A THEORETICAL STUDY ON THE TRANSFER EQUATION FOR THE SCATTERING OF POLARIZED LIGHT IN A PLANE-PARALLEL MEDIUM

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A THEORETICAL STUDY ON THE TRANSFER

EQUATION FOR THE SCATTERING OF

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PARALLEL MEDIUM

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PREFACE

In this research, the one-dimensional radiative transfer problem including polarization effects was investigated. Beginning with the fundamental transport equation for polarized light in a one-dimensional plane-parallel medium, the derivation of the exact solution, diffusion approximation with numerical results, and the numerical results for the derived exact solution are presented in Chapters I, II, and III, respectively. Chapters I to III are in the format of three papers to be published. Chapter I has already been published in the Journal of Quantitative Spectroscopy and Radiative Transfer. Chapter II will be presented at the National Heat Transfer Conference in August of 1999; and plans are to submit Chapter III for journal publication in the future.

The problem which we need to focus on is the one-dimensional radiative transfer problem for polarized light without reflective boundaries. One of the major reasons for examining the present one-dimensional case when solutions for some cases exist is that most of the previous studies cannot handle elliptically polarized light as the incident source, while the present work does not have this restriction. This work is the first step in solving realistic problems with polarization. Thus, we have concentrated on the polarization, but simplified the geometry and interfaces by choosing a one-dimensional case with non-reflective boundaries. Future plans are to generalize this solution after demonstrating the ability to handle polarization effects. In this study, we assume that collimated polarized incident radiation at an angle θ_o exists only at the top boundary and is a sheet of laser-like beams which simplifies our study to a one-dimensional problem. Moreover, no radiation is incident at the lower boundary, and there are no refractive index effects at either boundary. Furthermore, absorption and scattering without emission are assumed in the medium (which applies only to a low temperature medium). Note that the polarized phase matrix applied to the medium is general, requiring only that the scattering particles be randomly oriented and have one plane of symmetry.

In Chapter I, starting with the fundamental transport equation for polarized light of Eq. (1-1), the exact expressions are derived for the general source matrix of Eq. (1-24), fundamental source matrix of Eq. (1-23), reflection and transmission matrices of Eqs. (1-44) and (1-43), reflected and transmitted intensity matrices of Eqs. (1-56) and (1-57), and reflected and transmitted flux matrices of Eqs. (1-59) and (1-60). The procedure used is similar to that which Ambarzumian applied to the scalar problem. The principle of superposition as well as Ambarzumian's method are used in the solution process.

In addition, a procedure that modifies the classical P_1 approximation which has been applied to the scalar problem is introduced in Chapter II. Beginning with the fundamental diffuse transport equation for polarized light of Eq. (2-1), the expression for intensity is derived by using the classical P_1 approximation with both Mark's and Marshak's boundary conditions as well as the modified P_1 method with Marshak's boundary conditions for one-dimensional radiative transfer problem including polarization effects. The plane-parallel medium of interest scatters, absorbs, and is exposed to collimated incident polarized radiation. Numerical results are presented for five optical thicknesses (5, 10, 15, 20, and 30), five albedoes (0.5, 0.9, 0.95, 0.99, and 1), and three selected sets of the scattering coefficients. These solutions are compared with the classical

P_1 approximation and with the exact scalar results. Qualitatively good agreement for intensity is shown between the modified P_1 and the exact scalar solutions, while the classical P_1 approximation predictions are poor.

In Chapter III, numerical results are presented for radiative transfer in one-dimensional finite media without reflective boundaries but which scatters, absorbs, and is exposed to a incident polarized radiation. For these solutions, only information at the boundaries is obtained. The polarized phase matrix of the medium is general, requiring only that the scattering particles be randomly oriented and have one plane of symmetry. Numerical solutions are presented for various optical thicknesses (up to 10), two albedoes (0.5 and 0.99), two selected sets of the scattering coefficients, and four different incident polarized light boundary conditions which can be utilized to superpose and represent any incident polarized radiation.

In addition, these results are compared with the solution of the diffusion approximation for polarized light as well as with the exact results for the scalar problem. The comparison shows that the diffusion approximation can predict the state of the polarization qualitatively well; while the intensity for the scalar problem is equal to the intensity including polarization effects when the number of Legendre polynomials in the polarization phase matrix is one. Furthermore, the scalar results estimate the intensity very well for three of the chosen incident polarized light boundary conditions, but do poorly for one chosen incident polarized light boundary condition when the number of Legendre polynomials is greater than one.

Finally, a summary of conclusions and recommendations are provided in Chapter IV.

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TABLE OF CONTENTS

Chapter Page

LIST OF TABLES

LIST OF FIGURES

Figure Page

Figure | Page

Figure Page

Figure		Page

NOMENCLATURE

a_i	coefficients of the scattering matrix, Eq. (1-4), where i is from one to four
A	function defined in Eq. (2-17)
$\mathbb{A}^m$	matrix defined in Eq. (1-7c)
b_i	coefficients of the scattering matrix, Eq. (1-4), where i is from one to two
Bi	functions defined in Eqs. (2-24), where i is from one to two
$\mathbb{B}_i$	matrix of scattering coefficients defined in Eq. (1-7f)
$C_{i\text{-Mark}}$	coefficients defined in Eqs. (2-23a), (2-23b), (2-25a), and (2-25b), where i is from one to four
$C_{i\text{-Marshak}}$	coefficients defined in Eqs. (2-27a), (2-27b), (2-29a), and (2-29b), where i is from one to four
$\mathbb{C}^m$	matrix defined in Eq. (1-7a)
Di	functions defined in Eqs. (2-28), where i is from one to two
$\mathbb{D}_2$	matrix defined in Eq. (1-7d)
Ei	functions defined in Eqs. (2-28), where i is from one to two
$\mathbb{F}$	input flux matrix defined as $[F_I\ F_Q\ F_U\ F_V]^T$ (W/m^2-Hz)
Gi	functions defined in Eqs. (2-35), where i is from one to seven
Hi	functions defined in Eqs. (2-35), where i is from one to four
$\mathbb{H}$	matrix defined in Eq. (1-11)
i_a	angles in Fig. 1-2, where a is one or two

I	first Stokes parameter which represents the intensity of a polarized beam (W/m^2-Steradian-Hz)
I_d	diffuse first Stokes parameter (W/m^2-Steradian-Hz)
$\mathbb{I}$	polarized intensity matrix (W/m^2-Steradian-Hz)
$\mathbb{I}^+$	polarized intensity matrix in the positive τ direction (W/m^2-Steradian-Hz)
$\mathbb{I}^-$	polarized intensity matrix in the negative τ direction (W/m^2-Steradian-Hz)
$\mathbb{I}_d$	diffuse intensity matrix (W/m^2-Steradian-Hz)
$\mathbb{I}_d^+$	diffuse intensity matrix in the positive τ direction (W/m^2-Steradian-Hz)
$\mathbb{I}_d^-$	diffuse intensity matrix in the negative τ direction (W/m^2-Steradian-Hz)
$\mathbb{I}_o^+$	incident polarized intensity matrix outside the top boundary in the positive τ direction (W/m^2-Steradian-Hz)
$\mathbb{K}_{yijm}$	matrices defined in Eqs. (1-20) and (1-31b), where y is one or three
L	number of Legendre polynomials
$L(\pi - \alpha)$	linear transform matrix defined in Eq. (1-3)
n	the index of refraction of the material bounding the top and bottom boundaries of the medium as well as the index of refraction of the medium
$P_a(\mu, \phi)$	points on Fig. 1-2, where a is one or two
P_i	Legendre polynomials defined in Eq. (1-a7)
P_i^m	Associated Legendre function defined in Eq. (1-a8)
$\mathbb{P}$	general phase matrix defined in Eq. (1-6)
$\mathbb{P}_{imy}$	source matrices defined in Eqs. (1-19a) and (1-19b), where y is from one to two
$\mathbb{PP}_{imy}$	source matrices defined in Eqs. (1-23) and (1-30), where y is one or three
$\overline{\mathbb{PP}_{kmy}}$	matrices defined in Eqs. (1-42a) and (1-42c), where y is one or three

$\overline{\mathbb{PPI}}_{kmy}$	matrices defined in Eqs. (1-42b) and (1-42d), where y is one or three
Q	second Stokes parameter
Q_d	diffuse second Stokes parameter
R	scattering matrix defined in Eq. (1-4)
R_i^m	combination of generalized spherical functions
$\mathbb{S}$	general source matrix defined in Eq. (1-14)
$\mathbb{S}^m$	matrix defined in Eq. (1-7b)
Ti	functions defined in Eqs. (2-37), where i is from one to five
T_i^m	combination of generalized spherical functions
U	third Stokes parameter
U_d	diffuse third Stokes parameter
V	fourth Stokes parameter
V_d	diffuse fourth Stokes parameter

Greek

α	designated angle in Eq. (1-3)
α_i	phase matrix constant defined in Eq. (1-a6)
β_i	phase matrix constant defined in Eq. (1-a1)
γ_i	phase matrix constant defined in Eq. (1-a3)
δ	Dirac delta function
δ_i	phase matrix constant defined in Eq. (1-a2)
δ_{0m}	Kronecker delta function

ε_i	phase matrix constant defined in Eq. (1-a4)
ζ_i	phase matrix constant defined in Eq. (1-a5)
θ	polar angle of polarized intensity inside the medium
θ_{in}	polar angle of incident polarized intensity just inside the top boundary of the medium
θ_o	incident [polar] angle
Θ	angle between light rays before and after scattering
μ	cosine of the polar angle inside the medium, $\cos\theta$
μ_{in}	cosine of the polar angle just inside the top boundary, $\cos\theta_{in}$
μ_o	cosine of the incident [polar] angle of the incident polarized intensity, $\cos\theta_o$
μ_s	$= \cos\Theta$
Π	matrix defined in Eq. (1-7e)
Π'	matrix defined in Eq. (1-31a)
τ	optical location
τ_0	finite optical thickness
ϕ	azimuthal angle inside the medium
ϕ_o	azimuthal angle outside the medium
ω	single scattering albedo

Superscripts

+	in the positive τ direction
-	in the negative τ direction

Subscripts

d	diffuse term
in	inside the medium
o	outside the medium

CHAPTER I

DEVELOPMENT OF RADIATIVE TRANSFER EQUATIONS FOR THE SCATTERING OF POLARIZED LIGHT IN A PLANE-PARALLEL MEDIUM

Abstract

The objective of the present work is to demonstrate that the exact solutions of one-dimensional radiative transfer source function, intensity, and flux for polarized radiation can be obtained by using a procedure similar to that which Ambarzumian applied to the scalar problem. In this paper, the exact expressions are derived for the source matrices, reflection and transmission matrices, reflected and transmitted intensity matrices, and reflected and transmitted flux matrices at the boundaries of a plane-parallel medium which scatters, absorbs, and is exposed to incident polarized radiation. The polarized phase matrix of the medium is general, requiring only that the scattering particles be randomly oriented and have one plane of symmetry. In the future, the equations formulated herein will be solved numerically for various polarized boundary conditions, albedoes, and optical thicknesses.

Introduction

Extensive studies for radiative transfer exist in the literature. However, although some solutions are available, the exact numerical solutions for the scattering of polarized light in a one-dimensional finite medium exposed to elliptically polarized incident light without reflective boundaries are not available. Many researchers have simplified the effect of polarization due to its mathematical complexity, while others have formulated equations that become very difficult to solve numerically. Some interesting and related studies, mainly focused on polarization, will be reviewed in the following.

Some typical studies focused on the effect of polarization in plane-parallel media were conducted by Chandrasekhar [1-1], Reguigui [1-2, 1-3], Hovenier [1-4], Siewert [1-5], Hovenier and van der Mee [1-6], Benassi et al. [1-7, 1-8], Zege and Chaikovskaya [1-9], and Mishchenko [1-10, 1-11]. Several studies concentrate on the derivation of radiative transfer equations without numerical results while others present numerical solutions with the incident radiation being unpolarized, circularly polarized, or linearly polarized in order to simplify the numerical process.

The fundamental radiative transfer equation including polarization effects was derived by Chandrasekhar [1-1] for a plane-parallel atmosphere with Rayleigh scattering. The exact solutions for the parallel and perpendicular components (I_l and I_r) of polarized radiation were also presented by Chandrasekhar for a plane-parallel axisymmetric atmosphere with Rayleigh's law and unpolarized incident radiation. Most of the work after Chandrasekhar has tried to extend his work in order to handle a general scattering matrix.

The derivation of the correlation transfer equation for dynamic light scattering (very similar to that of radiative transfer) was presented by Reguigui [1-2, 1-3]. By using various radiative transfer solution approaches, the numerical results for correlation (which is comparable to radiative intensity) were obtained for both finite and semi-infinite media with the incident radiation being unpolarized. The effects of polarization and other important parameters on the correlation function were considered and discussed. It was found that polarization effects cannot be ignored for low optical thickness ($\tau_0 \le 5$), but are less important for high optical thickness ($\tau_0 \ge 20$).

An extension of the doubling method [per Hovenier (van de Hulst, 1963)] was presented by Hovenier [1-4] to solve the radiative transfer problem in plane-parallel atmospheres including polarization effects. He presented numerical intensity results for four different phase matrices: (a) Rayleigh scattering, (b) two simple test matrices with unit albedo, of which one of these two models was designed to simulate the scattering of water vapor at a wavelength of 0.7 μm, and (c) for comparison purposes, a scalar phase function (i.e., no polarization). The numerical results for the two simple test matrices suggested that ignoring polarization is not very important for intensity but obviously loses the degree of polarization. His simple test matrices predicted the state of polarization with relative accuracy. Unpolarized unidirectional incident light, modeling that coming from the sun, was used for this research.

The radiative transfer problem for a finite plane-parallel medium exposed to incident elliptically polarized radiation was considered by Siewert [1-5]. The problem was reduced to a group of radiative transfer equations, formulated in terms of the four classical

Stokes parameters, by using a Fourier decomposition in the azimuthal angle. No numerical solutions were presented.

Hovenier and van der Mee [1-6] have found the relationships between the Stokes parameters and several complex polarization parameters. The polarized transport equation and phase matrix for a plane-parallel atmosphere were discussed and formulated by using both Stokes parameters and complex polarization parameters. By using the addition theorem of generalized spherical functions, the phase matrix and all its Fourier components were expressed analytically. No numerical results were provided.

A dispersion matrix, which was used to get the elementary solutions for the polarized radiative transfer equation, was given in various representations by Benassi et al. [1-7]. They discussed how to compute the zeros of the determinant of the dispersion matrix in order to get the analytical solutions. Furthermore, numerical results were given for three different scattering models with the incident radiation being unpolarized or circularly polarized.

Starting with an analytical representation of the phase matrix, Benassi et al. [1-8] presented the solution for scattering of polarized light in a plane-parallel medium with the assumption that the intensity is independent of the azimuthal angle (i.e. azimuthally symmetric). Numerical results were given for the incident radiation being either unpolarized or circularly polarized in order to satisfy the azimuthally symmetric assumption.

Zege and Chaikovskaya [1-9] presented an approximate method to solve the radiative transfer problem including polarization effects. Instead of the originally complicated vector radiative transfer equations (VRTEs), which were sets of four simultaneous equations based on the Stokes parameters, a simplified new set of VRTEs, based on an approximate Green's function matrix, were derived with the major assumption that the scattering matrix of the medium was isotropic. The advantages for this isotropic medium approximation were: (a) the set of four simultaneous equations for the original VRTEs can be simplified to either sets of two simultaneous equations or the scalar equations, (b) the new VRTEs have simpler kernels than the kernels of the original VRTEs, (c) some complicated functions can be eliminated from the original VRTEs, and (d) the new VRTEs give quick convergence as well as high accuracy. No numerical solutions were provided.

Mishchenko [1-10, 1-11] formulated exact reflected radiation equations by using an extension of the invariant imbedding method for a finite plane-parallel atmosphere including polarization effects. However, the formulated equations were numerically complex, requiring double integration, thus the author numerically solved two simplified problems, for unpolarized incident radiation and for linearly polarized incident radiation.

For the diffuse scattering of polarized light, Herman et al. [1-12] presented numerical results for both spherical and plane-parallel atmospheres by using the Gauss-Seidel calculation method. Comparisons between the polarized spherical Gauss-Seidel method and Monte Carlo calculations of other published studies for both spherical and plane-parallel media were also made. When all scattering terms were considered, the four Stokes parameters (I, Q, U, and V) were in good agreement between the polarized spherical Gauss-Seidel method and the Monte Carlo method. The solar radiation incident at the top of the atmosphere was assumed to be a completely unpolarized parallel beam.

Special work, mainly on the phase matrix, for the scattering of polarized light was performed by Siewert [1-13], Vestrucci and Siewert [1-14], de Rooij and van der Stap [1-15], and Hovenier [1-16].

An analytical phase matrix corresponding to a Stokes representation of the polarized scattering matrix, which allowed the components of phase matrix to be expressed by a Fourier decomposition, was reported by Siewert [1-13]. The fundamental constants and matrices of this phase matrix were deduced by using a set of orthogonality and recursive relations. Three symmetry relationships of this phase matrix were also provided at the end. No numerical solutions were given.

An analytical phase matrix (components in a Fourier decomposition) for scattering of polarized light was presented by Vestrucci and Siewert [1-14]. Some values of the fundamental constants required for this phase matrix were provided for different scattering matrices. No numerical results for intensity were presented.

The polarized scattering matrix, which can be expanded in generalized spherical functions, was considered by de Rooij and van der Stap [1-15]. The expansion coefficients of this scattering matrix, which represented scattering by homogeneous spherical particles, were calculated in two ways: (1) Domke's [per de Rooij and van der Stap (Domke, 1975)] explicit expressions, and (2) numerical angular integration. Furthermore, four sets of expansion coefficients were given according to four specific scattering matrices. No numerical results for intensity were provided.

Hovenier [1-16] discussed the symmetry relationships based on two different polarized scattering matrices for which particles were randomly oriented and: (1) had a plane of symmetry, or (2) did not have a plane of symmetry. He presented the symmetry relations for the phase matrix and for the reflection and transmission matrices, based on a scattering matrix which represented particles having a plane of symmetry. For the scattering matrix representing particles not having a plane of symmetry, birefringence and dichroism might occur, and the symmetry relations for only the phase matrix were considered. No numerical results were provided.

Assuming a semi-infinite scattering medium that was homogeneous with randomly oriented polydisperse scattering spheres having a plane of symmetry, Mishchenko [1-17] has presented the Stokes reflection matrix which can be used to find radar reflectivity, polarization ratios, and enhancement factors. Some graphical results for the effects of particle size parameters, as well as the real and imaginary parts of the index of refraction, on the photometric and polarization characteristics of the radar return were also provided. No numerical results for intensity were given.

Mueller and Crosbie [1-18] presented a polarized phase matrix for the three-dimensional radiative transfer problem based on a scattering matrix which represented randomly oriented scattering particles having a plane of symmetry. In that paper, the geometry was finite in the z-direction and infinite in the x- and y-directions, with elliptically polarized radiation incident only on the top boundary. Great effort was expended to reduce this three-dimensional problem to a one-dimensional problem which depended on two parameters. A general four by four source matrix was derived by using the method of superposition. Some symmetry relationships were developed. Moreover, an extensive review of a wide variety of radiative transfer literature was also provided. No numerical results were presented.

The purpose of this chapter of the current study is to obtain the exact expressions for the source matrices, reflection and transmission matrices, reflected and transmitted intensity matrices, and reflected and transmitted flux matrices only at the boundaries of a one-dimensional plane-parallel medium which scatters and absorbs, with polarization fully included. The polarized phase matrix of the medium requires only that the scattering particles be randomly oriented and have one plane of symmetry [1-16]. Moreover, these exact expressions are relatively straightforward and should be numerically simpler to solve than those of previous researchers. Therefore, the present work extends previous research because the numerical solutions will allow the incident radiation to be elliptically polarized, which implies that the solutions depend on the azimuthal angle. Future research will be directed toward the inclusion of refractive index effects and multi-dimensionality, after numerical solutions of the current work are obtained.

Development of General Equations for Polarized Light

In this section, beginning with the fundamental transport equation for polarized light in a one-dimensional plane-parallel medium, the diffuse transport equation, general source matrix, the sub-source matrices, and the fundamental source matrix will be derived. Absorption and scattering without emission are assumed in the medium, and refractive index effects at the boundaries are neglected.

Problem Description

As mentioned earlier, the problem which we are interested in and which needs to be solved first is the one-dimensional radiative transfer problem for polarized light without reflective boundaries. One of the major reasons to do the present one-dimensional case when solutions for some cases exist is that the previous studies either cannot handle elliptically polarized light as the incident source or they are extremely difficult to solve numerically, while the present work does not have these restrictions. The geometry for this problem is shown in Fig. 1-1.

In this research, we assume that collimated polarized incident radiation at an angle θ_o exists only at the top boundary and is a sheet of laser-like beams which simplifies our study to a one-dimensional problem. Moreover, no radiation is incident at the lower boundary and there are no refractive index effects at either boundary. Furthermore, absorption and scattering without emission are assumed in the medium (which applies only as a good approximation to a low temperature medium). Note that the probability of scattering in the various directions depends on the phase matrix function, which will be discussed later.

Fundamental Equations

The transport equation, modified slightly from Chandrasekhar [1-1], for polarized light in a plane-parallel atmosphere (Fig. 1-1) can be written in the form

$$\mu \frac{d\,\mathbb{I}(\tau,\mu,\phi)}{d\tau} + \mathbb{I}(\tau,\mu,\phi) = \frac{\omega}{4\pi} \int_0^{2\pi} \int_{-1}^{1} \mathbb{P}(\mu,\mu',\phi,\phi')\,\mathbb{I}(\tau,\mu',\phi')\,d\mu'\,d\phi', \qquad (1\text{-}1)$$

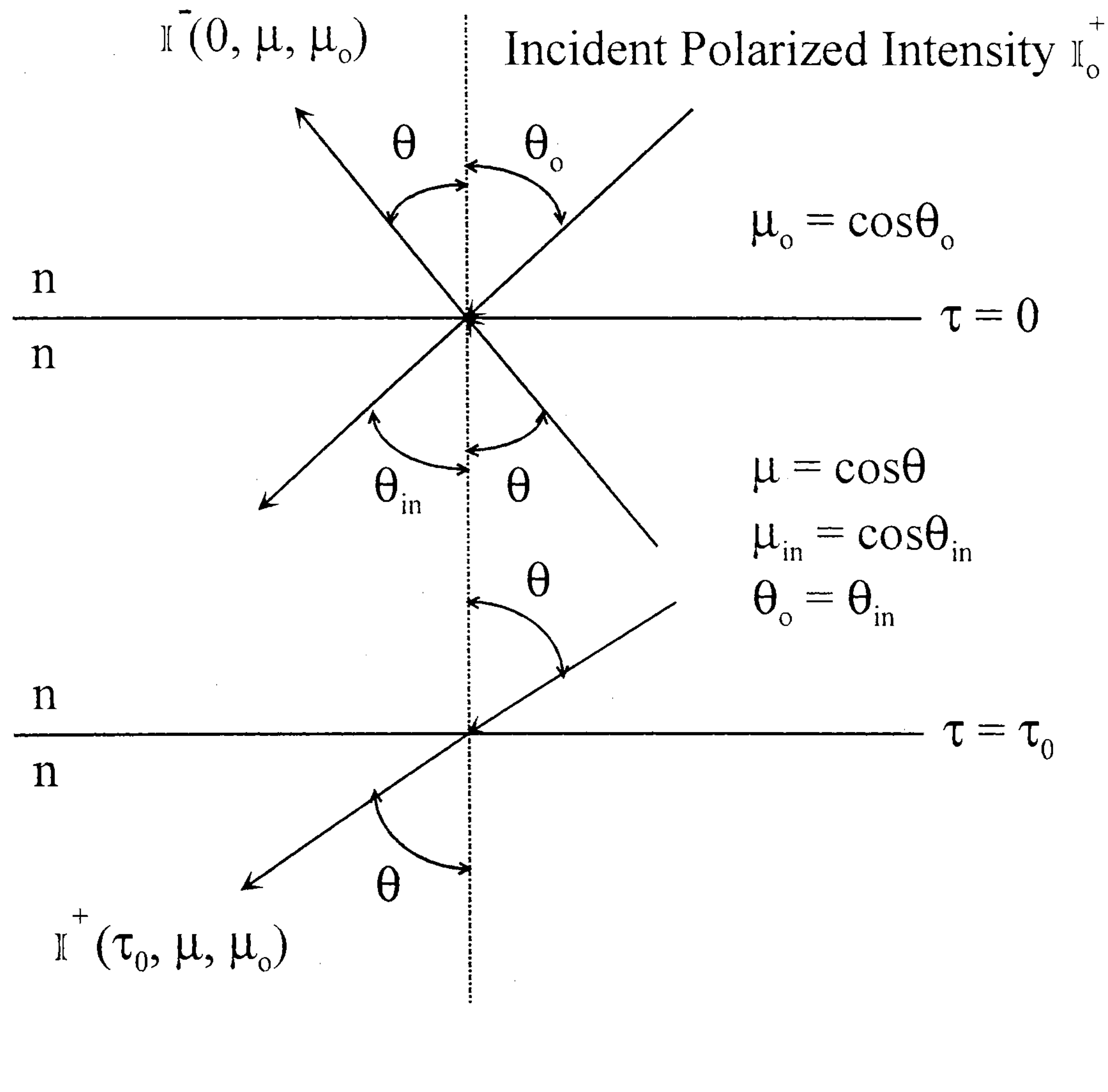

Figure 1-1. Geometry of a One-Dimensional Medium without Reflective Boundaries

where τ is the normal optical thickness, μ is the direction cosine of the propagation angle of the radiation, ω is the single scattering albedo, and the intensity vector $\mathbb{I}(\tau, \mu, \phi)$ consists of the four Stokes parameters, that is, $\mathbb{I}(\tau, \mu, \phi) = [I(\tau, \mu, \phi)\ \ Q(\tau, \mu, \phi)\ \ U(\tau, \mu, \phi)\ \ V(\tau, \mu, \phi)]^T$. Furthermore, $\mathbb{P}(\mu, \mu', \phi, \phi')$ is the phase matrix [1-16], given by

$$\mathbb{P}(\mu, \mu', \phi, \phi') = L(\pi - i_2)\, R(\cos\Theta)\, L(-i_1), \tag{1-2}$$

where $L(-i_1)$ and $L(\pi - i_2)$ are the linear transformation matrices [1-16], which are required to rotate meridian planes before and after scattering onto a local scattering plane (Fig. 1-2), given by

$$L(\pi - \alpha) = L(-\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\alpha & -\sin 2\alpha & 0 \\ 0 & \sin 2\alpha & \cos 2\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{1-3}$$

Moreover, i_1 represents the angle between the meridian plane OP_1Z and the scattering plane OP_1P_2, while i_2 represents the angle between the planes OP_1P_2 and OP_2Z, as shown in Fig. 1-2. Here, Θ is the angle between the light rays before and after scattering.

In addition, $R(\cos\Theta)$ is the scattering matrix [1-16], and can be expressed in general as

$$R(\mu_s) = \begin{bmatrix} a_1(\mu_s) & b_1(\mu_s) & 0 & 0 \\ b_1(\mu_s) & a_2(\mu_s) & 0 & 0 \\ 0 & 0 & a_3(\mu_s) & b_2(\mu_s) \\ 0 & 0 & -b_2(\mu_s) & a_4(\mu_s) \end{bmatrix}, \tag{1-4}$$

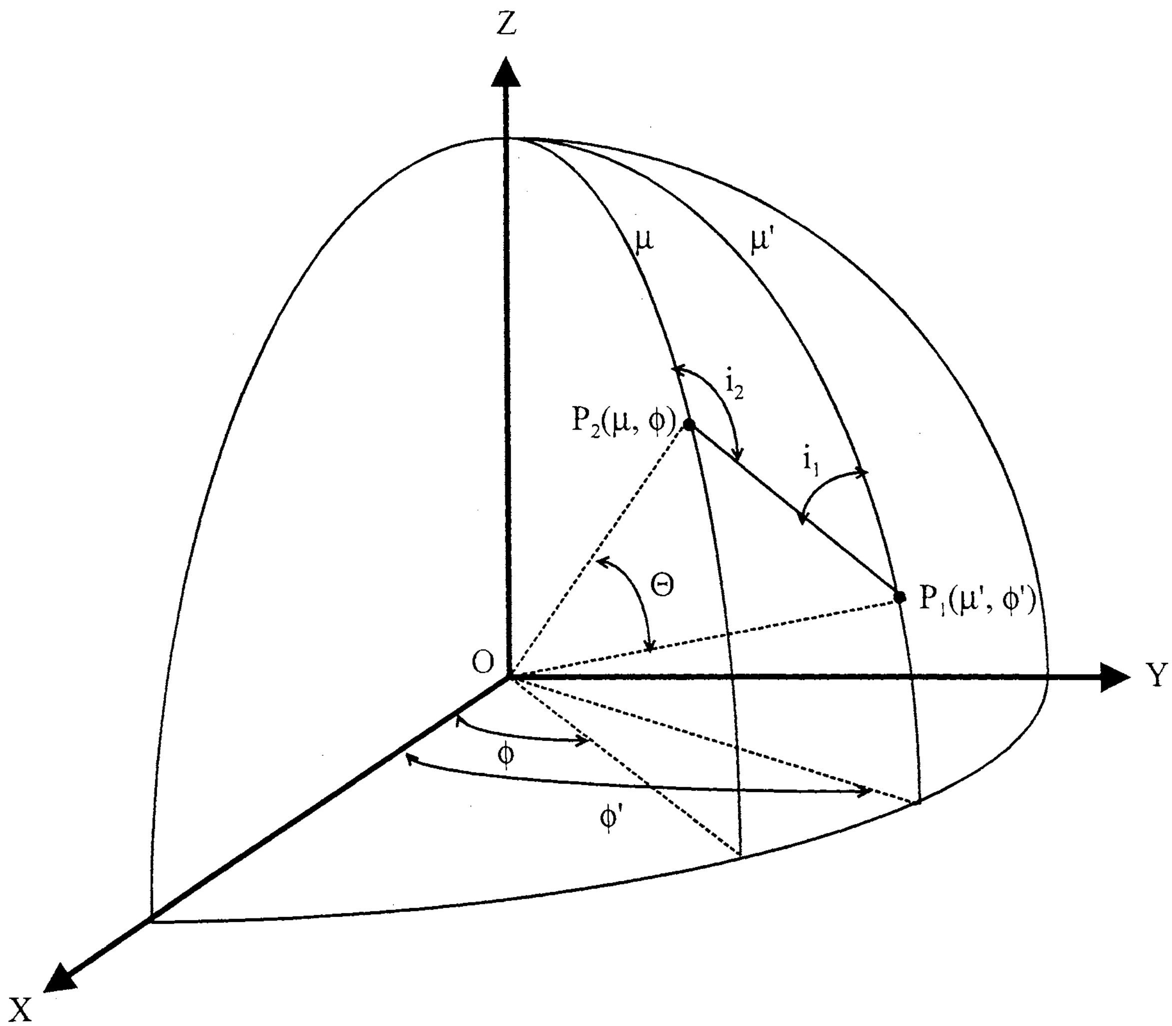

Figure 1-2. Geometry of Scattering Plane with Respect to Meridian Planes

where $\mu_s = \cos\Theta$. Notice that the scattering matrix of Eq. (1-4) assumes that the scattering particles are randomly oriented, and have at least one plane of symmetry [1-16]. Normalization [1-13] requires that

$$\int_{-1}^{1} a_1(\mu_s)\, d\mu_s = 2. \tag{1-5}$$

The six functions $a_1(\mu_s)$, $a_2(\mu_s)$, $a_3(\mu_s)$, $a_4(\mu_s)$, $b_1(\mu_s)$, and $b_2(\mu_s)$ appearing in Eq. (1-4) are real valued for $\mu_s \in [-1, 1]$ and must be known inputs.

Before developing the transport equation further, it will be useful to express the phase matrix in a general form. Following the same procedure as Siewert [1-13], the phase matrix of Eq. (1-2) can be expanded in a Fourier series [1-8] as

$$\mathbb{P}(\mu, \mu', \phi, \phi') = \sum_{m=0}^{L} \frac{1}{1 + \delta_{0m}} \{ \mathbb{C}^m(\mu, \mu') \cos[m(\phi - \phi')]$$

$$+ \mathbb{S}^m(\mu, \mu') \sin[m(\phi - \phi')] \}, \tag{1-6}$$

where δ_{0m} is the Kronecker delta function, and other functions in Eq. (1-6) are defined as

$$\mathbb{C}^m(\mu, \mu') = \mathbb{A}^m(\mu, \mu') + \mathbb{D}_2\, \mathbb{A}^m(\mu, \mu')\, \mathbb{D}_2, \tag{1-7a}$$

$$\mathbb{S}^m(\mu, \mu') = \mathbb{A}^m(\mu, \mu')\, \mathbb{D}_2 - \mathbb{D}_2\, \mathbb{A}^m(\mu, \mu'), \tag{1-7b}$$

$$\mathbb{A}^m(\mu, \mu') = \sum_{i=m}^{L} \frac{(i - m)!}{(i + m)!} \Pi_i^m(\mu)\, \mathbb{B}_i\, \Pi_i^m(\mu'), \tag{1-7c}$$

$$\mathbb{D}_2 = \mathrm{diag}\{1, 1, -1, -1\}, \tag{1-7d}$$

$$\Pi_i^m(\mu)=\begin{bmatrix} P_i^m(\mu) & 0 & 0 & 0 \\ 0 & R_i^m(\mu) & -T_i^m(\mu) & 0 \\ 0 & -T_i^m(\mu) & R_i^m(\mu) & 0 \\ 0 & 0 & 0 & P_i^m(\mu) \end{bmatrix}, \tag{1-7e}$$

and the matrix of scattering coefficients is given by

$$\mathbb{B}_i = \begin{bmatrix} \beta_i & \gamma_i & 0 & 0 \\ \gamma_i & \alpha_i & 0 & 0 \\ 0 & 0 & \zeta_i & -\varepsilon_i \\ 0 & 0 & \varepsilon_i & \delta_i \end{bmatrix}. \tag{1-7f}$$

The matrix $\mathbb{B}_i$ is specified in detail in the Appendix I. In addition, in Eq. (1-7e), $P_i^m(\mu)$ denotes associated Legendre functions while $R_i^m(\mu)$ and $T_i^m(\mu)$ are combinations of generalized spherical functions. These functions are also defined in the Appendix I.

Boundary Conditions

We will consider a finite layer with optical location $\tau \in [0, \tau_0]$. The non-reflecting surface $\tau = 0$ is illuminated by a parallel beam incident at θ_o to the normal surface, and is bounded at τ_0 by another non-reflecting interface upon which no radiation is incident. We will find the solution of Eq. (1-1) subject to the boundary conditions, for $\mu \in [0, 1]$ and $\phi \in [0, 2\pi]$,

$$\mathbb{I}(0, \mu, \phi) = \pi\,\delta(\mu - \mu_o)\,\delta(\phi - \phi_o)\,\mathbb{F} \tag{1-8a}$$

and $\mathbb{I}(\tau_0, -\mu, \phi) = 0,$ (1-8b) where δ is the Dirac delta function and the vector $\mathbb{F} = [F_I \; F_Q \; F_U \; F_V]^T$, presumed given, specifies the state of polarization of the incident intensity at the upper boundary.

Diffuse Intensity Vector

We can now separate the intensity vector into unscattered (with $\mathbb{F}$) and diffuse ($\mathbb{I}_d$) components [1-8] by writing

$$\mathbb{I}(\tau, \mu, \phi) = \pi \, \delta(\mu - \mu_o) \, \delta(\phi - \phi_o) \exp(-\tau / \mu) \, \mathbb{F} + \mathbb{I}_d(\tau, \mu, \phi) \tag{1-9a}$$

and

$$\mathbb{I}(\tau, -\mu, \phi) = \mathbb{I}_d(\tau, -\mu, \phi). \tag{1-9b}$$

After we substitute Eqs. (1-9) into Eqs. (1-1) and (1-8), we see that the diffuse field is defined by

$$\mu \frac{d \, \mathbb{I}_d(\tau, \mu, \phi)}{d\tau} + \mathbb{I}_d(\tau, \mu, \phi) = \frac{\omega}{4\pi} \int_0^{2\pi} \int_{-1}^{1} \mathbb{P}(\mu, \mu', \phi, \phi') \, \mathbb{I}_d(\tau, \mu', \phi') \, d\mu' \, d\phi'$$
$$+ \, \mathbb{H}(\tau, \mu, \phi), \tag{1-10}$$

where $\mathbb{H}(\tau, \mu, \phi) = \dfrac{\omega}{4} \, \mathbb{P}(\mu, \mu_o, \phi, \phi_o) \exp(-\tau / \mu_o) \, \mathbb{F}$, (1-11)

and the boundary conditions, for $\mu \in [0, 1]$ and $\phi \in [0, 2\pi]$, are

$$\mathbb{I}_d(0, \mu, \phi) = 0 \tag{1-12a}$$

and $\mathbb{I}_d(\tau_0, -\mu, \phi) = 0.$ (1-12b)

There are at least three possible exact techniques which could be used to solve for the diffuse intensity vector of Eq. (1-10). One is the approach similar to Ambarzumian [1-19] (used by Liu [1-20]), another is that of invariant imbedding used by Mishchenko [1-10, 1-11], while the third is a direct solution method similar to Benassi et al [1-8]. Because an Ambarzumian [1-19] like procedure has certain advantages, such as obtaining a relatively simple solution only at the boundaries without it being necessary to solve for intensity inside the boundaries, an approach similar to that of Refs. [1-19] and [1-20] will be utilized.

Source Matrix

Equation (1-10) can be written in another form as

$$\mu \frac{d\,\mathbb{I}_d(\tau, \mu, \phi)}{d\tau} + \mathbb{I}_d(\tau, \mu, \phi) = \mathbb{S}(\tau, \mu, \phi), \tag{1-13}$$

where

$$\mathbb{S}(\tau, \mu, \phi) = \frac{\omega}{4\pi} \int_0^{2\pi} \int_{-1}^{1} \mathbb{P}(\mu, \mu', \phi, \phi')\, \mathbb{I}_d(\tau, \mu', \phi')\, d\mu'\, d\phi' + \mathbb{H}(\tau, \mu, \phi), \tag{1-14}$$

is the general source matrix. Equations (1-12) are modified to convenient forms as

$$\mathbb{I}_d^+(0, \mu, \phi) = 0 \tag{1-15a}$$

and $\mathbb{I}_d^-(\tau_0, \mu, \phi) = 0,$ (1-15b) where the superscripts + and - on Eqs. (1-15) denote that the intensity is generally propagating in the positive and negative τ directions (see Fig. 1-1), respectively.

Solving Eq. (1-13) for $\mathbb{I}_d^+$ and $\mathbb{I}_d^-$, using an integrating factor, yields

$$\mathbb{I}_d^+(\tau, \mu, \phi) = \mathbb{I}_d^+(0, \mu, \phi) \exp(-\tau / \mu) + \int_0^\tau \mathbb{S}(t, \mu, \phi) \exp[-(\tau - t) / \mu] \frac{dt}{\mu} \tag{1-16a}$$

and

$$\mathbb{I}_d^-(\tau, \mu, \phi) = \mathbb{I}_d^-(\tau_0, \mu, \phi) \exp[-(\tau_0 - \tau) / \mu]$$

$$+ \int_\tau^{\tau_0} \mathbb{S}(t, -\mu, \phi) \exp[-(t - \tau) / \mu] \frac{dt}{\mu}. \tag{1-16b}$$

Our next goal is to find a more explicit form for the general source matrix in Eq. (1-14). First, substituting Eqs. (1-6), (1-7a), (1-7b), (1-7c), (1-11), (1-16), and (1-15) into Eq. (1-14) with the assumption that the incident azimuthal angle, ϕ_o, is equal to zero, we have

$$\begin{aligned}
\mathbb{S}(\tau, \mu, \mu_o, \phi; \tau_0) = {} & \frac{\omega}{4} \exp(-\tau / \mu_o) \sum_{m=0}^{L} \frac{1}{1 + \delta_{0m}} \sum_{i=m}^{L} \frac{(i - m)!}{(i + m)!} \{ [\Pi_i^m(\mu) \mathbb{B}_i \\
& \times \Pi_i^m(\mu_o) + \mathbb{D}_2 \Pi_i^m(\mu) \mathbb{B}_i \Pi_i^m(\mu_o) \mathbb{D}_2] \cos(m\phi) + [\Pi_i^m(\mu) \\
& \times \mathbb{B}_i \Pi_i^m(\mu_o) \mathbb{D}_2 - \mathbb{D}_2 \Pi_i^m(\mu) \mathbb{B}_i \Pi_i^m(\mu_o)] \sin(m\phi) \} \mathbb{F} \\
& + \frac{\omega}{4\pi} \sum_{m=0}^{L} \frac{1}{1 + \delta_{0m}} \sum_{i=m}^{L} \frac{(i - m)!}{(i + m)!} \int_0^{2\pi} \int_0^1 \int_0^{\tau_0} \langle \{ \Pi_i^m(\mu) \mathbb{B}_i \\
& \times \Pi_i^m[\mathrm{sign}(\tau - t)\mu'] + \mathbb{D}_2 \Pi_i^m(\mu) \mathbb{B}_i \Pi_i^m[\mathrm{sign}(\tau - t)\mu'] \mathbb{D}_2 \} \\
& \times [\cos(m\phi) \cos(m\phi') + \sin(m\phi) \sin(m\phi')] + \{ \Pi_i^m(\mu) \mathbb{B}_i \\
& \times \Pi_i^m[\mathrm{sign}(\tau - t)\mu'] \mathbb{D}_2 - \mathbb{D}_2 \Pi_i^m(\mu) \mathbb{B}_i \Pi_i^m[\mathrm{sign}(\tau - t)\mu'] \} \\
& \times [\sin(m\phi) \cos(m\phi') - \cos(m\phi) \sin(m\phi')] \rangle
\end{aligned}$$

$$\times\ \mathbb{S}[t,\ \mathrm{sign}(\tau - t)\mu',\ \mu_o,\ \phi';\ \tau_0]\ \exp[-|\tau - t|/\mu']\ dt\ \frac{d\mu'}{\mu'}\ d\phi', \quad (1\text{-}17)$$

where sign(τ - t) is 1 if $\tau \geq t$, and is -1 if $\tau < t$.

Examination of Eq. (1-17) suggests that a reasonable expansion of the general source matrix is

$$\mathbb{S}(\tau,\ \mu,\ \mu_o,\ \phi;\ \tau_0) = \frac{\omega}{4\pi} \sum_{m=0}^{L} \frac{1}{1 + \delta_{0m}} \sum_{i=m}^{L} \frac{(i - m)!}{(i + m)!} \{ \cos(m\phi)\ [\Pi_i^m(\mu)$$

$$\times\ \mathbb{B}_i\ \mathbb{P}_{im1}(\tau,\ \mu_o;\ \tau_0) + \mathbb{D}_2\ \Pi_i^m(\mu)\ \mathbb{B}_i\ \mathbb{P}_{im2}(\tau,\ \mu_o;\ \tau_0)]$$

$$+ \sin(m\phi)\ [\Pi_i^m(\mu)\ \mathbb{B}_i\ \mathbb{P}_{im2}(\tau,\ \mu_o;\ \tau_0) - \mathbb{D}_2\ \Pi_i^m(\mu)$$

$$\times\ \mathbb{B}_i\ \mathbb{P}_{im1}(\tau, \mu_o; \tau_0)]\}, \quad (1\text{-}18)$$

where $\mathbb{P}_{im1}$ and $\mathbb{P}_{im2}$ are matrices yet to be determined.

The use of Eq. (1-18) in Eq. (1-17) results in two independent sets of matrix equations for $\mathbb{P}_{im1}$ and $\mathbb{P}_{im2}$

$$\mathbb{P}_{im1}(\tau,\ \mu_o;\ \tau_0) = \pi \exp(-\tau / \mu_o)\ \Pi_i^m(\mu_o)\ \mathbb{F}$$

$$+ \frac{\omega}{2} \sum_{j=m}^{L} \frac{(j - m)!}{(j + m)!} \int_0^{\tau_0} \mathbb{K}_{1ijm}(\tau - t)\ \mathbb{P}_{jm1}(t,\ \mu_o;\ \tau_0)\ dt \quad (1\text{-}19a)$$

and

$$\mathbb{P}_{im2}(\tau,\ \mu_o;\ \tau_0) = \pi \exp(-\tau / \mu_o)\ \Pi_i^m(\mu_o)\ \mathbb{D}_2\ \mathbb{F}$$

$$+ \frac{\omega}{2} \sum_{j=m}^{L} \frac{(j - m)!}{(j + m)!} \int_0^{\tau_0} \mathbb{K}_{1ijm}(\tau - t)\ \mathbb{P}_{jm2}(t, \mu_o; \tau_0)\ dt, \quad (1\text{-}19b)$$

where

$$\mathbb{K}_{1ijm}(\tau - t) = \int_0^1 \Pi_i^m[\mathrm{sign}(\tau - t)\mu']\,\Pi_j^m[\mathrm{sign}(\tau - t)\mu']\,\mathbb{B}_j \exp[-|\tau - t|/\mu']\frac{d\mu'}{\mu'}. \qquad (1\text{-}20)$$

Now, by inspecting Eqs. (1-19), it can be seen that

$$\mathbb{P}_{im1}(\tau, \mu_o; \tau_0) = \mathbb{PP}_{im1}(\tau, \mu_o; \tau_0)\,\mathbb{F} \qquad (1\text{-}21)$$

and $\mathbb{P}_{im2}(\tau, \mu_o; \tau_0) = \mathbb{PP}_{im1}(\tau, \mu_o; \tau_0)\,\mathbb{D}_2\,\mathbb{F}$, (1-22)

where $\mathbb{P}_{im1}(\tau, \mu_o; \tau_0)$ and $\mathbb{P}_{im2}(\tau, \mu_o; \tau_0)$ are four by one matrices while $\mathbb{PP}_{im1}(\tau, \mu_o; \tau_0)$ is a four by four matrix. Substituting Eqs. (1-21) and (1-22) into Eqs. (1-19a) and (1-19b), respectively, Eqs. (1-19) can be reduced to a single equation

$$\mathbb{PP}_{im1}(\tau, \mu_o; \tau_0) = \pi \exp(-\tau/\mu_o)\,\Pi_i^m(\mu_o) + \frac{\omega}{2}\sum_{j=m}^{L}\frac{(j - m)!}{(j + m)!}\int_0^{\tau_0} \mathbb{K}_{1ijm}(\tau - t)\,\mathbb{PP}_{jm1}(t, \mu_o; \tau_0)\,dt. \qquad (1\text{-}23)$$

At this point in the derivation, we have reduced the problem to solve for only the fundamental source matrix $\mathbb{PP}_{im1}(\tau, \mu_o; \tau_0)$. One of the major advantages in this simplification is that we can apply different boundary conditions ($\mathbb{F}$'s) after we get the solution for $\mathbb{PP}_{im1}(\tau, \mu_o; \tau_0)$, even though $\mathbb{PP}_{im1}$ is a four by four matrix as opposed to the two simpler four by one matrices of $\mathbb{P}_{im1}$ and $\mathbb{P}_{im2}$.

Now, by substituting Eqs. (1-21) and (1-22) into Eq. (1-18), the general source matrix can be written in terms of the fundamental source matrix $\mathbb{PP}_{im1}(\tau, \mu_o; \tau_0)$ as

$$\mathbb{S}(\tau, \mu, \mu_o, \phi; \tau_0) = \frac{\omega}{4\pi} \sum_{m=0}^{L} \frac{1}{1 + \delta_{0m}} \sum_{i=m}^{L} \frac{(i - m)!}{(i + m)!} \{ \cos(m\phi) [\Pi_i^m(\mu) \mathbb{B}_i \times \mathbb{PP}_{im1}(\tau, \mu_o; \tau_0) + \mathbb{D}_2 \Pi_i^m(\mu) \mathbb{B}_i \mathbb{PP}_{im1}(\tau, \mu_o; \tau_0) \mathbb{D}_2] + \sin(m\phi) [\Pi_i^m(\mu) \mathbb{B}_i \mathbb{PP}_{im1}(\tau, \mu_o; \tau_0) \mathbb{D}_2 - \mathbb{D}_2 \Pi_i^m(\mu) \mathbb{B}_i \times \mathbb{PP}_{im1}(\tau, \mu_o; \tau_0)] \} \mathbb{F}. \tag{1-24}$$

Thus far, we have derived an equation for the fundamental source matrix $\mathbb{PP}_{im1}(\tau, \mu_o; \tau_0)$. The assumptions made to this point are: no incident radiation entering from the lower boundary, no index of refraction effects at both boundaries, the particles are randomly oriented and have at least one plane of symmetry, and ϕ_o is equal to zero. The fundamental source matrix will be manipulated in the next section in order to get the solution of general source matrix $\mathbb{S}$ in Eq. (1-24).

Solution of the Plane-Parallel Polarized Light Problem

Starting with the fundamental source matrix, the reflection and transmission matrices, reflected and transmitted intensity matrices, and reflected and transmitted flux matrices will be determined by using Ambarzumian's approach [1-19, 1-20] and superposition.

Derivative of the Fundamental Source Matrix with Respect to Optical Location

Ambarzumian's approach [1-19, 1-20] will be used to solve for the fundamental source matrix $\mathbb{PP}_{im1}(\tau, \mu_o; \tau_0)$. During this process, the derivative of $\mathbb{PP}_{im1}$ with respect to τ is needed. Therefore, this derivative will be obtained first.

Equation (1-23) can be first written in expanded form as

$$\mathbb{P}\mathbb{P}_{im1}(\tau,\, \mu_o;\, \tau_0) = \pi \exp(-\tau / \mu_o)\, \Pi_i^m(\mu_o) + \frac{\omega}{2} \sum_{j=m}^{L} \frac{(j - m)!}{(j + m)!} \int_0^\tau \mathbb{K}_{1ijm}(\tau - t)$$
$$\times\, \mathbb{P}\mathbb{P}_{jm1}(t,\, \mu_o;\, \tau_0)\, dt + \frac{\omega}{2} \sum_{j=m}^{L} \frac{(j - m)!}{(j + m)!} \int_\tau^{\tau_0} \mathbb{K}_{1ijm}(\tau - t)$$
$$\times\, \mathbb{P}\mathbb{P}_{jm1}(t,\, \mu_o;\, \tau_0)\, dt, \tag{1-25}$$

where $\mathbb{K}_{1ijm}(\tau - t) = \int_0^1 \Pi_i^m(\mu')\, \Pi_j^m(\mu')\, \mathbb{B}_j \exp[-(\tau - t) / \mu'] \frac{d\mu'}{\mu'}$, (1-26a)

for the first integral in Eq. (1-25) where $t < \tau$, and

$$\mathbb{K}_{1ijm}(\tau - t) = \int_0^1 \Pi_i^m(-\mu')\, \Pi_j^m(-\mu')\, \mathbb{B}_j \exp[-(t - \tau) / \mu'] \frac{d\mu'}{\mu'}, \tag{1-26b}$$

for the second integral in Eq. (1-25) where $t > \tau$. Thus, mathematical manipulations will be straightforward in Eq. (1-25).

Using the substitution $\bar{t} = \tau - t$ in the first integral and $\bar{t} = t - \tau$ in the second integral of Eq. (1-25), and using Leibnitz rule to take the derivative of this equation with respect to τ yields

$$\frac{d\, \mathbb{P}\mathbb{P}_{im1}(\tau,\, \mu_o;\, \tau_0)}{d\tau} = -(\pi / \mu_o) \exp(-\tau / \mu_o)\, \Pi_i^m(\mu_o) + \frac{\omega}{2} \sum_{j=m}^{L} \frac{(j - m)!}{(j + m)!} \mathbb{K}_{1ijm}(\tau)$$
$$\times\, \mathbb{P}\mathbb{P}_{jm1}(0,\, \mu_o;\, \tau_0) - \frac{\omega}{2} \sum_{j=m}^{L} \frac{(j - m)!}{(j + m)!} \mathbb{K}_{1ijm}(\tau - \tau_0)$$
$$\times\, \mathbb{P}\mathbb{P}_{jm1}(\tau_0,\, \mu_o;\, \tau_0) + \frac{\omega}{2} \sum_{j=m}^{L} \frac{(j - m)!}{(j + m)!} \int_0^\tau \mathbb{K}_{1ijm}(\bar{t})$$
$$\times\, \frac{d\, \mathbb{P}\mathbb{P}_{jm1}(\tau - \bar{t},\, \mu_o;\, \tau_0)}{d\tau}\, d\bar{t} + \frac{\omega}{2} \sum_{j=m}^{L} \frac{(j - m)!}{(j + m)!}$$

$$\times \int_0^{\tau_0 - \tau} \mathbb{K}_{1ijm}(-\bar{t}) \frac{d\, \mathbb{PP}_{jm1}(\tau + \bar{t}, \mu_o; \tau_0)}{d\tau} d\bar{t}. \tag{1-27}$$

Using the substitution $t = \tau - \bar{t}$ in the first integral and $t = \bar{t} + \tau$ in the second integral, Eq. (1-27) may be written as

$$\frac{d\, \mathbb{PP}_{im1}(\tau, \mu_o; \tau_0)}{d\tau} = -(\pi / \mu_o) \exp(-\tau / \mu_o) \Pi_i^m(\mu_o) + \frac{\omega}{2} \sum_{j=m}^{L} \frac{(j - m)!}{(j + m)!} \mathbb{K}_{1ijm}(\tau)$$
$$\times \mathbb{PP}_{jm1}(0, \mu_o; \tau_0) - \frac{\omega}{2} \sum_{j=m}^{L} \frac{(j - m)!}{(j + m)!} \mathbb{K}_{1ijm}(\tau - \tau_0)$$
$$\times \mathbb{PP}_{jm1}(\tau_0, \mu_o; \tau_0) + \frac{\omega}{2} \sum_{j=m}^{L} \frac{(j - m)!}{(j + m)!} \int_0^{\tau_0} \mathbb{K}_{1ijm}(\tau - t)$$
$$\times \frac{d\, \mathbb{PP}_{jm1}(t, \mu_o; \tau_0)}{dt} dt. \tag{1-28}$$

The solution of Eq. (1-28) can be found by the method of superposition. Replacing μ_o with μ' and j with k in Eq. (1-23); then post-multiplying the resulting equation by $\frac{\omega}{2\pi} \frac{(j - m)!}{(j + m)!} \Pi_j^m(\mu') \mathbb{B}_j \mathbb{PP}_{jm1}(0, \mu_o; \tau_0) / \mu'$, integrating from zero to one with respect to μ', and summing from j = m to L as well as making use of Eq. (1-20), we obtain

$$\frac{\omega}{2\pi} \sum_{j=m}^{L} \frac{(j - m)!}{(j + m)!} \int_0^1 \mathbb{PP}_{im1}(\tau, \mu'; \tau_0) \Pi_j^m(\mu') \frac{d\mu'}{\mu'} \mathbb{B}_j \mathbb{PP}_{jm1}(0, \mu_o; \tau_0)$$
$$= \frac{\omega}{2} \sum_{j=m}^{L} \frac{(j - m)!}{(j + m)!} \mathbb{K}_{1ijm}(\tau) \mathbb{PP}_{jm1}(0, \mu_o; \tau_0) + \frac{\omega}{2} \sum_{k=m}^{L} \frac{(k - m)!}{(k + m)!}$$
$$\times \int_0^{\tau_0} \mathbb{K}_{1ikm}(\tau - t) \{ \frac{\omega}{2\pi} \sum_{j=m}^{L} \frac{(j - m)!}{(j + m)!} \int_0^1 \mathbb{PP}_{km1}(t, \mu'; \tau_0) \Pi_j^m(\mu') \frac{d\mu'}{\mu'}$$
$$\times \mathbb{B}_j \mathbb{PP}_{jm1}(0, \mu_o; \tau_0) \} dt. \tag{1-29}$$

Equation (1-29) can be used to superpose for part of the solution of Eq. (1-28). In order to complete the use of superposition, we need to define a new matrix function (sub-fundamental source matrix) which is

$$\mathbb{PP}_{im3}(\tau, \mu_o; \tau_0) = \pi \exp(-\tau / \mu_o) \Pi'^{m}_{i}(\mu_o)$$

$$+ \frac{\omega}{2} \sum_{j=m}^{L} \frac{(j - m)!}{(j + m)!} \int_0^{\tau_0} \mathbb{K}_{3ijm}(\tau - t)\, \mathbb{PP}_{jm3}(t, \mu_o; \tau_0)\, dt, \qquad (1\text{-}30)$$

where

$$\Pi'^{m}_{i}(\mu) = \begin{bmatrix} P_i^m(\mu) & 0 & 0 & 0 \\ 0 & R_i^m(\mu) & T_i^m(\mu) & 0 \\ 0 & T_i^m(\mu) & R_i^m(\mu) & 0 \\ 0 & 0 & 0 & P_i^m(\mu) \end{bmatrix}, \qquad (1\text{-}31a)$$

and

$$\mathbb{K}_{3ijm}(\tau - t) = \int_0^1 \Pi'^{m}_{i}[\mathrm{sign}(\tau - t)\mu']\, \Pi'^{m}_{j}[\mathrm{sign}(\tau - t)\mu']\, \mathbb{B}_j \exp(-|\tau - t|/\mu') \frac{d\mu'}{\mu'}. \qquad (1\text{-}31b)$$

By comparing Eq. (1-31a) and Eq. (1-7e) with the help of Siewert [1-13], we have the following relation, for $i \geq m$,

$$\Pi_i^m(-\mu) = (-1)^{i-m}\, \Pi'^{m}_{i}(\mu). \qquad (1\text{-}32a)$$

Equation (1-32a) also implies that

$$\mathbb{K}_{1ijm}(t - \tau) = (-1)^{i+j}\, \mathbb{K}_{3ijm}(\tau - t). \qquad (1\text{-}32b)$$

Replacing μ_o by μ', j by k, τ by $\tau_0 - \tau$, and t by τ_0 - t in Eq. (1-30); then post-multiplying the resulting equation by $-\frac{\omega}{2\pi}(-1)^{i+j}\frac{(j-m)!}{(j+m)!}\Pi'^{m}_{j}(\mu')\,\mathbb{B}_j$ $\times\,\mathbb{PP}_{jm1}(\tau_0,\mu_o;\tau_0)/\mu'$, integrating from zero to one with respect to μ', and summing from j = m to L as well as making use of Eq. (1-20), gives

$$
\begin{aligned}
&-\frac{\omega}{2\pi}\sum_{j=m}^{L}(-1)^{j+i}\frac{(j-m)!}{(j+m)!}\int_0^1 \mathbb{PP}_{im3}(\tau_0-\tau,\mu';\tau_0)\,\Pi'^{m}_{j}(\mu')\,\frac{d\mu'}{\mu'}\,\mathbb{B}_j\,\mathbb{PP}_{jm1}(\tau_0,\mu_o;\tau_0)\\
&= -\frac{\omega}{2}\sum_{j=m}^{L}\frac{(j-m)!}{(j+m)!}\mathbb{K}_{1ijm}(\tau-\tau_0)\,\mathbb{PP}_{jm1}(\tau_0,\mu_o;\tau_0)+\frac{\omega}{2}\sum_{k=m}^{L}\frac{(k-m)!}{(k+m)!}\\
&\quad\times\int_0^{\tau_0}\mathbb{K}_{1ikm}(\tau-t)\,\Big\{-\frac{\omega}{2\pi}\sum_{j=m}^{L}(-1)^{j+k}\frac{(j-m)!}{(j+m)!}\int_0^1\mathbb{PP}_{km3}(\tau_0-t,\mu';\tau_0)\\
&\quad\times\Pi'^{m}_{j}(\mu')\,\frac{d\mu'}{\mu'}\,\mathbb{B}_j\,\mathbb{PP}_{jm1}(\tau_0,\mu_o;\tau_0)\Big\}\,dt. \qquad (1\text{-}33)
\end{aligned}
$$

Finally, replacing k by j and j by k in the second term of the right hand side for both Eqs. (1-29) and (1-33), multiplying Eq. (1-23) by $-1/\mu_o$, and adding all three of these equations together; then comparing the resulting equation with Eq. (1-28), the solution of Eq. (1-28) by superposition is found to be

$$
\begin{aligned}
\frac{d\,\mathbb{PP}_{im1}(\tau,\mu_o;\tau_0)}{d\tau} &= -(1/\mu_o)\,\mathbb{PP}_{im1}(\tau,\mu_o;\tau_0)+\frac{\omega}{2\pi}\sum_{j=m}^{L}\frac{(j-m)!}{(j+m)!}\\
&\quad\times\int_0^1\mathbb{PP}_{im1}(\tau,\mu';\tau_0)\,\Pi^{m}_{j}(\mu')\,\frac{d\mu'}{\mu'}\,\mathbb{B}_j\,\mathbb{PP}_{jm1}(0,\mu_o;\tau_0)-\frac{\omega}{2\pi}\\
&\quad\times\sum_{j=m}^{L}(-1)^{i+m}\frac{(j-m)!}{(j+m)!}\int_0^1\mathbb{PP}_{im3}(\tau_0-\tau,\mu';\tau_0)\,\Pi^{m}_{j}(-\mu')\,\frac{d\mu'}{\mu'}\\
&\quad\times\mathbb{B}_j\,\mathbb{PP}_{jm1}(\tau_0,\mu_o;\tau_0). \qquad (1\text{-}34)
\end{aligned}
$$

A similar procedure for $\mathbb{PP}_{im1}(\tau_0-\tau,\mu_o;\tau_0)$ yields

$$\frac{d\,\mathbb{P}\mathbb{P}_{im1}(\tau_0-\tau,\mu_o;\tau_0)}{d\tau} = (1/\mu_o)\,\mathbb{P}\mathbb{P}_{im1}(\tau_0-\tau,\mu_o;\tau_0) + \frac{\omega}{2\pi}\sum_{j=m}^{L}(-1)^{i+m}\frac{(j-m)!}{(j+m)!}$$
$$\times \int_0^1 \mathbb{P}\mathbb{P}_{im3}(\tau,\mu';\tau_0)\,\Pi_j^m(-\mu')\,\frac{d\mu'}{\mu'}\,\mathbb{B}_j\,\mathbb{P}\mathbb{P}_{jm1}(\tau_0,\mu_o;\tau_0)$$
$$-\frac{\omega}{2\pi}\sum_{j=m}^{L}\frac{(j-m)!}{(j+m)!}\int_0^1 \mathbb{P}\mathbb{P}_{im1}(\tau_0-\tau,\mu';\tau_0)\,\Pi_j^m(\mu')\,\frac{d\mu'}{\mu'}$$
$$\times\,\mathbb{B}_j\,\mathbb{P}\mathbb{P}_{jm1}(0,\mu_o;\tau_0), \qquad (1\text{-}35)$$

From Eqs. (1-34) and (1-35), we see that $\mathbb{P}\mathbb{P}_{im3}$ must be known before the equations can be solved. Applying the same procedure to $\mathbb{P}\mathbb{P}_{im3}$ that was applied to $\mathbb{P}\mathbb{P}_{im1}$ gives

$$\frac{d\,\mathbb{P}\mathbb{P}_{im3}(\tau,\mu_o;\tau_0)}{d\tau} = -(1/\mu_o)\,\mathbb{P}\mathbb{P}_{im3}(\tau,\mu_o;\tau_0) + \frac{\omega}{2\pi}\sum_{j=m}^{L}(-1)^{m-j}\frac{(j-m)!}{(j+m)!}$$
$$\times \int_0^1 \mathbb{P}\mathbb{P}_{im3}(\tau,\mu';\tau_0)\,\Pi_j^m(-\mu')\,\frac{d\mu'}{\mu'}\,\mathbb{B}_j\,\mathbb{P}\mathbb{P}_{jm3}(0,\mu_o;\tau_0) - \frac{\omega}{2\pi}$$
$$\times \sum_{j=m}^{L}(-1)^{j+i}\frac{(j-m)!}{(j+m)!}\int_0^1 \mathbb{P}\mathbb{P}_{im1}(\tau_0-\tau,\mu';\tau_0)\,\Pi_j^m(\mu')\,\frac{d\mu'}{\mu'}$$
$$\times\,\mathbb{B}_j\,\mathbb{P}\mathbb{P}_{jm3}(\tau_0,\mu_o;\tau_0), \qquad (1\text{-}36)$$

and

$$\frac{d\,\mathbb{P}\mathbb{P}_{im3}(\tau_0-\tau,\mu_o;\tau_0)}{d\tau} = (1/\mu_o)\,\mathbb{P}\mathbb{P}_{im3}(\tau_0-\tau,\mu_o;\tau_0) + \frac{\omega}{2\pi}\sum_{j=m}^{L}(-1)^{i+j}$$
$$\times \frac{(j-m)!}{(j+m)!}\int_0^1 \mathbb{P}\mathbb{P}_{im1}(\tau,\mu';\tau_0)\,\Pi_j^m(\mu')\,\frac{d\mu'}{\mu'}\,\mathbb{B}_j$$
$$\times\,\mathbb{P}\mathbb{P}_{jm3}(\tau_0,\mu_o;\tau_0) - \frac{\omega}{2\pi}\sum_{j=m}^{L}(-1)^{m-j}\frac{(j-m)!}{(j+m)!}$$
$$\times \int_0^1 \mathbb{P}\mathbb{P}_{im3}(\tau_0-\tau,\mu';\tau_0)\,\Pi_j^m(-\mu')\,\frac{d\mu'}{\mu'}\,\mathbb{B}_j$$
$$\times\,\mathbb{P}\mathbb{P}_{jm3}(0,\mu_o;\tau_0). \qquad (1\text{-}37)$$

Equations (1-34), (1-35), (1-36), and (1-37) are four dependent integro-differential matrix equations. In order to make use of these four integro-differential matrix equations, we need to first rewrite the matrix equations for $\mathbb{PP}_{jm1}(0, \mu_o; \tau_0)$, $\mathbb{PP}_{jm1}(\tau_0, \mu_o; \tau_0)$, $\mathbb{PP}_{jm3}(0, \mu_o; \tau_0)$, and $\mathbb{PP}_{jm3}(\tau_0, \mu_o; \tau_0)$ in appropriate forms. Thus, we will focus our attention next on these four matrix equations.

Fundamental and Sub-Fundamental Source Matrix Equations at the Boundaries

Replacing τ by $\tau_0 - \tau$ and t by τ_0 - t in Eqs. (1-23) and (1-30), Eqs. (1-23) and (1-30) can be rewritten as

$$\mathbb{PP}_{im1}(\tau_0 - \tau, \mu_o; \tau_0) = \pi \exp[(\tau - \tau_0)/\mu_o]\, \Pi_i^m(\mu_o) + \frac{\omega}{2} \sum_{j=m}^{L} \frac{(j-m)!}{(j+m)!} \times \int_0^{\tau_0} \mathbb{K}_{1ijm}(t-\tau)\, \mathbb{PP}_{jm1}(\tau_0 - t, \mu_o; \tau_0)\, dt \qquad (1\text{-}38)$$

and

$$\mathbb{PP}_{im3}(\tau_0 - \tau, \mu_o; \tau_0) = \pi \exp[(\tau - \tau_0)/\mu_o]\, \Pi_i'^m(\mu_o) + \frac{\omega}{2} \sum_{j=m}^{L} \frac{(j-m)!}{(j+m)!} \times \int_0^{\tau_0} \mathbb{K}_{3ijm}(t-\tau)\, \mathbb{PP}_{jm3}(\tau_0 - t, \mu_o; \tau_0)\, dt. \qquad (1\text{-}39)$$

By setting $\tau = 0$, and replacing i by j and j by k in Eqs. (1-23), (1-38), (1-30), and (1-39), respectively, we have the following expressions in order

$$\mathbb{PP}_{jm1}(0, \mu_o; \tau_0) \doteq \pi\, \Pi_j^m(\mu_o)$$

$$+\ \frac{\omega}{2}\sum_{k=m}^{L}\frac{(k-m)!}{(k+m)!}\int_0^{\tau_0}\mathbb{K}_{1jkm}(-t)\,\mathbb{PP}_{kml}(t,\mu_o;\tau_0)\,dt, \tag{1-40a}$$

$$\mathbb{PP}_{jml}(\tau_0,\mu_o;\tau_0)=\pi\exp(-\tau_0/\mu_o)\,\Pi_j^m(\mu_o)$$

$$+\frac{\omega}{2}\sum_{k=m}^{L}\frac{(k-m)!}{(k+m)!}\int_0^{\tau_0}\mathbb{K}_{1jkm}(t)\,\mathbb{PP}_{kml}(\tau_0-t,\mu_o;\tau_0)\,dt, \tag{1-40b}$$

$$\mathbb{PP}_{jm3}(0,\mu_o;\tau_0)=\pi\,\Pi'^m_j(\mu_o)$$

$$+\frac{\omega}{2}\sum_{k=m}^{L}\frac{(k-m)!}{(k+m)!}\int_0^{\tau_0}\mathbb{K}_{3jkm}(-t)\,\mathbb{PP}_{km3}(t,\mu_o;\tau_0)\,dt, \tag{1-40c}$$

and

$$\mathbb{PP}_{jm3}(\tau_0,\mu_o;\tau_0)=\pi\exp(-\tau_0/\mu_o)\,\Pi'^m_j(\mu_o)$$

$$+\frac{\omega}{2}\sum_{k=m}^{L}\frac{(k-m)!}{(k+m)!}\int_0^{\tau_0}\mathbb{K}_{3jkm}(t)\,\mathbb{PP}_{km3}(\tau_0-t,\mu_o;\tau_0)\,dt. \tag{1-40d}$$

Substituting Eq. (1-20) into Eqs. (1-40a) and (1-40b), and Eq. (1-31b) into Eqs. (1-40c) and (1-40d), then interchanging the order of integration, Eqs. (1-40) may be written as

$$\mathbb{PP}_{jml}(0,\mu_o;\tau_0)=\pi\,\Pi_j^m(\mu_o)+\frac{\omega}{2}\sum_{k=m}^{L}\frac{(k-m)!}{(k+m)!}\int_0^1\Pi_j^m(-\mu')\,\Pi_k^m(-\mu')$$

$$\times\ \mathbb{B}_k\ \overline{\mathbb{PP}_{kml}(\mu',\mu_o;\tau_0)}\ \frac{d\mu'}{\mu'}, \tag{1-41a}$$

$$\mathbb{PP}_{jml}(\tau_0,\mu_o;\tau_0)=\pi\exp(-\tau_0/\mu_o)\,\Pi_j^m(\mu_o)+\frac{\omega}{2}\sum_{k=m}^{L}\frac{(k-m)!}{(k+m)!}\int_0^1\Pi_j^m(\mu')$$

$$\times\ \Pi_k^m(\mu')\,\mathbb{B}_k\,\overline{\mathbb{PPI}_{km1}(\mu',\,\mu_o;\,\tau_0)}\,\frac{d\mu'}{\mu'}, \tag{1-41b}$$

$$\mathbb{PP}_{jm3}(0,\,\mu_o;\,\tau_0) = \pi\,\Pi'^m_j(\mu_o) + \frac{\omega}{2}\sum_{k=m}^{L}\frac{(k-m)!}{(k+m)!}\int_0^1 \Pi'^m_j(-\mu')\,\Pi'^m_k(-\mu')$$

$$\times\ \mathbb{B}_k\,\overline{\mathbb{PP}_{km3}(\mu',\,\mu_o;\,\tau_0)}\,\frac{d\mu'}{\mu'}, \tag{1-41c}$$

and

$$\mathbb{PP}_{jm3}(\tau_0,\,\mu_o;\,\tau_0) = \pi\,\exp(-\,\tau_0\,/\,\mu_o)\,\Pi'^m_j(\mu_o) + \frac{\omega}{2}\sum_{k=m}^{L}\frac{(k-m)!}{(k+m)!}\int_0^1 \Pi'^m_j(\mu')$$

$$\times\ \Pi'^m_k(\mu')\,\mathbb{B}_k\,\overline{\mathbb{PPI}_{km3}(\mu',\,\mu_o;\,\tau_0)}\,\frac{d\mu'}{\mu'}, \tag{1-41d}$$

where the transform functions of Eqs. (1-41) are defined as

$$\overline{\mathbb{PP}_{km1}(\mu',\,\mu_o;\,\tau_0)} = \int_0^{\tau_0} \mathbb{PP}_{km1}(t,\,\mu_o;\,\tau_0)\,\exp(-\,t\,/\,\mu')\,dt, \tag{1-42a}$$

$$\overline{\mathbb{PPI}_{km1}(\mu',\,\mu_o;\,\tau_0)} = \int_0^{\tau_0} \mathbb{PP}_{km1}(\tau_0\,-\,t,\,\mu_o;\,\tau_0)\,\exp(-\,t\,/\,\mu')\,dt, \tag{1-42b}$$

$$\overline{\mathbb{PP}_{km3}(\mu',\,\mu_o;\,\tau_0)} = \int_0^{\tau_0} \mathbb{PP}_{km3}(t,\,\mu_o;\,\tau_0)\,\exp(-\,t\,/\,\mu')\,dt, \tag{1-42c}$$

and $\overline{\mathbb{PPI}_{km3}(\mu',\,\mu_o;\,\tau_0)} = \int_0^{\tau_0} \mathbb{PP}_{km3}(\tau_0\,-\,t,\,\mu_o;\,\tau_0)\,\exp(-\,t\,/\,\mu')\,dt.$ (1-42d)

$\overline{\mathbb{PP}_{km1}}$ and $\overline{\mathbb{PPI}_{km1}}$ are the transforms of fundamental source matrices, while $\overline{\mathbb{PP}_{km3}}$ and $\overline{\mathbb{PPI}_{km3}}$ are the transforms of sub-fundamental source matrices. The matrices $\overline{\mathbb{PP}_{km1}}$ and $\overline{\mathbb{PPI}_{km1}}$ are also the so-called reflection and transmission matrices, respectively.

Therefore, we have found $\mathbb{PP}_{jm1}(0, \mu_o; \tau_0)$, $\mathbb{PP}_{jm1}(\tau_0, \mu_o; \tau_0)$, $\mathbb{PP}_{jm3}(0, \mu_o; \tau_0)$, and $\mathbb{PP}_{jm3}(\tau_0, \mu_o; \tau_0)$ in terms of the transforms $\overline{\mathbb{PP}_{km1}}$, $\overline{\mathbb{PPI}_{km1}}$, $\overline{\mathbb{PP}_{km3}}$, and $\overline{\mathbb{PPI}_{km3}}$, respectively. If these transforms were available, we could now determine the fundamental and sub-fundamental source matrices at the boundaries, then get the intensities and fluxes at the boundaries. Thus, our next objective will be to find the expressions for these transforms all of which involve $\mathbb{PP}_{jm1}(0, \mu_o; \tau_0)$, $\mathbb{PP}_{jm1}(\tau_0, \mu_o; \tau_0)$, $\mathbb{PP}_{jm3}(0, \mu_o; \tau_0)$, and $\mathbb{PP}_{jm3}(\tau_0, \mu_o; \tau_0)$.

<u>Solving for the Transforms of the Fundamental and Sub-Fundamental Source Matrices</u>

To transform Eq. (1-35), multiply it by $\exp(-\tau/\mu)$ and integrate over τ from zero to τ_0, which yields

$$\begin{aligned}
\overline{\mathbb{PPI}_{im1}(\mu, \mu_o; \tau_0)} = {} & [1/(1/\mu - 1/\mu_o)] \{ \mathbb{PP}_{im1}(\tau_0, \mu_o; \tau_0) - \exp(-\tau_0/\mu) \\
& \times \mathbb{PP}_{im1}(0, \mu_o; \tau_0) + \frac{\omega}{2\pi} \sum_{j=m}^{L} (-1)^{i+m} \frac{(j-m)!}{(j+m)!} \\
& \times \int_0^1 \overline{\mathbb{PP}_{im3}(\mu, \mu'; \tau_0)}\, \Pi_j^m(-\mu') \frac{d\mu'}{\mu'} \mathbb{B}_j \mathbb{PP}_{jm1}(\tau_0, \mu_o; \tau_0) \\
& - \frac{\omega}{2\pi} \sum_{j=m}^{L} \frac{(j-m)!}{(j+m)!} \int_0^1 \overline{\mathbb{PPI}_{im1}(\mu, \mu'; \tau_0)}\, \Pi_j^m(\mu') \\
& \times \frac{d\mu'}{\mu'} \mathbb{B}_j \mathbb{PP}_{jm1}(0, \mu_o; \tau_0) \},
\end{aligned} \tag{1-43}$$

where $\overline{\mathbb{PPI}_{im1}(\mu, \mu_o; \tau_0)}$ and $\overline{\mathbb{PP}_{im3}(\mu, \mu'; \tau_0)}$ are defined as in Eqs. (1-42b) and (1-42c), respectively.

Then, following the similar procedure for the other transforms, we get

$$
\begin{aligned}
\overline{\mathbb{PP}_{im1}(\mu, \mu_o; \tau_0)} = {} & [1/(1/\mu + 1/\mu_o)]\{\mathbb{PP}_{im1}(0, \mu_o; \tau_0) - \exp(-\tau_0/\mu) \\
& \times \mathbb{PP}_{im1}(\tau_0, \mu_o; \tau_0) + \frac{\omega}{2\pi} \sum_{j=m}^{L} \frac{(j-m)!}{(j+m)!} \int_0^1 \overline{\mathbb{PP}_{im1}(\mu, \mu'; \tau_0)} \\
& \times \Pi_j^m(\mu') \frac{d\mu'}{\mu'} \mathbb{B}_j \, \mathbb{PP}_{jm1}(0, \mu_o; \tau_0) - \frac{\omega}{2\pi} \sum_{j=m}^{L} (-1)^{i+m} \frac{(j-m)!}{(j+m)!} \\
& \times \int_0^1 \overline{\mathbb{PPI}_{im3}(\mu, \mu'; \tau_0)} \, \Pi_j^m(-\mu') \frac{d\mu'}{\mu'} \mathbb{B}_j \, \mathbb{PP}_{jm1}(\tau_0, \mu_o; \tau_0)\}, \qquad (1\text{-}44)
\end{aligned}
$$

$$
\begin{aligned}
\overline{\mathbb{PPI}_{im3}(\mu, \mu_o; \tau_0)} = {} & [1/(1/\mu - 1/\mu_o)]\{\mathbb{PP}_{im3}(\tau_0, \mu_o; \tau_0) - \exp(-\tau_0/\mu) \\
& \times \mathbb{PP}_{im3}(0, \mu_o; \tau_0) + \frac{\omega}{2\pi} \sum_{j=m}^{L} (-1)^{i+j} \frac{(j-m)!}{(j+m)!} \\
& \times \int_0^1 \overline{\mathbb{PP}_{im1}(\mu, \mu'; \tau_0)} \, \Pi_j^m(\mu') \frac{d\mu'}{\mu'} \mathbb{B}_j \, \mathbb{PP}_{jm3}(\tau_0, \mu_o; \tau_0) \\
& - \frac{\omega}{2\pi} \sum_{j=m}^{L} (-1)^{m-j} \frac{(j-m)!}{(j+m)!} \int_0^1 \overline{\mathbb{PPI}_{im3}(\mu, \mu'; \tau_0)} \\
& \times \Pi_j^m(-\mu') \frac{d\mu'}{\mu'} \mathbb{B}_j \, \mathbb{PP}_{jm3}(0, \mu_o; \tau_0)\}, \qquad (1\text{-}45)
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{\mathbb{PP}_{im3}(\mu, \mu_o; \tau_0)} = {} & [1/(1/\mu + 1/\mu_o)]\{\mathbb{PP}_{im3}(0, \mu_o; \tau_0) - \exp(-\tau_0/\mu) \\
& \times \mathbb{PP}_{im3}(\tau_0, \mu_o; \tau_0) + \frac{\omega}{2\pi} \sum_{j=m}^{L} (-1)^{m-j} \frac{(j-m)!}{(j+m)!} \\
& \times \int_0^1 \overline{\mathbb{PP}_{im3}(\mu, \mu'; \tau_0)} \, \Pi_j^m(-\mu') \frac{d\mu'}{\mu'} \mathbb{B}_j \, \mathbb{PP}_{jm3}(0, \mu_o; \tau_0) \\
& - \frac{\omega}{2\pi} \sum_{j=m}^{L} (-1)^{j+i} \frac{(j-m)!}{(j+m)!} \int_0^1 \overline{\mathbb{PPI}_{im1}(\mu, \mu'; \tau_0)} \\
& \times \Pi_j^m(\mu') \frac{d\mu'}{\mu'} \mathbb{B}_j \, \mathbb{PP}_{jm3}(\tau_0, \mu_o; \tau_0)\}. \qquad (1\text{-}46)
\end{aligned}
$$

Equations (1-41), and (1-43) to (1-46) could be solved now for all functions. However, a very important point to note is that Eqs. (1-43) and (1-45) cannot be easily solved numerically due to the term $1/(1/\mu - 1/\mu_o)$, which will approach infinity when $\mu = \mu_o$. Therefore, a different method will be used to solve for $\mathbb{PP}_{im1}(0, \mu_o; \tau_0)$, $\mathbb{PP}_{im1}(\tau_0, \mu_o; \tau_0)$, $\mathbb{PP}_{im3}(0, \mu_o; \tau_0)$, and $\mathbb{PP}_{im3}(\tau_0, \mu_o; \tau_0)$ as shown in the following section. Then Eqs. (1-43) to (1-46) will be used to solve for the reflection and transmission matrices, $\overline{\mathbb{PP}_{km1}}$ and $\overline{\mathbb{PPI}_{km1}}$, which will in turn be used to solve for the reflected and transmitted intensity, and flux matrices later.

Alternate Approach to Solve for the Fundamental and Sub-Fundamental Source Matrix Equations at the Boundaries

This approach requires the derivatives of the fundamental and sub-fundamental source matrices with respect to optical thickness τ_0 [1-19, 1-20]. Using Leibnitz rule to take the derivative of Eq. (1-25) with respect to τ_0 yields

$$\frac{d\,\mathbb{PP}_{im1}(\tau, \mu_o; \tau_0)}{d\tau_0} = \frac{\omega}{2} \sum_{j=m}^{L} \frac{(j - m)!}{(j + m)!} \mathbb{K}_{1ijm}(\tau - \tau_0)\, \mathbb{PP}_{jm1}(\tau_0, \mu_o; \tau_0)$$

$$+ \frac{\omega}{2} \sum_{k=m}^{L} \frac{(k-m)!}{(k+m)!} \int_0^{\tau_0} \mathbb{K}_{1ikm}(\tau - t) \frac{d\,\mathbb{PP}_{km1}(t, \mu_o; \tau_0)}{d\tau_0}\, dt, \qquad (1\text{-}47)$$

where index j has been replaced by k in the second summation.

The solution of Eq. (1-47) is found by the method of superposition. Multiplying Eq. (1-33) by -1, and comparing the resulting equation with Eq. (1-47), we find that

$$\frac{d\,\mathbb{PP}_{im1}(\tau, \mu_o; \tau_0)}{d\tau_0} = \frac{\omega}{2\pi} \sum_{j=m}^{L} (-1)^{i+m} \frac{(j - m)!}{(j + m)!} \int_0^1 \mathbb{PP}_{im3}(\tau_0 - \tau, \mu'; \tau_0)\, \Pi_j^m(-\mu')$$

$$\times \frac{d\mu'}{\mu'} \mathbb{B}_j \, \mathbb{PP}_{jm1}(\tau_0, \mu_o; \tau_0). \tag{1-48}$$

Following a similar procedure for the other fundamental and sub-fundamental source matrices, we have

$$\frac{d\, \mathbb{PP}_{im1}(\tau_0 - \tau, \mu_o; \tau_0)}{d\tau_0} = -(1/\mu_o)\, \mathbb{PP}_{im1}(\tau_0 - \tau, \mu_o; \tau_0) + \frac{\omega}{2\pi} \sum_{j=m}^{L} \frac{(j - m)!}{(j + m)!}$$

$$\times \int_0^1 \mathbb{PP}_{im1}(\tau_0 - \tau, \mu'; \tau_0)\, \Pi_j^m(\mu') \frac{d\mu'}{\mu'} \mathbb{B}_j$$

$$\times \mathbb{PP}_{jm1}(0, \mu_o; \tau_0), \tag{1-49}$$

$$\frac{d\, \mathbb{PP}_{im3}(\tau, \mu_o; \tau_0)}{d\tau_0} = \frac{\omega}{2\pi} \sum_{j=m}^{L} (-1)^{j+i} \frac{(j - m)!}{(j + m)!} \int_0^1 \mathbb{PP}_{im1}(\tau_0 - \tau, \mu'; \tau_0)\, \Pi_j^m(\mu')$$

$$\times \frac{d\mu'}{\mu'} \mathbb{B}_j \, \mathbb{PP}_{jm3}(\tau_0, \mu_o; \tau_0), \tag{1-50}$$

and

$$\frac{d\, \mathbb{PP}_{im3}(\tau_0 - \tau, \mu_o; \tau_0)}{d\tau_0} = -(1/\mu_o)\, \mathbb{PP}_{im3}(\tau_0 - \tau, \mu_o; \tau_0) + \frac{\omega}{2\pi} \sum_{j=m}^{L} (-1)^{m-j}$$

$$\times \frac{(j - m)!}{(j + m)!} \int_0^1 \mathbb{PP}_{im3}(\tau_0 - \tau, \mu'; \tau_0)\, \Pi_j^m(-\mu') \frac{d\mu'}{\mu'} \mathbb{B}_j$$

$$\times \mathbb{PP}_{jm3}(0, \mu_o; \tau_0). \tag{1-51}$$

Finally, letting $\tau = 0$, Eqs. (1-48), (1-49), (1-50), and (1-51) become

$$\frac{d\, \mathbb{PP}_{im1}(0, \mu_o; \tau_0)}{d\tau_0} = \frac{\omega}{2\pi} \sum_{j=m}^{L} (-1)^{i+m} \frac{(j - m)!}{(j + m)!} \int_0^1 \mathbb{PP}_{im3}(\tau_0, \mu'; \tau_0)\, \Pi_j^m(-\mu')$$

$$\times \frac{d\mu'}{\mu'} \mathbb{B}_j \, \mathbb{PP}_{jm1}(\tau_0, \mu_o; \tau_0), \tag{1-52}$$

$$\frac{d\,\mathbb{PP}_{im1}(\tau_0, \mu_o; \tau_0)}{d\tau_0} = -(1/\mu_o)\,\mathbb{PP}_{im1}(\tau_0, \mu_o; \tau_0) + \frac{\omega}{2\pi} \sum_{j=m}^{L} \frac{(j-m)!}{(j+m)!}$$

$$\times \int_0^1 \mathbb{PP}_{im1}(\tau_0, \mu'; \tau_0)\, \Pi_j^m(\mu') \frac{d\mu'}{\mu'} \mathbb{B}_j \, \mathbb{PP}_{jm1}(0, \mu_o; \tau_0), \tag{1-53}$$

$$\frac{d\,\mathbb{PP}_{im3}(0, \mu_o; \tau_0)}{d\tau_0} = \frac{\omega}{2\pi} \sum_{j=m}^{L} (-1)^{j+i} \frac{(j-m)!}{(j+m)!} \int_0^1 \mathbb{PP}_{im1}(\tau_0, \mu'; \tau_0)\, \Pi_j^m(\mu')$$

$$\times \frac{d\mu'}{\mu'} \mathbb{B}_j \, \mathbb{PP}_{jm3}(\tau_0, \mu_o; \tau_0), \tag{1-54}$$

and

$$\frac{d\,\mathbb{PP}_{im3}(\tau_0, \mu_o; \tau_0)}{d\tau_0} = -(1/\mu_o)\,\mathbb{PP}_{im3}(\tau_0, \mu_o; \tau_0) + \frac{\omega}{2\pi} \sum_{j=m}^{L} (-1)^{m-j} \frac{(j-m)!}{(j+m)!}$$

$$\times \int_0^1 \mathbb{PP}_{im3}(\tau_0, \mu'; \tau_0)\, \Pi_j^m(-\mu') \frac{d\mu'}{\mu'} \mathbb{B}_j \, \mathbb{PP}_{jm3}(0, \mu_o; \tau_0). \tag{1-55}$$

Equations (1-52), (1-53), (1-54), and (1-55) are four dependent integro-differential matrix equations for $\mathbb{PP}_{im1}(0, \mu_o; \tau_0)$, $\mathbb{PP}_{im1}(\tau_0, \mu_o; \tau_0)$, $\mathbb{PP}_{im3}(0, \mu_o; \tau_0)$, and $\mathbb{PP}_{im3}(\tau_0, \mu_o; \tau_0)$. These four matrix equations can be solved simultaneously by any reliable scheme, such as the Runge-Kutta numerical calculation method. Moreover, Eqs. (1-43), (1-46) and (1-44), (1-45) are two pairs of dependent integral equations. These equations can be solved by the successive approximation method, once $\mathbb{PP}_{im1}(0, \mu_o; \tau_0)$, $\mathbb{PP}_{im1}(\tau_0, \mu_o; \tau_0)$, $\mathbb{PP}_{im3}(0, \mu_o; \tau_0)$, and $\mathbb{PP}_{im3}(\tau_0, \mu_o; \tau_0)$ are available. Thus, all equations have been derived to allow the calculation of all fundamental and sub-fundamental source matrices at the boundaries, and the transforms of these matrices. In the next section, the reflected and transmitted intensity matrices will be derived in terms of the reflection matrix $\overline{\mathbb{PP}_{kml}}$ and transmission matrix $\overline{\mathbb{PPI}_{kml}}$, respectively.

Reflected and Transmitted Intensity Matrices

Substituting Eqs. (1-15b), (1-18), (1-21), (1-22), and (1-42a) into Eq. (1-16b) as well as setting $\tau = 0$ yields the following reflected intensity matrix

$$\begin{aligned} \mathbb{I}_d^-(0, \mu, \mu_o, \phi; \tau_0) = \frac{\omega}{4\pi} \sum_{m=0}^{L} \frac{1}{1+\delta_{0m}} \sum_{i=m}^{L} \frac{(i-m)!}{(i+m)!} \{ \cos(m\phi) [\Pi_i^m(-\mu) \\ \times \mathbb{B}_i \overline{\mathbb{PP}_{iml}(\mu, \mu_o; \tau_0)} + \mathbb{D}_2 \Pi_i^m(-\mu) \mathbb{B}_i \overline{\mathbb{PP}_{iml}(\mu, \mu_o; \tau_0)} \\ \times \mathbb{D}_2] + \sin(m\phi) [\Pi_i^m(-\mu) \mathbb{B}_i \overline{\mathbb{PP}_{iml}(\mu, \mu_o; \tau_0)} \mathbb{D}_2 \\ - \mathbb{D}_2 \Pi_i^m(-\mu) \mathbb{B}_i \overline{\mathbb{PP}_{iml}(\mu, \mu_o; \tau_0)}] \} \frac{\mathbb{F}}{\mu}, \end{aligned} \tag{1-56}$$

where $\overline{\mathbb{PP}_{iml}}$ is defined by Eq. (1-42a).

Then, substituting Eqs. (1-15a), (1-18), (1-21), (1-22), and (1-42b) into Eq. (1-16a) as well as setting $\tau = \tau_0$ yields the following transmitted intensity matrix

$$\begin{aligned} \mathbb{I}_d^+(\tau_0, \mu, \mu_o, \phi; \tau_0) = \frac{\omega}{4\pi} \sum_{m=0}^{L} \frac{1}{1+\delta_{0m}} \sum_{i=m}^{L} \frac{(i-m)!}{(i+m)!} \{ \cos(m\phi) [\Pi_i^m(\mu) \mathbb{B}_i \\ \times \overline{\mathbb{PPI}_{iml}(\mu, \mu_o; \tau_0)} + \mathbb{D}_2 \Pi_i^m(\mu) \mathbb{B}_i \overline{\mathbb{PPI}_{iml}(\mu, \mu_o; \tau_0)} \\ \times \mathbb{D}_2] + \sin(m\phi) [\Pi_i^m(\mu) \mathbb{B}_i \overline{\mathbb{PPI}_{iml}(\mu, \mu_o; \tau_0)} \mathbb{D}_2 \\ - \mathbb{D}_2 \Pi_i^m(\mu) \mathbb{B}_i \overline{\mathbb{PPI}_{iml}(\mu, \mu_o; \tau_0)}] \} \frac{\mathbb{F}}{\mu}, \end{aligned} \tag{1-57}$$

where $\overline{\mathbb{PPI}_{iml}}$ is defined by Eq. (1-42b).

Equations (1-56) and (1-57) can be solved directly to obtain the reflected and transmitted intensity matrices, after $\overline{\mathbb{PP}_{im1}}$ and $\overline{\mathbb{PPI}_{im1}}$ have been determined. In the next section, the reflected and transmitted flux matrices will be derived in terms of the reflection matrix $\overline{\mathbb{PP}_{km1}}$ and transmission matrix $\overline{\mathbb{PPI}_{km1}}$, respectively.

<u>Reflected and Transmitted Flux Matrices</u>

The general flux equation is [1-1]

$$q = \int \mathbb{I}\, \mu\, d\Omega, \tag{1-58}$$

where Ω denotes the solid angle. Then, by using Eqs. (1-56) and (1-57) with the general flux definition of Eq. (1-58), the reflected and transmitted flux matrices are

$$\begin{aligned} q_d^-(0, \mu_o; \tau_0) = \frac{\omega}{4} \sum_{i=0}^{L} \int_0^1 [\, &\Pi_i^0(-\mu)\, \mathbb{B}_i\, \overline{\mathbb{PP}_{i01}(\mu, \mu_o; \tau_0)} \\ &+ \mathbb{D}_2\, \Pi_i^0(-\mu)\, \mathbb{B}_i\, \overline{\mathbb{PP}_{i01}(\mu, \mu_o; \tau_0)}\, \mathbb{D}_2 \,]\, d\mu\ \mathbb{F} \end{aligned} \tag{1-59}$$

and

$$\begin{aligned} q_d^+(\tau_0, \mu_o; \tau_0) = \frac{\omega}{4} \sum_{i=0}^{L} \int_0^1 [\, &\Pi_i^0(\mu)\, \mathbb{B}_i\, \overline{\mathbb{PPI}_{i01}(\mu, \mu_o; \tau_0)} \\ &+ \mathbb{D}_2\, \Pi_i^0(\mu)\, \mathbb{B}_i\, \overline{\mathbb{PPI}_{i01}(\mu, \mu_o; \tau_0)}\, \mathbb{D}_2 \,]\, d\mu\ \mathbb{F}, \end{aligned} \tag{1-60}$$

respectively. Equations (1-59) and (1-60) can be solved to obtain the reflected and transmitted flux matrices, after $\overline{\mathbb{PP}_{im1}}$ and $\overline{\mathbb{PPI}_{im1}}$ have been determined. Note that Eqs. (1-59) and (1-60) without the $\mathbb{F}$ vector would yield generic four by four flux matrices independent of boundary conditions. These generic equations are related to the reduced one-dimensional flux equations from Muller [1-21].

Conclusions

The exact expressions were derived for the general source matrix, fundamental source matrix, reflection and transmission matrices, reflected and transmitted intensity matrices, and reflected and transmitted flux matrices for polarized light in a plane-parallel medium without reflective boundaries. The work was done by using a procedure similar to that of Ambarzumian [1-19] and Liu [1-20] with the assumptions being: collimated polarized incident radiation at angle θ_o exists at the top boundary and is a sheet of laser-like beams; no incident radiation enters from the lower boundary; scattering and absorption but no emission occur in the medium; no index of refraction effects exist at the boundaries; the particles are randomly oriented and have at least one plane of symmetry; and the azimuthal angle of the incident radiation is equal to zero.

As discussed earlier, we need to solve Eqs. (1-52)-(1-55) simultaneously first, and then solve Eqs. (1-43) and (1-46) simultaneously and Eqs. (1-44) and (1-45) simultaneously, and finally solve Eqs. (1-56) and (1-57) and Eqs. (1-59) and (1-60), respectively, in order to get the numerical results for the reflected and transmitted intensities and fluxes

Near term future work will be focused on numerical solution for a wide range of phase matrices, albedoes, optical thicknesses, incident angles, and incident polarizations.

Moreover, in order to compare more easily with experimental data, it is planned that long term future research will include refractive index effects.

Addendum to Conclusions

This paper was published in 1999 in the Journal of Quantitative Spectroscopy and Radiative Transfer, Vol. 61, No. 1, pp. 1-18, before completing any numerical results. Some of the near term future work mentioned in the conclusions has been done in Chapters II and III. In the next chapter, the derivation and numerical results for the diffusion approximation will be discussed.

Acknowledgements

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Appendix I

The purpose of this appendix is to define the coefficients of the $\mathbb{B}_i$ matrix which must be known in order to solve the present one-dimensional problem numerically.

The matrix $\mathbb{B}_i$ is given in Eq. (1-7f) as

$$\mathbb{B}_i = \begin{bmatrix} \beta_i & \gamma_i & 0 & 0 \\ \gamma_i & \alpha_i & 0 & 0 \\ 0 & 0 & \zeta_i & -\varepsilon_i \\ 0 & 0 & \varepsilon_i & \delta_i \end{bmatrix}. \tag{1-7f}$$

Furthermore, the coefficients of $\mathbb{B}_i$ are given as [1-13]

$$\beta_i = [(2i + 1)/2] \int_{-1}^{1} P_i(\mu_s)\, a_1(\mu_s)\, d\mu_s, \tag{1-a1}$$

$$\delta_i = [(2i + 1)/2] \int_{-1}^{1} P_i(\mu_s)\, a_4(\mu_s)\, d\mu_s, \tag{1-a2}$$

$$\gamma_i = [(2i + 1)/2] \left[\frac{(i - 2)!}{(i + 2)!}\right]^{1/2} \int_{-1}^{1} P_i^2(\mu_s)\, b_1(\mu_s)\, d\mu_s, \tag{1-a3}$$

$$\varepsilon_i = -[(2i + 1)/2] \left[\frac{(i - 2)!}{(i + 2)!}\right]^{1/2} \int_{-1}^{1} P_i^2(\mu_s)\, b_2(\mu_s)\, d\mu_s, \tag{1-a4}$$

$$\zeta_i = [(2i + 1)/2] \left[\frac{(i - 2)!}{(i + 2)!}\right]^{1/2} \int_{-1}^{1} [a_3(\mu_s)\, R_i^2(\mu_s) + a_2(\mu_s)\, T_i^2(\mu_s)]\, d\mu_s, \tag{1-a5}$$

and

$$\alpha_i = [(2i + 1)/2] \left[\frac{(i - 2)!}{(i + 2)!}\right]^{1/2} \int_{-1}^{1} [a_2(\mu_s)\, R_i^2(\mu_s) + a_3(\mu_s)\, T_i^2(\mu_s)]\, d\mu_s, \tag{1-a6}$$

where $P_i(\mu_s)$ is used to denote the Legendre polynomials, defined as [1-22]

$$P_i(\mu_s) = \left[\frac{d^i(\mu_s^2 - 1)^i}{d\mu_s^i}\right] / (2^i\ i!), \tag{1-a7}$$

$P_i^m(\mu_s)$ are associated Legendre functions, defined as [1-22]

$$P_i^m(\mu_s) = (1 - \mu_s^2)^{m/2} \frac{d^m}{d\mu_s^m} P_i(\mu_s), \tag{1-a8}$$

$\mu_s = \cos\Theta$, and $R_i^m(\mu_s)$ and $T_i^m(\mu_s)$ are combinations of generalized spherical functions which are defined as [1-13]

$$R_i^m(\mu) = -(1/2)\,(i)^m\,[\frac{(i+m)!}{(i-m)!}]^{1/2}\,\{P_{m,2}^i(\mu) + P_{m,-2}^i(\mu)\} \tag{1-a9}$$

and

$$T_i^m(\mu) = -(1/2)\,(i)^m\,[\frac{(i+m)!}{(i-m)!}]^{1/2}\,\{P_{m,2}^i(\mu) - P_{m,-2}^i(\mu)\}, \tag{1-a10}$$

where for $i \geq \sup(|m|, |n|)$, the generalized spherical functions are

$$P_{m,n}^i(\mu) = A_{m,n}^i\,(1-\mu)^{-(n-m)/2}\,(1+\mu)^{-(n+m)/2}\,\frac{d^{i-n}}{d\mu^{i-n}}[(1-\mu)^{i-m}\,(1+\mu)^{i+m}] \tag{1-a11}$$

with $A_{m,n}^i = (-1)^{i-m}\,(i)^{n-m}\,[\frac{(i-m)!\,(i+n)!}{(i+m)!\,(i-n)!}]^{1/2}/[2^i\,(i-m)!]$. (1-a12)

Notice that $\sup(|m|,|n|)$ is the larger of the two numbers $|m|$ and $|n|$, and $P_{m,n}^i(\mu) = 0$ if $i < \sup(|m|, |n|)$.

CHAPTER II

DIFFUSION APPROXIMATION OF RADIATIVE TRANSFER FOR THE SCATTERING OF POLARIZED LIGHT IN A PLANE-PARALLEL MEDIUM

Abstract

A procedure that modifies the classical P_1 approximation is introduced here. The objective of the present work is to illustrate that the solutions of the one-dimensional radiative transfer problem for polarized radiation can be obtained by using this modified P_1 method which has been applied to the scalar problem. In this paper, the expression for intensity is derived by using the classical P_1 approximation with both Mark's and Marshak's boundary conditions as well as the modified P_1 method with Marshak's boundary conditions. The plane-parallel medium of interest scatters, absorbs, and is exposed to collimated incident polarized radiation. Numerical results are presented for five optical thicknesses (5, 10, 15, 20, and 30), five albedoes (0.5, 0.9, 0.95, 0.99, and 1), and three selected sets of the scattering coefficients. These solutions are compared with the classical P_1 approximation and with the exact scalar results. Qualitatively good agreement for intensity is shown between the modified P_1 and the exact scalar solutions, while the classical P_1 approximation predictions are poor.

Introduction

Various computational techniques for solving the radiative transfer problem in a plane-parallel medium including polarization effects are available. Nevertheless, there are no results for the diffusion approximation for the scattering of polarized light in a one-dimensional finite medium exposed to polarized incident light without reflective boundaries. Many researchers have simplified the effect of polarization due to its mathematical complexity, while others have formulated equations that become very difficult to solve numerically. Some interesting and related studies, mainly focused on polarization, will be reviewed in the following.

Some typical studies examining the effect of polarization in plane-parallel media were conducted by Chandrasekhar [2-1], Reguigui [2-2, 2-3], Hovenier [2-4], Wauben and Hovenier [2-5], Wauben et al. [2-6], Siewert [2-7], Garcia and Siewert [2-8, 2-9], Hovenier and van der Mee [2-10], Benassi et al. [2-11, 2-12], Zege and Chaikovskaya [2-13], Mishchenko [2-14, 2-15], and Ambirajan and Look [2-16, 2-17]. Several studies concentrate on the derivation of radiative transfer equations without numerical results while others present numerical solutions with the incident radiation being unpolarized, circularly polarized, or linearly polarized in order to simplify the numerical process.

The fundamental radiative transfer equation including polarization effects was derived by Chandrasekhar [2-1] for a plane-parallel atmosphere with Rayleigh scattering. The exact solutions for the parallel and perpendicular components (I_l and I_r) of polarized radiation were also presented by Chandrasekhar for a plane-parallel axisymmetric atmosphere with Rayleigh's law of scattering and unpolarized incident radiation. Most of the work after Chandrasekhar has tried to extend his work in order to handle a general scattering matrix.

The derivation of the correlation transfer equation for dynamic light scattering (very similar to that of radiative transfer) was presented by Reguigui [2-2, 2-3]. By using various radiative transfer solution approaches, the numerical results for correlation (which is comparable to radiative intensity) were obtained for both finite and semi-infinite media with the incident radiation being unpolarized. The effects of polarization and other important parameters on the correlation function were considered and discussed. It was found that polarization effects cannot be ignored for low optical thickness ($\tau_0 \leq 5$), but are less important for high optical thickness ($\tau_0 \geq 20$).

An extension of the doubling method [per Hovenier (van de Hulst, 1963)] was presented by Hovenier [2-4] to solve the radiative transfer problem in plane-parallel atmospheres including polarization effects. He presented numerical intensity results for four different phase matrices: (a) Rayleigh scattering, (b) two simple test matrices with unit albedo, of which one of these two models was designed to simulate the scattering of water vapor at a wavelength of 0.7 μm, and (c) for comparison purposes, a scalar phase function (i.e., no polarization). The numerical results for the two simple test matrices suggested that ignoring polarization is not very important for intensity but obviously loses the degree of polarization. Unpolarized unidirectional incident light, modeling that coming from the sun, was used for this research.

Wauben and Hovenier [2-5] utilized the adding/doubling method as well as the F_N method to solve the radiative transfer problem in a plane-parallel homogeneous atmosphere including polarization effects. The medium was illuminated by unpolarized incident light and bounded by a black lower surface. Numerical results for all four Stokes parameters were presented for three different kind of randomly-oriented spheroids.

The adding principle was employed by Wauben et al. [2-6] to calculate the radiative transfer in a plane-parallel inhomogeneous atmosphere including polarization effects. The medium was illuminated by unpolarized incident light on the upper boundary, by isotropically radiating internal sources, and by an isotropically radiating lower surface. Numerical results of all Stokes parameters were presented for all three kinds of illumination.

The radiative transfer problem for a finite plane-parallel medium exposed to incident elliptically polarized radiation was considered by Siewert [2-7]. The problem was reduced to a group of radiative transfer equations, formulated in terms of the four classical Stokes parameters, by using a Fourier decomposition in the azimuthal angle. No numerical solutions were presented.

Two different methods were applied to solve the radiative transfer problem including polarization effects by Garcia and Siewert [2-8, 2-9]. The medium was plane-parallel with unpolarized incident light on the top and a reflective lower boundary. Numerical results of all Stokes parameters were tabulated by using the generalized spherical harmonics method [2-8] and the F_N method [2-9].

Hovenier and van der Mee [2-10] have found the relationships between the Stokes parameters and several complex polarization parameters. The polarized transport equation and phase matrix for a plane-parallel atmosphere were discussed and formulated by using both Stokes parameters and complex polarization parameters. By using the addition theorem of generalized spherical functions, the phase matrix and all of its Fourier components were expressed analytically. No numerical results were provided.

A dispersion matrix, which was used to get the elementary solutions for the polarized radiative transfer equation, was given in various representations by Benassi et al. [2-11]. They discussed how to compute the zeros of the determinant of the dispersion matrix in order to get the analytical solutions. Furthermore, numerical results were given for three different scattering models with the incident radiation being unpolarized or circularly polarized.

Starting with an analytical representation of the phase matrix, Benassi et al. [2-12] presented the solution for scattering of polarized light in a plane-parallel medium with the assumption that the intensity is independent of the azimuthal angle (i.e. azimuthally symmetric). Numerical results were given for the incident radiation being either unpolarized or circularly polarized in order to satisfy the azimuthally symmetric assumption.

Zege and Chaikovskaya [2-13] presented an approximate method to solve the radiative transfer problem including polarization effects. Instead of the originally complicated vector radiative transfer equations (VRTEs), which were sets of four simultaneous equations based on the Stokes parameters, a simplified new set of VRTEs, based on an approximate Green's function matrix, were derived with the major assumption that the scattering matrix of the medium was isotropic. The advantages for this isotropic medium approximation were: (a) the set of four simultaneous equations for the original VRTEs could be simplified to either sets of two simultaneous equations or the scalar equations, (b) the new VRTEs had simpler kernels than the kernels of the original VRTEs,

(c) some complicated functions could be eliminated from the original VRTEs, and (d) the new VRTEs gave quick convergence as well as high accuracy. No numerical solutions were provided.

Mishchenko [2-14, 2-15] formulated exact reflected radiation equations by using an extension of the invariant imbedding method for a finite plane-parallel atmosphere including polarization effects. However, the formulated equations were numerically complex, requiring double integration. Thus the author numerically solved two simplified problems, for unpolarized incident radiation and for linearly polarized incident radiation.

For a plane-parallel medium with incident light being circularly polarized, Ambirajan and Look [2-16, 2-17] studied the radiative transfer problem both theoretically [2-16] and experimentally [2-17]. A backward Monte Carlo method was introduced by Ambirajan and Look [2-16] to numerically solve for the backscattered intensity, while experimental data was presented by Ambirajan and Look [2-17] for transmitted intensity as well as the degree of linear and circular polarization versus optical radius.

For the diffuse scattering of polarized light, Herman et al. [2-18] presented numerical results for both spherical and plane-parallel atmospheres by using the Gauss-Seidel calculation method. Comparisons between the polarized spherical Gauss-Seidel method and Monte Carlo calculations of other published studies for both spherical and plane-parallel media were also made. When all scattering terms were considered, the four Stokes parameters (I, Q, U, and V) were in good agreement, comparing the polarized spherical Gauss-Seidel method and the Monte Carlo method. The solar radiation incident at the top of the atmosphere was assumed to be a completely unpolarized parallel beam.

Special work, mainly on the phase matrix, for the scattering of polarized light has been performed by Siewert [2-19], Vestrucci and Siewert [2-20], de Rooij and van der Stap [2-21], Hovenier [2-22], Kuik et al. [2-23], and Mishchenko and Travis [2-24]. None of that work provided numerical results for intensity.

An analytical phase matrix corresponding to a Stokes representation of the polarized scattering matrix, which allowed the components of phase matrix to be expressed by a Fourier decomposition, was reported by Siewert [2-19]. The fundamental constants and matrices of this phase matrix were deduced by using a set of orthogonality and recursive relations. Three symmetry relationships of this phase matrix were also provided at the end. No numerical solutions were given.

An analytical phase matrix (components in a Fourier decomposition) for scattering of polarized light was presented by Vestrucci and Siewert [2-20]. Some values of the fundamental constants required for this phase matrix were provided for three different scattering models.

The polarized scattering matrix, which can be expanded in generalized spherical functions, was considered by de Rooij and van der Stap [2-21]. The expansion coefficients of this scattering matrix, which represented scattering by homogeneous spherical particles, were calculated in two ways: (1) Domke's [per de Rooij and van der Stap (Domke, 1975)] explicit expressions, and (2) numerical angular integration. Furthermore, four sets of expansion coefficients were given according to four specific scattering matrices.

Hovenier [2-22] discussed the symmetry relationships based on two different polarized scattering matrices for which particles were randomly oriented and: (1) had a plane of symmetry, or (2) did not have a plane of symmetry. He presented the symmetry relations for the phase matrix and for the reflection and transmission matrices, based on a scattering matrix which represented particles having a plane of symmetry. For the scattering matrix representing particles not having a plane of symmetry, birefringence and dichroism might occur, and the symmetry relations for only the phase matrix were considered. No numerical results were provided.

A T-matrix method, based on numerically solving Maxwell's equations, was applied by Kuik et al. [2-23] to calculate the expansion coefficients for the scattering matrix. Numerical results of the expansion coefficients were given for three different cases.

Two FORTRAN codes to get mainly the scattering coefficients of the scattering matrix were described in detail by Mishchenko and Travis [2-24]. The T-matrix method was used by both codes with the assumption that scattering particles have a plane of symmetry perpendicular to the rotational axis. These two codes are available on the Web at http://www.giss.nasa.gov/~crmin.

Assuming a semi-infinite scattering medium that was homogeneous with randomly oriented polydisperse scattering spheres having a plane of symmetry, Mishchenko [2-25] has presented the Stokes reflection matrix which can be used to find radar reflectivity, polarization ratios, and enhancement factors. Some graphical results for the effects of particle size parameters, as well as the real and imaginary parts of the index of refraction, on the photometric and polarization characteristics of the radar return were also provided. No numerical results for intensity were given.

Haferman et al. [2-26] solved a multi-dimensional radiative transfer problem including polarization effects by using the discrete-ordinates method. Numerical results for backscattered Stokes parameters were provided only for a one-dimensional plane-parallel medium illuminated by an unpolarized intensity from the top boundary.

Mueller and Crosbie [2-27] presented a polarized phase matrix for the three-dimensional radiative transfer problem based on a scattering matrix which represented randomly oriented scattering particles having a plane of symmetry. In that paper, the geometry was finite in the z-direction and infinite in the x- and y-directions, with elliptically polarized radiation incident only on the top boundary. Great effort was expended to reduce this three-dimensional problem to a one-dimensional problem which depended on two parameters. A general four by four source matrix was derived by using the method of superposition. Some symmetry relationships were developed. Moreover, an extensive review of a wide variety of radiative transfer literature was also provided. No numerical results were presented.

The purpose of this chapter of the current study is to numerically solve the intensity matrix for diffusion approximation by using the classical P_1 approximation as well as the modified P_1 method, for a one-dimensional plane-parallel medium which scatters and absorbs, with polarization fully included. The present work will allow the incident radiation to be elliptically polarized, which implies that the solutions depend on the azimuthal angle. Moreover, these expressions for intensity are straightforward and numerically simpler to solve than those of previous researchers. Future research will be directed toward the exact numerical solutions, defined in Chapter III, of the current work.

Diffusion Approximation of Diffuse Transport Equation for Polarized Light

In this section, beginning with the diffuse transport equation for polarized light in a one-dimensional plane-parallel medium, the expression for the intensity of the diffusion approximation will be derived by using the classical P_1 approximation as well as the modified P_1 method. Absorption and scattering without emission are assumed in the medium, and refractive index effects at the boundaries are neglected.

Problem Description

As mentioned earlier, the problem in which we are interested is the diffusion approximation of one-dimensional radiative transfer for polarized light without reflective boundaries. This work is the first step in solving realistic problems with polarization. Thus, we have concentrated on the polarization, but simplified the geometry and interfaces by choosing a one-dimensional case with non-reflective boundaries. Future plans are to generalize this solution after demonstrating the ability to handle polarization effects. Another reason for working on the one-dimensional case is to get approximate results which can be later compared with the exact solutions, which will be presented in a future paper. The geometry for this problem is shown in Fig. 2-1.

In this research, we assume that collimated polarized incident radiation at an angle θ_o exists only at the top boundary and is a sheet of laser-like beams which simplifies our study to a one-dimensional problem. Moreover, no radiation is incident at the lower boundary and there are no refractive index effects at either boundary. Furthermore, absorption and scattering without emission are assumed in the medium (which applies only

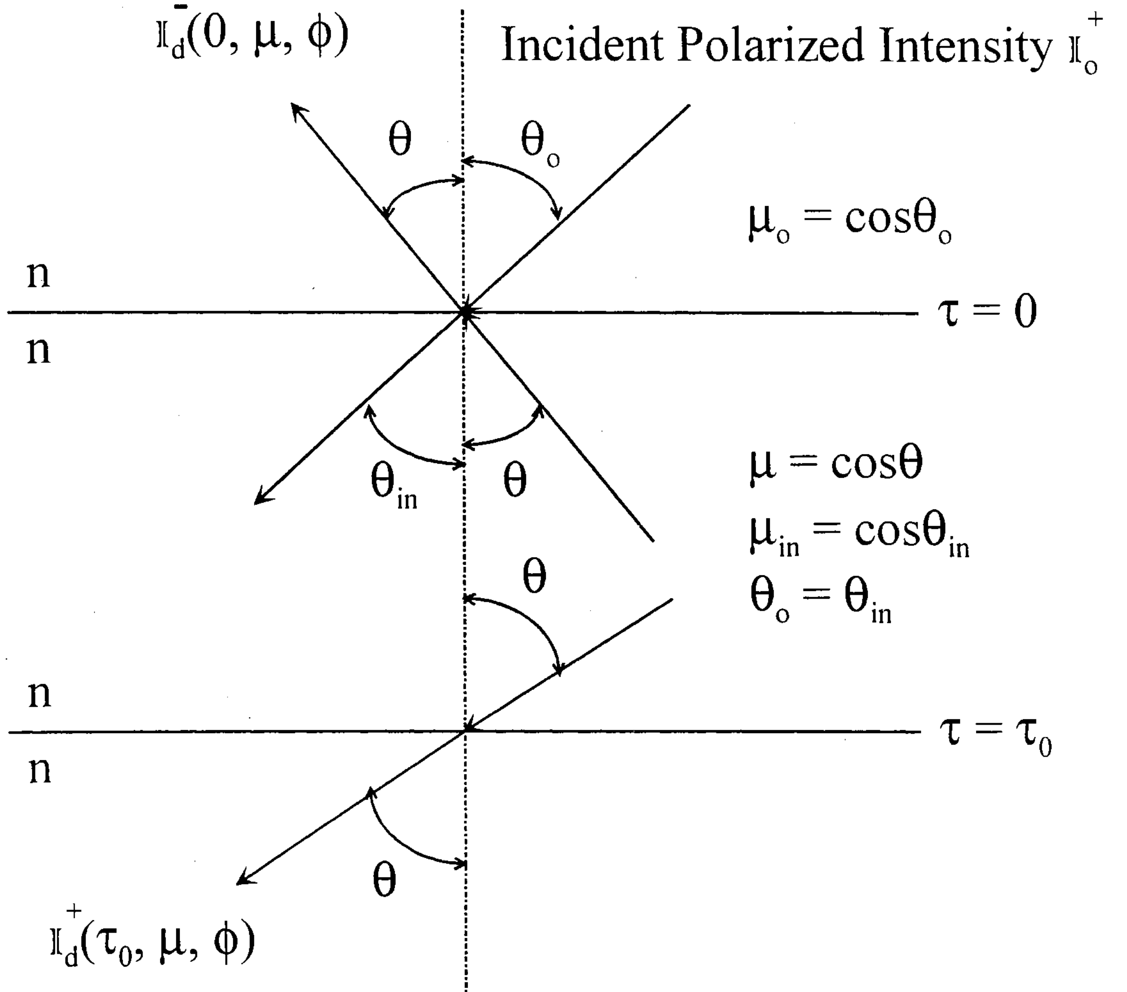

Figure 2-1. Geometry of a One-Dimensional Medium without Reflective Boundaries

as a good approximation to a low temperature medium). Note that the probability of scattering in the various directions depends on the phase matrix function, which will be discussed later.

Fundamental Equations

In a recent work [2-28], we have derived the diffuse transport equation for polarized light in a plane-parallel atmosphere (Fig. 2-1) as follows

$$\mu \frac{d\,\mathbb{I}_d(\tau,\mu,\phi)}{d\tau} + \mathbb{I}_d(\tau,\mu,\phi) = \frac{\omega}{4\pi}\int_0^{2\pi}\int_{-1}^{1} \mathbb{P}(\mu,\mu',\phi,\phi')\,\mathbb{I}_d(\tau,\mu',\phi')\,d\mu'\,d\phi'$$

$$+\frac{\omega}{4}\,\mathbb{P}(\mu,\mu_o,\phi,\phi_o)\exp(-\tau/\mu_o)\,\mathbb{F}, \tag{2-1}$$

with the boundary conditions, for $\mu \in [0, 1]$ and $\phi \in [0, 2\pi]$, to be

$$\mathbb{I}_d(0, \mu, \phi) = 0 \tag{2-2a}$$

and $\mathbb{I}_d(\tau_0, -\mu, \phi) = 0$, (2-2b)

where τ is the normal optical thickness, μ is the direction cosine of the propagation angle of the radiation, ω is the single scattering albedo, the vector $\mathbb{F} = [F_I\ F_Q\ F_U\ F_V]^T$ which is presumed given specifies the state of polarization of the incident intensity at the upper boundary, and the diffuse intensity vector $\mathbb{I}_d(\tau, \mu, \phi)$ consists of the four Stokes parameters, that is, $\mathbb{I}_d(\tau, \mu, \phi) = [I_d(\tau, \mu, \phi)\ \ Q_d(\tau, \mu, \phi)\ \ U_d(\tau, \mu, \phi)\ \ V_d(\tau, \mu, \phi)]^T$. Furthermore, $\mathbb{P}(\mu, \mu', \phi, \phi')$ is the phase matrix that can be expanded in general in a Fourier series [2-12] as

$$\mathbb{P}(\mu, \mu', \phi, \phi') = \sum_{m=0}^{L} \frac{1}{1+\delta_{0m}} \{\mathbb{C}^m(\mu, \mu') \cos[m(\phi - \phi')]$$

$$+ \mathbb{S}^m(\mu, \mu') \sin[m(\phi - \phi')]\}, \tag{2-3}$$

where δ_{0m} is the Kronecker delta function, and other functions in Eq. (2-3) are defined as

$$\mathbb{C}^m(\mu, \mu') = \mathbb{A}^m(\mu, \mu') + \mathbb{D}_2 \mathbb{A}^m(\mu, \mu') \mathbb{D}_2, \tag{2-4a}$$

$$\mathbb{S}^m(\mu, \mu') = \mathbb{A}^m(\mu, \mu') \mathbb{D}_2 - \mathbb{D}_2 \mathbb{A}^m(\mu, \mu'), \tag{2-4b}$$

$$\mathbb{A}^m(\mu, \mu') = \sum_{i=m}^{L} \frac{(i-m)!}{(i+m)!} \Pi_i^m(\mu) \mathbb{B}_i \Pi_i^m(\mu'), \tag{2-4c}$$

$$\mathbb{D}_2 = \mathrm{diag}\{1, 1, -1, -1\}, \tag{2-4d}$$

$$\Pi_i^m(\mu) = \begin{bmatrix} P_i^m(\mu) & 0 & 0 & 0 \\ 0 & R_i^m(\mu) & -T_i^m(\mu) & 0 \\ 0 & -T_i^m(\mu) & R_i^m(\mu) & 0 \\ 0 & 0 & 0 & P_i^m(\mu) \end{bmatrix}, \tag{2-4e}$$

and the matrix of scattering coefficients is given by

$$\mathbb{B}_i = \begin{bmatrix} \beta_i & \gamma_i & 0 & 0 \\ \gamma_i & \alpha_i & 0 & 0 \\ 0 & 0 & \zeta_i & -\varepsilon_i \\ 0 & 0 & \varepsilon_i & \delta_i \end{bmatrix}. \tag{2-4f}$$

The matrix $\mathbb{B}_i$ will be specified in detail later. In addition, in Eq. (2-4e), $P_i^m(\mu)$ denotes associated Legendre functions while $R_i^m(\mu)$ and $T_i^m(\mu)$ are combinations of generalized spherical functions [2-19]. Notice that the phase matrix of Eq. (2-3) assumes that the scattering particles are randomly oriented, and have at least one plane of symmetry [2-22].

Diffusion Approximation

Substituting Eqs. (2-4) into Eq. (2-3) with the assumption that L is equal to one, the phase matrix for diffusion approximation can be simplified as

$$\mathrm{P}(\mu,\mu',\phi,\phi') = \mathrm{diag}\{\ \beta_0 + \beta_1\ \mu\,\mu' + \beta_1\,(1-\mu^2)^{1/2}\,(1-\mu'^2)^{1/2}\cos(\phi-\phi'), 0, 0,$$

$$\delta_0 + \delta_1\ \mu\,\mu' + \delta_1\,(1-\mu^2)^{1/2}\,(1-\mu'^2)^{1/2}\cos(\phi-\phi')\ \}. \tag{2-5}$$

Then, substituting Eq. (2-5) into Eq. (2-1) with the help of the definitions of vector F and $\mathrm{I_d}(\tau, \mu, \phi)$ which we defined earlier, the diffuse transport equation for the diffusion approximation can be written as

$$\mu\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix}\mathrm{I_d}(\tau,\mu,\phi)\\ \mathrm{Q_d}(\tau,\mu,\phi)\\ \mathrm{U_d}(\tau,\mu,\phi)\\ \mathrm{V_d}(\tau,\mu,\phi)\end{bmatrix}+\begin{bmatrix}\mathrm{I_d}(\tau,\mu,\phi)\\ \mathrm{Q_d}(\tau,\mu,\phi)\\ \mathrm{U_d}(\tau,\mu,\phi)\\ \mathrm{V_d}(\tau,\mu,\phi)\end{bmatrix}$$

$$=\frac{\omega}{4\pi}\int_0^{2\pi}\int_{-1}^{1}\begin{bmatrix}\{\ \beta_0+\beta_1\ \mu\,\mu'+\beta_1\,(1-\mu^2)^{1/2}\,(1-\mu'^2)^{1/2}\cos(\phi-\phi')\ \}\mathrm{I_d}(\tau,\mu',\phi')\\ 0\\ 0\\ \{\ \delta_0+\delta_1\ \mu\,\mu'+\delta_1\,(1-\mu^2)^{1/2}\,(1-\mu'^2)^{1/2}\cos(\phi-\phi')\ \}\mathrm{V_d}(\tau,\mu',\phi')\end{bmatrix}\mathrm{d}\mu'\,\mathrm{d}\phi'$$

$$+\frac{\omega}{4}\exp(-\tau/\mu_o)\begin{bmatrix}\{\ \beta_0+\beta_1\ \mu\,\mu_o+\beta_1\,(1-\mu^2)^{1/2}\,(1-\mu_o{}^2)^{1/2}\cos(\phi-\phi_o)\ \}\mathrm{F_I}\\ 0\\ 0\\ \{\ \delta_0+\delta_1\ \mu\,\mu_o+\delta_1\,(1-\mu^2)^{1/2}\,(1-\mu_o^2)^{1/2}\cos(\phi-\phi_o)\ \}\mathrm{F_V}\end{bmatrix}. \tag{2-6}$$

Equation (2-6) allows us to simplify the problem to consider only two independent equations, I_d and V_d, instead of solving the whole matrix equation due to the reason that the vector components of Eq. (2-6) are decoupled and the Q_d and U_d are always zero by applying the boundary conditions of Eqs. (2-2) to Eq. (2-6). This also implies that the reflected and transmitted intensities will be either circularly polarized or unpolarized light, independent of the kind of polarized incident radiation that we apply at the top boundary. By observing Eq. (2-6), we also realize that we need to solve for only the diffuse intensity I_d, since the equations for I_d and V_d have very similar form. Therefore, once we get the expression for the diffuse intensity I_d, we can get the expression for V_d by changing I_d, β_0, β_1, and F_I, to V_d, δ_0, δ_1, and F_V, respectively. Note that omitting the cosine terms in I_d equation of Eq. (2-6) gives us the classical P_1 approximation equation for scalar problem.

In this research, three techniques will be used to solve for the diffuse intensity I_d of Eq. (2-6). The first two approaches are the classical P_1 approximation by applying Mark's and Marshak's Boundary conditions, while the third approach is the so-called modified P_1 method.

Classical P_1 Approximation

The diffuse intensity I_d of Eq. (2-6) can be rewritten as

$$\mu \frac{d\, I_d(\tau,\mu,\phi)}{d\tau} + I_d(\tau,\mu,\phi)$$

$$= \frac{\omega}{4\pi} \int_0^{2\pi} \int_{-1}^{1} \{\ \beta_0\, P_0(\mu') + \beta_1\, \mu\, P_1(\mu') + \beta_1\, (1-\mu^2)^{1/2}\, P_1^1(\mu') \cos(\phi-\phi')\ \}$$

$$\times I_d(\tau,\mu',\phi')\, d\mu'\, d\phi' + \frac{\omega}{4} \exp(-\tau/\mu_o) \{\ \beta_0\, P_0(\mu) + \beta_1\, \mu_o\, P_1(\mu) + \beta_1\, (1-\mu_o{}^2)^{1/2}$$

$$\times P_1^1(\mu)\cos(\phi-\phi_o)\ \}F_I, \tag{2-7}$$

where $P_0(\mu)=1$, (2-8a)

$$P_1(\mu)=\mu, \tag{2-8b}$$

and $P_1^1(\mu)=(1-\mu^2)^{1/2}$. (2-8c)

First, let

$$I_d(\tau,\mu,\phi)=I_{d0}(\tau)\,P_0(\mu)+I_{d1}(\tau)\,P_1(\mu)+I_{d2}(\tau)\,P_1^1(\mu)\cos\phi\,. \tag{2-9}$$

Then, substituting Eq. (2-9) into Eq. (2-7), we get

$$\begin{aligned}
&P_0(\mu)\{\ (1/3)\frac{d\,I_{d1}(\tau)}{d\tau}+(1-\omega\,\beta_0)\,I_{d0}(\tau)-\frac{\omega}{4}\exp(-\tau/\mu_o)\,\beta_0\,F_I\ \}\\
&+P_1(\mu)\{\ \frac{d\,I_{d0}(\tau)}{d\tau}+(1-\frac{\omega}{3}\beta_1)\,I_{d1}(\tau)-\frac{\omega}{4}\exp(-\tau/\mu_o)\,\beta_1\,\mu_o\,F_I\ \}\\
&+P_1^1(\mu)\{\ (1-\frac{\omega}{3}\beta_1)\,I_{d2}(\tau)\cos\phi-\frac{\omega}{4}\exp(-\tau/\mu_o)\,\beta_1\,(1-\mu_o^2)^{1/2}\cos(\phi-\phi_o)\,F_I\ \}\\
&+P_2(\mu)\{\ (2/3)\frac{d\,I_{d1}(\tau)}{d\tau}\ \}+P_2^1(\mu)\{\ (1/3)\frac{d\,I_{d2}(\tau)}{d\tau}\cos\phi\ \}=0\,.
\end{aligned} \tag{2-10}$$

By observing Eq. (2-10) with the assumption ϕ_o is zero, we find that

$$\frac{d\,I_{d1}(\tau)}{d\tau}+3(1-\omega\,\beta_0)\,I_{d0}(\tau)=\frac{3\omega}{4}\exp(-\tau/\mu_o)\,\beta_0\,F_I, \tag{2-11}$$

$$\frac{d\,I_{d0}(\tau)}{d\tau}+(1-\frac{\omega}{3}\beta_1)\,I_{d1}(\tau)=\frac{\omega}{4}\exp(-\tau/\mu_o)\,\beta_1\,\mu_o\,F_I, \tag{2-12}$$

and

$$I_{d2}(\tau) = \frac{\omega}{4}(1-\frac{\omega}{3}\beta_1)^{-1}\exp(-\tau/\mu_o)\,\beta_1\,(1-\mu_o^2)^{1/2}\,F_I\,. \tag{2-13}$$

Now, substituting Eqs. (2-12) and (2-13) into Eq. (2-9), the diffuse intensity I_d can be expressed in terms of only $I_{d0}(\tau)$ as

$$I_d(\tau,\mu,\phi) = I_{d0}(\tau) - \mu(1-\frac{\omega}{3}\beta_1)^{-1}\frac{d\,I_{d0}(\tau)}{d\tau} + \frac{\omega}{4}(1-\frac{\omega}{3}\beta_1)^{-1}\exp(-\tau/\mu_o)\,\beta_1\,F_I$$

$$\times\{\,\mu\,\mu_o + (1-\mu^2)^{1/2}(1-\mu_o^2)^{1/2}\cos\phi\,\}. \tag{2-14}$$

So far, we have simplified the expression of the diffuse intensity I_d in terms of $I_{d0}(\tau)$ only. Thus, the solution of the diffuse intensity I_d can be obtained once the solution of $I_{d0}(\tau)$ is found by solving Eqs. (2-11) and (2-12) simultaneously with desired boundary conditions. Notice that the term containing $I_{d0}(\tau)$ will be canceled out in Eq. (2-11) when ω is unity, due to the fact that β_0 is always equal to 1 [2-20]. Therefore, we need to consider two different cases for the solution of $I_{d0}(\tau)$ before applying the boundary conditions. One is the case for ω less than one, and the other is the special case for ω equal to one.

For $\omega \leq 1$. In order to get the general expression of $I_{d0}(\tau)$ for ω less than one, we need to first take the derivative of Eq. (2-12) with respect to τ, which with the help of Eq. (2-11) yields

$$\frac{d^2 I_{d0}(\tau)}{d\tau^2} - (3-\omega\,\beta_1)(1-\omega\,\beta_0)\, I_{d0}(\tau) = \frac{\omega}{4}\exp(-\tau/\mu_o)\, F_I\, (-3\beta_0 + \omega\,\beta_0\,\beta_1 - \beta_1)\,. \quad (2\text{-}15)$$

Then, the solution of Eq. (2-15) can be obtained by adding its homogeneous and particular solutions together as

$$I_{d0}(\tau) = C_1 \exp(A\,\tau) + C_2 \exp(-A\,\tau) + \frac{\omega}{4}\exp(-\tau/\mu_o)\, F_I \\ \times (-3\beta_0 + \omega\,\beta_0\,\beta_1 - \beta_1)\,/\,[(1/\mu_o^2) - A^2], \quad (2\text{-}16)$$

where

$$A = (3 - \omega\,\beta_1 - 3\,\omega\,\beta_0 + \omega^2 \beta_0\,\beta_1)^{1/2}\,. \quad (2\text{-}17)$$

Finally, substituting Eq. (2-16) into Eq. (2-14), the general expression for the diffuse intensity I_d for both Mark's and Marshak's boundary conditions when ω less than one is

$$I_d(\tau,\mu,\phi) = C_1 \exp(A\,\tau)[1 - \mu\, A\,(1-\frac{\omega}{3}\beta_1)^{-1}] + C_2 \exp(-A\,\tau)[1 + \mu\, A\,(1-\frac{\omega}{3}\beta_1)^{-1}] \\ + \frac{\omega}{4}\exp(-\tau/\mu_o)\, F_I \left\langle \{\, (-3\beta_0 + \omega\,\beta_0\,\beta_1 - \beta_1)\,/\,[(1/\mu_o^2) - A^2]\,\}[1 + (\mu/\mu_o) \right. \\ \left. \times (1-\frac{\omega}{3}\beta_1)^{-1}] + \beta_1 (1-\frac{\omega}{3}\beta_1)^{-1}\, [\mu\,\mu_o + (1-\mu^2)^{1/2}\,(1-\mu_o^2)^{1/2}\cos\phi] \right\rangle. \quad (2\text{-}18)$$

<u>For $\omega = 1$.</u> By solving Eq. (2-11) with ω equal to one, we have

$$I_{d1}(\tau) = C_3 - \frac{3\,\mu_o}{4}\exp(-\tau/\mu_o)\, F_I\,. \quad (2\text{-}19)$$

Then, substituting Eq. (2-19) into Eq. (2-12), the general expression of $I_{d0}(\tau)$ for ω equal to one is

$$I_{d0}(\tau) = C_3\,[(\beta_1/3) - 1]\,\tau + C_4 - \frac{3\,\mu_o^2}{4}\exp(-\tau/\mu_o)\,F_I\,. \tag{2-20}$$

Finally, substituting Eq. (2-20) into Eq. (2-14), the general expression of the diffuse intensity I_d for both Mark's and Marshak's boundary conditions when ω is equal to one is

$$I_d(\tau,\mu,\phi) = C_3\,\{\,[(\beta_1/3) - 1]\,\tau + \mu\,\} + C_4 + \frac{1}{4}\exp(-\tau/\mu_o)\,F_I\,\{\,-3\,\mu_o(\mu_o + \mu)$$
$$+\,\beta_1(1 - \frac{\beta_1}{3})^{-1}\,(1-\mu^2)^{1/2}\,(1-\mu_o^2)^{1/2}\cos\phi\,\}. \tag{2-21}$$

Equations (2-18) and (2-21) are the general expressions of the diffuse intensity I_d for ω less than one and ω equal to one, respectively. The unknown coefficients C_1 to C_4 in these two equations will be determined later by applying the desired boundary conditions.

<u>Mark's Boundary Conditions</u>

Let $P_2^1(\mu) = 3\,\mu\,(1-\mu^2)^{1/2} = 0$ [2-28], which is the highest order in Eq. (2-10); then the solution is μ equal to 0, -1, and 1. In this paper, μ equal to one is chosen to apply Mark's boundary conditions.

Now, substituting Eq. (2-14) with μ equal to one into Eqs. (2-2), we have the following Mark's boundary conditions for any ω

$$I_d(0,1,\phi) = I_{d0}(0) - (1-\frac{\omega}{3}\beta_1)^{-1}\frac{d\,I_{d0}(0)}{d\tau} + \frac{\omega}{4}(1-\frac{\omega}{3}\beta_1)^{-1}\beta_1 F_I\,\mu_o = 0 \tag{2-22a}$$

and

$$I_d(\tau_0,-1,\phi) = I_{d0}(\tau_0) + (1-\frac{\omega}{3}\beta_1)^{-1}\frac{d\,I_{d0}(\tau_0)}{d\tau} - \frac{\omega}{4}(1-\frac{\omega}{3}\beta_1)^{-1}$$

$$\times \exp(-\tau_0/\mu_o)\,\beta_1\,F_I\,\mu_o = 0\,. \tag{2-22b}$$

The Mark's boundary conditions of Eqs. (2-22) will be used to solve for the unknown coefficients of $I_{d0}(\tau)$ in Eq. (2-16) for ω less than one and in Eq. (2-20) for ω equal to one, respectively.

Applying Mark's Boundary Conditions with $\omega \le 1$. By substituting Eq. (2-16) into Eqs. (2-22) and solving for the coefficients $C_{1\text{-Mark}}$ and $C_{2\text{-Mark}}$, these undetermined coefficients can be expressed as

$$C_{1\text{-Mark}} = B2/[B2(A-1+\frac{\omega}{3}\beta_1)+B1(A+1-\frac{\omega}{3}\beta_1)]\langle\ \{\ \frac{\omega}{4}F_I\,(-3\beta_0+\omega\,\beta_0\,\beta_1-\beta_1)$$

$$/[(1/\mu_o^2)-A^2]\ \}\{\ [2\exp(-\tau_0/\mu_o)/B1](1-\frac{\omega}{3}\beta_1)(A-1+\frac{\omega}{3}\beta_1)+1-\frac{\omega}{3}\beta_1$$

$$+(1/\mu_o)\ \}+\frac{\omega}{4}\beta_1\,F_I\,\mu_o\ \rangle-(\omega/2\,B1)\{\ (-3\beta_0+\omega\,\beta_0\,\beta_1-\beta_1)/[(1/\mu_o^2)-A^2]\ \}$$

$$\times(1-\frac{\omega}{3}\beta_1)\exp(-\tau_0/\mu_o)\,F_I \tag{2-23a}$$

and

$$C_{2\text{-Mark}} = (-1)/\{\,[B2\,(A-1+\frac{\omega}{3}\beta_1)/B1] + A + 1 - \frac{\omega}{3}\beta_1\,\}\langle\ \{\,\frac{\omega}{4}F_I\,(-3\beta_0 + \omega\,\beta_0\,\beta_1 - \beta_1)$$
$$/[(1/\mu_o^2) - A^2]\,\}\{\,[2\,(1-\frac{\omega}{3}\beta_1)\exp(-\tau_0/\mu_o)/B1](A-1+\frac{\omega}{3}\beta_1) + 1 - \frac{\omega}{3}\beta_1$$
$$+ (1/\mu_o)\,\} + \frac{\omega}{4}\beta_1\,F_I\,\mu_o\ \rangle, \tag{2-23b}$$

where

$$B1 = [(1-A-\frac{\omega}{3}\beta_1)\exp(-\tau_0/\mu_o) + (1+A-\frac{\omega}{3}\beta_1)\exp(A\,\tau_0)], \tag{2-24a}$$

$$B2 = [(1+A-\frac{\omega}{3}\beta_1)\exp(-\tau_0/\mu_o) + (1-A-\frac{\omega}{3}\beta_1)\exp(-A\,\tau_0)], \tag{2-24b}$$

and A is defined in Eq. (2-17).

Applying Mark's Boundary Conditions with $\omega = 1$. Similarly, by substituting Eq. (2-20) into Eqs. (2-22), the unknown coefficients $C_{3\text{-Mark}}$ and $C_{4\text{-Mark}}$ in Eq. (2-20) can be obtained as

$$C_{3\text{-Mark}} = \frac{3\mu_o}{4}F_I\,\{\,[\exp(-\tau_0/\mu_o)](1-\mu_o) + \mu_o + 1\,\}/[(-\beta_1\,\tau_0)/3 + \tau_0 + 2] \tag{2-25a}$$

and

$$C_{4\text{-Mark}} = [\exp(-\tau_0/\mu_o) + 1]^{-1}\langle\ (3/2)\,\mu_o^2\,F_I\,\exp(-\tau_0/\mu_o) - \frac{3\mu_o}{4}F_I\,\{\,[\exp(-\tau_0/\mu_o)]$$
$$\times(1-\mu_o) + \mu_o + 1\,\}[(\beta_1\,\tau_0)/3 - \tau_0 + \exp(-\tau_0/\mu_o) - 1]$$
$$/[(-\beta_1\,\tau_0)/3 + \tau_0 + 2]\ \rangle. \tag{2-25b}$$

Marshak's Boundary Conditions

Marshak's boundary conditions are defined as

$$\int_0^{2\pi} \int_0^1 I_d(0,\mu,\phi)\,\mu\, d\mu\, d\phi = 0 \qquad \text{(2-26a)}$$

and

$$\int_0^{2\pi} \int_0^1 I_d(\tau_0,-\mu,\phi)\,\mu\, d\mu\, d\phi = 0\,. \qquad \text{(2-26b)}$$

The unknown coefficients of $I_{d0}(\tau)$ in Eq. (2-16) for ω less than one and in Eq. (2-20) for ω equal to one will now be determined by applying Marshak's boundary conditions of Eqs. (2-26).

Applying Marshak's Boundary Conditions with $\omega \leq 1$. By substituting Eqs. (2-14) and (2-16) into Eqs. (2-26) and solving for the unknown coefficients $C_{1\text{-Marshak}}$ and $C_{2\text{-Marshak}}$, these coefficients can be written as

$$\begin{aligned} C_{1\text{-Marshak}} = (-1/D1)\Big\langle\, & (-D1/D2)\Big\{\, \frac{\omega}{4} F_I\,(-3\beta_0 + \omega\,\beta_0\,\beta_1 - \beta_1)[-2\,E2\,(1-\frac{\omega}{3}\beta_1) \\ & \times \exp(-\tau_0/\mu_o)\,/\,D1 + (1-\frac{\omega}{3}\beta_1 + \frac{2}{3\mu_o})]\,/\,[(1/\mu_o^2) - A^2] + \frac{\omega}{6}\beta_1\,F_I\,\mu_o\,\Big\} \\ & \times [E2\exp(-A\,\tau_0) + E1\exp(-\tau_0/\mu_o)] + (\omega/2)\Big\{\,(-3\beta_0 + \omega\,\beta_0\,\beta_1 - \beta_1) \\ & /[(1/\mu_o^2) - A^2]\,\Big\}(1-\frac{\omega}{3}\beta_1)\exp(-\tau_0/\mu_o)\,F_I\,\Big\rangle \end{aligned} \qquad \text{(2-27a)}$$

and

$$C_{2\text{-Marshak}} = (-D1/D2)\{\ \frac{\omega}{4} F_I\,(-3\beta_0 + \omega\,\beta_0\,\beta_1 - \beta_1)\,[-2\,E2\,(1 - \frac{\omega}{3}\beta_1)\exp(-\tau_0/\mu_o)$$

$$/D1 + (1 - \frac{\omega}{3}\beta_1 + \frac{2}{3\mu_o})]/[(1/\mu_o^2) - A^2] + \frac{\omega}{6}\beta_1\,F_I\,\mu_o\ \}, \tag{2-27b}$$

where

$$D1 = E1\exp(A\,\tau_0) + E2\exp(-\tau_0/\mu_o), \tag{2-28a}$$

$$D2 = (E1)^2\exp(A\,\tau_0) - (E2)^2\exp(-A\,\tau_0), \tag{2-28b}$$

$$E1 = [1 - (\omega\,\beta_1)/3 + (2\,A)/3], \tag{2-28c}$$

$$E2 = [1 - (\omega\,\beta_1)/3 - (2\,A)/3], \tag{2-28d}$$

and A is defined in Eq. (2-17).

<u>Applying Marshak's Boundary Conditions with ω = 1.</u> In a like manner, by substituting Eqs. (2-14) and (2-20) into Eqs. (2-26), the unknown coefficients $C_{3\text{-Marshak}}$ and $C_{4\text{-Marshak}}$ in Eq. (2-20) can be obtained as

$$C_{3\text{-Marshak}} = -\mu_o\,F_I\,[(\beta_1/3 - 1)\,\tau_0 - 4/3]^{-1}\,\{\ (3/4)\,\mu_o\,[1 - \exp(-\tau_0/\mu_o)] + \exp(-\tau_0/\mu_o)$$

$$\times (1/2 - \beta_1/6)/(1 - \beta_1/3) + 1/2\ \} \tag{2-29a}$$

and

$$C_{4\text{-Marshak}} = (2/3)\,\mu_o\,F_I\,[(\beta_1/3 - 1)\,\tau_0 - 4/3]^{-1}\,\{\ (3/4)\,\mu_o\,[1 - \exp(-\tau_0/\mu_o)] + \exp(-\tau_0/\mu_o)$$

$$\times(1/2-\beta_1/6)/(1-\beta_1/3)+1/2\ \}+\mu_o\, F_I\, [1/2+(3/4)\,\mu_o]. \qquad (2\text{-}29b)$$

Thus far, the unknown coefficients C_1 and C_2 in Eq. (2-16) for ω less than one and C_3 and C_4 in Eq. (2-20) for ω equal to one have been derived explicitly for both Mark's and Marshak's boundary conditions. In the next section, a modified P_1 method, modified from classical P_1 approximation, will be introduced and applied with the Marshak's boundary conditions only.

The Modified P_1 Method

Equation (2-6) can be rewritten in another form as

$$\mu\frac{d\,\mathbb{I}_d(\tau,\mu,\phi)}{d\tau}+\mathbb{I}_d(\tau,\mu,\phi)=\mathbb{S}(\tau,\mu,\phi), \qquad (2\text{-}30)$$

where

$$\mathbb{S}(\tau,\mu,\phi)=\begin{bmatrix} S_I(\tau,\mu,\phi)\\ S_Q(\tau,\mu,\phi)\\ S_U(\tau,\mu,\phi)\\ S_V(\tau,\mu,\phi)\end{bmatrix}$$

$$=\frac{\omega}{4\pi}\int_0^{2\pi}\int_{-1}^{1}\begin{bmatrix}\{\ \beta_0+\beta_1\ \mu\,\mu'+\beta_1\,(1-\mu^2)^{1/2}\,(1-\mu'^2)^{1/2}\cos(\phi-\phi')\ \}I_d(\tau,\mu',\phi')\\ 0\\ 0\\ \{\ \delta_0+\delta_1\ \mu\,\mu'+\delta_1\,(1-\mu^2)^{1/2}\,(1-\mu'^2)^{1/2}\cos(\phi-\phi')\ \}V_d(\tau,\mu',\phi')\end{bmatrix}d\mu'\,d\phi'$$

$$+\frac{\omega}{4}\exp(-\tau/\mu_o)\begin{bmatrix}\{\ \beta_0+\beta_1\ \mu\ \mu_o+\beta_1\ (1-\mu^2)^{1/2}\ (1-\mu_o{}^2)^{1/2}\cos(\phi-\phi_o)\ \}F_I\\ 0\\ 0\\ \{\ \delta_0+\delta_1\ \mu\ \mu_o+\delta_1\ (1-\mu^2)^{1/2}\ (1-\mu_o^2)^{1/2}\cos(\phi-\phi_o)\ \}F_V\end{bmatrix} \tag{2-31}$$

is the source matrix. Equations (2-2) are modified to convenient forms as

$$\mathbb{I}_d^+(0,\mu,\phi) = 0 \tag{2-32a}$$

and $\mathbb{I}_d^-(\tau_0,\mu,\phi) = 0,$ (2-32b)

where the superscripts + and - on Eqs. (2-32) denote that the intensity is generally propagating in the positive and negative τ directions (see Fig. 2-1), respectively.

Solving Eq. (2-30) for $\mathbb{I}_d^+$ and $\mathbb{I}_d^-$, using an integrating factor with the help of Eqs. (2-31) and (2-32), yields

$$\begin{bmatrix} I_d^+(\tau,\mu,\phi)\\ Q_d^+(\tau,\mu,\phi)\\ U_d^+(\tau,\mu,\phi)\\ V_d^+(\tau,\mu,\phi)\end{bmatrix}=\int_0^\tau\begin{bmatrix} S_I(t,\mu,\phi)\\ S_Q(t,\mu,\phi)\\ S_U(t,\mu,\phi)\\ S_V(t,\mu,\phi)\end{bmatrix}\exp[-(\tau-t)/\mu]\frac{dt}{\mu} \tag{2-33a}$$

and

$$\begin{bmatrix} I_d^-(\tau,\mu,\phi)\\ Q_d^-(\tau,\mu,\phi)\\ U_d^-(\tau,\mu,\phi)\\ V_d^-(\tau,\mu,\phi)\end{bmatrix}=\int_\tau^{\tau_0}\begin{bmatrix} S_I(t,-\mu,\phi)\\ S_Q(t,-\mu,\phi)\\ S_U(t,-\mu,\phi)\\ S_V(t,-\mu,\phi)\end{bmatrix}\exp[-(t-\tau)/\mu]\frac{dt}{\mu}\ . \tag{2-33b}$$

At this point in the derivation, the general equations of diffuse intensity have been derived in Eqs. (2-33) for the modified P_1 method. Our next goal is to apply the solution of the classical P_1 approximation with Marshak's Boundary conditions to Eqs. (2-33) in order to get the solution for diffuse intensity I_d.

The Modified P_1 Method with $\omega \leq 1$. By substituting Eqs. (2-18) and (2-31) into Eqs. (2-33) with the previous assumption of ϕ_o equal to zero and solving for the diffuse intensities I_d^+ and I_d^-, we have

$$
\begin{aligned}
I_d^+(\tau,\mu,\phi) = {} & C_{1\text{-Marshak}}\,[\omega \exp(-\tau/\mu)/(\mu\,\mathrm{H1})]\,\mathrm{G1}\,(\beta_0 - \mathrm{G7}\,\beta_1\,\mu\,\mathrm{A}) \\
& + C_{2\text{-Marshak}}\,[\omega \exp(-\tau/\mu)/(\mu\,\mathrm{H2})]\,\mathrm{G2}\,(\beta_0 + \mathrm{G7}\,\beta_1\,\mu\,\mathrm{A}) \\
& + \frac{\omega^2}{4} F_I\,[\exp(-\tau/\mu)/(\mu\,\mathrm{H4})]\,(-3\beta_0 + \omega\,\beta_0\,\beta_1 - \beta_1)\,\mathrm{G3}\,(\beta_0 + \mathrm{G7}\,\beta_1\,\mu/\mu_o) \\
& /[(1/\mu_o^2) - \mathrm{A}^2] + \frac{\omega}{4}[\exp(-\tau/\mu)/(\mu\,\mathrm{H4})]\,\mathrm{G3}\,[\beta_0 + \beta_1\,\mu\,\mu_o + \mathrm{G7}\,\omega\,\beta_1^2 \\
& \times \mu\,\mu_o + \beta_1\,(1-\mu^2)^{1/2}\,(1-\mu_o^2)^{1/2}\cos\phi + \mathrm{G7}\,\omega\,\beta_1^2\,(1-\mu^2)^{1/2} \\
& \times (1-\mu_o^2)^{1/2}\cos\phi]\,F_I
\end{aligned}
\qquad (2\text{-}34a)
$$

and

$$
\begin{aligned}
I_d^-(\tau,\mu,\phi) = {} & - C_{1\text{-Marshak}}\,[\omega \exp(\tau/\mu)/(\mu\,\mathrm{H2})]\,\mathrm{G4}\,(\beta_0 + \mathrm{G7}\,\beta_1\,\mu\,\mathrm{A}) \\
& - C_{2\text{-Marshak}}\,[\omega \exp(\tau/\mu)/(\mu\,\mathrm{H1})]\,\mathrm{G5}\,(\beta_0 - \mathrm{G7}\,\beta_1\,\mu\,\mathrm{A}) \\
& - \frac{\omega^2}{4} F_I\,[\exp(\tau/\mu)/(\mu\,\mathrm{H3})]\,(-3\beta_0 + \omega\,\beta_0\,\beta_1 - \beta_1)\,\mathrm{G6}\,(\beta_0 - \mathrm{G7}\,\beta_1\,\mu/\mu_o) \\
& /[(1/\mu_o^2) - \mathrm{A}^2] - \frac{\omega}{4}[\exp(\tau/\mu)/(\mu\,\mathrm{H3})]\,\mathrm{G6}\,[\beta_0 - \beta_1\,\mu\,\mu_o - \mathrm{G7}\,\omega\,\beta_1^2 \\
& \times \mu\,\mu_o + \beta_1\,(1-\mu^2)^{1/2}\,(1-\mu_o^2)^{1/2}\cos\phi + \mathrm{G7}\,\omega\,\beta_1^2\,(1-\mu^2)^{1/2} \\
& \times (1-\mu_o^2)^{1/2}\cos\phi]\,F_I,
\end{aligned}
\qquad (2\text{-}34b)
$$

where

$$G1 = \exp(H1\,\tau) - 1\,, \tag{2-35a}$$

$$G2 = \exp(H2\,\tau) - 1\,, \tag{2-35b}$$

$$G3 = \exp(H4\,\tau) - 1\,, \tag{2-35c}$$

$$G4 = \exp(-H2\,\tau_0) - \exp(-H2\,\tau)\,, \tag{2-35d}$$

$$G5 = \exp(-H1\,\tau_0) - \exp(-H1\,\tau)\,, \tag{2-35e}$$

$$G6 = \exp(-H3\,\tau_0) - \exp(-H3\,\tau)\,, \tag{2-35f}$$

$$G7 = (1/3)\,(1 - \frac{\omega}{3}\beta_1)^{-1}\,, \tag{2-35g}$$

$$H1 = 1/\mu + A\,, \tag{2-35h}$$

$$H2 = 1/\mu - A\,, \tag{2-35i}$$

$$H3 = 1/\mu + 1/\mu_o\,, \tag{2-35j}$$

$$H4 = 1/\mu - 1/\mu_o\,, \tag{2-35k}$$

and $C_{1\text{-Marshak}}$, $C_{2\text{-Marshak}}$, and A are defined in Eqs. (2-27a), (2-27b), and (2-17).

The Modified P_1 Method with $\omega = 1$. By substituting Eqs. (2-21) and (2-31) into Eqs. (2-33) with the assumption of ϕ_o equal to zero and solving for the diffuse intensity I_d^+ and I_d^-, we get

$$I_d^+(\tau,\mu,\phi) = C_{3\text{-Marshak}} \{ [(\beta_1/3) - 1]\tau + T1\,\mu \} + C_{4\text{-Marshak}}\, T1 + T5\, F_I\, (T2/H4) \times [(-3+\beta_1)\,\mu_o + (\beta_1/\mu)(1-\mu^2)^{1/2}(1-\mu_o^2)^{1/2}\cos\phi] + [1/(4\,\mu)] \times (T2/H4)[1 + \beta_1\mu\,\mu_o - 3\,\mu_o^2 + \beta_1 (1-\mu^2)^{1/2}(1-\mu_o^2)^{1/2}\cos\phi]\, F_I \qquad \text{(2-36a)}$$

and

$$I_d^-(\tau,\mu,\phi) = C_{3\text{-Marshak}} \{ [(-\beta_1/3+1)\,\tau_0 + \mu]\, T3 + (\beta_1/3 - 1)\,\tau - \mu \} + C_{4\text{-Marshak}}\,(-T3+1) + (T5/H3)\, F_I\, [-T4 + \exp(-\tau/\mu_o)] \times [(3-\beta_1)\,\mu_o + (\beta_1/\mu)(1-\mu^2)^{1/2}(1-\mu_o^2)^{1/2}\cos\phi] + (1/4) \times \{ [-T4 + \exp(-\tau/\mu_o)]/(\mu\, H3) \} [1 - \beta_1\,\mu\,\mu_o - 3\,\mu_o^2 + \beta_1 (1-\mu^2)^{1/2}(1-\mu_o^2)^{1/2}\cos\phi]\, F_I, \qquad \text{(2-36b)}$$

where

$$T1 = 1 - \exp(-\tau/\mu), \qquad \text{(2-37a)}$$

$$T2 = \exp(-\tau/\mu_o) - \exp(-\tau/\mu), \qquad \text{(2-37b)}$$

$$T3 = \exp[(-\tau_0 + \tau)/\mu], \qquad \text{(2-37c)}$$

$$T4 = \exp(-\tau_0/\mu - \tau_0/\mu_o + \tau/\mu), \qquad \text{(2-37d)}$$

$$T5 = (\beta_1/12)/(1-\beta_1/3), \qquad \text{(2-37e)}$$

and $C_{3\text{-Marshak}}$, $C_{4\text{-Marshak}}$, H3, and H4 are defined in Eqs. (2-29a), (2-29b), (2-35j) and (2-35k).

Summary of Equations for the Diffuse Intensity I_d

So far, all required equations that allow the numerical calculation of the diffuse intensity I_d have been derived. The assumptions made are: no incident radiation entering from the lower boundary, no index of refraction effects at either boundary, the particles are randomly oriented and have at least one plane of symmetry [2-22], diffusion approximation, and ϕ_o is equal to zero. As mentioned earlier, these equations for diffuse intensity I_d can also be used to get the numerical solution for V_d by simply changing I_d, β_0, β_1, and F_I, to V_d, δ_0, δ_1, and F_V in these equations, respectively. These equations for diffuse intensity I_d are now summarized as follows.

For the classical P_1 approximation, the general equation of the diffuse intensity I_d is in Eq. (2-18) for ω less than one with the coefficients in Eqs. (2-23) for Mark's and Eqs. (2-27) for Marshak's boundary conditions; and in Eq. (2-21) for ω equal to one with the coefficients in Eqs. (2-25) for Mark's and Eqs. (2-29) for Marshak's boundary conditions.

For the modified P_1 method, the diffuse intensity I_d in the backward and forward directions for ω less than one are in Eqs. (2-34b) and (2-34a), respectively; and for ω equal to one are in Eqs. (2-36b) and (2-36a), respectively.

Numerical Results

Some selected figures are presented in this section for the comparison between the analytical results derived herein for the P_1 (classical and modified) approximations and for the scalar reduction of the polarized diffusion problem – both results presented herein being exact numerical solutions after taking account of their respective analytical assumptions and approximations. The examination of Eq. (2-6) suggests that the exact scalar results [2-29] are also the exact solution for the polarized light diffusion problem due to the reason that I_d and V_d are two independent equations which simplify the polarized light diffusion problem to a scalar problem. For these figures, the reflected and transmitted diffuse intensities are calculated only at the boundaries. The reflected diffuse intensity represents only the intensity reflected from the medium due to scattering within the medium; while the transmitted diffuse intensity excludes any part of the incident intensity that directly reaches the lower interface undisturbed. For normalization reasons, both diffuse intensities are divided by the incident polarized radiation, πF_I for I_d and πF_V for V_d. In addition, the numerical data for all of these figures is tabulated in Appendix II.

Due to this work being applied to the diffusion case, only optical thicknesses of 5 and above will be addressed. It is felt that the diffusion approximation along with the modified P_1 approximation would not be as accurate for smaller optical thickness. Future plans are to solve this problem exactly and present results for all optical thicknesses.

For the examples presented herein, three different sets of the scattering coefficients, in Eq. (2-4f), are used. The first set of the scattering coefficients, denoted as P and suggested by Vestrucci and Siewert [2-20], is suitable for the polarized light problem for scattering albedo equal to one. With the wave number multiplied by the radius of the small, absorbing spherical particles equal to 0.5 and the index of refraction of the particles with respect to the surrounding medium being 1.33, we have a β_0 of 1, a β_1 of 0.1400343465, a δ_0 of 0.06399215408, a δ_1 of 1.5, and γ_i, α_i, ζ_i, and ε_i are all zero. Although the scattering coefficients P were calculated for ω equal to one, to avoid recalculating these coefficients from the Mie theory, we use these coefficients with other ω values for demonstration purposes. The second and third sets of the scattering coefficients, denoted as F and B, are proper for the scalar radiation problem. F represents a strong forward scattering phase function with β_0 equal to one, β_1 equal to one, and all of the rest of the scattering coefficients are zero; while B represents a strong backward scattering phase function with β_0 equal to one, β_1 equal to negative one, and all of the rest of the scattering coefficients are zero. The phase functions F and B are appropriate only for the scalar problem and used for testing the critical conditions for our classical and modified P_1 approximations as well as making the comparison with the first Stokes parameter of the polarized light problem.

On the figure legends, SC refers to the scattering coefficients, MP1 refers to the modified P_1 solution, and EX refers to the exact scalar results.

Figures 2-2 to 2-16 on the following pages are based on the incident polarized radiation normal to the upper boundary with $\mathbb{F} = [1 \quad 0 \quad 0 \quad 1]^T$, which is a circularly polarized incident light.

Figures 2-2 to 2-7 give the comparison between the classical P_1 approximation with Mark's boundary conditions, the classical P_1 approximation with Marshak's boundary conditions, the modified P_1 method with Marshak's boundary conditions, and the exact solution with a scattering albedo (ω) of one and an optical thickness (τ_0) of 5. Figures 2-2, 2-4, and 2-6 represent the reflected diffuse intensity with the scattering coefficients being B, F, and P, respectively; while Figs. 2-3, 2-5, and 2-7 represent the transmitted diffuse intensity with the scattering coefficients being B, F, and P, respectively.

Figures 2-2, 2-4, and 2-6 show that the classical P_1 approximation with both boundary conditions predicts the solution for the reflected diffuse intensity poorly, while the modified P_1 method predicts the results for the reflected diffuse intensity qualitatively well. However, Figs 2-3, 2-5, and 2-7 show that the classical P_1 approximation with Mark's boundary conditions cannot estimate the solution for the transmitted diffuse intensity well, while both the classical P_1 approximation with Marshak's boundary conditions and the modified P_1 method provide good estimates of the results for the transmitted diffuse intensity. Therefore, from this point in the paper, we will only consider the comparison between the results of the modified P_1 method and the exact solution.

The scattering coefficients of B, F, and P provide similar trends when comparing the modified P_1 results and the exact solution, as shown in Fig. 2-8 for the reflected diffuse intensity and in Fig. 2-9 for the transmitted diffuse intensity with ω equal to one and τ_0 equal to 5. Thus, the specific set of scattering coefficients appears less important for the predictions by the modified P_1 method.

Furthermore, as expected, the reflected diffuse intensity for B (strong backward) is larger than that for F (strong forward) in Fig. 2-8, while the transmitted diffuse intensity for B is smaller than that for F in Fig. 2-9. This is due to the fact that the backward scattering phase function tends to scatter toward the upper boundary and scatter away from the lower boundary, while the forward scattering phase function has an opposite tendency. We can also see that the results for P (for polarized light) are right between those of B and F in both Figs. 2-8 and 2-9, due to the fact that P represents a slightly forward scattering phase matrix.

Figures 2-10 to 2-15 not only give the comparison between the modified P_1 method with Marshak's boundary conditions and the exact solution with the exit angle (μ) equal to 0.9 but also demonstrate the effects of scattering albedo (ω) and optical thickness (τ_0). Figures 2-10, 2-12, and 2-14 present the reflected diffuse intensity with the scattering coefficients being B, F, and P, respectively; while Figs. 2-11, 2-13, and 2-15 present the transmitted diffuse intensity with the scattering coefficients being B, F, and P, respectively.

As expected, the predictions of the modified P_1 method for the reflected diffuse intensity, in Figs. 2-10, 2-12, and 2-14, agree qualitatively with those of the exact solution; while the predictions of the modified P_1 method for the transmitted diffuse intensity, in Figs. 2-11, 2-13, and 2-15, yield very good comparisons. For these three figures, note that the close proximity of the logarithmic curves of the modified P_1 to the exact solution curves demonstrates the high accuracy of the modified P_1 solution for transmission.

Moreover, the reflected and transmitted diffuse intensities increase as the single scattering albedo increases in Figs. 2-10 to 2-15. The reason is that a larger percentage of radiation will be scattered in the medium when the single scattering albedo becomes greater. This effect will contribute to the diffuse intensities at both boundaries.

In addition, the reflected diffuse intensity increases as optical thickness increases, as revealed by Figs. 2-10, 2-12, and 2-14. The reason is due to the reduced chances of scattering outside through the lower boundary when optical thickness becomes larger, with improved chances of scattering to the outside of the medium through the upper boundary. On other hand, the transmitted diffuse intensity decreases as optical thickness increases, as revealed by Figs. 2-11, 2-13, and 2-15. This is due to the fact that the increased number of scattering events causes more intensity to scatter out of the medium through the upper boundary than through the lower boundary.

Finally, Fig. 2-16 provides the state of polarization for both the reflected and transmitted diffuse intensity vectors by using the modified P_1 method with scattering coefficients = P, ω equal to one and τ_0 equal to 5. Figure 2-16 shows that both the reflected and transmitted diffuse intensity vectors are circularly polarized light, while the incident radiation is also circularly polarized light.

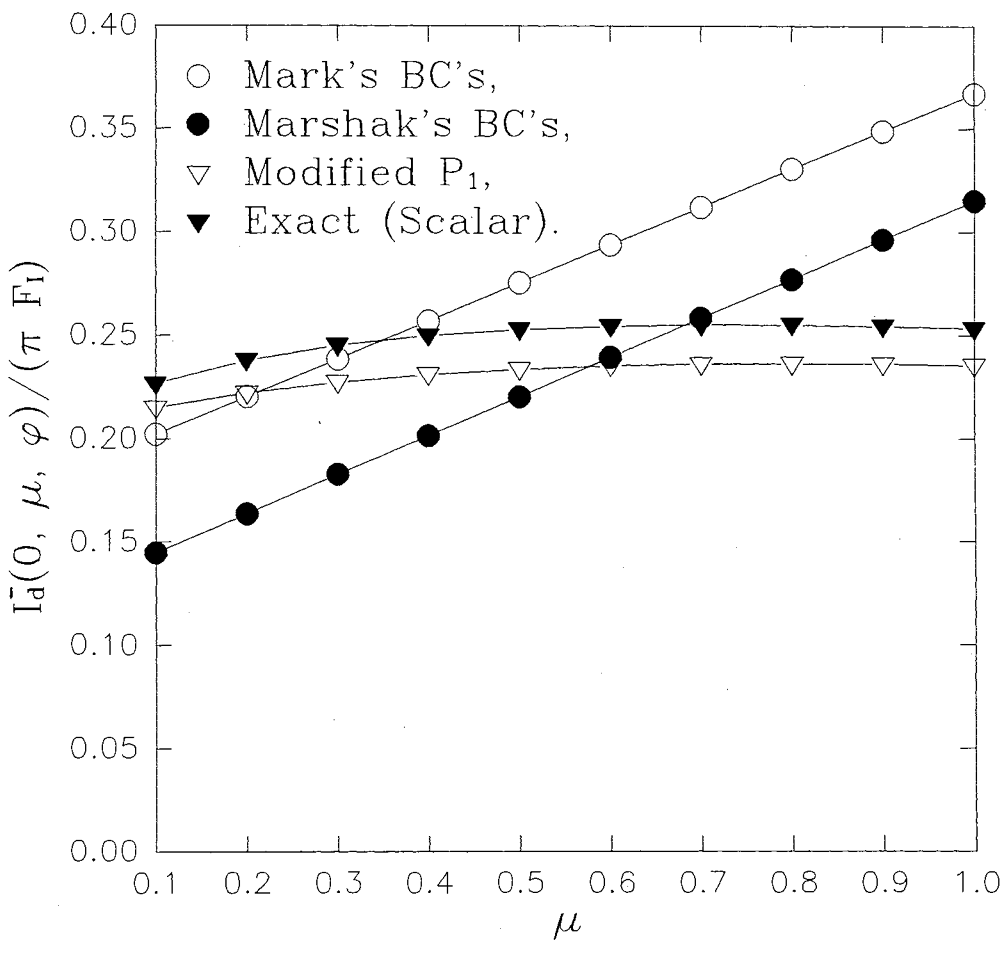

Figure 2-2. Comparison of the Non-Dimensional Reflected Diffuse Intensity among Various P_1 Approximations and Exact Solutions for Scattering Coefficients B ($\mu_o = 1$, $\omega = 1$, $\tau_0 = 5$)

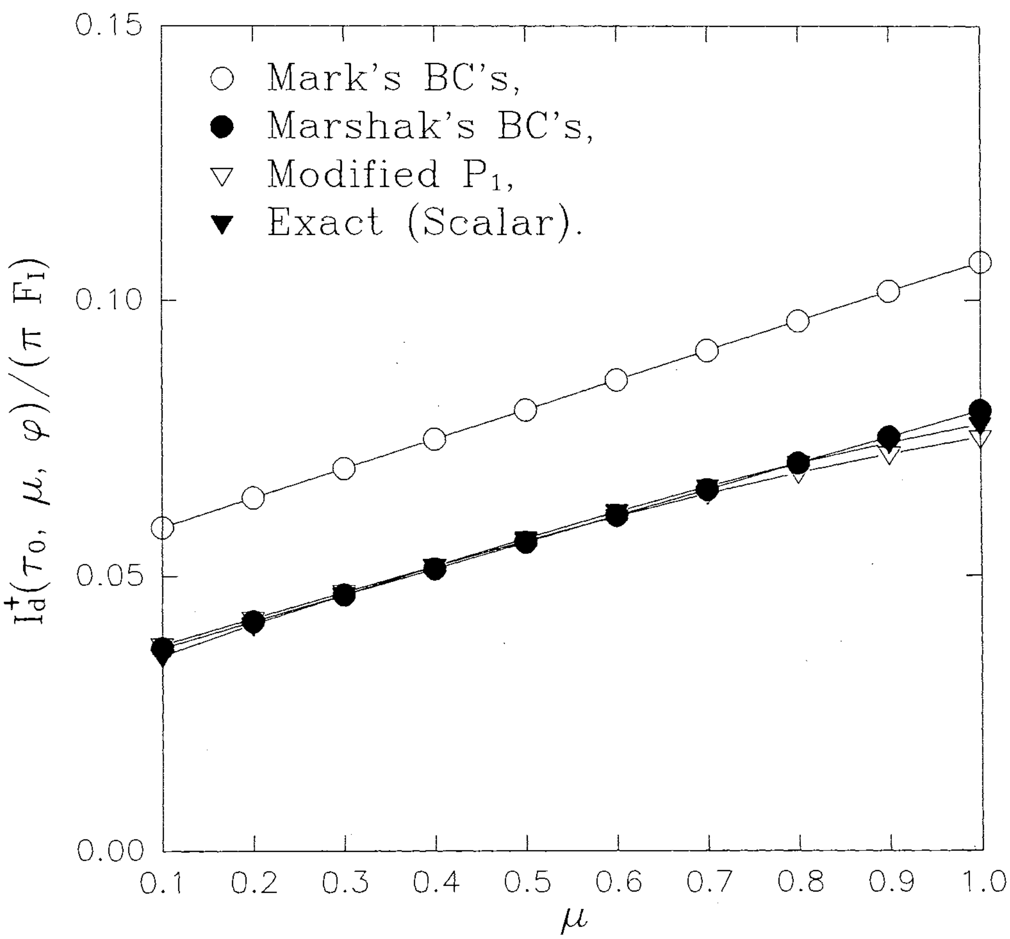

Figure 2-3. Comparison of the Non-Dimensional Transmitted Diffuse Intensity among Various P_1 Approximations and Exact Solutions for Scattering Coefficients B ($\mu_o = 1$, $\omega = 1$, $\tau_0 = 5$)

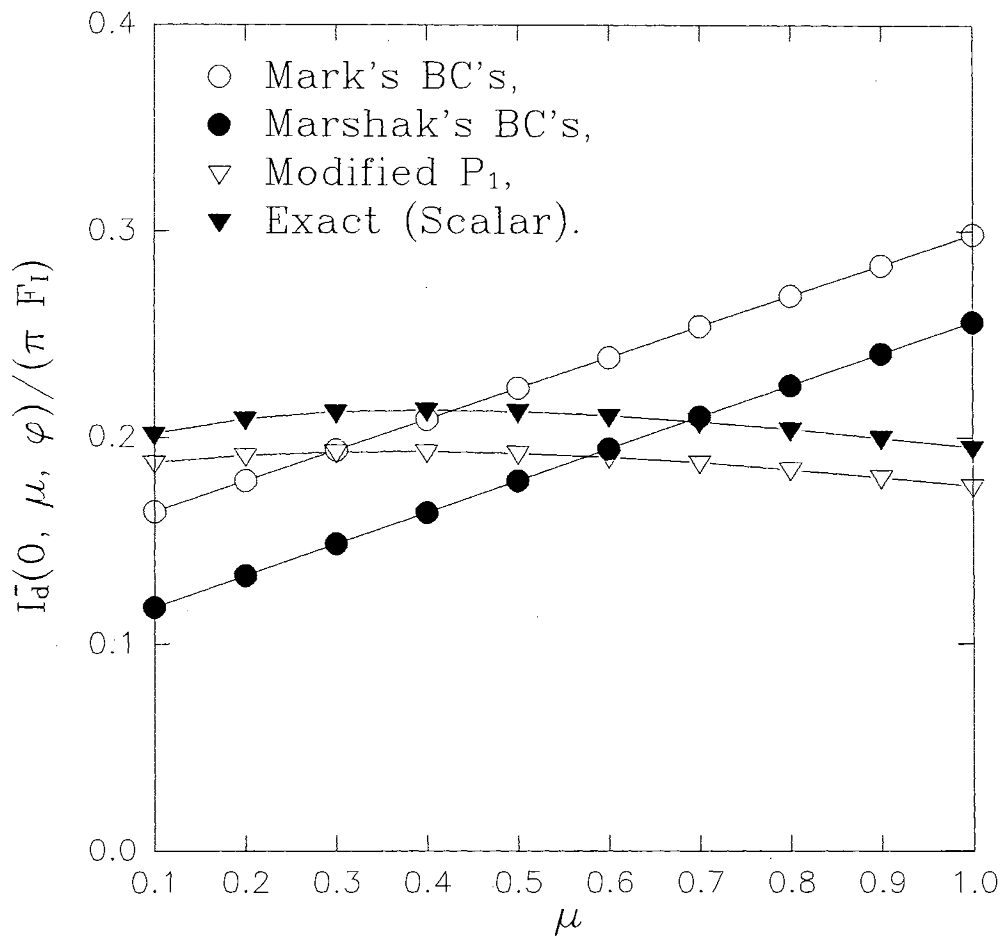

Figure 2-4. Comparison of the Non-Dimensional Reflected Diffuse Intensity among Various P_1 Approximations and Exact Solutions for Scattering Coefficients F ($\mu_o = 1$, $\omega = 1$, $\tau_0 = 5$)

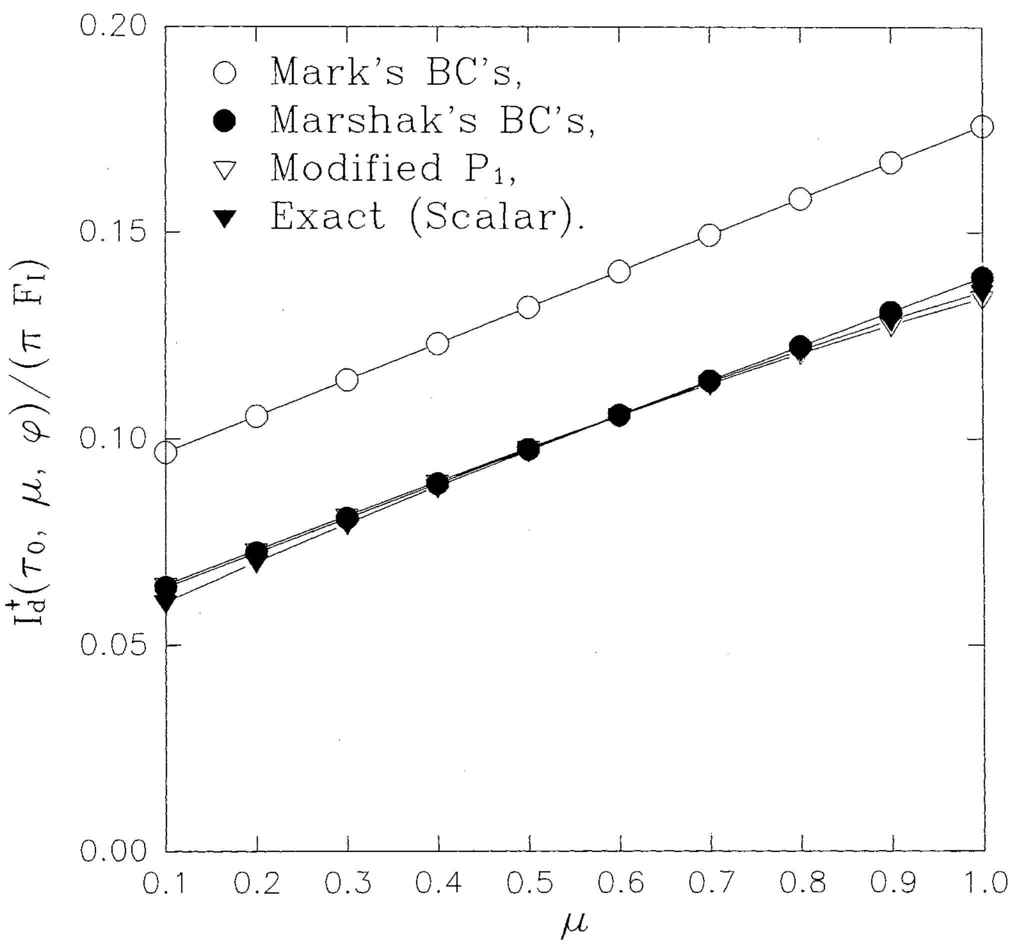

Figure 2-5. Comparison of the Non-Dimensional Transmitted Diffuse Intensity among Various P_1 Approximations and Exact Solutions for Scattering Coefficients F ($\mu_o = 1$, $\omega = 1$, $\tau_0 = 5$)

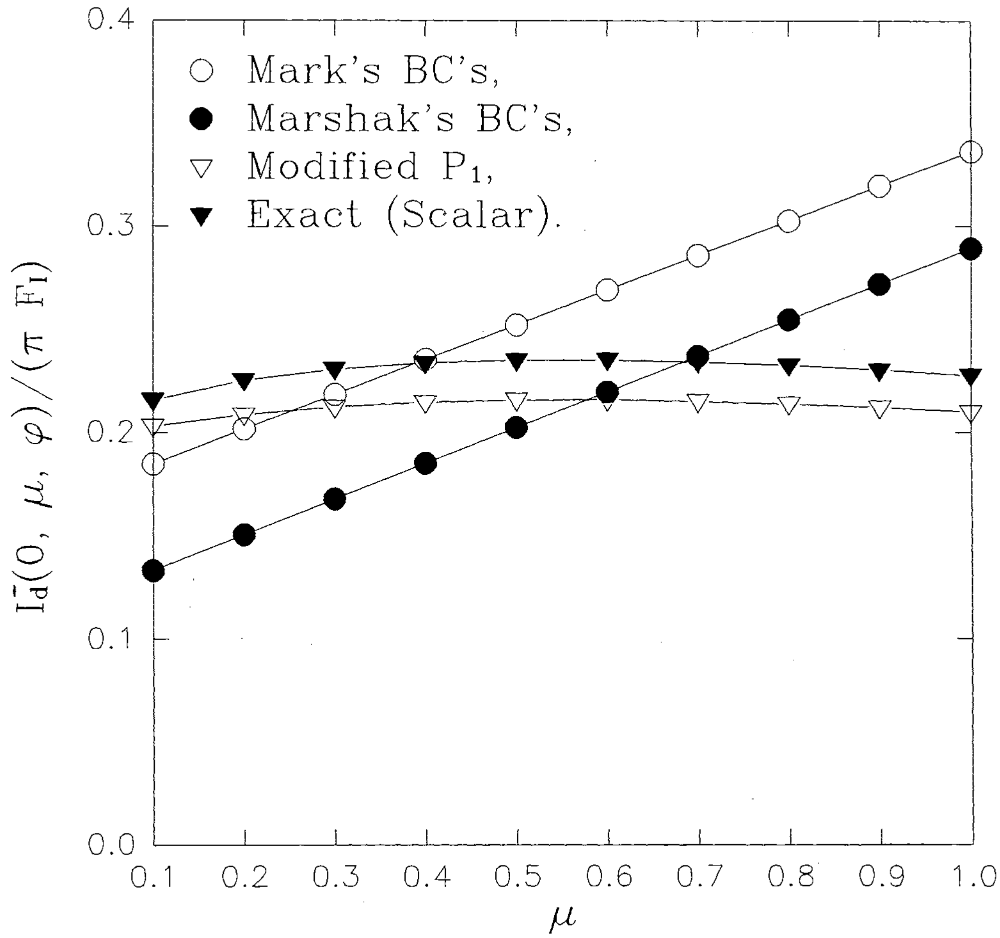

Figure 2-6. Comparison of the Non-Dimensional Reflected Diffuse Intensity among Various P_1 Approximations and Exact Solutions for Scattering Coefficients P ($\mu_o = 1$, $\omega = 1$, $\tau_0 = 5$)

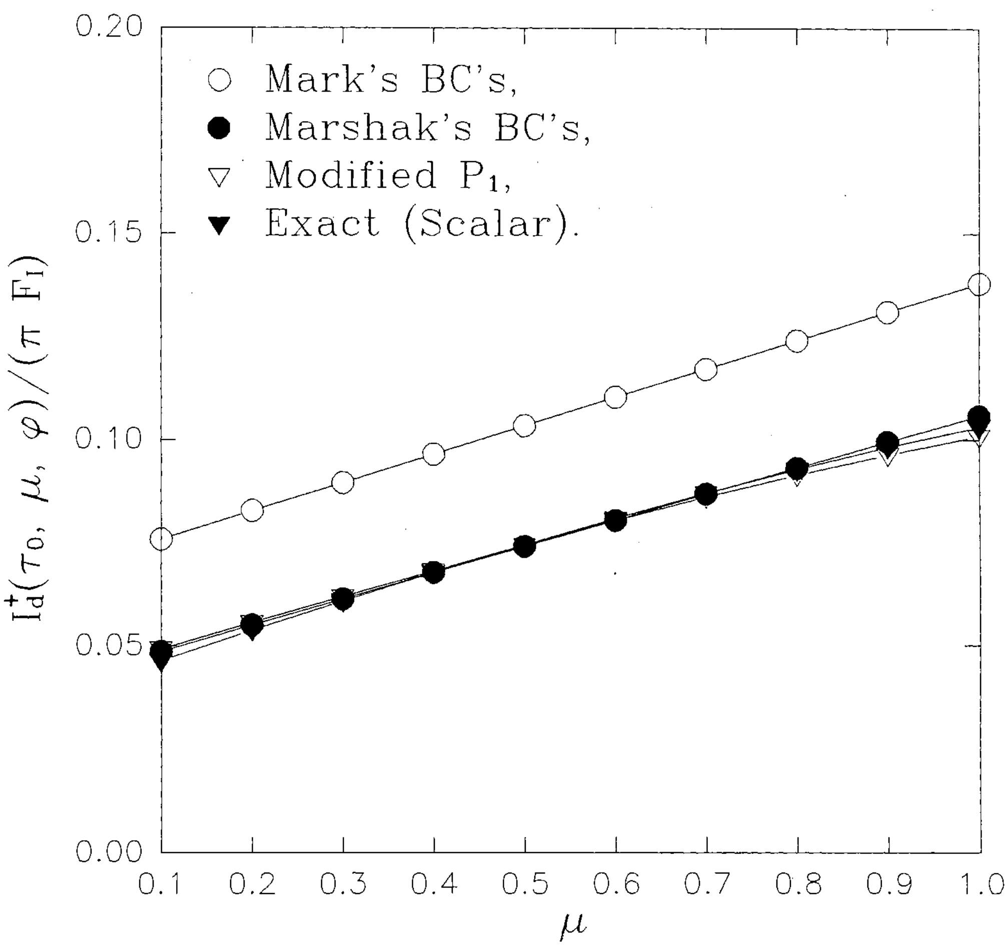

Figure 2-7. Comparison of the Non-Dimensional Transmitted Diffuse Intensity among Various P_1 Approximations and Exact Solutions for Scattering Coefficients P ($\mu_o = 1$, $\omega = 1$, $\tau_0 = 5$)

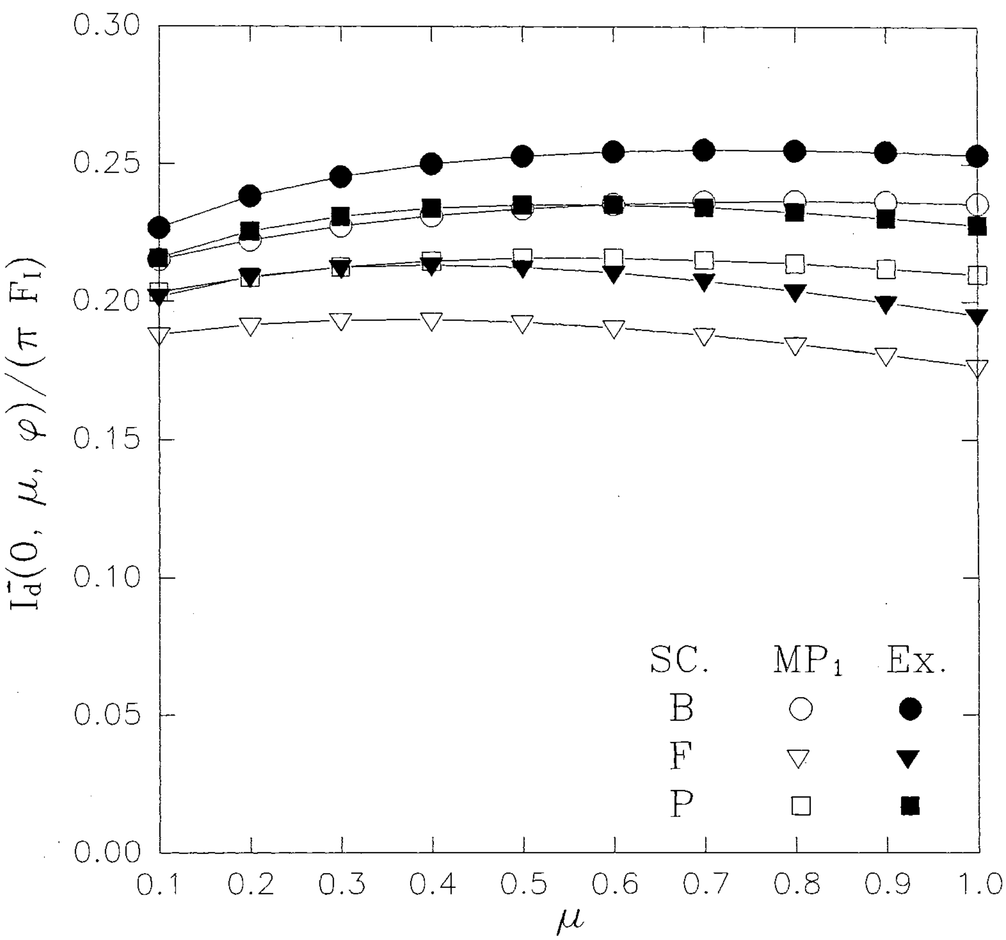

Figure 2-8. Comparison of the Non-Dimensional Reflected Diffuse Intensity among the Modified P_1 Method and the Exact Solutions for Scattering Coefficients B, F, and P ($\mu_o = 1$, $\omega = 1$, $\tau_0 = 5$)

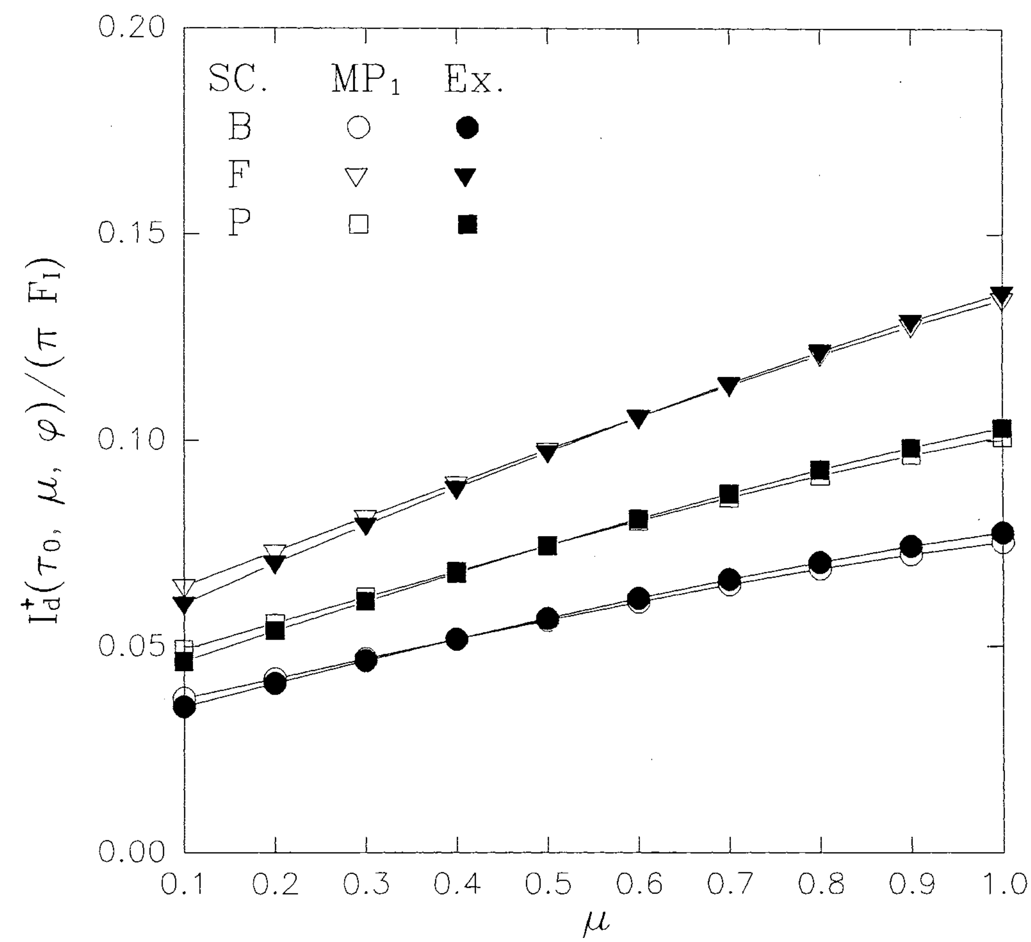

Figure 2-9. Comparison of the Non-Dimensional Transmitted Diffuse Intensity among the Modified P_1 Method and the Exact Solutions for Scattering Coefficients B, F, and P ($\mu_o = 1$, $\omega = 1$, $\tau_0 = 5$)

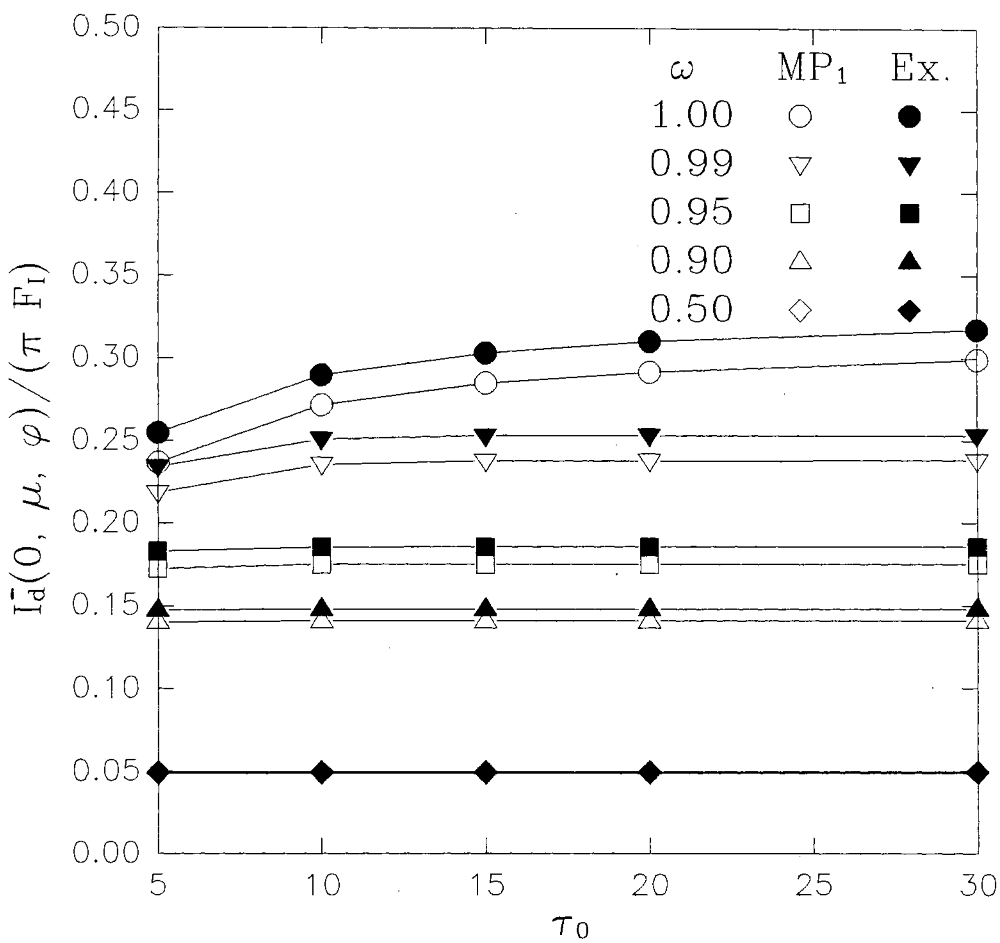

Figure 2-10. Comparison of the Non-Dimensional Reflected Diffuse Intensity between the Modified P_1 Method and the Exact Solution for Scattering Coefficients B ($\mu_o = 1$, $\mu = 0.9$)

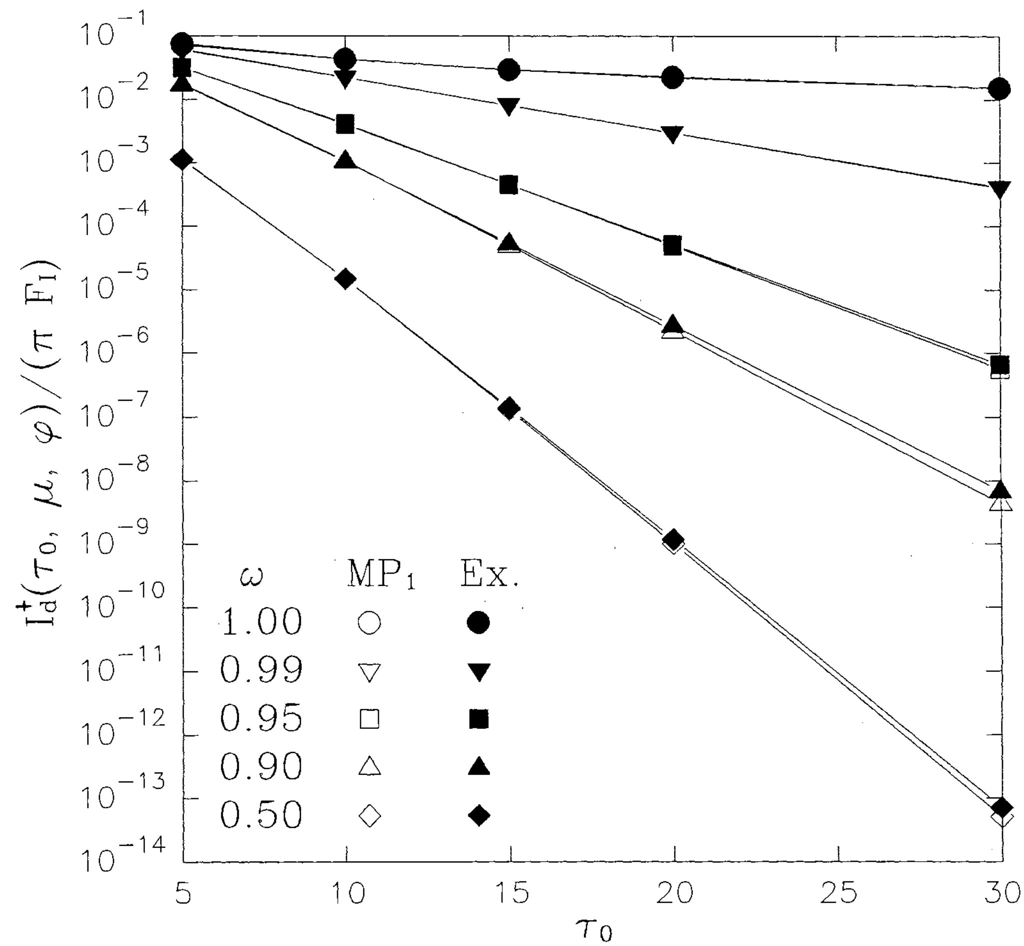

Figure 2-11. Comparison of the Non-Dimensional Transmitted Diffuse Intensity between the Modified P_1 Method and the Exact Solution for Scattering Coefficients B ($\mu_o = 1$, $\mu = 0.9$)

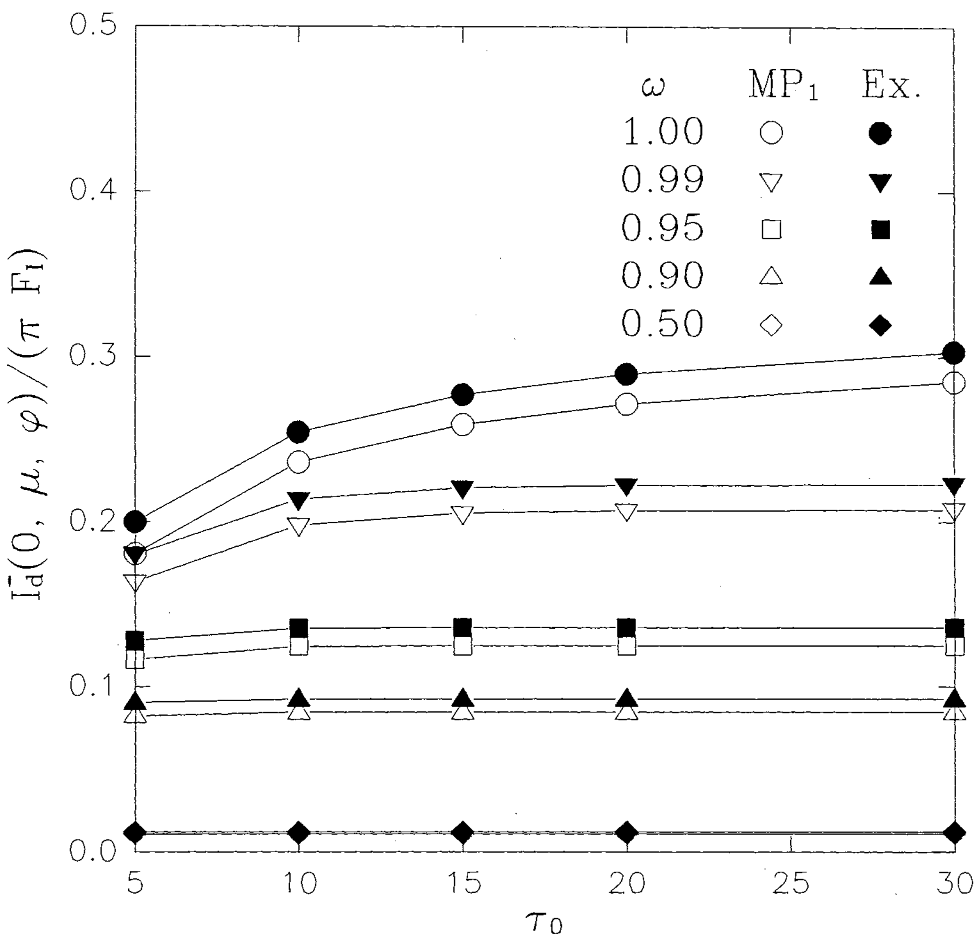

Figure 2-12. Comparison of the Non-Dimensional Reflected Diffuse Intensity between the Modified P_1 Method and the Exact Solution for Scattering Coefficients F ($\mu_o = 1$, $\mu = 0.9$)

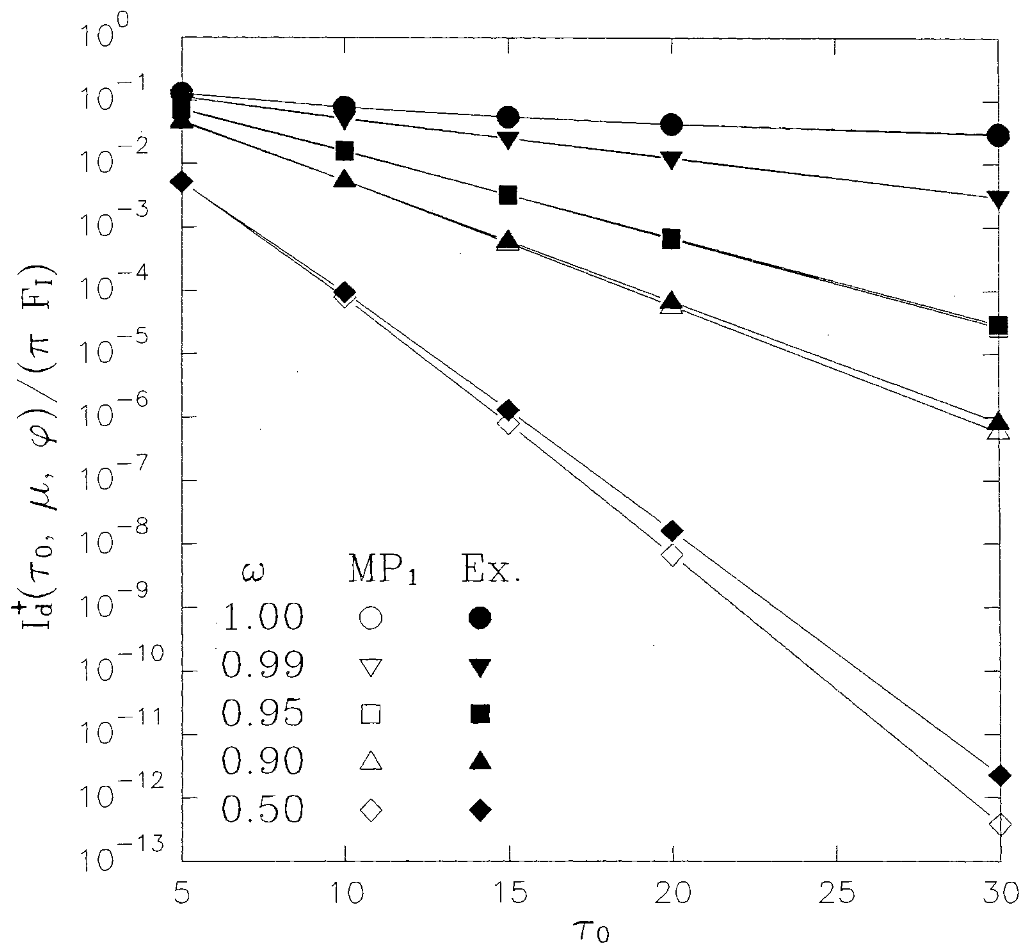

Figure 2-13. Comparison of the Non-Dimensional Transmitted Diffuse Intensity between the Modified P_1 Method and the Exact Solution for Scattering Coefficients F ($\mu_o = 1$, $\mu = 0.9$)

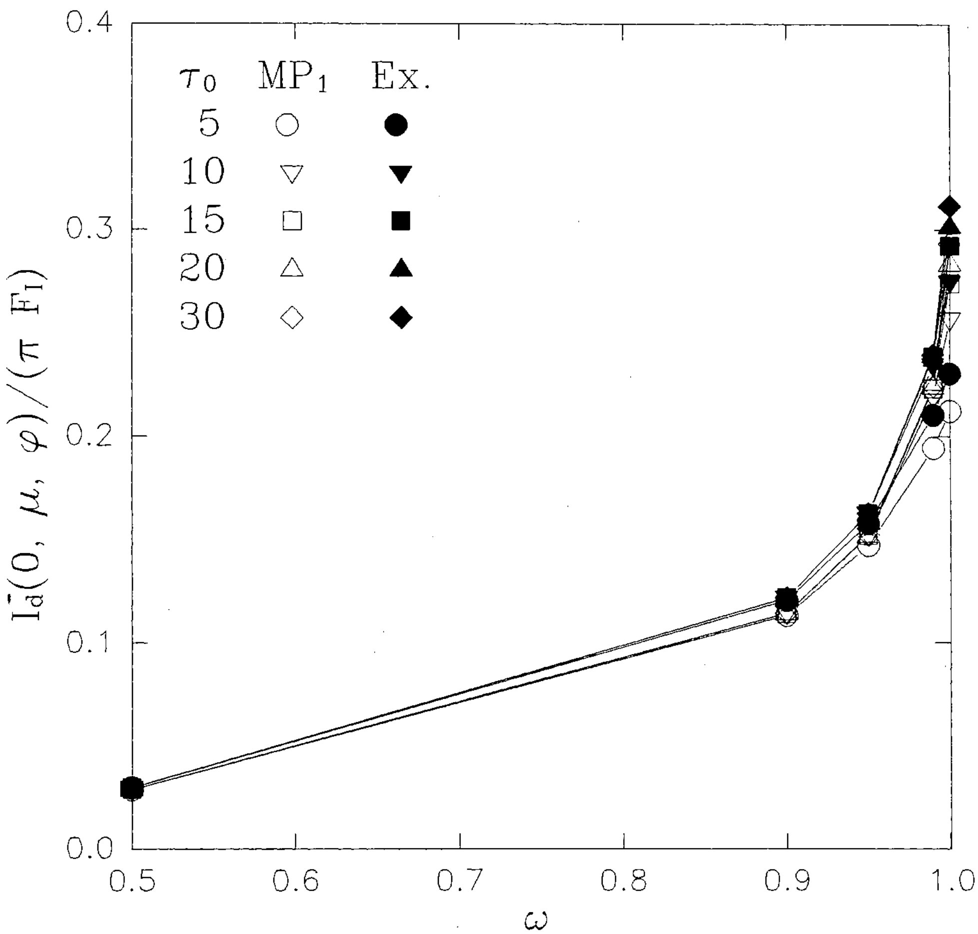

Figure 2-14. Comparison of the Non-Dimensional Reflected Diffuse Intensity between the Modified P_1 Method and the Exact Solution for Scattering Coefficients P ($\mu_o = 1$, $\mu = 0.9$)

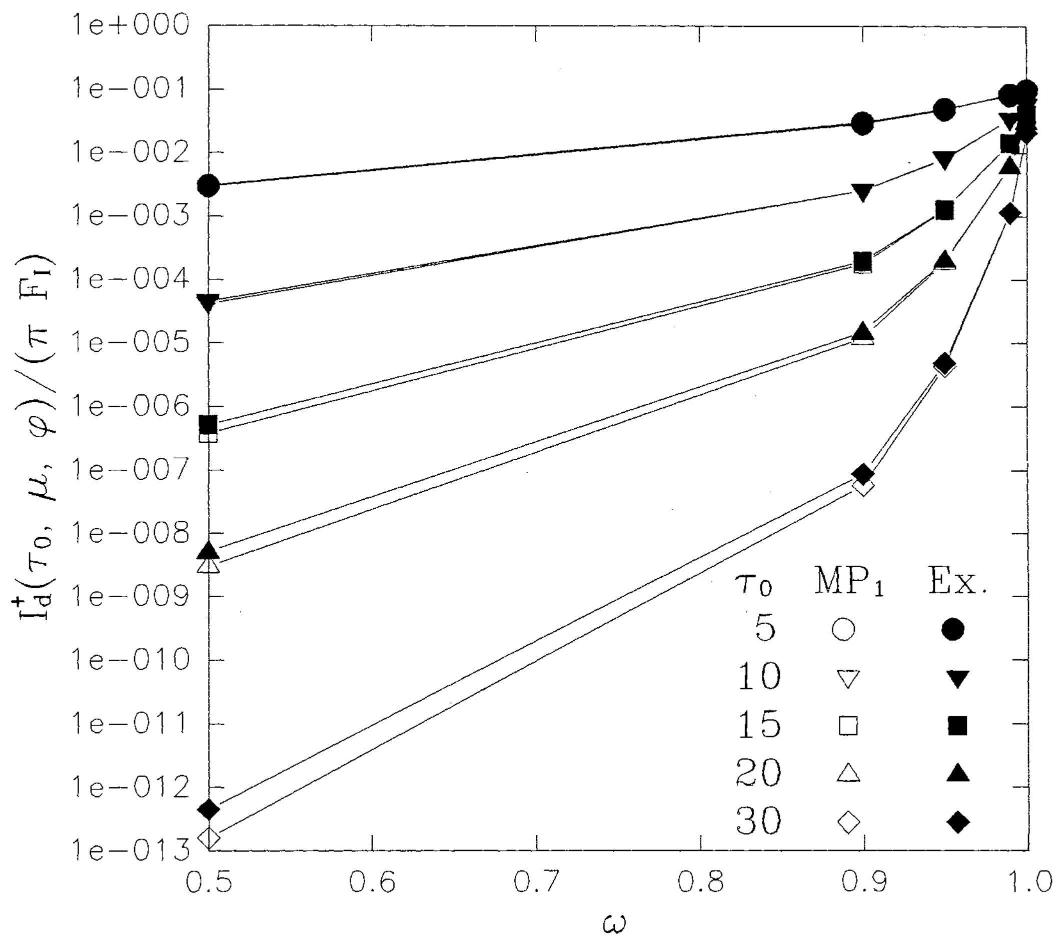

Figure 2-15. Comparison of the Non-Dimensional Transmitted Diffuse Intensity between the Modified P_1 Method and the Exact Solution for Scattering Coefficients P ($\mu_o = 1$, $\mu = 0.9$)

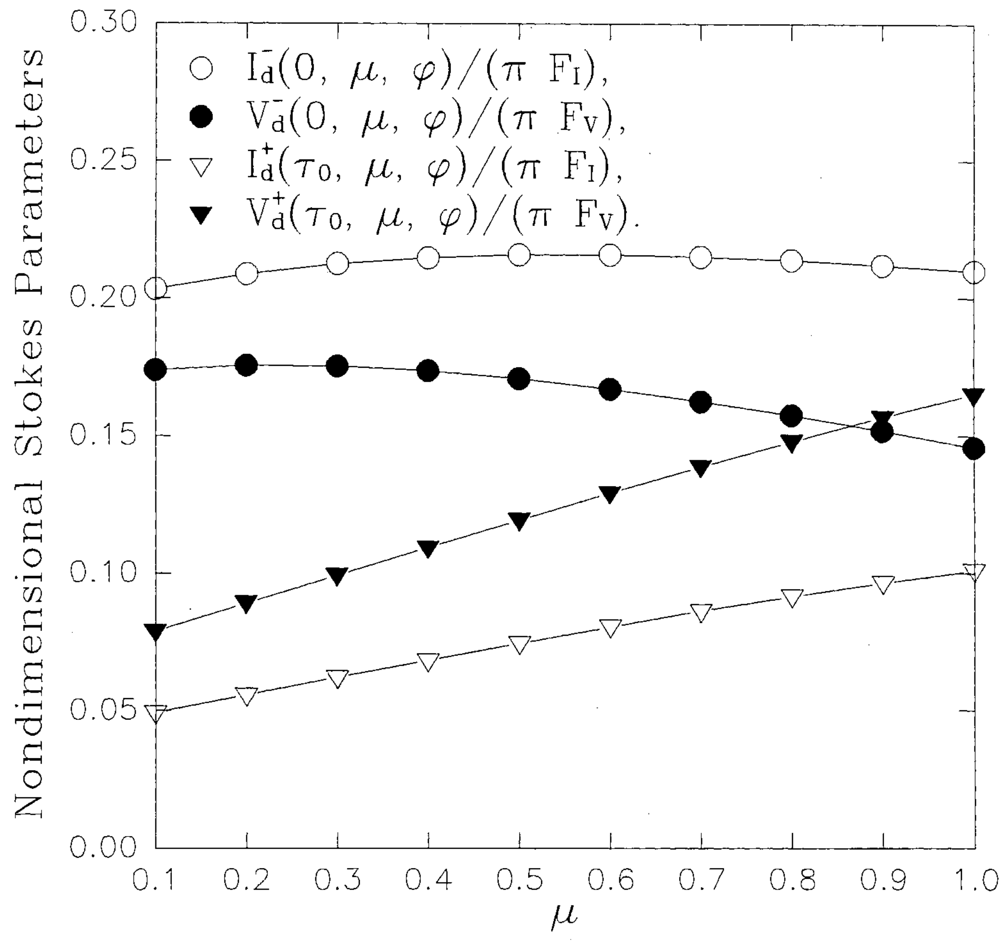

Figure 2-16. Non-Dimensional Reflected and Transmitted Stokes Parameters for the Modified P_1 Method with Scattering Coefficients P ($\mu_o = 1$, $\omega = 1$, $\tau_0 = 5$)

Conclusions

The expression for the diffuse intensity I_d for the diffusion approximation was derived by using the classical P_1 approximation with both Mark's and Marshak's boundary conditions as well as the modified P_1 method for polarized light in a plane-parallel medium without reflective boundaries. The work was done with the assumptions being: collimated polarized incident radiation at angle θ_o exists at the top boundary and is a sheet of laser-like beams; no incident radiation enters from the lower boundary; scattering and absorption but no emission occur in the medium; no index of refraction effects exist at the boundaries; the particles are randomly oriented and have at least one plane of symmetry; diffusion approximation; and the azimuthal angle of the incident radiation is equal to zero. As discussed earlier, the expression for V_d (the fourth Stokes parameter) can be easily obtained by changing I_d, β_0, β_1, and F_1 from the expression of the diffuse intensity I_d, to V_d, δ_0, δ_1, and F_V, respectively.

Some selected numerical results are included in order to not only make the comparison between the various P_1 approximations and the exact scalar results, but also to observe the effects of the albedo (0.5, 0.9, 0.95, 0.99, and 1), and optical thickness (5, 10, 15, 20, and 30). A qualitatively good agreement is found between the results from the modified P_1 method and the exact scalar solutions. However, as might be expected, the predictions from the classical P_1 approximation are poor.

Near term future work will be directed toward getting the exact numerical solution for the current polarized light problem. Moreover, it is planned that long term research will include refractive index effects in order to compare with experimental data in the future.

Addendum to Conclusions

This paper was submitted to the National Heat Transfer Conference in August of 1999 before completing the exact numerical solution with the polarization fully included. The exact numerical results (near term future work of the Conclusions section) for polarization will be presented in the next chapter.

Acknowledgements

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Appendix II

Tables 2-a2 through 2-a16 on the following pages give the numerical results used to plot Figs. 2-2 to 2-16.

TABLE 2-a2

DATA FOR FIG. 2-2

	Mark's BC's	Marshak's BC's	Modified P_1	Exact
μ	$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^-(0, \mu, \phi)/(\pi F_I)$
1	0.367280628	0.315105855	0.235639942	0.253349505
0.9	0.348916596	0.296199504	0.236351901	0.254488841
0.8	0.330552565	0.277293153	0.236633123	0.255162706
0.7	0.312188534	0.258386801	0.236387355	0.255246009
0.6	0.293824502	0.23948045	0.235500813	0.254578502
0.5	0.275460471	0.220574099	0.233836643	0.252945917
0.4	0.257096439	0.201667747	0.231225045	0.250043347
0.3	0.238732408	0.182761396	0.227447205	0.245399974
0.2	0.220368377	0.163855045	0.222211856	0.238202391
0.1	0.202004345	0.144948693	0.215121553	0.226728281

TABLE 2-a3

DATA FOR FIG. 2-3

	Mark's BC's	Marshak's BC's	Modified P_1
μ	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$
1	0.106967056	0.080100547	0.07516056
0.9	0.101618703	0.075294515	0.072196814
0.8	0.09627035	0.070488482	0.068777332
0.7	0.090921997	0.065682449	0.064945953
0.6	0.085573645	0.060876416	0.060769974
0.5	0.080225292	0.056070383	0.056335241
0.4	0.074876939	0.05126435	0.051728602
0.3	0.069528586	0.046458317	0.047017477
0.2	0.064180233	0.041652285	0.042242092
0.1	0.058831881	0.036846252	0.037424144

Exact
$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$
0.077549888
0.074223646
0.070428306
0.066209278
0.061636531
0.056793229
0.051752294
0.046547659
0.041151199
0.035423618

TABLE 2-a4

DATA FOR FIG. 2-4

	Mark's BC's	Marshak's BC's	Modified P_1
μ	$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^-(0, \mu, \phi)/(\pi F_I)$
1	0.29841551	0.255976219	0.176589987
0.9	0.283494734	0.240617646	0.180802045
0.8	0.268573959	0.225259073	0.184608175
0.7	0.253653183	0.209900499	0.187901988
0.6	0.238732408	0.194541926	0.190561721
0.5	0.223811632	0.179183353	0.192445629
0.4	0.208890857	0.16382478	0.193382042
0.3	0.193970081	0.148466207	0.193152015
0.2	0.179049306	0.133107634	0.191464445
0.1	0.16412853	0.117749061	0.18792192

Exact
$I_d^-(0, \mu, \phi)/(\pi F_I)$
0.195099944
0.199756168
0.203982175
0.207646652
0.210587066
0.212592784
0.213370387
0.212471536
0.209126766
0.201722438

TABLE 2-a5

DATA FOR FIG. 2-5

	Mark's BC's	Marshak's BC's	Modified P_1	Exact
μ	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$
1	0.175832173	0.139230184	0.13420703	0.13579945
0.9	0.167040565	0.130876373	0.127746669	0.128956318
0.8	0.158248956	0.122522562	0.120802281	0.121608837
0.7	0.149457347	0.114168751	0.11343132	0.113808636
0.6	0.140665739	0.10581494	0.105709067	0.105627968
0.5	0.13187413	0.097461129	0.097726255	0.097146362
0.4	0.123082521	0.089107318	0.089571605	0.088425253
0.3	0.114290913	0.080753507	0.081312667	0.079476096
0.2	0.105499304	0.072399696	0.072989503	0.070226824
0.1	0.096707695	0.064045885	0.064623777	0.060429461

TABLE 2-a6

DATA FOR FIG. 2-6

	Mark's BC's	Marshak's BC's	Modified P_1
μ	$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^-(0, \mu, \phi)/(\pi F_I)$
1	0.336341016	0.289320438	0.209889272
0.9	0.319523966	0.271961211	0.212127564
0.8	0.302706915	0.254601985	0.213945938
0.7	0.285889864	0.237242759	0.21524372
0.6	0.269072813	0.219883533	0.215903648
0.5	0.252255762	0.202524306	0.215786733
0.4	0.235438711	0.18516508	0.214722362
0.3	0.218621661	0.167805854	0.212491662
0.2	0.20180461	0.150446628	0.208803438
0.1	0.184987559	0.133087401	0.203260261

Exact
$I_d^-(0, \mu, \phi)/(\pi F_I)$
0.227837543
0.230517194
0.232746816
0.23439859
0.235311269
0.235272177
0.233981438
0.230978079
0.225467939
0.215776299

TABLE 2-a7

DATA FOR FIG. 2-7

	Mark's BC's	Marshak's BC's	Modified P_1
μ	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$
1	0.137906667	0.105885965	0.10090971
0.9	0.131011334	0.099532807	0.09642115
0.8	0.124116	0.093179649	0.091464518
0.7	0.117220667	0.086826491	0.086089588
0.6	0.110325334	0.080473333	0.08036714
0.5	0.10343	0.074120176	0.07438515
0.4	0.096534667	0.067767018	0.068231285
0.3	0.089639334	0.06141386	0.06197302
0.2	0.082744	0.055060702	0.055650509
0.1	0.075848667	0.048707544	0.049285436

Exact
$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$
0.103061851
0.098195292
0.092844196
0.087056697
0.080903764
0.074466969
0.067814203
0.060969554
0.053885651
0.0463756

TABLE 2-a8

DATA FOR FIG. 2-8

	Modified P_1	Exact	Modified P_1
	B(Backward)	B(Backward)	F(Forward)
μ	$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^-(0, \mu, \phi)/(\pi F_I)$
1	0.235639942	0.253349505	0.176589987
0.9	0.236351901	0.254488841	0.180802045
0.8	0.236633123	0.255162706	0.184608175
0.7	0.236387355	0.255246009	0.187901988
0.6	0.235500813	0.254578502	0.190561721
0.5	0.233836643	0.252945917	0.192445629
0.4	0.231225045	0.250043347	0.193382042
0.3	0.227447205	0.245399974	0.193152015
0.2	0.222211856	0.238202391	0.191464445
0.1	0.215121553	0.226728281	0.18792192

Exact	Modified P_1	Exact
F(Forward)	P(Polarized)	P(Polarized)
$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^-(0, \mu, \phi)/(\pi F_I)$
0.195099944	0.209889272	0.227837543
0.199756168	0.212127564	0.230517194
0.203982175	0.213945938	0.232746816
0.207646652	0.21524372	0.23439859
0.210587066	0.215903648	0.235311269
0.212592784	0.215786733	0.235272177
0.213370387	0.214722362	0.233981438
0.212471536	0.212491662	0.230978079
0.209126766	0.208803438	0.225467939
0.201722438	0.203260261	0.215776299

TABLE 2-a9

DATA FOR FIG. 2-9

	Modified P_1	Exact	Modified P_1
	B(Backward)	B(Backward)	F(Forward)
μ	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$
1	0.07516056	0.077549888	0.13420703
0.9	0.072196814	0.074223646	0.127746669
0.8	0.068777332	0.070428306	0.120802281
0.7	0.064945953	0.066209278	0.11343132
0.6	0.060769974	0.061636531	0.105709067
0.5	0.056335241	0.056793229	0.097726255
0.4	0.051728602	0.051752294	0.089571605
0.3	0.047017477	0.046547659	0.081312667
0.2	0.042242092	0.041151199	0.072989503
0.1	0.037424144	0.035423618	0.064623777

Exact	Modified P_1	Exact
F(Forward)	P(Polarized)	P(Polarized)
$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$
0.13579945	0.10090971	0.103061851
0.128956318	0.09642115	0.098195292
0.121608837	0.091464518	0.092844196
0.113808636	0.086089588	0.087056697
0.105627968	0.08036714	0.080903764
0.097146362	0.07438515	0.074466969
0.088425253	0.068231285	0.067814203
0.079476096	0.06197302	0.060969554
0.070226824	0.055650509	0.053885651
0.060429461	0.049285436	0.0463756

TABLE 2-a10

DATA FOR FIG. 2-10

	Modified P_1	Exact	Modified P_1
	$\omega = 1$	$\omega = 1$	$\omega = 0.99$
τ_0	$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^-(0, \mu, \phi)/(\pi F_I)$
5	0.236351901	0.254488841	0.218367302
10	0.271620467	0.289726693	0.23547506
15	0.284901739	0.303054595	0.23760447
20	0.291858845	0.310055747	0.237889917
30	0.299040376	0.317297066	0.237933825

Exact	Modified P_1	Exact	Modified P_1
$\omega = 0.99$	$\omega = 0.95$	$\omega = 0.95$	$\omega = 0.9$
$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^-(0, \mu, \phi)/(\pi F_I)$
0.234173161	0.172525911	0.183102541	0.140277302
0.250906971	0.175061193	0.185505322	0.140811307
0.253001686	0.17509172	0.185536428	0.14081244
0.253284761	0.17509208	0.185536829	0.140812442
0.253328698	0.175092084	0.185536834	0.140812442

Exact	Modified P_1	Exact
$\omega = 0.9$	$\omega = 0.5$	$\omega = 0.5$
$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^-(0, \mu, \phi)/(\pi F_I)$
0.147608299	0.048623801	0.049419759
0.148115487	0.048626975	0.049422993
0.148116801	0.048626975	0.049422994
0.148116804	0.048626975	0.049422994
0.148116804	0.048626975	0.049422994

TABLE 2-a11

DATA FOR FIG. 2-11

	Modified P_1	Exact	Modified P_1
	$\omega = 1$	$\omega = 1$	$\omega = 0.99$
τ_0	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$
5	0.072196814	0.074223646	0.059452576
10	0.042450319	0.042797759	0.022112174
15	0.029219456	0.029496618	0.008076833
20	0.022262742	0.02249564	0.002970869
30	0.015081214	0.015254322	0.000402981

Exact	Modified P_1	Exact	Modified P_1
$\omega = 0.99$	$\omega = 0.95$	$\omega = 0.95$	$\omega = 0.9$
$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$
0.060408594	0.031773444	0.031370361	0.017600284
0.021925902	0.004103399	0.003986492	0.001077934
0.008033143	0.000450773	0.000455204	5.01481E-05
0.002966615	4.89043E-05	5.16502E-05	2.22927E-06
0.000405645	5.74552E-07	6.6448E-07	4.33257E-09

Exact	Modified P_1	Exact
$\omega = 0.9$	$\omega = 0.5$	$\omega = 0.5$
$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$
0.017096965	0.001126452	0.001104364
0.001054013	1.47445E-05	1.47298E-05
5.42284E-05	1.30433E-07	1.4021E-07
2.73006E-06	1.00362E-09	1.17692E-09
6.87926E-09	5.07528E-14	7.06948E-14

TABLE 2-a12

DATA FOR FIG. 2-12

	Modified P_1	Exact	Modified P_1
	$\omega = 1$	$\omega = 1$	$\omega = 0.99$
τ_0	$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^-(0, \mu, \phi)/(\pi F_I)$
5	0.180802045	0.199756168	0.163721854
10	0.236202865	0.25442581	0.198016974
15	0.259119519	0.277227547	0.205289739
20	0.271619987	0.289726528	0.20699651
30	0.284901737	0.303054594	0.207506083

Exact	Modified P_1	Exact	Modified P_1
$\omega = 0.99$	$\omega = 0.95$	$\omega = 0.95$	$\omega = 0.9$
$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^-(0, \mu, \phi)/(\pi F_I)$
0.180373995	0.116498384	0.12776406	0.082337698
0.21358354	0.12449799	0.135283025	0.084419155
0.220674288	0.12482548	0.135607871	0.084441275
0.222349176	0.124838809	0.135621958	0.084441502
0.222852714	0.124839374	0.135622596	0.084441505

Exact	Modified P_1	Exact
$\omega = 0.9$	$\omega = 0.5$	$\omega = 0.5$
$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^-(0, \mu, \phi)/(\pi F_I)$
0.090331898	0.010820309	0.011997806
0.092302641	0.010824037	0.012002812
0.092327136	0.010824037	0.012002813
0.092327437	0.010824037	0.012002813
0.092327441	0.010824037	0.012002813

TABLE 2-a13

DATA FOR FIG. 2-13

	Modified P_1	Exact	Modified P_1
	$\omega = 1$	$\omega = 1$	$\omega = 0.99$
τ_0	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$
5	0.127746669	0.128956318	0.112859253
10	0.077867921	0.078098642	0.052159046
15	0.055001677	0.055323666	0.025045619
20	0.0425016	0.042824859	0.012252223
30	0.029219852	0.029496794	0.002962768

Exact	Modified P_1	Exact	Modified P_1
$\omega = 0.99$	$\omega = 0.95$	$\omega = 0.95$	$\omega = 0.9$
$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$
0.112856697	0.073495194	0.071720788	0.047432819
0.05166076	0.015956729	0.015619663	0.005514697
0.024883137	0.003229209	0.003257529	0.000564089
0.012208858	0.000651534	0.00067849	5.70984E-05
0.002969144	2.65166E-05	2.94322E-05	5.8407E-07

Exact	Modified P_1	Exact
$\omega = 0.9$	$\omega = 0.5$	$\omega = 0.5$
$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$
0.045659003	0.00514687	0.005091369
0.005509706	7.89033E-05	9.40803E-05
0.000613429	7.9329E-07	1.2944E-06
6.79708E-05	6.69527E-09	1.60988E-08
8.34048E-07	3.77443E-13	2.23558E-12

TABLE 2-a14

DATA FOR FIG. 2-14

	Modified P_1	Exact	Modified P_1
	$\tau_0 = 5$	$\tau_0 = 5$	$\tau_0 = 10$
ω	$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^-(0, \mu, \phi)/(\pi F_I)$
1	0.212127564	0.230517194	0.256757507
0.99	0.194407149	0.210474628	0.219249066
0.95	0.147086366	0.15782419	0.151809635
0.9	0.113199529	0.120664856	0.114318328
0.5	0.029012701	0.029904921	0.029016511

Exact	Modified P_1	Exact	Modified P_1
$\tau_0 = 10$	$\tau_0 = 15$	$\tau_0 = 15$	$\tau_0 = 20$
$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^-(0, \mu, \phi)/(\pi F_I)$
0.274870285	0.274247582	0.292359361	0.283564539
0.234682543	0.223344711	0.23869787	0.224083935
0.162297387	0.151918261	0.162406967	0.151920727
0.121733345	0.114323883	0.121739696	0.114323909
0.029909454	0.029016511	0.029909454	0.029016511

Exact	Modified P_1	Exact
$\tau_0 = 20$	$\tau_0 = 30$	$\tau_0 = 30$
$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^-(0, \mu, \phi)/(\pi F_I)$
0.301710491	0.293296512	0.311504217
0.239427764	0.224244592	0.239587647
0.16240965	0.151920784	0.162409717
0.121739734	0.114323909	0.121739734
0.029909454	0.029016511	0.029909454

TABLE 2-a15

DATA FOR FIG. 2-15

	Modified P_1	Exact	Modified P_1	Exact	Modified P_1	Exact	Modified P_1	Exact	Modified P_1	Exact
	$\tau_0 = 5$	$\tau_0 = 5$	$\tau_0 = 10$	$\tau_0 = 10$	$\tau_0 = 15$	$\tau_0 = 15$	$\tau_0 = 20$	$\tau_0 = 20$	$\tau_0 = 30$	$\tau_0 = 30$
ω	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$
1	0.09642115	0.098195292	0.057313279	0.057654167	0.039873614	0.040191852	0.030557048	0.030840896	0.020825078	0.021047171
0.99	0.08272016	0.083351639	0.034375443	0.03408973	0.014521705	0.014444497	0.006213438	0.006201538	0.001143925	0.0011495
0.95	0.049590997	0.048645903	0.008365137	0.008177192	0.001268165	0.001283484	0.000191062	0.000200842	4.33393E-06	4.91667E-06
0.9	0.030111002	0.029090121	0.002546171	0.002530741	0.000178748	0.000195455	1.22823E-05	1.49465E-05	5.77013E-08	8.72521E-08
0.5	0.002996526	0.002938929	4.14622E-05	4.55305E-05	3.84121E-07	5.14068E-07	3.04614E-09	5.15834E-09	1.58684E-13	4.48734E-13

TABLE 2-a16

DATA FOR FIG. 2-16

μ	$I_d^-(0, \mu, \phi)/(\pi F_I)$	$I_d^+(\tau_0, \mu, \phi)/(\pi F_I)$	$V_d^-(0, \mu, \phi)/(\pi F_V)$
1	0.209889272	0.10090971	0.145781316
0.9	0.212127564	0.09642115	0.151819511
0.8	0.213945938	0.091464518	0.157464724
0.7	0.21524372	0.086089588	0.162605275
0.6	0.215903648	0.08036714	0.167115238
0.5	0.215786733	0.07438515	0.170850317
0.4	0.214722362	0.068231285	0.173637867
0.3	0.212491662	0.06197302	0.175258872
0.2	0.208803438	0.055650509	0.175422317
0.1	0.203260261	0.049285436	0.173730807

$V_d^+(\tau_0,\mu,\phi)/(\pi F_V)$
0.165013884
0.156729203
0.147945732
0.138728034
0.129155549
0.119321566
0.10931578
0.09920581
0.089031631
0.07881489

CHAPTER III

NUMERICAL CALCULATION OF RADIATIVE TRANSFER FOR THE SCATTERING OF POLARIZED LIGHT IN A PLANE-PARALLEL MEDIUM

Abstract

The objective of the present work is to obtain exact numerical solutions for radiative transfer in one-dimensional finite media without reflective boundaries. In this paper, the exact expressions are presented for the fundamental source matrix, reflection and transmission matrices, reflected and transmitted intensity matrices, and reflected and transmitted flux matrices at the boundaries of a plane-parallel medium which scatters, absorbs, and is exposed to a incident polarized radiation. The problem is simplified by solving for the desired matrix functions only at boundaries. The principle of superposition as well as Ambarzumian's method are used in the solution process. The polarized phase matrix of the medium is general, requiring only that the scattering particles be randomly oriented and have one plane of symmetry. Numerical results are presented for various optical thicknesses (up to 10), two albedoes (0.5 and 0.99), two selected sets of the scattering coefficients (i.e., phase matrices), and four different incident polarized light boundary conditions which can be utilized to superpose and represent any incident polarized radiation. In addition, these results are compared with the solution of the

diffusion approximation for polarized light as well as with the exact results for the scalar problem. The comparison shows that the diffusion approximation can predict the state of the polarization well qualitatively; while the intensity for the scalar problem is equal to the intensity including polarization effects when the number of Legendre polynomials in the polarization phase matrix is one. Furthermore, the scalar results estimate the intensity very well for three of the chosen incident polarized light boundary conditions, but do poorly for one chosen incident polarized light boundary condition when the number of Legendre polynomials is greater than one.

Introduction

The rapid improvement of the personal computer in recent years has allowed radiation researchers to numerically solve mathematically complicated problems, such as those including polarization effects, on their desktop. However, although some solutions are available, exact numerical solutions with assumptions made for the scattering of polarized light in a one-dimensional finite medium exposed to elliptically polarized incident light without reflective boundaries are insufficient. Most researchers have simplified the effects of polarization due to its mathematical complexity, while others have formulated equations that become very difficult to solve numerically. Some related studies, mainly focused on polarization, will be reviewed in the following paragraphs.

Some typical studies focussing on the effect of polarization in plane-parallel media were conducted by Chandrasekhar [3-1], Reguigui [3-2, 3-3], Hovenier [3-4], Wauben and Hovenier [3-5], Wauben et al. [3-6], Siewert [3-7], Garcia and Siewert [3-8, 3-9], Hovenier and van der Mee [3-10], Benassi et al. [3-11, 3-12], Zege and Chaikovskaya [3-13], Mishchenko [3-14, 3-15], and Ambirajan and Look [3-16, 3-17]. Several studies concentrate on the derivation of radiative transfer equations without numerical results while others present numerical solutions with the incident radiation being unpolarized, circularly polarized, or linearly polarized in order to simplify the numerical process. Some of the relevant papers will be discussed herein while others were discussed in Chapter II.

Wauben and Hovenier [3-5] utilized the adding/doubling method as well as the F_N method to solve the radiative transfer problem in a plane-parallel homogeneous atmosphere including polarization effects. The medium was illuminated by unpolarized incident light and bounded by a black lower surface. Numerical results of all Stokes parameters were presented for three different kinds of randomly-oriented spheroids.

The adding principle was employed by Wauben et al. [3-6] to calculate the radiative transfer in a plane-parallel inhomogeneous atmosphere including polarization effects. The medium was illuminated by an unpolarized incident light on the upper boundary, by isotropically radiating internal sources, and by an isotropically radiating lower surface. Numerical results of all Stokes parameters were presented for all three kinds of illumination.

Two different methods were applied to solve the radiative transfer problem including polarization effects by Garcia and Siewert [3-8, 3-9]. The medium was plane-parallel with unpolarized incident light on the top and a reflective lower boundary. Numerical results of all Stokes parameters were tabulated by using the generalized spherical harmonics method [3-8] and the F_N method [3-9], respectively.

Mishchenko [3-14, 3-15] formulated exact reflected radiation equations by using an extension of the invariant imbedding method for a finite plane-parallel atmosphere including polarization effects. However, the formulated equations were numerically complex, requiring double integration. Thus, the author numerically solved two simplified problems, one for unpolarized incident radiation and the other for linearly polarized incident radiation.

For a plane-parallel medium with incident light being circularly polarized, Ambirajan and Look [3-16, 3-17] studied the radiative transfer problem both theoretically [3-16] and experimentally [3-17]. A backward Monte Carlo method was introduced by Ambirajan and Look [3-16] to numerically solve for the backscattered intensity, while experimental data was presented by Ambirajan and Look [3-17] for transmitted intensity as well as the degree of linear and circular polarization versus optical radius.

For the diffuse scattering of polarized light, Herman et al. [3-18] presented numerical results for both spherical and plane-parallel atmospheres by using the Gauss-Seidel calculation method. Comparisons between the polarized spherical Gauss-Seidel method and Monte Carlo calculations of other published studies for both spherical and plane-parallel media were also made. When all scattering terms were considered, the four Stokes parameters (I, Q, U, and V) were in good agreement between the polarized spherical Gauss-Seidel method and the Monte Carlo method. The solar radiation incident at the top of the atmosphere was assumed to be a completely unpolarized parallel beam.

Special work, mainly on the phase matrix, for the scattering of polarized light was performed by Siewert [3-19], Vestrucci and Siewert [3-20], de Rooij and van der Stap [3-

21], Hovenier [3-22], Kuik et al. [3-23], and Mishchenko and Travis [3-24]. None of these publications provided numerical results for intensity.

Two FORTRAN codes to get mainly the scattering coefficients of the scattering matrix were described in detail by Mishchenko and Travis [3-24]. The T-matrix method was used by both codes with the assumption that scattering particles have a plane of symmetry perpendicular to the rotational axis. These two codes are available on the Web at http://www.giss.nasa.gov/~crmin.

Assuming a semi-infinite scattering medium that was homogeneous with randomly oriented polydisperse scattering spheres having a plane of symmetry, Mishchenko [3-25] has presented the Stokes reflection matrix which can be used to find radar reflectivity, polarization ratios, and enhancement factors. Some graphical results for the effects of particle size parameter, as well as the real and imaginary parts of the index of refraction, on the photometric and polarization characteristics of the radar return were also provided. No numerical results for intensity were given.

Haferman et al. [3-26] solved a multi-dimensional radiative transfer problem including polarization effects by using the discrete-ordinates method. Numerical results for backscattered Stokes parameters were provided only for a one-dimensional plane-parallel atmosphere (having a diffusely reflecting lower surface) illuminated by an unpolarized intensity from the top boundary with single scattering albedo and optical thickness being 0.99 and 1, respectively.

Mueller and Crosbie [3-27] presented a polarized phase matrix for the three-dimensional radiative transfer problem based on a scattering matrix which represented randomly oriented scattering particles having a plane of symmetry. In that paper, the geometry was finite in the z-direction and infinite in the x- and y-directions, with elliptically polarized radiation incident only on the top boundary. Great effort was expended to reduce this three-dimensional problem to a one-dimensional problem which depended upon two parameters. A general four-by-four source matrix was derived by using the method of superposition. Some symmetry relationships were developed. Moreover, an extensive review of a wide variety of radiative transfer literature was also provided. No numerical results were presented.

By using a procedure similar to that which Ambarzumian applied to the scalar problem, Liu and Dougherty [3-28] derived exact expressions for the source matrices, reflection and transmission matrices, reflected and transmitted intensity matrices, and reflected and transmitted flux matrices at the boundaries of a one-dimensional plane-parallel medium which scattered, absorbed, and was exposed to incident polarized radiation. The polarized phase matrix of the medium was general, requiring only that the scattering particles be randomly oriented and have one plane of symmetry. No numerical results were given.

A procedure that modified the classical P_1 approximation, which had been applied to the scalar problem, was introduced by Liu and Dougherty [3-29] to solve a one-dimensional radiative transfer problem including polarization effects. In this paper, the expression for intensity was derived by using the classical P_1 approximation with both Mark's and Marshak's boundary conditions as well as the modified P_1 method with Marshak's boundary conditions. The plane-parallel medium of interest scattered, absorbed, and was exposed to collimated incident polarized radiation. Numerical results were presented for five optical thicknesses (5, 10, 15, 20, and 30), five albedoes (0.5, 0.9, 0.95, 0.99, and 1), and three selected sets of the scattering coefficients. These solutions were compared with the classical P_1 approximation and with the exact scalar results. Qualitatively good agreement for intensity was shown between the modified P_1 and the exact scalar solutions, while the classical P_1 approximation predictions were poor.

The purpose of this chapter of the current study is to numerically solve the intensity matrices, for a one-dimensional plane-parallel medium which scatters and absorbs, with polarization fully included [3-28]. The polarized phase matrix of the medium requires only that the scattering particles be randomly oriented and have one plane of symmetry [3-22]. Moreover, these exact expressions for intensity are relatively straightforward and should be numerically simpler to solve than those of previous researchers. Therefore, the present work extends previous research because the numerical solutions will allow the incident radiation to be elliptically polarized, which implies that the solutions depend upon the azimuthal angle. Future research will be directed toward the inclusion of refractive index effects and multi-dimensionality.

Solution of the Plane-Parallel Polarized Light Problem

Starting with the diffuse transport equation for polarized light in a one-dimensional plane-parallel medium, exact expressions for the intensity and flux matrices will be presented in terms of the reflection or transmission matrix in this section (derived in Ref. [3-28]), by using Ambarzumian's approach [3-30] as well as the principle of superposition. Absorption and scattering without emission are assumed in the medium, and refractive index effects at the boundaries are neglected.

Problem Description

As mentioned earlier, the problem which we need to focus on first is the one-dimensional radiative transfer problem for polarized light without reflective boundaries. One of the major reasons to do the present one-dimensional case when solutions for some cases exist is that most of the previous studies cannot handle elliptically polarized light as the incident source while the present work does not have this restriction. In addition, the present one-dimensional work can help to determine how accurate the previous diffusion approximation results [3-29] for polarized light will be for future applications. The geometry for this problem is shown in Fig. 3-1.

In this research, we assume that collimated polarized incident radiation at an angle θ_o exists only at the top boundary and is a sheet of laser-like beams which simplifies our study to a one-dimensional problem. Moreover, no radiation is incident at the lower boundary and there are no refractive index effects at either boundary. Furthermore, absorption and scattering without emission are assumed in the medium (which applies only as a good approximation to a low temperature medium). Note that the probability of scattering in the various directions depends upon the phase matrix function, which will be discussed in detail in the next section.

Fundamental Equations

In a recent work (see Ref. [3-28] and also Chapter I), we have derived the diffuse transport equation for polarized light in a plane-parallel atmosphere (Fig. 3-1) as follows

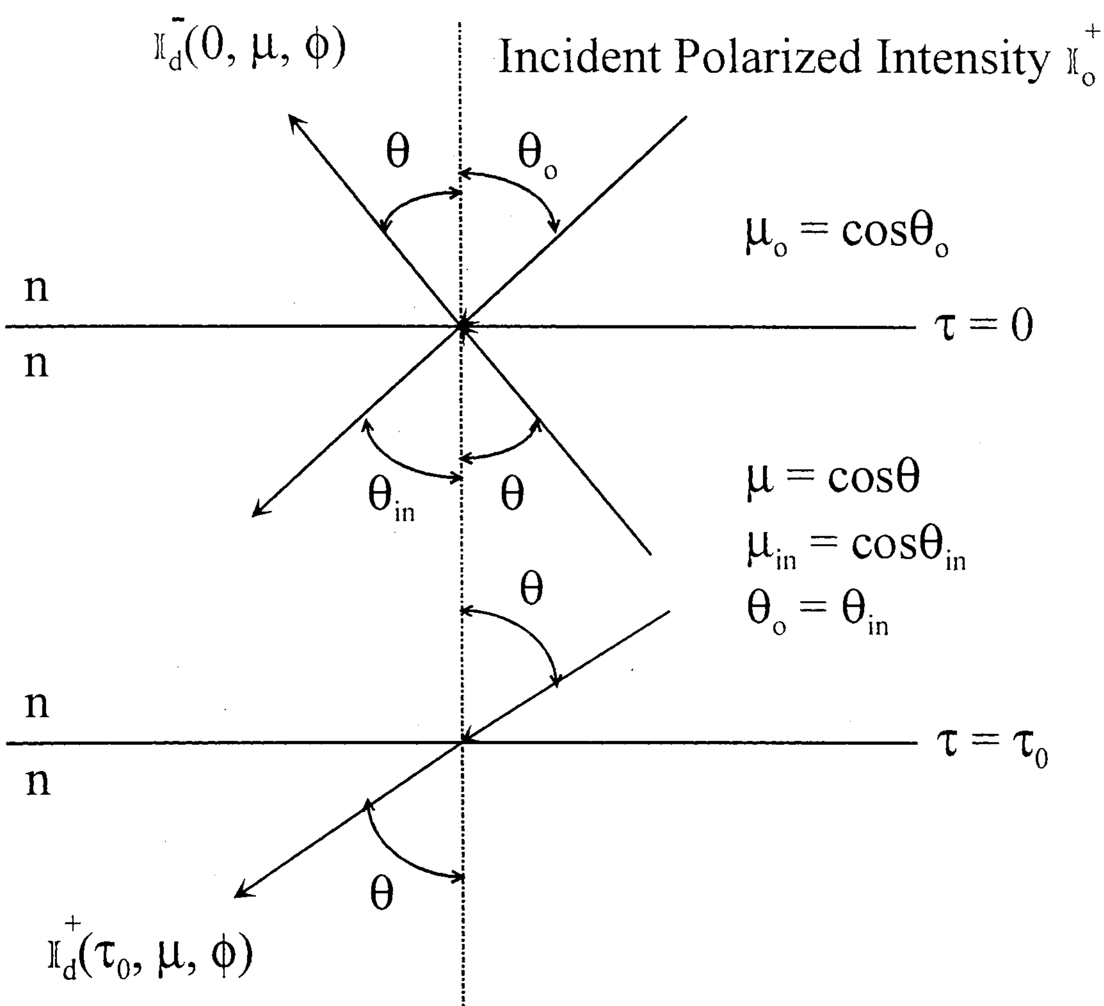

Figure 3-1. Geometry of a One-Dimensional Medium without Reflective Boundaries

$$\mu \frac{d\mathbb{I}_d(\tau,\mu,\phi)}{d\tau} + \mathbb{I}_d(\tau,\mu,\phi) = \frac{\omega}{4\pi} \int_0^{2\pi} \int_{-1}^{1} \mathbb{P}(\mu,\mu',\phi,\phi')\, \mathbb{I}_d(\tau,\mu',\phi')\, d\mu'\, d\phi'$$

$$+ \frac{\omega}{4} \mathbb{P}(\mu,\mu_o,\phi,\phi_o) \exp(-\tau/\mu_o)\, \mathbb{F}, \tag{3-1}$$

with the boundary conditions, for $\mu \in [0, 1]$ and $\phi \in [0, 2\pi]$, to be

$$\mathbb{I}_d(0, \mu, \phi) = 0 \tag{3-2a}$$

$$\text{and } \mathbb{I}_d(\tau_0, -\mu, \phi) = 0, \tag{3-2b}$$

where τ is the optical thickness, μ is the direction cosine of the propagation angle of the radiation, ω is the single scattering albedo, the vector $\mathbb{F} = [F_I \;\; F_Q \;\; F_U \;\; F_V]^T$ which is presumed given specifies the state of polarization of the incident intensity at the upper boundary, and the diffuse intensity vector $\mathbb{I}_d(\tau, \mu, \phi)$ consists of the four Stokes parameters, that is, $\mathbb{I}_d(\tau, \mu, \phi) = [I_d(\tau, \mu, \phi) \;\; Q_d(\tau, \mu, \phi) \;\; U_d(\tau, \mu, \phi) \;\; V_d(\tau, \mu, \phi)]^T$. Note that in this diffuse formulation, the original boundary condition, $\mathbb{F}$, appears as a source term. Furthermore, $\mathbb{P}(\mu, \mu', \phi, \phi')$ is the phase matrix that can be expanded in general in a Fourier series [3-12] as

$$\mathbb{P}(\mu,\mu',\phi,\phi') = \sum_{m=0}^{L} \frac{1}{1+\delta_{0m}} \{\mathbb{C}^m(\mu,\mu') \cos[m(\phi-\phi')]$$

$$+ \mathbb{S}^m(\mu,\mu') \sin[m(\phi-\phi')]\}, \tag{3-3}$$

where δ_{0m} is the Kronecker delta function, and other functions in Eq. (3-3) are defined as

$$\mathbb{C}^m(\mu, \mu') = \mathbb{A}^m(\mu, \mu') + \mathbb{D}_2\, \mathbb{A}^m(\mu, \mu')\, \mathbb{D}_2, \tag{3-4a}$$

$$\mathbb{S}^m(\mu,\mu') = \mathbb{A}^m(\mu,\mu')\,\mathbb{D}_2 - \mathbb{D}_2\,\mathbb{A}^m(\mu,\mu'), \tag{3-4b}$$

$$\mathbb{A}^m(\mu,\mu') = \sum_{i=m}^{L} \frac{(i-m)!}{(i+m)!}\,\Pi_i^m(\mu)\,\mathbb{B}_i\,\Pi_i^m(\mu'), \tag{3-4c}$$

$$\mathbb{D}_2 = \mathrm{diag}\{1, 1, -1, -1\}, \tag{3-4d}$$

$$\Pi_i^m(\mu) = \begin{bmatrix} P_i^m(\mu) & 0 & 0 & 0 \\ 0 & R_i^m(\mu) & -T_i^m(\mu) & 0 \\ 0 & -T_i^m(\mu) & R_i^m(\mu) & 0 \\ 0 & 0 & 0 & P_i^m(\mu) \end{bmatrix}, \tag{3-4e}$$

and the matrix of scattering coefficients is given by

$$\mathbb{B}_i = \begin{bmatrix} \beta_i & \gamma_i & 0 & 0 \\ \gamma_i & \alpha_i & 0 & 0 \\ 0 & 0 & \zeta_i & -\varepsilon_i \\ 0 & 0 & \varepsilon_i & \delta_i \end{bmatrix}. \tag{3-4f}$$

The matrix $\mathbb{B}_i$ will be specified later. In addition, in Eq. (3-4e), $P_i^m(\mu)$ denotes associated Legendre functions while $R_i^m(\mu)$ and $T_i^m(\mu)$ are combinations of generalized spherical functions.[19] Notice that the phase matrix of Eq. (3-3) assumes that the scattering particles are randomly oriented, and have at least one plane of symmetry [3-22].

<u>Reflected and Transmitted Intensity Matrices</u>

By applying Ambarzumian's approach [3-30] and superposition to Eq. (3-1), the reflected and transmitted intensity matrices for the current one-dimensional plane-parallel problem with polarization included are [3-28]

$$\mathbb{I}_d^-(0,\ \mu,\ \mu_o,\ \phi;\ \tau_0) = \frac{\omega}{4\pi} \sum_{m=0}^{L} \frac{1}{1+\delta_{0m}} \sum_{i=m}^{L} \frac{(i-m)!}{(i+m)!} \{ \cos(m\phi) [\Pi_i^m(-\mu)$$
$$\times\ \mathbb{B}_i\ \overline{\mathbb{PP}_{im1}(\mu,\ \mu_o;\ \tau_0)} + \mathbb{D}_2\ \Pi_i^m(-\mu)\ \mathbb{B}_i\ \overline{\mathbb{PP}_{im1}(\mu,\ \mu_o;\ \tau_0)}$$
$$\times\ \mathbb{D}_2] + \sin(m\phi) [\Pi_i^m(-\mu)\ \mathbb{B}_i\ \overline{\mathbb{PP}_{im1}(\mu,\ \mu_o;\ \tau_0)}\ \mathbb{D}_2$$
$$-\ \mathbb{D}_2\ \Pi_i^m(-\mu)\ \mathbb{B}_i\ \overline{\mathbb{PP}_{im1}(\mu,\ \mu_o;\ \tau_0)}] \} \frac{\mathbb{F}}{\mu} \tag{3-5a}$$

and

$$\mathbb{I}_d^+(\tau_0,\ \mu,\ \mu_o,\ \phi;\ \tau_0) = \frac{\omega}{4\pi} \sum_{m=0}^{L} \frac{1}{1+\delta_{0m}} \sum_{i=m}^{L} \frac{(i-m)!}{(i+m)!} \{ \cos(m\phi) [\Pi_i^m(\mu)\ \mathbb{B}_i$$
$$\times\ \overline{\mathbb{PPI}_{im1}(\mu,\ \mu_o;\ \tau_0)} + \mathbb{D}_2\ \Pi_i^m(\mu)\ \mathbb{B}_i\ \overline{\mathbb{PPI}_{im1}(\mu,\ \mu_o;\ \tau_0)}$$
$$\times\ \mathbb{D}_2] + \sin(m\phi) [\Pi_i^m(\mu)\ \mathbb{B}_i\ \overline{\mathbb{PPI}_{im1}(\mu,\ \mu_o;\ \tau_0)}\ \mathbb{D}_2$$
$$-\ \mathbb{D}_2\ \Pi_i^m(\mu)\ \mathbb{B}_i\ \overline{\mathbb{PPI}_{im1}(\mu,\ \mu_o;\ \tau_0)}] \} \frac{\mathbb{F}}{\mu}, \tag{3-5b}$$

where the superscripts - and + in Eqs. (3-5) denote that the intensity is generally propagating in the negative and positive τ directions (see Fig. 3-1), respectively. In addition, $\overline{\mathbb{PP}_{im1}}$ and $\overline{\mathbb{PPI}_{im1}}$ in Eqs. (3-5) are the reflection and transmission matrices, respectively, and defined as [3-28]

$$\overline{\mathbb{PP}_{im1}(\mu, \mu_o; \tau_0)} = \int_0^{\tau_0} \mathbb{PP}_{im1}(t, \mu_o; \tau_0) \exp(-t/\mu)\, dt \tag{3-6a}$$

and

$$\overline{\mathbb{PPI}_{im1}(\mu, \mu_o; \tau_0)} = \int_0^{\tau_0} \mathbb{PP}_{im1}(\tau_0 - t, \mu_o; \tau_0) \exp(-t/\mu)\, dt\,. \tag{3-6b}$$

Furthermore, $\mathbb{PP}_{im1}(\tau,\ \mu_o;\ \tau_0)$ in Eqs. (3-6) is the fundamental source matrix and defined as [3-28]

$$\mathbb{PP}_{iml}(\tau, \mu_o; \tau_0) = \pi \exp(-\tau / \mu_o) \Pi_i^m(\mu_o) + \frac{\omega}{2} \sum_{j=m}^{L} \frac{(j - m)!}{(j + m)!} \int_0^{\tau_0} \mathbb{K}_{lijm}(\tau - t) \mathbb{PP}_{jml}(t, \mu_o; \tau_0) \, dt, \quad (3\text{-}7)$$

where

$$\mathbb{K}_{lijm}(\tau - t) = \int_0^1 \Pi_i^m[\mathrm{sign}(\tau - t)\mu'] \Pi_j^m[\mathrm{sign}(\tau - t)\mu'] \mathbb{B}_j \exp[-|\tau - t|/\mu'] \frac{d\mu'}{\mu'}, \quad (3\text{-}8)$$

with sign(τ - t) being 1 if $\tau \geq t$, and -1 if $\tau < t$.

So far, the intensity matrices are represented in terms of the fundamental source matrix $\mathbb{PP}_{iml}(\tau, \mu_o; \tau_0)$. Therefore, Eqs. (3-5) can be numerically solved to obtain the reflected and transmitted intensity matrices directly, after $\mathbb{PP}_{iml}(\tau, \mu_o; \tau_0)$, $\overline{\mathbb{PP}_{iml}}$, and $\overline{\mathbb{PPI}_{iml}}$ have been determined. The assumptions made to this point are: no incident radiation entering from the lower boundary, no index of refraction effects at both boundaries, the particles are randomly oriented and have at least one plane of symmetry, and ϕ_o is equal to zero. In next section, the reflected and transmitted flux matrices will be expressed in terms of the reflection matrix $\overline{\mathbb{PP}_{kml}}$ and transmission matrix $\overline{\mathbb{PPI}_{kml}}$, respectively.

Reflected and Transmitted Flux Matrices

The general flux equation is [3-1]

$$\mathbb{q} = \int \mathbb{I} \, \mu \, d\Omega, \quad (3\text{-}9)$$

where Ω denotes the solid angle. By using Eqs. (3-5) with the above general flux definition, the reflected and transmitted flux matrices are [3-28]

$$q_d^-(0, \mu_o; \tau_0) = \frac{\omega}{4} \sum_{i=0}^{L} \int_0^1 [\, \Pi_i^0(-\mu)\, \mathbb{B}_i\, \overline{\mathbb{PP}_{i01}(\mu, \mu_o; \tau_0)}$$

$$+ \mathbb{D}_2\, \Pi_i^0(-\mu)\, \mathbb{B}_i\, \overline{\mathbb{PP}_{i01}(\mu, \mu_o; \tau_0)}\, \mathbb{D}_2 \,]\, d\mu\ \mathbb{F} \qquad (3\text{-}10a)$$

and

$$q_d^+(\tau_0, \mu_o; \tau_0) = \frac{\omega}{4} \sum_{i=0}^{L} \int_0^1 [\, \Pi_i^0(\mu)\, \mathbb{B}_i\, \overline{\mathbb{PPI}_{i01}(\mu, \mu_o; \tau_0)}$$

$$+ \mathbb{D}_2\, \Pi_i^0(\mu)\, \mathbb{B}_i\, \overline{\mathbb{PPI}_{i01}(\mu, \mu_o; \tau_0)}\, \mathbb{D}_2 \,]\, d\mu\ \mathbb{F}, \qquad (3\text{-}10b)$$

respectively. Equations (3-10) can be solved to obtain the reflected and transmitted flux matrices, after $\overline{\mathbb{PP}_{im1}}$ and $\overline{\mathbb{PPI}_{im1}}$ have been determined.

Thus far, all equations that are required to numerically solve the present one-dimensional plane-parallel problem including polarization effects have been expressed explicitly. Note that Eqs. (3-5) and (3-10) without the $\mathbb{F}$ vector would yield generic four by four intensity and flux matrices, respectively, independent of boundary conditions. Therefore, we can apply any boundary conditions, once we get these generic intensity and flux matrices.

Numerical Results

Some selected figures are presented in this section to show the comparison among the exact numerical results with assumptions made for polarized light (labeled Pol in the figure legend), the diffusion approximation results [3-29] for polarized light (labeled DA in the figure legend), and the exact scalar results (labeled Sc in the figure legend) [3-30]. The Runge-Kutta numerical calculation method as well as the successive approximation method are used to solve the current polarized problem numerically. For these solutions, the reflected and transmitted intensity vectors are calculated only at the boundaries. The reflected intensity vector represents only the intensity vector reflected from the medium due to scattering within the medium; while the transmitted intensity vector excludes any part of the incident intensity vector that directly reaches the lower interface undisturbed. For normalization reasons, both intensity vectors are divided by the incident radiation, π F_I.

Moreover, for these examples, two sets of the scattering coefficients, in Eq. (3-4f), suitable for polarized light problems and suggested by Vestrucci and Siewert [3-20], are chosen to use. With the wave number multiplied by the radius of the small absorbing spherical particles equal to 0.5, the index of refraction of the particles with respect to the surrounding medium being 1.33, and the single scattering albedo being one, the matrices of scattering coefficients for phase matrix I are given as [3-20]

$$\mathbb{B}_0 = \begin{bmatrix} 1.0000000000 & 0 & 0 & 0 \\ 0 & 0.00000000 & 0 & 0 \\ 0 & 0 & 0.00000000 & 0 \\ 0 & 0 & 0 & 0.06399215408 \end{bmatrix}, \qquad (3\text{-}11a)$$

$$\mathbb{B}_1 = \begin{bmatrix} 0.1400343465 & 0 & 0 & 0 \\ 0 & 0.0000000000 & 0 & 0 \\ 0 & 0 & 0.0000000000 & 0 \\ 0 & 0 & 0 & 1.50000000 \end{bmatrix}, \qquad (3\text{-}11b)$$

$$\mathbb{B}_2 = \begin{bmatrix} 0.5 & 1.224744871 & 0 & 0 \\ 1.224744871 & 3 & 0 & 0 \\ 0 & 0 & 0.2409165905 & 0 \\ 0 & 0 & 0 & 0.1054062694 \end{bmatrix}, \qquad (3\text{-}11c)$$

$$\text{and } \mathbb{B}_3 = \begin{bmatrix} 0.02936407694 & -0.05361122441 & 0 & 0 \\ -0.05361122441 & 0.09788025648 & 0 & 0 \\ 0 & 0 & 0.00000 & 0 \\ 0 & 0 & 0 & 0.00000 \end{bmatrix}. \qquad (3\text{-}11d)$$

By considering the Mie scattering of light at a wavelength of 0.951 μm, and by using a gamma distribution of spherical particles with an effective radius of 0.2 μm, an effective variance of 0.07, and an index of refraction of 1.44, the matrices of scattering coefficients for phase matrix II (for scattering albedo equal to one) are given as [3-20]

$$\mathbb{B}_0 = \begin{bmatrix} 1.000000000 & 0 & 0 & 0 \\ 0 & 0.0000000000 & 0 & 0 \\ 0 & 0 & 0.0000000000 & 0 \\ 0 & 0 & 0 & 0.7120634246 \end{bmatrix}, \qquad (3\text{-}12a)$$

$$\mathbb{B}_1 = \begin{bmatrix} 1.4552931819 & 0 & 0 & 0 \\ 0 & 0.0000000000 & 0 & 0 \\ 0 & 0 & 0.0000000000 & 0 \\ 0 & 0 & 0 & 1.760141193 \end{bmatrix}, \qquad (3\text{-}12b)$$

$$\mathbb{B}_2 = \begin{bmatrix} 1.0540263128 & -0.7552491518 & 0 & 0 \\ -0.7552491518 & 3.3091220464 & 0 & 0 \\ 0 & 0 & 2.5773207443 & -0.0420726875 \\ 0 & 0 & 0.0420726875 & 1.0668243107 \end{bmatrix}, \qquad (3\text{-}12c)$$

$$\mathbb{B}_3 = \begin{bmatrix} 0.3975899378 & -0.3619934319 & 0 & 0 \\ -0.3619934319 & 0.9633758276 & 0 & 0 \\ 0 & 0 & 0.7574437604 & -0.0850671555 \\ 0 & 0 & 0.0850671555 & 0.3965110389 \end{bmatrix}, \quad (3\text{-}12d)$$

$$\mathbb{B}_4 = \begin{bmatrix} 0.1165930161 & -0.1155748816 & 0 & 0 \\ -0.1155748816 & 0.2474124256 & 0 & 0 \\ 0 & 0 & 0.1638177665 & -0.0154318420 \\ 0 & 0 & 0.0154318420 & 0.0957641237 \end{bmatrix}, \quad (3\text{-}12e)$$

$$\text{and } \mathbb{B}_5 = \begin{bmatrix} 0.0238747702 & -0.0249815879 & 0 & 0 \\ -0.0249815879 & 0.0452636955 & 0 & 0 \\ 0 & 0 & 0.0278314781 & -0.0031534874 \\ 0 & 0 & 0.0031534874 & 0.0176508810 \end{bmatrix}. \quad (3\text{-}12f)$$

Note that phase matrix II is a modification of that from Vestrucci and Siewert [3-20], formed by using only the first 6 of their original 14 matrices. In addition, although the scattering coefficients for phase matrices I and II were originally calculated for ω equal to one [3-8], to avoid recalculating these coefficients from the Mie theory, we use these coefficients with other ω values for demonstration purposes.

All of the results presented in Figs. 3-2 to 3-19 on the following pages are based on the calculation of any iterative fractional error to be less than or equal to 10^{-6}, a total number of Gauss-Legendre integration quadrature points equal to 74, the exit azimuthal angle (ϕ) being equal to zero, optical thickness step size equal to 0.0005, and the use of phase matrix I (except for Figs. 3-18 and 3-19). The iterative error 10^{-6} was utilized to get good accuracy for future application. Wherein, to get two- or three-dimensional solutions which will be accurate, the current one-dimensional solution must be numerically summed or integrated, resulting in a degraded final accuracy for the two- or three-dimensional solutions.

The Gauss-Legendre quadrature was used to integrate, for example, over μ' in Eq. (3-8). This number of Gauss-Legendre quadrature was determined to be sufficient by running a case (L = 2, ω = 0.99, and τ_0 = 0.01) with 160 quadrature, and finding that the worst error between the 74 point quadrature and the 160 point quadrature was a 10^{-6} fractional error. In addition, the optical thickness step size was determined to be sufficiently accurate by running a case (L = 2, ω = 0.99, and τ_0 = 0.1), and finding that there was no difference (to 10^{-6}) in the results from using either 0.0005 or 0.00005 as the step size. Furthermore, four different incident polarized light boundary conditions ($\mathbb{F}$'s), which can be used to superpose any incident polarized radiation [3-8], were used to obtain numerical results. These incident intensity vectors ($\mathbb{F}$'s) are designated as B.C. 1 = $[1\ \ 0\ \ 0\ \ 0]^T$, B.C. 2 = $[1\ \ 1\ \ 0\ \ 0]^T$, B.C. 3 = $[1\ \ 0\ \ 1\ \ 0]^T$, and B.C. 4 = $[1\ \ 0\ \ 0\ \ 1]^T$, where B.C. 1 represents unpolarized light, B.C.'s 2 and 3 represent linearly polarized light with different polarization angles (horizontal and 45°), and B.C. 4 represents circularly polarized light.

Figures 3-2 to 3-5 provide a comparison between the exact scalar solution [3-30], the diffusion approximation [3-29], and the exact polarization solution with the number of Legendre polynomials (L) equal to 1, a scattering albedo (ω) of 0.99, an optical thickness (τ_0) of 0.5, and incident intensity vectors ($\mathbb{F}$) of B.C. 4. Figures 3-2 and 3-4 represent the reflected Stokes parameters with the incident radiation angle (μ_o) being 1 and 0.9, respectively; while Figs. 3-3 and 3-5 represent the transmitted Stokes parameters with the incident radiation angle (μ_o) being 1 and 0.9, respectively.

As expected, Figs. 3-2 to 3-5 show that the solution of scalar intensity is equal to that of the intensity from full polarization; while the solution from the diffusion approximation predicts the state of the polarization qualitatively well. The reason that the scalar results can predict the fully polarized solution so well for intensity is that the transport equation of Eq. (3-1) will decouple to become two independent equations for I (intensity) and V when the number of Legendre polynomials (L) is one [3-29]. However, the use of the scalar results will obviously lose the state of polarization. By comparing Figs. 3-2 and 3-4 or Figs. 3-3 and 3-5, we can also deduce that the diffusion approximation estimates the fourth Stokes parameters (V) relatively poorly when μ_o is not equal to 1. Furthermore, the I (intensity) changes only a little, however, V changes dramatically between μ_o equal to one and μ_o equal to 0.9 by comparing Figs. 3-2 and 3-4 or Figs. 3-3 and 3-5. The major reason for this is that the $\mathbb{B}_1$ of Eq. (3-11b) will be more important when μ_o equal to 0.9 [3-29]; therefore, V will change more than I (intensity) does due to δ_1 being much larger than β_1 in Eq. (3-11b). In addition, the peak in the intensity at μ approximately equal to 0.1 in Figs 3-3 and 3-5 gives the preferred direction of the phase matrix I. Moreover, the sign of V is negative between μ values of 0.1 and one in Fig. 3-2 due to that we view the reflected Stokes parameters in an opposite direction from that of the incident radiation. Besides, the fact that V changes from positive to negative between μ values of zero and one in Fig. 3-4 implies that the direction of circularly polarized light changes.

By using all four incident polarized light boundary conditions, Figs. 3-6 and 3-7 provide the exact reflected and transmitted Stokes parameters, respectively, with number of Legendre polynomials (L) equal to 2, a scattering albedo (ω) of 0.99, an optical thickness (τ_0) of 0.5, and the exit angle (μ) equal to 0.6.

Important information revealed from Figs. 3-6 and 3-7 is that the I and Q for boundary conditions 1, 3, and 4 are equal to each other. Therefore, the results from boundary condition 2 will be the dominant component for intensity if the incident polarized radiation needs to be superposed by combining the results from boundary conditions 1 to 4. Furthermore, Fig. 3-7 shows the effects of different boundary conditions for intensity by observing that the peak of intensity for B.C. 2 is at a different incident angle than for B.C.'s 1, 3, and 4. Moreover, U changing from positive to negative between μ_o equal to zero and μ_o equal to one in Fig. 3-6 reveals that the polarization angle of linearly polarized light varies from one (45°) to the other (-45°). In addition, Figs. 3-6 and 3-7 illustrate the relationship between the state of polarization and the incident radiation angle (μ_o).

Stokes parameters are provided for four different incident boundary conditions in Figs. 3-8 to 3-11. With the number of Legendre polynomials (L) equal to 3, a scattering albedo (ω) of 0.5, and an optical thickness (τ_0) of 0.5, Figs. 3-8 and 3-10 present the reflected Stokes parameters with the cosine of the incident radiation angle (μ_o) being 1 and 0.6, respectively; while Figs. 3-9 and 3-11 present the transmitted Stokes parameters with the cosine of the incident radiation angle (μ_o) being 1 and 0.6, respectively.

By observing Figs. 3-8 to 3-11, we find out that not only the I and Q for B.C.'s 1, 3, and 4 are the same, but also all Stokes parameters for B.C.'s 1, 3, and 4 are more sensitive than the ones for B.C. 2 to variation in the incident radiation angle (μ_o). Note that the curves of I and Q for B.C. 2 are close to each other for all four figures; and this is due to the combination effects from the chosen phase matrix and the boundary conditions. Moreover, Q for B.C. 2 is equal to U for B.C. 3 when both μ and μ_o are equal to one in

Fig. 3-9, while Q for B.C. 2 is equal to the opposite of U for B.C. 3 when both μ and μ_o are equal to one in Fig. 3-8. This is due to the physical symmetry for B.C. 2 and B. C. 3 when both μ and μ_o are equal to one, since, physically, for those two B.C.s at those μ and μ_o values, it should be impossible to distinguish between the two cases. The opposite sign of Q for B.C. 2 and U for B.C. 3 in Fig. 3-8 is due to the fact that we look at the reflected Stokes parameters in an opposite direction from that of the incident radiation, and due to the difficulty of distinguishing between "positive" and "negative" linear polarization directions.

Three-dimensional Figs. 3-12 and 3-13 tell us how the intensity (I) for B.C.'s 1, 3, and 4 changes with respect to the incident radiation angle (μ_o) under the same conditions as for Figs. 3-8 to 3-11. By viewing from the direction of the μ axis, we can see that the reflected intensity will curve up (up to a μ_o of 0.2) and then curve down in Fig. 3-12; while the transmitted intensity will curve up (up to a μ_o of 0.7) and then curve down in Fig. 3-13.

Figures 3-14 to 3-17 not only give the comparison between the exact scalar intensity and the intensity including polarization effects with incident radiation angle (μ_o) equal to 1 but also demonstrate the effects of optical thickness (τ_0) and number of Legendre polynomials (L). Figures 3-14 and 3-15 present the reflected and transmitted intensities with a scattering albedo (ω) of 0.99 as well as an exit angle (μ) of 0.6, respectively; while Figs. 3-16 and 3-17 present the reflected and transmitted intensities with a scattering albedo (ω) of 0.5 as well as an exit angle (μ) of 0.9, respectively.

Similar to the previous cases, the intensities (I) for incident B.C.'s 1, 3, and 4 are equal to one another with a fixed number of Legendre polynomials (L) in Figs. 3-14 to 3-17. In addition, the number of Legendre polynomials (L) is shown to have a small effect on the intensity results for both scalar and polarization solutions in Figs. 3-14 to 3-17 when L varies from 2 to 3. This can be explained by observing $\mathbb{B}_3$ of Eq. (3-11d), where all of the nonzero terms of $\mathbb{B}_3$ are relatively smaller than those of $\mathbb{B}_2$ in Eq. (3-11c). Figures 3-14 to 3-17 also reveal that the intensity for B.C. 2 always has the highest values. The reason for this is due to the zero terms in positions (1, 3) and (1, 4) in Eqs. (3-11) which will not contribute to intensity for B.C.'s 3 and 4; while the nonzero terms in position (1, 2) in Eqs. (3-11) will make a significant contribution to the intensity for B.C. 2.

Moreover, for both the reflected and transmitted intensities of Figs 3-14 to 3-17, the intensities with and without polarization effects show similar trends. Clearly, the largest difference is always between the intensity for B.C. 2 and the scalar intensity; while the difference between the intensity for B.C.'s 1, 3, and 4 and the scalar intensity is relatively small. Thus, the scalar results may approximate the intensity for B.C.'s 1, 3, and 4 reasonably well, but will definitely lose the state of the polarization.

Furthermore, the reflected intensity increases and plateaus as optical thickness increases, which is revealed by Figs. 3-14 and 3-16. The reason for this is due to the reduced chances of scattering outside through the lower boundary when optical thickness becomes larger, thus resulting in improved chances of scattering outside through the upper boundary. On other hand, Figs. 3-15 and 3-17 show an interesting but expected fact. Initially, the transmitted intensity increases as optical thickness increases. The explanation for this is that the intensity has a greater chance of scattering as optical thickness increases. However, after a certain optical thickness, the transmitted intensity decreases. This is due to the fact that the increased number of scattering events causes more intensity to scatter out from the upper boundary than from the lower boundary. Note that the transmitted intensity herein excludes any part of the incident intensity that directly reaches the lower interface undisturbed. Also note that the reflected intensity for larger optical thickness ($\tau_0 > 5$) in Fig. 3-16 is flatter than that in Fig. 3-14. This is due to the fact that the scattering albedo in Fig. 3-16 is smaller than the one in Fig. 3-14, and thus the Fig. 3-16 case will require a smaller optical thickness in order to become optically thick. For further information on phase matrix I, more figures are included in Appendix III.

Finally, Figs. 3-18 for reflected intensity and 3-19 for transmitted intensity provide a comparison between the exact scalar intensity and the intensity including polarization effects with scattering albedo (ω) equal to 0.5, an optical thickness (τ_0) of 0.1, incident radiation angle (μ_o) equal to 1, and phase matrix II. Figures 3-18 and 3-19 also show that the intensities for B.C.'s 1, 3, and 4 with a fixed number of Legendre polynomials (L) are the same. In addition, the scalar results yield a good estimate of the intensity for B.C.'s 1, 3, and 4, but predict the intensity for B.C. 2 poorly. Moreover, Figs. 3-18 and 3-19 reveal that the intensity for B.C. 2 always has the lowest value. The reason is the same as in Figs. 3-14 to 3-17; however, the nonzero terms in position (1, 2) in Eqs. (3-12) will make a negative contribution to the intensity for B.C. 2. Furthermore, there is a greater difference for intensity when changing from an L of 3 to 4 as compared to the difference for intensity when changing from an L of 4 to 5 for the both scalar and polarized intensities in Figs. 3-18 and 3-19. This information tells us that the intensity results for phase matrix II can be relatively accurately estimated using an L of 4.

Very important information, which simplified the numerical process, was found during development of the exact polarization computer program. It was found that, $\mathbb{D}_2$ $\mathbb{PP}_{im1}(\tau, \mu_o; \tau_0)$ $\mathbb{D}_2$ = $\mathbb{PP}_{im3}(\tau, \mu_o; \tau_0)$, $\overline{\mathbb{PPI}_{im1}(\mu, \mu_o; \tau_0)} = \mathbb{D}_2 \overline{\mathbb{PPI}_{im3}(\mu, \mu_o; \tau_0)} \mathbb{D}_2$, and $\overline{\mathbb{PP}_{im3}(\mu, \mu_o; \tau_0)} = \mathbb{D}_2 \overline{\mathbb{PP}_{im1}(\mu, \mu_o; \tau_0)} \mathbb{D}_2$ (see Ref. [3-28]). After applying the above simplification, the computational run time is about 60 hours (on a Pentium 350 MHz computer) for the case in which L is equal to 2, ω is equal to 0.9,and τ_0 is equal to 10 for phase matrix I.

In addition, a numerical difficulty, which resulted in computer program convergence problems for either large scattering albedo or large number of Legendre polynomials, was found during this research. A great effort to improve this convergence problem was expended but the problem still existed for scattering albedo (ω) larger than 0.5 or number of Legendre polynomials (L) greater than two. The improvement techniques included: (a) the integration interval from zero to one over μ' in Eq. (3-8) was divided into 11 regions, (b) a different number of Gauss-Legendre quadrature points was used in each of the 11 regions, (c) the results of the previous optical thickness were used as a first guess to start the iteration process for the next optical thickness. This numerical problem can also be solved by using an iterative error of 10^{-2} instead of 10^{-6}. However, the results will definitely lose accuracy. Therefore, this convergence problem needs to be considered and solved to expand the current results in the future.

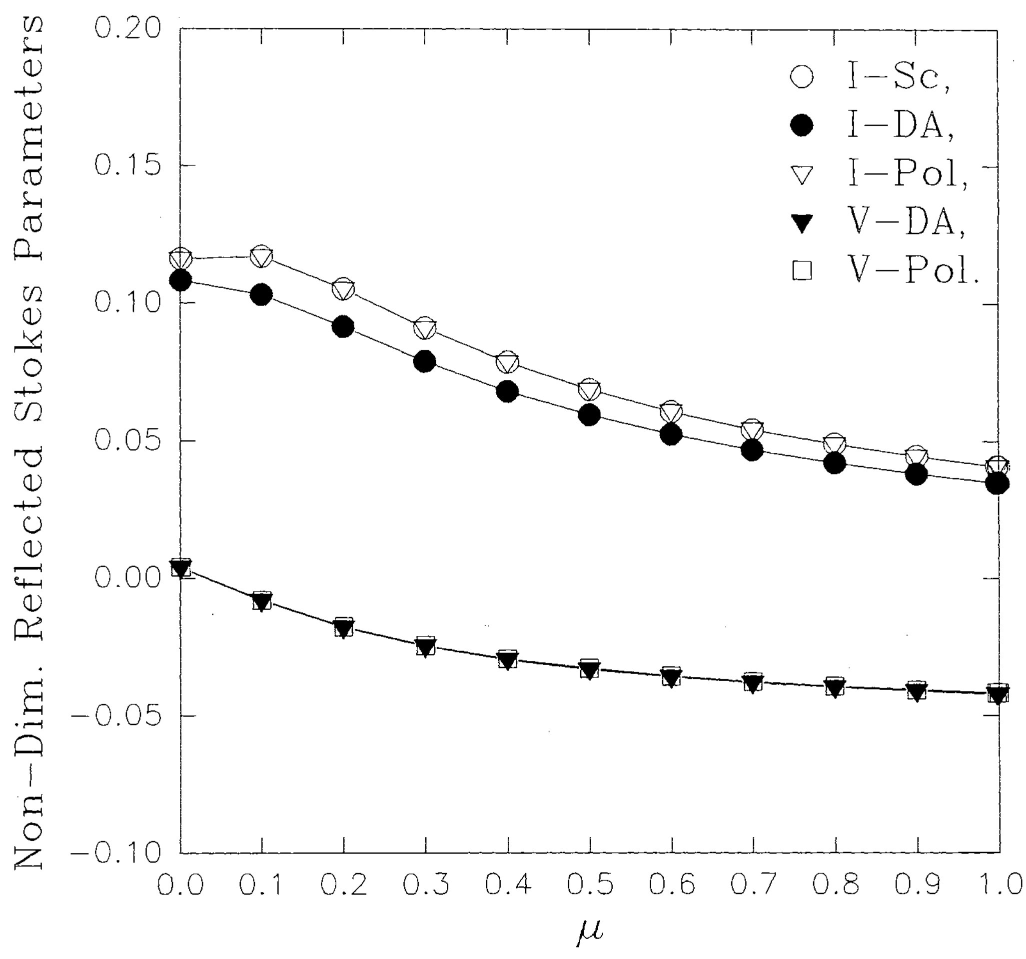

Figure 3-2. Comparison of the Non-Dimensional Reflected Stokes Parameters from Diffusion Approximation, Exact Scalar Solution, and Exact Polarization Solution for Phase Matrix I ($L = 1$, $\omega = 0.99$, $\tau_0 = 0.5$, $\mu_o = 1$, B.C. 4)

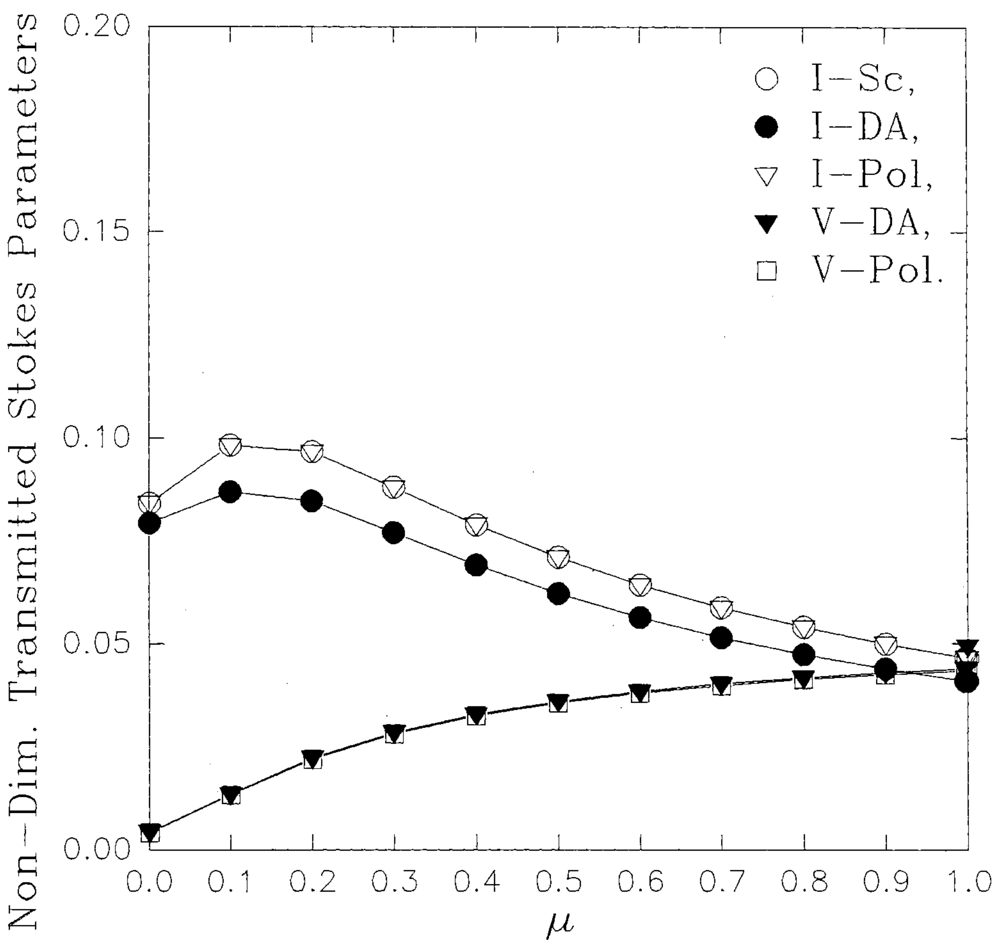

Figure 3-3. Comparison of the Non-Dimensional Transmitted Stokes Parameters from Diffusion Approximation, Exact Scalar Solution, and Exact Polarization Solution for Phase Matrix I ($L = 1$, $\omega = 0.99$, $\tau_0 = 0.5$, $\mu_o = 1$, B.C. 4)

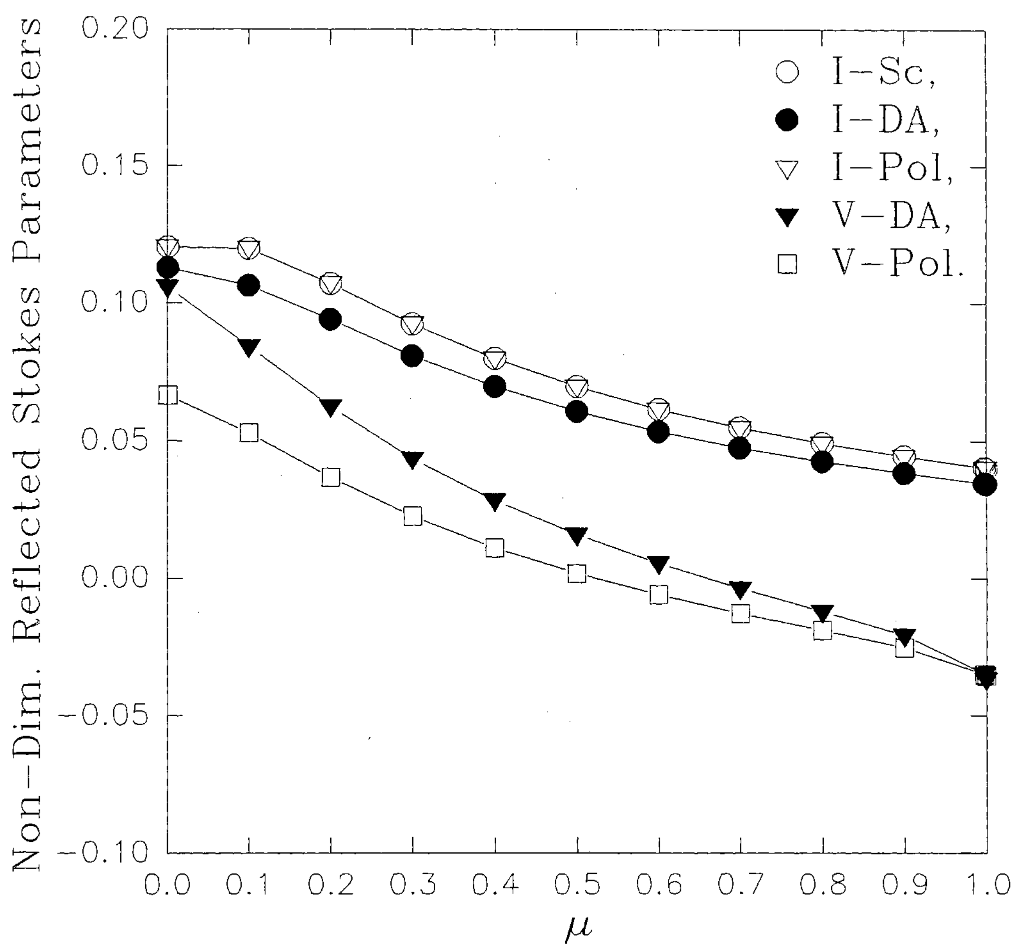

Figure 3-4. Comparison of the Non-Dimensional Reflected Stokes Parameters from Diffusion Approximation, Exact Scalar Solution, and Exact Polarization Solution for Phase Matrix I ($L = 1$, $\omega = 0.99$, $\tau_0 = 0.5$, $\mu_o = 0.9$, B.C. 4)

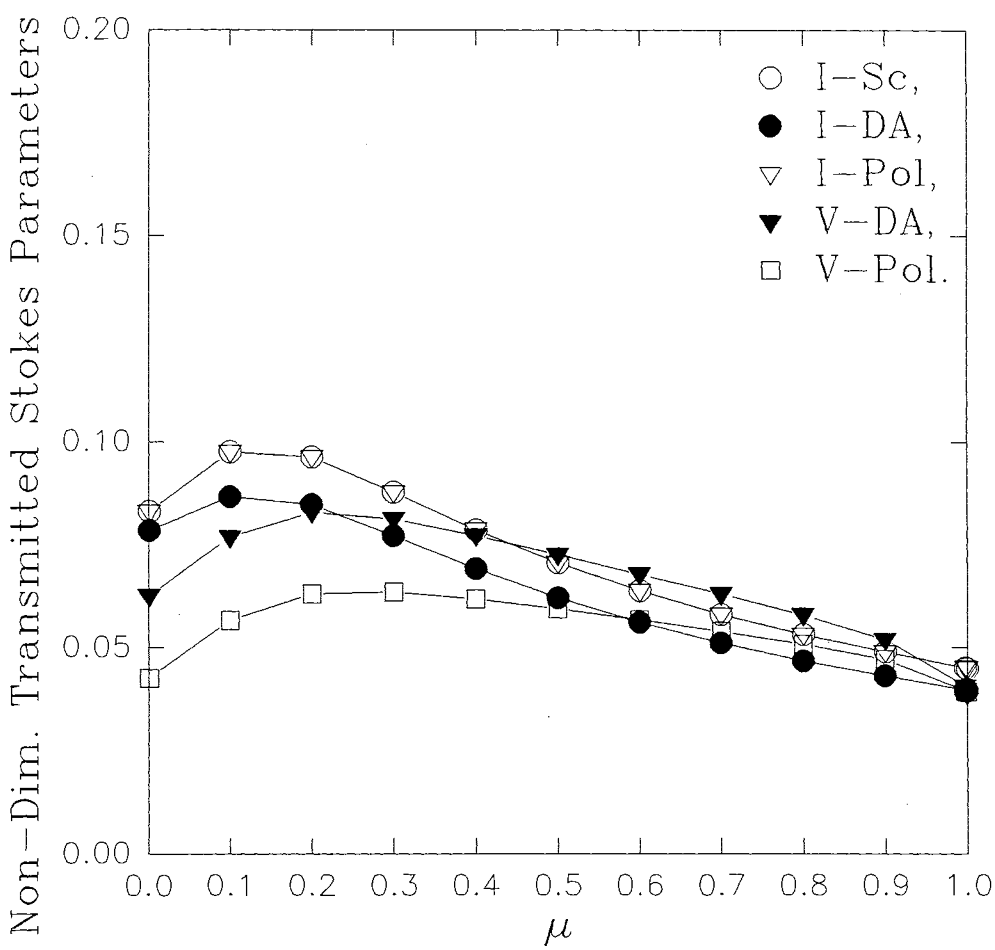

Figure 3-5. Comparison of the Non-Dimensional Transmitted Stokes Parameters from Diffusion Approximation, Exact Scalar Solution, and Exact Polarization Solution for Phase Matrix I (L = 1, $\omega = 0.99$, $\tau_0 = 0.5$, $\mu_o = 0.9$, B.C. 4)

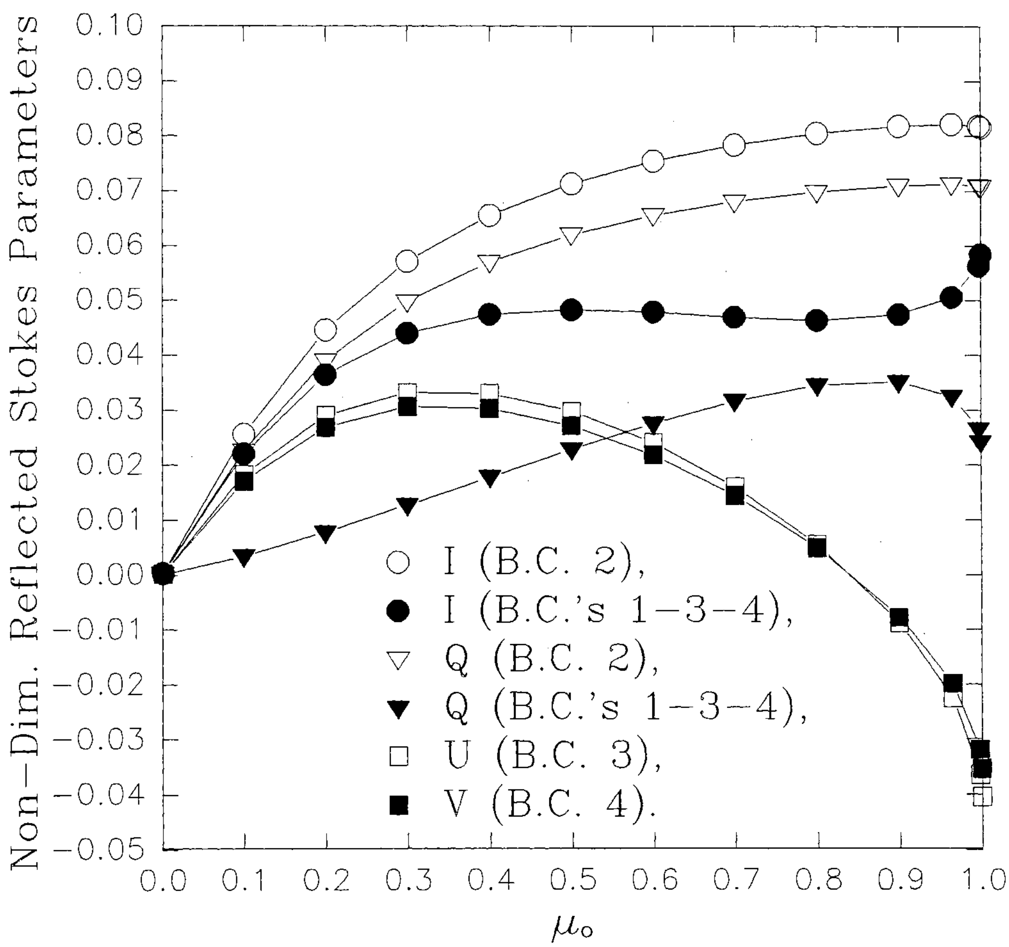

Figure 3-6. Non-Dimensional Reflected Stokes Parameters versus Incident Radiation Angle for Phase Matrix I with Various Boundary Conditions ($L = 2$, $\omega = 0.99$, $\tau_0 = 0.5$, $\mu = 0.6$)

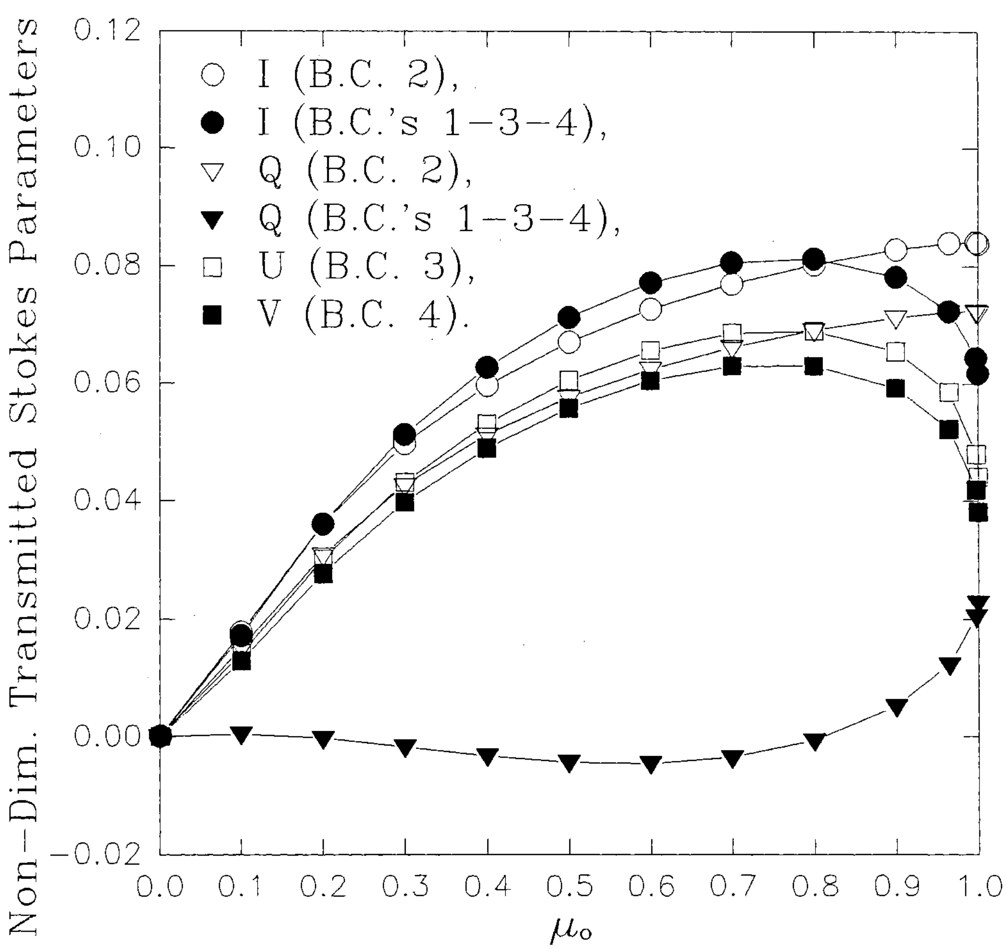

Figure 3-7. Non-Dimensional Transmitted Stokes Parameters versus Incident Radiation Angle for Phase Matrix I with Various Boundary Conditions ($L = 2$, $\omega = 0.99$, $\tau_0 = 0.5$, $\mu = 0.6$)

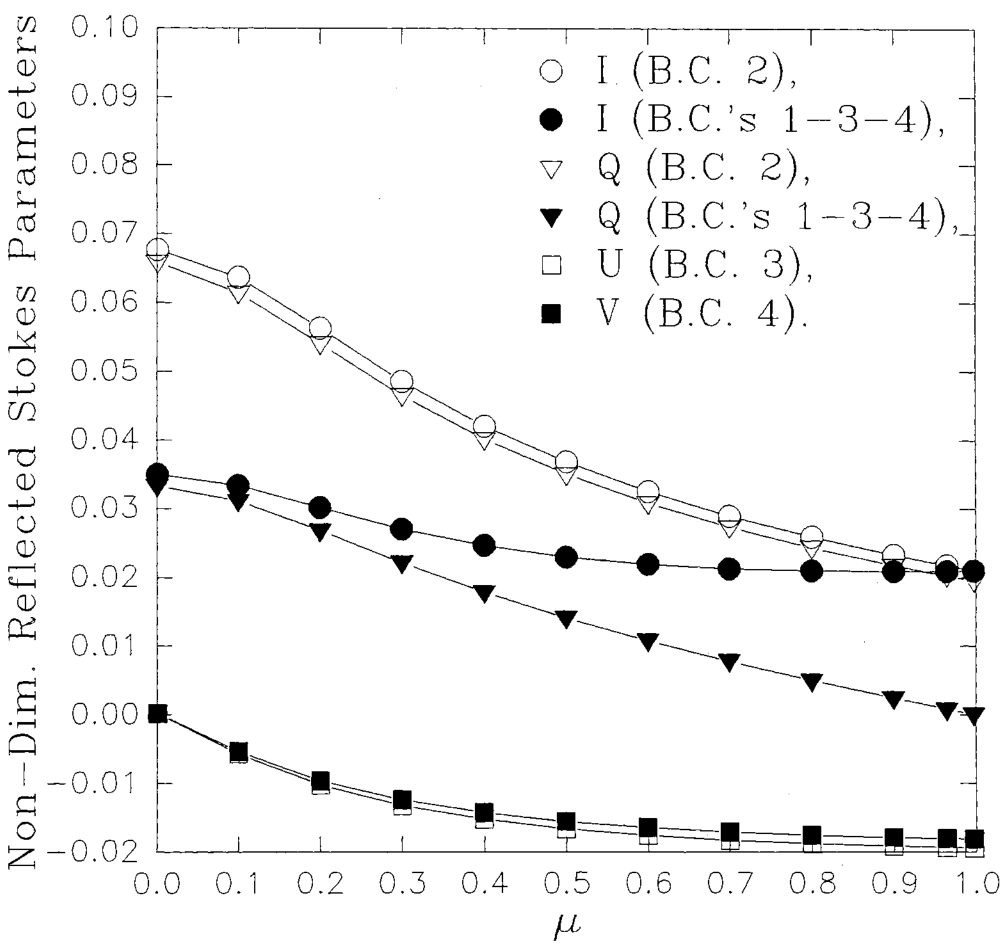

Figure 3-8. Non-Dimensional Reflected Stokes Parameters versus Exit Angle for Phase Matrix I with Various Boundary Conditions ($L = 3$, $\omega = 0.5$, $\tau_0 = 0.5$, $\mu_o = 1$)

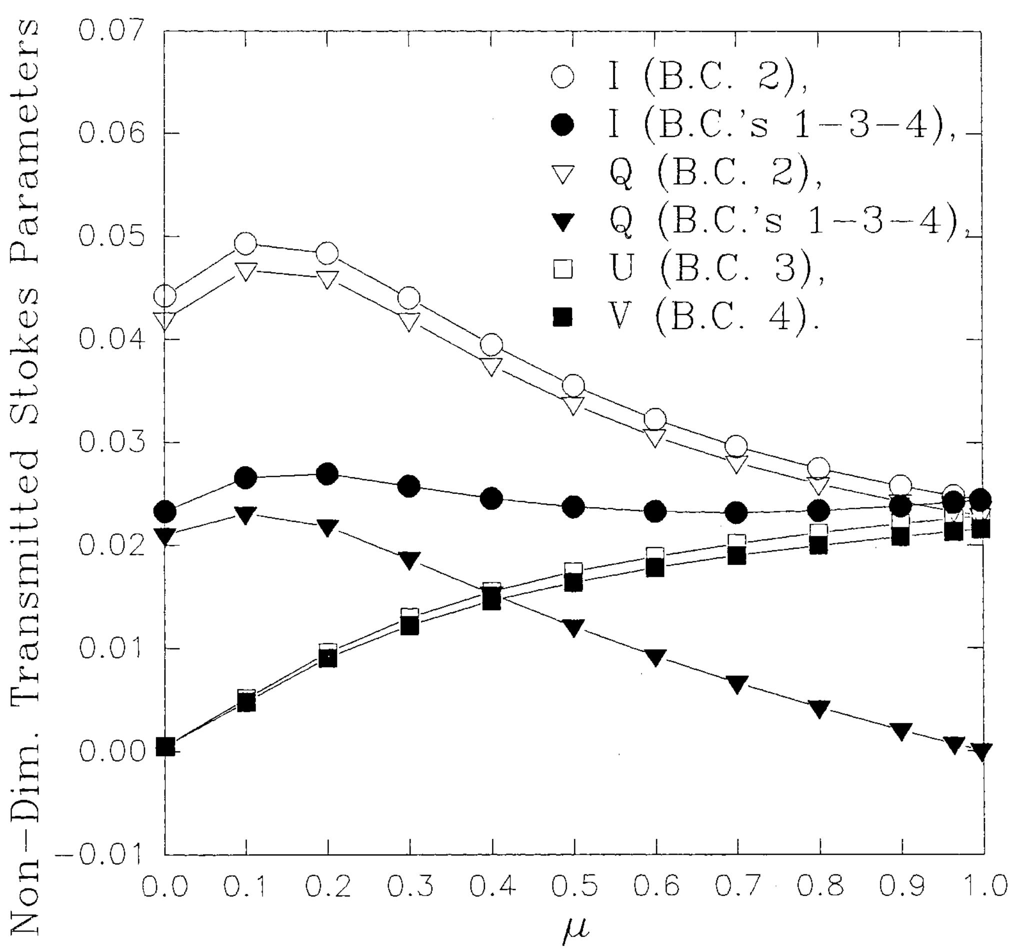

Figure 3-9. Non-Dimensional Transmitted Stokes Parameters versus Exit Angle for Phase Matrix I with Various Boundary Conditions ($L = 3$, $\omega = 0.5$, $\tau_0 = 0.5$, $\mu_o = 1$)

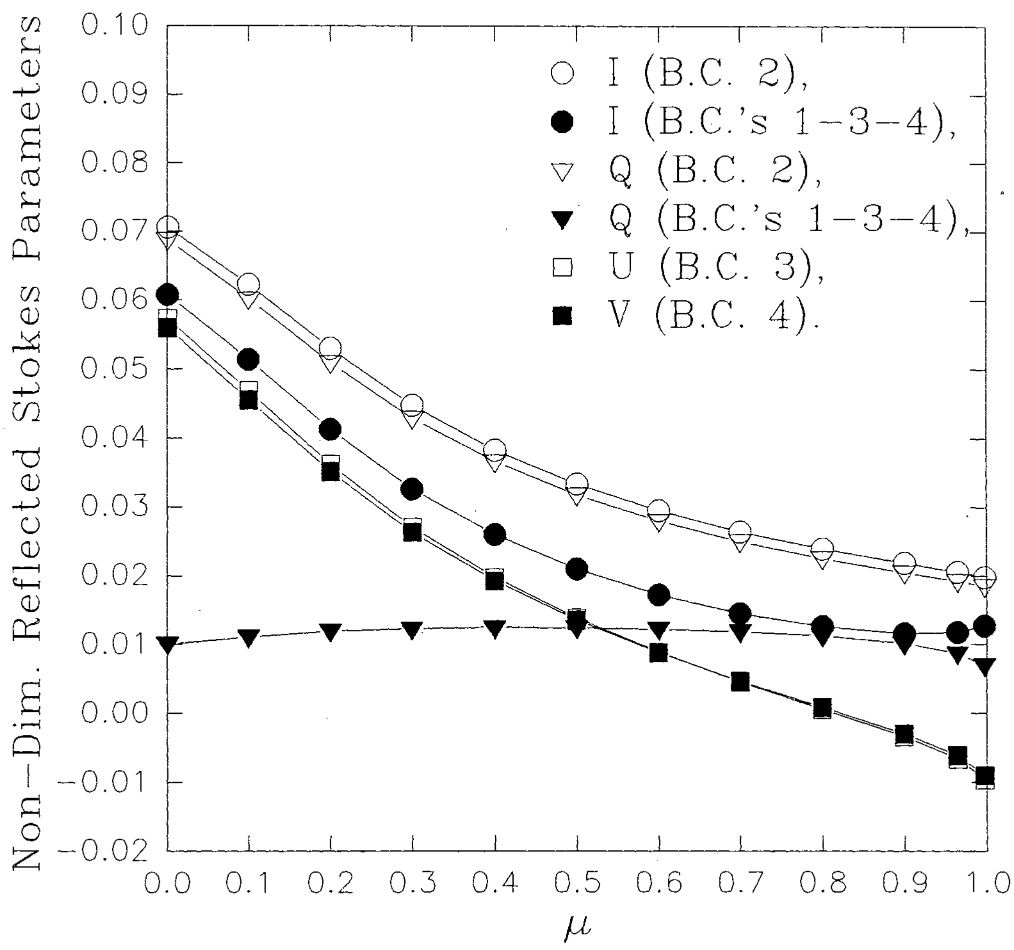

Figure 3-10. Non-Dimensional Reflected Stokes Parameters versus Exit Angle for Phase Matrix I with Various Boundary Conditions ($L = 3$, $\omega = 0.5$, $\tau_0 = 0.5$, $\mu_o = 0.6$)

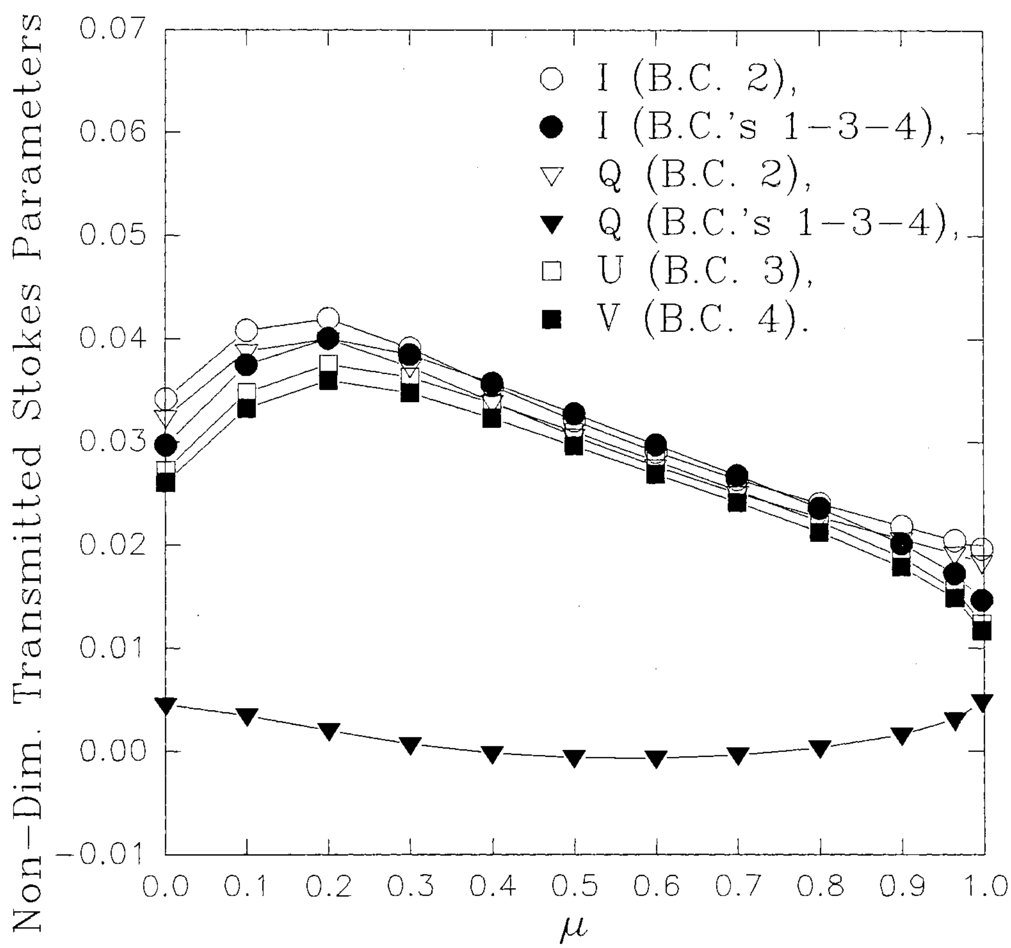

Figure 3-11. Non-Dimensional Transmitted Stokes Parameters versus Exit Angle for Phase Matrix I with Various Boundary Conditions ($L = 3$, $\omega = 0.5$, $\tau_0 = 0.5$, $\mu_o = 0.6$)

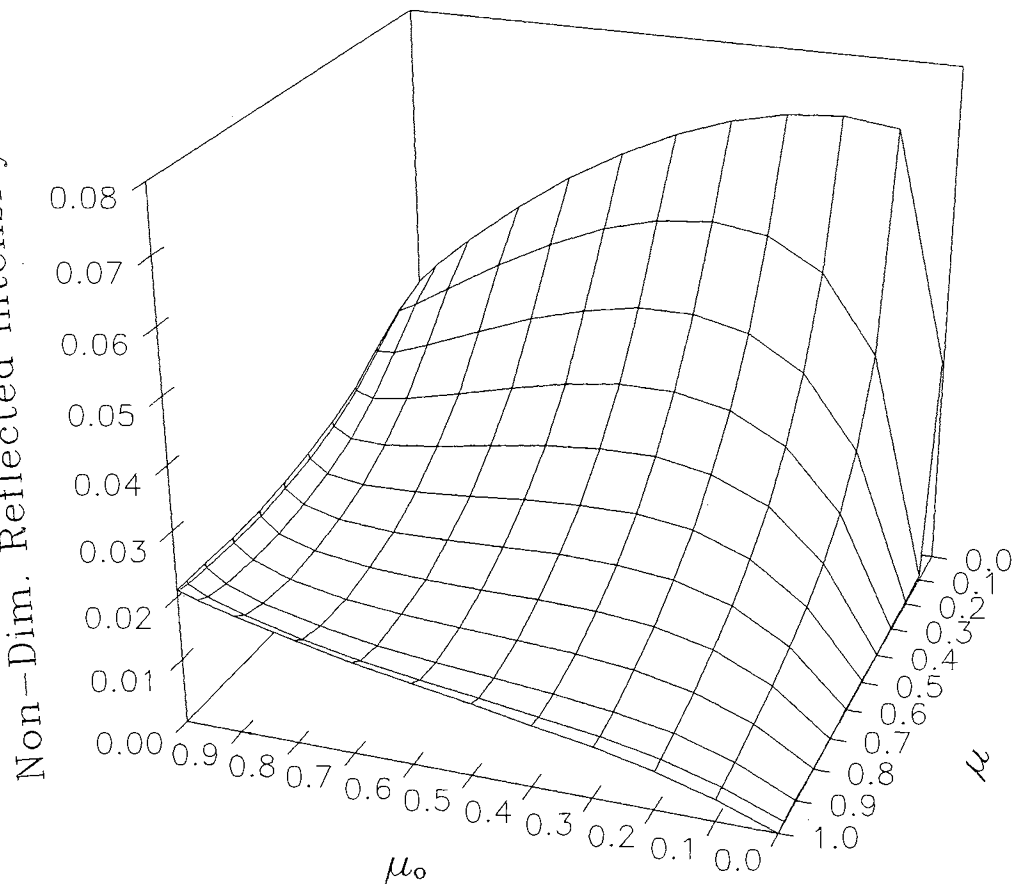

Figure 3-12. Non-Dimensional Reflected Intensity versus Incident Radiation Angle and Exit Angle for Phase Matrix I ($L = 3$, $\omega = 0.5$, $\tau_0 = 0.5$, B.C.'s 1, 3, 4)

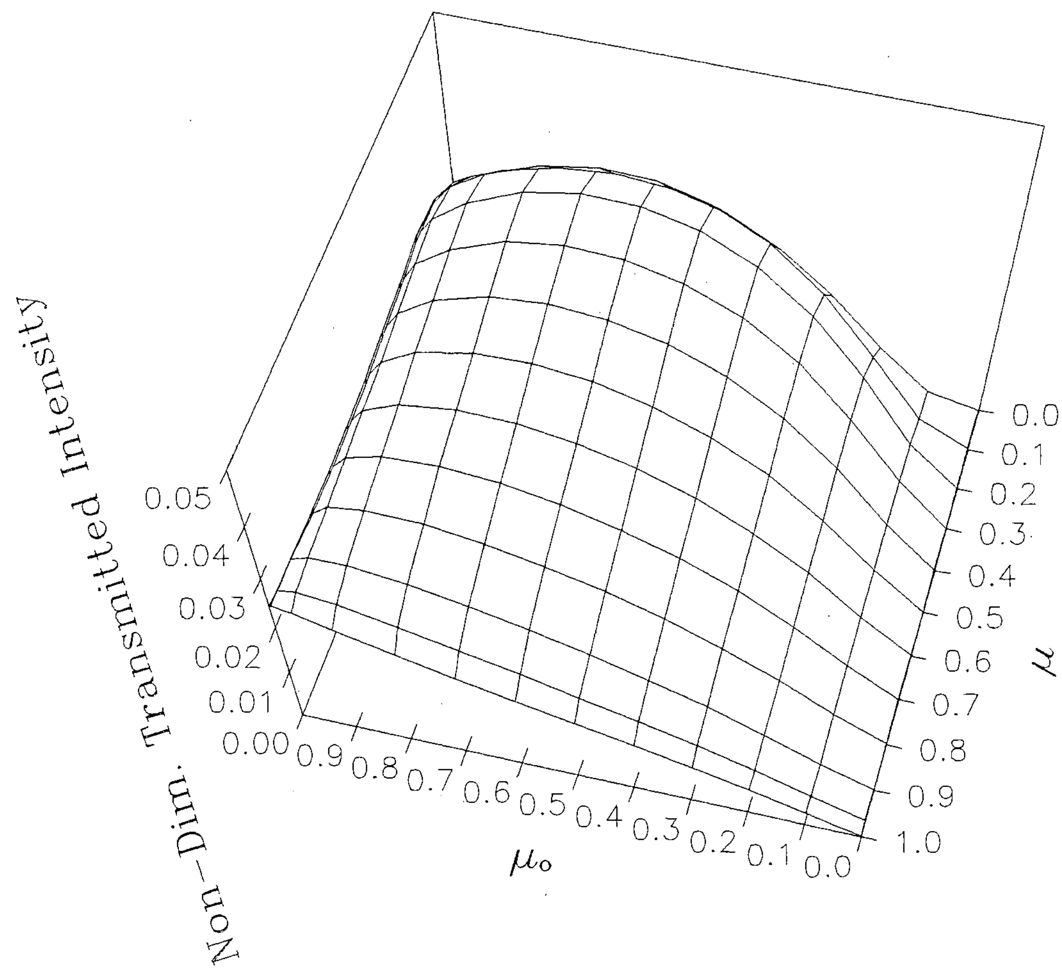

Figure 3-13. Non-Dimensional Transmitted Intensity versus Incident Radiation Angle and Exit Angle for Phase Matrix I ($L = 3$, $\omega = 0.5$, $\tau_0 = 0.5$, B.C.'s 1, 3, 4)

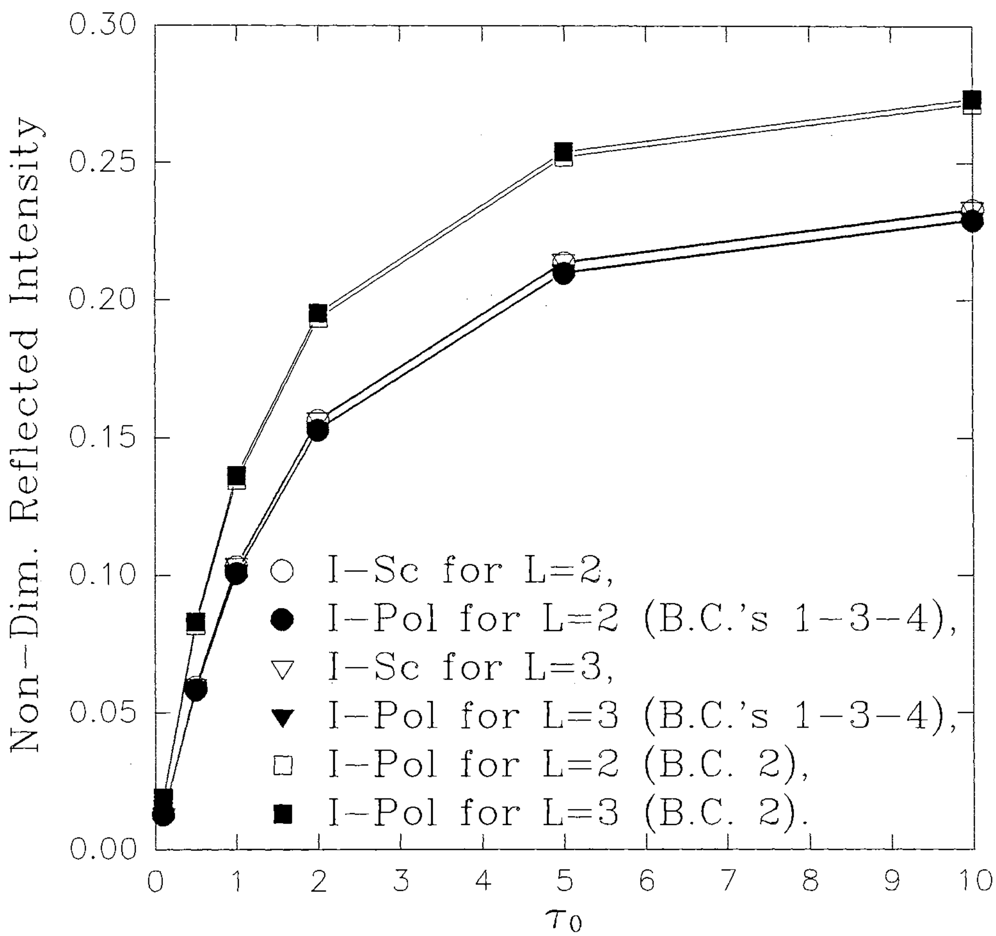

Figure 3-14. Comparison of the Non-Dimensional Reflected Intensity between Exact Scalar Solution and Exact Polarization Solution for Phase Matrix I with Various Boundary Conditions ($\omega = 0.99$, $\mu_o = 1$, $\mu = 0.6$)

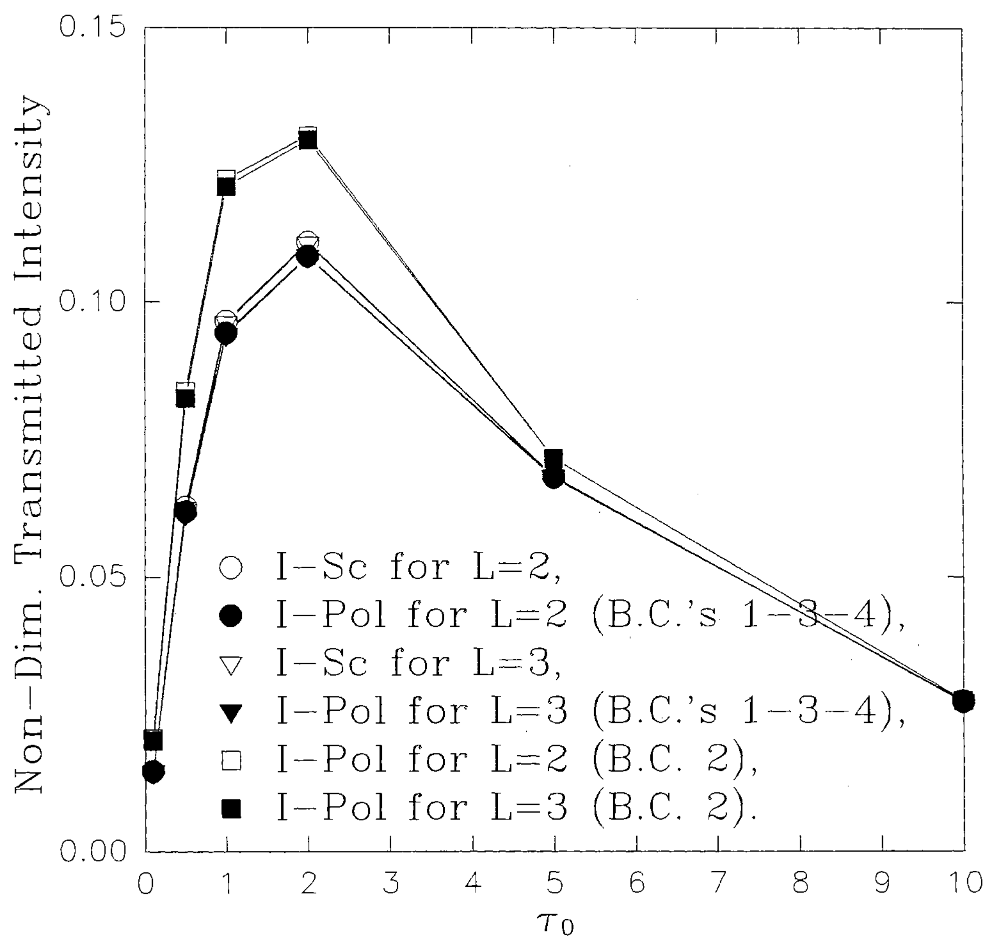

Figure 3-15. Comparison of the Non-Dimensional Transmitted Intensity between Exact Scalar Solution and Exact Polarization Solution for Phase Matrix I with Various Boundary Conditions ($\omega = 0.99$, $\mu_o = 1$, $\mu = 0.6$)

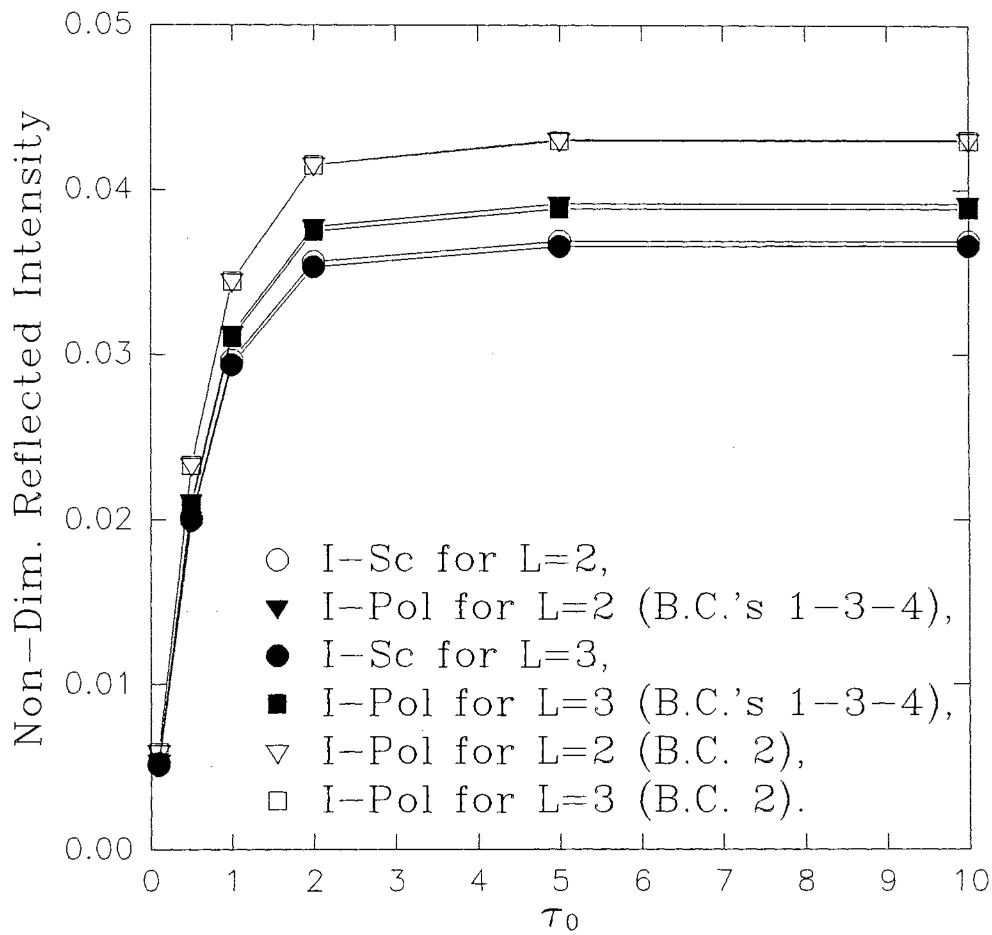

Figure 3-16. Comparison of the Non-Dimensional Reflected Intensity between Exact Scalar Solution and Exact Polarization Solution for Phase Matrix I with Various Boundary Conditions ($\omega = 0.5$, $\mu_o = 1$, $\mu = 0.9$)

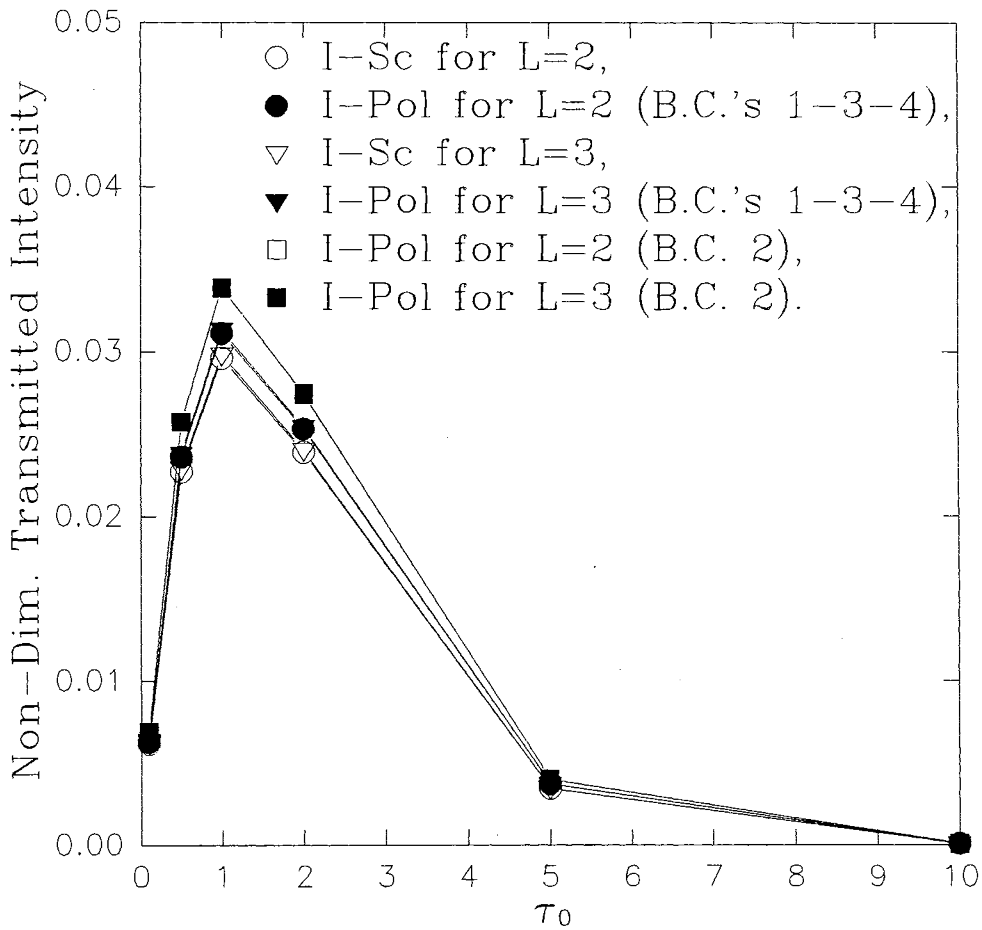

Figure 3-17. Comparison of the Non-Dimensional Transmitted Intensity between Exact Scalar Solution and Exact Polarization Solution for Phase Matrix I with Various Boundary Conditions ($\omega = 0.5$, $\mu_o = 1$, $\mu = 0.9$)

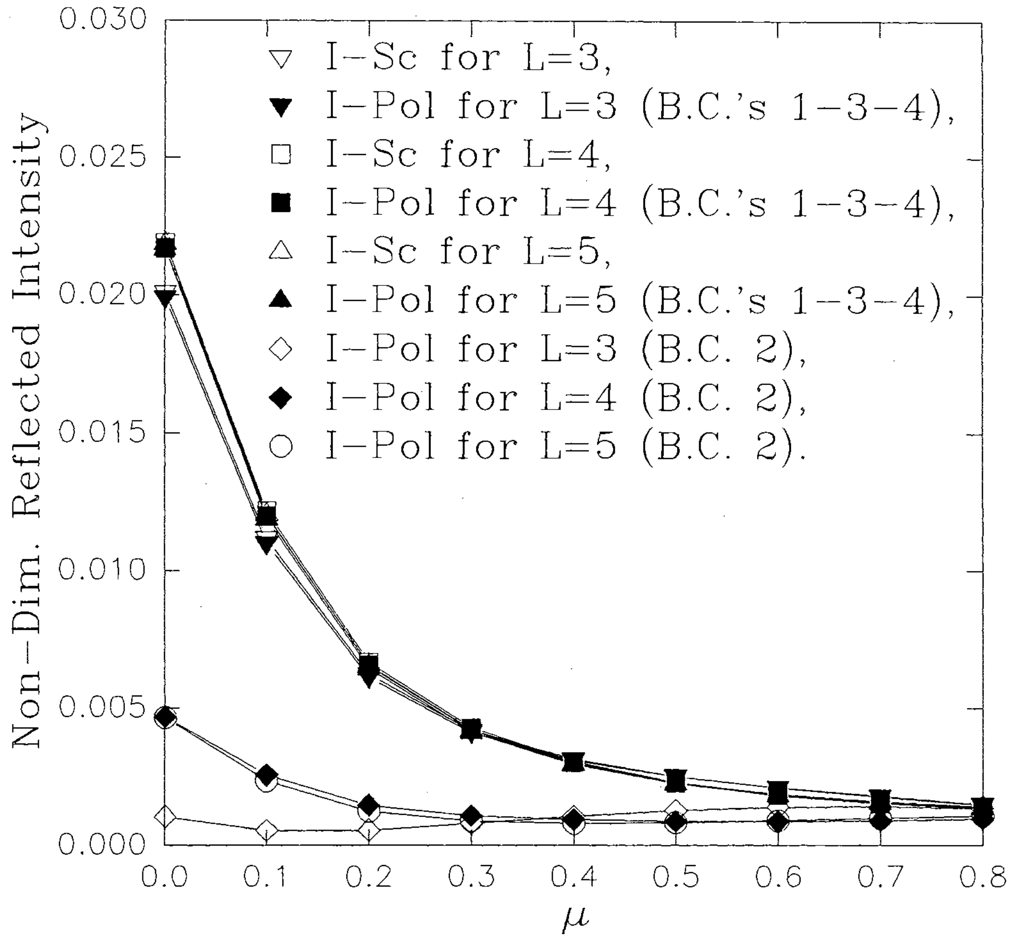

Figure 3-18. Comparison of the Non-Dimensional Reflected Intensity between Exact Scalar Solution and Exact Polarization Solution for Phase Matrix II with Various Boundary Conditions ($\omega = 0.5$, $\tau_0 = 0.1$, $\mu_o = 1$)

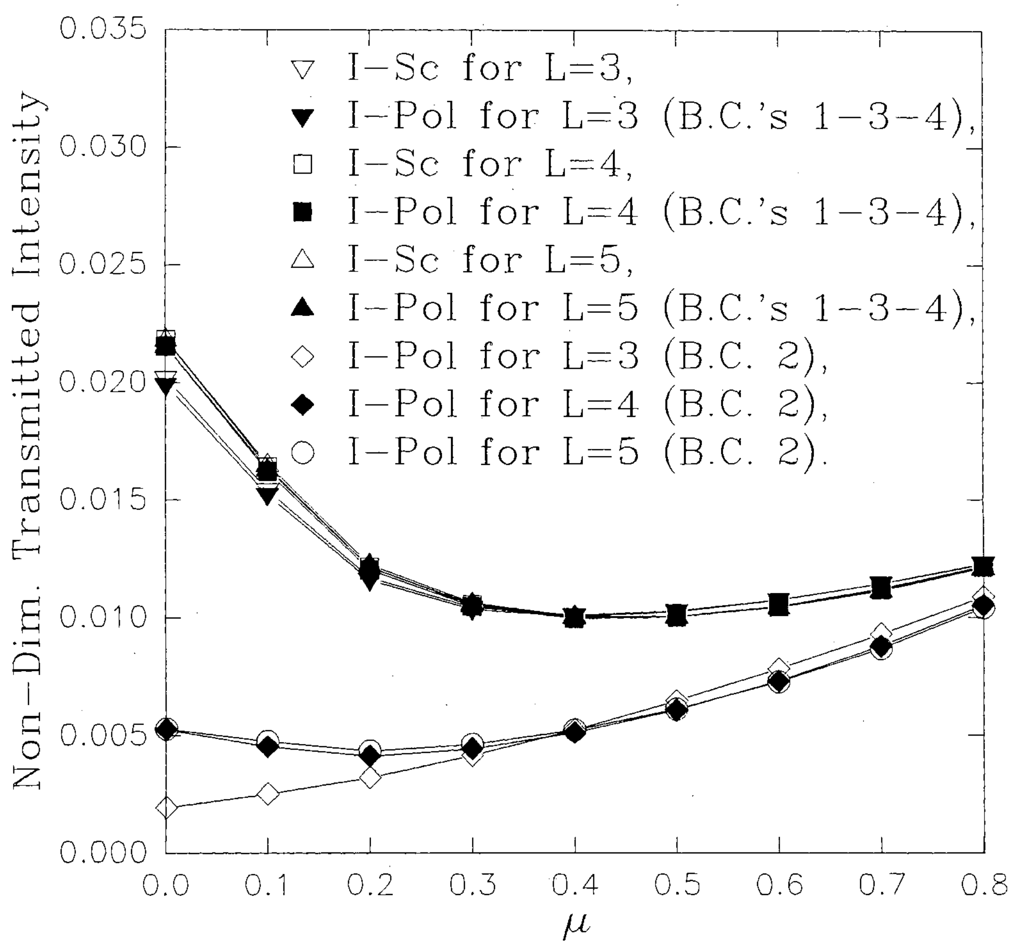

Figure 3-19. Comparison of the Non-Dimensional Transmitted Intensity between Exact Scalar Solution and Exact Polarization Solution for Phase Matrix II with Various Boundary Conditions ($\omega = 0.5$, $\tau_0 = 0.1$, $\mu_o = 1$).

Conclusions

Exact expressions were numerically solved for the fundamental source matrix, reflection and transmission matrices, and reflected and transmitted intensity matrices for polarized light in a plane-parallel medium without reflective boundaries. The Runge-Kutta method and the successive approximation method were applied to obtain numerical solutions, with the assumptions being: collimated polarized incident radiation at angle θ_o exists at the top boundary as a sheet of laser-like beams; no incident radiation enters from the lower boundary; scattering and absorption but no emission occur in the medium; no index of refraction effects exist at the boundaries; the particles are randomly oriented and have at least one plane of symmetry; and the azimuthal angle of the incident radiation (ϕ_o) and the exit azimuthal angle (ϕ) are both equal to zero.

With the phase matrices that were used, the numerical results show that the diffusion approximation can predict the state of the polarization qualitatively well; while the scalar intensity is equal to the intensity including polarization effects for number of Legendre polynomials (L) equal to one. Moreover, with the phase matrices studied, the scalar results estimate the intensity for B.C.'s 1 (unpolarized), 3 (linearly polarized with polarization angle being 45°), and 4 (circularly polarized) very well, but predict the intensity for B.C. 2 (linearly polarized with polarization angle being 0°) poorly for the number of Legendre polynomials (L) greater than one. However, the use of the scalar results will obviously lose the state of polarization.

Long term future work will be focused on including refractive index effects in order to compare more easily with experimental data. Multi-dimensional geometry will also be considered once the convergence problem is solved.

Acknowledgements

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Medium," accepted for presentation at 1999 National Heat Transfer Conference, Albuquerque, New Mexico, August 14-17.

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Appendix III

Figures 3-a1 to 3-a8 are included herein and presented on the following pages. Figures 3-a1 to 3-a6 cover other combination of τ_0's, ω's, and μ's in addition to those of Figs. (3-6) and (3-7), while Figs. 3-a7 to 3-a8 give the other Stokes parameters of Figs. (3-16) and (3-17) with L equal to two.

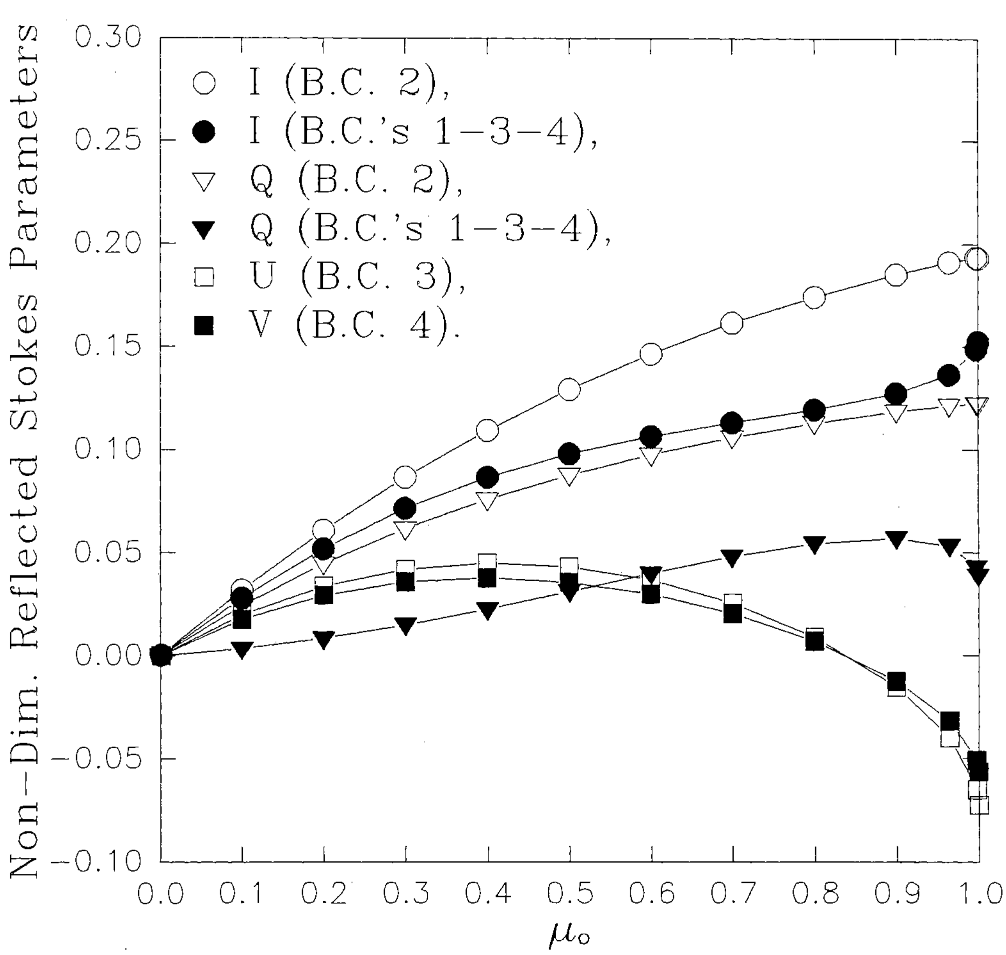

Figure 3-a1. Non-Dimensional Reflected Stokes Parameters versus Incident Radiation Angle for Phase Matrix I with Various Boundary Conditions (L = 2, ω = 0.99, τ_0 = 2.0, μ = 0.6)

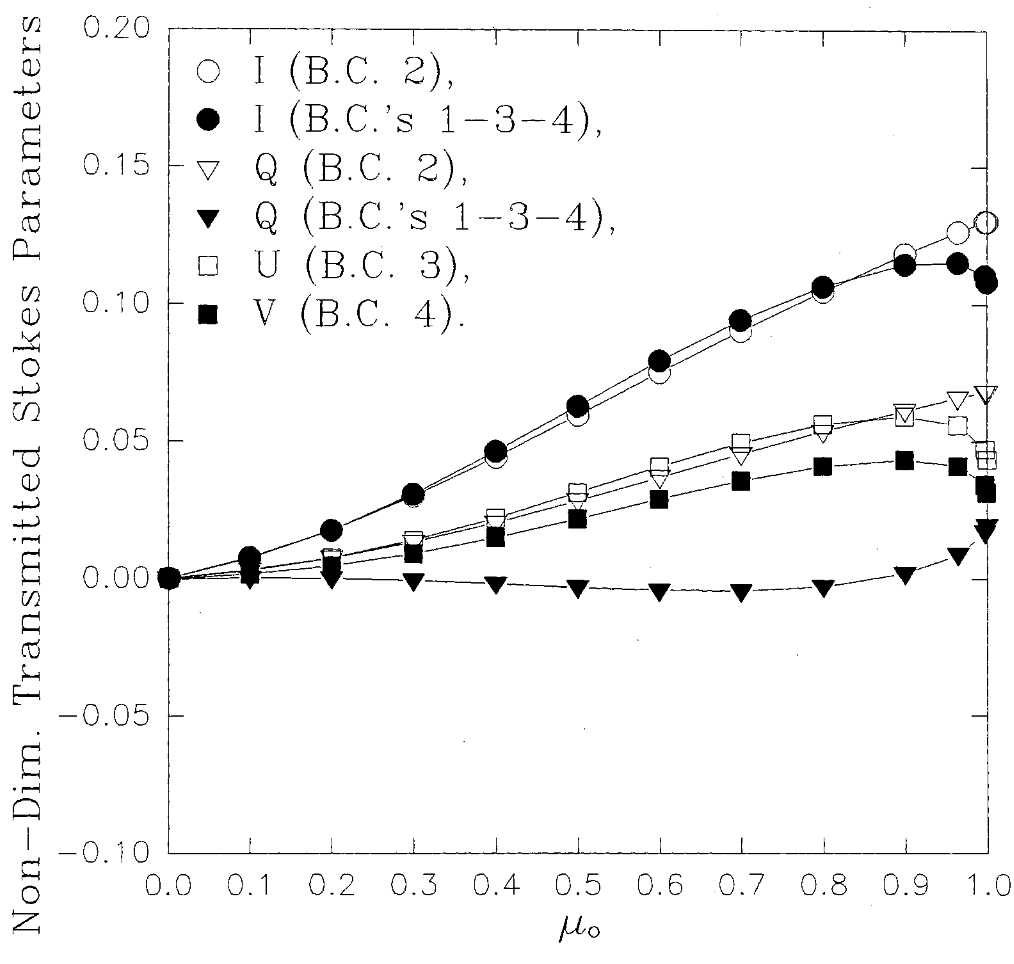

Figure 3-a2. Non-Dimensional Transmitted Stokes Parameters versus Incident Radiation Angle for Phase Matrix I with Various Boundary Conditions ($L = 2$, $\omega = 0.99$, $\tau_0 = 2.0$, $\mu = 0.6$)

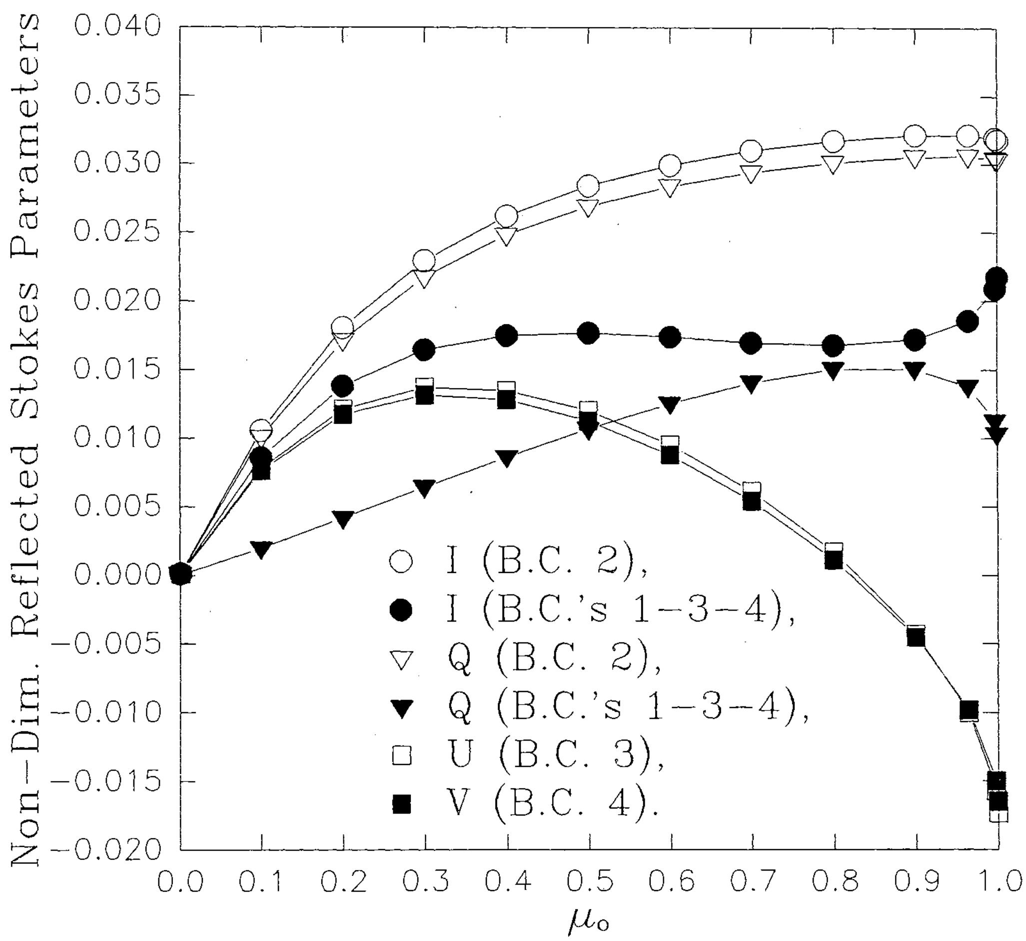

Figure 3-a3. Non-Dimensional Reflected Stokes Parameters versus Incident Radiation Angle for Phase Matrix I with Various Boundary Conditions ($L = 2$, $\omega = 0.5$, $\tau_0 = 0.5$, $\mu = 0.6$)

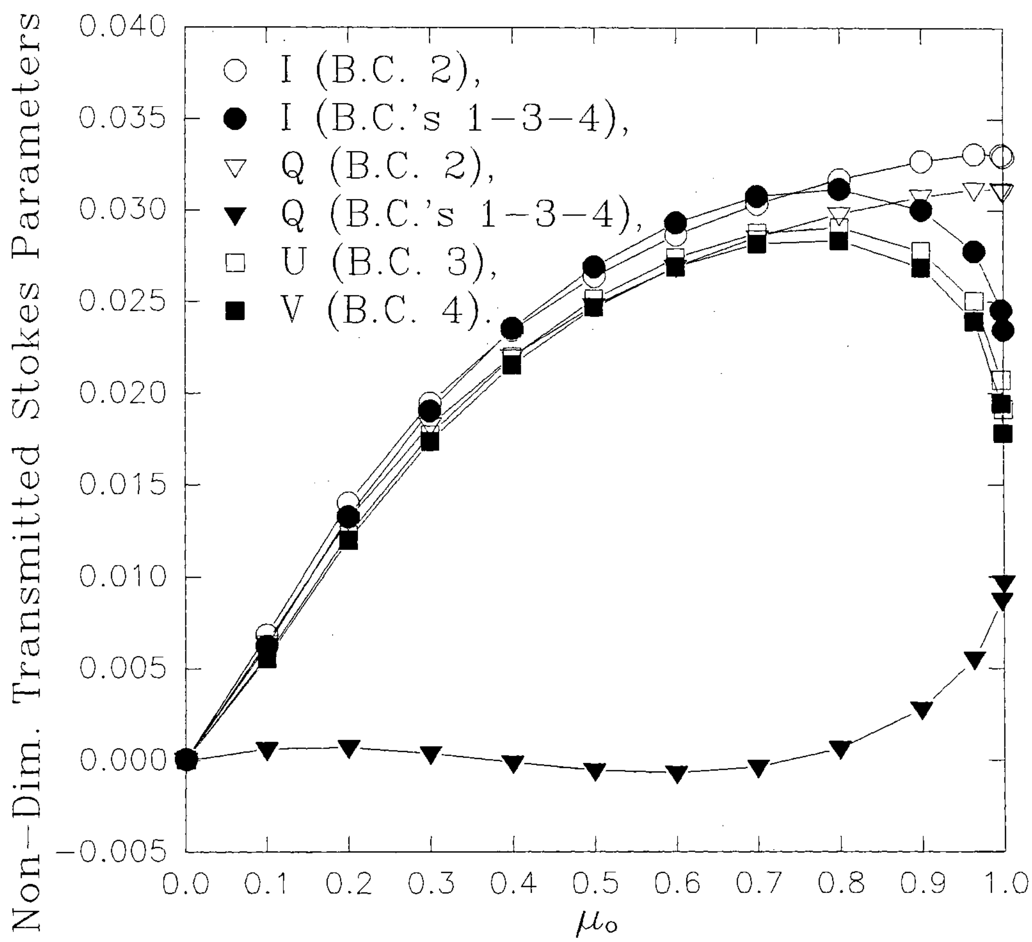

Figure 3-a4. Non-Dimensional Transmitted Stokes Parameters versus Incident Radiation Angle for Phase Matrix I with Various Boundary Conditions ($L = 2$, $\omega = 0.5$, $\tau_0 = 0.5$, $\mu = 0.6$)

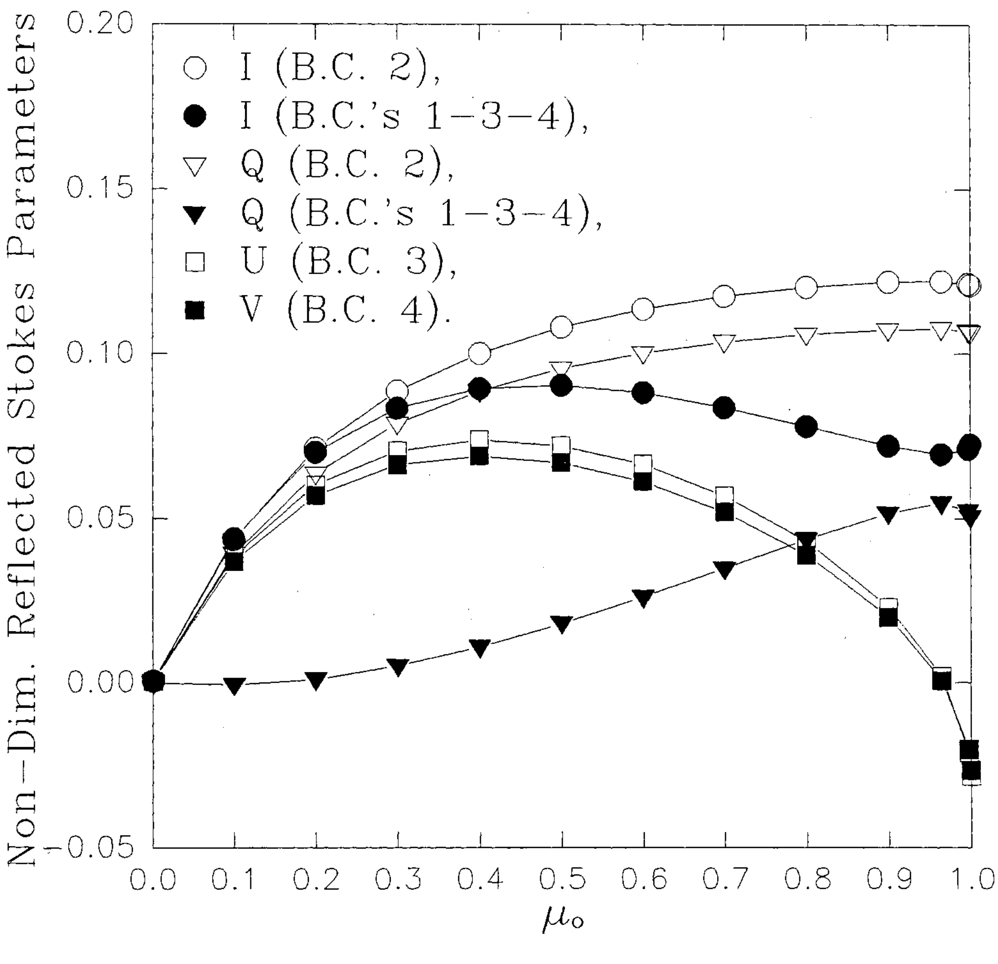

Figure 3-a5. Non-Dimensional Reflected Stokes Parameters versus Incident Radiation Angle for Phase Matrix I with Various Boundary Conditions ($L = 2$, $\omega = 0.99$, $\tau_0 = 0.5$, $\mu = 0.3$)

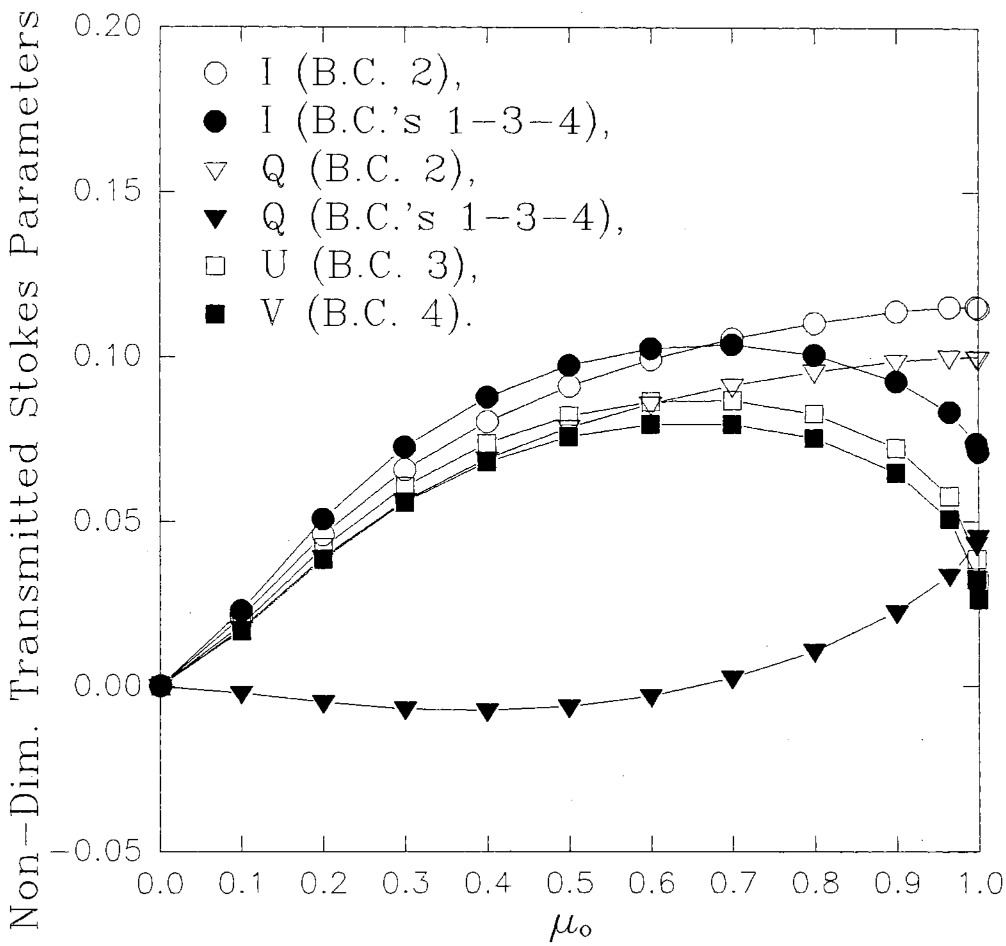

Figure 3-a6. Non-Dimensional Transmitted Stokes Parameters versus Incident Radiation Angle for Phase Matrix I with Various Boundary Conditions ($L = 2$, $\omega = 0.99$, $\tau_0 = 0.5$, $\mu = 0.3$)

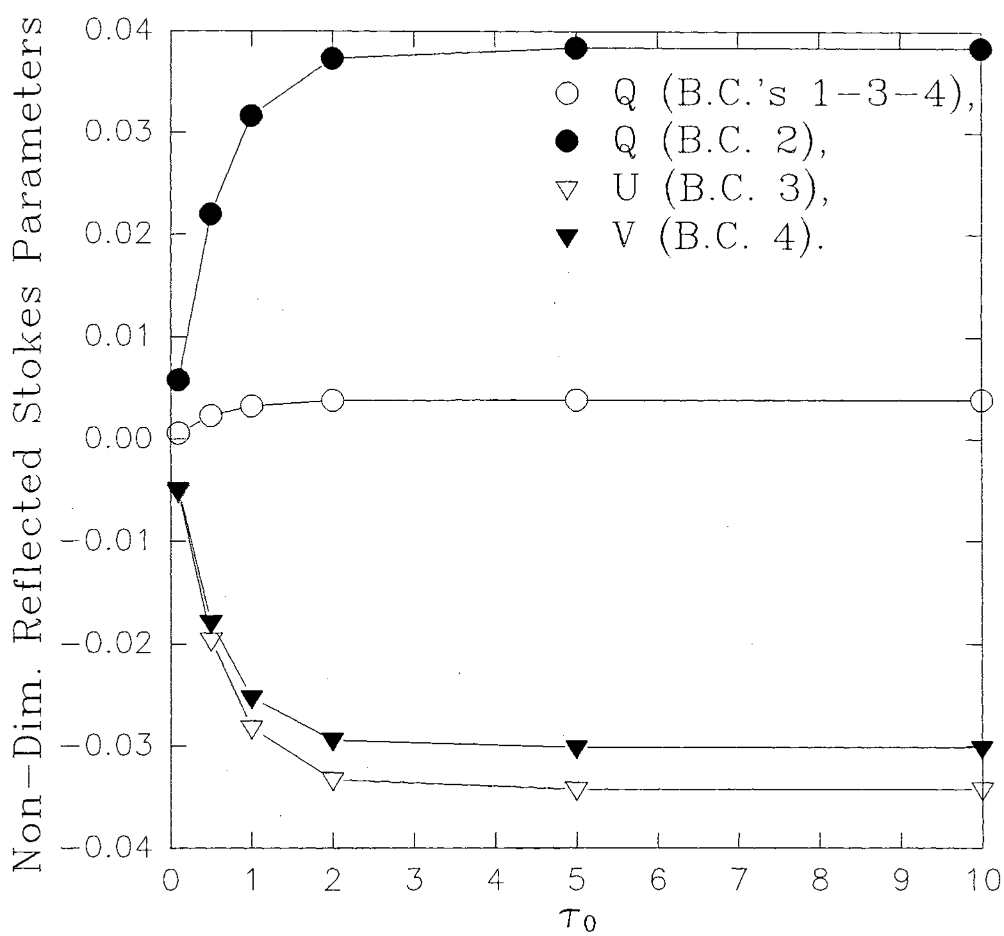

Figure 3-a7. Non-Dimensional Reflected Stokes Parameters versus Optical Thickness for Phase Matrix I with Various Boundary Conditions ($L = 2$, $\omega = 0.5$, $\mu_o = 1$, $\mu = 0.9$)

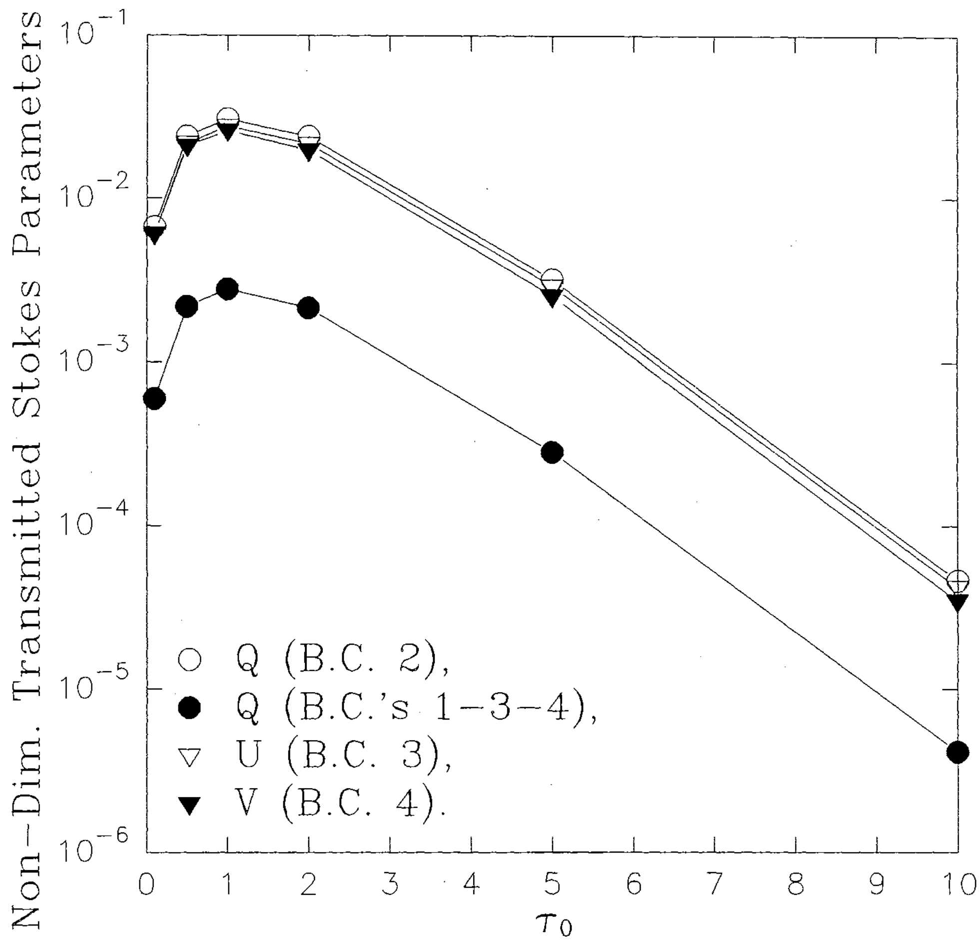

Figure 3-a8. Non-Dimensional Transmitted Stokes Parameters versus Optical Thickness for Phase Matrix I with Various Boundary Conditions ($L = 2$, $\omega = 0.5$, $\mu_o = 1$, $\mu = 0.9$)

CHAPTER IV

SUMMARY OF CONCLUSIONS

The expression for the diffuse intensity I_d for the diffusion approximation was derived by using the classical P_1 approximation with both Mark's and Marshak's boundary conditions, as well as using the modified P_1 method, for polarized light in a plane-parallel medium without reflective boundaries. The work was done with the assumptions being: collimated polarized incident radiation at angle θ_o exists at the top boundary and is a sheet of laser-like beams; no incident radiation enters from the lower boundary; scattering and absorption but no emission occur in the medium; no index of refraction effects exist at the boundaries; the particles are randomly oriented and have at least one plane of symmetry; diffusion approximation; and the azimuthal angle of the incident radiation is equal to zero.

Some selected numerical results are included in order to not only make the comparison between the various P_1 approximations and the exact scalar results, but also to observe the effects of the albedo (0.5, 0.9, 0.95, 0.99, and 1), and optical thickness (5, 10, 15, 20, and 30). A qualitatively good agreement is found between the results from the modified P_1 method and the exact scalar solutions. However, as might be expected, the predictions from the classical P_1 approximation are poor.

In addition, the exact expressions were derived for the general source matrix, fundamental source matrix, reflection and transmission matrices, reflected and transmitted intensity matrices, and reflected and transmitted flux matrices for polarized light in a plane-parallel medium without reflective boundaries. The work was done by using a procedure similar to that of Ambarzumian [1-19] with the assumptions being: collimated polarized incident radiation at angle θ_o exists at the top boundary and is a sheet of laser-like beams; no incident radiation enters from the lower boundary; scattering and absorption but no emission occur in the medium; no index of refraction effects exist at the boundaries; the particles are randomly oriented and have at least one plane of symmetry; and the azimuthal angle of the incident radiation is equal to zero.

Exact expressions were numerically solved for the fundamental source matrix, reflection and transmission matrices, and reflected and transmitted intensity matrices for polarized light in a plane-parallel medium without reflective boundaries. The Runge-Kutta method and the successive approximation method were applied to obtain these exact numerical solutions. With the phase matrices that were used, the numerical results show that the diffusion approximation can predict the state of the polarization qualitatively well; while the scalar intensity is equal to the intensity including polarization effects for number of Legendre polynomials (L) equal to one. Moreover, with the phase matrices studied, the scalar results estimate the intensity for B.C.'s 1 (unpolarized), 3 (linearly polarized with polarization angle being 45°), and 4 (circularly polarized) very well, but predict the intensity for B.C. 2 (linearly polarized with polarization angle being 0°) poorly for the number of Legendre polynomials (L) greater than one. However, the use of the scalar results will obviously lose the state of polarization.

Note that, under the assumptions made in this thesis, the numerical solutions are exact to 4 or 5 decimal places. This exactness of numerical solution does not imply that the solutions “exactly” represent physical reality; but implies only that the numerical solution for the problem as posed is precise with no approximations used in the numerical process.

Future work will be focused first on the convergence problem, and then on including refractive index effects in order to compare more easily with experimental data. In addition, multi-dimensional geometry will also be considered.

APPENDIXES

APPENDIX A

COMPUTER PROGRAM FOR DIFFUSION APPROXIMATION

The computer program of the diffusion approximation for a one-dimensional plane-parallel medium without reflective boundaries is included in this appendix.

```
C-----PROGRAM FOR DIFFUSION APPROXIMATION AND IMPROVING P1
C-----APPROXIMATION WITH BOUNDARY CONDITIONS BEING MARSHAK'S
C-----BC'S.
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 MUO, MU, MUA, MUB, MUC, MUD
      DIMENSION MUO(10), MU(10), BETA(2), RI1(10,10), TI1(10,10),
     *  RI2(10,10), TI2(10,10), TAU0(30)
      OPEN(UNIT=4,FILE='DIFAPXN.DAT')
      OPEN(UNIT=9,FILE='DIFAPXN.OUT')
C-----READ IN AND PRINT OUT THE VALUE OF L(MAXIMUM VALUE OF L=1).
      READ(4,*) L
      WRITE(9,1) L
    1 FORMAT(1X,'NUMBER OF LEGENDRE POLYNOMIALS (L)=',I3)
C-----READ IN AND PRINT OUT THE VALUE OF W.
      READ(4,*) W
      WRITE(9,2) W
    2 FORMAT(1X,'SCATTERING ALBEDO (W)=',F19.17)
C-----READ IN AND PRINT OUT THE VALUES OF EXPANSION COEFFICIENTS
C-----BETA'S.
      READ(4,*) (BETA(I),I=1,L+1)
      WRITE(*,*)(BETA(I),I=1,L+1)
      WRITE(9,*) 'EXPANSION COEFFICIENTS :'
      DO 50 I=1, L+1
      WRITE(9,3) I-1, BETA(I)
    3 FORMAT(6X,'BETA(',I2,')=',F12.10)
   50 CONTINUE
C-----READ IN AND PRINT OUT THE VALUE OF PHI.
```

```
      READ(4,*) PHI
      WRITE(9,4) PHI
    4 FORMAT(1X,'AZIMUTHAL ANGLE (PHI)=',F8.3)
      PAI=3.141592654
      PHI=PHI*PAI/180.D0
C-----READ IN THE VALUE OF NMUOS.
      READ(4,*) NMUOS
      WRITE(*,*) NMUOS
C-----READ IN THE VALUES OF MUO'S.
      READ(4,*) (MUO(J),J=1, NMUOS)
      WRITE(*,*)(MUO(J),J=1, NMUOS)
C-----LET MUO=MU.
      DO 60 I=1, NMUOS
      MU(I)=MUO(I)
   60 CONTINUE
C-----READ IN THE VALUE OF NTAU0S.
      READ(4,*) NTAU0S
      WRITE(*,*) NTAU0S
C-----READ IN THE VALUES OF TAU0'S.
      READ(4,5) (TAU0(I),I=1,NTAU0S)
    5 FORMAT(5F5.2)
C-----CALCULATING THE REFLECTED AND TRANSMITTED INTENSITES FOR
C-----DIFFUSION(RI1 & TI1) AND IMPROVING P1(RI2 & TI2)
C-----APPROXIMATIONS WITH MARSHAK'S BC'S.
      IF(W .EQ. 1.0) THEN
      A1=BETA(2)/3.D0-1.D0
      DO 100 I=1,NTAU0S
      TAUA=TAU0(I)
      WRITE(9,*)
      WRITE(9,*) '----------------------------------------------------'
      WRITE(9,6) TAUA
    6 FORMAT(1X,'OPTICAL THICKNESS =',F12.8)
      WRITE(9,*)
      DO 200 J=1,NMUOS
      MUA=MUO(J)
      DO 300 K=1,NMUOS
      MUB=MU(K)
      B1=DEXP(-TAUA/MUA)
      C1=DEXP(-TAUA/MUB)
      D1=1.D0-B1
      E1=1.D0-C1
      F1=(1.D0-MUA**2)**0.5
      G1=(1.D0-MUB**2)**0.5
      H1=1.D0/MUB+1.D0/MUA
      O1=DEXP(-H1*TAUA)
      P1=1.D0/MU(K)-1.D0/MUO(J)
```

```
      RI1(K,J)=(-MUO(J)/(A1*TAUA-4.D0/3.D0)*(3.D0/4.D0*MUO(J)*D1-B1/A1
     *  *(1.D0/2.D0-BETA(2)/6.D0)+1.D0/2.D0)*(-MU(K)-2.D0/3.D0)-1.D0/4.D0/A1
     *  *(3.D0*MU(K)*MUO(J)-BETA(2)*MU(K)*MUO(J)+BETA(2)*G1*F1
     *  *DCOS(PHI))+MUO(J)/2.D0)/PAI
      TI1(K,J)=(-MUO(J)/(A1*TAUA-4.D0/3.D0)*(3.D0/4.D0*MUO(J)*D1-B1/A1
     *  *(1.D0/2.D0-BETA(2)/6.D0)+1.D0/2.D0)*(A1*TAUA+MU(K)-2.D0/3.D0)
     *  -1.D0/4.D0/A1*B1*(-3.D0*MU(K)*MUO(J)+BETA(2)*MU(K)*MUO(J)
     *  +BETA(2)*G1*F1*DCOS(PHI))+MUO(J)*(1.D0/2.D0+3.D0/4.D0*MUO(J)
     *  *D1))/PAI
      RI2(K,J)=(-MUO(J)/(A1*TAUA-4.D0/3.D0)*(3.D0/4.D0*MUO(J)*D1+B1/2.D0
     *  +1.D0/2.D0)*((-A1*TAUA+MU(K))*C1-MU(K))+(2.D0/3.D0*MUO(J)/(A1*
     *  TAUA-4.D0/3.D0)*(3.D0/4.D0*MUO(J)*D1+B1/2.D0+1.D0/2.D0)+MUO(J)
     *  *(1.D0/2.D0+3.D0/4.D0*MUO(J)))*E1-BETA(2)/12.D0/A1*((-O1+1.D0)
     *  /H1)*(3.D0*MUO(J)-BETA(2)*MUO(J)+BETA(2)/MU(K)*F1*G1*
     *  DCOS(PHI))+1.D0/4.D0/MU(K)*((-O1+1.D0)/H1)*(1.D0-BETA(2)*
     *  MU(K)*MUO(J)-3.D0*(MUO(J)**2)+BETA(2)*G1*F1*DCOS(PHI)))/PAI
      IF(P1 .EQ. 0.0) THEN
      TI2(K,J)=999999.D0
      ELSE
      TI2(K,J)=(-MUO(J)/(A1*TAUA-4.D0/3.D0)*(3.D0/4.D0*MUO(J)*D1+B1/2.D0
     *  +1.D0/2.D0)*(A1*TAUA+E1*MU(K))+(2.D0/3.D0*MUO(J)/(A1*TAUA-4.D0/
     *  3.D0)*(3.D0/4.D0*MUO(J)*D1+B1/2.D0+1.D0/2.D0)+MUO(J)*(1.D0/2.D0
     *  +3.D0/4.D0*MUO(J)))*E1-BETA(2)/12.D0/A1*((B1-C1)/P1)*(-3.D0*
     *  MUO(J)+BETA(2)*MUO(J)+BETA(2)/MU(K)*F1*G1*DCOS(PHI))+1.D0
     *  /4.D0/MU(K)*((B1-C1)/P1)*(1.D0+BETA(2)*MU(K)*MUO(J)-3.D0*
     *  (MUO(J)**2)+BETA(2)*G1*F1*DCOS(PHI)))/PAI
      ENDIF
  300 CONTINUE
  200 CONTINUE
      WRITE(9,*) 'FOR DIFFUSION APPROXIMATION :'
      DO 500 JJ=1, NMUOS, 2
      WRITE(9,*)
      WRITE(9,30) MUO(JJ), MUO(JJ+1)
   30 FORMAT(34X,'MUO=',F15.12,25X,'MUO=',F15.12)
      WRITE(9,31)
   31 FORMAT(9X,'MU',20X,'RI',20X,'TI',20X,'RI',20X,'TI')
      WRITE(9,32) (MU(II),RI1(II,JJ),TI1(II,JJ),RI1(II,JJ+1),
     *  TI1(II,JJ+1),II=1,NMUOS)
   32 FORMAT(5(1PE20.12,2X))
  500 CONTINUE
      WRITE(9,*)
      WRITE(9,*) 'FOR IMPROVING P1 APPROXIMATION :'
      DO 600 KK=1, NMUOS, 2
      WRITE(9,*)
      WRITE(9,33) MUO(KK), MUO(KK+1)
   33 FORMAT(34X,'MUO=',F15.12,25X,'MUO=',F15.12)
```

```
      WRITE(9,34)
   34 FORMAT(9X,'MU',20X,'RI',20X,'TI',20X,'RI',20X,'TI')
      WRITE(9,35) (MU(II),RI2(II,KK),TI2(II,KK),RI2(II,KK+1),
     *  TI2(II,KK+1),II=1,NMUOS)
   35 FORMAT(5(1PE20.12,2X))
  600 CONTINUE
      WRITE(9,*)
  100 CONTINUE
      ELSE
      A2=(3.D0-W*BETA(2)-3.D0*W*BETA(1)+W**2*BETA(1)*BETA(2))**0.5
      B2=1.D0-W/3.D0*BETA(2)
      C2=-3*BETA(1)+W*BETA(1)*BETA(2)-BETA(2)
      E1=B2+2.D0/3.D0*A2
      E2=B2-2.D0/3.D0*A2
      DO 1000 I=1,NTAU0S
      TAUB=TAU0(I)
      WRITE(9,*)
      WRITE(9,*) '--------------------------------------------------'
      WRITE(9,10) TAUB
   10 FORMAT(1X,'OPTICAL THICKNESS =',F12.8)
      WRITE(9,*)
      DO 1100 J=1,NMUOS
      MUC=MUO(J)
      DO 1200 K=1,NMUOS
      MUD=MU(K)
      D2=DEXP(-TAUB/MUC)
      D2A=DEXP(-TAUB/MUD)
      G2=DEXP(A2*TAUB)
      H2=DEXP(-A2*TAUB)
      O2=1.D0/MUD+A2
      P2=1.D0/MUD-A2
      Q2=1.D0/MUD-1.D0/MUC
      R2=1.D0/MUD+1.D0/MUC
      S2=DEXP(O2*TAUB)
      S2A=DEXP(-O2*TAUB)
      T2=DEXP(P2*TAUB)
      T2A=DEXP(-P2*TAUB)
      U2=DEXP(Q2*TAUB)
      V2=DEXP(-R2*TAUB)
      W2=(1.D0-MUD**2)**0.5
      X2=(1.D0-MUC**2)**0.5
      Y2=E1*G2+E2*D2
      Z2=E1*E1*G2-E2*E2*H2
      F2=E2*H2+E1*D2
      RI1(K,J)=(-Y2/Z2*(W/4.D0*C2/(1.D0/(MUO(J)**2)-A2**2)*(-2.D0*E2*
     *  B2*D2/Y2+B2+2.D0/3.D0/MUO(J))+W/6.D0*BETA(2)*MUO(J))*(-F2*(1.D0
```

```
     *  +MU(K)*A2/B2)/Y2+(1.D0-MU(K)*A2/B2))+W/4.D0*(C2/(1.D0/(MUO(J)
     *  **2)-A2**2)*(1.D0-MU(K)/MUO(J)/B2-2.D0*(B2+MU(K)*A2)*D2/Y2)+
     *  BETA(2)/B2*(-MU(K)*MUO(J)+W2*X2*DCOS(PHI))))/PAI
        TI1(K,J)=(-Y2/Z2*(W/4.D0*C2/(1.D0/(MUO(J)**2)-A2**2)*(-2.D0*E2*
     *  B2*D2/Y2+B2+2.D0/3.D0/MUO(J))+W/6.D0*BETA(2)*MUO(J))*(-F2*G2*
     *  (1.D0-MU(K)*A2/B2)/Y2+H2*(1.D0+MU(K)*A2/B2))+W/4.D0*(C2*D2/
     *  (1.D0/(MUO(J)**2)-A2**2)*(1.D0+MU(K)/MUO(J)/B2-2.D0*(B2-MU(K)
     *  *A2)*G2/Y2)+BETA(2)*D2/B2*(MU(K)*MUO(J)+W2*X2*DCOS(PHI))))/PAI
        RI2(K,J)=(W/MU(K)/Y2*(-Y2/Z2*(W/4.D0*C2/(1.D0/(MUO(J)**2)-A2**2)*
     *  (-2.D0*E2*B2*D2/Y2+B2+2.D0/3.D0/MUO(J))+W/6.D0*BETA(2)*MUO(J))*
     *  F2+W/2.D0*C2*D2*B2/(1.D0/(MUO(J)**2)-A2**2))*(T2A-1.D0)/P2*
     *  (BETA(1)+1.D0/3.D0*BETA(2)*MU(K)*A2/B2)+Y2/Z2*(W/4.D0*C2/
     *  (1.D0/(MUO(J)**2)-A2**2)*(-2.D0*E2*B2*D2/Y2+B2+2.D0/3.D0/MUO(J))
     *  +W/6.D0*BETA(2)*MUO(J))*W/MU(K)*(S2A-1.D0)/O2*(BETA(1)-1.D0/3.D0
     *  *BETA(2)*MU(K)*A2/B2)-W**2/4.D0/MU(K)*C2/(1.D0/(MUO(J)**2)-A2
     *  **2)*(V2-1.D0)/R2*(BETA(1)-1.D0/3.D0*BETA(2)*MU(K)/B2/MUO(J))
     *  -W/4.D0/MU(K)*(V2-1.D0)/R2*(BETA(1)-BETA(2)*MU(K)*MUO(J)-W/3.D0
     *  *(BETA(2)**2)*MU(K)*MUO(J)/B2+BETA(2)*W2*X2*DCOS(PHI)+W/3.D0*
     *  (BETA(2)**2)*W2*X2*DCOS(PHI)/B2))/PAI
        IF(Q2 .EQ. 0.0) THEN
        TI2(K,J)=999999.D0
        ELSE
        TI2(K,J)=(-W/MU(K)/Y2*(-Y2/Z2*(W/4.D0*C2/(1.D0/(MUO(J)**2)-A2**2)*
     *  (-2.D0*E2*B2*D2/Y2+B2+2.D0/3.D0/MUO(J))+W/6.D0*BETA(2)*MUO(J))*
     *  F2+W/2.D0*C2*D2*B2/(1.D0/(MUO(J)**2)-A2**2))*D2A*(S2-1.D0)/O2*
     *  (BETA(1)-1.D0/3.D0*BETA(2)*MU(K)*A2/B2)-Y2/Z2*(W/4.D0*C2/
     *  (1.D0/(MUO(J)**2)-A2**2)*(-2.D0*E2*B2*D2/Y2+B2+2.D0/3.D0/MUO(J))
     *  +W/6.D0*BETA(2)*MUO(J))*W/MU(K)*D2A*(T2-1.D0)/P2*(BETA(1)+1.D0/
     *  3.D0*BETA(2)*MU(K)*A2/B2)+W**2/4.D0/MU(K)*D2A*C2/Q2/
     *  (1.D0/(MUO(J)**2)-A2**2)*(U2-1.D0)*(BETA(1)+1.D0/3.D0*BETA(2)*
     *  MU(K)/B2/MUO(J))+W/4.D0/MU(K)*D2A*(U2-1.D0)/Q2*(BETA(1)+BETA(2)
     *  *MU(K)*MUO(J)+W/3.D0*(BETA(2)**2)*MU(K)*MUO(J)/B2+BETA(2)*W2
     *  *X2*DCOS(PHI)+W/3.D0*(BETA(2)**2)*W2*X2*DCOS(PHI)/B2))/PAI
        ENDIF
1200    CONTINUE
1100    CONTINUE
        WRITE(9,*) 'FOR DIFFUSION APPROXIMATION :'
        DO 1300 JJ=1, NMUOS, 2
        WRITE(9,*)
        WRITE(9,36) MUO(JJ), MUO(JJ+1)
  36    FORMAT(34X,'MUO=',F15.12,25X,'MUO=',F15.12)
        WRITE(9,37)
  37    FORMAT(9X,'MU',20X,'RI',20X,'TI',20X,'RI',20X,'TI')
        WRITE(9,38) (MU(II),RI1(II,JJ),TI1(II,JJ),RI1(II,JJ+1),
     *  TI1(II,JJ+1),II=1,NMUOS)
  38    FORMAT(5(1PE20.12,2X))
```

```
1300 CONTINUE
     WRITE(9,*)
     WRITE(9,*) 'FOR IMPROVING P1 APPROXIMATION :'
     DO 1400 KK=1, NMUOS, 2
     WRITE(9,*)
     WRITE(9,39) MUO(KK), MUO(KK+1)
  39 FORMAT(34X,'MUO=',F15.12,25X,'MUO=',F15.12)
     WRITE(9,40)
  40 FORMAT(9X,'MU',20X,'RI',20X,'TI',20X,'RI',20X,'TI')
     WRITE(9,41) (MU(II),RI2(II,KK),TI2(II,KK),RI2(II,KK+1),
   *  TI2(II,KK+1),II=1,NMUOS)
  41 FORMAT(5(1PE20.12,2X))
1400 CONTINUE
     WRITE(9,*)
1000 CONTINUE
     ENDIF
     STOP
     END
```

APPENDIX B

SAMPLE OF OUTPUT DATA FOR DIFFUSION APPROXIMATION COMPUTER PROGRAM

An example of output data from the diffusion approximation computer program is as follows, where RI = reflected intensity and TI = transmitted intensity.

NUMBER OF LEGENDRE POLYNOMIALS (L)= 1
SCATTERING ALBEDO (W)= .99000000000000000
EXPANSION COEFFICIENTS :
BETA(0)=1.0000000000
BETA(1)= .1400343465
AZIMUTHAL ANGLE (PHI)= .000

OPTICAL THICKNESS = .10000000

FOR DIFFUSION APPROXIMATION :

MUO= 1.000000000000

MU	RI	TI
1.000000000000E+00	1.751965800288E-02	1.992730590439E-02
9.980000000000E-01	1.749863441327E-02	1.990339313731E-02
9.000000000000E-01	1.646847852270E-02	1.873166755013E-02
8.000000000000E-01	1.541729904253E-02	1.753602919586E-02
7.000000000000E-01	1.436611956236E-02	1.634039084160E-02
6.000000000000E-01	1.331494008219E-02	1.514475248734E-02
5.000000000000E-01	1.226376060201E-02	1.394911413307E-02
4.000000000000E-01	1.121258112184E-02	1.275347577881E-02
3.000000000000E-01	1.016140164167E-02	1.155783742455E-02
2.000000000000E-01	9.110222161496E-03	1.036219907028E-02
1.000000000000E-01	8.059042681323E-03	9.166560716020E-03

1.000000000000E-03	7.018374995952E-03	7.982878745299E-03

MUO= .700000000000

MU	RI	TI
1.000000000000E+00	1.752418099609E-02	1.914477944435E-02
9.980000000000E-01	1.802531433357E-02	1.957445671437E-02
9.000000000000E-01	2.007328654150E-02	2.111733545215E-02
8.000000000000E-01	2.037742501678E-02	2.114377912117E-02
7.000000000000E-01	2.026882176433E-02	2.081242610813E-02
6.000000000000E-01	1.992657187733E-02	2.027852999123E-02
5.000000000000E-01	1.942050689042E-02	1.960262618458E-02
4.000000000000E-01	1.878611350743E-02	1.881547732747E-02
3.000000000000E-01	1.804379449362E-02	1.793477012618E-02
2.000000000000E-01	1.720592622091E-02	1.697123338400E-02
1.000000000000E-01	1.627996117152E-02	1.593132749309E-02
1.000000000000E-03	1.528042567728E-02	1.483001708200E-02

MUO= .300000000000

MU	RI	TI
1.000000000000E+00	1.646563759094E-02	1.699827861067E-02
9.980000000000E-01	1.714337393429E-02	1.747765776052E-02
9.000000000000E-01	2.028725800597E-02	1.942458127148E-02
8.000000000000E-01	2.111009167193E-02	1.970215933284E-02
7.000000000000E-01	2.138159234046E-02	1.958469003899E-02
6.000000000000E-01	2.134099202498E-02	1.924359061804E-02
5.000000000000E-01	2.108157043706E-02	1.874569890396E-02
4.000000000000E-01	2.065073007921E-02	1.812498027401E-02
3.000000000000E-01	2.007572459934E-02	1.740096282024E-02
2.000000000000E-01	1.937308613309E-02	1.658549233546E-02
1.000000000000E-01	1.855276957256E-02	1.568570181157E-02
1.000000000000E-03	1.763001322182E-02	1.471563005078E-02

MUO= .100000000000

MU	RI	TI
1.000000000000E+00	1.249205374309E-02	1.238222436345E-02
9.980000000000E-01	1.320457177109E-02	1.263500111188E-02
9.000000000000E-01	1.675904566753E-02	1.348476369130E-02
8.000000000000E-01	1.789821232025E-02	1.343664040609E-02
7.000000000000E-01	1.846232198265E-02	1.317696547666E-02
6.000000000000E-01	1.870090087985E-02	1.279753447123E-02
5.000000000000E-01	1.871124257206E-02	1.233413969038E-02
4.000000000000E-01	1.854278930302E-02	1.180496991910E-02
3.000000000000E-01	1.822396745172E-02	1.122048263781E-02
2.000000000000E-01	1.777202053561E-02	1.058702138207E-02
1.000000000000E-01	1.719733180822E-02	9.908405937394E-03
1.000000000000E-03	1.651298601902E-02	9.194121906861E-03

FOR IMPROVING P1 APPROXIMATION :

MUO= 1.000000000000

MU	RI	TI
1.000000000000E+00	6.859749353592E-03	9.999990000000E+05
9.980000000000E-01	6.874839499603E-03	8.870347003539E-03
9.000000000000E-01	7.691939096736E-03	9.674400859806E-03
8.000000000000E-01	8.719463644014E-03	1.068511331397E-02
7.000000000000E-01	1.002010313588E-02	1.196380869104E-02
6.000000000000E-01	1.171912912645E-02	1.363302202969E-02
5.000000000000E-01	1.403190917756E-02	1.590305429202E-02
4.000000000000E-01	1.736201994692E-02	1.916699368959E-02
3.000000000000E-01	2.256117103924E-02	2.425101633833E-02
2.000000000000E-01	3.176617060695E-02	3.321136756403E-02
1.000000000000E-01	5.186618587259E-02	5.252033621018E-02
1.000000000000E-03	8.565084389081E-02	7.925892476269E-02

MUO= .700000000000

MU	RI	TI
1.000000000000E+00	7.014072195117E-03	8.374666083725E-03
9.980000000000E-01	7.075278215506E-03	8.435564794722E-03
9.000000000000E-01	8.181293495304E-03	9.529906498097E-03
8.000000000000E-01	9.376528767878E-03	1.071024725103E-02
7.000000000000E-01	1.083839310522E-02	9.999990000000E+05
6.000000000000E-01	1.271488469524E-02	1.400124060850E-02
5.000000000000E-01	1.524211797448E-02	1.648763588707E-02
4.000000000000E-01	1.885574184675E-02	2.003536394069E-02
3.000000000000E-01	2.447201420497E-02	2.553056613262E-02
2.000000000000E-01	3.439070607147E-02	3.517203012933E-02
1.000000000000E-01	5.606080387081E-02	5.584102991467E-02
1.000000000000E-03	9.386744440639E-02	8.315524911880E-02

MUO= .300000000000

MU	RI	TI
1.000000000000E+00	6.768351311772E-03	7.275304901498E-03
9.980000000000E-01	6.838535960585E-03	7.344984910811E-03
9.000000000000E-01	7.942996157156E-03	8.436369984254E-03
8.000000000000E-01	9.093464187059E-03	9.570193387434E-03
7.000000000000E-01	1.048800608785E-02	1.094167119970E-02
6.000000000000E-01	1.227029129882E-02	1.269008928230E-02
5.000000000000E-01	1.466527452485E-02	1.503175361888E-02
4.000000000000E-01	1.808709010082E-02	1.836127608891E-02
3.000000000000E-01	2.340897844245E-02	9.999990000000E+05
2.000000000000E-01	3.283800503147E-02	3.246496540795E-02
1.000000000000E-01	5.367557312589E-02	5.142415287637E-02

1.000000000000E-03	9.604709436824E-02	7.131218262620E-02

MUO= .100000000000

MU	RI	TI
1.000000000000E+00	5.186618587259E-03	5.252033621018E-03
9.980000000000E-01	5.240929166633E-03	5.305305538785E-03
9.000000000000E-01	6.080756416981E-03	6.121597186431E-03
8.000000000000E-01	6.952121612141E-03	6.963016856030E-03
7.000000000000E-01	8.008686267259E-03	7.977289987811E-03
6.000000000000E-01	9.361042985745E-03	9.266155525191E-03
5.000000000000E-01	1.118304591831E-02	1.098579847789E-02
4.000000000000E-01	1.379717328434E-02	1.341826703377E-02
3.000000000000E-01	1.789185770863E-02	1.714138429212E-02
2.000000000000E-01	2.524830504034E-02	2.353586426940E-02
1.000000000000E-01	4.215502386046E-02	9.999990000000E+05
1.000000000000E-03	9.435043440696E-02	3.846212004166E-02

APPENDIX C

COMPUTER PROGRAM FOR THE EXACT

POLARIZATION SOLUTION

The computer program of the exact polarization solution for a one-dimensional plane-parallel medium without reflective boundaries is included herein.

```
C--------PROGRAM FOR RADIATIVE TRANSFER IN A FINITE MEDIUM WITH
C--------POLARIZATION (FOR L=5 AND NQTOT=328*2).
C-----------------------------------------------------------------------------------------------------
C       NQ: NUMBER OF QUADRATURE POINTS.
C       PAI: PAI MATRIX WITH MU VALUES (THE FIRST HALF IS FOR
C            POSITIVE MU VAIUES AND THE REST IS FOR NEGATIVE MU
C            VALUES).
C       PAIQ: PAI MATRIX WITH QUADRATURE POINTS (THE FIRST HALF IS
C             FOR POSITIVE QUADRATURE POINTS AND THE REST IS FOR
C             NEGATIVE QUADRATURE POINTS).
C       PP1T: PPim1 AT 0 FOR MU VALUES.
C       PP1B: PPim1 AT TAU0 FOR MU VALUES.
C       PP3T: PPim3 AT 0 FOR MU VALUES.
C       PP3B: PPim3 AT TAU0 FOR MU VALUES.
C       PP1TQ: PPim1 AT 0 FOR QUADRATURE POINTS.
C       PP1BQ: PPim1 AT TAU0 FOR QUADRATURE POINTS.
C       PP3TQ: PPim3 AT 0 FOR QUADRATURE POINTS.
C       PP3BQ: PPim3 AT TAU0 FOR QUADRATURE POINTS.
C       PPI1BAR: PPIim1 BAR FOR MU VALUES.
C       PP1BAR: PPim1 BAR FOR MU VALUES.
C       PPI3BAR: PPIim3 BAR FOR MU VALUES.
C       PP3BAR: PPim3 BAR FOR MU VALUES.
C       PPI1BARQ: PPIim1 BAR FOR QUADRATURE POINTS.
C       PP1BARQ: PPim1 BAR FOR QUADRATURE POINTS.
C       PPI3BARQ: PPIim3 BAR FOR QUADRATURE POINTS.
C       PP3BARQ: PPim3 BAR FOR QUADRATURE POINTS.
C       RI: RFFLECTED INTENSITY, DIVIDED BY PAIG FOR NORMALIZATION
```

```
          BUT WITHOUT MULTIPLYING BY THE INCIDENT POLARIZED
          LIGHT (F).
C       TI: TRANSMITTED INTENSITY, DIVIDED BY PAIG FOR
C          NORMALIZATION BUT WITHOUT MULTIPLYING BY THE
C          INCIDENT POLARIZED LIGHT (F).
C       RIF: NORMALIZED REFLECTED INTENSITY.
C       TIF: NORMALIZED TRANSMITTED INTENSITY.
C       F: INCIDENT POLARIZED LIGHT.
C-----------------------------------------------------------------------------------------------
        IMPLICIT REAL*8 (A-H,O-Z)
        REAL*8 MU, MUX
        DIMENSION INDX(4),IPRINT(20),HSTEP(20),X(6,6,4,4),XX(4,4),
     * XINV(6,6,4,4),XXINV(4,4),Y(6,6,4,4),Z(6,6,4,4),
     * DUM1(26),DUM2(6,6,4,4,26),DUM3(6,6,4,4,26),DUM4(26),
     * DUM5(6,6,4,4,26),DUM6(6,6,4,4,26),DUM7(6,6,4,4,26),DUM8(26),
     * DUM9(6,6,4,4,26),DUM10(6,6,4,4,26),DUM11(6,6,4,4,26),AQ(328),
     * XD(16),AD(16),DUM12(656),DUM13(6,6,4,4,656),DUM14(6,6,4,4,656),
     * DUM15(656),DUM16(6,6,4,4,656),DUM17(6,6,4,4,656),
     * DUM18(6,6,4,4,656),DUM19(656),DUM20(6,6,4,4,656),
     * DUM21(6,6,4,4,656),DUM22(6,6,4,4,656),YY(168,4,4,328), MUX(13),
     * FLAG(4,4,13), FLAGR(4,4), FLAG1(4,4,13), FLAGR1(4,4)
        COMMON/BLK1/MU(26),XQ(656)
        COMMON/BLK2/LF(6,6,8)
        COMMON/BLK3/PP1T(6,6,4,4,13),PP1B(6,6,4,4,13)
        COMMON/BLK4/PP3T(6,6,4,4,13),PP3B(6,6,4,4,13)
        COMMON/BLK5/PP1TQ(6,6,4,4,328),PP1BQ(6,6,4,4,328)
        COMMON/BLK6/PP3TQ(6,6,4,4,328),PP3BQ(6,6,4,4,328)
        COMMON/BLK7/PAIQ(6,6,4,4,656),B(6,4,4),A(328)
        COMMON/BLK8/W,PAIG
        COMMON/BLK9/N,L,NMUS,NQTOT
        COMMON/BLK15/ERROR
        COMMON/BLK16/RELAX
        COMMON/BLK17/PAI(6,6,4,4,26)
        COMMON/BLK18/PHI
        COMMON/BLK19/RI(13,13,4,4),TI(13,13,4,4)
        COMMON/BLK20/F1(4),F2(4),F3(4),F4(4)
        COMMON/BLK21/LIU
        OPEN(UNIT=4,FILE='pfinitnd.d')
        OPEN(UNIT=5,FILE='pfnd2f.o')
        OPEN(UNIT=6,FILE='pfnd3f.o')
        OPEN(UNIT=7,FILE='pfnd4f.o')
        OPEN(UNIT=8,FILE='pfndi.o')
        OPEN(UNIT=9,FILE='pfnd1f.o')
C--------READ IN AND PRINT OUT THE VALUE OF NUMBER OF LEGENDRE
C--------POLYNOMIALS L.
        READ(4,*) L
```

```
      WRITE(9,1) L
    1 FORMAT(1X,'NUMBER OF LEGENDRE POLYNOMIALS (L)=',I3)
C--------READ IN AND PRINT OUT THE VALUE OF SCATTERING ALBEDO W.
      READ(4,*) W
      WRITE(9,2) W
    2 FORMAT(1X,'SCATTERING ALBEDO (W)=',F14.12)
C--------READ IN AND PRINT OUT THE VALUE OF AZIMUTHAL ANGLE PHI.
      READ(4,*) PHI
      WRITE(9,3) PHI
    3 FORMAT(1X,'AZIMUTHAL ANGLE (PHI)=',F14.12)
      PAIG=3.141592654D0
      PHI=PHI*PAIG/180.D0
C--------READ IN AND PRINT OUT THE VALUE OF ERROR.
      READ(4,*) ERROR
      WRITE(9,4) ERROR
    4 FORMAT(1X,'ERROR=',F14.12)
C--------READ IN THE VALUE OF NMUS.
      READ(4,*) NMUS
      WRITE(*,*) NMUS
C--------READ IN THE VALUES OF MU'S.
      READ(4,*) (MU(J),J=1, NMUS)
      WRITE(*,*) (MU(J),J=1, NMUS)
C--------READ IN AND PRINT OUT THE NUMBER OF QUADRATURE POINTS
C--------FOR NQ1, NQ2, NQ3, NQ4, NQ5, NQ6, NQ7, NQ8, NQ9, NQ10, NQ11,
C--------NQ12, NQ13, NQ14, NQ15, NQ16, NQ17, NQ18, NQ19, AND NQ20.
      READ(4,*) NQ1
      READ(4,*) NQ2
      READ(4,*) NQ3
      READ(4,*) NQ4
      READ(4,*) NQ5
      READ(4,*) NQ6
      READ(4,*) NQ7
      READ(4,*) NQ8
      READ(4,*) NQ9
      READ(4,*) NQ10
      READ(4,*) NQ11
      READ(4,*) NQ12
      READ(4,*) NQ13
      READ(4,*) NQ14
      READ(4,*) NQ15
      READ(4,*) NQ16
      READ(4,*) NQ17
      READ(4,*) NQ18
      READ(4,*) NQ19
      READ(4,*) NQ20
      WRITE(9,5) NQ1,NQ2,NQ3,NQ4,NQ5,NQ6,NQ7,NQ8,NQ9,NQ10
```

```
      5 FORMAT(1X,'NQ1=',I2,2X,'NQ2=',I2,2X,'NQ3=',I2,2X,'NQ4=',I2,2X,
     *  'NQ5=',I2,2X,'NQ6=',I2,2X,'NQ7=',I2,2X,'NQ8=',I2,2X,
     *  'NQ9=',I2,2X,'NQ10=',I2)
        WRITE(9,6) NQ11,NQ12,NQ13,NQ14,NQ15,NQ16,NQ17,NQ18,NQ19,NQ20
      6 FORMAT(1X,'NQ11=',I2,1X,'NQ12=',I2,1X,'NQ13=',I2,1X,'NQ14=',I2,1X,
     *  'NQ15=',I2,1X,'NQ16=',I2,1X,'NQ17=',I2,1X,'NQ18=',I2,1X,
     *  'NQ19=',I2,1X,'NQ20=',I2)
C--------READ IN AND PRINT OUT THE BOUNDARY VALUES FOR TWENTY
C--------INTERVALS AA, BB, CC, DD, EE, FF, GG, HH, OO, PP, QQ, RR, SS1, SS2,
C--------SS3, SS4, SS5, SS6, SS7, SS8, AND SS9.
        READ(4,*) AA
        READ(4,*) BB
        READ(4,*) CC
        READ(4,*) DD
        READ(4,*) EE
        READ(4,*) FF
        READ(4,*) GG
        READ(4,*) HH
        READ(4,*) OO
        READ(4,*) PP
        READ(4,*) QQ
        READ(4,*) RR
        READ(4,*) SS1
        READ(4,*) SS2
        READ(4,*) SS3
        READ(4,*) SS4
        READ(4,*) SS5
        READ(4,*) SS6
        READ(4,*) SS7
        READ(4,*) SS8
        READ(4,*) SS9
        WRITE(9,*) 'INTERVALS ARE :'
        WRITE(9,*) AA,BB,CC,DD,EE,FF,GG,HH,OO,PP,QQ,RR,
     *   SS1,SS2,SS3,SS4,SS5,SS6,SS7,SS8,SS9
C--------READ IN THE VALUE OF NUMBER OF PRINTING NPRINT.
        READ(4,*) NPRINT
        WRITE(*,*) NPRINT
C--------READ IN THE VALUES OF PRINTING STEPS IPRINT'S.
        READ(4,7) (IPRINT(I),I=1,NPRINT)
      7 FORMAT(5I10)
C--------READ IN THE VALUES OF H STEPS HSTEP'S.
        READ(4,8) (HSTEP(I),I=1,NPRINT)
      8 FORMAT(5F17.12)
        WRITE(9,*) '   IPRINT                 HSTEP'
        DO 190 I=1, NPRINT
        WRITE(9,9) IPRINT(I), HSTEP(I)
```

```
    9 FORMAT(I10,13X,F20.12)
  190 CONTINUE
C--------READ IN AND PRINT OUT THE VALUE OF THE RELAXATION.
      READ(4,*) RELAX
      WRITE(9,*)
      WRITE(9,10) RELAX
   10 FORMAT(1X,'RELAXATION=',F14.12)
      WRITE(9,*)
C--------SET UP SOME NECESSARY PARAMETERS TO USE FOR CALLING
C--------SUBROUTINE.
      LP=L+1
      LMAX=6
      N=4
      NP=4
      NMUSMAX=26
      NQTOTMAX=656
      SIGNP=1.D0
      SIGNM=-1.D0
C--------READ IN AND PRINT OUT THE VALUES OF INCIDENT POLARIZED
C--------LIGHT (F).
      READ(4,*) (F1(I),I=1,N)
      READ(4,*) (F2(I),I=1,N)
      READ(4,*) (F3(I),I=1,N)
      READ(4,*) (F4(I),I=1,N)
      WRITE(9,*) ' F1     F2     F3     F4'
      WRITE(9,*) '------------------------------'
      DO 200 I=1, N
      WRITE(9,11) F1(I), F2(I), F3(I), F4(I)
   11 FORMAT(1X,F5.2,3X,F5.2,3X,F5.2,3X,F5.2)
  200 CONTINUE
      WRITE(9,*)
C--------CALL SUBROUTINE BMATRIX TO READ IN AND PRINT OUT THE
C--------SCATTERING COEFFICIENTS OF B MATRIX.
      CALL BMATRIX(LP,LMAX,N,NP,B)
C--------CALL SUBROUTINE DXA.
      CALL DXA(NQ1,AA,BB,XQ,AQ)
      CALL DXA(NQ2,BB,CC,XD,AD)
      DO 210 I=1, NQ2
      XQ(NQ1+I)=XD(I)
      AQ(NQ1+I)=AD(I)
  210 CONTINUE
      CALL DXA(NQ3,CC,DD,XD,AD)
      DO 220 I=1, NQ3
      XQ(NQ1+NQ2+I)=XD(I)
      AQ(NQ1+NQ2+I)=AD(I)
  220 CONTINUE
```

```
      CALL DXA(NQ4,DD,EE,XD,AD)
      DO 230 I=1, NQ4
      XQ(NQ1+NQ2+NQ3+I)=XD(I)
      AQ(NQ1+NQ2+NQ3+I)=AD(I)
  230 CONTINUE
      CALL DXA(NQ5,EE,FF,XD,AD)
      DO 240 I=1, NQ5
      XQ(NQ1+NQ2+NQ3+NQ4+I)=XD(I)
      AQ(NQ1+NQ2+NQ3+NQ4+I)=AD(I)
  240 CONTINUE
      CALL DXA(NQ6,FF,GG,XD,AD)
      DO 250 I=1, NQ6
      XQ(NQ1+NQ2+NQ3+NQ4+NQ5+I)=XD(I)
      AQ(NQ1+NQ2+NQ3+NQ4+NQ5+I)=AD(I)
  250 CONTINUE
      CALL DXA(NQ7,GG,HH,XD,AD)
      DO 260 I=1, NQ7
      XQ(NQ1+NQ2+NQ3+NQ4+NQ5+NQ6+I)=XD(I)
      AQ(NQ1+NQ2+NQ3+NQ4+NQ5+NQ6+I)=AD(I)
  260 CONTINUE
      CALL DXA(NQ8,HH,OO,XD,AD)
      DO 270 I=1, NQ8
      XQ(NQ1+NQ2+NQ3+NQ4+NQ5+NQ6+NQ7+I)=XD(I)
      AQ(NQ1+NQ2+NQ3+NQ4+NQ5+NQ6+NQ7+I)=AD(I)
  270 CONTINUE
      CALL DXA(NQ9,OO,PP,XD,AD)
      DO 280 I=1, NQ9
      XQ(NQ1+NQ2+NQ3+NQ4+NQ5+NQ6+NQ7+NQ8+I)=XD(I)
      AQ(NQ1+NQ2+NQ3+NQ4+NQ5+NQ6+NQ7+NQ8+I)=AD(I)
  280 CONTINUE
      CALL DXA(NQ10,PP,QQ,XD,AD)
      DO 290 I=1, NQ10
      XQ(NQ1+NQ2+NQ3+NQ4+NQ5+NQ6+NQ7+NQ8+NQ9+I)=XD(I)
      AQ(NQ1+NQ2+NQ3+NQ4+NQ5+NQ6+NQ7+NQ8+NQ9+I)=AD(I)
  290 CONTINUE
      CALL DXA(NQ11,QQ,RR,XD,AD)
      DO 300 I=1, NQ11
      XQ(NQ1+NQ2+NQ3+NQ4+NQ5+NQ6+NQ7+NQ8+NQ9+NQ10+I)=XD(I)
      AQ(NQ1+NQ2+NQ3+NQ4+NQ5+NQ6+NQ7+NQ8+NQ9+NQ10+I)=AD(I)
  300 CONTINUE
      CALL DXA(NQ12,RR,SS1,XD,AD)
      DO 310 I=1, NQ12
      XQ(NQ1+NQ2+NQ3+NQ4+NQ5+NQ6+NQ7+NQ8+NQ9+NQ10+NQ11+I)=
     *  XD(I)
      AQ(NQ1+NQ2+NQ3+NQ4+NQ5+NQ6+NQ7+NQ8+NQ9+NQ10+NQ11+I)=
     *  AD(I)
```

```
310 CONTINUE
    CALL DXA(NQ13,SS1,SS2,XD,AD)
    DO 320 I=1, NQ13
    XQ(NQ1+NQ2+NQ3+NQ4+NQ5+NQ6+NQ7+NQ8+NQ9+NQ10+NQ11+
  *  NQ12+I)=XD(I)
    AQ(NQ1+NQ2+NQ3+NQ4+NQ5+NQ6+NQ7+NQ8+NQ9+NQ10+NQ11+
  *  NQ12+I)=AD(I)
320 CONTINUE
    CALL DXA(NQ14,SS2,SS3,XD,AD)
    DO 330 I=1, NQ14
    XQ(NQ1+NQ2+NQ3+NQ4+NQ5+NQ6+NQ7+NQ8+NQ9+NQ10+NQ11+
  *  NQ12+NQ13+I)=XD(I)
    AQ(NQ1+NQ2+NQ3+NQ4+NQ5+NQ6+NQ7+NQ8+NQ9+NQ10+NQ11+
  *  NQ12+NQ13+I)=AD(I)
330 CONTINUE
    CALL DXA(NQ15,SS3,SS4,XD,AD)
    DO 340 I=1, NQ15
    XQ(NQ1+NQ2+NQ3+NQ4+NQ5+NQ6+NQ7+NQ8+NQ9+NQ10+NQ11+
  *  NQ12+NQ13+NQ14+I)=XD(I)
    AQ(NQ1+NQ2+NQ3+NQ4+NQ5+NQ6+NQ7+NQ8+NQ9+NQ10+NQ11+
  *  NQ12+NQ13+NQ14+I)=AD(I)
340 CONTINUE
    CALL DXA(NQ16,SS4,SS5,XD,AD)
    DO 350 I=1, NQ16
    XQ(NQ1+NQ2+NQ3+NQ4+NQ5+NQ6+NQ7+NQ8+NQ9+NQ10+NQ11+
  *  NQ12+NQ13+NQ14+NQ15+I)=XD(I)
    AQ(NQ1+NQ2+NQ3+NQ4+NQ5+NQ6+NQ7+NQ8+NQ9+NQ10+NQ11+
  *  NQ12+NQ13+NQ14+NQ15+I)=AD(I)
350 CONTINUE
    CALL DXA(NQ17,SS5,SS6,XD,AD)
    DO 360 I=1, NQ17
    XQ(NQ1+NQ2+NQ3+NQ4+NQ5+NQ6+NQ7+NQ8+NQ9+NQ10+NQ11+
  *  NQ12+NQ13+NQ14+NQ15+NQ16+I)=XD(I)
    AQ(NQ1+NQ2+NQ3+NQ4+NQ5+NQ6+NQ7+NQ8+NQ9+NQ10+NQ11+
  *  NQ12+NQ13+NQ14+NQ15+NQ16+I)=AD(I)
360 CONTINUE
    CALL DXA(NQ18,SS6,SS7,XD,AD)
    DO 370 I=1, NQ18
    XQ(NQ1+NQ2+NQ3+NQ4+NQ5+NQ6+NQ7+NQ8+NQ9+NQ10+NQ11+
  *  NQ12+NQ13+NQ14+NQ15+NQ16+NQ17+I)=XD(I)
    AQ(NQ1+NQ2+NQ3+NQ4+NQ5+NQ6+NQ7+NQ8+NQ9+NQ10+NQ11+
  *  NQ12+NQ13+NQ14+NQ15+NQ16+NQ17+I)=AD(I)
370 CONTINUE
    CALL DXA(NQ19,SS7,SS8,XD,AD)
    DO 380 I=1, NQ19
    XQ(NQ1+NQ2+NQ3+NQ4+NQ5+NQ6+NQ7+NQ8+NQ9+NQ10+NQ11+
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     *   NQ12+NQ13+NQ14+NQ15+NQ16+NQ17+NQ18+I)=XD(I)
         AQ(NQ1+NQ2+NQ3+NQ4+NQ5+NQ6+NQ7+NQ8+NQ9+NQ10+NQ11+
     *   NQ12+NQ13+NQ14+NQ15+NQ16+NQ17+NQ18+I)=AD(I)
  380 CONTINUE
      CALL DXA(NQ20,SS8,SS9,XD,AD)
      DO 390 I=1, NQ20
      XQ(NQ1+NQ2+NQ3+NQ4+NQ5+NQ6+NQ7+NQ8+NQ9+NQ10+NQ11+
     *   NQ12+NQ13+NQ14+NQ15+NQ16+NQ17+NQ18+NQ19+I)=XD(I)
      AQ(NQ1+NQ2+NQ3+NQ4+NQ5+NQ6+NQ7+NQ8+NQ9+NQ10+NQ11+
     *   NQ12+NQ13+NQ14+NQ15+NQ16+NQ17+NQ18+NQ19+I)=AD(I)
  390 CONTINUE
C--------CALCULATE AND PRINT OUT THE TOTAL NUMBER OF
C--------QUADRATURE POINTS NQTOT.
      NQTOT=NQ1+NQ2+NQ3+NQ4+NQ5+NQ6+NQ7+NQ8+NQ9+NQ10+NQ11
     *  +NQ12+NQ13+NQ14+NQ15+NQ16+NQ17+NQ18+NQ19+NQ20
      WRITE(9,12) NQTOT
   12 FORMAT(1X,'NUMBER OF QUADRATURE POINTS (N)=',I3)
C--------PRINT OUT THE QUADRATURE POINTS XQ.
      WRITE(9,*) 'QUADRATURE POINTS ARE:'
      WRITE(9,13) (XQ(I),I=1, NQTOT)
   13 FORMAT(4(1PE18.11,2X))
C--------CALCULATE THE NEGATIVE QUADRATURE POINTS AND MU
C--------VALUES (DOUBLE THE NMUS AND NQTOT).
      DO 400 I=1, NQTOT
      XQ(NQTOT+I)=XQ(I)*(-1.D0)
  400 CONTINUE
      DO 450 I=1, NMUS
      MU(NMUS+I)=MU(I)*(-1.D0)
  450 CONTINUE
      NQTOT=NQTOT*2
      NMUS=NMUS*2
C--------CALL SUBROUTINE XPAI TO GET THE VALUES OF X COEFFICIENTS
C--------FOR PAI MATRIX.
      CALL XPAI(LP,LMAX,N,NP,X)
C--------SET UP AN IDENTITY MATRIX.
      DO 550 I=1, N
      DO 600 J=1, N
      XXINV(I,J)=0.D0
  600 CONTINUE
      XXINV(I,I)=1.D0
  550 CONTINUE
C--------CALCULATE THE INVERSE OF X COEFFICIENTS FOR PAI MATRIX.
      DO 650 M=1, L+1
      IF(M .LE. 3) THEN
      MI=3
      ELSE
```

```
      MI=M
      ENDIF
      DO 700 I=MI, L+1
      DO 750 J=1,N
      DO 800 K=1,N
      XX(J,K)=X(M,I,J,K)
  800 CONTINUE
  750 CONTINUE
      CALL LUDCMP(XX,N,NP,INDX,D)
      DO 850 II=1,N
      CALL LUBKSB(XX,N,NP,INDX,XXINV(1,II))
  850 CONTINUE
      DO 900 JJ=1, N
      DO 950 KK=1, N
      XINV(M,I,JJ,KK)=XXINV(JJ,KK)
C--------SET UP AN IDENTITY MATRIX AGAIN.
      XXINV(JJ,KK)=0.D0
  950 CONTINUE
      XXINV(JJ,JJ)=1.D0
  900 CONTINUE
  700 CONTINUE
  650 CONTINUE
C--------CALL SUBROUTINE YPAI TO GET THE VALUES OF Y COEFFICIENTS
C--------FOR PAI MATRIX.
      CALL YPAI(LP,LMAX,N,NP,Y)
C--------CALL SUBROUTINE ZPAI TO GET THE VALUES OF Z COEFFICIENTS
C--------FOR PAI MATRIX.
      CALL ZPAI(LP,LMAX,N,NP,Z)
C--------CALCULATE THE PAI MATRIX WITH MU VALUES.
C--------FOR L .LE. 2.
      CALL PAI2(MU,LP,LMAX,N,NP,NMUS,NMUSMAX,PAI)
      IF(L+1 .LT. 4) GO TO 2100
C--------FOR L .GT. 2.
C--------FOR M=0 AND I .GT. 2.
      DO 1050 I=3, L
      DO 1100 KK=1, NMUS
      DUM1(KK)=(2.D0*I-1.D0)*MU(KK)
 1100 CONTINUE
      CALL MMULT1(DUM1,PAI,1,I,LMAX,N,NP,NMUS,NMUSMAX,DUM2)
      CALL MMULT2(Y,PAI,1,I,LMAX,N,NP,NMUS,NMUSMAX,DUM3)
      CALL MAOS1(DUM2,DUM3,1,I,LMAX,N,NP,NMUS,NMUSMAX,SIGNM,
     *  DUM2)
      CALL MMULT3(XINV,DUM2,1,I,LMAX,N,NP,NMUS,NMUSMAX,DUM2)
      DO 1150 MM=1, NMUS
      DO 1200 J=1, N
      DO 1250 K=1, N
```

```
      PAI(1,I+1,J,K,MM)=DUM2(1,I,J,K,MM)
 1250 CONTINUE
 1200 CONTINUE
 1150 CONTINUE
 1050 CONTINUE
C--------FOR M=1 AND I .GT. 2.
      DO 1300 I=3, L
      DO 1350 KK=1, NMUS
      DUM4(KK)=(2.D0*I-1.D0)*MU(KK)
 1350 CONTINUE
      CALL MMULT1(DUM4,PAI,2,I,LMAX,N,NP,NMUS,NMUSMAX,DUM5)
      CALL MMULT2(Y,PAI,2,I,LMAX,N,NP,NMUS,NMUSMAX,DUM6)
      CALL MMULT3(Z,PAI,2,I,LMAX,N,NP,NMUS,NMUSMAX,DUM7)
      CALL MAOS1(DUM5,DUM6,2,I,LMAX,N,NP,NMUS,NMUSMAX,SIGNM,
     *  DUM5)
      CALL MAOS1(DUM5,DUM7,2,I,LMAX,N,NP,NMUS,NMUSMAX,SIGNP,
     *  DUM5)
      CALL MMULT3(XINV,DUM5,2,I,LMAX,N,NP,NMUS,NMUSMAX,DUM5)
      DO 1400 MM=1, NMUS
      DO 1450 J=1, N
      DO 1500 K=1, N
      PAI(2,I+1,J,K,MM)=DUM5(2,I,J,K,MM)
 1500 CONTINUE
 1450 CONTINUE
 1400 CONTINUE
 1300 CONTINUE
C--------FOR M .GE. 2 AND M=I.
      DO 1550 M=3, L+1
      CALL FACTOR(M-3,1,FACTA)
      CALL FACTOR(M+1,1,FACTB)
      MDUMMY=2*M-2
      CALL FACTOR(MDUMMY,1,FACTC)
      CALL FACTOR(MDUMMY,2,FACTD)
      DK=FACTC/(2.D0**(M-1.D0))*(FACTA*FACTB)**(-0.5)
      DO 1600 KK=1,NMUS
      DO 1650 I=1, N
      DO 1700 J=1, N
      PAI(M,M,I,J,KK)=0.D0
 1700 CONTINUE
 1650 CONTINUE
      PAI(M,M,1,1,KK)=FACTD*(1.D0-MU(KK)**2)*
     *  (1.D0-MU(KK)**2)**((M-1.D0)/2.D0-1.D0)
      PAI(M,M,2,2,KK)=DK*(1.D0+MU(KK)**2)*(1.D0-MU(KK)**2)
     *  **((M-1.D0)/2.D0-1.D0)
      PAI(M,M,2,3,KK)=-DK*(2.D0*MU(KK))*(1.D0-MU(KK)**2)**
     *  ((M-1.D0)/2.D0-1.D0)
```

```
      PAI(M,M,3,2,KK)=-DK*(2.D0*MU(KK))*(1.D0-MU(KK)**2)**
     *  ((M-1.D0)/2.D0-1.D0)
      PAI(M,M,3,3,KK)=DK*(1.D0+MU(KK)**2)*(1.D0-MU(KK)**2)
     *  **((M-1.D0)/2.D0-1.D0)
      PAI(M,M,4,4,KK)=FACTD*(1.D0-MU(KK)**2)*
     *  (1.D0-MU(KK)**2)**((M-1.D0)/2.D0-1.D0)
 1600 CONTINUE
 1550 CONTINUE
C--------FOR M .GE. 2 AND I .GT. M.
      DO 1750 M=3, L+1
      DO 1800 I=M, L
      DO 1850 KK=1, NMUS
      DUM8(KK)=(2.D0*I-1.D0)*MU(KK)
 1850 CONTINUE
      CALL MMULT1(DUM8,PAI,M,I,LMAX,N,NP,NMUS,NMUSMAX,DUM9)
      CALL MMULT2(Y,PAI,M,I,LMAX,N,NP,NMUS,NMUSMAX,DUM10)
      CALL MMULT3(Z,PAI,M,I,LMAX,N,NP,NMUS,NMUSMAX,DUM11)
      CALL MAOS1(DUM9,DUM10,M,I,LMAX,N,NP,NMUS,NMUSMAX,
     *  SIGNM,DUM9)
      CALL MAOS1(DUM9,DUM11,M,I,LMAX,N,NP,NMUS,NMUSMAX,
     *  SIGNP,DUM9)
      CALL MMULT3(XINV,DUM9,M,I,LMAX,N,NP,NMUS,NMUSMAX,DUM9)
      DO 1900 MM=1, NMUS
      DO 1950 J=1, N
      DO 2000 K=1, N
      PAI(M,I+1,J,K,MM)=DUM9(M,I,J,K,MM)
 2000 CONTINUE
 1950 CONTINUE
 1900 CONTINUE
 1800 CONTINUE
 1750 CONTINUE
      GO TO 2100
C--------CALCULATE THE PAI MATRIX WITH QUADRATURE POINTS.
C--------FOR L .LE. 2.
 2100 CALL PAI2(XQ,LP,LMAX,N,NP,NQTOT,NQTOTMAX,PAIQ)
      IF(L+1 .LT. 4) GO TO 3200
C--------FOR L .GT. 2.
C--------FOR M=0 AND I .GT. 2.
      DO 2150 I=3, L
      DO 2200 KK=1, NQTOT
      DUM12(KK)=(2.D0*I-1.D0)*XQ(KK)
 2200 CONTINUE
      CALL MMULT1(DUM12,PAIQ,1,I,LMAX,N,NP,NQTOT,NQTOTMAX,
     *  DUM13)
      CALL MMULT2(Y,PAIQ,1,I,LMAX,N,NP,NQTOT,NQTOTMAX,DUM14)
      CALL MAOS1(DUM13,DUM14,1,I,LMAX,N,NP,NQTOT,NQTOTMAX,
```

```
     *  SIGNM,DUM13)
        CALL MMULT3(XINV,DUM13,1,I,LMAX,N,NP,NQTOT,NQTOTMAX,
     *  DUM13)
        DO 2250 MM=1, NQTOT
        DO 2300 J=1, N
        DO 2350 K=1, N
        PAIQ(1,I+1,J,K,MM)=DUM13(1,I,J,K,MM)
 2350   CONTINUE
 2300   CONTINUE
 2250   CONTINUE
 2150   CONTINUE
C--------FOR M=1 AND I .GT. 2.
        DO 2400 I=3, L
        DO 2450 KK=1, NQTOT
        DUM15(KK)=(2.D0*I-1.D0)*XQ(KK)
 2450   CONTINUE
        CALL MMULT1(DUM15,PAIQ,2,I,LMAX,N,NP,NQTOT,NQTOTMAX,
     *  DUM16)
        CALL MMULT2(Y,PAIQ,2,I,LMAX,N,NP,NQTOT,NQTOTMAX,DUM17)
        CALL MMULT3(Z,PAIQ,2,I,LMAX,N,NP,NQTOT,NQTOTMAX,DUM18)
        CALL MAOS1(DUM16,DUM17,2,I,LMAX,N,NP,NQTOT,NQTOTMAX,
     *  SIGNM,DUM16)
        CALL MAOS1(DUM16,DUM18,2,I,LMAX,N,NP,NQTOT,NQTOTMAX,
     *  SIGNP,DUM16)
        CALL MMULT3(XINV,DUM16,2,I,LMAX,N,NP,NQTOT,NQTOTMAX,
     *  DUM16)
        DO 2500 MM=1, NQTOT
        DO 2550 J=1, N
        DO 2600 K=1, N
        PAIQ(2,I+1,J,K,MM)=DUM16(2,I,J,K,MM)
 2600   CONTINUE
 2550   CONTINUE
 2500   CONTINUE
 2400   CONTINUE
C--------FOR M .GE. 2 AND M=I.
        DO 2650 M=3, L+1
        CALL FACTOR(M-3,1,FACTA)
        CALL FACTOR(M+1,1,FACTB)
        MDUMMY=2*M-2
        CALL FACTOR(MDUMMY,1,FACTC)
        CALL FACTOR(MDUMMY,2,FACTD)
        DK=FACTC/(2.D0**(M-1.D0))*(FACTA*FACTB)**(-0.5)
        DO 2700 KK=1,NQTOT
        DO 2750 I=1, N
        DO 2800 J=1, N
        PAIQ(M,M,I,J,KK)=0.D0
```

```
  2800 CONTINUE
  2750 CONTINUE
       PAIQ(M,M,1,1,KK)=FACTD*(1.D0-XQ(KK)**2)*
     *  (1.D0-XQ(KK)**2)**((M-1.D0)/2.D0-1.D0)
       PAIQ(M,M,2,2,KK)=DK*(1.D0+XQ(KK)**2)*(1.D0-XQ(KK)**2)
     *  **((M-1.D0)/2.D0-1.D0)
       PAIQ(M,M,2,3,KK)=-DK*(2.D0*XQ(KK))*(1.D0-XQ(KK)**2)
     *  **((M-1.D0)/2.D0-1.D0)
       PAIQ(M,M,3,2,KK)=-DK*(2.D0*XQ(KK))*(1.D0-XQ(KK)**2)
     *  **((M-1.D0)/2.D0-1.D0)
       PAIQ(M,M,3,3,KK)=DK*(1.D0+XQ(KK)**2)*(1.D0-XQ(KK)**2)
     *  **((M-1.D0)/2.D0-1.D0)
       PAIQ(M,M,4,4,KK)=FACTD*(1.D0-XQ(KK)**2)*
     *  (1.D0-XQ(KK)**2)**((M-1.D0)/2.D0-1.D0)
  2700 CONTINUE
  2650 CONTINUE
C--------FOR M .GE. 2 AND I .GT. M.
       DO 2850 M=3, L+1
       DO 2900 I=M, L
       DO 2950 KK=1, NQTOT
       DUM19(KK)=(2.D0*I-1.D0)*XQ(KK)
  2950 CONTINUE
       CALL MMULT1(DUM19,PAIQ,M,I,LMAX,N,NP,NQTOT,NQTOTMAX,
     *  DUM20)
       CALL MMULT2(Y,PAIQ,M,I,LMAX,N,NP,NQTOT,NQTOTMAX,DUM21)
       CALL MMULT3(Z,PAIQ,M,I,LMAX,N,NP,NQTOT,NQTOTMAX,DUM22)
       CALL MAOS1(DUM20,DUM21,M,I,LMAX,N,NP,NQTOT,NQTOTMAX,
     *  SIGNM,DUM20)
       CALL MAOS1(DUM20,DUM22,M,I,LMAX,N,NP,NQTOT,NQTOTMAX,
     *  SIGNP,DUM20)
       CALL MMULT3(XINV,DUM20,M,I,LMAX,N,NP,NQTOT,NQTOTMAX,
     *  DUM20)
       DO 3000 MM=1, NQTOT
       DO 3050 J=1, N
       DO 3100 K=1, N
       PAIQ(M,I+1,J,K,MM)=DUM20(M,I,J,K,MM)
  3100 CONTINUE
  3050 CONTINUE
  3000 CONTINUE
  2900 CONTINUE
  2850 CONTINUE
       GO TO 3200
C--------CALCULATING NUMBER OF FUNCTIONS WHICH NEED TO BE
C--------COMPUTED BY RK5.
  3200 NFUN=0
       DO 3250 I=1, L+1
```

```
      NFUN=NFUN+I
 3250 CONTINUE
      NFUN=4*NFUN
      NFUNP1=NFUN+1
      INFUN=2*NFUN
C--------TRANSFER FROM FIVE DIMENSIONS TO FOUR DIMENSIONS IN
C--------ORDER TO USE RK5.
      LL=0
      DO 3300 M=1, L+1
      DO 3350 I=M, L+1
      LL=LL+1
      LF(I,M,1)=LL
      LF(I,M,2)=LL+1
      LF(I,M,3)=LL+2
      LF(I,M,4)=LL+3
      LF(I,M,5)=NFUN+LL
      LF(I,M,6)=NFUN+LL+1
      LF(I,M,7)=NFUN+LL+2
      LF(I,M,8)=NFUN+LL+3
      LL=LL+3
 3350 CONTINUE
 3300 CONTINUE
C--------INITIAL VALUES FOR PP1T, PP1B, PP3T, PP3B, PP1TQ, PP1BQ,
C--------PP3TQ, AND PP3BQ WHICH TRANSFER FROM FIVE DIMENSIONS TO
C--------FOUR DIMENSIONS IN ORDER TO USE RK5.
      DO 3400 M=1, L+1
      DO 3450 I=M, L+1
      DO 3500 J=1, 2
      DO 3550 MM=1, NMUS/2
      DO 3600 JR1=1, N
      DO 3650 JC1=1, N
      YY(LF(I,M,J),JR1,JC1,MM)=PAIG*PAI(M,I,JR1,JC1,MM)
 3650 CONTINUE
 3600 CONTINUE
 3550 CONTINUE
 3500 CONTINUE
      DO 3700 JJ=5, 6
      DO 3750 MMM=1, NQTOT/2
      DO 3800 JR2=1, N
      DO 3850 JC2=1, N
      YY(LF(I,M,JJ),JR2,JC2,MMM)=PAIG*PAIQ(M,I,JR2,JC2,MMM)
 3850 CONTINUE
 3800 CONTINUE
 3750 CONTINUE
 3700 CONTINUE
 3450 CONTINUE
```

```
      NFUN=NFUN+I
 3250 CONTINUE
      NFUN=4*NFUN
      NFUNP1=NFUN+1
      INFUN=2*NFUN
C--------TRANSFER FROM FIVE DIMENSIONS TO FOUR DIMENSIONS IN
C--------ORDER TO USE RK5.
      LL=0
      DO 3300 M=1, L+1
      DO 3350 I=M, L+1
      LL=LL+1
      LF(I,M,1)=LL
      LF(I,M,2)=LL+1
      LF(I,M,3)=LL+2
      LF(I,M,4)=LL+3
      LF(I,M,5)=NFUN+LL
      LF(I,M,6)=NFUN+LL+1
      LF(I,M,7)=NFUN+LL+2
      LF(I,M,8)=NFUN+LL+3
      LL=LL+3
 3350 CONTINUE
 3300 CONTINUE
C--------INITIAL VALUES FOR PP1T, PP1B, PP3T, PP3B, PP1TQ, PP1BQ,
C--------PP3TQ, AND PP3BQ WHICH TRANSFER FROM FIVE DIMENSIONS TO
C--------FOUR DIMENSIONS IN ORDER TO USE RK5.
      DO 3400 M=1, L+1
      DO 3450 I=M, L+1
      DO 3500 J=1, 2
      DO 3550 MM=1, NMUS/2
      DO 3600 JR1=1, N
      DO 3650 JC1=1, N
      YY(LF(I,M,J),JR1,JC1,MM)=PAIG*PAI(M,I,JR1,JC1,MM)
 3650 CONTINUE
 3600 CONTINUE
 3550 CONTINUE
 3500 CONTINUE
      DO 3700 JJ=5, 6
      DO 3750 MMM=1, NQTOT/2
      DO 3800 JR2=1, N
      DO 3850 JC2=1, N
      YY(LF(I,M,JJ),JR2,JC2,MMM)=PAIG*PAIQ(M,I,JR2,JC2,MMM)
 3850 CONTINUE
 3800 CONTINUE
 3750 CONTINUE
 3700 CONTINUE
 3450 CONTINUE
```

```
      PP3BQ(I,M,JR2,JC2,MMM)=YY(LF(I,M,8),JR2,JC2,MMM)
 4750 CONTINUE
 4700 CONTINUE
 4650 CONTINUE
 4450 CONTINUE
 4400 CONTINUE
CC    CALL OUTPUT1(XN,LP,N,NMUS/2,NQTOT/2)
C--------CALCULATE THE COMMON FACTOR FOR DERIVATIVES OF PP1T,
C--------PP1B, PP3T, PP3B, PP1TQ, PP1BQ, PP3TQ, AND PP3BQ.
      DO 4800 I=1, NQTOT/2
      A(I)=W/2.D0/PAIG*AQ(I)/XQ(I)
 4800 CONTINUE
C--------DO LOOP FOR RK5.
      LIU=1
      DO 5000 J=1, NPRINT
      NIPRINT=IPRINT(J)
      H=HSTEP(J)
      DO 5100 I=1, NIPRINT
      CALL RK5(NMUS/2,NQTOT/2,N,NFUN,NFUNP1,INFUN,H,XN,YY)
 5100 CONTINUE
C--------TRANSFER FROM FOUR DIMENSIONS TO FIVE DIMENSIONS.
      DO 5200 M=1, L+1
      DO 5250 I=M, L+1
      DO 5300 MM=1, NMUS/2
      DO 5350 JR1=1, N
      DO 5400 JC1=1, N
      PP1T(I,M,JR1,JC1,MM)=YY(LF(I,M,1),JR1,JC1,MM)
      PP1B(I,M,JR1,JC1,MM)=YY(LF(I,M,2),JR1,JC1,MM)
      PP3T(I,M,JR1,JC1,MM)=YY(LF(I,M,3),JR1,JC1,MM)
      PP3B(I,M,JR1,JC1,MM)=YY(LF(I,M,4),JR1,JC1,MM)
 5400 CONTINUE
 5350 CONTINUE
 5300 CONTINUE
      DO 5450 MMM=1, NQTOT/2
      DO 5500 JR2=1, N
      DO 5550 JC2=1, N
      PP1TQ(I,M,JR2,JC2,MMM)=YY(LF(I,M,5),JR2,JC2,MMM)
      PP1BQ(I,M,JR2,JC2,MMM)=YY(LF(I,M,6),JR2,JC2,MMM)
      PP3TQ(I,M,JR2,JC2,MMM)=YY(LF(I,M,7),JR2,JC2,MMM)
      PP3BQ(I,M,JR2,JC2,MMM)=YY(LF(I,M,8),JR2,JC2,MMM)
 5550 CONTINUE
 5500 CONTINUE
 5450 CONTINUE
 5250 CONTINUE
 5200 CONTINUE
C--------PRINT OUT THE PP1T, PP1B, PP3T, PP3B, PP1TQ, PP1BQ, PP3TQ,
```

```
C--------AND PP3BQ FOR THE DESIRED OPTICAL THICKNESSES.
CC      IF(LIU .EQ. 1 .OR. J .EQ. NPRINT) THEN
CC      CALL OUTPUT1(XN,LP,N,NMUS/2,NQTOT/2)
CC      ELSE
CC      ENDIF
C--------CALL SUBROUTINE PPBARFUN TO CALCULATE PPI1BAR, PP1BAR,
C--------PPI1BARQ, PP1BARQ, PPI3BARQ, AND PP3BARQ.
        CALL PPBARFUN(XN)
C--------CALL SUBROUTINE INTEN TO CALCULATE THE REFLECTED AND
C--------TRANSMITTED INTENSITIES (RI AND TI).
        CALL INTEN
C--------INTERPOLATE THE VALUES FOR RI AND TI AT MU=1.0 BY USING
C--------LAGRANGE'S POLYNOMIAL APPROXIATION METHOD WHEN L=0
C--------AND 1.
        IF (L .LE. 1) THEN
        DO 6000 I=1, NMUS/2
        MUX(I)=MU(I)
   6000 CONTINUE
        DO 6100 JA=1, NMUS/2
        DO 6150 JB=1, NMUS/2
        DO 6200 JR1=1, N
        DO 6250 JC1=1, N
        FLAG(JR1,JC1,JB)=RI(JB,JA,JR1,JC1)
        FLAG1(JR1,JC1,JB)=TI(JB,JA,JR1,JC1)
   6250 CONTINUE
   6200 CONTINUE
   6150 CONTINUE
        CALL LAGR(MUX,FLAG,1.D0,6,2,N,NMUS/2,FLAGR)
        CALL LAGR(MUX,FLAG1,1.D0,6,2,N,NMUS/2,FLAGR1)
        DO 6350 JR2=1, N
        DO 6400 JC2=1, N
        RI(1,JA,JR2,JC2)=FLAGR(JR2,JC2)
        TI(1,JA,JR2,JC2)=FLAGR1(JR2,JC2)
   6400 CONTINUE
   6350 CONTINUE
   6100 CONTINUE
        ELSE
        ENDIF
C--------CALL SUBROUTINE INTENF TO CALCULATE THE NORMALIZED
C--------REFLECTED AND TRANSMITTED INTENSITIES (RIF AND TIF).
        CALL INTENF
        LIU=LIU+1
        WRITE(*,*) LIU-1
        WRITE(*,*)
        IF(LIU .LE. 5) THEN
        WRITE(9,*)
```

```
      WRITE(9,*)
      ELSE IF(LIU .GT. 5 .AND. LIU .LE. 10) THEN
      WRITE(5,*)
      WRITE(5,*)
      ELSE IF(LIU .GT. 10 .AND. LIU .LE. 15) THEN
      WRITE(6,*)
      WRITE(6,*)
      ELSE IF(LIU .GT. 15) THEN
      WRITE(7,*)
      WRITE(7,*)
      ENDIF
      WRITE(8,*)
      WRITE(8,*)
 5000 CONTINUE
      STOP
      END


C--------SUBROUTINE TO READ IN AND PRINT OUT THE SCATTERING
C--------COEFFICIENTS OF B MATRIX (INPUT L=L+1).
      SUBROUTINE BMATRIX(L,LP,N,NP,B)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION B(LP,NP,NP)
      DO 100 I=1, L
      WRITE(9,1) I-1
    1 FORMAT('B(',I1,')=')
      DO 150 J=1, N
      READ(4,*) (B(I,J,K), K=1,N)
  150 CONTINUE
      DO 200 JJ=1,N
      WRITE(9,2) (B(I,JJ,KK),KK=1,N)
    2 FORMAT(4(1PE18.11,2X))
  200 CONTINUE
  100 CONTINUE
      WRITE(9,*)
      RETURN
      END


C--------SUBROUTINE TO DECOMPOSE A MATRIX TO A LOWER AND UPPER
C--------TRIANGULAR MATRICES.
      SUBROUTINE LUDCMP(A,N,NP,INDX,D)
      IMPLICIT REAL*8 (A-H,O-Z)
      PARAMETER (NMAX=100,TINY=1.0E-20)
      DIMENSION A(NP,NP),INDX(N),VV(NMAX)
      D=1.
```

```
      DO 12 I=1,N
      AAMAX=0.
      DO 11 J=1,N
      IF (ABS(A(I,J)).GT.AAMAX) AAMAX=ABS(A(I,J))
   11 CONTINUE
      IF (AAMAX.EQ.0.) PAUSE 'Singular matrix.'
      VV(I)=1./AAMAX
   12 CONTINUE
      DO 19 J=1,N
      IF (J.GT.1) THEN
      DO 14 I=1,J-1
      SUM=A(I,J)
      IF (I.GT.1)THEN
      DO 13 K=1,I-1
      SUM=SUM-A(I,K)*A(K,J)
   13 CONTINUE
      A(I,J)=SUM
      ENDIF
   14 CONTINUE
      ENDIF
      AAMAX=0.
      DO 16 I=J,N
      SUM=A(I,J)
      IF (J.GT.1)THEN
      DO 15 K=1,J-1
      SUM=SUM-A(I,K)*A(K,J)
   15 CONTINUE
      A(I,J)=SUM
      ENDIF
      DUM=VV(I)*ABS(SUM)
      IF (DUM.GE.AAMAX) THEN
      IMAX=I
      AAMAX=DUM
      ENDIF
   16 CONTINUE
      IF (J.NE.IMAX)THEN
      DO 17 K=1,N
      DUM=A(IMAX,K)
      A(IMAX,K)=A(J,K)
      A(J,K)=DUM
   17 CONTINUE
      D=-D
      VV(IMAX)=VV(J)
      ENDIF
      INDX(J)=IMAX
      IF(J.NE.N)THEN
```

```
      IF(A(J,J).EQ.0.)A(J,J)=TINY
      DUM=1./A(J,J)
      DO 18 I=J+1,N
      A(I,J)=A(I,J)*DUM
   18 CONTINUE
      ENDIF
   19 CONTINUE
      IF(A(N,N).EQ.0.)A(N,N)=TINY
      RETURN
      END

C--------SUBROUTINE TO CALCULATE THE MATRIX INVERSION AFTER LU
C--------DECOMPOSITION (LUDCMP).
      SUBROUTINE LUBKSB(A,N,NP,INDX,B)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(NP,NP),INDX(N),B(N)
      II=0
      DO 12 I=1,N
      LL=INDX(I)
      SUM=B(LL)
      B(LL)=B(I)
      IF (II.NE.0)THEN
      DO 11 J=II,I-1
      SUM=SUM-A(I,J)*B(J)
   11 CONTINUE
      ELSE IF (SUM.NE.0.) THEN
      II=I
      ENDIF
      B(I)=SUM
   12 CONTINUE
      DO 14 I=N,1,-1
      SUM=B(I)
      IF(I.LT.N)THEN
      DO 13 J=I+1,N
      SUM=SUM-A(I,J)*B(J)
   13 CONTINUE
      ENDIF
      B(I)=SUM/A(I,I)
   14 CONTINUE
      RETURN
      END

C--------SUBROUTINE TO CALCULATE THE VALUES OF X COEFFICIENTS FOR
C--------PAI MATRIX (INPUT L=L+1).
```

```
      SUBROUTINE XPAI(L,LP,N,NP,X)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION X(LP,LP,NP,NP)
      DO 100 M=1, L
      DO 150 I=M, L
      DO 200 J=1, N
      DO 250 K=1, N
      X(M,I,J,K)=0.D0
  250 CONTINUE
  200 CONTINUE
      IF(I-1 .GE. 2) THEN
      X(M,I,1,1)=I-M+1.D0
      X(M,I,2,2)=(I-M+1.D0)/I*((I+2.D0)*(I-2.D0))**0.5
      X(M,I,3,3)=(I-M+1.D0)/I*((I+2.D0)*(I-2.D0))**0.5
      X(M,I,4,4)=I-M+1.D0
      ELSE
      ENDIF
  150 CONTINUE
  100 CONTINUE
      RETURN
      END


C--------SUBROUTINE TO CALCULATE THE VALUES OF Y COEFFICIENTS FOR
C--------PAI MATRIX (INPUT L=L+1).
      SUBROUTINE YPAI(L,LP,N,NP,Y)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION Y(LP,LP,NP,NP)
      DO 100 M=1, L
      DO 150 I=M, L
      DELTA=1.D0
      IF(M .EQ. I) DELTA=0.D0
      DO 200 J=1, N
      DO 250 K=1, N
      Y(M,I,J,K)=0.D0
  250 CONTINUE
  200 CONTINUE
      IF(I-1 .GE. 2) THEN
      Y(M,I,1,1)=DELTA*(I+M-2.D0)
      Y(M,I,2,2)=DELTA*((I+M-2.D0)/(I-1.D0)*((I-1.D0)**2-4.D0)**0.5)
      Y(M,I,3,3)=DELTA*((I+M-2.D0)/(I-1.D0)*((I-1.D0)**2-4.D0)**0.5)
      Y(M,I,4,4)=DELTA*(I+M-2.D0)
      ELSE
      ENDIF
  150 CONTINUE
  100 CONTINUE
```

```
      RETURN
      END

C--------SUBROUTINE TO CALCULATE THE VALUES OF Z COEFFICIENTS FOR
C--------PAI MATRIX (INPUT L=L+1).
      SUBROUTINE ZPAI(L,LP,N,NP,Z)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION Z(LP,LP,NP,NP)
      DO 100 M=1, L
      DO 150 I=M, L
      DO 200 J=1, N
      DO 250 K=1, N
      Z(M,I,J,K)=0.D0
  250 CONTINUE
  200 CONTINUE
      IF(I-1 .GE. 2) THEN
      Z(M,I,3,2)=2.D0*(M-1.D0)*(2.D0*(I-1.D0)+1.D0)/(I-1.D0)/I
      Z(M,I,2,3)=2.D0*(M-1.D0)*(2.D0*(I-1.D0)+1.D0)/(I-1.D0)/I
      ELSE
      ENDIF
  150 CONTINUE
  100 CONTINUE
      RETURN
      END

C--------SUBROUTINE FOR ADDING OR SUBTRACTING TWO MATRICES(5)
C--------(INPUT L=L+1).
C--------(ONE M AND ONE I FOR EACH CALL).
      SUBROUTINE MAOS1(A,B,M,I,LP,N,NP,NMUS,NMUSP,SIGN,C)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(LP,LP,NP,NP,NMUSP), B(LP,LP,NP,NP,NMUSP),
     *  C(LP,LP,NP,NP,NMUSP)
      DO 50 MM=1, NMUS
      DO 100 J=1, N
      DO 150 K=1, N
      C(M,I,J,K,MM)=A(M,I,J,K,MM)+B(M,I,J,K,MM)*SIGN
  150 CONTINUE
  100 CONTINUE
   50 CONTINUE
      RETURN
      END

C--------SUBROUTINE FOR MULTIPLICATION OF SCALAR NUMBERS WITH A
```

```
C--------MATRIX(5) (INPUT L=L+1).
C--------(ONE M AND ONE I FOR EACH CALL).
         SUBROUTINE MMULT1(A,B,M,I,LP,N,NP,NMUS,NMUSP,C)
         IMPLICIT REAL*8 (A-H,O-Z)
         DIMENSION A(NMUSP), B(LP,LP,NP,NP,NMUSP),
     *   C(LP,LP,NP,NP,NMUSP)
         DO 50 MM=1, NMUS
         DO 100 J=1, N
         DO 150 K=1, N
         C(M,I,J,K,MM)=A(MM)*B(M,I,J,K,MM)
     150 CONTINUE
     100 CONTINUE
      50 CONTINUE
         RETURN
         END


C--------SUBROUTINE FOR MULTIPLICATION OF TWO MATRICES WITH
C--------DIFFERENT DIMENSIONS(4*5) (INPUT L=L+1).
C--------(ONE M AND ONE I FOR EACH CALL).
C--------(I*(I-1)).
         SUBROUTINE MMULT2(A,B,M,I,LP,N,NP,NMUS,NMUSP,C)
         IMPLICIT REAL*8 (A-H,O-Z)
         DIMENSION A(LP,LP,NP,NP),B(LP,LP,NP,NP,NMUSP),
     *   C(LP,LP,NP,NP,NMUSP)
         DO 50 MM=1, NMUS
         DO 100 J=1, N
         DO 150 K=1, N
         SUM=0.D0
         DO 200 KK=1, N
         SUM=A(M,I,J,KK)*B(M,I-1,KK,K,MM)+SUM
     200 CONTINUE
         C(M,I,J,K,MM)=SUM
     150 CONTINUE
     100 CONTINUE
      50 CONTINUE
         RETURN
         END


C--------SUBROUTINE FOR MULTIPLICATION OF TWO MATRICES WITH
C--------DIFFERENT DIMENSIONS(4*5) (INPUT L=L+1).
C--------(ONE M AND ONE I FOR EACH CALL).
         SUBROUTINE MMULT3(A,B,M,I,LP,N,NP,NMUS,NMUSP,C)
         IMPLICIT REAL*8 (A-H,O-Z)
         DIMENSION A(LP,LP,NP,NP),B(LP,LP,NP,NP,NMUSP),
```

```
     *   C(LP,LP,NP,NP,NMUSP)
         DO 50 MM=1, NMUS
         DO 100 J=1, N
         DO 150 K=1, N
         SUM=0.D0
         DO 200 KK=1, N
         SUM=A(M,I,J,KK)*B(M,I,KK,K,MM)+SUM
  200    CONTINUE
         C(M,I,J,K,MM)=SUM
  150    CONTINUE
  100    CONTINUE
   50    CONTINUE
         RETURN
         END


C--------SUBROUTINE TO CALCULATE FACTORIAL
C--------(IOP .NE. 1 WILL CALCULATE 1*3*5....(2M-1)).
         SUBROUTINE FACTOR(M,IOP,FACT)
         IMPLICIT REAL*8 (A-H,O-Z)
         IF (IOP .NE. 1) THEN
         FACT1=1.D0
         DO 50 I=1, M-1, 2
         FACT1=I*FACT1
   50    CONTINUE
         ELSE
         FACT1=1.D0
         DO 100 I=1, M
         FACT1=I*FACT1
  100    CONTINUE
         ENDIF
         FACT=FACT1
         RETURN
         END


C--------SUBROUTINE TO CALCULATE THE PAI MATRIX FOR L .LE. 2
C---------(INPUT L=L+1).
         SUBROUTINE PAI2(A,L,LP,N,NP,NMUS,NMUSP,PAI)
         IMPLICIT REAL*8 (A-H,O-Z)
         DIMENSION A(NMUSP), PAI(LP,LP,NP,NP,NMUSP)
         DO 100 K=1,NMUS
         DO 150 I=1, N
         DO 200 J=1, N
         PAI(1,1,I,J,K)=0.D0
  200    CONTINUE
```

```
  150 CONTINUE
      PAI(1,1,1,1,K)=1.D0
      PAI(1,1,4,4,K)=1.D0
  100 CONTINUE
      IF(L .LT. 2) GO TO 1000
      DO 250 K=1,NMUS
      DO 300 I=1, N
      DO 350 J=1, N
      PAI(1,2,I,J,K)=0.D0
      PAI(2,2,I,J,K)=0.D0
  350 CONTINUE
  300 CONTINUE
      PAI(1,2,1,1,K)=A(K)
      PAI(1,2,4,4,K)=A(K)
      PAI(2,2,1,1,K)=(1.D0-A(K)**2)**0.5
      PAI(2,2,4,4,K)=(1.D0-A(K)**2)**0.5
  250 CONTINUE
      IF(L .LT. 3) GO TO 1000
      DO 400 K=1,NMUS
      DO 450 I=1, N
      DO 500 J=1, N
      PAI(1,3,I,J,K)=0.D0
      PAI(2,3,I,J,K)=0.D0
      PAI(3,3,I,J,K)=0.D0
  500 CONTINUE
  450 CONTINUE
      PAI(1,3,1,1,K)=0.5D0*(3.D0*A(K)**2-1.D0)
      PAI(1,3,2,2,K)=(6.D0)**0.5/4.D0*(1.D0-A(K)**2)
      PAI(1,3,3,3,K)=(6.D0)**0.5/4.D0*(1.D0-A(K)**2)
      PAI(1,3,4,4,K)=0.5D0*(3.D0*A(K)**2-1.D0)
      PAI(2,3,1,1,K)=3.D0*A(K)*(1.D0-A(K)**2)**0.5
      PAI(2,3,2,2,K)=-(6.D0)**0.5/2.D0*A(K)*(1.D0-A(K)**2)**0.5
      PAI(2,3,2,3,K)=(6.D0)**0.5/2.D0*(1.D0-A(K)**2)**0.5
      PAI(2,3,3,2,K)=(6.D0)**0.5/2.D0*(1.D0-A(K)**2)**0.5
      PAI(2,3,3,3,K)=-(6.D0)**0.5/2.D0*A(K)*(1.D0-A(K)**2)**0.5
      PAI(2,3,4,4,K)=3.D0*A(K)*(1.D0-A(K)**2)**0.5
      PAI(3,3,1,1,K)=3.D0*(1.D0-A(K)**2)
      PAI(3,3,2,2,K)=6.D0*(24.D0)**(-0.5)*(1.D0+A(K)**2)
      PAI(3,3,2,3,K)=-12.D0*(24.D0)**(-0.5)*A(K)
      PAI(3,3,3,2,K)=-12.D0*(24.D0)**(-0.5)*A(K)
      PAI(3,3,3,3,K)=6.D0*(24.D0)**(-0.5)*(1.D0+A(K)**2)
      PAI(3,3,4,4,K)=3.D0*(1.D0-A(K)**2)
  400 CONTINUE
 1000 RETURN
 END
```

```
C--------SUBROUTINE TO PRINTOUT THE PP1T, PP1B, PP3T, PP3B, PP1TQ,
C--------PP1BQ, PP3TQ, AND PP3BQ (INPUT L=L+1).
      SUBROUTINE OUTPUT1(XN,L,N,NMUS,NQTOT)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 MU
      COMMON/BLK1/MU(26),XQ(656)
      COMMON/BLK3/PP1T(6,6,4,4,13),PP1B(6,6,4,4,13)
      COMMON/BLK4/PP3T(6,6,4,4,13),PP3B(6,6,4,4,13)
      COMMON/BLK5/PP1TQ(6,6,4,4,328),PP1BQ(6,6,4,4,328)
      COMMON/BLK6/PP3TQ(6,6,4,4,328),PP3BQ(6,6,4,4,328)
      WRITE(9,*)
      WRITE(9,*) '=============================================='
      WRITE(9,1) XN
    1 FORMAT(1X,'OPTICAL THICKNESS=',F12.8)
      WRITE(9,*)
      WRITE(9,*)'--------------------------------------------------'
      DO 100 MM=1, NMUS
      WRITE(9,2) MU(MM)
    2 FORMAT('MUO=',1PE18.11)
      WRITE(9,*)'--------------------------------------------------'
      DO 200 M=1, L
      WRITE(9,3) M-1
    3 FORMAT('M=',I2)
      DO 300 I=M, L
      WRITE(9,4) I-1
    4 FORMAT('I=',I2)
      WRITE(9,*) 'PP1T='
      DO 400 JR1=1, N
      WRITE(9,5) (PP1T(I,M,JR1,JC1,MM),JC1=1,N)
    5 FORMAT(4(1PE18.11,2X))
  400 CONTINUE
      WRITE(9,*) 'PP1B='
      DO 500 JR2=1, N
      WRITE(9,5) (PP1B(I,M,JR2,JC2,MM),JC2=1,N)
  500 CONTINUE
      WRITE(9,*) 'PP3T='
      DO 600 JR3=1, N
      WRITE(9,5) (PP3T(I,M,JR3,JC3,MM),JC3=1,N)
  600 CONTINUE
      WRITE(9,*) 'PP3B='
      DO 700 JR4=1, N
      WRITE(9,5) (PP3B(I,M,JR4,JC4,MM),JC4=1,N)
  700 CONTINUE
  300 CONTINUE
  200 CONTINUE
```

```
      WRITE(9,*)'-------------------------------------------------'
      WRITE(9,*)
  100 CONTINUE
      WRITE(9,*)'================================================================'
      WRITE(9,*)
      DO 800 MM=1, NQTOT
      WRITE(9,6) XQ(MM)
    6 FORMAT('XQ=',1PE18.11)
      WRITE(9,*)'-------------------------------------------------'
      DO 825 M=1, L
      WRITE(9,3) M-1
      DO 850 I=M, L
      WRITE(9,4) I-1
      WRITE(9,*) 'PP1TQ='
      DO 875 JR1=1, N
      WRITE(9,5) (PP1TQ(I,M,JR1,JC1,MM),JC1=1,N)
  875 CONTINUE
      WRITE(9,*) 'PP1BQ='
      DO 900 JR2=1, N
      WRITE(9,5) (PP1BQ(I,M,JR2,JC2,MM),JC2=1,N)
  900 CONTINUE
      WRITE(9,*) 'PP3TQ='
      DO 925 JR3=1, N
      WRITE(9,5) (PP3TQ(I,M,JR3,JC3,MM),JC3=1,N)
  925 CONTINUE
      WRITE(9,*) 'PP3BQ='
      DO 950 JR4=1, N
      WRITE(9,5) (PP3BQ(I,M,JR4,JC4,MM),JC4=1,N)
  950 CONTINUE
  850 CONTINUE
  825 CONTINUE
      WRITE(9,*)'-------------------------------------------------'
      WRITE(9,*)
  800 CONTINUE
      WRITE(9,*)'================================================================'
      RETURN
      END


C--------SUBROUTINE TO PRINTOUT THE PPI1BARQ AND PP3BARQ.
      SUBROUTINE OUTPUT2(I,M,JJJ)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 MU
      COMMON/BLK1/MU(26),XQ(656)
      COMMON/BLK9/N,L,NMUS,NQTOT
      COMMON/BLK13/PPI1BARQ(6,6,13,4,4,328),PP1BARQ(6,6,13,4,4,328)
```

```
      COMMON/BLK14/PPI3BARQ(6,6,13,4,4,328),PP3BARQ(6,6,13,4,4,328)
      DO 100 JJ=1, NQTOT/2
      WRITE(8,1) XQ(JJ)
    1 FORMAT('XQ=',1PE18.11)
      WRITE(8,*) 'PPI1BARQ='
      DO 200 JR1=1, N
      WRITE(8,2) (PPI1BARQ(I,M,JJJ,JR1,JC1,JJ),JC1=1,N)
    2 FORMAT(4(1PE18.11,2X))
  200 CONTINUE
      WRITE(8,*) 'PP3BARQ='
      DO 300 JR2=1, N
      WRITE(8,2) (PP3BARQ(I,M,JJJ,JR2,JC2,JJ),JC2=1,N)
  300 CONTINUE
      WRITE(8,*)'---------------------------'
  100 CONTINUE
      RETURN
      END


C--------SUBROUTINE TO PRINTOUT THE PP1BARQ AND PPI3BARQ.
      SUBROUTINE OUTPUT3(I,M,JJJ)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 MU
      COMMON/BLK1/MU(26),XQ(656)
      COMMON/BLK9/N,L,NMUS,NQTOT
      COMMON/BLK13/PPI1BARQ(6,6,13,4,4,328),PP1BARQ(6,6,13,4,4,328)
      COMMON/BLK14/PPI3BARQ(6,6,13,4,4,328),PP3BARQ(6,6,13,4,4,328)
      DO 100 JJ=1, NQTOT/2
      WRITE(8,1) XQ(JJ)
    1 FORMAT('XQ=',1PE18.11)
      WRITE(8,*) 'PP1BARQ='
      DO 200 JR1=1, N
      WRITE(8,2) (PP1BARQ(I,M,JJJ,JR1,JC1,JJ),JC1=1,N)
    2 FORMAT(4(1PE18.11,2X))
  200 CONTINUE
      WRITE(8,*) 'PPI3BARQ='
      DO 300 JR2=1, N
      WRITE(8,2) (PPI3BARQ(I,M,JJJ,JR2,JC2,JJ),JC2=1,N)
  300 CONTINUE
      WRITE(8,*)'---------------------------'
  100 CONTINUE
      RETURN
      END


C--------SUBROUTINE TO PRINTOUT THE PPI1BAR.
```

```
      SUBROUTINE OUTPUT4(I,M,JJJ)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 MU
      COMMON/BLK1/MU(26),XQ(656)
      COMMON/BLK9/N,L,NMUS,NQTOT
      COMMON/BLK11/PPI1BAR(6,6,13,4,4,13),PP1BAR(6,6,13,4,4,13)
      DO 100 JJ=1, NMUS/2
      WRITE(8,1) MU(JJ)
    1 FORMAT('MUO=',1PE18.11)
      DO 200 JR1=1, N
      WRITE(8,2) (PPI1BAR(I,M,JJJ,JR1,JC1,JJ),JC1=1,N)
    2 FORMAT(4(1PE18.11,2X))
  200 CONTINUE
      WRITE(8,*)'---------------------------'
  100 CONTINUE
      RETURN
      END


C--------SUBROUTINE TO PRINTOUT THE PP1BAR.
      SUBROUTINE OUTPUT5(I,M,JJJ)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 MU
      COMMON/BLK1/MU(26),XQ(656)
      COMMON/BLK9/N,L,NMUS,NQTOT
      COMMON/BLK11/PPI1BAR(6,6,13,4,4,13),PP1BAR(6,6,13,4,4,13)
      DO 100 JJ=1, NMUS/2
      WRITE(8,1) MU(JJ)
    1 FORMAT('MUO=',1PE18.11)
      DO 200 JR1=1, N
      WRITE(8,2) (PP1BAR(I,M,JJJ,JR1,JC1,JJ),JC1=1,N)
    2 FORMAT(4(1PE18.11,2X))
  200 CONTINUE
      WRITE(8,*)'---------------------------'
  100 CONTINUE
      RETURN
      END


C--------SUBROUTINE FOR RUNGE-KUTTA METHOD.
      SUBROUTINE RK5(NPIC,NZMU,N,NFUN,NFUNP1,INFUN,H,XN,YN)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION C(6), Z(6), A(6,5), YN(168,4,4,328), Y(168,4,4,328),
     *  AK(168,6,4,4,328), SUM1(4,4), SUM2(4,4), PHI1(4,4), PHI2(4,4)
      COMMON/BLK10/DER(168,4,4,328)
      DATA C(2),C(3),C(4),C(5),C(6) /.25D0,.25D0,.5D0,.75D0,1.0D0/
```

```
      DATA Z(1),Z(2),Z(3),Z(4),Z(5),Z(6) /7.0D0,0.0D0,32.0D0,12.0D0,
    1 32.0D0,7.0D0/
      DATA A(2,1),A(3,1),A(3,2),A(4,1),A(4,2),A(4,3) /.25D0,.125D0,
    1 .125D0,0.0D0,-.50D0,1.0D0/
      DATA A(5,1),A(5,2),A(5,3),A(5,4) /.1875D0,0.0D0,0.0D0,.5625D0/
      DATA A(6,1),A(6,2),A(6,3),A(6,4),A(6,5) /-.428571428571429D0,
    1 .285714285714285D0,1.71428571428571D0,-1.71428571428571D0,
    2 1.14285714285714D0/
      CALL DERV(XN,YN)
      DO 200 L=1,NFUN
      DO 300 I=1,NPIC
      DO 400 JR1=1, N
      DO 500 JC1=1, N
      AK(L,1,JR1,JC1,I)=DER(L,JR1,JC1,I)*H
  500 CONTINUE
  400 CONTINUE
  300 CONTINUE
  200 CONTINUE
      DO 600 L=NFUNP1,INFUN
      DO 700 I=1,NZMU
      DO 800 JR1=1, N
      DO 900 JC1=1, N
      AK(L,1,JR1,JC1,I)=DER(L,JR1,JC1,I)*H
  900 CONTINUE
  800 CONTINUE
  700 CONTINUE
  600 CONTINUE
      DO 1000 K=2,6
      DO 1100 L=1,NFUN
      DO 1200 I=1,NPIC
      K1=K-1
      DO 1300 JR1=1, N
      DO 1400 JC1=1, N
      SUM1(JR1,JC1)=0.D0
 1400 CONTINUE
 1300 CONTINUE
      DO 1500 J=1,K1
      DO 1600 JR2=1, N
      DO 1700 JC2=1, N
      SUM1(JR2,JC2)=A(K,J)*AK(L,J,JR2,JC2,I)+SUM1(JR2,JC2)
 1700 CONTINUE
 1600 CONTINUE
 1500 CONTINUE
      DO 1800 JR3=1, N
      DO 1900 JC3=1, N
      Y(L,JR3,JC3,I)=YN(L,JR3,JC3,I)+SUM1(JR3,JC3)
```

```
1900 CONTINUE
1800 CONTINUE
1200 CONTINUE
1100 CONTINUE
     DO 2000 L=NFUNP1,INFUN
     DO 2100 I=1,NZMU
     K1=K-1
     DO 2200 JR4=1, N
     DO 2300 JC4=1, N
     SUM2(JR4,JC4)=0.D0
2300 CONTINUE
2200 CONTINUE
     DO 2400 J=1,K1
     DO 2425 JR5=1, N
     DO 2450 JC5=1, N
     SUM2(JR5,JC5)=A(K,J)*AK(L,J,JR5,JC5,I)+SUM2(JR5,JC5)
2450 CONTINUE
2425 CONTINUE
2400 CONTINUE
     DO 2500 JR6=1, N
     DO 2600 JC6=1, N
     Y(L,JR6,JC6,I)=YN(L,JR6,JC6,I)+SUM2(JR6,JC6)
2600 CONTINUE
2500 CONTINUE
2100 CONTINUE
2000 CONTINUE
     X=XN+C(K)*H
     CALL DERV(X,Y)
     DO 2700 L=1,NFUN
     DO 2800 I=1,NPIC
     DO 2900 JR7=1, N
     DO 3000 JC7=1, N
     AK(L,K,JR7,JC7,I)=DER(L,JR7,JC7,I)*H
3000 CONTINUE
2900 CONTINUE
2800 CONTINUE
2700 CONTINUE
     DO 3100 L=NFUNP1,INFUN
     DO 3200 I=1,NZMU
     DO 3300 JR8=1, N
     DO 3400 JC8=1, N
     AK(L,K,JR8,JC8,I)=DER(L,JR8,JC8,I)*H
3400 CONTINUE
3300 CONTINUE
3200 CONTINUE
3100 CONTINUE
```

```
1000 CONTINUE
     DO 3500 L=1,NFUN
     DO 3600 I=1,NPIC
     DO 3700 JR1=1, N
     DO 3800 JC1=1, N
     PHI1(JR1,JC1)=0.D0
3800 CONTINUE
3700 CONTINUE
     DO 3900 K7=1,6
     DO 4000 JR2=1, N
     DO 4100 JC2=1, N
     PHI1(JR2,JC2)=PHI1(JR2,JC2)+Z(K7)*AK(L,K7,JR2,JC2,I)
4100 CONTINUE
4000 CONTINUE
3900 CONTINUE
     DO 4200 JR3=1, N
     DO 4300 JC3=1, N
     YN(L,JR3,JC3,I)=YN(L,JR3,JC3,I)+PHI1(JR3,JC3)/90.D0
4300 CONTINUE
4200 CONTINUE
3600 CONTINUE
3500 CONTINUE
     DO 4400 L=NFUNP1,INFUN
     DO 4500 I=1,NZMU
     DO 4600 JR1=1, N
     DO 4700 JC1=1, N
     PHI2(JR1,JC1)=0.D0
4700 CONTINUE
4600 CONTINUE
     DO 4800 K7=1,6
     DO 4900 JR2=1, N
     DO 5000 JC2=1, N
     PHI2(JR2,JC2)=PHI2(JR2,JC2)+Z(K7)*AK(L,K7,JR2,JC2,I)
5000 CONTINUE
4900 CONTINUE
4800 CONTINUE
     DO 5100 JR3=1, N
     DO 5200 JC3=1, N
     YN(L,JR3,JC3,I)=YN(L,JR3,JC3,I)+PHI2(JR3,JC3)/90.D0
5200 CONTINUE
5100 CONTINUE
4500 CONTINUE
4400 CONTINUE
     XN=XN+H
     RETURN
     END
```

```
C--------SUBROUTINE FOR DERIVATIVES TO USE IN RK5.
      SUBROUTINE DERV(XN,Y)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 MU
      DIMENSION Y(168,4,4,328), SUMA(4,4), SUMA1(4,4), SUMA2(4,4),
     *  SUMB(4,4), SUMB1(4,4), SUMB2(4,4), SUMC(4,4), SUMC1(4,4),
     *  SUMC2(4,4), SUMC3(4,4), SUMD(4,4), SUMD1(4,4), SUMD2(4,4),
     *  SUMD3(4,4), SUME(4,4), SUME1(4,4), SUME2(4,4), SUME3(4,4),
     *  SUMF(4,4), SUMF1(4,4), SUMF2(4,4), SUMF3(4,4), SUMAN(4,4),
     *  SUMA1N(4,4), SUMA2N(4,4), SUMBN(4,4), SUMB1N(4,4), SUMB2N(4,4),
     *  SUMCN(4,4), SUMC1N(4,4), SUMC2N(4,4), SUMC3N(4,4), SUMDN(4,4),
     *  SUMD1N(4,4), SUMD2N(4,4), SUMD3N(4,4), SUMEN(4,4), SUME1N(4,4),
     *  SUME2N(4,4), SUME3N(4,4), SUMFN(4,4), SUMF1N(4,4), SUMF2N(4,4),
     *  SUMF3N(4,4)
      COMMON/BLK1/MU(26),XQ(656)
      COMMON/BLK2/LF(6,6,8)
      COMMON/BLK7/PAIQ(6,6,4,4,656),B(6,4,4),A(328)
      COMMON/BLK8/W,PAIG
      COMMON/BLK9/N,L,NMUS,NQTOT
      COMMON/BLK10/DER(168,4,4,328)
C--------DERIVATIVES FOR QUADRATURE POINTS.
      DO 100 M=1, L+1
      DO 200 I=M, L+1
      DO 300 JJ=1, NQTOT/2
      DO 400 JR1=1, N
      DO 500 JC1=1, N
      SUMC(JR1,JC1)=0.D0
      SUMC1(JR1,JC1)=0.D0
      SUMC2(JR1,JC1)=0.D0
      SUMC3(JR1,JC1)=0.D0
      SUMD(JR1,JC1)=0.D0
      SUMD1(JR1,JC1)=0.D0
      SUMD2(JR1,JC1)=0.D0
      SUMD3(JR1,JC1)=0.D0
      SUME(JR1,JC1)=0.D0
      SUME1(JR1,JC1)=0.D0
      SUME2(JR1,JC1)=0.D0
      SUME3(JR1,JC1)=0.D0
      SUMF(JR1,JC1)=0.D0
      SUMF1(JR1,JC1)=0.D0
      SUMF2(JR1,JC1)=0.D0
      SUMF3(JR1,JC1)=0.D0
  500 CONTINUE
  400 CONTINUE
```

```
      DO 600 J=M, L+1
      DO 700 JR2=1, N
      DO 800 JC2=1, N
      SUMA(JR2,JC2)=0.D0
      SUMA1(JR2,JC2)=0.D0
      SUMA2(JR2,JC2)=0.D0
      SUMB(JR2,JC2)=0.D0
      SUMB1(JR2,JC2)=0.D0
      SUMB2(JR2,JC2)=0.D0
  800 CONTINUE
  700 CONTINUE
      DO 900 KK=1, NQTOT/2
      DO 950 JR3=1, N
      DO 1000 JC3=1, N
      SUM1=0.D0
      SUM2=0.D0
      DO 1050 K1=1, N
      SUM1=Y(LF(I,M,8),JR3,K1,KK)*PAIQ(M,J,K1,JC3,NQTOT/2+KK)+SUM1
      SUM2=Y(LF(I,M,6),JR3,K1,KK)*PAIQ(M,J,K1,JC3,KK)+SUM2
 1050 CONTINUE
      SUMA1(JR3,JC3)=SUM1
      SUMB1(JR3,JC3)=SUM2
 1000 CONTINUE
  950 CONTINUE
      DO 1100 JR4=1, N
      DO 1150 JC4=1, N
      SUMA2(JR4,JC4)=SUMA1(JR4,JC4)*A(KK)
      SUMB2(JR4,JC4)=SUMB1(JR4,JC4)*A(KK)
 1150 CONTINUE
 1100 CONTINUE
      DO 1200 JR5=1, N
      DO 1250 JC5=1, N
      SUMA(JR5,JC5)=SUMA2(JR5,JC5)+SUMA(JR5,JC5)
      SUMB(JR5,JC5)=SUMB2(JR5,JC5)+SUMB(JR5,JC5)
 1250 CONTINUE
 1200 CONTINUE
  900 CONTINUE
      CALL FACTOR(J-M,1,FACTT)
      CALL FACTOR(J+M-2,1,FACTB)
      DO 1300 JR6=1, N
      DO 1350 JC6=1, N
      SUMC1(JR6,JC6)=(-1.D0)**(I+M)*FACTT/FACTB*SUMA(JR6,JC6)
      SUMD1(JR6,JC6)=FACTT/FACTB*SUMB(JR6,JC6)
      SUME1(JR6,JC6)=(-1.D0)**(J+I)*FACTT/FACTB*SUMB(JR6,JC6)
      SUMF1(JR6,JC6)=(-1.D0)**(M-J)*FACTT/FACTB*SUMA(JR6,JC6)
 1350 CONTINUE
```

```
1300 CONTINUE
     DO 1400 JR7=1, N
     DO 1450 JC7=1, N
     SUM3=0.D0
     SUM4=0.D0
     SUM5=0.D0
     SUM6=0.D0
     DO 1500 K2=1, N
     SUM3=SUMC1(JR7,K2)*B(J,K2,JC7)+SUM3
     SUM4=SUMD1(JR7,K2)*B(J,K2,JC7)+SUM4
     SUM5=SUME1(JR7,K2)*B(J,K2,JC7)+SUM5
     SUM6=SUMF1(JR7,K2)*B(J,K2,JC7)+SUM6
1500 CONTINUE
     SUMC2(JR7,JC7)=SUM3
     SUMD2(JR7,JC7)=SUM4
     SUME2(JR7,JC7)=SUM5
     SUMF2(JR7,JC7)=SUM6
1450 CONTINUE
1400 CONTINUE
     DO 1550 JR8=1, N
     DO 1600 JC8=1, N
     SUM7=0.D0
     SUM8=0.D0
     SUM9=0.D0
     SUM10=0.D0
     DO 1650 K3=1, N
     SUM7=SUMC2(JR8,K3)*Y(LF(J,M,6),K3,JC8,JJ)+SUM7
     SUM8=SUMD2(JR8,K3)*Y(LF(J,M,5),K3,JC8,JJ)+SUM8
     SUM9=SUME2(JR8,K3)*Y(LF(J,M,8),K3,JC8,JJ)+SUM9
     SUM10=SUMF2(JR8,K3)*Y(LF(J,M,7),K3,JC8,JJ)+SUM10
1650 CONTINUE
     SUMC3(JR8,JC8)=SUM7
     SUMD3(JR8,JC8)=SUM8
     SUME3(JR8,JC8)=SUM9
     SUMF3(JR8,JC8)=SUM10
1600 CONTINUE
1550 CONTINUE
     DO 1700 JR9=1, N
     DO 1750 JC9=1, N
     SUMC(JR9,JC9)=SUMC3(JR9,JC9)+SUMC(JR9,JC9)
     SUMD(JR9,JC9)=SUMD3(JR9,JC9)+SUMD(JR9,JC9)
     SUME(JR9,JC9)=SUME3(JR9,JC9)+SUME(JR9,JC9)
     SUMF(JR9,JC9)=SUMF3(JR9,JC9)+SUMF(JR9,JC9)
1750 CONTINUE
1700 CONTINUE
 600 CONTINUE
```

```
      DO 1800 JR10=1, N
      DO 1850 JC10=1, N
      DER(LF(I,M,5),JR10,JC10,JJ)=SUMC(JR10,JC10)
      DER(LF(I,M,6),JR10,JC10,JJ)=(-1.D0/XQ(JJ))*
     *  Y(LF(I,M,6),JR10,JC10,JJ)+SUMD(JR10,JC10)
      DER(LF(I,M,7),JR10,JC10,JJ)=SUME(JR10,JC10)
      DER(LF(I,M,8),JR10,JC10,JJ)=(-1.D0/XQ(JJ))*
     *  Y(LF(I,M,8),JR10,JC10,JJ)+SUMF(JR10,JC10)
 1850 CONTINUE
 1800 CONTINUE
  300 CONTINUE
  200 CONTINUE
  100 CONTINUE
C--------DERIVATIVES FOR MU VALUES.
      DO 2100 M=1, L+1
      DO 2200 I=M, L+1
      DO 2300 JJ=1, NMUS/2
      DO 2400 JR1=1, N
      DO 2500 JC1=1, N
      SUMCN(JR1,JC1)=0.D0
      SUMC1N(JR1,JC1)=0.D0
      SUMC2N(JR1,JC1)=0.D0
      SUMC3N(JR1,JC1)=0.D0
      SUMDN(JR1,JC1)=0.D0
      SUMD1N(JR1,JC1)=0.D0
      SUMD2N(JR1,JC1)=0.D0
      SUMD3N(JR1,JC1)=0.D0
      SUMEN(JR1,JC1)=0.D0
      SUME1N(JR1,JC1)=0.D0
      SUME2N(JR1,JC1)=0.D0
      SUME3N(JR1,JC1)=0.D0
      SUMFN(JR1,JC1)=0.D0
      SUMF1N(JR1,JC1)=0.D0
      SUMF2N(JR1,JC1)=0.D0
      SUMF3N(JR1,JC1)=0.D0
 2500 CONTINUE
 2400 CONTINUE
      DO 2600 J=M, L+1
      DO 2700 JR2=1, N
      DO 2800 JC2=1, N
      SUMAN(JR2,JC2)=0.D0
      SUMA1N(JR2,JC2)=0.D0
      SUMA2N(JR2,JC2)=0.D0
      SUMBN(JR2,JC2)=0.D0
      SUMB1N(JR2,JC2)=0.D0
      SUMB2N(JR2,JC2)=0.D0
```

```
2800 CONTINUE
2700 CONTINUE
     DO 2900 KK=1, NQTOT/2
     DO 2950 JR3=1, N
     DO 3000 JC3=1, N
     SUM11=0.D0
     SUM12=0.D0
     DO 3050 K1=1, N
     SUM11=Y(LF(I,M,8),JR3,K1,KK)*PAIQ(M,J,K1,JC3,NQTOT/2+KK)+SUM11
     SUM12=Y(LF(I,M,6),JR3,K1,KK)*PAIQ(M,J,K1,JC3,KK)+SUM12
3050 CONTINUE
     SUMA1N(JR3,JC3)=SUM11
     SUMB1N(JR3,JC3)=SUM12
3000 CONTINUE
2950 CONTINUE
     DO 3100 JR4=1, N
     DO 3150 JC4=1, N
     SUMA2N(JR4,JC4)=SUMA1N(JR4,JC4)*A(KK)
     SUMB2N(JR4,JC4)=SUMB1N(JR4,JC4)*A(KK)
3150 CONTINUE
3100 CONTINUE
     DO 3200 JR5=1, N
     DO 3250 JC5=1, N
     SUMAN(JR5,JC5)=SUMA2N(JR5,JC5)+SUMAN(JR5,JC5)
     SUMBN(JR5,JC5)=SUMB2N(JR5,JC5)+SUMBN(JR5,JC5)
3250 CONTINUE
3200 CONTINUE
2900 CONTINUE
     CALL FACTOR(J-M,1,FACTT1)
     CALL FACTOR(J+M-2,1,FACTB1)
     DO 3300 JR6=1, N
     DO 3350 JC6=1, N
     SUMC1N(JR6,JC6)=(-1.D0)**(I+M)*FACTT1/FACTB1*SUMAN(JR6,JC6)
     SUMD1N(JR6,JC6)=FACTT1/FACTB1*SUMBN(JR6,JC6)
     SUME1N(JR6,JC6)=(-1.D0)**(J+I)*FACTT1/FACTB1*SUMBN(JR6,JC6)
     SUMF1N(JR6,JC6)=(-1.D0)**(M-J)*FACTT1/FACTB1*SUMAN(JR6,JC6)
3350 CONTINUE
3300 CONTINUE
     DO 3400 JR7=1, N
     DO 3450 JC7=1, N
     SUM13=0.D0
     SUM14=0.D0
     SUM15=0.D0
     SUM16=0.D0
     DO 3500 K2=1, N
     SUM13=SUMC1N(JR7,K2)*B(J,K2,JC7)+SUM13
```

```
      SUM14=SUMD1N(JR7,K2)*B(J,K2,JC7)+SUM14
      SUM15=SUME1N(JR7,K2)*B(J,K2,JC7)+SUM15
      SUM16=SUMF1N(JR7,K2)*B(J,K2,JC7)+SUM16
3500  CONTINUE
      SUMC2N(JR7,JC7)=SUM13
      SUMD2N(JR7,JC7)=SUM14
      SUME2N(JR7,JC7)=SUM15
      SUMF2N(JR7,JC7)=SUM16
3450  CONTINUE
3400  CONTINUE
      DO 3550 JR8=1, N
      DO 3600 JC8=1, N
      SUM17=0.D0
      SUM18=0.D0
      SUM19=0.D0
      SUM20=0.D0
      DO 3650 K3=1, N
      SUM17=SUMC2N(JR8,K3)*Y(LF(J,M,2),K3,JC8,JJ)+SUM17
      SUM18=SUMD2N(JR8,K3)*Y(LF(J,M,1),K3,JC8,JJ)+SUM18
      SUM19=SUME2N(JR8,K3)*Y(LF(J,M,4),K3,JC8,JJ)+SUM19
      SUM20=SUMF2N(JR8,K3)*Y(LF(J,M,3),K3,JC8,JJ)+SUM20
3650  CONTINUE
      SUMC3N(JR8,JC8)=SUM17
      SUMD3N(JR8,JC8)=SUM18
      SUME3N(JR8,JC8)=SUM19
      SUMF3N(JR8,JC8)=SUM20
3600  CONTINUE
3550  CONTINUE
      DO 3700 JR9=1, N
      DO 3750 JC9=1, N
      SUMCN(JR9,JC9)=SUMC3N(JR9,JC9)+SUMCN(JR9,JC9)
      SUMDN(JR9,JC9)=SUMD3N(JR9,JC9)+SUMDN(JR9,JC9)
      SUMEN(JR9,JC9)=SUME3N(JR9,JC9)+SUMEN(JR9,JC9)
      SUMFN(JR9,JC9)=SUMF3N(JR9,JC9)+SUMFN(JR9,JC9)
3750  CONTINUE
3700  CONTINUE
2600  CONTINUE
      DO 3800 JR10=1, N
      DO 3850 JC10=1, N
      DER(LF(I,M,1),JR10,JC10,JJ)=SUMCN(JR10,JC10)
      DER(LF(I,M,2),JR10,JC10,JJ)=(-1.D0/MU(JJ))*
     *  Y(LF(I,M,2),JR10,JC10,JJ)+SUMDN(JR10,JC10)
      DER(LF(I,M,3),JR10,JC10,JJ)=SUMEN(JR10,JC10)
      DER(LF(I,M,4),JR10,JC10,JJ)=(-1.D0/MU(JJ))*
     *  Y(LF(I,M,4),JR10,JC10,JJ)+SUMFN(JR10,JC10)
3850  CONTINUE
```

```
 3800 CONTINUE
 2300 CONTINUE
 2200 CONTINUE
 2100 CONTINUE
      RETURN
      END


C--------SUBROUTINE TO CALCULATE PPI1BAR, PP1BAR, PPI1BARQ,
C--------PP1BARQ, PPI3BARQ, AND PP3BARQ.
      SUBROUTINE PPBARFUN(XN)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 MU
      DIMENSION SUMA(4,4), SUMA1(4,4), SUMA2(4,4), SUMB(4,4),
     *  SUMB1(4,4), SUMB2(4,4), SUME(4,4), SUME1(4,4), SUME2(4,4),
     *  SUME3(4,4), SUMF(4,4), SUMF1(4,4), SUMF2(4,4), SUMF3(4,4),
     *  SUMK(4,4), SUMK1(4,4), SUMK2(4,4), SUMK3(4,4), SUML(4,4),
     *  SUML1(4,4), SUML2(4,4), SUML3(4,4), SUMC(4,4), SUMC1(4,4),
     *  SUMC2(4,4), SUMD(4,4), SUMD1(4,4), SUMD2(4,4), SUMG(4,4),
     *  SUMG1(4,4), SUMG2(4,4), SUMG3(4,4), SUMH(4,4), SUMH1(4,4),
     *  SUMH2(4,4), SUMH3(4,4), PPI1BARQN(6,6,13,4,4,328),
     *  PP3BARQN(6,6,13,4,4,328), FLAG(4,4,328), FLAGR(4,4), XX(328),
     *  MINUM(13)
      COMMON/BLK1/MU(26),XQ(656)
      COMMON/BLK3/PP1T(6,6,4,4,13),PP1B(6,6,4,4,13)
      COMMON/BLK4/PP3T(6,6,4,4,13),PP3B(6,6,4,4,13)
      COMMON/BLK5/PP1TQ(6,6,4,4,328),PP1BQ(6,6,4,4,328)
      COMMON/BLK6/PP3TQ(6,6,4,4,328),PP3BQ(6,6,4,4,328)
      COMMON/BLK7/PAIQ(6,6,4,4,656),B(6,4,4),A(328)
      COMMON/BLK9/N,L,NMUS,NQTOT
      COMMON/BLK11/PPI1BAR(6,6,13,4,4,13),PP1BAR(6,6,13,4,4,13)
      COMMON/BLK13/PPI1BARQ(6,6,13,4,4,328),PP1BARQ(6,6,13,4,4,328)
      COMMON/BLK14/PPI3BARQ(6,6,13,4,4,328),PP3BARQ(6,6,13,4,4,328)
      COMMON/BLK15/ERROR
      COMMON/BLK16/RELAX
      COMMON/BLK21/LIU
      WRITE(9,*) '=============================================='
      WRITE(8,*) '=============================================='
      WRITE(9,1) XN
      WRITE(8,1) XN
    1 FORMAT(1X,'OPTICAL THICKNESS=',F12.8)
      WRITE(9,*)
      WRITE(8,*)
      WRITE(9,*)'---------------------------------------------------'
      WRITE(8,*)'---------------------------------------------------'
      IF(LIU .GT. 5 .AND. LIU .LE. 10) THEN
```

```
      WRITE(5,*) '=================================================='
      WRITE(5,1) XN
      WRITE(5,*)
      WRITE(5,*)'--------------------------------------------------'
      ELSE IF(LIU .GT. 10 .AND. LIU .LE. 15) THEN
      WRITE(6,*) '=================================================='
      WRITE(6,1) XN
      WRITE(6,*)
      WRITE(6,*)'--------------------------------------------------'
      ELSE IF(LIU .GT. 15) THEN
      WRITE(7,*) '=================================================='
      WRITE(7,1) XN
      WRITE(7,*)
      WRITE(7,*)'--------------------------------------------------'
      ENDIF
C--------DO LOOP OF SUCCESSIVE APPROXIMATIONS FOR QUADRATURE
C--------POINTS BEGINS.
      DO 50 M=1, L+1
      WRITE(9,2) M-1
      WRITE(8,2) M-1
    2 FORMAT('M=',I2)
      DO 100 I=M, L+1
      WRITE(9,3) I-1
      WRITE(8,3) I-1
    3 FORMAT('I=',I2)
      DO 150 JJJ=1, NMUS/2
      WRITE(9,4) MU(JJJ)
      WRITE(8,4) MU(JJJ)
    4 FORMAT('MU=',1PE18.11)
      WRITE(9,*)'--------------------------------------------------'
      WRITE(8,*)'--------------------------------------------------'
      IF(LIU .GT. 1) GO TO 310
C--------INITIAL GUESSING FOR PPI1BARQ AND PP3BARQ.
      DO 200 MM=1, NQTOT/2
      DO 250 JJR1=1, N
      DO 300 JJC1=1, N
      PPI1BARQ(I,M,JJJ,JJR1,JJC1,MM)=MU(JJJ)*XQ(MM)/(XQ(MM)-MU(JJJ))*
     *  (PP1BQ(I,M,JJR1,JJC1,MM)-DEXP(-XN/MU(JJJ))*
     *  PP1TQ(I,M,JJR1,JJC1,MM))
      PP3BARQ(I,M,JJJ,JJR1,JJC1,MM)=MU(JJJ)*XQ(MM)/(XQ(MM)+MU(JJJ))*
     *  (PP3TQ(I,M,JJR1,JJC1,MM)-DEXP(-XN/MU(JJJ))*
     *  PP3BQ(I,M,JJR1,JJC1,MM))
  300 CONTINUE
  250 CONTINUE
  200 CONTINUE
  310 WRITE(8,*)
```

```
C---------PRINT OUT THE VALUES OF INITIAL GUESSING FOR PPI1BARQ AND
C---------PP3BARQ.
CC        WRITE(8,*) 'THE VALUES OF INITIAL GUESSING ARE:'
CC        CALL OUTPUT2(I,M,JJJ)
C---------ITERATIONS BEGIN FOR PPI1BARQ AND PP3BARQ.
          IF(JJJ .EQ. 1) GO TO 9999
          E1=10000000000.D0
          EMAX=100000.D0
          ITER=0
          DO 350 JJCONV=1, 1200
          E2=E1
          E1=EMAX
          ITER=ITER+1
          EMAX=0.D0
          DO 375 JJ=1, NQTOT/2
          DO 400 JR1=1, N
          DO 500 JC1=1, N
          SUME(JR1,JC1)=0.D0
          SUME1(JR1,JC1)=0.D0
          SUME2(JR1,JC1)=0.D0
          SUME3(JR1,JC1)=0.D0
          SUMF(JR1,JC1)=0.D0
          SUMF1(JR1,JC1)=0.D0
          SUMF2(JR1,JC1)=0.D0
          SUMF3(JR1,JC1)=0.D0
          SUMK(JR1,JC1)=0.D0
          SUMK1(JR1,JC1)=0.D0
          SUMK2(JR1,JC1)=0.D0
          SUMK3(JR1,JC1)=0.D0
          SUML(JR1,JC1)=0.D0
          SUML1(JR1,JC1)=0.D0
          SUML2(JR1,JC1)=0.D0
          SUML3(JR1,JC1)=0.D0
      500 CONTINUE
      400 CONTINUE
          DO 600 J=M, L+1
          DO 700 JR2=1, N
          DO 800 JC2=1, N
          SUMA(JR2,JC2)=0.D0
          SUMA1(JR2,JC2)=0.D0
          SUMA2(JR2,JC2)=0.D0
          SUMB(JR2,JC2)=0.D0
          SUMB1(JR2,JC2)=0.D0
          SUMB2(JR2,JC2)=0.D0
      800 CONTINUE
      700 CONTINUE
```

```
      DO 900 KK=1, NQTOT/2
      DO 950 JR3=1, N
      DO 1000 JC3=1, N
      SUM1=0.D0
      SUM2=0.D0
      DO 1050 K1=1, N
      SUM1=PP3BARQ(I,M,JJJ,JR3,K1,KK)*PAIQ(M,J,K1,JC3,NQTOT/2+KK)
     *  +SUM1
      SUM2=PPI1BARQ(I,M,JJJ,JR3,K1,KK)*PAIQ(M,J,K1,JC3,KK)+SUM2
1050  CONTINUE
      SUMA1(JR3,JC3)=SUM1
      SUMB1(JR3,JC3)=SUM2
1000  CONTINUE
 950  CONTINUE
      DO 1100 JR4=1, N
      DO 1150 JC4=1, N
      SUMA2(JR4,JC4)=SUMA1(JR4,JC4)*A(KK)
      SUMB2(JR4,JC4)=SUMB1(JR4,JC4)*A(KK)
1150  CONTINUE
1100  CONTINUE
      DO 1200 JR5=1, N
      DO 1250 JC5=1, N
      SUMA(JR5,JC5)=SUMA2(JR5,JC5)+SUMA(JR5,JC5)
      SUMB(JR5,JC5)=SUMB2(JR5,JC5)+SUMB(JR5,JC5)
1250  CONTINUE
1200  CONTINUE
 900  CONTINUE
      CALL FACTOR(J-M,1,FACTT)
      CALL FACTOR(J+M-2,1,FACTB)
      DO 1300 JR6=1, N
      DO 1350 JC6=1, N
      SUME1(JR6,JC6)=(-1.D0)**(I+M)*FACTT/FACTB*SUMA(JR6,JC6)
      SUMF1(JR6,JC6)=FACTT/FACTB*SUMB(JR6,JC6)
      SUMK1(JR6,JC6)=(-1.D0)**(M-J)*FACTT/FACTB*SUMA(JR6,JC6)
      SUML1(JR6,JC6)=(-1.D0)**(J+I)*FACTT/FACTB*SUMB(JR6,JC6)
1350  CONTINUE
1300  CONTINUE
      DO 1400 JR7=1, N
      DO 1450 JC7=1, N
      SUM5=0.D0
      SUM6=0.D0
      SUM11=0.D0
      SUM12=0.D0
      DO 1500 K2=1, N
      SUM5=SUME1(JR7,K2)*B(J,K2,JC7)+SUM5
      SUM6=SUMF1(JR7,K2)*B(J,K2,JC7)+SUM6
```

```
      SUM11=SUMK1(JR7,K2)*B(J,K2,JC7)+SUM11
      SUM12=SUML1(JR7,K2)*B(J,K2,JC7)+SUM12
1500  CONTINUE
      SUME2(JR7,JC7)=SUM5
      SUMF2(JR7,JC7)=SUM6
      SUMK2(JR7,JC7)=SUM11
      SUML2(JR7,JC7)=SUM12
1450  CONTINUE
1400  CONTINUE
      DO 1550 JR8=1, N
      DO 1600 JC8=1, N
      SUM13=0.D0
      SUM14=0.D0
      SUM19=0.D0
      SUM20=0.D0
      DO 1650 K3=1, N
      SUM13=SUME2(JR8,K3)*PP1BQ(J,M,K3,JC8,JJ)+SUM13
      SUM14=SUMF2(JR8,K3)*PP1TQ(J,M,K3,JC8,JJ)+SUM14
      SUM19=SUMK2(JR8,K3)*PP3TQ(J,M,K3,JC8,JJ)+SUM19
      SUM20=SUML2(JR8,K3)*PP3BQ(J,M,K3,JC8,JJ)+SUM20
1650  CONTINUE
      SUME3(JR8,JC8)=SUM13
      SUMF3(JR8,JC8)=SUM14
      SUMK3(JR8,JC8)=SUM19
      SUML3(JR8,JC8)=SUM20
1600  CONTINUE
1550  CONTINUE
      DO 1700 JR9=1, N
      DO 1750 JC9=1, N
      SUME(JR9,JC9)=SUME3(JR9,JC9)+SUME(JR9,JC9)
      SUMF(JR9,JC9)=SUMF3(JR9,JC9)+SUMF(JR9,JC9)
      SUMK(JR9,JC9)=SUMK3(JR9,JC9)+SUMK(JR9,JC9)
      SUML(JR9,JC9)=SUML3(JR9,JC9)+SUML(JR9,JC9)
1750  CONTINUE
1700  CONTINUE
 600  CONTINUE
      DO 1800 JR10=1, N
      DO 1850 JC10=1, N
      PPI1BARQN(I,M,JJJ,JR10,JC10,JJ)=MU(JJJ)*XQ(JJ)/(XQ(JJ)-MU(JJJ))*
     *  (PP1BQ(I,M,JR10,JC10,JJ)-DEXP(-XN/MU(JJJ))*
     *  PP1TQ(I,M,JR10,JC10,JJ)+SUME(JR10,JC10)-SUMF(JR10,JC10))
      PP3BARQN(I,M,JJJ,JR10,JC10,JJ)=MU(JJJ)*XQ(JJ)/(XQ(JJ)+MU(JJJ))*
     *  (PP3TQ(I,M,JR10,JC10,JJ)-DEXP(-XN/MU(JJJ))*
     *  PP3BQ(I,M,JR10,JC10,JJ)+SUMK(JR10,JC10)-SUML(JR10,JC10))
1850  CONTINUE
1800  CONTINUE
```

```
      DO 1900 JR11=1, N
      DO 1950 JC11=1, N
      ERR1=DABS(PPI1BARQN(I,M,JJJ,JR11,JC11,JJ)-
     *  PPI1BARQ(I,M,JJJ,JR11,JC11,JJ))
      IF(ERR1 .GT. EMAX) EMAX=ERR1
      ERR4=DABS(PP3BARQN(I,M,JJJ,JR11,JC11,JJ)-
     *  PP3BARQ(I,M,JJJ,JR11,JC11,JJ))
      IF(ERR4 .GT. EMAX) EMAX=ERR4
 1950 CONTINUE
 1900 CONTINUE
  375 CONTINUE
      WRITE(8,*) EMAX
      IF(EMAX .LT. ERROR) GO TO 3000
      IF(EMAX .GT. 3000)THEN
      WRITE(*,*) 'DIVERGE !  DO NOT WASTE TIME !!'
      WRITE(9,*) 'DIVERGE !  DO NOT WASTE TIME !!'
      WRITE(8,*) 'DIVERGE !  DO NOT WASTE TIME !!'
      WRITE(8,5) EMAX
    5 FORMAT(1X,'MAXIUM ERROR=',1PE18.11)
      GO TO 3500
      ELSE
      ENDIF
      IF(E1 .GT. EMAX) GO TO 2500
      IF(E2 .LT. E1) THEN
      WRITE(*,*) 'DOES NOT CONVERGE !'
      WRITE(9,*) 'DOES NOT CONVERGE !'
      WRITE(8,*) 'DOES NOT CONVERGE !'
      WRITE(8,5) EMAX
      GO TO 3500
      ELSE
      ENDIF
 2500 DO 2000 JN=1, NQTOT/2
      DO 2050 JR12=1, N
      DO 2100 JC12=1, N
C     PPI1BARQ(I,M,JJJ,JR12,JC12,JN)=PPI1BARQN(I,M,JJJ,JR12,JC12,JN)
C     PP3BARQ(I,M,JJJ,JR12,JC12,JN)=PP3BARQN(I,M,JJJ,JR12,JC12,JN)
      PPI1BARQ(I,M,JJJ,JR12,JC12,JN)=RELAX*
     *  PPI1BARQ(I,M,JJJ,JR12,JC12,JN)+(1.D0-RELAX)*
     *  PPI1BARQN(I,M,JJJ,JR12,JC12,JN)
      PP3BARQ(I,M,JJJ,JR12,JC12,JN)=RELAX*
     *  PP3BARQ(I,M,JJJ,JR12,JC12,JN)+(1.D0-RELAX)*
     *  PP3BARQN(I,M,JJJ,JR12,JC12,JN)
 2100 CONTINUE
 2050 CONTINUE
 2000 CONTINUE
  350 CONTINUE
```

```
      WRITE(*,*) 'NEED MORE ITERATIONS !'
      WRITE(9,*)'***********************'
      WRITE(9,*) 'NEED MORE ITERATIONS !'
      WRITE(9,*)'***********************'
      WRITE(8,*)'***********************'
      WRITE(8,*) 'NEED MORE ITERATIONS !'
      WRITE(8,*)'***********************'
      GO TO 3500
 3000 WRITE(*,*) '----------GOOD----------'
C--------PRINT OUT THE FINAL VALUES FOR PPI1BARQ AND PP3BARQ.
 3500 DO 3800 JF=1, NQTOT/2
      DO 3900 JR13=1, N
      DO 4000 JC13=1, N
      PPI1BARQ(I,M,JJJ,JR13,JC13,JF)=PPI1BARQN(I,M,JJJ,JR13,JC13,JF)
      PP3BARQ(I,M,JJJ,JR13,JC13,JF)=PP3BARQN(I,M,JJJ,JR13,JC13,JF)
 4000 CONTINUE
 3900 CONTINUE
 3800 CONTINUE
CC    WRITE(8,*) 'THE VALUES AFTER ITERATIONS ARE:'
      WRITE(9,6) ITER
      WRITE(8,6) ITER
    6 FORMAT(1X,'NUMBER OF ITERATIONS=',I6)
      WRITE(9,*)'--------------------------------------------------'
      WRITE(8,*)'--------------------------------------------------'
CC    CALL OUTPUT2(I,M,JJJ)
CC    WRITE(8,*)'--------------------------------------------------'
C--------CALCULATING THE VALUES OF PP1BARQ AND PPI3BARQ BY
C--------KNOWING PPI1BARQ = D2*PPI3BARQ*D2 AND PP3BARQ =
C--------D2*PP1BARQ*D2.
 9999 DO 4375 JJ=1, NQTOT/2
      DO 4400 JR1=1, N
      DO 4500 JC1=1, N
      PP1BARQ(I,M,JJJ,JR1,JC1,JJ)=PP3BARQ(I,M,JJJ,JR1,JC1,JJ)
      PPI3BARQ(I,M,JJJ,JR1,JC1,JJ)=PPI1BARQ(I,M,JJJ,JR1,JC1,JJ)
 4500 CONTINUE
 4400 CONTINUE
      PP1BARQ(I,M,JJJ,1,3,JJ)=(-1.D0)*PP3BARQ(I,M,JJJ,1,3,JJ)
      PP1BARQ(I,M,JJJ,1,4,JJ)=(-1.D0)*PP3BARQ(I,M,JJJ,1,4,JJ)
      PP1BARQ(I,M,JJJ,2,3,JJ)=(-1.D0)*PP3BARQ(I,M,JJJ,2,3,JJ)
      PP1BARQ(I,M,JJJ,2,4,JJ)=(-1.D0)*PP3BARQ(I,M,JJJ,2,4,JJ)
      PP1BARQ(I,M,JJJ,3,1,JJ)=(-1.D0)*PP3BARQ(I,M,JJJ,3,1,JJ)
      PP1BARQ(I,M,JJJ,3,2,JJ)=(-1.D0)*PP3BARQ(I,M,JJJ,3,2,JJ)
      PP1BARQ(I,M,JJJ,4,1,JJ)=(-1.D0)*PP3BARQ(I,M,JJJ,4,1,JJ)
      PP1BARQ(I,M,JJJ,4,2,JJ)=(-1.D0)*PP3BARQ(I,M,JJJ,4,2,JJ)
      PPI3BARQ(I,M,JJJ,1,3,JJ)=(-1.D0)*PPI1BARQ(I,M,JJJ,1,3,JJ)
      PPI3BARQ(I,M,JJJ,1,4,JJ)=(-1.D0)*PPI1BARQ(I,M,JJJ,1,4,JJ)
```

```
      PPI3BARQ(I,M,JJJ,2,3,JJ)=(-1.D0)*PPI1BARQ(I,M,JJJ,2,3,JJ)
      PPI3BARQ(I,M,JJJ,2,4,JJ)=(-1.D0)*PPI1BARQ(I,M,JJJ,2,4,JJ)
      PPI3BARQ(I,M,JJJ,3,1,JJ)=(-1.D0)*PPI1BARQ(I,M,JJJ,3,1,JJ)
      PPI3BARQ(I,M,JJJ,3,2,JJ)=(-1.D0)*PPI1BARQ(I,M,JJJ,3,2,JJ)
      PPI3BARQ(I,M,JJJ,4,1,JJ)=(-1.D0)*PPI1BARQ(I,M,JJJ,4,1,JJ)
      PPI3BARQ(I,M,JJJ,4,2,JJ)=(-1.D0)*PPI1BARQ(I,M,JJJ,4,2,JJ)
 4375 CONTINUE
C--------PRINT OUT THE VALUES FOR PP1BARQ AND PPI3BARQ.
CC    WRITE(8,*) 'THE VALUES ARE:'
CC    CALL OUTPUT3(I,M,JJJ)
CC    WRITE(8,*)'----------------------------------------------------'
  150 CONTINUE
  100 CONTINUE
   50 CONTINUE
C--------CALCULATING THE STARTING POINT MINUM(J) TO INTERPOLATE
C--------THE VALUES OF PPI1BAR BY USING LAGRANGE'S POLYNOMIAL
C--------APPROXIATION METHOD WHEN MU(JJJ)=MU(JJ).
      IDEG=6
      DO 8800 J=1, NMUS/2
      IF(MU(J) .GT. XQ(NQTOT/2)) THEN
      MINUM(J)=NQTOT/2-IDEG
CC    WRITE(9,*) J, MINUM(J)
      GO TO 8800
      ELSE
      ENDIF
      DO 8900 I=1, NQTOT/2
      IF(MU(J) .GT. XQ(I)) GO TO 8900
      MINUM(J)=I-IDEG/2-1
      IF(MINUM(J) .LT. 1) MINUM(J)=1
      IF(MINUM(J) .GT. NQTOT/2-IDEG) MINUM(J)=NQTOT/2-IDEG
CC    WRITE(9,*) J, MINUM(J)
      GO TO 8800
 8900 CONTINUE
 8800 CONTINUE
C--------BEGIN TO CALCULATE THE PPI1BAR AND PP1BAR FOR MU VALUES.
      DO 9000 M=1, L+1
      WRITE(8,2) M-1
      DO 9050 I=M, L+1
      WRITE(8,3) I-1
      DO 9100 JJJ=1, NMUS/2
      WRITE(8,4) MU(JJJ)
      WRITE(8,*)'----------------------------------------------------'
C--------BEGIN TO CALCULATE THE PPI1BAR FOR MU VALUES.
      DO 9150 JJ=1, NMUS/2
      IF(MU(JJJ) .EQ. MU(JJ)) GO TO 11000
      DO 9200 JR1=1, N
```

```
      DO 9250 JC1=1, N
      SUME(JR1,JC1)=0.D0
      SUME1(JR1,JC1)=0.D0
      SUME2(JR1,JC1)=0.D0
      SUME3(JR1,JC1)=0.D0
      SUMF(JR1,JC1)=0.D0
      SUMF1(JR1,JC1)=0.D0
      SUMF2(JR1,JC1)=0.D0
      SUMF3(JR1,JC1)=0.D0
9250  CONTINUE
9200  CONTINUE
      DO 9300 J=M, L+1
      DO 9350 JR2=1, N
      DO 9400 JC2=1, N
      SUMA(JR2,JC2)=0.D0
      SUMA1(JR2,JC2)=0.D0
      SUMA2(JR2,JC2)=0.D0
      SUMB(JR2,JC2)=0.D0
      SUMB1(JR2,JC2)=0.D0
      SUMB2(JR2,JC2)=0.D0
9400  CONTINUE
9350  CONTINUE
      DO 9450 KK=1, NQTOT/2
      DO 9500 JR3=1, N
      DO 9550 JC3=1, N
      SUM1=0.D0
      SUM2=0.D0
      DO 9600 K1=1, N
      SUM1=PP3BARQ(I,M,JJJ,JR3,K1,KK)*PAIQ(M,J,K1,JC3,NQTOT/2+KK)+
     * SUM1
      SUM2=PPI1BARQ(I,M,JJJ,JR3,K1,KK)*PAIQ(M,J,K1,JC3,KK)+SUM2
9600  CONTINUE
      SUMA1(JR3,JC3)=SUM1
      SUMB1(JR3,JC3)=SUM2
9550  CONTINUE
9500  CONTINUE
      DO 9650 JR4=1, N
      DO 9700 JC4=1, N
      SUMA2(JR4,JC4)=SUMA1(JR4,JC4)*A(KK)
      SUMB2(JR4,JC4)=SUMB1(JR4,JC4)*A(KK)
9700  CONTINUE
9650  CONTINUE
      DO 9750 JR5=1, N
      DO 9800 JC5=1, N
      SUMA(JR5,JC5)=SUMA2(JR5,JC5)+SUMA(JR5,JC5)
      SUMB(JR5,JC5)=SUMB2(JR5,JC5)+SUMB(JR5,JC5)
```

```
 9800 CONTINUE
 9750 CONTINUE
 9450 CONTINUE
      CALL FACTOR(J-M,1,FACTT)
      CALL FACTOR(J+M-2,1,FACTB)
      DO 9850 JR6=1, N
      DO 9900 JC6=1, N
      SUME1(JR6,JC6)=(-1.D0)**(I+M)*FACTT/FACTB*SUMA(JR6,JC6)
      SUMF1(JR6,JC6)=FACTT/FACTB*SUMB(JR6,JC6)
 9900 CONTINUE
 9850 CONTINUE
      DO 9950 JR7=1, N
      DO 10000 JC7=1, N
      SUM5=0.D0
      SUM6=0.D0
      DO 10050 K2=1, N
      SUM5=SUME1(JR7,K2)*B(J,K2,JC7)+SUM5
      SUM6=SUMF1(JR7,K2)*B(J,K2,JC7)+SUM6
10050 CONTINUE
      SUME2(JR7,JC7)=SUM5
      SUMF2(JR7,JC7)=SUM6
10000 CONTINUE
 9950 CONTINUE
      DO 10100 JR8=1, N
      DO 10150 JC8=1, N
      SUM13=0.D0
      SUM14=0.D0
      DO 10200 K3=1, N
      SUM13=SUME2(JR8,K3)*PP1B(J,M,K3,JC8,JJ)+SUM13
      SUM14=SUMF2(JR8,K3)*PP1T(J,M,K3,JC8,JJ)+SUM14
10200 CONTINUE
      SUME3(JR8,JC8)=SUM13
      SUMF3(JR8,JC8)=SUM14
10150 CONTINUE
10100 CONTINUE
      DO 10250 JR9=1, N
      DO 10300 JC9=1, N
      SUME(JR9,JC9)=SUME3(JR9,JC9)+SUME(JR9,JC9)
      SUMF(JR9,JC9)=SUMF3(JR9,JC9)+SUMF(JR9,JC9)
10300 CONTINUE
10250 CONTINUE
 9300 CONTINUE
      DO 10350 JR10=1, N
      DO 10400 JC10=1, N
      PPI1BAR(I,M,JJJ,JR10,JC10,JJ)=MU(JJJ)*MU(JJ)/(MU(JJ)-MU(JJJ))*
     *  (PP1B(I,M,JR10,JC10,JJ)-DEXP(-XN/MU(JJJ))*
```

```
     *  PP1T(I,M,JR10,JC10,JJ)+SUME(JR10,JC10)-SUMF(JR10,JC10))
 10400 CONTINUE
 10350 CONTINUE
       GO TO 9150
 11000 DO 11050 JN=1, NQTOT/2
       DO 11100 JR11=1, N
       DO 11150 JC11=1, N
       FLAG(JR11,JC11,JN)=PPI1BARQ(I,M,JJJ,JR11,JC11,JN)
 11150 CONTINUE
 11100 CONTINUE
 11050 CONTINUE
       DO 11200 I1=1, NQTOT/2
       XX(I1)=XQ(I1)
 11200 CONTINUE
       CALL LAGR(XX,FLAG,MU(JJ),IDEG,MINUM(JJ),N,NQTOT/2,FLAGR)
       DO 11250 JR12=1, N
       DO 11300 JC12=1, N
       PPI1BAR(I,M,JJJ,JR12,JC12,JJ)=FLAGR(JR12,JC12)
 11300 CONTINUE
 11250 CONTINUE
  9150 CONTINUE
C--------PRINT OUT THE VALUES FOR THE PPI1BAR.
       WRITE(8,*) '************'
       WRITE(8,*) 'PPI1BAR ARE:'
       WRITE(8,*) '************'
       WRITE(8,*)'----------------------------------------------------'
       CALL OUTPUT4(I,M,JJJ)
       WRITE(8,*)'----------------------------------------------------'
C--------BEGIN TO CALCULATE THE PP1BAR FOR MU VALUES.
       DO 14375 JJ=1, NMUS/2
       DO 14400 JR1=1, N
       DO 14500 JC1=1, N
       SUMG(JR1,JC1)=0.D0
       SUMG1(JR1,JC1)=0.D0
       SUMG2(JR1,JC1)=0.D0
       SUMG3(JR1,JC1)=0.D0
       SUMH(JR1,JC1)=0.D0
       SUMH1(JR1,JC1)=0.D0
       SUMH2(JR1,JC1)=0.D0
       SUMH3(JR1,JC1)=0.D0
 14500 CONTINUE
 14400 CONTINUE
       DO 14600 J=M, L+1
       DO 14700 JR2=1, N
       DO 14800 JC2=1, N
       SUMC(JR2,JC2)=0.D0
```

```
      SUMC1(JR2,JC2)=0.D0
      SUMC2(JR2,JC2)=0.D0
      SUMD(JR2,JC2)=0.D0
      SUMD1(JR2,JC2)=0.D0
      SUMD2(JR2,JC2)=0.D0
14800 CONTINUE
14700 CONTINUE
      DO 14900 KK=1, NQTOT/2
      DO 14950 JR3=1, N
      DO 15000 JC3=1, N
      SUM3=0.D0
      SUM4=0.D0
      DO 15050 K1=1, N
      SUM3=PP1BARQ(I,M,JJJ,JR3,K1,KK)*PAIQ(M,J,K1,JC3,KK)+SUM3
      SUM4=PPI3BARQ(I,M,JJJ,JR3,K1,KK)*PAIQ(M,J,K1,JC3,NQTOT/2+KK)+
     *  SUM4
15050 CONTINUE
      SUMC1(JR3,JC3)=SUM3
      SUMD1(JR3,JC3)=SUM4
15000 CONTINUE
14950 CONTINUE
      DO 15100 JR4=1, N
      DO 15150 JC4=1, N
      SUMC2(JR4,JC4)=SUMC1(JR4,JC4)*A(KK)
      SUMD2(JR4,JC4)=SUMD1(JR4,JC4)*A(KK)
15150 CONTINUE
15100 CONTINUE
      DO 15200 JR5=1, N
      DO 15250 JC5=1, N
      SUMC(JR5,JC5)=SUMC2(JR5,JC5)+SUMC(JR5,JC5)
      SUMD(JR5,JC5)=SUMD2(JR5,JC5)+SUMD(JR5,JC5)
15250 CONTINUE
15200 CONTINUE
14900 CONTINUE
      CALL FACTOR(J-M,1,FACTT)
      CALL FACTOR(J+M-2,1,FACTB)
      DO 15300 JR6=1, N
      DO 15350 JC6=1, N
      SUMG1(JR6,JC6)=FACTT/FACTB*SUMC(JR6,JC6)
      SUMH1(JR6,JC6)=(-1.D0)**(I+M)*FACTT/FACTB*SUMD(JR6,JC6)
15350 CONTINUE
15300 CONTINUE
      DO 15400 JR7=1, N
      DO 15450 JC7=1, N
      SUM7=0.D0
      SUM8=0.D0
```

```
      DO 15500 K2=1, N
      SUM7=SUMG1(JR7,K2)*B(J,K2,JC7)+SUM7
      SUM8=SUMH1(JR7,K2)*B(J,K2,JC7)+SUM8
15500 CONTINUE
      SUMG2(JR7,JC7)=SUM7
      SUMH2(JR7,JC7)=SUM8
15450 CONTINUE
15400 CONTINUE
      DO 15550 JR8=1, N
      DO 15600 JC8=1, N
      SUM15=0.D0
      SUM16=0.D0
      DO 15650 K3=1, N
      SUM15=SUMG2(JR8,K3)*PP1T(J,M,K3,JC8,JJ)+SUM15
      SUM16=SUMH2(JR8,K3)*PP1B(J,M,K3,JC8,JJ)+SUM16
15650 CONTINUE
      SUMG3(JR8,JC8)=SUM15
      SUMH3(JR8,JC8)=SUM16
15600 CONTINUE
15550 CONTINUE
      DO 15700 JR9=1, N
      DO 15750 JC9=1, N
      SUMG(JR9,JC9)=SUMG3(JR9,JC9)+SUMG(JR9,JC9)
      SUMH(JR9,JC9)=SUMH3(JR9,JC9)+SUMH(JR9,JC9)
15750 CONTINUE
15700 CONTINUE
14600 CONTINUE
      DO 15800 JR10=1, N
      DO 15850 JC10=1, N
      PP1BAR(I,M,JJJ,JR10,JC10,JJ)=MU(JJJ)*MU(JJ)/(MU(JJ)+MU(JJJ))*
     *  (PP1T(I,M,JR10,JC10,JJ)-DEXP(-XN/MU(JJJ))*
     *  PP1B(I,M,JR10,JC10,JJ)+SUMG(JR10,JC10)-SUMH(JR10,JC10))
15850 CONTINUE
15800 CONTINUE
14375 CONTINUE
C--------PRINT OUT THE VALUES FOR THE PP1BAR.
      WRITE(8,*) '************'
      WRITE(8,*) 'PP1BAR ARE:'
      WRITE(8,*) '************'
      WRITE(8,*)'----------------------------------------------------'
      CALL OUTPUT5(I,M,JJJ)
      WRITE(8,*)'----------------------------------------------------'
 9100 CONTINUE
 9050 CONTINUE
 9000 CONTINUE
      RETURN
```

```
      END

C--------SUBROUNTINE TO INTERPOLATE THE VALUES OF PPI1BAR BY
C--------USING LAGRANGE'S POLYNOMIAL APPROXIATION METHOD WHEN
C--------MU(JJJ)=MU(JJ) IN SUBROUNTINE PPBARFUN. (OR) INTERPOLATE
C--------THE VALUES FOR RI AND TI AT MU=1.0 BY USING LAGRANGE'S
C--------POLYNOMIAL APPROXIATION METHOD WHEN L=0 AND 1.
      SUBROUTINE LAGR(X,Y,XARG,IDEG,MIN,N,M,FLAGR)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION X(M), Y(N,N,M), FLAGR(N,N), YEST(N,N), TERM(N,N)
      FACTOR=1.D0
      MAX=MIN+IDEG
      DO 2 J=MIN, MAX
      IF(XARG .NE. X(J)) GO TO 2
      DO 100 JR1=1, N
      DO 200 JC1=1, N
      FLAGR(JR1,JC1)=Y(JR1,JC1,J)
  200 CONTINUE
  100 CONTINUE
      RETURN
    2 FACTOR=FACTOR*(XARG-X(J))
      DO 300 JR2=1, N
      DO 400 JC2=1, N
      YEST(JR2,JC2)=0.D0
  400 CONTINUE
  300 CONTINUE
      DO 5 I=MIN, MAX
      DO 500 JR3=1, N
      DO 600 JC3=1, N
      TERM(JR3,JC3)=Y(JR3,JC3,I)*FACTOR/(XARG-X(I))
  600 CONTINUE
  500 CONTINUE
      DO 4 J=MIN, MAX
      IF(I .NE. J) THEN
      DO 700 JR4=1, N
      DO 800 JC4=1, N
      TERM(JR4,JC4)=TERM(JR4,JC4)/(X(I)-X(J))
  800 CONTINUE
  700 CONTINUE
      ELSE
      ENDIF
    4 CONTINUE
      DO 900 JR5=1, N
      DO 1000 JC5=1, N
      YEST(JR5,JC5)=YEST(JR5,JC5)+TERM(JR5,JC5)
```

```
 1000 CONTINUE
  900 CONTINUE
    5 CONTINUE
      DO 1100 JR6=1, N
      DO 1200 JC6=1, N
      FLAGR(JR6,JC6)=YEST(JR6,JC6)
 1200 CONTINUE
 1100 CONTINUE
      RETURN
      END


C--------SUBROUTINE TO CALCULATE THE REFLECTED AND TRANSMITTED
C--------INTENSITIES (RI AND TI).
      SUBROUTINE INTEN
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 MU
      DIMENSION D2(4,4), SUMA(4,4), SUMA1(4,4), SUMB(4,4), SUMC(4,4),
     *  SUMD(4,4), SUME(4,4), SUMF(4,4), SUMAN(4,4), SUMA1N(4,4),
     *  SUMBN(4,4), SUMCN(4,4), SUMDN(4,4), SUMEN(4,4), SUMFN(4,4)
      COMMON/BLK1/MU(26),XQ(656)
      COMMON/BLK7/PAIQ(6,6,4,4,656),B(6,4,4),A(328)
      COMMON/BLK8/W,PAIG
      COMMON/BLK9/N,L,NMUS,NQTOT
      COMMON/BLK11/PPI1BAR(6,6,13,4,4,13),PP1BAR(6,6,13,4,4,13)
      COMMON/BLK17/PAI(6,6,4,4,26)
      COMMON/BLK18/PHI
      COMMON/BLK19/RI(13,13,4,4),TI(13,13,4,4)
C--------SET UP THE CONSTANT MATRIX D2.
      DO 100 JRA=1, N
      DO 200 JCA=1, N
      D2(JRA,JCA)=0.D0
  200 CONTINUE
  100 CONTINUE
      D2(1,1)=1.D0
      D2(2,2)=1.D0
      D2(3,3)=-1.D0
      D2(4,4)=-1.D0
C--------BEGIN TO CALCULATE THE RI FOR MU VALUES.
CC    WRITE(8,*)'**************************'
CC    WRITE(8,*)'REFLECTED INTENSITIES ARE:'
CC    WRITE(8,*)'**************************'
      DO 300 JJJ=1, NMUS/2
CC    WRITE(8,1) MU(JJJ)
CC  1 FORMAT('MU=',1PE18.11)
CC    WRITE(8,*)'----------------------------------------------------'
```

```
      DO 400 JJ=1, NMUS/2
CC    WRITE(8,2) MU(JJ)
CC  2 FORMAT('MUO=',1PE18.11)
CC    WRITE(8,*)'--------------------------------------------------'
      DO 500 JRB=1, N
      DO 550 JCB=1, N
      SUMF(JRB,JCB)=0.D0
  550 CONTINUE
  500 CONTINUE
      DO 600 M=1, L+1
      DELTA=1.D0
      IF(M .EQ. 1) DELTA=0.5D0
      DO 700 JRC=1, N
      DO 800 JCC=1, N
      SUMA(JRC,JCC)=0.D0
      SUMA1(JRC,JCC)=0.D0
      SUMB(JRC,JCC)=0.D0
      SUMC(JRC,JCC)=0.D0
      SUMD(JRC,JCC)=0.D0
      SUME(JRC,JCC)=0.D0
  800 CONTINUE
  700 CONTINUE
      DO 900 I=M, L+1
      DO 1000 JR1=1, N
      DO 1100 JC1=1, N
      SUM1=0.D0
      DO 1200 K1=1, N
      SUM1=PAI(M,I,JR1,K1,NMUS/2+JJJ)*B(I,K1,JC1)+SUM1
 1200 CONTINUE
      SUMA1(JR1,JC1)=SUM1
 1100 CONTINUE
 1000 CONTINUE
      DO 1300 JR2=1, N
      DO 1400 JC2=1, N
      SUM2=0.D0
      DO 1500 K2=1, N
      SUM2=SUMA1(JR2,K2)*PP1BAR(I,M,JJJ,K2,JC2,JJ)+SUM2
 1500 CONTINUE
      SUMA(JR2,JC2)=SUM2
 1400 CONTINUE
 1300 CONTINUE
      DO 1600 JR3=1, N
      DO 1700 JC3=1, N
      SUM3=0.D0
      DO 1800 K3=1, N
      SUM3=SUMA(JR3,K3)*D2(K3,JC3)+SUM3
```

```
 1800 CONTINUE
      SUMC(JR3,JC3)=SUM3
 1700 CONTINUE
 1600 CONTINUE
      DO 1900 JR4=1, N
      DO 2000 JC4=1, N
      SUM4=0.D0
      DO 2100 K4=1, N
      SUM4=D2(JR4,K4)*SUMA(K4,JC4)+SUM4
 2100 CONTINUE
      SUMD(JR4,JC4)=SUM4
 2000 CONTINUE
 1900 CONTINUE
      DO 2200 JR5=1, N
      DO 2300 JC5=1, N
      SUM5=0.D0
      DO 2400 K5=1, N
      SUM5=SUMD(JR5,K5)*D2(K5,JC5)+SUM5
 2400 CONTINUE
      SUMB(JR5,JC5)=SUM5
 2300 CONTINUE
 2200 CONTINUE
      CALL FACTOR(I-M,1,FACTT)
      CALL FACTOR(I+M-2,1,FACTB)
      DO 2500 JR6=1, N
      DO 2600 JC6=1, N
      SUME(JR6,JC6)=W/4.D0/PAIG/PAIG*FACTT/FACTB*(DCOS(M*PHI)*
     *  (SUMA(JR6,JC6)+SUMB(JR6,JC6))+DSIN(M*PHI)*(SUMC(JR6,JC6)
     *  -SUMD(JR6,JC6)))/MU(JJJ)+SUME(JR6,JC6)
 2600 CONTINUE
 2500 CONTINUE
  900 CONTINUE
      DO 2700 JR7=1, N
      DO 2800 JC7=1, N
      SUMF(JR7,JC7)=DELTA*SUME(JR7,JC7)+SUMF(JR7,JC7)
 2800 CONTINUE
 2700 CONTINUE
  600 CONTINUE
      DO 2900 JR8=1, N
      DO 3000 JC8=1, N
      RI(JJJ,JJ,JR8,JC8)=SUMF(JR8,JC8)
 3000 CONTINUE
 2900 CONTINUE
CC    DO 3100 JR9=1, N
CC    WRITE(8,3) (RI(JJJ,JJ,JR9,JC9),JC9=1,N)
CC  3 FORMAT(4(1PE18.11,2X))
```

```
C 3100 CONTINUE
CC     WRITE(8,*)'--------------------------------------------------'
   400 CONTINUE
CC     WRITE(8,*)'--------------------------------------------------'
   300 CONTINUE
C--------BEGIN TO CALCULATE THE TI FOR MU VALUES.
CC     WRITE(8,*)'****************************'
CC     WRITE(8,*)'TRANSMITTED INTENSITIES ARE:'
CC     WRITE(8,*)'****************************'
       DO 3300 JJJ=1, NMUS/2
CC     WRITE(8,1) MU(JJJ)
CC     WRITE(8,*)'--------------------------------------------------'
       DO 3400 JJ=1, NMUS/2
CC     WRITE(8,2) MU(JJ)
CC     WRITE(8,*)'--------------------------------------------------'
       DO 3500 JRB=1, N
       DO 3550 JCB=1, N
       SUMFN(JRB,JCB)=0.D0
  3550 CONTINUE
  3500 CONTINUE
       DO 3600 M=1, L+1
       DELTA=1.D0
       IF(M .EQ. 1) DELTA=0.5D0
       DO 3700 JRC=1, N
       DO 3800 JCC=1, N
       SUMAN(JRC,JCC)=0.D0
       SUMA1N(JRC,JCC)=0.D0
       SUMBN(JRC,JCC)=0.D0
       SUMCN(JRC,JCC)=0.D0
       SUMDN(JRC,JCC)=0.D0
       SUMEN(JRC,JCC)=0.D0
  3800 CONTINUE
  3700 CONTINUE
       DO 3900 I=M, L+1
       DO 4000 JR1=1, N
       DO 4100 JC1=1, N
       SUM1=0.D0
       DO 4200 K1=1, N
       SUM1=PAI(M,I,JR1,K1,JJJ)*B(I,K1,JC1)+SUM1
  4200 CONTINUE
       SUMA1N(JR1,JC1)=SUM1
  4100 CONTINUE
  4000 CONTINUE
       DO 4300 JR2=1, N
       DO 4400 JC2=1, N
       SUM2=0.D0
```

```
      DO 4500 K2=1, N
      SUM2=SUMA1N(JR2,K2)*PPI1BAR(I,M,JJJ,K2,JC2,JJ)+SUM2
4500  CONTINUE
      SUMAN(JR2,JC2)=SUM2
4400  CONTINUE
4300  CONTINUE
      DO 4600 JR3=1, N
      DO 4700 JC3=1, N
      SUM3=0.D0
      DO 4800 K3=1, N
      SUM3=SUMAN(JR3,K3)*D2(K3,JC3)+SUM3
4800  CONTINUE
      SUMCN(JR3,JC3)=SUM3
4700  CONTINUE
4600  CONTINUE
      DO 4900 JR4=1, N
      DO 5000 JC4=1, N
      SUM4=0.D0
      DO 5100 K4=1, N
      SUM4=D2(JR4,K4)*SUMAN(K4,JC4)+SUM4
5100  CONTINUE
      SUMDN(JR4,JC4)=SUM4
5000  CONTINUE
4900  CONTINUE
      DO 5200 JR5=1, N
      DO 5300 JC5=1, N
      SUM5=0.D0
      DO 5400 K5=1, N
      SUM5=SUMDN(JR5,K5)*D2(K5,JC5)+SUM5
5400  CONTINUE
      SUMBN(JR5,JC5)=SUM5
5300  CONTINUE
5200  CONTINUE
      CALL FACTOR(I-M,1,FACTT)
      CALL FACTOR(I+M-2,1,FACTB)
      DO 5500 JR6=1, N
      DO 5600 JC6=1, N
      SUMEN(JR6,JC6)=W/4.D0/PAIG/PAIG*FACTT/FACTB*(DCOS(M*PHI)*
     *  (SUMAN(JR6,JC6)+SUMBN(JR6,JC6))+DSIN(M*PHI)*
     *  (SUMCN(JR6,JC6)-SUMDN(JR6,JC6)))/MU(JJJ)+SUMEN(JR6,JC6)
5600  CONTINUE
5500  CONTINUE
3900  CONTINUE
      DO 5700 JR7=1, N
      DO 5800 JC7=1, N
      SUMFN(JR7,JC7)=DELTA*SUMEN(JR7,JC7)+SUMFN(JR7,JC7)
```

```
  5800 CONTINUE
  5700 CONTINUE
  3600 CONTINUE
       DO 5900 JR8=1, N
       DO 6000 JC8=1, N
       TI(JJJ,JJ,JR8,JC8)=SUMFN(JR8,JC8)
  6000 CONTINUE
  5900 CONTINUE
CC     DO 6100 JR9=1, N
CC     WRITE(8,3) (TI(JJJ,JJ,JR9,JC9),JC9=1,N)
C 6100 CONTINUE
CC     WRITE(8,*)'--------------------------------------------------'
  3400 CONTINUE
CC     WRITE(8,*)'--------------------------------------------------'
  3300 CONTINUE
CC     WRITE(8,*)'=================================================='
       RETURN
       END

C--------SUBROUTINE TO CALCULATE THE NORMALIZED REFLECTED AND
C--------TRANSMITTED INTENSITIES (RIF AND TIF).
       SUBROUTINE INTENF
       IMPLICIT REAL*8 (A-H,O-Z)
       REAL*8 MU
       DIMENSION RIF1(13,13,4), RIF2(13,13,4), RIF3(13,13,4),
     *  RIF4(13,13,4), TIF1(13,13,4), TIF2(13,13,4), TIF3(13,13,4),
     *  TIF4(13,13,4)
       COMMON/BLK1/MU(26),XQ(656)
       COMMON/BLK9/N,L,NMUS,NQTOT
       COMMON/BLK19/RI(13,13,4,4),TI(13,13,4,4)
       COMMON/BLK20/F1(4),F2(4),F3(4),F4(4)
       COMMON/BLK21/LIU
       IF(LIU .LE. 5) THEN
       WRITE(9,*)'=================================================='
       ELSE IF(LIU .GT. 5 .AND. LIU .LE. 10) THEN
       WRITE(5,*)'=================================================='
       ELSE IF(LIU .GT. 10 .AND. LIU .LE. 15) THEN
       WRITE(6,*)'=================================================='
       ELSE IF(LIU .GT. 15) THEN
       WRITE(7,*)'=================================================='
       ENDIF
       DO 100 JJJ=1, NMUS/2
       IF(LIU .LE. 5) THEN
       WRITE(9,1) MU(JJJ)
     1 FORMAT('MU=',1PE18.11)
       WRITE(9,*)'--------------------------------------------------'
```

```
      ELSE IF(LIU .GT. 5 .AND. LIU .LE. 10) THEN
      WRITE(5,1) MU(JJJ)
      WRITE(5,*)'---------------------------------------------------'
      ELSE IF(LIU .GT. 10 .AND. LIU .LE. 15) THEN
      WRITE(6,1) MU(JJJ)
      WRITE(6,*)'---------------------------------------------------'
      ELSE IF(LIU .GT. 15) THEN
      WRITE(7,1) MU(JJJ)
      WRITE(7,*)'---------------------------------------------------'
      ENDIF
      DO 200 JJ=1, NMUS/2
      IF(LIU .LE. 5) THEN
      WRITE(9,2) MU(JJ)
    2 FORMAT('MUO=',1PE18.11)
      WRITE(9,*)'---------------------------------------------------'
      WRITE(9,*)'*************************************'
      WRITE(9,*)'NORMALIZED REFLECTED INTENSITIES ARE:'
      WRITE(9,*)'*************************************'
      WRITE(9,*)
      ELSE IF(LIU .GT. 5 .AND. LIU .LE. 10) THEN
      WRITE(5,2) MU(JJ)
      WRITE(5,*)'---------------------------------------------------'
      WRITE(5,*)'*************************************'
      WRITE(5,*)'NORMALIZED REFLECTED INTENSITIES ARE:'
      WRITE(5,*)'*************************************'
      WRITE(5,*)
      ELSE IF(LIU .GT. 10 .AND. LIU .LE. 15) THEN
      WRITE(6,2) MU(JJ)
      WRITE(6,*)'---------------------------------------------------'
      WRITE(6,*)'*************************************'
      WRITE(6,*)'NORMALIZED REFLECTED INTENSITIES ARE:'
      WRITE(6,*)'*************************************'
      WRITE(6,*)
      ELSE IF(LIU .GT. 15) THEN
      WRITE(7,2) MU(JJ)
      WRITE(7,*)'---------------------------------------------------'
      WRITE(7,*)'*************************************'
      WRITE(7,*)'NORMALIZED REFLECTED INTENSITIES ARE:'
      WRITE(7,*)'*************************************'
      WRITE(7,*)
      ENDIF
C--------BEGIN TO CALCULATE THE RIF FOR MU VALUES.
      DO 1000 JR1=1, N
      SUM1=0.D0
      DO 1100 K1=1, N
      SUM1=RI(JJJ,JJ,JR1,K1)*F1(K1)+SUM1
```

```
 1100 CONTINUE
      RIF1(JJJ,JJ,JR1)=SUM1
 1000 CONTINUE
      DO 2000 JR2=1, N
      SUM2=0.D0
      DO 2100 K2=1, N
      SUM2=RI(JJJ,JJ,JR2,K2)*F2(K2)+SUM2
 2100 CONTINUE
      RIF2(JJJ,JJ,JR2)=SUM2
 2000 CONTINUE
      DO 3000 JR3=1, N
      SUM3=0.D0
      DO 3100 K3=1, N
      SUM3=RI(JJJ,JJ,JR3,K3)*F3(K3)+SUM3
 3100 CONTINUE
      RIF3(JJJ,JJ,JR3)=SUM3
 3000 CONTINUE
      DO 4000 JR4=1, N
      SUM4=0.D0
      DO 4100 K4=1, N
      SUM4=RI(JJJ,JJ,JR4,K4)*F4(K4)+SUM4
 4100 CONTINUE
      RIF4(JJJ,JJ,JR4)=SUM4
 4000 CONTINUE
      IF(LIU .LE. 5) THEN
      WRITE(9,*) '     RIF1              RIF2              RIF3
     *  RIF4'
      WRITE(9,*)
      DO 4200 I4=1, N
      WRITE(9,3) RIF1(JJJ,JJ,I4), RIF2(JJJ,JJ,I4), RIF3(JJJ,JJ,I4),
     *  RIF4(JJJ,JJ,I4)
 4200 CONTINUE
    3 FORMAT(4(1PE18.11,2X))
      WRITE(9,*)
      WRITE(9,*)'****************************************'
      WRITE(9,*)'NORMALIZED TRANSMITTED INTENSITIES ARE:'
      WRITE(9,*)'****************************************'
      WRITE(9,*)
      ELSE IF(LIU .GT. 5 .AND. LIU .LE. 10) THEN
      WRITE(5,*) '     RIF1              RIF2              RIF3
     *  RIF4'
      WRITE(5,*)
      DO 4201 I4=1, N
      WRITE(5,3) RIF1(JJJ,JJ,I4), RIF2(JJJ,JJ,I4), RIF3(JJJ,JJ,I4),
     *  RIF4(JJJ,JJ,I4)
 4201 CONTINUE
```

```
      WRITE(5,*)
      WRITE(5,*)'****************************************'
      WRITE(5,*)'NORMALIZED TRANSMITTED INTENSITIES ARE:'
      WRITE(5,*)'****************************************'
      WRITE(5,*)
      ELSE IF(LIU .GT. 10 .AND. LIU .LE. 15) THEN
      WRITE(6,*) '    RIF1            RIF2            RIF3
     *  RIF4'
      WRITE(6,*)
      DO 4202 I4=1, N
      WRITE(6,3) RIF1(JJJ,JJ,I4), RIF2(JJJ,JJ,I4), RIF3(JJJ,JJ,I4),
     *  RIF4(JJJ,JJ,I4)
 4202 CONTINUE
      WRITE(6,*)
      WRITE(6,*)'****************************************'
      WRITE(6,*)'NORMALIZED TRANSMITTED INTENSITIES ARE:'
      WRITE(6,*)'****************************************'
      WRITE(6,*)
      ELSE IF(LIU .GT. 15) THEN
      WRITE(7,*) '    RIF1            RIF2            RIF3
     *  RIF4'
      WRITE(7,*)
      DO 4203 I4=1, N
      WRITE(7,3) RIF1(JJJ,JJ,I4), RIF2(JJJ,JJ,I4), RIF3(JJJ,JJ,I4),
     *  RIF4(JJJ,JJ,I4)
 4203 CONTINUE
      WRITE(7,*)
      WRITE(7,*)'****************************************'
      WRITE(7,*)'NORMALIZED TRANSMITTED INTENSITIES ARE:'
      WRITE(7,*)'****************************************'
      WRITE(7,*)
      ENDIF
C--------BEGIN TO CALCULATE THE TIF FOR MU VALUES.
      DO 5000 JR5=1, N
      SUM5=0.D0
      DO 5100 K5=1, N
      SUM5=TI(JJJ,JJ,JR5,K5)*F1(K5)+SUM5
 5100 CONTINUE
      TIF1(JJJ,JJ,JR5)=SUM5
 5000 CONTINUE
      DO 6000 JR6=1, N
      SUM6=0.D0
      DO 6100 K6=1, N
      SUM6=TI(JJJ,JJ,JR6,K6)*F2(K6)+SUM6
 6100 CONTINUE
      TIF2(JJJ,JJ,JR6)=SUM6
```

```
6000 CONTINUE
      DO 7000 JR7=1, N
      SUM7=0.D0
      DO 7100 K7=1, N
      SUM7=TI(JJJ,JJ,JR7,K7)*F3(K7)+SUM7
7100 CONTINUE
      TIF3(JJJ,JJ,JR7)=SUM7
7000 CONTINUE
      DO 8000 JR8=1, N
      SUM8=0.D0
      DO 8100 K8=1, N
      SUM8=TI(JJJ,JJ,JR8,K8)*F4(K8)+SUM8
8100 CONTINUE
      TIF4(JJJ,JJ,JR8)=SUM8
8000 CONTINUE
      IF(LIU .LE. 5) THEN
      WRITE(9,*) '     TIF1              TIF2              TIF3
     *  TIF4'
      WRITE(9,*)
      DO 8200 I8=1, N
      WRITE(9,3) TIF1(JJJ,JJ,I8), TIF2(JJJ,JJ,I8), TIF3(JJJ,JJ,I8),
     *  TIF4(JJJ,JJ,I8)
8200 CONTINUE
      WRITE(9,*)
      WRITE(9,*)'----------------------------------------------------'
      ELSE IF(LIU .GT. 5 .AND. LIU .LE. 10) THEN
      WRITE(5,*) '     TIF1              TIF2              TIF3
     *  TIF4'
      WRITE(5,*)
      DO 8201 I8=1, N
      WRITE(5,3) TIF1(JJJ,JJ,I8), TIF2(JJJ,JJ,I8), TIF3(JJJ,JJ,I8),
     *  TIF4(JJJ,JJ,I8)
8201 CONTINUE
      WRITE(5,*)
      WRITE(5,*)'----------------------------------------------------'
      ELSE IF(LIU .GT. 10 .AND. LIU .LE. 15) THEN
      WRITE(6,*) '     TIF1              TIF2              TIF3
     *  TIF4'
      WRITE(6,*)
      DO 8202 I8=1, N
      WRITE(6,3) TIF1(JJJ,JJ,I8), TIF2(JJJ,JJ,I8), TIF3(JJJ,JJ,I8),
     *  TIF4(JJJ,JJ,I8)
8202 CONTINUE
      WRITE(6,*)
      WRITE(6,*)'----------------------------------------------------'
      ELSE IF(LIU .GT. 15) THEN
```

```
      WRITE(7,*) '      TIF1              TIF2              TIF3
     *  TIF4'
      WRITE(7,*)
      DO 8203 I8=1, N
      WRITE(7,3) TIF1(JJJ,JJ,I8), TIF2(JJJ,JJ,I8), TIF3(JJJ,JJ,I8),
     *  TIF4(JJJ,JJ,I8)
 8203 CONTINUE
      WRITE(7,*)
      WRITE(7,*)'---------------------------------------------------'
      ENDIF
  200 CONTINUE
      IF(LIU .LE. 5) THEN
      WRITE(9,*)'================================================'
      ELSE IF(LIU .GT. 5 .AND. LIU .LE. 10) THEN
      WRITE(5,*)'================================================'
      ELSE IF(LIU .GT. 10 .AND. LIU .LE. 15) THEN
      WRITE(6,*)'================================================'
      ELSE IF(LIU .GT. 15) THEN
      WRITE(7,*)'================================================'
      ENDIF
  100 CONTINUE
      RETURN
      END


C--------SUBROUTINE DXA FOR NUMERICAL INTEGRATION.
      SUBROUTINE DXA(N,AA,BB,X,A)
```

APPENDIX D

SAMPLE OF OUTPUT DATA FOR THE EXACT POLARIZATION COMPUTER PROGRAM

An example of the output data from the exact polarization computer program is as follows, where HSTEP is optical thickness step size; NQs are number of Gauss-Legendre quadrature; and F1, F2, F3, and F4 appended to TI (transmitted intensity) and RI (reflected intensity) are for the boundary conditions discussed on p. 128 of this document.

```
NUMBER OF LEGENDRE POLYNOMIALS (L)= 1
SCATTERING ALBEDO (W)= .500000000000
AZIMUTHAL ANGLE (PHI)= .000000000000
ERROR= .000001000000
NQ1=10   NQ2= 6   NQ3= 6   NQ4= 6   NQ5= 6   NQ6= 6   NQ7= 6   NQ8= 6
NQ9= 6   NQ10=10  NQ11= 6  NQ12= 0  NQ13= 0  NQ14= 0  NQ15= 0  NQ16= 0
NQ17= 0  NQ18= 0  NQ19= 0  NQ20= 0
INTERVALS ARE :
0.000000000000000E+000  1.000000000000000E-001  2.000000000000000E-001
3.000000000000000E-001  4.000000000000000E-001  5.000000000000000E-001
6.000000000000000E-001  7.000000000000000E-001  8.000000000000000E-001
9.000000000000000E-001  9.600000000000000E-001  1.000000000000000
1.000000000000000       1.000000000000000       1.000000000000000
1.000000000000000       1.000000000000000       1.000000000000000
1.000000000000000       1.000000000000000       1.000000000000000
IPRINT        HSTEP
 200        .000500000000
 200        .000500000000
 200        .000500000000
 200        .000500000000
 200        .000500000000
```

200	.000500000000
200	.000500000000
200	.000500000000
200	.000500000000
200	.000500000000
2000	.000500000000
2000	.000500000000
2000	.000500000000
2000	.000500000000
2000	.000500000000
2000	.000500000000
2000	.000500000000
2000	.000500000000
2000	.000500000000
2000	.000500000000

RELAXATION= .000000000000

F1	F2	F3	F4
1.00	1.00	1.00	1.00
0.00	1.00	0.00	0.00
0.00	0.00	1.00	0.00
0.00	0.00	0.00	1.00

B(0)=

1.00000000000E+00	0.00000000000E+00	0.00000000000E+00	0.00000000000E+00
0.00000000000E+00	0.00000000000E+00	0.00000000000E+00	0.00000000000E+00
0.00000000000E+00	0.00000000000E+00	0.00000000000E+00	0.00000000000E+00
0.00000000000E+00	0.00000000000E+00	0.00000000000E+00	6.39921540800E-02

B(1)=

1.40034346500E-01	0.00000000000E+00	0.00000000000E+00	0.00000000000E+00
0.00000000000E+00	0.00000000000E+00	0.00000000000E+00	0.00000000000E+00
0.00000000000E+00	0.00000000000E+00	0.00000000000E+00	0.00000000000E+00
0.00000000000E+00	0.00000000000E+00	0.00000000000E+00	1.50000000000E+00

NUMBER OF QUADRATURE POINTS (N)= 74
QUADRATURE POINTS ARE:

1.30467357414E-03	6.74683166555E-03	1.60295215850E-02	2.83302302935E-02
4.25562830509E-02	5.74437169491E-02	7.16697697065E-02	8.39704784150E-02
9.32531683344E-02	9.86953264259E-02	1.03376524290E-01	1.16939530677E-01
1.38069040696E-01	1.61930959304E-01	1.83060469323E-01	1.96623475710E-01
2.03376524290E-01	2.16939530677E-01	2.38069040696E-01	2.61930959304E-01
2.83060469323E-01	2.96623475710E-01	3.03376524290E-01	3.16939530677E-01
3.38069040696E-01	3.61930959304E-01	3.83060469323E-01	3.96623475710E-01
4.03376524290E-01	4.16939530677E-01	4.38069040696E-01	4.61930959304E-01

4.83060469323E-01	4.96623475710E-01	5.03376524290E-01	5.16939530677E-01
5.38069040696E-01	5.61930959304E-01	5.83060469323E-01	5.96623475710E-01
6.03376524290E-01	6.16939530677E-01	6.38069040696E-01	6.61930959304E-01
6.83060469323E-01	6.96623475710E-01	7.03376524290E-01	7.16939530677E-01
7.38069040696E-01	7.61930959304E-01	7.83060469323E-01	7.96623475710E-01
8.03376524290E-01	8.16939530677E-01	8.38069040696E-01	8.61930959304E-01
8.83060469323E-01	8.96623475710E-01	9.00782804144E-01	9.04048098999E-01
9.09617712951E-01	9.16998138176E-01	9.25533769831E-01	9.34466230169E-01
9.43001861824E-01	9.50382287049E-01	9.55951901001E-01	9.59217195856E-01
9.61350609716E-01	9.66775812271E-01	9.75227616278E-01	9.84772383722E-01
9.93224187729E-01	9.98649390284E-01		

==

OPTICAL THICKNESS= .10000000

==

==

MU= 1.00000000000E+00

MUO= 1.00000000000E+00

****

NORMALIZED REFLECTED INTENSITIES ARE:

****

RIF1	RIF2	RIF3	RIF4
3.42002683225E-03	3.42002683225E-03	3.42002683225E-03	3.42002683225E-03
0.00000000000E+00	0.00000000000E+00	0.00000000000E+00	0.00000000000E+00
0.00000000000E+00	0.00000000000E+00	0.00000000000E+00	0.00000000000E+00
0.00000000000E+00	0.00000000000E+00	0.00000000000E+00	-5.27424650210E-03

****

NORMALIZED TRANSMITTED INTENSITIES ARE:

****

TIF1	TIF2	TIF3	TIF4
4.42542233768E-03	4.42542233768E-03	4.42542233768E-03	4.42542233768E-03
0.00000000000E+00	0.00000000000E+00	0.00000000000E+00	0.00000000000E+00
0.00000000000E+00	0.00000000000E+00	0.00000000000E+00	0.00000000000E+00
0.00000000000E+00	0.00000000000E+00	0.00000000000E+00	5.72928639014E-03

MUO= 7.00000000000E-01

****

NORMALIZED REFLECTED INTENSITIES ARE:

****

RIF1	RIF2	RIF3	RIF4
3.51512056038E-03	3.51512056038E-03	3.51512056038E-03	3.51512056038E-03
0.00000000000E+00	0.00000000000E+00	0.00000000000E+00	0.00000000000E+00
0.00000000000E+00	0.00000000000E+00	0.00000000000E+00	0.00000000000E+00
0.00000000000E+00	0.00000000000E+00	0.00000000000E+00	-3.36363796019E-03

****
NORMALIZED TRANSMITTED INTENSITIES ARE:
****

TIF1	TIF2	TIF3	TIF4
4.20013004690E-03	4.20013004690E-03	4.20013004690E-03	4.20013004690E-03
0.00000000000E+00	0.00000000000E+00	0.00000000000E+00	0.00000000000E+00
0.00000000000E+00	0.00000000000E+00	0.00000000000E+00	0.00000000000E+00
0.00000000000E+00	0.00000000000E+00	0.00000000000E+00	4.17828745191E-03

MUO= 3.00000000000E-01

****
NORMALIZED REFLECTED INTENSITIES ARE:
****

RIF1	RIF2	RIF3	RIF4
3.39850421964E-03	3.39850421964E-03	3.39850421964E-03	3.39850421964E-03
0.00000000000E+00	0.00000000000E+00	0.00000000000E+00	0.00000000000E+00
0.00000000000E+00	0.00000000000E+00	0.00000000000E+00	0.00000000000E+00
0.00000000000E+00	0.00000000000E+00	0.00000000000E+00	-1.04591290692E-03

****
NORMALIZED TRANSMITTED INTENSITIES ARE:
****

TIF1	TIF2	TIF3	TIF4
3.65235252536E-03	3.65235252536E-03	3.65235252536E-03	3.65235252536E-03
0.00000000000E+00	0.00000000000E+00	0.00000000000E+00	0.00000000000E+00
0.00000000000E+00	0.00000000000E+00	0.00000000000E+00	0.00000000000E+00
0.00000000000E+00	0.00000000000E+00	0.00000000000E+00	1.90245806487E-03

MUO= 1.00000000000E-01

****
NORMALIZED REFLECTED INTENSITIES ARE:
****

RIF1	RIF2	RIF3	RIF4
2.60615427590E-03	2.60615427590E-03	2.60615427590E-03	2.60615427590E-03
0.00000000000E+00	0.00000000000E+00	0.00000000000E+00	0.00000000000E+00
0.00000000000E+00	0.00000000000E+00	0.00000000000E+00	0.00000000000E+00
0.00000000000E+00	0.00000000000E+00	0.00000000000E+00	-3.80753903344E-05

****
NORMALIZED TRANSMITTED INTENSITIES ARE:
****

TIF1	TIF2	TIF3	TIF4
2.63544023907E-03	2.63544023907E-03	2.63544023907E-03	2.63544023907E-03
0.00000000000E+00	0.00000000000E+00	0.00000000000E+00	0.00000000000E+00
0.00000000000E+00	0.00000000000E+00	0.00000000000E+00	0.00000000000E+00
0.00000000000E+00	0.00000000000E+00	0.00000000000E+00	6.89198623080E-04

===

VITA

Cho-Chun Liu

Candidate for the Degree of

Doctor of Philosophy

Thesis: A THEORETICAL STUDY ON THE TRANSFER EQUATION FOR THE SCATTERING OF POLARIZED LIGHT IN A PLANE-PARALLEL MEDIUM

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