SUBSPACES OF REARRANGEMENT INVARIANT SPACES NOT CONTAINING l_2

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CHAPTER 1: INTRODUCTION

In 1974 W. Johnson and E. Odell examined subspaces of L_p not containing l_2. They showed that if Y is a subspace of $L_p (2 < p < \infty)$ such that no subspace of Y is isomorphic to l_2, then Y is isomorphic to a subspace of l_p.

In the dissertation we will prove similar results for a wider class of spaces than the L_p spaces. We consider rearrangement invariant function spaces X on $[0,1]$ which satisfy the following four properties.

(1) X has Boyd indices satisfying $2 < p_X < q_X < \infty$

(2) X does not contain an isomorphic copy of l_1.

(3) The Haar system is an unconditional basis of X.

(4) X has an upper l_2-estimate as a Banach lattice with respect to the ordering induced by the Haar system.

It is known that the fourth hypothesis implies the second, but for convenience we list all four hypotheses.

We show that if X is a rearrangement invariant space satisfying these requirements and Y is a subspace of X not containing l_2, then Y embeds in a space Z which is defined as a certain sum of finite dimensional subspaces. In the case that $X = L_p$, this result reduces to the theorem of Johnson and Odell.

First, in Chapter 2, we introduce many of the definitions that will be used in this paper and present many of the miscellaneous theorems we will use in our proof. Chapter 3 lists for reference some ofthe most important theorems proved in [KP] and [JO], for it is these theorems that we generalize in this paper. In Chapter 4, we prove some preliminary results, most of which are generalizations of Kadec and Pelczynski's work. Chapter 5 is devoted to the construction of a Banach space which will perform the same role for us that l_p did in Johnson and Odell's paper. Finally, in Chapter 6 we prove the main result of the paper and observe that this result is indeed a generalization of Johnson and Odell's theorem.

We now define some essential concepts of Banach space theory. Let X be an infinite dimensional Banach space. A sequence $\{e_i : i \in \mathbf{N}\}$ is called a **Schauder basis** for X if for each element x of X, there is a unique sequence of scalars $(a_i)_{i=1}^{\infty}$ such that $x = \sum_{i=1}^{\infty} a_i x_i$, with the right hand side converging in the norm of X.

If $\{e_i : i \in \mathbf{N}\}$ is a basis and $P_n(x)$ is defined by $P_n(x) = \sum_{i=1}^{n} a_i e_i$, then $\sup_n \|P_n\|$ is finite. This constant $\sup_n \|P_n\|$ is called the **basis constant** of the basis $\{e_i : i \in \mathbf{N}\}$.

Let X be a Banach space with a basis $\{e_i : i \in \mathbf{N}\}$. Consider the set of all sequences $(\epsilon_i)_{i=1}^{\infty}$ where each ϵ_i is either 1 or -1. For each sequence $(\epsilon_i)_{i=1}^{\infty}$, define $P_{(\epsilon_i)} : X \to X$ by $P_{(\epsilon_i)}(\sum_{i=1}^{\infty} a_i e_i) = \sum_{i=1}^{\infty} \epsilon_i a_i e_i$. Let $K = \sup \|P_{(\epsilon_i)}\|$, where the supremum is taken over all possible sequences $(\epsilon_i)_{i=1}^{\infty}$. If $K < \infty$ then we say that the basis $\{e_i : i \in \mathbf{N}\}$ is **unconditional** and K is the **unconditional basis**

constant.

There are several equivalent useful ways of defining an unconditional basis; we will describe another one of them. For any subset $\sigma \subset \mathbf{N}$, we define

$$P_\sigma(\sum_{n=1}^{\infty} a_n e_n) = \sum_{n\in\sigma} a_n e_n.$$

If $\sup_\sigma \|P_\sigma\| < \infty$, then we say that $\{e_n : n \in \mathbf{N}\}$ is an unconditional basis, and that $\sup_\sigma \|P_\sigma\|$ is the **suppression unconditional basis constant**. A Schauder basis is unconditional if and only if its suppression unconditional basis constant is finite.

A sequence of elements (x_n) is called a **basic sequence** if it is a basis for its closed linear span. Two basic sequences (x_n) and (y_n) are called **equivalent** if for every sequence of scalars (a_n), $\sum_{n=1}^{\infty} a_n x_n$ converges if and only if $\sum_{n=1}^{\infty} a_n y_n$ converges.

There are two important systems of functions on $[0,1]$ which we will need in our study. The first, the **Haar system**, is extremely important to us because it forms an unconditional basis for the Banach space X. There are two styles of notation for describing the Haar functions, and we will use both styles in this paper. The first system is a double subscript system. Let $h_\emptyset(t) = 1$ for all t on the interval. Then for each nonnegative integer i we define functions $h_{i,j}$ for each integer j in the range $0 \le j \le 2^i - 1$ as follows:

$$h_{i,j}(t) = \begin{cases} 1, & 2^{-i} \cdot j \quad \le t \; < 2^{-i} \cdot (j + \frac{1}{2}) \\ -1, & 2^{-i} \cdot (j + \frac{1}{2}) \le t \; < 2^{-i} \cdot (j+1) \\ 0, & \text{otherwise} \end{cases}$$

Note that each function $h_{i,j}$ has support of measure 2^{-i}. Furthermore the supports of the functions $h_{i,0}, h_{i,1} \ldots$, and $h_{i,2^i-1}$ are disjoint with union $[0,1]$. We call the functions $\{h_{i,0}, h_{i,1} \ldots$, and $h_{i,2^i-1}\}$ a generation (or level) of the Haar system.

We will use a single subscript style to index the Haar system through the majority of this paper. In this style we let $h_0(t) = h_\emptyset(t)$ and let $h_k(t) = h_{i,j}(t)$, where $k = 2^i + j$. Thus we define h_k for every nonnegative integer k.

We will also be making use of the **Rademacher** system of functions. Since we have already defined the Haar functions by formulas, we can describe the Rademacher functions conveniently. Let $r_i(t)$ be the sum of the functions $h_{i,j}(t)$ for $0 \le j \le 2^i - 1$. Thus $r_i(t)$ takes the value 1 if $h_{i,j}(t) = 1$ for some j, and -1 if $h_{i,j}(t) = -1$ for some j. Incidentally, we consider the Haar and Rademacher functions to be defined almost everywhere, and pay no attention to the values of these functions at dyadic points.

The Banach space $L_p[0,1]$, also written as L_p, will play a central role. By definition, it is the space of all equivalence classes of functions (modulo equality almost everywhere) of functions $x(t)$ which are defined on the interval $[0,1]$ and for which $\int_0^1 |x|^p < \infty$. The L_p norm of x is $\|x\| = (\int_0^1 |x|^p)^{\frac{1}{p}}$. Similarly, $L_p[0,\infty)$ is the set of all functions on $[0, \infty)$ for which $\int_0^\infty |x|^p < \infty$, with norm $\|x\| = (\int_0^\infty |x|^p)^{\frac{1}{p}}$. Finally, l_p is the spaces of all sequences $(x_n)_{n=1}^\infty$ of scalars such that $\sum_{i=1}^\infty |x_i|^p < \infty$, with norm $\|x\| = (\sum_{i=1}^\infty |x_i|^p)^{\frac{1}{p}}$.

For further information on Banach space theory the reader may consult the books of Diestel [D], Guerre [G], Habala, Hajek, and Zizler [HHZ], and Lindenstrauss and Tzafriri [LT]. Diestel's book is not only well written and relatively easy to understand but very entertaining as well. Lindenstrauss and Tzafriri's book is the most comprehensive reference of these books, and includes a large amount of information on Banach lattices and rearrangement invariant spaces.

CHAPTER 2: BACKGROUND THEOREMS AND DEFINITIONS

In this chapter we present the majority of the definitions and theorems that will be cited and used in the main body of the paper, Chapters 4-6. We first introduce some notation.

If A is a set we let A^{-1} be its set theoretic complement.

(Ω, Σ, μ) will always be a measure space with σ - algebra Σ and measure μ.

If X is a Banach space of functions on $[0,1]$, we define $M_X(\epsilon) = \{x \in X : \mu(t : |x(t)| \geq \epsilon \|x\|_X) \geq \epsilon\}$, where μ is Lebesgue measure. If $x \in X$ we denote the norm of x in X by $\|x\|_X$, or in the case that $X = L_p[0,1]$, we write $\|x\|_p$.

Throughout this paper, we will assume that $0 < \epsilon \leq 1$.

Let $\lfloor x \rfloor$ be the greatest integer which is less than or equal to x.

Paley [P] proved the following theorem, which will play a key role for us.

Theorem 2.1 *The Haar system is an unconditional basis of $L_p[0,1]$ for every p , $1 < p < \infty$.*

We are also interested in another property of L_p, namely that under the pointwise (almost everywhere) ordering, L_p has a lattice structure. More precisely it is a Banach lattice. The following definitions may be found in [LT].

Definition: A partially ordered Banach space X over the reals is called a **Banach lattice** provided

(1) $x \leq y$ implies $x + z \leq y + z$ for every $x, y, z \in X$.

(2) $ax \geq 0$, for every $x \geq 0$ and every nonnegative real a.

(3) For all $x, y \in X$ there exists a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$.

(4) $\|x\| \leq \|y\|$ whenever $|x| < |y|$, where the absolute value $|x|$ of $x \in X$ is defined by $|x| = x \vee (-x)$.

It is important to note that a Banach space can easily have more than one lattice structure, that is, more than one partial ordering which satisfies the axioms (perhaps under an equivalent norm). In particular, the space X which with we will be dealing in this paper has two natural orderings. There is the pointwise ordering where $f \leq g$ if $f(x) \leq g(x)$ for all $x \in [0,1]$ (except on a set of measure 0), and the Haar ordering where $\sum_{i=1}^{\infty} a_i h_i \leq \sum_{i=1}^{\infty} b_i h_i$ if $a_i \leq b_i$ for all i. Generally, X with the Haar ordering may fail property (4) of the definition of Banach lattice under the rearrangement invariant norm but nonetheless we will employ the lattice language when referring to X in the Haar ordering; we will implicitly be using property (4) with a constant. To distinguish between the orderings we will always mean the pointwise ordering unless we explicitly specify the Haar ordering.

We denote the dual of X by X^*. Every measurable function g on Ω so that $gf \in L_1(\mu)$, for every $f \in X$, defines an element x_g^* in X^* by

$$x_g^*(f) = \int_\Omega f g d\mu.$$

The set of all functionals of this form is denoted X' and forms a linear subspace of X^*.

Definition: A Banach lattice X is said to satisfy an **upper l_p estimate** if there exists a constant $M, M < \infty$, such that for every choice of pairwise disjoint elements $\{x_i\}_{i=1}^n$ in X, we have

$$\|\sum_{i=1}^{n} x_i\| \leq M(\sum_{i=1}^{n} \|x_i\|^p)^{\frac{1}{p}}.$$

Definition: Let (Ω, Σ, μ) be a complete σ- finite measure space.

A Banach space X consisting of equivalence classes, modulo equality almost everywhere, of locally integrable real valued functions on Σ is called a **Köthe function space** if the following conditions hold.

1. If $|f(w)| \leq |g(w)|$ a.e. on Ω, with f measurable and $g \in X$, then $f \in X$ and $\|f\| \leq \|g\|$.

2. For every $\sigma \in \Sigma$ with $\mu(\sigma) < \infty$ the characteristic function of σ ,χ_σ, belongs to X.

Definition: Let (Ω, Σ, μ) be one of the measure spaces $\{1, 2, \ldots\}$, $[0, 1]$ or $[0, \infty]$ (with the natural measure).

A Köthe function space X on (Ω, Σ, μ) is said to be a **rearrangement invariant space** if the following hold:

1. If τ is an automorphism of the measure space Ω onto itself and f is a measurable function on Ω then $f \in X$ if and only if $f(\tau^{-1}(w)) \in X$ and if this is the case then $\|f(w)\| = \|f(\tau^{-1}(w))\|$.

2. X' is a norming subspace of X^*, (where "norming" means that $\|x\| = \sup\{|x^*(x)| : x^* \in X', \|x^*\| = 1\}$ for every $x \in X$), and thus X is order isometric to a subspace of X''. As a subspace of X'', X is either maximal (i.e. $X = X''$) or minimal (i.e. X is the closed linear span of the simple integrable functions of X'').

3. (a) If $\Omega = \{1, 2, \ldots\}$, then as sets,

$$l_1 \subset X \subset l_\infty$$

and the inclusion maps are of norm one, i.e., if $f \in l_1$ then $\|f\|_X \leq \|f\|_1$ and if $f \in X$ then $\|f\|_\infty \leq \|f\|_X$.

(b) If $\Omega = [0, 1]$ then, as sets,

$$L_\infty[0, 1] \subset X \subset L_1[0, 1]$$

and the inclusion maps are of norm one, i.e., if $f \in L_\infty$ then $\|f\|_X \leq \|f\|_\infty$ and if $f \in X$ then $\|f\|_1 \leq \|f\|_X$.

(c) If $\Omega = [0, \infty)$ then, as sets,

$$L_\infty[0, \infty) \cap L_1[0, \infty) \subset X \subset L_\infty[0, \infty) + L_1[0, \infty).$$

and the inclusion maps are of norm one with respect to the natural norms in these spaces.

Some remarks on the above definition are in order. The first requirement of the definition is in some sense the most important. Its effect is that, for any

$f \in X$, the norm of f depends only on the distribution function d_f, which is given by

$$d_f(t) = \mu(\{w \in \Omega : f(w) > t\}), -\infty < t < \infty.$$

Specifically, if $f \in X$ and g is a measurable function with $d_g(t) = d_f(t)$, then also $g \in X$ and $\|f\|_X = \|g\|_X$. The second requirement is a technical one which, in particular, all separable spaces satisfy. Finally, the third part of the definition is actually a normalization condition. (3b) and (3c) each imply that $\|\chi_{[0,1]}\|_X = 1$.

Rearrangement invariant spaces are natural interpolation spaces and we will have need of a set of interpolation indices, the Boyd indices. In order to define the Boyd indices of a rearrangement invariant space X, we first define for every $s, 0 < s < \infty$, a linear operator D_s. If X is defined on the interval $I = [0, \infty)$ and $f \in X$, then we let

$$(D_s f)(t) = f(\frac{t}{s}), 0 < s < \infty, 0 \leq t < \infty.$$

If $I = [0, 1]$, then we let

$$(D_s f)(t) = \begin{cases} f(\frac{t}{s}), & t \leq \min(1, s) \\ 0, & s < t \leq 1 \text{ (in case } s < 1). \end{cases}$$

Definition: Let X be a rearrangement invariant function space on an interval I which is either $[0, 1]$ or $[0, \infty)$. The **Boyd indices** p_X and q_X are defined by

$$p_X = \lim_{s \to \infty} \frac{\log(s)}{\log \|D_s\|} = \sup_{s > 1} \frac{\log(s)}{\log \|D_s\|}$$

$$q_X = \lim_{s \to 0^+} \frac{\log(s)}{\log \|D_s\|} = \inf_{0<s<1} \frac{\log(s)}{\log \|D_s\|}$$

It is a routine matter to show that $1 \leq p_X \leq q_X \leq \infty$ for any rearrangement invariant space X. Also one can show that if $X = L_p[0,1]$, then $p_X = q_X = p$. With knowledge of the Boyd indices of X, we can use some interpolation theorems, and the following fundamental fact.

Theorem 2.2 *([LT],p.132) Let X be a rearrangement invariant function space on an interval I which is either $[0,1]$ or $[0,\infty)$. Then for every p and q that satisfy $1 \leq p < p_X$ and $q_X < q \leq \infty$, we have*

$$L_p(I) \cap L_q(I) \subset X \subset L_p(I) + L_q(I),$$

with the inclusion maps being continuous.

We will also need some standard facts from basis theory.

Theorem 2.3 *([BP],p.153) Let (x_n) be a basic sequence in a Banach space X with biorthogonal functionals (x_n^*). If the sequence (y_n) in X satisfies the condition*

$$\sum_{n=1}^{\infty} \|x_n - y_n\| \|x_n^*\| < 1,$$

then (y_n) is a basic sequence, and (x_n) and (y_n) are equivalent.

Bessaga and Pelczynski's theorem can be combined with a subsequence argument to yield the following corollary, which we will later use to prove Theorem 4.8.

Corollary 2.4 *([G], p.13) Let (x_n) be a basic sequence in X not converging to* 0 *in norm. If the sequence (y_n) in X satisfies the condition*

$$\sum_{n=1}^{\infty} \|x_n - y_n\| < \infty,$$

then there exists an infinite subset $M \subset \mathbf{N}$ such that the subsequences $(x_n)_{n\in M}$ and $(y_n)_{n\in M}$ are equivalent.

The following theorem was published by Bessaga and Pelczynski in 1958.

Theorem 2.5 *([BP],p.156) Let X be a Banach space with an unconditional basis, and suppose the sequence (y_n) in X converges weakly to* 0 *but not in norm. Then there is a subsequence (y_{n_k}) which is an unconditional basic sequence.*

This next lemma is a well known result, and a proof can be found in Rosenthal's paper [R].

Lemma 2.6 *If $(x_i)_{i=1}^{\infty}$ is a normalized unconditional basic sequence in L_p with unconditional basis constant λ, and if $2 < p < \infty$, then*

$$(\sum_{n=1}^{k} |\alpha_n|^p)^{\frac{1}{p}} \leq \lambda \| \sum_{n=1}^{k} \alpha_n x_n \|_p. \tag{2.1}$$

It will be convenient to use the space $X(l_2)$.

Definition: Let X be a rearrangement invariant function space on $[0,1]$. $\mathbf{X(l_2)}$ is defined to be the completion of all sequences $(x_1, x_2, \ldots)$ of elements of x which are eventually zero, with respect to the norm

$$\|(x_1, x_2, \ldots)\|_{\mathbf{X(l_2)}} = \|(\sum_{i=1}^{\infty} |x_i|^2)^{\frac{1}{2}}\|_X$$

B. Mitjagin [M1] showed that any separable rearrangement invariant space X on $[0,1]$ with $1 < p_x \leq q_X < \infty$ is isomorphic to $X(l_2)$, so in particular our space has this property.

CHAPTER 3:WORK OF KADEC, PELCZYNSKI, JOHNSON, AND ODELL

In 1958 M. Kadec and A. Pelczynski published a paper on the L_p spaces whose methods we draw upon very heavily here. They introduced the notation $M_p(\epsilon)$ which in our notation is $M_X(\epsilon)$, where $X = L_p$. Some of their results are listed below for comparison with our versions for X.

Theorem 3.1 (KP) *Let $p \geq 1$ and let (x_n) be a sequence in $L_p[0,1]$ such that for every $\epsilon > 0$ there is an index n_ϵ such that x_{n_ϵ} does not belong to $M_p(\epsilon)$. Then there exists a subsequence of (x_n) which, when normalized, is a basic sequence equivalent to the unit vector basis of l_p.*

Theorem 3.2 (KP) *Let $p > 2$ and let (x_n) be an unconditional basic sequence in L_p with $0 < \inf_n \|x_n\|_p \leq \sup_n \|x_n\|_p < \infty$. Then (x_n) is equivalent to the unit vector basis in l_2 iff there is an $\epsilon > 0$ such that x_n is in $M_p(\epsilon)$ for $n = 1, 2, \ldots$.*

Theorem 3.3 (KP) *Let $p > 2$ and let (x_n) be a sequence in L_p satisfying the following conditions:*

(1) (x_n) *converges weakly to* 0

(2) $\limsup_n \|x_n\|_p > 0$.

Then there is a subsequence (x_{n_k}) which is equivalent either (a) to the unit vector basis in l_p or (b)to the unit vector basis in l_2. Moreover, (b) holds iff there is $\epsilon > 0$ such that x_n is in $M_p(\epsilon)$ for infinitely many n.

In 1974 Johnson and Odell examined subspaces of L_p not containing l_2. Their study of subspaces of $L_p, 2 < p < \infty$, was the main source of inspiration for this paper. Some of the results of [JO] are included below.

Lemma 3.4 (JO) *If Y is a subspace of $L_p(2 < p < \infty)$ such that no subspace of Y is isomorphic to l_2, then for any $\delta > 0$, there exists n such that if $y = \sum \alpha_i h_i \in Y$ and $\|y\|_p \leq 1$, then*

$$\|\sum_{i=n}^{\infty} \alpha_i h_i\|_2 \leq \delta.$$

Theorem 3.5 (JO) *If Y is a subspace of $L_p(2 < p < \infty)$ such that no subspace of Y is isomorphic to l_2, then Y is isomorphic to a subspace of l_p.*

CHAPTER 4 : PRELIMINARY THEOREMS

In Chapters 4,5, and 6, we assume that X is a rearrangement invariant space on [0,1] satisfying the four hypotheses listed on page 1. In this chapter we build up to Theorem 4.10, where we prove an analog of Theorem 3.3. We first prove a general lemma showing that the norm in X is absolutely continuous. Then we adapt the methods Kadec and Pelczynski used with L_p to prove similar theorems for X. Our eventual goal is to prove an analog of Kadec and Pelczynski's dichotomy result. Specifically, we will prove at the end of this chapter that if a sequence approaches 0 weakly but not in norm, then either there is a subsequence equivalent to the unit vector basis of l_2, or there is a subsequence equivalent to some disjointly supported sequence in X.

Lemma 4.1 *Suppose E is a finite dimensional subspace of X of the form $E = [h_i]_{i=1}^n$. Then for every $\epsilon > 0$, there exists some δ, $\delta > 0$, such that if $\mu(A) < \delta$ then for all $x \in X$,*

$$\frac{\|x \cdot 1_A\|_X}{\|x\|_X} < \epsilon.$$

Proof: Clearly this lemma works in the case that $X = L_p (1 < p < \infty)$ and $dim(E) = 1$. We will first prove the lemma in the case that $X = L_p$ and $dim(E) = m$.

Let $\{e_1, e_2, \ldots, e_m\}$ be a normalized basis for E , and endow $\mathbf{R}^m$ with the l_1^m norm. Let $T : E \to \mathbf{R}^m$ be the natural isomorphism and let $\|T\| = c$. Fix $\epsilon > 0$. Let $\epsilon_0 = \frac{\epsilon}{c}$, and choose δ small enough that the theorem is satisfied for ϵ_0 and each of the one-dimensional subspaces $[e_1], \ldots, [e_m]$.

Suppose

$$x = \sum_{i=1}^{m} a_i e_i.$$

Then

$$\|x \cdot 1_A\|_p \leq \sum_{i=1}^{m} |a_i| \cdot \|e_i \cdot 1_A\|_p \leq \sum_{i=1}^{m} |a_i| \cdot \epsilon_0 \cdot \|e_i\|_p \leq \epsilon_0 c \|\sum_{i=1}^{m} a_i e_i\|_p,$$

which is to say that

$$\|x \cdot 1_A\|_p \leq \epsilon \cdot \|x\|_p. \tag{4.2}$$

Therefore this lemma holds in the case $X = L_q$, where $q = q_X$. Let $f \in X \cap L_q$. Suppose (by Theorem 2.2) that

$$\|f\|_X \leq a_1 \|f\|_q. \tag{4.3}$$

Let S_E be the unit sphere of E with respect to the X norm, $S_E = \{x \in E : \|x\|_X = 1\}$. Let $\{e_1, \ldots, e_m\}$ be a normalized basis for E with basis constant v.

Since the closure of $L_\infty \cap S_X$ contains S_E, we can choose an $\frac{\epsilon}{8}$ net $\{z_1, \ldots, z_k\}$ in $L_\infty \cap S_X$, i.e., for every $y \in S_E$ there exists i such that $\|z_i - y\|_X \leq \frac{\epsilon}{8}$.

Let $W = Span\{z_1, \ldots, z_k\}$. Since W is a finite dimensional subspace of $X \cap L_q$, there is some constant a_2 such that for $f \in W$,

$$\|f\|_q \leq a_2 \|f\|_X. \tag{4.4}$$

Because inequality 4.2 holds for $p = q$ in the finite dimensional subspace W, there is some $\delta, \delta > 0$, such that for all $f \in W$, $\mu(A) < \delta$ then

$$\|f \cdot 1_A\|_q \leq \frac{\epsilon}{8a_1 a_2}\|f\|_q.$$

If $\mu(A) < \delta$, then, we have

$$\|f \cdot 1_A\|_X \leq a_1\|f \cdot 1_A\|_q \leq \frac{\epsilon}{8a_2}\|f\|_q \leq \frac{\epsilon}{8}\|f\|_X.$$

Finally, if $x \in S_E$, then there exists some z_0 chosen from the set $\{z_1, \ldots, z_k\}$ such that $\|z_0 - x\|_X \leq \frac{\epsilon}{8}$. So

$$\begin{aligned}
\|x \cdot 1_A\|_X &\leq \|(z_0 - x) \cdot 1_A\|_X + \|z_0 \cdot 1_A\|_X \\
&\leq \|z_0 - x\|_X + \|z_0 \cdot 1_A\|_X \\
&\leq \frac{\epsilon}{8} + \frac{\epsilon}{8} \\
&< \epsilon.
\end{aligned}$$

Remark: The lemma was actually proved for every finite dimensional subspace of X.

We now proceed to develop the theory required for Theorem 4.10, beginning with two facts which should be fairly clear from the definitions but which nevertheless are important.

Observation 4.2 *If* $\epsilon_1 \leq \epsilon_2$*, then* $M_X(\epsilon_1) \supseteq M_X(\epsilon_2)$.

Observation 4.3 $\bigcup_{\epsilon>0} M_X(\epsilon) = X$.

Theorem 4.4 *If* $x \notin M_X(\epsilon)$ *then there is a set* A *(depending on* x*) such that* $\mu(A) \leq \epsilon$,

$$\frac{\|x \cdot 1_{A^{-1}}\|_X}{\|x\|_X} \leq \epsilon$$

and consequently,

$$\frac{\|x \cdot 1_A\|_X}{\|x\|_X} \geq 1 - \epsilon.$$

Proof:

Let $A = \{t : |x(t)| \geq \epsilon \|x\|_X\}$ Then $\mu(A) \leq \epsilon$. For $t \in A^{-1}$, $|x(t)| \leq \epsilon \|x\|_X$. By the lattice properties,

$$\|x \cdot 1_{A^{-1}}\|_X \leq \left\| \|x\|_X \cdot \epsilon \cdot 1_{[0,1]} \right\|_X = \epsilon \|x\|_X.$$

Since

$$\|x \cdot 1_A\|_X + \|x \cdot 1_{A^{-1}}\|_X \geq \|x\|_X,$$

we have

$$\frac{\|x \cdot 1_A\|_X}{\|x\|_X} \geq 1 - \epsilon$$

as well.

Notation: In the next two theorems, we will use the need to refer frequently to the set $\{t : |x(t)| \geq \epsilon \|x\|_X\}$, we call it S; that is, $S = \{t : |x(t)| \geq \epsilon \|x\|_X\}$.

Theorem 4.5 $\|x\|_2 \geq \epsilon^{\frac{3}{2}} \, \|x\|_X$ *for every* $x \in M_X(\epsilon)$

Proof:

Since $x \in M_X(\epsilon)$, we have that $\mu(S) \geq \epsilon$.

So

$$\begin{aligned}
\|x\|_2 &= \left(\int_0^1 |x(t)|^2 dt\right)^{\frac{1}{2}} \\
&\geq \left(\int_S |x(t)|^2 dt\right)^{\frac{1}{2}} \\
&\geq \left(\epsilon^2 \|x\|_X^2 \mu(S)\right)^{\frac{1}{2}} \\
&\geq \epsilon^{\frac{3}{2}} \|x\|_X.
\end{aligned}$$

Theorem 4.6 *Let* $\theta = \frac{p_X + 2}{2}$. *Suppose* c *and* k *satisfy*

$$\|x\|_\theta \leq k \cdot \|x\|_X$$

and

$$c \cdot \|x\|_2 \geq \|x\|_X.$$

Then $x \in M_X(\epsilon)$ *, where* $\epsilon = (c(1+k))^{\frac{-2(p_X+2)}{p_X-2}}$.

Proof:

Suppose that $x \notin M_X(\epsilon)$, where ϵ is the number above, so that $\mu(S) < \epsilon$.

For any set $E \subseteq [0,1]$, we can apply Holder's inequality to the functions $f(x) = x^2$ and $g(x) = 1_E$ to get

$$(\int_E |x(t)|^2)^{\frac{1}{2}} < (\mu(E))^{\frac{\theta-2}{2\theta}} \cdot (\int_E |x(t)|^\theta)^{\frac{1}{\theta}}.$$

Let $p = p_X$ in this proof to simplify notation. Then

$$\begin{aligned}
\|x\|_2 &= \left(\int_S |x(t)|^2 dt + \int_{[0,1]-S} |x(t)|^2 dt\right)^{\frac{1}{2}} \\
&\leq \left(\int_S |x(t)|^2 dt\right)^{\frac{1}{2}} + \left(\int_{[0,1]-S} |x(t)|^2 dt\right)^{\frac{1}{2}} \\
&\leq (\mu(S))^{\frac{\theta-2}{2\theta}} \cdot \|x\|_\theta + \epsilon\|x\|_X \\
&< (\epsilon)^{\frac{\theta-2}{2\theta}} \cdot k\|x\|_X + \epsilon\|x\|_X \\
&= (\epsilon + \epsilon^{\frac{p-2}{2(p+2)}} \cdot k)\|x\|_X.
\end{aligned}$$

Now since

$$\|x\|_X \leq c\|x\|_2 < c(\epsilon + \epsilon^{\frac{p-2}{2(p+2)}} \cdot k)\|x\|_X \leq c \cdot \epsilon^{\frac{p-2}{2(p+2)}}(1+k)\|x\|_X,$$

we see that

$$\epsilon^{\frac{p-2}{2(p+2)}} \cdot c(1+k) > 1,$$

and therefore

$$\epsilon > (c(1+k))^{\frac{-2(p+2)}{p-2}}.$$

Thus if we have

$$\epsilon \leq (c(1+k))^{\frac{-2(p+2)}{p-2}},$$

then $x \in M_X(\epsilon)$.

Corollary 4.7 *If Y is a subset of X and there is a constant K such that*

$$\frac{\|y\|_X}{\|y\|_2} < K$$

for every $y \in Y$, then Y is contained in $M_X(\epsilon)$ for some fixed ϵ.

Proof: Apply Theorem 4.6 and Theorem 2.2.

Theorem 4.8 *Let (x_n) be a sequence in X such that for every $\epsilon \geq 0$ there is an index n_ϵ such that x_{n_ϵ} does not belong to $M_X(\epsilon)$. Then there is a subsequence (x'_n)*

such that $\frac{x_n}{\|x_n'\|_X}$ *is a basic sequence equivalent to some disjointly supported sequence in* X.

Proof:

Given (x_n), we will find a subsequence (x_n') and a disjoint sequence of sets (A_n'), which we will use to define the disjointly supported sequence (z_n). Then we can apply a standard perturbation theorem of Bessaga and Pelczynski to show these sequences are equivalent.

First recall that if $x \in X$, then the set function $\tau_x(A) = \|x \cdot 1_A\|_X$ is absolutely continuous, by Lemma 4.1.

Choose x_1' so that $x_1' \notin M_X(4^{-1})$. Then choose A_1 with $\mu(A_1) < \epsilon$ so that

$$\frac{\|x_1' \cdot 1_{A_1}\|_X}{\|x_1'\|_X} \geq 1 - 4^{-1}.$$

By Lemma 4.1 there exists ϵ_2 such that

$$\mu(A) < \epsilon_2 \Rightarrow \frac{\|x_1' \cdot 1_A\|_X}{\|x_1'\|_X} \leq 4^{-2}.$$

Now choose $x_2' \notin M_X(\min(\epsilon_2, 4^{-2}))$, so that $x_2' \notin M_X(\epsilon_2)$ and $x_2' \notin M_X(4^{-2})$. By Theorem 4.4, there exists a set A_2 with $\mu(A_2) < \min(\epsilon_2, 4^{-2})$ and

$$\frac{\|x_2' \cdot 1_{A_2}\|_X}{\|x_2'\|_X} \geq 1 - 4^{-2}.$$

It follows automatically from the choice of ϵ_2 that

$$\frac{\|x_1' \cdot 1_{A_2}\|_X}{\|x_1'\|_X} < 4^{-2}.$$

We can continue in this way choosing (x_n') and A_n for each integer n such that $\frac{\|x_n' \cdot 1_{A_n}\|_X}{\|x_n'\|_X} \geq 1 - 4^{-(n+1)}$, and $\max_{i \leq n} \frac{\|x_i \cdot 1_{A_{n+1}}\|_X}{\|x_i\|_X} < 4^{-(n+1)}$.

Notation: For each $n \in \mathbf{N}$ we make the following definitions:

$$\begin{aligned} A_n' &= A_n \setminus \bigcup_{i=n+1}^{\infty} A_i \\ z_n(t) &= \frac{x_n'(t)}{\|x_n'\|_X} \cdot 1_{A_n'} \\ w_n(t) &= \frac{x_n'(t)}{\|x_n'\|_X} \end{aligned}$$

Note that for $m \neq n, A_m' \cap A_n' = \emptyset$.

We now see that

$$\begin{aligned} \|w_n - z_n\|_X &= \|(w_n - z_n) \cdot 1_{(A_{n'})^C}\|_X \\ &= \|w_n \cdot 1_{(A_{n'})^C}\|_X \\ &\leq \|w_n \cdot 1_{(A_n - A_{n'})}\|_X + \|w_n \cdot 1_{(A_n)^C}\|_X \\ &\leq \|\sum_{i=n+1}^{\infty} w_n \cdot 1_{A_i}\|_X + 4^{-(n+1)} \\ &\leq \sum_{i=n+1}^{\infty} 4^{-(i+1)} + 4^{-(n+1)} \\ &\leq 4^{-n} \end{aligned}$$

We also know that

$$\begin{aligned}
\|z_n\|_X &= \|w_n \cdot 1_{A_{n'}}\|_X && (4.5)\\
&\geq \|w_n \cdot 1_{A_n}\|_X - \sum_{j=n+1}^{\infty} \|w_n \cdot 1_{A_j}\|_X && (4.6)\\
&\geq 1 - 4^{-(n+1)} - \sum_{j=n+1}^{\infty} 4^{-(j+1)} && (4.7)\\
&\geq 1 - 4^{-n}. && (4.8)
\end{aligned}$$

This then shows that (z_n) does not converge to 0 in X and we saw above that $\sum_{n=1}^{\infty} \|w_n - z_n\|_X < \infty$. The disjointly supported sequence (z_n) is clearly basic. Corollary 2.4 therefore tells us that a subsequence of (w_n) is a basic sequence equivalent to a subsequence of (z_n), which is just what we needed.

Theorem 4.9 *Suppose (x_n) is an unconditional basic sequence in X with $0 < \inf \|x_n\|_X \leq \sup \|x_n\|_X < \infty$, with the (x_n) disjointly supported with respect to the Haar basis. Suppose there is some ϵ such that all for all $n, x_n \in M_X(\epsilon)$. Then (x_n) is equivalent to the unit vector basis of l_2.*

Proof:

Assume without loss of generality that $\|x_n\|_X = 1$ for all n. The fact that there exists C_1 with

$$\|\sum_{n=1}^{\infty} a_n x_n\|_X \leq C_1 \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{\frac{1}{2}}$$

follows from the upper l_2 estimate hypothesis.

Now by Theorem 4.5 we know that $\|x_n\|_2 \geq \epsilon^{\frac{3}{2}}$ for all n. Suppose β satisfies $\|x\|_X \geq \beta\|x\|_2$ for all $x \in [x_n]$. Let C_2 be the unconditional basis constant of (x_n). It isn't difficult to see that the functions $\{r_n(t)a_nx_n\}_{n=1}^{\infty}$ are orthogonal in $L_2(I \times I)$, and the equation $\|\sum_{n=1}^{\infty} f_n\|_2 = (\sum_{n=1}^{\infty} \|f_n\|^2)^{\frac{1}{2}}$ for orthogonal functions will give us equation 4.10 below, so we get

$$\begin{aligned}
\|\sum_{n=1}^{\infty} a_nx_n\|_X &\geq \frac{1}{C_2} \sup_{t\in[0,1]} \|\sum_{n=1}^{\infty} r_n(t)a_nx_n\|_X && (4.9)\\
&\geq \frac{\beta}{C_2}\|\sum_{n=1}^{\infty} r_n(t)a_nx_n\|_{L_2(I\times I)} \\
&= \frac{\beta}{C_2}\left(\sum_{n=1}^{\infty} \|r_na_nx_n\|^2_{L_2(I\times I)}\right)^{\frac{1}{2}} && (4.10)\\
&= \frac{\beta}{C_2}\left(\sum_{n=1}^{\infty} \|a_nx_n\|_2^2\right)^{\frac{1}{2}} \\
&\geq \epsilon^{\frac{3}{2}} \cdot \frac{\beta}{C_2}(\sum_{n=1}^{\infty} |a_n|^2)^{\frac{1}{2}}
\end{aligned}$$

We are now ready for our analog of Theorem 3.3.

Theorem 4.10 *Suppose (x_n) is a sequence in X, disjoint with respect to the Haar system, which converges weakly to 0 but does not converge to 0 in norm. Then there is a subsequence (x_{n_k}) which is equivalent either to the unit vector basis of l_2 or to some sequence of disjointly supported functions in X. Specifically, if there is some ϵ, $\epsilon > 0$, such that every for infinitely many n, we have $x_n \in M_X(\epsilon)$, then a subsequence is equivalent to the unit vector basis of l_2.*

Proof: Suppose first that for some ϵ, infinitely many x_n are in $M_X(\epsilon)$; since we are only trying to draw conclusions for a subsequence, we may assume that all x_n are in this $M_X(\epsilon)$. Since the sequence converges weakly, it is bounded in norm. We then apply Theorem 2.5 to find an unconditional basic subsequence (y_n). By Theorem 4.9 , then, (y_n) is equivalent to the unit vector basis of l_2.

Now suppose that there is no ϵ such that infinitely many x_n are in $M_X(\epsilon)$. We can then apply Theorem 4.8 to conclude that (x_n) is equivalent to a disjointly supported sequence as desired.

CHAPTER 5: CONSTRUCTION OF A DISJOINT SUM BANACH SPACE

Let $\tilde{X}$ be a rearrangement invariant space on $[0, \infty]$ and let X be the restriction of $\tilde{X}$ to $[0, 1]$. We assume that X satisfies the four hypotheses enumerated at the beginning of Chapter 1. Let $H_i = [h_{i,j}]_{j=0}^{2^i-1}$. Let (F_n) be disjoint finite subsets of $\mathbf{N}$ with $\max(F_n) < \min(F_{n+1})$, where every positive integer is contained in one of these sets. We let X_k be the subspace spanned by $\{h_{i,j} : i \in F_k, 0 \leq j \leq 2^i - 1\}$, or equivalently, $X_k = Span\{H_i : i \in F_k\}$. We now introduce spaces of the following form:

$$Z = \sum_{i=1}^{\infty} \oplus X_i,$$

where we define the norm on Z as follows:

$$\|(x_i)_{i=1}^{\infty}\|_Z = \|\sum_{i=1}^{\infty} x_i \circ \tau_i\|_{\tilde{X}},$$

where $\tau_i : [i-1, i] \to [0, 1]$ is defined by $\tau_i(x) = x - (i-1)$. Let Z be the set of all sequences (x_i) with $x_i \in X_i$ such that $\|(x_i)_{i=1}^{\infty}\|_Z < \infty$

We can think of Z as arising from a rearrangement invariant space $\tilde{X}$, together with an increasing sequence of integers $(m_i)_{i=1}^{\infty}$, where $m_i = \min(F_i)$. We will therefore denote the space by $Z(\tilde{X}, (m_i))$ if we wish to be explicit.

There is a close connection between the sequence $(x_i)_{i=1}^{\infty}$ in Z and the function $\sum_{i=1}^{\infty} x_i \circ \tau_i$ in $\tilde{X}$. We will sometimes identify the one with the other in the subsequent pages.

Theorem 5.1 *Z is a Banach space.*

Proof: A well known principle (cf. [F],p.144) says that it suffices to show that if (y_m) is a sequence of elements of Z, and $\sum_{m=1}^{\infty} \|y_m\|_Z < \infty$ then $\sum_{m=1}^{\infty} y_m$ converges in Z .

Fix an integer n. By definition

$$\|\sum_{i=1}^{\infty} y_{n,i} \circ \tau_i\|_{\tilde{X}} = \|y_n\|_Z.$$

Thus there is some constant B such that

$$\sum_{n=1}^{\infty} \|\sum_{i=1}^{\infty} y_{n,i} \circ \tau_i\|_{\tilde{X}} = \sum_{n=1}^{\infty} \|y_n\|_Z = B < \infty.$$

Since $\tilde{X}$ is a Banach space, we know by this principle that the function

$$\tilde{w}(x) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} y_{n,i} \circ \tau_i(x)$$

is in $\tilde{X}$.

Similarly, since $\tilde{X}$ is a lattice,

$$\tilde{w}'(x) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} |y_{n,i} \circ \tau_i(x)|$$

is in $\tilde{X}$.

Note that for all n, we know that $y_{n,l}$ is in the span of $\{H_j : j \in F_l\}$ which is a finite dimensional subspace of X. Since

$$\sum_{n=1}^{\infty} \|y_{n,l}\|_X \leq \sum_{n=1}^{\infty} \|y_n\|_X < \infty,$$

we know by the principle that for every l, the sum $w_l(x)$ defined by

$$w_l(x) = \sum_{n=1}^{\infty} y_{n,l}(x)$$

converges in X to an element in $\{H_j : j \in F_l\}$. Define w in Z by $w = (w_1, w_2, w_3, \ldots)$, noting that $\|w\|_Z = \|\tilde{w}\|_{\tilde{X}}$.

Let $v_{N,i} = \sum_{m=1}^{N} y_{m,i}$ and let $\theta_{N,i} = w_i - v_{N,i} = \sum_{m=N+1}^{\infty} y_{m,i}$, and define v_N and θ_N in the obvious way by these sequences.

Since

$$\left|\sum_{i=1}^{\infty}(\theta_{N,i} \circ \tau_i)(x)\right| \leq |\tilde{w}'(x)|$$

for all x, we know that $\theta_N \in Z$.

Now

$$\|w - v_N\|_Z = \|\sum_{n=N+1}^{\infty} y_n\|_Z,$$

and this quantity approaches 0 as $N \to \infty$, so $v_N \to w$ in Z as desired.

Remark: The proof can be viewed as showing that the natural image of Z in $\tilde{X}$ is closed.

Theorem 5.2 *For any two sequences $(m_i)_{i=1}^{\infty}$ and $(n_i)_{i=1}^{\infty}$, $Z(\tilde{X}, (m_i)_{i=1}^{\infty})$ is isomorphic to $Z(\tilde{X}, (n_i)_{i=1}^{\infty})$*

Proof: Let $Z_1 = Z(\tilde{X}, (m_i)_{i=1}^{\infty})$ and let $Z_2 = Z(\tilde{X}, (n_i)_{i=1}^{\infty})$. We will proceed as follows. First we show that Z_2 embeds into Z_1 as a complemented subspace, and by symmetry of course Z_1 embeds into Z_2 as a complemented subspace as well.

We will use the double-subscript style of indexing the Haar system here, so that elements $x \in X$, take the form

$$x = a_\emptyset h_\emptyset + \sum_{i=0}^{\infty} \sum_{j=0}^{2^n-1} a_{i,j} h_{i,j}.$$

Let

$$H_i = [h_{i,j}]_{j=0}^{2^i-1}.$$

For $i = 1$ or 2, we let

$$Z_i = \sum_{k=1}^{\infty} \oplus X_k^i,$$

where

$$X_k^1 = [H_i]_{i=m_k}^{m_{k+1}-1},$$

$$X_k^2 = [H_i]_{i=n_k}^{n_{k+1}-1},$$

and in the special case $k = 1$ we have $H_1^1 = Span\{h_\emptyset, [H_i]_{i=m_1}^{m_2-1}\}$. and $H_1^2 = Span\{h_\emptyset, [H_i]_{i=n_1}^{n_2-1}\}$. We will define $\theta : Z_1 \to Z_2$ by defining the function on each $h_{i,j}$ and extending linearly.

We first choose a subsequence (m'_k) of (m_k) such that $m'_{(k+1)} - m'_k \geq n_{k+1} - n_k$.

We then map the finite dimensional subspace $X_k^2 = [H_i]_{i=n_k}^{n_{k+1}-1}$ of Z_2 into the finite dimensional subspace $X_k^1 = [H_i]_{i=m'_k}^{m'_{(k+1)}-1}$ of Z_1 as follows.

We first show how to map H_{n_k} into $H_{m'_k}$.

Let

$$\theta(h_{n_k,s}) = \sum_{p=s\cdot 2^{m'_k-n_k}}^{((s+1)\cdot 2^{m'_k-n_k})-1} h_{m'_k,p}.$$

We now map H_{n_k+t} into $H_{m'_k+t}$ for each $t \in \{0, \ldots, n_{k+1} - n_k\}$.

Let

$$\theta(h_{n_k+t,s}) = \sum_p h_{m'_k+t,2^{m_k-n_k}\cdot\lfloor\frac{s}{2^t}\rfloor+p},$$

where the sum is taken over all p such that $0 \leq p \leq 2^{m'_k-n_k}$ and $p \equiv s\, mod 2^t$.

One can see that because θ preserves the joint distribution of the Haar functions, we have $x \in X_k^2$ then $\|\theta(x)\|_{Z_1} = \|x\|_{Z_2}$. Then the rearrangement invariant property of $\tilde{X}$ makes it clear that for any $x \in Z_2$, we have $\|\theta(x)\|_{Z_1} = \|x\|_{Z_2}$. Furthermore one can see that the image $\theta(Z_2)$ is complemented in Z_1. Indeed, let $\mathcal{B}_k$ be the $\sigma-$ algebra generated by $\{\theta(h_{n_{k+1}-1,s}) : 0 \leq s \leq 2^{(n_{k+1}-1)} - 1\}$, and define $P : Z_1 \to \theta(Z_2)$ by $P(x_k)_{k=1}^\infty = (E(x_k|\mathcal{B}_k))_{k=1}^\infty$, where E is the conditional expectation operator ([LT],p.122).

By the same arguments, of course, Z_1 embeds as a complemented subspace of Z_2.

We now introduce a universal space U which has the property that any space $Z(\tilde{X}, (p_i)_{i=1}^\infty)$ can embed as a complemented subspace of U by the above argument. First partition $\mathbf{N}$ into infinitely many infinite subsets N_i, $1 \leq i \leq \infty$. Then consider the σ -algebra on the interval $[n_{i,j}, n_{i,j}+1]$ which is generated by the 2^j sets each of measure 2^{-j}, $[n_{i,j}, n_{i,j}+2^{-j}], \ldots, [n_{i,j}+1-2^{-j}, n_{i,j}+1]$. Let $\mathcal{A}$ be the σ -algebra generated by the union of all such sets on $[0,\infty)$. We let U be the closure of the simple $\mathcal{A}$- measurable functions on $[0,\infty)$ under the $\tilde{X}$ norm. Note that U is complemented in $\tilde{X}$ by the generalized conditional expectation operator E induced by $\mathcal{A}$([LT],p.122).

For any spaces (W_i) whose norms are calculated based on the $\tilde{X}$ norm, it is possible to define a sum $\sum \oplus W_i$ based on the $\tilde{X}$ norm. To calculate the norm of $(w_1, w_2, w_3, \ldots)$ in $\sum \oplus W_i$, we let $w(x)$ be a function such that $d_w(t) = \sum_{i=1}^{\infty} d_{w_i}(t)$ and then let $\|(w_1, w_2, w_3, \ldots)\|_W = \|w\|_{\tilde{X}}$. We define $\sum \oplus W_i$ to be the set of all sequences $(w_1, w_2, w_3, \ldots)$ with $w_i \in W_i$ for all i such that $\|(w_1, w_2, w_3, \ldots)\|_W < \infty$. Observe that U is isomorphic to $\sum_{i=1}^{\infty} \oplus U$ and that $\sum \oplus_{i=1}^{\infty} U$ is complemented in $\tilde{X}$. Because U has the same structure as $Z(\tilde{X}, (m_i))$, the proof given above that $\theta(Z_2)$ is complemented in Z_1 can be adapted to show that $\theta(Z_1)$ is complemented in U, where θ is defined analogously. Let T be the corresponding projection defined analogously to the projection P above. Let Q be the kernel of this projection, so that $U = Q \oplus \theta(Z_1)$. Similarly, there is a map $\phi : U \to Z_1$ so that $\phi(U)$ is complemented in Z_1; let $\phi(U) = W$ and let W_1 be its complement in Z_1, so that $Z_1 \sim W \oplus W_1$.

We now use Pelczynski's decomposition argument to show that Z_1 is isomorphic to Z_2. We will first show that Z_1 is isomorphic to U, from which it will follow that Z_2 is isomorphic to U, and so it will follow that Z_1 is isomorphic to Z_2. We write $A \sim B$ to mean that A is isomorphic to B.

Then

$$\begin{aligned} U \oplus Z_1 &\sim U \oplus (W \oplus W_1) \\ &\sim (U \oplus W) \oplus W_1 \\ &\sim (U \oplus U) \oplus W_1 \end{aligned}$$

$$
\begin{aligned}
&\sim U \oplus W_1 \\
&\sim W \oplus W_1 \\
&\sim Z_1
\end{aligned}
$$

On the other hand we will show that $U \oplus Z_1 \sim U$ as well. We will need to define an isomorphism $\phi : (U \oplus U \oplus \ldots) \oplus Z_1 \to (Q \oplus Q \oplus \cdots) \oplus (\theta(Z_1) \oplus \theta(Z_1) \oplus \cdots) \oplus Z_1$. Proceeding formally, let $\phi((f_1, f_2, \ldots), g) = ((I - \tilde{T})f_1, (I - \tilde{T})f_2, \ldots), (\tilde{T}f_1, \tilde{T}f_2, \ldots), g)$, where $\tilde{T} : (U \oplus U \oplus \cdots) \to (\theta(Z_1) \oplus \theta(Z_1) \oplus \cdots)$ is defined by $\tilde{T}(x_1, x_2, \ldots) = (Tx_1, Tx_2, \ldots)$. Extend this map to $\tilde{X}$ by composing with the projection E. Now in the case that $U = L_p[0, \infty)$ for any p, we know that $\tilde{T} \circ E$ is a bounded projection. By interpolation $\tilde{T} \circ E$ is bounded on $\tilde{X}$, and hence $\tilde{T}$ is a bounded projection on $\sum_{i=1}^{\infty} \oplus U$. Thus the operator ϕ is an isomorphism as required.

Therefore we obtain that

$$
\begin{aligned}
U \oplus Z_1 &\sim (U \oplus U \oplus \ldots) \oplus Z_1 \\
&\sim (Q \oplus Q \oplus \cdots) \oplus (\theta(Z_1) \oplus \theta(Z_1) \oplus \cdots) \oplus Z_1 \\
&\sim (Q \oplus Q \oplus \cdots) \oplus (\theta(Z_1) \oplus \theta(Z_1) \oplus \cdots) \\
&\sim (U \oplus U \oplus \cdots) \\
&\sim U,
\end{aligned}
$$

and this completes the proof.

Remark: We assumed at the beginning of the chapter that there is a rearrangement invariant space $\tilde{X}$ on $[0,\infty]$ such that $X = \{f \in \tilde{X} : supp(f) \subset [0,1]\}$. We did not assume that the Boyd indices of $\tilde{X}$ were the same as X, only that $\tilde{X}$ is an interpolation space in the L_p scale. Thus there are many possible choices of $\tilde{X}$ and corresponding $Z(\tilde{X},(m_i))$. In the next chapter we assume that $\tilde{X}$ has been fixed.

CHAPTER 6: MAIN THEOREM

Lemma 6.1 is critical to our main theorem.

Notation: Here we let (h_i) be the normalized Haar basis in X, and for any $x \in X$, we let $h_i^*(x)$ be the coefficient of h_i in the expansion of x, so that

$$x = \sum_{i=0}^{\infty} h_i^*(x) h_i.$$

Lemma 6.1 *If Y is a subspace of X containing no subspace isomorphic to l_2, then for any $\delta \geq 0$, there exists n such that if $y = \sum_{i=1}^{\infty} \alpha_i h_i \in Y$, then*

$$\frac{\| \sum_{i=n}^{\infty} \alpha_i h_i \|_2}{\|y\|_X} \leq \delta$$

Let $T : Y \to L_2[0,1]$ be the identity map. First we will show that T is compact. Suppose that T is *not* compact. Then there is a sequence (z_n) in B_Y such that $(T(z_n))$ has no convergent subsequence. Now choose a subsequence (z'_n) of (z_n) such that there exists a constant C such that $\|Tz'_m - Tz'_n\|_2 \geq C$ for all integers m and n. Since X contains no copy of l_1, Rosenthal's l_1 theorem (cf. [D],p. 201) tells us that (z'_n) has a weakly Cauchy subsequence (z''_n). For each natural number n, we define $y_n = z''_{n+1} - z''_n$. Then $y_n \to 0$ weakly, and yet $\|Ty_n\|_2 \geq C$ for all integers n.

Since $\|y_n\|_X \leq 2$, we know that $\frac{\|y_n\|_X}{\|Ty_n\|_2}$ is bounded. Then Corollary 4.7 tells us that all the y_n are in some fixed $M_X(\epsilon)$ space. Then by a standard perturbation argument and Theorem 4.10, we know that a subsequence of (y_n) is equivalent to the unit vector basis of l_2. This contradicts our assumption that l_2 is not contained in Y, and therefore we know that T must be compact.

Suppose the lemma fails for some $\delta = \sigma_0$. Then for every n there exists $y_n \in B_Y$ such that

$$\|\sum_{i=n}^{\infty} h_i^*(y_n)h_i\|_2 > \sigma_0.$$

Since T is compact, we can find $\{x_1, \ldots, x_m\}$, a $\frac{\sigma_0}{8}$ -net in $T(B_Y)$. Choose $\{x_1' \ldots x_m'\}$ and an integer N such that

$$\|x_i - x_i'\|_2 < \frac{\sigma_0}{4}$$

and $h_j^*(x_i') = 0$ for all $j > N$ and i, $1 \leq i \leq m$.

(To accomplish that, we first choose for each i an integer N_i such that $\|\sum_{j=N_i}^{\infty} h_j^*(x_i)h_j\|_2 <$ $\frac{\sigma_0}{4}$ and then let N be the maximum of these N_i. Let $x_i' = \sum_{j=1}^{N} h_j^*(x_i)h_j$.)

Now since $y_{N+1} \in B_Y$, there is some x_i chosen from the set $\{x_1, \ldots, x_m\}$ such that

$$\|x_i - Ty_{N+1}\|_2 \leq \frac{\sigma_0}{2},$$

and so

$$\|Ty_{N+1} - x_i'\|_2 \leq \frac{3}{4}\sigma_0.$$

Yet y_{N+1} was chosen such that

$$\|\sum_{i=N+1}^{\infty} h_i^*(y_{N+1})h_i\|_2 > \sigma_0,$$

meaning that

$$\|Ty_{N+1} - x_i'\|_2 \geq \|\sum_{i=N+1}^{\infty} h_i^*(y_{N+1})h_i\|_2 \geq \sigma_0,$$

which is a contradiction.

Theorem 6.2 *Let $\tilde{X}$ be a rearrangement invariant space on $[0, \infty)$ and let X be the restriction of $\tilde{X}$ to $[0, 1]$. If Y is a subspace of X such that no subspace of Y is isomorphic to l_2, then Y is isomorphic to a subspace of the space $Z = Z(\tilde{X}, (m_n)_{n=1}^{\infty})$.*

Proof:

Let c be the suppression unconditional basis constant of the Haar system in X.

Inductively, we will choose sequences (p_i), (ϵ_i), and (σ_i) to satisfy the following four properties.

(1) If $x \in [h_i]_{i=1}^{p_n-1}$ and $\mu(A) < \epsilon_n$, then

$$\frac{\|x \cdot 1_A\|_X}{\|x\|_X} < \epsilon_{n-1}$$

(2)

$$\sigma_n \leq (\frac{\epsilon_n}{2^{n+1}})^{\frac{3}{2}} \cdot \frac{1}{2^{n+5}c^2},$$

so that

$$\sum_{n=1}^{\infty}(\frac{\epsilon_n}{2^{n+1}})^{\frac{-3}{2}}\sigma_n \leq \frac{1}{16c^2}$$

(3) If $y \in Y$ and $y = \sum_{i=1}^{\infty} \alpha_i h_i$, then

$$\frac{\| \sum_{i=p_{n+1}}^{\infty} \alpha_i h_i \|_2}{\|y\|_X} \leq \sigma_n.$$

(4) $\sum_{i=1}^{\infty} \epsilon_i \leq k$, where $k = \frac{1}{8c}$

We can accomplish these goals as follows: first choose $p_1 = 1, \epsilon_1 = \frac{k}{2}$, and $0 < \sigma_1 < (\frac{\epsilon_1}{4})^{\frac{3}{2}} \cdot \frac{1}{64c^4}$.

Let $n \geq 2$ and assume p_i, ϵ_i, and σ_i have been chosen to the above specifications for $1 \leq i \leq n-1$. We can then use Lemma 6.1 to choose p_n such that it is an integer power of 2 (in order that h_{p_n} might be the first Haar function in its generation) and large enough that if $y = \sum_{i=1}^{\infty} \alpha_i h_i \in Y$, then

$$\| \sum_{i=p_n}^{\infty} \alpha_i h_i \|_2 \leq \sigma_{n-1} \|y\|_X.$$

Then choose ϵ_n small enough that $\epsilon_n \leq 2^{-(n+1)} k$ and (by Lemma 4.1) so that if $x \in [h_i]_{i=1}^{p_n - 1}$ and $\mu(A) < \epsilon_n$, then

$$\frac{\|x \cdot 1_A\|_X}{\|x\|_X} \leq \epsilon_{n-1}.$$

Finally choose σ_n such that

$$0 < \sigma_n \leq (\frac{\epsilon_n}{2^{n+1}})^{\frac{3}{2}} \cdot \frac{1}{2^{n+5} c^2}.$$

Notation:

If

$$y = \sum_{i=1}^{\infty} \alpha_i h_i,$$

then for each $k \in \mathbf{N}$ we let

$$y_k = \sum_{i=p_k}^{p_{k+1}-1} \alpha_i h_i,$$

and

$$X_k = [h_i]_{i=p_k}^{p_{k+1}-1},$$

so that

$$y = \sum_{i=1}^{\infty} y_i.$$

Define

$$I = \{n : y_n \in M_X(\frac{\epsilon_{n-1}}{2^n})\}$$

and

$$J = \{n : y_n \notin M_X(\frac{\epsilon_{n-1}}{2^n})\}$$

Now let $n \in I$.

By Theorem 4.5 and property (3),

$$\begin{aligned} \|y_n\|_X &\leq (\frac{\epsilon_{n-1}}{2^n})^{\frac{-3}{2}} \|y_n\|_2 \\ &\leq (\frac{\epsilon_{n-1}}{2^n})^{\frac{-3}{2}} \|\sum_{i=n}^{\infty} y_i\|_2 \\ &\leq (\frac{\epsilon_{n-1}}{2^n})^{\frac{-3}{2}} \cdot \sigma_{n-1}\|y\|_X \end{aligned}$$

Therefore

$$\|\sum_{n\in I} y_n\|_X \leq \sum_{n=1}^{\infty} \left[(\frac{\epsilon_{n-1}}{2^n})^{\frac{-3}{2}} \cdot \sigma_{n-1}\|y\|_X\right] \leq \frac{1}{16c^2}\|y\|_X.$$

So letting $y_I = \sum_{n\in I} y_n$, we have shown that

$$\|y_I\|_X \leq \frac{1}{16c^2}\|y\|_X.$$

For any $y \in Y$, recall that $y = \sum_{i=1}^{\infty} y_i$, where $y_i \in X_i$. Then define a map $\psi : Y \to Z$ by $\psi(y) = (y_1, y_2, y_3, \ldots)$.

Noting that $\|\psi(y_n)\|_Z = \|y_n\|_X$, one can use the same argument to show that

$$\|\psi(y_I)\|_Z \leq \frac{1}{16c^2}\|y\|_X$$

as well.

We will now show that ψ is an isomorphism onto its image. Fix $y \in Y$.

By Theorem 4.4, for each $i \in J$, we can choose a set Λ_i such that

$$\mu(\Lambda_i) \leq \frac{\epsilon_{i-1}}{2^i}.$$

and

$$\frac{\|y_i \cdot \Lambda_i\|_X}{\|y_i\|_X} \geq 1 - \frac{\epsilon_{i-1}}{2^i}. \tag{6.11}$$

(If $i \notin J$, we let Λ_i be the empty set.)

For the sake of convenience, we now introduce a plethora of notation.

First define sets A_i as follows:

$$A_i = \Lambda_i \setminus \bigcup_{j=1}^{\infty} \Lambda_{i+2j}.$$

The set of all A_i with i odd (resp. even) is then pairwise disjoint.

We next define y_n^C to agree with y_n on the small set Λ_n where the support of y_n is concentrated.

$$\begin{aligned} y_n^C &= y_n \cdot 1_{\Lambda_n} \\ y_n^R &= y - y_n^C \\ y^C &= \sum_{n=1}^{\infty} y_n^C \\ y^R &= \sum_{n=1}^{\infty} y_n^R \end{aligned}$$

We now define y_E, then define y_E^C and y_E^R in a natural way.

$$y_E = \sum_{2n \in J} y_{2n}$$

$$y_E^C = \sum_{2n \in J} y_{2n}^C$$

$$y_E^R = \sum_{2n \in J} y_{2n}^R$$

Thus we have

$$y_E = y_E^C + y_E^R$$

We define functions y_n^D as follows:

$$y_n^D = y_n \cdot 1_{A_n}$$

The functions y_n^D are disjoint in the sense that all y_n with even (respectively odd) subscripts are pairwise disjointly supported.

Naturally, we define y_E^D as

$$y_E^D = \sum_{2n \in J} y_{2n} \cdot 1_{A_{2n}}.$$

Now

$$\frac{\|y\|_X - \|y^R\|_X}{\|y\|_X} \leq \frac{\|y^C\|_X}{\|y\|_X} \leq \frac{\|y\|_X + \|y^R\|_X}{\|y\|_X}$$

and

$$\begin{aligned}
\|y_E^R\|_X &= \|\sum_{2n \in J} y_{2n}^R\|_X && (6.12)\\
&\leq \sum_{2n \in J} \|y_{2n}^R\|_X && (6.13)\\
&\leq \sum_{n=1}^{\infty} \frac{\epsilon_{2n-1}}{2^{2n}} \|y_{2n}\|_X \qquad \text{by (6.11)} && (6.14)\\
&\leq k \cdot \sup_{n \in J} \|y_{2n}\|_X && (6.15)\\
&\leq c \cdot k \cdot \|y_E\|_X && (6.16)
\end{aligned}$$

Thus

$$1 - ck \leq \frac{\|y_E^C\|}{\|y_E\|} \leq 1 + ck, \tag{6.17}$$

(That is,

$$1 - ck \leq \frac{\|\sum_{2n \in J} y_{2n} \cdot 1_{\Lambda_{2n}}\|}{\|\sum_{2n \in J} y_{2n}\|} \leq 1 + ck.)$$

Now let us define

$$S_n = \Lambda_n \backslash A_n = \Lambda_n \cap (\Lambda_{n+2} \cup \Lambda_{n+4} \cup \ldots).$$

Then

$$\left| \| \sum_{2n \in J} y_{2n} \cdot 1_{\Lambda_{2n}} \|_X - \| \sum_{2n \in J} y_{2n} \cdot 1_{A_{2n}} \|_X \right| \leq \| \sum_{2n \in J} y_{2n} \cdot 1_{S_{2n}} \|_X .$$

Since

$$\mu(S_{2n}) \leq \sum_{j=n+1}^{\infty} \mu(A_{2j}) \leq \sum_{j=n+1}^{\infty} \frac{\epsilon_{2j-1}}{2^{2j}} \leq \epsilon_{2n+1},$$

we then have, by (1), that

$$\frac{\|y_{2n} \cdot 1_{S_{2n}}\|_X}{\|y_{2n}\|_X} \leq \epsilon_{2n}.$$

Then

$$\begin{aligned} \| \sum_{2n \in J} y_{2n} \cdot 1_{S_{2n}} \|_X &\leq \sum_{2n \in J} \|y_{2n} \cdot 1_{S_{2n}}\|_X \\ &\leq \sum_{2n \in J} \epsilon_{2n} \|y_{2n}\|_X \\ &\leq c \cdot k \cdot \| \sum_{2n \in J} y_{2n} \|_X \\ &= ck \|y_E\|_X \end{aligned}$$

Combining the above equations with equation (6.17), then,

$$1 - 2ck \leq \frac{\| \sum_{2n \in J} y_{2n} \cdot 1_{A_{2n}} \|_X}{\| \sum_{2n \in J} y_{2n} \|_X} \leq 1 + 2ck,$$

in other words,

$$1 - 2ck \leq \frac{\|y_E^D\|_X}{\|y_E\|_X} \leq 1 + 2ck.$$

At this time it will be necessary for technical reasons to define two new functions Φ_1 and Φ_2 with the property that $\Phi_2 \circ \Phi_1 = \Psi$ on the subspace Y. These functions have the advantage that they will enable us to work more easily with restriction operators of the form $f \longmapsto f \cdot 1_A$.

Let $x \in X$ with $x = \sum_{i=1}^{\infty} x_i$, where $x_i \in X_i$. Define $\Phi_1 : X \to X(l_2)$ by

$$\Phi_1(\sum_{i=1}^{\infty} x_i) = (x_1, x_2, x_3, \ldots).$$

It is well known that this map is continuous ([LT],p.172). We then define $\Phi_2 :$

$\Phi_1(X) \to Z$ by

$$\Phi_2(x_1, x_2, x_3, \ldots) = (x_1, x_2, x_3, \ldots).$$

Formally we can identify Z with its image in $\tilde{X}$ and extend Φ_1 from $X(l_2)$ to $\tilde{X}$.

Observe that if only one of the x_i in the expansion $x = \sum_{i=1}^{\infty} x_i$ is nonzero, then the situation is very simple. In particular, we have in this case that

$$\|\Phi_2(\Phi_1(x))\|_Z = \|\Phi_1(x)\|_{X(l_2)} = \|x\|_X.$$

We now return to the main estimation argument. By the triangle inequality

$$\|\Phi_2(\sum_{2n \in J} \Phi_1(y_{2n}) \cdot 1_{A_{2n}})\|_{\tilde{X}} - \|\Phi_2(\sum_{2n \in J} \Phi_1(y_{2n}) \cdot 1_{A_{2n}^{-1}})\|_{\tilde{X}} \leq \|\Phi_2(\Phi_1(\sum_{2n \in J} y_{2n}))\|_Z$$

$$\leq \|\Phi_2(\sum_{2n\in J} \Phi_1(y_{2n}) \cdot 1_{A_{2n}})\|_{\tilde{X}} + \|\Phi_2(\sum_{2n\in J} \Phi_1(y_{2n}) \cdot 1_{A_{2n}^{-1}})\|_{\tilde{X}} \tag{6.18}$$

We implicitly showed above in equations 6.13 to 6.16 that

$$\sum_{2n\in J} \|\Phi_2(\Phi_1(y_{2n}) \cdot 1_{A_{2n}^{-1}})\|_Z \leq 2ck\|y_E\|_X \tag{6.19}$$

Of course, since the sets (A_{2n}) are disjoint,

$$\|\Phi_2(\sum_{2n\in J} \Phi_1(y_{2n}) \cdot 1_{A_{2n}})\|_{\tilde{X}} = \|\sum_{2n\in J} y_{2n} \cdot 1_{A_{2n}}\|_X$$

We showed that

$$(1-2ck)\|y_E\|_X \leq \|y_E^D\|_X \leq (1+2ck)\|y_E\|_X. \tag{6.20}$$

(6.18) and (6.19) give us

$$\|\Phi_2(\sum_{2n\in J} \Phi_1(y_{2n}) \cdot 1_{A_{2n}})\|_{\tilde{X}} - 2ck\|y_E\|_X \leq \|\psi y_E\|_Z$$

$$\leq \|\Phi_2(\sum_{2n\in J} \Phi_1(y_{2n}) \cdot 1_{A_{2n}})\|_{\tilde{X}} + 2ck\|y_E\|_X. \tag{6.21}$$

Thus

$$(1-4ck)\|y_E\|_X \leq \|\psi(y_E)\|_Z \leq (1+4ck)\|y_E\|_X. \tag{6.22}$$

Let us define y_O analogously to the way we defined y_E. Let

$$y_O = \sum_{2n-1\in J} y_{2n-1}.$$

The same arguments which led to equation 6.22, then, would give us that

$$(1-4ck)\|y_O\|_X \leq \|\psi(y_O)\|_Z \leq (1+4ck)\|y_O\|_X. \tag{6.23}$$

By the triangle inequality and the unconditionality of the Haar system

$$\frac{\|y_E\|_X + \|y_O\|_X}{2c} \leq \|y_E + y_O\|_X \leq \|y_E\|_X + \|y_O\|_X \tag{6.24}$$

Since the images of $\psi(y_E)$ and $\psi(y_O)$ are disjointly supported when considered as functions on $[0, \infty)$,

$$\frac{\|\psi(y_E)\|_Z + \|\psi(y_O)\|_Z}{2} \leq \|\psi(y_E + y_O)\|_Z \leq \|\psi(y_E)\|_Z + \|\psi(y_O)\|_Z \tag{6.25}$$

From (6.22),(6.23) and (6.25) above, we get

$$\frac{1-4ck}{2}(\|y_E\|_X + \|y_O\|_X) \leq \|\psi(y_E + y_O)\|_Z \leq (1+4ck)(\|y_E\|_X + \|y_O\|_X). \tag{6.26}$$

By (6.24)

$$\frac{1-4ck}{4c}(\|y_E + y_O\|_X) \leq \|\psi(y_E + y_O)\|_Z \leq (1+4ck)(\|y_E + y_O\|_X). \tag{6.27}$$

We have already shown that

$$\|y_I\|_X \leq \frac{1}{16c^2}\|y\|_X. \tag{6.28}$$

and also that

$$\|\psi(y_I)\|_Z \leq \frac{1}{16c^2}\|y\|_X. \tag{6.29}$$

Noting that

$$y = y_O + y_E + y_I,$$

we can easily get from equation (6.28) that

$$\frac{3}{4}\|y\|_X \leq \|y_E + y_O\|_X \leq \frac{5}{4}\|y\|_X. \tag{6.30}$$

By equations (6.27) and (6.30),

$$\frac{3(1-4ck)}{16c}\|y\|_X \leq \|y_E + y_O\|_Z \leq (1+4ck)\frac{5}{4}\|y\|_X \tag{6.31}$$

Of course

$$\|\psi(y_E + y_O)\|_Z - \|\psi(y_I)\|_Z \leq \|\psi(y)\|_Z \leq \|\psi(y_E + y_O)\|_Z + \|\psi(y_I)\|_Z. \tag{6.32}$$

Combining equations (6.31), (6.32), and (6.29), we get

$$\left[\frac{3(1-4ck)}{16c} - \frac{1}{16c^2}\right](\|y\|_X) \leq \|\psi(y)\|_Z \leq \left[\frac{5(1+4ck)}{4} + \frac{1}{16c^2}\right](\|y\|_X). \tag{6.33}$$

Since $k = \frac{1}{16c}$ and $c < c^2$, this means

$$\frac{1}{32c^2}\|y\|_X \leq \|\psi(y)\|_Z \leq \frac{15}{8}\|y\|_X. \tag{6.34}$$

Finally, we conclude this paper by demonstrating that Theorem 3.5 is a special case of Theorem 6.2. More precisely, if we let $\tilde{X} = L_p[0,\infty)(2 < p < \infty)$ and $X = L_p[0,1]$, in Theorem 6.2 (the main theorem of this paper), we obtain Theorem 3.5 (Johnson and Odell's main theorem) . This fact can be proven easily with the aid of the following well known lemma.

Lemma 6.3 *Suppose $(X_n)_{n=1}^{\infty}$ are finite dimensional subspaces of X and there are projections $P_n : L_p \to X_n$ and a constant c such that $\|P_n\| \leq c$ for all n. Then $(\sum \oplus X_n)_{l_p}$ is isomorphic to l_p.*

Proof:

We first choose, for each n, some dyadic $\sigma-$ algebra $\mathcal{D}_{k_n}$ such that

$$\|E(x|\mathcal{D}_{k_n}) - x\|_p \leq \frac{1}{2(c+1)}\|x\|_p,$$

where $E(x|\mathcal{D}_{k_n})$ is the conditional expectation operator. For simplicity we write $E_n(x)$ for $E(x|\mathcal{D}_{k_n})$.

Let $\tilde{X_n} = P_n(L_p)$. Let $x \in X_n$, and let $f = E_n(x)$.

Now

$$\begin{aligned}
\|P_n(f) - f\|_p &= \|P_n(E_n(x)) - E_n(x)\|_p \\
&\leq \|P_n(E_n(x)) - P_n(x)\|_p + \|P_n(x) - E_n(x)\|_p \\
&\leq \|P_n(E_n(x)) - x\|_p + \|x - E_n(x)\|_p \\
&\leq (\|P_n\| + 1) \cdot \|x - E_n(x)\|_p
\end{aligned}$$

$$
\begin{aligned}
&\leq (c+1)\cdot\frac{\|x\|_p}{2(c+1)} \\
&= \frac{1}{2}\|x\|_p \\
&\leq \frac{c+1}{2c+1}\|f\|_p
\end{aligned}
$$

Let R_n be the restriction of P_n to $\tilde{X}_n$. Then we can see that R_n is an isomorphism from $\tilde{X}_n$ onto X_n.

Define $Q_n : L_p(\mathcal{D}_{k_n}) \rightarrow \tilde{X}_n$ by $Q_n = R_n^{-1} \circ P_n$, and observe that Q_n is a projection.

Now define

$$Q : (\sum_{n=1}^{\infty} \oplus L_p(\mathcal{D}_{k_n}))_{l_p} \rightarrow (\sum_{n=1}^{\infty} \oplus \tilde{X}_n)_{l_p}$$

by

$$Q(x_1, x_2, x_3, \ldots) = (Q_1(x_1), Q_2(x_2), Q_3(x_3), \ldots).$$

Now since $(\sum_{n=1}^{\infty} \oplus L_p(\mathcal{D}_{k_n}))_{l_p}$ is isomorphic to l_p, we know that $(\sum_{n=1}^{\infty} \oplus \tilde{X}_n)_{l_p}$ is a complemented subspace of l_p. By a famous theorem of Pelczynski, $(\sum_{n=1}^{\infty} \oplus \tilde{X}_n)_{l_p}$ is isomorphic to l_p, and hence so is $(\sum_{n=1}^{\infty} \oplus X_n)_{l_p}$ as well.

Theorem 6.4 *Suppose $X = L_p[0,1](2 < p < \infty)$. If Y is a subspace of X such that no subspace of Y is isomorphic to l_2, then Y is isomorphic to a subspace of l_p.*

Proof:

We apply Theorem 6.2. with $\tilde{X} = L_p[0,\infty)(2 < p < \infty)$ and $X = L_p[0,1]$.

We first demonstrate that $X = L_p$ satisfies the hypotheses. First, we know that $2 < p_X = q_X = p < \infty$. Theorem 2.1 says that the Haar system is an unconditional basis for L_p. It is well known that L_p has an upper l_2 estimate (cf. [LT],p.73). Since $L_p[0,1]$ is reflexive and l_1 is not, it is clear that $L_p[0,1]$ cannot contain a copy of l_1.

Now let us check that the conclusion of Johnson and Odell's theorem is satisfied, that is, let us verify that Y embeds in l_p. Theorem 6.2 tells us in general that Y embeds in $\sum \oplus X_n$. If $\tilde{X} = L_p[0,\infty)$, then this means that Y embeds in $(\sum \oplus X_n)_{l_p}$. If $P_n : L_p \to X_n$ are the basis projections, then $\|P_n\| \leq c$, where c is the suppression unconditional basis constant of X. By Lemma 6.3,then, we have that $(\sum \oplus X_n)_{l_p}$ is isomorphic to l_p, which concludes the proof.

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V

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