2}^{\infty} x^n$ for all $|x| < 1$. Now, $g^*(t) = \sum_{n=2}^{\infty} t^n$ for all $|t| < 1$, and $i(\mathbf{0}, f \cap g) = 2$ as expected.

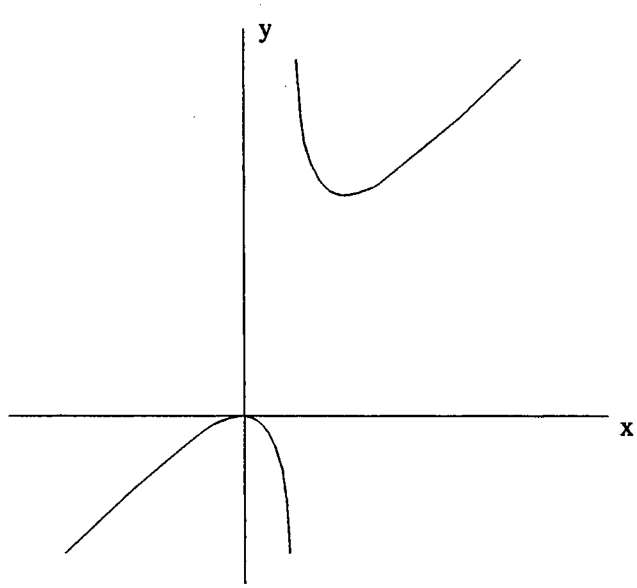


Figure 2.2

$f(x,y) = x^2 - xy + y$

This definition for $i(\mathbf{p}, f \cap g)$ is unacceptable as a general method of calculation because it is too specialized. It does not allow both of f and g to be singular at $\mathbf{p}$ and requires $f(\mathbf{p}) = 0$. To define intersection multiplicity in general we need some definitions.

2.3. Quotient Vector Spaces

The general definition for intersection multiplicity given below is defined as the dimension of vector spaces over the complex numbers. Let us first recall some facts from linear algebra about quotient vector spaces.

2.3.1. *The Definition.*

If V is a vector space over the field $\mathbb{C}$ and $\sim$ is an equivalence relation on V, define the *quotient set of* V *over* $\sim$, denoted $V/\sim$, to be the set of equivalence classes $\{[v]: v \in V\}$ defined by $\sim$. In certain cases, such as in the theorem below, $V/\sim$ is a vector space over $\mathbb{C}$.

Proposition 2.1. *Let V be a vector space over the field $\mathbb{C}$, and let W be a subspace of V. Then the relation $\sim$ on the set V, defined by $v \sim v'$ if and only if $v - v' \in W$, is an equivalence relation. Addition and scalar multiplication can be defined on the set $V/\sim$ by*

$$[v] + [v'] = [v + v'] \text{ for all } v,\ v' \in V$$

and

$$\alpha[v] = [\alpha v] \text{ for all } v \in V \text{ and } \alpha \in \mathbb{C}.$$

Under these operations the quotient set $V/\sim$ is a vector space over $\mathbb{C}$.

Proof: To show $\sim$ is an equivalence relation we must show $\sim$ is reflexive, symmetric and transitive. For all $v \in V$, $v \sim v$ since $v - v = 0$ and W is a subspace of V. If $v \sim v'$, then $v - v' \in W$. Again, since W is a subspace of V, $v' - v = -(v - v') \in W$ so $v' \sim v$. If $v \sim v'$

and $v' \sim v''$ then $v - v' \in W$ and $v' - v'' \in W$. Thus $v - v'' = (v - v') + (v' - v'') \in W$ since W is closed under addition. Therefore $\sim$ is an equivalence relation.

To show $V/\sim$ is a vector space we must first show the definition of addition and scalar multiplication are well defined. Suppose $[v] = [v_1]$ and $[v'] = [v_1']$. Now we have $v - v_1 \in W$ and $v' - v_1' \in W$, and hence $(v+v') - (v_1 + v_1') = (v - v_1) + (v' - v_1') \in W$ because W is closed under addition. Therefore $[v + v'] = [v_1 + v_1']$. Also, $\alpha v - \alpha v' = \alpha(v - v') \in W$ because W is closed under scalar multiplication, hence $[\alpha v] = [\alpha v']$. Therefore the definitions of addition and scalar multiplication are well defined. It is easily shown, with these definitions, the vector space axioms hold for $V/\sim$. Therefore $V/\sim$ is a vector space over $\mathbb{C}$.

There could be several subspaces of V for which the quotient space is of interest. To eliminate confusion, denote $V/\sim$ by V/W for the subspace W and call this vector space the *quotient space of* V *over* W.

2.3.2. *Calculating the Dimension of Quotient Vector Spaces.*

There are many tools which can be used to calculate the dimensions of vector spaces. For example, if $W \subseteq V$ then

$$\dim W \leq \dim V. \tag{2.1}$$

If $\dim(V/W) = 0$ then the only element of V/W is the zero element. In other words, if $[r] \in V/W$ then $[r] = [0]$. Thus, for all $r \in V$, $r = r - 0 \in W$. Therefore $V = W$. Conversely, if $V = W$ and $[r] \in V/W$, then $r - 0 = r \in W$ and $[r] = [0]$. Therefore

$$\dim V/W = 0 \text{ if and only if } V = W.$$

The dimension of vector spaces can easily be calculated when there are linear maps involving vector spaces of known dimensions. Let $T: V \to W$ be a linear map. Then the following are true:

$$\dim V = \dim \operatorname{Ker}(T) + \dim \operatorname{Im}(T) \tag{2.2}$$

and if T is bijective, then

$$\dim V = \dim W. \tag{2.3}$$

The first statement is a familiar result from linear algebra and the second statement is an easy consequence of the first.

It often happens there is a natural way to assign a linear map which in turn can be used to calculate the dimensions of the vector spaces involved. For example, let V be a vector space over $\mathbb{C}$ and let U and W be subspaces of V such that $W \subseteq U \subseteq V$. Define the map

$$T: V/W \to V/U$$

by $T([a]) = [a]$. Note the two different meanings of $[a]$ in this assignment. Is this a well-defined map? Suppose $[a] = [b] \in V/W$. Then $a - b \in W \subseteq U$ so $[a] = [b] \in V/U$, and the map is well-defined. It is clear from the definitions of addition and scalar multiplication this is a linear map. It is also true this map is surjective. For each $[a] \in V/U$ there is a corresponding $[a] \in V/W$ which is its preimage. That means $\operatorname{Im} T = V/U$ so we have $\dim V/W = \dim \operatorname{Ker} T + \dim V/U$ and in particular

$$\dim V/W \geq \dim V/U.$$

With the same sets we can define another natural map

$$S: U/W \to V/W$$

by $S([a]) = [a]$. If $[a] = [b] \in U/W$ then $a, b \in U \subseteq V$ and $a - b \in W$. So $[a] = [b] \in V/W$ and the map is well defined. Again it is clear this map is linear. What does this map imply about the dimensions of U/W and V/W? In this case, the map is injective: if $S([a]) = [0] \in V/W$ then $a \in W$ so $[a] = [0] \in U/W$. Now $\operatorname{Ker} S = \{[0]\}$ and $\dim U/W = \dim \operatorname{Im} S$. We can go on to say

$$\dim U/W \leq \dim V/W.$$

since $\operatorname{Im} S \subseteq V/W$.

2.4. Defining Intersection Multiplicity on $\mathbb{C}^2$

We are now prepared to define intersection multiplicity in general and state useful properties of it. This is done in the statement and proof of the following theorem.

Theorem 2.2. *Given 2 curves f and g on $\mathbb{C}^2$ and a point $\mathbf{p} \in \mathbb{C}^2$ there exists a unique intersection multiplicity $i(\mathbf{p}, f \cap g)$ which has the following properties for all curves f, f_i, g, and g_i on $\mathbb{C}^2$:*

$$i(\mathbf{p}, f \cap g) = i(\mathbf{p}, g \cap f); \tag{2.4}$$

$$i(\mathbf{p}, f \cap g_1 g_2) = i(\mathbf{p}, f \cap g_1) + i(\mathbf{p}, f \cap g_2); \tag{2.5}$$

$$i(\mathbf{p}, f \cap g_1 + f g_2) = i(\mathbf{p}, f \cap g_1); \tag{2.6}$$

$$i(\mathbf{p}, f \cap g) = 0 \textit{ if and only if } \mathbf{p} \notin f \cap g; \tag{2.7.1}$$

$$i(\mathbf{p}, f \cap g) \textit{ is a positive integer if and only if } f \textit{ and } g \textit{ intersect properly at } \mathbf{p}; \tag{2.7.2}$$

$$i(\mathbf{p}, f \cap g) = \infty \textit{ if and only if } f \textit{ and } g \textit{ intersect improperly at } \mathbf{p}; \tag{2.7.3}$$

$$i(\mathbf{p}, x - a \cap y - b) = 1; \textit{and} \tag{2.8}$$

$$i(\mathbf{p}, f \cap g) = i(T^{-1}(\mathbf{p}), f \circ T \cap g \circ T)) \textit{ where } T\colon \mathbb{C}^2 \to \mathbb{C}^2 \textit{ is a translation.} \tag{2.9}$$

The proof of this theorem comes in several parts. The existence is proven by defining $i(\mathbf{p}, f \cap g)$ and showing that the definition has the properties specified. The definition is given below and the properties are verified in Appendix A. Proof of existence will allow us to talk about the intersection multiplicity but does not help in calculating those numbers. The uniqueness is proven by showing that any numbers satisfying the properties specified in Theorem 2.2 have to be intersection multiplicities as defined. The uniqueness of $i(\mathbf{p}, f \cap g)$ will allow us to calculate intersection multiplicities from the properties in Theorem 2.2 without resorting to the definition each time. Uniqueness is proven in Section 2.5.

Intersection multiplicity is defined by creating a vector space over $\mathbb{C}$ which depends on f, g and $\mathbf{p}$ and defining $i(\mathbf{p}, f \cap g)$ to be the dimension of that vector space over $\mathbb{C}$. Let $\mathcal{O}_\mathbf{p}$ be the set of all rational functions in 2 variables which are defined at $\mathbf{p}$. Note $\mathcal{O}_\mathbf{p}$ is a vector space over $\mathbb{C}$ with the usual addition and scalar multiplication. Let f and g be algebraic curves on $\mathbb{C}^2$. Define $(f, g) = \{af + bg : a, b \in \mathcal{O}_\mathbf{p}\}$. We want to show (f, g) is a subspace of $\mathcal{O}_\mathbf{p}$. For all $af + bg$, $a'f + b'g \in (f, g)$ and all $\alpha \in \mathbb{C}$,

$$(af + bg) + (a'f + b'g) = (a + a')f + (b + b')g$$

and

$$\alpha(af + bg) = (\alpha a)f + (\alpha b)g.$$

Thus (f, g) is a subspace of $\mathcal{O}_\mathbf{p}$.

Now $\mathcal{O}_\mathbf{p}/(f, g)$ is a vector space and we can define $i(\mathbf{p}, f \cap g) = \dim \mathcal{O}_\mathbf{p}/(f, g)$. Note there are no restrictions on the polynomials f and g in this new definition, so it applies to all polynomials f and g and all points $\mathbf{p}$.

Example 2.2: Let us look at an example where $\dim \mathcal{O}_\mathbf{p}/(f, g)$ is calculated. According to Theorem 2.2, $i(\mathbf{0}, x \cap y) = 1$. This is proven here. First, we will find a bijective linear map

$$T: \mathcal{O}_\mathbf{0}/(x, y) \to \mathbb{C}$$

from which we know $i(\mathbf{0}, x \cap y) = \dim \mathcal{O}_\mathbf{0}/(x, y) = \dim \mathbb{C} = 1$ (2.3).

This proof depends on being able to describe the elements of (x,y). Suppose $f/g \in \mathcal{O}_{\mathbf{0}}$. Then $g(\mathbf{0}) \neq 0$ because f/g must be defined at $\mathbf{0}$. On the other hand, $f/g = ax + by$ for some $a, b \in \mathcal{O}_{\mathbf{0}}$, so $(f/g)(\mathbf{0}) = 0$. Conversely, $f/g \in (x,y)$ if $(f/g)(\mathbf{0}) = 0$ and $g(\mathbf{0}) \neq 0$. Thus $(x,y) = \{f/g : (f/g)(\mathbf{0}) = 0 \text{ and } g(\mathbf{0}) \neq 0\}$

Define T by putting $T([a]) = a(\mathbf{0})$ for all $[a] \in \mathcal{O}_{\mathbf{0}}/(x,y)$. Is T well defined? Suppose $T([a]) = T([b])$. Then $(a-b)(\mathbf{0}) = a(\mathbf{0}) - b(\mathbf{0}) = 0$ so $a - b \in (x,y)$. Therefore T is well defined.

We must also show that T is a bijective map. For each $\alpha \in \mathbb{C}$, $[\alpha] \in \mathcal{O}_{\mathbf{0}}/(x,y)$ and $\alpha(\mathbf{0}) = \alpha$. Hence, T is surjective. If $T([f/g]) = 0$, then $(f/g)(\mathbf{0}) = 0$ and $f/g \in (x,y)$. Thus, the kernel of T is $\{[0]\}$ and T is bijective.

Therefore, $i(\mathbf{0}, x \cap y) = 1$. Note the same proof works for $i(\mathbf{p}, x - a \cap y - b)$. Simply write f and g as polynomials in $x - a$ and $y - b$ and replace x, y and $\mathbf{0}$ by $x - a$, $y - b$ and $\mathbf{p} = (a,b)$, respectively, and the proof is the same. We have verified (2.8).

Example 2.1: (continued) To show $i(\mathbf{0}, x^2 - xy + y \cap y) = 2$ using this definition we need to show that $\dim \mathcal{O}_{\mathbf{0}}/(x^2 - xy + y, y) = 2$. First, note that a general element of $(x^2 - xy + y, y)$ can be written $a(x^2 - xy + y) + by = ax^2 + (b + a - ax)y$ where a and b are in $\mathcal{O}_{\mathbf{0}}$. Therefore, $(x^2 - xy + y, y) \subseteq (x^2, y)$. The inequality can also be shown in the other direction. Thus, it suffices to show that $\dim \mathcal{O}_{\mathbf{0}}/(x^2, y) = 2$. Note that the statement

$$\dim \mathcal{O}_{\mathbf{0}}/(x^2 - xy + y, y) = \dim \mathcal{O}_{\mathbf{p}}/(x^2, y)$$

is equivalent to the statement

$$i(\mathbf{0}, x^2 - xy + y \cap y) = i(\mathbf{0}, x^2 \cap y).$$

The claim is that $V = \mathcal{O}_{\mathbf{0}}/(x^2, y)$ has a basis $B = \{[1], [x]\}$. If this claim is true, then $\dim V = 2$. Clearly, $[1]$ and $[x]$ are linearly independent as elements of V. To show that

B spans V we need to show that for every $[r/s] \in V$, $[r/s] = a[1] + b[x] = [a + bx]$ for some $a, b \in \mathbb{C}$. Equivalently, $(r - as - bxs)/s \in (x^2, y)$. An element of (x^2, y) is a rational function f/g defined at $\mathbf{0}$ where f has no constant term and no first degree term in x. Let r/s be any element of $\mathcal{O}_\mathbf{0}$ and choose $a = r(\mathbf{0})/s(\mathbf{0})$ and $b = \left[\frac{\partial r}{\partial x}(\mathbf{0}) - a\frac{\partial s}{\partial x}(\mathbf{0})\right]/s(\mathbf{0})$. To show that $(r - as - bxs)/s \in (x^2, y)$ it suffices to show that $(r - as - bxs)(\mathbf{0}) = 0$ and $\frac{\partial}{\partial x}(r - as - bxs)(\mathbf{0}) = 0$. The choices of a and b above insure that this is true.

Again we have shown $i(\mathbf{0}, x^2 - xy + y \cap y) = 2$.

2.5. An Algorithm for Calculating Intersection Multiplicity

To show the properties Theorem 2.2 uniquely define $i(\mathbf{p}, f \cap g)$ it suffices to give an algorithm for calculating $i(\mathbf{p}, f \cap g)$ only using these properties. First we will give a procedure for calculating $i(\mathbf{p}, f \cap g)$ then, in Section 2.5.2, we will show this procedure is indeed an algorithm.

2.5.1. *The Algorithm.*

Algorithm 2.1.

1. Replace f and g with polynomials also called f and g obtained by translating $\mathbf{p}$ to the origin.
2. Determine if f and g have a common factor containing $\mathbf{p}$. If so, put $i(\mathbf{p}, f \cap g) = \infty$ and stop. Otherwise, put $i(\mathbf{p}, f \cap g) = 0$ and continue.
3. If $f(0,0) \neq 0$ or $g(0,0) \neq 0$ stop. Otherwise, continue.
4. Calculate the polynomials $f(x,0)$ and $g(x,0)$. Put d_f equal to the degree of $f(x,0)$ and d_g equal to the degree of $g(x,0)$.
   a. If $g(x,0) \equiv 0$ or $d_f > d_g$, switch f and g and go to 4.
   b. Otherwise, if $f(x,0) \equiv 0$, increase $i(\mathbf{p}, f \cap g)$ by the largest degree of x which divides $g(x,0)$, replace f by f/y and go to 3.

c. Otherwise, replace f by f/a and g by g/b where a and b are scalars such that the new $f(x,0)$ and $g(x,0)$ will be monic polynomials. Replace g again by $g - x^{d_g - d_f} f$ and go to 3.

If it can be shown this procedure correctly calculates $i(\mathbf{p}, f \cap g)$ and terminates for all polynomials f and g assuming the properties above, we will have shown these properties uniquely determine $i(\mathbf{p}, f \cap g)$. Note step 2 is quite involved and requires an algorithm for finding common factors for polynomials in 2 variables.

Example 2.1: (continued) By following the algorithm above we obtain the data in Table 2.1. Again $i(\mathbf{0}, f \cap g) = 2$. The algorithm terminates at step 3 because f no longer contains $\mathbf{0}$.

TABLE 2.1

ALGORITHMIC CALCULATION OF $i(\mathbf{0}, x^2 - xy + y \cap y)$

	f	g	$f(x,0)$	$g(x,0)$	$i(\mathbf{p}, f \cap g)$
Before step 4(a)	$x^2 - xy + y$	y	x^2	0	0
After step 4(b)	y	$x^2 - xy + y$	0	x^2	2
After step 3	1	$x^2 - xy + y$			2

2.5.2. *Why the Algorithm Works.*

Here we will justify each of the steps in the algorithm and verify that the algorithm does terminate for all polynomials.

We may assume $\mathbf{p}$ is the origin. If $\mathbf{p}$ were not the origin, by (2.9) a translation can be applied to f and g to obtain new polynomials, also called f and g, whose intersection multiplicity at the origin is the same as the original $i(\mathbf{p}, f \cap g)$.

Exactly one of the following is true: $\mathbf{0} \notin f \cap g$, f and g intersect properly at $\mathbf{0}$ or f and g intersect improperly at $\mathbf{0}$. If f and g intersect improperly, then $i(\mathbf{0}, f \cap g) = \infty$ and we are done by (2.7.3). If $\mathbf{0} \notin f \cap g$, then $i(\mathbf{0}, f \cap g) = 0$ and we are done by (2.7.1). Thus we may assume f and g intersect properly at $\mathbf{0}$. Note we now know f and g have no constant terms.

The rest of the proof follows by induction. The case of $i(\mathbf{0}, f \cap g) = 0$ is taken care of above, so assume $i(\mathbf{0}, f \cap g) = n > 0$. Further assume $i(\mathbf{0}, a \cap b) = j$ can be calculated for all curves a and $b \in \mathbb{C}^2$ and all $j < n$.

Consider the two polynomials, $f(x,0)$ and $g(x,0)$, and let d_f and d_g be the degrees of these polynomials, respectively. By (2.4), we may assume either $f(x,0) \equiv 0$ or $1 \leq d_f \leq d_g$.

Case 1: If $f(x,0) \equiv 0$, then f has no terms strictly in x, hence is divisible by y. Let $f = yf_1$. On the other hand, y cannot divide g since f and g have no common factors. By (2.5), $i(\mathbf{0}, f \cap g) = i(\mathbf{0}, f_1 \cap g) + i(\mathbf{0}, y \cap g)$. Let m be the highest power of x dividing $g(x,0)$. Since g has no constant term and y does not divide g, $m > 0$. Now, $g(x,0) = x^m(\alpha_0 + \alpha_1 x + \ldots + \alpha_j x^j) = x^m g_1$ and $g(x,y) = x^m g_1 + y g_2$ with g_2 a polynomial in x and y. Since

$$\begin{aligned} i(\mathbf{0}, y \cap g) &= i(\mathbf{0}, y \cap x^m g_1) \\ &= m\, i(\mathbf{0}, y \cap x) + i(\mathbf{0}, y \cap g_1) \\ &= m + i(\mathbf{0}, y \cap g_1) \\ &= m > 0, \end{aligned}$$

then $i(\mathbf{0}, f \cap g) = i(\mathbf{0}, f_1 \cap g) + m$. Now $i(\mathbf{0}, f_1 \cap g) < n$ and this case is finished by the induction hypothesis.

Case 2: What happens when $f(x,0) \not\equiv 0$? Put $M = d_f + d_g$. If $\alpha \in \mathbb{C}$ and a, b are curves, then $i(\mathbf{p}, a \cap \alpha b) = i(\mathbf{p}, a \cap b)$ by properties 2.5 and 2.7.1. Thus, we may assume $f(x,0)$ and

$g(x,0)$ are monic. Put $h = g - x^{d_g - d_f} f$ and by (2.6) we have $i(\mathbf{0}, f \cap g) = i(\mathbf{0}, f \cap h)$. Also, $h(x,0) = g(x,0) - x^{d_g - d_f} f(x,0)$ and either $h(x,0) \equiv 0$ or the degree of $h(x,0) < d_g$. Put d_h equal to the degree of $h(x,0)$.

If $h(x,0) \equiv 0$, switch g and h and the calculation of $i(\mathbf{0}, f \cap g) = i(\mathbf{0}, f \cap h)$ falls under case 1 above. Otherwise, $1 \leq d_h < d_g$. Replace g by h and then interchange f and g, if necessary, such that $d_f \leq d_g$. Now the calculation of $i(\mathbf{0}, f \cap g)$ falls under the case of $f(x,0) \not\equiv 0$. Since the new $d_f + d_g < M$, a finite number of repetitions of this process will finally result in $h(x,0) \equiv 0$ which can be handled by case 1 above. Therefore the proof is done.

It is easily seen (2.9) of Theorem 2.2 was not needed in this proof. Instead of assuming $\mathbf{p} = \mathbf{0}$, write f and g as polynomials in $x - a$ and $y - b$. Then, using $x - a$, $y - b$, and $\mathbf{p} = (a, b)$ in place of x, y and $\mathbf{0}$ above, the proof is the same using polynomial expansions about $\mathbf{p}$.

2.6. Intersection Multiplicity on Complex Manifolds

If $\{(U_i, f_i)\}$ and $\{(U_i, g_i)\}$ are 2 algebraic curves on a 2-dimensional manifold x and $\mathbf{p} \in U_i$, the intersection multiplicity $i(\mathbf{p}, f_i \cap g_i)$ can be calculated as above. To show that intersection multiplicity is well defined for curves on complex manifolds, we need to show that

$$i(\mathbf{p}, f_i \cap g_i) = i(\phi_{ji}(\mathbf{p}), f_j \cap g_j)$$

whenever $\mathbf{p} \in U_i \cap U_j$. By definition, we need to show

$$\dim \mathcal{O}_{\mathbf{p}}/(f_i, g_i) = \dim \mathcal{O}_{\phi_{ji}(\mathbf{p})}/(f_j, g_j). \tag{2.10}$$

First note $\mathcal{O}_{\mathbf{p}}$ and $\mathcal{O}_{\phi_{ij}(\mathbf{p})}$ are actually the same set. Since intersection multiplicity is a local property, we need only consider what happens on a neighborhood of $\mathbf{p}$. The vector spaces

$\mathcal{O}_{\mathbf{p}}$, $\mathcal{O}_{\phi_{ji}(\mathbf{p})}$, (f_i, g_i), and (f_j, g_j) are all defined on $U_i \cap U_j$ and this is the neighborhood of $\mathbf{p}$ we will consider. The elements of $\mathcal{O}_{\mathbf{p}}$ are rational functions defined at $\mathbf{p}$. If we knew the coordinates of ϕ_{ij} were also rational functions, then for each $r \in \mathcal{O}_{\mathbf{p}}$, $r \circ \phi_{ij}$ is a rational function defined at $\phi_{ji}(\mathbf{p})$. Thus every element of $\mathcal{O}_{\mathbf{p}}$ gives an element of $\mathcal{O}_{\phi_{ij}}$. Similarly, every element of $\mathcal{O}_{\phi_{ij}(\mathbf{p})}$ gives an element of $\mathcal{O}_{\mathbf{p}}$. Thus, to establish (2.10) we need only show (f_j, g_j) is the same subspace as (f_i, g_i). But again, these can be considered the same set. For each $af_i + bg_i \in (f_i, g_i)$, $(af_i + bg_i) \circ \phi_{ij} \in (f_j, g_j)$ and vice versa. Therefore, (2.10) is true and intersection multiplicity is well defined on X if the gluing maps have rational coordinates. All gluing maps used in this paper have rational coordinates.

It should be noted that the proof of property 2.9 is a special case of this argument.

2.7. Conclusion

Note that using the notation of this chapter, Bezout's Theorem for $\mathbb{C}^2$ can be restated with the inequality

$$\sum_{\mathbf{p} \in f \cap g} i(\mathbf{p}, f \cap g) \leq mn$$

for curves f and g which meet properly. Although there is no Bezout's theorem for general 2-dimensional manifolds X, the number of points in the intersection of 2 curves C and D which meet properly can be counted and is

$$\sum_{\mathbf{p} \in C \cap D} i(\mathbf{p}, C \cap D). \tag{2.11}$$

The manifolds in this paper are all compact, hence, every open cover $\{U_i\}$ of X can be considered a finite cover. On each U_i, $\sum_{\mathbf{p} \in C \cap D \cap U_i} i(\mathbf{p}, C \cap D)$ is finite by Bezout's Theorem. Therefore, (2.11) is finite.

Intersection multiplicity will be used in several ways in Chapters 4 and 5. For instance, if F is a meromorphic function and C is a curve on a complex manifold X, intersection multiplicities can be used to count the zeros and poles of F on C. The zeros F_0 and poles F_∞ of F define curves on X. Then the zeros of F on C is the set $F_0 \cap C$ and the multiplicity of each zero $\mathbf{p} \in F_0 \cap C$ is $i(\mathbf{p}, F_0 \cap C)$. Similarly, the multiplicity of each pole $\mathbf{p}$ on C is $i(\mathbf{p}, F_\infty \cap C)$.

CHAPTER THREE

REMOVING BASE POINTS

3.1. Introduction

Consider the triangular surface defined by the image of the rational map

$$\psi : \mathbb{P}^2 - - \longrightarrow \mathbb{P}^3$$

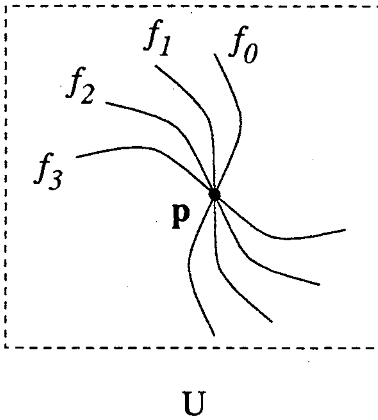
where

$$\psi(\mathbf{p}) = (f_0(\mathbf{p}) : f_1(\mathbf{p}) : f_2(\mathbf{p}) : f_3(\mathbf{p})) \tag{3.1}$$

(see Section 1.8). In this chapter we will find a new parameter space X and map

$$\psi' : X \longrightarrow \mathbb{P}^3$$

such that ψ' has no base points and $\mathrm{Im}\,(\psi) \subseteq \mathrm{Im}\,(\psi')$. This will be accomplished by blowing up $\mathbb{P}^2$ at all the base points of ψ. The effect of blowing up a point will be to "pull apart" a disk around the point according to the different directions through the point. Another way to think of it is to replace the point $\mathbf{p}$ with a line E such that all the lines which once passed through $\mathbf{p}$ now intersect the line E and do not intersect each other (see Figure 3.2). This process will remove base points by "pulling apart" the curves $f_i = 0$. If there is a base point $\mathbf{p}$, all $f_i(\mathbf{p}) = 0$ and the curves f_i may look like Figure 3.1 if the curves meet transversally at $\mathbf{p}$. After blowing up $\mathbb{P}^2$ at $\mathbf{p}$, these curves will no longer intersect but instead will meet E at four different points corresponding to the four tangent directions through $\mathbf{p}$. Thus, there is no longer a base point.



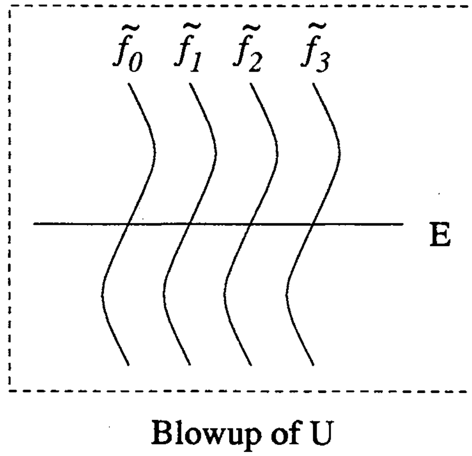


Figure 3.1

Blow-up of the Curves f_i

Of course, this is a simplistic view, but accurate all the same. We will begin by looking at what happens to curves locally by blowing up points in $\mathbb{C}^2$. Later, we will blow up points in $\mathbb{P}^2$ and investigate the complications involved when trying to remove all the base points of a surface.

3.2. Blow ups on the Complex Plane

The blow up of $\mathbb{C}^2$ at the origin will be a 2-dimensional manifold which is a subset of the 3-dimensional manifold $\mathbb{C}^2 \times \mathbb{P}^1$ with coordinates $(r, s; t\colon u)$. The complex coordinates (r, s) allow us to assign one point $(r, s; r\colon s)$ in the blow up for every point in $\mathbb{C}^2$ away from $\mathbf{0}$. The homogeneous coordinates $(t\colon u)$ allow us to assign one point $(0, 0; t\colon u)$ for each direction through $\mathbf{0}$. The surface in $\mathbb{C}^2 \times \mathbb{P}^1$ which contains these points is defined by the equation

$$ru = st$$

and is denoted $\tilde{\mathbb{C}}^2$. This surface is a 2-dimensional manifold with the following coordinate neighborhoods:

$$B_0 = \{(r, s; t\colon u) : ru = st \text{ and } t \neq 0\}$$

and

$$B_1 = \{(r, s; t\colon u) : ru = st \text{ and } u \neq 0\}.$$

The local coordinates on B_0 are defined by $(r, u/t) = \psi_0(r, s; t\colon u)$ and on B_1 by $(s, t/u) = \psi_1(r, s; t\colon u)$. The gluing maps are $\psi_{10} : B_0 \longrightarrow B_1$ defined by

$$\psi_{10}(r, u) = (ru, 1/u)$$

and $\psi_{10} : B_1 \longrightarrow B_0$ defined by

$$\psi_{01}(s, t) = (st, 1/t)$$

which are holomorphic on $\psi_0(B_0 \cap B_1)$ and $\psi_1(B_0 \cap B_1)$, respectively.

There is also a projection map

$$\pi : \tilde{\mathbb{C}}^2 \longrightarrow \mathbb{C}^2$$

defined by $\pi(r, s; t\colon u) = (r, s)$. The set $\tilde{\mathbb{C}}^2$ together with the projection map π is called the *blow up of* $\mathbb{C}^2$ *at* $\mathbf{0}$. The following proposition describes the relationship of $\mathbb{C}^2$ and $\tilde{\mathbb{C}}^2$ and how $\mathbb{C}^2$ has been "pulled apart." (See Figure 3.2)

Proposition 3.1. *If π and $\tilde{\mathbb{C}}^2$ are defined as above the the following are true:*

1. *$\pi^{-1}(\mathbf{0})$ is the curve $E = \{(r, s; t\colon u) : r = 0$ and $s = 0\}$.*
2. *The topological spaces $\mathbb{C}^2 - \mathbf{0}$ and $\tilde{\mathbb{C}}^2 - E$ are homeomorphic.*
3. *If 2 different lines in $\mathbb{C}^2$, l_1 and l_2, intersect at the origin, then $\pi^{-1}(l_1) \cup \pi^{-1}(l_2)$ contains the line E and two other lines, $\tilde{l_1}$ and $\tilde{l_2}$, with $\tilde{l_1} \cap \tilde{l_2} = \emptyset$.*

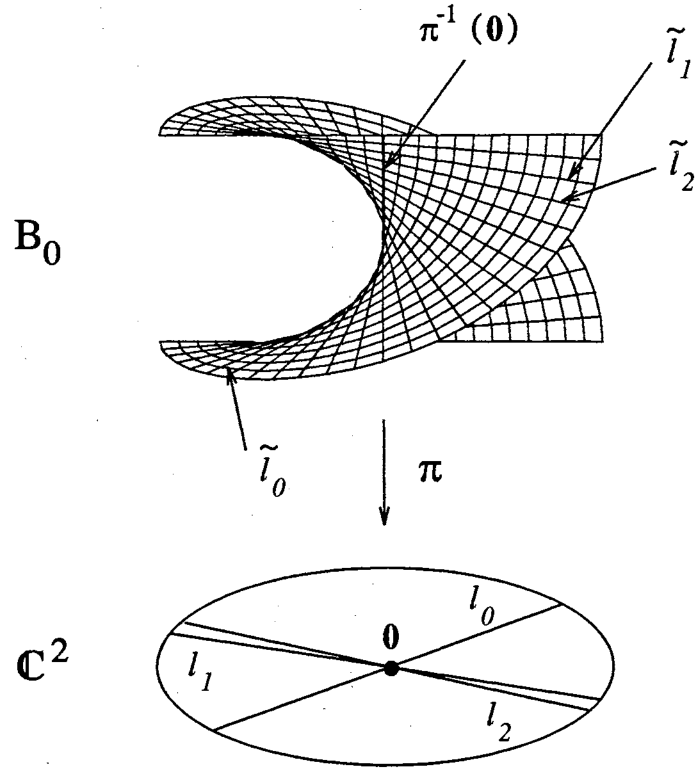


Figure 3.2

Blowing Up the Origin in $\mathbb{C}^2$

Proof:

1. If $\mathbf{p} = \mathbf{0}$, every point $(0, 0; t{:}\, u)$ is mapped to $\mathbf{p}$ by π. The local equations of this curve are $r = 0$ on B_0 and $s = 0$ on B_1. Thus $\pi^{-1}(\mathbf{0})$ is the line $\{(r, s; t{:}\, u) : r = 0 \text{ and } s = 0\}$ in $\tilde{\mathbb{C}}^2$. We will call this line the *exceptional curve* and denote it E.
2. If $\mathbf{p} \in \mathbb{C}^2 - \mathbf{0}$, $\mathbf{p} = (r, s)$ where one of r or s is nonzero. Any element of $\pi^{-1}(\mathbf{p})$ is a point $(r, s; t{:}\, u)$ for some t and u. If r is nonzero and t is chosen arbitrarily, $u = st/r$ is uniquely determined. In the same manner, the homogeneous coordinates are uniquely determined if s is nonzero instead. Thus $\pi^{-1}(\mathbf{p})$ consists of the single point $(r, s; r{:}\, s)$. On the other hand, if $\mathbf{p} = (r, s; t{:}\, u) \in \tilde{\mathbb{C}}^2 - E$, $\pi(\mathbf{p}) = (r, s) \neq \mathbf{0}$. Therefore, there is a one-to-one correspondence between the points of $\mathbb{C}^2 - \mathbf{0}$ and $\tilde{\mathbb{C}}^2 - E$ under π. Also, the maps π and π^{-1} are continuous on $\tilde{\mathbb{C}}^2 - E$ and $\mathbb{C}^2 - \mathbf{0}$, respectively. Therefore, these topological spaces are homeomorphic.
3. Let l_1 be a line in $\mathbb{C}^2$ containing $\mathbf{0}$ parameterized by $r = a\rho$ and $s = b\rho$. Since $\mathbf{0} \in l_1$, we know $E \subseteq \pi^{-1}(l_1)$ from part 1 of the proof. In fact, $\pi^{-1}(l_1) = \{(a\rho, b\rho; a{:}\, b)\} \cup E$. Denote the line $\{(a\rho, b\rho; a{:}\, b)\}$ by $\tilde{l_1}$. If l_2 is a different line parameterized by $r = a'\sigma$ and $s = b'\sigma$ then $\tilde{l_2} = \{(a'\sigma, b'\sigma; a'{:}\, b')\}$. Since l_1 and l_2 are different lines, $(a{:}\, b) \neq (a'{:}\, b')$ as homogeneous coordinates. Thus $\tilde{l_1} \cap \tilde{l_2}$ is empty.

We will call $\tilde{l_i}$ the strict transform of l_i. In general, the *strict transform of a curve D in $\mathbb{C}^2$* will be the closure of the set $\pi^{-1}(D) - E$ and will be denoted $\tilde{D}$. The set $\pi^{-1}(D)$ is called the *total transform of D*.

The projection map π is said to be a *birational equavalence* because of part 2 of this proposition [Har: p. 493]. This will be useful later in calculating the genus of curves which are blown up. Our first purpose is to use blowing up to separate the curves $f_i = 0$, but it is important also to see what happens to each individual curve $f_i = 0$. In this example we investigate what happens to a singular curve blown up at its singularity.

Example 3.1: The plane curve $C \subseteq \mathbb{C}^2$ defined by the zeros of the polynomial

$$y^2 = x^2(x+1) \tag{3.2}$$

has a double point at the origin (see Figure 3.3). Also, there are two distinct tangents to C at the origin:

$$l_1 = \{(x,y) : x = y\}$$

and

$$l_2 = \{(x,y) : x = -y\}.$$

Let $(x, y; t\colon u)$ be the coordinates of $\mathbb{C}^2 \times \mathbb{P}^1$. The exceptional curve is defined by the local equations $x = 0$ in B_0 and $y = 0$ in B_1. From Proposition 3.1(3) we know the strict transforms of the tangent lines are the curves

$$\tilde{l_1} = \{(x, x; 1\colon 1)\}$$

and

$$\tilde{l_2} = \{(x, -x; 1\colon -1)\}$$

but how do we represent $\pi^{-1}(C)$ and $\tilde{C}$? The relationship between the local coordinates of B_0 and the coordinates of $\mathbb{C}^2$ is defined by $\pi \circ \psi_0^{-1}(x,u) = \pi(x, xu; 1\colon u) = (x, xu) \in \mathbb{C}^2$. Thus, we can calculate the local equations for $\pi^{-1}(C)$ by making the substitutions x for x and xu for y in equation (3.2) to obtain

$$(xu)^2 = x^2(x+1)$$

or, equivalently,

$$\{x^2 = 0\} \cup \{u^2 = x+1\}.$$

The strict transform of C can now be calculated by removing E which has local equation $x = 0$. There are, in fact, two copies of E defined by $x^2 = 0$. After removing E and taking the closure of what is left, we obtain the local equation of $\tilde{C}$ in B_0, the parabola

$$u^2 = x + 1.$$

In the same way we calculate the local equations of $\pi^{-1}(C)$ and $\tilde{C}$ in B_1. The substitutions this time are y for y and yt for x. The local equation of $\pi^{-1}(C)$ is

$$y^2 = y^2t^2(yt + 1)$$

or, equivalently,

$$\{y^2 = 0\} \cup \{1 = t^2(yt + 1)\}.$$

Again there are two copies of E defined by $y^2 = 0$. The closure of $\pi^{-1}(C) - E$ is

$$1 = t^2(yt + 1)$$

and is the local equation for $\tilde{C}$ in B_1.

Therefore, $\pi^{-1}(C)$ is defined by data

$$\{(B_0, x^2(u^2 - x - 1)), (B_1, y^2(t^2(yt + 1) - 1))\},$$

and $\tilde{C}$ is defined by data

$$\{(B_i, g_i)\} = \{(B_0, u^2 - x - 1), (B_1, t^2(yt + 1) - 1)\}.$$

The exceptional curve E occurs with multiplicity 2 in $\pi^{-1}(C)$. It is no coincidence this is the same as $m_{\mathbf{0}}(C)$ as will be shown in Proposition 3.2.

The original curve C is nonsingular except at $\mathbf{0}$. We can easily check to see the curve $\tilde{C}$ is nonsingular everywhere. The partial of g_0 with respect to x is -1 so g_0 has no singular points on B_0. The first partials of g_1 are only zero if $t = 0$, but $(y, t) = (y, 0)$ is not in the zero set of g_1. Therefore, g_1 also has no singular points on B_1. In effect, the blow up removed the singularity of the curve C. How did this work? The blow up pulled $\mathbb{C}^2$ apart according to the directions of lines through $\mathbf{0}$. Since C had two distinct tangents at $\mathbf{0}$, the curve C was also pulled apart there. Instead of the curve meeting itself, $\tilde{C}$ now meets E at two distinct points, $\mathbf{p}_1 = (0,0;1:1)$ and $\mathbf{p}_2 = (0,0;1:-1)$ (see Figure 3.3).

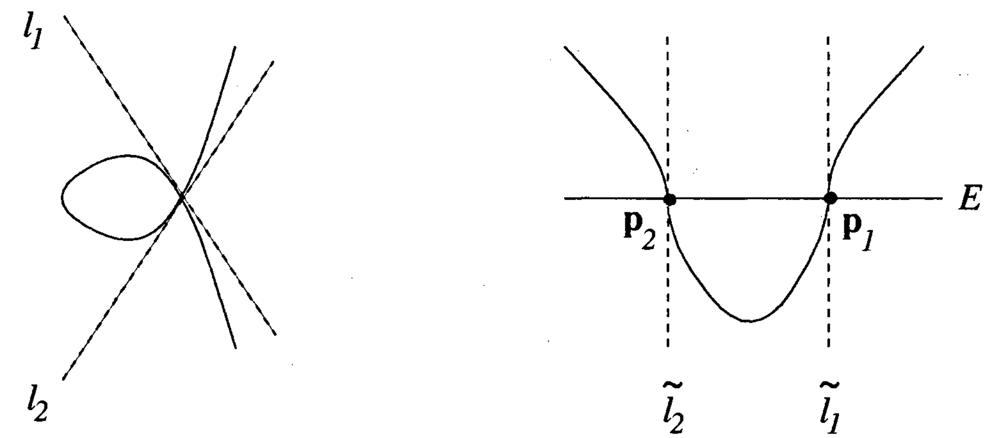


Figure 3.3

$C = y^2 - x^2(x+1)$ and $\pi^{-1}(C)$

It should be clear from this example that an important application of blowing up is to resolve singularities. To calculate implicit degree we will use blow ups to remove base points. Later we will use blow ups to produce nonsingular curves in order to calculate the genus of curves.

3.3. Blow ups on the Projective Plane

To blow up points in $\mathbb{P}^2$ we will simply blow up a point in one of U_0, U_1, or U_2 and glue the result back together with the other two sets. Since U_i is a subset of $\mathbb{C}^2$, the blow up there is exactly as in Section 3.2. First, consider the blow up of $\mathbf{p} = (1:0:0) \in \mathbb{P}^2$ and note

$\mathbf{p} \in U_0$ and $\mathbf{p} \notin U_1 \cup U_2$. The local coordinates of U_0 are (y_0, z_0), so the blow up of U_0 at $\mathbf{p}$ is the set $\tilde{U_0} = \{(y_0, z_0; t\colon u) : y_0 u = z_0 t\}$ together with the projection map $\pi\colon \tilde{U_0} \to U_0$. The set $\tilde{U_0}$ can be covered by the coordinate neighborhoods

$$(\tilde{U_0})_0 = \{(y_0, z_0; t\colon u) : y_0 u = z_0 t \text{ and } t \neq 0\}$$

and

$$(\tilde{U_0})_1 = \{(y_0, z_0; t\colon u) : y_0 u = z_0 t \text{ and } u \neq 0\}.$$

In $(\tilde{U_0})_0$ the local coordinates are $\psi_0(y_0, z_0; t\colon u) = (y_0, u/t)$ and in $(\tilde{U_0})_1$ the local coordinates are $\psi_1(y_0, z_0; t\colon u) = (z_0, t/u)$. The local coordinates of U_1 and U_2 remain unchanged. The sets $(\tilde{U_0})_0$ and $(\tilde{U_0})_1$ will be used repeatedly, so rename these sets U_{00} and U_{01}, respectively. This notation will be used throughout this chapter, i.e., when a set A is blown up at a point it will be denoted $\tilde{A}$ and its coordinate neighborhoods will be A_0 and A_1. For convenience of notation, the subscripts 00 and 01 will be treated as a single index (see gluing maps below).

The manifold $\tilde{\mathbb{P}}^2$, the blow up of $\mathbb{P}^2$, is the first manifold introduced in this paper without a global set of coordinates. This makes moving from one set of local coordinates to another rather clumsy. Figure 3.4, illustrates the relationship between the local neighborhoods U_{00}, U_{01}, U_1, U_2 and the various coordinate charts, gluing maps, and the projection map π. It should be noted that in Figure 3.4 that gluing maps such as ψ_{10} are not defined on all of $\psi(U_{01})$ but only on $\psi(U_{01} \cap U_{00})$. We will not write down what each of the coordinate charts are for a general element of $\tilde{\mathbb{P}}^2$, but we do need to use the gluing maps to convert from the local coordinates of one cover element to another.

The gluing maps for U_{00} and U_{01} come from the manifold structure of $\tilde{U_0}$:

$$\phi_{01,00} = \psi_{10} \text{ and}$$

$$\phi_{00,01} = \psi_{01}.$$

$$
\begin{array}{ccccccccccccc}
 & & & & & \phi_0(U_0) & \xrightarrow{\phi_{10}} & \phi_1(U_1) & \xrightarrow{\phi_{21}} & \phi_2(U_2) & & \\
 & & & & & \uparrow \phi_0 & & \uparrow \phi_1 & & \uparrow \phi_2 & & \\
\psi(U_{01}) & \xrightarrow{\psi_{10}} & \psi(U_{00}) & & & U_0 & \cup & U_1 & \cup & U_2 & = & \mathbb{P}^2 \\
\uparrow \psi_1 & & \uparrow \psi_0 & & \nearrow \pi & & & & & & & \\
U_{01} & \cup & U_{00} & = & \tilde{U}_0 & & & & & & &
\end{array}
$$

Figure 3.4

Some Maps Used to Define $\tilde{\mathbb{P}}^2$

The gluing maps ϕ_{12} and ϕ_{21} for U_1 and U_2 are unchanged. The remaining gluing maps are found by tracing through Figure 3.4:

$$
\begin{array}{ll}
\phi_{00,1} = \psi_0 \circ \pi^{-1} \circ \phi_0^{-1} \circ \phi_{01} & \phi_{1,00} = \phi_{10} \circ \phi_0 \circ \pi \circ \psi_0^{-1} \\
\phi_{00,2} = \psi_0 \circ \pi^{-1} \circ \phi_0^{-1} \circ \phi_{02} & \phi_{2,00} = \phi_{20} \circ \phi_0 \circ \pi \circ \psi_0^{-1} \\
\phi_{01,1} = \psi_1 \circ \pi^{-1} \circ \phi_0^{-1} \circ \phi_{01} & \phi_{1,01} = \phi_{10} \circ \phi_0 \circ \pi \circ \psi_1^{-1} \\
\phi_{01,2} = \psi_1 \circ \pi^{-1} \circ \phi_0^{-1} \circ \phi_{02} & \phi_{2,01} = \phi_{20} \circ \phi_0 \circ \pi \circ \psi_1^{-1}
\end{array}
$$

The maps ψ_i, ψ_{ij}, and ϕ_{ij} are homeomorphisms on their domains and π is a homeomorphism off of E. Thus, the maps above are well defined gluing maps. To calculate one of these, say $\phi_{1,00}$, simply compose the maps:

$$
\begin{aligned}
\phi_{1,00}(y_0, u) &= \phi_{10} \circ \pi(y_0, y_0u; 1\!:\!u) \\
&= \phi_{10}(y_0, y_0u) \\
&= (1/y_0, u).
\end{aligned}
$$

The other gluing maps can be found in the same way.

Define a new projection map,

$$\pi\colon \tilde{\mathbb{P}}^2 \longrightarrow \mathbb{P}^2$$

by defining it on the open cover $\{U_{00}, U_{01}, U_1, U_2\}$ in the following way. For points in U_{00}

$$\pi(y_0, u) = (1, y_0, y_0u),$$

for points in U_{01}

$$\pi(z_0, t) = (1, z_0 t, z_0),$$

for points in U_1

$$\pi(x_1, z_1) = (x_1, 1, z_1), \text{ and}$$

for points in U_2

$$\pi(x_2, y_2) = (x_2, y_2, 1).$$

This map simply projects the local coordinates in each element of the cover of $\tilde{\mathbb{P}}^2$ onto its corresponding point in $\mathbb{P}^2$. The exceptional curve of $\tilde{\mathbb{P}}^2$ is $E = \pi^{-1}(\mathbf{p})$ and is defined by data

$$\{(U_{00}, y_0), (U_{01}, z_0), (U_1, 1), (U_2, 1)\}. \tag{3.3}$$

In the same way as in Proposition 3.1, the map π is homeomorphic away from $\mathbf{p}$ and E, and π^{-1} pulls $\mathbb{P}^2$ apart at $\mathbf{p}$ according to the directions of lines through $\mathbf{p}$.

Let us quickly look at a curve blown up at its singularity in $\mathbb{P}^2$.

Example 3.2: The curve C in $\mathbb{P}^2$ defined by $xy^2 = z^3$ has a double point at $\mathbf{p} = (1{:}0{:}0)$ and is defined by local data

$$\{(U_0, y_0^2 - z_0^3), (U_1, x_1 - z_1^3), (U_2, x_2 y_2^2 - 1)\}.$$

Blow up $\mathbb{P}^2$ at $\mathbf{p}$ and the total transform $\pi^{-1}(C)$ is defined by data

$$\{(U_{00}, y_0^2(1 - y_0 u^3)), (U_{01}, z_0^2(t^2 - z_0)), (uo, x_1 - z_1^3), (U_2, x_2 y_2^2 - 1)\}.$$

These local equations were found in the same way as in Example 3.1. Since U_1 and U_2 are left unchanged by the blow up, the local equations for these sets are the same for $\mathbb{P}^2$ and $\tilde{\mathbb{P}}^2$. The local equations for U_{00} and U_{01} were found by making the substitutions $y_0 u$ for

z_0 and $z_0 t$ for y_0, respectively. The local equations for E are y_0 in U_{00} and z_0 in U_{01}. By removing E we get the strict transform $\tilde{C}$ which is defined by data

$$\{(U_{00}, 1 - y_0 u_0^3), (U_{01}, t^2 - z_0), (U_1, x_1 - z_1^3), (U_2, x_2 y_2^2 - 1)\}.$$

Notice that the curve $\tilde{C}$ is defined the same as C on U_1 and U_2. There were two copies of E in $\pi^{-1}(C)$ as expected. The curve $\tilde{C}$ is nonsingular since each local equation is nonsingular locally.

Here we found the blow up of $\mathbb{P}^2$ at $\mathbf{p} = (1\!:\!0\!:\!0)$. If $\mathbf{p}$ had been any other point in $\mathbb{P}^2$ there is an linear change of coordinates $T\colon \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ such that $T(1\!:\!0\!:\!0) = \mathbf{p}$. The map

$$T \circ \pi\colon \tilde{\mathbb{P}}^2 \longrightarrow \mathbb{P}^2$$

gives the blow up at $\mathbf{p}$. In this way, $\mathbb{P}^2$ can be blown up at any point. See Appendix C for an example which uses a linear change of coordinates to move a base point to a convenient location before blowing up.

3.4. Meromorphic Functions on $\tilde{\mathbb{P}}^2$

The manifold $\tilde{\mathbb{P}}^2$ is the first manifold introduced in this paper with no global coordinates. Thus, it is also the first manifold for which meromorphic functions cannot be defined globally by one expression. The following example illustrates some meromorphic functions on $\tilde{\mathbb{P}}^2$.

Example 3.3: Consider the curves $f = x^2 - y^2$ and $g = xy$ on $\mathbb{P}^2$. Blow up $\mathbb{P}^2$ at the point $(1\!:\!0\!:\!0)$ to get $\pi\colon \tilde{\mathbb{P}}^2 \to \mathbb{P}^2$. The total transform of these curves are defined by

$$F = \{(U_{00}, 1 - y_0^2), (U_{01}, 1 - t^2 z_0^2), (U_1, x_1^2 - 1), (U_2, x_2^2 - y_2^2)\}$$

and

$$G = \{(U_{00}, y_0), (U_{01}, t z_0), (U_1, x_1), (U_2, x_2 y_2)\}.$$

To construct a meromorphic function M on $\tilde{\mathbb{P}}^2$ we will take the quotients of the local equations for F and G on each element of the cover to get

$$M = \left\{ \left(U_{00}, \frac{1-y_0^2}{y_0} \right), \left(U_{01}, \frac{1-t^2 z_0^2}{t z_0} \right), \left(U_1, \frac{x_1^2 - 1}{x_1} \right), \left(U_2, \frac{x_2^2 - y_2^2}{x_2 y_2} \right) \right\}.$$

We need to check this is, in fact, a meromorphic function. The only part which needs to be checked is that it well defined for all points in $\tilde{\mathbb{P}}^2$. This is easily done with the gluing maps and here it is done for the points $\mathbf{p} \in U_{00} \cap U_1$. If $\mathbf{p} = (x_1, z_1)$ in the local coordinates of U_1, $\mathbf{p} = (y_0, t) = (1/x_1, z_1)$ in the local coordinates of U_{00}. Since $x_1 \neq 0$ on $U_{00} \cap U_1$,

$$\frac{1-y_0^2}{y_0} = \frac{1 - \frac{1}{x_1^2}}{\frac{1}{x_1}} = \frac{x_1^2 - 1}{x_1}$$

and M is well defined on $U_{00} \cap U_1$. Similarly, this can be done for all points in $\tilde{\mathbb{P}}^2$. The zero locus M_0 of the meromorphic function M is a well-defined curve on $\tilde{\mathbb{P}}^2$; in fact, it is exactly the curve F. The poles M_∞ of M define the curve G.

In general, this construction does not give a meromorphic function for all curves f and g. For instance, if $f = x^2 - y^2$ and $g = x$ define curves on $\mathbb{P}^2$, then

$$M = \left\{ \left(U_{00}, 1 - y_0^2 \right), \left(U_{01}, 1 - t^2 z_0^2 \right), \left(U_1, \frac{x_1^2 - 1}{x_1} \right), \left(U_2, \frac{x_2^2 - y_2^2}{x_2} \right) \right\}$$

is not a meromorphic function on $\tilde{\mathbb{P}}^2$ since it is not well defined at the point $\mathbf{p}$ with local coordinates $(2,3)$ on U_1 and local coordinates $(2/3, 1/3)$ on U_2. At this point the value of $M(\mathbf{p})$ is $3/2$ on U_1 and $1/2$ on U_2. This construction will work if f/g is meromorphic on $\mathbb{P}^2$ and this is shown in Section 4.5.1.

The data F itself does not define a meromorphic function. In particular, there is a point $\mathbf{p} \in \tilde{\mathbb{P}}^2 - E$ which has local coordinates $(y_0, t) = (3/2, 4/3)$ in U_{00} and $F(y_0, t) = 1 - y_0^2 = -5/4$. The local coordinates of this point are $(x_1, z_1) = (2/3, 4/3)$ in U_1 and $F(x_1, z_1) = x_1^2 - 1 = -5/9$. So F is not a meromorphic function on $U_{00} \cap U_1$.

A meromorphic function can also be constructed by extending local data on any element of the cover. For example, given (U_{00}, f_{00}) a meromorphic function is defined by

$$\{(U_{00}, f_{00}), (U_{01}, f_{00} \circ \phi_{00,01}), (U_1, f_{00} \circ \phi_{00,1}), (U_2, f_{00} \circ \phi_{00,2})\}.$$

Local data $(U_{00}, 1/y_0)$ extends to the meromorphic function

$$\{(U_{00}, 1/y_0), (U_{01}, 1/(tz_0)), (U_1, x_1), (U_2, y_2)\}$$

on $\tilde{\mathbb{P}}^2$. The data for the other elements of the cover were found by using the gluing maps, e.g., $x_1 = 1/y_0$ on $U_{00} \cap U_1$ since $\phi_{00,1}(x_1, z_1) = (1/x_1, z_1)$.

3.5. Removing Base Points

3.5.1. *One Base Point.*

We have looked at the local equations of one curve blown up on $\tilde{\mathbb{P}}^2$, but it was claimed at the beginning of this chapter that base points could be removed with blow ups. How does this happen? The rational map ψ from Section 3.1 has a base point if the curves $f_i = 0$ share a common point in $\mathbb{P}^2$. If the tangent directions of the f_i at $\mathbf{p}$ are all distinct, when we blow up $\mathbb{P}^2$ at $\mathbf{p}$ the curves $f_i = 0$ will be pulled apart and will no longer intersect. Therefore, there are no base points. This is illustrated in the following example.

Example 3.4: Define a triangular surface by

$$\psi(x{:}\, y{:}\, z) = (f_0{:}\, f_1{:}\, f_2{:}\, f_3) = (x^2z{:}\, x^2y{:}\, y^3{:}\, z^3).$$

There is a base point of ψ at $\mathbf{p} = (1{:}\,0{:}\,0)$ since $f_i(\mathbf{p}) = 0$ for all i, and this is the only base point of ψ. The curves $f_i = 0$ intersect at $\mathbf{p}$ and might look something like the four curves in Figure 3.1 since each f_i is nonsingular and f_i has a distinct tangent at $\mathbf{p}$.

Let $\pi : \tilde{\mathbb{P}}^2 \longrightarrow \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at $\mathbf{p}$. A rational map $\psi' : \tilde{\mathbb{P}}^2 - - \rightarrow \mathbb{P}^3$ using $\tilde{\mathbb{P}}^2$ as the parameter space would be defined by data

$$\{(U_{00}, (f_{00,0}, f_{00,1}, f_{00,2}, f_{00,3})),$$

$$(U_{01}, (f_{01,0}, f_{01,1}, f_{01,2}, f_{01,3})),$$

$$(U_1, (f_{1,0}, f_{1,1}, f_{1,2}, f_{1,3})),$$

$$(U_2, (f_{2,0}, f_{2,1}, f_{2,2}, f_{2,3})),$$

That leaves the question of what polynomials $f_{i,j}$ to use. If we simply use the total transform of each $f_i = 0$ to get

$$\{(U_{00}, (y_0u\colon y_0\colon y_0^3\colon y_0^3u^3)), (U_{01}, (z_0\colon z_0t\colon z_0^3t^3\colon z_0^3)),$$

$$(U_1, (x_1^2z_1\colon x_1^2\colon 1\colon z_1^3)), (U_2, (x_2^2\colon x_2^2y_2\colon y_2^3\colon 1))\} \tag{3.4}$$

we see that the points on the exceptional curve E given by (3.3) are base points. So this definition is unsatisfactory. On U_{00} note that points $(y_0u\colon y_0\colon y_0^3\colon y_0^3u^3) \in \mathbb{P}^3$ and $(u\colon 1\colon y_0^2\colon y_0^2u^3) \in \mathbb{P}^3$ are the same when $y_0 \neq 0$. Similarly, $(z_0\colon z_0t\colon z_0^3t^3\colon z_0^3) = (1\colon t\colon z_0^2t^3\colon z_0^2) \in \mathbb{P}^3$ when $z_0 \neq 0$. Thus, by factoring out one copy of the exceptional curve on U_{00} and U_{01} we can get data

$$\{(U_{00}, (u\colon 1\colon y_0^2\colon y_0^2u^3)), (U_{01}, (1\colon t\colon z_0^2t^3\colon z_0^2)),$$

$$(U_1, (x_1^2z_1\colon x_1^2\colon 1\colon z_1^3)), (U_2, (x_2^2\colon x_2^2y_2\colon y_2^3\colon 1))\} \tag{3.5}$$

which defines the same points as (3.4) off of E and is well defined on E. We say the base point $\mathbf{p}$ has multiplicity 1 because one copy of the exceptional divisor was factored out to remove the base points.

Define $\psi' : \tilde{\mathbb{P}}^2 \longrightarrow \mathbb{P}^3$ using (3.5). In Section 4.9 we will check that ψ' satisfies the conditions for defining a surface on $\tilde{\mathbb{P}}^2$ as required in Section 1.8.

Does $\mathrm{Im}\,(\psi')$ define the same surface as $\mathrm{Im}\,(\psi)$? This is easily checked. Let $\mathbf{q} \in \mathbb{P}^2 - \mathbf{p}$, i.e., let $\mathbf{q}$ be a point in $\mathbb{P}^2$ for which ψ is defined. Now, $\pi^{-1}(\mathbf{q}) \in \tilde{\mathbb{P}}^2 - E$ and $\psi'(\pi^{-1}(\mathbf{q})) =$

$\psi(\mathbf{q})$. For example, if $\mathbf{q} \in U_0 - \mathbf{p}$ and $\pi^{-1}(\mathbf{q}) \in U_{00} - E$, then $\psi(\mathbf{q}) = \psi(1{:}y_0{:}z_0) = (z_0{:}y_0{:}y_0^3{:}z_0^3)$ while

$$\begin{aligned}\psi'(\mathbf{q}) &= \psi'(y_0, z_0/y_0)\\ &= (y_0 z_0/y_0{:}\,y_0{:}\,y_0^3{:}\,y_0^3 z_0^3/y_0^3)\\ &= (z_0{:}\,y_0{:}\,y_0^3{:}\,z_0^3)\end{aligned}$$

also. This can be done for all points in $\tilde{\mathbb{P}}^2 - E$. Therefore, $\operatorname{Im}(\psi) \subseteq \operatorname{Im}(\psi')$.

Exactly how are $\operatorname{Im}(\psi)$ and $\operatorname{Im}(\psi')$ different? The set $\operatorname{Im}(\psi')$ contains more points than $\operatorname{Im}(\psi)$ and these points lie on a certain curve. If $(X_0{:}X_1{:}X_2{:}X_3)$ are the coordinates of $\mathbb{P}^3$, then $\operatorname{Im}(\psi)$ does not contain any points where $X_2 = X_3 = 0$. This is seen from the parametric equations for ψ. If $X_2 = X_3 = 0$, then $y = z = 0$, so X_0 and X_1 must also be zero, but $(0{:}0{:}0{:}0)$ is not a point in $\mathbb{P}^3$. In fact, $\operatorname{Im}(\psi)$ is missing the points where $\operatorname{Im}(\psi')$ meets the line $\{(X_0{:}X_1{:}0{:}0)\}$. In a way, the base point $\mathbf{p}$ "maps" to the points $\operatorname{Im}(\psi') \cap \{(X_0{:}X_1{:}0{:}0)\}$.

3.5.2. *More than One Base Point.*

The rational map ψ may have more than one base point. If so, the method used to remove the base point in Example 3.3 is not convenient. Blowing up a point $\mathbf{p}$ in $\mathbb{P}^2$ is a local phenomenom, i.e., it only affects points arbitrarily close to $\mathbf{p}$. To remove several base points, it will be convenient to blow up subsets of $\mathbb{P}^2$ smaller than U_0.

Let's go back to $\mathbb{C}^2$. In Section 3.2, the blow up of $\mathbb{C}^2$ at $\mathbf{0}$ "tore" $\mathbb{C}^2$ apart near the origin but did not change the structure of $\mathbb{C}^2$ away from the origin. This is more clearly seen here. First, cover $\mathbb{C}^2$ by 2 sets:

$$W = \mathbb{C}^2 - \{\mathbf{0}\}$$

and

$$\Delta = \Delta(\mathbf{0}, \delta)$$

where δ is any positive real number. Now, blow up Δ at the origin. Again consider a subset of $\mathbb{C}^2 \times \mathbf{P}^1$, but this time let

$$\tilde{\Delta} = \{(r, s; t{:}\, u) : ru = st \text{ and } (r, s) \in \Delta\}$$

and cover $\tilde{\Delta}$ by the sets

$$\Delta_0 = \tilde{\Delta} \cap \{(r, s; t{:}\, u) : ru = st \text{ and } t \neq 0\}$$

and

$$\Delta_1 = \tilde{\Delta} \cap \{(r, s; t{:}\, u) : ru = st \text{ and } u \neq 0\}.$$

The sets Δ_0 and Δ_1 have the same coordinate charts here which B_0 and B_1 had in Section 3.2.

Create the manifold $\tilde{\mathbb{C}}^2$ by gluing the sets W, Δ_0 and Δ_1 together. The gluing maps $\psi_{01} : \Delta_1 \longrightarrow \Delta_0$ and $\psi_{10} : \Delta_0 \longrightarrow \Delta_1$ still work. The second part of the proof of Proposition 3.1 describes the gluing maps for $\tilde{\Delta}$ and W. In fact, the set obtained by gluing W and $\tilde{\Delta}$ together here is homeomorphic to the set B in Section 3.3. We could blow up the curve in Example 3.1 using the sets W, Δ_0, and Δ_1. There would be three local equations for the strict transform, one for each of W, Δ_0 and Δ_1 : $y^2 = x^2(x+1)$, $u^2 = x+1$ and $1 = t^2(ty+1)$, respectively.

Let's remove the base point from ψ in Example 3.4 by blowing up U_0 at $\mathbf{p}$ in this way. First, cover $\mathbb{P}^2$ with $\{(U_0, U_1, U_2)\}$, and then replace U_0 with the sets

$$W = \{\mathbf{p} \in U_0 : \mathbf{p} \neq (1{:}0{:}0)\}$$

and

$$\Delta = \{\mathbf{p} \in U_0{:}\, \mathbf{p} = (y_0, x_0) \in B(\mathbf{0}, \delta)\}.$$

Blow up Δ as above and glue U_1, U_2, W, Δ_0 and Δ_1 together.

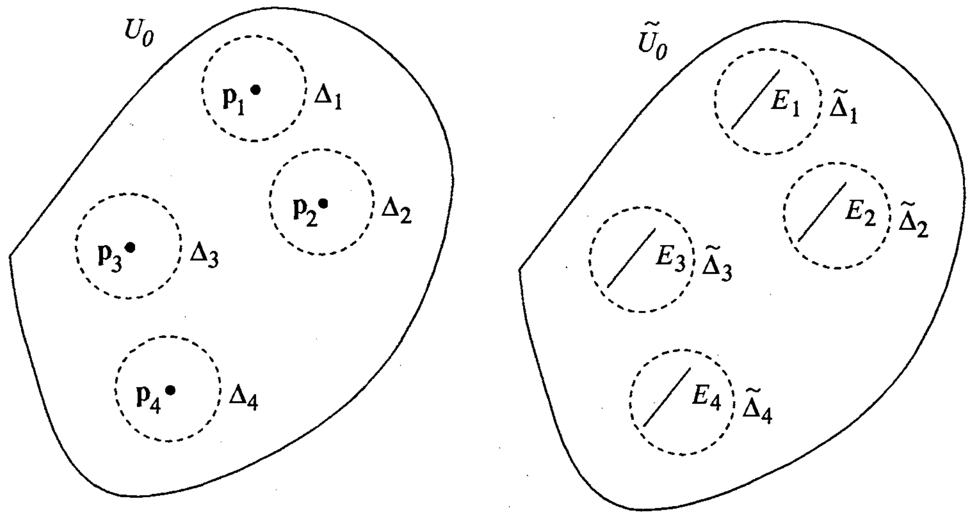


Figure 3.5

Blow-up of 4 Points in U_0

Using this scheme, we can remove several base points at the same time. Suppose ψ had base points $\{\mathbf{p}_1, \cdots, \mathbf{p}_m\}$. Using a linear change of coordinates, we may assume all $\mathbf{p}_i$ are in U_0, that is, the points $\mathbf{p}_i$ do not lie on the line $x = 0$. Choose positive real numbers δ_i such that the sets $\Delta(\mathbf{p}_i, \delta_i) \subseteq U_0$ are pairwise disjoint (See Figure 3.5.). Cover U_0 with the sets

$$W = U_0 - \{\mathbf{p}_1, \cdots, \mathbf{p}_m\}$$

and

$$\Delta_i = \{\mathbf{p} \in U_0 : \mathbf{p} = (y_0, z_0) \in \Delta(\mathbf{p}_i, \delta_i)\}$$

for all $i = 1, \cdots, m$. Since the $\mathbf{p}_i$ may be in U_1 and U_2 also we create new open sets

$$V_i = U_i - \{\mathbf{p}_1, \cdots, \mathbf{p}_m\}$$

for $i = 1$ and 2. Now, blow up each Δ_i at $\mathbf{p}_i$ using a linear change of coordinates as in Section 3.3 to get the sets $\tilde{\Delta}_i$. Glue the sets W, $\tilde{\Delta}_i$, V_j together and we have the blow up of $\mathbb{P}^2$ at $\{\mathbf{p}_1, \cdots, \mathbf{p}_m\}$. Whenever we refer to the *blow up of a manifold* X we will mean the blow up of X at finitely many points. See Appendix C for a surface requiring the blow up of 2 base points simultaneously.

Notice for each point $\mathbf{p}_i$ we will have an exceptional curve E_i and $E_i \subset \tilde{\Delta}_i$. But, since $\Delta_i \cap \Delta_j = \emptyset$ whenever $i \neq j$, $\tilde{\Delta}_i \cap \tilde{\Delta}_j = \emptyset$ also. Therefore, $E_i \cap E_j = \emptyset$ whenever $i \neq j$. This will be useful later when calculating intersections of curves on blow ups of $\mathbb{P}^2$.

3.5.3. *Removing All Base Points.*

Besides the problem of more than one base point it is also possible to remove a base point only to discover another one. This will occur if the curves $f_i = 0$ have a common tangent direction at the base point, because, given a common tangent direction, the strict transforms of these curves will share a point on the exceptional curve. This happens twice in the following example.

Example 3.5: Define a surface by

$$\psi(x{:}\,y{:}\,z) = (f_0{:}\,f_1{:}\,f_2{:}\,f_3) = (xz^2{:}\,y^2z{:}\,y^3{:}\,z^3)$$

which has a base point at $\mathbf{p}_1 = (1{:}0{:}0)$. Let $\pi : \tilde{\mathbb{P}}^2 \longrightarrow \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at $\mathbf{p}_1$. Removing the base point $\mathbf{p}_1$ will only affect the points in U_0. As in Example 3.4 we can define $\psi' : \tilde{\mathbb{P}}^2 - - \rightarrow \mathbb{P}^3$ by removing two copies of E from the total transforms of the $f_i = 0$ to yield

$$\begin{aligned}
&\{(U_{00}, (u^2{:}\,y_0u{:}\,y_0{:}\,y_0u^3)),\\
&(U_{01}, (1{:}\,z_0t^2{:}\,z_0t^3{:}\,z_0)),\\
&(U_1, (z_1^2{:}\,z_1{:}\,1{:}\,z_1^3)),\\
&(U_2, (x_2{:}\,y_2^2{:}\,y_2^3{:}\,1))\}.
\end{aligned}$$

The base point $\mathbf{p}_1$ has multiplicity 2. But ψ' is not defined for $\mathbf{p}_2 = (y_0, u_0) = (0,0) \in U_{00}$ and thus not a map. We say $\mathbf{p}_2$ is also a base point of ψ and $\mathbf{p}_2$ *is infinitely near* $\mathbf{p}_1$ since

$\pi(\mathbf{p}_2) = \mathbf{p}_1$. Our goal is to create a map whose image is the closure of $\text{Im}(\psi)$ and to do this we must continue to remove base points until there are none.

We will remove the base point $\mathbf{p}_2$ by blowing up $\tilde{\mathbb{P}}^2$ at $\mathbf{p}_2$. The point $\mathbf{p}_1 \in U_{00}$ but is not in $U_{01} \cap U_1 \cap U_2$, so to blow up $\tilde{\mathbb{P}}^2$ we will blow up U_{00} at $\mathbf{p}_2$ and glue the result together with U_{01},U_1, and U_2. Denote the blow up of U_{00} at $\mathbf{p}_1$ by $\tilde{U}_{00}$, and let

$$\pi_1\colon X \longrightarrow \tilde{\mathbb{P}^2}$$

be the blow up of $\tilde{\mathbb{P}}^2$ at $\mathbf{p}_2$. Let the coordinates of $\tilde{U}_{00}$ be $(y_0, u; r\colon s)$ and cover $\tilde{U}_{00}$ with U_{000} and U_{001} with local coordinate (y_0, s) and (u, r), respectively (see Section 3.3). The relationship of these coordinates is $y_0 s = ur$ and the local equations of the exceptional curve, named E_1, are $y_0 = 0$ in U_{000} and $u = 0$ in U_{001}. The data will not be changed on U_{01}, U_1 and U_2. Substitute $y_0 s$ for u on U_{000} and ur for y_0 on U_{001}. Then $\psi'' : X - - \rightarrow \mathbb{P}^3$ is defined by

$$\begin{aligned}&\{(U_{000}, (y_0 s^2\colon y_0 s\colon 1\colon y_0^3 s^3)),\\ &(U_{001}, (u\colon ru\colon r\colon ru^3)),\\ &(U_{01}, (1\colon z_0 t^2\colon z_0 t^3\colon z_0)),\\ &(U_1, (z_1^2\colon z_1\colon 1\colon z_1^3)),\\ &(U_2, (x_2\colon y_2^2\colon y_2^3\colon 1))\}\end{aligned}$$

by removing one copy of E_1 from the data on U_{000} and U_{001}. The multiplicity of the base point $\mathbf{p}_2$ is also 1. Again, ψ'' is not a map because there is a base point at $\mathbf{p}_3 = (u, r) = (0, 0) \in U_{001}$. This base point must also be removed.

The process above will be repeated to blow up X at $\mathbf{p}_3$ by blowing up U_{001} to be $\tilde{U}_{001}$. The points in U_{000},U_{01},U_1 and U_2 will remain unchanged. Let the coordinates of $\tilde{U}_{001}$ be $(u, r; v\colon w)$; cover $\tilde{U}_{001}$ with the sets U_{0010} and U_{0011} with coordinates (u, w) and (r, v), respectively. The relationship of the variables is $uw = rv$ and the local equations of the exceptional curve, E_2, are $u = 0$ in U_{0010} and $r = 0$ in U_{0011}. Then $\pi_2 : \tilde{X} \longrightarrow X$ be the blow up of X at $\mathbf{p}_3$. Again, find the strict transforms of the curves defining ψ''' by making the appropriate substitutions and factoring out E_2 to get $\psi''' : \tilde{X} \longrightarrow \mathbb{P}^3$ defined by

$$\begin{aligned}
&\{(U_{000}, (y_0 s^2 : y_0 s : 1 : y_0^3 s^3)),\\
&(U_{0010}, (1 : wu : w : wu^3)),\\
&(U_{0011}, (v : rv : 1 : r^3 v^3)),\\
&(U_{01}, (1 : z_0 t^2 : z_0 t^3 : z_0)),\\
&(U_1, (z_1^2 : z_1 : 1 : z_1^3)),\\
&(U_2, (x_2 : y_2^2 : y_2^3 : 1))\}.
\end{aligned}$$

This time ψ''' is a map. It can be checked that $\text{Im}\,(\psi''')$ is the closure of $\text{Im}\,(\psi)$ as in Section 3.5.1.

What does the set $\tilde{X}$ look like? If we think of blowing-in up $\mathbb{P}^2$ as replacing a point with a line, then we have done that three times here. Figure 3.6 gives an idea of what $\tilde{X}$ looks like. Actually, there is some checking to be done to see if the picture of the last blow up is accurate. If $\mathbf{p}_3$ had not fallen on the intersection of E_1 and $\tilde{E}$, the blow up of X would not have separated $\tilde{E}$ and E_1, as in Figure 3.7.

How do we determine whether Figure 3.6 or Figure 3.7 correctly displays the blow up of $\mathbf{p}_3$? We need to find the local coordinates of $E_1 \cap \tilde{E}$ in U_{001} and see if the point is $\mathbf{p}_3$. In U_{00}, the local equation for E is $y_0 = 0$ and the local equation for $\pi^{-1}(E)$ in U_{001} is $ur = 0$. Thus $\tilde{E}$ has local equation $r = 0$ in U_{001}. Recall the local equation of E_1 in U_{001} is $u = 0$. Thus, $\tilde{E} \cap E_1 = (r, u) = (0, 0) = \mathbf{p}_3$ as shown in Figure 3.6 and the blow up of X at $\mathbf{p}_3$ does separate $\tilde{E}$ and E_1. The situation in Figure 3.7 can occur but we will find out eventually that we need not make a distinction between these two cases.

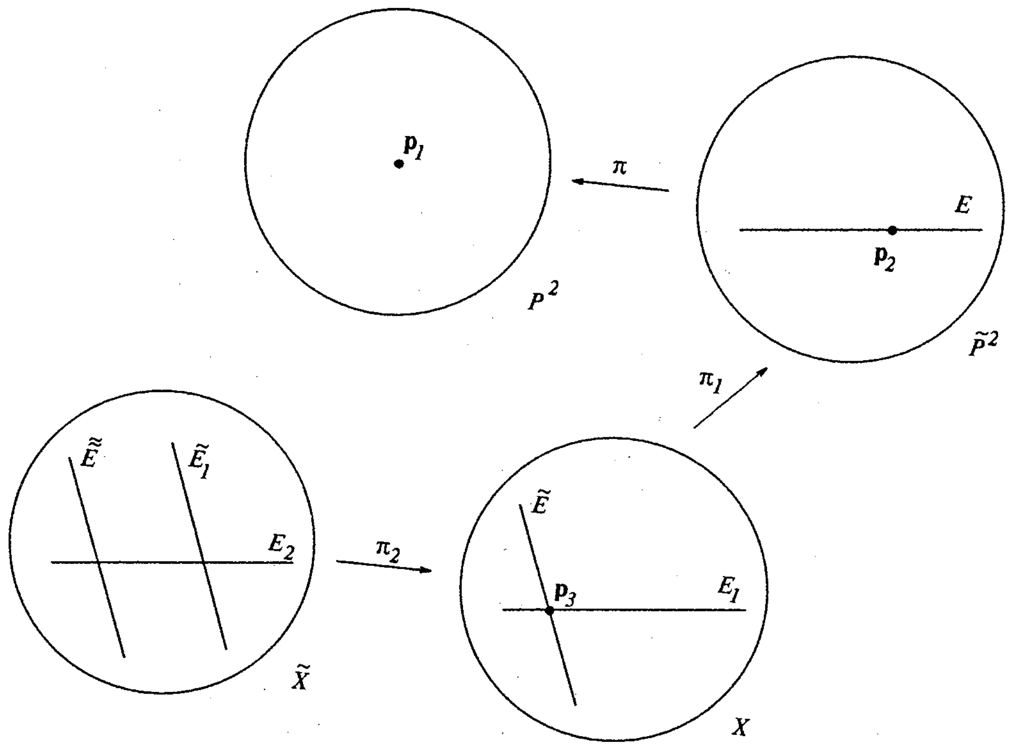


Figure 3.6

Blow-up Maps for Example 3.5

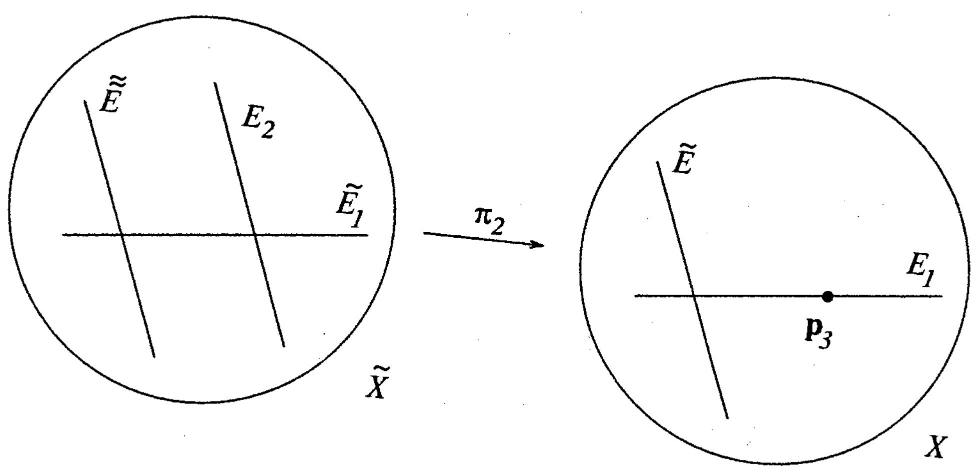


Figure 3.7

Alternate Case for Third Blow-up in Example 3.5

3.6. How Do Blow ups Affect Curves?

The curves in Examples 3.1 and 3.2 were singular but their strict transforms were not. This will not always be the case. It is true, however, that blow ups do not increase the multiplicity of any point on the curve. This is proven in Proposition 3.3 below. First, we will prove an assertion made in Section 3.2.

Proposition 3.2. *Let $\pi : \tilde{X} \longrightarrow X$ be the blow up of X at the points $\mathbf{p}_1, \ldots, \mathbf{p}_s$ with exceptional curves $\pi^{-1}(\mathbf{p}_i) = E_i$. If C is a curve on X and $m_{\mathbf{p}_i}(C) = r_i$, then there are r_i copies of E_i in $\pi^{-1}(C)$.*

Proof: Fix i and let U be a coordinate neighborhood of $\mathbf{p}_i$. Choose a ball $\Delta \subseteq U$ such that Δ does not contain $\mathbf{p}_j$ for $j \neq i$. Let $f = 0$ be the local equation for C in U and (x, y) the local coordinates of U. By a linear change of coordinates we may assume $\mathbf{p}_i = (0,0)$ in the local coordinates of U. Now, $f = f^{(m)} + f^{(m+1)} + \cdots + f^{(m+k)}$ where each $f^{(n)}$ is a form of degree n, $f^{(m)} \neq 0$, and $m = m_{\mathbf{p}_i}$.

Blow up Δ at $\mathbf{p}_i$ and let $(x, y; t{:}u)$ be the coordinates of $\tilde{\Delta}$. The local equations of $\pi^{-1}(C)$ in $\tilde{\Delta}$ are

$$\begin{aligned} f(x, xu) &= f^{(m)}(x, xu) + f^{(m+1)}(x, xu) + \cdots + f^{(m+k)}(x, xu) \\ &= x^m(f^{(m)}(1, u) + xf^{(m+1)}(1, u) + \cdots + x^k f^{(m+k)}(1, u)) \end{aligned}$$

on Δ_0 and

$$\begin{aligned} f(yt, y) &= f^{(m)}(yt, y) + f^{(m+1)}(yt, y) + \cdots + f^{(m+k)}(yt, y) \\ &= y^m(f^{(m)}(t, 1) + yf^{(m+1)}(t, 1) + \cdots + y^k f^{(m+k)}(t, 1)) \end{aligned}$$

on Δ_1. The local equations of E_i in $\tilde{\Delta}$ are $x = 0$ and $y = 0$ on Δ_0 and Δ_1, respectively. Therefore, there are $m = r_i$ copies of E_i in $\pi^{-1}(C)$ defined by $x^m = 0$ on Δ_0 and by $y^m = 0$ on Δ_1.

Now we will prove blow ups never make singularities worse.

Proposition 3.3. *Let X be a finite blow up of $\mathbb{P}^2$ and C a curve on X and $\mathbf{p}$ a point on C. Let $\pi : \tilde{X} \longrightarrow X$ be the blow up of X at $\mathbf{p}$ and $E = \pi^{-1}(\mathbf{p})$ the exceptional curve. If $\mathbf{q} \in \tilde{C} \cap E$, i.e., $\mathbf{q}$ is a point on the exceptional curve and the strict transform of a curve $C \subseteq X$, then $m_{\mathbf{q}}(\tilde{C}) \leq m_{\mathbf{p}}(C)$.*

Proof: Let U be a coordinate neighborhood of $\mathbf{p}$ with local coordinates (x, y). By a linear change of coordinates we may assume $\mathbf{p} = (0,0)$ and C is not tangent to either $x = 0$ or $y = 0$. Let the coordinates of $\tilde{U}$ be $(x, y; t{:}u)$. In Proposition 3.2 we found the local equations of $\pi^{-1}(C)$, and by factoring out x^m and y^m we obtain the local equations of $\tilde{C}$ which are

$$f_0(x,y) = f^{(m)}(1,u) + xf^{(m+1)}(1,u) + \cdots + x^k f^{(m+k)}(1,u)$$

and

$$f_1(y,t) = f^{(m)}(t,1) + yf^{(m+1)}(t,1) + \cdots + y^k f^{(m+k)}(t,1)$$

To show $m_{\mathbf{q}}(\tilde{C}) \leq m$ we will show the mth order partials of f_0 and f_1 are not all zero at $\mathbf{q}$. In particular,

$$\frac{\partial^m f_0}{\partial u^m} = \frac{\partial^m f^{(m)}}{\partial u^m} + x\frac{\partial f^{(m+1)}}{\partial u^m} + \cdots + x^m \frac{\partial^m f^{(m+k)}}{\partial u^m}.$$

The coordinates of $\mathbf{q}$ must be $(0,0;t{:}u)$ for it to be on E, so locally $\mathbf{q} = (x,u) = (0,u)$. Thus

$$\frac{\partial^m f_0}{\partial u^m}\bigg|_{\mathbf{q}} = \frac{\partial^m f^{(m)}}{\partial u^m}\bigg|_{(0,u)}.$$

Now $\dfrac{\partial^m f^{(m)}}{\partial u^m}\bigg|_{(0,u)} \neq 0$; if it were, C would be tangent to $x = 0$ at $\mathbf{p}$. Therefore, $f^{(m)}(1,u)$ has a u^m term, say au^m, and

$$\frac{\partial^m f_0}{\partial u^m}\bigg|_{\mathbf{q}} = am! \neq 0.$$

Similarly, $\dfrac{\partial^m f_1}{\partial t^m}\bigg|_{\mathbf{p}} \neq 0$ since $y = 0$ is not tangent to C. This completes the proof.

The multiplicity of the points on a curve away from the points blown up has not changed because the curve does not change except arbitrarily near the point blown up. The multiplicity of points on the exceptional curve is not increased by Proposition 3.3. Therefore, a blow up does not make singularities worse. It is true all singularities of a curve can be removed by applying enough blow ups to the curve[Har p. 390]. We have not proven that here.

3.7. Conclusion

In this paper we will only blow up a point on a surface but it is possible to blow up curves on a surface, surfaces in a 3-dimensional manifold, etc. The process of blowing up also goes by other names in other contexts: monoidal transformation, σ-process, dilatation, locally quadratic transformation, Hopf map, and others. Here, a blow up is always the blow up of a surface at a finite number of points.

In Chapter 5 we will use the information obtained from removing base points to calculate the implicit degree of $\operatorname{Im}(\psi)$. Surprisingly, the only information needed in the calculation of implicit degree is the parametric degree (or parametric bidegree) and the multiplicity of each base point. The focus of this chapter has been on triangular surfaces. Removing base points works the same way on tensor product surfaces and Appendix C contains an example of removing base points on a tensor product surface. Resolution of the singularities of a curve is essential in the calculation of the genus of a curve. An example of the resolution of singularities is in the proof of Proposition 6.1.

CHAPTER FOUR

DIVISORS

4.1. Motivation

The calculations of implicit degree and genus depend on the intersections of curves on two-dimensional manifolds such as the blow ups of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$. Why this is true for implicit degree calculations is explained below. Just as Bezout's Theorem classifies the intersection of curves on $\mathbb{P}^2$, there is a way to classify the intersection of curves on finite blow ups of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$. To do this we will look at a larger class of objects called divisors which includes curves. The remainder of the chapter is dedicated to defining divisors on the two-dimensional manifolds which result from blowing up $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$.

4.1.1. *Degree of a Triangular Surface With No Base Points.*

First, let's see how the degree of a surface can be calculated by intersecting the right curves. Consider the triangular surface which is the image of

$$\psi: \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$$

where $\psi(x:y:z) = (f_0:f_1:f_2:f_3)$ and the homogeneous degree of each f_i is n. Let S be the closure of $\operatorname{Im}(\psi)$. Every rational surface is algebraic so there is a polynomial, $F : \mathbb{P}^3 \longrightarrow \mathbb{C}$, whose zero locus is the closure of the rational surface $\operatorname{Im}(\psi)$. Define the *degree of the rational surface* $\operatorname{Im}(\psi)$ to be the degree of the polynomial F and call it d. Assume ψ has a one-to-one parameterization.

The degree of F can be determined by counting the number of times it is intersected by a general line in the following way (see Figure 4.1). Let L be any line in $\mathbb{P}^3$ which is not contained in F. We shall see the set $F \cap L$ is a finite set of d elements. Let the coordinates of $\mathbb{P}^3$ be $(X_0 : X_1 : X_2 : X_3)$ and let the homogeneous parametric equations for L be $X_i(s:t) = \alpha_i s + \beta_i t$ for $i = 0, 1, 2, 3$. Since F is a homogeneous polynomial we can substitute the equations for the line L into F to get another homogeneous polynomial $F \circ L$ of degree d in indeterminates s and t. Counting multiplicities, there are d zeros of $F \circ L$, say $(s_1 : t_1), \ldots, (s_d : t_d)$. The d points $(X_0(s_i : t_i), X_1(s_i : t_i), X_2(s_i : t_i), X_3(s_i : t_i)) \in \mathbb{P}^3$ are exactly the points in $F \cap L$. Therefore, the number of points in $F \cap L$, counting multiplicities, is equal to the degree of F.

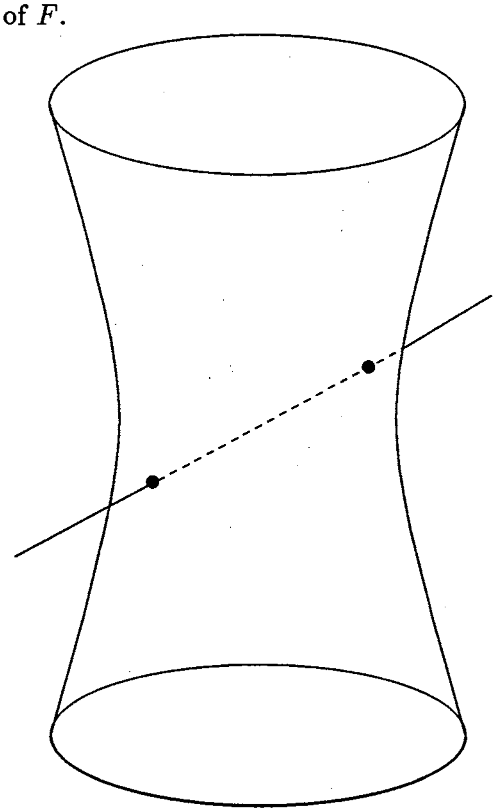


Figure 4.1

Intersection of a Surface and a Line

It can be seen that the degree of F is equal to the number of points in the intersection of F with a general line, but we have not shown a method of calculating this number. We can actually count the number of points in $F \cap L$ easily if we to represent L in a different way. For now suppose ψ has no base points. Let

$$H = a_0X_0 + a_1X_1 + a_2X_2 + a_3X_3$$

and

$$K = b_0X_0 + b_1X_1 + b_2X_2 + b_3X_3$$

be the equations of two distinct planes in $\mathbb{P}^3$ whose intersection is the line L. These planes are not subsets of F since $L \not\subseteq F$ and the intersection of each plane with F can be represented in $\mathbb{P}^2$ by the curves

$$A(x{:}\,y{:}\,z) = a_0f_0 + a_1f_1 + a_2f_2 + a_3f_3 = 0 \tag{4.1}$$

and

$$B(x{:}\,y{:}\,z) = b_0f_0 + b_1f_1 + b_2f_2 + b_3f_3 = 0 \tag{4.2}$$

by substituting the definitions $X_i = f_i$ for ψ into the equations for H and K. Thus A and B are each homogeneous of degree n. These curves A and B are called *plane sections* of the surface F. On the one hand, the curves A and B meet in n^2 points by applying Bezout's theorem. On the other hand, the two curves A and B meet in exactly the same points as L and F. Thus the degree of the surface, and of its implicit equation $F = 0$, is $d = n^2$ and we have calculated the implicit degree of a surface $F \subset \mathbb{P}^3$ by calculating the number of points in the intersection of two curves in $\mathbb{P}^2$. Note that the set of all plane sections is a linear system.

4.1.2. *Degree of a Triangular Surface With Base Points.*

We can do the same thing for surfaces with base points, the only difference is the curves to intersect lie on a blow up of $\mathbb{P}^2$. Suppose ψ has finitely many base points $\mathbf{p}_b = (x_b{:}\,y_b{:}\,z_b)$.

Since $f_i(x_b\colon y_b\colon z_b) = 0$ for each i, these base points necessarily lie on the plane sections A and B. Unfortunately, there is no point in $F \cap L$ which corresponds to these base points because there is no such point as $(f_0(\mathbf{p}_b)\colon f_1(\mathbf{p}_b)\colon f_2(\mathbf{p}_b)\colon f_3(\mathbf{p}_b)) = (0\colon 0\colon 0\colon 0)$ in $\mathbb{P}^3$. So, even though there are n^2 points in $A \cap B \subseteq \mathbb{P}^2$, there are fewer than n^2 points in $F \cap L \subseteq \mathbb{P}^3$. In some way, the presence of base points diminishes the degree of the surface. By removing all the base points we can define a map

$$\psi'\colon X \longrightarrow \mathbb{P}^3$$

where X is a finite blow up of $\mathbb{P}^2$ and $S = \mathrm{Im}\,(\psi')$ is the closure of $\mathrm{Im}\,(\psi)$. Let the image of ψ' be defined by data $\{(V_i, (f_{i0}, f_{i1}, f_{i2}, f_{i3}))\}$. The set X can now be considered the parameter space and the intersection of the planes H and K with the surface S can be written as curves on X defined by data

$$\{(V_i, (a_0 f_{i0} + a_1 f_{i1} + a_2 f_{i2} + a_3 f_{i3}))\} \tag{4.3}$$

and

$$\{(V_i, (b_0 f_{i0} + b_1 f_{i1} + b_2 f_{i2} + b_3 f_{i3}))\}, \tag{4.4}$$

respectively. Counting multiplicities, the number of points in the intersection of these two curves is the number of points in $S \cap L$ and is exactly the degree of the surface $\mathrm{Im}\,(\psi')$. There are no points in the curves (4.3) and (4.4) which do not correspond to points in $S \cap L$ because we have removed all base points. The problem now is that curves do not intersect on finite blow ups of $\mathbb{P}^2$ as they do in $\mathbb{P}^2$. In particular, Bezout's Theorem no longer applies. However, the intersection of curves on finite blow ups of $\mathbb{P}^2$ can be calculated with the use of divisors, as we shall see.

4.2. The Group Div (X) of Divisors on the Manifold X

Let X be an n-dimensional complex manifold. A *Weil divisor on* X is a finite sum

$$D = \sum_{i=1}^{r} n_i(D_i)$$

where each D_i is an irreducible hypersurface in X and $n_i \in \mathbb{Z}$. On a 2-dimensional manifold, each D_i is a curve and on a 1-dimensional manifold each D_i is a point. Here we will concentrate on 2-dimensional manifolds X.

If C is a curve on X and $C_1, \ldots, C_r$ are the irreducible components of C with multiplicities $n_1, \ldots, n_r > 0$, define the *divisor of the curve* C to be

$$(C) = \sum_{i=1}^{r} n_i(C_i).$$

In general, a divisor $D = \sum_{i=1}^{r} n_i(C_i)$ with each C_i an irreducible curve and $n_i > 0$ for all i is called an *effective divisor.* Note that the divisor of a curve is always an effective divisor. If C is an irreducible curve, (C) is called a *prime divisor.* Let the curve defined by the zeros of a meromorphic function F on X be named F_0 and the curve of poles F_∞. Define the *divisor of a meromorphic function* F to be

$$(F) = (F_0) - (f_\infty)$$

and call it a *principal divisor.*

An *additive group* G is a set which is closed under addition and has these properties:

1. addition is associative
2. there exists an additive identity $0 \in G$; and
3. for all $g \in G$ there is an additive inverse $-g$.

The set of all divisors on X, denoted Div (X), is an additive group. For every divisor $D \in$ Div (X), its inverse $-D$ is also a divisor and $D + (-D) = 0$ is the identity on Div (X) given by an empty sum.

We can also define divisors in a different way. A *Cartier divisor on X* is defined by data $\{(U_i, f_i)\}$ where

1. f_i is a meromorphic function and not identically 0 on U_i and
2. $f_i = hf_j$ where h is a nonvanishing holomorphic function on $U_i \cap U_j$; and
3. 2 sets of data $\{(U_i, f_i)\}$ and $\{(U_i, f_i')\}$ define the same divisor if f_i/f_i' is holomorphic and nonzero on U_i.

The only difference between this definition and the definition for curves on X is that for curves each f_i was holomorphic on U_i. If $D = \sum_{i=1}^{r} n_i(D_i)$ is a Weil divisor on X and the data for each irreducible curve D_i is $\{(U_i, f_i)\}$, then $\{(U_i, \pi_{i=1}^{r} f_i^{n_i})\}$ is the data for the corresponding Cartier divisor. Conversely, if $D = \{(U_i, f_i)\}$ is the data for any Cartier divisor and D_0 and D_∞ are the zeros and poles of this data, then $D = (D_0) - (D_\infty)$ as a Weil divisor. Thus, the 2 definitions are the same on a complex manifold. Note that this also shows that any divisor can be written as the difference of 2 effective divisors.

If a Weil divisor D is principal, the associated Cartier divisor can be defined by a meromorphic function. Similarly, if a Weil divisor is effective, the associated Cartier divisor is defined by a curve.

The group Div(X) can be partitioned into equivalence classes in the following way: Put $D \sim D'$ if $D - D'$ is a principal divisor. It is easiest to show that $\sim$ is an equivalence relation using Cartier divisors. Let D, D' and D'' be given by data $\{(U_i, f_i)\}$, $\{(U_i', f_i')\}$, and $\{(U_i'', f_i'')\}$, respectively. Then $D - D$ is given by $\{(U_i, f_i/f_i)\}$. Since each f_i is meromorphic and not identically 0 on U_i, the data for $D - D$ defines a meromorphic function which is 1 everywhere f_i is not 0. If $D \sim D'$ then $\{(U_i, f_i/f_i')\}$ defines a meromorphic function. Clearly, $\{(U_i, f_i'/f_i)\}$ also defines a meromorphic function so $D' \sim D$. Finally, assume $D \sim D'$ and $D' \sim D''$ from which we know $\{(U_i, f_i/f_i')\}$ and $\{(U_i, f_i'/f_i'')\}$ are meromorphic functions. Now, $D - D'' = (D - D') - (D'' - D')$, so $D - D''$ can be written with data

$\{(U_i, f_i f_i'/f_i' f_i'')\}$. If the f_i/f_i' and f_i'/f_i'' satisfy the requirements for a global meromorphic function, their product must also, and $D \sim D''$. Therefore, $\sim$ is reflexive, symmetric, and transitive and, hence, an equivalence relation. Denote the equivalence class containing D by $[D]$.

We have already seen many divisors. For example, in Section 3.3, the projective plane was blown up at the point $\mathbf{p} = (1{:}\,0{:}\,0)$ and the exceptional curve E was found to be defined by data

$$\{(U_{00}, y_0), (U_{01}, z_0), (U_1, 1), (U_2, 1)\}.$$

This data also defines the Cartier divisor (E) which will appropriately be called the *exceptional divisor.* The equivalence classes of divisors on a manifold will be of more use to us than specific divisors. In the next 3 sections we investigate the nature of these equivalence classes on the manifolds we will use most.

4.3. Equivalence Classes of Div ($\mathbb{P}^2$)

All curves on $\mathbb{P}^2$ can be classified by the homogeneous degree of the polynomial equation of the curve. The classification of divisors on $\mathbb{P}^2$ is an extension of this idea. Let $D \in \mathrm{Div}(\mathbb{P}^2)$ and write $D = D_1 - D_2$ where each D_i is an effective divisor. Each of D_i is a divisor of a curve so let f_i be a homogeneous polynomial with $D_i = (f_i)$. Note that $D_1 - D_2 = (f_1/f_2)$.

We want to find another divisor linearly equivalent to D. To do this we will find a divisor $D' = (f)$ where f is a homogeneous polynomial to subtract from D such that $D - D' = \left(\frac{f_1}{f_2}\right) - (f) = \left(\frac{f_1}{f_2 f}\right)$ is principal, i.e., $\frac{f_1}{f_2 f}$ is a meromorphic function on $\mathbb{P}^2$. Let the degree of f_i be n_i. The only meromorphic functions on $\mathbb{P}^2$ are quotients of homogeneous polynomials of like degree, so let $f = h^n$ where h is any line in $\mathbb{P}^2$ and $n = n_1 - n_2$. Now $(h^n) = n(h)$ is a divisor on $\mathbb{P}^2$ and the divisor $D - n(h) = \left(\frac{f_1}{f_2 h^n}\right)$ is principal. Therefore,

$D \sim n(h)$ and each equivalence class of divisors on $\mathbb{P}^2$ can be represented by $n(h)$ where $n \in \mathbb{Z}$. The class of divisors linearly equivalent to $0(h)$ is exactly the class of principal divisors on $\mathbb{P}^2$.

These equivalence classes can be used to find the number of points in the intersection of general curves on $\mathbb{P}^2$. Suppose f and g are two homogeneous polynomials of degree n and m, respectively, and f and g have no common factors. Now $(f) \sim n(h)$ and $(g) \sim m(h)$. The number of points in $f \cap g$ is equal to nm. Thus all information needed to calculate the intersection of any two curves f and g on $\mathbb{P}^2$ which intersect properly is contained in the linear equivalence classes of their divisors. Note that all lines belong to the equivalence class $[(h)]$ where h is any line; all conics belong to $[(c)]$ where c is any conic; and so on.

4.4. Equivalence Classes of Div $(\mathbb{P}^1 \times \mathbb{P}^1)$

The divisors in Div $(\mathbb{P}^1 \times \mathbb{P}^1)$ can also be classified but this works a little differently than on $\mathbb{P}^2$ because is intersections on $\mathbb{P}^1 \times \mathbb{P}^1$ do not work the same as in $\mathbb{P}^2$; Bezout's Theorem does not apply here. For instance, using the coordinates introduced in Section 1.3, the curves $\rho_0 = 0$ and $\rho_1 = 0$ never intersect in $\mathbb{P}^1 \times \mathbb{P}^1$ since there is no such point with coordinates $(0\colon 0; \sigma_0\colon \sigma_1)$. In fact, there are two distinct classes of curves on $\mathbb{P}^1 \times \mathbb{P}^1$ with these intersection properties: two curves within the same class do not intersect and two curves from different classes intersect at one point. One class contains all the curves of the form $\alpha_0\rho_0 + \alpha_1\rho_1 = 0$ with $\alpha_i \in \mathbb{C}$ and the other contains the curves of the form $\beta_0\sigma_0 + \beta_1\sigma_1 = 0$ with $\beta_i \in \mathbb{C}$. Two curves from the same class are either the same curve or have no intersection since the only solution to a nonsingular homogeneous system of 2 equations is $(0,0)$ and $(\rho_0\colon \rho_1; 0\colon 0)$ and $(0\colon 0; \sigma_0\colon \sigma_1)$ are not points in $\mathbb{P}^1 \times \mathbb{P}^1$. On the other hand, the only point in the intersection of $\alpha_0\rho_0 + \alpha_0\rho_1 = 0$ and $\beta_0\sigma_0 + \beta_1\sigma_1 = 0$ is $(-\alpha_1\colon \alpha_0; -\beta_1\colon \beta_0)$ so curves of different classes meet in exactly one point. The curves from these 2 classes are called *lines on* $\mathbb{P}^1 \times \mathbb{P}^1$ because they are parameterized by the projective line $\mathbb{P}^1$. These classes of lines are distinguished by their bidegree: lines from the first class have bidegree $(1,0)$ and lines from the second class have bidegree $(0,1)$.

We will now classify all divisors in $\mathbb{P}^1 \times \mathbb{P}^1$ in the same way we did for $\mathbb{P}^2$. Let $D = \left(\frac{f_1}{f_2}\right)$ be any divisor in $\mathbb{P}^1 \times \mathbb{P}^1$ where the bidegree of f_i is (n_i, m_i). To find another divisor linearly equivalent to D we will find a divisor D' such that $D - D' = (f)$ where f is a meromorphic function. Let k be a line of bidegree $(1,0)$ and l a line of bidegree $(0,1)$. Put $n = n_1 - n_2$, $m = m_1 - m_2$, and $D' = (k^n l^m) = n(k) + m(l)$. Now, $D - D' = \left(\frac{f_1}{f_2 k^n l^m}\right)$ is a principal divisor. Thus, $D \sim n(k) + m(l)$ where $n, m \in \mathbb{Z}$ and k and l are any lines of bidegree $(1,0)$ and $(0,1)$, respectively. Keep in mind that m and n may be any integer.

Note a significant difference between the equivalence classes in $\text{Div}\,(\mathbb{P}^1 \times \mathbb{P}^1)$ and those of $\text{Div}\,(\mathbb{P}^2)$. In $\mathbb{P}^1 \times \mathbb{P}^1$, not all lines are in the same equivalence class; in fact, there are 2 linear equivalence classes of lines as described above. Later it will be shown that the number of points in the intersection of any two curves in $\mathbb{P}^1 \times \mathbb{P}^1$ which intersect properly can be calculated from their divisor classes just as in $\mathbb{P}^2$.

4.5. Equivalence Classes of Div (X) for Other Manifolds X

In this section we will classify all divisors in $\text{Div}\,(X)$ where X is a finite blow up of either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. We will begin by looking at the relationship between divisors on a 2-dimensional manifold X and the blow up of X at finitely many points. From this it will be easy to see exactly what divisors there are on one blow up of $\mathbb{P}^2$ and then several blow ups of $\mathbb{P}^2$. Finally, the same will be done for $\mathbb{P}^1 \times \mathbb{P}^1$.

4.5.1. *Projection Maps and Pullback Maps.*

Let $\pi : \tilde{X} \longrightarrow X$ be the blow up of the 2-dimensional manifold X at the points $\mathbf{p}_1, \ldots, \mathbf{p}_s$ with exceptional curves $E_i = \pi^{-1}(\mathbf{p}_i)$. There is a natural map called the *pullback* *map*

$$\pi^* : \text{Div}\,(X) \longrightarrow \text{Div}\,(\tilde{X})$$

which arises from the projection map and associates the divisor of an irreducible curve in X to the divisor of its total transform in $\tilde{X}$. This map π^* can be defined in the following way. Consider an irreducible curve $C = \{(U_i, f_i)\}$ on X and its associated divisor (C). The set $\{U_i\}$ covers X and, because of the definition of π, the set $\{\pi^{-1}(U_i)\}$ covers $\tilde{X}$. Let π be defined on $\tilde{X}$ by $\{(\pi^{-1}(U_i), \pi_i)\}$. On each $\pi^{-1}(U_i)$ define $\pi^* f_i : \pi^{-1}(U_i) \longrightarrow \mathbb{C}$ to be $f_i \circ \pi_i$. Finally, define

$$(\pi^* C) = \{(\pi^{-1}(U_i), \pi^* f_i)\}.$$

Does this definition do what is was supposed to do, i.e., is $(\pi^* C)$ the divisor of the total inverse image of C? On each $\pi^{-1}(U_i)$ the zero locus of $\pi^* f_i$ is $\pi^{-1}(\{\mathbf{p} \in U_i : f_i(\mathbf{p}) = 0\})$ which is the total transform of the curve C in $\pi^{-1}(U_i)$.

This definition of $(\pi^* C)$ is given in terms of local equations but it will be more convenient to define it in terms of divisors of curves. From Proposition 3.2, which calculates the total transform of a curve on X, we can write

$$(\pi^* C) = (\tilde{C}) + \sum m_i(E_i) \tag{4.5}$$

where $\tilde{C}$ is the strict transform of C and $m_i = m_{\mathbf{p}_i}(C)$.

We still need to define $(\pi^* D)$ for a general divisor $D \in \text{Div}\,(X)$. If $D = \sum n_i(C_i)$ is any divisor on X where each (C_i) is prime, define $(\pi^* D) = \sum n_i(\pi^* C)$ and call it the *pullback divisor of D.* Clearly, $(\pi^*(D + D')) = (\pi^* D) + (\pi^* D')$ for all $D, D' \in \text{Div}\,(X)$. Hence π^* is a linear map.

It will be important later that π^* also preserve linear equivalence which is a stronger property than linearity. Let $C, D \in \text{Div}\,(X)$ with $C \sim D$. Then $C - D$ is the divisor of a meromorphic function, say $F = \{(U_i, f_i)\}$. Since π^* is linear,

$$(\pi^*C) - (\pi^*D) = (\pi^*(C - D)).$$

Our goal is to show that $(\pi^*C) - (\pi^*D)$ is the divisor of a meromorphic function on $\tilde{X}$ so that $(\pi^*C) \sim (\pi^*D)$. Thus, we need only show that (π^*F) is the divisor of a meromorphic function. The data for (π^*F) on $\tilde{X}$ is

$$\{(\pi^{-1}(U_i), f_i \circ \pi_i)\}.$$

To show that (π^*F) is the divisor of a meromorphic function, this data must satisfy the requirements in Section 1.5.3. Both π_i and f_i are holomorphic on $\pi^{-1}(U_i)$ and U_i, respectively. Also, $f_i(\mathbf{p}) = f_j(\mathbf{p})$ for all $\mathbf{p} \in U_i \cap U_j$. Thus, $(f_i \circ \pi_i)(\pi^{-1}(\mathbf{p})) = (f_j \circ \pi_j)(\pi^{-1}(\mathbf{p}))$ for all points $\pi^{-1}(\mathbf{p}) \in \pi^{-1}(U_i) \cap \pi^{-1}(U_j)$. However, each point of $\pi^{-1}(U_i) \cap \pi^{-1}(U_j)$ is equal to $\pi^{-1}(\mathbf{p})$ for some $\mathbf{p} \in U_i \cap U_j$. Therefore, (π^*F) does define a meromorphic function on $\tilde{X}$ and π^* preserves linear equivalence class.

Now we will use (4.5) to write divisors of $\tilde{X}$ in terms of divisors in X and the pullback map.

Proposition 4.1. *If $\pi : \tilde{X} \longrightarrow X$ is the blow up of the 2-dimensional manifold X at the points $\mathbf{p}_1, \ldots, \mathbf{p}_s$ with exceptional curves $E_i = \pi^{-1}(\mathbf{p}_i)$, then the divisor of an irreducible curve C in $\tilde{X}$ is either equal to (E_i) for some i or*

$$(\pi^*(\pi(C))) - \sum_{i=1}^{s} m_i(E_i)$$

where $n, m_i \in \mathbb{Z}$ with $n, m_i > 0$.

Proof: Let C be an irreducible curve in $\tilde{X}$ and assume $(C) \neq (E_i)$ for any i. The projection of C, $\pi(C)$, is a curve on X and so defines a divisor there. If the curve $\pi(C)$ contains any of the points $\mathbf{p}_i$, then in general, C may or may not contain any of the exceptional curves. Under the assumptions that C is irreducible and $C \neq E_i$, however, $E_i \not\subseteq C$ for any i. Therefore, C must be the strict transform of $\pi(C)$. Let $m_i = m_{\mathbf{p}_i}(C)$ and we can write

$$\pi^*(\pi(C)) = (C) + \sum m_i(E_i)$$

from (4.5). Now

$$(C) = \pi^*(\pi(C)) - \sum m_i(E_i).$$

Since each divisor in $\tilde{X}$ is the formal sum of divisors of irreducible curves in $\tilde{X}$,

$$\begin{aligned} D &= \sum_j n_j(C_j) \\ &= \sum_j n_j \left[\pi^*(\pi(C_j)) - \sum_i m_{ij}(E_i)\right] \\ &= \sum_j n_j \pi^*(\pi(C_j)) - \sum_j \sum_i n_j m_{ij}(E_i) \\ &= \pi^*\left(\sum_j n_j(\pi(C_j))\right) - \sum_j \sum_i n_j m_{ij}(E_i) \end{aligned}$$

so every divisor in $\tilde{X}$ can be written as the sum of the pullback of a divisor in X and multiples of the exceptional curves.

We will make this more precise by looking at blow ups of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$

4.5.2. *Equivalence Classes of* Div $(\tilde{\mathbb{P}}^2)$.

Proposition 4.1 can be rewritten to apply specifically to blow ups of $\mathbb{P}^2$.

Corollary 4.1.1. *If $\pi : \tilde{\mathbb{P}}^2 \longrightarrow \mathbb{P}^2$ is the blow up of $\mathbb{P}^2$ at the points $\mathbf{p}_1, \ldots, \mathbf{p}_s$ with exceptional curves $E_i = \pi^{-1}(\mathbf{p}_i)$, then the divisor of an irreducible curve C in $\mathbb{P}^2$ is either equal to (E_i) for some i or is linearly equivalent to*

$$n(\pi^* h) - \sum_{i=1}^{s} m_i(E_i)$$

where h is any line in $\mathbb{P}^2$ and $n, m_i \in \mathbb{Z}$ with $n, m_i > 0$.

The major difference here is that instead of finding exactly what the divisor is in $\tilde{\mathbb{P}}^2$, this proposition gives the linear equivalence class of the divisor which is all that will be needed later. The proof follows from Proposition 4.1. Let C be an irreducible curve in $\tilde{\mathbb{P}}^2$. Then

$$(C) = \pi^*(\pi(C)) - \sum_{i=1}^{s} m_i(E_i)$$

where $m_i = m_{\mathbf{p}_i}(C)$. Since $\pi(C)$ is a curve in $\mathbb{P}^2$, $(\pi(C)) \sim n(h)$ for some integer n and any line $h \in \mathbb{P}^2$. This, and the fact that π^* is linear yields

$$(C) \sim n(\pi^* h) - \sum_{i=1}^{s} m_i(E_i).$$

If $D = \sum_{j=1}^{r} k_j(C_j)$ is any divisor in $\tilde{\mathbb{P}}^2$ where each (C_j) is a prime divisor, then

$$\begin{aligned} D &= \sum_{j=1}^{r} k_j(C_j) \\ &\sim \sum_{j=1}^{r} k_j \left[n_j(\pi^* h) - \sum_{i=1}^{s} m_{ij}(E_i) \right] \\ &= \sum_{j=1}^{r} k_j n_j (\pi^* h) - \sum_{j=1}^{r} \sum_{i=1}^{s} k_j m_{ij}(E_i) \\ &= K(\pi^* h) - \sum_{i=1}^{s} M_i(E_i) \end{aligned}$$

with $K, M_i \in \mathbb{Z}$. Note that K and M_i need not be nonnegative here.

All divisors in $\tilde{\mathbb{P}}^2$ can be classified in this way, but we have already seen that several blow ups are necessary to remove all base points in some cases. When this happens, simply apply Proposition 4.1 for each blow up.

Example 4.1: Blow up $\mathbb{P}^2$ at $\mathbf{p}_1$ and $\mathbf{p}_2$ to get X and blow up X at $\mathbf{p}_3$ to get $\tilde{X}$. Let E_1, E_2, and E_3 be the exceptional curves and

$$\tilde{X} \xrightarrow{\pi_2} X \xrightarrow{\pi_1} \mathbb{P}^2$$

the projection maps. There are 2 pullback maps

$$\mathrm{Div}\,(\mathbb{P}^2) \xrightarrow{\pi_1^*} \mathrm{Div}\,(X) \xrightarrow{\pi_2^*} \mathrm{Div}\,(\tilde{X}).$$

All elements of Div (X) are linearly equivalent to

$$n_1\pi_1^*(h) - m_1(E_1) - m_2(E_2) \tag{4.6}$$

from Corollary 4.1.1 where $m_i \in \mathbb{Z}$. Now begin with a divisor $D \in \mathrm{Div}\,(\tilde{X})$ with $D = \sum_{j=1}^r k_j(D_j)$ where each (D_j) is prime. From Proposition 4.1, with $m_3 \in \mathbb{Z}$,

$$\begin{aligned} D &= \sum_{j=1}^r k_j(\pi_2^*(\pi_2(D_j)) - m_3(E_3) \\ &= \pi_2^*\left(\sum_{j=1}^r k_j\pi_2(D_j)\right) - m_3(E_3) \end{aligned}$$

since π_2^* is linear. The divisor $\sum_{j=1}^r k_j\pi_2(D_j)$ on X is linearly equivalent to (4.6) for some n and π_2^* preserves linear equivalence class, so

$$\begin{aligned} D &\sim \pi_2^*[n(\pi_1^*h) - m_1(E_1) - m_2(E_2)] - m_3(E_3) \\ &= n\pi_2^*(\pi_1^*h) - m_1(\pi_2^*E_1) - m_2(\pi_2^*E_2) - m_3(E_3). \end{aligned}$$

Classifying divisors in this way can be done for any finite succession of blow ups on $\mathbb{P}^2$.

4.5.3. *Equivalence Classes on Blow Ups of* $\mathbb{P}^1 \times \mathbb{P}^1$.

Again there is a corollary to Proposition 4.1 but this time it applies to blow ups of $\mathbb{P}^1 \times \mathbb{P}^1$.

Corollary 4.1.2. *If $\pi : X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the points $\mathbf{p}_1, \ldots, \mathbf{p}_s$ with exceptional curves $E_i = \pi^{-1}(\mathbf{p}_i)$, then the divisor of an irreducible curve C in $\mathbb{P}^1 \times \mathbb{P}^1$ is either equal to (E_i) for some i or is linearly equivalent to*

$$n_1(\pi^*k) + n_2(\pi^*l) - \sum_{i=1}^s m_i(E_i)$$

where k and l are lines in $\mathbb{P}^1 \times \mathbb{P}^1$ with bidegree (1,0) and (0,1), respectively, and $n_i, m_i \in \mathbb{Z}$ with $n_i, m_i > 0$.

The proof works exactly like that of Corollary 4.1.1 except it uses the information developed in Section 4.4.

4.6. Classifying Plane Sections of a Triangular Surface

We can now get back to the main purpose of this chapter: classifying plane sections of surfaces. After we have done this, all that is left in the process of calculating implicit degree is to see how intersections can be found from divisor classes and this is done in Chapter 5.

In this section we will classify plane sections of triangular surfaces. Let $\psi: \mathbb{P}^2 - - \rightarrow \mathbb{P}^3$ define a surface as in Section 4.1. After removing all base points we obtain a map $\psi': X \rightarrow \mathbb{P}^3$ and $\mathrm{Im}\,(\psi')$ is the closure of $\mathrm{Im}\,(\psi)$. The plane sections of S are given by (4.3) and (4.4), but these are simply the strict transforms of the curves (4.1) and (4.2) in $\mathbb{P}^2$ which are of degree n. So to classify (4.3) and (4.4) we need to classify strict transforms of curves of degree n. We will do this for the surfaces in Examples 3.4 and 3.5.

Example 4.2: In Example 3.4 all base points were removed with the blow up $\pi: \tilde{\mathbb{P}}^2 \rightarrow \mathbb{P}^2$. Here we will work with only one plane section, (4.3). The divisor class for the other plane section is the same. The curve (4.1) for this example is

$$A = a_0 x^2 z + a_1 x^2 y + a_2 y^3 + a_3 z^3 = 0.$$

The degree of A is 3 and is equal to the degree of f_i. Thus, $(A) \sim 3(h) \in \mathrm{Div}\,(\mathbb{P}^2)$ where h

is any line in $\mathbb{P}^2$. The curve (4.3) is defined by

$$\{(U_{00}, a_0u + a_1 + a_2y_0^2 + a_3y_0^2u^3),$$

$$(U_{01}, a_0 + a_1t + a_2z_0^2t^3 + a_3z_0^2),$$

$$(U_1, a_0x_1^2z_1 + a_1x_1^2 + a_2 + a_3z_1^3),$$

$$(U_2, a_0x_2^2 + a_1x_2^2y_2 + a_2y_2^3 + a_3)\}$$

and is the strict transform of A. From Corollary 4.1.1 the divisor of this curve is linearly equivalent to

$$(3\pi^*h) - E$$

of Div $(\tilde{\mathbb{P}}^2)$. The only information needed to find this divisor class was the parametric degree of the surface and the multiplicity of the base points.

In Example 3.5, 3 successive blow ups were required to remove all base points:

$$\tilde{X} \xrightarrow{\pi_2} X \xrightarrow{\pi_1} \tilde{\mathbb{P}}^2 \xrightarrow{\pi} \mathbb{P}^2.$$

So there are three pullback maps

$$\text{Div}\,(\mathbb{P}^2) \xrightarrow{\pi^*} \text{Div}\,(\tilde{\mathbb{P}}^2) \xrightarrow{\pi_1^*} \text{Div}\,(X) \xrightarrow{\pi_2^*} \text{Div}\,(\tilde{X}).$$

If $m_i = m_{\mathbf{p}_i}(A)$, then $m_1 = 2$, $m_2 = 1$, and $m_3 = 1$. Again the degree of A is equal to the degree of f_i and is 3. On $\mathbb{P}^2$, $A \sim 3(h)$ as above. Following Example 4.1, the divisor associated to a general plane section of S would be

$$\pi_2^*\{\pi_1^*[3(\pi^*h) - 2(E)] - (E_1)\} - (E_2)$$

in Div (X).

4.7. Classifying Plane Sections of a Tensor Product Surface

Consider the tensor product surface which is the image of

$$\psi : \mathbb{P}^1 \times \mathbb{P}^1 - - \rightarrow \mathbb{P}^3$$

where $\psi(\rho_0 : \rho_1 ; \sigma_0 : \sigma_1) = (f_0 : f_1 : f_2 : f_3)$ and the bidegree of each f_i is (n_1, n_2). After removing all the base points we obtain a map $\psi : X \longrightarrow \mathbb{P}^3$ and $\text{Im}\,(\psi')$ is the closure of $\text{Im}\,(\psi)$. Just as in the case of triangular surfaces, the number of points in the intersection of two general plane sections is represented in X by (4.3) and (4.4). Also these plane sections are the strict transforms of (4.1) and (4.2) which, in the case of a tensor product surface, are curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree (n_1, n_2). If ψ requires a single blow up to remove the base points $\mathbf{p}_1, \ldots, \mathbf{p}_s$, then (4.3) and (4.4) are linearly equivalent to

$$n_1(\pi^* k) + n_2(\pi^* l) - \sum m_i E_i$$

where k and l are as in Section 4.4 and m_i is the multiplicity of the base point $\mathbf{p}_i$.

4.8. Holomorphic Line Bundles.

So far we have seen divisors which arise naturally from curves and plane sections of surfaces. Divisors also arise naturally from another place: meromorphic sections of holomorphic line bundles. Line bundles actually serve two purposes in this paper. One is to develop a canonical divisor class which is essential for finding the genus of a curve in Chapter 6. The other is to allow an easy method for finding other divisors in a given divisors class. This will allow us to calculate self intersections in Chapter 5.

In a way, line bundles are not much different than manifolds. A manifold is a topological space which looks locally like U where $U \subseteq \mathbb{C}^n$; a line bundle is a topological space which looks locally like $U \times \mathbb{C}$ where U is a subset of $\mathbb{C}^n$.

Let T be any topological space, $\{U_i\}$ an open cover of T, and

$$\psi_{ij} : U_i \cap U_j \longrightarrow \mathbb{C} - \{0\}$$

holomorphic maps such that for each point $\mathbf{p} \in U_i \cap U_j$

$$(\psi_{ij} \cdot \psi_{ji})(\mathbf{p}) = 1 \tag{4.7}$$

and for each point $\mathbf{p} \in U_i \cap U_j \cap U_k$

$$(\psi_{ij} \cdot \psi_{jk} \cdot \psi_{ki})(\mathbf{p}) = 1 \tag{4.8}$$

Then $L = \cup(U_i \times \mathbb{C})$ is a *holomorphic line bundle over* T where the points $(\mathbf{p}, z) \in U_i \times \mathbb{C}$ and $(\mathbf{p}, z \cdot \psi_{ij}(\mathbf{p})) \in U_j \times \mathbb{C}$ are identified whenever $\mathbf{p} \in U_i \cap U_j$. The maps ψ_{ij} are called *transition maps* and change the coordinates of points as you move from one $U_i \times \mathbb{C}$ to the next. Two line bundles $(L, \{\phi_{ij}\})$ and $(L', \{\phi'_{ij}\})$ are said to be equivalent if there are functions f_i holomorphic on U_i such that

$$\phi'_{ij} = \frac{f_i}{f_j}\phi_{ij}$$

for all i, j.

In this paper the topological space T will be a 1 or 2-dimensional manifold. So L will look like a curve or a surface with a complex line through each point.

Let $(L, \{\phi_{ij}\})$ and $(L', \{\phi'_{ij}\})$ be two line bundles on X. Define $L \otimes L'$ and L^{-1} to be the line bundles with transition functions $\phi_{ij}\phi'_{ij}$ and $1/\phi_{ij}$, respectively. Then the set of all holomorphic line bundles over X is a group with the operation $\otimes$. This is called the *Picard Group of* X and written $\mathrm{Pic}\,(X)$. The identity element of this group is called the *trivial bundle* and is any line bundle L_{id} with transition functions ϕ_{ij} such that there are holomorphic functions f_i on V_i with

$$1 = \frac{f_i}{f_j}\phi_{ij}.$$

In the next section we will see how line bundles arise from divisor classes.

4.9. The Relationship of Line Bundles and Divisors

One purpose of introducing line bundles is to have other machinery which we can use along with divisors. We will see here that in a way holomorphic line bundles are exactly the same as divisors, but the way in which line bundles are defined give us more versatility. We will first show for each divisor we can associate a line bundle. Then we will show this line bundle is unique. We will also show that to each line bundle we can associate a linear equivalence class of divisors.

Let $D \in \operatorname{Div}(X)$ where D is associated to the data $\{(U_i, f_i)\}$ and f_i is meromorphic on U_i. The functions

$$\psi_{ij} = f_i / f_j$$

are holomorphic and nonzero on $U_i \cap U_j$ because f_i and f_j define the same meromorphic function on $U_i \cap U_j$, hence, have the same zeros and poles. It is clear the ψ_{ij} satisfy conditions (4.7) and (4.8). The line bundle $\cup\{U_i \times \mathbb{C}\}$ with transition functions ψ_{ij} is called the *associated line bundle of* D and written $[D]$.

It is necessary to check that $[D]$ is well defined regardless of the data chosen to represent D. Suppose D could also be represented by the data $\{(U_i, f_i')\}$. If $h_i = f_i/f_i'$, then h_i is holomorphic and never vanishes on U_i because the zeros and poles of f_i and f_i' coincide there. Now

$$\psi'_{ij} = \frac{f_i'}{f_j'} = \psi_{ij} \frac{h_j}{h_i}$$

for each i,j. Therefore, ψ_{ij} and ψ'_{ij} are transition functions for the same line bundle, $[D]$.

Now, let's start with a line bundle $L \in \operatorname{Pic}(X)$ with transition functions ψ_{ij} and determine a divisor associated to L. Define a *holomorphic section of the line bundle* L to be data $\{(U_i, s_i)\}$ where s_i is holomorphic on U_i and satisfying

$$s_i = \psi_{ij} s_j \qquad (4.9)$$

on $U_i \cap U_j$. A *meromorphic section* is defined in the same way except that each s_i is meromorphic on U_i. Call the section given by $\{(U_i, s_i)\}$ simply s.

What sort of object does $\{(U_i, s_i)\}$ represent when s is a meromorphic section? The s_i can have poles, so s does not necessarily represent a curve. There is also no guarantee that s_i is a well defined meromorphic function. It is known that on $U_i \cap U_j$

$$s_i/s_j = \psi_{ij}$$

and ψ_{ij} is holomorphic and nonvanishing. Thus, s_i and s_j have the same poles and zeros on $U_i \cap U_j$. This is enough to say that s is the data for a Cartier divisor on X. This divisor is called the *divisor of the meromorphic section* s and is denoted (s). Note that if s is a holomorphic section, then (s) is an effective divisor.

Is (s) unique to L? The answer here is no. Another meromorphic section s' can give a different divisor (s'). It is a simple matter, however, to show that $(s) \sim (s')$. For s and s' to be sections they must satisfy (4.8). Thus, on $U_i \cap U_j$

$$\frac{s_i}{s'_i} = \frac{s_i\psi_{ij}}{s'_i\psi_{ij}} = \frac{s_j}{s'_j}.$$

Therefore, $\{(U_i, s_i/s'_i)\}$ defines a global meromorphic function on X and $(s) - (s')$ is a principal divisor.

There are 2 other facts that are easily checked. Given a meromorphic section s of a holomorphic line bundle L, the line bundle associated to (s) is exactly the line bundle L. Also, given 2 linearly equivalent $D \sim D'$, they are associated to the same line bundle, i.e., $[D] = [D']$. Using all this, there is a one-to-one correspondence between the linear equivalence classes of $\mathrm{Div}\,(X)$ and the elements of $\mathrm{Pic}\,(X)$. Thus, it is reasonable to use $[D]$ for both a divisor class and its associated line bundle. We can see in the following example partly how this would be useful to us.

Example 4.4: Consider the exceptional divisor (E) given by (3.3). The line bundle associated to this divisor, $[(E)]$, has transition functions

$$\psi_{01,00} = z_0/y_0,\ \psi_{1,00} = 1/y_0,\ \psi_{2,00} = 1/y_0$$

$$\psi_{1,01} = 1/z_0,\ \psi_{2,01} = 1/z_0, \text{ and } \psi_{2,1} = 1.$$

To find a meromorphic section s of $[(E)]$ we can simply choose one of the s_i to be some rational function on U_i and determine the others from it using (4.8). Let $s_{00} = 1$ and then s is defined by data

$$\{(U_{00}, 1), (U_{01}, 1/t), (U_1, x_1), (U_2, x_2/y_2)\}$$

after making the appropriate coordinate changes. The divisor (s) is linearly equivalent to (E).

Example 4.5: In Section 1.8 a rational parametric surface was defined as data

$$\{(U_i, (F_{i0}\colon F_{i1}\colon F_{i2}\colon F_{i3}))\}$$

where

$$\{(U_i, F_{ij})\}$$

for each j defines a meromorphic section of the same holomorphic line bundle L. In Example 3.4 it was claimed that (3.5) defined a rational parametric surface. Here we will partially show that (3.5) satisfies this condition.

Fix $j = 0$ and consider the functions $\psi_{kl} = F_{k0}/F_{l0}$ for $k, l = 00, 01, 1, 2$ with $k \neq l$. By construction the ψ_{kl} satisfy (4.7) and (4.8). Thus, the ψ_{kl} are the transition functions for a line bundle L on $\tilde{\mathbb{P}}^2$. In particular,

$$\psi_{01,00}(z_0, t) = t,$$

$$\psi_{1,00}(x_1, z_1) = x_1^2, \text{ and}$$

$$\psi_{2,00}(x_2, y_2) = x_2^2 y_2$$

which are found by taking quotients F_{k0}/F_{l0}. These functions are holomorphic and nonvanishing on their domains.

Let $s_{00} = 1$ and use the transition functions to extend this to a meromorphic section of L to get

$$\{(U_{00}, 1), (U_{01}, t), (U_1, x_1^2), (U_2, x_2^2 y_2)\}.$$

Thus,

$$\{(U_{00}, F_{00,0}), (U_{01}, F_{01,0}), (U_1, F_{1,0}), (U_2, F_{2,0})\}$$

of (3.5) is a meromorphic section of L. It can also be shown that

$$\{(U_i, F_{ij})\}$$

is a meromorphic section of L for each j.

In this section we have seen that there is a one-to-one correspondence between the holomorphic line bundles on X and the linear equivalence classes of divisors on X. It was also noted that if s is a holomorphic section of a line bundle L, then the divisor (s) is an effective divisor. In fact, L is the line bundle of an effective divisor D if and only if L has a nontrivial holomorphic section s with $(s) = D$. If D is effective then the data for D as a Cartier divisor also defines a holomorphic section of $[D]$. Conversely, if L has a holomorphic section s, then it is associated to the effective divisor (s). In other words, there is a one-to-one correspondence between holomorphic line bundles on X with holomorphic sections and the linear equivalence classes of effective divisors on X. This correspondence will be useful later.

4.10. Canonical Divisors

All specific divisors found so far have arisen from the plane sections of surfaces. There is another natural divisor class on each manifold which will be needed in the calculation of genus in Chapter 6. This divisor class is found by creating a line bundle using the Jacobians of the gluing maps and is called the canonical divisor class.

4.10.1. *Canonical Divisors on* $\mathbb{P}^2$.

The meromorphic sections of a holomorphic line bundle on $\mathbb{P}^2$ are associated to divisors on $\mathbb{P}^2$. Here we will show that a line bundle can be found using the gluing maps for $\mathbb{P}^2$. Put $\psi_{ij} = \mathcal{J}\phi_{ji}$ for $i, j = 1, 2, 3$, the Jacobian of each gluing map. These ψ_{ij} define transition functions for a holomorphic line bundle which is easily checked by seeing that the ψ_{ij} satisfy (4.7) and (4.8). The transition functions are

$$\psi_{01} = \mathcal{J}\phi_{10}(y_0, z_0) = -1/y_0^3,$$

$$\psi_{10} = \mathcal{J}\phi_{01}(x_1, z_1) = -1/x_1^3,$$

$$\psi_{02} = \mathcal{J}\phi_{02}(y_0, z_0) = 1/z_0^3,$$

$$\psi_{20} = \mathcal{J}\phi_{20}(x_2, y_2) = 1/x_2^3,$$

$$\psi_{12} = \mathcal{J}\phi_{21}(x_1, z_1) = -1/z_1^3, \text{ and}$$

$$\psi_{21} = \mathcal{J}\phi_{12}(x_2, y_2) = -1/y_2^3.$$

To find a meromorphic section of this line bundle put $s_1 = 1$. Using (4.8) we can find the data for the section

$$\{(U_0, -1/y_0^3), (U_1, 1), (U_2, -1/y_2^3)\}.$$

Globally this is the divisor of $-1/y^3$. Thus, this line bundle defines the set of divisors linearly equivalent to $-3(h)$ where h is any line in $\mathbb{P}^2$.

In general, the line bundle whose transition functions are the Jacobians of the gluing maps on a manifold X is called the *canonical line bundle*. The class of divisors given by meromorphic sections of this line bundle is called the *canonical divisor class.* Elements of the canonical divisor class are usually denoted K_X and in this case $[K_{\mathbb{P}^2}] = [-3(h)]$. It should be noted that using the construction of the canonical line bundle presented here, it would be difficult to show that the line bundle does not depend on the coordinate structure of the manifold. Using a different construction, this is shown in [Har: p. 146]. The canonical line bundle for $\mathbb{P}^2$ found there is the same as the one presented here.

4.10.2. *Canonical Divisors on* $\tilde{\mathbb{P}}^2$.

Let $\tilde{\mathbb{P}}^2$ be the blow up of $\mathbb{P}^2$ at $\mathbf{p} = (1{:}\,0{:}\,0)$ as in Section 3.3. Using the gluing maps

$$\begin{aligned}\phi_{1,00}(y_0, u) &= (1/y_0, u),\\ \phi_{1,01}(z_0, t) &= (1/(z_0 t), 1/t), \text{ and}\\ \phi_{1,2}(x_2, y_2) &= (x_2/y_2, 1/y_2)\end{aligned}$$

the transition functions can easily be calculated by taking the Jacobian of each gluing map:

$$\begin{aligned}\psi_{00,1}(y_0, u) &= -1/y_0^2,\\ \psi_{01,1}(z_0, t) &= 1/(z_0^2 t^3), \text{ and}\\ \psi_{2,1}(x_2, y_2) &= -1/y_2^3.\end{aligned}$$

The other transition maps can be calculated for the canonical bundle on $\tilde{\mathbb{P}}^2$.

Put $s_1 = 1/x_1^3$ and extend this data to a meromorphic section s of $[K_{\tilde{\mathbb{P}}^2}]$ to get

$$\{(U_{00}, -y_0), (U_{01}, z_0), (U_1, 1/x_1^3), (U_2, -1/x_2^3)\}.$$

The divisor associated to this data can be found by determining its zeros and poles. On U_{00} and U_{01}, this section has zeros on $y_0 = 0$ and $z_0 = 0$, but this is simply the exceptional curve E. On U_1 and U_2, this section has a pole of order 3 along $x_1 = 0$ and $x_2 = 0$, respectively. The curve given by $x_1 = 0$ and $x_2 = 0$ is the pullback of the curve $x = 0$ in $\mathbb{P}^2$, so the divisor of the poles is linearly equivalent to $-3(\pi^* h)$ where h is any line in $\mathbb{P}^2$.

Note that the pullback of $x = 0$ does not intersect U_{00} or U_{01}. Now, the canonical divisor class for $\tilde{\mathbb{P}}^2$ is $[K_{\tilde{\mathbb{P}}^2}] = [-3(\pi^*h) + E]$. If any other point of $\mathbb{P}^2$ had been blown up the result would have been the same.

4.10.3. *Canonical Divisors on* X *and* $\tilde{X}$*.*

In general, do the Jacobians of the gluing maps for a 2-dimensional complex manifold X satisfy the conditions for transitions functions on a line bundle as claimed in 4.10.1? Let $\{\phi_{ij}\}$ be the gluing maps for X. Recall that $\phi_{ij} = \phi_{ji}^{-1}$. Also note that $\phi_{ij} = \phi_{jk} \circ \phi_{kj}$. Since $\mathcal{J}\phi^{-1} = (\mathcal{J}\phi)^{-1}$ and $\mathcal{J}(\phi_1 \circ \phi_2) = \mathcal{J}(\phi_1)\mathcal{J}(\phi_2)$, the Jacobians of the gluing maps do satisfy (4.7) and (4.8). Therefore, the definition of the canonical line bundle on X makes sense.

Although $[K_{\mathbb{P}^2}]$ and $[K_{\tilde{\mathbb{P}}^2}]$ were calculated independently, there is a relationship between these 2 divisor classes. In fact, $[K_{\tilde{X}}] = [(\pi^* K_X) + E]$ whenever $\tilde{X}$ is X blown up at one point. This can be shown by looking at the relationship of the gluing maps for X and $\tilde{X}$.

Let $\{U_i\}_{i=0}^n$ be a cover for X such that $\mathbf{p} \in U_0$ and $\mathbf{p} \notin U_i$ for $i \neq 0$ (see Section 3.5.2). Blow up X at $\mathbf{p}$ to get $\tilde{X}$. The cover for $\tilde{X}$ will be $\{U_{00}, U_{01}, U_i\}_{i=1}^n$. Since the sets U_i for $i \neq 0$ are the same for X and $\tilde{X}$, these manifolds also share the same gluing maps ϕ_{ij} where $i \neq 0$ and $j \neq 0$. Let (x_i, y_i) be the local coordinates for each U_i. If $(x_0, y_0; t{:}\,u)$ are the coordinates for $\tilde{U}_0$, the local coordinates for U_{00} and U_{01} can be assumed to be (x_0, u) and (y_0, t), respectively. Using Figure 3.4 as a guide we can calculate the following gluing maps for $\tilde{X}$ where $i \neq 0$.

$$\phi_{00,i}(x_i, y_i) = (\phi_{0i}^{(1)}, \phi_{0i}^{(2)}/\phi_{0i}^{(1)}),$$

$$\phi_{01,i}(x_i, y_i) = (\phi_{0i}^{(2)}, \phi_{0i}^{(1)}/\phi_{0i}^{(2)}),$$

$$\phi_{i,00}(x_0, u) = \phi_{i0}(x_0, x_0 u), \text{ and}$$

$$\phi_{i,01}(y_0, t) = \phi_{i0}(y_0 t, y_0)$$

where the coordinates of ϕ_{ij} are given by $(\phi_{ij}^{(1)}, \phi_{ij}^{(2)})$.

The transition functions for the canonical bundle on X are $\mathcal{J}\phi_{ji}$. The transition functions for the canonical bundle on $\tilde{X}$ agree with those for X when $i \neq 0$ and $j \neq 0$ because the gluing maps agree. The remaining transition functions for the canonical bundle on $\tilde{X}$ are

$$\mathcal{J}\phi_{00,i} = \frac{1}{x_0}\mathcal{J}\phi_{0i},$$
$$\mathcal{J}\phi_{01,i} = \frac{1}{y_0}\mathcal{J}\phi_{0i},$$
$$\mathcal{J}\phi_{i,00} = x_0\mathcal{J}\phi_{i0}, \text{ and}$$
$$\mathcal{J}\phi_{i,01} = y_0\mathcal{J}\phi_{i0}$$

for $i \neq 0$.

Let $\{(U_i, s_i)\}$ be data for any meromorphic section of $[K_X]$. Fix $i \neq 0$ and use $s'_i = s_i$ on U_i to calculate a meromorphic section s' of $[K_{\tilde{X}}]$. For $j \neq 0$,

$$s'_j = \mathcal{J}\phi_{ji}s'_i = s_j.$$

On U_{00} and U_{01}, respectively,

$$s'_{00} = \mathcal{J}\phi_{i,00}s'_i = x_0\mathcal{J}\phi_{i0}s_i = x_0s_0$$

and

$$s'_{01} = \mathcal{J}\phi_{i,01}s'_i = y_0\mathcal{J}\phi_{i0}s_i = y_0s_0.$$

Thus, a meromorphic section of $[K_{\tilde{X}}]$ is $\{(U_{00}, x_0s_0), (U_{01}, y_0s_0), (U_i, s_i)_{i\neq 0}\}$. The divisor associated to this section is the pullback of (s) plus the extra zeros on the exceptional curve E. Therefore, $[K_{\tilde{X}}] = [(\pi^*K_X) + E]$ as claimed.

4.11. Divisors and Linear Systems

Linear systems were defined in Section 1.7. A set of divisors which can be linearly parameterized by $\mathbb{P}^N$ for some N is also a linear system since effective divisors are divisors of curves. Define a *complete linear system*, denoted $|D|$, to be the space of all effective divisors linearly equivalent to D. It is not immediately apparent that this set of divisors is a linear system. Let $[D]$ be the line bundle associated to the divisor D. Every effective divisor linearly equivalent to D is the zero set of a holomorphic section of $[D]$. If a specific $D' \in |D|$ is the zero set of two holomorphic sections s and s' of $[D]$, then the zeros of s and s' are the same. It was show in Section 4.9 that $(s) - (s')$ is a principal divisor so s/s' defines a meromorphic function. However, s and s' have the same poles, so s/s' is actually a holomorphic function and must be constant. Therefore, $s = \alpha s'$ for some complex number α. Therefore, there is a one-to-one correspondence between the elements of $|D|$ and the holomorphic sections of $[D]$ up to nonzero scalar multiples. The set of holomorphic sections of a holomorphic line bundle on a compact complex manifold is a finite dimensional vector space [Har p. 100]. Say the holomorphic sections of $[D]$ are isomorphic to $\mathbb{C}^N$. As described in Section 1.7.1, throw out the zero section and equate 2 sections if they are nonzero scalar multiples of each other, and on the one hand, we have $\mathbb{P}^{N-1}$ while on the other hand we have $|D|$. Therefore, $|D|$ is a linear system.

The construction above allows us to prove the following.

Proposition 4.2. *Let $C_1, \ldots, C_r$ be curves on X and $|D|$ a complete linear system in* $\mathrm{Div}\,(X)$ *with no base points. Then the general element of $|D|$ meets each $C_1, \ldots, C_r$ properly.*

Proof: If any C_i is not irreducible, C_i can be replaced in the list of curves by all of its irreducible components. Thus, we may assume each C_i is irreducible. Let $\mathbf{p}_i \in C_i$ for each i. If a curve D' does not contain any $\mathbf{p}_i$, then D' must meet each C_i properly because C_i is irreducible. Therefore, it suffices to show that a general element of $|D|$ does not contain any of the $\mathbf{p}_i$.

The complete linear system $|D|$ is isomorphic to $\mathbb{P}^K$ for some K. Fix $\mathbf{p} \in X$. Consider the set of effective divisors in $|D|$ which contain $\mathbf{p}$ and call this set $H_{\mathbf{p}}$. Let $\{s_0, \ldots, s_K\}$ be a basis for the vector space of holomorphic sections of $[D]$. Let $D' \in H_{\mathbf{p}}$. Then there is a holomorphic section s of $[D]$ associated to D' and

$$s = \alpha_0 s_0 + \cdots + \alpha_k s_k$$

for some $\alpha_i \in \mathbb{C}$. Since $\mathbf{p} \in D'$, $\mathbf{p}$ is a zero of s and

$$s(\mathbf{p}) = \alpha_0 s_0(\mathbf{p}) + \cdots + \alpha_k s_k(\mathbf{p}) = 0.$$

Since $\mathbf{p}$ is not a base point of $|D|$, at least one $s_i(\mathbf{p}) \neq 0$, so the set of $\{(\alpha_0 : \cdots : \alpha_k)\} \subseteq \mathbb{P}^K$ satisfying this equation is a hyperplane in $\mathbb{P}^K$, i.e., $H_{\mathbf{p}}$ is a subspace of dimension $K-1$. Therefore, the elements of $|D|$ containing a fixed point is a hyperplane in $|D|$.

Now $H = \cup_{i=1}^{r} H_{\mathbf{p}_i}$ is the set of all elements of $|D|$ containing any of the points $\mathbf{p}_i$ and is the union of a finite number of hyperplanes in $|D|$. Therefore, H has dimension strictly less than $|D|$ and the proof is done.

Note that the results of Bertini's Theorem and this one can be combined to get this Corollary.

Corollary 4.2.1. *Let $C_1, \ldots, C_r$ be curves on X and $|D|$ a complete linear system in* Div (X) *with no base points. Then the general element of $|D|$ is nonsingular and meets each $C_1, \ldots, C_r$ properly.*

Since the set of singular curves in $|D|$ and the set of curves meeting a C_i improperly both have dimension strictly less than $|D|$, so does their union. This corollary will be used in Section 5.2.

4.12. More Divisors: Divisors on Curves

Divisors were defined for general complex manifolds but up until now we have only considered divisors on a 2-dimensional manifold. A nonsingular curve on a 2-dimensional complex manifold inherits the manifold structure and is itself a 1-dimensional manifold. In Chapter 5, divisors on a nonsingular curve play an important role in defining intersection numbers.

4.12.1. *Nonsingular Curves as Complex Manifolds.*

The particular 1-dimensional manifolds in this paper are nonsingular curves. The following proposition illustrates how the manifold structure of a surface is inherited by a curve on that surface.

Proposition 4.3. *Suppose X is a 2-dimensional complex manifold. Then any nonsingular curve on X is a 1-dimensional complex manifold.*

Proof: Let $\{(U_i, \phi_i)\}$ be the coordinate charts for X with $\phi_i : U_i \longrightarrow V_i \subseteq \mathbb{C}^2$ homeomorphic and the local coordinates (x_i, y_i) for each U_i. Let C be a nonsingular curve on X given by data $\{(U_i, f_i)\}$.

Fix i and let $\mathbf{p} \in U_i$. Create a coordinate chart for a neighborhood of $\mathbf{p}$ on C in the following way. Assume for now that $\frac{\partial f_i \circ \phi_i}{\partial y_i}(\mathbf{p}) \neq 0$. If the local coordinates of $\mathbf{p}$ are $\phi_i(\mathbf{p}) = (p_1, p_2)$ then by the Implicit Function Theorem there is a neighborhood V of $p_1 \in \mathbb{C}$, a unique holomorphic function $y(x)$ on V such that $y(p_1) = p_2$ and $f_i \circ \phi_i(x, y(x)) = 0$ for all $x \in V$, and there is a neighborhood W of $\phi_i(\mathbf{p}) \in \mathbb{C}^2$ such that $y(x) = y$ for all $(x, y) \in W \cap \phi_i(C) \subseteq \mathbb{C}^2$. Define a function $\psi_{i,\mathbf{p}} = (x, y(x))$. The inverse of this function is simply the projection onto the first coordinate: $\psi_{i,\mathbf{p}}^{-1}(x, y) = x$. Both $\psi_{i,\mathbf{p}}$ and $\psi_{i,\mathbf{p}}^{-1}$ are holomorphic on their domains.

Now, on $U_{i,\mathbf{p}} = \phi_i^{-1}(W \cap \phi_i(C))$ define the coordinate map $\phi_{\mathbf{p},i} : U_{i,\mathbf{p}} \longrightarrow V_{i,\mathbf{p}} = V_i \subseteq \mathbb{C}$ to be

$$\phi_{\mathbf{p},i} = \psi_{\mathbf{p},i}^{-1} \circ \phi_i.$$

This is a homeomorphism. Do this for each point $\mathbf{p}$ in each U_i. IF, for a particular point $\mathbf{p} \in U_j$, $\frac{\partial f_i \circ \phi_i}{\partial y_i}(\mathbf{p}) = 0$, use the implicit function theorem to find a neighborhood of p_2 and a holomorphic function $x(y)$ instead. Then $\psi_{i,\mathbf{p}}(y) = (x(y), y)$ and $\psi_{i,\mathbf{p}}^{-1}$ would be projection onto the second coordinate.

Cover C with the open sets $U_{i,\mathbf{p}}$. The gluing maps $\phi(i,\mathbf{p})(j,\mathbf{q}) = \psi_{i,\mathbf{p}}^{-1} \circ \phi_{ij} \circ \psi_{j,\mathbf{q}}$ are holomorphic on $\phi^{-1}(U_{i,\mathbf{p}} \cap U_{j,\mathbf{q}})$ being the composition of holomorphic functions.

The definitions of holomorphic functions, meromorphic functions and even curves given in Section 1.5 all work when the underlying manifold is 1-dimensional. In fact, if $C = \{(U_i, f_i)\}$ is a nonsingular curve on X, then curves, holomorphic and meromorphic functions on C are all restrictions of curves, holomorphic functions and meromorphic functions on X. For example, data $\{(U_i, g_i)\}$ which defines a holomorphic function on X restricts to $\{(U_i \cap C, g_i|_C)\}$ and defines a holomorphic function on C.

4.12.2. *Divisors on a Nonsingular Curve.*

All definitions given in Section 4.2 for Weil divisors, Cartier divisors, prime divisors, etc., also work on an irreducible curve C. A Cartier divisor now is a finite formal sum of point on C. If $D = \{(U_i \cap C, g_i|_C)\}$ defines a Weil divisor, then the associated Cartier divisor is the sum of the zeros of D on C minus the sum of the poles of D on C.

Line bundles and their relationship to divisors is still the same, also. As with functions and curves, a line bundle L on X can be restricted to a line bundle $L|_C$ on C. Consider this example.

Example 4.6: Let X be $\mathbb{P}^2$ blown up at $(1:0:0)$ and consider the exceptional curve E given by (3.3) and the nonsingular curve $\tilde{C}$ from Example 3.2 given by

$$\{(U_{00}, 1 - y_0 u_0^3), (U_{01}, t^2 - z_0), (U_1, x_1 - z_1^3), (U_2, x_2 y_2^2 - 1)\}.$$

The transition functions ψ_{ij} for the line bundle $[E]$ on X were calculated in Example 4.4.

The data for E defines a Cartier divisor when restricted to the curve $\tilde{C}$. The associated Weil divisor can be calculated by finding the zeros and poles of E on $\tilde{C}$. On $U_{00} \cup U_1 \cup U_2$, $E \cap \tilde{C} = \emptyset$. On U_{01}, E has a second order zero at $\mathbf{p} = (z_0, t) = (0,0)$ on $\tilde{C}$. Thus $(E)|_{\tilde{C}} = 2\mathbf{p}$.

Restricting the line bundle $[E]$ to $\tilde{C}$ by restricting each of the transition functions to $\tilde{C}$ yields a line bundle $[E]|_{\tilde{C}}$. The data

$$s = \{(U_{00}, 1), (U_{01}, 1/t), (U_1, x_1), (U_2, x_2/y_2)\}$$

defines a meromorphic section of $[E]|_{\tilde{C}}$ as well as $[E]$ (see Example 4.4). Now, s has no zeros or poles on $\tilde{C}$ on $U_{00} \cup U_2$. However, s does have a pole along the curve $t = 0$ in U_{01} and $t = 0$ meets $\tilde{C}$ at $\mathbf{p}$. Since $i(\mathbf{p}, t \cap t^2 - z_0) = 1$ the order of this pole is 1. Similarly, $\mathbf{q} = (x_1, z_1) = (0,0)$ is a zero of order 3 on $\tilde{C}$ in U_1 since $i(\mathbf{q}, x_1 \cap z_1 - x_1^3) = 3$. Thus, the Cartier divisor on $\tilde{C}$ associated to s is $3\mathbf{q} - \mathbf{p}$.

Both s and E are meromorphic sections of $[E]|_{\tilde{C}}$, so $(s) \sim (E)$. Therefore, $3\mathbf{q} - \mathbf{p} \sim 2\mathbf{p}$ as divisors on $\tilde{C}$.

4.12.3. *The Degree of a Divisor on a Nonsingular Curve.*

Let $D = \sum_{i=1}^{r} n_i \mathbf{p}_i$ be a Weil divisor on the nonsingular curve C. Define the *degree of* D to be the sum $\deg(D) = \sum_{i=1}^{r} n_i$ Although this could be defined for divisors in general, it is only well defined for divisors on 1-dimensional manifolds. If $D \sim D'$ as divisors on C, then the claim is $\deg(D) = \deg(D')$. Clearly, degree is additive so $\deg(D) - \deg(D') = \deg(D - D')$. Thus, the claim is true if we can show that $\deg(D) = 0$ if D is a principal divisor. The degree of a divisor D counts the number of zeros of D on C and subtracts the number of poles of D on C. Therefore, deg (D) will be zero for a principal divisor if we can show that the number of poles equals the number of zeros for a meromorphic function on C.

Proposition 4.4. *Suppose C is a nonsingular irreducible curve on a 2-dimensional complex manifold X. Every meromorphic function F on C has the same number of zeros and poles on C, counting multiplicities.*

Proof: Let $F : C \longrightarrow \mathbb{C}^2$ be a meromorphic function defined by $\{(U_i, f_i/g_i)\}$ on a nonsingular, irreducible curve C on a 2-dimensional complex manifold. Further assume each of f_i and g_i is holomorphic on U_i. Create a new function $G : C \longrightarrow \mathbb{P}^1$ defined by $\{(U_i, (f_i/g_i{:}\,1))\}$ where f_i/g_i is defined and $\{(U_i, (1{:}\,0))\}$ where g_i is zero. The advantage of writing G in this way is that the zeros of F are now those points where $G(\mathbf{p}) = (0{:}\,1)$ and the poles of F are those points where $G(\mathbf{p}) = (1{:}\,0)$. This is a holomorphic mapping on C since it can be written $\{(U_i, (f_i{:}\,g_i))\}$.

Let $z_0 \in U_i$. The coordinates of U_i and $\mathbb{P}^1$ can be chosen so that there is a neighborhood Δ_{z_0} of z_0 where $G(z) = (z^k{:}\,1)$ for $z \in \Delta_{z_0}$ and $(0{:}\,1)$ is not in $G(\Delta_{z_0})$. Now, $G^{-1}(0{:}\,1) \cap \Delta_{z_0}$ is 0 with multiplicity k and $G^{-1}(\beta{:}\,1) \cap \Delta_{z_0} = \{\zeta^i \sqrt[k]{\beta} : \zeta = e^{2\pi i/k} \text{ for } i = 0, \ldots, k-1\}$ for all other points $(\beta{:}\,1)$ in $G(\Delta_{z_0})$

Define a new function $H : \mathbb{P}^1 \longrightarrow \mathbb{Z}$ by assigning to each $(\alpha{:}\,\beta)$ in $\mathbb{P}^1$ the number of points in $G^{-1}(\alpha{:}\,\beta)$. Fix $(\alpha{:}\,\beta)$. For each point $z_i \in G^{-1}(\alpha{:}\,\beta)$ choose a neighborhood $\Delta_i = \Delta_{z_i}$ as above with k_i the number of points in the image of z_i under G. There are $H(\alpha{:}\,\beta) = \sum_i k_i$ points in $G^{-1}(\alpha{:}\,\beta)$.

For any open neighborhood Δ of $(\alpha{:}\,\beta)$, $G^{-1}(\Delta) \cap \Delta_i$ is an open neighborhood of z_i and there are k_i elements in $G^{-1}(x{:}\,y) \cap \Delta_i$ for all $(x{:}\,y) \in \Delta$. Therefore, $H(x{:}\,y) = \sum_i k_i$

for all $(x:y) \in \Delta$. Since H is constant on an open neighborhood of each point in $\mathbb{P}^1$ and $\mathbb{P}^1$ is connected, H is a constant on $\mathbb{P}^1$.

Now, the number of zeros of F is $H(0:1)$ and the number of poles of F is $H(1:0)$. But H is constant so these numbers are the same and F has the same number of poles and zeros.

4.12.4. *Cartier and Weil Divisors.*

Let $C \subset X$ be an irreducible, nonsingular curve. A divisor on C can be found by restricting certain divisors of X to C. Let $D = \sum n_i D_i$ be a divisor on X with each D_i prime. If D_i meets C properly for each D_i then we can define the divisor $D|_C$ to be

$$D|_C = \sum_{\mathbf{p} \in C \cap D} i(\mathbf{p}, C \cap D)\mathbf{p}. \tag{4.10}$$

The divisor D can also be represented on X as a Cartier divisor by data $\{(U_i, f_i)\}$. Earlier it was stated that

$$D|_C = \{(U_i \cap C, f_i|_C)\} \tag{4.11}$$

was a divisor on C. Are these 2 definitions consistent? The relationship of line bundles and divisors is the same for 1-dimensional manifolds as they are for 2-dimensional manifolds. As a formal sum of points on C, $D|_C$ in (4.10) is a divisor on C and is the sum of zeros and poles of some meromorphic section of the line bundle L associated to that divisor. On the other hand, the data given in (4.11) is also the data of some meromorphic section of $[D|_C]$. But the zeros and poles of (4.11) are exactly the points listed in the sum (4.10) with the same multiplicities. Thus, the 2 definitions are consistent.

4.13. Conclusion

Throughout this chapter there has been a great deal of notational abuse. But there was good reason for all of it. On a complex manifold X there is a one-to-one correspondence between the set of Cartier divisors and Weil divisors. For this reason we will simply call the elements of each set divisors. Also, there is a one-to-one correspondence between the equivalence classes on $\mathrm{Div}\,(X)$ and the holomorphic line bundles on X. Thus $[D]$ is used denote elements of each of these sets. Finally, curves, effective divisors, and holomorphic sections of line bundles are all given by the same data $\{(U_i, f_i/g_i)\}$. By assuming f_i and g_i have no nonconstant common factor, the data for curves, effective divisors and holomorphic sections of holomorphic line bundles can be used interchangeably. For example, the complete linear system $|D|$ can be represented by the set $\{s\}$ of nonzero holomorphic sections of the line bundle $[D]$.

In Chapter 5 we investigate complete linear systems in more detail and use the machinery developed in this chapter to define intersection numbers.

CHAPTER FIVE

CALCULATING THE DEGREE OF A RATIONAL SURFACE

5.1. Introduction

We want to calculate the number of points in the intersection of the general plane sections (4.3) and (4.4). To do this we will define an integer for every pair of divisors C and D in Div (X) which will only depend on the linear equivalence classes of C and D. Although not all divisors are the divisors of curves, if C and D are the divisors of curves with no common component, the intersection number will be the number of points of intersection of those curves.

5.2. More on Complete Linear Systems

Bertini's Theorem says that almost all divisors are nonsingular away from the base locus of the system. If a complete linear system has no base points, by Proposition 4.2 there are curves in the linear system meeting a fixed set of curves properly. In this section we will investigate linear systems which have no base points so that we can apply Bertini's Theorem and Proposition 4.2. The first concern is whether a complete linear system exists which has no base points.

Proposition 5.1. *Let X be a 2-dimensional manifold which can be embedded into $\mathbb{P}^k$. Then there exists an effective divisor $H \in$ Div (X) such that $|H|$ has no base points.*

Proof: Name the embedding $\Psi : X \longrightarrow \mathbb{P}^k$. Let $H = \alpha_0 X_0 + \ldots + \alpha_k X_k$ by any hyperplane in $\mathbb{P}^k$. If $S = \Psi(X) \subseteq \mathbb{P}^k$, then $S \cap H$ is a hyperplane section of S and is a curve in $\mathbb{P}^k$ (the intersection of any two hypersurfaces is a curve). Hence $\Psi^{-1}(S \cap H)$ is a curve on X and is an effective divisor there. Call this divisor H also.

The claim is $|H| \in \mathrm{Div}\,(X)$ has no base points. We will show this by showing that for every point $\mathbf{p} \in X$ there is a divisor in $|H|$ which does not contain $\mathbf{p}$. Let $\mathbf{q} = \Psi(\mathbf{p}) \in \mathbb{P}^k$ and choose a hyperplane H' in $\mathbb{P}^k$ so that $\mathbf{q} \notin H'$. Consider the divisor on X which is defined by $\Psi^{-1}(H')$ and call it H'. Let $F = 0$ and $F' = 0$ be the equations of H and H', respectively. On $\mathbb{P}^k$, F/F' is a meromorphic function hence, $\Psi \circ F/\Psi \circ F'$ must be a meromorphic function on X. Therefore, in $\mathrm{Div}\,(X)$, $H \sim H'$. But $\mathbf{p} \notin H'$ so $|H|$ has no base points.

There is at least one complete linear system on X which has no base points. The following proposition and corollary allow us to create other linear systems with no base points.

Proposition 5.2. *If the complete linear systems $|C|$ and $|D|$ have no base points, then $|C + D|$ has no base points.*

Proof: We proceed by contradiction. Suppose $|C + D|$ has a base point $\mathbf{p}$ and $|D|$ has no base points. Let $s = \{(U_i, s_i)\}$ be a holomorphic section of $[D]$ which does not have $\mathbf{p}$ as a zero and $s' = \{(U_i, s_i{}')\}$ be any holomorphic section of $[C]$. If transition functions for $[D]$ and $[C]$ are ψ_{ij} and $\psi_{ij}{}'$, respectively, then $\psi_{ij}\psi_{ij}{}'$ are transition functions for $[C + D]$. Therefore, $ss' = \{(U_i, ss_i{}')\}$ is a holomorphic section of $[C + D]$. If U_i is a cover element containing $\mathbf{p}$, then $s_i s_i{}'(\mathbf{p}) = 0$ but $s_i(\mathbf{p}) \neq 0$. Therefore, $s_i{}'(\mathbf{p}) = 0$ and every holomorphic section of $[C]$ has $\mathbf{p}$ as a zero. Thus, $|C|$ has a base point.

Corollary 5.2.1. *If $|C|$ has no base points, then $|nC|$ has no base points for any positive integer n.*

Thus we know $|nH|$ has no base points for any positive integer n. Finally, given any complete linear system we can create another from it which has no base points.

Proposition 5.3. *Let X be a 2-dimensional complex manifold which can be embedded in $\mathbb{P}^k$ and C a divisor on X. There is a positive integer n large enough so that $|C + nH|$ has no base points where H is as in Proposition 5.2.*

Proof: Name the embedding $\Psi : X \longrightarrow \mathbb{P}^k$. We will begin with $k = 3$. Let X be a 2-dimensional complex manifold which can be embedded into $\mathbb{P}^3$ as a surface and call the embedding Ψ. Since X and $S = \Psi(X)$ are homeomorphic, the sets Div (X) and Div (S) are isomorphic. Thus, to prove this proposition for X we need only prove it for surfaces in $\mathbb{P}^3$.

Let C be a curve on S in $\mathbb{P}^3$ and $I(C) = \{f : f(\mathbf{p}) = 0 \text{ for all } \mathbf{p} \in C\}$, the *ideal of* C. Choose $G \in I(C)$ such that the zero locus of G is not a subset of S and let m be the homogeneous degree of G. On the one hand, the divisor of the curve $G \cap S$ is linearly equivalent to the divisor mH where H is any plane in $\mathbb{P}^3$. On the other hand, the curve C is a subset of the curve $G \cap S$, so $G \cap S = C \cup D$ for some curve D on S. Therefore, $C + D \sim mH$ as divisors. For any nonnegative integer n, the linear system

$$|C + nH|$$

is equal to

$$|(n+m)H - D|$$

since $C + nH \sim (n+m)H - D$. Thus, it suffices to show that there is an n large enough so that $|(n+m)H - D|$ has no base points. The advantage is that we can find a subset of the latter complete linear system which can be shown to have no base points. If a subset of a linear system has no base points, then the linear system has no base points.

Write $D = \sum r_i D_i$ where each D_i is a prime divisor and let n be any nonnegative integer. Consider a homogeneous polynomial F of degree $n + m$ on $\mathbb{P}^3$ and let D_F be the divisor on S defined by the curve $F \cap S$. Now, consider only those polynomials F of degree $n+m$ where $D_F = D + E_F$ for some effective divisor E_F on S. Let L_n be the set of all divisors E_F found in this way. Since $D + E_F \sim (n+m)H$ and each E_F is effective, L_n is a subset of the complete linear system $|(n+m)H - D|$. We will show that there is a n large enough so that L_n has no base points.

Let I be the set of all polynomials f such that $D_f = D + E_f$ for some effective divisor E_f where D_f is the divisor defined by the curve $f \cap S$. This set is an ideal. The zero set $Z(J)$ of an ideal J is the intersection of the zero loci of all elements of J. In this case, $Z(I)$ is exactly D.

The Hilbert Basis Theorem can be used to show that I is generated by a finite set of polynomials, that is $I = \{\sum_{i=1}^{s} \alpha_i f_i : \alpha_i \in \mathbb{C}[X_0, X_1, X_2, X_3]\}$. Let M be the maximum degree of a fixed generating set $\{f_i\}$. The claim is that L_n has no base points when $n+m \geq M$.

Suppose $\mathbf{p} \notin D$. Choose $f \in I$ with $f(\mathbf{p}) \neq 0$ and $\deg(f) \leq M$. This is possible because $Z(I)$ is exactly D so there must be some f in I which is not zero at $\mathbf{p}$; otherwise, $\mathbf{p}$ would be in the zero set of I. If $\mathbf{p} = (p_0{:}p_1{:}p_2{:}p_3)$, then one of the p_i is not zero, say $p_j \neq 0$. Put $g = X_j^s f$ where $s = n + m - \deg(f) \geq 0$. Now $g(\mathbf{p}) \neq 0$ and $\deg(g) = n+m$. Since $\mathbf{p}$ is not in the zero set of the divisor (g), $\mathbf{p}$ cannot be in the zero set of the divisor $(g) - D$ because subtracting off an effective divisor doesn't add any points to the zero set of the divisor. Therefore, $(g) - D$ is in L_n and points off D are not base points of L_n.

Suppose $\mathbf{p} \in D$ and let $D_1, \ldots, D_s$ be the components of D which contain $\mathbf{p}$. Assume $\mathbf{p} = (p_0{:}p_1{:}p_2{:}p_3)$ and $p_j \neq 0$ and let U be the open subset of $\mathbb{P}^3$ where $X_j \neq 0$. Consider the irreducible curves c_i in U defined by $D_i \cap U$ for $i = 1, \ldots, s$. For each i, there must be a polynomial g_i defined on U where the intersection of $S \cap U$ and the zero locus of g_i is exactly the curve c_i. Homogenize the polynomial $\prod g_i^{r_i}$ with respect to X_j to obtain a homogeneous polynomial f_1 on $\mathbb{P}^3$. Let f_2 be a homogeneous polynomial on $\mathbb{P}^3$ which does not contain $\mathbf{p}$ in its zero set and the divisor obtained by intersecting the zero set of f_2 with S is $D_{f_2} = \sum_{i=s+1}^{t} r_i D_i + E_{f_2}$ for some effective divisor E_{f_2}. Let $f = f_1 f_2$. The polynomial f is in now in I. If $E_f = D - D_f$ as defined above, then E_f does not contain $\mathbf{p}$ because D and D_f contain the components D_i for $i = 1, \ldots, s$ with exactly the same multiplicities. If $\deg(f) < M$, put $g = X_j^s f$ where X_j^s is not zero at $\mathbf{p}$ and so that $\deg(g) = M$. Now, $(g) = D + (X_j^s) + E_f$ and $(g) - D = (X_j^s) + E_f$. Neither (X_j^s) nor E_f contain $\mathbf{p}$. Therefore, points on D are not base points of L_n.

Thus, L_n has no base points and neither do $|(n+m)H - D|$ and $|C + mH|$.

The proof follows in the same way for $k > 3$. The only difference is that the polynomials will be homogeneous on $\mathbb{P}^k$ instead of $\mathbb{P}^3$ and S would be called a hypersurface instead of a surface.

Suppose we had divisors $C_1, \ldots, C_r$. For each i choose n_i so that $|C_i + n_i H|$ has no base points. Put $n = \max\{n_i\}$. Now $|C_i + nH| = |C_i + n_i H + (n - n_i)H|$ has no base points for all i by Proposition 5.2. This idea is used in the proof of Theorem 5.4.

All results in this section require that the manifold X be embedded in $\mathbb{P}^k$ for some k. Clearly, this is true for $\mathbb{P}^2$, but is is also true for $\mathbb{P}^1 \times \mathbb{P}^1$ and all blow ups of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ [Har: pp. 161-163]. Therefore, these results apply to all manifolds used in this paper.

5.3. Definition of Intersection Number

We are now ready to define intersection numbers by assigning an integer to each pair of divisors C and D on X. Denote this integer by $C \cdot D$ and call it the *intersection number of C and D.* The intersection number of a divisor paired with itself is denoted C^2 and is called the *self intersection of C.* A typical divisor for a plane section of a triangular surface looks like $n\pi^* h - \sum m_i E_i$. To find the self intersection of this divisor we first need to expand the product $(n\pi^*h - \sum m_i E_i)^2$. Thus it will be helpful if intersection product were linear and commutative. Also, intersection numbers should be independent of divisor class representative. Finally, if 2 divisors C and D are effective and intersect properly, the intersection number should simply give the numbers of points in $C \cap D$ counting multiplicities. Not only are these properties necessary but the following proposition shows they are enough to uniquely determine the intersection number.

Theorem 5.4. *For each pair of divisors C and D in* $\mathrm{Div}(X)$ *there is a unique integer called the intersection number of C and D denoted by $C \cdot D$ which has the following properties: if C, D, C_i, D_i are elements of* $\mathrm{Div}(X)$ *then*

1. $C \cdot D = D \cdot C$;
2. $C_1 \cdot D = C_2 \cdot D$ *whenever* $C_1 \sim C_2$;
3. $C \cdot (D_1 + D_2) = C \cdot D_1 + C \cdot D_2$; *and*
4. *whenever C and D are effective divisors which meet properly then*

$$C \cdot D = \sum_{\mathbf{p} \in C \cap D} i(\mathbf{p}, C \cap D).$$

Proof: Uniqueness: Let C and D be any 2 divisors on X. Using Propositions 5.1 and 5.3 choose $n > 0$ large enough and $H \in \mathrm{Div}(X)$ so that $|C + nH|$, $|D + nH|$, and $|H|$ have no base points. By Corollary 4.2.1 we can find nonsingular effective divisors

$C' \in |C + nH|$,

$D' \in |D + nH|$ meeting C' properly,

$E' \in |nH|$ meeting D' properly, and

$F' \in |nH|$ meeting C' and E' properly.

Now $C \sim C' - E'$ and $D \sim D' - F'$ so

$$
\begin{aligned}
C \cdot D &= (C' - E') \cdot (D' - F') && \text{by (2)} \\
&= C' \cdot (D' - F') - E' \cdot (D' - F') && \text{by (3)} \\
&= (D' - F') \cdot C' - (D' - F') \cdot E' && \text{by (1)} \\
&= D' \cdot C' - F' \cdot C' - D' \cdot E' + F' \cdot E' && \text{by (3)} \\
&= \sum_{\mathbf{p} \in D' \cap C'} i(\mathbf{p}, D' \cap C') - \sum_{\mathbf{p} \in F' \cap C'} i(\mathbf{p}, F' \cap C') - \\
&\quad \sum_{\mathbf{p} \in D' \cap E'} i(\mathbf{p}, D' \cap E') + \sum_{\mathbf{p} \in F' \cap E'} i(\mathbf{p}, F' \cap E') && \text{by (4).}
\end{aligned}
$$

Therefore $C \cdot D$ is uniquely determined by the 4 properties of this proposition. Once it is proven that $C \cdot D$ exists, these properties can be used to calculate $C \cdot D$ for any divisors in X.

Existence: First we will define $C \cdot D$ for a subset of $\text{Div}\,(X)$ then show the definition can be extended to all of $\text{Div}\,(X)$ by the same process as above. Let $\mathcal{B}(X)$ be the set of divisors $C \in \text{Div}\,(X)$ such that $|C|$ has no base points. By Proposition 5.1 we know that $\mathcal{B}(X)$ is nonempty. For every $C, D \in \mathcal{B}(X)$, $C + D \in \mathcal{B}(X)$ from Proposition 5.2.

Define $C \cdot D$ for $C, D \in \mathcal{B}(X)$ as follows. Choose $C' \in |C|$ nonsingular and $D' \in |D|$ nonsingular and meeting C' properly and put

$$
\begin{aligned}
C \cdot D &= \sum_{\mathbf{p} \in C' \cap D'} i(\mathbf{p}, C' \cap D') \\
&= \sum_{\mathbf{p} \in D' \cap C'} i(\mathbf{p}, D' \cap C').
\end{aligned}
$$

It needs to be shown that this definition does not depend on the choices of C' and D' and that it satisfies the 4 properties in the theorem.

Fix C' and choose an alternate $D'' \in |D|$ also nonsingular and meeting C' properly.

Now

$$\sum_{\mathbf{p}\in C'\cap D'} i(\mathbf{p}, C'\cap D') = \deg([D']|_{C'})$$

$$= \deg([D'']|_{C'})$$

$$= \sum_{\mathbf{p}\in C'\cap D''} i(\mathbf{p}, C'\cap D'')$$

since $[D']|_{C'} = [D'']|_{C''}$. Thus the value of $C \cdot D$ does not depend on the choice of D'. On the other hand, choose $C'' \in |C|$ also nonsingular. Since the choice of D' does not matter, choose $D' \in |D|$ such that D' is nonsingular and meets both C' and C'' properly. Then

$$\sum_{\mathbf{p}\in D'\cap C'} i(\mathbf{p}, D'\cap C') = \deg([C']|_{D'})$$

$$= \deg([C'']|_{D'})$$

$$= \sum_{\mathbf{p}\in D'\cap C''} i(\mathbf{p}, D'\cap C'').$$

Therefore, the definition is well defined for $C, D \in \mathcal{B}(X)$.

It is immediately clear that conditions (1) and (2) hold for this definition. Also, (4) is true by construction. Finally, to show (3) is true, choose $C_i' \in |C_i|$ so that C_i' meets D properly for $i = 1, 2$. Now, using (2),

$$(C_1 + C_2) \cdot D = (C_1' + C_2') \cdot D$$

$$= \sum_{\mathbf{p}\in C_1'C_2'} (\mathbf{p}, C_1'C_2' \cap D)$$

$$= \sum_{\mathbf{p}\in C_1'} (\mathbf{p}, C_1' \cap D) + \sum_{\mathbf{p}\in C_2'} (\mathbf{p}, C_2' \cap D)$$

$$= C_1' \cdot D + C_2' \cdot D$$

$$= C_1 \cdot D + C_2 \cdot D$$

by Theorem 2.2(2.1).

To define $C \cdot D$ on $\operatorname{Div}(X)$ in general choose C', D', E' and F' as in the proof for uniqueness and put

$$C \cdot D = C' \cdot D' - C' \cdot F' - E' \cdot D' + E' \cdot F'.$$

Is this definition well defined? Each of $C', D', E', F' \in \mathcal{B}(X)$ so

$$\begin{aligned} C \cdot D &= C' \cdot D' - C' \cdot F' - E' \cdot D' + E' \cdot F' \\ &= D' \cdot C' - D' \cdot E' - F' \cdot C' + F' \cdot E' \\ &= D \cdot C. \end{aligned}$$

This yields 2 results. First, if this definition is well defined, we have shown it satisfies condition (1). Also, to show this definition is well defined it suffices to show that it does not depend on the choices of D' and F'. Fix C' and let $D', D'' \in |D+nH|$ both nonsingular and meeting C' properly. Choose E' as before and $F', F'' \in |nH|$ both nonsingular and meeting C' and E' properly. All the divisors $C', D', D'', E', F', F'' \in \mathcal{B}(X)$ and $D' + F'' \sim D'' + F'$ so

$$\begin{aligned} C' \cdot (D' + F'') &= C' \cdot (D'' + F'), \\ C' \cdot D' + C' \cdot F'' &= C' \cdot D'' + C' \cdot F', \text{ and} \\ C' \cdot D' - C' \cdot F' &= C' \cdot D'' - C' \cdot F''. \end{aligned}$$

Similarly,

$$-E' \cdot D' + E' \cdot F' = -E' \cdot D'' + E' \cdot F''.$$

Thus

$$\begin{aligned} C \cdot D &= C' \cdot D' - C' \cdot F' - E' \cdot D' + E' \cdot F' \\ &= C' \cdot D'' - C' \cdot F'' - E' \cdot D'' + E' \cdot F'' \end{aligned}$$

and the definition is well defined.

Clearly, this definition depends only on divisor class. By a tedious calculation and using the properties for divisors in $\mathcal{B}(X)$, linearity can also be shown. Again, condition (4) is true by construction.

5.4. Calculating Intersection Number in Div $(\mathbb{P}^2)$

At this point we are prepared to calculate $C \cdot D$ for any C and D in Div $(\mathbb{P}^2)$. Let's start by calculating h^2 where h is the divisor of any line in $\mathbb{P}^2$. Let h' be the divisor of any other line in $\mathbb{P}^2$. By property 1, $h^2 = h \cdot h'$. But h and h' are divisors of distinct lines in $\mathbb{P}^2$ and those lines meet at exactly one point. Thus, by property 4, $h^2 = 1$. Now, let C and D be any divisors in Div $(\mathbb{P}^2)$. In section 4.3 we classified all divisors of $\mathbb{P}^2$ as nh for some $n \in \mathbb{Z}$ and for any line h in $\mathbb{P}^2$. Thus $C \cdot D = nh \cdot mh = nm(h^2)$ by the linearity of intersection numbers. Since $h^2 = 1$, $C \cdot D = nm$.

What have we done here? As was said before, if C and D are divisor of curves of degrees n and m in $\mathbb{P}^2$ with no common divisor, then $C \cdot D = nm$ is the number of points in the intersection of those curves counting multiplicities. If C and D are divisors of curves *with* a common factor, then $C \cdot D$ is still nm. So $C \cdot D$ does not calculate the number of points in the intersection of two specific curves, but instead calculates the number of points in the intersection of two general curves with the same degree as the curves C and D.

What about divisors which are not effective? The divisors C and D do not have to be effective divisors for us to calculate $C \cdot D$: consider $C = \left(\frac{1}{x^3}\right)$ and $D = \left(y^2\right)$. Then $C \sim -3h$ and $D \sim 2h$ and $C \cdot D = -6$. There is no interpretation here which relates to curves since C is not a curve and is not linearly equivalent to any curve. There is an interpretation in terms of poles and zeros. Consider C as a divisor on the curve $y^2 = 0$. The number $C \cdot D = -6$ indicates C has 6 more poles than zeros on $y^2 = 0$ counting multiplicities. The conclusion here is C is not the divisor of a meromorphic function on $y^2 = 0$, because meromorphic functions on curves have the same number of poles and zeroes, i.e., if a divisor C arises from a meromorphic function on a curve D, then $C \cdot D = 0$.

5.5. Calculating Intersection Numbers on Div (X)

Our goal is to be able to calculate the number of points in the intersection of two general plane sections (4.3) and (4.4). In this section we will develop further properties of the intersection number which will allow to perform this calculation. Let us first look at an example to see what other properties we need.

5.5.1. *One Base Point.*

In this section we consider curves on $\tilde{\mathbb{P}}^2$ where $\mathbb{P}^2$ is blown up at 1 point.

Example 5.1: If ψ has one base point with multiplicity m, then the plane sections (4.3) and (4.4) are linearly equivalent to

$$n\pi^* h - mE.$$

Using the linearity and commutativity of intersection numbers we can expand this product to get

$$n^2(\pi^* h)^2 - 2nm(\pi^* h) \cdot E + m^2 E^2.$$

Hence, for triangular surfaces, we need to know how to calculate intersection numbers $(\pi^* h)^2$, E^2, and $(\pi^* h) \cdot E$.

Proposition 5.5. *If X is $\mathbb{P}^2$ blown up at $\mathbf{p}$, E is the exceptional curve, and $\pi: X \longrightarrow \mathbb{P}^2$ is the projection map, then*

1. $(\pi^* h)^2 = 1$ *for any line h in $\mathbb{P}^2$,*
2. $(\pi^* h) \cdot E = 0$ *for any line h in $\mathbb{P}^2$, and*
3. $E^2 = -1$.

A formal proof of Proposition 5.5 is in Appendix B. Property 1 follows because we can choose two lines h', h'' in $\mathbb{P}^2$ which don't at meet $\mathbf{p}$ and each is linearly equivalent to h. Then $(\pi^* h') \cdot (\pi^* h'') = 1$ because the strict transforms of h' and h'' meet at one point in X. Then $(\pi^* h)^2 = (\pi^* h') \cdot (\pi^* h'')$ because the pullback preserves equivalence class and intersection numbers only depend on equivalence class. Using the line h', property 5.5(2) is clear because the pullback of h' does not meet E. Thus $(\pi^* h) \cdot E = (\pi^* h') \cdot E = 0$.

The final property reflects the fact that blowing up a point in $\mathbb{P}^2$ pulls apart curves according to distinct tangent directions. Consider the following example.

Example 5.2: Let C and D be two curves of degree two in $\mathbb{P}^2$ such that $C \cap D$ consists of four distinct points, one of which is $\mathbf{p}$. The curves must meet at $\mathbf{p}$ with distinct tangent directions (see Figure 5.1). Let $\pi: X \longrightarrow \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at $\mathbf{p}$ and $E = \pi^{-1}(\mathbf{p})$. Away from E, the strict transforms of C and D will meet in three distinct points exactly as C and D do away from $\mathbf{p}$. Both $\tilde{C}$ and $\tilde{D}$ meet E but at distinct points corresponding to the distinct tangent directions of C and D at $\mathbf{p}$.

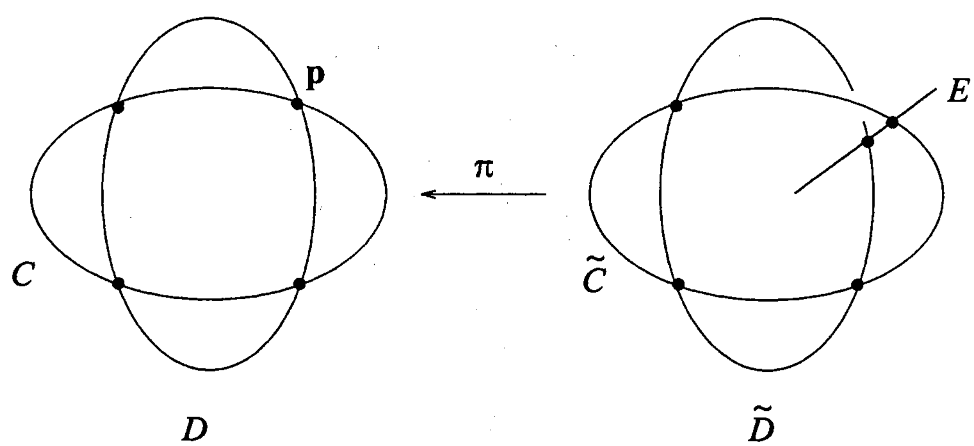


Figure 5.1

Blowup of Two Quadrics in $\mathbb{P}^2$

Looking at this calculation from another direction, the divisors C and D in Div $(\mathbb{P}^2)$ have intersection number 4 while the strict transforms $\tilde{C}$ and $\tilde{D}$ in Div (X) have intersection number 3 because both pairs satisfy the hypotheses for Theorem 5.4(4). On the other hand, we can calculate $C \cdot D$ and $\tilde{C} \cdot \tilde{D}$ using Proposition 5.5. On $\mathbb{P}^2$, $C \cdot D = (2h)^2 = 4$. On X, both $\tilde{C}$ and $\tilde{D}$ are each linearly equivalent to $2\pi^*h - E$ since $m_{\mathbf{p}}(C) = m_{\mathbf{p}}(D) = 1$. Thus

$$\begin{aligned}\tilde{C} \cdot \tilde{D} =& (2\pi^*h - E)^2 \\ =& 4(\pi^*h)^2 - 4(\pi^*h) \cdot E + E^2 \\ =& 4 + E^2\end{aligned}$$

using properties 1 and 2 from Proposition 5.5. But, $4 + E^2$ has to be 3 so E^2 must be -1.

Example 5.1 can now be completed using Proposition 5.5:

$$n^2(\pi^*h)^2 - 2nm(\pi^*h) \cdot E + m^2E^2 = n^2(1) - 2mn(0) + m^2(-1) = n^2 - m^2$$

and $\tilde{A} \cdot \tilde{B} = n^2 - m^2$. Thus, the degree of a triangular surface with one base point of multiplicity m is $n^2 - m^2$.

5.5.2. *Any Number of Base Points.*

We need to extend this idea for when there are any number of base points. The following proposition provides us with the other tools we need.

Proposition 5.6. *Let X be a 2-dimensional manifold, $\tilde{X}$ a blow up of X at the points $\mathbf{p}_1, \ldots, \mathbf{p}_r$, $E_1, \cdots, E_r$ the exceptional curves, and $\pi: \tilde{X} \longrightarrow X$ be the projection map. Then for all C and $D \in \operatorname{Div}(X)$*

1. $(\pi^*C) \cdot (\pi^*D) = C \cdot D$,
2. $(\pi^*C) \cdot E_i = 0$,
3. $E_i^2 = -1$, *and*
4. $E_i \cdot E_j = 0$ *if* $i \neq j$.

A formal proof is in Appendix B in which we first assume C and D are divisors of irreducible curves. This is possible because both π^* and the intersection number are linear. Property 1 follows from choosing $C' \sim C$ and $D' \sim D$ such that they contain none of the points blown up. Then the number of points in the intersection of the pullbacks of C and D is the same as the number of points in the intersection of the original curves. Also the proper transform of C' does not meet any E_i so property 2 also follows. Property 3 again reflects the fact that the blow up pulls X apart at each $\mathbf{p}_i$. We saw in Section 3.5.2 that $E_i \cap E_j = \emptyset$ whenever $i \neq j$. This makes property 4 true.

5.6. Triangular Surfaces

So far, intersection numbers have been calculated when one blow up was necessary. The following proposition shows that successive blow ups do not complicate matters. Also, a general formula is given for the degree of a triangular surface based on the parametric degree of the surface and the multiplicity of the base points.

Proposition 5.7. *Suppose $\psi : \mathbb{P}^2 - - \rightarrow \mathbb{P}^3$ defines a triangular parametric surface with parametric degree n. If ψ has s base points each of multiplicity m_i, then the degree of the surface* $\mathrm{Im}\,(\psi)$ *is* $n^2 - \sum_{i=1}^{s} m_i^2$.

Proof: We will proceed by induction on the numbers of blow ups necessary to remove all the base points. If ψ only requires one blow up then the plane sections of the surface are linearly equivalent to $n\pi^*h - \sum_{i=1}^{r} m_i E_i$ and the self intersection of these divisors is

$$\left(n\pi^*h - \sum_{i=1}^{r} m_i E_i\right)^2 = n^2 - \sum_{i=1}^{r} m_i^2$$

by applying the propositions above.

Suppose this proposition is true whenever there are k blow ups and suppose ψ requires $k+1$ blow ups. Let $m_1, \cdots, m_{s-1}$ be the multiplicities of the base points removed by the first k blow ups and $m_s, \cdots, m_r$ be the multiplicities of the base points removed by the $k+1$st blow up. Let π_i be the projection map for the ith blow up. The degree of $\mathrm{Im}\,(\psi)$ is

$$d = \left(\pi_{k+1}^* C - \sum_{i=s}^{r} m_i E_i\right)^2$$

where C is the strict transform of a plane section of $\mathrm{Im}(\psi)$ after the first k blow ups (by Proposition 5.5). Now

$$d = (\pi_{k+1}^* C)^2 - 2\sum_{i=s}^{r} m_i(\pi_{k+1}^* C)\cdot E_i + \left(\sum_{i=s}^{r} m_i E_i\right)^2$$

after applying Proposition 5.5. But $(\pi_{k+1}^* C)^2 = C^2$, $(\pi_{k+1}^* C)\cdot E_i = 0$, and $\left(\sum_{i=s}^{r} m_i E_i\right)^2 = -\sum_{i=s}^{r} m_i^2$. Therefore,

$$d = C^2 - \sum_{i=s}^{r} m_i^2.$$

The induction hypothesis yields $C^2 = n^2 - \sum_{i=1}^{s-1} m_i^2$ and the proof is done.

Example 5.3: Let's calculate the degree of the surface in Example 3.5. In Section 4.5.2 we found the plane sections of the surface were linearly equivalent to

$$\pi_2^*[\pi_1^*(3\pi^* h - 2E) - E_1] - E_2.$$

Simply using Propositions 5.5 and 5.6,

$$\begin{aligned}\{\pi_2^*[\pi_1^*(3\pi^* h - 2E) - E_1] - E_2\}^2 &= [\pi_1^*(3\pi^* h - 2E) - E_1]^2 + E_2^2 \\ &= (3\pi^* h - 2E)^2 + E_1^2 + E_2^2 \\ &= 9h^2 + 4E^2 + E_1^2 + E_2^2 \\ &= 9 - 4 - 1 - 1 = 3.\end{aligned}$$

This is exactly what is found by applying Proposition 5.7 instead with $n = 3$, $m_1 = 2$, $m_2 = 1$ and $m_3 = 1$.

5.7. Tensor Product Surfaces

This chapter has focused on triangular surfaces but little needs to be added to give a general formula for the degree of a tensor product surface. Theorem 5.4 and Proposition 5.6 are applicable to $\mathbb{P}^1 \times \mathbb{P}^1$ and blow ups of $\mathbb{P}^1 \times \mathbb{P}^1$. Proposition 5.5 however, is specific to $\mathbb{P}^2$. The following is the analogous statement for $\mathbb{P}^1 \times \mathbb{P}^1$.

Proposition 5.8. *Let X is $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at* p*, E is the exceptional curve, and $\pi: X \longrightarrow \mathbb{P}^2$ is the projection map. If k is any curve of bidegree $(1,0)$ and l is any curve of bidegree $(0,1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$, then*

1. $(\pi^* k)^2 = (\pi^* l)^2 = 0$;
2. $(\pi^* k) \cdot (\pi^* l) = 1$;
3. $(\pi^* k) \cdot E = (\pi^* l) \cdot E = 0$; *and*
4. $E^2 = -1$.

Again, the formal proof of this proposition is in Appendix B. Property 3 actually follows from the more general result in Proposition 5.6(2). Blowing up a point is a local phenomenon, so $E^2 = -1$ here as on any manifold. The first 2 properties are proven similarly to Proposition 5.5.

Under the hypotheses of Proposition 5.8, any divisor in Div (X) is linearly equivalent to

$$D = \pi^*(n_1 k + n_2 l) - mE.$$

The self intersection of D is

$$\begin{aligned} D^2 &= (\pi^*(n_1 k + n_2 l) - mE)^2 \\ &= (n_1 k + n_2 l)^2 + m^2 E^2 \\ &= 2n_1 n_2 - m^2. \end{aligned}$$

This illustrates the difference between Proposition 5.7 and the following proposition.

Proposition 5.9. *Suppose $\psi : \mathbb{P}^1 \times \mathbb{P}^1 - - \rightarrow \mathbb{P}^3$ defines a tensor product parametric surface with parametric bidegree (n_1, n_2). If ψ has s base points each of multiplicity m_i, then the degree of the surface* $\operatorname{Im}(\psi)$ *is* $2n_1 n_2 - \sum_{i=1}^{s} m_i^2$.

Thus, we have a general formula for the degree of a tensor product surface based only on the parametric bidegree and the multiplicity of the base points. The proof of Proposition 5.9 is similar to the proof of Proposition 5.7 and will be omitted.

5.8. Conclusion

The formula for the implicit degree of a triangular surface is

$$n^2 - \sum_{i=1}^{s} m_i^2$$

where n is the parametric degree of the surface and m_i is the multiplicity of each base point. The formula for the implicit degree of a tensor product surface where the parametric bidegree is (n_1, n_2) is

$$2n_1n_2 - \sum_{i=1}^{s} m_i^2$$

again where m_i is the multiplicity of each base point. In particular, if a triangular surface has no base points the degree of the surface and n^2 and the degree of a tensor product surface is $2n_1n_2$ if it has no base points.

CHAPTER SIX

SOME GENUS FORMULAS

6.1. Introduction and Definition

It was stated in Chapter 1 that all rational curves are algebraic. It is not true, however, that all algebraic curves are rational, and, in particular, the curve of intersection of two rational surfaces in $\mathbb{P}^3$ is not always rational. The purpose of this chapter is to present a method for determining when the curve of intersection is rational.

All compact 2-dimensional orientable real manifolds can be classified with a nonnegative integer called the *topological genus* which is a topological invariant. Only manifolds with topological genus 0 are rational curves, that is, only manifolds of genus 0 have a rational parameterization as described in Section 1.1. An irreducible nonsingular complex curve is a 2-dimensional real manifold and can also be classified by its topological genus. Irreducible singular complex curves are not manifolds, but are birationally equivalent to a nonsingular curve.

In this chapter we will give formulas for calculating the topological genus of the curve of intersection of two rational surfaces when that curve is nonsingular and the topological genus of a birationally equivalent curve when the curve of intersection is singular. If that formula yields 0 for two surfaces, the intersection curve is rational. Throughout this chapter, a curve will mean an irreducible complex curve. The relationship between genus and intersection numbers given by the adjunction formula below will not be proven in this paper. The relationship between genus and rationality will also not be proven here. Instead, we will concentrate on the process of going from these results to the formula for the genus of the curve of intersection of two surfaces.

The topological genus is only defined for nonsingular curves and the curve of intersection of two surfaces may be singular. Here we define the geometric genus. For nonsingular curves the topological genus and the geometric genus are the same. The advantage of using the geometric genus is that it can be defined for singular curves because it is a birational invariant. The *geometric genus of a nonsingular curve* C on a 2-dimensional complex manifold X is given by the adjunction formula

$$p_g(C) = \frac{C \cdot (C + K_X)}{2} + 1 \tag{6.1}$$

where K_X is any canonical divisor of X [Har p. 361]. If C is a singular curve, there is a seires of blow ups of X, $\pi : Y \longrightarrow X$, such that $\tilde{C}$, the strict transform of C, is nonsingular on Y. The *geometric genus of a singular curve* C is given by

$$p_g(C) = \frac{\tilde{C} \cdot (\tilde{C} + K_Y)}{2} + 1.$$

The number $p_g(C)$ is a birational invariant [Har p. 181]. Thus, if there exist another series of blow ups $\pi' : Y' \longrightarrow X$, such that $\tilde{C}'$, the strict transform of C under this transformation, is nonsingular on Y', then

$$\frac{\tilde{C}' \cdot (\tilde{C}' + K_{Y'})}{2} + 1 = \frac{\tilde{C} \cdot (\tilde{C} + K_Y)}{2} + 1.$$

Therefore, the geometric genus of a singular curve is well-defined. The geometric genus will simply be referred to as the *genus*.

The number

$$p_a(C) = \frac{C \cdot (C + K_X)}{2} + 1 \tag{6.2}$$

can be calculated for any divisor C but is not a birational invariant. This number is called the *arithmetic genus of* C if C is an effective divisor and is called the *virtual genus of* C

for other divisors. In the case that C is nonsingular on X, the arithmetic genus and the geometric genus on X are the same.

6.2. Plane Curves

6.2.1. *Plane Curve Genus Formulas.*

The genus of a nonsingular curve C in $\mathbb{P}^2$ is completely determined by its degree. If d is the degree of C,then C is linearly equivalent to $d(h)$ where h is any line in $\mathbb{P}^2$. Therefore, the genus of C is

$$\begin{aligned} p_g(C) &= \frac{dh \cdot (d-3)h}{2} + 1 \\ &= \frac{(d-1)(d-2)}{2} \end{aligned}$$

since $K_{\mathbb{P}^2}$ is linearly equivalent to $-3(h)$.

For any plane curve, the arithmetic genus is

$$p_a(C) = \frac{(d-1)(d-2)}{2}. \tag{6.3}$$

The singularities of a plane curve cause the geometric genus to be less than the arithmetic genus, i.e., the genus of a singular curve C is

$$p_g(C) = p_a(C) - \kappa \tag{6.4}$$

for some correction term $\kappa > 0$ which is a function of the singularities of C. This correction term is given for all singular plan curves in the following proposition.

Proposition 6.1. *Let C be any irreducible curve in $\mathbb{P}^2$. Resolve the singularities of C with a series of blowups*

$$X_n \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} \mathbb{P}^2.$$

Then

$$p_g(C) = p_a(C) - \frac{1}{2}\sum m_{\mathbf{p}_i}(m_{\mathbf{p}_i} - 1)$$

where the sum extends over all the singularities of C in $\mathbb{P}^2$ and over all singularities of strict transforms of C in each X_i and $m_{\mathbf{p}_i}$ is the multiplicity of the singularity $\mathbf{p}_i$.

Here we will only look at the proof of a special case of this proposition. If the curve C has ordinary singularities, resolution of the singularities requires only one blowup $\pi : X \longrightarrow \mathbb{P}^2$. Let the singularities of C be $\mathbf{p}_1, \dots, \mathbf{p}_s$ and the exceptional curves be $E_1, \dots, E_s$. Now $\tilde{C}$, the strict transform of C, meets the exceptional curve E_i transversally at $m_{\mathbf{p}_i} = m_{\mathbf{p}_i}(C)$ distinct points, there are no singularities of $\tilde{C}$, and

$$p_g(C) = \frac{\tilde{C} \cdot (\tilde{C} + K_X)}{2} + 1.$$

The curve C is linearly equivalent to $d(h)$ where d is the degree of C and h is any line in $\mathbb{P}^2$, so $\tilde{C} \sim d(\pi^* h) - \sum m_{\mathbf{p}_i} E_i$. On X, $K_X \sim -3(\pi^* h) + \sum E_i$. Therefore, the genus of C is

$$p_g(C) = 1 + \frac{1}{2}\left(d(\pi^* h) - \sum m_{\mathbf{p}_i} E_i\right) \cdot \left((d-3)(\pi^* h) - \sum (m_{\mathbf{p}_i} - 1) E_i\right).$$

Simplifying and using (6.3) the geometric genus becomes

$$p_g(C) = p_a(C) - \frac{1}{2} \sum m_{\mathbf{p}_i}(m_{\mathbf{p}_i} - 1).$$

Thus, in the case of ordinary singularities, the correction term is

$$\kappa = \frac{1}{2} \sum m_{\mathbf{p}_i}(m_{\mathbf{p}_i} - 1).$$

A curve with singularities which are not ordinary will require several blowups to resolve all singularities. Proposition 6.1 is actually a corollary of Proposition 6.3.

In another special case, the correction term κ for a plane curve with only ordinary double points is simply the number of double points. If C has only δ ordinary double points, then Proposition 6.1 can be used to calculate

$$\begin{aligned} p_g(C) &= p_a(C) - \frac{1}{2} \sum_{\delta} 2(2-1) \\ &= p_a(C) - \delta. \end{aligned}$$

The problem of calculating the genus of the curve of intersection of two rational surfaces in Sections 6.7 and 6.8 will be reduced to finding the genus of a plane curve with only ordinary double points. The genus will be

$$p_g(C) = p_a(C) - \kappa$$

where the correction term κ is the number of double points.

6.2.2. *Further Porperties of the Genus of a Plane Curve.*

There is a clear relationship between the singularities of a plane curve and the genus. In fact, the genus can be used to find an upper bound on the numbers of singularities of a curve in $\mathbb{P}^2$. Using (6.4), Proposition 6.1, and the fact that $p_g(C) \geq 0$, we find that

$$\sum m_{\mathbf{p}_i}(m_{\mathbf{p}_i} - 1) \leq (d-1)(d-2)$$

and a plane curve has genus 0 if and only if this is an equality. For example, an irreducible plane conic cannot have any singularities and an irreducible plane cubic can have at most one double point. Table 6.1 gives the genus of some irreducible plane curves.

TABLE 6.1

GENUS OF SOME PLANE CURVES

Curve	Singularities	Genus
Line	None	0
Conic	None	0
Cubic	None	1
Cubic	One double point	0
Quartic	None	3
Quartic	Three double points	0
Quartic	One triple point	0

It is reasonable to ask if curves of any degree can be found. Since the genus of a nonsingular plane curve is $p_g(C) = \frac{(d-1)(d-2)}{2}$ and the degree d is a positive integer, there are no nonsingular plane curves of genus 2, 4, 5, 7, 8, 9, 11, 12, 13, 14, 16, ... On the other hand, if we allow the curve to be singular there are plane curves of any genus as the following proposition shows.

Proposition 6.2. *Let p be a positive integer. There exists a curve in $\mathbb{P}^2$ with $p_g(C) = p$.*

Proof: Consider homogeneous polynomials

$$F(x{:}\,y{:}\,z) = z^{2p}y^2 - h(x,z)$$

where

$$h(x,z) = x^{2p+2} + a_1x^{2p+1}z + \cdots + a_{2p+1}xz^{2p+1} + a_{2p+2}z^{2p+2},$$

$h(1,0) \neq 0$, and the zeros of h are all of multiplicity 1, i.e., h has $2p+2$ distinct zeros. The only singularity of F is $(0{:}\,1{:}\,0)$ and has multiplicity $2p$. Proposition 6.1 can be used to calculate the genus of F. To do so the singularities of F must be resolved and the multiplicity of the singularities of F and of all the strict transforms of F must be recorded. For this purpose the multiplicities will be labeled m_i with $m_1 = 2p$.

Find the local equation on U_1 for F by dehomogenizing F with respect to y to get $f(x,z) = z^{2p} - h(x,z)$. Blowup U_1 at the origin by letting the global coordinates of $\tilde{U}_{01}$ be $(x,z;u{:}\,v)$. There are $2p$ copies of the exceptional curve to factor out and the local equations of $\tilde{f}$ are

$$\{(U_{10}, f_0 = v^{2p} - x^2h(1,v)), (U_{11}, f_1 = 1 - z^2h(u,1))\}.$$

By construction the partial derivative $h_u(u,1)$ cannot be zero where $h(u,1)$ is zero, and for that reason, f_1 has no singularities on U_{11}. Since $p > 0$ and $h(1,0) \neq 0$, f_0 has a singularity of order 2 at $(x, v) = (0,0)$. Assume $p > 0$, set $m_2 = 2$, and blowup U_{10} at the origin to resolve this singularity.

Let the global coordinates of $\tilde{U_{10}} = (x, v; s{:}t)$. There are 2 copies of the exceptional curve to factor out and the local equations of $\tilde{f_0}$ are

$$\{(U_{100}, f_{00} = x^{2p-2}t^{2p} - h(1, xt)), (U_{101}, f_{01} = v^{2p-2} - s^2h(1, v))\}.$$

Again, by construction f_{00} has no singularities on U_{100}. If $p = 1$, the curve f_{01} is nonsingular and we have found all the m_i. However, if $p > 1$, f_{01} has a singularity of order 2 at $(v, s) = (0,0)$.

This pattern continues with each blowup. On part of the blowup, the strict transform is nonsingular because of the choice of zeros of h, and on the other part of the blowup the strict transform is of the form $a^{2(p-k)} - b^2h(1, a)$ where (a, b) are the local coordinates and $k + 1$ is the total number of blowups. All singularities are resolved at the $(p + 1)st$ blowup and $m_2 = \cdots = m_{p+1} = 2$. Therefore, by Proposition 6.1,

$$\begin{aligned} p_g(F) &= p_a(F) - \frac{1}{2}\sum_{i}^{p+1} m_i(m_i - 1) \\ &= \frac{1}{2}(2p + 1)(2p) - \frac{1}{2}\left(2p(2p - 1) + \sum_{i}^{p} 2(2 - 1)\right) \\ &= p. \end{aligned}$$

6.3. Genus Formulas for Curves on $\mathbb{P}^1 \times \mathbb{P}^1$

The genus of a nonsingular curve C in $\mathbb{P}^1 \times \mathbb{P}^1$ is completely determined by its bidegree in much the same way the genus of a plane curve is determined by its degree. If (n_1, n_2) is the bidegree of C, then C is linearly equivalent to $n_1(k) + n_2(l)$ where k and l are lines of bidegree $(1,0)$ and $(0,1)$, respectively. Therefore, the genus of C is

$$\begin{aligned} p_g(C) &= \frac{(n_1k + n_2l) \cdot ((n_1 - 2)k + (n_2 - 2)l)}{2} + 1 \\ &= (n_1 - 1)(n_2 - 1) \end{aligned}$$

because $K_{\mathbb{P}^1 \times \mathbb{P}^1} \sim -2(k) - 2(l)$. The arithmetic genus is calculated for any curve in $\mathbb{P}^1 \times \mathbb{P}^1$ in the same way and is

$$p_a(C) = (n_1 - 1)(n_2 - 1). \tag{6.5}$$

The singularities of a curve in $\mathbb{P}^1 \times \mathbb{P}^1$ cause the genus to be less than the arithmetic genus just as for plane curves and the genus of a singular curve C is

$$p_g(C) = p_a(C) - \kappa$$

for some correction term $\kappa > 0$ which is a function of the singularities of C. This correction term is calculated in the same way as for plane curves which is proven in Proposition 6.3.

The existence of curves on $\mathbb{P}^1 \times \mathbb{P}^1$ with a certain genus is not as restrictive as in $\mathbb{P}^2$. In fact, there is a nonsingular curve on $\mathbb{P}^1 \times \mathbb{P}^1$ of genus p for any nonnegative integer p. Let C be a nonsingular curve with bidegree $(p+1, 2)$ or $(2, p+1)$ and the genus of C is p.

6.4. Genus Formulas for curves on other 2-Dimensional Manifolds

The correction term κ in $p_g(C) = p_a(C) - \kappa$ is defined here for curves on any 2-dimensional complex manifold.

Proposition 6.3. *Let C be any irreducible curve on a 2-dimensional complex manifold X. Resolve the singularities of C with a series of blowups*

$$X_n \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X.$$

Then

$$p_g(C) = p_a(C) - \frac{1}{2} \sum m_{\mathbf{p}_i}(m_{\mathbf{p}_i} - 1)$$

where the sum extends over all the singularities of C in X and over all singularities of strict transforms of C in each X_i and $m_{\mathbf{p}_i}$ is the multiplicity of the singularity $\mathbf{p}_i$.

The proof is an induction on the number of blowups. The basis step of the induction proof is similar to the discussion after Proposition 6.1.

6.5. Genus Formulas for Plane Curves in $\mathbb{P}^3$

Although the curves mentioned in later sections will lie in 3-dimensional manifolds, the genus will always be calculated for each of these by finding a representation of those curves in a 2-dimensional manifold. Here we will develop a formula for the genus of plane curves in $\mathbb{P}^3$ by using the projection of the plane onto $\mathbb{P}^2$. First, however, we will introduce a couple of necessary definitions and a generalization of Bezout's Theorem for $\mathbb{P}^3$.

The *degree of a curve in* $\mathbb{P}^3$ is defined to be the number of points in the intersection of that curve and a general plane in $\mathbb{P}^3$. Also, a curve and a surface are said to meet *properly in* $\mathbb{P}^3$ if no component of the curve is a subset of any component of the surface.

Bezout's Theorem for $\mathbb{P}^3$. *If a curve C of degree n and a surface S of degree m meet properly in $\mathbb{P}^3$, then the number of points in the intersection $C \cap S$, counting multiplicities, is nm [Sh p. 198].*

Consider a curve C of degree k which is a subset of the plane H in $\mathbb{P}^3$. There is a homeomorphic map π from the plane H in $\mathbb{P}^3$ to $\mathbb{P}^2$. Under this map the image of the intersection of H and a general plane K onto $\mathbb{P}^2$ is a general line l in $\mathbb{P}^2$. Since C is of degree k in $\mathbb{P}^3$ there are k points, counting multiplicities, in $C \cap K$ in $\mathbb{P}^3$, and so there are k points, counting multiplicities, in $\pi(C) \cap l$ in $\mathbb{P}^2$. Therefore, $\pi(C)$ is a curve of degree k in $\mathbb{P}^2$. The genus of C and its homeomorphic image $\pi(C)$ are the same. If C is nonsingular,

$$p_g(C) = \frac{(k-1)(k-2)}{2}.$$

If C is singular,

$$p_g(C) = \frac{(k-1)(k-2)}{2} - \kappa$$

and we need only calculate the correction term κ. Of more use later,

$$p_a(C) = \frac{(k-1)(k-2)}{2} \tag{6.6}$$

regardless of the singularities of C.

Consider a general curve C of degree k in $\mathbb{P}^3$ and a homeomorphic map from some subset of $\mathbb{P}^3$ containing C to $\mathbb{P}^2$. In this case, if the image of C is a curve in $\mathbb{P}^2$, the degree of this curve need not be k. In other words, degree is *not* a topological invariant.

Example 6.1: Bezout's Theorem can be used to find the degree of the curve of intersection of 2 surfaces R and S. Suppose the implicit degrees of R and S are a and b, respectively. The degree of the curve $R \cap S$ in $\mathbb{P}^3$ is the number of points, counting multiplicities, in the intersection of $R \cap S$ and a general plane H. Now, $H \cap R$ is a plane curve in $\mathbb{P}^3$ and the degree of $H \cap R$ is equal to the number of points in $(H \cap R) \cap K$ where K is a general plane in $\mathbb{P}^3$. However, $(H \cap R) \cap K = (H \cap K) \cap R$ and $H \cap K$ is a general line. By Bezout's Theorem there are a, not necessarily distinct, points in $(H \cap K) \cap R$, so $H \cap R$ is of degree a. Now, $(R \cap S) \cap H = (H \cap R) \cap S$. Since $H \cap R$ is a curve of degree a and S is a surface of degree b, there are ab, not necessarily distinct, points in $(H \cap R) \cap S$. Therefore, the degree of the curve of intersection of R and S is ab.

6.6. Double Curves

A general surface in $\mathbb{P}^3$ has singularities which will complicate the calculation of the genus of the curve of intersection of that surface and a second surface. There are at most two types of singularities of a general rational surface in $\mathbb{P}^3$: double points along a curve of self-intersection called the *double curve of the surface,* and isolated triple points. There are no higher order singularities and the singularities that do exist are ordinary in the sense that the intersection of the surface with a general plane results in a curve with ordinary singularities[GH: pp. 611-618]. Later, we will only be interested in the general intersection of two rational surfaces, so we may assume the intersection curve does not contain the triple points of either surface. However, one surface necessarily meets the double curve of a second surface. The points on the curve of intersection which contain points on the double curve of one of the surfaces are double points on the curve of intersection and will decrease the genus of the curve of intersection.

The purpose of this section is to derive a formula for the degree of a double curve. From this degree and using Bezout's Theorem we will be able to count the number of double points on the curve of intersection of two surfaces which arise from the double curve of one of the surfaces.

6.6.1. *Degree of the Double Curve on a Triangular Surface.*

The degree of the double curve D of a triangular surface R is the number of points, counting multiplicities, in the intersection of that curve and a general plane H. But, this is also the number of points in the intersection of the plane section $R \cap H$ and the double curve D. A plane section $R \cap H$ will have singularities at all points where H meets the double curve of R, where H contains a triple point of R, and where H is tangent to R. However, a general plane does not contain any of the triple points of R and is not tangent to R and the singularities which come from the intersection of the $R \cap H$ with D will all be ordinary double points. Thus, to count the number of points in $D \cap H$, we need only count the number of double points in $R \cap H$. The genus of $R \cap H$ gives us a convenient way of doing this.

The plan is to calculate the geometric genus and arithmetic genus of the plane section and then solve the equation

$$p_g(R \cap H) = p_a(R \cap H) - \delta$$

for δ, the number of ordinary double points. If the parametric degree of a surface R is m and the ρ base points of R have multiplicities $k_1, \ldots, k_\rho$, then a plane section of R can be represented by the divisor

$$m(\pi^* h) - \sum_{\rho} k_i E_i$$

on a blowup of $\mathbb{P}^2$ (see Section 4.6). Therefore,

$$p_g(R \cap H) = \frac{1}{2}(m-1)(m-2) - \frac{1}{2}\sum_{\rho} k_i(k_i - 1)$$

by using Proposition 6.1 and the fact that geometric genus is a birational invariant. Notice that if the base points are simple, this formula reduces to

$$p_g(R \cap H) = \frac{1}{2}(m-1)(m-2).$$

From Section 6.5 we know that to calculate the arithmetic genus of the plane curve $R \cap H$ it suffices to know the degree of the plane section. The implicit degree of the surface R in $\mathbb{P}^3$ is $m^2 - \sum_\rho k_i^2$ from Proposition 5.7. The discussion of Section 4.1 showed that the number of points in the intersection of a general plane section of a rational surface and a general plane is equal to the implicit degree of the surface. This argument also show that the degree of the general plane section of a rational surface is the same as the implicit degree of the surface. Therefore, the degree of the plane section as a curve in $\mathbb{P}^3$ is $m^2 - \sum_\rho k_i^2$ and the arithmetic genus from (6.6) is

$$p_a(R \cap H) = \frac{1}{2}\left(m^2 - \sum_{\rho} k_i^2 - 1\right)\left(m^2 - \sum_{\rho} k_i^2 - 2\right).$$

Again, if the base points are simple, this formula reduces to

$$p_a(R \cap H) = \frac{1}{2}\left(m^2 - \rho - 1\right)\left(m^2 - \rho - 2\right).$$

Now, the number of double points on $R \cap H$, and hence, the degree of the double curve of R, is

$$\delta = \frac{1}{2}\left(m^2 - 1 - \sum_{\rho} k_i^2\right)\left(m^2 - 2 - \sum_{\rho} k_i^2\right) - \frac{1}{2}(m-1)(m-2) + \frac{1}{2}\sum_{\rho} k_i(k_i - 1)$$

for any set of base points or

$$\delta = \frac{1}{2}\left(m^2 - 1 - \rho\right)\left(m^2 - 2 - \rho\right) - \frac{1}{2}(m-1)(m-2) \tag{6.7}$$

if the base points are simple.

Note, that certain curves necessarily do not have a double curve. For instance, $\delta = 0$ if R is of parametric degree $m = 2$ and has 2 simple base points.

6.6.2. *Degree of the Double Curve on a Tensor Product Surface.*

Again, the number of double points on a plane section is equal to the degree of the double curve. The number of double points is

$$\delta = p_a(R \cap H) - p_g(R \cap H).$$

If the parametric bidegree of R is (m_1, m_2) and the ρ base points of R have multiplicities $k_1, \ldots, k_\rho$, then a plane section of R can be represented by the divisor

$$m_1(\pi^* k) + m_2(\pi^* l) - \sum_\rho k_i E_i$$

on some blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ (see Section 4.7). Thus,

$$p_g(R \cap H) = (m_1 - 1)(m_2 - 1) - \frac{1}{2}\sum_\rho k_i(k_i - 1)$$

for any base points and

$$p_g(R \cap H) = (m_1 - 1)(m_2 - 1)$$

if the base points are simple.

The degree of the plane curve $R \cap H$ in $\mathbb{P}^3$ is the same as the implicit degree of the surface R which is $2m_1m_2 - \sum_\rho k_i^2$ from Proposition 5.8. The arithmetic genus of $R \cap H$ in $\mathbb{P}^3$ is

$$p_a(R \cap H) = \frac{1}{2}\left(2m_1m_2 - \sum_\rho k_i^2 - 1\right)\left(2m_1m_2 - \sum_\rho k_i^2 - 2\right)$$

from (6.6).

Therefore, the degree of the double curve is

$$\delta = \frac{1}{2}\left(2m_1m_2 - \sum_{\rho} k_i^2 - 1\right)\left(2m_1m_2 - \sum_{\rho} k_i^2 - 2\right) - (m_1-1)(m_2-1) + \frac{1}{2}\sum_{\rho} k_i(k_i-1)$$

for any type of base points and

$$\delta = \frac{1}{2}(2m_1m_2 - \rho - 1)(2m_1m_2 - \rho - 2) - (m_1-1)(m_2-1) \tag{6.8}$$

for simple base points.

6.7. Genus of the Intersection of Triangular Surfaces

6.7.1. *General Intersection.*

Consider two general triangular surfaces R and S with only simple base points in $\mathbb{P}^3$ and the curve of intersection $C = R \cap S$. Let R be given by the parametric equations

$$(f_0 : f_1 : f_2 : f_3)$$

on $\mathbb{P}^2$ each of homogeneous degree m and with ρ simple base points $\mathbf{p}_1, \ldots, \mathbf{p}_\rho$. The implicit equation of R in $\mathbb{P}^3$ is a polynomial equation $F(X_0 : X_1 : X_2 : X_3) = 0$ with homogeneous degree $m^2 - \rho$ using the formulas in Chapter 5. Similarly, if S is given by the parametric equations

$$(g_0 : g_1 : g_2 : g_3)$$

on $\mathbb{P}^2$ each of homogeneous degree n and with σ simple base points $\mathbf{q}_1, \ldots, \mathbf{q}_\sigma$, then the implicit equation of S in $\mathbb{P}^3$ is a polynomial equation $G(X_0 : X_1 : X_2 : X_3) = 0$ with homogeneous degree $n^2 - \sigma$.

To calculate the genus of the curve of intersection $R \cap S$ we will first calculate the arithmetic genus and then the correction term κ from the singular points of the curve. The only singular points for the general curve of intersection are double points along the double curve of one of the surfaces. Other multiple points could occur if the surfaces were not general. For instance, the surfaces could be tangent to each other or there could be a triple point on one surface which lies on the curve of intersection. In general, these two events will not occur. Also, if the base points were not simple the singularities which arise from those base points would not necessarily be ordinary.

First, we will find a representation of $R \cap S$ in the parameter space of one of the surfaces in order to calculate the arithmetic genus of $R \cap S$. Substitute the parametric equations for S into the implicit equation for R to yield the homogeneous polynomial $F(g_0 : g_1 : g_2 : g_3) = 0$ of degree $n(m^2 - \rho)$ in $\mathbb{P}^2$. The divisor of this curve in $\mathbb{P}^2$ is $n(m^2 - \rho)h$ where h is any line in $\mathbb{P}^2$. However, S has base points and $\mathbb{P}^2$ does not serve as an appropriate parameter space for S and the polynomial $F(g_0 : g_1 : g_2 : g_3) = 0$ has multiple points at each base point $\mathbf{q}_1, \ldots, \mathbf{q}_\sigma$ of S. The multiplicity of these singularities depends only on the multiplicity of the base points and in general these singularities are ordinary. Let $\pi : X \longrightarrow \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ which removes all base points of S. Since $m_{\mathbf{q}_i}(g_j) = 1$ for all i and and at least one j and the homogeneous degree of F is $m^2 - \rho$, $m_{\mathbf{q}_i}(F(g_0 : g_1 : g_2 : g_3)) = m^2 - \rho$ for general R. Thus, the divisor of the intersection curve $\tilde{C}$ in X is

$$\tilde{C} = n(m^2 - \rho)h - \sum_{\sigma}(m^2 - \rho)E_i.$$

The arithmetic genus of the curve of intersection in the parameter space of S is

$$\begin{aligned} p_a(\tilde{C}) &= \frac{\tilde{C} \cdot (\tilde{C} - K_X)}{2} + 1 \\ &= \frac{1}{2}[n(m^2 - \rho) - 1][n(m^2 - \rho) - 2] - \frac{1}{2}\sigma(m^2 - \rho)(m^2 - \rho - 1) \end{aligned}$$

where $K_X \sim -3\pi^*(h) + \sum_\sigma E_i$.

The geometric genus of the curve of intersection is

$$p_g(\tilde{C}) = p_a(\tilde{C}) - \kappa$$

where κ is the correction due to the singularities of $\tilde{C}$. These singularities come only from the double curve of R. From (6.7), the double curve of R has degree

$$\delta = \frac{1}{2}\left(m^2 - 1 - \rho\right)\left(m^2 - 2 - \rho\right) - \frac{1}{2}(m-1)(m-2).$$

Using Bezout's Theorem, we know there are $(n^2 - \sigma)\delta$ points in the intersection of the surface S and the double curve of R. Each of these points is a double point on the curve of intersection. Thus, the correction term attributed to the double curve of R is

$$\kappa = (n^2 - \sigma)\delta.$$

Now, the geometric genus of the curve of intersection is

$$\begin{aligned} p_g(\tilde{C}) &= p_a(\tilde{C}) - \kappa \\ &= \frac{1}{2}[n(m^2 - \rho) - 1][n(m^2 - \rho) - 2] \\ &- \frac{1}{2}\sigma(m^2 - \rho)(m^2 - \rho - 1) \\ &- (n^2 - \sigma)\left[\frac{1}{2}\left(m^2 - 1 - \rho\right)\left(m^2 - 2 - \rho\right) - \frac{1}{2}(m-1)(m-2)\right] \end{aligned}$$

which simplifies to

$$p_g(\tilde{C}) = \frac{1}{2}[3(m^2 - \rho)(n^2 - \sigma) - 3m(n^2 - \sigma) - 3n(m^2 - \rho) + m^2n^2 - \rho\sigma + 2].$$

Although this calculation was done by finding a representation for the curve of intersection in the parameter space of one of the surfaces, it is clear from this symmetric form that the same result would be found by using the parameter space of the other surface.

6.7.2. *Assumptions.*

Let's review the assumptions made in this calculation. First of all, the curve of intersection was assumed to be irreducible which allows us to use the formulas presented in this chapter.

The correction term due to singularities depends on finding all singularities of the curve of intersection. Singularities occur at where the surfaces are tangent and where the curve of intersection meets singularities on a surface. It was assumed the surfaces were not tangent and the curve of intersection does not meet triple points of either curve. This narrowed the sources of singularities to one place: points on the double curve. Finally, all singularities were assume to be ordinary.

6.7.3. *Other Genus Formulas for Intersections.*

Suppose the surfaces above were tangent at some point. As long as this point does not coincide with any other singularities on the intersection curve it is an ordinary double point on the curve of intersection. Thus, for each such tangency, the genus of the curve of intersection is one less than in the general case.

Example 6.2: Let R and S be 2 general surfaces with parametric degree $m = n = 2$ and with 2 simple base points each. Each surface is a quadric in $\mathbb{P}^3$ and the intersection curve $R \cap S$ has degree 4 in $\mathbb{P}^3$. In Section 6.6.1 it was shown that neither R nor S has a double curve since the degree of the double curve is $\delta = 0$. The genus of the curve of intersection is $p_g(\tilde{C}) = 1$.

If R and S have one general point of tangency then $p_g(\tilde{C}) = 0$. Since $p_g(\tilde{C}) \geq 0$ there cannot be 2 points where R and S are tangent, at least not with an irreducible intersection curve. Actually, R and S can have 2 points of tangency but the intersection curve reduces to two curves of degree 2.

6.8. Genus of the Intersection of Tensor Product Surfaces

The formula for the intersection of two general tensor product surfaces is found in much the same way as for triangular surfaces with the same assumptions. Let R be a surface given by the parametric equations

$$(f_0 \colon f_1 \colon f_2 \colon f_3)$$

on $\mathbb{P}^1 \times \mathbb{P}^1$ each of bidegree (m_1, m_2) and with ρ simple base points $\mathbf{p}_1, \ldots, \mathbf{p}_\rho$. The implicit equation of R in $\mathbb{P}^3$ is a polynomial equation $F(X_0 \colon X_1 \colon X_2 \colon X_3) = 0$ with homogeneous degree $2m_1m_2 - \rho$ using the formulas in Chapter 5. Similarly, if S is given by the parametric equations

$$(g_0 \colon g_1 \colon g_2 \colon g_3)$$

on $\mathbb{P}^2$ each of bidegree (n_1, n_2) and with σ simple base points $\mathbf{q}_1, \ldots, \mathbf{q}_\sigma$, then the implicit equation of S in $\mathbb{P}^3$ is a polynomial equation $G(X_0 \colon X_1 \colon X_2 \colon X_3) = 0$ with homogeneous degree $2n_1n_2 - \sigma$.

First, we will find a representation of $C = R \cap S$ in the parameter space of S by substituting the parametric equations for S into the implicit equation for R to yield the polynomial $F(g_0 \colon g_1 \colon g_2 \colon g_3) = 0$ of bidegree $(n_1(2m_1m_2 - \rho), n_2(2m_1m_2 - \rho))$ in $\mathbb{P}^1 \times \mathbb{P}^1$. The divisor of this curve in $\mathbb{P}^2$ is $n_1(2m_1m_2 - \rho)k + n_2(2m_1m_2 - \rho)l$ where k and l are lines of bidegree $(1,0)$ and $(0,1)$, respectively. Again, $\mathbb{P}^1 \times \mathbb{P}^1$ does not serve as an appropriate parameter space because of the base points. Let $\pi : X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be the blow up which removes all the base points of S. The polynomial $F(g_0 \colon g_1 \colon g_2 \colon g_3) = 0$ has multiple points at each base point $\mathbf{q}_1, \ldots, \mathbf{q}_\sigma$ of S. The multiplicity of these singularities depends only on the multiplicity of the base points and in general these singularities are ordinary. Since $m_{\mathbf{q}_i}(g_j) = 1$ for all i and at least one j and the homogeneous degree of F is $2m_1m_2 - \rho$, $m_{\mathbf{q}_i}(F(g_0 \colon g_1 \colon g_2 \colon g_3)) = 2m_1m_2 - \rho$ for general R. The divisor of $\tilde{C}$ is

$$\tilde{C} = n_1(2m_1m_2 - \rho)k + n_2(2m_1m_2 - \rho)L - \sum_\sigma (2m_1m_2 - \rho)E_i.$$

The arithmetic genus of the curve of intersection in the parameter space of S is

$$\begin{aligned}p_a(\tilde{C}) =& \frac{\tilde{C}\cdot(\tilde{C}-K_X)}{2}+1\\ =& [n_1(2m_1m_2-\rho)-1][n_2(2m_1m_2-\rho)-1]\\ &-\frac{1}{2}\sigma(2m_1m_2-\rho)(2m_1m_2-\rho-1)\end{aligned}$$

where $K_X \sim -2(\pi^*k) - 2(\pi^*l) + \sum_\sigma E_i$.

The geometric genus of the curve of intersection is

$$p_g(\tilde{C}) = p_a(\tilde{C}) - \kappa$$

where κ is the correction due to ordinary singularities on $\tilde{C}$.

The polynomial $F(g_0{:}\, g_1{:}\, g_2{:}\, g_3) = 0$ has multiple points at each base point $\mathbf{q}_1, \ldots, \mathbf{q}_\sigma$ of S and the multiplicity of these singularities depends only on the multiplicity of the base points. Also, in general these singularities are ordinary. Since $m_{\mathbf{q}_i}(g_j) = 1$ for all i and at least one j and the homogeneous degree of F is $2m_1m_2 - \rho$, $m_{\mathbf{q}_i}(F(g_0{:}\, g_1{:}\, g_2{:}\, g_3)) = 2m_1m_2 - \rho$ for general R.

From (6.8), the double curve of R has degree

$$\delta = \frac{1}{2}(2m_1m_2-\rho-1)(2m_1m_2-\rho-2)-(m_1-1)(m_2-1).$$

Using Bezout's Theorem, we know there are $(2n_1n_2 - \sigma)\delta$ points in the intersection of the surface S and the double curve of R. Each of these points is a double point on the curve of intersection. Thus, the correction term attributed to the double curve of R is

$$\kappa = (2n_1n_2 - \sigma)\delta.$$

Now, the geometric genus of the curve of intersection is

$$\begin{aligned}p_g(\tilde{C}) &= p_a(\tilde{C}) - \kappa\\ &= [n_1(2m_1m_2-\rho)-1][n_2(2m_1m_2-\rho)-1]\\ &-\frac{1}{2}\sigma(2m_1m_2-\rho)(2m_1m_2-\rho-1)\\ &-(2n_1n_2-\sigma)\left[\frac{1}{2}(2m_1m_2-\rho-1)(2m_1m_2-\rho-2)-(m_1-1)(m_2-1)\right]\end{aligned}$$

which simplifies to

$$p_g(\tilde{C}) = 2m_1m_2n_1n_2$$

$$- (n_1+n_2)(2m_1m_2-\rho) - (m_1+m_2)(2n_1n_2-\sigma)$$

$$+ \frac{3}{2}(2n_1n_2-\sigma)(2m_1m_2-\rho) - \frac{1}{2}\rho\sigma + 1.$$

Again, this formula is symmetric in the parameters of the two surfaces.

6.9. Genus of Plane Sections

Using a development similar to that of the genus of general intersection curves, we can find formulas for the genus of general plane sections of triangular and tensor product surfaces. Actually, the formulas developed below are for the general intersection of a parametric surface with any implicit surface with no singular curve.

Let R be a general triangular surface with parametric representation

$$(f_0 \colon f_1 \colon f_2 \colon f_3),$$

parametric degree m, and ρ simple base points $\mathbf{p}_1, \ldots, \mathbf{p}_\rho$. Let S be an implicit surface with equation

$$G(X_0 \colon X_1 \colon X_2 \colon X_3) = 0$$

of degree n.

The intersection, $C = R \cap S$ of these surfaces can be represented by

$$G(f_0 \colon f_1 \colon f_2 \colon f_3) = 0$$

which is a homogeneous polynomial in $(x_0 \colon x_1 \colon x_2 \colon x_3)$ of degree mn. This curve in $\mathbb{P}^2$ has singularities at each of the base points of R of multiplicity n. If $\pi : X \longrightarrow \mathbb{P}^2$ is the blow up that removes all these singularities (which are assumed to be ordinary), then the divisor representing this curve in X is

$$\tilde{C} = mn\pi^*(h) - \sum_{i=1}^{\rho} nE_i.$$

Therefore, the arithmetic genus of $\tilde{C}$ is

$$\begin{aligned} p_a(\tilde{C}) =& \frac{\tilde{C}\cdot(\tilde{C}+K_X)}{2}+1 \\ =& \frac{1}{2}(mn-1)(mn-2)-\frac{1}{2}\rho n(n-1). \end{aligned}$$

If we assume that S has no singular curve, the blow up removes all singularities of this curve so

$$p_g(\tilde{C}) = p_a(\tilde{C}).$$

In the same way we can show that the genus of the general curve of intersection of a tensor product surface of parametric degree (m_1, m_2) with ρ simple base points with a surface of implicit degree n is

$$p_g(\tilde{C}) = (m_1 n - 1)(m_2 n - 1) - \frac{1}{2}\rho n(n-1).$$

From these two genus formulas, the genus of a general plane section of a triangular surface is

$$p_g = \frac{1}{2}(m-1)(m-2)$$

and the genus of a general plane section of a tensor product surface is

$$p_g = (m_1 - 1)(m_2 - 1).$$

6.10. Conclusion

The formulas for the genus of general curves of intersection for triangular and tensor product surfaces with simple base points are

$$p_g(\tilde{C}) = \frac{1}{2}[3(m^2-\rho)(n^2-\sigma) - 3m(n^2-\sigma) - 3n(m^2-\rho) + m^2n^2 - \rho\sigma + 2]$$

and

$$\begin{aligned} p_g(\tilde{C}) =& 2m_1m_2n_1n_2 \\ & - (n_1+n_2)(2m_1m_2-\rho) - (m_1+m_2)(2n_1n_2-\sigma) \\ & + \frac{3}{2}(2n_1n_2-\sigma)(2m_1m_2-\rho) - \frac{1}{2}\rho\sigma + 1 \end{aligned}$$

respectively.

TABLE 6.2

DEGREE AND GENUS OF THE INTERSECTION OF TWO TRIANGULAR SURFACES WITH ρ AND σ BASE POINTS, RESPECTIVELY

Parametric Degree of Each Surface	Curve of Intersection Degree	Genus
1	$(1-\rho)(1-\sigma)$	$(0-\rho)(0-\sigma)$
2	$(4-\rho)(4-\sigma)$	$(3-\rho)(3-\sigma)$
3	$(9-\rho)(9-\sigma)$	$(9-\rho)(9-\sigma)+1$
4	$(16-\rho)(16-\sigma)$	$(18-\rho)(18-\sigma)-3$

TABLE 6.3

DEGREE AND GENUS OF THE INTERSECTION OF TWO TENSOR PRODUCT SURFACES WITH ρ AND σ BASE POINTS, RESPECTIVELY

Parametric Bidegree of Each Surface	Curve of Intersection Degree	Genus
(1,1)	$(2-\rho)(2-\sigma)$	$(1-\rho)(1-\sigma)$
(2,2)	$(8-\rho)(8-\sigma)$	$(8-\rho)(8-\sigma)+1$
(3,3)	$(18-\rho)(18-\sigma)$	$(21-\rho)(21-\sigma)-8$
(4,4)	$(32-\rho)(32-\sigma)$	$(40-\rho)(40-\sigma)-63$

In Table 6.2 we can see that although the degree and the genus of curves of intersection for triangular surfaces both grow quickly with the parametric degree, the genus increases faster. For surfaces of parametric degree 1 and no base points, the implicit degree of the surface is 1 and the genus is 0. The implicit degree and genus are about the same when the parametric degree is 3 and the surfaces have no base points: the degree is 81 and the genus is 82. However, when the parametric degree is 4, the implicit degree is 256 and the genus is 321 when there are no base points.

The rate of growth of the genus with respect to degree is higher for tensor product surfaces. The degree and genus are about the same (64 and 65, respectively) when the bidegree is $(2, 2)$ and there are no base points, but when the bidegree is $(3, 3)$ the degree is 324 and the genus is 433 for surfaces with no base points. The genus of the curve of intersection for general tensor product surfaces of bidegree $(4, 4)$ with no base points is 1537. The degree and genus can be controlled by the introduction of base points. These formulas only indicate the effect of simple base points. Further reduction in degree and genus will result from base points of higher multiplicities.

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APPENDICES

APPENDIX A

PROOF OF EXISTENCE FOR THEOREM 2.2

In Section 2.6 a general definition was given for $i(\mathbf{p}, f \cap g)$ which was claimed to have properties (2.5) through (2.9) of Theorem 2.2. Here we prove this assertion.

Proof of 2.2(2.5): Obviously, the sets (f, g) and (g, f) are the same so

$$\dim \mathcal{O}_{\mathbf{p}}/(f, g) = \dim \mathcal{O}_{\mathbf{p}}/(g, f).$$

Proof of 2.2(2.6): There are three cases to consider: (i) $\mathbf{p} \notin f \cap gh$, (ii) f and gh intersect improperly at $\mathbf{p}$, and (iii) f and gh intersect properly at $\mathbf{p}$.

(i). If $\mathbf{p} \notin f \cap gh$ then either $\mathbf{p} \notin f$ or $\mathbf{p} \notin gh$. If $\mathbf{p} \notin f$ then $i(\mathbf{p}, f \cap g) = i(\mathbf{p}, f \cap h) = i(\mathbf{p}, f \cap gh) = 0$ by (2.7.1). On the other hand, if $\mathbf{p} \notin gh$, then $\mathbf{p} \notin g$ and $\mathbf{p} \notin h$ so again $i(\mathbf{p}, f \cap g) = i(\mathbf{p}, f \cap h) = i(\mathbf{p}, f \cap gh) = 0$. In either case $i(\mathbf{p}, f \cap g) + i(\mathbf{p}, f \cap h) = i(\mathbf{p}, f \cap gh)$.

(ii) If f and gh intersect improperly at $\mathbf{p}$, they have some common factor, l, containing $\mathbf{p}$. Now l divides at least one of g or h so f intersects one of g or h improperly. Either way both sides of $i(\mathbf{p}, f \cap g) + i(\mathbf{p}, f \cap h) = i(\mathbf{p}, f \cap gh)$ are equal to ∞ by (2.7.3).

(iii) Suppose f and gh intersect properly at $\mathbf{p}$. To establish

$$\dim(\mathcal{O}_{\mathbf{p}}/(f, gh)) = \dim(\mathcal{O}_{\mathbf{p}}/(f, g)) + \dim(\mathcal{O}_{\mathbf{p}}/(f, h))$$

we will use the linear maps

$$T : \mathcal{O}_{\mathbf{p}}/(f, h) \to \mathcal{O}_{\mathbf{p}}/(f, gh)$$

and

$$S : \mathcal{O}_{\mathbf{p}}/(f, gh) \to \mathcal{O}_{\mathbf{p}}/(f, g)$$

where the first map is defined by $T([r]) = [gr]$ and S is the natural map. We will show T is injective, hence, $\dim(\mathcal{O}_{\mathbf{p}}/(f,h)) = \dim(\mathrm{Im}(T))$ (2.6). We already know S is surjective and $\dim(\mathcal{O}_{\mathbf{p}}/(f,gh)) = \dim(\mathrm{Ker}(S)) + \dim(\mathcal{O}_{\mathbf{p}}/(f,g))$ (2.5). Finally, we will show $\mathrm{Im}(T) = \mathrm{Ker}(S)$.

T is injective: Suppose $T([\frac{r}{s}]) = [\frac{gr}{s}] = [0] \in \mathcal{O}_{\mathbf{p}}/(f,gh)$. If we show $\frac{r}{s} \in (f,h)$ then $[\frac{r}{s}] = [0] \in \mathcal{O}_{\mathbf{p}}/(f,h)$ and T is injective. So we need to show $\frac{r}{s} = a'f + b'h$ for some $a', b' \in \mathcal{O}_{\mathbf{p}}$. We know

$$\frac{gr}{s} = a(f) + b(gh) \tag{A.1}$$

for some $a, b \in \mathcal{O}_{\mathbf{p}}$. Thus $\frac{r}{s} = \frac{a}{g}(f) + b(h)$ and it suffices to show $\frac{a}{g} \in \mathcal{O}_{\mathbf{p}}$ or, in other words, $\frac{a}{g}$ is defined at $\mathbf{p}$.

Let's clear equation (A.1) of fractions. Choose q such that qa, qb, and $q\left(\frac{r}{s}\right)$ are polynomials and $q(\mathbf{p}) \neq 0$. Both sides of

$$q\left(\frac{gr}{s}\right) = q(af + bgh)$$

are polynomials and

$$qaf = g\left(\frac{qr}{s} - bh\right).$$

Since f and gh have no common factors, each factor of f cannot divide g so f must divide $\frac{qr}{s} - bh$. Let $df = \frac{qr}{s} - bh$ and then $gdf = qaf$ and $\frac{a}{g} = \frac{d}{q}$ which is defined at $\mathbf{p}$ because $q(\mathbf{p}) \neq 0$.

$\mathrm{Im}(T) = \mathrm{Ker}(S)$: Let $[\frac{gr}{s}] \in \mathrm{Im}(T)$. Then $S([\frac{gr}{s}]) = [\frac{gr}{s}] \in \mathcal{O}_{\mathbf{p}}/(f,g)$. But $\frac{gr}{s} = 0(f) + \frac{r}{s}(g) \in (f,g)$ so $S([\frac{gr}{s}]) = [0]$. Therefore, $\mathrm{Im}(T) \subset \mathrm{Ker}(S)$.

Conversely, let $S[\frac{r}{s}] = [0]$. So $\frac{r}{s} = af + bg$ for some $a, b \in \mathcal{O}_{\mathbf{p}}$. Since $\frac{r}{s} - bg = af + 0gh \in (f,gh)$, $[\frac{r}{s}] = [bg]$. But $[bg] = T[b] \in \mathrm{Im}(T)$. Therefore, $\mathrm{Im}(T) = \mathrm{Ker}(S)$.

Proof of 2.2(2.6): To prove $i(\mathbf{p}, f \cap g_1 + fg_2) = i(\mathbf{p}, f \cap g_1)$ we will prove $(f, g_1 + fg_2) =$

(f,g_1). Since $g_1 = (-g_2)f + 1(g_1 + fg_2) \in (f, g_1 + fg_2)$ and $f = (1)f + 0(g_1 + fg_2) \in (f, g_1 + fg_2)$ it is clear $(f,g_1) \subseteq (f, g_1 + fg_2)$. On the other hand, $f = (1)f + 0(g_1) \in (f, g_1)$ and $g_1 + fg_2 = (g_2)f + (1)g_1 \in (f,g_1)$ so $(f, g_1 + fg_2) \subseteq (f, g_1)$. Therefore, the two sets are the same.

Proof of 2.2(2.7.1): If $f(\mathbf{p}) \neq 0$ then $\mathbf{p} \notin f \cap g$ so suppose $i(\mathbf{p}, f \cap g) = 0$ and $f(\mathbf{p}) = 0$. Since $\dim \mathcal{O}_{\mathbf{p}}/(f,g) = 0$, $\mathcal{O}_{\mathbf{p}} = (f,g)$. In particular, $1 \in \mathcal{O}_{\mathbf{p}}$ so $1 \in (f,g)$ and there exist $a, b \in \mathcal{O}_{\mathbf{p}}$ with $af + bg = 1$. Now $(af + bg)(\mathbf{p}) = b(\mathbf{p})g(\mathbf{p}) = 1$ so $g(\mathbf{p})$ cannot be 0. Therefore, if $i(\mathbf{p}, f \cap g) = 0$, $\mathbf{p} \notin f \cap g$.

Conversely, suppose $\mathbf{p} \notin f \cap g$. Then one of $f(\mathbf{p}) \neq 0$ or $g(\mathbf{p}) \neq 0$. Assume, without loss of generality, $f(\mathbf{p}) \neq 0$. Thus, $\frac{1}{f} \in \mathcal{O}_{\mathbf{p}}$. For each $r \in \mathcal{O}_{\mathbf{p}}$, $r = \left(r\frac{1}{f}\right) f + 0(g) \in (f,g)$. Thus, $(f,g) = \mathcal{O}_{\mathbf{p}}$ and $\dim \mathcal{O}_{\mathbf{p}}/(f,g) = 0$.

Proof of 2.2(2.7.2): Since $i(\mathbf{p}, f \cap g)$ is either a nonnegative integer or ∞, to verify (2.7.2) it suffices to verify (2.7.1) and (2.7.3). If (2.7.1) and (2.7.3) are true $i(\mathbf{p}, f \cap g) = 0$ or ∞ if and only if $\mathbf{p} \notin f \cap g$ or f and g intersect improperly, so it follows that $i(\mathbf{p}, f \cap g)$ must be a positive integer otherwise.

Proof of 2.2(2.7.3): This is the trickiest property to prove. To do so we will create a chain of inequalities

$$\dim \mathcal{O}_{\mathbf{p}}/(f,g) \geq \dim \mathcal{O}_{\mathbf{p}}/(h) \geq \dim \mathbb{C}[x,y]/(h)$$

where h is a common factor of f and g containing $\mathbf{p}$. Finally, it will be shown that $\dim \mathbb{C}[x,y]/(h) = \infty$. Therefore we will have shown $i(\mathbf{p}, f \cap g) = \dim \mathcal{O}_{\mathbf{p}}/(f,g) = \infty$.

Let $f = f_1 h$ and $g = g_1 h$ with $\mathbf{p} \in h$. If $af + bg \in (f,g)$, then $af + bg = (af_1 + bg_1)h \in (h)$. Therefore, $(f,g) \subseteq (h)$ and $\dim \mathcal{O}_{\mathbf{p}}/(f,g) \geq \dim \mathcal{O}_{\mathbf{p}}/(h)$ (2.5).

Define a map $T : \mathbb{C}[x,y]/(h) \to \mathcal{O}_{\mathbf{p}}/(h)$ by $T([r]) = [r]$. Note the different meanings of not only $[r]$ in the definition of T but also of (h). In Section 2.5.2 it was shown that

T is a well-defined injective linear map since $(h) \subseteq \mathbb{C}[x,y] \subseteq \mathcal{O}_{\mathbf{p}}$. Thus $\dim \mathcal{O}_{\mathbf{p}}/(h) \geq \dim \mathbb{C}[x,y]/(h)$.

We will now show that $\dim \mathbb{C}[x,y]/(h) = \infty$ by demonstrating a linearly independent set with r elements for all $r \in \mathbf{N}$. Since h is a nonzero polynomial in 2 variables, h has infinitely many zeros in $\mathbb{C}^2$. Fix $r \in \mathbf{N}$ and choose r distinct points $\mathbf{p}_i \in h$. Choose r polynomials $h_i \in \mathbb{C}[x,y]$ such that $h_i(\mathbf{p}_i) = 1$ and $h_i(\mathbf{p}_j) = 0$ whenever $i \neq j$. Note that $[h_i] \neq [h_j] \in \mathcal{O}_{\mathbf{p}}/(h)$ for $i \neq j$. For, if $h_i - h_j \in (h)$, there is some $a \in \mathcal{O}_{\mathbf{p}}$ with $h_i - h_j = ah$ and $1 = (h_i - h_j)(\mathbf{p}_i) \neq (ah)(p_i) = 0$. Thus, the $[h_i]$ are r distinct elements in $\mathcal{O}_{\mathbf{p}}/(h)$. Suppose $\sum \lambda_i[h_i] = [\sum \lambda_i h_i] = [0] \in \mathbb{C}[x,y]/(h)$. Then $\sum \lambda_i h_i \in (h)$. But $\lambda_i = \sum \lambda_i h_i(\mathbf{p}_i) = 0$ for each i since $\sum \lambda_i h_i = ah$ for some $a \in \mathbb{C}[x,y]$. Thus, the $\{h_i\}$ are linearly independent in $\mathbb{C}[x,y]/(h)$ and $\dim \mathbb{C}[x,y]/(h) \geq r$ for all $r \in \mathbf{N}$. Therefore, $\dim \mathbb{C}[x,y]/(h) = \infty$.

Proof of 2.2(2.8): This was proven in Example 2.2.

Proof of 2.2(2.9): This is a special case of the argument in Section 2.7.

APPENDIX B

PROOFS OF PROPOSITIONS IN CHAPTER FIVE

These are the proofs to Propositions 5.5, 5.6 and 5.8.

Proof of 5.5(1): Recall that the map π^* preserves equivalence classes. If $h' \sim h$ and $h'' \sim h$, then

$$(\pi^* h)^2 = (\pi^* h') \cdot (\pi^* h'')$$

by 5.4(2). Let h be any line in $\mathbb{P}^2$. Choose h' and h'' such that $h' \neq h''$ and neither contains the point $\mathbf{p}$. There is exactly one point in the intersection of h' and h'' in $\mathbb{P}^2$ and so there must also be exactly one point in the intersection of the curves $\pi^* h'$ and $\pi^* h''$ in X. Therefore, by 5.4(4),

$$(\pi^* h)^2 = 1.$$

Proof of 5.5(2): For any 2 lines h and h' in $\mathbb{P}^2$, $h \sim h'$ and $\pi^* h \sim \pi^* h'$. Choose h' a line in $\mathbb{P}^2$ such that h' does not contain $\mathbf{p}$. The curve $\pi^* h'$ is simply the strict transform of h' and does not meet the exceptional curve E. Thus

$$(\pi^* h) \cdot E = (\pi^* h') \cdot E = 0.$$

Proof of 5.5(3): Follows from 5.6(3).

Proof of 5.6(1): Let $C = \sum_{i=1}^{r} n_i C_i$ and $D = \sum_{i=1}^{s} m_i D_i$ where each C_i and D_i is a prime divisor. Then

$$C \cdot D = \sum_{i,j} n_i m_j (C_i \cdot D_j)$$

and, since π^* is linear,

$$(\pi^*C)\cdot(\pi^*D)=\sum_{i,j} n_i m_j[(\pi^*C_i)\cdot(\pi^*D_j)].$$

Thus, it suffices to verify this property for prime divisors C and D.

Let C and D be prime divisors. Choose $n>0$ large enough and $H\in \operatorname{Div}(X)$ so that $|C+nH|$, $|D+nH|$, and $|nH|$ have no base points. Let $C'\in|C+nH|$ so that $\mathbf{p}_i\notin C'$ for any i. Choose $D'\in|D+nH|$ meeting C' properly and $\mathbf{p}_i\notin D'$ for any i. Choose $E'\in|nH|$ meeting D' properly and $F'\in|nH|$ meeting C' and E' properly and $\mathbf{p}_i\notin E'$ and $\mathbf{p}_i\notin F'$ for any i. Now $C\sim C'-E'$ and $D\sim D'-F'$ and

$$\begin{aligned}C\cdot D&=(C'-E')\cdot(D'-F')\\&=C'\cdot D'-C'\cdot F'-E'\cdot D'+E'\cdot F'\\&=\sum_{\mathbf{p}\in C'\cap D'} i(\mathbf{p},C'\cap D')-\sum_{\mathbf{p}\in C'\cap F'} i(\mathbf{p},C'\cap F')-\\&\quad\sum_{\mathbf{p}\in E'\cap D'} i(\mathbf{p},E'\cap D')+\sum_{\mathbf{p}\in E'\cap F'} i(\mathbf{p},E'\cap F')\end{aligned}$$

by Theorem 5.4. But C', D', E', and F' contain none of the points which were blown up. Since $X-\{\mathbf{p}_i\}$ is homeomorphic to $\tilde{X}-\{E_i\}$, $C'\cap D'$, $C'\cap F'$, $E'\cap D'$, and $E'\cap F'$, have the same number of points counting multiplicities as $(\pi^*C')\cap(\pi^*D')$, $(\pi^*C')\cap(\pi^*F')$, $(\pi^*E')\cap(\pi^*D')$, and $(\pi^*E')\cap(\pi^*F')$, respectively. Thus,

$$\begin{aligned}C\cdot D=&\sum_{\mathbf{p}\in(\pi^*C')\cap(\pi^*D')} i(\mathbf{p},(\pi^*C')\cap(\pi^*D'))-\sum_{\mathbf{p}\in(\pi^*C')\cap(\pi^*F')} i(\mathbf{p},(\pi^*C')\cap(\pi^*F'))-\\&\sum_{\mathbf{p}\in(\pi^*E')\cap(\pi^*D')} i(\mathbf{p},(\pi^*E')\cap(\pi^*D'))+\sum_{\mathbf{p}\in(\pi^*E')\cap(\pi^*F')} i(\mathbf{p},(\pi^*E')\cap(\pi^*F'))\\&=(\pi^*C')\cdot(\pi^*D')-(\pi^*C')\cdot(\pi^*F')-(\pi^*E')\cdot(\pi^*D')+(\pi^*E')\cdot(\pi^*F')\\&=((\pi^*C')-(\pi^*E'))\cdot((\pi^*D')-(\pi^*F'))\\&=(\pi^*(C'-E'))\cdot(\pi^*(D'-F'))\\&=(\pi^*C)\cdot(\pi^*D)\end{aligned}$$

by Theorem 5.4 and the fact that the pullback map preserves linear equivalence class. Therefore,

$$C \cdot D = (\pi^* C) \cdot (\pi^* D)$$

for all divisors C and D.

Proof of 5.6(2): This works very much like the 5.6(4). First of all

$$(\pi^* C) \cdot E_i = \sum_j m_j (\pi^* C_j) \cdot E_i$$

where $C = \sum_j m_j C_j$ with C_j prime so we may assume C is prime. Choose $n > 0$ large enough and $H \in \text{Div}(X)$ so that $|C+nH|$ and $|nH|$ have no base points. Let $C' \in |C+nH|$ so that $\mathbf{p}_i \notin C'$ for any i. Choose $D' \in |nH|$ so that $\mathbf{p}_i \notin D'$ for any i. Now $C \sim C' - D'$ and

$$\begin{aligned}(\pi^* C) \cdot E_i &= \pi^*(C' - D') \cdot E_i \\ &= (\pi^* C') \cdot E_i - (\pi^* D') \cdot E_i \\ &= \sum_{\mathbf{p} \in (\pi^* C') \cap E_i} i(\mathbf{p}, (\pi^* C') \cap E_i) - \sum_{\mathbf{p} \in (\pi^* D') \cap E_i} i(\mathbf{p}, (\pi^* D') \cap E_i) \\ &= 0\end{aligned}$$

since $\pi^* C'$ and $\pi^* D'$ do not meet any E_i.

Proof of 5.6(3): Let $\mathbf{p}_i \in U_i \subseteq X$ and the local coordinates of U_i be (x, y). We may assume $\mathbf{p}_i = (0,0)$ in the local coordinates. Further assume that U_i does not contain $\mathbf{p}_j$ and no U_j contains $\mathbf{p}_i$ for $j \neq i$. In $\tilde{X}$ the exceptional divisor is defined by

$$\{(U_{i0}, y_0), (U_{i1}, z_0)\} \cup \{(U_j, 1) : j \neq i\}.$$

To find another divisor linearly equivalent to E we will find another meromorphic section of the line bundle E as in Example 4.5. By putting $s_{i0} = 1$ we get the section

$$E' = \{(U_{i0}, 1), (U_{i1}, 1/t)\} \cup \{(U_j, s_j) : j \neq i\}.$$

$E_i^2 = E_i \cdot E' = \deg[E']|_{E_i}$ On U_{i0} the function $s_{00} = 1$ has no poles or zeros; on U_{i1} the function $s_{01} = 1/t$ has a pole of order 1 at $t = 0$ on E_i which is the point $\mathbf{q} = (x, t) = (0, 0)$; the curve E_i does not intersect U_i for $j \neq i$ therefore E' does not have any poles or zeros on E_i in $\cup_{j \neq i} U_j$. Therefore, $E_i^2 = \deg[-1(\mathbf{q})] = -1$.

Proof of 5.6(4): We can apply Proposition 5.4(4) here and recall that in Example 3.5 we showed that $E_i \cap E_j = \emptyset$. Therefore,

$$E_i \cdot E_j = \sum_{\mathbf{p} \in E_i \cap E_j} i(\mathbf{p}, E_i \cap E_j) = 0.$$

Proof of 5.8(1): Choose $k', k'' \in [k]$ such that neither contains $\mathbf{p}$. Following the proof of 5.5(1)

$$(\pi^* k)^2 = (\pi^* k') \cdot (\pi^* k'') = 0$$

since k' and k'' do not meet in $\mathbb{P}^1 \times \mathbb{P}^1$. Similarly, $(\pi^* l)^2 = 0$.

Proof of 5.8(2): Choose $k' \in [k]$ and $l' \in [l]$ such that neither contains $\mathbf{p}$. Now k' and l' meet at exactly one point so

$$(\pi^* k) \cdot (\pi^* l) = (\pi^* k') \cdot (\pi^* l') = 1.$$

Proof of 5.8(3) and 5.8(4): Follow from 5.6(2) and 5.6(3), respectively.

APPENDIX C

EXAMPLES OF TRIANGULAR AND TENSOR PRODUCT SURFACES

In *Techniques for Cubic Algebraic Surfaces*[S] there is an example which begins with an implicit equation of a surface in $\mathbb{C}^3$, finds a parameterization on $\mathbb{C}^2$, and finally shows there are 2 base points on the surface. Here this example is explored in much more (gory) detail. First, the surface is considered as a triangular surface $\psi : \mathbb{P}^2 - - \rightarrow \mathbb{P}^3$ and later as a tensor product surface $\phi : \mathbb{P}^1 \times \mathbb{P}^1 - - \rightarrow \mathbb{P}^3$. The first example involves a linear change of coordinates to process a point before blowing up, and the second blows-up a point other than the origin of $\mathbb{C}^2$. Both examples require the blow up of more than one point simultaneously.

C.1. The Parameterization

Consider the surface in $\mathbb{C}^3$ with implicit equation

$$x^2 + y^2 + xy - z^2 - x - y + z = 0.$$

To find a parameterization , make the substitutions $x = sz$ and $y = tz$. This yields

$$z^2(s^2 + t^2 + st - 1) - z(s + t - 1) = 0$$

from which we get

$$x = \frac{s(s+t-1)}{s^2+t^2+st-1},$$
$$y = \frac{t(s+t-1)}{s^2+t^2+st-1}, \text{ and}$$
$$z = \frac{s+t-1}{s^2+t^2+st-1}.$$

Let the homogeneous coordinates of $\mathbb{P}^2$ be $(s:t:u)$ and of $\mathbb{P}^3$ be $(X:Y:Z:W)$. Put $W = s^2+t^2+st-1$, $X = xW$, $Y = yW$, and $Z = zW$, and homogenize each of these to get

$$X = s(s+t-u),$$

$$Y = t(s+t-u),$$

$$Z = u(s+t-u), \text{ and}$$

$$W = s^2+t^2+st-u^2.$$

It is easy to check these equations satisfy

$$X^2+Y^2+XY-Z^2-W(X+Y-Z)=0 \tag{C.1}$$

which is the homogeneous implicit equation of the surface. Now the triangular surface is the closure of the image of $\psi(s:t:u) = (X:Y:Z:W)$.

For the tensor product parameterization, let the coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$ be $(s_0:s_1;t_0:t_1)$. Homogenize xW, yW, zW and W above so they all have bidegree $(2,2)$ to get

$$X' = s_0t_1(s_0t_1+s_1t_0-s_1t_1),$$

$$Y' = s_1t_0(s_0t_1+s_1t_0-s_1t_1),$$

$$Z' = s_1t_1(s_0t_1+s_1t_0-s_1t_1),$$

$$W' = s_0^2t_1^2+s_1^2t_0^2+s_0t_0s_1t_1-s_1^2t_1^2.$$

Again these equations satisfy (C.1) and the closure of the image of $\phi(s_0:s_1;t_0:t_1) = (X':Y':Z':W')$ is the tensor product surface.

C.2. The Triangular Surface

By setting each of X, Y, Z, and W to 0 we find ψ has base points at $\mathbf{p}_1 = (0:1:1)$ and $\mathbf{p}_2 = (1:0:1)$. This in inconvenient since $\mathbf{p}_1 \in U_1 \cap U_2$ and $\mathbf{p}_2 \in U_0 \cap U_2$. There are two ways this problem can be solved. One is to simply use the cover $\{U_0, U_1, U_2 - \{\mathbf{p}_1, \mathbf{p}_2\}\}$ instead of $\{U_0, U_1, U_2\}$. The other is to use a linear change of coordinates on $\mathbb{P}^2$ such that the base points are more conveniently located. We will do the latter using $T(s\colon t\colon u) = (t+u\colon s+u\colon s+t)$. Now

$$\begin{aligned} X \circ T &= 2u(t+u), \\ Y \circ T &= 2u(s+u), \\ Z \circ T &= 2u(s+t), \text{ and} \\ W \circ T &= (t+u)^2 + (s+u)^2 + (t+u)(s+u) - (s+t)^2 \\ &= 3u^2 + 3u(t+s) - ts. \end{aligned}$$

The base points can either be recalculated directly from the new parametric equations or by noting

$$\begin{aligned} T(\mathbf{q}_1) &= T(1\colon 0\colon 0) = \mathbf{p}_1, \text{ and} \\ T(\mathbf{q}_2) &= T(0\colon 1\colon 0) = \mathbf{p}_2. \end{aligned}$$

The local parameterization of the surface is

$$\begin{aligned} &\{(U_0,(2u(t+u), 2u(1+u), 2u(1+t), 3u^2 + 3u(t+1) - t)), \\ &(U_1,(2u(1+u), 2u(s+u), 2u(s+1), 3u^2 + 3u(s+1) - s)), \\ &(U_2,(2(t+1), 2(s+1), 2(s+t), 3 + 3(t+s) - ts)))\}. \end{aligned}$$

Use coordinates $(t, u; a\colon b)$ for $\tilde{U}_0$ and $(s, u; c\colon d)$ for $\tilde{U}_1$ and blowup at $\mathbf{q}_1$ and $\mathbf{q}_2$. After the blow up the local parameterization is

$$\begin{aligned} &\{(U_{00},(2tb(1+b), 2b(1+tb), 2b(1+t), 3tb + 3tb^2 + 3b - 1)), \\ &(U_{01},(2u(a+1), 2(1+u), 2(1+ua), 3ua + 3u + 3 - a)), \\ &(U_{10},(2d(1+sd), 2sd(1+d), 2d(s+1), 3d + 3sd^2 + 3sd - 1)), \\ &(U_{11},(2(1+u), 2u(c+1), 2(uc+1), 3 + 3u + 3uc - c)), \\ &(U_2,(2(t+1), 2(s+1), 2(s+t), 3s + 3t - st + 3))\}. \end{aligned}$$

There are no more base points and the multiplicity of each of the base points is 1. Therefore the implicit degree of this surface is $n^2 - \sum m_i^2 = 2^2 - 1^2 - 1^2 = 2$ as was already known.

C.3. The Tensor Product Surface

Before we calculate the base points, let's see how many there have to be. The bidegree of the parameterization is $(n_1, n_2) = (2, 2)$ and the degree of the surface is 2 so

$$2 = 2n_1n_2 - \sum_{i=1}^{s} m_i^2 = 8 - \sum_{i=1}^{s} m_i^2.$$

The only two ways $\sum_{i=1}^{s} m_i^2$ can be 6 is for $m_i = 1$ for $i = 1, \ldots, 6$ or for $m_1 = 1$, $m_2 = 1$, and $m_3 = 2$. (Ouch!) Thus there are either six simple base points or 2 simple base points and a double base point. After some tedious arithmetic we find 3 base points at $\mathbf{p}_1 = (0{:}1; 1{:}1)$, $\mathbf{p}_2 = (1{:}0; 1{:}0)$, and $\mathbf{p}_3 = (1{:}1; 0{:}1)$. Again, these are inconveniently located but this time the problem will be solved by using the cover

$$\{W_{00}, W_{01}, W_{10}, W_{11} - \{\mathbf{p}_1, \mathbf{p}_3\}\}.$$

There is still a slight inconvenience. In W_{00}, $\mathbf{p}_2$ is locally the origin, but $\mathbf{p}_1 \in W_{10}$ and $\mathbf{p}_3 \in W_{01}$ are represented locally by $(0, 1)$ and $(1, 0)$, respectively. This will change the process of blowing up these points slightly. The local parameterization of ϕ is

$$\{(W_{00}, (t_1(t_1 + s_1 - s_1t_1){:}\, s_1(t_1 + s_1 - s_1t_1); s_1t_1(t_1 + s_1 - s_1t_1){:}\, t_1^2 + s_1^2 + s_1t_1 - s_1^2t_1^2)),$$

$$(W_{01}, (1 + s_1t_0 - s_1{:}\, s_1t_0(1 + s_1t_0 - s_1); s_1(1 + s_1t_0 - s_1){:}\, 1 + s_1^2t_0^2 + s_1t_0 - s_1^2))$$

$$(W_{10}, (s_0t_1(s_0t_1 + 1 - t_1){:}\, s_0t_1 + 1 - t_1; t_1(s_0t_1 + 1 - t_1){:}\, s_0^2t_1^2 + 1 + s_0t_1 - t_1^2))$$

$$(W_{11} - \{\mathbf{p}_1, \mathbf{p}_2\}, (s_0(s_0 + t_0 - 1){:}\, t_0(s_0 + t_0 - 1); s_0 + t_0 - 1{:}\, s_0^2 + t_0^2 + s_0t_0 - 1))\}.$$

Let the global coordinates of $\tilde{W}_{00}$ be $(s_1, t_1; a{:}b)$, global coordinates of $\tilde{W}_{01}$ be $(s_1, t_0; c{:}d)$, and global coordinates of $\tilde{W}_{10}$ be $(s_0, t_1; e{:}f)$. To blow up $\mathbf{p}_1$ and $\mathbf{p}_3$, the surfaces $\tilde{W}_{01} \subseteq W_{01} \times \mathbb{C}$ and $\tilde{W}_{10} \subseteq W_{10} \times \mathbb{C}$ will be defined by

$$(s_1 - 1)d = t_0c$$

and

$$s_0f = (t_1 - 1)e,$$

respectively, and the exceptional curves are

$$\{((W_{01})_0, s_1 - 1), ((W_{01})_1, t_0)\} \subseteq \tilde{W}_{01}$$

and

$$\{((W_{10})_0, s_o), ((W_{10})_1, t_1 - 1)\} \subseteq \tilde{W}_{10}.$$

After blowing up the 3 base points the local parameterization of the surface is

$$\{((W_{00})_0,(b(b+1-s_1b): b+1-s_1b; s_1b(b+1-s_1b): b^2+1+b-s_1^2b^2)),$$

$$((W_{00})_1,(1+a-at_1: a(1+a-at_1); at_1(1+a-at_1): 1+a^2+a-a^2t_1^2)),$$

$$((W_{01})_0,(s_1d-1: s_1d(s_1-1)(s_1d-1); s_1(s_1d-1): s_1^2d^2(s_1-1)+s_1d-s_1-1)),$$

$$((W_{01})_1,(ct_0-c+1: t_0(t_0c+1)(ct_0-c+1);$$

$$(t_0c+1)(ct_0-c+1): (t_0c+1)^2t_0-t_0c^2+t_0c-2c+1)),$$

$$((W_{10})_0,(s_0(s_0f+1)(s_0f+1-f): s_0f+1-f;$$

$$(s_0f+1)(s_0f+1-f): s_0(s_0f+1)^2+s_0f+1-s_0f^2-2f)),$$

$$((W_{10})_1,(t_1e(t_1-1)(t_1e-1): t_1e-1; t_1(t_1e-1): e^2t_1^2(t_1-1)-t_1+t_1e-1)),$$

$$(W_{11}-\{\mathbf{p}_1,\mathbf{p}_2\},(s_0(s_0+t_0-1): t_0(s_0+t_0-1); s_0+t_0-1: s_0^2+t_0^2+s_0t_0-1))\}.$$

Fortunately, $m_1 = 1$, $m_2 = 2$ and $m_3 = 1$ and there are no more base points. Thus, the implicit degree is $8 - 2^2 - 1^2 - 1^2 = 2$ as expected.

APPENDIX D

THE 27 LINES ON A CUBIC SURFACE

Divisors are used for a great many other applications. One of those is presented here. Given a cubic surface in $\mathbb{P}^3$ we want to show there are exactly 27 lines on the surface. To do this we need to come up with a cubic surface and then find the 27 lines.

Let $(f_0 : f_1 : f_2 : f_3)$ be a triangular surface S on $\mathbb{P}^3$ with parametric degree 3. Also let S have 6 base points $\mathbf{p}_1, \ldots, \mathbf{p}_6$ each with multiplicity 1 such that no 3 of the base points are colinear and the 6 base points do not all lie on a conic curve in $\mathbb{P}^2$. The implicit degree of the surface S is $3^2 - \sum_{i=1}^{6} 1^2 = 3$ from Proposition 5.4. It was stated in Section 1.8 that all nonsingular cubic surfaces in $\mathbb{P}^3$ can be defined this way. The claim here is that there are at least 27 lines on such a surface.

Let $\pi : X \longrightarrow \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at $\mathbf{p}_1, \ldots, \mathbf{p}_6$ and $E_1, \ldots, E_6$ the exceptional curves. Now define a map $\psi' : X \longrightarrow \mathbb{P}^3$ with $\mathrm{Im}\,(\psi') = S$.

By Bezout's Theorem, the intersection of a line and a general plane in $\mathbb{P}^3$ has exactly one point. Let C be any curve on a cubic surface S in $\mathbb{P}^3$. Then the number of points in $C \cap H$, counting multiplicities, is the same as the number of points in C intersect the plane section $S \cap H$. Thus C is a line if and only if $C \cap (S \cap H)$ has exactly one point, counting multiplicities. Let D_C be the divisor in X which represents C and D_H the divisor in X which represents the plane section of S. Then $D_C \cdot D_H = 1$ if and only if C is a line on S.

D.1. The 27 Lines

First, let us identify these lines by their divisors. Consider the following curves in $\mathbb{P}^2$.

(1) The sets $\psi'(E_i)$ are curves in $\mathbb{P}^3$. Identify these three six curves by e_i.

(2) Let $\mathbf{p}_i$ and $\mathbf{p}_j$ be two base points and L_{ij} the line containing them in $\mathbb{P}^2$. There are 15 such lines. The strict transform $\tilde{L}_{ij}$ of each is a curve on X and $\psi'(\tilde{L}_{ij})$ is a curve on $\mathbb{P}^3$. Identify these 15 curves in $\mathbb{P}^3$ by l_{ij}.

(3) Any 5 points completely determine a conic curve in $\mathbb{P}^2$. Let C_i be the conic in $\mathbb{P}^2$ which passes through all $\mathbf{p}_j$ except $\mathbf{p}_i$. Again, the strict transform $\tilde{C}_i$ is a curve in X and $\psi'(\tilde{C}_i)$ is a curve in $\mathbb{P}^3$. There are six such curves and we will call them c_i.

The inverse image of each of these curves under ψ' is

$$(\psi')^{-1}(e_i) = E_i,$$

$$(\psi')^{-1}(l_{ij}) = \tilde{L}_{ij}, \text{ and}$$

$$(\psi')^{-1}(c_i) = \tilde{C}_i.$$

Thus each of these curves in $\mathbb{P}^3$ on S can be represented by curves in X, hence have divisors in Div (X). The curve L_{ij} is the line in $\mathbb{P}^2$ which contains $\mathbf{p}_i$ and $\mathbf{p}_j$, so $\tilde{L}_{ij} \sim \pi^*h - E_i - E_j$ where h is any line in $\mathbb{P}^2$. Similary $\tilde{C}_i \sim 2\pi^*h - \sum_{j\neq i} E_j$ since $C_i \sim 2h$.

To show e_i, l_{ij}, and c_i are all lines we also need the divisor in Div (X) for the plane section $S \cap H$ where H is a plane in $\mathbb{P}^3$. In chapter 4 we found this divisor to be linearly equivalent to $3\pi^*h - \sum_{k=1}^{6} E_k$.

Now test each curve to see if it is a line:

(1) $E_i \cdot \left(3\pi^*h - \sum_{k=1}^{6} E_k\right) = -E_i^2 = 1;$

(2) $(\pi^*h - E_i - E_j) \cdot \left(3\pi^*h - \sum_{k=1}^{6} E_k\right) = 3(\pi^*h)^2 + E_i^2 + E_j^2 = 1;$ and

(3) $\left(2\pi^*h - \sum_{j\neq i} E_j\right) \cdot \left(3\pi^*h - \sum_{k=1}^{6} E_k\right) = 6(\pi^*h)^2 + \sum_{j\neq i} E_j^2 = 1.$

Therefore these 27 curves are lines.

D.2. No Other Lines on S

A line is a smooth irreducible curve so we will start with C, a smooth irreducible curve on S. Further assume that $C \neq E_i$ for any $i = 1, \ldots, 6$. We will proceed by first finding an effective divisor D_C on X which represents C. The genus of a line L in $\mathbb{P}^3$ is 0 because L is homeomorphic to $\mathbb{P}^1$. Thus, if C is a line, $p_g(C) = p_g(D_C)$ must be 0. Also, if C is a line, then the intersection of C and a general plane must contain exactly one point by Bezout's Theorem. Thus the intersection number $D_C \cdot H$ must be 1 for a line C. Using these two conditions, we will show that C must be represented by $\tilde{L}_{ij} \sim (\pi^*h) - E_i - E_j$ or $\tilde{C}_i \sim 2(\pi^*h) - \sum_{j\neq i} E_j$.

The curve $\psi^{-1}(C)$ is represented in X by the divisor $D_C = n\pi^*h - \sum_i m_i E_i$ for h any line in $\mathbb{P}^2$, $n > 0$, and $m_i \geq 0$ (see Section 4.5). The genus of the smooth curve C is

$$p_g(C) = \frac{D_C \cdot (D_C + K_X)}{2} - 1$$

where

$$K_X \sim -3h + \sum_{i=1}^{6} E_i.$$

The divisor of a general plane section of S is

$$H \sim 3h - \sum_{i=1}^{6} E_i.$$

Note $K_X \sim -H$. This is only true because the multiplicity of each base point is 1. For C to be a line $p_g(C) = 0$ and $D_C \cdot H = 1$. Applying these equations to g above we get

$$\frac{D_C^2 - D_C \cdot H}{2} - 1 = \frac{D_C^2 - 1}{2} - 1 = 0.$$

Therefore, $D_C^2 = -1$, i.e., only curves with self intersection -1 can be lines on S.

Look at the 2 intersection numbers

$$D_C^2 = n^2 - \sum m_i^2 = -1 \tag{D.1}$$

and

$$D_C \cdot H = 3n - \sum m_i = 1. \tag{D.2}$$

Recall Schwarz's inequality which says if $x_1, x_2, \ldots, x_n$ are $y_1, y_2, \ldots, y_n$ and 2 sequences of real numbers, then

$$\left|\sum x_i y_i\right|^2 \leq \left|\sum x_i^2\right| \left|\sum y_i^2\right|.$$

Letting $x_i = 1$ and $y_i = m_i$ for $i = 1, \ldots, 6$ we get

$$\left(\sum m_i\right)^2 \leq 6\left(\sum m_i^2\right).$$

By substituting the values of $\sum m_i$ and $\sum m_i^2$ from (D.1) and (D.2), this inequality becomes

$$(3n-1)^2 \leq 6(n^2+1)$$

or

$$3n^2 - 6n - 5 \leq 0.$$

The only positive integers which satisfy this inequality are $n = 1$ and 2.

If $n = 1$ the only way to satisfy both $D_C^2 = -1$ and $D_C \cdot H = 1$ is for $m_i = m_j = 1$ for some $i \neq j$ and for $m_k = 0$ for $k \neq i$ and $k \neq j$. This is $\tilde{L}_{ij}$. If $n = 2$, $D_C^2 = -1$ and $D_C \cdot H = 1$ are only true when $m_i = 0$ for some i and $m_j = 1$ if $j \neq i$. This is $\tilde{C}_i$. Therefore, the 27 lines described in above are the only lines on a cubic surface.

VITA 2

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