# RANKING SMALL PROGRESSIVELY CENSORED SAMPLES TO ESTIMATE EXPERIMENT TIME AND TEST FOR GOODNESS-OF-FIT

By

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# CHAPTER ONE

## INTRODUCTION

*Censoring* with respect to lifetime data refers to samples where exact lifetimes of some items in the sample are not fully measured. A censored lifetime $x$ in the sample is *right censored* if all that is known is $x \geq T$, for a time $T$. A censored lifetime $x$ in the sample is *left censored* if all that is known is $x \leq T$. A sample is *singly right censored* if only the largest lifetimes are censored. A sample is *singly left censored* if only the smallest lifetimes are censored. *Truncated* samples differ from censored samples in that the population values are restricted prior to sampling, but the sample itself is fully measured. For example, a singly left truncated sample occurs when all sample items are fully measured, but the population from which the sample is taken consists of only items with lifetimes greater than a time $T$, a point of truncation.

A common feature in life testing experiments is the occurrence of singly right censored samples due to termination of the experiment prior to failure of all items in the sample. For example, Nelson (1982) describes a life test of a sample of twelve Type B7 electric cords. The twelve cords are flexed using a machine to simulate actual use, though at an accelerated rate. The lifetimes for the first nine electric cords to fail are recorded. The ninth failure occurs at 141.9 hours. Testing is then discontinued at 164.1 hours. Three of the twelve electric cords in the sample did not fail prior to the termination of the test. These three censored lifetimes are known only to exceed 164.1 hours and are described as *time censored*.

Singly right censored samples also occur when live items are withdrawn at a time corresponding to the failure time of the last fully measured item in the sample. An example in Mann and Fertig (1973) describes a life test where a sample of thirteen airplane components are put on test. The lifetimes of the first ten airplane components to fail are recorded. The three remaining airplane components, though still functioning, are removed from test at the time of the tenth failure. These three censored lifetimes are known only to exceed the tenth failure time and are described as *failure censored.*

Formally in the statistical literature, the preceding two types of censoring are considered as Type I and Type II censoring. A sample is classified as *Type I* (time censored) if observations are removed from the life test at pre-specified times or *Type II* (failure censored) if items are removed corresponding to failure times of other fully measured items in the study. For Type I censoring, the time T is fixed in advance, and the number of failed items $m$ is a random variable; for Type II censoring, T is a random variable, and $m$ is fixed. Nelson (1982) comments, "Time censoring is more common in practice; failure censoring is more common in the literature as it is mathematically more tractable."

Censored data also occurs when live items, in addition to failed items, are removed at several stages during the course of the life test. Nelson (1982) describes an example where nineteen items consisting of insulating fluid are tested at 34 kilovolts until breakdown in an accelerated test. Of the nineteen sampled, only eight are fully measured. The other eleven lifetimes in the sample are censored in the following stages: three are removed at the third failure, three are removed at the fifth failure, and five have not failed when the experiment ends at the eighth failure. Such an experiment is considered to be *multiply* or *progressively censored* in that live items are eliminated at various stages of an experiment from further observation, and the sample items remaining after each stage of censoring continue until failure or until a subsequent stage of censoring.

Formally, a sample of size $n$ is progressively censored if $r_1, r_2, \ldots, r_m$ number of items in a sample are removed from further observation at points $T_1 < T_2 < \ldots < T_m$ ($m < n$), and for the $r_j$ items removed at time $T_j$, the lifetime of the censored items are known only to have lifetimes greater than $T_j$. Consequently, the censoring "progresses" through "multiple" stages. Other names for progressively censored samples include "hyper" or "multi-" censored (Cohen, 1991).

In life testing experiments where the experimenter wishes to reduce the experiment time in which to observe $m$ failure times, a censored sampling plan rather than a complete sampling plan may be used. That is, the experimenter may shorten the expected experiment time by sampling $n$ ( $n > m$ ) items and stopping after $m$ failures are observed rather than sampling $m$ items and completely measuring all $m$

lifetimes. The smallest expected experiment time is achieved by censoring the $n$-$m$ largest lifetimes. This is single right censoring and is also a special case of progressive censoring where $r_1 = r_2 = \ldots = r_{m-1} = 0$ and $r_m = n$-$m$. Any progressive censoring plan, though, will result in smaller expected experiment times than complete sampling plans. Hence, progressive censoring plans may be considered as a compromise between complete sampling and single right censored sampling to achieve a shortened life test.

Epstein and Sobel (1953) contend that the only justification for a single right censoring procedure over a complete sampling procedure is to save time. Thomas and Wilson (1972) comment that progressive censoring not only saves time but also permits the experimenter the flexibility to examine live items in addition to failed items and still allows some of the more extreme lifetimes to remain in the sample. As discussed by Balasooriya and Saw (1998) , this multi-stage censoring allows the experimenter to save expensive test specimens or resources and release them for other use. The progressive censoring procedure is considered as attractive to the experimenter where life expectancies are high, rapid results are required, testing is expensive, or live items removed at various stages in the experiment may be informative, conserved, or redirected for other use.

One concern about using progressively censored life testing is that the parameter estimators obtained may be less precise than those obtained when using single right censoring plans. It is, however, the conjecture of Viveros and Balakrishnan (1994) that the loss in precision in many practical applications is not greater than that which occurs with ordinary Type II single right censored experiments. Two other concerns to the experimenter about progressive censoring plans which have been identified by Viveros and Balakrishnan (1994) are: (1) a lack of design guidance in selecting Type II progressive censoring schemes and (2) a belief that the statistical analysis is more complicated that traditional methods.

In the area of progressive censoring, much has been published about the derivation of parameter estimates. A smaller area of research extends to estimating experiment times. In life testing experiments, obtaining estimates of experiment time is useful when implementing a sampling plan.

Previously, Hsieh (1994) and Tse and Yuen (1998) are the only authors to offer numerical studies which provide design guidance in selecting Type II progressive censoring schemes, with respect to expected experiment time. Hsieh (1994) considered expected experiment time for Type II Weibull-distributed singly right censored lifetimes (a special case of progressive censoring), and Tse and Yuen (1998) considered expected experiment times for Type II Weibull-distributed progressively censored samples where removals are random and occur according to a uniform discrete probability distribution.

This research contains two new numerical studies of expected experiment time for Type II progressively censored samples. One is a numerical study for a special 50% fixed removal scheme, and the other is a numerical study for a random removal scheme if all removal schemes are equally likely. The numerical studies are generated by an alternative formula to the formula of Tse and Yuen (1998). The formula is validated by repeating the study of Tse and Yuen (1998).

The expected experiment time formulas used in this research are derived by a conditional procedure suggested by Thomas and Wilson (1972) for finding means, variances, and covariances of Type II progressively censored order statistics. In this instance, the conditional procedure is applied to only the largest Type II progressively censored order statistic , which is experiment time. This alternative formula is then adapted to finding moments of experiment time. By finding moments of experiment time, the standard deviations associated with the experiment times may also be investigated. There are no previous studies of the standard deviations associated with expected experiment time values. In this research, numerical studies of the standard deviations associated with experiment time accompany the numerical studies of expected experiment time.

The conditional procedure described by Wilson and Thomas (1972), which lead to the formula for finding moments of experiment time, further suggests a correlation-type test for goodness-of-fit test for Type II progressively censored samples. A description of the test and illustrations as to how the test may be used to investigate different distributional assumptions concerning progressively censored data is included in this research.

# CHAPTER TWO

# TERMINOLOGY AND LITERATURE REVIEW

In this chapter, terminology and definitions relating to experiment time for censored samples are provided. Comparisons among complete samples are then made to Type II singly right censored samples and progressively censored samples. A literature review is also included.

## 2.1 Terminology and Definitions

Let $m$ be the number of failures observed before the termination of a life test of $n$ items and $r_i$ denote the number of items removed at the time of the $i$th failure (Type II censoring). A *Type II progressively censored sample of size m* consists of $m$ observed ordered lifetimes (order statistics) $x_{1,n} \le x_{2,n} \le \ldots \le x_{m,n}$ from a complete sample of size $n$ where

$$r_m = n - \sum_{j=1}^{m-1}(r_j + 1) - 1,$$

$0 \le r_1 \le n-1$, and $0 \le r_i \le \sum_{j=1}^{m-1} (r_j + 1) - 1$ for $i = 2,3,\ldots,m-1$.

(Johnson, Kotz, and Balakrishnan, 1994)

If lifetimes, $X$, are continuously distributed with a cumulative distribution function (cdf) F $(x)$ and probability density function (pdf) f $(x)$, *the joint pdf of a Type II progressively censored sample*, $x_{1,n}, x_{2,n}, \ldots, x_{m,n}$, with corresponding censoring scheme $r_1, r_2, \ldots r_m$ is considered to be

$$f(x_{1,n}, x_{2,n}, \ldots x_{m,n}) = c \prod_{i=1}^{m} f(x_{i,n})\,[1 - F(x_{i,n})]^{r_i}, \quad 0 < x_{1,n} < \ldots < x_{m,n} < \infty\ .$$

where c is an ordering constant given by

$$c = n \prod_{j=2}^{m} (n - \sum_{i=1}^{j-1} r_i - j + 1)$$

$$= n(n - r_1 - 1)(n - r_1 - r_2 - 2)\ldots(n - \sum_{i=1}^{m-1}(r_i + 1))\ .$$

(Cohen, 1991)

The joint pdf of a *general* Type II progressively censored sample, differs slightly, and is given by

$$f(x_{r_o+1,n}, x_{2,n}, \ldots x_{m,n}) = c_o \left[F(x_{r_o+1})\right]^{r_o} \prod_{i=r_o+1}^{m} f(x_{i,n})\,[1 - F(x_{i,n})]^{r_i}$$

where $c_o = \binom{n}{r_o}(n - r_o) \prod_{j=2}^{m} (n - \sum_{i=1}^{j-1} r_i - j + 1)$

and $r_o$ is the number of removals withdrawn prior to the first observed failure time $x_{r_{o+1}}$.

(Balakrishnan and Sandhu, 1996)

The Type II progressive censoring as first defined differs from *general* Type II progressive censoring in that the general case allows for removals to occur prior to the first failure. Therefore, the Type II progressive censoring considered in this paper can be viewed as a special case of general Type II progressive censoring in which $r_o = 0$.

The joint pdf of a Type II progressively censored sample, $f(x_{1,n}, x_{2,n}, \ldots x_{m,n})$, is a legitimate pdf as $f(x_{1,n}, x_{2,n}, \ldots x_{m,n}) \geq 0$ for every, $0 < x_{1,n} < \ldots < x_{m,n} < \infty$ and

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_{1,n}, x_{2,n}, \ldots x_{m,n})\ dx_{1,n} \ldots dx_{m-1,n}\ d\,x_{m,n} = 1 \text{ since}$$

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_{1,n}, x_{2,n}, \ldots x_{m,n})\, dx_{1,n} \ldots dx_{m-1,n}\, d x_{m,n}$$

$$= \int_{0}^{\infty} \int_{x_{1,n}}^{\infty} \cdots \int_{x_{m-1,n}}^{\infty} c \prod_{i=1}^{m} f(x_{i,n}) [1 - F(x_{i,n})]^{r_i} \, dx_{1,n} \ldots dx_{m-1,n}\, d x_{m,n}$$

$$= c \int_{0}^{\infty} \int_{x_{1,n}}^{\infty} \cdots \int_{x_{m-2,n}}^{\infty} \left[ \int_{x_{m-1,n}}^{\infty} f(x_{m,n})[1 - F(x_{m,n})]^{r_m} dx_{m,n} \right]$$

$$\times \prod_{i=1}^{m-1} f(x_{i,n}) [1 - F(x_{i,n})]^{r_i} \, dx_{1,n} \ldots dx_{m-1,n}$$

$$= c \int_{0}^{\infty} \int_{x_{1,n}}^{\infty} \cdots \int_{x_{m-2,n}}^{\infty} \left[ - \frac{\left[1 - F(x_{m,n})\right]^{r_m + 1}}{r_m + 1} \Bigg|_{x_{m-1,n}}^{\infty} \right]$$

$$\times \prod_{i=1}^{m-1} f(x_{i,n}) [1 - F(x_{i,n})]^{r_i} \, dx_{1,n} \ldots dx_{m-1,n}$$

$$= c \int_{0}^{\infty} \int_{x_{1,n}}^{\infty} \cdots \int_{x_{m-2,n}}^{\infty} \frac{[1 - F(x_{m-1,n})]^{r_m + 1}}{r_m + 1}$$

$$\times \prod_{i=1}^{m-1} f(x_{i,n}) [1 - F(x_{i,n})]^{r_i} \, dx_{1,n} \ldots dx_{m-1,n} .$$

Substituting

$$r_m = n - \sum_{j=1}^{m-1} (r_j + 1) - 1$$

and continuing integration with respect to $x_{m-1,n}, \ldots, x_{1,n}$, the right hand side of the equation then becomes

$$\frac{c}{(n - \sum_{j=1}^{m-1}(r_j + 1))(n - \sum_{j=1}^{m-2}(r_j + 1)) \ldots (n - (r_1 + 1))n} = 1.$$

The joint pdf of the Type II progressively censored sample may also be expressed as

$$f(x_{1,n}, x_{2,n}, \ldots x_{m,n}) = c\prod_{i=1}^{m} f(x_{i,n}) \sum_{\ell_i=0}^{r_i} \binom{r_i}{\ell_i} (-1)^{\ell_i} [F(x_{i,n})]^{\ell_i},$$

$$0 < x_{1,n} < \ldots < x_{m,n} < \infty,$$

since

$$[1 - F(x_{i,n})]^{r_i} = \sum_{\ell_i=0}^{r_i} \binom{r_i}{\ell_i} (-1)^{\ell_i} [F(x_{i,n})]^{\ell_i}.$$

The *expected experiment time for a Type II progressively censored sample* is denoted $E[X_{m,n} \mid r_1, r_2, \ldots r_m]$ and is the expected time to observe *m* failures in a progressively censored sample of size *n* with the censoring removals $r_1, r_2, \ldots r_m$. The *expected experiment time for a Type II singly right censored sample* is denoted $E[X_{m,n}]$ and is a special case of Type II progressive censoring where $r_1 = r_2 = \ldots = r_{m-1} = 0$ and $r_m = n - m$. The *expected experiment time for a complete sample* of *size m*, is denoted $E[X_{m,m}]$, or simply $E[X_m]$.

The *ratio of expected experiment times* (REET) is the ratio of expected experiment time under a censoring plan to the expected experiment time under a complete sampling plan. For the Type II singly right censoring plan,

$$\text{REET} = \frac{E[X_{m,n}]}{E[X_{m,m}]}.$$

(Hsieh, 1994)

For a Type II progressive censoring plan with removals $r_1, r_2, \ldots r_m$,

$$\text{REET} = \frac{E[X_{m,n} \mid r_1, r_2, \ldots, r_m]}{E[X_{m,m}]}.$$

The REET values provide information as to how much experiment time may be saved if a particular censoring plan is employed compared to a complete sampling plan.

Let $X_{m,n}$ be the *m*th order statistic in a sample of size *n* from a distribution with pdf $f(x)$. Then, the *pdf of the largest order statistic* $X_{m,n}$ is given by

$$g\ (x_{m,n}) = n\binom{n-1}{m-1}[F(x_{m,n})]^{m-1}[1-F(x_{m,n})]^{n-m}\, f\,(x_{m,n}).$$

(Serfling, 1980)

The pdf of the largest order statistic $X_{m,n}$ is derived by taking the derivative of the cdf.

Let G be the cdf of the largest order statistic $X_{m,n}$ , then

$$G\,(x_{m,n}) = P(\text{ at least } m \text{ of the } X_i \text{ are less than or equal to } x_{m,n})$$

$$= \sum_{i=m}^{n}\binom{n}{i}[F(x_{m,n})]^{i}[1-F(x_{m,n})]^{n-i}\ ,$$

(David, 1981, 2.1.3)

and the pdf is

$$g(x_{m,n}) = \frac{\partial G(x_{m,n})}{\partial x_{m,n}} = \sum_{i=m}^{n}\binom{n}{i}i[F(x_{m,n})]^{i-1}[1-F(x_{m,n})]^{n-i}\ \ f\,(x_{m,n})$$

$$- \sum_{i=m}^{n}\binom{n}{i}(n-i)[F(x_{m,n})]^{i}[1-F(x_{m,n})]^{n-i-1}\ \ f\,(x_{m,n})$$

$$= \binom{n}{m}m[F(x_{m,n})]^{m-1}[1-F(x_{m,n})]^{n-m}\ \ f\,(x_{m,n})$$

$$+ \sum_{i=m+1}^{n}\binom{n}{i}i[F(x_{m,n})]^{i-1}[1-F(x_{m,n})]^{n-i}\ \ f\,(x_{m,n})$$

$$- \sum_{i=m}^{n-1}\binom{n}{i}(n-i)[F(x_{m,n})]^{i}[1-F(x_{m,n})]^{n-i-1}\ \ f\,(x_{m,n})$$

$$- \binom{n}{n}(n-n)[F(x_{m,n})]^{n}[1-F(x_{m,n})]^{n-n-1}\ \ f\,(x_{m,n})$$

$$= \binom{n}{m}m[F(x_{m,n})]^{m-1}[1-F(x_{m,n})]^{n-m}\ \ f\,(x_{m,n}).$$

For any continuous distribution, *the expected experiment time under Type II single right censoring* is

$$E[X_{m,n}] = \int_{-\infty}^{\infty} x_{m,n}\, g\ (x_{m,n})\ \ dx_{m,n}$$

$$= \int_{-\infty}^{\infty} x_{m,n} \; n\binom{n-1}{m-1}[F(x_{m,n})]^{m-1}[1-F(x_{m,n})]^{n-m}\, dF(x_{m,n})$$

$$= \text{n} \binom{n-1}{m-1}\int_0^1 F^{-1}(u)\;\; u^{m-1}\;\; (1-u)^{n-m}\;\; du \;\;,\; \text{ where } \text{u} = \text{F}\,(\,x_{m,n}\;)\,.$$

(David, 1981, 3.1.1)

The expected experiment time under Type II singly right censoring may also be expressed as a function of a largest order statistic $E[X_{i,i}]$, $i = m,...,n$,

$$\text{E}[\,X_{m,n}\,] \;=\; \sum_{i=m}^{n}\binom{i-1}{m-1}\binom{n}{i}(-1)^{i-m}\,E[X_{i,i}]\,,$$

(David, 1981, 3.4.3)

or as a function of a smallest order statistic $E[X_{1,j+n-m+1}]$, $j = 0,...m-1$,

$$\text{E}[\,X_{m,n}\,] = n\binom{n-1}{m-1}\sum_{j=0}^{m-1}\binom{m-1}{j}(-1)^j\,\frac{1}{(j+n-m+1)}\,E[X_{1,j+n-m+1}]\,.$$

For a continuous lifetime distribution with $F(0) = 0$ and removals $r_1, r_2,...,r_m$ the *expected experiment time under Type II progressive censoring* is given by

$$\text{E}[\,X_{m,n}\,|\,r_1, r_2,...,r_m,] \;=\; \int_0^{\infty} x_{m,n} \int_0^{x_{m,n}} \cdots \int_0^{x_{2,n}} \text{f}\,(\,x_{1,n},...,x_{m,n}\,)\;\, dx_{1,n}...dx_{m-1,n}\;\, \text{d}\,x_{m,n}\,,$$

$$0 < x_{1,n} < ... < x_{m,n} < \infty\,,$$

$$= \int_0^{\infty} x_{m,n} \int_0^{x_{m,n}} \cdots \int_0^{x_{2,n}} \text{c}\prod_{i=1}^{m}\;\text{f}\,(\,x_i\,)\;\sum_{\ell_i=0}^{r_i}\binom{r_i}{\ell_i}(-1)^{\ell_i}\,[\text{F}(x_i\,)]^{\ell_i}$$

$$dx_{1,n}...dx_{m-1,n}\,\text{d}\,x_{m,n}\,;$$

which becomes

$$\text{E}[\,X_{m,n}\,|\,r_1, r_2,...,r_m\,] = \text{c}\sum_{\ell_1=0}^{r_1}\;...\sum_{\ell_m=0}^{r_m}(-1)^{\ell_1+...+\ell_m}\,\frac{\binom{r_1}{\ell_1}...\binom{r_m}{\ell_m}}{\prod_{i=1}^{m-1}(\ell_1+...+\ell_i+i)}\,.$$

$$\times \int_0^{\infty} x_{m,n}\, f\,(x_{m,n})[F(x_{m,n})]^{\ell_1+\ldots+\ell_m+m-1}\, dx_{m,n}\,.$$

The last integral is a multiple of the expected value of the largest order statistic for a sample of size $\ell_1+\ldots+\ell_m+m$; hence, the formula for the expected experiment time can be represented as

$$\mathrm{E}[\,X_{m,n}\,|\,r_1,r_2,\ldots,r_m\,] \qquad = c \sum_{\ell_1=0}^{r_1} \ldots \sum_{\ell_m=0}^{r_m} (-1)^{\ell_1+\ldots+\ell_m} \frac{\binom{r_1}{\ell_1}\cdots\binom{r_m}{\ell_m}}{\prod_{i=1}^{m-1}(\ell_1+\ldots+\ell_i+i)}$$

$$\times \frac{E[X_{\ell_1+\ldots+\ell_m+m,\ \ell_1+\ldots+\ell_m+m}]}{\ell_1+\ldots+\ell_m+m},$$

or, alternatively, as

$$\mathrm{E}[\,X_{m,n}\,|\,r_1,r_2,\ldots,r_m\,] = c \sum_{\ell_1=0}^{r_1} \ldots \sum_{\ell_m=0}^{r_m} (-1)^{\ell_1+\ldots+\ell_m} \frac{\binom{r_1}{\ell_1}\cdots\binom{r_m}{\ell_m}}{\prod_{i=1}^{m-1}(\ell_1+\ldots+\ell_i+i)}$$

$$\times \sum_{k=0}^{\ell_1+\ldots+\ell_m+m-1} (-1)^k \binom{\ell_1+\ldots+\ell_m+m-1}{k} \frac{E[X_{1,k+1}]}{(k+1)}\,.$$

In life testing, if $n$ items are placed under observation and $n$ failure times are observed, the life test is referred to as a *complete sample* test (Lemon, 1975). A method to obtain means, variances, and covariances of Type II progressively censored order statistics by using the complete sample rankings of the progressively censored sample is described in an appendix of an article by Thomas and Wilson (1972).

Let $x_{1,n} \le x_{2,n} \le \ldots \le x_{m,n}$ be the $m$ ordered observations in a censored sample of size $n$, and $z_{1,n} \le z_{2,n} \le \ldots \le z_{n,n}$ be the $n$ ordered observations in the same sample were all $n$ observations fully measured. Let $R_i$, $i=1,2,\ldots,m$, denote the ranks in the complete sample of the ith ordered observations in the censored sample, i.e., $x_{i,n} = z_{R_i,n}$. The *complete sample ranks of the Type II progressively censored sample* $x_{1,n} \le x_{2,n} \le \ldots \le x_{m,n}$ are these values $R_1, R_2,\ldots,R_m$. For a Type II progressively censored sample, the complete sample rank of the first ordered observation $x_{1,n}$ is known as $x_{1,n} = z_{1,n}$, that is, $R_i = 1$. But, the complete sample ranks of $x_{2,n} \le x_{3,n} \le \ldots \le x_{m,n}$ are not necessarily known.

The different complete sample rankings $\boldsymbol{R} = (R_1, R_2, \ldots, R_m)$ of the progressively censored order statistics can be described using the recursive equation

$$R_i = R_{i-1} + 1,\ R_{i-1} + 2, \ldots, i + r_1 + r_2 + \ldots + r_{i-1}$$

for $i = 2, \ldots, m$ with $R_1 = 1$.

(Thomas and Wilson, 1972, A1)

For fixed removals $r_1, r_2, \ldots, r_m$, the *probability function for complete sample ranks* may be found by letting

$$p = P(R_1, R_2, \ldots, R_m) = P(R_1) \prod_{i=2}^{m} P(R_i \mid R_1, \ldots R_{i-1})$$

(Thomas and Wilson, 1972, A2)

where $P(R_1 = 1) = 1$ and $P(R_i \mid, R_1, \ldots, R_{i-1})$ is the probability function

$$P(R_i \mid, R_1, \ldots, R_{i-1}) = \frac{\begin{pmatrix} n - R_i \\ \sum_{j=1}^{i-1} (r_j + 1) - R_i + 1 \end{pmatrix}}{\begin{pmatrix} n - R_{i-1} \\ \sum_{j=1}^{i-1} (r_j + 1) - R_{i-1} \end{pmatrix}}, i = 2, \ldots, m.$$

(Thomas and Wilson, 1972, A3)

If $\boldsymbol{X} = (X_{1,n}, \ldots, X_{m,n})'$ is the vector of the Type II progressively censored order statistics, $\boldsymbol{Z} = (Z_{1,n}, \ldots, Z_{n,n})'$ is the vector of the corresponding complete sample order statistics, $\boldsymbol{R} = (R_1, R_2, \ldots, R_m)$ is the vector complete sample rankings of the Type II progressively censored ordered statistics, and $\boldsymbol{D}$ is a matrix defined by

$$d_{ij} = \begin{cases} 1, & j = R_i \\ 0, & j \ne R_i \end{cases} \quad j = 1, \ldots, n, \quad i = 1, \ldots, m,$$

then $\boldsymbol{X} = \boldsymbol{DZ}$.

The *means of Type II progressively censored order statistics* may be written in the form

$$\mu_x = \mathrm{E}[\boldsymbol{X}] = \mathrm{E}_\ell \, \mathrm{E}_{\mathbf{D}_\ell} [\boldsymbol{D}_\ell \ \ \boldsymbol{Z} \mid \boldsymbol{D}_\ell \ ] = E_\ell [\boldsymbol{D}_\ell{}' \ \mu_Z] = \left( \sum_{\ell=1}^{M} D_\ell p_\ell \right) \mu_Z$$

where $M$ denotes the number of different possible vectors of complete sample rankings, $p_1, p_2, \ldots, p_M$ are the probabilities of the $M$ possible complete sample ranking schemes, and $\boldsymbol{D}_\ell$ is defined as

$$d_{ij}^{\ell} = \begin{cases} 1, & j = R_i \\ 0, & j \neq R_i \end{cases}$$

for $i = 1,2,\ldots,m$, $j = 1,2,\ldots n$, and $\ell = 1,2,\ldots M$.

(Thomas and Wilson, 1972, A7)

The *variances and covariances of Type II progressively censored order statistics* are obtained similarly as

$$\begin{aligned} \Sigma_x &= \mathrm{E}[\mathbf{X}\mathbf{X}'] - \mu_x \mu_x{}' \\ &= E_\ell E_{\mathbf{D}_\ell} [\boldsymbol{D}_\ell \ \ \boldsymbol{Z}\boldsymbol{Z}' \ \boldsymbol{D}_\ell{}' \mid \boldsymbol{D}_\ell] - \mu_x \mu_x{}' \\ &= E_\ell E_{\mathbf{D}_\ell} [[\boldsymbol{D}_\ell \ \ (\Sigma_z + \mu_z \mu_z{}') \boldsymbol{D}_\ell{}' \mid \boldsymbol{D}_\ell] - \mu_x \mu_x{}' \\ &= \sum_{\ell=1}^{M} \boldsymbol{D}_\ell \ \ (\Sigma_z + \mu_z \mu_z{}') \ \boldsymbol{D}_\ell{}' p_\ell - \mu_x \mu_x{}'. \end{aligned}$$

(Thomas and Wilson, 1972, A8)

Viveros and Balakrishnan (1994) describe existing methods of calculating means, variances, and covariances of the progressively censored order statistics as “cumbersome”.

## 2.2 Lifetime Distributions

A parametric approach to analyzing Type II progressively censored data is to consider lifetime distributions for F(x). Possible parametric models for lifetime distributions are discussed below. Each has the common property of known expectations of first order statistics. Additionally, inverses of their distribution functions exist in simple closed forms making the distributions especially applicable for simulation studies.

### Weibull Distribution

A common model for lifetime distributions is the *three parameter Weibull* lifetime distribution. The distribution was originally used to model fatigue data and is widely used when the "weakest link" model is most appropriate, i.e., when an item experiences failure when any of its component parts fail (Nelson, 1982).

If $X$ is a random variable with a three parameter Weibull lifetime distribution $X \sim$ Weibull ($\gamma$, $\theta$, $\beta$) where $\gamma$ is the threshold parameter, $\theta$ is the scale parameter, sometimes referred to as the characteristic life, and $\beta$ is the shape parameter, then the cdf of $X$ is given by

$$F(x;\gamma,\theta,\beta) = 1-\exp\left\{-\left[\frac{x-\gamma}{\theta}\right]^{\beta}\right\}, \quad 0<\gamma \le x < \infty,\ \theta > 0,\ \beta > 0,$$

and the pdf of $X$ is given by

$$f(x;\gamma,\theta,\beta) = \frac{\beta}{\theta^{\beta}}(x-\gamma)^{\beta-1}\exp\left\{-\left[\frac{x-\gamma}{\theta}\right]^{\beta}\right\}, \quad 0<\gamma \le x < \infty,\ \theta > 0,\ \beta > 0.$$

The kth moment of $X$ is given by $E[X^{(k)}] = \gamma + \theta^{k}\Gamma\left(1+\frac{k}{\beta}\right)$ where

$\Gamma(k) = \int_0^{\infty} u^{k-1}\exp(-u)du,\ k > 0$, is the gamma function (Meeker and Escobar, 1998).

For the three parameter Weibull, the pdf of the first order statistic $X_{1,n}$ is

$$f(x_1\ ; \gamma, \theta, \beta) = \frac{\beta}{\theta^{*\beta}}(x_1-\gamma)^{\beta-1}\exp\left\{-\left[\frac{x_1-\gamma}{\theta^*}\right]^{\beta}\right\}, \quad \gamma \le x_1 < \infty,\ \theta^* > 0,\ \beta > 0$$

where $\theta^* = \dfrac{\theta}{n^{1/\beta}}$.

(Balakrishnan and Cohen, 1991, 8.2.6 )

Therefore, if $X \sim$ Weibull $(\gamma, \theta, \beta)$, then $X_{1,n} \sim$ Weibull $(\gamma, \dfrac{\theta}{n^{1/\beta}}, \beta)$.

The *two parameter Weibull* distribution is a special case of the three parameter Weibull distribution in which the threshold parameter, $\gamma$, is zero. The two parameter Weibull is denoted simply $X \sim$ Weibull $(\theta, \beta)$. Other special cases of the two parameter Weibull are the exponential and Rayleigh distributions. If $\beta = 1$, the two parameter Weibull distribution is the exponential $(\theta)$ distribution; if $\beta = 2$, the two parameter Weibull distribution is the Rayleigh distribution. For $3 \le \beta \le 4$, the Weibull resembles the normal distribution in shape; for $\beta \ge 10$, the Weibull resembles the least extreme value distribution in shape (Nelson, 1982).

For any univariate distribution function $F$, and $0 < p < 1$, the quantity $\eta_p = F^{-1}(p)$ $= \inf\{x : F(x) \ge p\}$ is the *100pth* percentile. For a Weibull distribution, $\eta_p$ can be expressed as

$$\eta_p = F^{-1}(p) = \theta\left[1 - \log(1-p))^{1/\beta}\right].$$

where $\eta_p$ is the time by which *100p%* of all items will have failed. The median is the value equal to $\eta_{.50}$. As $100\,(1-e^{-1}) \cong 63.2$, the characteristic life parameter $\theta$ may be approximated by $\eta_{.632}$.

The *hazard function* (failure rate) associated with a random variable $X$ is defined to be

$$h(x) = f(x) \,/\, [\ 1 - F(x)\ ].$$

For surviving items in a sample, the hazard function indicates the propensity of an item to fail over a small unit of time. For a Weibull-distributed random variable $X$, the hazard function is

$$h(x) = \frac{\beta}{\theta}\left(\frac{x-\gamma}{\theta}\right)^{\beta-1}$$

which is a power function of $x$. Accordingly, the failure rate for the distribution is decreasing if $\beta < 1$, increasing if $\beta > 1$, and constant for $\beta = 1$.

Type I Extreme Value Distribution

Frequently, when working with Weibull distributions, it is more convenient to consider Type I extreme value distributions because the Type I extreme value distribution possess a location-scale structure which the Weibull distribution does not. The Type I extreme value distribution is directly related to the two parameter Weibull distribution in that if $X$ is a Weibull random variable with shape parameter $\beta$ and scale parameter $\theta$, then $Y = \log(X)$ is a Type I extreme value random variable with location parameter $\mu = \log\theta$ and scale parameter $\sigma = 1/\beta$, and accordingly, the results which are attributable to Type I extreme value distributions are directly transferable to Weibull distributions. Because of this relationship to the Weibull, the Type I extreme value distribution is sometimes referred to as the "Log-Weibull".

If $X$ is distributed as a Type I extreme value random variable, then the cdf is given by

$$F(x;\mu,\sigma) = \exp\left[-\exp\left(-\frac{x-\mu}{\sigma}\right)\right], \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \ \sigma > 0,$$

and the pdf is given by

$$f(x;\mu,\sigma) = \frac{1}{\sigma}\exp\left(-\frac{x-\mu}{\sigma}\right)\exp\left[-\exp\left(-\frac{x-\mu}{\sigma}\right)\right], \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \ \sigma > 0.$$

The "Type I" as it appears in the Type I least extreme value distribution name refers to one of three possible asymptotic types of distributions ( Type I, II, or II) of the smallest order statistic and does not refer to Type I (failure) censoring.

Burr Type XII distribution

Another model of lifetime distributions is the *Burr Type XII distribution*. The Burr Type XII distribution is considered as a tentative model for lifetime distributions where distribution's shape is L-shaped or unimodal . Tadikamalla ( 1980) summarizes many of the relationships between the Burr Type XII distribution and other distributions including the Weibull, the Lomax, Compound Weibull, the Weibull-Exponential, the logistic, the log logistic, and the Kappa family of distributions.

If $X$ is a random variable with a Burr Type XII ($d$, $c$) lifetime distribution, denoted by $X \sim$ Burr Type XII ($d$, $c$), then the cdf of X is given by $F(x) = 1-(x^c+1)^{-d}$ , x > 0, $c > 0$, $d > 0$.

and the pdf of $X$ is given by $f(x) = \dfrac{cdx^{c-1}}{(x^c+1)^{d+1}}$ , x > 0, $c > 0$, $d > 0$.

The pdf is unimodal at the point $x = \dfrac{(c-1)}{(dc+1)^{1/c}}$ if $c > 1$ and L-shaped if $c \le 1$, and the *kth* moment of $X$ is expressed as

$$E[X^{(k)}] = d\ B\left(\frac{k}{c}+1, d-\frac{k}{c}\right) , k < cd$$

where B($a,b$) is the beta function $\int_0^1 x^{a-1}(1-x)^{b-1}dx$ for $a > 0$, $b > 0$ and B($a,b$) = $\dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

(Tadikamalla, 1980)

If $X \sim$ Burr Type XII ($d$, $c$), then the pdf of the first order statistic $X_{1,n}$ is distributed Burr Type XII ($dn$, $c$); hence,

$$E[X_{1,n}^{(k)}] = dn\ B\left(\frac{k}{c}+1, dn-\frac{k}{c}\right) , k < cdn .$$

The possible parametric models under progressive censoring schemes considered in this research have distribution functions which exist in closed form and have known expectations of first order statistics. The Weibull and Burr Type XII distributions are shown to have these properties. Other distributions such as the Pareto distribution could have been considered.

## 2.3 Expected Experiment Time

Early research concerning experiment times centered on finding expected experiment time for the case of Type II singly right censored samples. Expected experiment times for exponentially distributed lifetimes were first considered by Epstein and Sobel (1953) and Weibull-distributed lifetimes were later considered by Hsieh (1994).

For exponentially distributed lifetimes, Epstein and Sobel (1953) give the expected experiment time of Type II singly right samples as

$$E[X_{m,n}] = n\binom{n-1}{m-1}\sum_{j=0}^{m-1}\binom{m-1}{j}(-1)^j \frac{1}{(j+n-m+1)^2}\,.$$

This when simplified becomes

$$E[X_{m,n}] = \theta\sum_{j=1}^{m}\frac{1}{(n-j+1)}$$

(Mann, 1974, 6.27)

which implies

$$\text{REET} = \frac{E[X_{m,n}]}{E[X_{m,m}]} = \frac{\sum_{j=1}^{m}\frac{1}{(n-j+1)}}{\sum_{j=1}^{m}\frac{1}{(m-j+1)}}\,.$$

Alternatively, Mann (1974) suggests that, for Type II singly right censored samples from an exponential distribution, the *ratio of median experiment times* results in essentially the same values as the ratio of expected experiment time and eliminates extensive calculations when $m$ is large.

For two parameter Weibull distributed lifetimes, Hsieh (1994) gives the expected experiment for Type II singly right censored samples as

$$E[X_{m,n}] = n\binom{n-1}{m-1}\theta\int_0^1[-\ln(1-x)]^{1/\beta}x^{m-1}(1-x)^{n-m}\,dx$$

$$= m\theta\Gamma(1+1/\beta)\binom{n}{m}\sum_{j=0}^{m-1}\binom{m-1}{j}(-1)^j \frac{1}{(j+n-m+1)^{1+1/\beta}}$$

which implies the REET value is

$$\text{REET} = \frac{E[X_{m,n}]}{E[X_{m,m}]} = \frac{\binom{n}{m}\psi(j : m, n-m, 1/\beta)}{\psi(j; m, 0, 1/\beta)}$$

where $\psi(j : m, x, b) = \sum_{j=0}^{m-1}\binom{m-1}{j}(-1)^j \frac{1}{(j+x+1)^{1+b}}$.

The REET value is then observed to depend on the shape parameter $\beta$ and values of $n$ and $m$ but not the scale parameter $\theta$ for Type II singly right censored Weibull-distributed lifetimes.

Formulas of expected experiment time for Type II progressively censored samples are more recent developments. Formulas for Weibull-distributed lifetimes are found in Tse and Yuen (1998) for both random and fixed removals. Tse and Yuen (1998) provides a general formula for expected experiment time where removals are fixed and the sample is Type II progressively censored (previously the research of Hsieh (1994) considered only single right censoring ). The formula given by Tse and Yuen (1998) may be written as

$$E[X_{m,n} \mid r_1, r_2, \ldots, r_m\ ] = c \sum_{\ell_1=0}^{r_1} \cdots \sum_{\ell_m=0}^{r_2} (-1)^{\ell_1 + \ldots \ell_m} \frac{\binom{r_1}{\ell_1} \cdots \binom{r_m}{\ell_m}}{\prod_{i=1}^{m-1} h(\ell_i)}$$

$$\times \int_0^\infty x_m\ (x_{m,n})[F(x_{m,n})]^{h(\ell_m)-1}\, dx_{m,n}$$

where $h(\ell_m) = \ell_1 + \ldots + \ell_m + m$.

This formula then becomes, for Weibull distributed lifetimes,

$$E[X_{m,n} \mid r_1, r_2, \ldots, r_m\ ] = c \sum_{\ell_1=0}^{r_1} \cdots \sum_{\ell_m=0}^{r_2} (-1)^{\ell_1 + \ldots \ell_m} \frac{\binom{r_1}{\ell_1} \cdots \binom{r_m}{\ell_m}}{\prod_{i=1}^{m-1} h(\ell_i)}$$

$$\times\ \theta\Gamma(1+1/\beta)\sum_{j=0}^{h(\ell_m)-1}\binom{h(\ell_m)-1}{j}(-1)^j\frac{1}{(j+1)^{1+1/\beta}}.$$

(The formula as originally published contained errors.)

Tse and Yuen (1998) also gives a formula for expected experiment time where removals are random and the sample is Type II progressively censored. The formula is as follows:

$$E[X_{m,n}] = E_r[E[X_{m,n} \mid r_1, r_2, \ldots, r_m]]$$

$$= \sum_{r_1=0}^{g(r_1)} \cdots \sum_{r_m=0}^{g(r_{m-1})} P(r_1, r_2, \ldots, r_m)\ E[X_{m,n} \mid r_1, r_2, \ldots, r_m]$$

where

$$g(r_i) = n - m - r_1 - \ldots - r_{i-1};\quad 0 \le r_i \le n - m - (r_1 + \ldots + r_{i-1}),\ i = 1,2,\ldots,m-1,$$

$$P(r_1, r_2, \ldots, r_m) = P(r_1)\prod_{i=2}^{m-1} P(r_i \mid r_1, \ldots, r_{i-1});$$

$$P(r_1) = \frac{1}{n-m+1}, \text{ and } P(r_i \mid r_1, \ldots, r_{i-1}) = \frac{1}{n-m-(r_1+\ldots+r_{i-1})+1} \text{ for } i = 1,2,\ldots,m-1.$$

A formula of expected experiment time for Type II progressively censored samples for two parameter exponentially distributed lifetimes, i.e., exponential lifetimes with a threshold parameter, is offered in Balasooriya and Saw (1998). Using the notation in this chapter, if $\gamma$ is the threshold parameter and $\theta$ is the scale parameter, the formula in Balasooriya and Saw (1998) of expected experiment time of a two parameter exponential life test is

$$E[X_{m,n}] = \gamma + \left[\frac{1}{n} + \sum_{i=2}^{m}\frac{1}{n - \sum_{j=1}^{i-1} r_j - i + 1}\right]\theta.$$

This formula applies to fixed removals only.

Under the general case of Type II progressive censoring for two parameter exponential distributed lifetimes, the formula becomes

$$E[X_{m,n}] = \gamma + \left[ \frac{1}{n} + \sum_{i=r_o+2}^{m} \frac{1}{n - \sum_{j=r_o+1}^{i-1} r_j - i + 1} \right] \theta$$

(Balakrishnan and Sandhu, 1996)

where $r_o$ is the number of the first failure times which are not observed.

Investigations into experiment time often require simulating partial sets of order statistics. Simulating Type II progressively censored samples is easily achievable using a four step algorithm of Balakrishnan and Sandhu (1995) if the cdf for the lifetime distribution exists in simple closed form. This is true with the Weibull, Type I extreme value, and Burr Type XII lifetime distributions. Other methods to generate partial sets of order statistics may be found in the references of Balakrishan and Sandhu (1995).

## 2.4 Parameter Estimation

While few articles consider experiment time of progressively censored samples, and then only with respect to Weibull (or Exponential)-distributed data, methods of parameter estimation using progressively censored samples have been described by many authors. In many instances, the parameter estimation technique employed is maximum likelihood estimation.

Under Type II progressive censoring, the likelihood function is defined to be

$$L = L(x_{1,n}, x_{2,n}, \ldots, x_{m,n}; r_1, r_2, \ldots, r_m) = c \prod_{i=1}^{m} \left\{ n - \sum_{j=1}^{i-1} r_j - i + 1 \right\} f(x_i) \prod_{i=1}^{m} [1 - F(x_i)]^{r_i} .$$

(Cohen, 1963)

This likelihood function is then employed in an iterative procedure to obtain the maximum likelihood estimates (MLEs) of the parameters.

In independent research, Wingo (1973), Cohen (1975) and Lemon(1975) are credited with first obtaining MLEs for three parameter progressively censored Weibull-distributed lifetimes. Wingo (1993) also describes the methodology for obtaining MLEs for progressively censored Burr Type XII-distributed lifetimes. Various authors have investigated maximum likelihood estimation using progressively censored data for other distributions including Cohen (1976), Cohen and Norgaard (1977), and Davis (1979). Cohen (1991) summarizes many of the techniques obtain MLEs and other point estimators of parameters for a wide range of censored distributions.

As an example, if the sample is obtained from a Type I least extreme value distribution, then

$$L = c\prod_{i=1}^{m} \frac{1}{\sigma}\exp\left(-\frac{x-\mu}{\sigma}\right)\exp\left[-\exp\left(-\frac{x-\mu}{\sigma}\right)\right]\prod_{i=1}^{m} \exp\left[-\exp\left(-\frac{x-\mu}{\sigma}\right)\right]^{r_i}.$$

Taking the $\frac{\partial \log L}{\partial\mu}$ and $\frac{\partial \log L}{\partial\sigma}$ and equating both to zero implies

$$e^{(-\mu/\sigma)} = m \Big/ \left[\sum_{j=1}^{m}\exp(x_i/\sigma) + \sum_{i=1}^{m} r_i \exp(x_i/\sigma)\right]$$

and

$$\sigma = \frac{\sum_{i=1}^{m} x_i \exp(x_i/\sigma) + \sum_{i=1}^{m} r_i \exp(x_i/\sigma)}{\sum_{i=1}^{m}\exp(x_i/\sigma) + \sum_{i=1}^{m} r_i \exp(x_i/\sigma)} - \frac{\sum_{i=1}^{m} x_i}{m}.$$

A maximum likelihood estimator for $\sigma$, $\hat{\sigma}_{MLE}$, is found by equating

$$\mathrm{f}(\sigma) = \sigma - \frac{\sum_{i=1}^{m} x_i \exp(x_i/\sigma) + \sum_{i=1}^{m} r_i \exp(x_i/\sigma)}{\sum_{i=1}^{m}\exp(x_i/\sigma) + \sum_{i=1}^{m} r_i \exp(x_i/\sigma)} + \frac{\sum_{i=1}^{m} x_i}{m} = 0$$

and using the iterative method such as Newton's method:

$$\sigma_{n+1} = \sigma_n - \mathrm{f}(\sigma_n)/\mathrm{f}'(\sigma_n).$$

right-censored, and progressively censored data. An example using the procedure is described in Dodson (1994).

Frequently, a graphical rather than a quantitative method is used to investigate goodness-of-fit. Most of the literature relating to probability plotting or graphical tests of goodness-of-fit are concerned with complete (uncensored ) data. A procedure for a graphical test for goodness-of-fit of progressively censored data is contained in Johnson (1964) and O'Connor (1981). Johnson (1964) refers to the tests not as "progressively censored" experiments but as "incomplete" tests with "suspended items".

A visual inspection as to whether or not the data are linearly related usually serves as a sufficient test of goodness-of-fit for graphical tests. A correlation-type test statistic, however, may provide a more quantitative measure of the goodness-of-fit for progressively censored data. Correlation-type tests for goodness-of-fit are developed by Ryan and Joiner (1974) and Filliben (1974) for complete samples and adapted to apply to Type II single right censored data by Smith and Bain (1976). In this research the correlation-type test of goodness-of-fit is adapted further to apply to progressively censored data.

# CHAPTER THREE

# EXPERIMENT TIME

The purpose of this chapter is to examine $k$th moments of experiment time under Type II progressive censoring schemes with fixed and random removals. A formula is presented which expresses moments of experiment time in terms of simpler moments of the smallest order statistic. The formula is applicable for lifetime distributions where $k$th moments of the smallest order statistic are known. In this chapter, the previous numerical studies of Tse and Yuen (1998) are extended to consider standard deviations associated with expected experiment time for Weibull-distributed data. The experiment time for the Burr Type XII-distributed lifetimes under Type II progressive censoring is also investigated.

## 3.1 Introduction

In life and fatigue studies, a complete sample test occurs if $n$ items are placed under observation, and $n$ failure times are observed. If the life test experiment ends with only $m$ ($m < n$) failure times observed, then the test is a censored life test. In a typical censored life test, censoring occurs after the first $m$ failures and the largest $n - m$ lifetimes are censored. An experiment with this single stage of censoring is referred to as a single right censored experiment. For progressively censored experiments, the removals may occur at multiple stages as the experiment progresses. That is, $r_1, r_2, \ldots, r_m$ number of sample items are removed from the life test at times $T_i$, $i = 1, 2, \ldots, m$.

Another distinction is made between Type I (time) censoring where the removals occur at pre-specified times and Type II (failure) censoring where removals occur coinciding with failure times of other uncensored items in the sample. In the case of Type I censoring, the times $T_i$ are pre-specified; in the case of Type II censoring, the times $T_i$ are the failure times of the uncensored sample items. The progressively censored sample is considered to be the *m* observed (uncensored) lifetimes $x_{1,n} \le x_{2,n} \le ... \le x_{m,n}$ from the complete sample of size *n*.

A major consideration in any life testing experiment is the total duration of test time. An attractive feature of a progressively censored experiment is the potential time and cost savings which the progressively censored experiment allows (Balakrishnan and Sandhu, 1996).

Thomas and Wilson (1972), in an appendix to their article, describe a procedure to obtain the means, variances, and covariances of the *m* progressively censored order statistics by conditioning on the complete sample ranks of the Type II progressively censored sample. Since experiment time is the largest order statistic , $X_{m,n}$, the procedure of Thomas and Wilson (1972) may be modified to consider only the mean of $X_{m,n}$, the expected value of experiment time, $E[X_{m,n}]$.

If $X_{1,n} \le X_{2,n} \le ... \le X_{m,n}$ are the *m* order statistics of a progressively censored sample and $Z_{1,n} \le Z_{2,n} \le ... \le Z_{n,n}$ are the *n* order statistics of the complete sample had the sample not been subjected to censoring, then $X_{i,n} = Z_{R_i,n}$ for some $R_i$, $i = 1,2,...,m$ . The complete sample ranks of the Type II progressively *censored sample* $x_{1,n} \le x_{2,n} \le ... \le x_{m,n}$ are denoted by $R_1, R_2, ..., R_m$.

For a Type II progressively censored sample, the complete sample rank of the first ordered observation $x_{1,n}$ is known as $x_{1,n} = z_{1,n}$ , that is, $R_1 = 1$. The complete sample ranks of $x_{2,n} \le x_{3,n} \le ... \le x_{m,n}$ are not necessarily known. The sample space for the rank vector $\boldsymbol{R}$ = ( $R_1, R_2, ..., R_m$ ) can be described recursively using the equation

$$R_i = R_{i-1} + 1, \ R_{i-1} + 2, \dots, i + r_1 + r_2 + \dots + r_{i-1}$$

for $i = 2,...,m$ with $R_1 = 1$.

(Thomas and Wilson, 1972, A1)

For fixed removals $r_1, r_2, ..., r_m$, the *probability function for complete sample ranks* is found by letting

$$p = P(R_1, R_2, ..., R_m) = P(R_1)\prod_{i=2}^{m} P(R_i \mid R_1, ..R_{i-1})$$

(Thomas and Wilson, 1972, A2)

where $P(R_1 = 1) = 1$ and $P(R_i \mid, R_1, ..., R_{i-1})$ is the probability function

$$P(R_i \mid, R_1, ..., R_{i-1}) = \frac{\begin{pmatrix} n - R_i \\ \sum_{j=1}^{i-1}(r_j + 1) - R_i + 1 \end{pmatrix}}{\begin{pmatrix} n - R_{i-1} \\ \sum_{j=1}^{i-1}(r_j + 1) - R_{i-1} \end{pmatrix}}, i = 2, ..., m.$$

(Thomas and Wilson, 1972, A3)

If $\boldsymbol{X} = (X_{1,n}, ..., X_{m,n})'$ is the vector of the Type II progressively censored order statistics, $\boldsymbol{Z} = (Z_{1,n}, ..., Z_{n,n})'$ is the vector of the corresponding complete sample order statistics, $\boldsymbol{R} = (R_1, R_2, ..., R_m)$ is the vector complete sample rankings of the Type II progressively censored ordered statistics, and $\boldsymbol{D}$ is a matrix defined by

$$d_{ij} = \begin{cases} 1, & j = R_i \\ 0, & j \neq R_i \end{cases} \quad j = 1, ..., n, \quad i = 1, ..., m,$$

then $\boldsymbol{X} = \boldsymbol{DZ}$.

Using this matrix notation, the *means of Type II progressively censored order statistics* may be written in the form

$$\mu_x = \mathrm{E}[\boldsymbol{X}] = \mathrm{E}_\ell \mathrm{E}_{\mathbf{D}_\ell}[\boldsymbol{D}_\ell \;\; \boldsymbol{Z} \mid \boldsymbol{D}_\ell \;] = E_\ell[\boldsymbol{D}_\ell{}' \;\mu_Z] = \left(\sum_{\ell=1}^{M} D_\ell p_\ell\right)\mu_Z$$

where $M$ denotes the number of different possible vectors of complete sample rankings, $p_1, p_2, ..., p_M$ are the probabilities of the $M$ possible complete sample ranking schemes, and $\boldsymbol{D}_\ell$ is defined as

$$d_{ij}^{\ell} = \begin{cases} 1, & j = R_i \\ 0, & j \neq R_i \end{cases}$$

for $i = 1, 2, ..., m$, $j = 1, 2, ...n$, and $\ell = 1, 2, ...M$.

(Thomas and Wilson, 1972, A7)

The *variances and covariances of Type II progressively censored order statistics* are obtained similarly as

$$\begin{aligned}\Sigma_x &= \mathrm{E}[\mathbf{X}\mathbf{X}'] - \mu_x \mu_x' \\ &= E_\ell E_{\mathbf{D}_\ell}[\boldsymbol{D}_\ell \ \boldsymbol{Z}\boldsymbol{Z}' \ \boldsymbol{D}_\ell' \mid \boldsymbol{D}_\ell] - \mu_x \mu_x' \\ &= E_\ell E_{\mathbf{D}_\ell}[\boldsymbol{D}_\ell \ (\Sigma_z + \mu_z \mu_z')\boldsymbol{D}_\ell' \mid \boldsymbol{D}_\ell] - \mu_x \mu_x' \\ &= \sum_{\ell=1}^{M} \boldsymbol{D}_\ell \ (\Sigma_z + \mu_z \mu_z') \ \boldsymbol{D}_\ell' p_\ell - \mu_x \mu_x'.\end{aligned}$$

(Thomas and Wilson, 1972, A8)

Thomas and Wilson (1972) employ matrix notation to obtain means, variances, and covariances of all order statistics $X_{1,n} \le X_{2,n} \le \ldots \le X_{m,n}$. Their matrix notation is adapted to summation notation in this chapter to consider only expected experiment time, $X_{m,n}$, the largest order statistic of the Type II progressively censored sample, and further modified to consider moments of experiment time.

One previous numerical study by Tse and Yuen (1998) investigates the expected experiment time for Type II progressively censored Weibull-distributed lifetimes with random removals. Their study does not consider the variability associated with the experiment time estimate. The research in this chapter seeks to examine the variability associated with expected experiment time for different fixed and random Type II progressively censored removal schemes.

For Type II progressive censored data with predetermined removals $r_1, r_2, \ldots, r_m$, the formula by Tse and Yuen (1998) for expected experiment time for the Weibull distribution under progressive censoring is derived by conditioning on the number of removals $r_1, r_2, \ldots, r_m$ at each stage of censoring. If $x_{1,n} < x_{2,n} < \ldots < x_{m,n}$ are the $m$ ordered observed (uncensored) lifetimes of a progressively censored sample with removals $r_1, r_2, \ldots, r_m$, then for $X$ a two parameter Weibull random variable with cdf

$$F(x; \beta, \theta) = 1 - \exp\left[-(x/\theta)^{\beta}\right] \ , \ x > 0, \ \theta > 0, \beta > 0,$$

the formula used by Tse and Yuen (1998) for expected experiment time may be written as

$$E[X_{m,n} \mid r_1, r_2, \ldots, r_m] = c \sum_{\ell_1=0}^{r_1} \cdots \sum_{\ell_m=0}^{r_2} (-1)^{\ell_1+\ldots+\ell_m} \frac{\binom{r_1}{\ell_1} \cdots \binom{r_m}{\ell_m}}{\prod_{i=1}^{m-1} h(\ell_i)}$$

$$\times \int_0^\infty x_m \, f(x_{m,n}) [F(x_{m,n})]^{h(\ell_m)-1} \, dx_{m,n}$$

$$= c \sum_{\ell_1=0}^{r_1} \cdots \sum_{\ell_m=0}^{r_2} (-1)^{\ell_1+\ldots \ell_m} \frac{\binom{r_1}{\ell_1} \cdots \binom{r_m}{\ell_m}}{\prod_{i=1}^{m-1} h(\ell_i)}$$

$$\times \, \theta \Gamma(1+1/\beta) \sum_{j=0}^{h(\ell_m)-1} \binom{h(\ell_m)-1}{j} (-1)^j \frac{1}{(j+1)^{1+1/\beta}}$$

where $h(\ell_m) = \ell_1 + \ldots + \ell_m + m$, $c$ is an ordering constant, and the removals $r_1, r_2, \ldots, r_m$ are fixed.

(Tse and Yuen, 1998, 8)

(The formula as originally published contained errors.)

For random removal schemes, their formula for expected experiment time becomes

$$E[X_{m,n}] = E_r[E[X_m \mid r_1, r_2, \ldots, r_m]]$$

$$= \sum_{r_1=0}^{g(r_1)} \sum_{r_2=0}^{g(r_2)} \cdots \sum_{r_m=0}^{g(r_{m-1})} P(r_1, r_2, \ldots, r_m) \; E[X_{m,n} \mid r_1, r_2, \ldots, r_m]$$

where

$$g(r_i) = n - m - r_1 - \ldots - r_{i-1}; \quad 0 \le r_i \le n - m - (r_1 + \ldots + r_{i-1}), \; i = 1, 2, \ldots, m.$$

(Tse and Yuen, 1998)

The numerical study of Tse and Yuen (1998) obtains first moments of experiment time by conditioning experiment time on removal schemes. In this research, $k$th moments of experiment time are obtained by conditioning on the complete sample rankings in a manner suggested by Thomas and Wilson (1972). The earlier study of Tse and Yuen (1998) is then extended to consider standard deviations of experiment time for Type II progressively censored data.

### 3.2 A Formula for Moments of Experiment Time

The *k*th moment of the experiment time is denoted as $E[X_{m,n}{}^{(k)}]$ and is derived below for fixed and random removals for a Type II progressively censored sample.

Fixed Removals

Suppose under Type II progressive censoring the number of removals at each stage of censoring $r_1, r_2, \ldots, r_m$ are fixed and subject to the conditions that $r_1 + r_2 + \ldots + r_m = n - m$. That is, the experiment is progressively censored with fixed numbers of removals coinciding with each failure time, and the experiment ends at the time of the *m*th failure. The *k*th moment of experiment time $E[X_m{}^{(k)}]$ may be derived by conditioning on the complete sample ranking $R_m$ of the *m*th order statistic in the progressively censored sample. A formula for the *kth moment of the experiment time for progressively censored samples with fixed removals* can be expressed as

$$E[X_{m,n}{}^{(k)}] \quad = \sum_{all R_m} E[X_{m,n}{}^{(k)} \mid R_m] P[R_m]$$

$$= \sum_{all R_m} E[X_{m,n}{}^{(k)} \mid R_m] \sum_{all R_1, R_2, \ldots, R_{m-1}} P[R_1, R_2, \ldots, R_m]$$

$$= \sum_{all R_m} E[X_{m,n}{}^{(k)} \mid R_m] \sum_{all R_1, R_2, \ldots, R_{m-1}} P[R_1] \prod_{i=2}^{m} P[R_i \mid R_1, R_2, \ldots, R_{i-1}].$$

Substituting $\prod_{i=2}^{m} \frac{\binom{n-R_i}{\sum_{j=1}^{i-1}(r_j+1) - R_i + 1}}{\binom{n-R_{i-1}}{\sum_{j=1}^{i-1}(r_j+1) - R_{i-1}}}$ in the equation for $P[R_1] \prod_{i=2}^{m} P[R_i \mid R_1, R_2, \ldots, R_{i-1}]$.

(Thomas and Wilson, 1972)

$$E[X_{m,n}{}^{(k)}] = \sum_{R_m = m}^{m + \sum_{i=1}^{m-1} r_i} E[X_{R_m,n}{}^{(k)}]$$

$$\times \sum_{R_{m-1}=m-1}^{h(m-1)} \sum_{R_{m-2}=m-2}^{h(m-2)} \cdots \sum_{R_2=2}^{h(2)} \prod_{i=2}^{m} \frac{\binom{n-R_i}{\sum_{j=1}^{i-1}(r_j+1)-R_i+1}}{\binom{n-R_{i-1}}{\sum_{j=1}^{i-1}(r_j+1)-R_{i-1}}}.$$

Substituting $n\binom{n-1}{R_m-1}\sum_{j=0}^{R_m-1}\binom{R_m-1}{j}(-1)^j \frac{1}{(j+n-R_m+1)} E[X_{1,j+n-R_m+1}{}^{(k)}]$ in the equation for

$E[X_{R_m,n}{}^{(k)}]$,

(David, 1981, 3.4.3)

$$E[X_{m,n}{}^{(k)}] = \sum_{R_m=m}^{m+\sum_{i=1}^{m-1} r_i} n\binom{n-1}{R_m-1}\sum_{j=0}^{R_m-1}\binom{R_m-1}{j}(-1)^j \frac{1}{(j+n-R_m+1)} E[X_{1,j+n-R_m+1}{}^{(k)}]$$

$$\times \sum_{R_{m-1}=m-1}^{h(m-1)} \sum_{R_{m-2}=m-2}^{h(m-2)} \cdots \sum_{R_2=2}^{h(2)} \prod_{i=2}^{m} \frac{\binom{n-R_i}{\sum_{j=1}^{i-1}(r_j+1)-R_i+1}}{\binom{n-R_{i-1}}{\sum_{j=1}^{i-1}(r_j+1)-R_{i-1}}} \qquad (1)$$

where $h(k) = \min\left[(k+\sum_{i=1}^{k-1} r_i),(R_{k+1}-1)\right],\ k = 2,3,\ldots,m-1.$

Therefore, by conditioning on complete sample rankings, an equation for the $k$th moments of experiment time for a Type II progressively censored sample is expressed as a function of the $k$th moment of the smallest order statistic.

One fixed removal scheme for Type II progressively censored data to which equation (1) will apply is the 50% fixed removal scheme where alternate sample items are censored. This is done by letting $r_i = 1$, for all $i = 1,2,\ldots,m$, in equation (1). For this 50% fixed removal scheme, the formula for the $k$th moment of experiment time becomes

$$E[X_{m,n}{}^{(k)}] = \sum_{R_m=m}^{m+\sum_{i=1}^{m-1} r_i} n\binom{n-1}{R_m-1}\sum_{j=0}^{R_m-1}\binom{R_m-1}{j}(-1)^j \frac{1}{(j+n-R_m+1)} E[X_{1,j+n-R_m+1}{}^{(k)}]$$

$$\times \sum_{R_{m-1}=m-1}^{h(m-1)} \sum_{R_{m-2}=m-2}^{h(m-2)} \cdots \sum_{R_2=2}^{h(2)} \prod_{i=2}^{m} \frac{\binom{n-R_i}{2i-1-R_i}}{\binom{n-R_{i-1}}{2i-2-R_{i-1}}}. \qquad (1a)$$

Many other fixed removal schemes are possible.

Random Removals

Suppose under Type II progressive censoring the $r_1, r_2, \ldots, r_m$ removals are random and subject to the condition that $r_1 + r_2 + \ldots + r_m = n - m$. That is, the experiment is progressively censored with random removals coinciding with each failure time, and the experiment ends at the time of the $m$th failure. The formula for the *kth moment of the experiment time with random removals* may then be expressed as

$$E[X_{m,n}{}^{(k)}] = \sum_{all\ r_1, r_2, \ldots, r_m} P(r_1, r_2, \ldots, r_m)$$

$$\times \sum_{R_m=m}^{m+\sum_{i=1}^{m-1} r_i} n\binom{n-1}{R_m-1}\sum_{j=0}^{R_m-1}\binom{R_m-1}{j}(-1)^j \frac{1}{(j+n-R_m+1)} E[X_{1,j+n-R_m+1}{}^{(k)}]$$

$$\times \sum_{R_{m-1}=m-1}^{h(m-1)} \sum_{R_{m-2}=m-2}^{h(m-2)} \cdots \sum_{R_2=2}^{h(2)} \prod_{i=2}^{m} \frac{\binom{n-R_i}{\sum_{j=1}^{i-1}(r_j+1)-R_i+1}}{\binom{n-R_{i-1}}{\sum_{j=1}^{i-1}(r_j+1)-R_{i-1}}} \qquad (2).$$

The approach of Tse and Yuen (1998) assumes removals occur at random, coinciding with each failure time, and that the number of each of the removals $r_i$ follows a uniform discrete probability distribution such that

$$P(r_1, r_2, ..., r_m) = P(r_1) \prod_{i=2}^{m-1} P(r_i \mid r_1, r_2, ..., r_{i-1})$$

where $P(r_1) = \dfrac{1}{n-m+1}$ and $P(r_i \mid r_1, r_2, ..., r_{i-1}) = \dfrac{1}{n-m+(r_1+...+r_{i-1})+1}$, $i = 2, ..., m-1$.

Therefore, making the appropriate substitution for $P(r_1, r_2, ..., r_m)$, equation (2) becomes

$$E[X_{m,n}{}^{(k)}] = \sum_{r_1=0}^{g(r_1)} \sum_{r_2=0}^{g(r_2)} \cdots \sum_{r_{m-1}=0}^{g(r_{m-1})} \frac{1}{n-m+1} \prod_{i=2}^{m-1} \frac{1}{(n-m-(r_1+...+r_{i-1})+1)}$$

$$\times \sum_{R_m=m}^{m+\sum_{i=1}^{m-1} r_i} n \binom{n-1}{R_m-1} \sum_{j=0}^{R_m-1} \binom{R_m-1}{j} (-1)^j \frac{1}{(j+n-R_m+1)} E[X_{1,j+n-R_m+1}{}^{(k)}]$$

$$\times \sum_{R_{m-1}=m-1}^{h(m-1)} \sum_{R_{m-2}=m-2}^{h(m-2)} \cdots \sum_{R_2=2}^{h(2)} \prod_{i=2}^{m} \frac{\binom{n-R_i}{\sum_{j=1}^{i-1}(r_j+1)-R_i+1}}{\binom{n-R_{i-1}}{\sum_{j=1}^{i-1}(r_j+1)-R_{i-1}}} \tag{2a}$$

where $g(r_1) = n-m$ and $g(r_i) = n-m-r_1-r_2-...-r_{i-1}$, $i = 2, ..., m-1$.

A second approach is to assume the experimenter selects a removal scheme at random from all possible removal schemes and that all removal schemes are equally likely. Letting

$P(r_1, r_2, ..., r_m) = \dfrac{1}{\binom{n-1}{m-1}}$ for all removals $r_1, r_2, ..., r_m$, since the number of nonnegative integer

solutions for the equation $r_1 + r_2 + ... + r_m = n-m$ is $\binom{n-1}{m-1}$,

(van Lint and Wilson, 1992, 13.21)

equation (2) becomes

$$E[X_{m,n}{}^{(k)}] = \sum_{r_1=0}^{g(r_1)} \sum_{r_2=0}^{g(r_2)} \cdots \sum_{r_{m-1}=0}^{g(r_{m-1})} \frac{1}{\binom{n-1}{m-1}}$$

$$\times \sum_{R_m=m}^{m+\sum_{i=1}^{m-1} r_i} n \binom{n-1}{R_m-1} \sum_{j=0}^{R_m-1} \binom{R_m-1}{j} (-1)^j \frac{1}{(j+n-R_m+1)} E[X_{1,j+n-R_m}{}^{(k)}]$$

$$\times \sum_{R_{m-1}=m-1}^{h(m-1)} \sum_{R_{m-2}=m-2}^{h(m-2)} \cdots \sum_{R_2=2}^{h(2)} \prod_{i=2}^{m} \frac{\binom{n-R_i}{\sum_{j=1}^{i-1}(r_j+1)-R_i+1}}{\binom{n-R_{i-1}}{\sum_{j=1}^{i-1}(r_j+1)-R_{i-1}}} . \qquad (2b)$$

Equations (2a) and (2b) are best used where sample sizes are small, $n < 20$, or where differences between the number of items fully measured, $m$, and the sample size, $n$, is small. This is because of the extensive computer time to consider all possible rankings to obtain the expected values. The same limitation is noted for the formulas used by Tse and Yuen (1998). Tse and Yuen (1998) consider the dominant factor, in terms of computing time, to be the value of $n$. Examples using the equations are given in the following sections. All computations in the following sections are performed using *SAS System for Windows, Release 6.12* (1996).

## 3.2 Examples

### Random Removals

#### The Weibull Example with Random Removals

A common model for lifetime data is the two parameter Weibull distribution. Where $X$ is a random variable that is Weibull $(\theta, \beta)$, the first order statistic from a sample of size $n$, $X_{1,n}$, is Weibull $(\theta / n^{1/\beta}, \beta)$ with

$$E[X_{1,n}^{(k)}] = \left(\frac{\theta}{n^{1/\beta}}\right)^k \Gamma\left(1+\frac{k}{\beta}\right). \qquad (3)$$

(Balakrishnan and Cohen, 1991, 8.2 )

Therefore, the first or second moments of experiment time for progressively censored Weibull-distributed lifetimes can be determined by noting that

$$E[X^{(k)}_{1,j+n-R_m+1}] = \left(\frac{\theta}{(j+n-R_m+1)^{1/\beta}}\right)^k \Gamma\left(1+\frac{k}{\beta}\right).$$

For Weibull-distributed lifetimes defined by a shape parameter $\beta$, it is well known that values of $\beta<1$ imply a declining failure rate, $\beta=1$ imply a constant failure rate, and $\beta>1$ imply an increasing failure rate.

Tables 3.2.1 - 3.2.3 show first and second moments and standard deviations of experiment time for different values of $\beta$ and combinations of *m* and *n* using equation (2a). Without loss of generality, the calculations for the tables for Weibull-distributed data in this section are performed with $\theta=1$ (see equation (3)). The expected experiment times reported in Table 3.2.1 appear in close agreement to the numerical values reported by Tse and Yuen (1998, Table 1). No differences were expected as equation (2a), when used to find the first moment of experiment time for Weibull-distributed lifetimes, is equivalent to the equation offered by Tse and Yuen (1998).

Any examination of experiment time estimates, however, should also consider the variability associated with the estimator. Therefore, the second moments of experiment time in Table 3.2.2, in conjunction with the first moments in Table 3.2.1, are used to obtain the standard deviations that are reported in Table 3.2.3. The values in Table 3.2.3 suggest that standard deviations associated with experiment time values are especially large in instances where $\beta<1$. Consequently, the expected experiment times reported in Table 3.2.1 should be used cautiously for Weibull-distributed lifetimes with decreasing failure rates. Furthermore, the standard deviations of experiment time in Table 3.2.3 suggest:

(1) for fixed values of *m*,

  (a) if $\beta\leq 1$, the standard deviations of experiment time tend to decrease as *n* increases,

  (b) if $\beta>1$, the standard deviations may increase or decrease as *n* increases, but, overall, are of approximately the same magnitude, and

(2) for fixed values of *m* and *n*, as $\beta$ increases, the standard deviations decrease.

## Table 3.2.4

$E[X_{m,n}]/\theta$ Using Equation (2a) for Two Parameter Weibull Samples

| | | β | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | n | 0.25 | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 2.00 | 3.00 | 5.00 |
| 3 | 3 | 67.7963 | 4.7222 | 2.4296 | 1.8333 | 1.5761 | 1.4361 | 1.2904 | 1.1718 | 1.0936 |
| | 4 | 44.1186 | 3.5069 | 1.9499 | 1.5417 | 1.3666 | 1.2724 | 1.1759 | 1.0999 | 1.0522 |
| | 5 | 32.7637 | 2.8093 | 1.6479 | 1.3481 | 1.2228 | 1.1573 | 1.0931 | 1.0463 | 1.0204 |
| | 6 | 26.1682 | 2.3552 | 1.4377 | 1.2080 | 1.1160 | 1.0703 | 1.0289 | 1.0037 | 0.9947 |
| | 10 | 14.8535 | 1.4642 | 0.9832 | 0.8858 | 0.8601 | 0.8557 | 0.8646 | 0.8901 | 0.9238 |
| | 15 | 9.9016 | 1.0183 | 0.7272 | 0.6893 | 0.6955 | 0.7122 | 0.7489 | 0.8058 | 0.8687 |
| 5 | 5 | 107.5326 | 6.6772 | 3.1816 | 2.2833 | 1.8958 | 1.6842 | 1.4620 | 1.2784 | 1.1545 |
| | 6 | 92.8072 | 6.0341 | 2.9518 | 2.1521 | 1.8055 | 1.6157 | 1.4160 | 1.2508 | 1.1392 |
| | 8 | 74.8683 | 5.1664 | 2.6246 | 1.9594 | 1.6702 | 1.5117 | 1.3450 | 1.2074 | 1.1147 |
| | 10 | 64.0825 | 4.5921 | 2.3959 | 1.8202 | 1.5703 | 1.4338 | 1.2908 | 1.1735 | 1.0953 |
| | 12 | 56.7300 | 4.1748 | 2.2229 | 1.7122 | 1.4916 | 1.3716 | 1.2468 | 1.1456 | 1.0791 |
| | 15 | 49.1011 | 3.7174 | 2.0261 | 1.5865 | 1.3984 | 1.2972 | 1.1934 | 1.1111 | 1.0587 |
| 6 | 6 | 126.2316 | 7.4939 | 3.4735 | 2.4500 | 2.0105 | 1.7711 | 1.5203 | 1.3134 | 1.1739 |
| | 8 | 106.3499 | 6.6583 | 3.1817 | 2.2857 | 1.8986 | 1.6869 | 1.4643 | 1.2801 | 1.1556 |
| | 10 | 93.9611 | 6.0892 | 2.9734 | 2.1653 | 1.8150 | 1.6232 | 1.4213 | 1.2542 | 1.1411 |
| | 12 | 85.2763 | 5.6653 | 2.8128 | 2.0704 | 1.7483 | 1.5719 | 1.3862 | 1.2327 | 1.1290 |
| | 15 | 76.0130 | 5.1884 | 2.6262 | 1.9580 | 1.6682 | 1.5096 | 1.3431 | 1.2059 | 1.1138 |
| 9 | 9 | 178.6700 | 9.5428 | 4.1621 | 2.8290 | 2.2651 | 1.9508 | 1.6446 | 1.3863 | 1.2135 |
| | 10 | 173.1245 | 9.3454 | 4.0990 | 2.7953 | 2.2429 | 1.9445 | 1.6341 | 1.3802 | 1.2103 |
| | 12 | 164.3024 | 9.0221 | 3.9942 | 2.7388 | 2.2055 | 1.9169 | 1.6163 | 1.3699 | 1.2048 |
| | 15 | 154.5767 | 8.6518 | 3.8718 | 2.6721 | 2.1610 | 1.8840 | 1.5948 | 1.3575 | 1.1980 |
| 10 | 10 | 195.1260 | 10.1286 | 4.3493 | 2.9290 | 2.3309 | 2.0092 | 1.6757 | 1.4042 | 1.2231 |
| | 11 | 190.5550 | 9.9719 | 4.3002 | 2.9030 | 2.3140 | 1.9968 | 1.6678 | 1.3996 | 1.2207 |
| | 12 | 186.5931 | 9.8340 | 4.2566 | 2.8799 | 2.2988 | 2.9857 | 1.6607 | 1.3956 | 1.2185 |
| | 13 | 183.1066 | 9.7109 | 4.2175 | 2.8591 | 2.2851 | 2.2851 | 1.6542 | 1.3919 | 1.2166 |
| | 14 | 180.0002 | 9.5998 | 4.1819 | 2.8401 | 2.2726 | 2.2726 | 1.6483 | 1.3885 | 1.2147 |
| | 15 | 177.2036 | 9.4987 | 4.1494 | 2.8226 | 2.2611 | 1.9580 | 1.6428 | 1.3853 | 1.2130 |
| 15 | 15 | 271.4854 | 12.5911 | 5.0988 | 3.3182 | 2.5827 | 2.1919 | 1.7914 | 1.4696 | 1.2576 |
| | 16 | 269.1996 | 12.5234 | 5.0791 | 3.3082 | 2.5764 | 2.1873 | 1.7886 | 1.4680 | 1.2568 |
| | 17 | 267.1240 | 12.4616 | 5.0610 | 3.2990 | 2.5705 | 2.1831 | 1.7859 | 1.4665 | 1.2560 |
| | 18 | 265.2260 | 12.4048 | 5.0443 | 3.2906 | 2.5651 | 2.1792 | 1.7835 | 1.4652 | 1.2553 |

Table 3.2.1

$E[X_{m,n}]/\theta$ Using Equation (2a) for Two Parameter Weibull Samples

| | | β | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | n | 0.25 | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 2.00 | 3.00 | 5.00 |
| 3 | 3 | 67.7963 | 4.7222 | 2.4296 | 1.8333 | 1.5761 | 1.4361 | 1.2904 | 1.1718 | 1.0936 |
| | 4 | 44.1186 | 3.5069 | 1.9499 | 1.5417 | 1.3666 | 1.2724 | 1.1759 | 1.0999 | 1.0522 |
| | 5 | 32.7637 | 2.8093 | 1.6479 | 1.3481 | 1.2228 | 1.1573 | 1.0931 | 1.0463 | 1.0204 |
| | 6 | 26.1682 | 2.3552 | 1.4377 | 1.2080 | 1.1160 | 1.0703 | 1.0289 | 1.0037 | 0.9947 |
| | 10 | 14.8535 | 1.4642 | 0.9832 | 0.8858 | 0.8601 | 0.8557 | 0.8646 | 0.8901 | 0.9238 |
| | 15 | 9.9016 | 1.0183 | 0.7272 | 0.6893 | 0.6955 | 0.7122 | 0.7489 | 0.8058 | 0.8687 |
| 5 | 5 | 107.5326 | 6.6772 | 3.1816 | 2.2833 | 1.8958 | 1.6842 | 1.4620 | 1.2784 | 1.1545 |
| | 6 | 92.8072 | 6.0341 | 2.9518 | 2.1521 | 1.8055 | 1.6157 | 1.4160 | 1.2508 | 1.1392 |
| | 8 | 74.8683 | 5.1664 | 2.6246 | 1.9594 | 1.6702 | 1.5117 | 1.3450 | 1.2074 | 1.1147 |
| | 10 | 64.0825 | 4.5921 | 2.3959 | 1.8202 | 1.5703 | 1.4338 | 1.2908 | 1.1735 | 1.0953 |
| | 12 | 56.7300 | 4.1748 | 2.2229 | 1.7122 | 1.4916 | 1.3716 | 1.2468 | 1.1456 | 1.0791 |
| | 15 | 49.1011 | 3.7174 | 2.0261 | 1.5865 | 1.3984 | 1.2972 | 1.1934 | 1.1111 | 1.0587 |
| 6 | 6 | 126.2316 | 7.4939 | 3.4735 | 2.4500 | 2.0105 | 1.7711 | 1.5203 | 1.3134 | 1.1739 |
| | 8 | 106.3499 | 6.6583 | 3.1817 | 2.2857 | 1.8986 | 1.6869 | 1.4643 | 1.2801 | 1.1556 |
| | 10 | 93.9611 | 6.0892 | 2.9734 | 2.1653 | 1.8150 | 1.6232 | 1.4213 | 1.2542 | 1.1411 |
| | 12 | 85.2763 | 5.6653 | 2.8128 | 2.0704 | 1.7483 | 1.5719 | 1.3862 | 1.2327 | 1.1290 |
| | 15 | 76.0130 | 5.1884 | 2.6262 | 1.9580 | 1.6682 | 1.5096 | 1.3431 | 1.2059 | 1.1138 |
| 9 | 9 | 178.6700 | 9.5428 | 4.1621 | 2.8290 | 2.2651 | 1.9508 | 1.6446 | 1.3863 | 1.2135 |
| | 10 | 173.1245 | 9.3454 | 4.0990 | 2.7953 | 2.2429 | 1.9445 | 1.6341 | 1.3802 | 1.2103 |
| | 12 | 164.3024 | 9.0221 | 3.9942 | 2.7388 | 2.2055 | 1.9169 | 1.6163 | 1.3699 | 1.2048 |
| | 15 | 154.5767 | 8.6518 | 3.8718 | 2.6721 | 2.1610 | 1.8840 | 1.5948 | 1.3575 | 1.1980 |
| 10 | 10 | 195.1260 | 10.1286 | 4.3493 | 2.9290 | 2.3309 | 2.0092 | 1.6757 | 1.4042 | 1.2231 |
| | 11 | 190.5550 | 9.9719 | 4.3002 | 2.9030 | 2.3140 | 1.9968 | 1.6678 | 1.3996 | 1.2207 |
| | 12 | 186.5931 | 9.8340 | 4.2566 | 2.8799 | 2.2988 | 2.9857 | 1.6607 | 1.3956 | 1.2185 |
| | 13 | 183.1066 | 9.7109 | 4.2175 | 2.8591 | 2.2851 | 2.2851 | 1.6542 | 1.3919 | 1.2166 |
| | 14 | 180.0002 | 9.5998 | 4.1819 | 2.8401 | 2.2726 | 2.2726 | 1.6483 | 1.3885 | 1.2147 |
| | 15 | 177.2036 | 9.4987 | 4.1494 | 2.8226 | 2.2611 | 1.9580 | 1.6428 | 1.3853 | 1.2130 |
| 15 | 15 | 271.4854 | 12.5911 | 5.0988 | 3.3182 | 2.5827 | 2.1919 | 1.7914 | 1.4696 | 1.2576 |
| | 16 | 269.1996 | 12.5234 | 5.0791 | 3.3082 | 2.5764 | 2.1873 | 1.7886 | 1.4680 | 1.2568 |
| | 17 | 267.1240 | 12.4616 | 5.0610 | 3.2990 | 2.5705 | 2.1831 | 1.7859 | 1.4665 | 1.2560 |
| | 18 | 265.2260 | 12.4048 | 5.0443 | 3.2906 | 2.5651 | 2.1792 | 1.7835 | 1.4652 | 1.2553 |

Table 3.2.2

$E[X_{m,n}^{(2)}]/\theta^2$ Using Equation (2a) for Two Parameter Weibull Samples

| | | β | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | n | 0.25 | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 2.00 | 3.00 | 5.00 |
| 3 | 3 | 120493.650 | 67.7963 | 10.3553 | 4.7222 | 3.1206 | 2.4296 | 1.8333 | 1.4361 | 1.2163 |
| | 4 | 73984.104 | 44.1186 | 7.2519 | 3.5069 | 2.4220 | 1.9499 | 1.5417 | 1.2724 | 1.1280 |
| | 5 | 53767.836 | 32.7637 | 5.5998 | 2.8093 | 1.9990 | 1.6479 | 1.3481 | 1.1573 | 1.0630 |
| | 6 | 42515.391 | 26.1682 | 4.5781 | 2.3552 | 1.7130 | 1.4377 | 1.2080 | 1.0703 | 1.0119 |
| | 10 | 23836.477 | 14.8535 | 2.7018 | 1.4642 | 1.1204 | 0.9832 | 0.8858 | 0.8557 | 0.8778 |
| | 15 | 15831.932 | 9.9016 | 1.8288 | 1.0183 | 0.8039 | 0.7272 | 0.6893 | 0.7122 | 0.7803 |
| 5 | 5 | 200083.480 | 107.5326 | 15.4427 | 6.6772 | 4.2282 | 3.1816 | 2.2833 | 1.6842 | 1.3477 |
| | 6 | 168907.700 | 92.8072 | 13.6775 | 6.0341 | 3.8786 | 2.9518 | 2.1521 | 1.6157 | 1.3132 |
| | 8 | 133038.210 | 74.8683 | 11.3961 | 5.1664 | 3.3921 | 2.6246 | 1.9594 | 1.5117 | 1.2593 |
| | 10 | 112417.230 | 64.0825 | 9.9483 | 4.5921 | 3.0599 | 2.3959 | 1.8202 | 1.4338 | 1.2176 |
| | 12 | 98708.375 | 56.7300 | 8.9265 | 4.1748 | 2.8130 | 2.2229 | 1.7122 | 1.3716 | 1.1835 |
| | 15 | 84748.908 | 49.1011 | 7.8355 | 3.7174 | 2.5367 | 2.0261 | 1.5865 | 1.2972 | 1.1416 |
| 6 | 6 | 239671.77 | 126.2316 | 17.6842 | 7.4939 | 4.6722 | 3.4735 | 2.4500 | 1.7711 | 1.3914 |
| | 8 | 196793.08 | 106.3499 | 15.3514 | 6.6583 | 4.2237 | 3.1817 | 2.2857 | 1.6869 | 1.3497 |
| | 10 | 171372.94 | 93.9611 | 13.8210 | 6.0892 | 3.9100 | 2.9734 | 2.1653 | 1.6232 | 1.3173 |
| | 12 | 154065.61 | 85.2763 | 12.7108 | 5.6653 | 3.6726 | 2.8128 | 2.0704 | 1.5719 | 1.2907 |
| | 15 | 136020.15 | 76.0119 | 11.4919 | 5.1887 | 3.3988 | 2.6262 | 1.9579 | 1.5096 | 1.2577 |
| 9 | 9 | 357659.91 | 178.6700 | 23.5822 | 9.5428 | 5.7477 | 4.1621 | 2.8290 | 1.9608 | 1.4834 |
| | 10 | 344489.43 | 173.1244 | 22.9911 | 9.3454 | 5.6470 | 4.0990 | 2.7953 | 1.9445 | 1.4758 |
| | 12 | 323931.28 | 164.3024 | 22.0345 | 9.0221 | 5.4807 | 3.9942 | 2.7388 | 1.9169 | 1.4627 |
| | 15 | 301794.03 | 154.5767 | 20.9560 | 8.6518 | 5.2881 | 3.8718 | 2.6721 | 1.8840 | 1.4469 |
| 10 | 10 | 396742.46 | 195.1260 | 25.3362 | 10.1286 | 6.0465 | 4.3493 | 2.9290 | 2.0092 | 1.5062 |
| | 11 | 385624.10 | 190.5551 | 24.8597 | 9.9719 | 5.9674 | 4.3002 | 2.9030 | 1.9968 | 1.5004 |
| | 12 | 376089.91 | 186.5931 | 24.4428 | 9.8340 | 5.8976 | 4.2566 | 2.8799 | 1.9857 | 1.4953 |
| | 13 | 367776.73 | 183.1066 | 24.0729 | 9.7109 | 5.8349 | 4.2175 | 2.8591 | 1.9756 | 1.4905 |
| | 14 | 360428.78 | 180.0002 | 23.7408 | 9.5998 | 5.7782 | 4.1819 | 2.8401 | 1.9664 | 1.4862 |
| | 15 | 353860.09 | 177.2036 | 23.4400 | 9.4987 | 5.7265 | 4.1494 | 2.8226 | 1.9580 | 1.4822 |
| 15 | 15 | 590441.92 | 271.4854 | 33.0123 | 12.5911 | 7.2675 | 5.0988 | 3.3182 | 2.1919 | 1.5900 |
| | 16 | 584291.33 | 269.1996 | 32.7940 | 12.5234 | 7.2348 | 5.0791 | 3.3082 | 2.1873 | 1.5880 |
| | 17 | 578730.95 | 267.1240 | 32.5959 | 12.4616 | 7.2048 | 5.0610 | 3.2990 | 2.1831 | 1.5861 |
| | 18 | 573666.46 | 265.2260 | 32.4124 | 12.4048 | 7.1773 | 5.0443 | 3.2906 | 2.1792 | 1.5843 |

Table 3.2.3

*Std.Dev.*$[X_{m,n}]/\theta$ Using Equation (2a) Using Two Parameter Weibull Samples

| | | β | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | n | 0.25 | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 2.00 | 3.00 | 5.00 |
| 3 | 3 | 340.4369 | 6.7451 | 2.1101 | 1.1667 | 0.7978 | 0.6059 | 0.4102 | 0.2510 | 0.1424 |
| | 4 | 268.3983 | 5.6409 | 1.8574 | 1.0631 | 0.7446 | 0.5753 | 0.3987 | 0.2501 | 0.1449 |
| | 5 | 229.5526 | 4.9871 | 1.6983 | 0.9959 | 0.7097 | 0.5554 | 0.3916 | 0.2503 | 0.1474 |
| | 6 | 204.5253 | 4.5410 | 1.5846 | 0.9466 | 0.6837 | 0.5405 | 0.3864 | 0.2508 | 0.1497 |
| | 10 | 153.6745 | 3.5651 | 1.3172 | 0.8244 | 0.6169 | 0.5009 | 0.3719 | 0.2518 | 0.1560 |
| | 15 | 125.4348 | 2.9774 | 1.1402 | 0.7370 | 0.5659 | 0.4690 | 0.3584 | 0.2509 | 0.1601 |
| 5 | 5 | 434.1892 | 7.9339 | 2.3065 | 1.2098 | 0.7962 | 0.5875 | 0.3821 | 0.2234 | 0.1217 |
| | 6 | 400.3680 | 7.5098 | 2.2281 | 1.1843 | 0.7866 | 0.5842 | 0.3833 | 0.2262 | 0.1242 |
| | 8 | 356.9775 | 6.9409 | 2.1230 | 1.1521 | 0.7762 | 0.5825 | 0.3876 | 0.2323 | 0.1292 |
| | 10 | 329.1059 | 6.5571 | 2.0513 | 1.1310 | 0.7707 | 0.5832 | 0.3925 | 0.2382 | 0.1339 |
| | 12 | 309.0147 | 6.2691 | 1.9963 | 1.1149 | 0.7669 | 0.5844 | 0.3971 | 0.2435 | 0.1436 |
| | 15 | 286.9460 | 5.9398 | 1.9315 | 1.0957 | 0.7623 | 0.5859 | 0.4029 | 0.2504 | 0.1590 |
| 6 | 6 | 473.0088 | 8.3710 | 2.3706 | 1.2212 | 0.7937 | 0.5801 | 0.3725 | 0.2147 | 0.1155 |
| | 8 | 430.6771 | 7.8751 | 2.2865 | 1.1974 | 0.7868 | 0.5798 | 0.3763 | 0.2194 | 0.1192 |
| | 10 | 403.1678 | 7.5421 | 2.2315 | 1.1835 | 0.7847 | 0.5819 | 0.3810 | 0.2243 | 0.1228 |
| | 12 | 383.1365 | 7.2925 | 2.1906 | 1.1741 | 0.7849 | 0.5848 | 0.3858 | 0.2289 | 0.1263 |
| | 15 | 360.8908 | 7.0066 | 2.1436 | 1.1640 | 0.7848 | 0.5893 | 0.3924 | 0.2354 | 0.1310 |
| 9 | 9 | 570.7337 | 9.3597 | 2.5018 | 1.2409 | 0.7856 | 0.5634 | 0.3526 | 0.1976 | 0.1038 |
| | 10 | 560.8185 | 9.2622 | 2.4878 | 1.2377 | 0.7851 | 0.5639 | 0.3536 | 0.1986 | 0.1045 |
| | 12 | 544.9184 | 9.1051 | 2.4660 | 1.2333 | 0.7851 | 0.5653 | 0.3557 | 0.2005 | 0.1058 |
| | 15 | 527.1623 | 8.9288 | 2.4424 | 1.2294 | 0.7861 | 0.5678 | 0.3588 | 0.2032 | 0.1076 |
| 10 | 10 | 598.8892 | 9.6196 | 2.5337 | 1.2449 | 0.7831 | 0.5590 | 0.3477 | 0.1936 | 0.1011 |
| | 11 | 591.0270 | 9.5454 | 2.5235 | 1.2427 | 0.7829 | 0.5595 | 0.3485 | 0.1944 | 0.1016 |
| | 12 | 584.1857 | 9.4808 | 2.5147 | 1.2410 | 0.7829 | 0.5601 | 0.3493 | 0.1951 | 0.1021 |
| | 13 | 578.1425 | 9.4237 | 2.5071 | 1.2396 | 0.7830 | 0.5607 | 0.3501 | 0.1958 | 0.1026 |
| | 14 | 572.7379 | 9.3725 | 2.5004 | 1.2385 | 0.7833 | 0.5613 | 0.3509 | 0.1965 | 0.1031 |
| | 15 | 567.8547 | 9.3262 | 2.4945 | 1.2375 | 0.7835 | 0.5619 | 0.3517 | 0.1971 | 0.1035 |
| 15 | 15 | 718.8446 | 10.6278 | 2.6485 | 1.2572 | 0.7727 | 0.5426 | 0.3302 | 0.1796 | 0.0920 |
| | 16 | 715.4180 | 10.6002 | 2.6452 | 1.2566 | 0.7728 | 0.5429 | 0.3305 | 0.1799 | 0.0922 |
| | 17 | 712.3031 | 10.5751 | 2.6422 | 1.2562 | 0.7729 | 0.5431 | 0.3309 | 0.1802 | 0.0923 |
| | 18 | 709.4516 | 10.5521 | 2.6396 | 1.2558 | 0.7730 | 0.5434 | 0.3312 | 0.1804 | 0.0925 |

Values of the coefficient of variation ( $CV$ ) of experiment time are reported in Table 3.2.4. The coefficient of variation, also referred to as the standardized second central moment, is defined to be

$$CV = Std.Dev.[X_{m,n}] / E[X_{m,n}] \ .$$

The $CV$ is applicable when the variable measured is on a ratio scale (i.e., has an absolute zero). The $CV$ is a simple method to compare two or more populations with respect to variability. This ratio of standard deviation to expected value is a unitless measure of spread. The CV is used to compare the relative amount of variability of expected experiment time for different values of β or $m$ and $n$.

As one would expect, the $CV$ for a complete sample ($m = n$ ) is the smaller than the $CV$'s for the censored sample ( $m<n$ ). Further, the $CV$s for experiment time in Table 3.2.4 suggest:

(1) for fixed values of $m$ and β, the coefficient of variation increases as $n$ increases, and

(2) for fixed values of $m$ and $n$, the coefficient of variation decreases as β increases

( i.e., for fixed values of $m$ and $n$, more precise estimates of $E[X_{m,n}]$ may be obtained for large values of β).

Table 3.2.4

Coefficient of Variation of $X_{m,n}$ Using Equation (2a) for Two Parameter Weibull Samples

| | | β | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | n | 0.25 | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 2.00 | 3.00 | 5.00 |
| 3 | 3 | 5.0215 | 1.4284 | 0.8685 | 0.6364 | 0.5062 | 0.4219 | 0.3179 | 0.2142 | 0.1302 |
| | 4 | 6.0836 | 1.6085 | 0.9525 | 0.6896 | 0.5449 | 0.4522 | 0.3391 | 0.2274 | 0.1377 |
| | 5 | 7.0063 | 1.7752 | 1.0305 | 0.7387 | 0.5804 | 0.4799 | 0.3583 | 0.2393 | 0.1445 |
| | 6 | 7.8158 | 1.9281 | 1.1022 | 0.7836 | 0.6127 | 0.5050 | 0.3755 | 0.2499 | 0.1505 |
| | 10 | 10.3460 | 2.4349 | 1.3397 | 0.9307 | 0.7172 | 0.5854 | 0.4301 | 0.2828 | 0.1688 |
| | 15 | 12.6681 | 2.9239 | 1.5681 | 1.0693 | 0.8138 | 0.6585 | 0.4786 | 0.3113 | 0.1843 |
| 5 | 5 | 4.0377 | 1.1882 | 0.7249 | 0.5298 | 0.4200 | 0.3488 | 0.2614 | 0.1747 | 0.1054 |
| | 6 | 4.3140 | 1.2446 | 0.7548 | 0.5503 | 0.4357 | 0.3615 | 0.2707 | 0.1808 | 0.1090 |
| | 8 | 4.7681 | 1.3435 | 0.8089 | 0.5880 | 0.4648 | 0.3854 | 0.2882 | 0.1924 | 0.1159 |
| | 10 | 5.1357 | 1.4279 | 0.8562 | 0.6213 | 0.4908 | 0.4068 | 0.3041 | 0.2029 | 0.1223 |
| | 12 | 5.4471 | 1.5017 | 0.8981 | 0.6512 | 0.5141 | 0.4260 | 0.3185 | 0.2125 | 0.1331 |
| | 15 | 5.8440 | 1.5978 | 0.9533 | 0.6906 | 0.5451 | 0.4517 | 0.3376 | 0.2254 | 0.1502 |
| 6 | 6 | 3.7472 | 1.1170 | 0.6825 | 0.4985 | 0.3947 | 0.3275 | 0.2451 | 0.1635 | 0.0984 |
| | 8 | 4.0496 | 1.1828 | 0.7286 | 0.5238 | 0.4144 | 0.3437 | 0.2570 | 0.1714 | 0.1032 |
| | 10 | 4.2929 | 1.2386 | 0.7505 | 0.5466 | 0.4323 | 0.3585 | 0.2681 | 0.1788 | 0.1076 |
| | 12 | 4.4929 | 1.2872 | 0.7788 | 0.5671 | 0.4489 | 0.3720 | 0.2783 | 0.1857 | 0.1119 |
| | 15 | 4.7477 | 1.3504 | 0.8162 | 0.5945 | 0.4704 | 0.3903 | 0.2922 | 0.1952 | 0.1176 |
| 9 | 9 | 3.1943 | 0.9808 | 0.6011 | 0.4386 | 0.3468 | 0.2873 | 0.2144 | 0.1426 | 0.0855 |
| | 10 | 3.2394 | 0.9911 | 0.6069 | 0.4428 | 0.3501 | 0.2900 | 0.2164 | 0.1439 | 0.0863 |
| | 12 | 3.3166 | 1.0092 | 0.6174 | 0.4503 | 0.3560 | 0.2949 | 0.2201 | 0.1464 | 0.0878 |
| | 15 | 3.4104 | 1.0320 | 0.6308 | 0.4601 | 0.3638 | 0.3014 | 0.2250 | 0.1967 | 0.0899 |
| 10 | 10 | 3.0692 | 0.9497 | 0.5825 | 0.4250 | 0.3359 | 0.2782 | 0.2075 | 0.1379 | 0.0827 |
| | 11 | 3.1016 | 0.9572 | 0.5868 | 0.4281 | 0.3383 | 0.2802 | 0.2090 | 0.1389 | 0.0833 |
| | 12 | 3.1308 | 0.9641 | 0.5908 | 0.4309 | 0.3406 | 0.2821 | 0.2104 | 0.1398 | 0.0838 |
| | 13 | 3.1574 | 0.9704 | 0.5945 | 0.4336 | 0.3427 | 0.2838 | 0.2117 | 0.1407 | 0.0843 |
| | 14 | 3.1819 | 0.9763 | 0.5979 | 0.4361 | 0.3446 | 0.2854 | 0.2129 | 0.1415 | 0.0849 |
| | 15 | 3.2045 | 0.9818 | 0.6012 | 0.4384 | 0.3465 | 0.2870 | 0.2141 | 0.1423 | 0.0853 |
| 15 | 15 | 2.6478 | 0.84407 | 0.5194 | 0.3789 | 0.2992 | 0.2475 | 0.1843 | 0.1222 | 0.0731 |
| | 16 | 2.6576 | 0.8464 | 0.5208 | 0.3798 | 0.3000 | 0.2482 | 0.1848 | 0.1226 | 0.0733 |
| | 17 | 2.6666 | 0.8486 | 0.5221 | 0.3808 | 0.3007 | 0.2488 | 0.1853 | 0.1229 | 0.0735 |
| | 18 | 2.6749 | 0.8506 | 0.5233 | 0.3817 | 0.3014 | 0.2494 | 0.1857 | 0.1231 | 0.0737 |

Different table values from those reported by Tse and Yuen (1998) are obtained by using equation (2b) where the experimenter assumes that all removal schemes are equally likely. Another way to view the resulting values using equation (2b) are as the averages of the moments of experiment time of all Type II progressive censoring removal schemes.

Tables 3.2.5 - 3.2.8 display the values for the first and second moments, standard deviations, and *CVs* of experiment time, respectively, using equation (2b). For both equation (2a) and equation (2b), the expected experiment time does not decrease appreciably as $n$ increases if $\beta$ is large. Since shorter experiment times are a major reason why progressive censoring schemes are used, this result should be considered: for both probability assignments to random removals, Type II progressive censoring schemes did not result in appreciably shortened experiments in the for Weibull-distributed lifetimes with increasing failure rates. However, only two probability assignments for random removals are considered in this research. Many other probability assignments for random removals are possible. For example, the removals may occur according to any number of multinomial probability distributions.

For all values of $\beta$, the expected experiment time values in Table 3.2.5, using equation (2b), are smaller than the corresponding expected experiment time values in Table 3.2.1, using equation (2a). This is because the probabilities assigned with equation (2a) give heavier weight to removal schemes with early removals; consequently, the larger expected experiment time values follow.

The standard deviations reported in Table 3.2.7, using equation (2b), are still large if $\beta<1$ and are approximately of the same magnitude if $\beta\geq1$. Therefore, for both probability assignments for random removals, the expected experiment time values should be used cautiously in instances of Weibull-distributed lifetimes with decreasing failure rates.

The *CVs* in Table 3.2.8, using equation (2b) suggest

(1) for fixed values of $m$ and $\beta$, the coefficient of variation increases as $n$ increases and,

(2) for fixed values of $m$ and $n$, the coefficient of variation decreases as $\beta$ increases .

These are the same relationships suggested in Table 3.2.4, using equation (2a).

Table 3.2.5

$E[X_{m,n}]/\theta$ Using Equation (2b) for Two Parameter Weibull Samples

| | | β | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | n | 0.25 | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 2.00 | 3.00 | 5.00 |
| 3 | 3 | 67.7963 | 4.7222 | 2.4296 | 1.8333 | 1.5761 | 1.4361 | 1.2904 | 1.1718 | 1.0936 |
| | 4 | 38.4811 | 3.2176 | 1.8357 | 1.4722 | 1.3167 | 1.2333 | 1.1486 | 1.0828 | 1.0423 |
| | 5 | 25.7517 | 2.4064 | 1.4789 | 1.2417 | 1.1446 | 1.0952 | 1.0487 | 1.0178 | 1.0037 |
| | 6 | 18.9450 | 1.9059 | 1.2402 | 1.0800 | 1.0202 | 0.9932 | 0.9730 | 0.9672 | 0.9730 |
| | 10 | 8.7053 | 1.0096 | 0.7582 | 0.7287 | 0.7365 | 0.7526 | 0.7861 | 0.8363 | 0.8904 |
| | 12 | 6.7530 | 0.8094 | 0.6362 | 0.6324 | 0.6547 | 0.6807 | 0.7276 | 0.7933 | 0.8622 |
| | 15 | 5.0149 | 0.6200 | 0.5131 | 0.5131 | 0.5661 | 0.6013 | 0.6613 | 0.7432 | 0.8285 |
| 6 | 6 | 126.2316 | 7.4939 | 3.4735 | 2.4500 | 2.0105 | 1.7711 | 1.5203 | 1.3134 | 1.1739 |
| | 8 | 71.5299 | 5.1355 | 2.6393 | 1.9770 | 1.6867 | 1.5266 | 1.3571 | 1.2160 | 1.1202 |
| | 10 | 47.1520 | 3.8405 | 2.1312 | 1.6716 | 1.4698 | 1.3585 | 1.2413 | 1.1445 | 1.0796 |
| | 12 | 34.0434 | 3.0340 | 1.7885 | 1.4560 | 1.3119 | 1.2338 | 1.1530 | 1.0885 | 1.0471 |
| | 15 | 23.2475 | 2.2781 | 1.4419 | 1.2279 | 1.1401 | 1.0952 | 1.0524 | 1.0230 | 1.0082 |
| 9 | 9 | 178.6700 | 9.5428 | 4.1621 | 2.8290 | 2.2651 | 1.9608 | 1.6446 | 1.3863 | 1.2135 |
| | 10 | 143.4000 | 8.2873 | 3.7608 | 2.6146 | 2.1239 | 1.8571 | 1.5778 | 1.3479 | 1.1930 |
| | 12 | 98.9301 | 6.4957 | 3.1537 | 2.2796 | 1.8987 | 1.6892 | 1.4678 | 1.2834 | 1.1580 |
| | 15 | 63.8395 | 4.8260 | 2.5395 | 1.9247 | 1.6530 | 1.5024 | 1.3421 | 1.2078 | 1.1160 |
| 15 | 15 | 271.4855 | 12.5911 | 5.0988 | 3.3182 | 2.5827 | 2.1919 | 1.7914 | 1.4696 | 1.2576 |
| | 16 | 235.2294 | 11.5179 | 4.7857 | 3.1595 | 2.4817 | 2.1193 | 1.7461 | 1.4443 | 1.2445 |
| | 17 | 205.8252 | 10.5933 | 4.5086 | 3.0170 | 2.3900 | 2.0532 | 1.7045 | 1.4210 | 1.2322 |
| | 18 | 181.6622 | 9.7897 | 4.2616 | 2.8813 | 2.3065 | 1.9925 | 1.6660 | 1.3992 | 1.2207 |

Table 3.2.6

$E[X_{m,n}^{(2)}]/\theta^2$ Using Equation (2b) for Two Parameter Weibull Samples

| | | β | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | n | 0.25 | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 2.00 | 3.00 | 5.00 |
| 3 | 3 | 120493.65 | 67.7963 | 10.3553 | 4.7222 | 3.1206 | 2.4296 | 1.8333 | 1.4361 | 1.2163 |
| | 4 | 62910.404 | 38.4811 | 6.5131 | 3.2176 | 2.2557 | 1.8357 | 1.4722 | 1.2334 | 1.1070 |
| | 5 | 40776.812 | 25.7517 | 4.6193 | 2.4064 | 1.7591 | 1.4789 | 1.2417 | 1.0952 | 1.0284 |
| | 6 | 29572.171 | 18.9450 | 3.5227 | 1.9059 | 1.4383 | 1.2402 | 1.0800 | 0.9932 | 0.9678 |
| | 10 | 13401.926 | 8.7053 | 1.7189 | 1.0096 | 0.8234 | 0.7582 | 0.7287 | 0.7526 | 0.8140 |
| | 12 | 10388.991 | 6.7530 | 1.3484 | 0.8094 | 0.6761 | 0.6362 | 0.6324 | 0.6807 | 0.7645 |
| | 15 | 7717.1611 | 5.0149 | 1.0103 | 0.6200 | 0.5317 | 0.5131 | 0.5312 | 0.6013 | 0.7074 |
| 6 | 6 | 239671.77 | 126.2316 | 17.6842 | 7.4939 | 4.6722 | 3.4735 | 2.4500 | 1.7711 | 1.3914 |
| | 8 | 123523.32 | 71.5299 | 11.1687 | 5.1355 | 3.3972 | 2.6393 | 1.9770 | 1.5266 | 1.2694 |
| | 10 | 77671.749 | 47.1520 | 7.8864 | 3.8405 | 2.6539 | 2.1312 | 1.6716 | 1.3585 | 1.1810 |
| | 12 | 54613.741 | 34.0434 | 5.9676 | 3.0339 | 2.1691 | 1.7885 | 1.4560 | 1.2338 | 1.1125 |
| | 15 | 36494.41 | 23.2475 | 4.2716 | 2.2781 | 1.6946 | 1.4419 | 1.2279 | 1.0952 | 1.0332 |
| 9 | 9 | 357659.91 | 178.6700 | 23.5822 | 9.5428 | 5.7477 | 4.1621 | 2.8290 | 1.9608 | 1.4834 |
| | 10 | 273894.37 | 143.4000 | 19.8229 | 8.2873 | 5.1073 | 3.7608 | 2.6146 | 1.8571 | 1.4346 |
| | 12 | 176822.89 | 98.9301 | 14.7182 | 6.4957 | 4.1618 | 3.1537 | 2.2796 | 1.6892 | 1.3532 |
| | 15 | 107476.39 | 63.8395 | 10.2747 | 4.8260 | 3.2377 | 2.5395 | 1.9247 | 1.5024 | 1.2587 |
| 15 | 15 | 590441.92 | 271.4855 | 33.0123 | 12.5911 | 7.2675 | 5.0988 | 3.3182 | 2.1919 | 1.5900 |
| | 16 | 492887.52 | 235.2294 | 29.5485 | 11.5179 | 6.7482 | 4.7857 | 3.1595 | 2.1193 | 1.5575 |
| | 17 | 417500.76 | 205.8252 | 26.6334 | 10.5933 | 6.2937 | 4.5086 | 3.0170 | 2.0532 | 1.5274 |
| | 18 | 358150.84 | 181.6622 | 24.1548 | 9.7897 | 5.8928 | 4.2616 | 2.8881 | 1.9925 | 1.4996 |

Table 3.2.7

*Std.Dev.*$[X_{m,n}]/\theta$ Using Equation (2b) for Two Parameter Weibull Samples

| | | β | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | n | 0.25 | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 2.00 | 3.00 | 5.00 |
| 3 | 3 | 340.4369 | 6.7451 | 2.1101 | 1.1667 | 0.7978 | 0.6058 | 0.4102 | 0.2510 | 0.1424 |
| | 4 | 247.8500 | 5.3036 | 1.7729 | 1.0248 | 0.7225 | 0.5608 | 0.3910 | 0.2468 | 0.1438 |
| | 5 | 200.2840 | 4.4678 | 1.5595 | 0.9299 | 0.6702 | 0.5287 | 0.3767 | 0.2435 | 0.1447 |
| | 6 | 170.9189 | 3.9131 | 1.4088 | 0.8599 | 0.6305 | 0.5037 | 0.3652 | 0.2405 | 0.1453 |
| | 10 | 115.4389 | 2.7724 | 1.0696 | 0.6919 | 0.5301 | 0.4380 | 0.3327 | 0.2196 | 0.1459 |
| | 12 | 101.7025 | 2.4694 | 0.9715 | 0.6399 | 0.4975 | 0.4157 | 0.3209 | 0.2267 | 0.1456 |
| | 15 | 87.7041 | 2.1519 | 0.8643 | 0.5812 | 0.4596 | 0.3893 | 0.3064 | 0.2214 | 0.1450 |
| 6 | 6 | 473.0088 | 8.3710 | 2.3704 | 1.2212 | 0.7937 | 0.5801 | 0.3725 | 0.2147 | 0.1155 |
| | 8 | 344.1031 | 6.7199 | 2.0501 | 1.1076 | 0.7432 | 0.5557 | 0.3677 | 0.2188 | 0.1209 |
| | 10 | 274.6788 | 5.6923 | 1.8288 | 1.0228 | 0.7028 | 0.5345 | 0.3618 | 0.2206 | 0.1244 |
| | 12 | 231.2029 | 4.9839 | 1.6640 | 0.9560 | 0.6693 | 0.5160 | 0.3558 | 0.2212 | 0.1268 |
| | 15 | 189.6153 | 4.2494 | 1.4807 | 0.8776 | 0.6282 | 0.4923 | 0.3469 | 0.2209 | 0.1292 |
| 9 | 9 | 570.7337 | 9.3597 | 2.5018 | 1.2409 | 0.7856 | 0.5634 | 0.3526 | 0.1976 | 0.1038 |
| | 10 | 503.3198 | 8.6441 | 2.3831 | 1.2046 | 0.7721 | 0.5587 | 0.3537 | 0.2007 | 0.1065 |
| | 12 | 408.6872 | 7.5323 | 2.1846 | 1.1398 | 0.7462 | 0.5480 | 0.3538 | 0.2049 | 0.1106 |
| | 15 | 321.5601 | 6.36785 | 1.9559 | 1.0590 | 0.7109 | 0.5314 | 0.3513 | 0.2086 | 0.1150 |
| 15 | 15 | 660.1046 | 10.6278 | 2.6485 | 1.2572 | 0.7727 | 0.5426 | 0.3302 | 0.1796 | 0.0920 |
| | 16 | 661.4791 | 10.1275 | 2.5779 | 1.2391 | 0.7679 | 0.5423 | 0.3325 | 0.1823 | 0.0939 |
| | 17 | 612.4841 | 9.6751 | 2.5112 | 1.2212 | 0.7625 | 0.5414 | 0.3342 | 0.1846 | 0.0957 |
| | 18 | 570.2190 | 9.2641 | 2.4482 | 1.2035 | 0.7568 | 0.5400 | 0.3355 | 0.1865 | 0.9725 |

Table 3.2.8

Coefficient of Variation of $X_{m,n}$ Using Equation (2b) for Two Parameter Weibull Samples

| | | β | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | n | 0.25 | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 2.00 | 3.00 | 5.00 |
| 3 | 3 | 5.0214 | 1.4284 | 0.8685 | 0.6364 | 0.5062 | 0.4219 | 0.3179 | 0.2142 | 0.1302 |
| | 4 | 6.4408 | 1.6483 | 0.9658 | 0.6961 | 0.5487 | 0.4547 | 0.3404 | 0.2279 | 0.1379 |
| | 5 | 7.7775 | 1.8566 | 1.0545 | 0.7489 | 0.5855 | 0.4827 | 0.3592 | 0.2392 | 0.1442 |
| | 6 | 9.0219 | 2.0532 | 1.1359 | 0.7962 | 0.6180 | 0.5071 | 0.3753 | 0.2487 | 0.1493 |
| | 10 | 13.2608 | 2.7460 | 1.4106 | 0.9495 | 0.7198 | 0.5819 | 0.4232 | 0.2617 | 0.1638 |
| | 12 | 15.0603 | 3.0507 | 1.5270 | 1.0119 | 0.7600 | 0.6107 | 0.4411 | 0.2858 | 0.1689 |
| | 15 | 17.4886 | 3.4709 | 1.6845 | 1.0942 | 0.8119 | 0.6475 | 0.4634 | 0.2979 | 0.1750 |
| 6 | 6 | 3.7472 | 1.1170 | 0.6824 | 0.4985 | 0.3947 | 0.3275 | 0.2451 | 0.1635 | 0.0984 |
| | 8 | 4.8107 | 1.3085 | 0.7767 | 0.5602 | 0.4406 | 0.3640 | 0.2710 | 0.1800 | 0.1079 |
| | 10 | 5.8254 | 1.4822 | 0.8581 | 0.6119 | 0.4782 | 0.3935 | 0.2915 | 0.1928 | 0.1152 |
| | 12 | 6.7914 | 1.6427 | 0.9304 | 0.6566 | 0.5102 | 0.4183 | 0.3085 | 0.2032 | 0.1211 |
| | 15 | 8.1564 | 1.8654 | 1.0269 | 0.7147 | 0.5510 | 0.4495 | 0.3297 | 0.2159 | 0.1282 |
| 9 | 9 | 3.1943 | 0.9808 | 0.6011 | 0.4386 | 0.3468 | 0.2873 | 0.2144 | 0.1426 | 0.0855 |
| | 10 | 3.5099 | 1.0431 | 0.6337 | 0.4607 | 0.3635 | 0.3008 | 0.2242 | 0.1489 | 0.0892 |
| | 12 | 4.1289 | 1.1596 | 0.6927 | 0.5000 | 0.3930 | 0.324 | 0.2411 | 0.1597 | 0.0955 |
| | 15 | 5.0370 | 1.3195 | 0.7702 | 0.5502 | 0.4300 | 0.3537 | 0.2618 | 0.1727 | 0.1030 |
| 15 | 15 | 2.4315 | 0.8441 | 0.5194 | 0.3789 | 0.2992 | 0.2475 | 0.1843 | 0.1222 | 0.0731 |
| | 16 | 2.8121 | 0.8793 | 0.5387 | 0.3922 | 0.3094 | 0.2559 | 0.1904 | 0.1262 | 0.0755 |
| | 17 | 2.9758 | 0.9133 | 0.5570 | 0.4048 | 0.3190 | 0.2637 | 0.1961 | 0.1299 | 0.0776 |
| | 18 | 3.1389 | 0.9463 | 0.5745 | 0.4167 | 0.3281 | 0.2710 | 0.2014 | 0.1333 | 0.0797 |

The Burr Type XII Example with Random Removals

The formulas may also be applied to other than Weibull-distributed lifetimes since the formula developed for this research is applicable to any distribution where the moments of the smallest order statistic are known. The Burr Type XII distribution is one possible model of lifetime distributed data if the lifetime distribution is L-shaped or unimodal. Consider $X$, a random variable for a Burr Type XII $(d, c)$ lifetime distribution, with cdf given by

$$F(x) = 1 - (x^c + 1)^{-d}, \quad x > 0, c > 0, d > 0$$

and pdf given by

$$f(x) = \frac{cdx^{c-1}}{(x^c + 1)^{d+1}}, \quad x > 0, c > 0, d > 0.$$

If $X$ is distributed as Burr Type XII $(d, c)$, then the pdf of the first order statistic, $X_{1,n}$, is distributed Burr Type XII $(dn, c)$ and

$$E[X_{1,n}^{(k)}] = dnB\left(\frac{k}{c}+1, dn-\frac{k}{c}\right) = \frac{\Gamma\left(\frac{k}{c}+1\right)\Gamma\left(dn-\frac{k}{c}\right)}{\Gamma(dn+1)}, \quad k < cd.$$

Moments of experiment time may be obtained by substituting

$$E[X_{1,j+n-R_m+1}] = d(j+n-R_m+1)B\left(\frac{1}{c}+1, d(j+n-R_m+1)-\frac{1}{c}\right)$$

$$= d(j+n-R_m+1)\,\frac{\Gamma\left(\frac{1}{c}+1\right)\Gamma\left(d(j+n-R_m+1)-\frac{1}{c}\right)}{\Gamma(d(j+n-R_m+1)+1)},$$

if $1 < cd$, in the formulas in this section.

As an example to illustrate this procedure, expected experiment times for Burr Type XII-distributed data are reported in Table 3.2.9 for $m = 6$ and $n = 6,8,10,12$.

This example also illustrates some limitations to using the formulas for expected experiment time with Burr Type XII-distributed data. Table 3.2.9 does not appear complete for all values of $c$ and $d$. This is because either

(a) the expectations $E[X_{1,j+n-R_m+1}]$ do not exist when $cd \leq 1$ (the upper left-hand corner of the table) or

(b) the expectations $E[X_{1,j+n-R_m+1}]$ exist, but the gamma function calculations required are too large to compute the beta function values necessary (the lower right-hand corner of the table).

Table 3.2.9

$E[X_{m,n}]$ Using Equation (2a ) Two Parameter Burr Type XII Samples

| | | | *d* | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| m | n | *c* | 0.25 | 0.50 | 1.00 | 2.0 | 5.0 | 10.0 | 15.0 |
| 6 | 6 | 0.25 | | | | | 5.8976 | 0.0444 | 0.0055 |
| | 8 | | | | | | 3.0577 | 0.0240 | 0.0030 |
| | 10 | | | | | | 1.9291 | 0.0155 | |
| | 12 | | | | | | 1.3590 | | |
| 6 | 6 | 0.50 | | | | | 0.7347 | 0.1125 | 0.0433 |
| | 8 | | | | | | 0.4587 | 0.0738 | 0.0289 |
| | 10 | | | | | | 0.3239 | 0.0537 | |
| | 12 | | | | | | 0.2460 | | |
| 6 | 6 | 1.0 | | | | 3.4329 | 0.6879 | 0.2877 | 0.1815 |
| | 8 | | | | | 2.3639 | 0.5269 | 0.2266 | 0.1441 |
| | 10 | | | | | 1.8065 | 0.4308 | 0.1886 | |
| | 12 | | | | | 1.4654 | 0.3664 | | |
| 6 | 6 | 2.0 | | | 4.2522 | 1.6288 | 0.7882 | 0.4551 | 0.4114 |
| | 8 | | | | 3.1363 | 1.3475 | 0.6854 | 0.4551 | 0.3642 |
| | 10 | | | | 2.5408 | 1.1741 | 0.6164 | 0.4129 | |
| | 12 | | | | 2.1681 | 1.0541 | 0.5658 | | |
| 6 | 6 | 5.0 | 19.5097 | 3.0980 | 1.6436 | 1.1851 | 0.8984 | 0.7605 | 0.6950 |
| | 8 | | 11.9097 | 2.4908 | 1.4640 | 1.0982 | 0.8484 | 0.7221 | 0.6610 |
| | 10 | | 8.5375 | 2.1542 | 1.3512 | 1.0390 | 0.8123 | 0.6937 | |
| | 12 | | 6.6883 | 1.9381 | 1.2719 | 0.9948 | 0.7842 | | |
| 6 | 6 | 10.0 | 3.1099 | 1.6831 | 1.2686 | 1.0844 | 0.9460 | 0.8707 | 0.8325 |
| | 8 | | 2.5109 | 1.5180 | 1.1983 | 1.0439 | 0.9191 | 0.8483 | 0.8117 |
| | 10 | | 2.1825 | 1.4176 | 1.1520 | 1.0153 | 0.8991 | 0.8313 | |
| | 12 | | 1.9742 | 1.3487 | 1.1181 | 0.9934 | 0.8833 | | |
| 6 | 6 | 15.0 | 2.0450 | 1.4032 | 1.1693 | 1.0546 | 0.9632 | 0.9115 | 0.8847 |
| | 8 | | 1.7836 | 1.3115 | 1.1260 | 1.0282 | 0.9487 | 0.8957 | 0.8698 |
| | 10 | | 1.6314 | 1.2540 | 1.0969 | 1.0093 | 0.9311 | 0.8837 | |
| | 12 | | 1.5308 | 1.2138 | 1.0754 | 0.9947 | 0.9201 | | |

Fixed Removals

A Weibull (1,1) Example with $m$=3 and $n$=4

Table 3.2.10 illustrates the differences between the equations for fixed removals and random removals. Consider Type II progressively censored data where $m$=3 and $n$=4 from a Weibull (1.1.) distribution.

For fixed removals, using equation (1a), the expected experiment time , $E[X_{3,4}]$, may be found for each of three possible fixed removal schemes:

(1) if $r_1 = 0, r_2 = 0, r_3 = 1$, then $E[X_{3,4}]$= **1.0833**,

(2) if $r_1 = 0, r_2 = 1, r_3 = 0$ $E[X_{3,4}]$ = **1.5833**, and

(3) if $r_1 = 1, r_2 = 0, r_3 = 0$, then $E[X_{3,4}]$= **1.7500**.

For random removals, using equation (2a) , since $P(r_1 = 0, r_2 = 0, r_3 = 1) = \frac{1}{4}$, $P(r_1 = 0, r_2 = 1, r_3 = 0) = \frac{1}{4}$, and $P(r_1 = 1, r_2 = 0, r_3 = 0) = \frac{1}{2}$, the expected experiment time in Table 3.2.1is

$$E[X_{3,4}] = \frac{1}{4}1.0833 + \frac{1}{4}1.0833 + \frac{1}{4}1.0833 = \mathbf{1.5417}.$$

For random removals, using equation (2b), since $P(r_1 = 0, r_2 = 0, r_3 = 1) = \frac{1}{3}$, $P(r_1 = 0, r_2 = 1, r_3 = 0) = \frac{1}{3}$, and $P(r_1 = 1, r_2 = 0, r_3 = 0) = \frac{1}{3}$, the expected experiment time value in Table 2.4.5 is

$$E[X_{3,4}] = \frac{1}{3}1.0833 + \frac{1}{3}1.5833 + \frac{1}{3}1.7500 = \mathbf{1.4722}$$

Table 3.2.10

$E[X_{3,4}]$ Using Equations (1a), (2a), and (2b) for Weibull (1,1) Samples

| Fixed Removal Schemes | Complete Sample Ranking Schemes | $E[X_{3,4} \mid R_3] \times P(R_3)$ | $E[X_{3,4}]$ |
|---|---|---|---|
| $r_1 = 0, r_2 = 0, r_3 = 1$ | $R_1 = 1, R_2 = 2, R_3 = 3$ | $1.0833 \times 1 = 1.0833$ | **1.0833** |
| $r_1 = 0, r_2 = 1, r_3 = 0$ | $R_1 = 1, R_2 = 2, R_3 = 3$ | $1.0833 \times \frac{1}{2} = 0.5416$ | |
| | $R_1 = 1, R_2 = 2, R_3 = 4$ | $2.0833 \times \frac{1}{2} = 1.0416$ | **1.5833** |
| $r_1 = 1, r_2 = 0, r_3 = 0$ | $R_1 = 1, R_2 = 2, R_3 = 3$ | $1.0833 \times \frac{1}{3} = 0.3611$ | |
| | $R_1 = 1, R_2 = 2, R_3 = 4$ | $2.0833 \times \frac{1}{3} = 0.6944$ | |
| | $R_1 = 1, R_2 = 3, R_3 = 4$ | $2.0833 \times \frac{1}{3} = 0.6944$ | **1.7500** |

A Weibull Example with a 50% Fixed Removal Scheme

A 50% fixed removal scheme which is considered by Montanari, et al. (1998) for progressive stress tests of electrical breakdown data of Weibull-distributed lifetimes is one example of a fixed removal scheme. The sample is described as "FCFCFC..." (F=failed and C=censored) , i.e., alternate items are censored. The equation for experiment time for this fixed removal scheme is given previously as equation (1a).

First and second moments of expected experiment time for this progressive censoring scheme may be obtained by noting, as before, that for Type II progressively censored Weibull-distributed data

$$E[X^{(k)}_{1,j+n-R_m+1}] = \left( \frac{\theta}{(j+n-R_m+1)^{1/\beta}} \right)^k \Gamma\left(1+\frac{k}{\beta}\right)$$

and substituting this value in equation (1a).

Table 3.2.11 gives expected experiment time values using equation (1a) for progressively censored Weibull-distributed lifetimes. Tables 3.2.12 and 3.2.13 provide second moments and standard deviation values using equation (1a). Table 3.2.14 gives coefficients of variation, using equation (1a).

The expected experiment times, standard deviations , and $CV$'s using the 50% censoring approach are smaller in comparison to corresponding values where random progressive censoring schemes are used. Although the standard deviations of experiment time are smaller with this 50% removal scheme than with either of the other two random removal schemes, the standard deviations still appear to be large when the value of the shape parameter $\beta$ is small.

Table 3.2.11

$E[X_{m,n}]/\theta$ Using Equation (1a) for Two Parameter Weibull Samples

| | | β | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | n | 0.25 | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 2.00 | 3.00 | 5.00 |
| 3 | 6 | 4.2373 | 1.1806 | 0.9642 | 0.9167 | 0.9052 | 0.9047 | 0.9124 | 0.9301 | 0.9521 |
| 4 | 8 | 5.5057 | 1.4410 | 1.1273 | 1.0417 | 1.0084 | 0.9932 | 0.9818 | 0.9790 | 0.9830 |
| 5 | 10 | 6.7208 | 1.6693 | 1.2626 | 1.1417 | 1.0889 | 1.0610 | 1.0338 | 1.0147 | 1.0051 |
| 6 | 12 | 7.8895 | 1.8735 | 1.3785 | 1.2250 | 1.1548 | 1.1158 | 1.0750 | 1.0424 | 1.0220 |
| 7 | 14 | 9.0172 | 2.0587 | 1.4799 | 1.2964 | 1.2104 | 1.1616 | 1.1090 | 1.0651 | 1.0355 |
| 8 | 16 | 10.1085 | 2.2285 | 1.5702 | 1.3589 | 1.2586 | 1.2009 | 1.1379 | 1.0840 | 1.0468 |
| 9 | 18 | 11.1669 | 2.3857 | 1.6517 | 1.4145 | 1.3009 | 1.2352 | 1.1629 | 1.1003 | 1.0564 |
| 10 | 20 | 12.1954 | 2.5322 | 1.7260 | 1.4645 | 1.3388 | 1.2657 | 1.1849 | 1.1145 | 1.0648 |

Table 3.2.12

$E[X_{m,n}^{(2)}]/\theta^2$ Using Equation (1a) for Two Parameter Weibull Samples

| | | β | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | n | 0.25 | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 2.00 | 3.00 | 5.00 |
| 3 | 6 | 470.6783 | 4.2373 | 1.6309 | 1.1806 | 1.0294 | 0.9642 | 0.9167 | 0.9047 | 0.9218 |
| 4 | 8 | 626.4022 | 5.5057 | 2.0498 | 1.4410 | 1.2269 | 1.1273 | 1.0417 | 0.9932 | 0.9791 |
| 5 | 10 | 781.5761 | 6.7208 | 2.4321 | 1.6693 | 1.3948 | 1.2626 | 1.1417 | 1.0610 | 1.0214 |
| 6 | 12 | 936.2179 | 7.8895 | 2.7851 | 1.8735 | 1.5412 | 1.3785 | 1.2250 | 1.1158 | 1.0545 |
| 7 | 14 | 1090.3440 | 9.0172 | 3.1140 | 2.0587 | 1.6715 | 1.4799 | 1.2964 | 1.1616 | 1.0817 |
| 8 | 16 | 1243.9697 | 10.1085 | 3.4227 | 2.2285 | 1.7889 | 1.5702 | 1.3589 | 1.2009 | 1.1045 |
| 9 | 18 | 1397.1090 | 11.1669 | 3.7140 | 2.3857 | 1.8960 | 1.6517 | 1.4145 | 1.2352 | 1.1242 |
| 10 | 20 | 1549.7752 | 12.1954 | 3.9902 | 2.5322 | 1.9946 | 1.7260 | 1.4645 | 1.2657 | 1.4150 |

Table 3.2.13

$Std.Dev.[X_{m,n}]/\theta$ Using Equation (1a) for Two Parameter Weibull Samples

| | | β | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | n | 0.25 | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 2.00 | 3.00 | 5.00 |
| 3 | 6 | 21.2773 | 1.6863 | 0.8374 | 0.5833 | 0.4582 | 0.3817 | 0.2901 | 0.1992 | 0.1239 |
| 4 | 8 | 24.4949 | 1.8518 | 0.8825 | 0.5966 | 0.4584 | 0.3755 | 0.2787 | 0.1864 | 0.1132 |
| 5 | 10 | 27.1368 | 1.9835 | 0.9153 | 0.6049 | 0.4573 | 0.3701 | 0.2191 | 0.1773 | 0.1059 |
| 6 | 12 | 29.5631 | 2.0927 | 0.9407 | 0.6101 | 0.4558 | 0.3655 | 0.2634 | 0.1704 | 0.1006 |
| 7 | 14 | 31.7653 | 2.1861 | 0.9612 | 0.6148 | 0.4542 | 0.3615 | 0.2579 | 0.1650 | 0.0964 |
| 8 | 16 | 33.7904 | 2.2676 | 0.9783 | 0.6179 | 0.4527 | 0.3580 | 0.2533 | 0.1606 | 0.0931 |
| 9 | 18 | 35.6709 | 2.3399 | 0.9928 | 0.6204 | 0.4512 | 0.3549 | 0.2493 | 0.1569 | 0.0904 |
| 10 | 20 | 37.4306 | 2.4049 | 1.0055 | 0.6224 | 0.4498 | 0.3522 | 0.2459 | 0.1537 | 0.0749 |

Table 3.2.13

Coefficient of Variation of $X_{m,n}$ Using Equation (1a) for Two Parameter Weibull Samples

| | | β | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | n | 0.25 | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 2.00 | 3.00 | 5.00 |
| 3 | 6 | 5.0215 | 1.4284 | 0.8686 | 0.6364 | 0.5062 | 0.4219 | 0.3179 | 0.2207 | 0.1302 |
| 4 | 8 | 4.4419 | 1.1981 | 0.7828 | 0.5727 | 0.4546 | 0.3781 | 0.2839 | 0.1904 | 0.1151 |
| 5 | 10 | 4.0377 | 1.1882 | 0.7249 | 0.5298 | 0.4200 | 0.3488 | 0.2614 | 0.1747 | 0.1054 |
| 6 | 12 | 3.7472 | 1.1170 | 0.6807 | 0.4985 | 0.3947 | 0.3275 | 0.2451 | 0.1635 | 0.0984 |
| 7 | 14 | 3.5227 | 1.0619 | 0.6495 | 0.4742 | 0.3753 | 0.3112 | 0.2326 | 0.1549 | 0.0931 |
| 8 | 16 | 3.3428 | 1.0175 | 0.6230 | 0.4547 | 0.3597 | 0.2981 | 0.2226 | 0.1485 | 0.889 |
| 9 | 18 | 3.1943 | 0.9808 | 0.6011 | 0.4386 | 0.3468 | 0.2873 | 0.2144 | 0.1426 | 0.0855 |
| 10 | 20 | 3.0692 | 0.9497 | 0.5825 | 0.4250 | 0.3359 | 0.2782 | 0.2075 | 0.1379 | 0.0827 |

### 3.3 Other Approaches

Percentiles of empirical distributions of experiment time $X_{m,n}$ may also be informative regarding total duration of test-time because of the large standard deviations associated with some experiment time values.

As an example, simulated percentiles of Weibull-distributed progressively censored data when $\beta = 0.50$ are reported in Table 3.3.1. The empirical distributions of experiment time suggest that these distributions are unimodal and positively skewed. For this value of $\beta$, the resulting estimates of the median experiment time in Table 3.3.1 are smaller than the expected experiment time values in Table 3.2.1.

The percentiles reported in this example are based on 100,000 progressively censored sample data sets which are generated using the method of Balakrishnan and Sandhu (1995) to simulate Type II progressively censored samples.

Table 3.3.1

Percentiles of Experiment Time $X_{m,n}$ with Random Removals as in Equation (2a)

for Two Parameter Weibull Samples*

| | | | Probability of a Smaller Value | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| m | n | β | .05 | .10 | .25 | .50 | .75 | .90 | .95 |
| 3 | 4 | .50 | 0.1288 | 0.2410 | 0.6310 | 1.6539 | 4.0482 | 8.4129 | 12.8641 |
| | 5 | | 0.0855 | 0.1616 | 0.4300 | 1.1709 | 3.0268 | 6.8700 | 10.8725 |
| | 6 | | 0.0629 | 0.1195 | 0.3197 | 0.9010 | 2.4471 | 5.8077 | 9.4738 |
| | 10 | | 0.0242 | 0.0458 | 0.1280 | 0.3929 | 1.2496 | 3.5483 | 6.3005 |
| | 15 | | 0.0110 | 0.0213 | 0.0606 | 0.1945 | 0.7101 | 2.3886 | 4.7427 |
| 6 | 8 | .50 | 0.6812 | 1.0348 | 2.0193 | 4.1426 | 8.2671 | 14.8500 | 20.7956 |
| | 10 | | 0.5576 | 0.8614 | 1.7392 | 3.6926 | 7.5368 | 13.8782 | 19.7503 |
| | 12 | | 0.4562 | 0.7145 | 1.5055 | 3.2998 | 6.9064 | 13.0665 | 18.8642 |
| | 15 | | 0.3522 | 0.5747 | 1.2458 | 2.8805 | 6.2912 | 12.1491 | 17.7303 |
| 9 | 10 | .50 | 1.5190 | 2.1321 | 3.6636 | 6.5948 | 11.7094 | 19.5408 | 26.4656 |
| | 12 | | 1.4123 | 1.9831 | 3.4432 | 6.2528 | 11.2372 | 18.8872 | 25.6530 |
| | 15 | | 1.2803 | 1.8166 | 3.2263 | 5.9558 | 10.8085 | 18.3753 | 25.0780 |
| 15 | 16 | .50 | 2.8653 | 3.7656 | 5.8476 | 9.5378 | 15.6199 | 24.5775 | 32.3516 |
| | 17 | | 2.8773 | 3.7513 | 5.8128 | 9.5236 | 15.5799 | 24.3834 | 32.0289 |
| | 18 | | 2.8177 | 3.6833 | 5.7647 | 9.4628 | 15.4950 | 24.2794 | 31.9031 |

*Simulated percentiles based on 100,000 Type II progressively censored sample data sets

## 3.4 Conclusion

The formulas for moments of experiment time of Type II progressively censored data developed in this chapter are shown as generalizable to any distribution whose moments of a first order statistic are known. Tables generated using the formulas offer design guidance on selecting possible censoring schemes and sample sizes which result in shortened experiment times. Tables developed using the formula also demonstrate that expected experiment time information should be used cautiously as the variability associated with this estimator of experiment time may be large. In such instances, the experimenter may wish to consider other approaches such as examining the percentiles of an empirical distribution of experiment time.

# CHAPTER FOUR

# RANKING PROGRESSIVELY CENSORED SAMPLES TO TEST FOR GOODNESS-OF-FIT

Assumptions regarding the lifetime distribution of data are often explored using tests for goodness-of-fit. This chapter considers correlation-type tests for goodness-of-fit of progressively censored data. For small sample sizes, a conditional procedure is described which uses correlations between progressively censored sample values and their expected values conditioned on rankings of the progressively censored sample values in a complete sample. For all sample sizes, a second method is also suggested which employs a function to approximate the rankings of progressively censored data in a complete sample for a correlation-type test of goodness-of-fit. Examples are given to illustrate methods and applications of these correlation-type goodness-of-fit tests for Type II progressively censored samples.

## 4.1 Introduction

In life-testing experiments, censoring occurs when items are removed from life-test before all sample items have failed. Type I (time) censoring denotes censoring schemes where the removals occur at pre-specified times. Type II (failure) censoring denotes censoring schemes where the removals coincide with failure times of other measured items in the sample. In the instance of single right censoring, the experiment ends with a single stage of censoring where only the largest lifetimes are censored. A multiply or progressively censored sample occurs in life-testing when sample items are removed from life-test at various stages of an experiment, and the sample items remaining continue until failure or until a later stage of censoring.

Suppose an experiment begins with a complete sample of $n$ items and ends when $m$ number of failure times are observed. Suppose, also, that as the experiment progresses, $r_i$ items are removed from life-test at times $T_i$, $i = 1,2,\ldots,m$. Consequently, $n - m = r_1 + r_2 + \ldots + r_m$. The resulting $m$ ordered uncensored lifetimes, $x_{1,n} < x_{2,n} < \ldots < x_{m,n}$, from a complete sample of size $n$ are the progressively censored sample. If the removal times, $T_i$, $i = 1,2,\ldots,m$, are pre-specified times, the sample is a Type I progressively censored sample. If the removal times correspond to failure times of sample items $x_i$, $i = 1,2,\ldots,m$, the sample is a Type II progressively censored sample.

Two correlation-type goodness-of-fit tests for Type II progressively censored data are introduced in this chapter. The first test employs a conditional procedure to calculate a correlation test statistic. This first test, however, is applicable only to smaller sample sizes because of the extensive computational time required to consider all possible ranking schemes. The second test uses a rank function approximation to obtain a correlation-type test statistic. The second test is applicable to all sample sizes.

Both tests are similar in structure to the correlation-type tests for goodness-of-fit introduced by Filliben (1974) for complete samples and later adapted to apply to Type II single right censored data by Smith and Bain (1976). In this chapter, correlation-type tests are adapted further to apply to Type II progressive censoring schemes.

Examples of the two methods to test for goodness-of-fit of Type II progressively censored data are included to more fully illustrate these concepts introduced in this section.

## 4.2 A Conditional Method

Consider a Type II progressively censored sample, $x_{1,n}, x_{2,n}, \ldots, x_{m,n}$, where $r_1, r_2, \ldots, r_m$ are the number of removals at each stage of censoring. Suppose it is desired to test $H_o$ : $X \sim F(x)$, where F is a specified cumulative distribution function, based on the progressively censored sample.

In this section, to test for goodness-of-fit of progressively censored data, a test statistic is developed which uses the correlation coefficients $\hat{\rho}(x_{1,n},\ldots,x_{m,n}; k_{1,n},\ldots,k_{m,n} \mid R_1,\ldots,R_m)$ where

$x_{1,n}, x_{2,n},..., x_{m,n}$ are the ordered progressively censored sample values, the values $k_{1,n}, k_{2,n}...k_{m,n}$ are the corresponding expectations, and $R_1, R_2,..., R_m$ are the rankings in a complete sample of the ordered progressively censored sample values $x_{1,n}, x_{2,n},..., x_{m,n}$.

The procedure of conditioning on complete sample rankings of an ordered Type II progressively censored sample is described by Thomas and Wilson (1972). If $Z_1 < Z_2 < ... < Z_n$ are the $n$ order statistics of a complete sample of size $n$ and $X_{1,n} < X_{2,n} < ... < X_{m,n}$ are the $m$ progressively censored order statistics of the same complete sample of size $n$, then $Z_{R_i} = X_{i,n}$ for some $R_i$, $i = 1,2,...,m$. Thomas and Wilson used this conditional approach to obtain means, variances, and covariances of progressively censored order statistics.

If the complete sample rank $R_i$ of $X_{i,n}$ is known, then the expected value of $X_{i,n}$ is $k_{i,n} = F^{-1}(R_i/(n+1))$. The sample correlation test statistic $T$ is then obtained by summing the sample correlations, $\hat{\rho}(x_{1,n},..., x_{m,n}; k_{1,n},..., k_{m,n} \mid R_1, R_2,..R_m)$, weighted by the probability of the complete sample ranks, $P(R_1, R_2,..., R_m)$. By conditioning on $R_1, R_2,..., R_m$, all possible rankings are considered.

A formula for this goodness-of-fit test statistic of a Type II progressively censored sample $x_{1,n}, x_{2,n},..., x_{m,n}$ where $r_1, r_2,..., r_m$ are the number of removals at each stage of censoring is given by

$$T = \sum_{all R_1, R_2,..., R_m} P(R_1, R_2,..., R_m) \mid r_1, r_2,..., r_m)\hat{\rho}(x_1, x_2,..., x_m; k_1, k_2,..., k_m \mid R_1, R_2,..., R_m)$$

$$= \sum_{R_m=m}^{h(m)} \sum_{R_{m-1}=m-1}^{h(m-1)} ... \sum_{R_2=2}^{h(2)} P(R_1, R_2,...R_m \mid r_1, r_2,..., r_m)\hat{\rho}(x_1, x_2,..., x_m; k_1, k_2,..., k_m \mid R_1, R_2,..., R_m)$$

where

$$h(g) = \begin{cases} \min\left( (g + \sum_{i=1}^{g-1} r_i), (R_{g+1} - 1) \right), & g = 2,3,..., m-1 \\ m + \sum_{i=1}^{m-1} r_i & , g = m \end{cases}$$

and $P(R_1, R_2,..., R_m)$ is found using the identity

$$P(R_1, R_2, \ldots, R_m) = P(R_1)\prod_{i=2}^{m} P(R_i \mid R_1, R_2, \ldots, R_{i-1})$$

where

$$P(R_1 = 1) = 1 \text{ and } P(R_i \mid R_1, R_2, \ldots, R_m) = \prod_{i=2}^{m} \frac{\binom{n - R_i}{\sum_{j=1}^{i-1}(r_j + 1) - R_i + 1}}{\binom{n - R_{i-1}}{\sum_{j=1}^{i-1}(r_j + 1) - R_{i-1}}}, \quad i = 2, 3, \ldots, m.$$

(Thomas & Wilson , 1972)

This test for goodness-of-fit requires a null hypothesis, $H_o$: $X \sim F(X)$, with a specified distribution and specified parameters. Several distributions, however, possess a location-scale structure, i.e., the cumulative probability distribution functions (cdfs) are of the form $F(x) = G((x - \mu)/\sigma)$, and the probability density functions (pdfs) are of the form $f(x) = (1/\sigma)g((x-\mu)/\sigma)$. For members of a location-scale family, the null hypothesis is a test of a composite form with location and scale parameters unspecified. This is because correlation-type tests are invariant to changes in location and scale parameters. The exponential, normal, and Type I extreme-value distributions are examples of distributions having such location-scale structures.

The Weibull distribution is not strictly a member of a location-scale family. The parameters for a two-parameter Weibull distribution are a shape and scale parameter and not a location and scale parameter. So, for instances of two-parameter Weibull-distributed data, the null hypothesis of a correlation-type test of goodness-of-fit is of a composite form with respect to the shape parameter only.

When analyzing Weibull-distributed data, however, it is often more convenient to work with Type I extreme value distributions. The "Type I" in the name Type I extreme value distribution is in reference to one of three possible types of the asymptotic distributions of the smallest order statistic and not a reference to time censoring . The Weibull distribution is directly related to the Type I extreme value distribution in that if X is a Weibull random variable with shape parameter $\beta$ and scale parameter $\theta$, then $Y = \log(X)$ is a Type I extreme value random variable with location parameter $\mu = \log\theta$ and scale parameter $\sigma = 1 / \beta$. Therefore, if the assumed distribution is a two parameter Weibull distribution, the experimenter has the option to test a null hypothesis of (1) the life-times are two-parameter Weibull-distributed with a specified shape parameter and unspecified scale parameter or (2) the log-lifetimes are extreme-value distributed with no parameters specified.

The test procedures in this chapter use special tables of percentiles of the test statistic $T$ which are obtained by Monte Carlo simulations. Different simulated table percentile values are necessary for different combinations of removal schemes and values of $m$ and $n$. The algorithm of Balakrishnan and Sandhu (1995) is used to simulate 10,000 sets of progressively censored sample data. For each set, $T$ is calculated, and the percentiles of $T$ are reported. For a censored sample, the test statistic $T$ is compared to percentile values of $T$ for specified values of $m$, $n$, and the given removal scheme. The null hypothesis is rejected for small values of $T$. A program used to calculate the test statistic and the corresponding percentile values appears in the Appendix

Some examples are illustrated here.

## Examples

### The Thomas and Wilson Example - An Illustration of the Necessary Computations

The following example illustrates the computations required in calculating the test statistic $T$ for a test of goodness-of-fit test using the data of Thomas and Wilson (1972). The sample data of Thomas and Wilson (1972) consist of a simulated progressively censored sample of Weibull-distributed data. The log-times of the data and the removal scheme are reported in Table 4.2.1. Since lifetimes are Weibull-distributed, the log-times to failure are Type I extreme value-distributed. Therefore, the null hypothesis considered is $H_o: X \sim F(x) = 1 - \exp\{-\exp[(x-\mu)/\sigma]\}$, $-\infty < x < \infty$, $\sigma > 0$, and,

$$k_i = \ln\left\{-\ln\left(1 - \frac{R_i}{(n+1)}\right)\right\}.$$

Table 4.2.2 shows possible complete sample rankings, $R_1, R_2, R_3, R_4, R_5$, their probabilities, $P(R_1, R_2, R_3, R_4, R_5)$, and corresponding correlations,

$$\tilde{\rho}(x_{1,10},\ldots,x_{5,10};k_{1,10},\ldots,k_{5,10} \mid R_1, R_2, \ldots R_5) .$$

$T$ is calculated by summing the sample correlations weighted by the probabilities. In this example, $T$ = 0.9744. For this removal scheme, the simulated percentile values of $T$ are reported in Table 4.2.3. When $T = 0.9744$, Table 4.2.3 shows a significance level for the test between 0.75 and 0.90 as

$0.9712 < 0.9744 < 0.9812$

Table 4.2.1

Log-Times of a Progressively Censored Sample by Thomas and Wilson

| | | | | |
|---|---|---|---|---|
| $x_1 = 0.35$ | $x_2 = 2.50$ | $x_3 = 3.81$ | $x_4 = 4.19$ | $x_5 = 4.86$ |
| $r_1 = 0$ | $r_2 = 3$ | $r_3 = 0$ | $r_4 = 0$ | $r_5 = 2$ |

Table 4.2.2

Conditional Probabilities and Correlations Used in Calculating the Test Statistic $T$

for the Progressively Censored Sample by Thomas and Wilson

| $R_1$ | $R_2$ | $R_3$ | $R_4$ | $R_5$ | $P(R_1, R_2, R_3, R_4, R_5)$ | $\hat{\rho}(x_1,\ldots,x_5;k_1,\ldots,k_5)$ |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 0.1785714 | 0.9912336 |
| 1 | 2 | 3 | 4 | 6 | 0.1071429 | 0.9764191 |
| 1 | 2 | 3 | 5 | 6 | 0.1071429 | 0.9711818 |
| 1 | 2 | 4 | 5 | 6 | 0.1071429 | 0.9903400 |
| 1 | 2 | 3 | 4 | 7 | 0.0535714 | 0.9555094 |
| 1 | 2 | 3 | 5 | 7 | 0.0535714 | 0.9602282 |
| 1 | 2 | 4 | 5 | 7 | 0.0535714 | 0.9843209 |
| 1 | 2 | 3 | 6 | 7 | 0.0535714 | 0.9505764 |
| 1 | 2 | 4 | 6 | 7 | 0.0535714 | 0.9831560 |
| 1 | 2 | 5 | 6 | 7 | 0.0535714 | 0.9320347 |
| 1 | 2 | 3 | 4 | 8 | 0.0178571 | 0.9439653 |
| 1 | 2 | 3 | 5 | 8 | 0.0178571 | 0.9712894 |
| 1 | 2 | 4 | 5 | 8 | 0.0178571 | 0.9420170 |
| 1 | 2 | 4 | 6 | 8 | 0.0178571 | 0.9724934 |
| 1 | 2 | 5 | 6 | 8 | 0.0178571 | 0.9812810 |
| 1 | 2 | 3 | 7 | 8 | 0.0178571 | 0.9316549 |
| 1 | 2 | 4 | 7 | 8 | 0.0178571 | 0.9642552 |
| 1 | 2 | 5 | 7 | 8 | 0.0178571 | 0.9763856 |
| 1 | 2 | 6 | 7 | 8 | 0.0178571 | 0.9744436 |

Table 4.2.3

Percentiles of the Test Statistic $T$ for Progressively Censored Extreme Value-Distributed Lifetimes

with Removals $r_1 = 0, r_2 = 3, r_3 = 0, r_4 = 0, r_5 = 2$ *

| | | Approximate Probability of a Smaller Value | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | n | .01 | .05 | .10 | .25 | .50 | .75 | .90 | .95 | .99 |
| 5 | 10 | 0.8188 | 0.8703 | 0.8966 | 0.9300 | 0.9544 | 0.9712 | 0.9812 | 0.9854 | 0.9900 |

* Simulated percentiles based on 10,000 Type II progressively censored sample data sets

The Nelson Example - An Illustration of a Test of Goodness-of-Fit of the Type I Extreme Value Distribution

The following is a simulated Type II progressively censored sample of times to breakdown of insulating fluid tests by 34 kilovolts, attributed to Nelson (1982), cited in Viveros and Balakrishnan (1984). In the Nelson sample, eight sample items are fully measured from a complete sample of nineteen subjected to Type II progressive censoring. Table 4.2.4 reports the log-times of the progressively censored sample data and removal scheme of Nelson.

Suppose a test of goodness-of-fit test is considered where X is the log-time to failure and the null hypothesis is

$$H_o: X \sim F(x) = 1 - \exp\{-\exp[(x-\mu)/\sigma]\}, \ -\infty < x < \infty, \ \ \sigma > 0,$$

log-times to failure are Type I extreme value-distributed. Letting $k_i = \ln\left\{-\ln\left(1 - \frac{R_i}{n+1}\right)\right\}$, the test statistic $T = 0.9839$ for the Nelson data.

Simulated percentile values of $T$ are shown in Table 4.2.5. The information in Table 4.2.5 indicates a significance level between 0.90 and 0.95 as $0.9830 < 0.9839 < 0.9860$. A large significance level was expected because the Nelson data is a simulated sample of Weibull-distributed lifetimes subjected to Type II progressive censoring.

Table 4.2.4

Log-Times to Breakdown of a Progressively Censored Sample of Insulating Fluid

Tested by 34 Kilovolts by Nelson

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x_1$=-1.6608 | $x_2$=-.2485 | $x_3$ = -.0409 | $x_4$ = .2700 | $x_5$=1.0224 | $x_6$=1.5789 | $x_7$=1.8718 | $x_8$=1.9947 |
| $r_1$ = 0 | $r_2$ = 0 | $r_3$ = 3 | $r_4$ = 0 | $r_5$ = 3 | $r_6$ = 0 | $r_7$ = 0 | $r_8$ = 5 |

Table 4.2.5

Percentiles of the Test Statistic $T$ for Progressively Censored Extreme Value-Distributed Lifetimes

with a Removals $r_1 = 0, r_2 = 0, r_3 = 3, r_4 = 0, r_5 = 3, r_6 = 0, r_7 = 0, r_8 = 5$ *

| | | Approximate Probability of a Smaller Value | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | n | .01 | .05 | .10 | .25 | .50 | .75 | .90 | .95 | .99 |
| 8 | 19 | 0.8432 | 0.8910 | 0.9114 | 0.9402 | 0.9615 | 0.9750 | 0.9830 | 0.9860 | 0.9902 |

* Simulated percentiles based on 10,000 Type II progressively censored sample data sets

The Herd Example - An Illustration of a Test of Goodness-of-Fit of the Weibull Distribution

The next example of Type II progressively censored data is given by Herd (1956). In the life-testing experiment described by Herd, eleven gyroscopes are placed on life-test. Three are removed from the life-test at the first failure time, two are removed at the second failure time, two are removed at the third failure time, and the remaining gyroscope is allowed to continue on life-test until failure. The progressively censored sample values of Herd (1956) and the removal scheme are given in Table 4.2.6.

For the data in Table 4.2.6, Herd (1956) considered the problem of estimating the mean lifetime of gyroscopes assuming the lifetime to be exponentially distributed ( or Weibull with a shape parameter $\beta$=1). Consider, instead, the problem of testing the assumption that the lifetimes of the gyroscopes are Weibull-distributed with a specified shape parameter. The null hypothesis of the test is then

$H_o\ X \sim F(x) = 1-\exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\}$ , $x > 0, \theta > 0$, and $\beta = \beta_o$, and the test statistic $T$ is calculated using

the progressively censored sample values $x_1, x_2, x_3, x_4$ and $k_i = \frac{1}{\beta_o}\ln\left[1-\frac{R_i}{n+1}\right]$, $i = 1,2,3,4$.

Table 4.2.7 reports percentile values of $T$ for specified values of $\beta$. Table 4.2.8 reports calculated values of the test statistic T for the same specified values of $\beta$ and their corresponding OSL values. In the case where $\beta$ =1, Table 4.2.8 indicates a significance level between 0.75 and 0.90. Therefore, an assumption of exponential lifetimes of gyroscopes would not seem unreasonable based on the progressively censored sample values by Herd.

In modeling Weibull-distributed data, the failure rate of the random variable may also be examined. Values of the shape parameter $\beta < 1$ indicate a declining failure rate, values of $\beta > 1$ indicate an increasing failure rate, and values of $\beta = 1$ indicate a constant failure rate. Examining the OSL values in Table 4.2.8, the tests show a near constant or increasing failure rate for lifetimes of gyroscopes rather than a declining one. This example illustrates how the goodness-of-fit tests may be employed to examine failure rate.

Table 4.2.6

Lifetimes of a Progressively Censored Sample by Herd

| | | | |
|---|---|---|---|
| $x_1$=34 | $x_2$ = 113 | $x_3$ = 169 | $x_4$ =237 |
| $r_1$ = 3 | $r_2$=2 | $r_3$ = 2 | $r_4$= 0 |

Table 4.2.7

Percentiles of the Test Statistic $T$ for Progressively Censored Weibull-Distributed Lifetimes

with Removals $r_1 = 3, r_2 = 2, r_3 = 2, r_4 = 0$ *

| | | | Approximate Probability of a Smaller Value | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| m | n | β | .01 | .05 | .10 | .25 | .50 | .75 | .90 | .95 | .99 |
| 4 | 11 | 0.25 | 0.7236 | 0.8599 | 0.9500 | 0.9808 | 0.9815 | 0.9860 | 0.9917 | 0.9933 | 0.9939 |
| | | 0.50 | 0.8056 | 0.8751 | 0.9264 | 0.9317 | 0.9469 | 0.9689 | 0.9816 | 0.9840 | 0.9869 |
| | | 0.75 | 0.8232 | 0.8859 | 0.8922 | 0.9096 | 0.9384 | 0.9659 | 0.9778 | 0.9820 | 0.9861 |
| | | 1.00 | 0.8310 | 0.8689 | 0.8798 | 0.9042 | 0.9387 | 0.9648 | 0.9773 | 0.9821 | 0.9866 |
| | | 1.25 | 0.8336 | 0.8587 | 0.8750 | 0.9050 | 0.9410 | 0.9643 | 0.9780 | 0.9827 | 0.9875 |
| | | 1.50 | 0.8271 | 0.8535 | 0.8728 | 0.9073 | 0.9420 | 0.9646 | 0.9785 | 0.9836 | 0.9878 |
| | | 2.00 | 0.8161 | 0.8494 | 0.8712 | 0.9107 | 0.9440 | 0.9662 | 0.9797 | 0.9849 | 0.9892 |
| | | 3.00 | 0.8095 | 0.8488 | 0.8747 | 0.9158 | 0.9474 | 0.9690 | 0.9821 | 0.9867 | 0.9900 |
| | | 5.00 | 0.8019 | 0.8492 | 0.8771 | 0.9203 | 0.9502 | 0.9716 | 0.9837 | 0.9877 | 0.9911 |

* Simulated percentiles based on 10,000 Type II progressively censored sample data sets

Table 4.2.8

Values of the Test Statistic $T$ for the Progressively Censored Sample by Herd

| m | n | $\beta_o$ | $T$ Calculated | Observed Significance Level (OSL) |
|---|---|---|---|---|
| 4 | 11 | 0.25 | 0.8415 | 0.01<OSL<0.05 |
| | | 0.50 | 0.9175 | 0.05<OSL<0.10 |
| | | 0.75 | 0.9517 | 0.50<OSL<0.75 |
| | | 1.00 | 0.9680 | 0.75<OSL<0.90 |
| | | 1.25 | 0.9765 | 0.75<OSL<0.90 |
| | | 1.50 | 0.9815 | 0.90<OSL<0.95 |
| | | 2.00 | 0.9865 | 0.95<OSL<0.99 |
| | | 3.00 | 0.9901 | 0.99<OSL |
| | | 5.00 | 0.9917 | 0.99<OSL |

The Montanari, et al. Example- A Comparison of Specified versus Unspecified Parameters

The next example employs the 50% removal scheme considered by Montanari, et al. (1998). In this 50% removal scheme, a live item is removed from life-test at each failure time, i.e., $r_1 = 1, r_2 = 1, \ldots, r_m = 1$ and $m = 50\%\, n$. The censoring scheme is described as "FCFCFC... (F=failed, C=censored)". Suppose a goodness-of-fit test for small Type II progressively censored samples is considered where the number of items progressively censored in this manner.

Two approaches are outlined to test for goodness-of-fit. The first is a test that the data is Weibull-distributed with the value of the shape parameter specified; the second is that the log-lifetimes of the data is Type I Extreme Value-distributed with no parameters specified. Either of the following two tests of goodness-of-fit may be considered:

(1) $H_o: X \sim F(x) = 1 - \exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\}$ , $x > 0, \theta > 0, \ \beta = \beta_o$, lifetimes are Weibull-distributed

with the shape parameter $\beta$ specified. Using $k_i = \frac{1}{\beta_o}\ln\left[1 - \frac{R_i}{(n+1)}\right]$, the test statistic $T$ is

calculated for the sample values and the removal scheme $r_1 = 1, r_2 = 1, \ldots, r_m = 1$.

(2) $H_o: X \sim F(x) = 1 - \exp\{-\exp[(x - \mu)/\sigma]\}$, $-\infty < x < \infty, \ \ \sigma > 0$, log-lifetimes are Type I extreme value-distributed and parameters are unspecified. Using

$k_i = \ln\left\{-\ln\left(1 - \frac{R_i}{(n+1)}\right)\right\}$, the test statistic $T$ is calculated for sample values and the

removal scheme $r_1 = 1, r_2 = 1, \ldots, r_m = 1$.

The calculated value of the test statistic is compared to table percentile values of the test statistic. The null hypothesis is rejected for small values of $T$. Table 4.2.9 provides percentiles of the test statistic $T$ for case (1). Table 4.2.10 provides percentiles values for case (2).

Table 4.2.9

Percentiles of the Test Statistic $T$ for Progressively Censored Weibull-Distributed Lifetimes

with Removals $r_1 = 1, r_2 = 1, \ldots, r_m = 1$ *

| | | | Approximate Probability of a Smaller Value | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| m | n | β | .01 | .05 | .10 | .25 | .50 | .75 | .90 | .95 | .99 |
| 3 | 6 | 0.25 | 0.5973 | 0.7347 | 0.8560 | 0.9849 | 0.9953 | 0.9965 | 0.9978 | 0.9981 | 0.9982 |
| | | 0.50 | 0.6996 | 0.7760 | 0.8490 | 0.9662 | 0.9745 | 0.9862 | 0.9912 | 0.9921 | 0.9924 |
| | | 0.75 | 0.7554 | 0.8038 | 0.8606 | 0.9410 | 0.9627 | 0.9822 | 0.9887 | 0.9897 | 0.9900 |
| | | 1.00 | 0.7886 | 0.8280 | 0.8728 | 0.9281 | 0.9587 | 0.9809 | 0.9879 | 0.9890 | 0.9893 |
| | | 1.25 | 0.8090 | 0.8421 | 0.8799 | 0.9221 | 0.9566 | 0.9805 | 0.9878 | 0.9889 | 0.9892 |
| | | 1.50 | 0.8234 | 0.8519 | 0.8864 | 0.9180 | 0.9554 | 0.9805 | 0.9877 | 0.9888 | 0.9892 |
| | | 2.00 | 0.8404 | 0.8646 | 0.8836 | 0.9148 | 0.9553 | 0.9809 | 0.9881 | 0.9892 | 0.9895 |
| | | 3.00 | 0.8589 | 0.8697 | 0.8810 | 0.9139 | 0.9556 | 0.9810 | 0.9885 | 0.9896 | 0.9899 |
| | | 5.00 | 0.8529 | 0.8704 | 0.8816 | 0.9135 | 0.9566 | 0.9820 | 0.9890 | 0.9901 | 0.9904 |
| 4 | 8 | 0.25 | 0.6809 | 0.7570 | 0.8412 | 0.9654 | 0.9862 | 0.9903 | 0.9933 | 0.9941 | 0.9944 |
| | | 0.50 | 0.7134 | 0.8064 | 0.8484 | 0.9379 | 0.9584 | 0.9751 | 0.9827 | 0.9843 | 0.9859 |
| | | 0.75 | 0.7382 | 0.8421 | 0.8710 | 0.9179 | 0.9500 | 0.9701 | 0.9779 | 0.9812 | 0.9838 |
| | | 1.00 | 0.7571 | 0.8580 | 0.8796 | 0.9119 | 0.9463 | 0.9672 | 0.9845 | 0.9809 | 0.9838 |
| | | 1.25 | 0.7766 | 0.8598 | 0.8827 | 0.9133 | 0.9457 | 0.9657 | 0.9770 | 0.9809 | 0.9841 |
| | | 1.50 | 0.7870 | 0.8569 | 0.8818 | 0.9140 | 0.9455 | 0.9652 | 0.9771 | 0.9813 | 0.9846 |
| | | 2.00 | 0.8086 | 0.8575 | 0.8824 | 0.9169 | 0.9458 | 0.9660 | 0.9784 | 0.9823 | 0.9855 |
| | | 3.00 | 0.8128 | 0.8577 | 0.8859 | 0.9220 | 0.9468 | 0.9670 | 0.9793 | 0.9831 | 0.9863 |
| | | 5.00 | 0.8138 | 0.8563 | 0.8844 | 0.9207 | 0.9474 | 0.9678 | 0.9804 | 0.9843 | 0.9873 |
| 5 | 10 | 0.25 | 0.6987 | 0.7800 | 0.8413 | 0.9528 | 0.9771 | 0.9842 | 0.9888 | 0.9898 | 0.9903 |
| | | 0.50 | 0.7369 | 0.8311 | 0.8598 | 0.9209 | 0.9486 | 0.9680 | 0.9763 | 0.9788 | 0.9812 |
| | | 0.75 | 0.7619 | 0.8531 | 0.8805 | 0.9104 | 0.9436 | 0.9629 | 0.9726 | 0.9761 | 0.9796 |
| | | 1.00 | 0.7911 | 0.8586 | 0.8828 | 0.9136 | 0.9430 | 0.9619 | 0.9722 | 0.9756 | 0.9798 |
| | | 1.25 | 0.8000 | 0.8559 | 0.8824 | 0.9176 | 0.9438 | 0.9624 | 0.9725 | 0.9761 | 0.9807 |
| | | 1.50 | 0.8173 | 0.8613 | 0.8867 | 0.9193 | 0.9442 | 0.9626 | 0.9723 | 0.9766 | 0.9814 |
| | | 2.00 | 0.8120 | 0.8664 | 0.8897 | 0.9217 | 0.9467 | 0.9637 | 0.9735 | 0.9778 | 0.9828 |
| | | 3.00 | 0.8116 | 0.8654 | 0.8909 | 0.9215 | 0.9472 | 0.9636 | 0.9744 | 0.9790 | 0.9839 |
| | | 5.00 | 0.8120 | 0.8683 | 0.8923 | 0.9238 | 0.9485 | 0.9651 | 0.9757 | 0.9802 | 0.9853 |
| 6 | 12 | 0.25 | 0.7162 | 0.7955 | 0.8433 | 0.9465 | 0.9697 | 0.9793 | 0.9846 | 0.9859 | 0.9866 |
| | | 0.50 | 0.7557 | 0.8434 | 0.8729 | 0.9143 | 0.9447 | 0.9645 | 0.9725 | 0.9755 | 0.9781 |
| | | 0.75 | 0.7928 | 0.8577 | 0.8826 | 0.9119 | 0.9422 | 0.9607 | 0.9700 | 0.9733 | 0.9772 |
| | | 1.00 | 0.8157 | 0.8644 | 0.8867 | 0.9178 | 0.9439 | 0.9609 | 0.9697 | 0.9733 | 0.9777 |
| | | 1.25 | 0.8180 | 0.8681 | 0.8906 | 0.9216 | 0.9447 | 0.9615 | 0.9705 | 0.9743 | 0.9789 |
| | | 1.50 | 0.8207 | 0.8715 | 0.8935 | 0.9234 | 0.9465 | 0.9618 | 0.9711 | 0.9748 | 0.9798 |
| | | 2.00 | 0.8260 | 0.8759 | 0.8980 | 0.9270 | 0.9487 | 0.9633 | 0.9721 | 0.9758 | 0.9807 |
| | | 3.00 | 0.8304 | 0.8789 | 0.8988 | 0.9281 | 0.9498 | 0.9644 | 0.9734 | 0.9774 | 0.9826 |
| | | 5.00 | 0.8261 | 0.8760 | 0.9006 | 0.9290 | 0.9510 | 0.9655 | 0.9747 | 0.9789 | 0.9840 |

Table 4.2.9 Continued

Percentiles of the Test Statistic $T$ for Progressively Censored Weibull-Distributed Lifetimes

with Removals $r_1 = 1, r_2 = 1, \ldots, r_m = 1$ *

| | | | Approximate Probability of a Smaller Value | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| m | n | β | .01 | .05 | .10 | .25 | .50 | .75 | .90 | .95 | .99 |
| 7 | 14 | 0.25 | 0.7282 | 0.8080 | 0.8553 | 0.9362 | 0.9626 | 0.9748 | 0.9810 | 0.9825 | 0.9835 |
| | | 0.50 | 0.7699 | 0.8397 | 0.8754 | 0.9086 | 0.9422 | 0.9618 | 0.9708 | 0.9731 | 0.9760 |
| | | 0.75 | 0.8100 | 0.8568 | 0.8826 | 0.9179 | 0.9418 | 0.9598 | 0.9687 | 0.9724 | 0.9760 |
| | | 1.00 | 0.8152 | 0.8706 | 0.8903 | 0.9188 | 0.9438 | 0.9609 | 0.9694 | 0.9720 | 0.9766 |
| | | 1.25 | 0.8357 | 0.8704 | 0.8938 | 0.9250 | 0.9463 | 0.9621 | 0.9708 | 0.9740 | 0.9782 |
| | | 1.50 | 0.8395 | 0.8826 | 0.9048 | 0.9291 | 0.9483 | 0.9614 | 0.9702 | 0.9737 | 0.9797 |
| | | 2.00 | 0.8474 | 0.8880 | 0.9105 | 0.9331 | 0.9528 | 0.9649 | 0.9726 | 0.9758 | 0.9802 |
| | | 3.00 | 0.8421 | 0.8825 | 0.9018 | 0.9316 | 0.9520 | 0.9656 | 0.9735 | 0.9773 | 0.9824 |
| | | 5.00 | 0.8395 | 0.8884 | 0.9060 | 0.9345 | 0.9540 | 0.9671 | 0.9750 | 0.9780 | 0.9824 |
| 8 | 16 | 0.25 | 0.7557 | 0.8290 | 0.8634 | 0.9410 | 0.9582 | 0.9722 | 0.9781 | 0.9793 | 0.9810 |
| | | 0.50 | 0.7876 | 0.8582 | 0.8833 | 0.9092 | 0.9399 | 0.9601 | 0.9681 | 0.9708 | 0.9746 |
| | | 0.75 | 0.8171 | 0.8649 | 0.8876 | 0.9178 | 0.9424 | 0.9589 | 0.9686 | 0.9712 | 0.9747 |
| | | 1.00 | 0.8367 | 0.8692 | 0.8925 | 0.9216 | 0.9443 | 0.9599 | 0.9685 | 0.9722 | 0.9758 |
| | | 1.25 | 0.8302 | 0.8849 | 0.9031 | 0.9286 | 0.9492 | 0.9634 | 0.9711 | 0.9739 | 0.9772 |
| | | 1.50 | 0.8402 | 0.8884 | 0.9079 | 0.9308 | 0.9503 | 0.9631 | 0.9715 | 0.9746 | 0.9786 |
| | | 2.00 | 0.8451 | 0.8911 | 0.9144 | 0.9374 | 0.9539 | 0.9660 | 0.9723 | 0.9754 | 0.9806 |
| | | 3.00 | 0.8521 | 0.8947 | 0.9157 | 0.9379 | 0.9556 | 0.9674 | 0.9755 | 0.9784 | 0.9828 |
| | | 5.00 | 0.8454 | 0.8962 | 0.9182 | 0.9401 | 0.9576 | 0.9690 | 0.9760 | 0.9789 | 0.9834 |
| 9 | 18 | 0.25 | 0.7291 | 0.8323 | 0.8644 | 0.9347 | 0.9551 | 0.9686 | 0.9753 | 0.9770 | 0.9787 |
| | | 0.50 | 0.7809 | 0.8598 | 0.8809 | 0.9081 | 0.9410 | 0.9581 | 0.9664 | 0.9695 | 0.9730 |
| | | 0.75 | 0.8337 | 0.8659 | 0.8876 | 0.9171 | 0.9416 | 0.9593 | 0.9678 | 0.9708 | 0.9742 |
| | | 1.00 | 0.8367 | 0.8722 | 0.8953 | 0.9261 | 0.9471 | 0.9611 | 0.9688 | 0.9721 | 0.9762 |
| | | 1.25 | 0.8297 | 0.8826 | 0.9041 | 0.9298 | 0.9499 | 0.9615 | 0.9692 | 0.9732 | 0.9777 |
| | | 1.50 | 0.8486 | 0.8922 | 0.9120 | 0.9346 | 0.9517 | 0.9652 | 0.9724 | 0.9749 | 0.9793 |
| | | 2.00 | 0.8628 | 0.8990 | 0.9184 | 0.9397 | 0.9572 | 0.9676 | 0.9741 | 0.9770 | 0.9805 |
| | | 3.00 | 0.8731 | 0.9010 | 0.9190 | 0.9415 | 0.9576 | 0.9697 | 0.9762 | 0.9792 | 0.9834 |

* Simulated percentiles based on 10,000 Type II progressively censored sample data sets

Table 4.2.10

Percentiles of the Test Statistic $T$ for Progressively Censored Type I Extreme Value-Distributed Lifetimes with a Removals $r_1 = 1, r_2 = 1, \ldots, r_m = 1$ *

| | | Approximate Probability of a Smaller Value | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | n | .01 | .05 | .10 | .25 | .50 | .75 | .90 | .95 | .99 |
| 3 | 6 | 0.8289 | 0.8574 | 0.8896 | 0.9264 | 0.9643 | 0.9887 | 0.9958 | 0.9969 | 0.9973 |
| 4 | 8 | 0.7988 | 0.8667 | 0.8913 | 0.9229 | 0.9550 | 0.9751 | 0.9871 | 0.9910 | 0.9943 |
| 5 | 10 | 0.8098 | 0.8675 | 0.8922 | 0.9276 | 0.9540 | 0.9723 | 0.9828 | 0.9869 | 0.9919 |
| 6 | 12 | 0.8279 | 0.8759 | 0.8999 | 0.9323 | 0.9555 | 0.9719 | 0.9809 | 0.9849 | 0.9900 |
| 7 | 14 | 0.8290 | 0.8830 | 0.9057 | 0.9355 | 0.9572 | 0.9720 | 0.9806 | 0.9843 | 0.9890 |

* Simulated percentiles based on 10,000 Type II progressively censored sample data sets

## 4.3 A Second Method

The examples in Section 4.2 demonstrate that a conditional method of correlation-type goodness-of-fit tests may be employed for instances of small progressively censored samples. For larger sample sizes, the conditional method is not practical because of the extensive computational time required in considering all the possible ranking schemes. Therefore, a second correlation-type goodness-of-fit test is suggested in which is applicable to all sample sizes.

An alternative to conditioning on all possible complete sample rankings is to approximate the complete sample ranks $R_i$, $i = 1,2,\ldots,m$, of the progressively censored sample using a mean rank estimator $I(i)$, which is a function of the total number of failed and censored sample items at the time of the $i$th failure. The resulting second correlation-type test statistic then consists of only a single correlation between sample values $x_i$ and approximate expected values and not the many correlations and associated probabilities required of the conditional test.

Johnson (1964) provides a detailed example of calculating the function $I(i)$ which is referred to in the article as the "mean order number" for an "incomplete test" which has "suspended items". O'Connor (1981) further describes graphical tests of goodness-of-fit with Q-Q probability plots which rely on similar rank estimates of complete sample rankings. Mean rank and median rank functions are suggested by O'Connor (1981). The rank function $I(i)$ considered in this section is the mean rank function defined by

$$I(i) = \begin{cases} I(i-1) + \dfrac{n+1-I(i-1)}{n+2-C_i}, & i = 1,2,\ldots,m \\ 0 & , i = 0 \end{cases}$$

where $C_i$ is the total number of measured and censored lifetimes at the time of the $i$th failure of the progressively censored sample.

To test for goodness-of-fit using the second method, then, let $x_{1,n} < x_{2,n} < \ldots < x_{m,n}$ be a Type II progressively censored sample with $r_1, r_2, \ldots, r_m$ number of removals at each stage of censoring. Consider a null hypothesis $H_o: X \sim F(x)$ based on the progressively censored sample and a test statistic $T'$ such that

$$T' = \hat{\rho}(x_{1,n}, x_{2,n}, \ldots, x_{m,n}; \ell_1, \ell_2, \ldots, \ell_m)$$

where $\hat{\rho}$ is the sample correlation between the progressively censored sample values, $x_1, x_2, \ldots, x_m$ and $\ell_1, \ell_2, \ldots, \ell_m$, $\ell_i = F^{-1}(I(i)/(n+1))$, and where $C_i = \sum_{i=1}^{i-1} r_i + i$.

<u>The Thomas and Wilson Example - An Illustration of the Necessary Computations</u>

Consider again the data in Table 1 of Thomas and Wilson (1972) and a test that the log-lifetimes of the Type II progressively censored data are Type I extreme value-distributed. For the five progressively censored lifetimes given in Table 4.3.1 with removal scheme $r_1 = 0, r_2 = 3, r_3 = 0, r_4 = 0, r_5 = 2$, the approximations of their complete sample ranks $I(i), i = 1,2,\ldots,5$, using the rank function $I(i)$ are shown in Table 4.3.1.

Letting $\ell_i = \ln\left\{-\ln\left(1 - \frac{I(i)}{(n+1)}\right)\right\}$, the statistic for the test is

$$T' = \hat{\rho}(x_{1,10}, x_{2,10}, \ldots, x_{5,10}; \ell_{1,10}, \ell_{2,10}, \ldots, \ell_{5,10}) = 0.9819.$$

Table 4.3.2 shows the significance level for the test is between 0.75 and 0.90 as 0.9785 < 0.9819 < 0.9887. This result is similar to the finding using the conditional method.

Table 4.3.1

Rank Estimates for the Progressively Censored Sample by

Thomas and Wilson

| m | n | i | $r_i$ | $I(i-1)$ | $C_i$ | $\frac{n+1-I(i-1)}{n+2-C_i}$ | $I(i)=I(i-1)+\frac{n+1-I(i-1)}{n+2-C_i}$ |
|---|---|---|---|---|---|---|---|
| 5 | 10 | 1 | 0 | 0 | 1 | 1 | 1 |
| | | 2 | 3 | 1 | 2 | 1 | 2 |
| | | 3 | 0 | 2 | 6 | 1.5 | 3.5 |
| | | 4 | 0 | 3.5 | 7 | 1.5 | 5 |
| | | 5 | 2 | 5 | 8 | 1.5 | 6.5 |

Table 4.3.2

Percentiles of the Test Statistic $T'$ for Progressively Censored Extreme Value-Distributed Lifetimes

with the Removals $r_1 = 0, r_2 = 3, r_3 = 3, r_4 = 0, r_5 = 2$

| | | Approximate Probability of a Smaller Value | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | n | .01 | .05 | .10 | .25 | .50 | .75 | .90 | .95 | .99 |
| 5 | 10 | 0.8275 | 0.8758 | 0.9015 | 0.9368 | 0.961 | 0.9785 | 0.9887 | 0.9927 | 0.9974 |

* Simulated percentiles based on 10,000 Type II progressively censored sample data sets

The Cohen Example - An Illustration with Large Progressively Censored Samples

Cohen (1975) provides an example of a Type II progressively censored sample consisting of 68 items from a complete sample of 100 items. Ten items from the sample are removed at the sixth failure time, fifteen items are removed are the fortieth failure time, and the experiment ends at the sixty-eighth failure time at which time the seven remaining items were removed. Therefore, $r_6 = 10$, $r_{40} = 15$, $r_{68} = 7$, and otherwise $r_i = 0$, $i = 1,2,\ldots,68$. The lifetimes reported by Cohen are given in Table 4.3.3. These lifetimes are from a simulated Type II progressively censored sample from a three-parameter Weibull distribution with location parameter $\gamma = 100$, shape parameter $\beta = 2$, and scale parameter $\theta = 100$.

To test that the data in Table 4.3.3 is Weibull-distributed with the value of the shape parameter specified, the null hypothesis for the test is in the form

$$H_o\colon X \sim F(x) = 1 - \exp\left\{-\left(\frac{x-\gamma}{\theta}\right)^{\beta}\right\}, \quad x > 0,\ \theta > 0,\ \gamma > x,\ \beta = \beta_o.$$

lifetimes are Weibull-distributed with shape parameter $\beta = \beta_o$. Using $\ell_i = \dfrac{1}{\beta_o}\ln\left[1 - \dfrac{I(i)}{(n+1)}\right]$, the test statistic $T'$ is then calculated and compared to simulated table percentile values in Table 4.34. For the Cohen data, if $\beta = 2.0$, then $T' = 0.9973$ and $0.90 < OSL < 0.95$.

Table 4.3.3

Lifetimes of a Progressively Censored Sample by Cohen

| | | | | | | |
|---|---|---|---|---|---|---|
| $x_1$ =109.12 | $x_{11}$ = 130.53 | $x_{21}$ = 144.09 | $x_{31}$ =158.31 | $x_{41}$ =177.19 | $x_{51}$ =198.11 | $x_{61}$ =222.11 |
| $x_2$ =113.37 | $x_{12}$ =131.98 | $x_{22}$ =148.83 | $x_{32}$ =158.92 | $x_{42}$ =180.57 | $x_{52}$ =199.23 | $x_{62}$ =224.83 |
| $x_3$ =117.73 | $x_{13}$ =133.14 | $x_{23}$ =150.23 | $x_{33}$ =160.13 | $x_{43}$ =181.99 | $x_{53}$ =203.27 | $x_{63}$ =227.27 |
| $x_4$ =119.56 | $x_{14}$ =134.52 | $x_{24}$ =150.79 | $x_{34}$ =161.31 | $x_{44}$ =184.02 | $x_{54}$ =206.55 | $x_{64}$ =230.88 |
| $x_5$ =119.82 | $x_{15}$ =135.73 | $x_{25}$ =151.88 | $x_{35}$ =162.09 | $x_{45}$ =185.43 | $x_{55}$ =208.76 | $x_{65}$ =235.14 |
| $x_6$ =124.63 | $x_{16}$ =136.71 | $x_{26}$ =153.07 | $x_{36}$ =165.45 | $x_{46}$ =187.21 | $x_{56}$ =210.69 | $x_{66}$ =237.43 |
| $x_7$ =125.21 | $x_{17}$ =137.88 | $x_{27}$ =154.18 | $x_{37}$ =166.62 | $x_{47}$ =189.77 | $x_{57}$ =213.32 | $x_{67}$ =246.08 |
| $x_8$ =126.93 | $x_{18}$ =138.63 | $x_{28}$ =154.97 | $x_{38}$ =168.23 | $x_{48}$ =191.63 | $x_{58}$ =215.08 | $x_{68}$ =24935 |
| $x_9$ =128.25 | $x_{19}$ =141.11 | $x_{29}$ =155.26 | $x_{39}$ =169.98 | $x_{49}$ =194.88 | $x_{59}$ =218.43 | |
| $x_{10}$=129.41 | $x_{20}$ =142.33 | $x_{30}$ =156.82 | $x_{40}$ =174.22 | $x_{50}$ =196.91 | $x_{60}$ =219.37 | |

Table 4.3.4

Percentiles of the Test Statistic $T'$ for Progressively Censored Weibull-Distributed Lifetimes

with the Removal Scheme used by Cohen*

| | | | Approximate Probability of a Smaller Value | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| m | n | β | .01 | .05 | .10 | .25 | .50 | .75 | .90 | .95 | .99 |
| 68 | 100 | 2.0 | 0.9812 | 0.9868 | 0.9893 | 0.924 | 0.9947 | 0.9962 | 0.9972 | 0.9977 | 0.9983 |

* Simulated percentiles based on 10,000 Type II progressively censored sample data sets

The Montanari Example - An Illustration of a Test of Goodness-of-Fit for a Fixed 50% Removal Scheme

Estimating the rank $R_i$ with the mean rank function $I(i)$ also allows tests of goodness-of-fit for the larger values of $n$. An example of a large sample test is next considered for the 50% removal scheme used by Montanari, et al. (1998). Percentiles of the test statistic $T'$ for a null hypothesis that the data is Type I extreme-value distributed are provided in Table 4.3.5.

Table 4.3.5

Percentiles of the Test Statistic $T'$ for Progressively Censored Extreme Value-Distributed

Lifetimes with Removals $r_1 = 1, r_2 = 1, \ldots, r_m = 1$ *

| | | Approximate Probability of a Smaller Value | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| m | n | .01 | .05 | .10 | .25 | .50 | .75 | .90 | .95 | .99 |
| 3 | 6 | 0.8371 | 0.8656 | 0.8954 | 0.9281 | 0.9668 | 0.9916 | 0.9987 | 0.9997 | 0.9999 |
| 4 | 8 | 0.8046 | 0.8709 | 0.8956 | 0.9301 | 0.9606 | 0.9807 | 0.9921 | 0.9958 | 0.9992 |
| 5 | 10 | 0.8173 | 0.8705 | 0.8949 | 0.9317 | 0.9591 | 0.9774 | 0.9873 | 0.9920 | 0.9972 |
| 6 | 12 | 0.8265 | 0.8797 | 0.9037 | 0.9368 | 0.9617 | 0.9779 | 0.9869 | 0.9908 | 0.9962 |
| 7 | 14 | 0.8403 | 0.8873 | 0.9104 | 0.9400 | 0.9629 | 0.9777 | 0.9864 | 0.9901 | 0.9951 |
| 8 | 16 | 0.8445 | 0.8943 | 0.9155 | 0.9451 | 0.9660 | 0.9791 | 0.9869 | 0.9904 | 0.9948 |
| 9 | 18 | 0.8546 | 0.8987 | 0.9206 | 0.9472 | 0.9667 | 0.9796 | 0.9871 | 0.9900 | 0.9941 |
| 10 | 20 | 0.8530 | 0.9011 | 0.9245 | 0.9509 | 0.9692 | 0.9806 | 0.9875 | 0.9902 | 0.9941 |
| 20 | 40 | 0.8930 | 0.9310 | 0.9478 | 0.9665 | 0.9791 | 0.9863 | 0.9906 | 0.9924 | 0.9948 |
| 30 | 60 | 0.9089 | 0.9465 | 0.9595 | 0.9745 | 0.9840 | 0.9895 | 0.9927 | 0.9941 | 0.9958 |
| 40 | 80 | 0.9192 | 0.9525 | 0.9659 | 0.9790 | 0.9868 | 0.9912 | 0.9938 | 0.9949 | 0.9965 |
| 50 | 100 | 0.9291 | 0.9591 | 0.9699 | 0.9818 | 0.9885 | 0.9924 | 0.9946 | 0.9956 | 0.9969 |
| 100 | 200 | 0.9529 | 0.9727 | 0.9806 | 0.9884 | 0.9929 | 0.9953 | 0.9967 | 0.9972 | 0.9804 |

* Simulated percentiles based on 10,000 Type II progressively censored sample data sets

## 4.4 Power

In application, the power of a goodness-of-fit test should also have to be investigated. For progressively censored data, these tests may exhibit different power depending on the different combinations of removal schemes and values of $m$ and $n$.

For each combination of removal schemes and values of $m$ and $n$ test statistic selected, the power of these tests of goodness-of-fit may be examined by generating random samples in the simulation program from an alternative distribution instead of the hypothesized distribution. The power is then estimated by the proportions of values from the alternative distribution which are less than percentile values from the tables of the hypothesized distributions.

In practice, a researcher must performs power studies specific to the test statistic removal scheme, and values of $m$ and $n$. For a hypothesized distribution, many alternative distributions would also have to be examined. A small illustration of how one such study might begin follows.

As an example of investigating power, consider the test statistic $T'$ and the 50% removal scheme used by Montanari, et al. (1998). Table 4.4.1 reports the power of the test to detect a normal alternative distribution if the assumed distribution is the Type I extreme value distribution.

Table 4.4.1

Power of the Goodness-of-Fit of an Extreme Value Distribution against

a Normal Distribution for Progressively Censored Lifetimes

with Removals $r_1 = 1, r_2 = 1, \ldots, r_m = 1$ Using $T'$ *

| m | n | Nominal Significance Level of a Test .01 | .05 | .10 |
|---|---|---|---|---|
| 3 | 6 | 0.0121 | 0.0600 | 0.1280 |
| 4 | 8 | 0.0164 | 0.0738 | 0.1254 |
| 5 | 10 | 0.0199 | 0.0667 | 0.1161 |
| 6 | 12 | 0.0206 | 0.0686 | 0.1209 |
| 7 | 14 | 0.0199 | 0.0699 | 0.1301 |
| 8 | 16 | 0.0209 | 0.0780 | 0.1345 |
| 9 | 18 | 0.0188 | 0.0721 | 0.1351 |
| 10 | 20 | 0.0138 | 0.0691 | 0.1430 |
| 20 | 40 | 0.0120 | 0.0804 | 0.1723 |
| 30 | 60 | 0.0097 | 0.1116 | 0.2286 |
| 40 | 80 | 0.0107 | 0.1257 | 0.2956 |
| 50 | 100 | 0.0101 | 0.1537 | 0.3288 |
| 100 | 200 | 0.0318 | 0.3182 | 0.5935 |

* Simulated proportions based on 10,000 Type II progressively censored sample data sets

## 4.5 Conclusion

In instances of Type II progressive censoring, correlation-type tests of goodness-of-fit may be adapted to explore distribution assumptions concerning the data. For the Weibull distribution, the tests in this section may be used not only to determine if the data is Weibull-disributed, but also to examine the failure rate by testing the distribution for different values of the shape parameter $\beta$. This proposed test for goodness-of-fit of Type II progressively censored data is applicable to many different distributions, and the test statistic is of simple computational form. Computational time becomes an important issue with regard to Type II progressively censored data since tables of the test statistic must be generated specific to each removal scheme and number of items censored and uncensored and assumed distribution. Further, the examples in this section illustrate that the conditional correlation-type test approach is feasible if $n$ is small. A second approach, using rank approximation, is shown to be feasible for all values of $n$.

# CHAPTER FIVE

# CONCLUSIONS

The purpose of this study was to investigate the variability associated with the expected value estimates of experiment time for Type II progressively censored samples. For the first time, numerical studies of experiment time have also included studies of the standard deviations and coefficients of variation for different Type II progressive censoring plans.

Standard deviations of experiment time for Weibull-distributed lifetimes were quantified, examined, and found to be large, especially in instances where the Weibull-distributed lifetime data exhibited declining failure rates. Because of the high overall variability associated with expected value estimators of experiment time, a recommendation was made that the experimenter should also simulate empirical distributions of experiment time for a particular censoring scheme prior to conducting the experiment.

Numerical studies of coefficients of variation of experiment time were included and recommended as a method by which to compare the relative amount of variability associated with the expected experiment time estimates for different censoring plans.

All of the numerical studies where obtained using a formula for experiment time which proved to be generalizeable for moments of experiment time and shown to be applicable to distributions other than the Weibull distribution if the moments of the first order statistic are known. A conditional procedure of assigning complete sample rankings to the progressively censored order statistics which suggested the formula for moments of experiment time also suggested a correlation-type test statistic for goodness-of-fit of Type II progressively censored samples. The feasibility and suggested uses of this test were illustrated. For Weibull-distributed lifetimes, the experimenter may wish to apply the tests to investigate the failure rate of the distribution of the progressively censored sample.

The experiment time and goodness-of-fit topics presented here considered a special 50% fixed removal scheme for Type II progressive censoring. Future directions of this research include further investigations into comparisons of this censoring scheme to other progressive censoring schemes. The size and power of the goodness-of-fit test applied to samples subjected to 50% censoring can be further explored. This can be achieved by assuming a variety of combinations of distributions in the null and alternative hypotheses. Additionally, the experiment time formulas developed are applicable to investigations of skewness and kurtosis of the distribution of experiment time for Type II progressive censoring plans.

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## APPENDIX

## PROGRAM 1

```
**********************************************************************************************
* Program 1 to perform Chapter 2 calculations
**********************************************************************************************;
%macro censor(m,n,b);
proc iml;
start;
m = &m;
n = &n;
b = &b;
/*                                                    r1 r2       rm-1
removals r                                            1  2  ------ m-1
                                                    1
                                         count      |
                                                    |
                                                  n_1cm_1
*/;
n_1cm_1 = gamma(n)/(gamma(m)*gamma(n-m+1));
r = j(n_1cm_1,m-1,0);
gr = j(m+1,1,0);
count = 0;
* Needs m-1 do loops. Change this part if m changes;
                              gr[1] = n-m;
             do r1=0 to gr[1]; gr[2] = gr[1] - r1;
             do r2=0 to gr[2]; gr[3] = gr[2] - r2;
             do r3=0 to gr[3]; gr[4] = gr[3] - r3;
             do r4=0 to gr[4]; gr[5] = gr[4] - r4;
             do r5=0 to gr[5]; gr[6] = gr[5] - r5;
             do r6=0 to gr[6]; gr[7] = gr[6] - r6;
             do r7=0 to gr[7]; gr[8] = gr[7] - r7;
             do r8=0 to gr[8]; gr[9] = gr[8] - r8;
             do r9=0 to gr[9];
                        count = count + 1;
                        r[count,1] = r1;
                        r[count,2] = r2;
                        r[count,3] = r3;
                        r[count,4]=  r4;
                        r[count,5] = r5;
                        r[count,6] = r6;
                        r[count,7] = r7;
                        r[count,8] = r8;
                        r[count,9] = r9;
end; end; end; end; end; end; end; end; end;  *needs m-1 end statements;
```

```
/*
sum                           partial sums  of rj  , j = 1 to m-1
                                        1  2       ...      m-1
                                   1
                                   ...
                                 n_1cm_1

*/;
sum = j( n_1cm_1, m-1,0);
        do i = 1 to n_1cm_1;
        sum[i,1] = r[i,1];
        do j = 2 to m-1;
                sum[i,j] = sum[i,j-1] + r[i,j];
        end;
        end;
*  E1 = Expected value of the kth order statistic E[X 1,j+n-Rm+1] for WEIBULL data;
E1 = j(n-m+1,n,0);
do Rm = m to n;
        i = Rm-m+1;
        do j = 0 to n-1;
                E1[i,j+1] = gamma( 1 + 1/b) / ( j + n - Rm + 1 ) **(1/b);  * 1st moment;
               *E1[i,j+1] = gamma( 1 + 2/b) / ( j + n - Rm + 1 ) **(2/b);  * 2nd moment;

        end;
end;

/*
E2     =        (Rm - 1)!       *    (-1)**(j) *     1       * E1
             j! (Rm - 1 - j)!                   (j+n-Rm+1)


E3     =        E[Xm|Rm]        where   Rm = m + 0      E3[1]
                                                          ...
                                        Rm = m + n-m    E3[n-m+1]
*/;
E2 = j(n-m+1,n,0);
E3 = j(n-m+1,1,0);
c1 = j(n-m+1,1,0);
constant = 0;
do Rm = m to n;
        i = Rm-m+1;

        do j = 0 to Rm-1;
                E2[i,j+1] = (-1)**(j)* gamma(Rm)/(gamma(j+1)*gamma(Rm-j)
                                *(j+n-Rm+1))* E1[i,j+1];
        end;
        do j = 0 to Rm-1;
                E3[i] = E3[i] + E2[i,j+1];
        end;
        c1[i]= n*gamma(n)/(gamma(Rm)*gamma(n-Rm+1));
        E3[i] = c1[i]*E3[i];
end;
```

```
/*
rank - possible ranks of p.c. observations in complete sample
                                                                R1
                                                                ...
                                                                Rm

   h(k)  = min[ (k+ sum (k-1) , ( Rk+1 - 1 ) ]                  h(2)
                                                                ...
                                                                h(m-1)
                                                                h(m)
*/;
h = j(m,1,1);
rank = j(m,1,1);
ex = 0;
do i = 1 to n_1cm_1;
        c = n - m +1;                       * eq. (2a)
        do j = 2 to m-1;                    * eq. (2a)
             c = c*(n-m-sum[i,j-1]+1);      * eq. (2a)
        end;                                * eq. (2a)

        * c = n_1cm_1;    *eq. (2b);
        exnum=0;
* Needs do loops for Rm thru R2.  Change if m changes...;
                                                         h[m]   = m + sum[i,m-1];
     do Rm   = m to h[m];       rank[m]   = Rm;    h[m-1] = min(m-1 + sum[i,m-2], Rm   -1);
     do Rm_1 = m-1 to h[m-1];  rank[m-1] = Rm_1;  h[m-2] = min(m-2 + sum[i,m-3], Rm_1 -1);
     do Rm_2 = m-2 to h[m-2];  rank[m-2] = Rm_2;  h[m-3] = min(m-3 + sum[i,m-4], Rm_2 -1);
     do Rm_3 = m-3 to h[m-3];  rank[m-3] = Rm_3;  h[m-4] = min(m-4 + sum[i,m-5], Rm_3 -1);
     do Rm_4 = m-4 to h[m-4];  rank[m-4] = Rm_4;  h[m-5] = min(m-5 + sum[i,m-6], Rm_4 -1);
     do Rm_5 = m-5 to h[m-5];  rank[m-5] = Rm_5;  h[m-6] = min(m-6 + sum[i,m-7], Rm_5 -1);
     do Rm_6 = m-6 to h[m-6];  rank[m-6] = Rm_6;  h[m-7] = min(m-7 + sum[i,m-8], Rm_6 -1);
     do Rm_7 = m-7 to h[m-7];  rank[m-7] = Rm_7;  h[m-8] = min(m-8 + sum[i,m-9], Rm_7 -1);
     do Rm_8 = m-8 to h[m-8];  rank[m-8] = Rm_8;
                         p = 1;
                         do j = 2 to m;
                             pnum1 = n - rank[j];
                             pnum2 = sum[i,j-1] + ( j-1) - rank[j] + 1 ;
                             pnum3 = pnum1 - pnum2;
                             pdenom1 = n - rank[j-1];
                             pdenom2 = sum[i,j-1] + ( j-1) - rank[j-1];
                             pdenom3 = pdenom1 - pdenom2;

                             pnum = gamma( pnum1 + 1) / (gamma(pnum2 + 1)*gamma(pnum3+1));

                             pdenom = gamma(pdenom1+1)/(gamma(pdenom2+1)*gamma(pdenom3+1));
                              p = p*pnum/pdenom;
                         end;
                         exnum  = exnum + p*E3[Rm-m+1];
        end; end; end; end; end; end; end; end; end;    * needs m-1 end statements;
        ex = ex + exnum/c;
end;
print m n b ex '1st moment eq.(2a)' ;
finish;
run;
%mend;
* to call the macro;
%censor(10,15, 5);  * m = 10, n = 15, and beta = 5;
```

PROGRAM 2

```
*******************************************************************************************
* Program 2 to perform Chapter 4 Calculations
*******************************************************************************************;
%macro censor(m,n,b, iter);
proc iml;
start;
m =&m;
n =&n;
b = &b;
iter = &iter;  * number of iterations;
co = j(iter,1,0); * correlations;
do f = 1 to iter;
     w=j(m,1,0);  v = w;  s = w;   u = w; x = w;
     corrsum = 0;
* Part 1 of program 2 - Simulate Type II Progressively Censored Samples;
* generate x a progresssively censored sample for a specific fixed 50% removal scheme;
                r = j(m,1,1);

                s[1] = r[m];
                do i = 2 to m;
                        s[i] = s[i-1] + r[m-i+1];
                end;
* generate m indep uniform(0,1) observations;
                do i = 1 to m;
                        w[i] = uniform(-2);
                end;
* set v = w**(1/(i + rm+...+rm-i+1));
                v[1]= w[1]**(1/(i + s[1]) );
                do i = 1 to m;
                        v[i] = w[i] **( 1/(i + s[i]) );
                end;
* set u = 1-vm vm-1 ... vm-i+1;
                prod = 1;
                do i = 1 to m;
                        prod = prod*v[m-i+1];
                        u[i] = 1 - prod;
                end;
*set x = F**(-1) (u) ;
                do i = 1 to m;
               *         x[i] = (-log(1- u[i]))**(1/b); * Weibull ( shape b);
                         x[i] = log( (-1)*log(1-u[i])); * Extreme Value;
               *         x[i] = log(log(1-log(1-u[i]))); * Exponential Power;
               *         x[i] =  probit(u[i]) ;          * normal ;
                end;
* end of Part 1;
* Part 2 of Program 2;
*partial sums  of rj , j = 1 to m-1
                                           1  2      ...     m-1
                                   1
                                   ...
                                 n_1cm_1;
              sum = j( m,1,0);
              do j = 2 to m;
                        sum[j] = sum[j-1] + r[j];
              end;
```

```
*  rank - possible ranks of p.c. observations in a complete sample
                                                            R1
                                                            ...
                                                            Rm

      h(k)   = min[ (k+ sum (k-1) , ( Rk+1 - 1 ) ]          h(2)
                                                            ...
                                                            h(m-1)
                                                            h(m)
;

      h = j(m,1,1);
      rank = j(m,1,1);
          * Needs do loops for Rm thru R2.  Change if m changes...;
                                                         h[m] = m + sum[m-1];
   do Rm = m to h[m];            rank[m] = Rm;         h[m-1] = min( m-1 + sum[m-2], Rm -1);
   do Rm_1 = m-1 to h[m-1];      rank[m-1] = Rm_1;     h[m-2] = min( m-2 + sum[m-3], Rm -2);
   do Rm_2 = m-2 to h[m-2];      rank[m-2] = Rm_2;     h[m-3] = min( m-3 + sum[m-4], Rm -3);
   do Rm_3 = m-3 to h[m-3];      rank[m-3] = Rm_3;     h[m-4] = min( m-4 + sum[m-5], Rm -4);
   do Rm_4 = m-4 to h[m-4];      rank[m-4] = Rm_4;     h[m-5] = min( m-5 + sum[m-6], Rm -5);
   do Rm_5 = m-5 to h[m-5];      rank[m-5] = Rm_5;     h[m-6] = min( m-6 + sum[m-7], Rm -6);
   do Rm_6 = m-6 to h[m-6];      rank[m-6] = Rm_6;     h[m-7] = min( m-7 + sum[m-8], Rm -7);
   do Rm_7 = m-7 to h[m-7];      rank[m-7] = Rm_7;     h[m-8] = min( m-8 + sum[m-9], Rm -8);
   do Rm_8 = m-8 to h[m-8];      rank[m-8] = Rm_8;
                       p = 1;
                       do j = 2 to m;
                          pnum1 = n - rank[j];
                          pnum2 = sum[j-1] + ( j-1) - rank[j] + 1 ;
                          pnum3 = pnum1 - pnum2;
                          pdenom1 = n - rank[j-1];
                          pdenom2 = sum[j-1] + ( j-1) - rank[j-1];
                          pdenom3 = pdenom1 - pdenom2;
                          pnum = gamma( pnum1 + 1) / (gamma(pnum2 + 1)*gamma(pnum3+1));
                          pdenom = gamma(pdenom1+1)/(gamma(pdenom2+1)*gamma(pdenom3+1));
                           p = p*pnum/pdenom;
                       end;

                       k = j(m,1,1);
                       *set k = F**(-1) ( Ri/(n+1)) ;
                       do g = 1 to m;
                              dummy = rank[g]/(n+1);
                              *k[g] = (-log(1- dummy))**(1/b); * Weibull ( shape b);
                               k[g] = log( (-1)*log(1 - dummy)); * Extreme Value;
                       end;
                       sumx = x[+,];
                       sumk = k[+,];
                       meanx = sumx/m;
                       meank = sumk/m;
                       sx = t(x)*x - sumx**(2.0)/m;
                       sk = t(k)*k - sumk**(2.0)/m;
                       sumc = 0;
                       do a = 1 to m;
                         c = (x[a] - meanx)*(k[a] - meank);
                         sumc = sumc + c;
                       end;
                       corr = sumc/(sx*sk)**(.5);
```

```
                        corrsum  = corrsum + p*corr;
          end; end; end; end; end; end; end; end; end; * Needs m-1 end statements;
   co[f] = corrsum;  * correlations;
end;
print m n b ;
varnames={co};
create datacorr from co[colname=varnames];
append from co;
close datacorr;
finish;
run;
/* Find percentiles;
   PROC UNIVARIATE;
      OUTPUT OUT=LOCATION MEAN=MEAN MODE=MODE MEDIAN=MEDIAN
      Q1=Q1 Q3=Q3 P5=P5 P10=P10 P90=P90 P95=P95 MAX=MAX;
   PROC PRINT;
   run;
*/
/* Perform Power Analysis;
data power;
set datacorr;
iter = &iter;
count = 0;
count10 = 0;
count05 = 0;
count01 = 0;
do j = 1 to iter;
  if co  ge  .888804 then  count = count + 1;  *use 1st percentile value;
  else if .859313 le co lt .888804 then count10 = count10 +1;*use 5thand1stthpercentiles;
  else if .831809 le co lt .859313 then count05 = count05 + 1;*use 5thand10th percentiles;
  else if  0 le co lt .831809 then count01 = count01 + 1; *use 10th perentile;
end;
count = count/iter;
count10 = count10/iter;
count05 = count05/iter;
count01 = count01/iter;
proc means;
run;
*/;
%mend;
* call macro;
%censor(8,16, .25, 10000); * m = 8, n = 16, beta = .25 and no. of iterations = 10,000;
```

## VITA

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Thesis: RANKING SMALL PROGRESSIVELY CENSORED SAMPLES TO ESTIMATE EXPERIMENT TIME AND TEST FOR GOODNESS-OF-FIT

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