BOUNDARIES OF LOW-DIMENSIONAL TEICHMÜLLER SPACES OF RIEMANN SURFACES

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CHAPTER 1

INTRODUCTION

A *Riemann surface* is a topological surface which has a complex structure. It locally looks like the complex plane. Specifically, it is a complex one-dimensional connected manifold; it is a connected Hausdorff space, along with an open cover $\{U_\alpha\}$ and homeomorphisms $z_\alpha : U_\alpha \to \mathbb{C}$ such that

$$z_\alpha \circ z_\beta^{-1} : z_\beta(U_\alpha \cap U_\beta) \to z_\alpha(U_\alpha \cap U_\beta)$$

is holomorphic whenever $U_\alpha \cap U_\beta \neq \emptyset$ (see Figure 1.1). The z_α's are called *local coordinates* on the Riemann surface.

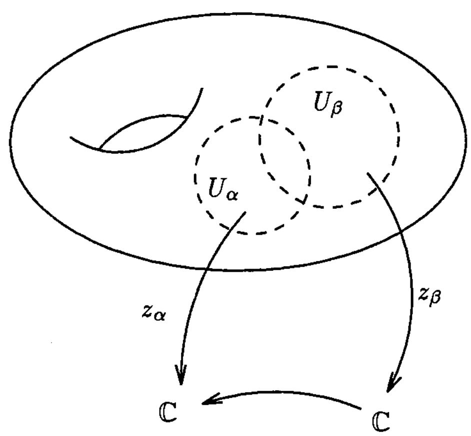


Figure 1.1: Local coordinates on a Riemann surface

A map between two Riemann surfaces $f : S \to T$ is *holomorphic* if for every local coordinate z on S and every local coordinate w on T, $w \circ f \circ z^{-1}$ is holomorphic. (This is a mapping from a subset of $\mathbb{C}$ into $\mathbb{C}$.) Two Riemann surfaces are *conformally equivalent* if there is a biholomorphic homeomorphism between the two. There may be many ways to put a complex structure on a topological surface, and we consider two Riemann surfaces as the same if they are conformally equivalent.

A Riemann surface has *type* (g,n) if the surface has genus g and n punctures. The *moduli* or *Riemann space* $R_{g,n}$ is the space of conformal equivalence classes of Riemann surfaces of type (g,n). If $3g-3+n>0$ then the conformal equivalence class of a Riemann surface of type (g,n) depends on $3g-3+n$ complex parameters called *moduli.* The basic goal of Teichmüller theory is to find a natural way to associate Riemann surfaces with their moduli; the lengths of geodesics, the meromorphic differentials, and similar objects associated with the Riemann surface should depend in an explicit way on the moduli. The moduli space is not (in general) a manifold, and is difficult to study from an analytic point of view. However, if each Riemann surface is *marked* by distinguishing a set of generators for its fundamental group, then the resulting space has a natural complex structure and can be realized as a bounded domain in $\mathbb{C}^m$. The space of marked Riemann surfaces of type (g,n) is called the *Teichmüller space* and is denoted by $T_{g,n}$. If $3g-3+n>0$ then $T_{g,n}$ is a space of complex dimension $3g-3+n$. The moduli space $R_{g,n}$ is the quotient of Teichmüller space $T_{g,n}$ by the modular group of surfaces of type (g,n). (The *modular group* of a surface is the group of homotopy classes of orientation-preserving homeomorphisms of the surface; it is also called the *mapping class group.*)

Riemann surfaces are closely related to Kleinian groups. A matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$ acts on $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ via the action

$$z \mapsto \frac{az+b}{cz+d}.$$

Since $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ yield the same action, we restrict our attention to the group $PSL(2,\mathbb{C})$ (which is isomorphic to the group of Möbius transformations). For a subgroup Γ of $PSL(2,\mathbb{C})$, the *regular* or *ordinary set* $\Omega(\Gamma)$ is the set of points $z \in \hat{\mathbb{C}}$ for which there is some neighborhood V such that $g(V) \cap V = \emptyset$ for all but finitely many $g \in \Gamma$. The *free regular set* $\Omega^\circ(\Gamma)$ is the set of points $z \in \hat{\mathbb{C}}$ for which there is some neighborhood V such that $g(V) \cap V = \emptyset$ for all nontrivial $g \in \Gamma$. The *limit set*

$\Lambda(\Gamma)$ is the complement of $\Omega(\Gamma)$ in $\hat{\mathbb{C}}$. A *Kleinian group* is a discrete subgroup G of $PSL(2,\mathbb{C})$ such that $\Omega(G) \neq \emptyset$.

The action of an element of $PSL(2,\mathbb{C})$ is characterized by the square of its trace. (The trace of an element in $PSL(2,\mathbb{C})$ is well-defined up to sign.) If $1 \neq g \in PSL(2,\mathbb{C})$, then g is *parabolic* if the square of its trace is 4, *hyperbolic* if the square of its trace is real and larger than 4, *elliptic* if the square of its trace is nonnegative, real and less than 4, and *loxodromic* otherwise. Parabolic transformations have exactly one fixed point, and all others (except the identity) have exactly two.

For a Kleinian group G, the quotient space $\Omega(G)/G$ is the set of equivalence classes of points $x \in \Omega(G)$, where two points $x, y \in \Omega(G)$ are equivalent if and only if there is some $g \in G$ with $g(x) = y$. The space $\Omega(G)/G$ is a (possibly disconnected) Riemann surface. The transition functions (the functions $z_\alpha \circ z_\beta^{-1}$) on $\Omega(G)/G$ are the elements of G. The group G *represents* the Riemann surface S if there is an open subset Ω_0 of $\Omega(G)$ which is invariant under the action of G such that Ω_0/G is conformally equivalent to S.

The components of $\Omega(G)$ are also called the *components* of G. The Kleinian group G is a *function group* if it has an invariant component $\Delta(G)$. A function group G is a *b-group* if $\Delta(G)$ is simply connected. A torsion-free b-group G is *terminal* if $(\Omega(G) - \Delta(G))/G$ is a union of thrice-punctured spheres. (There is only one conformal equivalence class of Riemann surfaces of type (0,3).)

A *fundamental domain* D for a Kleinian group G is an open subset of $\Omega(G)$ such that no two points in D are equivalent under the action of G, yet the closure of D contains a point from the equivalence class of every point in $\Omega(G)$, and such that the boundary of D consists of points in $\Lambda(G)$ and a collection of curves; the intersection of one of these curves with $\Omega(G)$ is a *side*, and for each side s, there is some $g \in G$ such that $g(s)$ is also a side of D.

Let G denote a b-group and let γ denote an oriented simple closed curve on

$\Delta(G)/G$. Let $\pi : \Delta(G) \to \Delta(G)/G$ denote the natural projection. Pick base points x_0 and $\tilde{x}_0$ on $\Delta(G)/G$ and $\Delta(G)$, respectively, such that $\pi(\tilde{x}_0) = x_0$. Now γ is homotopic to some loop γ_0 based at x_0. Let $\tilde{\gamma}$ denote the lift of γ_0 with starting point $\tilde{x}_0$. Let $\tilde{x}$ denote the final point of $\tilde{\gamma}$. Then there is some $g \in G$ such that $g(\tilde{x}_0) = \tilde{x}$. Furthermore, g is unique since $\Delta(G)$ is simply connected. Under the natural isomorphism $\pi_1(\Delta(G)/G, \tilde{x}_0) \to G$, the free homotopy class of γ gets mapped to g. We say that g *represents* γ. Since g depends upon the choice of the base point, the element representing γ is not unique; but it is unique up to conjugacy in G. If γ is not oriented, then g and g^{-1} represent γ. If G is any Kleinian group (not necessarily a b-group) representing the Riemann surface $S = \Omega_0/G$ and γ is a loop on S, then an element $g \in G$ *represents* γ if there is an arc $\tilde{\gamma}$ in Ω_0 invariant under $\langle g \rangle$ such that the projection of $\tilde{\gamma}$ onto S is freely homotopic to γ.

A *Fuchsian group* Γ is a Kleinian group which leaves some disc U in $\hat{\mathbb{C}}$ fixed; such groups are conjugate in $PSL(2, \mathbb{C})$ to subgroups of $PSL(2, \mathbb{R})$, which leave the upper half plane $\mathbb{H}$ fixed. The uniformization theorem states that every Riemann surface of type (g, n), $3g - 3 + n > 0$, is conformally equivalent to $\mathbb{H}/\Gamma$ for some Fuchsian group Γ which leaves $\mathbb{H}$ fixed.

The Bers embedding of Teichmüller space ([Ber70]) is an embedding into a bounded domain in the Banach space of cusp forms of weight -4 for a Fuchsian group Γ, defined in the lower half plane.

The embedding of $T_{g,n}$ with which we are concerned first appeared in [Mas74], and is sometimes called the Maskit embedding:

Theorem 1.0.1 *Let S be a marked Riemann surface of type (g, n), $3g - 3 + n > 0$. Then S can be realized as $\Delta(G)/G$, where G is a terminal b-group with invariant component $\Delta(G)$. The group G is unique up to conjugation in $PSL(2, \mathbb{C})$, and is generated by transformations which represent the elements of the fundamental group* of S *specified by the marking.*

The group G depends upon $3g-3+n$ complex parameters in the upper-half plane $\mathbb{H}$. Thus, $T_{g,n}$ is embedded in $\mathbb{H}^{3g-3+n}$.

In [Kra88] and [Kra90a] I. Kra shows that for $3g-3+n>1$, the group G can be algebraically constructed from simpler groups via amalgamated free products and HNN extensions using Maskit's First and Second Combination Theorems. Kra promises to show that this construction and the Bers and Maskit embeddings of Teichmüller space are "essentially the same" in the forthcoming second part of his paper [Kra90a]. Maskit's book *Kleinian Groups* ([Mas87]) contains a detailed description of his theorems. For the convenience of the reader we now present the theorems as they first appeared in [Mas65] and [Mas68].

Suppose two groups G_1 and G_2 have a common subgroup J, and $[G_1 : J] > 1$ and $[G_2 : J] > 1$. The word $g_1 \cdots g_n$ is a *normal form* if $g_i \in (G_1 \cup G_2) - J$ for all i, and whenever $g_i \in G_1$, g_{i-1} and g_{i+1} are in G_2, and whenever $g_i \in G_2$, g_{i-1} and g_{i+1} are in G_1. Two normal forms are *equivalent* if one can be written as $g_1 \cdots g_n$ and the other as $(g_1 j_1)(j_1^{-1} g_2 j_2) \cdots (j_{n-1}^{-1} g_n)$, where each $j_i \in J$. Defining multiplication of normal forms to be concatenation of words, the equivalence classes of normal forms make up a group denoted by $G_1 *_J G_2$, called the *amalgamated free product* of G_1 and G_2 across J, or the *free product* of G_1 and G_2 across the *amalgamated subgroup* J.

Maskit ([Mas65]) defines a *fundamental set* of a Kleinian group G to be a nonempty subset D of the free regular set such that $g(D) \cap D = \emptyset$ for all nontrivial $g \in G$ and such that the union over all $g \in G$ of $g(D)$ is the free regular set. The following theorem is Maskit's First Combination Theorem.

Theorem 1.0.2 *Let G_1 and G_2 be Kleinian groups with a common cyclic subgroup H. For each $m = 1, 2$ let D_m be a fundamental set for G_m; and let D_3 be a fundamental set for H. Let $E_m = \bigcup_{h \in H} h(D_m)$. Suppose $E_1 \cup E_2 = \Omega^\circ(H)$ and the* *interior of* $E_1 \cap E_2 \cap D_3$ *is nonempty. Suppose finally that there is some simple closed curve* γ *in the interior of* $E_1 \cup E_2 \cup \Lambda(H)$ *which is invariant under* H*, such that the closure of* $\gamma \cap D_3$ *is in the interior of* $E_1 \cap E_2$ *and such that* γ *separates* $E_1 - E_2$ *and* $E_2 - E_1$*. Then the group* G *generated by* G_1 *and* G_2 *is Kleinian,* $G = G_1 *_H G_2$*, and* $E_1 \cap E_2 \cap D_3$ *is a fundamental set for* G*.*

Now suppose J_1 and J_2 are subgroups of a group G, and suppose there is some element f such that $fJ_1f^{-1} = J_2$, but $\langle f \rangle \cap G = \{1\}$. The word $f^{\alpha_1}g_1f^{\alpha_2}g_2 \cdots f^{\alpha_n}g_n$ is a *normal form* if each $g_i \in G$; $g_i \neq 1$ for $i \neq n$; $\alpha_i \neq 0$ for $i \neq 1$; if $\alpha_i < 0$ and $g_{i-1} \in J_1$ then $\alpha_{i-1} \leq 0$; and if $\alpha_i > 0$ and $g_{i-1} \in J_2$ then $\alpha_{i-1} \geq 0$. The normal forms $f^{\alpha_1}g_1 \cdots f^{\alpha_i}g_if^kj_1f^{-k}j_2f^{\alpha_{i+3}}g_{i+3} \cdots f^{\alpha_n}g_n$ and $f^{\alpha_1}g_1 \cdots f^{\alpha_i}g_if^{\alpha_{i+3}}g_{i+3} \cdots f^{\alpha_n}g_n$ are *equivalent* if $j_2 = f^kj_1^{-1}f^{-k}$. Equivalence classes of normal forms form a group called the *HNN extension of* G *by* f, which is denoted by $G*_f$.

The following theorem is known as Maskit's Second Combination Theorem. Let $\overline{Y}$ denote the closure of the set Y.

Theorem 1.0.3 *Suppose* J_1 *and* J_2 *are cyclic subgroups of the Kleinian group* G_0*,* D *is a fundamental set for* G_0*,* f *is a Möbius transformation such that* $\langle f \rangle \cap G_0 = \{1\}$*, and* Y_1 *and* Y_2 *are disjoint Jordan domains with disjoint boundary curves* C_1 *and* C_2*, respectively. Let* $Y_3 = \hat{\mathbb{C}} - (\overline{Y_1} \cup \overline{Y_2})$*. Suppose that the interior of* $Y_3 \cap D$ *is nonempty,* $Y_3 \cup C_1$ *is a fundamental set for* $\langle f \rangle$*,* $fJ_1f^{-1} = J_2$*,* $f(C_1) = C_2$*, and for* $m = 1$ *and* 2*,* $\overline{D \cap C_m} \subset \Omega^\circ(G_0)$ *and* $\bigcup_{j \in J_m} j(D \cap \overline{Y_m}) = \overline{Y_m} \cap \Omega^\circ(G_0) = \overline{Y_m} \cap \Omega^\circ(J_m)$*. Let* $G = \langle G_0, f \rangle$*. Then* $G = G_0*_f$*,* G *is Kleinian, and* $D \cap (Y_3 \cup C_1)$ *is a fundamental set for* G*.*

The Maskit embeddings of Teichmüller spaces in $\mathbb{H}^{3g-3+n}$ can be studied by studying their boundaries. If a simple closed geodesic on a surface in $T_{g,n}$ is pinched to a point, the resulting surface lies on the boundary of $T_{g,n}$ and is called a *cusp*. If a maximal set of disjoint simple closed geodesics on the surface are simultaneously pinched to points, the resulting surface on the boundary of $T_{g,n}$ is a *maximal cusp.* L. Bers first conjectured in 1970 ([Ber70]) that cusps are dense in the boundary of his embedding of Teichmüller space; but he showed that, in the sense of dimension, most boundary points are not cusps. C. McMullen ([McM91a]) showed that maximal cusps are dense in the boundary of the Bers embedding; his proof extends to the Maskit embedding also. Thus, if we know the location of all the maximal cusps in the Maskit embedding, then we know the shape of the boundary.

A loxodromic or hyperbolic element representing a simple closed geodesic on $\Delta(G)/G$ becomes parabolic when the geodesic is pinched to a point. Hence, if we can tell where the elements representing these geodesics become parabolic, we will know where the cusps are.

The one-dimensional Teichmüller spaces are $T_{1,1}$ and $T_{0,4}$. A maximal set of disjoint simple closed geodesics on either a once-punctured torus or a four-times punctured sphere consists of exactly one geodesic. D. Wright ([Wri]) has found a nice way to construct the elements in the groups representing once-punctured tori from the simple closed geodesics. Rational numbers are assigned to the simple closed geodesics and cusps. C. McMullen has communicated to us that he can prove that this assignment can be extended to a homeomorphism from the real line to the boundary of $T_{1,1}$. The rational numbers are assigned to cusps in such a way that if p/q and r/s are Farey neighbors (that is, $qr - ps = \pm 1$), then all the cusps corresponding to rationals between p/q and r/s can be computed using only information about the simple closed curves corresponding to p/q and r/s. Adding the Farey neighbors in the Farey sense ($p/q \oplus r/s = (p+r)/(q+s)$) yields a new Farey neighbor for p/q and r/s. Hence the formation of rationals by Farey sequences reveals the simplicial structure of the boundary of $T_{1,1}$.

Our first main result is that there is a one-to-one correspondence of the simple closed geodesics on a once-punctured torus with the simple closed geodesics on a sphere with four punctures such that the representative elements in the group for the four-times punctured sphere are conjugate to the squares of the representative elements in the group for the once-punctured torus. This implies that the corresponding elements become parabolic at the same places, so the cusps are at the same places. It follows that the embeddings of $T_{1,1}$ and $T_{0,4}$ are the same.

The two-dimensional Teichmüller spaces are $T_{0,5}$ and $T_{1,2}$. The maximal sets of disjoint simple closed geodesics on five-times punctured spheres and twice-punctured tori consist of two geodesics. For the results of the one-dimensional cases to be generalized to these cases, we need to know when two simple closed geodesics are disjoint. In our second main result we find a suitable assignment of pairs of rational numbers to sets of disjoint simple closed geodesics on a five-times punctured sphere. These pairs of rationals are used to compute the number of intersection points of sets of simple closed geodesics on a five-times punctured sphere. In the future we hope to use this result to study the simplicial structure of the boundary of $T_{0,5}$.

In our final result we study the biholomorphic map from $T_{1,1}$ to the upper half plane $\mathbb{H}$. This abstract map has been known for some time, but no way had been seen to actually use the map to compute the image of specific points in $T_{1,1}$. The map involves integrating an abelian differential on a Riemann surface of type (1,1). We construct a specific Poncaré series and use it to construct the abelian differential. We then approximate the Poincaré series and use this approximation to compute the image of any point in $T_{1,1}$. An error bound for the approximation is specified.

CHAPTER 2

ONCE-PUNCTURED TORI

2.1 The embedding of $T_{1,1}$

This section follows Wright ([Wri]), where a more detailed description is presented.

Let Γ denote the Kleinian group generated by the parabolic transformations S_1 and S_2, where $S_1(z) = z + 2$ and $S_2(z) = \frac{z}{2z+1}$ (see Figure 2.1). Let $\mathbb{H}_L$ denote the

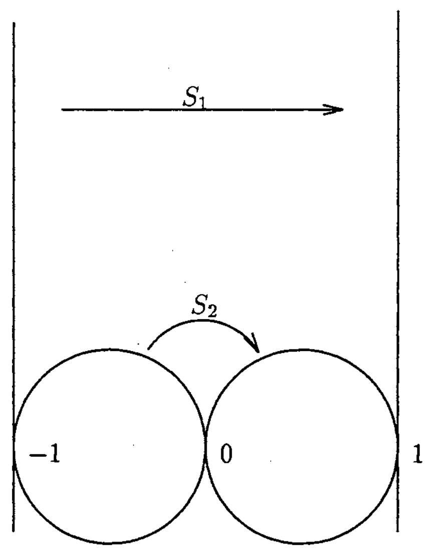


Figure 2.1: The action of the group Γ

lower half plane. Then the ordinary set $\Omega(\Gamma)$ is $\mathbb{H} \cup \mathbb{H}_L$, and the quotient space $\Omega(\Gamma)$ is the union of two triply-punctured spheres. To construct a surface of type (1,1) (that is, a once-punctured torus), cut off two punctures from $\mathbb{H}/\Gamma$ along simple closed curves, and glue the simple closed curves together. To achieve this algebraically we want to find a transformation T which conjugates S_2 to S_1. The assumption $TS_2T^{-1} = S_1$ implies that $T(z) = T_x(z) = \frac{1}{z} + x$ for some complex parameter x. In order for the surface $\mathbb{H}_L/\Gamma$ to remain unchanged, it is necessary to consider only those values of x for which $\mathrm{Im}(x) > 0$. Now let G_x denote the HNN extension of Γ

by T_x (see Figure 2.2). The group G_x is generated by S_1 and T_x. The Teichmüller

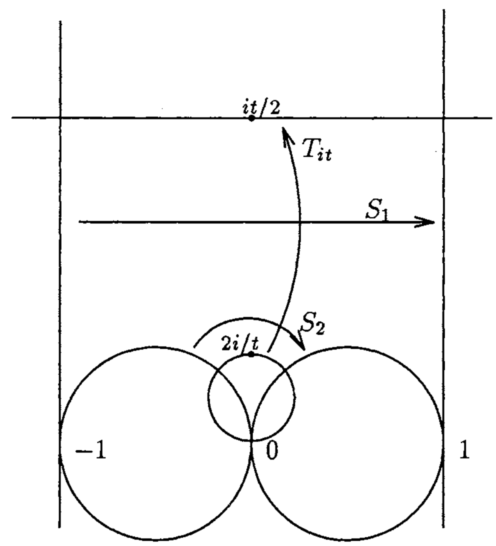


Figure 2.2: The action of the group G_{it}, $t > 2$

space $T_{1,1}$ is embedded in $\mathbb{H}$ as the set of all $x \in \mathbb{H}$ for which G_x is a terminal b-group and $\Delta(G_x)/G_x$ is a once-punctured torus; the marking on $\Delta(G_x)/G_x$ is the distinguished set of generators of $\pi_1(\Delta(G_x)/G_x)$ represented by the set of group elements $\{S_1, T_x\}$. Let $M_{1,1}$ denote the embedding of $T_{1,1}$ in $\mathbb{H}$. Wright ([Wri]) has shown that $\{z : \mathrm{Im}(z) > 2\} \subset M_{1,1} \subset \{z : \mathrm{Im}(z) > 1\}$, $x \in M_{1,1}$ if and only if $x + 2 \in M_{1,1}$, and $x \in M_{1,1}$ if and only if $-\overline{x} \in M_{1,1}$.

2.2 Words in the group G_x

In this section we develop a way to parametrize the conjugacy classes of elements of the group G_x which represent the simple closed geodesics on $\Delta(G_x)/G_x$. Let h be a homeomorphism from $\Delta(G_x)/G_x$ to the once-punctured torus $(\mathbb{C} - L_i)/\langle z \mapsto z+1, z \mapsto z+i\rangle$, where L_i denotes the lattice $\{n + mi : n, m \in \mathbb{Z}\}$. Then the homotopy class of any simple closed geodesic on $\Delta(G_x)/G_x$ is determined by the homotopy class of its image in $(\mathbb{C} - L_i)/\langle z \mapsto z+1, z \mapsto z+i\rangle$ under h. Every simple closed curve on $(\mathbb{C} - L_i)/\langle z \mapsto z+1, z \mapsto z+i\rangle$ is homotopic to a curve whose lifting to the unit square $\{z : 0 \le \mathrm{Re}(z) \le 1, 0 \le \mathrm{Im}(z) \le 1\}$ is the union of disjoint segments connecting different sides of the square. These segments will intersect the top and bottom sides of the square in the same number q of points, and they will intersect the left and right sides in the same number p of points. Hence it is possible to deform the curve so that the disjoint segments in its lifting are all line segments all with slopes either $\frac{q}{p}$ or $-\frac{q}{p}$. If the greatest common divisor of p and q is d, then the disjoint line segments will project to d curves on the once-punctured torus; therefore, we can assume p and q are relatively prime.

The elements of G_x can be thought of as words in the letters S_1, S_1^{-1}, T_x and T_x^{-1}. The words which represent simple closed geodesics on $\Delta(G_x)/G_x$ can be parametrized by pairs of relatively prime integers.

Let $\hat{\mathbb{Q}}$ denote the set of pairs of relatively prime integers (p,q) (hereafter written p/q) such that $q > 0$ unless $q = 0$ and $p = \pm 1$, and $p \neq 0$ unless $q = 1$. Give $\hat{\mathbb{Q}}$ the same ordering as the rationals, except that $-1/0 < p/q < 1/0$ whenever $q \neq 0$. We refer to the set $\hat{\mathbb{Q}}$ as the *extended rationals.* If p/q and n/m are in $\hat{\mathbb{Q}}$ and $qn - pm = \pm 1$, then p/q and n/m are called *Farey neighbors.* Define and addition $\oplus$ on Farey neighbors $(p/q, n/m)$ in $\hat{\mathbb{Q}}$ by $p/q \oplus n/m = (p+n)/(q+m)$. Every $p/q \in \hat{\mathbb{Q}}$ can be written as a finite sum $\oplus$ of $-1/0$, $0/1$ and $1/0$. Note that if p/q and n/m are Farey neighbors, then p/q and $p/q \oplus n/m$ are Farey neighbors, as are $p/q \oplus n/m$ and n/m. Note also that if $p/q < n/m$ then $p/q < p/q \oplus n/m < n/m$.

Let $S = S_1$ and $T = T_x$. For $p/q \in \hat{\mathbb{Q}}$, define words $W_{p/q} \in G_x$ in the following way. Let $\lceil y \rceil$ denote the smallest integer greater than or equal to y. For $0/1 < p/q \le 1/0$, define

$$W_{p/q} = X_1 X_2 \cdots X_{p+q},$$

where

$$X_j \in \{T^{-1}, S\} \text{ for } 1 \le j \le p+q,$$

and for $n = 0, 1, ..., p-1$, if $\lceil \frac{nq}{p} \rceil + n < j \le \lceil \frac{(n+1)q}{p} \rceil + n$, then $X_j = T^{-1}$; and if $j = \lceil \frac{(n+1)q}{p} \rceil + n + 1$, then $X_j = S$. Also define

$$W_{-p/q} = Y_1 Y_2 \cdots Y_{p+q},$$

where

$$Y_j = \begin{cases} T^{-1} & \text{if } X_{p+q+1-j} = T^{-1}; \\ S^{-1} & \text{if } X_{p+q+1-j} = S. \end{cases}$$

Finally define $W_{0/1} = T^{-1}$.

The words $W_{p/q}$ and $W_{-p/q}$ represent the simple closed geodesics on $\Delta(G_x)/G_x$ which correspond to the simple closed curves on $(\mathbb{C} - L_i)/\langle z \mapsto z+1, z \mapsto z+i \rangle$ whose liftings to the unit square cross the top and bottom sides of the square q times and cross the left and right sides of the square p times. (See Figure 2.3.) For $W_{p/q}$, $p/q > 0$, the segments of the liftings in the square can be deformed to line segments of slope $-\frac{q}{p}$; and for $W_{-p/q}$, $-p/q < 0$, they can be deformed to line segments of slope $\frac{q}{p}$. Hence every simple closed geodesic on $\Delta(G_x)/G_x$ is represented by a unique $W_{p/q}$, $p/q \in \hat{\mathbb{Q}}$, $p/q > -1/0$. (The words $W_{-1/0} = S^{-1}$ and $W_{1/0} = S$ represent the same geodesic.)

There is a nice concatenation law for the words $W_{p/q}$. We first prove two lemmas.

Lemma 2.2.1 *If r/s and p/q are Farey neighbors, $r/s < p/q$, $r > 0$, $0 \le n \le 2r$, and p does not divide n, then $\lceil \frac{nq}{p} \rceil = \lceil \frac{ns}{r} \rceil$. Also, $\lceil \frac{pq}{p} \rceil = q$ and $\lceil \frac{ps}{r} \rceil = q+1$.*

Proof: Since $sp - rq = 1$,

$$\begin{aligned} \frac{ns}{r} &= \frac{nsp}{rp} \\ &= \frac{nrq + n}{rp} \\ &= \frac{nq}{p} + \frac{n}{rp}. \end{aligned}$$

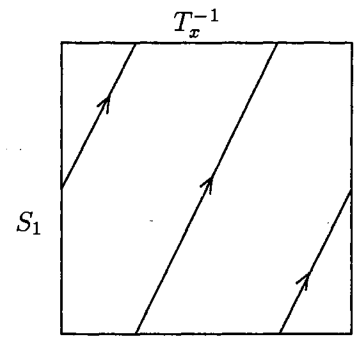


Figure 2.3: The geodesic represented by the word $W_{-1/2} = S_1^{-1}T_x^{-1}T_x^{-1}$

Thus

$$\lceil \frac{nq}{p} \rceil \leq \lceil \frac{ns}{r} \rceil.$$

First suppose $n = r$. Then

$$\lceil \frac{ns}{r} \rceil = s = \frac{nq}{p} + \frac{1}{p},$$

and since $p \geq 2$ we have $\frac{1}{p} < 1$, so

$$\lceil \frac{ns}{r} \rceil < \lceil \frac{nq}{p} \rceil + 1,$$

which implies

$$\lceil \frac{ns}{r} \rceil \leq \lceil \frac{nq}{p} \rceil.$$

Next suppose $n = 2r$. Then

$$\lceil \frac{ns}{r} \rceil = 2s = \frac{nq}{p} + \frac{2}{p}.$$

Now if $p = 2$, then $r = 1$ and $n = 2$, which contradicts the hypothesis that p does not divide n. If $p \geq 3$, then

$$\lceil \frac{ns}{r} \rceil < \lceil \frac{nq}{p} \rceil + 1$$

and

$$\lceil \frac{ns}{r} \rceil \leq \lceil \frac{nq}{p} \rceil.$$

Finally suppose r does not divide n. Then $\frac{nq}{p}$ and $\frac{ns}{r}$ are not integers. If $\lceil\frac{nq}{p}\rceil \neq \lceil\frac{ns}{r}\rceil$ then there is some integer K such that

$$\frac{nq}{p} < K < \frac{ns}{r}.$$

Clearly $K - \frac{nq}{p} \geq \frac{1}{p}$ and $\frac{ns}{r} - K \geq \frac{1}{r}$, which imply that

$$\frac{ns}{r} - \frac{nq}{p} \geq \frac{1}{p} + \frac{1}{r} > \frac{2}{p}.$$

But since

$$\frac{ns}{r} = \frac{nq}{p} + \frac{n}{rp}$$

and $n < 2r$,

$$\frac{ns}{r} - \frac{nq}{p} < \frac{2}{p}.$$

This contradiction shows $\lceil\frac{nq}{p}\rceil = \lceil\frac{ns}{r}\rceil$.

The second part of the lemma follows from the string of equalities $\lceil\frac{ps}{r}\rceil = \lceil\frac{1+rq}{r}\rceil = \lceil q + \frac{1}{r}\rceil = q + 1$.

q.e.d.

Lemma 2.2.2 *If $r > 0$, $p/q = r/s \oplus u/v$, and $r \leq n \leq p$ then $\lceil\frac{(n-r)v}{u}\rceil + s = \lceil\frac{nq}{p}\rceil$.*

Proof: Suppose u divides n. Then $\lceil\frac{(n-r)v}{u}\rceil = \frac{nv}{u} + \lceil\frac{-rv}{u}\rceil = \frac{nv}{u} + \lceil\frac{1-su}{u}\rceil = \frac{nv}{u} + 1 - s$. Also, $\lceil\frac{nq}{p}\rceil = \lceil\frac{nv}{u} + \frac{n}{pu}\rceil = \frac{nv}{u} + \lceil\frac{n}{pu}\rceil = \frac{nv}{u} + 1$.

Next suppose u does not divide n. Then by Lemma 2.2.1, if $r \leq n \leq p$ then $\lceil\frac{(n-r)q}{p}\rceil = \lceil\frac{(n-r)v}{u}\rceil$. Now since $\frac{ns}{r} = \frac{nq}{p} + \frac{n}{rp}$ for all n, $\frac{rq}{p} < s$ and $\lceil\frac{nq}{p}\rceil - s \leq \lceil\frac{nq}{p}\rceil - \lceil\frac{rq}{p}\rceil \leq \lceil\frac{nq}{p} - \frac{rq}{p}\rceil = \lceil\frac{(n-r)q}{p}\rceil$.

If $n = p$ the result is obvious. Otherwise, $\frac{nq}{p}$ and $\frac{(n-r)v}{u}$ are not integers. If $\lceil\frac{nq}{p}\rceil - s \neq \lceil\frac{(n-r)v}{u}\rceil$, then there is some integer K with

$$\frac{nq}{p} - s < K < \frac{(n-r)v}{u}.$$

Moreover $K-(\frac{nq}{p}-s)\geq\frac{1}{p}$ and $\frac{(n-r)v}{u}-K\geq\frac{1}{u}$, so $\frac{(n-r)v}{u}-(\frac{nq}{p}-s)\geq\frac{1}{p}+\frac{1}{u}$. On the other hand,

$$\begin{aligned}\frac{(n-r)v}{u}-(\frac{nq}{p}-s) &= \frac{(n-r)v}{u}-(\frac{nv}{u}+\frac{n}{pu}-s)\\ &= s-\frac{rv}{u}-\frac{n}{pu}.\end{aligned}$$

Now since $n\geq r$, $\frac{n}{pu}\geq\frac{r}{pu}=\frac{r+u-u}{pu}=\frac{1}{u}-\frac{1}{p}$. Thus

$$\begin{aligned}s-\frac{rv}{u}-\frac{n}{pu} &\leq s-\frac{rv}{u}+\frac{1}{p}-\frac{1}{u}\\ &= s-\frac{su-1}{u}+\frac{1}{p}-\frac{1}{u}\\ &= \frac{1}{p}.\end{aligned}$$

This contradiction finishes the proof.

q.e.d.

Proposition 2.2.3 *If $r/s,u/v\in\hat{\mathbb{Q}}$, $ru\geq 0$, and $su-rv=1$, then $W_{r/s\oplus u/v}=W_{r/s}W_{u/v}$.*

Proof: Let $p/q=r/s\oplus u/v$. First assume that $r>0$. Then $W_{r/s}W_{u/v}=X_1X_2\cdots X_{r+u+s+v}$, where for $n=0,1,...,r-1$, if $\lceil\frac{ns}{r}\rceil+n<j\leq\lceil\frac{(n+1)s}{r}\rceil+n$, $X_j=T^{-1}$; and if $j=\lceil\frac{(n+1)s}{r}\rceil+n+1$, $X_j=S$; and for $n=r,r+1,...,r+n-1$, if $\lceil\frac{(n-r)v}{u}\rceil+(n-r)<j-r-s\leq\lceil\frac{(n-r+1)v}{u}\rceil+(n-r)$, $X_j=T^{-1}$; and if $j-r-s=\lceil\frac{(n-r+1)v}{u}\rceil+(n-r+1)$, $X_j=S$. Now Lemma 2.2.1 implies that $\lceil\frac{ns}{r}\rceil=\lceil\frac{nq}{p}\rceil$ for $0\leq n\leq r$. Lemma 2.2.2 implies that $\lceil\frac{(n-r)v}{u}\rceil+s=\lceil\frac{nq}{p}\rceil$ for $r\leq n\leq p$. So $W_{p/q}=W_{r/s}W_{u/v}$.

Next suppose $r=0$. Then $u=s=p=1$, and so $\lceil\frac{pq}{p}\rceil=q=v+1$ and $\lceil\frac{pv}{u}\rceil=v$, and $W_{1/q}=W_{0/1}W_{1/v}$.

Finally, if $r<0$ and $u<0$, then the proposition follows from the first case and the fact that $-p/q=-u/v\oplus-r/s$.

q.e.d.

The rest of this section will be useful in the next chapter. For any word W, let $\ell(W)$ denote the length of W; that is, the number of letters in W.

Proposition 2.2.4 *Suppose $r/s, n/m \in \hat{\mathbb{Q}}$, $0/1 \leq r/s < n/m$, and $sn - rm = 1$. Then there are words W_1, W_2 in the letters T^{-1} and S such that*

$$W_{r/s}W_{n/m} = W_1T^{-1}SW_2,$$

$$W_{n/m}W_{r/s} = W_1ST^{-1}W_2,$$

and $\ell(W_1) = n + m - 1$.

Proof: We use induction. If $r/s = 0/1$ and $n/m = 1/0$, then $W_{r/s}W_{n/m} = T^{-1}S$ and $W_{n/m}W_{r/s} = ST^{-1}$ and $n + m - 1 = 0$, so the proposition is true in this case.

Now suppose the proposition is true for the Farey neighbors $(r/s, n/m)$; we prove the proposition for the Farey neighbors $(r/s, p/q)$ and $(p/q, n/m)$, where $p/q = r/s \oplus n/m$. Now

$$W_{r/s}W_{p/q} = W_{r/s}W_{r/s}W_{n/m} = W_{r/s}W_1T^{-1}SW_2,$$

and

$$W_{p/q}W_{r/s} = W_{r/s}W_{n/m}W_{r/s} = W_{r/s}W_1ST^{-1}W_2,$$

and $\ell(W_{r/s}W_1) = p + q - 1$, so the result holds for the Farey neighbors $(r/s, p/q)$. Similarly,

$$W_{p/q}W_{n/m} = W_{r/s}W_{n/m}W_{n/m} = W_1T^{-1}SW_2W_{n/m},$$

$$W_{n/m}W_{p/q} = W_{n/m}W_{r/s}W_{n/m} = W_1ST^{-1}W_2W_{n/m},$$

and $\ell(W_1) = n + m - 1$, so the result holds for $(p/q, n/m)$.

q.e.d.

Corollary 2.2.5 *Let $p/q = r/s \oplus n/m$, $0/1 \leq r/s$. Let*

$$(W_{p/q})^2 = X_1X_2 \cdots X_{2p+2q}$$

and

$$(W_{r/s})^2(W_{n/m})^2 = Y_1Y_2\cdots Y_{2p+2q},$$

where

$$X_j, Y_j \in \{T^{-1}, S\} \text{ for } 1 \leq j \leq 2p+2q.$$

Then $X_j = Y_j$ *for all* j *except* $j = p+q$ *and* $j = p+q+1$, *and* $X_{p+q} = S = Y_{p+q+1}$ *and* $X_{p+q+1} = T^{-1} = Y_{p+q}$.

Proof: Since

$$(W_{p/q})^2 = W_{r/s}W_{n/m}W_{r/s}W_{n/m}$$

and

$$(W_{r/s})^2(W_{n/m})^2 = W_{r/s}W_{r/s}W_{n/m}W_{n/m},$$

the result follows immediately from Proposition 2.2.4.

q.e.d.

Proposition 2.2.6 *For* $0/1 \leq p/q \leq 1/0$, *the word* $(W_{p/q})^2$ *can be uniquely decomposed into a product of the words* $X_M = T^{-2}$, $Y_{M,n} = T^{-1}S^nT^{-1}$, *and* S^n *for positive integers* n.

Proof: For $0/1 \leq p/q \leq 1/0$, the word $W_{p/q}$ is a product of the letters T^{-1} and S. Since there are an even number of letters T^{-1} in the word $(W_{p/q})^2$, the unique decomposition is clear.

q.e.d.

CHAPTER 3

SPHERES WITH FOUR PUNCTURES

3.1 The embedding of $T_{0,4}$

To construct a surface of type (0,4), take two thrice-punctured spheres, cut off a puncture from each along simple closed curves homotopic to the punctures, and glue the simple closed curves together. To achieve this algebraically, let A, B_1 and B_2 denote parabolic transformations with different fixed points, where AB_1 and AB_2 are parabolic. The groups $G_1 = \langle A, B_1 \rangle$ and $G_2 = \langle A, B_2 \rangle$ each represent two thrice-punctured spheres.

Conjugate so that $A(\infty) = \infty$, $A(0) = 4$, and $B_1(0) = 0$. Then $A = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$. (Here we make no distinction between Möbius transformations and the corresponding elements of $PSL(2, \mathbb{C})$.) Write $B_2 = \begin{pmatrix} 1-x & b \\ c & 1+x \end{pmatrix}$. Since AB_2 is parabolic, its trace must be -2 (if the trace were 2, then B_2 would fix ∞). This implies that $c = -1$. Since $B_2 \in PSL(2, \mathbb{C})$, $b = x^2$. So $B_2 = \begin{pmatrix} 1-x & x^2 \\ -1 & 1+x \end{pmatrix}$. Denote this element by $B_{2,x}$. Now B_1 must have the same form as B_2, but since $B_1(0) = 0$, $B_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$.

Let H_x denote the amalgamated free product $G_1 *_A G_2$ (see Figure 3.1). Then H_x is generated by A, B_1 and $B_{2,x}$. The embedding of $T_{0,4}$ into $\mathbb{H}$ is the set of $x \in \mathbb{H}$ for which H_x is a terminal b-group and $\Delta(H_x)/H_x$ is a four-times punctured sphere. Denote this set by $M_{0,4}$.

Proposition 3.1.1 *$x \in M_{0,4}$ if and only if $x + 2 \in M_{0,4}$.*

Proof: It is easy to check that $AB_{2,x} = B_{2,x+2}^{-1}$. Thus $H_x = \langle A, B_1, B_{2,x} \rangle = \langle A, B_1, AB_{2,x} \rangle = \langle A, B_1, B_{2,x+2}^{-1} \rangle = H_{x+2}$, so H_x and H_{x+2} represent the same Riemann surface, differently marked.

q.e.d.

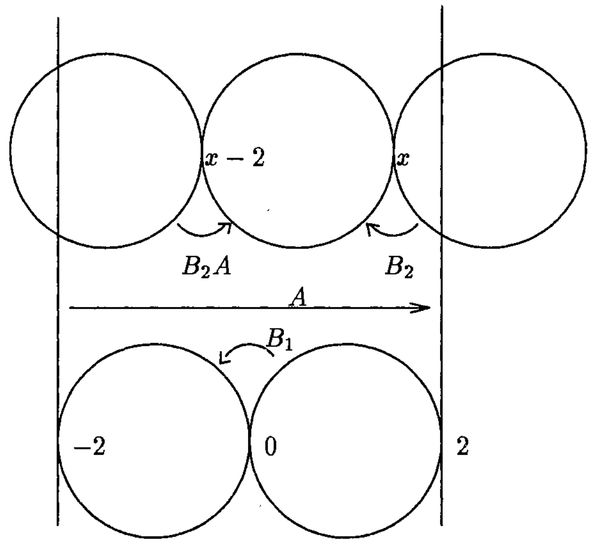


Figure 3.1: The action in the group H_x, $\mathrm{Im}(x) > 2$

Proposition 3.1.2 *$x \in M_{0,4}$ if and only if $-\overline{x} \in M_{0,4}$.*

Proof: Define $J(z) = -\overline{z}$. Then $J^{-1} = J$, and $JAJ = A^{-1}$, $JB_1J = B_1^{-1}$, and $JB_{2,x}J = B_{2,-\overline{x}}^{-1}$. Thus, J maps the limit set of H_x to the limit set of $H_{-\overline{x}}$.

q.e.d.

Proposition 3.1.3 *If $Im(x) > 2$ then $x \in M_{0,4}$.*

Proof: In light of Propositions 3.1.1 and 3.1.2, we assume $0 \leq \mathrm{Re}(x) \leq 1$. Let D_3 denote the vertical strip $\{z : -2 \leq \mathrm{Re}(z) < 2\}$; D_3 is a fundamental set for $\langle A \rangle$. The set

$$D_1 = D_3 \cap \{z : |z-1| \geq 1,\ |z+1| > 1\}$$

is a fundamental set for the group $G_1 = \langle A, B_1 \rangle$. Also, the set

$$D_2 = D_3 \cap \{z : |z-(x-1)| \geq 1,\ |z-(x+1)| > 1,\ |z-(x-3)| > 1\}$$

is a fundamental set for the group $G_2 = \langle A, B_{2,x} \rangle$.

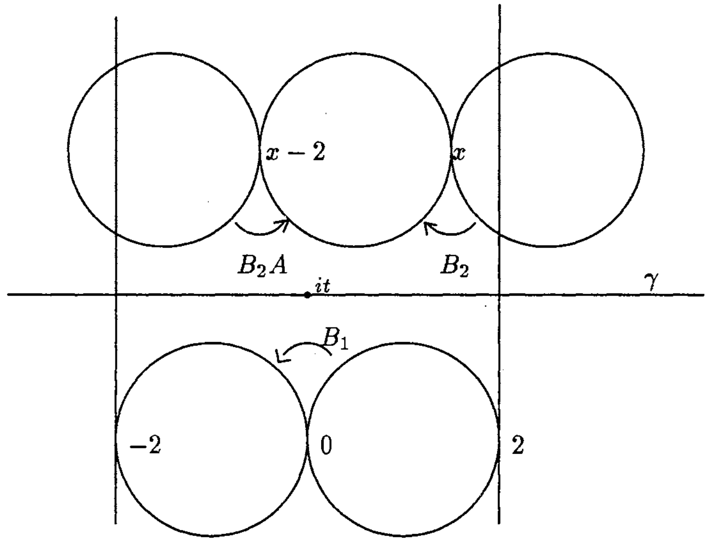


Figure 3.2: Using Maskit's First Combination Theorem

Choose a number t with $1 < t < \text{Im}(x) - 1$, and let $\gamma = \{z : \text{Im}(z) = t\}$. By Maskit's First Combination Theorem (Theorem 1.0.2), $H_x = \langle G_1, G_2 \rangle = G_1 *_{\langle A \rangle} G_2$ is a Kleinian group with fundamental set $D = D_1 \cap D_2$. The action of H_x on D clearly represents two triply-punctured spheres and one four-times punctured sphere.

q.e.d.

3.2 Words in the group H_x

We will now construct specific words in the group H_x which represent the simple closed geodesics on $\Delta(H_x)/H_x$. Label the four punctures on $\Delta(H_x)/H_x$ as q_1, q_2, q_3 and q_4, and connect the four punctures with four arcs as in Figure 3.3. The union of the arcs separates the surface into two components S^+ and S^-. Any simple closed geodesic on $\Delta(H_x)/H_x$ is the union of disjoint segments in S^+ and S^-. If any segment has two endpoints on the same arc, the geodesic can be deformed so that this segment vanishes. Since we are only concerned with the homotopy class of the simple closed geodesic, we can assume each segment connects different arcs.

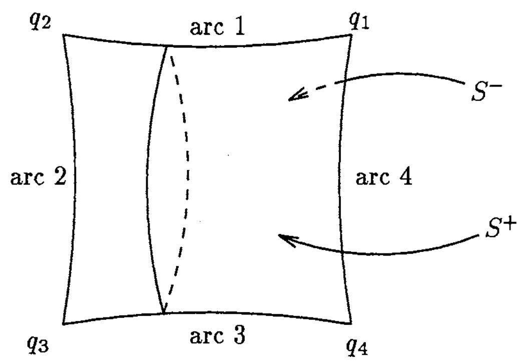


Figure 3.3: The surface $\Delta(H_x)/H_x$

Each crossing of an arc corresponds to a letter in the word in H_x representing the geodesic. Each letter is one of $A^{\pm 1}$, $B_1^{\pm 1}$, $B_2^{\pm 1}$ or I for the identity. The letters $B_1^{\pm 1}$ and $B_2^{\pm 1}$ correspond to opposite arcs as do the letters $A^{\pm 1}$ and I. It is not difficult to see that the number of crossings on opposite arcs is the same.

Now for each $p/q \in \hat{\mathbb{Q}}$, define a word $K_{p/q} \in H_x$ in the following way. For $0/1 < p/q \leq 1/0$, define

$$K_{p/q} = X_1 X_2 \cdots X_{p+2q},$$

where

$$X_j \in \{B_1, B_1^{-1}, B_2, B_2^{-1}, A, A^{-1}\} \text{ for } 1 \leq j \leq p+2q,$$

and for $n = 0, 1, ..., p-1$, if $\lceil \frac{2nq}{p} \rceil + n < j \leq \lceil \frac{(2n+1)q}{p} \rceil + n$, then

$$X_j = \begin{cases} B_1 & \text{if } j+n \text{ is odd} \\ B_2^{-1} & \text{if } j+n \text{ is even;} \end{cases}$$

if $\lceil \frac{(2n+1)q}{p} \rceil + n < j \leq \lceil \frac{(2n+2)q}{p} \rceil + n$, then

$$X_j = \begin{cases} B_1^{-1} & \text{if } j+n \text{ is odd} \\ B_2 & \text{if } j+n \text{ is even;} \end{cases}$$

and if $j = \lceil \frac{(2n+2)q}{p} \rceil + n + 1$, then

$$X_j = \begin{cases} A & \text{if } j+n \text{ is odd} \\ A^{-1} & \text{if } j+n \text{ is even.} \end{cases}$$

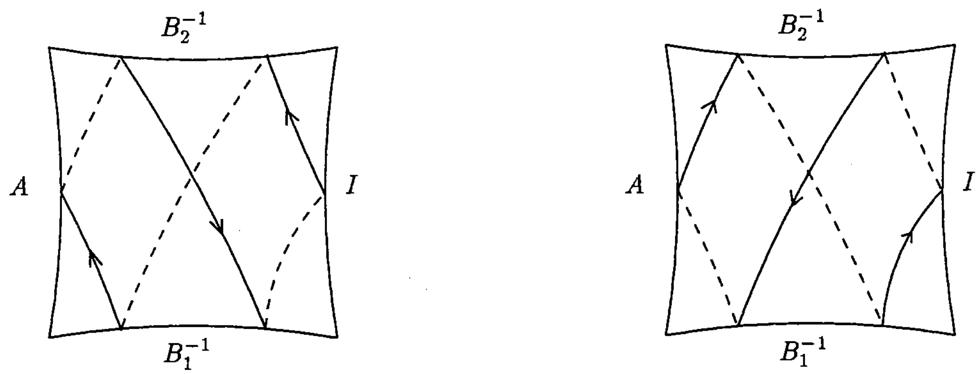
Also define

$$K_{-p/q} = Y_1 Y_2 \cdots Y_{p+2q},$$

where

$$Y_j = \begin{cases} B_1 & \text{if } X_{p+2q+1-j} = B_2^{-1}; \\ B_1^{-1} & \text{if } X_{p+2q+1-j} = B_2; \\ B_2 & \text{if } X_{p+2q+1-j} = B_1^{-1}; \\ B_2^{-1} & \text{if } X_{p+2q+1-j} = B_1; \\ A & \text{if } X_{p+2q+1-j} = A^{-1}; \\ A^{-1} & \text{if } X_{p+2q+1-j} = A. \end{cases}$$

Finally define $K_{0/1} = B_1 B_2^{-1}$.



$K_{1/2} = B_1 B_2^{-1} B_1^{-1} B_2 A$ $\qquad$ $K_{-1/2} = A^{-1} B_1^{-1} B_2 B_1 B_2^{-1}$

Figure 3.4: Simple closed curves on $\Delta(H_x)/H_x$

The words $K_{p/q}$ and $K_{-p/q}$ represent the simple closed curves which cross the arcs corresponding to $B_1^{\pm 1}$ and $B_2^{\pm 1}$ exactly q times each and which cross the arcs corresponding to $A^{\pm 1}$ and I exactly p times each. Clearly for $pq \neq 0$ there are exactly two homotopy classes of simple closed curves of this type which are homotopic to geodesics (so the curves are not homotopic to punctures or homotopically trivial). If $pq = 0$ then there is only one homotopy class. Hence every simple closed geodesic on $\Delta(H_x)/H_x$ is represented by a unique $K_{p/q}$ where $p/q \in \hat{\mathbb{Q}}$ and $p/q > -1/0$. See Figure 3.4.

Recall that $S(z) = S_1(z) = z + 2$, and define the new words $\tilde{K}_{p/q}$ in the following way. For $0/1 < p/q \leq 1/0$, let

$$\tilde{K}_{p/q} = X_1 X_2 \cdots X_{2p+2q},$$

where

$$X_j \in \{B_1, B_2^{-1}, S, S^{-1}\} \text{ for } 1 \leq j \leq 2p + 2q,$$

and for $n = 0, 1, ..., 2p - 1$, if $\lceil \frac{nq}{p} \rceil + n < j \leq \lceil \frac{(n+1)q}{p} \rceil + n$, then

$$X_j = \begin{cases} B_1 & \text{if } j + n \text{ is odd} \\ B_2^{-1} & \text{if } j + n \text{ is even;} \end{cases}$$

and if $j = \lceil \frac{(n+1)q}{p} \rceil + n + 1$, then

$$X_j = \begin{cases} S & \text{if } j + n \text{ is odd} \\ S^{-1} & \text{if } j + n \text{ is even.} \end{cases}$$

Also define

$$\tilde{K}_{-p/q} = Y_1 Y_2 \cdots Y_{2p+2q},$$

where

$$Y_j = \begin{cases} B_1 & \text{if } X_{2p+2q+1-j} = B_2^{-1}; \\ B_2^{-1} & \text{if } X_{2p+2q+1-j} = B_1; \\ S & \text{if } X_{2p+2q+1-j} = S^{-1}; \\ S^{-1} & \text{if } X_{2p+2q+1-j} = S. \end{cases}$$

Finally define $\tilde{K}_{0/1} = B_1 B_2^{-1}$.

Proposition 3.2.1 *For all $p/q \in \hat{\mathbb{Q}}$, $K_{p/q} = \tilde{K}_{p/q}$ as Möbius transformations.*

Proof: Write out the word $K_{p/q}$ as $X_1 X_2 \cdots X_{p+2q}$, where each X_j is one of the letters $A^{\pm 1}$, $B_1^{\pm 1}$ or $B_2^{\pm 1}$. First assume $0/1 \leq p/q$. By definition, for $n = 0, 1, \ldots, p - 1$, if $\lceil \frac{(2n+1)q}{p} \rceil + n < j \leq \lceil \frac{(2n+2)q}{p} \rceil + n$, then

$$X_j = \begin{cases} B_1^{-1} & \text{if } j + n \text{ is odd} \\ B_2 & \text{if } j + n \text{ is even;} \end{cases}$$

and if $j = \lceil \frac{(2n+2)q}{p} \rceil + n + 1$, then

$$X_j = \begin{cases} A & \text{if } j+n \text{ is odd} \\ A^{-1} & \text{if } j+n \text{ is even.} \end{cases}$$

Write each A as the product SS and each A^{-1} as $S^{-1}S^{-1}$. Then use the identities

$$B_2 S = S^{-1} B_2^{-1}$$

and

$$B_1^{-1} S^{-1} = S B_1$$

to move letters $S^{\pm 1}$ to the left and change all the letters B_2 and B_1^{-1} appearing in the word $K_{p/q}$ into the letters B_2^{-1} and B_1. The remaining word is $\tilde{K}_{p/q}$.

The proof for $p/q < 0/1$ is exactly the same, except we use the identities

$$S^{-1} B_1^{-1} = B_1 S$$

and

$$S B_2 = B_2^{-1} S^{-1}$$

to move letters $S^{\pm 1}$ to the right and change all the letters B_2 and B_1^{-1} into B_2^{-1} and B_1.

q.e.d.

Lemma 3.2.2 *If $r > 0$, $p/q = r/s \oplus u/v$, $2r \leq n \leq 2p$, and $n \neq p$, then $\lceil \frac{nq}{p} \rceil - 2s = \lceil \frac{(n-2r)v}{u} \rceil$. Also, if $p \geq 2r$, then $\lceil \frac{(p-2r)v}{u} \rceil = q - 2s + 1$.*

Proof: Suppose u divides n. Then $\lceil \frac{(n-2r)v}{u} \rceil = \frac{nv}{u} + \lceil -\frac{2rv}{u} \rceil = \frac{nv}{u} + \lceil \frac{2-2su}{u} \rceil = \frac{nv}{u} + \lceil \frac{2}{u} \rceil - 2s$. Now $\frac{nq}{p} = \frac{nv}{u} + \frac{n}{pu}$, so $\lceil \frac{nq}{p} \rceil = \frac{nv}{u} + \lceil \frac{n}{pu} \rceil$. If $u = 1$ and $p < n \leq 2p$, then $\lceil \frac{2}{u} \rceil = 2$ and $\lceil \frac{n}{pu} \rceil = \lceil \frac{n}{p} \rceil = 2$. If $u = 1$ and $n < p$, the statement is vacuous since $r \geq u$ implies $2r \geq p$. If $u \geq 2$ and $n \leq 2p$, then $1 \leq \lceil \frac{n}{pu} \rceil \leq \lceil \frac{2}{u} \rceil \leq 1$, and so $\lceil \frac{n}{pu} \rceil = \lceil \frac{2}{u} \rceil = 1$. Thus the result follows.

Next suppose u does not divide n. Lemma 2.2.1 implies that if $2r \leq n \leq 2p$ then $\lceil\frac{(n-2r)q}{p}\rceil = \lceil\frac{(n-2r)v}{u}\rceil$. Thus $\lceil\frac{nq}{p} - \frac{2rq}{p}\rceil = \lceil\frac{(n-2r)v}{u}\rceil$. Now since $\frac{ns}{r} = \frac{nq}{p} + \frac{n}{rp}$, $\frac{2rq}{p} < 2s$. Thus $\lceil\frac{nq}{p}\rceil - 2s \leq \lceil\frac{nq}{p}\rceil - \lceil\frac{2rq}{p}\rceil \leq \lceil\frac{nq}{p} - \frac{2rq}{p}\rceil = \lceil\frac{(n-2r)v}{u}\rceil$.

If $n = 2p$ then $\lceil\frac{nq}{p}\rceil - 2s = 2q - 2s = 2v$, and $\lceil\frac{(n-2r)v}{u}\rceil = \lceil\frac{2uv}{u}\rceil = 2v$ also.

If $n = 2r + u$ then $\lceil\frac{(n-2r)v}{u}\rceil = v$, and

$$\begin{aligned}
\lceil\frac{nq}{p}\rceil - 2s &= \lceil\frac{(2r+u)q}{p}\rceil - 2s \\
&= \lceil\frac{rq}{p} + \frac{pq}{p}\rceil - 2s \\
&= \lceil\frac{rq}{p} + q\rceil - 2s \\
&= \lceil\frac{sp-1}{p} + q\rceil - 2s \\
&= \lceil s + q - \frac{1}{p}\rceil - 2s \\
&= s + q - 2s \\
&= v.
\end{aligned}$$

Otherwise, $\frac{nq}{p}$ and $\frac{(n-2r)v}{u}$ are not integers. If $\lceil\frac{nq}{p}\rceil - 2s \neq \lceil\frac{(n-2r)v}{u}\rceil$, then there is some integer K with

$$\frac{nq}{p} - 2s < K < \frac{(n-2r)v}{u}.$$

Moreover $K - (\frac{nq}{p} - 2s) \geq \frac{1}{p}$ and $\frac{(n-2r)v}{u} - K \geq \frac{1}{u}$, so $\frac{(n-2r)v}{u} - (\frac{nq}{p} - 2s) \geq \frac{1}{p} + \frac{1}{u}$. On the other hand,

$$\begin{aligned}
\frac{(n-2r)v}{u} - (\frac{nq}{p} - 2s) &= \frac{(n-2r)v}{u} - (\frac{nv}{u} + \frac{n}{pu} - 2s) \\
&= 2s - \frac{2rv}{u} - \frac{n}{pu}.
\end{aligned}$$

Now since $n \geq 2r$, $\frac{n}{pu} \geq \frac{2r}{pu} = \frac{2(r+u-u)}{pu} = \frac{2(p-u)}{pu} = 2(\frac{1}{u} - \frac{1}{p}) > \frac{1}{u} - \frac{1}{p}$. Thus

$$\begin{aligned}
2s - \frac{2rv}{u} - \frac{n}{pu} &< 2s - \frac{2rv}{u} + \frac{1}{p} - \frac{1}{u} \\
&= 2s - \frac{2su-2}{u} + \frac{1}{p} - \frac{1}{u}
\end{aligned}$$

$$= 2s - 2s + \frac{2}{u} + \frac{1}{p} - \frac{1}{u}$$
$$= \frac{1}{u} + \frac{1}{p}.$$

Hence the result follows for $n \neq p$.

The second part of the lemma follows from the equalities $\lceil \frac{(p-2r)v}{u} \rceil = \lceil \frac{(-r+u)v}{u} \rceil = \lceil v - \frac{su-1}{u} \rceil = \lceil v - s + \frac{1}{u} \rceil = v - s + 1 = q - 2s + 1$.

q.e.d.

Proposition 3.2.3 *Suppose $p/q = r/s \oplus u/v$, $ru \geq 0$,*

$$\tilde{K}_{r/s}\tilde{K}_{u/v} = X_1 X_2 \cdots X_{2p+2q},$$

and

$$\tilde{K}_{p/q} = Y_1 Y_2 \cdots Y_{2p+2q},$$

where

$$X_j, Y_j \in \{B_1, B_2^{-1}, S, S^{-1}\} \text{ for } 1 \leq j \leq 2p+2q.$$

Then $X_j = Y_j$ for all j except $j = p+q$ and $j = p+q+1$, and $Y_{p+q} = X_{p+q+1}^{-1}$ and $Y_{p+q+1} = X_{p+q}$.

Proof: First suppose that $r > 0$. By definition, for $n = 0, 1, ..., 2r-1$, if $\lceil \frac{ns}{r} \rceil + n < j \leq \lceil \frac{(n+1)s}{r} \rceil + n$, then

$$X_j = \begin{cases} B_1 & \text{if } j+n \text{ is odd} \\ B_2^{-1} & \text{if } j+n \text{ is even;} \end{cases}$$

and if $j = \lceil \frac{(n+1)s}{r} \rceil + n + 1$, then

$$X_j = \begin{cases} S & \text{if } j+n \text{ is odd} \\ S^{-1} & \text{if } j+n \text{ is even;} \end{cases}$$

and for $n = 2r, 2r+1, ..., 2p-1$, if $\lceil \frac{(n-2r)v}{u} \rceil + n - 2r < j - 2r - 2s \leq \lceil \frac{(n-2r+1)v}{u} \rceil + n - 2r$, then

$$X_j = \begin{cases} B_1 & \text{if } j+n \text{ is odd} \\ B_2^{-1} & \text{if } j+n \text{ is even;} \end{cases}$$

and if $j - 2r - 2s = \lceil \frac{(n-2r+1)v}{u} \rceil + n - 2r + 1$, then

$$X_j = \begin{cases} S & \text{if } j+n \text{ is odd} \\ S^{-1} & \text{if } j+n \text{ is even.} \end{cases}$$

Since $p/q = r/s \oplus u/v$, r/s and p/q are Farey neighbors. Lemma 2.2.1 gives the result for $1 \leq j \leq 2r+2s$. If $2r < p$, then $Y_j = X_j$ for $1 \leq j \leq 2r+2s$. If $2r \geq p$, then $Y_j = X_j$ for $1 \leq j \leq 2r+2s$, except $X_{p+q}X_{p+q+1} = B_2^{-1}S$ or B_1S^{-1} and $Y_{p+q}Y_{p+q+1} = S^{-1}B_2^{-1}$ or SB_1, respectively. Similarly, Lemma 3.2.2 gives the result for $2r+2s+1 \leq j \leq 2p+2q$.

Now if $r = 0$, then $s = 1$, $u = 1$, and $p = 1$. Thus $\lceil \frac{pq}{p} \rceil = q = v+1$ and $\lceil \frac{pv}{u} \rceil = pv = v$; and $\lceil \frac{2pq}{p} \rceil - 2s = 2q - 2 = 2v$ and $\lceil \frac{2pv}{u} \rceil = 2v$. The conclusion thus holds for $r \geq 0$.

Note that if $p/q = r/s \oplus u/v$, then $-p/q = -u/v \oplus -r/s$. It follows from the definition of the words $\tilde{K}_{-p/q}$ that the proposition also holds for the cases where $r, u < 0$.

q.e.d.

3.3 The relationship between the words in G_x and the words in H_x

Let M denote the transformation $M(z) = z+1$.

Theorem 3.3.1 *If $0/1 \leq p/q \leq 1/0$, then $MK_{p/q}M^{-1} = (W_{p/q})^2$.*

Proof: We will show that $M\tilde{K}_{p/q}M^{-1} = (W_{p/q})^2$ using induction. The statement can easily be checked for $p/q = 0/1$ and $1/0$. Assume the statement is true for r/s and n/m, where $sn - rm = 1$ and $rn \geq 0$; we show the statement is true for $p/q = r/s \oplus n/m$. So, by assumption,

$$M\tilde{K}_{r/s}\tilde{K}_{n/m}M^{-1} = (W_{r/s})^2(W_{n/m})^2.$$

Write

$$(W_{r/s})^2(W_{n/m})^2 = U_1U_2\cdots U_{2p+2q},$$

where

$$U_j \in \{T^{-1}, S\} \; for \; 1 \leq j \leq 2p+2q.$$

Also, write

$$(W_{r/s})^2(W_{n/m})^2 = V_1V_2\cdots V_N,$$

where

$$V_j \in \{X_M, Y_{M,1}, Y_{M,2}, Y_{M,3}, ..., S, S^2, S^3, ...\} \; for \; 1 \leq j \leq N.$$

Define the words X_K and $Y_{K,n}$ by $X_K = B_1B_2^{-1}$ and $Y_{K,n} = B_1S^{-n}B_2^{-1}$. Then $X_M = MX_KM^{-1}$, $Y_{M,n} = MY_{K,n}M^{-1}$, and $S^n = MS^nM^{-1}$. Further, define the words $\tilde{V}_j$, $1 \leq j \leq N$, by

$$\tilde{V}_j = \begin{cases} X_K & \text{if } V_j = X_M, \\ Y_{K,n} & \text{if } V_j = Y_{M,n}, \\ S^n & \text{if } V_j = S^n. \end{cases}$$

Then

$$\tilde{K}_{r/s}\tilde{K}_{n/m} = \tilde{V}_1\tilde{V}_2\cdots\tilde{V}_N.$$

Now by Corollary 2.2.5, $U_{p+q+1} = S$. Suppose U_{p+q+1} appears in the word V_{j_0} in the decomposition

$$(W_{r/s})^2(W_{n/m})^2 = V_1V_2\cdots V_N.$$

Then $V_{j_0} = Y_{M,n}$ or $V_{j_0} = S^n$ for some n. If $V_{j_0} = Y_{M,n}$, then by Corollary 2.2.5

$$(W_{p/q})^2 = V_1V_2\cdots V_{j_0-1}ST^{-1}S^{n-1}T^{-1}V_{j_0+1}\cdots V_N.$$

But

$$\tilde{K}_{r/s}\tilde{K}_{n/m} = \tilde{V}_1\tilde{V}_2\cdots\tilde{V}_{j_0-1}B_1S^{-n}B_2^{-1}\tilde{V}_{j_0+1}\cdots\tilde{V}_N,$$

so by Proposition 3.2.3,

$$\tilde{K}_{p/q} = \tilde{V}_1\tilde{V}_2\cdots\tilde{V}_{j_0-1}SB_1S^{-n+1}B_2^{-1}\tilde{V}_{j_0+1}\cdots\tilde{V}_N.$$

So in this case $M\tilde{K}_{p/q}M^{-1} = (W_{p/q})^2$.

If, on the other hand, $V_{j_0} = S^n$, then by Corollary 2.2.5 and Proposition 2.2.6, $V_{j_0-1} = T^{-1}S^mT^{-1}$ for some m and

$$(W_{p/q})^2 = V_1V_2\cdots V_{j_0-2}T^{-1}S^{m+1}T^{-1}S^{n-1}V_{j_0+1}\cdots V_N.$$

In this case,

$$\tilde{K}_{r/s}\tilde{K}_{n/m} = \tilde{V}_1\tilde{V}_2\cdots\tilde{V}_{j_0-2}B_1S^{-m}B_2^{-1}S^n\tilde{V}_{j_0+1}\cdots\tilde{V}_N,$$

and by Proposition 3.2.3,

$$\tilde{K}_{p/q} = \tilde{V}_1\tilde{V}_2\cdots\tilde{V}_{j_0-2}B_1S^{-m+1}B_2^{-1}S^{n-1}\tilde{V}_{j_0+1}\cdots\tilde{V}_N.$$

Again, $M\tilde{K}_{p/q}M^{-1} = (W_{p/q})^2$.

q.e.d.

Corollary 3.3.2 *For all $p/q \in \hat{\mathbb{Q}}$, $MK_{p/q}M^{-1} = (W_{p/q})^2$.*

Proof: By Theorem 3.3.1, we need only show the result for $p/q < 0/1$. Again we use $\tilde{K}_{p/q}$. Assume $p/q > 0/1$, and write

$$(W_{p/q})^2 = U_1U_2\cdots U_N,$$

where

$$U_j \in \{X_M, Y_{M,1}, Y_{M,2}, Y_{M,3}, ..., S, S^2, S^3, ...\} \text{ for } 1 \le j \le N.$$

(Here, if $U_j = S^n$, then $U_{j+1} \neq S^m$.) Then by definition,

$$(W_{-p/q})^2 = V_1V_2\cdots V_N,$$

where

$$V_j = \begin{cases} X_M & \text{if } U_{N+1-j} = X_M; \\ Y_{M,-n} & \text{if } U_{N+1-j} = Y_{M,n}; \\ S^{-n} & \text{if } U_{N+1-j} = S^n. \end{cases}$$

Also, by the definition of $\tilde{K}_{-p/q}$, if

$$\tilde{K}_{p/q} = \tilde{U}_1 \tilde{U}_2 \cdots \tilde{U}_N,$$

where

$$\tilde{U}_j \in \{X_K, Y_{K,1}, Y_{K,2}, Y_{K,3}, ..., S, S^2, S^3, ...\} \; for \; 1 \le j \le N,$$

then

$$\tilde{K}_{-p/q} = \tilde{V}_1 \tilde{V}_2 \cdots \tilde{V}_N,$$

where

$$\tilde{V}_j = \begin{cases} X_K & \text{if } \tilde{U}_{N+1-j} = X_K; \\ Y_{K,-n} & \text{if } \tilde{U}_{N+1-j} = Y_{K,n}; \\ S^{-n} & \text{if } \tilde{U}_{N+1-j} = S^n. \end{cases}$$

Now, as in the proof of Theorem 3.3.1,

$$M \tilde{V}_j M^{-1} = V_j \; for \; 1 \le j \le N;$$

so $MK_{-p/q}M^{-1} = (W_{-p/q})^2$.

q.e.d.

Theorem 3.3.3 $M_{1,1} = M_{0,4}$.

Proof: Let ∂Y denote the boundary of the set Y. If c is a cusp on $\partial M_{1,1}$, then there is some word $W_{p/q}$ which becomes accidentally parabolic at $x = c$. But $(W_{p/q})^2$ is conjugate to $K_{p/q}$ by Corollary 3.3.2. Thus, c is a cusp on $\partial M_{0,4}$ also. Likewise, if c is a cusp on $\partial M_{0,4}$, then c is a cusp on $\partial M_{1,1}$.

Now C. McMullen ([McM91a]) has proven that cusps are dense in the boundary of Teichmüller space. Thus, $\partial M_{1,1} = \partial M_{0,4}$. Since both $M_{1,1}$ and $M_{0,4}$ contain the set $\{x : \mathrm{Im}(x) > 2\}$ (Proposition 3.1.3 and Proposition 2.3 of [Wri]), $M_{1,1} = M_{0,4}$.

q.e.d.

I. Kra (8.6 of [Kra90a]) gave a different proof of Theorem 3.3.3 using normalizers of the groups G_x and H_x to construct the identity map from $M_{0,4}$ to $M_{1,1}$. Corollary 3.3.2 does not follow from the arguments there.

CHAPTER 4

SPHERES WITH FIVE PUNCTURES

4.1 The embedding of $T_{0,5}$

To construct surfaces of type (0,5), take a surface of type (0,4) and a surface of type (0,3), cut off a puncture from each along simple closed curves homotopic to the punctures, and glue the simple closed curves together. This is the same basic construction used to create surfaces of type (0,4), and just like in that case, the algebraic building tools are the amalgamated free product and Maskit's First Combination Theorem. Here the group representing the surface of type (0,5) will be the amalgamated free product of H_x (the group representing a surface of type (0,4)) with a group representing a surface of type (0,3) across a common cyclic parabolic subgroup. Let P be a Möbius transformation such that $PAP^{-1} = B_1$ and $PB_1P^{-1} = A$. Then the matrix for P is $\begin{pmatrix} 0 & -2 \\ \frac{1}{2} & 0 \end{pmatrix}$ (and $P(z) = -\frac{4}{z}$). Also,

$$PB_{2,x}P^{-1} = \begin{pmatrix} 1+x & 4 \\ -\frac{x^2}{4} & 1-x \end{pmatrix} = B_{3,x}.$$

Thus, the following elements are parabolic: $A, B_1, B_{2,x}, B_{3,y}, AB_1, AB_{2,x}, B_1B_{3,y}$. (We now use the complex parameter y in the transformation B_3 to distinguish it from the parameter in B_2.) The group $\langle B_1, B_{3,y}\rangle$ represents a surface of type (0,3). Let $H_{x,y}$ denote the amalgamated free product of H_x with $\langle B_1, B_{3,y}\rangle$ across $\langle B_1\rangle$. (See Figure 4.1.) The embedding of $T_{0,5}$ is the set of all $(x,y) \in \mathbb{H}^2$ such that $H_{x,y}$ is a terminal b-group and $\Delta(H_{x,y})/H_{x,y}$ is a surface of type (0,5). We denote this set by $M_{0,5}$.

Note that since the fixed point of $B_{2,x}$ is x, the fixed point of $B_{3,y}$ is $-\frac{4}{y}$. Also, if $B_{2,x}$ takes the closed curve C_1 to C_2, then $B_{3,x}$ takes $P(C_1)$ to $P(C_2)$.

The next two propositions are the analogues to Propositions 3.1.1 and 3.1.2 for $M_{0,5}$.

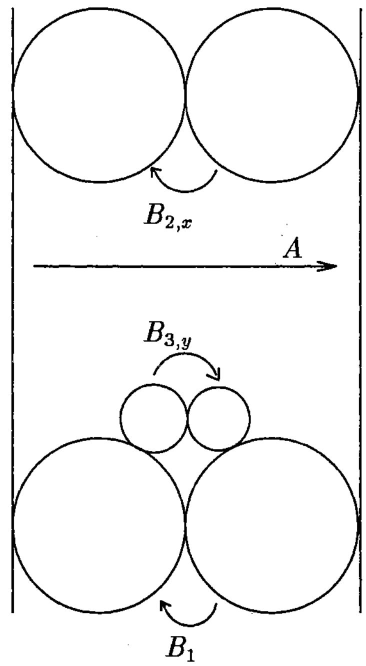


Figure 4.1: The action in the group $H_{x,y}$

Proposition 4.1.1 $(x, y) \in M_{0,5}$ *if and only if* $(x+2, y) \in M_{0,5}$ *if and only if* $(x, y+2) \in M_{0,5}$.

Proof: Since $AB_{2,x} = B_{2,x+2}^{-1}$, $H_{x,y} = H_{x+2,y}$. Likewise, $B_1 B_{3,y} = B_{3,y+2}^{-1}$, so $H_{x,y} = H_{x,y+2}$. Thus the groups $H_{x,y}$, $H_{x+2,y}$ and $H_{x,y+2}$ all represent the same Riemann surface, differently marked.

q.e.d.

Proposition 4.1.2 $(x, y) \in M_{0,5}$ *if and only if* $(-\overline{x}, -\overline{y}) \in M_{0,5}$.

Proof: Define $J(z) = -\overline{z}$. Then $J^{-1} = J$ and $JAJ = A^{-1}$, $JB_1J = B_1^{-1}$, $JB_{2,x}J = B_{2,-\overline{x}}^{-1}$, and $JB_{3,y}J = B_{3,-\overline{y}}^{-1}$. Hence J maps the limit set of $H_{x,y}$ to the limit set of $H_{-\overline{x},-\overline{y}}$.

q.e.d.

Proposition 4.1.3 $(x, y) \in M_{0,5}$ *if and only if* $(y, x) \in M_{0,5}$.

Proof: The proposition follows from the equalities $PAP^{-1} = B_1$, $PB_1P^{-1} = A$, $PB_{2,x}P^{-1} = B_{3,x}$ and $PB_{3,y}P^{-1} = B_{2,y}$.

q.e.d.

Proposition 3.1.3 has no single analogue for $M_{0,5}$, for there are infinitely many. For example, if we fix $x = 2i$, then it is easy to see using Maskit's First Combination Theorem that $(x, y) \in M_{0,5}$ if $\mathrm{Im}(y) > 4$.

4.2 Curves on spheres with five punctures

In order to understand the geometry of the cusps on the boundary of $T_{0,5}$ we must study simple closed curves on surfaces of type (0,5). Let S denote a surface of type (0,5). Label the punctures $q_1, \ldots, q_5$, and connect the 5 punctures with 5 arcs, labelled arc 1 to arc 5 in a counter-clockwise direction such that arc j connects q_j and q_{j+1} (see Figure 4.2). (Here the indices on q are taken modulo 5.) The union of the 5

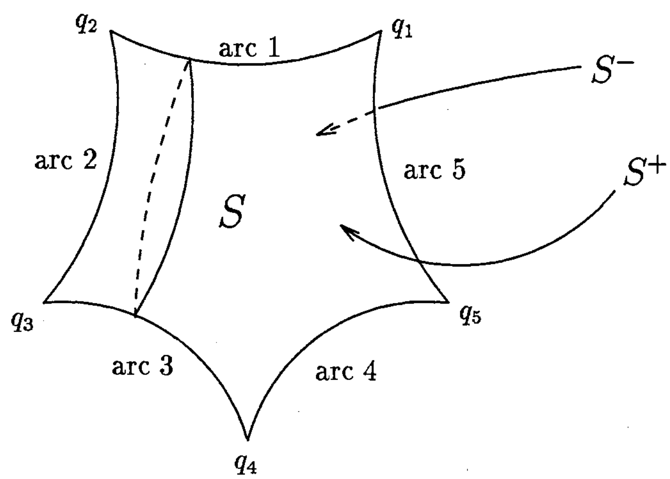


Figure 4.2: Punctures and arcs on S

arcs divides S into two components. Call these components S^+ and S^-. Let η be a simple closed curve on S which is not homotopic to a puncture and which is not homotopically trivial. We call such a curve *admissible*. The curve η divides S into two components, one which contains two punctures, and the other which contains three punctures. Call the component with two punctures $A(\eta)$.

The 5 arcs will intersect η in an even number N of points (since η is closed). From now on we will refer to these points as the *arc intersection points.* Label these points $P_1, ..., P_N$ in a counter-clockwise direction around the arcs, starting and ending at the puncture q_1.

The component $A(\eta)$ can be thought of as a thin strip on S, where the two sides of the strip meet near the two punctures inside $A(\eta)$. The sides make up the curve η, and if η has an orientation then the direction is opposite on different sides, and the component $A(\eta)$ consistently lies on the left or on the right of η. See the examples in Figure 4.3.

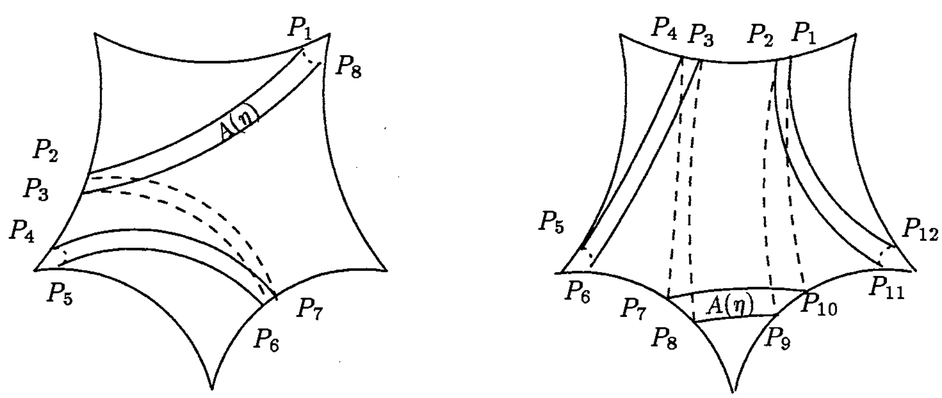


Figure 4.3: Simple closed curves on S

Let n_i denote the minimal number of arc intersection points in arc i, $1 \leq i \leq 5$, over all simple closed curves homotopic to η. Then since $N = \sum_{i=1}^{5} n_i$ is even, exactly 0, 2, or 4 of the integers n_i are odd. Since $A(\eta)$ contains exactly two of the five punctures, the arc intersection points occur in pairs on each arc except at the two ends of the strip $A(\eta)$ where the two punctures occur. We call the integers $(n_1, \ldots, n_5)$ the *arc intersection numbers* of η. (Henceforth, take all arc numbers and indices of punctures and arc intersection numbers modulo 5; and take all indices of arc intersection points modulo N.)

One can tell which two punctures $A(\eta)$ contains by the parity of the integers n_i. If n_i and n_{i+2} are the only odd integers of the five, $A(\eta)$ contains the punctures q_i and q_{i+1}. If n_j is the only even integer of the five, $A(\eta)$ contains the punctures q_{j+1} and q_{j+3}.

Proposition 4.2.1 *The component $A(\eta)$ of S contains the puncture q_j if and only if $n_j + n_{j+1} - n_{j+3}$ is odd.*

Proof: If $A(\eta)$ contains q_j, then either n_j and n_{j+2} are the only odd arc intersection numbers, or n_j and n_{j+1} are the only odd ones, or all arc intersection numbers are odd except either n_{j+4} or n_{j+2}. The proposition follows.

q.e.d.

Proposition 4.2.2 *For each j, $n_j \leq n_{j+2} + n_{j+3}$.*

Proof: Starting with an arc intersection point P_k on arc j, follow the oriented geodesic on which it lies. It is not difficult to see that the geodesic must pass through arc $j+2$ or arc $j+3$ before returning to arc j. The proposition follows.

q.e.d.

Proposition 4.2.3 *For some j, $n_j + n_{j+1} = N/2$. Furthermore, on one side S^ϵ of S (either S^+ or S^-), each segment of $\eta \cap S^\epsilon$ has one endpoint on either arc j or $j+1$ and the other on arc $j+2$, $j+3$ or $j+4$.*

Proof: Consider the disjoint segments of $\eta \cap S^+$ and $\eta \cap S^-$. If any one of these segments in S^+ connects arcs i and $i+1$, then there can be no such segment in S^- because η is simple and not homotopic to a puncture.

Suppose there is a pair of adjacent arcs such that there is no segment of either $\eta \cap S^+$ or $\eta \cap S^-$ which connects the adjacent arcs. Then if the puncture between these arcs is filled in, η cannot be deformed to make any arc intersection number smaller. Hence η can be thought of as a simple closed curve on a sphere with four punctures. Since the arc intersection numbers on a sphere with four punctures must be equal on opposite arcs, the index j must exist.

On the other hand, suppose there is no such pair of adjacent arcs. There is some segment in $\eta \cap S^+$; denote the endpoints of the segment by P_i and P_k, where $i < k$. If $k = i + 1$, then this segment connects adjacent arcs. Otherwise, there is a segment in $\eta \cap S^+$ with endpoints P_ℓ and P_m, where $i < \ell < m < k$. Continuing in this manner, we see that there must exist a segment in $\eta \cap S^+$ with endpoints P_n and P_{n+1}, and this segment connects adjacent arcs. Furthermore, by looking at the segments with endpoints P_m where $m < i$ and $m > k$, it is not difficult to see that there must be some other segment in $\eta \cap S^+$ which connects another pair of adjacent arcs. Hence at least two pairs of adjacent arcs are connected by segments in $\eta \cap S^+$, and likewise at least two pairs of adjacent arcs are connected by segments in $\eta \cap S^-$. Let S^ϵ denote the side on which exactly two pairs of adjacent arcs are connected by segments. If the pairs of arcs are $i, i+1$ and $i+2, i+3$ for some i, then clearly $n_{i+1} + n_{i+2} = N/2$ and each segment in S^ϵ has one endpoint on arc $i+1$ or $i+2$ and the other on arc $i+3$, $i+4$ or i. Otherwise, the pairs are i, $i+1$ and $i+1$, $i+2$ for some i. In this case, $n_{i+1} = N/2$, but this contradicts Proposition 4.2.2.

q.e.d.

Deform S so that the five punctures all lie in the same plane. Then let ρ denote the reflection in this plane.

Proposition 4.2.4 *If two admissible simple closed curves have the same arc intersection numbers then they are either homotopic or one is homotopic to the reflection ρ of the other.*

Proof: Suppose an admissible simple closed curve η on S has the arc intersection numbers $(n_1, \ldots, n_5)$. Let j denote the index such that $n_j + n_{j+1} = N/2$. Then on one side S^ϵ of S (either S^+ or S^-) each segment must have one endpoint on arc j or $j+1$ and the other endpoint on arc $j+2$, $j+3$ or $j+4$. Now start with an arc intersection point P on arc i, $i \neq j+2, j+4$. Following η in either direction, η must cross arc $i+2$ or $i+3$ before returning to arc i. If $n_{i+2} + n_{i+3} > n_i$, then η must cross arcs $i+2$ and $i+3$ exactly $n_{i+2} + n_{i+3} - n_i$ times more often than it crosses arc i before it returns to P. Hence there must be $n_{i+2} + n_{i+3} - n_i$ segments in $S^{-\epsilon}$ which help η to do so. There must be arc intersection points P_k and P_{k+1} on arcs $i+2$ and $i+3$, respectively, which are connected by a segment in $S^{-\epsilon}$, and the other $n_{i+2} + n_{i+3} - n_i - 1$ segments must connect endpoints of the form $P_{k-\ell}$, $P_{k+1+\ell}$. Since $(n_{j+2} + n_{j+3} - n_j) + (n_{j+3} + n_{j+4} - n_{j+1}) + (n_j + n_{j+1} - n_{j+3}) = N/2$, all the segments of $S^{-\epsilon}$ have been characterized.

Now if μ is another admissible simple closed curve with the same arc intersection numbers, then the segments of μ must adhere to the same characterizations as the segments of η, except the side S^ϵ might be different, in which case μ is homotopic to $\rho(\eta)$.

q.e.d.

We now want to consider pairs of disjoint simple closed curves on S. If each geodesic in a pair of disjoint simple closed geodesics on S is pinched to a point, the resulting surface is no longer a sphere with five punctures, and it is a maximal cusp on the boundary of $T_{0,5}$. We define the *arc intersection numbers* of a set of simple closed curves on S to be the sum of the arc intersection numbers of each of the curves in the set.

Each oriented admissible simple closed curve on S is represented by an element of $H_{x,y}$, unique up to conjugation in $H_{x,y}$. The letters in this word can be obtained by following the curve through the arcs 1 through 5. Each time an arc is crossed, one of the letters A, B_1, B_2, B_3, I, or an inverse of one of these letters is added to the word.

Figure 4.4 shows which arcs are associated to which letters and gives an example of the word obtained for a specific oriented curve.

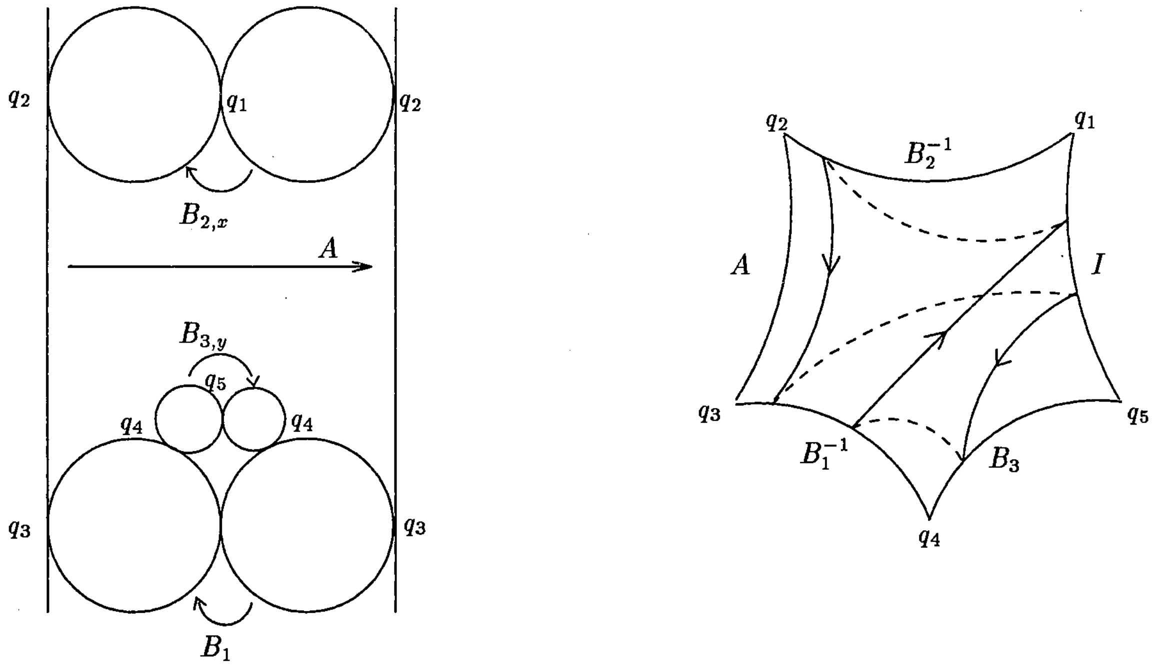


Figure 4.4: The curve represented by the word $B_2IB_1B_3IB_1^{-1} = B_2B_1B_3B_1^{-1}$

Theorem 4.2.5 *Let $(n_1, \ldots, n_5)$ be the arc intersection numbers for a pair of disjoint simple closed geodesics on S. If (x, y) is the maximal cusp on the boundary of $T_{0,5}$ corresponding to this pair of geodesics, then $(-\overline{x}, -\overline{y})$ is a maximal cusp which corresponds to the other pair of geodesics on S with arc intersection numbers $(n_1, \ldots, n_5)$.*

Proof: If $X_1X_2 \cdots X_I$ and $Y_1Y_2 \cdots Y_K$ are the words corresponding to one pair of disjoint simple closed geodesics with arc intersection numbers $(n_1, \ldots, n_5)$, then $X_1^{-1}X_2^{-1} \cdots X_I^{-1}$ and $Y_1^{-1}Y_2^{-1} \cdots Y_K^{-1}$ are the words corresponding to the other pair of geodesics with the same arc intersection numbers. Define $J(z) = -\overline{z}$. Then $JAJ = A^{-1}$, $JB_1J = B_1^{-1}$, $JB_{2,x}J = B_{2,-\overline{x}}^{-1}$, and $JB_{3,y}J = B_{3,-\overline{y}}^{-1}$. Since X is parabolic if and only if JXJ is parabolic, the theorem follows.

q.e.d.

Theorem 4.2.6 *If the maximal cusp (x, y) corresponds to the arc intersection numbers $(n_1, n_2, n_3, n_4, n_5)$, then (y, x) is a maximal cusp which corresponds to the arc intersection numbers $(n_4, n_3, n_2, n_1, n_5)$.*

Proof: Define $P(z) = -4/z$. Then $PAP^{-1} = B_1$, $PB_1P^{-1} = A$, $PB_{2,x}P^{-1} = B_{3,x}$, and $PB_{3,y}P^{-1} = B_{2,y}$. The theorem follows.

q.e.d.

Proposition 4.2.7 *Fix the nonnegative integers n_1, n_2, n_3, and n_4. Then there are at most two possible values of n_5 for which $(n_1, \ldots, n_5)$ satisfies the properties $n_j + n_{j+1} = N/2$ for some j and $n_i \leq n_{i+2} + n_{i+3}$ for all i.*

Proof: The integer n_5 is uniquely determined by the index j for which $n_j + n_{j+1} = N/2$ is satisfied.

Suppose first that $n_1 + n_2 \geq n_2 + n_3$ and $n_1 + n_2 \geq n_3 + n_4$. Then $j = 1$, 4, or 5. If $n_1 > n_4$, then $j \neq 4$, otherwise $n_5 = n_1 + n_2 + n_3 - n_4 > n_2 + n_3$, contradicting the second property. Likewise, if $n_1 < n_4$, then $j \neq 5$. If $n_1 = n_4$, then whether $j = 4$ or $j = 5$, $n_5 = n_2 + n_3$. So, if $n_1 + n_2 \geq n_2 + n_3$ and $n_1 + n_2 \geq n_3 + n_4$, then n_5 can assume at most two distinct values.

Suppose next that $n_2 + n_3 \geq n_1 + n_2$ and $n_2 + n_3 \geq n_3 + n_4$. Then $j = 2$, 4, or 5. If $n_1 > n_4$ then $j \neq 4$; if $n_1 < n_4$ then $j \neq 5$; and if $n_1 = n_4$ then whether $j = 4$ or 5, $n_5 = n_2 + n_3$.

The case $n_3 + n_4 \geq n_1 + n_2$ and $n_3 + n_4 \geq n_2 + n_3$ follows similarly.

q.e.d.

Suppose we are given a set of simple closed curves on S with arc intersection numbers $(n_1, \ldots, n_5)$. We associate to this set the pair of extended rationals $(\eta_2/n_1, \eta_3/n_4)$, where η_2 and η_3 are defined as follows. If $n_4 \leq n_1$, then $|\eta_2| = n_2$ and $|\eta_3| = (n_3 + n_4 - n_1)$. If $n_4 > n_1$, then $|\eta_2| = (n_1 + n_2 - n_4)$ and $|\eta_3| = n_3$. If there is a segment in S^- connecting arcs 1 and 2 then let η_2 be positive; if there is such a segment in S^+ then let η_2 be negative. If there is a segment in S^+ connecting arcs 3 and 4, then let η_3 be positive; if there is such a segment in S^- then let η_3 be negative.

Theorem 4.2.8 *If there are two simple closed curves on S which yield the same extended rationals $(\eta_2/n_1, \eta_3/n_4)$, then the curves are homotopic.*

Proof: It is clear how the arc intersection numbers n_1 through n_4 can be reconstructed from the pair of extended rationals. By Proposition 4.2.7, there are only two possibilities for n_5; but the signs of η_2 and η_3 (along with the integers n_1 through n_4) determine the index j for which $n_j + n_{j+1} = N/2$. By the proof of Proposition 4.2.7, the last arc intersection number n_5 is completely determined. Since the signs of η_2 and η_3 are known, the theorem follows from Proposition 4.2.4.

q.e.d.

CHAPTER 5

INTERSECTION NUMBERS

5.1 Formulas for intersection numbers of multiple curves

A *multiple curve* on a surface is a set of disjoint simple closed curves on the surface, none of which is homotopic to a puncture or homotopically trivial. The *intersection number* of two multiple curves η and μ, denoted by $\iota(\eta, \mu)$, is the minimum number of intersection points of η_1 and μ_1, where η_1 and μ_1 range over all multiple curves isotopic to η and μ, respectively. Define the binary operation $*$ on $\hat{\mathbb{Q}}$ by $\eta_2/n_1 * \mu_2/m_1 = n_1\mu_2 - m_1\eta_2$. Further define the binary operation $*$ on pairs of extended rationals by $(\eta_2/n_1, \eta_3/n_4) * (\mu_2/m_1, \mu_3/m_4) = (\eta_2/n_1 * \mu_2/m_1) + (\eta_3/n_4 * \mu_3/m_4)$.

Theorem 5.1.1 *Let η_2/n_1 and μ_2/m_1 denote the rational numbers of two multiple curves η and μ, respectively, on a sphere with four punctures. Then*

$$\iota(\eta, \mu) = 2|\eta_2/n_1 * \mu_2/m_1|.$$

Proof: Let L denote the lattice $\{n + mi : n, m \in \mathbb{Z}\}$, and consider the Riemann surface $S = (\mathbb{C} - L)/G$, where G is the group generated by $z \mapsto z + 2$, $z \mapsto z + 2i$, and $z \mapsto -\bar{z}$. The surface S is a sphere with four punctures. Thus we can consider the two multiple curves to lie on S.

If the theorem is true for admissible simple closed curves η and μ, then any multiple curve on S is represented by a rational number λ_2/ℓ_1, where the greatest common divisor of $|\lambda_2|$ and ℓ_1 is the number of components of the multiple curve. Thus, if the theorem is true for admissible simple closed curves, it is also true for multiple curves. Hence we can assume that η and μ are admissible simple closed curves.

Let x be a point in $\mathbb{C} - L$ which projects to a point on η. Then the loop η lifts to a curve $\tilde{\eta}$ from x to $x + 2\eta_2 + 2n_1 i$. The curve $\tilde{\eta}$ lies completely within the rectangle with vertices x, $x + 2\eta_2$, $x + 2n_1 i$, and $x + 2\eta_2 + 2n_1 i$. Consider all the liftings of the curve μ. The minimal number of intersection points of η and μ must equal the minimal number of intersection points of $\tilde{\eta}$ with the liftings of μ.

Now $2|\eta_2|m_1$ of the liftings of μ must cross the line segment from x to $x+2\eta_2$, and $2|\mu_2|n_1$ of the liftings of μ must cross the line segment from $x + 2\eta_2$ to $x + 2\eta_2 + 2n_1 i$.

Suppose $\operatorname{sgn}(\eta_2) = \operatorname{sgn}(\mu_2)$. Then all the liftings of μ within the rectangle connect the line segments from x to $x+2n_1 i$ and x to $x+2\eta_2$ to the line segments from $x+2\eta_2$ to $x + 2\eta_2 + 2n_1 i$ and $x + 2n_1 i$ to $x + 2\eta_2 + 2n_1 i$. In this case it is clear that the minimal number of intersection points of $\tilde{\eta}$ with the liftings of μ is $|2|\eta_2|m_1 - 2|\mu_2|n_1|$. Since $\operatorname{sgn}(\eta_2) = \operatorname{sgn}(\mu_2)$, this expression is equal to $2|\eta_2/n_1 * \mu_2/m_1|$.

On the other hand, suppose $\operatorname{sgn}(\eta_2) \neq \operatorname{sgn}(\mu_2)$. Then all the liftings of μ within the rectangle connect the line segments from x to $x+2n_1 i$ and $x+2n_1 i$ to $x+2n_1 i+2\eta_2$ to the line segments from x to $x + 2\eta_2$ and $x + 2\eta_2$ to $x + 2\eta_2 + 2n_1 i$. In this case the minimal number of intersection points is $|2|\eta_2|m_1 + 2|\mu_2|n_1| = 2|\eta_2/n_1 * \mu_2/m_1|$.

q.e.d.

We note that Theorem 5.1.1 implies that η_2/n_1 and μ_2/m_1 are Farey neighbors if and only if the intersection number of the corresponding simple closed curves is 2. This may be helpful when we try to generalize the notion of Farey neighbors for multiple curves on spheres with five punctures.

Theorem 5.1.2 *Suppose the arc intersection numbers of two multiple curves η, μ on a sphere S with five punctures are $(n_1, \ldots, n_5)$ and $(m_1, \ldots, m_5)$, with rational numbers $(\eta_2/n_1, \eta_3/n_4)$ and $(\mu_2/m_1, \mu_3/m_4)$, respectively. Suppose also that $n_1 + n_2 = n_4$ and $m_1 + m_2 = m_4$. Then*

$$\iota(\eta, \mu) = 2|\eta_3/n_4 * \mu_3/m_4|.$$

Proof: If $n_1 + n_2 = n_4$ and $m_1 + m_2 = m_4$ then by the proof of Proposition 4.2.4, there is no segment of either multiple curve in either S^+ or S^- which connects arcs 1 and 2. Thus, if we fill in the puncture q_2, the multiple curves cannot be deformed in such a way that their arc intersection numbers are reduced. The surface with q_2 filled in is a sphere with four punctures. Re-label the arcs between punctures so that the union of arcs 1 and 2 (along with the filled-in puncture) becomes arc 1, and arc j becomes arc $j-1$ for $3 \leq j \leq 5$. Then the rationals for the multiple curves are η_3/n_4 and μ_3/m_4, so the result follows from Theorem 5.1.1.

q.e.d.

Let η and μ be two multiple curves on a sphere S with four punctures. Consider the segments of $\eta \cap S^+$ and $\eta \cap S^-$. If any segment has an endpoint on arc 1, then orient the segment so that it is directed towards arc 1. If any segment has an endpoint on arc 3, direct the segment away from arc 3. If a segment has an endpoint on arc 2 or 4, direct the segment so that the direction does not change at the endpoint on that arc. If η only intersects arcs 2 and 4, then direct the segment in S^+ from arc 2 towards arc 4. Orient the segments of $\mu \cap S^+$ and $\mu \cap S^-$ using the same rules. The directions of the segments cannot in general be combined to give a direction on η or μ. At any intersection point of η and μ, traveling along η in the positive direction, if the segment of μ is directed from left to right, we call the intersection point *positively oriented* (with respect to η); and if the segment of μ is directed from right to left, we call the intersection point *negatively oriented* (with respect to η).

Lemma 5.1.3 *Deform η and μ so that the number of intersection points of the two curves is minimized. Suppose $\eta_2/n_1 > \mu_2/m_1$. Then all intersection points are positively oriented. If $\eta_2/n_1 < \mu_2/m_1$ then the all intersection points are negatively oriented.*

Proof: Consider S to be the surface $(\mathbb{C} - L)/G$, where L is the lattice $\{n + mi : n, m \in \mathbb{Z}\}$ and G is the group generated by $z \mapsto z + 2$, $z \mapsto z + 2i$ and $z \mapsto -\overline{z}$. Consider all the liftings of η and μ in the fundamental rectangle $0 \leq \mathrm{Re}(z) \leq 2$, $0 \leq \mathrm{Im}(z) \leq 1$. Direct each of these liftings in the same way the corresponding segments on S are directed.

Suppose $\eta_2/n_1 > \mu_2/m_1$. Then the ratio of the number of points on the vertical sides of the rectangle to the number of points on the horizontal sides is greater for the curve η than for μ. If $\mu_2 \geq 0$ then each of the lifts of the segments of η and μ is directed from the right vertical side or the bottom horizontal side of the rectangle to the left or top side. Thus the intersection points are positively oriented. If $\mu_2 < 0$ and $\eta_2 \geq 0$ then each lift of a segment of η is directed from the bottom or right side to the top or left side and each lift of a segment of μ is directed from the bottom or left side to the top or right side. Hence the intersection points are positively oriented. Likewise, if $\mu_2 < 0$ and $\eta_2 < 0$, each lift of any segment is directed from the bottom or left side to the right or top side of the rectangle, and so the intersection points are positively oriented.

If $\eta_2/n_1 < \mu_2/m_1$, the result follows by switching the roles of η and μ.

q.e.d.

Theorem 5.1.4 *Suppose the arc intersection numbers of two multiple curves η and μ on a sphere S with five punctures are $(n_1, \ldots, n_5)$ and $(m_1, \ldots, m_5)$, with rational numbers $(\eta_2/n_1, \eta_3/n_4)$ and $(\mu_2/m_1, \mu_3/m_4)$, respectively. Suppose $n_4 + n_5 = n_1 + n_2 + n_3$ and $m_4 + m_5 = m_1 + m_2 + m_3$. Then if $\eta_2/n_1 * \mu_2/m_1$ and $\eta_3/n_4 * \mu_3/m_4$ have the same sign, then*

$$\iota(\eta, \mu) = 2|(\eta_2/n_1, \eta_3/n_4) * (\mu_2/m_1, \mu_3/m_4)|.$$

If $\eta_2/n_1 * \mu_2/m_1$ *and* $\eta_3/n_4 * \mu_3/m_4$ *have opposite signs, then*

$$\begin{aligned}\iota(\eta,\mu) &= 2|\eta_2/n_1 * \mu_2/m_1| + 2|\eta_3/n_4 * \mu_3/m_4| \\ &- \; min\{4n_1m_1,\, 4|\eta_2/n_1 * \mu_2/m_1|,\, 4|\eta_3/n_4 * \mu_3/m_4|\}.\end{aligned}$$

Proof: Since $n_4+n_5 = n_1+n_2+n_3$ and $m_4+m_5 = m_1+m_2+m_3$, we have $n_4 \geq n_1$, $m_4 \geq m_1$, $n_5 = (n_1+n_2-n_4)+n_3 = |\eta_2|+n_3$, $m_5 = (m_1+m_2-m_4)+m_3 = |\mu_2|+m_3$, $\mathrm{sgn}(\eta_2) = \mathrm{sgn}(\eta_3)$, and $\mathrm{sgn}(\mu_2) = \mathrm{sgn}(\mu_3)$. Construct a simple closed curve C on S whose intersection numbers are (0,1,0,0,1), where the arc intersection point on arc 2 lies so that $|\eta_2|$ of the arc intersection points of η on arc 2 are closer to the puncture q_2 and $n_4 - n_1$ are closer to q_3; and where the C arc intersection point on arc 5 lies so that $|\eta_2|$ of the arc intersection points of η on arc 5 are closer to q_1 and n_3 are closer to q_5. Construct C so that $\rho(C) = C$, where $\rho : S \to S$ is the reflection in the plane through the punctures of S. Next deform the curve μ so that $|\mu_2|$ of the arc intersection points of μ on arc 2 are closer to q_2 than the arc intersection point of C on arc 2, and $m_4 - m_1$ are closer to q_3; and so that $|\mu_2|$ of the arc intersection points of μ on arc 5 are closer to q_1 than the arc intersection point of C, and m_3 are closer to q_5 (see Figures 5.1 and 5.2).

The multiple curves η and μ may intersect the curve C in several points, henceforth called *C-intersection points.* Now deform η and μ by fixing their arc intersection points but moving their C-intersection points so that ρ identifies each C-intersection point on η with another C-intersection point on η and each C-intersection point on μ with another C-intersection point on μ; we do this while keeping each curve simple.

Cut S along C, and identify $\rho(x)$ with x for each point x on C. By making punctures q_6 and q_7 at the points where C intersects arcs 2 and 5, we obtain a sphere S_4 with four punctures and a sphere S_5 with five punctures. The multiple curves η and μ have become multiple curves on each of these spheres, with arc intersection numbers $(n_1, |\eta_2|, n_1, |\eta_2|)$, $(m_1, |\mu_2|, m_1, |\mu_2|)$ on S_4 and $(n_1, n_4 - n_1, n_3, n_4, n_3)$,

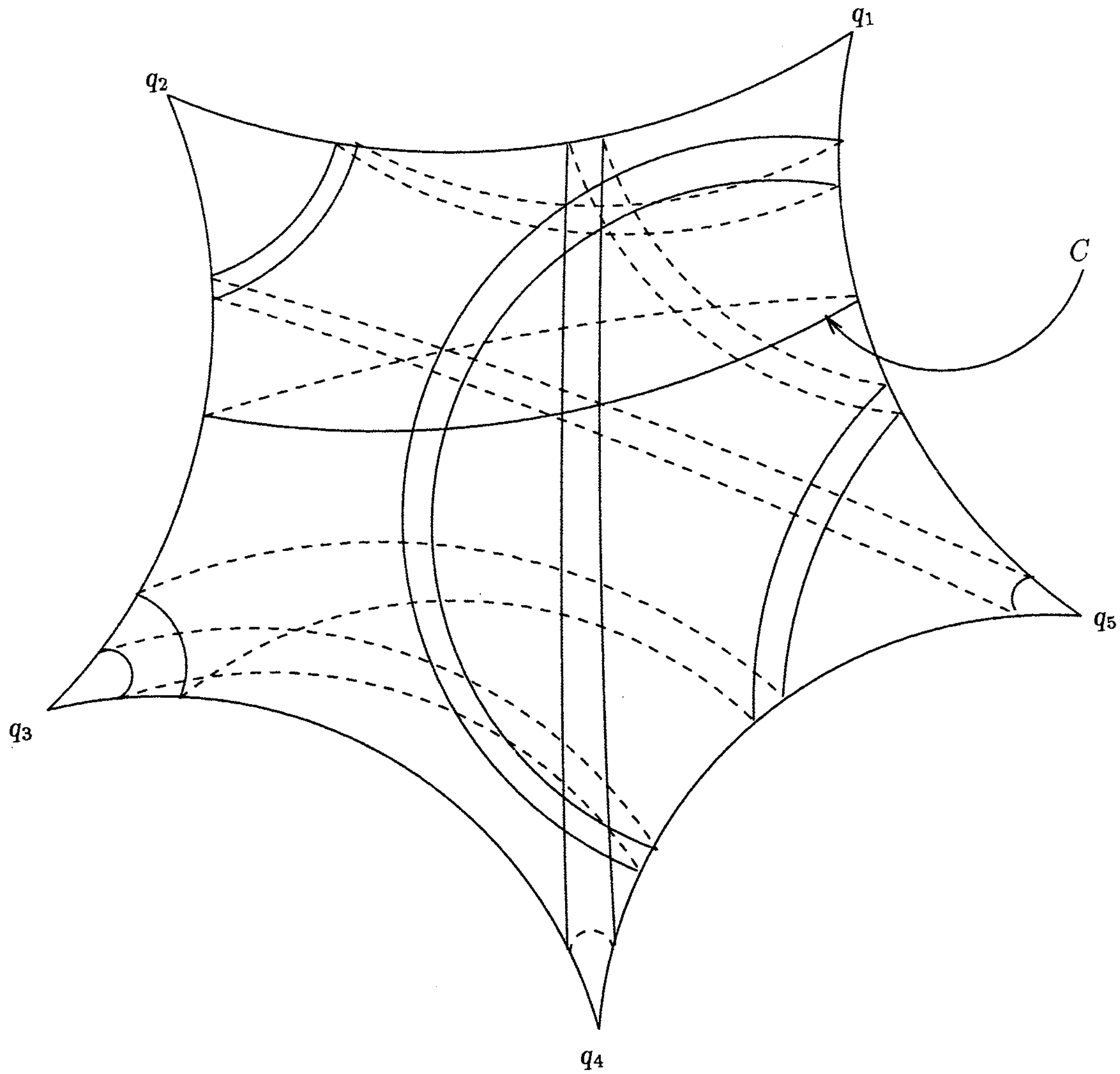


Figure 5.1: Cutting S along C

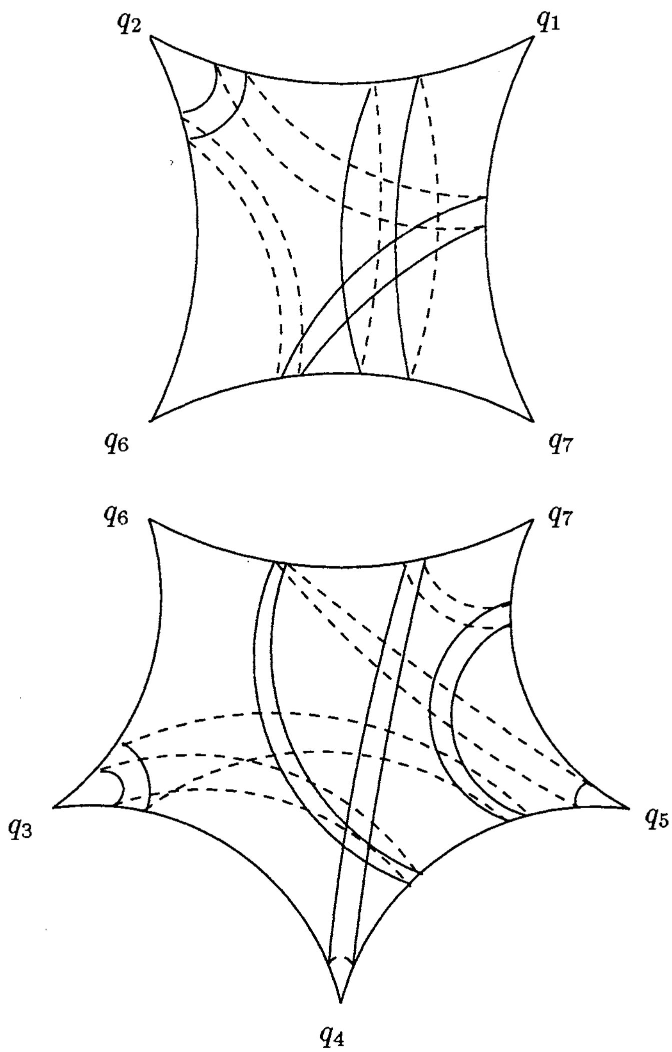


Figure 5.2: After the cut and paste

$(m_1, m_4 - m_1, m_3, m_4, m_3)$ on S_5.

By Theorem 5.1.1, there are a minimum of $2|\eta_2/n_1 * \mu_2/m_1|$ intersection points of the multiple curves on S_4, and by Theorem 5.1.2 there are a minimum of $2|\eta_3/n_4 * \mu_3/m_4|$ intersection points on S_5.

Now fill in the punctures q_6 and q_7 and unglue the identification performed on C and glue back S_4 and S_5 along C.

First suppose that $\eta_2/n_1 * \mu_2/m_1$ and $\eta_3/n_4 * \mu_3/m_4$ have the same sign. Then by Lemma 5.1.3, all intersection points of η with μ have the same orientation. Hence no deformation can cancel intersection points. To see this suppose it is possible to deform η and μ and cancel intersection points. Then there is at least one component of $S - \{\eta \cup \mu\}$ that disappears when this deformation is performed. Furthermore, the boundary of one of these disappearing components must contain exactly two of the intersection points of η and μ. We call a component of $S - \{\eta \cup \mu\}$ whose boundary contains exactly two intersection points of η with μ a *lens*. There can be no punctures contained in the disappearing lens, or the deformation could not take place.

Now delete the point of intersection of arc 2 with C from S to make a sphere S' with six punctures. Arc 2 can now be separated into two arcs, arcs $2a$ and $2b$, where arc $2a$ connects q_2 and the new puncture q_6, and arc $2b$ connects q_6 with q_3. Each time η and μ pass through arcs 1, $2a$, $2b$, 3, 4 or 5, either both their directions change or they both stay the same. Since the intersection points on the boundary of the lens in question have the same orientation, the lens must contain some puncture. The lens cannot contain any puncture except q_6, or η and μ could not be deformed on S across the puncture to make the lens disappear. But no lens can contain only the puncture q_6 by the construction of C.

Hence the minimal number of intersection points of η with μ is $2|\eta_2/n_1 * \mu_2/m_1| + 2|\eta_3/n_4 * \mu_3/m_4|$. Since $\eta_2/n_1 * \mu_2/m_1$ and $\eta_3/n_4 * \mu_3/m_4$ have the same sign, this expression is equal to $2|(\eta_2/n_1, \eta_3/n_4) * (\mu_2/m_1, \mu_3/m_4)|$.

Next suppose $\eta_2/n_1 * \mu_2/m_1$ and $\eta_3/n_4 * \mu_3/m_4$ have opposite signs. Then the intersection points in S_4 and S_5 are oriented differently. It also follows that $\mathrm{sgn}(\eta_2) = \mathrm{sgn}(\eta_3) = \mathrm{sgn}(\mu_2) = \mathrm{sgn}(\mu_3)$.

Suppose the intersection points in S_4 are negatively oriented with respect to η. Orient the curve C so that for any point α in $C \cap S^+$ and any point β in $C \cap S^-$, the points α, β and q_6 occur in that order.

Let x denote a C-intersection point on η and let y denote a C-intersection point on μ. Suppose the points x, y and q_6 occur in that order on C, and suppose there is no C-intersection point z on C such that x, z and y occur in that order on C. Since all the intersection points in S_4 are negatively oriented, and all the intersection points in S_5 are positively oriented, the component of $S - \{\eta \cup \mu\}$ containing the segment of C from x to y must be a lens containing no punctures. Hence η can be deformed to make this lens disappear. When this deformation is performed, the points x and y are interchanged on C. Cancelling positively and negatively oriented intersection points in this manner, we can continue until either all of the differently oriented intersection points have cancelled, or until for any C-intersection point x on η and any C-intersection point y on μ, the points y, x and q_6 occur in that order on C. The latter case happens when we have interchanged exactly $2n_1m_1$ C-intersection points, and we have cancelled $4n_1m_1$ of all the intersection points. At this stage, any lens whose boundary contains differently oriented intersection points must contain q_6, and it must also contain another puncture, since the intersection points are oriented differently, so no more cancellation can occur on the five-times punctured sphere S.

If the intersection points in S_4 are positively oriented, the same argument works by interchanging the roles of η and μ.

q.e.d.

If the arc intersection numbers of two multiple curves satisfy $n_j + n_{j+1} = n_{j+2} + n_{j+3} + n_{j+4}$ and $m_j + m_{j+1} = m_{j+2} + m_{j+3} + m_{j+4}$ for some j, then a cyclic permutation of the indices will set $j = 4$ so that Theorem 5.1.4 applies. This cyclic permutation corresponds to a rotation of the multiple curves on S. If there is no such j, the intersection number may be more difficult to compute. The existence of such a j implies that each of the two multiple curves is carried by one of two train tracks.

A *train track* on S is a graph τ on S consisting of edges called *branches* and vertices called *switches.* The branches meeting at a switch must be tangent there. Each branch is assigned a nonnegative integer called a *weight.* Each switch is oriented, and the sum of all weights on the branches leading to a switch with positive orientation must equal the sum of all the weights on the branches leading to the switch with negative orientation. A multiple curve η is *carried* by the train track τ if there is a map $\phi : S \to S$ homotopic to the identity such that $\phi(\eta) \subset \tau$.

One possible train track on S which carries a multiple curve whose arc intersection numbers satisfy $n_4 + n_5 = n_1 + n_2 + n_3$ is shown in Figure 5.3. The other possibility is the reflection ρ of this train track.

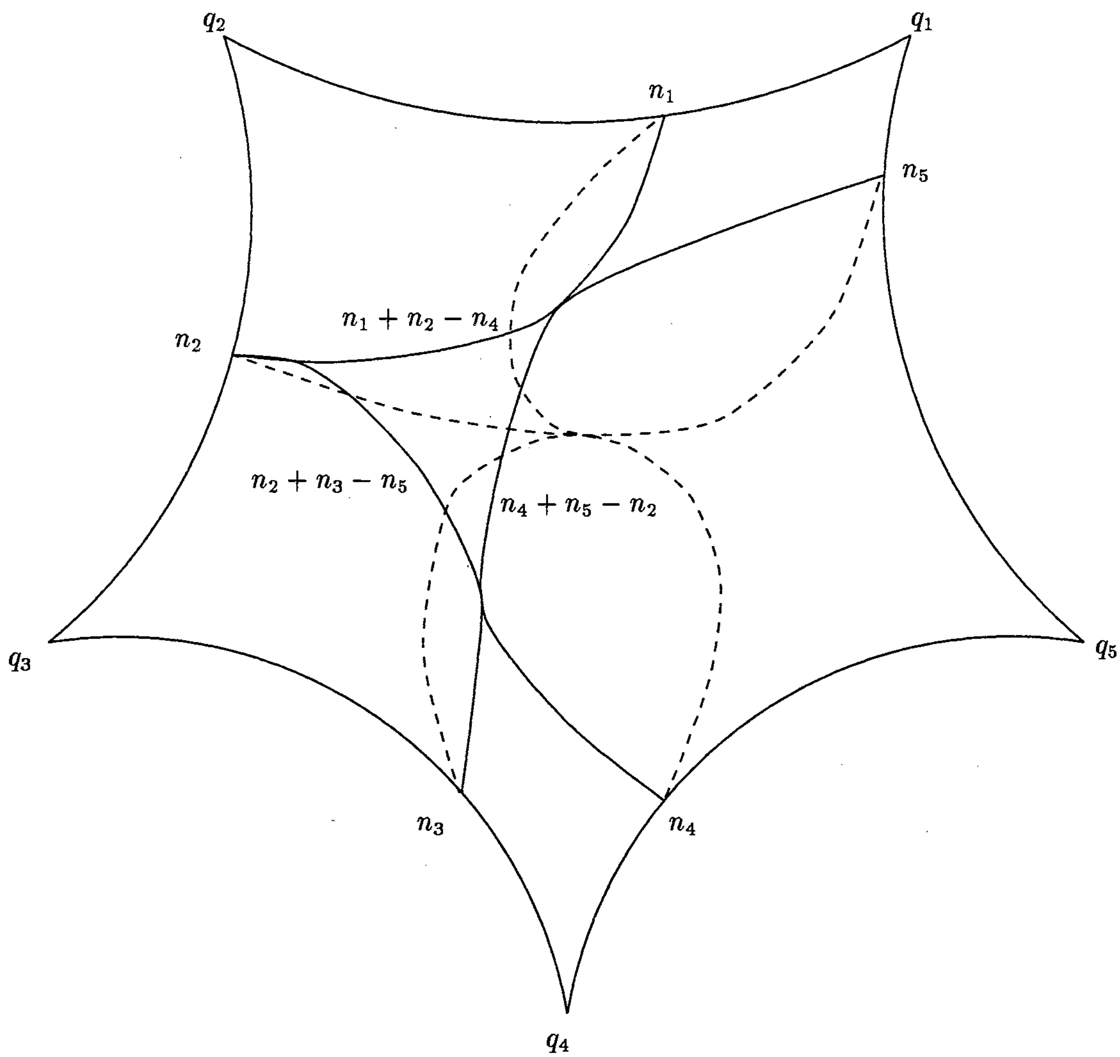


Figure 5.3: A train track

5.2 Actions of the modular group

The *modular group* of a surface is the group of homotopy classes of orientation-preserving homeomorphisms of the surface. In this section we study the effects of the modular group on the arc intersection numbers of simple closed curves on a surface of type (0,5).

Lemma 5.2.1 *Let $(n_1, \ldots, n_5)$ be the arc intersection numbers of one or more simple closed curves on S. Let S^ϵ denote either S^+ or S^-. Suppose there is no segment in S^ϵ connecting arcs i and $i+1$. Let σ denote an arc in S^ϵ from the puncture q_{i+1} to the puncture q_{i-1}. Then the minimal intersection number of σ with the curves is $n_i + n_{i+1}$.*

Proof: Each arc intersection point on arc i and arc $i+1$ must be the endpoint of a segment in S^ϵ. Since none of these arcs can connect arcs i and $i+1$, each one must intersect σ at least once. Clearly one can deform σ to intersect each of these segments exactly once.

q.e.d.

Lemma 5.2.2 *Let $(n_1, \ldots, n_5)$ be the arc intersection numbers of one or more disjoint simple closed curves on S. Let S^ϵ denote either S^+ or S^-. Suppose there is a segment in S^ϵ connecting arcs i and $i+1$. Let σ denote an arc in S^ϵ from q_{i+1} to q_{i-1}. Then the minimal intersection number of σ with the curves is the maximum of $2n_{i+3} - n_i - n_{i+1}$ and $|n_i - n_{i+1}|$.*

Proof: Suppose first that $n_{i+3} \geq n_i$ and $n_{i+3} \geq n_{i+1}$. Then $n_i + n_{i+1} - n_{i+3}$ is less than or equal to the minimum of n_i and n_{i+1}. There are $n_i + n_{i+1}$ arc intersection points on arcs i and $i+1$, $2(n_i + n_{i+1} - n_{i+3})$ of which are connected by segments in S^ϵ. Thus there are $n_i + n_{i+1} - 2(n_i + n_{i+1} - n_{i+3}) = 2n_{i+3} - n_i - n_{i+1}$ points on arcs i and $i+1$ which are endpoints of segments in S^ϵ not connecting arcs i and $i+1$. Each of these segments must intersect σ at least once, and clearly σ can be deformed to intersect these segments only once. Since $n_{i+3} \geq n_i$ and $n_{i+3} \geq n_{i+1}$, the maximum of $2n_{i+3} - n_i - n_{i+1}$ and $|n_i - n_{i+1}|$ is $2n_{i+3} - n_i - n_{i+1}$.

Suppose next that n_{i+3} is less than one of n_i and n_{i+1}. Without loss of generality, assume that $n_i \geq n_{i+1}$. Then of the $n_i + n_{i+1}$ arc intersection points on arcs i and $i+1$, exactly $2n_{i+1}$ of these are connected by segments in S^ϵ. Thus there are $n_i - n_{i+1}$ points on arcs i and $i+1$ which are endpoints of segments in S^ϵ not connecting arcs i and $i+1$. Hence in this case the minimal intersection number of σ with the curves is $n_i - n_{i+1}$.

q.e.d.

Let S' be the surface obtained by interchanging the punctures q_1 and q_5 via a half Dehn twist which takes arc 2 to an arc in S'^- and arc 5 to an arc in S'^+, and which leaves the other three arcs invariant (reversing the direction of arc 1). Label the arcs, punctures, and arc intersection numbers of S' as arc $1'$ through arc $5'$, q_1' through q_5' and n_1' through n_5' in the usual way. (So, for example, $q_1' = q_5$ and $q_2' = q_2$.)

Lemma 5.2.3 *Suppose there is no segment of the disjoint simple closed curves on S connecting arcs 1 and 2 in S^+. Then there is no segment of the corresponding curves connecting arcs $1'$ and $2'$ in S'^+.*

Proof: The inverse image of arc $2'$ in S is an arc α from q_2 to q_5 contained in S^+. Any arc intersection point on arc 1 must be an endpoint of a segment in S^+ which crosses arc α. This crossing point becomes an arc intersection point on arc $2'$, and the segment in S^+ connecting the arc intersection point on arc 1 to the crossing point on arc α becomes a segment in S'^- connecting arcs $1'$ and $2'$. Since the curves are disjoint, there is no segment connecting arcs $1'$ and $2'$ in S'^+. If there are no arc intersection points on arc 1, then the arc intersection numbers of S and S' are equal.

q.e.d.

Lemma 5.2.4 *Suppose there is a segment of one or more disjoint simple closed curves connecting arcs* 1 *and* 2 *in* S^+. *Suppose the arc intersection numbers satisfy* $n_4 > n_1$ *and* $n_4 > n_2$. *Then there is a segment of the corresponding curves connecting arcs* $1'$ *and* $2'$ *in* S'^-.

Proof: There are n_1 arc intersection points on arc 1, of which $n_1 + n_2 - n_4$ are connected to arc intersection points on arc 2 by segments of the curves in S^+. Thus there are $n_1 - (n_1 + n_2 - n_4) = n_4 - n_2$ arc intersection points on arc 1 which are endpoints of segments in S^+ which cross any arc α from q_2 to q_5 in S^+. Such an arc α is the inverse image of arc $2'$ in S. The $n_4 - n_2$ segments in S^+ connecting the arc intersection points on arc 1 to the crossing points on arc α become segments in S'^- connecting arcs $1'$ and $2'$.

q.e.d.

Lemma 5.2.5 *Suppose there is a segment of one or more disjoint simple closed curves connecting arcs* 1 *and* 2 *in* S^+. *Suppose the arc intersection numbers satisfy* $n_1 \geq n_4$ *and* $n_1 > n_2$. *Then there is a segment of the corresponding curves connecting arcs* $1'$ *and* $2'$ *in* S'^-.

Proof: Since $n_1 \geq n_4$, $n_1 + n_2 - n_4 \geq n_2$, so there are n_2 segments of the curves in S^+ connecting arcs 1 and 2. Thus, of the n_1 arc intersection points on arc 1, $n_1 - n_2$ of these are endpoints of segments in S^+ which must cross any arc α in S^+ from q_2 to q_5. The $n_1 - n_2$ segments in S^+ connecting the arc intersection points on arc 1 to the crossing points on arc α become segments in S'^- connecting arcs $1'$ and $2'$.

q.e.d.

Lemma 5.2.6 *Suppose there is a segment of one or more disjoint simple closed curves connecting arcs* 1 *and* 2 *in* S^+. *Suppose the arc intersection numbers satisfy* $n_2 \geq n_4$ *and* $n_2 \geq n_1$. *Then there are no segments of the corresponding curves connecting arcs* $1'$ *and* $2'$ *in* S'^-.

Proof: Since $n_2 \geq n_4$, $n_1 + n_2 - n_4 \geq n_1$, so there are n_1 segments of the curves in S^+ connecting arcs 1 and 2. Thus, none of the arc intersection points on arc 1 are endpoints of segments in S^+ which must intersect an arc α in S^+ from q_2 to q_5. Since there is a segment of the curves connecting arcs 1 and 2 in S^+, any segment connecting arc $1'$ and $2'$ in S'^- must correspond to segments in S^+ which connect arc 1 and such an arc α.

q.e.d.

Theorem 5.2.7 $\eta_2' = \eta_2 + n_1$.

Proof: Suppose first that $\eta_2 \geq 0$. Then there is no segment connecting arcs 1 and 2 in S^+. If $n_1 \geq n_4$, then $\eta_2 = n_2$, and by Lemmas 5.2.1 and 5.2.3, $\eta_2' = n_2' = n_1 + n_2 = \eta_2 + n_1$. If $n_1 < n_4$ then $\eta_2 = n_1 + n_2 - n_4$, and by the same lemmas, $\eta_2' = n_1' + n_2' - n_4' = n_1 + n_1 + n_2 - n_4 = \eta_2 + n_1$.

Suppose for the rest of the proof that $\eta_2 < 0$; we split this part of the proof into three cases. For the first case, suppose $n_4 > n_1$ and $n_4 > n_2$. Then $\eta_2 = -(n_1 + n_2 - n_4)$, and by Lemmas 5.2.2 and 5.2.4, $n_2' = 2n_4 - n_1 - n_2$ and $\eta_2' = n_1' + n_2' - n_4' = 2n_4 - n_1 - n_2 + n_1 - n_4 = n_4 - n_2 = \eta_2 + n_1$.

For the second case, suppose $n_1 \geq n_4$ and $n_1 > n_2$. Then $\eta_2 = -n_2$, and by Lemmas 5.2.2 and 5.2.5, $n_2' = n_1 - n_2$ and $\eta_2' = n_2' = n_1 - n_2 = \eta_2 + n_1$.

For the last case, suppose that $n_2 \geq n_4$ and $n_2 \geq n_1$. If $n_4 \geq n_1$, then $\eta_2 = -(n_1 + n_2 - n_4)$, and by Lemmas 5.2.2 and 5.2.6, $n_2' = n_2 - n_1$ and $\eta_2' = -(n_1' + n_2' - n_4') = -(n_1 + n_2 - n_1 - n_4) = n_4 - n_2 = \eta_2 + n_1$. If $n_4 < n_1$ then $\eta_2 = -n_2$, and by the same two lemmas, $n_2' = n_2 - n_1$ and $\eta_2' = -n_2' = \eta_2 + n_1$.

q.e.d.

Corollary 5.2.8 *Suppose there are two sets of disjoint admissible simple closed curves on S with rational numbers $(\eta_2/n_1, \eta_3/n_4)$ and $(\mu_2/m_1, \mu_3/m_4)$. Let $(\eta_2'/n_1', \eta_3'/n_4')$*

and $(\mu_2'/m_1', \mu_3'/m_4')$ *be the rational numbers of the associated sets of curves on* S'. *Then* $(\eta_2/n_1, \eta_3/n_4) * (\mu_2/m_1, \mu_3/m_4) = (\eta_2'/n_1', \eta_3'/n_4') * (\mu_2'/m_1', \mu_3'/m_4')$.

Proof: By the construction of S', $n_1' = n_1$, $\eta_3' = \eta_3$, $n_4' = n_4$, $m_1' = m_1$, $\mu_3' = \mu_3$, and $m_4' = m_4$. By Theorem 5.2.7, $\eta_2' = \eta_2 + n_1$ and $\mu_2' = \mu_2 + m_1$. Thus, $\eta_2'/n_1' * \mu_2'/m_1' = \eta_2/n_1 * \mu_2/m_1$, and the corollary follows.

q.e.d.

Lemma 5.2.9 *Suppose the arc intersection numbers of two disjoint sets of disjoint admissible simple closed curves on* S *are* $(n_1, \ldots, n_5)$ *and* $(m_1, \ldots, m_5)$. *For* $1 \leq i \leq 5$, *let* $\ell_i = n_i + m_i$. *Let* $N = \sum_{i=1}^{5} n_i$ *and* $L = \sum_{i=1}^{5} \ell_i$. *Then if* $\ell_j + \ell_{j+1} = L/2$, *then* $n_j + n_{j+1} = N/2$.

Proof: If $\ell_j + \ell_{j+1} = L/2$, then there is a side S^ϵ of S (either S^+ or S^-) on which each segment of the curves has one endpoint on arc j or $j+1$ and the other on one of the three other arcs. Since the curves with arc intersection numbers $(n_1, \ldots, n_5)$ form a subset of the curves with arc intersection numbers $(\ell_1, \ldots, \ell_5)$, each segment in S^ϵ of the curves with arc intersection numbers $(n_1, \ldots, n_5)$ must have one endpoint on arc j or $j+1$ and the other endpoint on one of the three other arcs. Hence $n_j + n_{j+1} = N/2$.

q.e.d.

Lemma 5.2.10 *Suppose the arc intersection numbers of two disjoint sets of disjoint admissible simple closed curves on* S *are* $(n_1, \ldots, n_5)$ *and* $(m_1, \ldots, m_5)$. *Then, for each* i, *if* $n_i > n_{i+3}$ *then* $m_i \geq m_{i+3}$, *and if* $n_i < n_{i+3}$ *then* $m_i \leq m_{i+3}$.

Proof: If $n_1 > n_4$ then there must be a piece of the curves with arc intersection numbers $(n_1, \ldots, n_5)$ which starts on arc 1, eventually passes through arc 3 and returns to arc 1 before touching arc 4. If $m_1 < m_4$ then there must be a piece of the curves with arc intersection numbers $(m_1, \ldots, m_4)$ which starts on arc 4, passes through arc 2 and returns to arc 4 before touching arc 1. But these two pieces must intersect. So the cases $n_1 > n_4$, $m_1 < m_4$ and $n_1 < n_4$, $m_1 > m_4$ cannot occur.

q.e.d.

Lemma 5.2.11 *Suppose the arc intersection numbers of two disjoint sets of disjoint admissible simple closed curves on S are $(n_1, \ldots, n_5)$ and $(m_1, \ldots, m_5)$, and the rational numbers of these sets are $(\eta_2/n_1, \eta_3/n_4)$ and $(\mu_2/m_1, \mu_3/m_4)$. Let j denote an index such that $n_j + n_{j+1} = n_{j+2} + n_{j+3} + n_{j+4}$ and $m_j + m_{j+1} = m_{j+2} + m_{j+3} + m_{j+4}$. Then $|(\eta_2/n_1, \eta_3/n_4) * (\mu_2/m_1, \mu_3/m_4)|$ is equal to:*

(i) $|(n_1m_2 - m_1n_2) + (n_3m_4 - m_3n_4) - (n_1m_4 - m_1n_4)|$, *if $j = 1$, 2, or 3; or*

(ii) $|(n_1m_2 - m_1n_2) - (n_3m_4 - m_3n_4) - (n_1m_4 - m_1n_4)|$, *if $j = 4$; or*

(iii) $|(n_1m_2 - m_1n_2) - (n_3m_4 - m_3n_4) + (n_1m_4 - m_1n_4)|$, *if $j = 5$.*

Proof: This is an easy computation using the definitions of the rational numbers and the operator $*$.

q.e.d.

Let $\tilde{S}$ be the surface obtained by rotating S so that arc $\tilde{i}$ = arc $i-1$ and $\tilde{q}_i = q_{i-1}$ for $1 \leq i \leq 5$. Let $\tilde{\eta}$ and $\tilde{\mu}$ denote the images of η and μ under this rotation.

Lemma 5.2.12 *Suppose the arc intersection numbers of two disjoint sets of disjoint admissible simple closed curves on S are $(n_1, \ldots, n_5)$ and $(m_1, \ldots, m_5)$, with rational numbers $(\eta_2/n_1, \eta_3/n_4)$ and $(\mu_2/m_1, \mu_3/m_4)$. Let $\tilde{S}$ denote the surface described above. Then*

$$|(\eta_2/n_1, \eta_3/n_4) * (\mu_2/m_1, \mu_3/m_4)| = |(\tilde{\eta}_2/\tilde{n}_1, \tilde{\eta}_3/\tilde{n}_4) * (\tilde{\mu}_2/\tilde{m}_1, \tilde{\mu}_3/\tilde{m}_4)|.$$

Proof: For notational purposes, let $\tilde{N}$ denote $|(\tilde{\eta}_2/\tilde{n}_1, \tilde{\eta}_3/\tilde{n}_4) * (\tilde{\mu}_2/\tilde{m}_1, \tilde{\mu}_3/\tilde{m}_4)|$. Suppose first that $j = 1$, 2, or 3. Then

$$|(\eta_2/n_1, \eta_3/n_4) * (\mu_2/m_1, \mu_3/m_4)| = |(n_1m_2 - m_1n_2) + (n_3m_4 - m_3n_4) - (n_1m_4 - m_1n_4)|$$

by Lemma 5.2.11, and if $j = 1$ or 2 then

$$\begin{aligned}\tilde{N} &= |(n_5m_1 - m_5n_1) + (n_2m_3 - m_2n_3) - (n_5m_3 - m_5n_3)| \\ &= |-(n_1m_2 - m_1n_2) - (n_3m_4 - m_3n_4) + (n_1m_4 - m_1n_4)|.\end{aligned}$$

If $j = 3$ then

$$\begin{aligned}\tilde{N} &= |(n_5m_1 - m_5n_1) - (n_2m_3 - m_2n_3) - (n_5m_3 - m_5n_3)| \\ &= |(n_1m_2 - m_1n_2) + (n_3m_4 - m_3n_4) - (n_1m_4 - m_1n_4)|.\end{aligned}$$

Next suppose $j = 4$. Then

$$|(\eta_2/n_1, \eta_3/n_4) * (\mu_2/m_1, \mu_3/m_4)| = |(n_1m_2 - m_1n_2) - (n_3m_4 - m_3n_4) - (n_1m_4 - m_1n_4)|$$

by Lemma 5.2.11, and

$$\begin{aligned}\tilde{N} &= |(n_5m_1 - m_5n_1) - (n_2m_3 - m_2n_3) + (n_5m_3 - m_5n_3)| \\ &= |-(n_1m_2 - m_1n_2) + (n_3m_4 - m_3n_4) + (n_1m_4 - m_1n_4)|.\end{aligned}$$

Finally suppose $j = 5$. Then

$$|(\eta_2/n_1, \eta_3/n_4) * (\mu_2/m_1, \mu_3/m_4)| = |(n_1m_2 - m_1n_2) - (n_3m_4 - m_3n_4) + (n_1m_4 - m_1n_4)|$$

by Lemma 5.2.11, and

$$\begin{aligned}\tilde{N} &= |(n_5m_1 - n_1m_5) + (n_2m_3 - m_2n_3) - (n_5m_3 - m_5n_3)| \\ &= |-(n_1m_2 - m_1n_2) + (n_3m_4 - m_3n_4) - (n_1m_4 - m_1n_4)|.\end{aligned}$$

q.e.d.

Magnus ([Mag34]) has proven the following theorem (see also [Bir74], p. 164). Let $q_1, \dots, q_n$ denote the punctures of an n-times punctured sphere S_n.

Theorem 5.2.13 *The modular group of S_n is generated by a half Dehn twist about a curve enclosing q_1 and q_n which interchanges q_1 and q_n and a rotation which takes q_i to q_{i+1} for $1 \leq i \leq n-1$ and which takes q_n to q_1.*

By this theorem, Corollary 5.2.8 and Lemma 5.2.12, we have proven the following

Theorem 5.2.14 *Suppose the arc intersection numbers of two disjoint sets of disjoint admissible simple closed curves on S are $(n_1, \ldots, n_5)$ and $(m_1, \ldots, m_5)$, with rational numbers $(\eta_2/n_1, \eta_3/n_4)$ and $(\mu_2/m_1, \mu_3/m_4)$. Then the formula*

$$|(\eta_2/n_1, \eta_3/n_4) * (\mu_2/m_1, \mu_3/m_4)|$$

is invariant under the modular group of S.

Corollary 5.2.15 *Suppose the arc intersection numbers of two disjoint admissible simple closed curves on S are $(n_1, \ldots, n_5)$ and $(m_1, \ldots, m_5)$, with rational numbers $(\eta_2/n_1, \eta_3/n_4)$ and $(\mu_2/m_1, \mu_3/m_4)$. Then*

$$|(\eta_2/n_1, \eta_3/n_4) * (\mu_2/m_1, \mu_3/m_4)| = 0.$$

Proof: There is an element of the modular group which takes the curve with rationals $(\eta_2/n_1, \eta_3/n_4)$ to the curve with rationals $(0/1, 0/0)$; thus we may assume $(\eta_2/n_1, \eta_3/n_4) = (0/1, 0/0)$ and $m_1 \geq m_4$. Hence $|(\eta_2/n_1, \eta_3/n_4) * (\mu_2/m_1, \mu_3/m_4)| = |\mu_2| = m_2$. Since the curves are disjoint, $m_2 = 0$.

q.e.d.

5.3 Tables of cusps on the boundaries of $M_{0,4}$ and $M_{0,5}$

Tables 5.1 and 5.2 show some of the cusps on the boundaries of $M_{0,4} = M_{1,1}$ and $M_{0,5}$. For the cusps on the boundary of $M_{0,4}$, the rational number represents the simple closed curve on the four-times punctured spheres which is pinched to a point.

For the cusps on the boundary of $M_{0,5}$, the pair of rationals is the sum of the pairs of rationals of the two disjoint simple closed curves on the five-times punctured spheres which are simultaneously pinched to points. All decimals are approximations.

p/q	$K_{p/q}$	cusp
0/1	$B_1B_2^{-1}$	$2i$
1/1	B_1B_2A	$2+2i$
1/2	$B_1B_2^{-1}B_1^{-1}B_2A$	$1+\sqrt{3}i$
1/3	$B_1B_2^{-1}B_1B_2B_1^{-1}B_2A$	$0.581+1.694i$
1/4	$B_1B_2^{-1}B_1B_2^{-1}B_1^{-1}B_2B_1^{-1}B_2A$	$0.352+1.721i$

Table 5.1: Cusps on the boundary of $M_{0,4}$

$(\eta_2/n_1, \eta_3/n_4)$	words in $H_{x,y}$	cusp
$(0/2, 0/1)$	$B_1B_2^{-1}$, $B_3^{-1}B_2^{-1}$	$(2i, 4i)$
$(0/1, 0/2)$	$B_3^{-1}B_2^{-1}$, $B_3^{-1}A$	$(4i, 2i)$
$(2/2, 0/1)$	B_1B_2A, $B_3^{-1}B_2A$	$(2+2i, 4i)$
$(0/2, 1/1)$	$B_1B_2^{-1}$, $B_2^{-1}B_3B_1$	$(2i, 2+4i)$
$(0/3, 1/2)$	$B_1B_2^{-1}$, $B_2^{-1}B_3B_2B_1^{-1}B_3^{-1}$	$(2i, 1+\sqrt{3}i+2i)$
$(0/3, -1/2)$	$B_1B_2^{-1}$, $B_2B_3^{-1}B_2^{-1}B_1B_3$	$(2i, -1+\sqrt{3}i+2i)$
$(1/2, 1/2)$	$B_2^{-1}B_3B_1$, $B_2^{-1}B_3A^{-1}$	$(1+\sqrt{7}i, 1+\sqrt{7}i)$
$(-1/2, 1/2)$	$B_2^{-1}B_3^{-1}$, $B_2B_3B_1A$	$(-1+\sqrt{7}i, 1+\sqrt{7}i)$
$(-1/2, -3/2)$	$B_2B_1B_3A$, $B_2B_1B_3B_1^{-1}$	$(-1+\sqrt{7}i, -3+\sqrt{7}i)$
$(0/4, 1/3)$	$B_1B_2^{-1}$, $B_2^{-1}B_3B_2B_3B_1B_2^{-1}B_3^{-1}$	$(2i, 0.581+1.694i+2i)$

Table 5.2: Cusps on the boundary of $M_{0,5}$

CHAPTER 6

HIGHER DIMENSIONAL TEICHMÜLLER SPACES

Kra's construction of surfaces of type (g, n) involves $3g - 3 + n$ amalgamated free products and HNN extensions (see [Kra88]). Let S denote a Riemann surface of type (g, n). A *maximal partition* P on S is a maximal set of nonhomotopic simple closed curves on S, none of which is homotopic to a puncture or homotopically trivial. There are $3g - 3 + n$ curves in any maximal partition on S. The set $S - P$ is topologically a union of $2g - 2 + n$ thrice-punctured spheres. (This is sometimes called the "pair of pants" decomposition of S, since a thrice-punctured sphere is homeomorphic to a "pair of pants".)

Hence, S can be constructed by gluing $2g - 2 + n$ pairs of pants together along the curves of the maximal partition. Start with any one of the pairs of pants, and pick one of its boundary curves. This boundary curve corresponds to one of the curves in the maximal partition; and there is some other boundary curve corresponding to the same curve in the partition. Glue the boundary curves together to make a new surface S'. If the boundary curves both lie on the same pair of pants then an HNN extension is used; if the boundary curves are on different pairs of pants an amalgamated free product is used. Next choose another boundary curve on S', and glue it to the corresponding boundary curve. If this boundary curve is also on S', an HNN extension is used; otherwise an amalgamated free product is used. Continuing in this way, the surface S is constructed from the $2g - 2 + n$ pairs of pants using g HNN extensions and $n - 3 + 2g$ amalgamated free products.

6.1 Surfaces of type (1,2)

Start with the group representing a surface of type (0,4): $H_x = \langle A, B_1, B_{2,x}\rangle$. We want to cut off two punctures and glue the boundary curves together to form a surface of type (1,2). As this problem is solved for the $T_{1,1}$ case, we need only conjugate; that is, we find a Möbius transformation P so that $PS_1P^{-1} = (AB_1)^{\pm 1}$ and $PS_2P^{-1} = B_1^{\pm 1}$. Then if $PS_1P^{-1} = AB_1$, then PS_2P^{-1} must be B_1 and

$$P = \begin{pmatrix} \sqrt{2}n & 0 \\ \frac{-1}{\sqrt{2}n} & \frac{1}{\sqrt{2}n} \end{pmatrix},$$

where $n = \pm i$. On the other hand, if $PS_1P^{-1} = (AB_1)^{-1}$, then PS_2P^{-1} must be B_1^{-1}, and

$$P = \begin{pmatrix} \sqrt{2}n & 0 \\ \frac{1}{\sqrt{2}n} & \frac{1}{\sqrt{2}n} \end{pmatrix},$$

where $n = \pm 1$. In the first case,

$$PT_xP^{-1} = \begin{pmatrix} -ix - i & 2i \\ -\frac{ix}{2} & i \end{pmatrix},$$

and in the second case,

$$PT_xP^{-1} = \begin{pmatrix} i - ix & -2i \\ -\frac{ix}{2} & -i \end{pmatrix}.$$

If we set $Q_x(z)$ to be the Möbius transformation determined by the first matrix, then $Q_{iv}(iy) = 2 - \frac{2iy}{vy+2}$, so Q_x takes circles in $\mathbb{H}$ tangent to $\mathbb{R}$ at 0 to circles in the lower half plane tangent to $\mathbb{R}$ at 2. Thus we discard this case and we set

$$Q_x = \begin{pmatrix} i - ix & -2i \\ -\frac{ix}{2} & -i \end{pmatrix}.$$

Then $Q_x(0) = 2$,

$$|Q_{u+iv}(\frac{4i}{v}) - (2 + \frac{2i}{v})| = \frac{2}{|v|},$$

and

$$|Q_{u+iv}(\frac{2i}{v} + \frac{2}{v}) - (2 + \frac{2i}{v})| = \frac{2}{|v|}.$$

Thus $Q_{u+iv}(z)$ takes the circle of radius $\frac{2}{v}$, center $\frac{2i}{v}$ $(v > 0)$ to the circle with radius $\frac{2}{v}$, center $2 + \frac{2i}{v}$. $Q_{u+iv}(z)$ takes the interior of the first circle to the exterior of the second, since $Q_{u+iv}(\frac{2i}{v}) = \frac{2(u-1)}{u} \in \mathbb{R} - \{2\}$.

Let $J_{x,y}$ denote the HNN extension of H_x by Q_y. (See Figure 6.1.) The embedding of $T_{1,2}$ is the set of all $(x,y) \in \mathbb{H}^2$ such that $J_{x,y}$ is a terminal b-group and $\Delta(J_{x,y})/J_{x,y}$ is a surface of type (1,2). Denote this set by $M_{1,2}$.

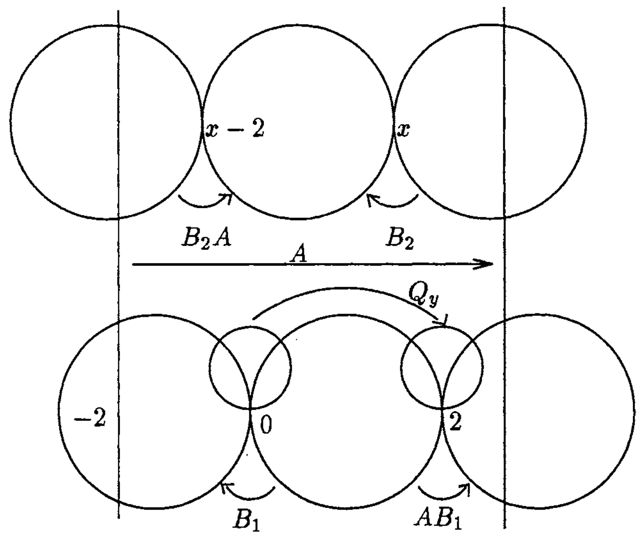


Figure 6.1: The action in the group $J_{x,y}$

Since $M_{1,1} = M_{0,4}$, one might wonder whether the same is true for $M_{0,5}$ and $M_{1,2}$. It is known that $T_{0,5}$ and $T_{1,2}$ are biholomorphic (see, for example, [Gar87]). However, it is not true that $M_{0,5}$ and $M_{1,2}$, as we have defined them here, are the same set in $\mathbb{H}^2$. For example, the point $(x,y) = (3.2i, 2.5i)$ is in $M_{1,2}$ but not $M_{0,5}$. Maskit's Second Combination Theorem can be used to show that $(3.2i, 2.5i) \in M_{1,2}$. This point is not in $M_{0,5}$ because $B_{2,3.2i}B_{3,2.5i}$ is parabolic; the point $(3.2i, 2.5i)$ is a cusp on the boundary of $M_{0,5}$. In fact, if $x = -8/y$, then $B_{2,x}B_{3,y}$ is parabolic.

In a forthcoming paper, L. Keen, J. Parker and C. Series ([KPS]) promise to generalize the Farey series enumeration of simple closed curves on once-punctured tori to an enumeration of the simple closed curves on twice-punctured tori. It will be interesting to see the relationship between their enumeration of simple closed curves on twice-punctured tori and the enumeration of simple closed curves on five-times punctured spheres presented in this thesis. (The Teichmüller spaces $T_{1,2}$ and $T_{0,5}$ are the only ones of complex dimension 2.)

6.2 Surfaces of type (2,0)

Start with the group representing a surface of type (1,2): $J_{x,y} = \langle A, B_1, B_{2,x}, Q_y \rangle$. To find a transformation which will cut off two punctures and glue the boundary curves together, we need only conjugate. Let $P(z) = x - z$, and define $R_{x,y} = PA^{-1}Q_yP^{-1}$. Let $J_{x,y,z}$ denote the HNN extension of $J_{x,y}$ by $R_{x,z}$. The embedding of $T_{2,0}$ is the set of all $(x, y, z) \in \mathbb{H}^3$ such that $J_{x,y,z}$ is a terminal b-group and $\Delta(J_{x,y,z})/J_{x,y,z}$ is a surface of type (2,0). The transformation $R_{x,z}$ cuts the surface of type (1,2) at the two punctures corresponding to the points x and $x+2$ and glues the boundary curves together to form the surface of type (2,0). Note that $R_{x,z}^{-1}AB_{2,x}R_{x,z} = B_{2,x}$ and

$$R_{x,z} = \begin{pmatrix} \frac{i(xz+2z+2)}{2} & \frac{-i(x^2z+2xz+4x+4)}{2} \\ \frac{iz}{2} & \frac{-i(xz+2)}{2} \end{pmatrix}.$$

See Figure 6.2.

A. Haas and P. Susskind ([HS92]) have studied simple closed curves on surfaces of type (2,0) from the viewpoint of train tracks. They have derived a formula for the number of components of a multiple curve determined by an integral weight train track on the surface. Similar methods might be used to determine intersection numbers of simple closed curves on surfaces of type (2,0).

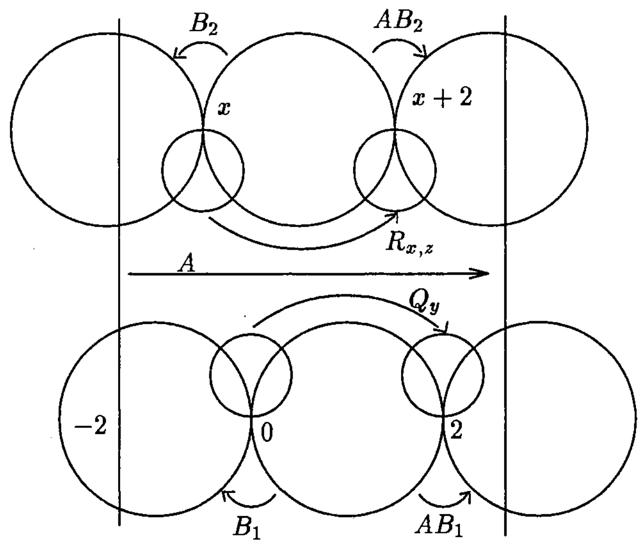


Figure 6.2: The action in the group $J_{x,y,z}$

CHAPTER 7

THE BIHOLOMORPHIC MAP FROM $T_{1,1}$ TO THE UPPER HALF PLANE

The purpose of this chapter is to construct and approximate the explicit biholomorphic map from $T_{1,1}$ to $\mathbb{H}$. This map involves the integration of an abelian differential on a Riemann surface. The abelian differential can be constructed using a cusp form for a Kleinian group. We start with a discussion of automorphic forms.

7.1 Automorphic forms

Let Γ be a Kleinian group. Suppose $F(z)$ is a function which is meromorphic in $\Omega(\Gamma)$ and satisfies $F(\gamma(z))\gamma'(z)^q = F(z)$ for all $\gamma \in \Gamma$. Let P denote a parabolic element of Γ with fixed point p. Then there is a constant c such that

$$\frac{1}{P(z)-p} = \frac{1}{z-p} + c$$

for all z. Thus,

$$\left(\frac{1}{P(z)-p}\right)' = \left(\frac{1}{z-p}\right)',$$

so

$$P'(z) = \left(\frac{P(z)-p}{z-p}\right)^2.$$

Therefore,

$$\begin{aligned}(P(z)-p)^{2q}F(P(z)) &= (P(z)-p)^{2q}\frac{F(z)}{P'(z)^q} \\ &= (z-p)^{2q}F(z),\end{aligned}$$

and so the function $(z-p)^{2q}F(z)$ is invariant under the group $\langle P\rangle$.

Since P is parabolic, there is a circular disc which is precisely invariant under $\langle P\rangle$. The map $z \mapsto t = \exp(\frac{2\pi i}{c(z-p)})$ sends this disc onto a punctured disc around the origin. Furthermore, $\exp(\frac{2\pi i}{c(z-p)}) = \exp(\frac{2\pi i}{c(w-p)})$ if and only if $z = P^n(w)$ for some integer n. Since $(z-p)^{2q}F(z)$ is invariant under $\langle P \rangle$ and meromorphic in $\Omega(\Gamma)$, there is a function $g(t)$, meromorphic in the punctured disc around the origin, so that $g(t) = (z-p)^{2q}F(z)$. Then F is *meromorphic (holomorphic) at* p if g is meromorphic (holomorphic) at the origin.

If $F(z)$ is meromorphic on $\Omega(\Gamma)$ and on the set of parabolic fixed points of Γ, and $F(\gamma(z))\gamma'(z)^q = F(z)$ for every $\gamma \in \Gamma$, then F is an *automorphic form of weight -2q for* Γ, or an *automorphic q-form for* Γ. Likewise, F is a *holomorphic automorphic q-form for* Γ if it is holomorphic on $\Omega(\Gamma)$ and on the set of parabolic fixed points of Γ, and $F(\gamma(z))\gamma'(z)^q = F(z)$ for all $\gamma \in \Gamma$.

A *cusped region* for the fixed point p of the parabolic element $P \in \Gamma$ is a circular disc D in $\Delta(\Gamma)$ which is precisely invariant under $\langle P \rangle$ such that $g(D) \cap D = \emptyset$ for all $g \in \Gamma - \langle P \rangle$. For a proof of the following theorem, see page 117 of [Kra72] and page 47 of [Kra84b].

Theorem 7.1.1 *Let* Γ *be a finitely generated, non-elementary Kleinian group with invariant component* $\Delta(\Gamma)$, *and let* $\lambda(z)$ *denote the Poincaré metric on* $\Delta(\Gamma)$. *Let* F *be a holomorphic automorphic q-form for* Γ. *Then the following conditions are equivalent:*

(i) $\iint_\omega \lambda^{2-q}(z)|F(z)|\,dx\,dy < \infty$, *where* ω *is any fundamental domain for the action of* Γ *on* $\Delta(\Gamma)$;

(ii) if p *is a fixed point of the parabolic element* $P \in \Gamma$, *and* $\{z_n\}$ *is a sequence of points in a cusped region for* p *with* $z_n \to p$, *then* $F(z_n) \to 0$;

(iii) the function $g(t) = (z-p)^{2q}F(z)$ *vanishes at* $t = 0$;

(iv) $sup\{\lambda^{-q}(z)|F(z)| : z \in \Delta(\Gamma)\} < \infty$.

If any of the above conditions hold, then $F(z)$ is a *cusp form of weight -2q for* Γ .

The surface $\Delta(G_\mu)/G_\mu$ is a punctured torus. (In this chapter we follow the notation of [Wri] and use G_μ instead of G_x.) This means there is a neighborhood of the puncture conformally equivalent to the punctured disk. Denote the puncture on $\Delta(G_\mu)/G_\mu$ by P_0, and let $\overline{\Delta(G_\mu)/G_\mu}$ denote the surface with the puncture filled in. That is, let p denote the fixed point of the parabolic element $P \in G_\mu$. There is a disc D contained in $\Delta(G_\mu)$ which is precisely invariant under $\langle P\rangle$ in G_μ, so that $D/\langle P\rangle$ is naturally embedded in $\Delta(G_\mu)/G_\mu$. Since P is parabolic, there must be a constant c so that

$$\frac{1}{P(z)-p} = \frac{1}{z-p} + c$$

for $z \in \mathbb{C}$. The map $z \mapsto \exp(\frac{2\pi i}{c(z-p)})$ is a conformal map from D onto a punctured disc around the origin $\Delta - \{0\}$. Furthermore, $\exp(\frac{2\pi i}{c(z-p)}) = \exp(\frac{2\pi i}{c(w-p)})$ if and only if $z = P^n(w)$ for some integer n. Hence, this map induces a conformal homeomorphism $\xi : D/\langle P\rangle \to \Delta - \{0\}$. Define $\xi(P_0) = 0$. Then $\overline{\Delta(G_\mu)/G_\mu}$ is the Riemann surface $\Delta(G_\mu)/G_\mu$, along with another point P_0 and a coordinate chart $(D/\langle P\rangle \cup P_0, \xi)$.

Given a Riemann surface S, a (holomorphic) *q-differential* ζ on S is an assignment of a (holomorphic) function f to each local coordinate z on S such that $f(z)(dz)^q$ is invariant under change of local coordinates. A 1-differential is called an *abelian differential*; a 2-differential is called a *quadratic differential.* If G is a Kleinian group, and F is a (holomorphic) automorphic form of weight $-2q$ for G, then F projects to a (holomorphic) q-differential on $\Omega(G)/G$.

Lemma 7.1.2 *If $F(z)$ is a cusp form of weight $-2q$ for G_μ, then the corresponding q-differential on $\overline{\Delta(G_\mu)/G_\mu}$ has a pole of order $\leq q-1$ at the puncture.*

Proof: Let p denote a parabolic fixed point of G_μ, and let $g(t) = (z-p)^{2q}F(z)$. We want to find the order of the q-differential $F(z)\,dz^q$ at the puncture P_0 of $\Delta(G_\mu)/G_\mu$.

Now since $t = \exp(\frac{2\pi i}{c(z-p)})$, $\frac{dz}{dt} = \frac{c(z-p)^2}{-2\pi i t}$. Thus,

$$\begin{aligned} F(z)\,dz^q &= \frac{(z-p)^{2q}}{(z-p)^{2q}} F(z)\,dz^q \\ &= \frac{g(t)}{(z-p)^{2q}} (\frac{dz}{dt})^q (dt)^q \\ &= \frac{g(t)}{(z-p)^{2q}} \left(\frac{c(z-p)^2}{-2\pi i t} \right)^q (dt)^q \\ &= g(t)(\frac{c}{-2\pi i t})^q dt^q. \end{aligned}$$

Hence, the order of $F(z)\,dz^q$ at P_0 is the order at $t = 0$ of $g(t)(\frac{c}{-2\pi i t})^q dt^q$, which is $\mathrm{ord}_0 g(t) - q$.

Since $F(z)\,dz^q$ is a cusp form, $\mathrm{ord}_0 g(t) \geq 1$, so the order of $F(z)\,dz^q$ at P_0 is $\geq 1 - q$; so $F(z)\,dz^q$ has a pole at P_0 of order $\leq q - 1$.

q.e.d.

Lemma 7.1.3 *The sum of the residues of an abelian differential over all points on a compact Riemann surface is zero.*

Proof: Let ζ be an abelian differential on a compact Riemann surface. Triangulate the surface so that no pole of ζ lies on the boundary of any triangle, and each triangle contains no more than one pole of ζ. Denote the triangles by T_1, ..., T_n, and their boundaries by ∂T_1, ..., ∂T_n. Then the sum of the residues of ζ over all points on the surface is $\frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\partial T_j} \zeta$. This sum is zero since each side of each triangle appears twice in opposite directions.

q.e.d.

Proposition 7.1.4 *If $F(z) \neq 0$ is a cusp form of weight -4 for G_μ, then F is nonzero on $\Delta(G_\mu)$.*

Proof: The cusp form F projects to a quadratic differential f on $\overline{\Delta(G_\mu)/G_\mu}$ which is holomorphic on $\Delta(G_\mu)/G_\mu$. Furthermore, by Lemma 7.1.2, f has at most a simple pole at the puncture P_0. Let $g \neq 0$ denote a holomorphic abelian differential on $\overline{\Delta(G_\mu)/G_\mu}$. Then g is nonzero, since it is defined on a compact Riemann surface of genus 1. (This follows from the Riemann-Roch Theorem; see [FK91].) Thus, $\frac{f}{g}$ is a meromorphic abelian differential on $\overline{\Delta(G_\mu)/G_\mu}$, holomorphic on $\Delta(G_\mu)/G_\mu$ and having at most a simple pole at P_0. By Lemma 7.1.3, $\frac{f}{g}$ must be a holomorphic abelian differential. Therefore, $\frac{f}{g}$ is nonzero on $\overline{\Delta(G_\mu)/G_\mu}$, and so F is nonzero on $\Delta(G_\mu)$.

q.e.d.

Lemma 7.1.5 *Suppose Ω is an open set in $\mathbb{C}$, and $\{f_n(z)\}_{n=1}^{\infty}$ is a sequence of holomorphic functions on Ω. If $\iint_\Omega \sum_{n=1}^{\infty} |f_n(z)|\, dx\, dy < \infty$, then $\sum_{n=1}^{\infty} f_n(z)$ converges absolutely uniformly on every compact subset of Ω.*

Proof: Let K be a compact set in Ω, and let $r > 0$ be less than the distance from K to the boundary of Ω. If $z_0 \in K$ and f is holomorphic on Ω then by the mean value property for holomorphic functions (see, for example [Con78]),

$$f(z_0) = \frac{1}{\pi r^2} \iint_{D(z_0,r)} f(z)\, dx\, dy,$$

where $D(z_0, r)$ denotes the disc around z_0 with radius r. Hence

$$\begin{aligned} |\sum_{n=j}^{k} f_n(z_0)| &\leq \frac{1}{\pi r^2} \iint_{D(z_0,r)} |\sum_{n=j}^{k} f_n(z)|\, dx\, dy \\ &\leq \frac{1}{\pi r^2} \iint_{K_r} |\sum_{n=j}^{k} f_n(z)|\, dx\, dy \end{aligned}$$

where K_r denotes the set of points $z \in \Omega$ such that $z \in K$ or the distance from z to K is less than r.

q.e.d.

Let N denote the subgroup of G_μ generated by S_1, and consider the set of right cosets of N in G_μ: $\{Ng : g \in G_\mu\}$. Two cosets Ng_1 and Ng_2 are the same if and only if $g_2 g_1^{-1} \in N$; that is, if and only if $g_2 = S_1^n g_1$ for some integer n. Since $(S_1 g)'(z) = S_1'(g(z))g'(z) = g'(z)$ for all $g \in G_\mu$, $g_1'(z) = g_2'(z)$ for any two elements g_1, g_2 in the same right coset. Let Υ denote any set consisting of exactly one element from each right coset. Then $P_q(z) = \sum_{g\in\Upsilon} g'(z)^q$ is well-defined for any integer q.

Proposition 7.1.6 *The series $\sum_{g\in\Upsilon} g'(z)^2$ converges absolutely uniformly on compact subsets of $\Delta(G_\mu)$ to a cusp form of weight -4 for G_μ.*

Proof: Let V denote the vertical strip $\{z : 0 < \mathrm{Re}(z) < 4\}$, and let ω be any fundamental domain for the action of G_μ on $\Delta(G_\mu)$. For each right coset of $N = \langle S_1 \rangle$ in G_μ, choose a representative g_j so that $g_j(\omega)$ is contained in V. Then

$$\begin{aligned}
\iint_\omega |\sum_{g_j\in\Upsilon} g_j'(z)^2|\, dx\, dy &\leq \iint_\omega \sum_{g_j\in\Upsilon} |g_j'(z)^2|\, dx\, dy \\
&= \sum_{g_j\in\Upsilon} \iint_\omega |g_j'(z)^2|\, dx\, dy \\
&= \sum_{g_j\in\Upsilon} \iint_{g_j(\omega)} dx\, dy \\
&< \iint_{\Delta(G_\mu)\cap V} dx\, dy \\
&< \infty.
\end{aligned}$$

By Lemma 7.1.5, the series $\sum_{g\in\Upsilon} g'(z)^2$ converges absolutely uniformly on compact subsets of ω. Since ω was an arbitrary fundamental domain, this series converges absolutely uniformly (to a holomorphic function) on compact subsets of $\Delta(G_\mu)$.

To show that $P_2(z)$ is an automorphic 2-form for G_μ, note that if $\gamma \in G_\mu$, then

$$\begin{aligned}
P_2(\gamma(z))\gamma'(z)^2 &= \sum_{g\in\Upsilon} g'(\gamma(z))^2 \gamma'(z)^2 \\
&= \sum_{g\in\Upsilon} ((g\circ\gamma)'(z))^2 \\
&= \sum_{h=g\circ\gamma\in\Upsilon} h'(z)^2 \\
&= P_2(z).
\end{aligned}$$

Since $\iint_\omega |P_2(z)|\, dx\, dy < \infty$ for any fundamental domain for the action of G_μ on $\Delta(G_\mu)$, $P_2(z)$ is a cusp form of weight -4 for G_μ.

q.e.d.

We prove in Section 7.2 that $P_2(z) = P_2(\mu; z)$ is not identically zero on $\Delta(G_\mu)$. By Proposition 7.1.4, $P_2(\mu; z)$ is never zero on $\Delta(G_\mu)$, so it has an analytic square root there. In fact, $\sqrt{P_2(\mu; z)}$ is a cusp form of weight -2 for G_μ. To prove this, let F be the lift to $\Delta(G_\mu)$ of the nontrivial holomorphic abelian differential on $\overline{\Delta(G_\mu)/G_\mu}$. Then F^2 is a cusp form of weight -4 for G_μ. By the Riemann-Roch Theorem (see, for example, [FK91], especially page 77), the complex dimension of cusp forms of weight -4 for G_μ is one. Hence there is some constant $c_1 \in \mathbb{C}$ with $F^2 = c_1 P_2(\mu; z)$. Thus, if $c_2^2 = c_1$, then either $F = c_2\sqrt{P_2(\mu; z)}$ or $F = -c_2\sqrt{P_2(\mu; z)}$. Since F projects to the holomorphic abelian differential on $\overline{\Delta(G_\mu)/G_\mu}$, it is a cusp form of weight -2 for G_μ.

Let ζ denote the abelian differential on $\overline{\Delta(G_\mu)/G_\mu}$ which is the projection of $\sqrt{P_2(\mu; z)}$. Then $\{\zeta\}$ is a basis for the space of holomorphic abelian differentials on $\overline{\Delta(G_\mu)/G_\mu}$. (This space has complex dimension 1 by the Riemann-Roch Theorem.) Choose a base point Q_0 on $\Delta(G_\mu)/G_\mu$, and let $\{a, b\}$ be the canonical basis for $\pi_1(\overline{\Delta(G_\mu)/G_\mu}, Q_0)$, so that the loops a and b have exactly one point in common and the angle from the positive direction on the a loop to the positive direction on the b loop at the point of intersection is positive and less than π radians. (Then if Q is any point in $\Delta(G_\mu)$, then any curve in $\Delta(G_\mu)$ from Q to $S_1(Q)$ projects to a loop on $\overline{\Delta(G_\mu)/G_\mu}$ in the homotopy class of a, and any curve in $\Delta(G_\mu)$ from Q to $T_\mu(Q)$ projects to a loop on $\overline{\Delta(G_\mu)/G_\mu}$ in the homotopy class of b.) Now define $\psi : M_{1,1} \to \mathbb{H}$ by $\psi(\mu) = \tau = \frac{\int_b \zeta}{\int_a \zeta}$.

Proposition 7.1.7 *Im*$(\psi(\mu)) > 0$.

Proof: We will show that $\mathrm{Im}(\overline{\int_a \zeta} \cdot \int_b \zeta) > 0$. First cut $\overline{\Delta(G_\mu)/G_\mu}$ along the loops a and b to produce a rectangle R with sides a_+, a_-, b_+, and b_-, where the sides a_+

and a_- are identified by G_μ to make the a loop (and similarly for b_+ and b_-). Let g denote an antiderivative of $\sqrt{P_2(z)}$ in R. Then if z_- and z_+ are points on b_- and b_+ identified by G_μ, then $g(z_+) - g(z_-) = \int_a \zeta$; similarly, if w_- and w_+ are points on a_- and a_+ identified by G_μ, then $g(w_+) - g(w_-) = \int_b \zeta$. Thus,

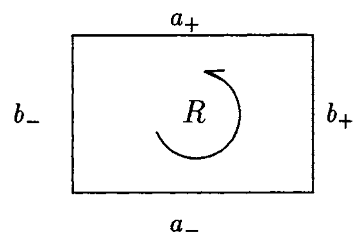


Figure 7.1: The orientation of R

$$
\begin{aligned}
\frac{1}{2i}\int_{\partial R} \overline{g}\zeta &= \frac{1}{2i}\left[\int_{a_-} \overline{g}\zeta - \int_{a_+} \overline{g}\zeta\right] + \frac{1}{2i}\left[\int_{b_+} \overline{g}\zeta - \int_{b_-} \overline{g}\zeta\right] \\
&= \frac{1}{2i}\left[\int_{a_-} (\overline{g}\zeta - (\overline{g} + \overline{\int_b \zeta})\zeta)\right] + \frac{1}{2i}\left[\int_{b_+} (\overline{g}\zeta - (\overline{g} - \overline{\int_a \zeta})\zeta)\right] \\
&= \frac{1}{2i}\left[-\int_a \zeta \overline{\int_b \zeta} + \int_b \zeta \overline{\int_a \zeta}\right] \\
&= \frac{1}{2i}\left[2i\mathrm{Im}(\overline{\int_a \zeta} \cdot \int_b \zeta)\right] \\
&= \mathrm{Im}(\overline{\int_a \zeta} \cdot \int_b \zeta).
\end{aligned}
$$

On the other hand, if we write $g = u + iv$, then $\zeta = du + i\,dv$ and

$$
\begin{aligned}
\frac{\overline{g}\zeta}{2i} &= \frac{1}{2i}(u\,du + v\,dv + iu\,dv - iv\,du) \\
&= \frac{u\,du + v\,dv}{2i} + \frac{u\,dv - v\,du}{2} \\
&= \frac{d(u^2 + v^2)}{4i} + \frac{u\,dv - v\,du}{2}.
\end{aligned}
$$

Thus, by Green's Theorem (see, for example, [Buc78]),

$$\begin{aligned}\frac{1}{2i}\int_{\partial R}\overline{g}\zeta &= \int_{\partial R}\frac{u\,dv - v\,du}{2}\\ &= \iint_R du\,dv > 0.\end{aligned}$$

q.e.d.

Let G_τ denote the group $\langle z \mapsto z+1, z \mapsto z+\tau\rangle$, and define $\overline{\alpha} : \overline{\Delta(G_\mu)/G_\mu} \to \mathbb{C}/G_\tau$ by $\overline{\alpha}(P) = \pi_\tau(\frac{\int_{Q_0}^P \zeta}{\int_a \zeta})$, where $\pi_\tau : \mathbb{C} \to \mathbb{C}/G_\tau$ is the natural projection. We will show that $\overline{\alpha}$ is a conformal homeomorphism. We use divisors to do so. A *divisor* on a Riemann surface is a formal symbol $\sum n_i P_i$, where each n_i is an integer and each P_i is a point on the surface. The *degree* of the divisor $\sum n_i P_i$ is the sum $\sum n_i$. The group of divisors on a surface is the free abelian group on the points of the surface. We choose to write the unity in this group as 0.

If f is a meromorphic function on a Riemann surface, the symbol (f) denotes the divisor $\sum n_i P_i$, where the P_i's are the zeros and poles of f, and n_i is the order of the zero or pole P_i; if P_i is a zero, then $n_i > 0$, and if P_i is a pole then $n_i < 0$. A divisor is *principal* if it is (f) for some f.

Now extend $\overline{\alpha}$ to a map $\overline{\alpha}_1$ from the group of divisors on $\overline{\Delta(G_\mu)/G_\mu}$ to the group of divisors on $\mathbb{C}/G_\tau$ by defining $\overline{\alpha}_1(\sum n_i P_i) = \sum n_i \overline{\alpha}(P_i)$.

A slightly more general statement than the following theorem is due to Abel; a proof appears in [FK91].

Theorem 7.1.8 *A divisor D on $\overline{\Delta(G_\mu)/G_\mu}$ is principal if and only if deg$(D) = 0$ and $\overline{\alpha}_1(D) = 0$.*

Proposition 7.1.9 *The map $\overline{\alpha}$ is a conformal homeomorphism.*

Proof: Let $\mathcal{L}_\tau$ denote the lattice $\{m + n\tau : n, m \in \mathbb{Z}\}$. Let c_1 and c_2 be two paths in $\overline{\Delta(G_\mu)/G_\mu}$ from Q_0 to P. Then $c_1 c_2^{-1}$ is homotopic to $ma + nb$ for some $m, n \in \mathbb{Z}$; so $\frac{\int_{c_1}\zeta}{\int_a \zeta} - \frac{\int_{c_2}\zeta}{\int_a \zeta} = \frac{\int_{c_1 c_2^{-1}}\zeta}{\int_a \zeta} = m + n\tau \in \mathcal{L}_\tau$. Thus, $\overline{\alpha}$ is well-defined.

Now let z be a local coordinate vanishing at P. Then $\overline{\alpha}(z) = \frac{\int_{Q_0}^{P}\zeta}{\int_a \zeta} + \frac{\int_P^z \zeta}{\int_a \zeta} = \frac{\int_{Q_0}^{P}\zeta}{\int_a \zeta} + \frac{\int_0^z \sqrt{P_2(z)}dz}{\int_a \zeta}$, so $\overline{\alpha}'(z) = \frac{\sqrt{P_2(z)}}{\int_a \zeta}$, and $\overline{\alpha}$ is holomorphic.

Since $\overline{\alpha}$ is holomorphic and non-constant, the Open Mapping Theorem (see [Con78], for example) guarantees that $\overline{\alpha}(\overline{\Delta(G_\mu)/G_\mu})$ is open in $\mathbb{C}/G_\tau$. Since $\overline{\Delta(G_\mu)/G_\mu}$ is compact, so is $\overline{\alpha}(\overline{\Delta(G_\mu)/G_\mu})$. Since $\mathbb{C}/G_\tau$ is Hausdorff, $\overline{\alpha}(\overline{\Delta(G_\mu)/G_\mu})$ is closed in $\mathbb{C}/G_\tau$. Thus, $\overline{\alpha}$ is surjective.

Finally, to show that $\overline{\alpha}$ is injective, suppose there are two distinct points P, P' on $\overline{\Delta(G_\mu)/G_\mu}$ such that $\overline{\alpha}(P) = \overline{\alpha}(P')$. Then by Abel's Theorem (Theorem 7.1.8), $P - P'$ is a principal divisor on $\overline{\Delta(G_\mu)/G_\mu}$; but by the Riemann-Roch Theorem there is no meromorphic function on a closed surface of genus 1 with a simple pole at one point and no other poles. (See page 269 of [Spr57], for example.) This contradiction finishes the proof.

q.e.d.

Let $\tau_1 = \overline{\alpha}(P_0)$, and let $\mathcal{L}_{\tau^*}$ denote the lattice $\{w + \tau_1 : w \in \mathcal{L}_\tau\}$. Then the map $\alpha : \Delta(G_\mu)/G_\mu \to (\mathbb{C} - \mathcal{L}_{\tau^*})/G_\tau$, which is the restriction of $\overline{\alpha}$ to the punctured surface, is also a conformal homeomorphism. Furthermore, it clearly preserves the marking.

Now choose a base point $Q \in \Delta(G_\mu)$ and define $\varphi : \Delta(G_\mu) \to \mathbb{C} - \mathcal{L}_{\tau^*}$ by

$$\varphi(z) = \frac{\int_Q^z \sqrt{P_2(\mu; z)}\,dz}{\int_Q^{Q+2} \sqrt{P_2(\mu; z)}\,dz}.$$

Let $\pi_\mu : \Delta(G_\mu) \to \Omega_0(G_\mu)/G_\mu$ be the natural projection. Then $\alpha \circ \pi_\mu = \pi_\tau \circ \varphi$, so φ must map any fundamental domain for the action of G_μ on $\Delta(G_\mu)$ onto a fundamental rectangle in $\mathbb{C} - \mathcal{L}_{\tau^*}$. It follows from the following proposition that φ is surjective.

Proposition 7.1.10 *The map φ satisfies the equations $\varphi(S_1(z)) = \varphi(z) + 1$ and $\varphi(T_\mu(z)) = \varphi(z) + \tau$.*

Proof: First note that $\int_{Q+2}^{z+2} \sqrt{P_2(\mu; z)}\,dz = \int_Q^z \sqrt{P_2(\mu; z+2)}\,dz = \int_Q^z \sqrt{P_2(\mu; z)}\,dz$

and

$$\begin{aligned}
\int_{T_\mu(Q)}^{T_\mu(z)} \sqrt{P_2(\mu;z)}\,dz &= \int_Q^z \sqrt{P_2(\mu;T_\mu(z))}T_\mu{}'(z)\,dz \\
&= \int_Q^z \sqrt{P_2(\mu;T_\mu(z))T_\mu{}'(z)^2}\,dz \\
&= \int_Q^z \sqrt{P_2(\mu;z)}\,dz.
\end{aligned}$$

Thus,

$$\begin{aligned}
\varphi(S_1(z)) &= \frac{\int_Q^{z+2}\sqrt{P_2(\mu;z)}\,dz}{\int_Q^{Q+2}\sqrt{P_2(\mu;z)}\,dz} \\
&= \frac{\int_Q^{Q+2}\sqrt{P_2(\mu;z)}\,dz}{\int_Q^{Q+2}\sqrt{P_2(\mu;z)}\,dz} + \frac{\int_{Q+2}^{z+2}\sqrt{P_2(\mu;z)}\,dz}{\int_Q^{Q+2}\sqrt{P_2(\mu;z)}\,dz} \\
&= 1+\varphi(z).
\end{aligned}$$

Also,

$$\begin{aligned}
\varphi(T_\mu(z)) &= \frac{\int_Q^{T_\mu(z)}\sqrt{P_2(\mu;z)}\,dz}{\int_Q^{Q+2}\sqrt{P_2(\mu;z)}\,dz} \\
&= \frac{\int_Q^{T_\mu(Q)}\sqrt{P_2(\mu;z)}\,dz}{\int_Q^{Q+2}\sqrt{P_2(\mu;z)}\,dz} + \frac{\int_{T_\mu(Q)}^{T_\mu(z)}\sqrt{P_2(\mu;z)}\,dz}{\int_Q^{Q+2}\sqrt{P_2(\mu;z)}\,dz} \\
&= \tau+\varphi(z).
\end{aligned}$$

q.e.d.

Since $\varphi'(z) = \frac{\sqrt{P_2(\mu;z)}}{\int_Q^{Q+2}\sqrt{P_2(\mu;z)}\,dz}$ is never zero, φ is a conformal map.

Proposition 7.1.11 *The map $\varphi : \Delta(G_\mu) \to \mathbb{C} - \mathcal{L}_{\tau^*}$ is a covering map.*

Proof: We have shown that φ is a continuous surjection. Let $z \in \mathbb{C} - \mathcal{L}_{\tau^*}$. Choose a fundamental rectangle R for $\mathbb{C} - \mathcal{L}_{\tau^*}$ such that z is in the interior of R. Then there is a connected open neighborhood N of z contained in R. Now pick any connected component of $\varphi^{-1}(N)$ in $\Delta(G_\mu)$ and any fundamental domain F for the action of G_μ on $\Delta(G_\mu)$ which contains that component in its interior. Then each connected component of $\varphi^{-1}(N)$ is an open set contained in a G_μ translate of F of the form $g(F)$, where $g = T_\mu^{n_1} S_1^{m_1} \cdots T_\mu^{n_r} S_1^{m_r}$ and $\sum_{i=1}^{r} n_i = \sum_{i=1}^{r} m_i = 0$. We have shown that every point in $\mathbb{C} - \mathcal{L}_{\tau^*}$ has an open neighborhood which is evenly covered by a union of disjoint open sets in $\Delta(G_\mu)$, each one to which the restriction of φ is a homeomorphism onto N. (These restrictions of φ are homeomorphisms by Proposition 7.1.9 and the fact that $\alpha \circ \pi_\mu = \pi_\tau \circ \varphi$.)

q.e.d.

Lemma 7.1.12 *Let $R_\mu(z) = \mu - z$. Then $R_\mu G_\mu R_\mu^{-1} = G_\mu$.*

Proof: Easy computations show that $R_\mu S_1 R_\mu^{-1} = S_1^{-1}$ and $R_\mu T_\mu R_\mu^{-1} = T_\mu^{-1}$.

q.e.d.

Recall that $P_2(z) = \sum_{g \in \Upsilon} g'(z)^2$, where Υ is any set of right coset representatives of $\langle S_1 \rangle$ in G_μ.

Lemma 7.1.13 *For $z \in \Delta(G_\mu)$, $P_2(z) = P_2(\mu - z)$.*

Proof: By Lemma 7.1.12, the set of right cosets of $\langle S_1 \rangle$ in G_μ equals the set of right cosets of $\langle S_1 \rangle$ in $R_\mu G_\mu R_\mu^{-1}$. Thus,

$$\begin{aligned}
P_2(z) &= \sum_{g \in \Upsilon} g'(z)^2 \\
&= \sum_{g \in \Upsilon} (R_\mu \circ g \circ R_\mu^{-1})'(z)^2 \\
&= \sum_{g \in \Upsilon} [(R_\mu \circ g)'(R_\mu^{-1}(z))(R_\mu^{-1})'(z)]^2 \\
&= \sum_{g \in \Upsilon} [(R_\mu \circ g)'(\mu - z)]^2 \\
&= \sum_{g \in \Upsilon} [R_\mu'(g(\mu - z))g'(\mu - z)]^2 \\
&= \sum_{g \in \Upsilon} g'(\mu - z)^2 \\
&= P_2(\mu - z).
\end{aligned}$$

q.e.d.

Lemma 7.1.14 $P_2(\mu; z) = \overline{P_2(-\overline{\mu}; -\overline{z})}$.

Proof: Define $J(z) = -\overline{z}$. Then $J^{-1} = J$, and $JT_\mu J = T_{-\overline{\mu}}$, and $JS_1J = S_1^{-1}$.

We first claim that for any $g \in G_\mu$, $[(JgJ)'(z)]^2 = [(Jg'J)(z)]^2$. Given any $g = T_\mu^{n_1} S_1^{m_1} \cdots T_\mu^{n_r} S_1^{m_r}$ in G_μ, define the *length* of g to be $\sum_{i=1}^{r}(|n_i| + |m_i|)$. We will use induction on the length of g to prove our claim. First, it is easy to compute that for $h = S_1$, S_1^{-1}, T_μ, or T_μ^{-1}, $[(JhJ)'(z)]^2 = [(Jh'J)(z)]^2$. Assume the claim is true for all words of length n, and let g have length $n+1$. Then $g = hg_1$, where $h = S_1$, S_1^{-1}, T_μ, or T_μ^{-1}, and g_1 has length n. Thus,

$$\begin{aligned}
[(JgJ)'(z)]^2 &= [(JhJJg_1J)'(z)]^2 \\
&= [(JhJ)'(Jg_1J)(z) \cdot (Jg_1J)'(z)]^2 \\
&= [(Jh'J)(Jg_1J)(z) \cdot (Jg_1{}'J)(z)]^2 \\
&= [(Jh'g_1J)(z) \cdot (Jg_1{}'J)(z)]^2 \\
&= [(Jh'g_1)(-\overline{z}) \cdot (Jg_1{}')(-\overline{z})]^2 \\
&= [J(h'g_1(-\overline{z}) \cdot g_1{}'(-\overline{z}))]^2 \\
&= [J((hg_1)'(-\overline{z}))]^2 \\
&= [Jg'J(z)]^2.
\end{aligned}$$

Now as g varies over all the right coset representatives of $\langle S_1 \rangle$ in G_μ, JgJ varies over all the right coset representatives of $\langle S_1 \rangle$ in $G_{-\overline{\mu}}$. Therefore, since for any $g \in G_\mu$, $[g'(z)]^2 = \overline{[(Jg'J)(-\overline{z})]^2} = \overline{[(JgJ)'(-\overline{z})]^2}$, the equality $P_2(\mu; z) = \overline{P_2(-\overline{\mu}; -\overline{z})}$ must hold.

q.e.d.

Lemma 7.1.15 *Fix $\mu = it$, $t > 2$. Let L_1 denote the horizontal line segment $x + i\frac{t}{2}$, $-1 \leq x \leq 1$; and let L_2 denote the vertical line segment iy, $\frac{2}{t} \leq y \leq \frac{t}{2}$. Then $P_2(z)$ is real on L_1 and L_2.*

Proof: By Lemmas 7.1.13 and 7.1.14, $\overline{P_2(z)} = P_2(-\bar{z}) = P_2(it + \bar{z})$. In particular, for $0 \le y \le \frac{t}{2} - \frac{2}{t}$,

$$\begin{aligned}
\overline{P_2(x + i(\frac{t}{2} - y))} &= P_2(it + \overline{x + i(\frac{t}{2} - y)}) \\
&= P_2(it + x + i(y - \frac{t}{2})) \\
&= P_2(x + i(\frac{t}{2} + y)),
\end{aligned}$$

and setting $y = 0$ into both sides of this equation yields $\overline{P_2(x + i\frac{t}{2})} = P_2(x + i\frac{t}{2})$. Thus, $P_2(z)$ is real on L_1. Likewise,

$$\begin{aligned}
\overline{P_2(x + i(\frac{t}{2} - y))} &= P_2(x + i(\frac{t}{2} + y)) \\
&= P_2(it - x - i(\frac{t}{2} + y)) \\
&= P_2(-x + i(\frac{t}{2} - y)).
\end{aligned}$$

Setting $x = 0$ into both sides here yields $\overline{P_2(i(\frac{t}{2} - y))} = P_2(i(\frac{t}{2} - y))$, which proves that $P_2(z)$ is real on L_2.

q.e.d.

Recall now that the map $\psi : M_{1,1} \to \mathbb{H}$ is defined by

$$\psi(\mu) = \frac{\int_b \sqrt{P_2(z)}\, dz}{\int_a \sqrt{P_2(z)}\, dz},$$

where $\{a, b\}$ is a basis for the fundamental group of $\overline{\Delta(G_\mu)/G_\mu}$, a corresponding to S_1 and b corresponding to T_μ.

Proposition 7.1.16 *The map $\psi : M_{1,1} \to \mathbb{H}$ takes the imaginary ray in $M_{1,1}$ to the imaginary ray in $\mathbb{H}$.*

Proof: Since $P_2(z)$ is nonzero on $\Delta(G_\mu)$, there are two possible cases: either $P_2(z) > 0$ on L_1 and L_2, or $P_2(z) < 0$ on L_1 and L_2. (The line segments L_1 and L_2 intersect at the point $\frac{t}{2}i$.) Thus, on L_1 and L_2, $\sqrt{P_2(z)}$ is either real or pure imaginary. In either case,

$$
\begin{aligned}
\overline{\psi(it)} &= \overline{\left(\frac{\int_b \sqrt{P_2(z)}\,dz}{\int_a \sqrt{P_2(z)}\,dz}\right)} \\
&= \overline{\left(\frac{\int_{\frac{t}{2}}^{\frac{t}{2}} \sqrt{P_2(iy)}\,i\,dy}{\int_{-1}^{1} \sqrt{P_2(x+i\frac{t}{2})}\,dx}\right)} \\
&= -\frac{\int_{\frac{t}{2}}^{\frac{t}{2}} \sqrt{P_2(iy)}\,i\,dy}{\int_{-1}^{1} \sqrt{P_2(x+i\frac{t}{2})}\,dx} \\
&= -\psi(it).
\end{aligned}
$$

q.e.d.

Proposition 7.1.17 *The map $\psi : M_{1,1} \to \mathbb{H}$ satisfies the equation $\psi(\mu+2n) = \psi(\mu)+n$, for any integer n.*

Proof: Let $P_2(\mu; z)$ denote the function $\sum_{g\in\Upsilon(\mu)} g'(z)^2$, where $\Upsilon(\mu)$ is any set of right coset representatives of $\langle S_1\rangle$ in G_μ. Since $T_{\mu+2n} = S_1^n T_\mu$, the set of right cosets of $\langle S_1\rangle$ in G_μ is the same as the set of right cosets of $\langle S_1\rangle$ in $G_{\mu+2n}$. Thus, $P_2(\mu; z) = P_2(\mu+2n; z)$ for any $\mu \in M_{1,1}$.

Now $\psi : M_{1,1} \to \mathbb{H}$ is defined by

$$
\psi(\mu) = \frac{\int_{b(\mu)} \sqrt{P_2(\mu; z)}\,dz}{\int_{a(\mu)} \sqrt{P_2(\mu; z)}\,dz},
$$

where $a(\mu)$ and $b(\mu)$ are curves in $\Delta(G_\mu)$ projecting to the generators of $\pi_1(\overline{\Delta(G_\mu)/G_\mu})$ corresponding to S_1 and T_μ, respectively. Thus,

$$
\begin{aligned}
\psi(\mu+2n) &= \frac{\int_{b(\mu+2n)} \sqrt{P_2(\mu+2n; z)}\,dz}{\int_{a(\mu+2n)} \sqrt{P_2(\mu+2n; z)}\,dz} \\
&= \frac{\int_{b(\mu+2n)} \sqrt{P_2(\mu; z)}\,dz}{\int_{a(\mu+2n)} \sqrt{P_2(\mu; z)}\,dz}.
\end{aligned}
$$

Since $T_{\mu+2n} = S_1^n T_\mu$, $G_{\mu+2n} = G_\mu$ and the set of curves of types $a(\mu)$ and $a(\mu+2n)$ are the same. Furthermore, $b(\mu+2n)$ can be chosen to be a curve of the type $b(\mu)$ followed by n curves of the type $a(\mu)$. Then

$$\begin{aligned}\psi(\mu+2n) &= \frac{\int_{b(\mu)} \sqrt{P_2(\mu;z)}\,dz}{\int_{a(\mu)} \sqrt{P_2(\mu;z)}\,dz} + n\frac{\int_{a(\mu)} \sqrt{P_2(\mu;z)}\,dz}{\int_{a(\mu)} \sqrt{P_2(\mu;z)}\,dz} \\ &= \psi(\mu)+n.\end{aligned}$$

q.e.d.

Lemma 7.1.18 $\Delta(G_{-\overline{\mu}}) = \{-\overline{z} : z \in \Delta(G_\mu)\}$.

Proof: Let $J(z) = -\overline{z}$. Then $JG_\mu J^{-1} = G_{-\overline{\mu}}$, so $\Delta(G_{-\overline{\mu}}) = J(\Delta(G_\mu))$.

q.e.d.

Proposition 7.1.19 $\psi(-\overline{\mu}) = -\overline{\psi(\mu)}$.

Proof: Let $b(-\overline{\mu})$ and $a(-\overline{\mu})$ be curves in $\Delta(G_{-\overline{\mu}})$ from some base point Q to $T_{-\overline{\mu}}(Q) = \frac{1}{Q} - \overline{\mu}$ and to $S_1(Q) = Q+2$, respectively. Then let $-\overline{b(-\overline{\mu})}$ and $-\overline{a(-\overline{\mu})}$ denote the negative conjugates of the curves $b(-\overline{\mu})$ and $a(-\overline{\mu})$; that is, $-\overline{b(-\overline{\mu})}$ and $-\overline{a(-\overline{\mu})}$ are curves from $-\overline{Q}$ to $\frac{1}{-\overline{Q}} + \mu$ and $-\overline{Q} - 2$, respectively. By Lemma 7.1.18, these curves lie in $\Delta(G_\mu)$. Thus,

$$\begin{aligned}\psi(-\overline{\mu}) &= \frac{\int_{b(-\overline{\mu})} \sqrt{P_2(-\overline{\mu};z)}\,dz}{\int_{a(-\overline{\mu})} \sqrt{P_2(-\overline{\mu};z)}\,dz} \\ &= \frac{\int_{-\overline{b(-\overline{\mu})}} \sqrt{P_2(-\overline{\mu};-\overline{z})}\,d(-\overline{z})}{\int_{-\overline{a(-\overline{\mu})}} \sqrt{P_2(-\overline{\mu};-\overline{z})}\,d(-\overline{z})} \\ &= \frac{\int_{-\overline{b(-\overline{\mu})}} \sqrt{\overline{P_2(\mu;z)}}\,d(\overline{z})}{\int_{-\overline{a(-\overline{\mu})}} \sqrt{\overline{P_2(\mu;z)}}\,d(\overline{z})} \\ &= \frac{\overline{\int_{-\overline{Q}}^{T_\mu(-\overline{Q})} \sqrt{P_2(\mu;z)}\,dz}}{\overline{\int_{-\overline{Q}}^{-\overline{Q}-2} \sqrt{P_2(\mu;z)}\,dz}}\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\overline{\int_{-\overline{Q}}^{T_\mu(-\overline{Q})}\sqrt{P_2(\mu;z)}\,dz}}{\overline{\int_{-\overline{Q}-2}^{-\overline{Q}}\sqrt{P_2(\mu;z)}\,dz}} \\
&= -\overline{\psi(\mu)}.
\end{aligned}
$$

q.e.d.

Notice that Proposition 7.1.16 follows immediately from Proposition 7.1.19.

Corollary 7.1.20 *ψ takes the ray $Re(\mu) = 1$ in $M_{1,1}$ to the vertical ray $Re(\tau) = \frac{1}{2}$ in $\mathbb{H}$.*

Proof: Propositions 7.1.17 and 7.1.19 imply that

$$
\begin{aligned}
\psi(it+1) &= -\overline{\psi(it-1)} \\
&= -\overline{\psi(it-1+2)-1} \\
&= -\overline{\psi(it+1)}+1.
\end{aligned}
$$

Thus, $\mathrm{Re}(\psi(it+1)) = \frac{1}{2}$.

q.e.d.

7.2 Eichler cohomology and the non-vanishing of the series

We begin by defining terms and notation. Throughout, we assume Γ is a non-elementary Kleinian group with $\infty \in \Omega(\Gamma)$. (A Kleinian group is *non-elementary* if its limit set contains more than two points; if this is true then the limit set must be uncountable.)

Let q be an integer ≥ 2. Let D be any Γ-invariant subset of the Riemann sphere. Then Γ acts on functions F on D by the formula $(F \cdot \gamma)(z) = F(\gamma(z))\gamma'(z)^{1-q}$.

Let Π_{2q-2} denote the vector space of polynomials in one complex variable of degree less than or equal to $2q-2$.

A mapping $\chi : \Gamma \to \Pi_{2q-2}$ is a *cocycle* for Γ if $\chi(\gamma_1\gamma_2) = \chi(\gamma_1) \cdot \gamma_2 + \chi(\gamma_2)$ for all $\gamma_1, \gamma_2 \in \Gamma$. A cocycle is a *coboundary* if there is some $p \in \Pi_{2q-2}$ with $\chi(\gamma) = p \cdot \gamma - p$ for all $\gamma \in \Gamma$. The *first Eichler cohomology space* $H^1(\Gamma, \Pi_{2q-2})$ is the set of cocycles modulo the coboundaries.

If $A \in \Gamma$ is parabolic, then a cocycle χ is *parabolic with respect to* A if there is some $p \in \Pi_{2q-2}$ with $\chi(A)(z) = (p \cdot A)(z) - p(z)$. Note that if χ is parabolic with respect to A, and $Ax = x$, then since $A'(x) = 1$, $\chi(A)(x) = 0$.

A function F on D is an *Eichler integral* if for each $\gamma \in \Gamma$ there is some $\chi(\gamma) \in \Pi_{2q-2}$ such that the restriction of $\chi(\gamma)$ to D is $F \cdot \gamma - F$. In this case χ is a cocycle for Γ, called the *period of the Eichler integral* F, and we write $\chi = pd(F)$.

We assume that x is a parabolic fixed point of Γ. Let P_x denote the set of elements of Γ which are parabolic and fix x, and let Γ_x denote all the elements of Γ which fix x. Let $n = [\Gamma_x : P_x]$. Then x is *q-admissible* if q is congruent to 0 modulo n. The point x is a *cusp* if P_x has rank 2, or if x represents two punctures on $\Omega(\Gamma)/\Gamma$, or if it represents one puncture on $\Omega(\Gamma)/\Gamma$ and there are two disjoint discs in $\Omega(\Gamma)$ which are precisely invariant under P_x in Γ.

The following theorem is due to L. Ahlfors ([Ahl82]) and D. Sullivan ([Sul81]).

Theorem 7.2.1 *Let $\{a_n\}$ and $\{b_n\}$ be sequences of complex numbers and assume $\sum_n |a_n|$ converges. Then the series $\sum_n \frac{a_n}{z-b_n}$ converges absolutely almost everywhere in $\mathbb{C}$. It converges in L^1 on compact sets. Also, if the b_n are distinct, and the limit function is zero almost everywhere, then $a_n = 0$ for all n.*

Proof: Fix a compact set K in $\mathbb{C}$. Then there is a constant $C(K)$ depending only on K such that

$$\iint_K \frac{|a|}{|z-b|} dx\, dy \le C(K)|a|.$$

So since $\sum_n |a_n|$ converges, the series $\sum \frac{a_n}{z-b_n}$ converges in L^1 on K. In particular, it converges absolutely almost everywhere.

In order to prove the last part of the proposition we use the idea of a distribution. A *distribution* is a linear functional on the space $\mathcal{C}_c^\infty(\mathbb{C})$. (Distributions must be continuous in a certain topology; see [Rud73], for example.) If λ is a distribution, then its partial derivative with respect to $\overline{z}$ is defined by

$$(\frac{\partial}{\partial \overline{z}}\lambda)(f) = -\lambda(\frac{\partial}{\partial \overline{z}}f).$$

Any locally integrable function g induces a distribution λ_g by the formula

$$\lambda_g(f) = \iint_{\mathbb{C}} g(z)\, f(z)\, dx\, dy.$$

So if g is a locally integrable function, then

$$(\frac{\partial}{\partial \overline{z}}(\lambda_g))(f) = -\iint_{\mathbb{C}} g(z) \frac{\partial}{\partial \overline{z}} f(z)\, dx\, dy.$$

Thinking of $\frac{a}{z-b}$ as a distribution, then, for $f \in \mathcal{C}_c^\infty(\mathbb{C})$,

$$(\frac{\partial}{\partial \overline{z}}(\frac{a}{z-b}))(f) = -\iint_{\mathbb{C}} \frac{a}{z-b}\frac{\partial}{\partial \overline{z}} f(z)\, dx\, dy = \pi a f(b).$$

(For a proof of the last of the above equalities, see [Rud87], for example.) Therefore, by Lebesgue's Dominated Convergence Theorem ([Rud87], page 29), for any $f \in \mathcal{C}_c^\infty(\mathbb{C})$,

$$\begin{aligned}
(\frac{\partial}{\partial \overline{z}}(\sum \frac{a_n}{z-b_n}))(f) &= -\iint_{\mathbb{C}} (\sum \frac{a_n}{z-b_n})(\frac{\partial}{\partial \overline{z}} f(z))\, dx\, dy \\
&= -\sum \iint_{\mathbb{C}} (\frac{a_n}{z-b_n})(\frac{\partial}{\partial \overline{z}} f(z))\, dx\, dy \\
&= \pi \sum a_n f(b_n).
\end{aligned}$$

So if $\sum \frac{a_n}{z-b_n} = 0$ almost everywhere, then $\sum a_n f(b_n) = 0$ for every $f \in \mathcal{C}_c^\infty(\mathbb{C})$. If the b_n are distinct, then $a_n = 0$ for each n. For example, to show that $a_1 = 0$, take a sequence $f_k \in \mathcal{C}_c^\infty(\mathbb{C})$ such that $f_k(b_1) = 1$, the support of f_k is contained in a ball around b_1 of radius $\frac{1}{k}$, and such that $|f_k(z)| \leq 1$. Then $0 = a_1 + E_k$, where $E_k \to 0$ as $k \to \infty$ since $\sum |a_n|$ converges.

q.e.d.

The following lemmas and theorem are due to I. Kra and appear in [Kra84a].

Lemma 7.2.2 *Let x be a fixed point of a parabolic element of Γ. Assume $\infty \in \Omega(\Gamma)$. If x is a cusp, then $\sum_{\gamma \in \Gamma / P_x} |\gamma'(x)|^2$ converges.*

Proof: Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$. Then $c = 0$ for only finitely many elements of Γ, and

$$\sum_{\gamma \in \Gamma / P_x}{}' \frac{1}{|cx+d|^4} = \sum_{\gamma \in \Gamma / P_x}{}' |c|^{-4} \frac{1}{|x - \gamma^{-1}(\infty)|^4},$$

where $\sum'$ means we sum over the elements where $c \neq 0$. We can choose representatives γ such that $\gamma^{-1}(\infty)$ is in the closure of a fundamental domain for P_x. Then, since x is a cusp, it is easy to see that $|x - \gamma^{-1}(\infty)|$ is bounded below by a positive number. Since $\infty \in \Omega(\Gamma)$, $\sum'_{\gamma \in \Gamma / P_x} |c|^{-4}$ converges. (See, for example, [For51], page 104.)

q.e.d.

Lemma 7.2.3 *Let Γ be a non-elementary Kleinian group with $\infty \in \Omega(\Gamma)$. Let x be a q-admissible fixed point of a parabolic element of Γ. Assume x is a cusp and $q \geq 2$. Let*

$$\psi(x, \zeta) = \sum_{\gamma \in \Gamma / P_x} \frac{\gamma'(x)^q}{\gamma(x) - \zeta}.$$

Then $\psi(x, \cdot)$ is a holomorphic Eichler integral for Γ which is not identically zero.

Proof: Theorem 7.2.1 and Lemma 7.2.2 imply that the series converges in L^1 on every compact subset of $\mathbb{C}$. Since each of the terms $\frac{\gamma'(x)^q}{\gamma(x)-\zeta}$ is holomorphic on $\Omega(\Gamma)$ and L^1 convergence of holomorphic functions on any subset B of $\mathbb{C}$ implies uniform convergence on compact subsets of B, $\psi(x, \cdot)$ must be holomorphic on every compact subset of $\Omega(\Gamma)$.

Recall that x is q-admissible if, for $n = [\Gamma_x : P_x]$, $q \equiv 0$ modulo n. Now for $g, h \in \Gamma/P_x$, $g(x) = h(x)$ if and only if $(h^{-1} \circ g) \in \Gamma_x/P_x$. Any generator γ of Γ_x/P_x satisfies $\gamma'(x) = e^{2\pi i/n}$, so if x is q-admissible, then $\gamma'(x)^q = 1$. Thus, if x is q-admissible, then

$$\psi(x,\zeta) = \sum_{\gamma\in\Gamma/\Gamma_x} \sum_{\gamma\in\Gamma_x/P_x} \frac{\gamma'(x)^q}{\gamma x - \zeta} = n \sum_{\gamma\in\Gamma/\Gamma_x} \frac{\gamma'(x)^q}{\gamma x - \zeta},$$

and all γx are distinct in Γ_x/P_x. Since $\gamma'(x)^q$ is not zero for any $\gamma \in \Gamma/P_x$, Theorem 7.2.1 implies that $\psi(x,\zeta)$ is not identically zero.

To show that $\psi(x,\cdot)$ is an Eichler integral for Γ on $\Omega(\Gamma)$, fix $g \in \Gamma$. Then

$$\begin{aligned}
\psi(x,g(\zeta))g'(\zeta)^{1-q} - \psi(x,\zeta) &= \sum_{\gamma\in\Gamma/P_x} \frac{\gamma'(x)^q}{\gamma x - g\zeta} g'(\zeta)^{1-q} - \sum_{\gamma\in\Gamma/P_x} \frac{\gamma'(x)^q}{\gamma x - \zeta} \\
&= \sum_{\gamma\in\Gamma/P_x} [\frac{(g\circ\gamma)'(x)^q}{g\gamma x - g\zeta} g'(\zeta)^{1-q} - \frac{\gamma'(x)^q}{\gamma x - \zeta}] \\
&= \sum_{\gamma\in\Gamma/P_x} [\frac{g'(\gamma x)^q \gamma'(x)^q g'(\zeta)^{1-q}}{(\gamma x - \zeta) g'(\gamma x)^{\frac{1}{2}} g'(\zeta)^{\frac{1}{2}}} - \frac{\gamma'(x)^q}{\gamma x - \zeta}] \\
&= \sum_{\gamma\in\Gamma/P_x} \frac{\gamma'(x)^q [g'(\gamma x)^{q-\frac{1}{2}} g'(\zeta)^{\frac{1}{2}-q} - 1]}{\gamma x - \zeta}.
\end{aligned}$$

Now

$$g'(\gamma x)^{q-\frac{1}{2}} g'(\zeta)^{\frac{1}{2}-q} - 1$$

is a polynomial in ζ of degree $2q-1$ that vanishes at $\zeta = \gamma x$. So for each $\gamma \in \Gamma$,

$$\frac{\gamma'(x)^q [g'(\gamma x)^{q-\frac{1}{2}} g'(\zeta)^{\frac{1}{2}-q} - 1]}{\gamma x - \zeta}$$

is an element of Π_{2q-2}. Since the series converges in L^1 on every compact subset of $\mathbb{C}$, it converges uniformly on compact subsets on $\mathbb{C}$. Thus, $\psi(x,g(\zeta))g'(\zeta)^{1-q} - \psi(x,\zeta)$ is in Π_{2q-2}.

q.e.d.

Let $\chi = pd(\psi(x,\cdot))$. Then the above proof shows that for any $g \in \Gamma$, $\chi(g)(\zeta)$ can be extended as a polynomial from $\Omega(\Gamma)$ to the whole complex plane. In particular, for $z \in \Lambda(\Gamma)$,

$$\chi(g)(z) = \lim_{\substack{\zeta \to z \\ \zeta \in \Omega(\Gamma)}} \chi(g)(\zeta).$$

Lemma 7.2.4 *Let $P \in \Gamma$ be parabolic with the q-admissible fixed point x. Let $\chi = pd(\psi(x,\cdot))$. Then $\chi(P)(x) \neq 0$.*

Proof: Since $\infty \in \Omega(\Gamma)$, there is some circle C whose interior contains $\Lambda(\Gamma)$. Since $\chi(P)$ is a polynomial on $\mathbb{C}$, and x is q-admissible,

$$\begin{aligned} \chi(P)(x) &= \frac{1}{2\pi i}\int_C \frac{\chi(P)(z)}{z-x}dz \\ &= \frac{1}{2\pi i}\int_C [n \sum_{\gamma \in \Gamma/\Gamma_x} (\frac{\gamma'(x)^q P'(x)^{1-q}}{(z-x)(\gamma x - Pz)} - \frac{\gamma'(x)^q}{(z-x)(\gamma x - z)})]dz, \end{aligned}$$

where $n = [\Gamma_x : P_x]$. Since the sum converges uniformly on C,

$$\chi(P)(x) = \frac{n}{2\pi i} \sum_{\gamma \in \Gamma/\Gamma_x} \int_C [\frac{\gamma'(x)^q P'(x)^{1-q}}{(z-x)(\gamma x - Pz)} - \frac{\gamma'(x)^q}{(z-x)(\gamma x - z)}]dz.$$

Each of the integrals in the sum above is the sum of the residues of

$$f(z) = \frac{\gamma'(x)^q P'(x)^{1-q}}{(z-x)(\gamma x - Pz)} - \frac{\gamma'(x)^q}{(z-x)(\gamma x - z)}$$

at the poles of $f(z)$. If γ is the identity, then it is easy to compute that if $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL_2(\mathbb{C})$ then

$$\begin{aligned} \mathrm{Res}(f;x) &= \frac{2c(q-\frac{1}{2})(cx+d)^{2q-2}}{(cx-a)} \\ &= (q-\frac{1}{2})P''(x), \end{aligned}$$

since P is parabolic and fixes x. If γ is not the identity, then straightforward computations yield the equalities

$$\mathrm{Res}(f;\gamma x) = \frac{\gamma'(x)^q}{\gamma x - x},$$

$$\mathrm{Res}(f;(P^{-1}\gamma)(x)) = -\frac{(P^{-1}\gamma)'(x)^q}{(P^{-1}\gamma)(x)-x},$$

and $\mathrm{Res}(f;x)=0$. Kra proves that $\sum[\frac{\gamma'(x)^q}{\gamma x - x} - \frac{(P^{-1}\gamma)'(x)^q}{(P^{-1}\gamma)(x)-x}]$ converges to 0, where the sum is over all representatives of Γ/Γ_x except the identity, in his paper [Kra84a]. Thus, $\chi(P)(x) = n(q-\frac{1}{2})P''(x) \neq 0$.

q.e.d.

Theorem 7.2.5 *Let x be a q-admissible parabolic fixed point of the non-elementary Kleinian group Γ. Assume $\infty \in \Omega(\Gamma)$, x is a cusp, and $q \geq 2$. Let*

$$\varphi(x,\zeta) = \varphi_\Gamma(x,\zeta) = (2q-1)! \sum_{\gamma\in\Gamma/P_x} \frac{\gamma'(x)^q}{(\gamma x - \zeta)^{2q}}.$$

Then $\varphi(x,\cdot)$ is a holomorphic automorphic form of weight $(-2q)$. Furthermore, $\varphi(x,\cdot)$ is not identically zero on any component Δ of Γ which is invariant under P_x.

Proof: First note that

$$\varphi(x,\zeta) = \frac{d^{2q-1}}{d\zeta^{2q-1}}\psi(x,\zeta).$$

So $\varphi(x,\cdot)$ is holomorphic by Lemma 7.2.3. Now $\varphi(x,\zeta)$ can be rewritten in the form

$$\varphi(x,\zeta) = (2q-1)! \sum_{\gamma\in\Gamma/P_x} \frac{(\gamma^{-1})'(\zeta)^q}{(x-\gamma^{-1}\zeta)^{2q}} = (2q-1)! \sum_{\gamma\in P_x\backslash\Gamma} \frac{\gamma'(\zeta)^q}{(x-\gamma\zeta)^{2q}}.$$

Written in this new way, it is easy to see that $\varphi(x,\zeta)$ is an automorphic form of weight $(-2q)$.

Next, suppose Δ is a component of Γ which is invariant under the parabolic element P, where P fixes x. Let $\chi = pd(\psi(x,\cdot))$. By Lemma 7.2.4, $\chi(P)(x) \neq 0$. Now

$$\chi(P)(x) = \lim_{\substack{\zeta\to x\\ \zeta\in\Delta}} \chi(P)(\zeta),$$

and for $\zeta \in \Delta$,

$$\chi(P)(\zeta) = \psi(x,P\zeta)P'(\zeta)^{1-q} - \psi(x,\zeta).$$

If $\psi(x,\zeta)$ is a polynomial for $\zeta \in \Delta$, then

$$\chi(P)(x) = \lim_{\substack{\zeta \to x \\ \zeta \in \Delta}} [\psi(x, P\zeta)P'(\zeta)^{1-q} - \psi(x,\zeta)] = 0.$$

This contradiction shows that $\psi(x,\cdot)$ is not a polynomial in Δ, and so $\frac{d^{2q-1}}{d\zeta^{2q-1}}\psi(x,\zeta)$ cannot be identically zero in Δ.

q.e.d.

Corollary 7.2.6 *For $q \geq 2$, the series*

$$\sum_{\gamma \in \langle S_1 \rangle \backslash G_\mu} \gamma'(\zeta)^q$$

is not identically zero on the invariant component of G_μ.

Proof: Choose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$ such that $A(\infty) \in \Omega(G_\mu)$. Then $\infty \in A^{-1}(\Omega(G_\mu)) = \Omega(A^{-1}G_\mu A)$, and $A^{-1}(\infty)$ is a q-admissible parabolic fixed point of $A^{-1}G_\mu A$. Also, $A^{-1}(\infty)$ is a cusp for $A^{-1}G_\mu A$. To simplify notation, let $\Gamma = A^{-1}G_\mu A$ and let $x = A^{-1}(\infty)$. Then $P_x = \langle A^{-1}S_1 A \rangle$, so

$$\begin{aligned}
\sum_{\gamma \in \langle S_1 \rangle \backslash G_\mu} \gamma'(\zeta)^q &= \sum_{\gamma \in P_x \backslash \Gamma} (A\gamma A^{-1})'(\zeta)^q \\
&= \sum_{\gamma \in P_x \backslash \Gamma} A'(\gamma A^{-1}\zeta)^q \gamma'(A^{-1}\zeta)^q (A^{-1})'(\zeta)^q \\
&= \sum_{\gamma \in P_x \backslash \Gamma} \frac{\gamma'(A^{-1}\zeta)^q (A^{-1})'(\zeta)^q}{(c\gamma A^{-1}\zeta + d)^{2q}} \\
&= \sum_{\gamma \in P_x \backslash \Gamma} \frac{\gamma'(A^{-1}\zeta)^q (A^{-1})'(\zeta)^q}{c^{2q}(-\gamma A^{-1}\zeta - \frac{d}{c})^{2q}} \\
&= \sum_{\gamma \in P_x \backslash \Gamma} \frac{\gamma'(A^{-1}\zeta)^q (A^{-1})'(\zeta)^q}{c^{2q}(x - \gamma A^{-1}\zeta)^{2q}} \\
&= \frac{(A^{-1})'(\zeta)^q}{c^{2q}(2q-1)!}\varphi_\Gamma(x, A^{-1}(\zeta)).
\end{aligned}$$

Since $A^{-1}(\Delta(G_\mu))$ is a component of Γ invariant under P_x, $\varphi_\Gamma(x, A^{-1}(\zeta))$ is not identically zero in $\Delta(G_\mu)$.

q.e.d.

7.3 The error in approximating the series

Proposition 7.3.1 *Let $g = g_1 g_2 \cdots g_n \in \langle S_1 \rangle \setminus G_\mu$, $g_1 = T_\mu^{\pm 1}$, $g_i g_{i+1} \neq 1$ for $1 \leq i \leq n-1$, and $g_i \in \{S_1, S_1^{-1}, T_\mu, T_\mu^{-1}\}$ for $1 \leq i \leq n$. Suppose $0 \leq Re(\mu) \leq 2$ and $Im(\mu) > 2$. Then*

(i) if $g_n = T_\mu$, then $g^{-1}(\infty)$ is inside $I(T_\mu)$;

(ii) if $g_n = T_\mu^{-1}$, then $g^{-1}(\infty)$ is inside $I(T_\mu^{-1})$;

(iii) if $g_n = S_1$, then $g^{-1}(\infty)$ is to the left of $Re(z) = 1$;

(iv) if $g_n = S_1^{-1}$, then $g^{-1}(\infty)$ is to the right of $Re(z) = 1$.

Proof: We use induction on the length of the word n. If $n = 1$, then $g = T_\mu^{\pm 1}$ and the proof is clear. Assume the proposition is true for all words of length $\leq n$; we want to show its truth for words g of length $n+1$. Write $g = g_1 g_2 \cdots g_{n+1}$. We divide the rest of the proof into cases.

First suppose $g_{n+1} = S_1$. Then if $g_n = S_1$, $g^{-1}(\infty)$ is to the left of $\mathrm{Re}(z) = -1$ by the induction hypothesis. If $g_n = T_\mu$, then $g_n^{-1} \cdots g_1^{-1}(\infty)$ is in $I(T_\mu)$, and so $g^{-1}(\infty)$ is to the left of $\mathrm{Re}(z) = -1$. If $g_n = T_\mu^{-1}$, then $g_n^{-1} \cdots g_1^{-1}(\infty)$ is in $I(T_\mu^{-1})$, and since $\mathrm{Re}(\mu) \leq 2$, $g^{-1}(\infty)$ is to the left of $\mathrm{Re}(z) = 1$.

The cases work similarly if $g_{n+1} = S_1^{-1}$. In this case, if $g_n = S_1^{-1}$ then $g^{-1}(\infty)$ is to the right of $\mathrm{Re}(z) = 3$. If $g_n = T_\mu$ then $g^{-1}(\infty)$ is to the right of $\mathrm{Re}(z) = 1$. Also, since $\mathrm{Re}(\mu) \geq 0$, if $g_n = T_\mu^{-1}$ then $g^{-1}(\infty)$ is to the right of $\mathrm{Re}(z) = 1$.

Next suppose $g_{n+1} = T_\mu$. Then if $g_n = T_\mu$ also, then $g_n^{-1} \cdots g_1^{-1}(\infty)$ is inside $I(T_\mu)$ by hypothesis. Since the isometric circles of T_μ and T_μ^{-1} do not intersect, $g^{-1}(\infty)$ is inside $I(T_\mu)$. If $g_n = S_1$, we must consider the previous letter also. If $g_{n-1} g_n = S_1 S_1$, then $g_n^{-1} \cdots g_1^{-1}(\infty)$ is to the left of $\mathrm{Re}(z) = -1$, which is outside $I(T_\mu^{-1})$, and so $g^{-1}(\infty)$ is inside $I(T_\mu)$. If $g_{n-1} g_n = T_\mu S_1$, then $g_{n-1}^{-1} \cdots g_1^{-1}(\infty)$ is inside $I(T_\mu)$, and

$g_n^{-1} \cdots g_1^{-1}(\infty)$ is to the left of $\mathrm{Re}(z) = -1$, and so $g^{-1}(\infty)$ is inside $I(T_\mu)$. If $g_{n-1}g_n = T_\mu^{-1}S_1$, then $g_{n-1}^{-1} \cdots g_1^{-1}(\infty)$ is inside $I(T_\mu^{-1})$, and $g_n^{-1} \cdots g_1^{-1}(\infty)$ is outside $I(T_\mu^{-1})$, so $g^{-1}(\infty)$ is inside $I(T_\mu)$. If $g_n = S_1^{-1}$ we must likewise consider the previous letter. If $g_{n-1}g_n = S_1^{-1}S_1^{-1}$, then $g_n^{-1} \cdots g_1^{-1}(\infty)$ is to the right of $\mathrm{Re}(z) = 3$, which is outside $I(T_\mu^{-1})$, so $g^{-1}(\infty)$ is inside $I(T_\mu)$. If $g_{n-1}g_n = T_\mu S_1^{-1}$, then $g_{n-1}^{-1} \cdots g_1^{-1}(\infty)$ is inside $I(T_\mu)$, and $g_n^{-1} \cdots g_1^{-1}(\infty)$ is below the horizontal line $\mathrm{Im}(z) = 1$ and since $\mathrm{Im}(\mu) > 2$, $g_n^{-1} \cdots g_1^{-1}(\infty)$ is outside $I(T_\mu^{-1})$; thus $g^{-1}(\infty)$ is inside $I(T_\mu)$. If $g_{n-1}g_n = T_\mu^{-1}S_1^{-1}$, then $g_{n-1}^{-1} \cdots g_1^{-1}(\infty)$ is inside $I(T_\mu^{-1})$, $g_n^{-1} \cdots g_1^{-1}(\infty)$ is outside $I(T_\mu^{-1})$, and $g^{-1}(\infty)$ is inside $I(T_\mu)$.

Finally suppose $g_{n+1} = T_\mu^{-1}$. Then if $g_n = T_\mu^{-1}$, then $g_n^{-1} \cdots g_1^{-1}(\infty)$ is inside $I(T_\mu^{-1})$, which is outside $I(T_\mu)$; thus $g^{-1}(\infty)$ is inside $I(T_\mu^{-1})$. If $g_{n-1}g_n = S_1S_1$, then $g_n^{-1} \cdots g_1^{-1}(\infty)$ is to the left of $\mathrm{Re}(z) = -1$, so $g^{-1}(\infty)$ is inside $I(T_\mu^{-1})$. If $g_{n-1}g_n = T_\mu S_1$, then again $g_n^{-1} \cdots g_1^{-1}(\infty)$ is to the left of $\mathrm{Re}(z) = -1$, and $g^{-1}(\infty)$ is inside $I(T_\mu^{-1})$. If $g_{n-1}g_n = T_\mu^{-1}S_1$, then $g_{n-1}^{-1} \cdots g_1^{-1}(\infty)$ is inside $I(T_\mu^{-1})$, and $g_n^{-1} \cdots g_1^{-1}(\infty)$ is above the line $\mathrm{Im}(z) = 1$ (because $\mathrm{Im}(\mu) > 2$), so $g_n^{-1} \cdots g_1^{-1}(\infty)$ is outside $I(T_\mu)$ and $g^{-1}(\infty)$ is inside $I(T_\mu^{-1})$. If $g_{n-1}g_n = S_1^{-1}S_1^{-1}$ then $g_n^{-1} \cdots g_1^{-1}(\infty)$ is to the right of $\mathrm{Re}(z) = 3$, which is outside $I(T_\mu)$, so $g^{-1}(\infty)$ is inside $I(T_\mu^{-1})$. If $g_{n-1}g_n = T_\mu S_1^{-1}$, then $g_{n-1}^{-1} \cdots g_1^{-1}(\infty)$ is inside $I(T_\mu)$, and $g_n^{-1} \cdots g_1^{-1}(\infty)$ is outside $I(T_\mu)$, so $g^{-1}(\infty)$ is inside $I(T_\mu^{-1})$. Finally, if $g_{n-1}g_n = T_\mu^{-1}S_1^{-1}$, then $g_{n-1}^{-1} \cdots g_1^{-1}(\infty)$ is inside $I(T_\mu^{-1})$ and $g_n^{-1} \cdots g_1^{-1}(\infty)$ is above the line $\mathrm{Im}(z) = 1$, and outside $I(T_\mu)$, so $g^{-1}(\infty)$ is inside $I(T_\mu^{-1})$.

q.e.d.

Corollary 7.3.2 *Under the hypotheses of Proposition 7.3.1, if $g_n = T_\mu$, S_1, or S_1^{-1}, then $g^{-1}(\infty)$ is outside $I(T_\mu^{-1})$; if $g_n = T_\mu^{-1}$, S_1, or S_1^{-1}, then $g^{-1}(\infty)$ is outside $I(T_\mu)$.*

Proof: Suppose $g_n = T_\mu$. Then by Proposition 7.3.1, $g^{-1}(\infty)$ is inside $I(T_\mu)$, which is disjoint from $I(T_\mu^{-1})$. By similar reasoning, if $g_n = T_\mu^{-1}$, then $g^{-1}(\infty)$ is outside $I(T_\mu)$.

Now if $g_n = S_1$ or S_1^{-1}, we consider the previous letter. If $g_{n-1}g_n = S_1S_1$ or $T_\mu S_1$, then by the proposition, $g^{-1}(\infty)$ is to the left of $\text{Re}(z) = -1$; so $g^{-1}(\infty)$ is outside $I(T_\mu)$ and $I(T_\mu^{-1})$. If $g_{n-1}g_n = T_\mu^{-1}S_1$, then $g^{-1}(\infty)$ is above the horizontal line $\text{Im}(z) = 1$ and to the left of the line $\text{Re}(z) = \text{Re}(\mu) - 1$; so $g^{-1}(\infty)$ is outside both isometric circles. If $g_{n-1}g_n = S_1^{-1}S_1^{-1}$ then $g^{-1}(\infty)$ is to the right of $\text{Re}(z) = 3$ and outside $I(T_\mu)$ and $I(T_\mu^{-1})$. If $g_{n-1}g_n = T_\mu^{-1}S_1^{-1}$, then $g^{-1}(\infty)$ is to the right of $\text{Re}(z) = \text{Re}(\mu) + 1$, which is outside $I(T_\mu)$ and $I(T_\mu^{-1})$. Finally, if $g_{n-1}g_n = T_\mu S_1^{-1}$, then $g^{-1}(\infty)$ is to the right of $\text{Re}(z) = 1$ and below $\text{Im}(z) = 1$; so $g^{-1}(\infty)$ is outside the isometric circles.

q.e.d.

Suppose we want to find a bound on $|\sum_H g'(z_0)^2|$, where H is some subset of $\langle S_1 \rangle \setminus G_\mu$ and $z_0 \in \Delta(G_\mu)$. By the mean value property for holomorphic functions (see [Con78], for example),

$$g'(z_0)^2 = \frac{1}{\pi r^2} \iint_{D(z_0,r)} g'(z)^2 \, dx \, dy,$$

where $D(z_0, r)$ is a disk around z_0 with radius r in which $g'(z)^2$ is holomorphic. Thus,

$$|g'(z_0)^2| \leq \frac{1}{\pi r^2} \iint_{D(z_0,r)} |g'(z)^2| \, dx \, dy.$$

The last integral equals $\frac{1}{\pi r^2}$ times the area of $g(D(z_0, r))$. The following proposition is now clear.

Proposition 7.3.3 *Suppose $D(z_0, r)$ is contained in some fundamental domain for G_μ. Then*

$$\sum_{g \in \langle S_1 \rangle \setminus G_\mu} |g'(z_0)^2| \leq \frac{2\, Im(\mu)}{\pi r^2}.$$

We need only estimate the series at points z_0 for which $0 \leq \mathrm{Re}(z_0) \leq 1$. (In order to integrate the square root of the series over the curves a and b, we can use the identity $P_2(\mu; z) = P_2(\mu; \mu - z)$ and integrate from $\frac{\mu}{2} - 1$ to $\frac{\mu}{2}$ for the curve a and from i to $\frac{\mu}{2}$ for the curve b. Since $\psi(\mu + 2) = \psi(\mu) + 1$, we can restrict our attention to the cases $0 \leq \mathrm{Re}(\mu) \leq 2$.) Suppose we choose r so that $D(z_0, r)$ lies inside the vertical strip $\{z : -3/2 + \mathrm{Re}(\mu)/2 \leq \mathrm{Re}(z) \leq \mathrm{Re}(\mu)/2 + 1/2\}$, inside $\Delta(G_\mu)$, and outside the isometric circle of T_μ^{-1}, and so that $T_\mu(D(z_0, r))$ lies outside the isometric circle of T_μ. Then we can bound $\sum_H |g'(z_0)^2|$, depending on H, as follows.

Suppose H consists of those words of the form $g_1 T_\mu^{-1} g$, where g_1 is fixed. Since T_μ^{-1} takes the outside of $I(T_\mu^{-1})$ to the inside of $I(T_\mu)$, $T_\mu^{-1} g(D(z_0, r))$ lies inside $I(T_\mu)$. By Corollary 7.3.2, $g_1^{-1}(\infty)$ lies outside $I(T_\mu)$; so g_1 takes the inside of $I(T_\mu)$ to the inside of $g_1(I(T_\mu))$. Thus, $\sum_H |g'(z_0)^2|$ is less then $\frac{1}{\pi r^2}$ times the area of the disk $g_1(I(T_\mu))$.

Next suppose H consists of words of the form $g_1 T_\mu S_1^{-1} g$ or $g_1 T_\mu T_\mu g$. Then $T_\mu S_1^{-1} g(D(z_0, r))$ and $T_\mu T_\mu g(D(z_0, r))$ are inside $I(T_\mu^{-1})$, and by Corollary 7.3.2, $g_1^{-1}(\infty)$ is outside $I(T_\mu^{-1})$. Thus, $\sum_H |g'(z_0)^2|$ is less than $\frac{1}{\pi r^2}$ times the area of the disk $g_1(I(T_\mu^{-1}))$.

Now suppose H consists of words of the form $g_1 S_1^{-1} S_1^{-1} g$. Then $S_1^{-1} S_1^{-1} g(D(z_0, r))$ is to the left of the line $\mathrm{Re}(z) = \mathrm{Re}(\mu) - 3$, and by Proposition 7.3.1, $g_1^{-1}(\infty)$ is inside $I(T_\mu)$, inside $I(T_\mu^{-1})$, or to the right of $\mathrm{Re}(z) = 1$. Thus $\sum_H |g'(z_0)^2|$ is less than $\frac{1}{\pi r^2}$ times the area of the disk $g_1(D_1)$, where $D_1 = \{z : \mathrm{Re}(z) < \mathrm{Re}(\mu) - 3\}$.

Suppose H consists of words of the form $g_1 S_1 S_1 g$. Since $S_1 S_1 g(D(z_0, r))$ is inside the half space $D_2 = \{z : \mathrm{Re}(z) > 5/2\}$, if g_1 ends in T_μ or S_1 then $\sum_H |g'(z_0)^2|$ is less than $\frac{1}{\pi r^2}$ times the area of the disk $g_1(D_2)$.

Finally suppose H consists of words of the form $g_1 S_1 T_\mu g$, where g_1 ends in S_1 or T_μ. Since $g_1^{-1}(\infty)$ is to the left of $\mathrm{Re}(z) = 1$ and $S_1 T_\mu g(D(z_0, r))$ is to the right of $\mathrm{Re}(z) = 1$, $\sum_H |g'(z_0)^2|$ is less than $\frac{1}{\pi r^2}$ times the area of the disk $g_1(D_3)$, where

$D_3 = \{z : \mathrm{Re}(z) > 1\}$.

We have considered sufficiently many cases for H; we have found error bounds for $\sum_H |g'(z_0)^2|$ in the cases where H contains words of the forms $g_1 T_\mu^{-1} g$, $g_1 T_\mu S_1^{-1} g$, $g_1 T_\mu T_\mu g$, $g_1 S_1^{-1} S_1^{-1} g$, $g_1 T_\mu S_1 S_1 g$, $g_1 S_1 S_1 S_1 g$, $g_1 S_1 S_1 T_\mu g$, and $g_1 T_\mu S_1 T_\mu g$.

7.4 The error in approximating the integral of the square root of the series

Lemma 7.4.1 *Let $z \in \Delta(G_\mu)$ and let $k \geq d(g^{-1}(\infty), z)$ for all $g \in \langle S_1 \rangle \setminus G_\mu$. Let r be any positive number such that there is a fundamental domain for G_μ containing the disk of radius r about z. Then*

$$\left| \left(\sqrt{\sum_{g \in \langle S_1 \rangle \setminus G_\mu} g'(z)^2} \right)'' \right| \leq \frac{16\ [Im(\mu)]^2}{\pi^2 k^2 r^4 |\sum g'(z)^2|^{3/2}} + \frac{20\ Im(\mu)}{\pi k^2 r^2 |\sum g'(z)^2|^{1/2}}.$$

Proof: To compute the second derivative, we compute

$$\left(\sqrt{\sum g'(z)^2} \right)' = \frac{1}{2} \left(\sum g'(z)^2 \right)^{-1/2} \cdot \sum \frac{-4c}{(cz+d)^5}$$

and

$$\begin{aligned} \left(\sqrt{\sum g'(z)^2} \right)'' &= -\ \frac{1}{4} \left(\sum g'(z)^2 \right)^{-3/2} \cdot \left(\sum \frac{-4c}{(cz+d)^5} \right)^2 \\ &\quad +\ \frac{1}{2} \left(\sum g'(z)^2 \right)^{-1/2} \cdot \left(\sum \frac{20c^2}{(cz+d)^6} \right). \end{aligned}$$

Thus,

$$\begin{aligned} \left| \left(\sqrt{\sum g'(z)^2} \right)'' \right| &\leq \frac{\left(\sum \left| \frac{4c}{(cz+d)^5} \right| \right)^2}{4 |\sum g'(z)^2|^{3/2}} + \frac{\sum \left| \frac{20c^2}{(cz+d)^6} \right|}{2 |\sum g'(z)^2|^{1/2}} \\ &= \frac{\left(\sum' \left| \frac{4}{(z+\frac{d}{c})} \cdot \frac{1}{(cz+d)^4} \right| \right)^2}{4 |\sum g'(z)^2|^{3/2}} + \frac{\sum' \left| \frac{20}{(z+\frac{d}{c})^2} \cdot \frac{1}{(cz+d)^4} \right|}{2 |\sum g'(z)^2|^{1/2}} \\ &\leq \frac{4}{k^2} \frac{\left(\sum' \frac{1}{|cz+d|^4} \right)^2}{|\sum g'(z)^2|^{3/2}} + \frac{10}{k^2} \frac{\sum' \frac{1}{|cz+d|^4}}{|\sum g'(z)^2|^{1/2}} \end{aligned}$$

$$\begin{aligned}
&\le \frac{4}{\pi^2 k^2 r^4}\frac{(2\cdot \mathrm{Im}(\mu))^2}{|\sum g'(z)^2|^{3/2}} + \frac{10\cdot 2\cdot \mathrm{Im}(\mu)}{\pi k^2 r^2\,|\sum g'(z)^2|^{1/2}} \\
&= \frac{16}{\pi^2 k^2 r^4}\frac{[\mathrm{Im}(\mu)]^2}{|\sum g'(z)^2|^{3/2}} + \frac{20\ \mathrm{Im}(\mu)}{\pi k^2 r^2\,|\sum g'(z)^2|^{1/2}}.
\end{aligned}$$

The symbol $\sum'$ denotes the sum over all $g(z) = \frac{az+b}{cz+d}$ for which $c \neq 0$.

q.e.d.

Lemma 7.4.2 *Let $z \in \Delta(G_\mu)$ and let $k \geq d(g^{-1}(\infty), z)$ for all $g \in \langle S_1\rangle \setminus G_\mu$. Let r be any positive number such that there is a fundamental domain for G_μ containing the disk of radius r about z. Then*

$$\begin{aligned}
\left|\left(\sqrt{\sum_{g\in\langle S_1\rangle\setminus G_\mu} g'(z)^2}\right)''''\right| &\le \frac{3840[Im(\mu)]^4}{\pi^4k^4r^8|\sum g'(z)^2|^{7/2}} + \frac{5760[Im(\mu)]^3}{\pi^3k^4r^6|\sum g'(z)^2|^{5/2}} \\
&+ \frac{3120[Im(\mu)]^2}{\pi^2k^4r^4|\sum g'(z)^2|^{3/2}} + \frac{840\ Im(\mu)}{\pi k^4r^2|\sum g'(z)^2|^{1/2}}.
\end{aligned}$$

Proof: Compute the derivatives

$$\begin{aligned}
\left(\sqrt{\sum g'(z)^2}\right)''' &= \frac{3}{8}\left(\sum g'(z)^2\right)^{-5/2}\left(\sum\frac{-4c}{(cz+d)^5}\right)^3 \\
&- \frac{3}{4}\left(\sum g'(z)^2\right)^{-3/2}\left(\sum\frac{-4c}{(cz+d)^5}\right)\left(\sum\frac{20c^2}{(cz+d)^6}\right) \\
&+ \frac{1}{2}\left(\sum g'(z)^2\right)^{-1/2}\left(\sum\frac{-120c^3}{(cz+d)^7}\right)
\end{aligned}$$

and

$$\begin{aligned}
\left(\sqrt{\sum g'(z)^2}\right)'''' &= -\frac{15}{16}\left(\sum g'(z)^2\right)^{-7/2}\left(\sum\frac{-4c}{(cz+d)^5}\right)^4 \\
&+ \frac{18}{8}\left(\sum g'(z)^2\right)^{-5/2}\left(\sum\frac{-4c}{(cz+d)^5}\right)^2\left(\sum\frac{20c^2}{(cz+d)^6}\right) \\
&- \frac{3}{4}\left(\sum g'(z)^2\right)^{-3/2}\left(\sum\frac{20c^2}{(cz+d)^6}\right)^2 \\
&- 1\left(\sum g'(z)^2\right)^{-3/2}\left(\sum\frac{-4c}{(cz+d)^5}\right)\left(\sum\frac{-120c^3}{(cz+d)^7}\right) \\
&+ \frac{1}{2}\left(\sum g'(z)^2\right)^{-1/2}\left(\sum\frac{840c^4}{(cz+d)^8}\right).
\end{aligned}$$

Then the proof is just like that of Lemma 7.4.1.

q.e.d.

Lemma 7.4.3 *Suppose $k \geq d(g^{-1}(\infty), z)$ for all $g \in \langle S_1 \rangle \setminus G_\mu$. Suppose $D(z,r)$ is contained in some fundamental domain for G_μ. Then*

$$\left| \left(\sum_{g \in \langle S_1 \rangle \setminus G_\mu} g'(z)^2 \right)' \right| \leq \frac{8}{\pi k r^2} \cdot Im(\mu).$$

Proof:

$$\begin{aligned} \left| \left(\sum_{g \in \langle S_1 \rangle \setminus G_\mu} g'(z)^2 \right)' \right| &= \left| \sum \left(\frac{1}{(cz+d)^4} \right)' \right| \\ &\leq \sum \left| \frac{-4c}{(cz+d)^5} \right| \\ &= 4 \sum{}' \frac{1}{|z + \frac{d}{c}|} \cdot \frac{1}{|cz+d|^4} \\ &\leq \frac{4}{k} \cdot \sum{}' \frac{1}{|cz+d|^4} \\ &< \frac{4}{k} \cdot \frac{2\,\mathrm{Im}(\mu)}{\pi r^2}. \end{aligned}$$

q.e.d.

Corollary 7.4.4 *Suppose $k \geq d(g^{-1}(\infty), z)$ for all $g \in \langle S_1 \rangle \setminus G_\mu$. Suppose the minimum possible value of $|\sum g'(z)^2|$ among $n \geq 2$ equidistant values along a line segment of length ℓ is η. Finally suppose that for any z on this line segment, $D(z,r)$ is contained in some fundamental domain of G_μ. Then, on this line segment,*

$$\left| \sum_{g \in \langle S_1 \rangle \setminus G_\mu} g'(z)^2 \right| \geq \eta - \frac{\ell}{2(n-1)} \cdot \frac{8}{\pi k r^2} \cdot Im(\mu).$$

Suppose we want to approximate $\int_a^b f(x)\,dx$. Take a partition of $[a,b]$ using points $a = x_1 < x_2 < \cdots < x_n = b$, where $x_i = a + (i-1)\frac{b-a}{n-1}$. Let $T_n(f)$ denote the approximation of $\int_a^b f(x)\,dx$ using the trapezoid rule with these points; that is,

$$T_n(f) = \frac{b-a}{2(n-1)} \left[f(a) + 2 \sum_{i=2}^{n-1} f(x_i) + f(b) \right].$$

Theorem 7.4.5 *Suppose $|f(x) - g(x)| \leq \epsilon$ for all $x \in [a, b]$. Then*

$$\left| \int_a^b f(x)\, dx - T_n(g) \right| \leq \max_{[a,b]} |f''(x)| \frac{(b-a)^3}{12(n-1)^2} + 2(b-a)\epsilon.$$

Proof:

$$\begin{aligned} \left| \int_a^b f(x)\, dx - T_n(g) \right| &\leq \left| \int_a^b f(x)\, dx - T_n(f) \right| + |T_n(f) - T_n(g)| \\ &\leq \max_{[a,b]} |f''(x)| \frac{(b-a)^3}{12(n-1)^2} + \frac{b-a}{(n-1)} \cdot 2(n-1)\epsilon. \end{aligned}$$

Here we use the standard error bound for the trapezoid rule (see, for example, [Kin84]).

q.e.d.

Since we are interested in integrating the square root of the series, we are interested in bounding $|\sqrt{z} - \sqrt{z + \epsilon e^{i\theta}}|$. It is clear that

$$\begin{aligned} \left| \sqrt{z} - \sqrt{z + \epsilon e^{i\theta}} \right| &\leq \epsilon \cdot \max \left| \left(\sqrt{z} \right)' \right| \\ &= \epsilon \cdot \max \left| \frac{1}{2\sqrt{z}} \right|, \end{aligned}$$

where the maximum is over the line segment from z to $z + \epsilon e^{i\theta}$. Thus,

$$\left| \sqrt{f(z)} - \sqrt{f(z) + \epsilon e^{i\theta}} \right| \leq \frac{\epsilon}{2 \min \sqrt{|f(z)|}}.$$

Corollary 7.4.6 *Suppose $|f(x) - g(x)| \leq \epsilon$ for all $x \in [a, b]$. Then*

$$\begin{aligned} \left| \int_a^b \sqrt{f(x)}\, dx - T_n(\sqrt{g}) \right| &\leq \max_{[a,b]} \left| \left(\sqrt{f(x)} \right)'' \right| \frac{(b-a)^3}{12(n-1)^2} \\ &+ \frac{\epsilon(b-a)}{\min_{[a,b]} \sqrt{|f(x)|}}. \end{aligned}$$

Proposition 7.4.7 *Suppose $K_a, K_b, M_a, M_b, \epsilon_a$, and ϵ_b are complex numbers such that $K_a = M_a + \epsilon_a$ and $K_b = M_b + \epsilon_b$ and $|\epsilon_a| \leq C_a$ and $|\epsilon_b| \leq C_b$. Suppose also that $C_a < |M_a|$. Then*

$$\left| \frac{K_b}{K_a} - \frac{M_b}{M_a} \right| \leq \frac{C_b}{|M_a| - C_a} + \frac{C_a |M_b|}{|M_a|^2 - C_a |M_a|}.$$

Proof:

$$
\begin{aligned}
\left|\frac{K_b}{K_a} - \frac{M_b}{M_a}\right| &= \left|\frac{K_b M_a - K_a M_b}{K_a M_a}\right| \\
&= \left|\frac{(M_b + \epsilon_b)M_a - (M_a + \epsilon_a)M_b}{(M_a + \epsilon_a)M_a}\right| \\
&\leq \left|\frac{\epsilon_b M_a}{M_a^2 + \epsilon_a M_a}\right| + \left|\frac{\epsilon_a M_b}{M_a^2 + \epsilon_a M_a}\right| \\
&\leq \frac{C_b}{|M_a| - C_a} + \frac{C_a|M_b|}{|M_a|^2 - C_a|M_a|}.
\end{aligned}
$$

q.e.d.

7.5 Computer considerations

When we integrate the square root of the Poincaré series, we must be careful to use a consistent branch of the square root. The FORTRAN package used for these computations uses the negative real axis for the branch cut for the square root function. Thus, if the Poincaré series stays away from the negative real axis on the curves a and b, the branch of square root is consistent. This will be guaranteed by the following corollary to Lemma 7.4.3.

Corollary 7.5.1 *Suppose $k \geq d(g^{-1}(\infty), z)$ for all $g \in \langle S_1 \rangle \setminus G_\mu$. Suppose the minimum possible value of $Re(\sum g'(z)^2)$ among $n \geq 2$ equidistant values along a line segment of length ℓ is η. Finally suppose that for any z on this line segment, $D(z, r)$ is contained in some fundamental domain of G_μ. Then, on this line segment,*

$$
Re\left(\sum_{g \in \langle S_1 \rangle \setminus G_\mu} g'(z)^2\right) \geq \eta - \frac{\ell}{2(n-1)} \cdot \frac{8}{\pi k r^2} \cdot Im(\mu).
$$

7.6 Examples

The FORTRAN program which implements the arguments used in this chapter to approximate the map $\psi : M_{1,1} \to \mathbb{H}$ is presented in the appendix. We used the program to compute the image of several points.

Our first goal was to test a conjecture concerning the image of real trace rays in $M_{1,1}$. L. Keen and C. Series ([KS93]) define the *real trace ray* of the word $W_{p/q}$ to be the unique branch of the set of x in $M_{1,1}$ for which the trace of $W_{p/q}$ is real and larger than 2, where the real part of this branch lies between $2p/q$ and $2p/q+2$ for $\mathrm{Im}(x)$ large enough. It was thought for some time that ψ took the real trace ray for $W_{p/q}$ to the vertical line $\mathrm{Re}(z)=p/q$ in $\mathbb{H}$. Our computer approximations show this to be false. For example, the point $.65388+4i$ is to the right of the real trace ray for $W_{1/3}$, but $\psi(.65388+4i)$ was computed to be $.32700+1.11749i$, with a maximum error in absolute value less than 0.00472. Hence, ψ must take this point to the left side of the ray $\mathrm{Re}(x)=1/3$ in $\mathbb{H}$.

It is also of some interest which point is mapped to i, which is the point in $\mathbb{H}$ representing the square torus. We know by Proposition 7.1.16 that $\psi^{-1}(i)=ti$ for some $t>2$. The computer program was used to prove that $3.75<t<3.78$. As the approximations in the program were probably better than the error bound suggested, we can guess that t is approximately 3.765.

7.7 Kra's biholomorphic map $M_{0,4}\to\mathbb{H}$

Recall that the map $M_{1,1}\to\mathbb{H}$ was given by

$$\mu\mapsto\frac{\int_{b(\mu)}\zeta_\mu}{\int_a\zeta_\mu},$$

where $\{\zeta_\mu\}$ is a basis for the space of holomorphic abelian differentials on $\Delta(G_\mu)/G_\mu$, and $\{a,b(\mu)\}$ is a canonical basis for the fundamental group of $\Delta(G_\mu)/G_\mu$. By the Riemann-Roch Theorem there are no holomorphic abelian differentials on a surface of type (0,4), so the map $M_{0,4}\to\mathbb{H}$ cannot be constructed in the same way.

Recall that, up to a constant multiple,

$$\zeta_\mu=\sqrt{\sum_{g\in\langle S_1\rangle\backslash G_\mu}g'(z)^2dz}.$$

When we try to do the same thing for the group $H_x = \langle A, B_1, B_{2,x}\rangle$, the error is that if

$$\theta(z) = \sqrt{\sum_{g\in\langle A\rangle\backslash H_x} g'(z)^2},$$

then $\theta(g(z))g'(z) = \theta(z)$ for most $g \in H_x$, but $\theta(g(z))g'(z) = -\theta(z)$ for all parabolic $g \in H_x$ which "correspond to punctures". Let Γ_0 denote the subgroup of H_x consisting of those $g \in H_x$ for which $\theta(g(z))g'(z) = \theta(z)$. Kra ([Kra88]) shows that

$$\Gamma_0 = \langle B_1^{-1}B_2, B_2^{-1}AB_2, B_2^{-2}, B_2AB_2A, A^{-1}B_1^{-1}A^{-1}B_1^{-1}, B_1^2\rangle.$$

Let $G_{01} = \langle A, B_1^2, B_1^{-1}AB_1\rangle$ and $G_{02} = \langle A, B_2^2, B_2^{-1}AB_2\rangle$. Then G_{01} and G_{02} represent surfaces of type (0,4). The amalgamated free product $G_{01}*_A G_{02}$ is a surface of type (0,6). The HNN extension of $G_{01}*_A G_{02}$ by $B_1^{-1}B_2$ is Γ_0; the element $B_1^{-1}B_2$ conjugates $B_2^{-1}AB_2$ to $B_1^{-1}AB_1$, and cuts off two punctures, gluing the boundary curves together to produce a surface of type (1,4). See Figure 7.2.

Since $[H_x : \Gamma_0] = 2$, $\Delta(\Gamma_0) = \Delta(H_x)$. Let $\{a, b(x)\}$ be a canonical basis for the fundamental group of $\overline{\Delta(\Gamma_0)/\Gamma_0}$, and let $\{\zeta_x\}$ be a basis for the space of holomorphic abelian differentials on $\Delta(\Gamma_0)/\Gamma_0$. Then Kra's map $M_{0,4} \to \mathbb{H}$ is given by

$$x \mapsto \frac{\int_{b(x)} \zeta_x}{\int_a \zeta_x}.$$

One such holomorphic abelian differential on $\Delta(\Gamma_0)/\Gamma_0$ is given by $\zeta_x = \theta(z)\,dz$, for this is invariant under change of local coordinates on $\Delta(\Gamma_0)/\Gamma_0$ (but not on $\Delta(H_x)/H_x$).

Define $\psi : M_{0,4} \to \mathbb{H}$ by

$$\psi(x) = \frac{\int_{b(x)} \zeta_x}{\int_a \zeta_x} = \tau.$$

Let G_τ denote the group generated by $z \mapsto z+1$ and $z \mapsto z+\tau$. Pick a base point $Q_0 \in \overline{\Delta(\Gamma_0)/\Gamma_0}$ and define $\overline{\alpha} : \overline{\Delta(\Gamma_0)/\Gamma_0} \to \mathbb{C}/G_\tau$ by

$$\overline{\alpha}(P) = \pi_\tau\left(\frac{\int_{Q_0}^P \zeta_x}{\int_a \zeta_x}\right),$$

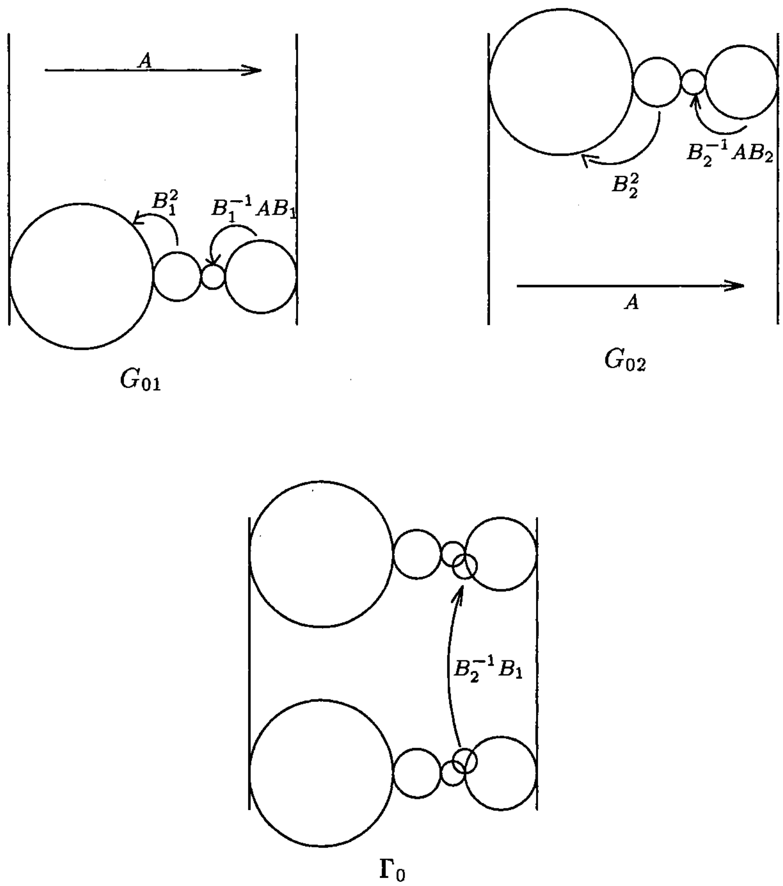


Figure 7.2: The geometry of Γ_0

where $\pi_\tau : \mathbb{C} \to \mathbb{C}/G_\tau$ is the natural projection.

For a proof of the following proposition, see the proof of Proposition 7.1.9.

Proposition 7.7.1 *$\overline{\alpha}$ is a conformal homeomorphism.*

Pick a base point $Q \in \Delta(\Gamma_0)$ and define $\varphi : \Delta(\Gamma_0) \to \mathbb{C}$ by

$$\varphi(z) = \frac{\int_Q^z \zeta_x}{\int_a \zeta_x} - \frac{\int_Q^{B_1 Q} \zeta_x}{\int_a \zeta_x}.$$

Proposition 7.7.2 *$\varphi(Az) = \varphi(z)+1$, $\varphi(B_2^{-1}B_1 z) = \varphi(z)+\tau$, and $\varphi(B_1 z) = -\varphi(z)$.*

Proof: Let

$$C = -\frac{\int_Q^{B_1 Q} \zeta_x}{\int_a \zeta_x}.$$

Then

$$\begin{aligned}
\varphi(Az) &= \frac{\int_Q^{Az} \zeta_x}{\int_a \zeta_x} + C \\
&= \frac{\int_Q^{AQ} \zeta_x}{\int_a \zeta_x} + \frac{\int_{AQ}^{Az} \zeta_x}{\int_a \zeta_x} + C \\
&= 1 + \varphi(z).
\end{aligned}$$

Next, we have

$$\begin{aligned}
\varphi(B_2^{-1}B_1 z) &= \frac{\int_Q^{B_2^{-1}B_1 z} \zeta_x}{\int_a \zeta_x} + C \\
&= \frac{\int_Q^{B_2^{-1}B_1 Q} \zeta_x}{\int_a \zeta_x} + \frac{\int_{B_2^{-1}B_1 Q}^{B_2^{-1}B_1 z} \zeta_x}{\int_a \zeta_x} + C \\
&= \tau + \varphi(z).
\end{aligned}$$

Finally, since B_1 is in H_x but not Γ_0, we have

$$\sqrt{\sum_{g \in \langle A \rangle \backslash H_x} g'(B_1(z))^2 B_1'(z)} = -\sqrt{\sum_{g \in \langle A \rangle \backslash H_x} g'(z)^2},$$

and so

$$\begin{aligned}
\varphi(B_1(z)) &= \frac{\int_Q^{B_1 z} \zeta_x}{\int_a \zeta_x} + C \\
&= \frac{\int_Q^{B_1 Q} \zeta_x}{\int_a \zeta_x} + C + \frac{\int_{B_1 Q}^{B_1 z} \zeta_x}{\int_a \zeta_x} \\
&= -\varphi(z).
\end{aligned}$$

q.e.d.

Corollary 7.7.3 *Let $\psi(x) = \tau$, and let G'_τ denote the group $\langle z \mapsto z+1, z \mapsto z+\tau, z \mapsto -z \rangle$. Let $L_{\tau/2}$ denote the lattice $\{n/2 + m\tau/2 : n, m \in \mathbb{Z}\}$. Then the surface $\Delta(H_x)/H_x$ is conformally equivalent to the surface $(\mathbb{C} - L_{\tau/2})/G'_\tau$.*

Proposition 7.7.4 *Let $P_2(x,z)$ denote $\sum_{g \in \langle A \rangle \backslash H_x} g'(z)^2$. Then $P_2(-\overline{x}, -\overline{z}) = \overline{P_2(x,z)}$.*

Proof: Define $J(z) = -\overline{z}$. Then $JAJ^{-1} = A^{-1}$; $JB_1J^{-1} = B_1^{-1}$; and $JB_{2,x}J^{-1} = B_{2,-\overline{x}}^{-1}$. Now the proof is the same as for Lemma 7.1.14.

q.e.d.

Proposition 7.7.5 $\psi(-\overline{x}) = -\overline{\psi(x)}$.

Proof: First note that if Q is any point in $\Delta(H_x)$, then

$$\begin{aligned}
\overline{\int_Q^{B_{2,x}B_1^{-1}(Q)} \sqrt{P_2(x,z)}\,dz} &= \overline{\int_{B_1(Q)}^{B_{2,x}(Q)} \sqrt{P_2(x,z)}\,dz} \\
&= \overline{\int_{B_1(Q)}^{B_{2,x}(Q)} -\sqrt{P_2(x, B_{2,x}^{-1}(z))}\,(B_{2,x}^{-1})'(z)\,dz} \\
&= \overline{-\int_{B_{2,x}^{-1}B_1(Q)}^{Q} \sqrt{P_2(x,z)}\,dz} \\
&= \overline{\int_Q^{B_{2,x}^{-1}B_1(Q)} \sqrt{P_2(x,z)}\,dz}.
\end{aligned}$$

Notice also that

$$-\overline{B_{2,-\overline{x}}B_1(Q)} = B_{2,x}B_1^{-1}(-\overline{Q}).$$

Thus,

$$
\begin{aligned}
\psi(-\overline{x}) &= \frac{\int_{b(-\overline{x})} \sqrt{P_2(-\overline{x}, z)}\, dz}{\int_a \sqrt{P_2(-\overline{x}, z)}\, dz} \\
&= \frac{\int_{-\overline{b(-\overline{x})}} \sqrt{P_2(-\overline{x}, -\overline{z})}\, d(-\overline{z})}{\int_{-\overline{a}} \sqrt{P_2(-\overline{x}, -\overline{z})}\, d(-\overline{z})} \\
&= \frac{\int_{-\overline{b(-\overline{x})}} \sqrt{\overline{P_2(x, z)}}\, d\overline{z}}{\int_{-\overline{a}} \sqrt{\overline{P_2(x, z)}}\, d\overline{z}} \\
&= \frac{\int_{-\overline{Q}}^{\overline{-B_{2,-\overline{x}}^{-1} B_1(Q)}} \overline{\sqrt{P_2(x, z)}\, dz}}{\int_{-\overline{Q}}^{-\overline{Q}-4} \overline{\sqrt{P_2(x, z)}\, dz}} \\
&= -\frac{\int_{-\overline{Q}}^{\overline{-B_{2,-\overline{x}} B_1(Q)}} \overline{\sqrt{P_2(x, z)}\, dz}}{\int_{-\overline{Q}-4}^{-\overline{Q}} \overline{\sqrt{P_2(x, z)}\, dz}} \\
&= -\frac{\int_{-\overline{Q}}^{B_{2,x} B_1^{-1}(-\overline{Q})} \overline{\sqrt{P_2(x, z)}\, dz}}{\int_{-\overline{Q}-4}^{-\overline{Q}} \overline{\sqrt{P_2(x, z)}\, dz}} \\
&= -\frac{\int_{-\overline{Q}}^{B_{2,x}^{-1} B_1(-\overline{Q})} \overline{\sqrt{P_2(x, z)}\, dz}}{\int_{-\overline{Q}-4}^{-\overline{Q}} \overline{\sqrt{P_2(x, z)}\, dz}} \\
&= -\overline{\psi(x)}.
\end{aligned}
$$

q.e.d.

Proposition 7.7.6 *$\psi(x + 2n) = \psi(x) + n$ for every integer n.*

Proof: Since $AB_{2,x} = B_{2,x+2}^{-1}$, the set of right cosets of $\langle A \rangle$ in H_x is the same as the set of right cosets of $\langle A \rangle$ in H_{x+2}. Thus, $P_2(x, z) = P_2(x + 2, z)$ for any $x \in M_{0,4}$. Hence

$$
\begin{aligned}
\psi(x+2) &= \frac{\int_{b(x+2)} \sqrt{P_2(x+2, z)}\, dz}{\int_a \sqrt{P_2(x+2, z)}\, dz} \\
&= \frac{\int_{b(x+2)} \sqrt{P_2(x, z)}\, dz}{\int_a \sqrt{P_2(x, z)}\, dz}.
\end{aligned}
$$

Since $B_{2,x+2}^{-1}B_1 = AB_{2,x}B_1$, the curve $b(x+2)$ is the projection of any curve in $\Delta(H_{x+2})$ from any point Q to $AB_{2,x}B_1(Q)$. Thus

$$\begin{aligned}
\psi(x+2) &= \frac{\int_Q^{AB_{2,x}B_1(Q)} \sqrt{P_2(x,z)}\,dz}{\int_Q^{AQ} \sqrt{P_2(x,z)}\,dz} \\
&= \frac{\int_Q^{B_{2,x}B_1(Q)} \sqrt{P_2(x,z)}\,dz + \int_{B_{2,x}B_1(Q)}^{AB_{2,x}B_1(Q)} \sqrt{P_2(x,z)}\,dz}{\int_Q^{AQ} \sqrt{P_2(x,z)}\,dz} \\
&= \frac{\int_Q^{B_{2,x}B_1(Q)} \sqrt{P_2(x,z)}\,dz}{\int_Q^{AQ} \sqrt{P_2(x,z)}\,dz} + 1.
\end{aligned}$$

But

$$\begin{aligned}
\int_Q^{B_{2,x}B_1(Q)} \sqrt{P_2(x,z)}\,dz &= \int_Q^{B_{2,x}B_1(Q)} \sqrt{P_2(x, B_{2,x}^{-2}(z))}\,(B_{2,x}^{-2})'(z)\,dz \\
&= \int_{B_{2,x}^{-2}(Q)}^{B_{2,x}^{-1}B_1(Q)} \sqrt{P_2(x,z)}\,dz \\
&= \int_{B_{2,x}^{-2}(Q)}^{Q} \sqrt{P_2(x,z)}\,dz + \int_Q^{B_{2,x}^{-1}B_1(Q)} \sqrt{P_2(x,z)}\,dz,
\end{aligned}$$

and $\int_{B_{2,x}^{-2}(Q)}^{Q} \sqrt{P_2(x,z)}\,dz = 0$ since

$$\begin{aligned}
\int_{B_{2,x}^{-2}(Q)}^{Q} \sqrt{P_2(x,z)}\,dz &= \int_{B_{2,x}^{-2}B_{2,x}(Q)}^{B_{2,x}(Q)} \sqrt{P_2(x,z)}\,dz \\
&= -\int_{B_{2,x}^{-1}(Q)}^{B_{2,x}(Q)} \sqrt{P_2(x, B_{2,x}^{-1}(z))}\,(B_{2,x}^{-1})'(z)\,dz \\
&= -\int_{B_{2,x}^{-2}(Q)}^{Q} \sqrt{P_2(x,z)}\,dz.
\end{aligned}$$

Thus, $\psi(x+2) = \psi(x) + 1$.

q.e.d.

Proposition 7.7.7 *Γ_0 is conjugate to a subgroup of G_x.*

Proof: Let $M(z) = z+1$. Then the following equalities hold:

$$MB_1^{-1}B_{2,x}M^{-1} = S_1T_x^{-2}S_1^{-1}$$

$$
\begin{aligned}
MB_{2,x}^{-1}AB_{2,x}M^{-1} &= S_1T_xS_1^{-2}T_x^{-1}S_1^{-1} \\
MB_{2,x}^{-2}M^{-1} &= S_1T_xS_1^{-1}T_x^{-1} \\
M(B_{2,x}A)^2M^{-1} &= T_xS_1^{-1}T_x^{-1}S_1 \\
M(A^{-1}B_1^{-1})^2M^{-1} &= S_1^{-1}T_x^{-1}S_1T_x \\
MB_1^2M^{-1} &= T_x^{-1}S_1T_xS_1^{-1}
\end{aligned}
$$

q.e.d.

Let $G_x^0 = M\Gamma_0M^{-1}$.

Proposition 7.7.8 $[G_\mu : G_\mu^0] = 4$.

Proof: The four distinct right cosets of G_μ^0 in G_μ are G_μ^0, $G_\mu^0 \cdot S_1$, $G_\mu^0 \cdot T_\mu$, and $G_\mu^0 \cdot S_1T_\mu$. To prove this assertion it suffices to show that G_μ^0 contains every word $g \in G_\mu$ which decomposes into an even number of letters $T_\mu^{\pm 1}$ and an even number of letters $S_1^{\pm 1}$. (For then, $G_\mu^0 \cdot S_1$ consists of those words consisting of an even number of letters $T_\mu^{\pm 1}$ and an odd number of letters $S_1^{\pm 1}$; and similarly for $G_\mu^0 \cdot T_\mu$ and $G_\mu^0 \cdot S_1T_\mu$.)

It is easy to see that every word $g \in G_\mu$ which has an even number of letters $T_\mu^{\pm 1}$ and an even number of letters $S_1^{\pm 1}$ is a product of the following words and their inverses: T_μ^2, S_1^2, $S_1T_\mu^2S_1$, $T_\mu S_1^2T_\mu$, $T_\mu S_1T_\mu S_1$, $T_\mu S_1^{-1}T_\mu S_1$, $T_\mu S_1T_\mu^{-1}S_1$, and $T_\mu S_1^{-1}T_\mu^{-1}S_1$. Since each of these words is in the group G_μ^0, the proof is complete.

q.e.d.

Note that Proposition 7.7.8 implies that $\Delta(G_\mu) = \Delta(G_\mu^0) = M(\Delta(\Gamma_0)) = M(\Delta(H_\mu))$; and $\Lambda(G_\mu) = M(\Lambda(H_\mu))$, where $M(z) = z + 1$.

Theorem 7.7.9 *The diagram in Figure 7.3 commutes.*

Proof: Since $\Delta(G_\mu) = \Delta(G_\mu^0)$, ζ_μ is a holomorphic abelian differential on $\Delta(G_\mu^0)/G_\mu^0$. Since $\Delta(G_\mu^0)/G_\mu^0$ and $\Delta(\Gamma_0)/\Gamma_0$ are conformally equivalent, ζ_μ and ζ_x differ by a constant multiple (that is, if $\zeta_\mu = f(z)\,dz$ and $\zeta_x = g(z)\,dz$, then $g(M^{-1}(z))\,dz = cf(z)\,dz$

$$
\begin{tikzcd}
M_{1,1} \arrow[rr, leftrightarrow, "\mathrm{id}"] \arrow[dr, leftrightarrow] & & M_{0,4} \arrow[dl, leftrightarrow] \\
& \mathbb{H} &
\end{tikzcd}
$$

Figure 7.3: The maps to $\mathbb{H}$

for some constant c). Pick base points Q on $\Delta(G_\mu^0)/G_\mu^0$ and $Q' = \pi_2 M^{-1}\pi_1^{-1}(Q)$ on $\Delta(\Gamma_0)/\Gamma_0$, where $\pi_1 : \Delta(G_\mu^0) \to \Delta(G_\mu^0)/G_\mu^0$ and $\pi_2 : \Delta(\Gamma_0) \to \Delta(\Gamma_0)/\Gamma_0$ are the natural projections. Then

$$
\begin{aligned}
\frac{\int_{b(\mu)} \zeta_\mu}{\int_{a(\mu)} \zeta_\mu} &= \frac{2\int_{b(\mu)} \zeta_\mu}{2\int_{a(\mu)} \zeta_\mu} \\
&= \frac{2\int_Q^{T_\mu(Q)} \zeta_\mu}{2\int_Q^{S_1(Q)} \zeta_\mu} \\
&= \frac{\int_Q^{T_\mu^2(Q)} \zeta_\mu}{\int_Q^{S_1^2(Q)} \zeta_\mu} \\
&= \frac{\int_{Q'}^{M^{-1}T_\mu^2 M(Q')} \zeta_x}{\int_{Q'}^{M^{-1}S_1^2 M(Q')} \zeta_x} \\
&= \frac{\int_{Q'}^{B_{2,x}B_1^{-1}(Q')} \zeta_x}{\int_{Q'}^{AQ'} \zeta_x} \\
&= \frac{-\int_{B_{2,x}B_1^{-1}(Q')}^{Q'} \zeta_x}{\int_{Q'}^{AQ'} \zeta_x} \\
&= \frac{\int_{B_1^{-1}(Q')}^{B_{2,x}^{-1}(Q')} \zeta_x}{\int_{Q'}^{AQ'} \zeta_x} \\
&= \frac{\int_{Q'}^{B_{2,x}^{-1}B_1(Q')} \zeta_x}{\int_{Q'}^{AQ'} \zeta_x} \\
&= \frac{\int_{b(x)} \zeta_x}{\int_{a(x)} \zeta_x}.
\end{aligned}
$$

q.e.d.

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APPENDIX

THE COMPUTER PROGRAM

```
c                                  PSI.F
c
c                        This program will sum the relative Poincare
c              Series at points along the "a" and "b" curves.  It will
c              output the error bound.


c       THRESHOLDS
        double precision eps, bnd
c       SUM OF THE SERIES
        complex*16 asum(100000), bsum(100000)
c       ERROR BOUND FOR THE SERIES AT A POINT
        double precision error
c       THE VALUE IN TEICHMULLER SPACE
        complex*16 mu
c       IMAGINARY PART OF MU
        double precision t
c       POINTS AT WHICH TO CALCULATE SERIES
        complex*16 z0
c       GENERATORS
        complex*16 y(4,4)
c       LEVEL IN TREE
        integer lev,maxlev,numterms
c       TAG LIST = 1, 2, 3, or 4
        integer tag(100000)
c       MATRIX LIST
        complex*16 x(4,100000)
c       ARE WE FAR ENOUGH IN THE TREE?
        logical farenuf
c       MAKE SURE TREE TRAVERSED ONCE
        integer treepart
c       NUMBER OF POINTS AT WHICH WE WILL CALCULATE SERIES
        real n
c       ANSWERS
        double precision ans1, ans2, relasum(100000), imasum(100000)
        double precision ans3,ans4,relbsum(100000),imbsum(100000)
        complex*16 ansa,ansb
c       RADIUS OF DISK, DISTANCES TO LIMIT SET, ERRORS IN INTEGRALS
        double precision r,ka,kb,totaerr,totberr
c       ERROR IN FINAL ANSWER
        double precision finerr
c       MINIMUM ABS VAL OF SERIES ON a AND b CURVES, AND MINIMUM REAL PARTS
        double precision minaabs, minbabs, minarel, minbrel
```

```
c       VALUE OF \pi
        double precision pi

        common /gens/ y
        common /tree/ x, tag
        common /level/ lev
        common /flag/ farenuf

c       READ INPUT VALUES
        write(*,*) 'Enter value of mu.'
        read(*,*) mu
        t = dimag(mu)
        write(*,*) 'Enter value for epsilon.'
        read(*,*) eps
        write(*,*) 'Enter bound on matrix size.'
        read(*,*) bnd
        write(*,*) 'Enter number of points (> 1) at which to '
        write(*,*) 'calculate the series. '
        read(*,*) n

c       CALCULATE GENERATORS AND INITIAL POINT
        error = 0.0
        call calgen(mu)
        r = (t - sqrt(t*t-4*t+8))/2
        if(r .gt. 0.5) r = 0.5
        kb = r
        ka = r + t/2 - 1
        write(*,*) 'mu = ',mu
        write(*,*) 'eps = ',eps
        write(*,*) 'bnd = ',bnd
        write(*,*) 'n = ',n
        write(*,*) 'r = ',r
        write(*,*) 'ka = ',ka
        write(*,*) 'kb = ',kb

c       INITIATE TREE
        tag(1)=2
        x(1,1)=y(1,2)
        x(2,1)=y(2,2)
        x(3,1)=y(3,2)
        x(4,1)=y(4,2)
        do 22 index = 1, n
           asum(index) = 1.0
           bsum(index) = 1.0
22      continue
```

```
      maxlev = 1
      lev = 1
      numterms = 1
      treepart = 0

c     MAIN BRANCHING


40    if(lev .eq. 1 .AND. tag(1) .eq. 2)then
           treepart = treepart + 1
           if(treepart .eq. 2)then
                tag(1)=4
                treepart = 0
                x(1,1)=y(1,4)
                x(2,1)=y(2,4)
                x(3,1)=y(3,4)
                x(4,1)=y(4,4)
           end if
      end if
      if(lev .eq. 1 .AND. tag(1) .eq. 4)then
           treepart = treepart + 1
           if(treepart .eq. 2)go to 999
      end if
      call test(eps,bnd,error,mu)
      if(farenuf) go to 417

      do 33 index = 1, n
         z0 = (mu/2)-1+(index-1)/(n-1)
c        (THESE ARE THE POINTS ALONG HALF THE a CURVE)
         asum(index)=asum(index)+1/((x(3,lev)*z0 + x(4,lev))**4)
         z0=(0,1)+(index-1)*((mu/2)-(0,1))/(n-1)
c        (THESE ARE THE POINTS ALONG HALF THE b CURVE)
         bsum(index)=bsum(index)+1/((x(3,lev)*z0 + x(4,lev))**4)
33    continue
      numterms = numterms+1
      call forwrd
      if(lev .gt. maxlev)maxlev = lev
      if(maxlev .gt. 99990) go to 666
      go to 40

c     GO BACK TO FIRST AVAILABLE TURN AND TURN ONCE
417   call bckwrd
      go to 40

999   error = error/(r*r)
```

```
c       (THE \pi WASN'T CALCULATED INTO AREAS OF DISKS.)

c       (NOW WE HAVE CALCULATED THE SERIES AT PTS ALONG THE a AND b CURVES)
        write(*,*)
        write(*,*) 'the error bound for the series is ',error
        write(*,*) 'maximum level attained was ',maxlev
        write(*,*) 'the number of terms summed was ', numterms

        do 66 index = 1, n
           relasum(index) = dreal(sqrt(asum(index)))
           imasum(index) = dimag(sqrt(asum(index)))
           relbsum(index)=dreal(((mu/2)-(0,1))*sqrt(bsum(index)))
           imbsum(index)=dimag(((mu/2)-(0,1))*sqrt(bsum(index)))
66      continue
c       CALCULATE REAL PART OF INTEGRAL OVER a CURVE
        call trapezoid(relasum, n, ans1)
c       CALCULATE IMAGINARY PART OF INTEGRAL OVER a CURVE
        call trapezoid(imasum, n, ans2)
        ansa=2*ans1+(0,2)*ans2
        write(*,*) 'The integral over the curve a equals ',ansa
c
        pi = 3.141592
        minaabs = cdabs(asum(1))
        minbabs = cdabs(bsum(1))
        minarel = dreal(asum(1))
        minbrel = dreal(bsum(1))
        do 122 index = 2,n
           if(cdabs(asum(index)) .lt. minaabs) minaabs = cdabs(asum(index))
           if(cdabs(bsum(index)) .lt. minbabs) minbabs = cdabs(bsum(index))
           if(dreal(asum(index)) .lt. minarel) minarel = dreal(asum(index))
           if(dreal(bsum(index)) .lt. minbrel) minbrel = dreal(bsum(index))
122     continue
        minaabs = minaabs - error
        minbabs = minbabs - error
        minarel = minarel - error
        minbrel = minbrel - error
        write(*,*) 'min. abs. val. on curve a before derivative: ',minaabs
        minaabs=minaabs-(4*t)/((n-1)*pi*ka*r*r)
        minarel = minarel - (4*t)/((n-1)*pi*ka*r*r)
        write(*,*) 'min. abs. val. on curve a after derivative: ', minaabs
        if(minaabs .le. 0)then
           write(*,*) 'Need more points (n) to get finite error bound.'
        end if
        write(*,*) 'The real part of the series on the a curve '
        write(*,*) 'is at least ',minarel
```

```
      call interr(n,error,r,ka,totaerr,t,minaabs)
      write(*,*)
      write(*,*) 'The error in the integral over a is less than ',totaerr

c     CALCULATE REAL PART OF INTEGRAL OVER b CURVE
      call trapezoid(relbsum, n, ans3)
c     CALCULATE IMAGINARY PART OF INTEGRAL OVER b CURVE
      call trapezoid(imbsum, n, ans4)
      ansb=2*ans3+(0,2)*ans4
      write(*,*) 'The integral over the curve b equals ',ansb
c
      write(*,*) 'min. abs. val. on curve b before derivative: ',minbabs
      minbabs = minbabs - (4*t*cdabs((mu/2.0)-(0,1)))/((n-1)*pi*kb*r*r)
      minbrel = minbrel - (4*t*cdabs((mu/2.0)-(0,1)))/((n-1)*pi*kb*r*r)
      write(*,*) 'min. abs. val. on curve b after derivative: ',minbabs
      write(*,*) 'The real part of the series on the b curve '
      write(*,*) 'is at least ',minbrel
      call interr(n,error,r,kb,totberr,t,minbabs)
      totberr=totberr*cdabs((mu/2)-(0,1))
      write(*,*)
      write(*,*) 'The error in the integral over b is less than ',totberr

      write(*,*)
      write(*,*) 'psi(',mu,') = ',ansb/ansa
      finerr=totaerr*cdabs(ansb)/(cdabs(ansa**2)-totaerr*cdabs(ansa))
      finerr=finerr+totberr/(cdabs(ansa)-totaerr)
      write(*,*) 'The modulus of the final error is less than ',finerr
666   stop
      end

c
c     GO FORWARD IN TREE
      subroutine forwrd
      complex*16 x(4,100000)
      integer tag(100000)
      complex*16 y(4,4)
      integer lev

      common /tree/ x, tag
      common /gens/ y
      common /level/ lev

      tag(lev+1) = tag(lev) +1
      if(tag(lev+1) .eq. 5) tag(lev+1) =1
      x(1,lev+1)=x(1,lev)*y(1,tag(lev+1))+x(2,lev)*y(3,tag(lev+1))
```

```
      x(2,lev+1)=x(1,lev)*y(2,tag(lev+1))+x(2,lev)*y(4,tag(lev+1))
      x(3,lev+1)=x(3,lev)*y(1,tag(lev+1))+x(4,lev)*y(3,tag(lev+1))
      x(4,lev+1)=x(3,lev)*y(2,tag(lev+1))+x(4,lev)*y(4,tag(lev+1))
      lev=lev+1


      if(lev .gt. 99990) write(*,*) 'level too high!!!'

      return
      end

c     GO BACKWARD IN TREE
      subroutine bckwrd
      integer lev
      logical endbr

      common /level/ lev

10    call branch(endbr)
      if(endbr .AND. lev .NE. 1) go to 10

      return
      end

c     GO TO NEXT BRANCH
      subroutine branch(endbr)
      logical endbr
      complex*16 x(4,100000)
      complex*16 y(4,4)
      integer tag(100000)
      integer lev, i, j

      common /tree/ x, tag
      common /gens/ y
      common /level/ lev

c     TERMINATE PROGRAM IF LEV IS 1
      if(lev .eq. 1) return

c     TURN AT LEV
      i=tag(lev)-1
      if(i .eq. 0) i=4
      j=i+2
      if(j .gt. 4) j=j-4
      if(j .ne. tag(lev-1)) go to 110
```

```
c       GO BACK ONE MORE LEVEL
                lev=lev-1
                endbr = .true.
                return
c       TURN AT LEV
110             tag(lev)=i
                x(1,lev)=x(1,lev-1)*y(1,tag(lev))+x(2,lev-1)*y(3,tag(lev))
                x(2,lev)=x(1,lev-1)*y(2,tag(lev))+x(2,lev-1)*y(4,tag(lev))
                x(3,lev)=x(3,lev-1)*y(1,tag(lev))+x(4,lev-1)*y(3,tag(lev))
                x(4,lev)=x(3,lev-1)*y(2,tag(lev))+x(4,lev-1)*y(4,tag(lev))
                endbr = .false.
                return

        end

c       CALCULATE GENERATORS AND INITIAL POINT
        subroutine calgen(mu)
        complex*16 mu
c       GENERATORS
        complex*16 y(4,4)

        common /gens/ y

        y(1,1) = (1,0)
        y(2,1) = (2,0)
        y(3,1) = (0,0)
        y(4,1) = (1,0)

        y(1,3) = y(4,1)
        y(2,3) = -y(2,1)
        y(3,3) = -y(3,1)
        y(4,3) = y(1,1)

        y(1,2) = (0,-1)*mu
        y(2,2) = (0,-1)
        y(3,2) = (0,-1)
        y(4,2) = (0,0)

        y(1,4) = y(4,2)
        y(2,4) = -y(2,2)
        y(3,4) = -y(3,2)
        y(4,4) = y(1,2)

        return
        end
```

```
c       TEST: SIZE OF MATRIX SMALL ENOUGH, FAR ENOUGH ALONG TREE;
c       AND CALCULATE ERROR BOUND FOR THIS BRANCH OF THE TREE

        subroutine test(eps,bnd,error,mu)

        double precision sizem, eps, bnd, error, radius, t
        complex*16 x(4,100000), mu, ta, tb, tc
        logical farenuf
        integer tag(100000), lev, nlev, mlev


        common /flag/ farenuf
        common /tree/ x, tag
        common /level/ lev

        sizem = cdabs(x(1,lev))+cdabs(x(2,lev))+cdabs(x(3,lev))
        sizem = sizem+cdabs(x(4,lev))
        farenuf = .FALSE.
c
c
        if(lev .le. 3) return
        mlev = lev - 2
        nlev = lev - 1
        if( (tag(lev) .eq. 1) .and. (tag(nlev) .eq. 1) )then
           if( (tag(mlev) .eq. 1) .or. (tag(mlev) .eq. 2) )then
c             (WORD ENDS IN SSS OR TSS)
              ta = x(1,mlev)/x(3,mlev)
              tb=(x(1,mlev)*(2.5)+x(2,mlev))/(x(3,mlev)*(2.5)+x(4,mlev))
              tc=(x(1,mlev)*(2.5,1)+x(2,mlev))
              tc=tc/(x(3,mlev)*(2.5,1)+x(4,mlev))
              if(radius(ta,tb,tc)**2 .lt. eps)then
                 error=error+radius(ta,tb,tc)**2
                 farenuf = .TRUE.
              end if
           end if
        end if
        if(tag(lev) .eq. 3)then
           if(tag(nlev) .eq. 3)then
c               (WORD ENDS IN S^{-1}S^{-1})
              t = dimag(mu)
              ta=x(1,mlev)/x(3,mlev)
              tb=(x(1,mlev)*(t-3.0)+x(2,mlev))
              tb=tb/(x(3,mlev)*(t-3.0)+x(4,mlev))
              tc=(x(1,mlev)*(t-3.0+(0,1))+x(2,mlev))
```

```
                tc=tc/(x(3,mlev)*(t-3.0+(0,1))+x(4,mlev))
                if(radius(ta,tb,tc)**2 .lt. eps)then
                   error=error+radius(ta,tb,tc)**2
                   farenuf = .TRUE.
                end if
             end if
             if(tag(nlev) .eq. 2)then
c                  (WORD ENDS IN TS^{-1})
                ta=x(1,mlev)*(mu+(1,0))+x(2,mlev)
                ta=ta/(x(3,mlev)*(mu+(1,0))+x(4,mlev))
                tb=x(1,mlev)*(mu-(1,0))+x(2,mlev)
                tb=tb/(x(3,mlev)*(mu-(1,0))+x(4,mlev))
                tc=x(1,mlev)*(mu-(0,1))+x(2,mlev)
                tc=tc/(x(3,mlev)*(mu-(0,1))+x(4,mlev))
                if(radius(ta,tb,tc)**2 .lt. eps)then
                   error=error+radius(ta,tb,tc)**2
                   farenuf = .TRUE.
                end if
             end if
          end if
          if(tag(lev) .eq. 2)then
             if(tag(nlev) .eq. 2)then
c               (WORD ENDS IN TT)
                ta=(x(1,mlev)*(mu+(1,0))+x(2,mlev))
                ta=ta/(x(3,mlev)*(mu+(1,0))+x(4,mlev))
                tb=(x(1,mlev)*(mu-(1,0))+x(2,mlev))
                tb=tb/(x(3,mlev)*(mu-(1,0))+x(4,mlev))
                tc=(x(1,mlev)*(mu-(0,1))+x(2,mlev))
                tc=tc/(x(3,mlev)*(mu-(0,1))+x(4,mlev))
                if(radius(ta,tb,tc)**2 .lt. eps)then
                   error = error + radius(ta,tb,tc)**2
                   farenuf = .TRUE.
                end if
             end if
             if(tag(nlev) .eq. 1)then
                if( (tag(mlev) .eq. 2) .or. (tag(mlev) .eq. 1) )then
c                 (WORD ENDS IN TST OR SST)
                   ta=x(1,mlev)/x(3,mlev)
                   tb=(x(1,mlev)+x(2,mlev))/(x(3,mlev)+x(4,mlev))
                   tc=(x(1,mlev)*(1,1)+x(2,mlev))/(x(3,mlev)*(1,1)+x(4,mlev))
                   if(radius(ta,tb,tc)**2 .lt. eps)then
                      error=error+radius(ta,tb,tc)**2
                      farenuf = .TRUE.
                   end if
                end if
```

```
            end if
         end if
         if(tag(lev) .eq. 4)then
c                (WORD ENDS IN T^{-1}; ASSUME D(z0,r) IS OUTSIDE I(T^{-1}).)
            ta=(x(1,nlev)+x(2,nlev))/(x(3,nlev)+x(4,nlev))
            tb=(x(2,nlev)-x(1,nlev))/(x(4,nlev)-x(3,nlev))
            tc=(x(1,nlev)*(0,1)+x(2,nlev))/(x(3,nlev)*(0,1)+x(4,nlev))
            if(radius(ta,tb,tc)**2 .lt. eps)then
               error=error+radius(ta,tb,tc)**2
               farenuf = .TRUE.
            end if
         end if
         if((sizem .GE. bnd) .and. (.not. farenuf))then
            farenuf = .TRUE.
            write(*,*) 'ERROR IN ERROR TERM; MATRIX SIZE TOO LARGE'
            write(*,*) 'MATRIX SIZE = ',sizem
         end if
         return
         end
c
c
c        OUPUT THE RADIUS OF THE CIRCLE THROUGH
c        THE COMPLEX POINTS A,B,C.  ASSUMES POINTS ARE NOT
c        COLINEAR.  USES INTERSECTION OF PERPENDICULAR
c        BISECTORS TO COMPUTE CENTER.
c
         double precision function radius(a,b,c)
         complex*16 a,b,c,center
         double precision ar,ai,br,bi,cr,ci,centr,centi,temp
c
         ar = dreal(a)
         ai = dimag(a)
         br = dreal(b)
         bi = dimag(b)
         cr = dreal(c)
         ci = dimag(c)
c
         if(ai .eq. bi)then
            temp = bi
            bi = ci
            ci = temp
            temp = br
            br = cr
            cr = temp
         end if
```

```
      if(bi .eq. ci)then
         temp = ai
         ai = bi
         bi = temp
         temp = ar
         ar = br
         br = temp
      end if
      centr = (ar-br)*(ar+br)/(2.0*(bi-ai))
      centr = centr - (br-cr)*(br+cr)/(2.0*(ci-bi))
      centr = centr + (ci-ai)/2.0
      centr = centr/((ar-br)/(bi-ai) - (br-cr)/(ci-bi) )
      centi = ((ar-br)/(bi-ai))*(centr - (ar+br)/2.0)+(ai+bi)/2.0
      center = centr + (0,1)*centi
      radius = cdabs(center - a)
c
      return
      end
c
c     TRAPEZOID RULE
      subroutine trapezoid(dat, n, ans)
c     function values
      double precision dat(100000)
      real n
c     answer = ans
      double precision ans

      ans = dat(1) + dat(n)
      do 55 index = 2, n-1
         ans = ans + 2*dat(index)
55    continue
      ans = (1/(2*n-2))*ans
      return
      end
c
c
c     CALCULATE ERROR IN INTEGRAL APPROXIMATION
      subroutine interr(n,error,r,k,toterr,t,minabs)

      real n
      double precision minabs,toterr,pi,t,r,k,error

      toterr = 0.0
      pi = 3.141592
      toterr=(16*t*t)/(pi*pi*k*k*(r**4)*(minabs**(1.5)))
```

```
      toterr=toterr+(20*t)/(pi*k*k*r*r*(sqrt(minabs)))
      toterr=toterr/(12*(n-1)*(n-1))
      toterr=toterr+error/(sqrt(minabs))
c     WE INTEGRATE OVER HALF OF THE CURVE a OR b
      toterr = 2*toterr
      return
      end
```

VITA 2

Charles Allen Matthews

Candidate for the Degree of

Doctor of Philosophy

Thesis: BOUNDARIES OF LOW-DIMENSIONAL TEICHMÜLLER SPACES OF RIEMANN SURFACES

Major Field: Mathematics

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