EXTENDING GIVENTAL'S RESULTS FOR CONCAVEX VECTOR BUNDLES ON PROJECTIVE SPACES

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1 Introduction

1.1 History

In 1991, a group of physicists [6] made some startling predictions on the number of rational curves on a quintic threefold X, i.e. a subvariety of $\mathbb{P}^4$ given by a degree five polynomial $\mathbb{F}_5(z_0, ..., z_4) = 0$. An equivalent form of this prediction goes as follows. Consider the following $H^*\mathbb{P}^4[[e^t]][t]$-valued functions

$$J(T) = e^{pT} + \frac{p^2}{5}\sum_{d=1}^{\infty} n_d d^3 \sum_{k=1}^{\infty} \frac{e^{(p+kd)T}}{(p+kd)^2},$$

$$I(t) = e^{pt}\sum_{d=0}^{\infty} e^{dt}\frac{\prod_{m=1}^{5d}(5p+m)}{\prod_{m=1}^{d}(p+m)^5} = I_0(t) + I_1(t)p + I_2(t)p^2 + I_3(t)p^3.$$

Here n_d is the number of degree d rational curves in $\mathbb{P}^4$ that are situated in the quintic threefold X and p is the hyperplane class.

Consider the following change of variables

$$T = \frac{I_1(t)}{I_0(t)}. \tag{1}$$

Then the prediction was that

$$\frac{I(T)}{I_0(T)} = J(T). \tag{2}$$

This allows us to compute as many numbers n_d as desired.

What was startling to mathematicians about these predictions is that the number of rational curves in a quintic threefold was not known to be finite. In fact the famous Clemens conjecture, which is still open, states precisely that. So, not only were the physicists claiming the finiteness of these numbers, they were able to compute them.

This phenomenon ignited an exciting and massive undertaking for mathematicians to try to lay the foundations and understand the content of these predictions. The end result was the Gromov-Witten theory and Mirror Theorems. The numbers n_d are in this context essentially the Gromov-Witten invariants of the quintic and the prediction is now formulated in the form of the mirror theorems. The intuitive idea that n_d is the number of degree d rational curves on $\mathbb{P}^4$ lying on the quintic X is more subtle than we first thought. They do not always coincide, even if the Clemens Conjecture is assumed. The mirror theorem then computes the Gromov-Witten invariants of the quintic X.

In [13] Givental formulated and proved a mirror theorem for Fano and Calabi-Yau complete intersections X in a projective space $\mathbb{P}^m$, i.e. for X given by a section of the bundle $V = \oplus_{i=1}^{r}\mathcal{O}(k_i)$ where $k_i > 0$ and $\sum_{i=1}^{r} k_i \leq m+1$. The idea was to construct two $H^*\mathbb{P}^m[[e^t]][t]$-valued functions J_V and I_V by solving some diferential equations. Even though J_V is defined on $\mathbb{P}^m$, it encodes gravitational descendants (which are generalized G-W invariants) of X [8]. Then the mirror theorem says that after a change of variables, $I_V = J_V$. The end result is that it allows us in principle to compute gravitational descendants (in particular G-W invariants) of X. Later on [12] Givental generalized this to a mirror theorem for complete intersections in toric varieties.

1.2 Statement of the problem and results

In this study we are interested in extending mirror theorems to the case of

$$V = (\oplus_{i=1}^{r} \mathcal{O}(k_i)) \oplus (\oplus_{j=1}^{s} \mathcal{O}(-l_j)) = V^+ \oplus V^- \tag{3}$$

for certain positive k_i and l_j. The vector bundle V induces a modified Gromov-Witten theory in $\mathbb{P}^m$. Suppose $X \overset{i}{\hookrightarrow} \mathbb{P}^m$ is given by a section of V^+ and Y is a projective manifold such that $X \overset{j}{\hookrightarrow} Y$ with $\mathcal{N}_{X/Y} = i^*(V^-)$. The pure Gromov-Witten theory of Y and the modified Gromov-Witten theory of $\mathbb{P}^m$ are closely related.

In Theorem 3.5.2 we describe one aspect of this relation, namely the relation between their quantum $\mathcal{D}$-modules. Under natural restrictions, the generator J_Y of the pure $\mathcal{D}$-module of Y pulls back to the generator J_V of the modified $\mathcal{D}$-module of $\mathbb{P}^m$. It follows that even though J_V is defined on $\mathbb{P}^m$, it encodes the gravitational descendants of Y supported in X.

In the chapter on applications, we give an example where the quantum product of Y pulls back to the modified quantum product in $\mathbb{P}^m$.

A natural hypergeometric series I_V is defined. Then in Theorem 4.1.1 we show that after a change of variables, $I_V = J_V$. This allows us to compute on $\mathbb{P}^m$ the gravitational descendants of Y supported on X.

The only way that X remembers the ambient variety Y in this context is by the normal bundle. Y can therefore be substituted by a local manifold. It suggests that there should be a local version of mirror symmetry. This was first realized by Katz, Klemm, and Vafa [18]. The principle of local mirror symmetry in general has yet to be understood. Recently a group of authors [7] have made some calculations that contribute towards it.

Theorem 4.1.1 was first proven by Lian, Liu and Yau in [21] using a different approach.

We prove the mirror theorem following the Givental scheme for complete intersections in the projective space. Our proof makes use of the localization theorem in the moduli space of stable maps that was first introduced by Kontsevich in [20] and of the Frobenius manifolds suggested by Givental in [13].

Several applications of the mirror theorem proven in this work are treated in the last chapter. The first one deals with computing the contributions of the multiple covers of a rigid rational curve in a Calabi-Yau threefold X to the corresponding Gromov-Witten invariant. In the second application we consider the case of a Calabi-Yau threefold X that contains a projective plane $\mathbb{P}^2$. In this case a quantum product is constructed in $\mathbb{P}^2$ that is a natural restriction of the quantum product in X. The mirror theorem for this example allows us to compute the virtual numbers of degree d rational plane curves in the Calabi-Yau X.

2 Preliminaries

In this section we give a brief overview of the moduli space of stable maps and the localization techniques for the moduli space of stable maps to projective space.

2.1 Moduli space of stable maps of genus zero

The notion of stable maps is due to Kontsevich [20].

Definition 2.1.1 *A genus zero stable map is a connected, nodal marked curve of arithmetic genus zero* $(C, x_1, x_2, ..., x_n)$ *together with a morphism* $f : C \to X$ *satisfying the following conditions:*

1. $x_1, x_2, ..., x_n$ *are ordered, smooth points of* C*.*
2. *If* f *is constant on a component* C_i *of* C *then that component contains three special points (i.e. marked or nodal).*

The second condition forces the stable map to have only finitely many automorphisms.

If $f_*(C) = \beta \in H_2(X, \mathbb{Z})$ we say that the stable map $(C, x_1, ..., x_n, f : C \to X)$ has class β.

Families of stable maps of class β over a scheme S can be defined naturally thus giving a contravariant functor from schemes to sets [11]. If X is projective variety, this moduli functor is coarse, i.e. the functor can be coarsely represented by a projective scheme over $\mathbb{C}$. We denote this coarse moduli space by $\overline{\mathcal{M}}_{0,n}(X, \beta)$. Its expected dimension is

$$-K_X \cdot \beta + \dim X + n - 3. \tag{4}$$

Another viewpoint which will not be pursued in this study is that the above contravariant functor is an algebraic stack.

This moduli space comes equipped with some natural morphisms due to the universal property.

- The evaluation maps $e_i : \overline{\mathcal{M}}_{0,n}(X, \beta) \to X$ which sends $f : (C, x_1, ..., x_n,) \to X$ to $f(x_i)$.

- The forgetful map $\pi_n : \overline{\mathcal{M}}_{0,n}(X, \beta) \to \overline{\mathcal{M}}_{0,n-1}(X, \beta)$ which forgets one of the marked points and then, if necessary collapses the incident component to satisfy condition 2 of stability.

  The following objects will be central to this study.

Definition 2.1.2 *A line bundle $\mathcal{L}$ on X is called* **convex** *(***concave***) if $H^1(C, f^*(\mathcal{L}) = 0$ ($H^0(C, f^*(\mathcal{L}) = 0$) for any genus zero stable map $(C, x_1, ...x_n, f)$.*

Definition 2.1.3 *A direct sum of convex and concave line bundles on X is called a* **concavex** *vector bundle.*

2.2 Deformation theory of the moduli space of stable maps

There is a deformation-obstruction theory for stable maps [23]. Naming sheaves by their fibres, the basic exact sequence for the tangent space and the obstruction space of $\overline{\mathcal{M}}_{0,n}(X,\beta)$ is

$$0 \to \mathrm{Ext}^0(\Omega_C(\sum_i x_i), \mathcal{O}_C) \to \mathrm{H}^0(C, f^*TX) \to \mathcal{T}_M \to$$

$$\to \mathrm{Ext}^1(\Omega_C(\sum_i x_i), \mathcal{O}_C) \to \mathrm{H}^1(C, f^*TX) \to \Upsilon \to 0. \tag{5}$$

Here $\mathcal{T}_M = \mathrm{Ext}^1(f^*\Omega_X \to \Omega_C, \mathcal{O}_C)$ is the tangent space to $\overline{\mathcal{M}}_{0,n}(X,\beta)$ at the point $\{f : (C, x_1, ..., x_n,) \to X\}$ and $\Upsilon = \mathrm{Ext}^2(f^*\Omega_X \to \Omega_C, \mathcal{O}_C)$ is the obstruction space at the same point. $\mathrm{Ext}^0(\Omega_C(\sum_i x_i), \mathcal{O}_C)$ represents infinitesimal automorphisms of the marked source curve and $\mathrm{Ext}^1(\Omega_C(\sum_i x_i), \mathcal{O}_C)$ its infinitesimal deformations. If $H^1(C, f^*TX) = 0$ for any genus zero stable map $(C, x_1, ..., x_n, f : C \to X)$ we say that X is **convex**. For a convex X, the obstruction bundle Υ vanishes and the moduli space is unobstructed of the expected dimension. Examples of convex varieties are homogeneous spaces G/P [11]. For nonconvex varieties, this moduli space may behave badly and have components of larger dimensions. In this case, a Chow homology class of the expected dimension has been constructed [3] [23]. It is called the **virtual fundamental class**. We will denote it by $[\overline{\mathcal{M}}_{0,n}(X,\beta)]^{\mathrm{virt}}$. This class behaves well. We will be using here the following facts.

- The virtual fundamental class is preserved when pulled back by the forgetful map π_n. A proof of this fact can be found in section 7.1.5 of [8].

- If the obstruction sheaf Υ is free, the virtual fundamental class refines the top Chern class of Υ. This fact is proven in Proposition 5.6 of [3].

2.3 Localization

2.3.1 Equivariant cohomology

The notion of equivariant cohomology and the localization theorem that we are about to explain is valid for any compact connected Lie group. We will only state without proof the results that we will be using. The main reference here is [1]. For a detailed exposition of this subject we suggest Chapter 9 of [8].

Let $T = (\mathbb{C}^*)^{s+1}$ which is classified by the principal T-bundle $(\mathbb{C}^{\infty+1} - \{0\})^{s+1} \to (\mathbb{CP}^\infty)^{s+1}$. If $\pi_i : (\mathbb{CP}^\infty)^{s+1} \to \mathbb{CP}^\infty$ is the i-th projection for $i = 0, 1, ..., s$, we let $\lambda_i = c_1(\pi_i^*(\mathcal{O}(1)))$. We will use $\mathcal{O}(\lambda_i)$ for the line bundle $\pi_i^*(\mathcal{O}(1))$. Clearly $H^*(\mathbb{CP}^\infty)^{s+1} = \mathbb{C}[\lambda_0, ..., \lambda_s]$. If X is a manifold with T-action, we let

$$X_T := X \times_T (\mathbb{C}^{\infty+1} - \{0\})^{s+1}. \tag{6}$$

Definition 2.3.1 *The equivariant cohomology of X is*

$$H_T^*(X) := H^*(X_T). \tag{7}$$

Obviously, if $X = \{\text{point}\}$ we have $X_T = (\mathbb{CP}^\infty)^{s+1}$. Therefore

$$H_T^*(\{point\}) = H^*(\mathbb{CP}^\infty)^{s+1} = \mathbb{C}[\lambda_0, ..., \lambda_s]. \tag{8}$$

The equivariant cohomology $H_T^*(X)$ is a $\mathbb{C}[\lambda_0, ..., \lambda_s]$-module via the equivariant morphism $X \to \{\text{point}\}$.

Let $X^T = \cup X_j$ be the decomposition of the fixed point locus into its connected components. X_j is smooth for all j. Let $i_j : X_j \to X$ be the inclusion. The normal bundle $\mathcal{N}_j$ of X_j in X is equivariant therefore it has an equivariant Euler class $\text{Euler}_T(\mathcal{N}_j)$. We will be using the following form of the localization theorem

Theorem 2.3.1 *Let $\alpha \in H_T^*(X) \otimes \mathbb{C}(\lambda_0, ..., \lambda_s)$. Then*

$$\int_{X_T} \alpha = \sum_{j \in J} \int_{(X_j)_T} \frac{i_j^*(\alpha)}{\mathrm{Euler}_T(\mathcal{N}_j)}. \tag{9}$$

In this study we will be interested in the case $X = \mathbb{P}^s$. Let $\chi_0, \chi_1, ..., \chi_s$ be characters of the torus T. Clearly a basis for the characters of the torus is given by $\varepsilon_i(t_0, ..., t_s) = t_i$. In terms of this basis let $\chi_i = (a_{ij})$. We will say that the weight of the character ε_i is λ_i. Similarly the weight of the character χ_i is $\sum_j a_{ij}\lambda_j$. Let $\mathcal{O}(\chi_i) = \mathcal{O}(\sum_j a_{ij}\lambda_j)$ be a line bundle over $(\mathbb{CP}^\infty)^{s+1}$. Consider the following action of T on $\mathbb{P}^s$

$$(t_0, t_1, ..., t_s) \cdot (z_0, z_1, ..., z_s) = (\chi_0(t)z_0, ..., \chi_s(t)z_s). \tag{10}$$

Then $\mathbb{P}_T^s = \mathbb{P}(\oplus_i \mathcal{O}(\chi_i))$. Let $p = c_1(\mathcal{O}_{\mathbb{P}_T^s}(1))$. Then

$$H_T^*\mathbb{P}^s = \mathbb{C}[\lambda_0, ..., \lambda_s, p] / \prod_i (p - \sum_j a_{ij}\lambda_j). \tag{11}$$

We will be interested in the case $\chi_j = \varepsilon_j$. For the corresponding T-action we have $H_T^*\mathbb{P}^s = \mathbb{C}[\lambda_0, ..., \lambda_s, p] / \prod_j (p - \lambda_j)$. Let us see what the localization theorem says in this case. The locus of the fixed points consists of points p_j for $j = 0, 1, ..., s$ where p_j is the point whose j-th coordinate is 1 and all the other ones are 0. Let $\phi_j = \prod_{k \neq j}(p - \lambda_k)$ for $j = 0, 1, ..., s$. Then for $\alpha, \beta \in H_T^*(\mathbb{P}^s) \otimes \mathbb{C}(\lambda_0, ..., \lambda_s)$ we have

$$\alpha = \beta \Leftrightarrow \int_{\mathbb{P}_T^s} \alpha \cup \phi_k = \int_{\mathbb{P}_T^s} \beta \cup \phi_k \tag{12}$$

for all k. Also $i_j^*(\phi_j) = \prod_{k \neq j}(\lambda_j - \lambda_k) = \mathrm{Euler}_T(\mathcal{N}_j)$. The localization theorem says that for any polynomial $F(p) \in \mathbb{C}(\lambda_0, ..., \lambda_s)[p]$ we have

$$\int_{\mathbb{P}_T^s} F(p) = \sum_j \frac{F(\lambda_j)}{\prod_{k \neq j}(\lambda_j - \lambda_k)}. \tag{13}$$

2.3.2 Localization techniques in $\overline{M}_{0,n}(\mathbb{P}^s, d)$

Since $\mathbb{P}^r$ is convex, the moduli space $\overline{M}_{0,n}(\mathbb{P}^s, d)$ is of the expected dimension. A simple calculation shows that dim $\overline{M}_{0,n}(\mathbb{P}^s, d) = s + d + sd + n - 3$. Let T act diagonally on V with weights $-\lambda_0, -\lambda_1, ..., -\lambda_s$. It gives rise to an action of T on $\mathbb{P}^s$. By translating the target of a map we get an action of T on $\overline{M}_{0,n}(\mathbb{P}^s, d)$. Kontsevich [20] has identified fixed point components of this action in terms of decorated graphs. Our treatment here follows that of [15].

If $f : (C, x_1, ..., x_n) \to \mathbb{P}^s$ is a fixed stable map then the image curve is a fixed curve in $\mathbb{P}^s$. Also all marked points, collapsed components and nodes are mapped to the fixed points p_i of the T-action on $\mathbb{P}^s$. A noncontracted component must be mapped to a fixed line $\overline{p_i p_j}$ on $\mathbb{P}^s$. The only branch points are the two fixed points p_i and p_j. It follows that the restriction of the map f to this component is determined by its degree. The graph Γ corresponding to the fixed point component containing such a map is constructed as follows. The vertices correspond to the connected components of $f^{-1}\{p_0, p_1, ..., p_s\}$. The edges correspond to the noncontracted components of the map. The graph is decorated as follows. Each edge is marked by the degree of the map on the corresponding component, and each vertex is marked by the fixed point of $\mathbb{P}^s$ where the corresponding component is mapped to. To each vertex we associate a leg for each marked point that belongs to the corresponding component. The fixed point component $\overline{M}_\Gamma$ corresponding to Γ is constructed as follows. For a vertex v, let $n(v)$ be the number of legs or edges incident to that vertex. Also for an edge e let d_e

be the degree of the stable map on the corresponding component. Let

$$\overline{\mathcal{M}}_\Gamma := \prod_v \overline{M}_{0,n(v)}. \tag{14}$$

There is a finite group of automorphisms G_Γ acting on $\overline{\mathcal{M}}_\Gamma$ [8][15]. This group fits in the following exact sequence

$$0 \to \prod_e \mathbb{Z}/d_e\mathbb{Z} \to \mathrm{G}_\Gamma \to \mathrm{Aut}(\Gamma) \to 0. \tag{15}$$

Here $\mathrm{Aut}(\Gamma)$ is the automorphism group of the decorated graph Γ. The fixed point component corresponding to the decorated graph Γ is

$$\overline{M}_\Gamma = \overline{\mathcal{M}}_\Gamma/\mathrm{G}. \tag{16}$$

The order of the automorphism group G is

$$a_\Gamma = \prod_e d_e \cdot |\mathrm{Aut}(\Gamma)|. \tag{17}$$

Let $i_\Gamma : \overline{M}_\Gamma \hookrightarrow \overline{M}_{0,n}(\mathbb{P}^s, d)$ be the inclusion of the fixed point component corresponding to Γ and N_Γ its normal bundle. This bundle is T-equivariant. We will be using the following variation of the Bott residue formula for orbifolds. Let α be an equivariant cohomology class in $H^*_T(\overline{M}_{0,n}(\mathbb{P}^s, d))$. Then

$$\int_{\overline{M}_{0,n}(\mathbb{P}^s,d)_T} \alpha = \sum_\Gamma \int_{\overline{M}_\Gamma} (\frac{i^*_\Gamma(\alpha)}{a_\Gamma \mathrm{Euler}_T(N_\Gamma)}). \tag{18}$$

To compute the equvariant Euler class of the normal bundle we use the restriction of the basic deformation-obstruction exact sequence on $\overline{M}_\Gamma$ (recall that $\mathbb{P}^s$ is convex).

$$0 \to \mathrm{Ext}^0(\Omega_C(\sum_i x_i), \mathcal{O}_C) \to H^0(C, f^*T\mathbb{P}^s) \to \mathcal{T}_M \to \mathrm{Ext}^1(\Omega_C(\sum_i x_i), \mathcal{O}_C) \to 0. \tag{19}$$

We use this sequence to compute the weights of the T-action on $\mathcal{T}_M$. The equivariant Euler class of the normal bundle N_Γ is the product of the nonzero weights (i.e. the moving part of the tangent space). Recall that $\mathrm{Ext}^0(\Omega_C(\sum_i x_i), \mathcal{O}_C)$ represents the infinitesimal automorphisms of the pointed curve and $\mathrm{Ext}^1(\Omega_C(\sum_i x_i), \mathcal{O}_C)$ its infinitesimal deformations. On the other hand $H^0(C, f^*T\mathbb{P}^s)$ represents the infinitesimal deformations of the map. From the above exact sequence we conclude that we can compute the weights as follows.

For $\mathrm{Ext}^0(\Omega_C(\sum_i x_i), \mathcal{O}_C)$ we only compute the weights of the reparametrizations of the marked curve.

For $\mathrm{Ext}^1(\Omega_C(\sum_i x_i), \mathcal{O}_C)$ we compute the weights for smoothing the nodes and moving the nodes and the marked points. We emphasize one important piece here, namely the Chern classes of the cotangent line bundles of the fixed point component. The fiber of such a line bundle at the i-th marked point of a stable map $(C, x_1, x_2, ..., x_n, f)$ is the cotangent space $T^*_{x_i}C$. They come from deforming the nodes where a contracted component meets an uncontracted one. We can integrate these classes by using the Witten-Kontsevich formulas [27].

For $H^0(C, f^*T\mathbb{P}^s)$ we use the normalization sequence at the nodes of the source curve to express the deformation of f in terms of the deformations of the restriction of f to the components of C. For example, let $(C = C_1 \cup C_2, f)$ with $C_i \simeq \mathbb{P}^1$ for $i = 1, 2$, be a stable map in $\mathcal{M}_{0,0}(\mathbb{P}^s, d)$. Let $x = C_1 \cap C_2$. The normalization sequence is

$$0 \to \mathcal{O}_C \to \oplus_i \mathcal{O}_{C_i} \to \mathcal{O}_x \to 0. \tag{20}$$

We twist this by $f^*(T\mathbb{P}^s)$ and take the cohomology sequence to obtain

$$0 \to H^0(C, f^*(T\mathbb{P}^s) \to \oplus_i H^0(C_i, f^*(T\mathbb{P}^s) \to T_{f(x)}\mathbb{P}^s 0. \tag{21}$$

The weights of the torus action on $H^0(C, f^*T\mathbb{P}^s)$ are obtained by taking the union of the weights of the torus action on $H^0(C_i, f^*(T\mathbb{P}^s)$ for $i = 1, 2$ and then substracting the weights of the action on $T_{f(x)}\mathbb{P}^s$. Both sets of latter weights are easier to compute.

Finally we multiply the nonzero weights to obtain the equivariant Euler class of the normal bundle N_Γ.

Remark 2.3.1 *The torus action and localization techniques described here will be used in proving the Mirror Theorem for the case of a concavex bundle in a projective space.*

3 The quantum product induced by a vector bundle

3.1 Notations.

We will consider the case when the vector bundle is a direct sum of line bundles. To make formulas and notations shorter, we will focus in the case $V = \mathcal{O}(k) \oplus \mathcal{O}(-l)$ on $\mathbb{P}^m$ with $k, l > 0$. We will consider the action of $\mathbb{C}^*$ on the total space of any line bundle $\mathcal{L}$ by scaling on the fiber and trivial on $\mathbb{P}^m$. It gives rise to an action of $T = (\mathbb{C}^*)^2$ on V. Then $\mathbb{P}^m_T = \mathbb{P}^m \times \mathbb{P}^\infty \times \mathbb{P}^\infty$. Let $\lambda_i = c_1({\pi_i}^* \mathcal{O}_{\mathbb{P}^\infty}(1))$ for $i = 1, 2$ and p the equivariant hyperplane class. Since the action of T on $\mathbb{P}^m$ is trivial

$$H^*_T\mathbb{P}^m := H^*\mathbb{P}^m_T = \mathbb{Q}[p, \lambda_1, \lambda_2]/(p^{m+1}) = H^*\mathbb{P}^m \otimes_{\mathbb{Q}} \mathbb{Q}[\lambda_1, \lambda_2] = H^*(\mathbb{P}^m, \mathbb{Q}[\lambda_1, \lambda_2]).$$

One computes the equivariant Euler classes of summand line bundles:

$$\mathrm{Euler}_T(\mathcal{O}(k)) = kp - \lambda_1$$

$$\mathrm{Euler}_T(\mathcal{O}(-l)) = -lp - \lambda_2.$$

Define

$$\mathcal{P} := H^*(\mathbb{P}^m, \mathbb{Q}[\lambda_1, \lambda_2])$$

$$\mathcal{R} := H^*_T(\mathbb{P}^m) \otimes_{\mathbb{Q}[\lambda_1, \lambda_2]} \mathbb{Q}(\lambda_1, \lambda_2) = H^*(\mathbb{P}^m, \mathbb{Q}(\lambda_1, \lambda_2)).$$

Let $T_0 = 1, T_1 = p, ..., T_m = p^m$ be a basis of $\mathcal{R}$ as a $\mathbb{Q}(\lambda_1, \lambda_2)$-vector space. Clearly $-lp - \lambda_2$ is invertible in $\mathcal{R}$. Let

$$\omega : \mathcal{R} \to \mathbb{Q}(\lambda_1, \lambda_2)$$

be given by

$$\omega(\alpha) := \int_{\mathbb{P}^m_T} \alpha \cup \frac{kp - \lambda_1}{-lp - \lambda_2}.$$

Define a pairing in $\mathcal{R}$ as follows:

$$\langle a, b \rangle := \omega(a \cup b).$$

This is a perfect pairing. Let $(g_{rs}) := (\langle T_r, T_s \rangle)$ the matrix of this pairing and (g^{rs}) its inverse. Let $T^i = \sum_{j=0}^m g^{ij} T_j$ be the dual basis with respect to this pairing. Clearly

$$T^i = T_{m-i} \cdot \left(\frac{-lp - \lambda_2}{kp - \lambda_1} \right). \tag{22}$$

This implies that in $H^*(\mathbb{P}^m \times \mathbb{P}^m) \otimes \mathbb{Q}(\lambda_1, \lambda_2)$ we have

$$\sum_{i=1}^m T_i \otimes T^i = \Delta \cdot \left(1 \otimes \frac{-lp - \lambda_2}{kp - \lambda_1} \right)$$

where $\Delta = \sum_{i=0}^m T_i \otimes T_{m-i}$ is the class of the diagonal in $\mathbb{P}^m \times \mathbb{P}^m$. Consider now the following equivariant diagram:

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,n+1}(\mathbb{P}^m, d) & \xrightarrow{e_{n+1}} & \mathbb{P}^m \\ \downarrow{\scriptstyle \pi} & & \\ \overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d) & & \end{array}$$

where

$$e_{n+1}(C, x_1, x_2, ..., x_{n+1}, f) = f(x_{n+1})$$

$$\pi(C, x_1, ..., x_n, x_{n+1}, f) = (\tilde{C}, x_1, ..., x_n, f).$$

The curve $\tilde{C}$ is obtained from C after collapsing the unstable components. Define the equivariant bundles and their equivariant Euler classes:

$$V_d^+ := \pi_* e_{n+1}^*(\mathcal{O}(k))$$

$$V_d^- := R^1\pi_*(e_{n+1}^*(\mathcal{O}(-l)))$$

$$V_d := V_d^+ \oplus V_d^-$$

$$E_d^+ := \mathrm{Euler}_T(V_d^+)$$

$$E_d^- := \mathrm{Euler}_T(V_d^-)$$

$$E_d := \mathrm{Euler}_T(V_d) = E_d^+ E_d^-. \tag{23}$$

Note that $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)_T = \overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d) \times \mathbb{P}^\infty \times \mathbb{P}^\infty$. Let π_i be the projection map to the i-th factor of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)_T$. It is clear that:

$$(V_d^-)_T = V_d^+ \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^\infty}(-\lambda_1)$$

$$(V_d^-)_T = V_d^- \otimes \pi_3^* \mathcal{O}_{\mathbb{P}^\infty}(-\lambda_2)$$

$$E_d^+ = \sum c_u(V_d^+)(-\lambda_1)^u$$

$$E_d^- = \sum c_u(V_d^-)(-\lambda_2)^u. \tag{24}$$

3.2 Modified equivariant Gromov-Witten invariants and quantum cohomology.

For $i = 1, 2, ..., n$ let $\gamma_i \in \mathcal{R}$. Introduce the following modified integrals:

$$\tilde{I}_d(\gamma_1, ..., \gamma_n) := \int_{\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m,d)} \prod_{i=1}^n e_i^*(\gamma_i) E_d \in \mathbb{Q}(\lambda_1, \lambda_2) \tag{25}$$

and consequently the following equivariant potential:

$$\tilde{\Phi}(t_0, t_1, ..., t_m) := \sum_{n\geq 3} \sum_{d \geq 0} \frac{1}{n!} \tilde{I}_d(\gamma^{\otimes n}) \tag{26}$$

where $\gamma = t_0 + t_1 p + ... + t_m p^m$ and $t_i \in \mathbb{Q}(\lambda_1, \lambda_2)$.

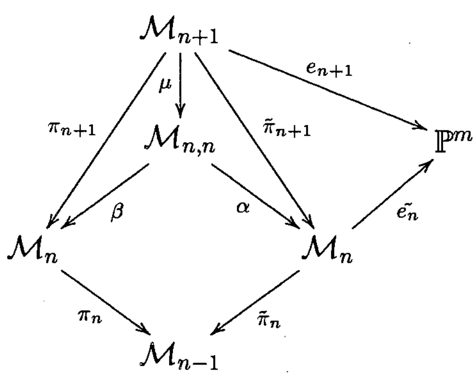
Remark 3.2.1 *Definitions (25) and (26) are slightly imprecise. The precise formulation would be that the integrals must be over $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)_T$. However, since the torus action on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)$ is trivial we can think of the standard integration with coefficients in $\mathbb{Q}(\lambda_1, \lambda_2)$.*

Consider the equivariant map that forgets the first point

$$\pi : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d) \to \overline{\mathcal{M}}_{0,n-1}(\mathbb{P}^m, d)$$

Lemma 3.2.1 $\pi^*(E_d) = E_d$.

Proof. Let $\mathcal{M}_k = \overline{\mathcal{M}}_{0,k}(\mathbb{P}^m, d)$ and $\mathcal{M}_{n,n} = \mathcal{M}_n \times_{\mathcal{M}_{n-1}} \mathcal{M}_n$. Consider the following equivariant commutative diagram:



The maps $\tilde{\pi}_{n+1}$ and π_{n+1} forget respectively the n-th and $n+1$-th marked points. We compute:

$$\pi_{n+1*}e^*_{n+1}\mathcal{O}(k) = \pi_{n+1*}\tilde{\pi}^*_{n+1}\tilde{e_n}^*\mathcal{O}(k) = \beta_*\mu_*\mu^*\alpha^*\tilde{e_n}^*\mathcal{O}(k). \tag{27}$$

But by the projection formula we have

$$\mu_*\mu^*\alpha^*\tilde{e}^*_n\mathcal{O}(k) = \alpha^*\tilde{e}^*_n\mathcal{O}(k) \otimes \mu_*(\mathcal{O}_{\mathcal{M}_{n+1}}). \tag{28}$$

Since the map μ is birational and $\mathcal{M}_{n+1}$ is normal

$$\mu_*(\mathcal{O}_{\mathcal{M}_{n+1}}) = \mathcal{O}_{\mathcal{M}_{n,n}},$$

therefore

$$\mu_*\mu^*\alpha^*\tilde{e_n^*}\mathcal{O}(k) = \alpha^*\tilde{e_n^*}\mathcal{O}(k). \tag{29}$$

The equation (27) becomes

$$\pi_{n+1_*}e_{n+1}^*\mathcal{O}(k) = \beta_*\alpha^*\tilde{e_n}^*\mathcal{O}(k) = \pi_n^*(\tilde{\pi}_{n*}\tilde{e_n^*}\mathcal{O}(k)). \tag{30}$$

The last equality follows from base extension properties since both maps π and $\tilde{\pi}_n$ are flat.

For the case of a negative line bundle we have

$$R^1\beta_*e_{n+1}^*\mathcal{O}(-l) = R^1\pi_{n+1_*}\tilde{\pi}_{n+1}^*\tilde{e_n}^*\mathcal{O}(-l) = R^1\pi_{n+1_*}\mu^*\alpha^*\tilde{e_n}^*\mathcal{O}(-l). \tag{31}$$

We now use the spectral sequence

$$R^p\beta_*(R^q\mu_*\mathcal{F}) \Longrightarrow R^{p+q}\pi_{n+1_*}\mathcal{F} \tag{32}$$

where $\mathcal{F}$ is a sheaf of $\mathcal{O}_{\mathcal{M}_{n+1}}$-modules. The map μ is birational. If we think of $\mathcal{M}_n$ as the universal map of $\mathcal{M}_{n-1}$, then the map μ has nontrivial fibers only over pairs of stable maps in $\mathcal{M}_n$ that represent the same special point (i.e. node or marked point) of a stable map in $\mathcal{M}_{n-1}$. These nontrivial fibers are isomorphic to $\mathbb{P}^1$. Since $\mathcal{F} = e_{n+1}^*\mathcal{O}(-l)$ we obtain $R^q\mu_*\mathcal{F} = 0$ for $q > 0$. It follows that this spectral sequence degenerates, giving

$$R^1\pi_{n+1_*}e_{n+1}^*\mathcal{O}(-l) = R^1\beta_*\mu_*\mu^*\alpha^*\tilde{e_n}^*\mathcal{O}(-l). \tag{33}$$

Now we proceed as in (29) to conclude

$$R^1\pi_{n+1_*}e_{n+1}^*\mathcal{O}(-l) = \pi_n^*(R^1\tilde{\pi}_{n*}\tilde{e_n^*}\mathcal{O}(-l)). \tag{34}$$

The lemma is proven.†

The above lemma is essential in proving that the modified correlators satisfy the same properties that the usual G-W invariants do. We now list and prove these properties.

Point mapping axiom. Let $d = 0$. Then $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, 0) = \overline{\mathcal{M}}_{0,n} \times \mathbb{P}^m$ and all the evaluation maps equal the projection to the second factor.

$$\tilde{I}_0(\gamma_1, ..., \gamma_n) = \int_{\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m,0)_T} \prod_{i=1}^{n} e_i^*(\gamma_i) \left(\frac{kp - \lambda_1}{-lp - \lambda_2} \right) =$$

$$= \int_{\overline{\mathcal{M}}_{0,n} \times \mathbb{P}^m} p_2^*(\gamma_1 \cup ... \cup \gamma_n) \cup \frac{kp - \lambda_1}{-lp - \lambda_2} = \int_{p_{2*}(\overline{\mathcal{M}}_{0,n} \times \mathbb{P}^m)} \gamma_1 \cup ... \cup \gamma_n p_{2*} \left(\frac{kp - \lambda_1}{-lp - \lambda_2} \right).$$

If $n > 3$ the dimension of $\overline{\mathcal{M}}_{0,n}$ is bigger than one. Therefore $p_{2*}(\overline{\mathcal{M}}_{0,n} \times \mathbb{P}^m) = 0$ i.e. $\tilde{I}_0(\gamma_1, ..., \gamma_n) = 0$. For $n = 3$, $\overline{\mathcal{M}}_{0,3}(\mathbb{P}^m, 0) = \mathbb{P}^m$ therefore

$$\tilde{I}_0(\gamma_1, \gamma_2, \gamma_3) = \int_{\mathbb{P}^m} \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \left(\frac{kp - \lambda_1}{-lp - \lambda_2} \right).$$

Fundamental class property. Let $\gamma_n = 1$ and $d \neq 0$. Consider the following equivariant morphism:

$$\pi : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d) \to \overline{\mathcal{M}}_{0,n-1}(\mathbb{P}^m, d)$$

which forgets the last marked point. Using Lemma 3.2.1 we obtain:

$$e_1^*(\gamma_1) \cup ... \cup e_{n-1}^*(\gamma_{n-1}) \cup e_n^*(1) E_d = \pi^*(e_1^*(\gamma_1) \cup ... \cup e_{n-1}^*(\gamma_{n-1}) E_d).$$

Therefore:

$$\tilde{I}_d(\gamma_1, ..., \gamma_{n-1}, 1) = \int_{\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m,d)} \pi^*(e_1^*(\gamma_1) \cup ... \cup e_{n-1}^*(\gamma_{n-1}) E_d) =$$

$$= \int_{\pi_*(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m,d))} e_1^*(\gamma_1) \cup ... \cup e_{n-1}^*(\gamma_{n-1}) E_d = 0.$$

The last equality is because the fibers of π are positive dimensional. If $d = 0$, by the first property we know that the integral is zero unless $n = 3$. In that case: $\tilde{I}_0(\gamma_1, \gamma_2, 1) = \langle \gamma_1, \gamma_2 \rangle$.

Divisor property. Let $\gamma_1 = tp$ be a divisor with $t \in \mathbb{Q}(\lambda_1, \lambda_2)$ and $d \neq 0$. Then

$$\tilde{I}_d(\gamma_1, \gamma_2, ..., \gamma_n) = (td) \cdot \tilde{I}_d(\gamma_2, ..., \gamma_n). \tag{35}$$

The proof of the divisor property in [11] works here as well. The only modification needed is to use Lemma 3.2.1.

Let

$$\tilde{\Phi}_{ijk} = \frac{\partial^3 \tilde{\Phi}}{\partial t_i \partial t_j \partial t_k}.$$

For $a, b \in \mathcal{R}$ define

$$a *_V b := \sum_{r=0}^{m} \tilde{\Phi}_{ijr} T^r. \tag{36}$$

Theorem 3.2.1 *$QH_T^*\mathbb{P}^m := (\mathcal{R}, *_V)$ is a commutative, associative algebra with unit T_0.*

Proof. Note that

$$\tilde{\Phi}_{ijk} = \sum_{n \geq 0} \sum_{d \geq 0} \frac{1}{n!} \tilde{I}_d(T_i, T_j, T_k, \gamma^{\otimes n}).$$

The commutativity follows from the symmetry of the new integrals. T_0 is the unit due to the fundamental class property for the modified Gromov-Witten invariants. We now turn our attention to proving the associativity. We proceed as in Theorem 4 in [11]. We have

$$(T_i *_V T_j) *_V T_k = \sum \sum \tilde{\Phi}_{ije} g^{ef} \tilde{\Phi}_{fkl} g^{ld} T_d$$

$$T_i *_V (T_j *_V T_k) = \sum \sum \tilde{\Phi}_{jke} g^{ef} \tilde{\Phi}_{fil} g^{ld} T_d.$$

Since the matrix (g^{ld}) is nonsingular, $(T_i *_V T_j) *_V T_k = T_i *_V (T_j *_V T_k)$ is equivalent to

$$\sum_{e,f} \tilde{\Phi}_{ije} g^{ef} \tilde{\Phi}_{fkl} = \sum_{e,f} \tilde{\Phi}_{jke} g^{ef} \tilde{\Phi}_{fil}. \tag{37}$$

Equation (37) is called **the WDVV equation for the modified potential** $\tilde{\Phi}$. To prove this equation we need a lemma. Before stating and proving it, we mention some divisors on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)$. Let A and B be a partition of the set of marked points and $d = d_1 + d_2$. Consider the closure $D = D(A, B, d_1, d_2)$ in $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)$ of stable maps (C, x_i, f) of the following type. The source curve is a union $C = C_1 \cup C_2$ with $C_i \simeq \mathbb{P}^1$ two curves meeting transversally at a node x. The marked points corresponding to A are on the curve C_1 and those corresponding to B are on C_2. The restriction of f to C_i has degree d_i. Then $D = D(A, B, d_1, d_2)$ is a divisor and

$$D = D(A, B, d_1, d_2) = \overline{\mathcal{M}}_{0,|A|+1}(\mathbb{P}^m, d_1) \times_{\mathbb{P}^m} \overline{\mathcal{M}}_{0,|B|+1}(\mathbb{P}^m, d_2) \tag{38}$$

where the extra marked point comes from the node x [11]. Let e_x and $\tilde{e}_x$ be the evaluation maps at the additional marked point in $\overline{\mathcal{M}}_{0,|A|+1}(\mathbb{P}^m, d_1)$ and $\overline{\mathcal{M}}_{0,|B|+1}(\mathbb{P}^m, d_2)$.

Lemma 3.2.2 *For any classes $\gamma_1, ..., \gamma_n$ in $\mathcal{R}$:*

$$\int_{D(A,B,d_1,d_2)} \prod_{i=1}^{n} e_i^*(\gamma_i) E_d = \sum_{a=0}^{m} \int_{\overline{\mathcal{M}}_{0,|A|+1}(\mathbb{P}^m,d)} \prod_{i \in A} e_i^*(\gamma_i) e_x^*(T_a) E_{d_1} \times$$
$$\times \int_{\overline{\mathcal{M}}_{0,|B|+1}(\mathbb{P}^m,d)} \prod_{j \in B} e_j^*(\gamma_j) \tilde{e}_x^*(T^a) E_{d_2}.$$

Proof. This lemma is the analogue of the Lemma 16 in [11]. The proof needs a minor modification.

Let $\iota : D(A,B,d_1,d_2) \to M_1 \times M_2 = \overline{\mathcal{M}}_{0,|A|+1}(\mathbb{P}^m, d_1) \times \overline{\mathcal{M}}_{0,|B|+1}(\mathbb{P}^m, d_2)$; $\alpha : D \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)$; $\mu = (e_x, \tilde{e}_x) : M_1 \times M_2 \to \mathbb{P}^m \times \mathbb{P}^m$ and $\delta : \mathbb{P}^m \to \mathbb{P}^m \times \mathbb{P}^m$ the diagonal embedding. Consider the following fibre square:

$$\begin{array}{ccc} D & \xrightarrow{\iota} & M_1 \times M_2 \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \mu} \\ \mathbb{P}^m & \xrightarrow{\delta} & \mathbb{P}^m \times \mathbb{P}^m \end{array}$$

In the above diagram ν is the evaluation map at the meeting point x in D. Consider the normalization sequence at x:

$$0 \to \mathcal{O}_C \to \mathcal{O}_{C'} \oplus \mathcal{O}_{C''} \to \mathcal{O}_x \to 0. \tag{39}$$

Twist it by $f^*(\mathcal{O}_{\mathbb{P}^m}(k))$ and take the cohomology sequence. We obtain:

$$0 \to H^0(C, f^*(\mathcal{O}_{\mathbb{P}^m}(k))) \to H^0(C', f|^*_{C'}(\mathcal{O}_{\mathbb{P}^m}(k))) \oplus H^0(C'', f|^*_{C''}(\mathcal{O}_{\mathbb{P}^m}(k))) \to$$

$$\to \mathcal{O}_{f(x)}(k) \to 0.$$

On D this sequence implies:

$$\alpha^*(E_d^+)\nu^*(kp - \lambda_1) = \iota^*(E_{d_1}^+ \times E_{d_2}^+). \tag{40}$$

We now twist the exact sequence (39) by $f^*(\mathcal{O}_{\mathbb{P}^m}(-l))$ and take cohomology to obtain:

$$\alpha^*(E_d^-) = \iota^*(E_{d_1}^- \times E_{d_2}^-)\nu^*(-lp - \lambda_2). \tag{41}$$

By combining equations (40) and (41) we obtain the restriction of E_d in the divisor $D(A,B,d_1,d_2)$:

$$\alpha^*(E_d) = \iota^*(E_{d_1} \times E_{d_2})\nu^* \left(\frac{-lp - \lambda_2}{kp - \lambda_1} \right). \tag{42}$$

Therefore

$$\int_{D(n_1,n_2,d_1,d_2)} \prod_{i=1}^{n} e_i^*(\gamma_i) E_d = \int_{M_1 \times M_2} \prod_{i=1}^{n} e_i^*(\gamma_i) E_{d_1} E_{d_2} \mu^* \left(1 \otimes \frac{-lp - \lambda_2}{kp - \lambda_1} \right) \mu^*(\Delta) =$$

$$= \int_{M_1 \times M_2} \prod_{i=1}^{n} e_i^*(\gamma_i) E_{d_1} E_{d_2} \mu^*(\sum_a T_a \otimes T^a) =$$

$$= \sum_{a=0}^{m} (\int_{\overline{\mathcal{M}}_{0,|A|+1}(\mathbb{P}^m,d)} \prod_{i=1}^{n_1} e_i^*(\gamma_i) e_x^*(T_a) E_{d_1}) \times (\int_{\overline{\mathcal{M}}_{0,|B|+1}(\mathbb{P}^m,d)} \prod_{j=1}^{n_2} e_j^*(\gamma_j) \tilde{e}_x^*(T^a) E_{d_2}).$$

The lemma is proven.†

We can now complete the proof of the associativity. Let q, r, s, t be four different integers in $\{1, 2, ...n\}$. Let $D(q, r, s, t) = \sum D(A, B, d_1, d_2)$ where the sum is over all partitions $A \cup B = \{1, 2, ..., n\}$ such that $q, r \in A$ and $s, t \in B$, and over all d_1 and d_2 that sum to d. There exists an equivariant morphism:

$$\pi : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d) \to \overline{\mathcal{M}}_{0,4}(\{pt\}, 0) = \mathbb{P}^1$$

that forgets the map and all the marked points but q, r, s, t. Obviously

$$[D(q, r, s, t)] = [D(q, s, r, t)]$$

in $\overline{\mathcal{M}}_{0,4}(\{pt\}, 0)$. Pulling back these linearly equivalent divisors in $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)$ we obtain $[D(q, r, s, t)] = [D(q, s, r, t)]$. Now integrate the class

$$\prod_{i=1}^{n-4} (e_i^*(\gamma)) \cup e_{n-3}^*(T_i) \cup e_{n-2}^*(T_j) \cup e_{n-1}^*(T_k) \cup e_n^*(T_l) \cup E_d$$

over $D(q, r, s, t)$ and use Lemma 3.2.2 to obtain the associativity.†

If we restrict $\tilde{\Phi}_{ijk}$ to the divisor classes $\gamma = tp$, and use the divisor property for the modified Gromov-Witten invariants, we obtain the small product:

$$T_i *_V T_j := T_i \cup T_j + \sum_{d>0} q^d \sum_{k=0}^{m} \tilde{I}_d(T_i, T_j, T_k) T^k. \qquad (43)$$

Here $q = e^t$. We extend this product to $\mathcal{R} \otimes_{\mathbb{Q}} \mathbb{Q}[q]$ to obtain the small equivariant quantum cohomology ring $SQH_T^* \mathbb{P}^m$.

We will use $*_V$ to denote both the small and the big quantum product. The difference will be clear from the context.

Remark 3.2.2 • *Equation (42) and Lemma 3.2.1 are the basis for translating properties of pure Gromov-Witten or gravitational correlators to the modified ones.*

- *One can see from (22) and (24) that if V is a pure negative line bundle, all ingredients in (43) are polynomials in λ_1 and λ_2. Therefore the nonequivariant limit of this product exists. An example of this situation is treated in the last chapter.*

3.3 Modified equivariant gravitational descendants.

Let $\mathcal{L}_i$ be the universal cotangent line at the i-th marked point. They can be defined as follows. Let $\pi_{n+1} : \overline{\mathcal{M}}_{0,n+1}(\mathbb{P}^m, d) \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)$ be the morphism that forgets the last marked point and contracts unstable components. Let s_i for $i = 1, 2, ..., n$ be the sections of the marked point. Then the cotangent line bundle at the i-th marked point is defined to be

$$\mathcal{L}_i := s_i^*(\omega_{\pi_{n+1}}). \tag{44}$$

We define the modified gravitational descendants to be

$$\tilde{I}_d(\tau_{k_1}\gamma_1, ..., \tau_{k_n}\gamma_n) := \int_{\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m,d)} \prod_{i=1}^{n} c_1^{k_i}(\mathcal{L}_i) \cup e_i^*(\gamma_i) E_d.$$

A combination of arguments used in proving the topological recursion relations for pure Gromov-Witten theory (see Lemma 10.2.2 in [8]) and in the proof of Lemma 3.2.2 (see Remark 3.2.2) can be used to show this:

Theorem 3.3.1 *The following modified topological recursion relations hold:*

$$\tilde{I}_d(\tau_{k_1}\gamma_1, \tau_{k_2}\gamma_2, \tau_{k_3}\gamma_3, \prod_{i\in I}\tau_{s_i}\omega_i) =$$

$$\sum \tilde{I}_{d_1}(\tau_{k_1-1}\gamma_1, \prod_{i\in I_1}\tau_{s_i}\omega_i, T_a)\tilde{I}_{d_2}(T^a, \tau_{k_2}\gamma_2, \tau_{k_3}\gamma_3, \prod_{i\in I_2}\tau_{s_i}\omega_i) \tag{45}$$

where the sum is over all splittings $d_1 + d_2 = d$ and partitions $I_1 \cup I_2 = I$ and over all indices a.

3.4 Equivariant quantum differential equations.

Consider the system of first order diferential equations on the big quantum cohomology ring $QH_T^*(\mathbb{P}^m)$

$$\hbar\frac{\partial}{\partial t_i} = T_i *_V : i = 1, ..., m. \tag{46}$$

Theorem 3.4.1 *The space of solutions of these equations has the following basis:*

$$s_a = T_a + \sum_{j=0}^{m}\sum_{n=0}^{\infty}\sum_{d=0}^{\infty}\sum_{k=0}^{\infty}\frac{\hbar^{-(k+1)}}{n!}\tilde{I}_d(\tau_k T_a, T_j, \gamma^{\otimes n})T^j =$$

$$= T_a + \sum_{j=0}^{m}\sum_{d=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{n!}\tilde{I}_d(\frac{T_a}{\hbar - c}, T_j, \gamma^{\otimes n})T^j.$$

Proof. We have

$$\hbar\frac{\partial s_a}{\partial t_i} = \sum_{j=0}^{m}\sum_{n=0}^{\infty}\sum_{d=0}^{\infty}\sum_{k=0}^{\infty}\frac{\hbar^{-k}}{n!}\tilde{I}_d(\tau_k T_a, T_j, T_i, \gamma^{\otimes n})T^j.$$

On the other hand

$$T_i * s_a = T_i * T_a + \sum_{j=0}^{m}\sum_{n_1=0}^{\infty}\sum_{d=0}^{\infty}\sum_{k=0}^{\infty}\frac{\hbar^{-(k+1)}}{n_1!}\tilde{I}_d(\tau_k T_a, T_j, \gamma^{\otimes n_1})(T_i * T^j) =$$

$$\sum_{n,d,e}\frac{1}{n!}\tilde{I}_d(T_i, T_a, T_e, \gamma^{\otimes n})T^e + \sum_{j=0}^{m}\sum_{n_1=0}^{\infty}\sum_{k=0}^{\infty}\sum_{d_1}\frac{\hbar^{-(k+1)}}{n_1!}\tilde{I}_{d_1}(\tau_k T_a, T_j, \gamma^{\otimes n_1})\times$$

$$\times \sum_{n_2,d_2,e} \frac{1}{n_2!} \tilde{I}_{d_2}(T_i, T^j, T_e, \gamma^{\otimes n_2}) T^e.$$

We use the topological recursion relations (45) and calculations similar to Proposition 10.2.1 of [8] to obtain:

$$\sum_{j=0}^{m} \sum_{n_1=0}^{\infty} \sum_{k=0}^{\infty} \frac{\hbar^{-(k+1)}}{n_1!} \tilde{I}_{d_1}(\tau_k T_a, T_j, \gamma^{\otimes n_1}) \times \sum_{n_2,d_2,e} \frac{1}{n_2!} \tilde{I}_{d_2}(T_i, T^j, T_e, \gamma^{\otimes n_2}) T^e =$$

$$= \sum_{e=0}^{m} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{\hbar^{-(k+1)}}{n!} \tilde{I}_d(\tau_{k+1} T_a, T_i, T_e, \gamma^{\otimes n}) T^e.$$

Substituting we get:

$$T_i * s_a = \sum_{e=0}^{m} \sum_{n=0}^{\infty} \sum_{d=0}^{\infty} \sum_{k=0}^{\infty} \frac{\hbar^{-k}}{n!} \tilde{I}_d(\tau_k T_a, T_e, T_i, \gamma^{\otimes n}) T^e.$$

The theorem is proven.†

Notation. We will use the following notation

$$\langle \tau_{k_1}\gamma_1, ..., \tau_{k_n}\gamma_n \rangle_d := \tilde{I}_d(\tau_{k_1}\gamma_1, ..., \tau_{k_n}\gamma_n). \tag{47}$$

We can restrict the section s_a to $\gamma \in H^0(\mathbb{P}^m) \oplus H^2(\mathbb{P}^m)$ to obtain solutions $\tilde{s}_a$ of

$$\hbar \frac{\partial}{\partial t_i} = T_i *_V : i = 0, 1.$$

A calculation similar to Proposition 10.2.3 in [8] shows that

$$\tilde{s}_a = e^{\frac{t_0 + pt_1}{\hbar}} \cup T_a + \sum_{d=1}^{\infty} \sum_{j=0}^{m} q^d \langle \frac{e^{\frac{t_0+pt_1}{\hbar}} \cup T_a}{\hbar - c}, T_j \rangle_d T^j. \tag{48}$$

Consider now the following $\mathcal{R}[[t_0, t_1, q]]$-valued function

$$\tilde{J}_V = \sum_{a=0}^{m} \langle \tilde{s}_i, 1 \rangle T^a \tag{49}$$

where the pairing is the one defined in the introduction. Substituting (48) in (49) we obtain:

$$\tilde{J}_V = \sum_a \langle e^{\frac{t_0+pt_1}{\hbar}} \cup T_a, 1\rangle T^a + \sum_{d=1}^{\infty}\sum_{a=0}^{m} q^d \langle \frac{e^{\frac{t_0+pt_1}{\hbar}} \cup T_a}{\hbar - c}, 1\rangle_d T^a.$$

Using the projection formula we obtain:

$$\tilde{J}_V = \exp\left(\frac{t_0 + pt_1}{\hbar}\right)\left(1 + \sum_{d>0} q^d e_{1*}\left(\frac{E_d}{\hbar(\hbar - c)}\right) \cup \left(\frac{-lp - \lambda_1}{kp - \lambda_2}\right)\right). \tag{50}$$

A nonequivariant counterpart of $\tilde{J}_V$ is desirable and the following lemma provides that.

Lemma 3.4.1 *$\tilde{J}_V$ has a nonequivariant limit.*

Proof. Consider the following exact sequence on $\mathcal{M}_{0,1}(\mathbb{P}^m, d)$:

$$0 \to V_d' \to V_d^+ \xrightarrow{\alpha} {e_1}^*(\mathcal{O}(k)) \to 0$$

where for a section $\sigma \in H^0(C, f^*((\mathcal{O}(k)))$ we have $\alpha(\sigma) = \sigma(x_1)$. Let

$$E_d' = \mathrm{Euler}_T(V_d'). \tag{51}$$

Then

$$E_d' \mathrm{Euler}_T({e_1}^*(\mathcal{O}(k))) = E_d^+. \tag{52}$$

We compute

$$e_{1*}\left(\frac{E_d}{\hbar(\hbar - c)}\right) \cup \left(\frac{-lp - \lambda_1}{kp - \lambda_2}\right) =$$

$$\sum_r \left(\int_{\mathbb{P}^m} e_{1*}\left(\frac{E_d}{\hbar(\hbar - c)}\right) \cup \left(\frac{-lp - \lambda_1}{kp - \lambda_2}\right) p^{m-r}\right) p^r =$$

$$\sum_r \left(\int_{\overline{\mathcal{M}}_{0,1}(\mathbb{P}^m, d)} \frac{e_1^*(p^{m-r})}{\hbar(\hbar - c)} e_1^* \left(\frac{-lp - \lambda_1}{kp - \lambda_2}\right) E_d\right) p^r.$$

We substitute (52) in the last line to obtain

$$e_{1*}\left(\frac{E_d}{\hbar(\hbar-c)}\right)\cup\left(\frac{-lp-\lambda_1}{kp-\lambda_2}\right)=\sum_r\left(\int_{\overline{\mathcal{M}}_{0,1}(\mathbb{P}^m,d)}\frac{e_1^*(p^{m-r})}{\hbar(\hbar-c)}e_1^*(-lp-\lambda_1)E'_dE_d^-\right)p^r.$$

We finally obtain

$$\tilde{J}_V=\exp\left(\frac{t_0+pt_1}{\hbar}\right)\left(1+\sum_{d>0}q^d e_{1*}\left(\frac{E'_dE_d^-}{\hbar(\hbar-c)}\right)\cup(-lp-\lambda_1)\right). \qquad (53)$$

In this formula we can see clearly that $\tilde{J}_V$ has a nonequivariant limit since all the objects are polynomials in λ. The lemma is proven.†

Let

$$J_V=\exp\left(\frac{t_0+pt_1}{\hbar}\right)\left(1+\sum_{d>0}q^d e_{1*}\left(\frac{E'_dE_d^-}{\hbar(\hbar-c)}\right)\cup(-lp)\right)$$

be the nonequivariant limit of $\tilde{J}_V$. In the next section we will see that J_V is a natural object.

3.5 Behaviour of the J-function

Assume that $\mathbb{P}^m$ is embedded in a smooth variety X with normal bundle $V=\mathcal{O}(-l)$ for some $l>0$. Let i denote the embedding.

Lemma 3.5.1 *The class $p^{m-1}=$ [line] of a line in $\mathbb{P}^m$ is an edge of the Mori cone MX of X.*

Proof. Let $C_1, C_2, ...C_n$ be irreducible curves in X and

$$[\text{line}]=[C_1]+...+[C_n].$$

We will show that $C_i \subset \mathbb{P}^m$ for all i which implies that $[l] = [C_i]$ for some i. Let $I = \{i : C_i \subset \mathbb{P}^m\}$ and $J = \{1, 2, ..., n\} - I$. We assume that J is nonempty. If

$$[\text{line}] - \sum_{i \in I} [C_i]$$

has nonpositive degree in $\mathbb{P}^m$, we intersect with an ample divisor in X to see that

$$[\text{line}] - \sum_{i \in I} [C_i] = \sum_{i \in J} [C_i]$$

is impossible. Otherwise, we intersect with $[\mathbb{P}^m]$ to get the same contradiction. All the curves C_i lie in $\mathbb{P}^m$. The lemma is proven.†

Lemma 3.5.2 *Let* $[C] = d[\text{line}]$ *be the homology class of a curve in* $\mathbb{P}^m$. *Then* $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d) = \overline{\mathcal{M}}_{0,n}(X, i_*([C]))$.

Proof. Let $[\mu] = (C, x_1, ...x_n, f : C \to X \in \overline{\mathcal{M}}_{0,n}(X, d)$ and $f(C) = C_1 \cup C_2 \cup ... \cup C_n$ be the irreducible decomposition. By Lemma 3.5.1, $[C_i] = m_i[\text{line}]$ in $H_2(\mathbb{P}^m, \mathbb{Z})$ for some $m_i > 0$. Therefore $C_i \cdot [\mathbb{P}^m] = -lm_i$ i.e. $C_i \subset \mathbb{P}^m$. It follows that f factors through $\mathbb{P}^m$ and therefore $(C, x_1, ...x_n, f : C \to \mathbb{P}^m) \in \overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)$.

On the other hand, an argument similar to the one used for multiple covers of rational curves in a Calabi-Yau threefold (Section 7.4.4. in [8]), shows that $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)$ is a component of $\overline{\mathcal{M}}_{0,n}(X, d)$. When combined the two arguments imply the lemma.†

Let $\overline{\mathcal{M}} := \overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d) = \overline{\mathcal{M}}_{0,n}(X, d)$. Recall from section 2 that on $\overline{\mathcal{M}}$ we have a vector bundle $V_d = R^1\pi_*(e_{n+1}^*(\mathcal{O}(-l)))$ and its top Chern class $E_d = c_{top}(V_d)$.

Lemma 3.5.3 $[\overline{\mathcal{M}}_{0,n}(X, d)]^{\text{virt}} = E_d \cdot [\overline{\mathcal{M}}]$.

Proof. We recall that the map $\pi : \overline{\mathcal{M}}_{0,n}(X,d) \to \overline{\mathcal{M}}_{0,n-1}(X,d)$ satisfies

$$\pi^*([\overline{\mathcal{M}}_{0,n-1}(X,d)]^{\text{virt}}) = [\overline{\mathcal{M}}_{0,n}(X,d)]^{\text{virt}} \tag{54}$$

Also recall from Lemma 3.2.1 that the map π in (54) for $X = \mathbb{P}^m$ satisfies the relation: $\pi^*(E_d) = E_d$. It follows that we only need to prove this lemma in the case $n = 0$.

The moduli space $\overline{\mathcal{M}}_{0,0}(X,d)$ is described locally at $(C, f : C \to X)$ by the following tangent-obstruction sequence:

$$0 \to \text{Ext}^0(\Omega_C, \mathcal{O}_C) \to H^0(f^*TX) \to \mathcal{T} \xrightarrow{\eta} \text{Ext}^1(\Omega_C, \mathcal{O}_C) \to$$

$$\to H^1(f^*TX) \to \Upsilon \to 0$$

where $\mathcal{T}$ is the tangent space and Υ is the obstruction space. We are continuing our earlier convention of describing sheaves by their fibers. Consider the following short exact sequence on $\mathbb{P}^m$

$$0 \to T\mathbb{P}^m \to TX|_{\mathbb{P}^m} \to \mathcal{O}(-l) \to 0.$$

Taking its corresponding cohomology sequence we obtain

$$H^0(f^*TX) \simeq H^0(f^*T\mathbb{P}^m)$$

and

$$H^1(f^*TX) \simeq H^1(f^*(\mathcal{O}(-l)).$$

Now the tangent-obstruction sequence reads

$$0 \to \text{Ext}^0(\Omega_C, \mathcal{O}_C) \to H^0(f^*T\mathbb{P}^m) \to \mathcal{T} \xrightarrow{\eta} \text{Ext}^1(\Omega_C, \mathcal{O}_C) \to$$

$$\to H^1(f^*(\mathcal{O}(-l)) \to \Upsilon \to 0.$$

The tangent-obstruction sequence for $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)$ implies that η is surjective. Thus $V_d \simeq \Upsilon$. Since the obstruction sheaf Υ is locally free and $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)$ is a smooth stack, Proposition 5.6 in [3] implies the lemma.† Let $m_1 = p^{m-1}, m_2, ..., m_r$ be a basis of MX. Denote ι the embedding of $\mathbb{P}^m$ in X. It gives rise to

$$\iota^* : H^*X \to H^*\mathbb{P}^m.$$

We can extend it to a homomorphism:

$$\iota^* : H^*X[[t_i', q_i]] \to H^*\mathbb{P}^m[[t_0, t_1, q]] \tag{55}$$

by defining $\iota^*(t_i') = t_i : i = 0, 1 : \iota^*(t_i) = 0 : i > 1 : \iota^*(q_1) = q : \iota^*(q_i) = 0 : i > 1$. In (55) we have $q_i = e^{t_i'}$ and $q = e^{t_1}$.

Let $D_1, ..., D_r$ be generators of the cone of nef divisors of X. It is shown in [13] that the generator of the quantum $\mathcal{D}$-module for the pure Gromov-Witten theory of X is

$$J_X = \exp\left(\frac{t_0 + tD}{\hbar}\right)\left(1 + \sum_{\beta \in MX} q^\beta e_{1*}\left(\frac{[\overline{\mathcal{M}}_{0,1}(X,\beta)]^{\text{virt}}}{\hbar(\hbar - c)}\right)\right)$$

Here, $q^\beta = q_1^{d_1} \cdot ... \cdot q_r^{d_r}$ where $\beta = d_1 m_1 + ... + d_r m_r$.

The following results show that J-function behaves nicely.

Theorem 3.5.1 $\iota^*(J_X) = J_V$.

Proof. Consider the following fiber diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,1}(\mathbb{P}^m, d) & \xrightarrow{\tilde{\iota}} & \overline{\mathcal{M}}_{0,1}(X, d) \\ \big\downarrow e_1 & & \big\downarrow e_1 \\ \mathbb{P}^m & \xrightarrow{\iota} & X \end{array}$$

The moduli spaces $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^m, d)$ and $\overline{\mathcal{M}}_{0,1}(X, d)$ are the same and $\tilde{\iota}$ is the identity map. Notice that for $\beta = dm_1$

$$\iota^*([\overline{\mathcal{M}}_{0,1}(X,\beta)]^{\mathrm{virt}}) = E_d.$$

By excess intersection theory:

$$\iota^*(e_{1*}\left(\frac{[\overline{\mathcal{M}}_{0,1}(X,\beta)]^{\mathrm{virt}}}{\hbar(\hbar - c)}\right)) = e_{1*}\left(\frac{E_d}{\hbar(\hbar - c)}\right) \cup \mathrm{Euler}(V).$$

The theorem is proven.†

Now, let X be a smooth hypersurface in $\mathbb{P}^m$ given by a section of $V = \mathcal{O}(k)$. If ι is the embeding of X in $\mathbb{P}^m$ and $\dim X > 2$, then it is shown in [8] that

$$\iota_*(J_X) = \mathrm{Euler}(\mathcal{O}(k)) J_V.$$

Combining these two results, we can prove the following theorem.

Theorem 3.5.2 *Let X be a smooth hypersurface of degree k in $\mathbb{P}^m$ and Y a smooth projective variety containing X such that $\mathcal{N} = \mathcal{N}_{X/Y} = \iota^*(\mathcal{O}(-l))$. Assume that MX is a face of MY and that if $C \subset Y$ is an irreducible curve with $[C] \in MX$ then $C \subset X$. Furthermore suppose that dim $X > 2$ (i.e. $m > 3$). Let j be the embedding of X in Y. Let $V = \mathcal{O}(k) \oplus \mathcal{O}(-l)$ on $\mathbb{P}^m$ and j^* be the map constructed similarly to ι^* in (55). Then*

$$\iota_*(j^*(J_Y)) = \mathrm{Euler}(\mathcal{O}(k)) J_V.$$

Proof. Let p be the hyperplane class in $\mathbb{P}^m$. The assumption $\dim X > 2$ implies that $H^2 X$ is generated by $\iota^*(p)$. Let $\{D_1 = i^*(p), D_2, ..., D_r\}$ be a set of generators of the cone of nef divisors of Y and $\{\beta_1, \beta_2, ...\beta_r\}$ generators of the Mori cone MY of Y. Here β_1 is a generator of MX. We establish the notation: $tD := t_1D_1 + ... + t_rD_r; d = (d_1, d_2, ..., d_r); q^d = q_1^{d_1} \cdot ... \cdot q_r^{d_r}$. Consider the following diagram:

$$\begin{array}{ccccc} \overline{\mathcal{M}}_{0,1}(Y, j_*(\beta)) & \stackrel{\tilde{j}}{\leftarrow} & \overline{\mathcal{M}}_{0,1}(X, \beta) & \stackrel{\tilde{\iota}}{\rightarrow} & \overline{\mathcal{M}}_{0,1}(\mathbb{P}^m, d) \\ \downarrow e_1 & & \downarrow e_1 & & \downarrow e_1 \\ Y & \stackrel{j}{\leftarrow} & X & \stackrel{\iota}{\rightarrow} & \mathbb{P}^m \end{array}$$

where the square on the left is a fibre square. An argument completely similar to Lemma 3.5.2 implies that

$$\overline{\mathcal{M}}_{0,1}(X, \beta) = \overline{\mathcal{M}}_{0,1}(Y, j_*(\beta)).$$

Let $\mathcal{M}(Y, \beta_1) := [\overline{\mathcal{M}}_{0,1}(Y, j_*(\beta))]^{\text{virt}}$ and $\mathcal{M}(X, \beta_1) := [\overline{\mathcal{M}}_{0,1}(X, \beta_1)]^{\text{virt}}$. We repeatedly use the projection formula:

$$\iota_*(j^*(J_Y)) = \iota_*\left(\exp\left(\frac{t_0 + t_1\iota^*(p)}{\hbar}\right)\sum_{d_1=0}^{\infty} q_1^{d_1} j^* e_{1*}\left(\frac{\mathcal{M}(Y, d_1\beta_1)}{\hbar(\hbar - c)}\right)\right)$$

$$= \iota_*\left(\exp\left(\frac{t_0 + t_1\iota^*(p)}{\hbar}\right)\sum_{d_1=0}^{\infty} q_1^{d_1} \iota^*(-lp) \cup e_{1*}\tilde{j}^*\left(\frac{\mathcal{M}(Y, d_1\beta_1)}{\hbar(\hbar - c)}\right)\right)$$

$$= \exp\left(\frac{t_0 + t_1 p}{\hbar}\right)\left(kp + \sum_{d_1=1}^{\infty} q_1^{d_1}(-lp) \cup \iota_* e_{1*}\tilde{j}^*\left(\frac{\mathcal{M}(Y, d_1\beta_1)}{\hbar(\hbar - c)}\right)\right)$$

$$= \exp\left(\frac{t_0 + t_1 p}{\hbar}\right)\{kp + \sum_{d_1=1}^{\infty} q_1^{d_1}(-lp) \cup e_{1*}\tilde{\iota}_*\tilde{j}^*\left(\frac{\mathcal{M}(Y, d_1\beta_1)}{\hbar(\hbar - c)}\right)\}. \tag{56}$$

The equality in the second row follows from excess intersection theory in the left square.

We now make use of an argument used in [26]. Recall that :

$$\overline{\mathcal{M}}_{0,1}(X, \beta) = \overline{\mathcal{M}}_{0,1}(Y, j_*(\beta)).$$

Denote this moduli space by $\overline{\mathcal{M}}$. Looking at $\overline{\mathcal{M}}$ in two different ways (i.e. considering the moduli problems of maps to X and Y) we conclude that there are two obstruction theories in $\overline{\mathcal{M}}$ which differ exactly by the bundle $R^1\tilde{\pi}_* e_2^*(\mathcal{N})$. It is shown in [26] that under these conditions:

$$\tilde{j}^*(\mathcal{M}(Y,\beta_1) = c_{\text{top}}(R^1\pi_* e_2^*(\mathcal{N})) \cap \mathcal{M}(X,\beta_1)$$

where

$$\pi : \overline{\mathcal{M}}_{0,2}(X,\beta) \to \overline{\mathcal{M}}_{0,1}(X,\beta)$$

is the map that forgets the second marked point. Consider the following commutative diagram:

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,2}(X,\beta) & \xrightarrow{e_2} & X \\ \downarrow{\scriptstyle \tilde{\iota}} & & \downarrow{\scriptstyle \iota} \\ \overline{\mathcal{M}}_{0,2}(\mathbb{P}^m,d) & \xrightarrow{\tilde{e_2}} & \mathbb{P}^m \end{array}$$

We compute:

$$e_2^*(\mathcal{N}) = e_2^*(\iota^*(\mathcal{O}(-l))) = \tilde{\iota}^* \tilde{e_2}^*(\mathcal{O}(-l)).$$

There is the following fibre square:

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,2}(X,\beta) & \xrightarrow{\tilde{\iota}} & \overline{\mathcal{M}}_{0,2}(\mathbb{P}^m,d) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \tilde{\pi}} \\ \overline{\mathcal{M}}_{0,1}(X,\beta) & \xrightarrow{\iota} & \overline{\mathcal{M}}_{0,1}(\mathbb{P}^m,d) \end{array}$$

We apply Proposition 9.3 in [16] to obtain:

$$R^1\pi_* e_2^*(\mathcal{N}) = R^1\pi_* \tilde{\iota}^* \tilde{e_2}^*(\mathcal{O}(-l)) = \iota^*(R^1\tilde{\pi}_* \tilde{e_2}^*(\mathcal{O}(-l))) = \iota^*(V_d^-).$$

Therefore:

$$\tilde{j}^*(\mathcal{M}(Y,d_1\beta_1)) = c_{\text{top}}(R^1\pi_* e_2^*(\mathcal{N})) \cap \mathcal{M}(X,d_1\beta_1) = \iota^*(E_d^-) \cap \mathcal{M}(X,\beta_1). \quad (57)$$

On the other hand, Proposition 11.2.3 of [8] says that :

$$\tilde{\iota}_* \mathcal{M}(X, \beta_1) = E_d^+. \tag{58}$$

Substituting (57) and (58) in (56) we obtain

$$\iota_*(j^*(J_Y)) = \exp\left(\frac{t_0 + t_1 p}{\hbar}\right)\left(kp + \sum_{d_1=1}^{\infty} q_1^{d_1} e_{1*}\left(\frac{E_d}{\hbar(\hbar - c)}\right) \cup (-lp)\right). \tag{59}$$

Recall that on $H^*(\overline{\mathcal{M}}_{0,1}(\mathbb{P}^m, d))$ we have:

$$E_d = E_d' E_d^- e_1^*(kp).$$

Substituting this in (59) and using the projection formula we get

$$\iota_*(j^*(J_Y)) = \exp\left(\frac{t_0 + t_1 p}{\hbar}\right)(kp)\left(1 + \sum_{d_1=1}^{\infty} q_1^{d_1} e_{1*}\left(\frac{E_d' E_d^-}{\hbar(\hbar - c)}\right) \cup (-lp)\right). \tag{60}$$

The theorem is proven.†

4 Mirror Theorem

This chapter is the heart of the work. We will formulate and prove the mirror theorem which in a slightly different form was first proven by Lian, Liu and Yau using a different approach [21].

4.1 Notations and set up.

Let $V = (\oplus_{i\in I}\mathcal{O}(k_i)) \oplus (\oplus_{j\in J}\mathcal{O}(-l_j)) = V^+ \oplus V^-$ with $k_i, l_j > 0$ for all $i \in I$ and $j \in J$.

Consider the hypergeometric series on $\mathbb{P}^s$:

$$I_V := \exp\left(\frac{t_0 + t_1 p}{\hbar}\right)\sum_{d=0}^{\infty} q^d \frac{\prod_{i\in I}\prod_{m=1}^{k_i d}(kp + m\hbar)\prod_{j\in J}\prod_{m=0}^{l_j d-1}(-l_j p - m\hbar)}{\prod_{m=1}^{d}(p + m\hbar)^{s+1}}. \quad (61)$$

Theorem 4.1.1 *Assume that $k + l \leq s + 1$ and that J is nonempty. There is a change of variables of the form $t_1 \to t_1 + I_1(q)$ which transforms I_V into J_V.*

We use Givental's approach for complete intersections in projective spaces [13] to prove an equivariant version of the theorem. The standard diagonal action of $T = (C^*)^{s+1}$ on $W = C^{s+1}$ gives rise to an action of $\mathbf{T}$ on $\mathbb{P}^s$. The representation for the equivariant cohomology with $\mathbb{Q}$-coefficients of the torus T is $\mathbb{Q}[\lambda_0, \lambda_1, ..., \lambda_s]$ where λ_i is first chern class of the representation of the i-th factor $\mathbb{C}^*$. Denote by the same letter p the equivariant hyperplane class. Then the equivariant cohomology of $\mathbb{P}^s$ is:

$$H^*_{\mathbf{T}}(\mathbb{P}^s) = \mathbb{Q}[p, \lambda]/\prod_{i=0}^{s}(p - \lambda_i).$$

Let $p_i : i = 0, 1, ..., s$ be the fixed points of this action and $\phi_i = \prod_{k\neq i}(p - \lambda_k)$ their corresponding equivariant Thom classes.

Remark 4.1.1 *For the remainder of this work whenever we refer to the equivariant cohomology, class or map it will be for the above torus action and not for the one introduced in the first few chapters.*

If all the maps and classes appearing in the function J_V are the equivariant ones then the equivariant J_V^{eq} is obtained.

$$J_V^{eq} = \exp\left(\frac{t_0 + pt_1}{\hbar}\right)\left(1 + \sum_{d>0} q^d e_{1*}\left(\frac{E_d' E_d^-}{\hbar(\hbar - c)}\right) \cup (-lp)\right) =$$

$$\exp\left(\frac{t_0 + t_1 p}{\hbar}\right) S(q, \hbar). \tag{62}$$

We will also consider an equivariant counterpart of the function I_V, namely:

$$I_V^{eq} = \exp\left(\frac{t_0 + t_1 p}{\hbar}\right) \sum_{d=0}^{\infty} q^d \frac{\prod_{m=1}^{kd}(kp + m\hbar)\prod_{m=0}^{ld-1}(-lp - m\hbar)}{\prod_{m=1}^{d}\prod_{i=0}^{s}(p - \lambda_i + m\hbar)} =$$

$$\exp\left(\frac{t_0 + t_1 p}{\hbar}\right) S'(q, \hbar). \tag{63}$$

Theorem 4.1.2 *A similar equivariant change of variables transforms I_V^{eq} into J_V^{eq}.*

Define correlators:

$$S_i := \langle S, \phi_i \rangle$$

and

$$S_i' := \langle S_i', \phi_i \rangle$$

where the pairing is the equivariant push-forward to a point. They determine S and S' according to these formulas:

$$S = \sum_{i=0}^{s} S_i \cup \phi^i$$

where ϕ^i is dual to ϕ_i. Similarly for S'. By the projection formula we have

$$S_i = 1 + \sum_{d=1}^{\infty} (e^t)^d \int_{\overline{\mathcal{M}}_{0,1}(\mathbb{P}^s,d)} \frac{e_1^*(-lp\phi_i)}{\hbar(\hbar - c)} E'_d E^-_d. \tag{64}$$

The proof of the equivariant mirror theorem is based on exhibiting similar properties of the correlators S_i and S'_i. S_i satisfies one more property which uniquely determines it. After the change of variables, that property is satisfied by S'_i as well, which implies $S_i = S'_i$.

We now proceed with displaying properties of the correlators S_i and S'_i.

4.2 Linear recursion relations

Lemma 4.2.1 *The correlators S_i satisfy the following linear recursion relations:*

$$S_i = 1 + \sum_{d=1}^{\infty} q^d R_{i,d}(\hbar^{-1}) + \sum_{d=1, j\neq i}^{\infty} q^d C_{i,j,d} S_j \left(q, \frac{\lambda_j - \lambda_i}{d}\right) \tag{65}$$

where $R_{i,d}(\hbar^{-1})$ are polynomials and

$$C_{i,j,d} = \frac{(\lambda_j - \lambda_i) \prod_{m=1}^{kd} (k\lambda_i + m\frac{\lambda_i - \lambda_j}{d}) \prod_{m=0}^{ld-1} (-l\lambda_i + m\frac{\lambda_i - \lambda_j}{d})}{d\hbar(d\hbar + \lambda_i - \lambda_j) \prod_{m=1}^{d} \prod_{k=0,(k,m)\neq(j,d)}^{s} (\lambda_i - \lambda_k + m\frac{\lambda_j - \lambda_i}{d})}. \tag{66}$$

Proof. It is not clear from the formulation of this lemma whether we can formally substitute $\hbar = \frac{\lambda_i - \lambda_j}{d}$. We will see during the proof of this lemma that the substitution makes sense. We will use the localization theorem to evaluate the integrals that appear in the formula for S_i. There are three types of fixed point components M_Γ of $\overline{\mathcal{M}}^{\mathbf{T}}_{0,1}(\mathbb{P}^s, d)$. The first one consists of those M_Γ where the component containing the marked point is collapsed to p_i. We denote the set of these components by $\mathcal{F}^{\mathbf{i}}_{\mathbf{1},\mathbf{d}}$. Let $\mathcal{F}^{\mathbf{i}}_{\mathbf{2},\mathbf{d}}$ be the set of those M_Γ in which the component containing the fixed point is a multiple cover of some line $\overline{p_i, p_j}$ for some $j \neq i$ with the marked point mapped to p_i. Finally let $\mathcal{F}^{\mathbf{i}}_{\mathbf{0,d}}$ be the rest of the fixed point components. Recall the localization theorem:

$$\int_{\overline{\mathcal{M}}_{0,1}(\mathbb{P}^s,d)} \frac{e_1^*(-lp\phi_i)}{\hbar(\hbar-c)} E'_d E_d^- = \sum_{\Gamma\in\ \mathcal{F}^i_{0,d}} \int_{M_\Gamma} \frac{1}{a_\Gamma \mathrm{Euler}(N_\Gamma)} \left(\frac{e_1^*(-lp\phi_i)}{\hbar(\hbar-c)} E'_d E_d^- \right)_\Gamma +$$

$$\sum_{\Gamma\in\ \mathcal{F}^i_{1,d}} \int_{M_\Gamma} \frac{1}{a_\Gamma \mathrm{Euler}(N_\Gamma)} \left(\frac{e_1^*(-lp\phi_i)}{\hbar(\hbar-c)} E'_d E_d^- \right)_\Gamma +$$

$$\sum_{\Gamma\in\ \mathcal{F}^i_{2,d}} \int_{M_\Gamma} \frac{1}{a_\Gamma \mathrm{Euler}(N_\Gamma)} \left(\frac{e_1^*(-lp\phi_i)}{\hbar(\hbar-c)} E'_d E_d^- \right)_\Gamma .$$

Here the Γ subscript means the restriction of the class to the fixed point component M_Γ. Notice first that:

$$\sum_{\Gamma\in\ \mathcal{F}^i_{0,d}} \int_{M_\Gamma} \frac{1}{a_\Gamma \mathrm{Euler}(N_\Gamma)} \left(\frac{e_1^*(-lp\phi_i)}{\hbar(\hbar-c)} E'_d E_d^- \right)_\Gamma = 0. \tag{67}$$

Indeed, let $\Gamma_j \in \mathcal{F}^{\mathbf{i}}_{\mathbf{0,d}}$ represent a fixed point component with the marked point mapped to the fixed point p_j for some $j \neq i$. Since $(e_1^*(\phi_i))_{\Gamma_j} = 0$ we are done.

Next, in each fixed point component that belongs to $\mathcal{F}^{\mathbf{i}}_{\mathbf{1,d}}$ the class c is nilpotent. Indeed, if Γ is the decorated graph that represents such a fixed point component, let $\overline{\mathcal{M}}_{0,k}$ correspond to the vertex of Γ that contains the marked point. Then $k \leq d+1$. There is a morphism:

$$\varphi : M_\Gamma \mapsto \overline{\mathcal{M}}_{0,k}$$

such that $\varphi^*(c) = c_\Gamma$. But clearly $c^{d-1} = 0$ on $\overline{\mathcal{M}}_{0,k}$. This implies that:

$$\sum_{\Gamma\in\ \mathcal{F}^i_{1,d}} \int_{M_\Gamma} \frac{1}{a_\Gamma \mathrm{Euler}(N_\Gamma)} \left(\frac{e_1^*(-lp\phi_i)}{\hbar(\hbar-c)} E'_d E_d^- \right)_\Gamma = R_{i,d}(\hbar^{-1}) \tag{68}$$

is a polynomial in $\hbar^{-1}$.

We now consider the fixed point components of $\mathcal{F}^{\mathbf{i}}_{\mathbf{2,d}}$. Again let Γ represent such a component. For a stable map (C, x_1, f) in Γ let C' be the component of C containing x_1, C'' the rest of the curve, $x = C' \cap C''$ and $f(x) = p_j$ for some $j \neq i$. Let d' be the degree of the map f on the component C'. Then $(C'', x, f|_{C''})$ is a fixed point in $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^s, d - d')$. Denote its decorated graph by Γ''. As Γ moves in $\mathcal{F}^{\mathbf{i}}_{\mathbf{2,d}}$, the set of all such Γ'' exhausts all the fixed points in $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^s, d - d')$ where the first marked point is mapped to p_j. Clearly $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma'')$. Recall from (17) the formula for the automorphism group G of the fixed point component M_Γ

$$a_\Gamma = |\mathrm{G}| = \prod_{\text{edges e}} d_e \cdot |\mathrm{Aut}(\Gamma)|.$$

This implies that:

$$a_\Gamma = d' a_{\Gamma''}.$$

In order to compute c_Γ we need to compute the weight of the T-action on $T^*_{x_1}C$. But $x_1 \in C'$ and f maps C' to the line $\overline{p_i p_j} \simeq \mathbb{P}^1$ with degree d'. The weight of the T-action on $T^*_{p_i}$ is $\lambda_j - \lambda_j$. It follows that $c_\Gamma = \dfrac{\lambda_j - \lambda_i}{d'}$.

We can split $\mathrm{Euler}(N_\Gamma)$ in three pieces: smoothing the node x, deforming the maps $f|_{C'}$ and $f|_{C''}$. Using localization techniques we obtain:

$$\mathrm{Euler}(N_\Gamma) = \left(\frac{\lambda_j - \lambda_i}{d'} - c''_\Gamma\right) \mathrm{Euler}(N_{\Gamma''}) \cdot$$
$$\prod_{m=0}^{d'-1} \prod_{k=0,(m,k)\neq(0,i)}^{2} \left(\lambda_i - \lambda_k + m\frac{\lambda_j - \lambda_i}{d'}\right).$$

Consider the normalization sequence at the node x:

$$0 \to \mathcal{O}_C \to \mathcal{O}_{C'} \oplus \mathcal{O}_{C''} \to \mathcal{O}_x \to 0. \tag{69}$$

Twist it by $f^*(\mathcal{O}(-l))$ and take the cohomology sequence. We obtain the following:

$$(E_d^-)_\Gamma = (-l\lambda_j)(E_{d-d'}^-)_{\Gamma''}\mathrm{Euler}_{d'}$$

where

$$\mathrm{Euler}_{d'} = \mathrm{Euler}(H^1(C', f|_{C'}^*\mathcal{O}(-l))).$$

The right side is a short notation for the Euler class of the bundle whose fiber is $H^1(C', f|_{C'}^*\mathcal{O}(-l))$. Let $(z_0, z_1, ..., z_s)$ be the coordinates on $\mathbb{P}^s$. Choose the coordinates (y_0, y_1) on C' such that $z_i(f|_{C'}) = y_0^{d'}$ and $z_j(f|_{C''}) = y_1^{d'}$. There is a basis for $H^1(C', f|_{C'}^*(\mathcal{O}(-l))) = H^1(\mathcal{O}_{\mathbb{P}^1}(-ld')$. It consists of

$$\frac{y_0^s y_1^{ld'-2-s}}{(y_0y_1)^{ld'-1}} = \frac{1}{y_0^{ld'-s-1}y_1^{1+s}} : s = 0, 1, ..., ld' - 2.$$

It allows us to compute:

$$\mathrm{Euler}_{d'} = \prod_{s=0}^{ld'-2}\left(\frac{1+s-ld'}{d'}\lambda_i - \frac{1+s}{d'}\lambda_j\right) = \prod_{s=1}^{ld'-1}\left(-l\lambda_i + s\frac{\lambda_i - \lambda_j}{d'}\right).$$

Therefore we have:

$$(E_d^-)_\Gamma = (-l\lambda_j)\prod_{s=1}^{ld'-1}\left(-l_t\lambda_i + s\frac{\lambda_i - \lambda_j}{d'}\right)(E_{d-d'}^-)_{\Gamma''}. \tag{70}$$

Now twist (69) by $f^*(\mathcal{O}(k))$ and take the cohomology sequence. We obtain:

$$0 \to H^0(C, f^*(\mathcal{O}(k))) \to H^0(C', f^*(\mathcal{O}(k))) \oplus H^0(C'', f^*(\mathcal{O}(k))) \to \mathcal{O}_{p_i}(k) \to 0.$$

We take the Euler classes of the corresponding bundles to obtain:

$$(E_d^+)_\Gamma = \frac{\mathrm{Euler}_{d'}^+(E_{d-d'}^+)_{\Gamma''}}{k\lambda_i} \tag{71}$$

where

$$\mathrm{Euler}_{d'}^+ = \mathrm{Euler}(H^0(C', f|_{C'}^*\mathcal{O}(k))).$$

Substituting $(E_d^+)_\Gamma = k\lambda_i (E_d')_\Gamma$ in (71) and computing $\mathrm{Euler}_{d'}^+$ we finally get:

$$(E_d')_\Gamma = \prod_{r=1}^{kd'} \left(k\lambda_i + r\frac{\lambda_j - \lambda_i}{d'} \right) (E_{d-d'}')_{\Gamma''}. \tag{72}$$

It follows from (67) and (68) that:

$$S_i = 1 + \sum_{d=1}^{\infty} q^d R_{i,d}(\hbar^{-1}) + \sum_{\Gamma \in \mathcal{F}_{2,d}^i} q^d \int_{M_\Gamma} \frac{(-l\lambda_i) \prod_{k \neq i}(\lambda_i - \lambda_k)(E_d' \cdot E_d^-)_\Gamma}{\hbar(\hbar - c_\Gamma) a_\Gamma \mathrm{Euler}(N_\Gamma)}. \tag{73}$$

From this representation of S_i it is clear that the coefficients of the power series $S_i = \sum_{d=0}^{\infty} S_{i,d} q^d$ belong to $\mathbb{Q}(\lambda, \hbar)$. Since $c_\Gamma = \dfrac{\lambda_j - \lambda_i}{d'}$ they do not have a pole at $\hbar = \dfrac{\lambda_i - \lambda_j}{d}$ for any $j \neq i$ and any $d > 0$. Therefore the substitution $\hbar = \dfrac{\lambda_i - \lambda_j}{d}$ in S_i makes sense. We use the equations (72) and (70) to obtain:

$$\sum_{\Gamma \in \mathcal{F}_{2,d}^i} q^d \frac{(-l\lambda_j) \prod_{k \neq i}(\lambda_i - \lambda_k)(E_d' \cdot E_d^-)_\Gamma}{\hbar(\hbar - c_\Gamma) a_\Gamma \mathrm{Euler}(N_\Gamma)} = \sum_{d'=1}^{\infty} \sum_{j \neq i} q^{d'} C_{i,j,d'}$$

$$\sum_{\Gamma'' \in \overline{\mathcal{M}}_{0,2}^{\mathrm{T}}(\mathbb{P}^2, d-d')} q^{d-d'} \int_{M_{\Gamma''}} \frac{-l\lambda_j \prod_{k \neq j}(\lambda_j - \lambda_k)(E_{d-d'}' \cdot E_{d-d'}^-)_{\Gamma''}}{(\frac{\lambda_j - \lambda_i}{d'})(\frac{\lambda_j - \lambda_i}{d'} - c_\Gamma'') a_{\Gamma''} \mathrm{Euler}(N_{\Gamma''})} =$$

$$\sum_{d'=1}^{\infty} \sum_{j \neq i} q^{d'} C_{i,j,d'} S_j \left(q, \frac{\lambda_j - \lambda_i}{d'} \right). \tag{74}$$

Substituting this into (73) we obtain

$$S_i = 1 + \sum_{d=1}^{\infty} q^d R_{i,d}(\hbar^{-1}) + \sum_{j \neq i} \sum_{d=1}^{\infty} q^d C_{i,j,d} S_j \left(q, \frac{\lambda_j - \lambda_i}{d} \right).$$

The lemma is proven.†

Lemma 4.2.2 *The correlators S_i' satisfy the same linear recursion relations.*

Proof. We recall that

$$S'_i = \sum_{d=0}^{\infty} (e^t)^d \frac{\prod_{m=1}^{kd}(k\lambda_i + m\hbar)\prod_{m=0}^{ld-1}(-l\lambda_i - m\hbar)}{\prod_{m=1}^{d}\prod_{j=0}^{s}(\lambda_i - \lambda_j + m\hbar)} =$$

$$\sum_{d=0}^{\infty} (e^t)^d \frac{\prod_{m=1}^{kd}(k\lambda_i + m\hbar)\prod_{m=0}^{ld-1}(-l\lambda_i - m\hbar)}{d\hbar\prod_{m=1}^{d}\prod_{j=0,(j,m)\neq(i,d)}^{s}(\lambda_i - \lambda_j + m\hbar)}. \tag{75}$$

By the calculus of residues in the variable $\hbar$ we have

$$\frac{\prod_{m=1}^{kd}(k\lambda_i + m\hbar)\prod_{m=0}^{ld-1}(-l\lambda_i - m\hbar)}{d\hbar\prod_{m=1}^{d}\prod_{j=0,(j,m)\neq(i,d)}^{s}(\lambda_i - \lambda_j + m\hbar)} = R_{i,d}(\hbar^{-1}) + \sum_{m=1,r\neq i}^{d} \frac{1}{d\hbar(\lambda_i - \lambda_r + m\hbar)}.$$

$$\frac{\prod_{n=1}^{kd}(k\lambda_i + n\frac{\lambda_r-\lambda_i}{m})\prod_{n=0}^{ld-1}(-l\lambda_i - n\frac{\lambda_r-\lambda_i}{m})}{\prod_{n=1,(j,n)\neq(r,m)}^{d}\prod_{j=0,(j,n)\neq(i,d)}^{s}(\lambda_i - \lambda_j + n\frac{\lambda_r-\lambda_i}{m})} \tag{76}$$

for some polynomials $R_{i,d}(\hbar^{-1})$ in $\hbar^{-1}$. We substitute equation (76) in (75) to obtain

$$S'_i = 1 + \sum_{d=1}^{\infty} q^d R_{i,d} + \sum_{d=1}^{\infty} q^d \sum_{r\neq i}\sum_{m=1}^{d} \frac{1}{d\hbar(\lambda_i - \lambda_r + m\hbar)}.$$

$$\frac{\prod_{n=1}^{kd}(k\lambda_i + n\frac{\lambda_r-\lambda_i}{m})\prod_{n=0}^{ld-1}(-l\lambda_i - n\frac{\lambda_r-\lambda_i}{m})}{\prod_{n=1,(j,n)\neq(r,m)}^{d}\prod_{j=0,(j,n)\neq(i,d)}^{s}(\lambda_i - \lambda_j + n\frac{\lambda_r-\lambda_i}{m})}. \tag{77}$$

We change the order of summation in the second summand of S'_i to obtain for a fixed $r \neq i$

$$\sum_{d=1}^{\infty} q^d \sum_{m=1}^{d} \frac{1}{d\hbar(\lambda_i - \lambda_r + m\hbar)} \frac{\prod_{n=1}^{kd}(k\lambda_i + n\frac{\lambda_r-\lambda_i}{m})\prod_{n=0}^{ld-1}(-l\lambda_i - n\frac{\lambda_r-\lambda_i}{m})}{\prod_{n=1,(j,n)\neq(r,m)}^{d}\prod_{j=0,(j,n)\neq(i,d)}^{s}(\lambda_i - \lambda_j + n\frac{\lambda_r-\lambda_i}{m})} =$$

$$\sum_{m=1}^{\infty} q^m \frac{1}{\hbar(\lambda_i - \lambda_r + m\hbar)}.$$

$$\sum_{d=m}^{\infty} q^{d-m} \frac{\prod_{n=1}^{kd}(k\lambda_i + n\frac{\lambda_r-\lambda_i}{m})\prod_{n=0}^{ld-1}(-l\lambda_i - n\frac{\lambda_r-\lambda_i}{m})}{d\prod_{n=1,(j,n)\neq(r,m)}^{d}\prod_{j=0,(j,n)\neq(i,d)}^{s}(\lambda_i - \lambda_j + n\frac{\lambda_r-\lambda_i}{m})}. \tag{78}$$

Now,

$$\sum_{d=m}^{\infty} q^{d-m} \frac{\prod_{n=1}^{kd}(k\lambda_i + n\frac{\lambda_r-\lambda_i}{m})\prod_{n=1}^{ld-1}(-l\lambda_i - n\frac{\lambda_r-\lambda_i}{m})}{d\prod_{n=1,(j,n)\neq(r,m)}^{d}\prod_{j=0,(j,n)\neq(i,d)}^{s}(\lambda_i - \lambda_j + n\frac{\lambda_r-\lambda_i}{m})} =$$

$$= \frac{\prod_{n=1}^{km}(k\lambda_i + n\frac{\lambda_r-\lambda_i}{m})\prod_{n=0}^{lm-1}(-l\lambda_i - n\frac{\lambda_r-\lambda_i}{m})}{\prod_{n=1,(j,n)\neq(r,m)}^{m}\prod_{j=0}^{s}(\lambda_i - \lambda_j + n\frac{\lambda_r-\lambda_i}{m})}.$$

$$\sum_{u=0}^{\infty} q^u \frac{\prod_{n=1}^{ku}(k\lambda_r - n\frac{\lambda_r-\lambda_i}{m})\prod_{n=0}^{lu-1}(-l\lambda_r - n\frac{\lambda_r-\lambda_i}{m})}{\prod_{n=1,(j,n)\neq(i,u)}^{u}\prod_{j=0}^{s}(\lambda_k - \lambda_j + n\frac{\lambda_r-\lambda_i}{m})(m+u)}. \tag{79}$$

The denominator in the last sum can be transformed as follows

$$\prod_{n=1,(j,n)\neq(i,u)}^{u}\prod_{j=0}^{s}\left(\lambda_r - \lambda_j + n\frac{\lambda_r-\lambda_i}{m}\right)(m+u)$$

$$= \prod_{n=1,(j,n)\neq(r,u)}^{u}\prod_{j=0}^{s}\left(\lambda_r - \lambda_j + n\frac{\lambda_r-\lambda_i}{m}\right)u.$$

Putting everything together we finally get

$$S_i' = 1 + \sum_{d=1}^{\infty} q^d R_{i,d} + \sum_{r\neq i}\sum_{m=1}^{\infty} q^m \frac{1}{\hbar(\lambda_i - \lambda_r + m\hbar)}.$$

$$\frac{\prod_{n=1}^{km}(k\lambda_i + n\frac{\lambda_r-\lambda_i}{m})\prod_{n=0}^{lm-1}(-l\lambda_i - n\frac{\lambda_r-\lambda_i}{m})}{\prod_{n=1,(j,n)\neq(r,m)}^{m}\prod_{j=0}^{s}(\lambda_i - \lambda_j + n\frac{\lambda_r-\lambda_i}{m})}.$$

$$\sum_{u=0}^{\infty} q^u \frac{\prod_{n=1}^{ku}(k\lambda_r - n\frac{\lambda_r-\lambda_i}{m})\prod_{n=0}^{lu-1}(-l\lambda_r - n\frac{\lambda_r-\lambda_i}{m})}{\prod_{n=1,(j,n)\neq(r,u)}^{u}\prod_{j=0}^{2}(\lambda_r - \lambda_j + n\frac{\lambda_r-\lambda_i}{m})u}. \tag{80}$$

The last summand is not yet $S_k'(q, \frac{\lambda_r - \lambda_i}{m})$ because the factor $\frac{\lambda_r - \lambda_i}{m}$ is missing in the denominator. Divide and multiply with it and we get

$$S_i' = 1 + \sum_{d=1}^{\infty} q^d R_{i,d} + \sum_{m=1,k\neq i}^{\infty} q^m C_{i,r,m} S'\left(q, \frac{\lambda_r - \lambda_i}{m}\right) \tag{81}$$

with

$$C_{i,r,m} = \frac{\lambda_r - \lambda_i}{m\hbar(\lambda_i - \lambda_r + m\hbar)} \frac{\prod_{n=1}^{km}(k\lambda_i + n\frac{\lambda_i-\lambda_r}{m})\prod_{n=0}^{lm-1}(-l\lambda_i - n\frac{\lambda_r-\lambda_i}{m})}{\prod_{n=1,(j,n)\neq(r,m)}^{m}\prod_{j=0}^{s}(\lambda_i - \lambda_j + n\frac{\lambda_r-\lambda_i}{m})}$$

i.e. the same linear recursion relations that were satisfied by S_i. The lemma is proven.†

4.3 Linear and nonlinear sigma models for a projective space.

We explain here some of the features that will be used in proving the double polynomiality property for the correlators S and S'.

Let

$$M_d := \overline{\mathcal{M}}_{0,0}(\mathbb{P}^s \times \mathbb{P}^1, (d,1)). \tag{82}$$

This moduli space compactifies the space of degree d maps $\mathbb{P}^1 \to \mathbb{P}^s$. We call M_d **the degree d nonlinear sigma model of $\mathbb{P}^s$.**

We will also consider

$$N_d := \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))^{s+1}) \tag{83}$$

which is also a compactification of degree d maps from $\mathbb{P}^1$ to $\mathbb{P}^s$. An element in $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))^{s+1}$ is an $s+1$-tuple of degree d homogeneous polynomials in two variables w_0 and w_1. As a vector space, $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))^{s+1}$ is generated by the vectors $v_{ir} = (0, ..., 0, w_0^r w_1^{d-r}, 0..., 0)$ for $i = 0, 1, ..., s$ and $r = 0, 1, ..., d$. The only nonzero component of v_{ir} is the i-th one. N_d is called **the degree d linear sigma model of the projective space $\mathbb{P}^s$.** The terminology for M_d and N_d comes from physics. For more on this subject, we recommend Appendix B in [8].

The action of $\mathbf{T} \times \mathbb{C}^*$ in $\mathbb{P}^s \times \mathbb{P}^1$ with weights λ in the $\mathbb{P}^s$ factor and $(\hbar, 0)$ in the $\mathbb{P}^1$ factor, gives rise to an action of $\mathbf{T} \times \mathbb{C}^*$ in M_d by translation of maps. There is also an action of $\mathbf{T} \times \mathbb{C}^*$ on N_d. Let $\bar{t} = (t_0, ..., t_s) \in \mathbf{T}$ and $t \in \mathbb{C}^*$. This action is

$$(\bar{t}, t) \cdot [P_0(w_0, w_1), ..., P_s(w_0, w_1)] = [t_0 P_0(t w_0, w_1), ..., t_s P_s(t w_0, w_1)] \tag{84}$$

Let κ be the equivariant hyperplane class in N_d. There is an equivariant map:

$$\psi : M_d \mapsto N_d.$$

We briefly describe this map set-theoretically and show that it is equivariant (for a proof that it is a morphism see [13] or [21]). A stable map $(C, f) \in M_d$ is given as follows:

$$C = C_0 \cup C_1 \cup ... \cup C_n$$

$$\deg((\pi_2 \circ f)|_{C_i}) = 0 : i = 1, 2, ..., n$$

$$\deg((\pi_1 \circ f)|_{C_i}) = k_i : i = 1, 2, ..., n$$

$$\deg((\pi_2 \circ f)|_{C_0}) = 1$$

$$\sum_{i=1}^{n} k_i = d.$$

We can choose coordinates on $C_0 \cong \mathbb{P}^1$ so that $\pi_2 \circ f : C_0 \mapsto \mathbb{P}^1$ is given by $f(y_0, y_1) = (y_1, y_0)$. Let $C_0 \cap C_i = (a_i, b_i)$ and $\pi_1 \circ f = [f_0 : f_1 : ... : f_s] : C_0 \mapsto \mathbb{P}^s$. Then the definition of ψ is

$$\psi(C, f) := \prod_{i=1}^{n} (b_i w_0 - a_i w_1)^{k_i} [f_0 : f_1 : ... : f_s]. \tag{85}$$

This map is equivariant. Indeed, let $\tau = (\bar{t}, t) \in \mathbf{T} \times \mathbb{C}^*$ with $\bar{t} = (t_0, t_1, ..., t_s)$. Let

$$\tau \cdot (C, f = (f^1, f^2)) = (C, \tilde{f} = (\tilde{f}^1, \tilde{f}^2)).$$

Then $\tilde{f}^2(y_0, y_1) = (ty_1, y_0)$ and $\tilde{f}^1(y_0, y_1) = \bar{t} \cdot f^1(y_0, y_1)$. To find $\psi(C, \tilde{f})$, we need to choose a representation of $(C, \tilde{f})$ such that $\tilde{f}^2$ permutes the coordinates. If we let α be an automorphism of C_0 given by $\alpha(y_0, y_1) = (y_0, t^{-1} y_1)$, that representation is

$$(\tilde{C}, \tilde{f} \circ \alpha = (\tilde{f}^1 \circ \alpha, \tilde{f}^2 \circ \alpha)) = (\tilde{C}, \bar{f}^1, \bar{f}^2).$$

The only difference between $\tilde{C}$ and C is that $C_0 \cap C_i = (a_i, tb_i)$. Also

$$\bar{f}^1 : C_0 \mapsto \mathbb{P}^s$$

is given by $\bar{f}^1(y_0, y_1) = \bar{t} \cdot f^1(y_0, t^{-1}y_1)$. Therefore

$$\psi(C, \tilde{f}) = \prod_{i=1}^{n} (b_i t w_0 - a_i w_1)^{k_i} [t_0 f_0(w_0, t^{-1}w_1) : \ldots : t_s f_s(w_0, t^{-1}w_1)].$$

Since the f_i's have the same degree we have:

$$[t_0 f_0(w_0, t^{-1}w_1) : t_1 f_1(w_0, t^{-1}w_1) : \ldots : t_s f_s(w_0, t^{-1}w_1)] =$$

$$= [t_0 f_0(tw_0, w_1) : t_1 f_1(tw_0, w_1) : \ldots : t_s f_s(tw_0, w_1)],$$

therefore

$$\psi(C, \tilde{f}) = \prod_{i=1}^{n} (b_i t w_0 - a_i w_1)^{k_i} [t_0 f_0(tw_0, w_1) : \ldots : t_s f_s(tw_0, w_1)] = \tau \cdot \psi(C, f).$$

We now describe the structure of the $\mathbf{T} \times \mathbb{C}^*$-fixed points in N_d and M_d.

Consider first the linear sigma model N_d. Let $p_{i,r}$ be the points of N_d corresponding to the vectors v_{ir}. The only fixed points of the $\mathbf{T} \times \mathbb{C}^*$-action on N_d are precisely the points p_{ir}. The weight of the hyperplane class at the fixed point p_{ir} is $\lambda_i + r\hbar$ and the Euler class of the tangent space T_{N_d} at p_{ir} is [21]

$$E_{ir} = \prod_{(j,t) \neq (i,r)} (\lambda_i - \lambda_j + r\hbar - t\hbar). \tag{86}$$

We now turn our attention to the nonlinear sigma model M_d. A $\mathbf{T} \times C^*$-fixed point component in M_d consists of stable maps (C, f) with $C = C_0 \cup C_1 \cup C_2$ and $f = (f^1, f^2) : C \mapsto \mathbb{P}^s \times \mathbb{P}^1$ such that $\deg(f|_{C_j}) = (d_j, 0)$ for $j = 1, 2$ with $d_1 + d_2 = d$ and $\deg(f|_{C_0}) = (0, 1)$. Also $f^1(C_0) = p_i$, a $\mathbf{T}$-fixed point in $\mathbb{P}^s$. Choose coordinates (y_0, y_1) in $C_0 \cong \mathbb{P}^1$ such that $C_1 \cap C_0 = x_1 = (1,0), C_2 \cap C_0 = x_2 = (0,1)$ and $f^2(y_0, y_1) = (y_1, y_0)$. We will denote this component by $\Gamma^i_{d_1,d_2}$. Note that this component is mapped by ψ to $p_{i,d_2} \in N_d$. Therefore, the weight of the $\mathbf{T} \times C^*$-action for the class $\psi^*(\kappa)$ in such a component is $\lambda_i + d_2\hbar$. There is a canonical isomorphism of this component to $\Gamma^i_{d_1} \times \Gamma^i_{d_2}$ where $\Gamma^i_{d_j}$ is a decorated graph representing a fixed point component in $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^s, d_j)$ with the image of the marked point being p_i. Let's find the normal bundle of this component in the above identification. Consider the normalization sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_{C_0} \oplus \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2} \to \mathcal{O}_{x_1} \oplus \mathcal{O}_{x_2} \to 0. \tag{87}$$

Twist (87) by $f^*(T\mathbb{P}^2 \times \mathbb{P}^1)$ and take the cohomology sequence. We obtain

$$0 \to H^0(C, f^*(T\mathbb{P}^s \times \mathbb{P}^1)) \to H^0(C_1, f_1^*(T\mathbb{P}^s)) \oplus T_{(0,1)}\mathbb{P}^1 \oplus H^0(C_2, f_1^*(T\mathbb{P}^s)) \oplus$$

$$T_{(1,0)}\mathbb{P}^1 \oplus T_{p_i}\mathbb{P}^s \oplus H^0(C_0, f_2^*(T\mathbb{P}^1)) \to T_{p_i}\mathbb{P}^s \oplus T_{(0,1)}\mathbb{P}^1 \oplus T_{p_i}\mathbb{P}^s \oplus T_{(1,0)}\mathbb{P}^1 \to 0.$$

In K-theory, this implies the following relations of the Euler classes of bundles

$$[H^0(C, f^*(T\mathbb{P}^s \times \mathbb{P}^1))] = [H^0(C_1, f_1^*(T\mathbb{P}^s))] + [H^0(C_2, f_1^*(T\mathbb{P}^s))] +$$

$$+[H^0(C_0, f_2^*(T\mathbb{P}^1))] - [T_{p_i}\mathbb{P}^s]. \tag{88}$$

The class $[H^0(C_1, f_1^*(T\mathbb{P}^s))]$ is part of $\mathcal{N}_{\Gamma^i_{d_1}}$ as it stands for the deformations of the restriction of the map f on C_1. Similarly for $[H^0(C_2, f_1^*(T\mathbb{P}^s))]$. The other pieces of the normal bundle $\mathcal{N}_{\Gamma^i_{d_1,d_2}}$ account for deforming the nodes and reparametrizations. Moving the node x_j as well as other nodes along C_j (for $j = 1, 2$) and reparametrizing components of C_j will be accounted for in $\mathcal{N}_{\Gamma^i_{d_j}}$. What is left then is moving the nodes

$x_j, j = 1, 2$ along C_0 and smoothing them and reparametrizing C_0. In other words,

$$[\mathcal{N}_{\Gamma^i_{d_1,d_2}}] = [\mathcal{N}_{\Gamma^i_{d_1}}] + [\mathcal{N}_{\Gamma^i_{d_2}}] + [H^0(C_0, f_2^*(T\mathbb{P}^1))]$$

$$+[T_{x_1}C_0] + [T_{x_2}C_0] - [T_{p_i}\mathbb{P}^s] - [TC_0] + [T_{x_1}C_1 \otimes T_{x_1}C_0] + [T_{x_2}C_2 \otimes T_{x_2}C_0] =$$

$$= [\mathcal{N}_{\Gamma^i_{d_1}}] + [\mathcal{N}_{\Gamma^i_{d_2}}] + [T_{x_1}C_0] + [T_{x_2}C_0] - [T_{p_i}\mathbb{P}^s] + [T_{x_1}C_1 \otimes T_{x_1}C_0] + [T_{x_2}C_2 \otimes T_{x_2}C_0]. \quad (89)$$

We substitute (4.3) into (89) to obtain

$$\frac{1}{E(\mathcal{N}_{\Gamma^i_{d_1,d_2}})} = \frac{1}{\prod_{k\neq i}(\lambda_i - \lambda_k)} \frac{1}{E(\mathcal{N}_{\Gamma^i_{d_1}})} \frac{1}{E(\mathcal{N}_{\Gamma^i_{d_2}})} \frac{e_1^*(\phi_i)}{-\hbar(-\hbar - c_1)} \frac{e_1^*(\phi_i)}{\hbar(\hbar - c_2)}$$

where $c_j, j = 1, 2$ is the first Chern class of the cotangent line bundle for $\mathcal{N}_{\Gamma^i_{d_j}}$.

4.4 Double polynomiality

Define the map

$$pf : H^*\mathbb{P}^s \otimes_{\mathbb{Q}[\lambda]} \mathbb{Q}(\lambda) \to \mathbb{Q}(\lambda)$$

as follows:

$$pf(a) := \int_{\mathbb{P}^s_{\mathbf{T}}} a \cup \left(\frac{kp}{-lp}\right).$$

The obvious cancellation is not carried out for pedagogical reasons.

Lemma 4.4.1 *If z is a variable, the expression:*

$$W'(z,h) = pf(S'(qe^{z\hbar},\hbar)e^{pz}S'(q,-\hbar)) = \int_{\mathbb{P}^s_{\mathbf{T}}} S'(qe^{z\hbar},\hbar)e^{pz}S'(q,-\hbar) \cup \left(\frac{kp}{-lp}\right) \quad (90)$$

belongs to $\mathbb{Q}(\lambda)[\hbar][[q,z]]$.

Proof. The lemma will follow from the identity

$$W'(z,h) = \sum_{d=0}^{\infty} q^d \int_{N_d} e^{z\kappa} \prod_{m=0}^{kd}(k\kappa - m\hbar) \prod_{m=1}^{ld-1}(-l\kappa + m\hbar). \quad (91)$$

The integral on the right side is a $\mathbf{T} \times C^*$-equivariant pushforward to a point. For $d = 0$ the convention

$$\int_{N_d} e^{z\kappa} \prod_{m=0}^{kd} (k\kappa + m\hbar) \prod_{m=1}^{ld-1} (-l\kappa + m\hbar) = \int_{\mathbb{P}^s_{\mathbf{T}}} e^{pz} \left(\frac{kp}{-lp}\right)$$

is taken.

Apply the localization formula to both integrals (90) and (91).

$$W'(z, \hbar) = \sum_{i=0}^{s} \frac{k\lambda_i e^{\lambda_i z}}{(-l\lambda_i) \prod_{j \neq i} (\lambda_i - \lambda_j)}.$$

$$\sum_{d_1=0}^{\infty} (qe^{z\hbar})^{d_1} \frac{\prod_{m=1}^{kd_1} (k\lambda_i + m\hbar) \prod_{m=0}^{ld_1-1} (-l\lambda_i - m\hbar)}{\prod_{m=1}^{d_1} \prod_{j=0}^{s} (\lambda_i - \lambda_j + m\hbar)}).$$

$$\sum_{d_2=0}^{\infty} q^{d_2} \frac{\prod_{m=1}^{kd_2} (k\lambda_i - m\hbar) \prod_{m=0}^{ld_2-1} (-l\lambda_i + m\hbar)}{\prod_{m=1}^{d_2} \prod_{j=0}^{s} (\lambda_i - \lambda_j - m\hbar)} =$$

$$\sum_{i=0}^{s} \frac{1}{\prod_{j \neq i} (\lambda_i - \lambda_j)} \sum_{d_1=0}^{\infty} q^{d_1 z} e^{(\lambda_i + d_1 \hbar) z} \frac{\prod_{m=0}^{kd_1} (k\lambda_i + m\hbar) \prod_{m=1}^{ld_1-1} (-l\lambda_i - m\hbar)}{\prod_{m=1}^{d_1} \prod_{j=0}^{s} (\lambda_i - \lambda_j + m\hbar)}.$$

$$\sum_{d_2=0}^{\infty} q^{d_2} \frac{\prod_{m=1}^{kd_2} (k\lambda_i - m\hbar) \prod_{m=0}^{ld_2-1} (-l\lambda_i + m\hbar)}{\prod_{m=1}^{d_2} \prod_{j=0}^{s} (\lambda_i - \lambda_j - m\hbar)}. \tag{92}$$

But now, for $d_1 > 0, d_2 > 0$

$$\frac{\prod_{m=0}^{kd_1} (k\lambda_i + m\hbar) \prod_{m=1}^{ld_1-1} (-l\lambda_i - m\hbar) \prod_{m=1}^{kd_2} (k\lambda_i - m\hbar) \prod_{m=0}^{ld_2-1} (-l\lambda_i + m\hbar)}{\prod_{j \neq i} (\lambda_i - \lambda_j) \prod_{m=1}^{d_1} \prod_{j=0}^{s} (\lambda_i - \lambda_j + m\hbar) \prod_{m=1}^{d_2} \prod_{j=0}^{s} (\lambda_i - \lambda_j - m\hbar)} =$$

$$\frac{\prod_{m=0}^{k(d_1+d_2)} (k(\lambda_i + d_1\hbar) - m\hbar) \prod_{m=1}^{l(d_1+d_2)-1} (-l(\lambda_i + d_1\hbar) + m\hbar)}{\prod_{j=0}^{s} \prod_{m=0,(j,m)\neq(i,d_1)}^{d_1+d_2} (\lambda_i + d_1\hbar - \lambda_j - m\hbar)}.$$

Therefore

$$W'(z,\hbar)=\sum_{d=0}^{\infty}q^d\sum_{d_1=0}^{d}\sum_{i=0}^{s}e^{(\lambda_i+d_1\hbar)z}$$

$$\frac{\prod_{m=0}^{kd}(k(\lambda_i+d_1\hbar)-m\hbar)\prod_{m=1}^{ld-1}(-l(\lambda_i+d_1\hbar)+m\hbar)}{\prod_{j=0}^{s}\prod_{m=0,(j,m)\neq(i,d_1)}^{d}(\lambda_i+d_1\hbar-\lambda_j-m\hbar)}.$$

By the localization formula in N_d and the formulas (86) we can see that

$$W'(z,\hbar)=\sum_{d=0}^{\infty}q^d\int_{N_d}e^{z\kappa}\prod_{m=0}^{kd}(k\kappa-m\hbar)\prod_{m=1}^{ld-1}(-l\kappa+m\hbar).$$

The lemma is proven.†

Lemma 4.4.2 *If z is a variable, the expression:*

$$W(z,\hbar)=pf(e^{pz}S(qe^{z\hbar},\hbar)S(q,-\hbar))$$

belongs to $\mathbb{Q}(\lambda)[\hbar][[q,z]]$.

Proof. Consider the following diagram

$$\begin{array}{ccc}\overline{\mathcal{M}}_{0,1}(\mathbb{P}^s\times\mathbb{P}^1,(d,1)) & \xrightarrow{e_1} & \mathbb{P}^s\times\mathbb{P}^1\\ \downarrow{\scriptstyle\pi} & & \\ \overline{\mathcal{M}}_{0,0}(\mathbb{P}^s\times\mathbb{P}^1,(d,1)) & & \end{array}$$

Define:

$$W^-{}_d:=R^1\pi_*((e_1)^*(\mathcal{O}_{\mathbb{P}^s}(-l)\otimes\mathcal{O}_{\mathbb{P}^1}))$$

$$W_d^+:=\pi_*((e_1)^*(\mathcal{O}_{\mathbb{P}^s}(k)\otimes\mathcal{O}_{\mathbb{P}^1}))$$

$$W_d=W_d^+\oplus W_d^-.$$

The lemma will follow from the identity:

$$pf(e^{pz}S(qe^{z\hbar},\hbar)S(q,-\hbar))=\sum_{d=0}^{\infty}q^d\int_{M_d}e^{z\psi^*\kappa}Euler(W_d).$$

Again we make use of the localization formula. The left side equals

$$\sum_{i=0}^{s} \frac{S_i(qe^{z\hbar},\hbar)e^{z\lambda_i}S_i(q,-\hbar)}{\prod_{k\neq i}(\lambda_i-\lambda_k)}\binom{k\lambda_i}{-l\lambda_i}.$$

Recall that

$$S_i = 1+\sum_{d=1}^{\infty}(e^t)^d\int_{\overline{\mathcal{M}}_{0,1}(\mathbb{P}^s,d)}\frac{e_1^*(-lp\phi_i)}{\hbar(\hbar-c)}E'_dE^-_d. \tag{93}$$

Let's compute the localization of Euler(W_d) in $\mathcal{N}_{\Gamma^i_{d_j}}$. If we twist the normalization sequence (87) by $f^*(\mathcal{O}(-l)\otimes\mathcal{O}_{\mathbb{P}^1})$ and take the corresponding long exact cohomology sequence, we get

$$0\to\mathbb{C}\to\mathcal{O}_{x_1}(-l)\oplus\mathcal{O}_{x_2}(-l)\to W^-_d\to W^-_{d_1}\oplus W^-_{d_2}\to 0.$$

The first piece is trivial. To compute the weights of the action, notice that we have an isomorphism

$$(\mathcal{O}_{\mathbb{P}^s}(-l)\times\mathcal{O}_{\mathbb{P}^1})|_{C_0}\cong\mathcal{O}_{C_0}\cong\mathbb{C}.$$

The left hand side is generated by say $\frac{1}{z_i{}^l}$ therefore the weight of that piece is $-l\lambda_i$. In K-theory then

$$\text{Euler}(W^-_d) = (-l\lambda_i)E^-_{d_1}E^-_{d_2}.$$

Similarly, twisting the normalization sequence (87) by $f^*(\mathcal{O}(k)\otimes\mathcal{O}_{\mathbb{P}^1})$ and taking the corresponding cohomology sequence we obtain:

$$\text{Euler}(W^+_d) = (k\lambda_i)E'_{d_1}E'_{d_2}.$$

Putting everything together we see that the contribution of such a component $\mathcal{N}_{\Gamma^i_{d_1,d_2}}$ to (93) is

$$(k\lambda_i)(-l\lambda_i)\frac{e^{z(\lambda_i+d_2\hbar)}}{\prod_{k\neq i}(\lambda_i-\lambda_k)}\int_{M_{\Gamma^i_{d_1}}}\frac{1}{E(\mathcal{N}_{\Gamma^i_{d_1}})}\frac{e_1^*(\phi_i)E'_{d_1}E^-_{d_1}}{-\hbar(-\hbar-c_1)}$$

$$\int_{M_{\Gamma^i_{d_2}}} \frac{1}{E(\mathcal{N}_{\Gamma^i_{d_2}})} \frac{e_1^*(\phi_i) E'_{d_2} E^-_{d_2}}{\hbar(\hbar - c_2)} = \frac{k\lambda_i}{-l\lambda_i} \frac{e^{z(\lambda_i + d_2\hbar)}}{\prod_{k\neq i}(\lambda_i - \lambda_k)}$$

$$\int_{M_{\Gamma^i_{d_1}}} \frac{1}{E(\mathcal{N}_{\Gamma^i_{d_1}})} \frac{e_1^*(-l\lambda_i\phi_i) E_{d_1}}{-\hbar(-\hbar - c_1)} \int_{M_{\Gamma^i_{d_2}}} \frac{1}{E(\mathcal{N}_{\Gamma^i_{d_2}})} \frac{e_1^*(-l\lambda_i\phi_i) E_{d_2}}{\hbar(\hbar - c_2)}.$$

Summing over all fixed point components, we get the lemma.†

4.5 Uniqueness

Lemma 4.5.1 *Let $S = \sum_{d=0}^{\infty} S_d q^d$ and $S' = \sum_{d=0}^{\infty} S"_d q^d$ be two power series with coefficients in $H^*_{\mathbf{T}}\mathbb{P}^s[[\hbar^{-1}]]$ that satisfy the following conditions:*

1. $S_0 = S'_0 = 1$

2. *They both satisfy the recursion relations of Lemma 4.2.1.*

3. *They both have the double polynomiality property of Lemma 4.4.1.*

4. *For any d, $S_d \equiv S'_d$ mod $(\hbar^{-2})$.*

Then $S = S'$.

Proof. Let $d_0 > 0$ be such that $S_d = S'_d$ for all $d < d_0$ and all $0 \leq i \leq s$. We want to show that $S_{d_0} - S'_{d_0} = 0$. By induction, this would prove this lemma. The recursion relations and the induction hypothesis imply that

$$S_{d_0,i} - S'_{d_0,i} = R_{d_0,i} - R'_{d_0,i} \in \mathbb{Q}(\lambda)[\hbar^{-1}]$$

where $R_{d,i}$ and $R'_{d,i}$ are the polynomials in the recursion relations for, respectively S_i and S'_i. By condition (4), $R_{d_0,i} - R'_{d_0,i}$ has a zero of order at least two at the origin. Let $R(\hbar^{-1}, \lambda)$ be an element in $H^*_{\mathbf{T}}\mathbb{P}^s \otimes \mathbb{Q}(\lambda)[\hbar^{-1}][[q]]$ such that $j_i^*(R) = R_{d_0,i} - R'_{d_0,i}$ for all i. By the localization formula in $\mathbb{P}^s$ such an $R(\hbar^{-1}, \lambda)$ can be found. The coeficient of q^{d_0} in $W(S) - W'(S')$ is

$$pf(e^{(p+d_0\hbar)z}R(\hbar, \lambda) + e^{pz}R(-\hbar, \lambda)).$$

By the condition (3), this coefficient should be a polynomial in $\hbar$. Let

$$R(\hbar, \lambda) = \sum_{k=2}^{r} \frac{a_k}{\hbar^k}.$$

Let $2s + 1$ be the largest odd number such that $a_{2s+1} \neq 0$. We have

$$R(\hbar, \lambda) = \frac{1}{\hbar^{2s+1}}(Bh^{-1} + A + O(\hbar))$$

where $A, B \in H^*_{\mathbf{T}}\mathbb{P}^s \otimes \mathbb{Q}(\lambda)$ and $O(\hbar)$ is a polynomial vanishing for $\hbar = 0$. If we expand the exponentials, we obtain

$$pf(e^{(p+d_0\hbar)z}R(\hbar, \lambda) + e^{pz}R(-\hbar, \lambda)) = \frac{1}{\hbar^{2s+1}}pf(2Ae^{pz} + d_0Bze^{pz} + O(\hbar)).$$

Since this should be a polynomial in $\hbar$, we obtain

$$pf(2Ae^{pz} + d_0Bze^{pz}) = 0$$

and this implies that $A = 0$ and $B = 0$. So $s = 0$ and this means that $R = 0$. The lemma is proven.†

4.6 Mirror transformation

We recall from the formulation of the mirror theorem that we are assuming that **there is at least one negative line bundle in V.** Recall that:

$$I_V^{eq} = \exp\left(\frac{t_0 + t_1 p}{\hbar}\right)\left(1 + \sum_{d=1}^{\infty} q^d \frac{\prod_{i\in I}\prod_{m=1}^{k_i d}(k_i p + m\hbar)\prod_{j\in J}\prod_{m=0}^{l_j d-1}(-l_j p - m\hbar)}{\prod_{m=1}^{d}\prod_{i=0}^{s}(p - \lambda_i + m\hbar)}\right).$$

Going back to the simplified presentation (i.e. $|I| = 1$ and $|J| = 1$), we can expand I^{eq} as follows:

$$I^{eq} = \exp\left(\frac{t_0 + pt}{\hbar}\right)\left(1 + I_1 \frac{p}{\hbar}\frac{1}{\hbar^{d(s+1-k-l)}} + o(\frac{1}{\hbar^2})\right)$$

where

$$I_1 = \sum_{d=1}^{\infty} q^d \frac{(-1)^{ld}(ld-1)!(kd)!}{(d!)^{s+1}}.$$

Notice the following consequences of Theorem 4.1.1 and Theorem 4.1.2.

- A negative line bundle produces a factor of $\frac{p}{\hbar}$. It implies that if V contains 2 or more negative line bundles, then $J^{eq} = I^{eq}$. This will be very important in the applications in the next section.
- If $k + l < s + 1$, then again $J^{eq} = I^{eq}$.

We obtain the following theorem.

Theorem 4.6.1 *Let $V = (\oplus_{i\in I}\mathcal{O}(k_i)) \oplus (\oplus_{j\in J}\mathcal{O}(-l_j))$ where $\sum_{i\in I} k_i + \sum_{j\in J} l_j \leq s+1$. If $|I| > 1$ or $k + l < s + 1$ then*

$$e_{1*}\left(\frac{E'_d E^-_d}{\hbar(\hbar - c)}\right) = \frac{\prod_{i\in I}\prod_{m=1}^{k_i d}(k_i p + m\hbar)\prod_{j\in J}\prod_{m=1}^{l_j d-1}(-l_j p - m\hbar)}{\prod_{m=1}^{d}(p + m\hbar)^{s+1}}$$

Proof. As mentioned above in this case we have $J^{eq} = I^{eq}$. Recall that

$$J_V^{eq} = \exp\left(\frac{t_0 + pt_1}{\hbar}\right)\left(1 + \sum_{d>0} q^d e_{1*}\left(\frac{E_d' E_d^-}{\hbar(\hbar - c)}\right) \cup \prod_{j\in J}(-l_j p)\right).$$

We obtain the equivariant identity:

$$e_{1*}\left(\frac{E_d' E_d^-}{\hbar(\hbar - c)}\right) \cup \prod_{j\in J}(-l_j p) = \frac{\prod_{i\in I}\prod_{m=1}^{k_i d}(k_i p + m\hbar)\prod_{j\in J}\prod_{m=0}^{l_j d-1}(-l_j p - m\hbar)}{\prod_{m=1}^{d}\prod_{k=0}^{s}(p - \lambda_k + m\hbar)}.$$

The restriction of p to any fixed point p_i is nonzero. This implies that p is invertible. Therefore we obtain

$$e_{1*}\left(\frac{E_d' E_d^-}{\hbar(\hbar - c)}\right) = \frac{\prod_{i\in I}\prod_{m=1}^{k_i d}(k_i p + m\hbar)\prod_{j\in J}\prod_{m=1}^{l_j d-1}(-l_j p - m\hbar)}{\prod_{m=1}^{d}\prod_{k=1}^{s+1}(p - \lambda_k + m\hbar)}.$$

We can take the nonequivariant limit of this identity to obtain :

$$e_{1*}\left(\frac{E_d' E_d^-}{\hbar(\hbar - c)}\right) = \frac{\prod_{i\in I}\prod_{m=1}^{k_i d}(k_i p + m\hbar)\prod_{j\in J}\prod_{m=1}^{l_j d-1}(-l_j p - m\hbar)}{\prod_{m=1}^{d}(p + m\hbar)^{s+1}}.$$

The theorem is proven.†

This theorem is particularly useful when $\mathrm{Euler}(V^-) = 0$ in $\mathbb{P}^s$. In that case the mirror theorem is true trivially. An example of such a situation when $V = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ is treated on the chapter on the examples.

The only remaining case is $k + l = s + 1$. We have:

$$I^{eq} = \exp\left(\frac{t_0 + pt}{\hbar}\right)\left(1 + I_1\frac{p}{\hbar} + o(\frac{1}{\hbar^2})\right).$$

Lemma 4.6.1 *Assume that $Z(q,\hbar)$ satisfies conditions a,b,c of Lemma 4.5.1. Then $\bar{Z} = \exp(\frac{I_1 p}{\hbar})Z(qe^{I_1},\hbar)$ also satisfies those conditions.*

Proof. The first condition is obvious. Let's prove the second condition. We write the recursion relations for Z and multiply by $e^{\frac{I_1\lambda_i}{\hbar}}$.

$$\bar{Z}_i = e^{\frac{I_1\lambda_i}{\hbar}} Z_i(qe^{I_1}, \hbar) = e^{\frac{I_1\lambda_i}{\hbar}} + \sum_{m=1}^{\infty}(qe^{I_1})^m (e^{\frac{I_1\lambda_i}{\hbar}}) R_{i,m} +$$

$$+ \sum_{k\neq i}(qe^{I_1})^m C_{i,k,m} e^{\frac{I_1\lambda_i}{\hbar}} Z_k\left(qe^{I_1}, \frac{\lambda_k - \lambda_i}{m}\right)$$

with

$$C_{i,k,m} = \frac{\lambda_r - \lambda_i}{m\hbar(\lambda_i - \lambda_r + m\hbar)} \frac{\prod_{n=1}^{km}(k\lambda_i + n\frac{\lambda_i - \lambda_r}{m}) \prod_{n=0}^{lm-1}(-l\lambda_i - n\frac{\lambda_r - \lambda_i}{m})}{\prod_{n=1,(j,n)\neq(k,m)}^{m} \prod_{j=0}^{s}(\lambda_i - \lambda_j + n\frac{\lambda_r - \lambda_i}{m})} =$$

$$= \frac{R(\lambda)}{m\hbar(\lambda_i - \lambda_k + m\hbar)}$$

with $R(\lambda) \in \mathbb{Q}(\lambda)$. We have the following identity

$$\frac{I_1\lambda_i}{\hbar} = \frac{I_1\lambda_k}{\frac{\lambda_k - \lambda_i}{m}} - mI_1 + (m\hbar + \lambda_i - \lambda_k)\frac{I_1\lambda_i}{\hbar(\lambda_i - \lambda_k)}.$$

The punch line here is that

$$\exp\left(-mI_1 + (m\hbar + \lambda_i - \lambda_k)\frac{I_1\lambda_k}{\hbar(\lambda_i - \lambda_k)}\right) = e^{-mI_1} + (m\hbar + \lambda_i - \lambda_k) I_2(q, \hbar)$$

where

$$I_2(q, \hbar) \in \mathbb{Q}(\lambda)[\hbar^{-1}][[q]].$$

Therefore

$$\exp\left(\frac{I_1\lambda_i}{\hbar}\right) = \exp\left(\frac{I_1\lambda_k}{\frac{\lambda_k - \lambda_i}{m}}\right)(e^{-mI_1} + (m\hbar + \lambda_i - \lambda_k) I_2(q, \hbar)).$$

Now,

$$(qe^{I_1})^m C_{i,k,m} \exp\left(\frac{I_1\lambda_i}{\hbar}\right) Z_k\left(qe^{I_1}, \frac{\lambda_k - \lambda_i}{m}\right) = q^m e^{I_1 m} C_{i,k,m} \exp\left(\frac{I_1\lambda_k}{\frac{\lambda_k - \lambda_i}{m}}\right) \cdot$$

$$(e^{-mI_1} + (m\hbar + \lambda_i - \lambda_k) I_2(q, \hbar)) Z_k\left(qe^{I_1}, \frac{\lambda_k - \lambda_i}{m}\right) =$$

$$= q^m C_{i,k,m} \exp\left(\frac{I_1\lambda_k}{\frac{\lambda_k-\lambda_i}{m}}\right) Z_k\left(qe^{I_1}, \frac{\lambda_k-\lambda_i}{m}\right) +$$

$$q^m e^{I_1 m} \exp\left(\frac{I_1\lambda_k}{\frac{\lambda_k-\lambda_i}{m}}\right) \frac{R(\lambda)}{m\hbar} I_2(q,\hbar) Z_k = q^m C_{i,k,m} \bar{Z}_k\left(q, \frac{\lambda_k-\lambda_i}{m}\right) + \sum_{u=1}^{\infty} q^u R'_{i,u}(\hbar^{-1})$$

where

$$R'_{i,u}(\hbar^{-1}) \in \mathbb{Q}(\lambda)[\hbar^{-1}].$$

If we substitute everything in the formula for $\bar{Z}_i$ we get the recursion relation with the same coefficients.

We now check double polynomiality for $\bar{Z}$.

$$pf(\bar{Z}(qe^{z\hbar},\hbar)e^{pz}\bar{Z}(q,-\hbar)) = pf\left(e^{\frac{I_1(qe^{z\hbar})p}{\hbar}} Z(qe^{I_1(qe^{z\hbar})}e^{z\hbar},\hbar)e^{pz}e^{\frac{I_1(q)p}{-\hbar}} Z(qe^{I_1(q)},-\hbar)\right)$$

$$= pf\left(Z(qe^{z\hbar}e^{I_1(qe^{z\hbar})},\hbar)e^{p(z+\frac{I_1(qe^{z\hbar})-I_1(q)}{\hbar})} Z(qe^{I_1(q)},-\hbar)\right)$$

$$= W\left(qe^{I_1(q)}, z + \frac{I_1(qe^{z\hbar})-I_1(q)}{\hbar}\right)$$

where

$$W(q,z) = pf(Z(qe^{z\hbar},\hbar)e^{pz}Z(q,-\hbar)).$$

Since $I_1(qe^{z\hbar}) - I_1(q)$ vanishes for $\hbar = 0$ we find that the coefficients of the power series

$$\frac{I_1(qe^{z\hbar}) - I_1(q)}{\hbar}$$

are polynomials with respect to $\hbar$. The lemma is proven.†

4.7 Completing the proof

Notice that

$$\bar{S} = \exp\left(\frac{I_1 p}{\hbar}\right) S(qe^I,\hbar) \equiv S'(q,\hbar), \mathrm{mod}(\hbar^{-2})$$

Since $\bar{S}$ and S' satisfy the conditions of the Lemma 4.5.1 they are equal. The "-" transformation has an inverse which is the one prescribed in the formulation of the theorem.†

5 Examples

In this chapter we will see some aplications of the mirror theorem.

6 $\mathbb{P}^1$ with $V = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$

Let C be a rational curve in a Calabi-Yau threefold X with normal bundle $N = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and $\beta = [C] \in H_2(X, \mathbb{Z})$. Since $K_X = \mathcal{O}_X$, the expected dimension of the moduli space $\overline{\mathcal{M}}_{0,0}(X, d\beta)$ is zero. However this moduli space contains a component of positive dimension, namely $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, d)$. Indeed, let $f : \mathbb{P}^1 \to C$ be an isomorphism, and $g : \mathbb{P}^1 \to \mathbb{P}^1$ a degree d multiple cover. Then $f \circ g$ is a stable map that belongs to $\overline{\mathcal{M}}_{0,0}(X, d\beta)$. For a proof of the fact that $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, d)$ is a component of $\overline{\mathcal{M}}_{0,0}(X, d\beta)$ see section 7.4.4 in [8]. Let N_d be the degree of $[\overline{\mathcal{M}}_{0,0}(X, d\beta)]^{\text{virt}}$. We want to compute the contribution n_d of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, d)$ to N_d. Kontsevich asserted in [20] and Behrend proved in [2] that the restriction of $[\overline{\mathcal{M}}_{0,0}(X, d\beta)]^{\text{virt}}$ to $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, d)$ is precisely E_d for $V = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Therefore:

$$n_d = \int_{\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, d)} E_d.$$

Note that dim $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, d) = 2d - 2$ and the rank of the bundle V_d is also $2d - 2$. We use the mirror theorem to compute numbers n_d. Since V contains two negative line bundle we can apply Theorem 3.4.1. It says:

$$e_{1*}\left(\frac{E_d}{\hbar(\hbar - c)}\right) = \frac{\prod_{m=1}^{d-1}(-p - m\hbar)^2}{\prod_{m=1}^{d}(p + m\hbar)^2} = \frac{1}{(p + d\hbar)^2}.$$

An expansion of the left hand side using the divisor property for the modified gravitational descendants (see for example section 10.1.2 of [8]) gives:

$$e_{1*}\left(\frac{E_d}{\hbar(\hbar - c)}\right) = \frac{dn_d}{\hbar^2} + \frac{p}{\hbar^3}\int_{\overline{\mathcal{M}}_{0,1}(\mathbb{P}^1,d)} cE_d.$$

On the other hand:

$$\frac{1}{(p+d\hbar)^2} = \frac{1}{d^2\hbar^2} - \frac{2p}{d^3\hbar^3}.$$

We obtain:

$$n_d = \frac{1}{d^3},$$

a well known formula that was first found by Voisin [28]. For a proof of the formula in this form see [24]. We also get

$$\int_{\overline{\mathcal{M}}_{0,1}(\mathbb{P}^1,d)} cE_d = -\frac{2}{d^3}.$$

7 $\mathbb{P}^2$ with $V = \mathcal{O}_{\mathbb{P}^2}(-3)$

Let X be a Calabi-Yau threefold containing a $\mathbb{P}^2$. By the adjunction formula the normal bundle of $\mathbb{P}^2$ in X is $K_{\mathbb{P}^2} = \mathcal{O}(-3)$. Let $C = d[l]$ be a rational curve in $\mathbb{P}^2$. Since $K_X = \mathcal{O}_X$, the expected dimension of the moduli space $\overline{\mathcal{M}}_{0,0}(X,[C])$ is zero. On the other hand, this moduli space has a component, namely $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2,d)$, of dimension $3d-1$. In fact, we have shown that these moduli spaces coincide (Lemma 3.5.2) and the virtual fundamental class of $\overline{\mathcal{M}}_{0,0}(X,[C])$ is the the refined top Chern class of the bundle $R^1\pi_*(e_1^*(K_{\mathbb{P}^2}))$ over $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2,d)$ (Lemma 3.5.3). Here:

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,1}(\mathbb{P}^2,d) & \xrightarrow{e_1} & \mathbb{P}^2 \\ \downarrow{\scriptstyle \pi} & & \\ \overline{\mathcal{M}}_{0,0}(\mathbb{P}^2,d) & & \end{array}$$

By definition the zero pointed Gromov-Witten invariant:

$$N_d := \deg[\overline{\mathcal{M}}_{0,0}(X,[C])]^{\mathrm{virt}} = \int_{\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2,d)} E_d$$

is called **the virtual number of degree d rational curves in X.**

Consider the quantum product for $\mathbb{P}^2$ with $V = \mathcal{O}_{\mathbb{P}^2}(-3)$. The pairing is:

$$\langle a,b\rangle := \int_{(\mathbb{P}^2)_{\mathbb{C}^*}} a \cup b \cup \left(\frac{1}{-3p-\lambda}\right)$$

We compute the intersection matrix:

$$(g_{rs}) = \begin{pmatrix} -9\lambda^{-3} & 3\lambda^{-2} & -\lambda^{-1} \\ 3\lambda^{-2} & -\lambda^{-1} & 0 \\ -\lambda^{-1} & 0 & 0 \end{pmatrix}$$

Its inverse is:

$$(g^{rs}) = \begin{pmatrix} 0 & 0 & -\lambda \\ 0 & -\lambda & -3 \\ -\lambda & -3 & 0 \end{pmatrix}$$

Consider the basis $1, p, p^2$ for $\mathcal{R}$ as a $\mathbb{Q}(\lambda)$-module. Let $-\lambda p^2, -3p^2-\lambda p, -3p-\lambda$ be its dual basis. Since both bases and E_d are polynomials in λ, we can restrict $*_V$ in $\mathcal{P} = H^*(\mathbb{P}^2, \mathbb{Q}[\lambda])$ and take the nonequivariant limit of $*_V$. We obtain the following quantum product on $H^*\mathbb{P}^2 \otimes \mathbb{Q}[[q]]$

$$a *_V b := a \cup b + \sum_{d=1}^{\infty} q^d T^k I_d(a,b,-3pT_k)$$

where p is the hyperplane class in $\mathbb{P}^2$, and $T_0 = p^2, T_1 = p, T_2 = 1$ a basis for $H^*\mathbb{P}^2$ with its dual $T^0 = 1, T^1 = p, T^2 = p^2$. Also I_d is the nonequivariant limit of $\tilde{I}_d$ i.e. for $\gamma_1, \gamma_2, ..., \gamma_n \in H^*\mathbb{P}^2$

$$I_d(\gamma_1,\gamma_2,...,\gamma_n) = \int_{\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2,d)} e_1^*\gamma_1 \cup e_2^*\gamma_2 \cup ... \cup e_n^*\gamma_n \cup E_d.$$

For example, using the divisor axiom we obtain

$$p *_V p = p^2(1 - 3\sum_{d>0} q^d d^3 N_d).$$

Theorem 3.2.1 implies this:

Theorem 7.0.1 *$(H^*\mathbb{P}^2, *_V)$ is an associative,commutative and unital ring with unity $1 = [\mathbb{P}^2]$.*

Denote by i the embedding $i : \mathbb{P}^2 \hookrightarrow X$.

Lemma 7.0.1 *The map $i^* : (H^*X, \mathbb{Q}) \to (H^*\mathbb{P}^2, \mathbb{Q})$ is surjective.*

Proof. Since the normal bundle of $\mathbb{P}^2$ in X is $\mathcal{O}_{\mathbb{P}^2}(-3)$, it follows that

$$i^*[X] = T^0$$

$$i^*(-\frac{1}{3}[\mathbb{P}^2]) = T^1$$

$$i^*(-\frac{1}{3}[l]) = T^2.$$

The lemma is proven.†

Let [line] $\in H_2(X, \mathbb{Z})$ be the class of a line $l \subset \mathbb{P}^2$ and $[C_1] = [l], [C_2], ...[C_k]$ the generators of MX. Consider the small quantum cohomology ring of X

$$QH_s^*X = (H^*X \otimes \mathbb{Q}[q_1, q_2, ..., q_k], *)$$

and the new small quantum cohomology ring of $\mathbb{P}^2$

$$(H^*\mathbb{P}^2 \otimes \mathbb{Q}[[q]], *_V)$$

where the products are given by three point correlators. Recall the extension of i^* to:

$$\tilde{i}^* : H^*X \otimes \mathbb{Q}[[q_1, q_2, ..., q_k]] \mapsto H^*\mathbb{P}^2 \otimes \mathbb{Q}[[q]]$$

as follows : $\tilde{i}^*(q_i) = 0 : i = 2, ..., k$ and $\tilde{i}^*(q_1) = q$.

Theorem 7.0.2 *The map $\tilde{i}^*$ is a ring homomorphism.*

Proof. Complete $\{\tau^0 = [X], \tau^1 = -\frac{1}{3}[\mathbb{P}^2], \tau^2 = -\frac{1}{3}p\}$ into a basis of $(H^*X, \mathbb{Q})$ by adding elements from $\mathrm{Ker}(i^*)$. Let $\{\tau_0 = [pt], \tau_1 = p, \tau_2 = [\mathbb{P}^2], ...\}$ be the dual basis. Let $a, b \in H^*X$. We want to show

$$\tilde{i}^*(a * b) = i^*(a) *_V i^*(b).$$

But

$$a * b = \sum_{\beta \in MX} \sum_k q^\beta \tau^k \int_{[\overline{\mathcal{M}}_{0,3}(X,\beta)]^{virt}} e_1^* a \cup e_2^* b \cup e_3^* \tau_k.$$

Note that this formula is true for a $\mathbb{Z}$-basis, but due to the uniqueness of the quantum product, it is true for any $\mathbb{Q}$-basis as well. Therefore,

$$\tilde{i}^*(a * b) = \sum_{d \geq 0} \sum_k q^d i^*(\tau^k) \int_{\overline{\mathcal{M}}_{0,3}(\mathbb{P}^2,d)} e_1^*(i^*a) \cup e_2^*(i^*b) \cup e_3^*(i^*\tau_k) E_d.$$

Now, $i^*(\tau^k) = T^k$ for $k = 0, 1, 2$ and for the rest of generators $i^*(\tau^k) = 0$. The theorem follows from the readily checked fact: $i^*(\tau_k) = -3pT_k$ for $k = 0, 1, 2$.†

Using the divisor and fundamental class properties of the modified gravitational descendants (see section 10.1.2 of [8]) it is easy to show that:

$$J_V = \exp\left(\frac{t_0 + t_1 p}{\hbar}\right)\left(1 - 3\frac{p^2}{\hbar^2}\sum_{d=1}^{\infty} q^d d N_d\right).$$

On the other hand, consider the hypergeometric series corresponding to the total space of $\mathcal{O}_{\mathbb{P}^2}(-3)$:

$$I := \exp\left(\frac{t_0 + t_1 p}{\hbar}\right) \sum_{d=0}^{\infty} q^d \frac{\prod_{m=0}^{3d-1}(-3p - m\hbar)}{\prod_{m=1}^{d}(p + m\hbar)^3}.$$

We expand this function and obtain

$$I = \exp\left(\frac{t_0 + t_1 p}{\hbar}\right) \left(1 + I_1 \frac{p}{\hbar} + o(\frac{1}{\hbar})\right)$$

where

$$I_1 = 3\sum_{d=1}^{\infty} q^d (-1)^d \frac{(3d-1)!}{(d!)^3}.$$

The mirror theorem for this case says that the formal functions I and J coincide up to the change of variables $T_1 = t_1 + I_1$. This theorem allows us to compute the virtual number of rational plane curves in the Calabi-Yau X. The first few numbers are $N_1 = 3$,$N_2 = \frac{-45}{8}$,$N_3 = \frac{244}{9}$.

8 A mirror conjecture for a split projective bundle

The problem that we address here is the following. Let X be a smooth projective variety over the field of complex numbers and $V = \oplus_{j=0}^{n} \mathcal{L}_j$ a direct sum of line bundles on X. Consider

$$\begin{array}{c} \mathbb{P}(V) \\ \downarrow \pi \\ X \end{array}$$

We want to relate the quantum $\mathcal{D}$-modules of X and of the projective bundle $\mathbb{P}(V)$ over X. Givental has shown that these modules are generated by a single formal vector-valued function J, therefore, equivalently the problem is to relate $J_{\mathbb{P}(V)}$ with J_X.

8.1 The formulation of the conjecture

Let $z = c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$. Without loss of generality we can assume that $\mathcal{L}_0 = \mathcal{O}_X$. Then $H^*\mathbb{P}(V)$ as a H^*X-module is generated by z with the single Grothendieck relation

$$z \prod_{i=1}^{n} (z - c_1(\mathcal{L}_i)) = 0.$$

There exists the following short exact sequence

$$0 \to H_2\mathbb{P}^n \to H_2\mathbb{P}(V) \overset{\pi_*}{\to} H_2X \to 0 \tag{94}$$

which splits as follows. For $\beta \in H_2(X, \mathbb{Z})$, let $i(\beta)$ be such that $\pi_* i(\beta) = \beta$ and

$$\int_{i(\beta)} z = 0. \tag{95}$$

For example, we can define

$$i(\beta) = \pi^*(p.d.\beta) \prod_{j=1}^{n} (z - c_1(L_j))$$

where *p.d.* means Poincare dual.

Lemma 8.1.1 *If the line bundles $\mathcal{L}_i$ are nonnegative then z is a nef divisor.*

Proof. Let C be a curve in $\mathbb{P}(V)$ with $f : C \to \mathbb{P}(V)$ the inclusion map. We have the following surjection

$$\pi^*(V) \to \mathcal{O}_{\mathbb{P}(V)}(z) \to 0. \tag{96}$$

Restricting this sequence to C we obtain

$$\oplus_i f^*(\mathcal{L}_i) \to \mathcal{O}_C(z \cdot C) \to 0. \tag{97}$$

Since $\deg f^*(\mathcal{L}_i) \geq 0$ for all i, we obtain that $z \cdot C \geq 0$.†

The basis for our conjecture is the following lemma.

Lemma 8.1.2 *If the line bundles $\mathcal{L}_i$ are nonnegative then*

$$M\mathbb{P}(V) = MX \oplus \mathbb{Z}_{\geq 0} \cdot [\text{line}] \tag{98}$$

where [line] *is the class of a line in the fiber of π.*

Proof. Let C be a curve in $\mathbb{P}(V)$. Then $C' = C - i(\pi_*(C))$ satisfies $\pi_*(C') = 0$. Therefore $C' = n \cdot [\text{line}]$. From the definition of i we conclude that $n = z \cdot C$. From Lemma 8.1.1 we get that $n \geq 0$.†

It follows that if $C \in \mathbb{P}(V)$ is a curve, there exist unique $\beta \in MX$ and d such that $[C] = \beta + d[\text{line}]$. We will use the notation $[C] = (d, \beta)$.

Clearly we have

$$\langle z - \pi^* c_1(\mathcal{L}_i), i(\beta)\rangle_{\mathbb{P}(V)} = -\langle c_1(\mathcal{L}_i), \beta\rangle_X.$$

Let $p_1, p_2, ..., p_k$ be the generators of the nef cone of X and $\beta_1, ..., \beta_k$ a $\mathbb{Z}_{\geq 0}$-basis for MX. Recall

$$\overline{\mathcal{M}}_{0,1}(X, \beta) \xrightarrow{e_1} X,$$

the evaluation map at the marked point. Let $tp = \sum_{i=1}^{k} t_i p_i$ and

$$J_\beta = e_{1*}\left(\frac{1}{\hbar(\hbar - c)}\right),$$

where c is the first Chern class of the cotangent line bundle at the marked point on $\overline{\mathcal{M}}_{0,1}(X, \beta)$. The generator of the quantum $\mathcal{D}$-module for the quantum cohomology of X is [13]

$$J_X = \exp\left(\frac{t_0 + tp}{\hbar}\right) \sum_{\beta \in MX} q^\beta J_\beta. \tag{99}$$

Define the following Givental type hypergeometric series

$$I_{\mathbb{P}(V)} := \exp\left(\frac{t_0 + tp + t_{k+1} z}{\hbar}\right) \cdot$$

$$\sum_{\beta \in MX; d \geq 0} {q_1}^d {q_2}^\beta \prod_{i=0}^{n} \frac{\prod_{m=-\infty}^{0} (z - c_1(\mathcal{L}_i) + m\hbar)}{\prod_{m=-\infty}^{d - \langle \beta, c_1(\mathcal{L}_i) \rangle_X} (z - c_1(\mathcal{L}_i) + m\hbar)} \pi^* J_\beta.$$

There exist the following exact sequences

$$0 \to \pi^* \Omega_X^1 \to \Omega_{\mathbb{P}(V)}^1 \to \Omega_{\mathbb{P}(V)/X}^1 \to 0$$

$$0 \to \Omega_{\mathbb{P}(V)/X}^1 \to \pi^* V \otimes \mathcal{O}_{\mathbb{P}(V)}(-z) \to \mathcal{O}_{\mathbb{P}(V)} \to 0.$$

These imply that

$$K_{\mathbb{P}(V)} = \pi^* K_X + \pi^* c_1(V) - (n+1) z. \tag{100}$$

Let (d, β) be a curve class in $H_2 \mathbb{P}(V)$. We obtain

$$-(d, \beta) \cdot K_{\mathbb{P}(V)} = (n+1)d - \beta \cdot K_X - \beta \cdot \pi^* c_1(V).$$

Therefore

$$\dim[\overline{\mathcal{M}}_{0,k}(\mathbb{P}(V),(d,\beta))]^{\text{virt}} = \dim[\overline{\mathcal{M}}_{0,k}(X,\beta)]^{\text{virt}} + n + (n+1)d - \beta \cdot \pi^* c_1(V) =$$

$$\dim[\overline{\mathcal{M}}_{0,k}(X,\beta)]^{\text{virt}} + n - \sum_i \int_\beta c_1(\mathcal{L}_i) + (n+1)d. \qquad (101)$$

Conjecture 8.1.1 *There exists a mirror transformation from $I_{\mathbb{P}(V)}$ to $J_{\mathbb{P}(V)}$ of the form*

$$t_0' = t_0 + f_0(q)\hbar + f(q)$$

$$t_i' = t_i + f_i(q).$$

8.2 Evidence for the conjecture

First, it follows from Givental's work [12] that the conjecture is true in the case of toric varieties where V is a direct sum of toric line bundles.

Before we mention another evidence for the conjecture, we need to state the Quantum Lefschetz Principle [19]. Let Y be convex (for simplicity) and $\beta \in H_2(Y,\mathbb{Z})$. Let $Z \subset Y$ be the zero locus of a vector bundle $W = \oplus_i \mathcal{L}_i$. We recall from Lemma 3.4.1 the bundle W_d' over $\mathcal{M}_{0,1}(Y,\beta)$ whose fiber over a stable map (C, x_1, f) is the sections of $H^0(C, f^*(W))$ that vanish at x_1. Let

$$J_W = \exp\left(\frac{t_0 + t_p}{\hbar}\right) \text{Euler}(W) \left(1 + \sum_{\beta \neq 0} q^\beta e_{1*}\left(\frac{W_\beta'}{\hbar(\hbar - c)}\right)\right). \qquad (102)$$

This is the generator of the quantum $\mathcal{D}$-module of Z. Consider also

$$I_W = \exp\left(\frac{t_0 + tp}{\hbar}\right) \text{Euler}(W) \times$$

$$\left(1+\sum_{\beta\neq 0} q^{\beta} \prod_{i} \frac{\prod_{m=-\infty}^{f_\beta(c_1(\mathcal{L}_i))}(c_1(\mathcal{L}_i)+m\hbar)}{\prod_{m=-\infty}^{0}(c_1(\mathcal{L}_i)+m\hbar)} e_1 \left(\frac{1}{\hbar(\hbar-c)}\right)\right). \tag{103}$$

The Quantum Lefschetz Principle (which is not yet proven in general) asserts that under suitable conditions J_W equals I_W after a change of variables of the form

$$t_0' = t_0 + f_0(q)\hbar + f(q)$$

$$t_i' = t_i + f_i(q).$$

The second evidence for the conjecture comes from this proposition.

Proposition 8.2.1 *Conjecture* 6.1.1 *together with the Quantum Lefschetz Principle produces the generator J_X of the quantum $\mathcal{D}$-module of X.*

Proof. Indeed, X_0 is the complete intersection of the divisors

$$z - c_1(\mathcal{L}_i) : i = 1, 2, ..., n$$

in $\mathbb{P}(V)$. Consider the following cohomology-valued function

$$I = \exp\left(\frac{t_0 + tp + t_{k+1}z}{\hbar}\right) \prod_{i=1}^{n}(z - c_1(L_i)) \sum_{\beta\in\Lambda; d\geq 0} {q_1}^d {q_2}^\beta \frac{1}{\prod_{m=1}^{d}(z+m\hbar)}\cdot$$

$$\prod_{k=1}^{n} \frac{\prod_{m=-\infty}^{d-<\beta, c_1(\mathcal{L}_k)>_X}(z - c_1(\mathcal{L}_k)+m\hbar)}{\prod_{m=-\infty}^{0}(z - c_1(\mathcal{L}_k)+m\hbar)} \prod_{i=0}^{n} \frac{\prod_{m=-\infty}^{0}(z - c_1(\mathcal{L}_i)+m\hbar)}{\prod_{m=-\infty}^{d-<\beta, c_1(\mathcal{L}_i)>_X}(z - c_1(\mathcal{L}_i)+m\hbar)} \pi^* J_\beta. \tag{104}$$

We can expand the factor

$$\frac{1}{\prod_{m=1}^{d}(z+m\hbar)}$$

and the exponential part as a power series in z. Using

$$z\prod_{i=1}^{n}(z - c_1(\mathcal{L}_i)) = 0$$

we find

$$I = J_{X_0} \sum_{d=0}^{\infty} (\frac{q_1}{\hbar})^d \frac{1}{d!} = J_{X_0} \exp(\frac{q_1}{\hbar}).$$

Make a change of variables ${t'}_0 = t_0 + q_1$. With this new variables we have $I = J_{X_0}$. The proposition is proven.†

8.3 An equivariant version of the conjecture

We will assume that there is a torus action T in X and the fixed point locus is finite. Furthermore, assume that the line bundles $\mathcal{L}_i$ are T-equivariant. Introduce an action of another torus T' of rank $n+1$ on the total space of the projective bundle. T' acts trivially on X and by scaling on the fiber of $\mathbb{P}(V)$. We have therefore an action of the big torus $T \times T'$ on $\mathbb{P}(V)$. One can see that $X_{T\times T'} = X_T \times (\mathbb{P}^\infty)^{n+1}$. Let π_i be the i-th projection from $(\mathbb{P}^\infty)^n$ and ${\mathcal{L}'}_i = (\mathcal{L}_i)_T \otimes \pi_i^* \mathcal{O}_{\mathbb{P}_i^\infty}(\lambda'_i)$ a line bundle over $X_{T\times T'}$. Then $\mathbb{P}_{T\times T'}(V) = \mathbb{P}(\oplus_{i=1}^n {\mathcal{L}'}_i)$. Therefore we have the following presentation of $H^*_{T\times T'}\mathbb{P}(V)$ as a $H^*_T X$-module:

$$H^*_{T\times T'}(\mathbb{P}V) = H^*_T X[z, \lambda'_0, \lambda'_1, ..., \lambda'_n] / \prod_{i=0}^{n} (z - c_1((\mathcal{L}_i)_T) - \lambda'_i). \tag{105}$$

We will also consider only the fibrewise action on $\mathbb{P}(V)$. In that case we have

$$H^*_{T'}(\mathbb{P}V) = H^* X[z, \lambda'_0, \lambda'_1, ..., \lambda'_n] / \prod_{i=0}^{n} (z - c_1(\mathcal{L}_i) - \lambda'_i). \tag{106}$$

Let p_r for $r = 1, 2, ..., s$ be the fixed points of the T-action on X with $\phi_1, \phi_2, ..., \phi_s$ their equivariant Thom classes. Let $H_{i,r}(\lambda)$ be the restriction of $c_1(\mathcal{L}_i)_T$ to the fixed point p_r. Also, let s_j be sections of π given by the natural projection

$$\oplus_{i=0}^n (\mathcal{L}_i) \to \mathcal{L}_j. \tag{107}$$

Denote by X_j the image of this section in $\mathbb{P}(V)$. Then the T' equivariant Euler class of the normal bundle $\mathcal{N}_j$ of the section X_j in $\mathbb{P}(V)$ is

$$\psi_j = \prod_{i \neq j} (c_1(\mathcal{L}_j)_T - c_1(\mathcal{L}_i)_T + \lambda'_j - \lambda'_i). \tag{108}$$

Let $p_{j,r}$ be the fixed points for the action of the big torus $T \times T'$ on $\mathbb{P}(V)$. The point $p_{j,r}$ corresponds to the T-fixed point p_r of X embedded on $\mathbb{P}(V)$ by the section s_j. The $T \times T'$-equivariant Euler class of the tangent space of $\mathbb{P}(V)$ at $p_{j,r}$ is

$$E_{i,r} = \phi_r \prod_{i \neq j} (H_{j,r}(\lambda) - H_{i,r}(\lambda) + \lambda'_j - \lambda'_i). \tag{109}$$

We formulate an equivariant version of the conjecture. Let

$$I^{eq}_{\mathbb{P}(V)} := \exp\left(\frac{t_0 + tp + t_{k+1}z}{\hbar}\right) \cdot$$

$$\sum_{\beta \in MX; d \geq 0} {q_1}^d {q_2}^\beta \prod_{i=0}^{n} \frac{\prod_{m=-\infty}^{0} (z - c_1(\mathcal{L}_i) - \lambda'_i + m\hbar)}{\prod_{m=-\infty}^{d - \langle \beta, c_1(\mathcal{L}_i) \rangle_X} (z - c_1(\mathcal{L}_i) - \lambda'_i + m\hbar)} \pi^* J^{eq}_\beta. \tag{110}$$

Conjecture 8.3.1 *There exists an equivariant mirror transformation from* $I^{eq}_{\mathbb{P}(V)}$ *to* $J^{eq}_{\mathbb{P}(V)}$.

One can hope to prove this conjecture by exhibiting similar properties of $I^{eq}_{\mathbb{P}(V)}$ and $J^{eq}_{\mathbb{P}(V)}$. Recently a group of authors [27] have suggested the extension of the mirror theorems to "balloon manifolds". It is not clear to us whether their approach would work in the case we are interested.

8.4 Double polynomiality

We here prove a general lemma.

Let Y be a smooth projective variety, D_1,D_2,...,D_k the generators of the cone of effective divisors, C_1, C_2,...,C_k generators of the Mori cone MY of Y. Assume that D_i is base point free for all i. Let $tD = \sum_i t_i D_i$. Recall

$$J_Y = \exp\left(\frac{t_0 + tD}{\hbar}\right) \sum_{\beta \in MY} q^\beta e_{1*}\left(\frac{[\mathcal{M}_{0,1}(Y,\beta)]^{\mathrm{virt}}}{\hbar(\hbar - c)}\right) = \exp\left(\frac{tD}{\hbar}\right) S(q,\hbar), \qquad (111)$$

the generator of the quantum $\mathcal{D}$-module of Y. Here $\beta = \sum_i d_i C_i$ and $q^\beta = e^{td} = e^{\sum_i t_i d_i}$. Let $z = (z_1, z_2, ..., z_k)$ and $(qe^{z\hbar})^\beta = \prod_i (q_i e^{z_i \hbar})^{d_i}$. For any $i = 1, 2, ..., k$ let the map

$$\psi_i : Y \to \mathbb{P}^{m_i} \qquad (112)$$

be given by the complete linear system $|D_i|$. Let $M_\beta Y := \overline{\mathcal{M}}_{0,0}(Y \times \mathbb{P}^1, \beta + 1)$. The map ψ_i gives rise to the map:

$$\phi_i : M_\beta Y \to M_{d_i}\mathbb{P}^{m_i} \to N_{d_i}\mathbb{P}^{m_i}. \qquad (113)$$

The $\mathbb{C}^*$-action on $\mathbb{P}^1$ with weights $(\hbar, 0)$ gives rise to a $\mathbb{C}^*$ action on the space $M_\beta Y$. Let $0 = [1,0] \in \mathbb{P}^1$ and $\infty = [0,1] \in \mathbb{P}^1$. The fixed point components of this action correspond to partitions $\beta_1 + \beta_2 = \beta$. For a partition (β_1, β_2) of β, the corresponding fixed point component M_{β_1,β_2} consists of stable maps of the following form

$$f : C = C_0 \cup C_\infty \cup C_1 \to Y \times \mathbb{P}^1 \qquad (114)$$

where f maps C_0 to $Y \times \{0\}$ with homology class β_1, C_∞ to $Y \times \{\infty\}$ with homology class β_2 and $C \simeq \mathbb{P}^1$ to $\{y\} \times \mathbb{P}^1$ with degree 1. Let $\beta_j = \sum_i d_{ji} C_i$ for $j = 1, 2$ and $\psi_i(y) = [z_0^y, z_1^y, ..., z_{m_i}^y] \in \mathbb{P}^{m_i}$. Let $x_0 = C_0 \cap C_1$ and $x_\infty = C_\infty \cap C_1$. On M_{β_1,β_2} the map ϕ_i has the following form

$$\phi_i|_{M_{\beta_1,\beta_2}} = [z_0^y w_0^{d_{1i}} w_1^{d_{2i}}, ..., z_{m_i}^y w_0^{d_{1i}} w_1^{d_{2i}}]. \qquad (115)$$

Consider the following diagram

$$\begin{array}{ccc} \mathcal{M}_{\beta_1,\beta_2} & \xrightarrow{\quad e \quad} & Y \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\delta} \\ \mathcal{M}_{0,1}(Y,\beta_1)\times\mathcal{M}_{0,1}(Y,\beta_2) & \xrightarrow[\phi]{} & Y\times Y \end{array}$$

where δ is the diagonal embedding and ρ is the natural inclusion.

Definition 8.4.1 *We will say that Y is* **semiconvex** *if it satisfies the following condition*

$$\rho^*([\mathcal{M}_{0,1}(Y,\beta_1)]^{\text{virt}}\otimes[\mathcal{M}_{0,1}(Y,\beta_2)]^{\text{virt}})=[\mathcal{M}_{\beta_1,\beta_2}]^{\text{virt}}. \tag{116}$$

Examples of semiconvex varieties are convex varieties (see Section 6 of [13]).

Let Δ be the diagonal in $Y\times Y$. Note that the fixed point component $\mathcal{M}_{\beta_1,\beta_2}$ is isomorphic to $\phi^{-1}(\Delta)$. It follows that for semiconvex varieties

$$\rho_*([\mathcal{M}_{\beta_1,\beta_2}]^{\text{virt}})=([\mathcal{M}_{0,1}(Y,\beta_1)]^{\text{virt}}\otimes[\mathcal{M}_{0,1}(Y,\beta_2)]^{\text{virt}})\cdot\phi^*(\Delta). \tag{117}$$

We believe that all varieties are semiconvex. However, in trying to prove this, we are presented with technical difficulties which we are not able to overcome at this point.

Recall that $N_{d_i}\mathbb{P}^{m_i}$ is the linear sigma model for the degree d_i stable maps to $\mathbb{P}^{m_i}$, i.e. the projective space of (m_i+1)-tuples of degree d_i-polynomials in two variables w_0 and w_1. Let κ_i be the equivariant hyperplane class in $N_{d_i}\mathbb{P}^{m_i}$. Let

$$P_i:=\phi_i^*(\kappa_i) \tag{118}$$

and $Pz:=\sum_i P_iz_i$.

Lemma 8.4.1 *Let Y be a semiconvex variety. The following holds*

$$\int_Y S(qe^{z\hbar})e^{Dz}S(q,-\hbar)=\sum_{\beta\in MY}q^\beta\int_{[M_\beta Y]^{virt}_{C^*}}e^{Pz} \tag{119}$$

where the integral on the right side is the equivariant push forward to a point.

Proof. We will apply the localization formula to the right side of (119) and show that it is equal to the left side. Consider the following diagram

$$\begin{array}{ccccc} & & \mathcal{M}_{\beta_1,\beta_2} & \xrightarrow{\phi_i} & N_{d_i}\mathbb{P}^{m_i} \\ & \swarrow^{\pi_2} & & & \uparrow \rho_i \\ \mathcal{M}_{0,1}(Y,\beta_1) & \xrightarrow[e_1]{} & Y & \xrightarrow[\psi_i]{} & \mathbb{P}^{m_i} \end{array}$$

Let $\chi = \rho_i \circ \psi_i \circ e_1$. The map $\rho_i : \mathbb{P}^{m_i} \to N_{d_i}\mathbb{P}^{m_i}$ is given by

$$\rho_i(z_0, z_1, ..., z_{m_i}) = [z_0 w_0^{d_{1i}} w_1^{d_{2i}}, ..., z_{m_i} w_0^{d_{1i}} w_1^{d_{2i}}]. \tag{120}$$

It can be written as the composition of the following three maps:

- $I_{d_{1i}} : \mathbb{P}^{m_i} \to N_{d_{1i}}\mathbb{P}^{m_i}$ given by $I_{d_{1i}}([z_0, ..., z_{m_i}]) = [z_0 w_0^{d_{1i}}, ..., z_{m_i} w_0^{d_{1i}}]$,
- $- : N_{d_{1i}}\mathbb{P}^{m_i} \to N_{d_{1i}}\mathbb{P}^{m_i}$ given by the permutation of variables i.e.

$$-[f_0(w_0, w_1), ...] = [f_0(w_1, w_0), ...],$$

- $I_{d_{2i}} : N_{d_{2i}}\mathbb{P}^{m_i} \to N_{d_i}\mathbb{P}^{m_i}$ given by $I_{d_{2i}}([f_0, ..., f_{m_i}]) = [f_0 w_0^{d_{2i}}, ..., f_{m_i} w_0^{d_{2i}}]$.

Therefore we obtain

$$\rho_i^*(\kappa_i) = H_i + d_{1i}\hbar \tag{121}$$

where H_i is the hyperplane class of $\mathbb{P}^{m_i}$. Consequently

$$\chi_i^*(\kappa_i) = e_1^*(D_i) + d_{1i}\hbar. \tag{122}$$

Let c_0 and c_∞ be the Chern classes of the cotangent line bundles at the marked point of $\overline{\mathcal{M}}_{0,1}(Y, \beta_1)$ and $\overline{\mathcal{M}}_{0,1}(Y, \beta_2)$ respectively. We recall the deformation-obstruction exact sequence for the moduli space $\mathcal{M}_\beta$

$$0 \to \mathrm{Ext}^0(\Omega_C, \mathcal{O}_C) \to \mathrm{H}^0(C, f^*TX) \to \mathcal{T}_M \to$$

$$\to \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C) \to \mathrm{H}^1(C, f^*TX) \to \Upsilon \to 0. \tag{123}$$

We restrict it to the fixed point component $\mathcal{M}_{\beta_1,\beta_2}$ to compute the Euler class of its virtual normal bundle. Consider the normalization sequence at the nodes x_0 and x_∞

$$0 \to \mathcal{O}_C \to \mathcal{O}_{C_0} \oplus \mathcal{O}_{C_\infty} \oplus \mathcal{O}_{C_1} \to \mathcal{O}_{x_0} \oplus \mathcal{O}_{x_\infty} \to 0. \tag{124}$$

Twist by $f^*(TX)$ and take the cohomology sequence. We obtain the following K-theory identity

$$\begin{gathered}
[H^0(C, f^*(TX))] - [H^1(C, f^*(TX))] = [H^0(C_0, f^*(TX))] - [H^1(C_0, f^*(TX))] + \\
[H^0(C_\infty, f^*(TX))] - [H^1(C_\infty, f^*(TX))] + [H^0(C_1, f^*(TX))] - \\
[H^1(C_1, f^*(TX))] - [T_{f(x_0)}X] - [T_{f(x_\infty)}X] = [H^0(C_0, f^*(TY))] - \\
[H^1(C_0, f^*(TY))] + [H^0(C_\infty, f^*(TY))] - [H^1(C_\infty, f^*(TY))] + [H^0(\mathbb{P}^1, T\mathbb{P}^1)] - [T_{f(x_0)}Y].
\end{gathered} \tag{125}$$

For the infinitesimal deformations of C we obtain

$$\begin{gathered}
[\mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)] = [\mathrm{Ext}^1(\Omega_{C_0}(x_0), \mathcal{O}_{C_0})] + [\mathrm{Ext}^1(\Omega_{C_\infty}(x_\infty), \mathcal{O}_{C_\infty})] + \\
[T_{x_0}C_0 \otimes T_0\mathbb{P}^1] + [T_{x_\infty}C_\infty \otimes T_\infty\mathbb{P}^1].
\end{gathered} \tag{126}$$

The last two terms correspond to smoothing the nodes x_0 and x_∞.

Finally, for the infinitesimal automorphisms of C we have

$$\begin{gathered}
[\mathrm{Ext}^0(\Omega_C, \mathcal{O}_C)] = [\mathrm{Ext}^0(\Omega_{C_0}(x_0), \mathcal{O}_{C_0})] + [\mathrm{Ext}^0(\Omega_{C_\infty}(x_\infty), \mathcal{O}_{C_\infty})] + \\
[\mathrm{Ext}^0(\Omega_{C_1}(x_0 + x_\infty), \mathcal{O}_{C_1})].
\end{gathered} \tag{127}$$

From the exact sequence (123) we obtain

$$[\text{Euler}(\mathcal{T}_M)] - [\text{Euler}(\Upsilon)] = [H^0(C, f^*(TX))] - [H^1(C, f^*(TX))] +$$

$$[\text{Ext}^1(\Omega_C, \mathcal{O}_C)] - [\text{Ext}^0(\Omega_C, \mathcal{O}_C)]. \tag{128}$$

We restrict to the moving part (in the terminology of [15]) and use (125), (126) and (127) to obtain

$$[\text{Euler}^m(\mathcal{T}_M)] - [\text{Euler}^m(\Upsilon)] = [H^0(\mathbb{P}^1, T\mathbb{P}^1)] +$$

$$[T_{x_0}C_0 \otimes T_0\mathbb{P}^1] + [T_{x_\infty}C_\infty \otimes T_\infty\mathbb{P}^1] - [\text{Ext}^0(\Omega_{C_1}(x_0 + x_\infty), \mathcal{O}_{C_1})]. \tag{129}$$

The third term is actually fixed. It will cancel with a fixed one dimensional piece coming from $[H^0(\mathbb{P}^1, T\mathbb{P}^1)]$. The only parts that survive correspond to smoothing the two nodes and moving them along C. We obtain

$$E([\mathcal{N}_{\beta_1,\beta_2}]^{\text{virt}}) = [\text{Euler}^m(\mathcal{T}_M)] - [\text{Euler}^m(\Upsilon)] = -\hbar(-\hbar - c_0)\hbar(\hbar - c_\infty). \tag{130}$$

Let $\{T_a\}$ and $\{T^a\}$ be dual basis in Y so that $[\Delta] = \sum_a T_a \otimes T^a$. Using (122) and the fact that Y is semiconvex, we compute

$$\sum_{\beta \in MY} q^\beta \int_{[M_\beta Y]^{\text{virt}}_{C^*}} e^{Pz} = \sum_{\beta_1 \in MY} \sum_{\beta_2 \in MY} q^{\beta_1} q^{\beta_2} \int_{[\mathcal{M}_{0,1}(Y,\beta_2)]^{\text{virt}}} \frac{e_1^*(T^a e^{(D+d_1\hbar)z})}{\hbar(\hbar - c)}.$$

$$\int_{[\mathcal{M}_{0,1}(Y,\beta_1)]^{\text{virt}}} \frac{e_1^*(T_a)}{-\hbar(-\hbar - c)} = \int_Y S(qe^{z\hbar}, \hbar) e^{Dz} S(q, \hbar).$$

The lemma is proven.†

8.5 Recursion relations

We will prove here some recursion relations for the hypergeometric series $I_{\mathbb{P}(V)}$ in general, i.e. without the assumption of the torus action on X. Let

$$S = \sum_{\beta \in MX; d \geq 0} {q_1}^d {q_2}^\beta \prod_{j=0}^{n} \frac{\prod_{m=-\infty}^{0} (z - c_1(\mathcal{L}_j) - \lambda'_j + m\hbar)}{\prod_{m=-\infty}^{d-\langle \beta, c_1(\mathcal{L}_j)\rangle_X} (z - c_1(\mathcal{L}_j) - \lambda'_j + m\hbar)} \pi^* J_\beta$$

and

$$S_i = \sum_{\beta \in MX; d \geq 0} {q_1}^d {q_2}^\beta \prod_{j=0}^{n} \frac{\prod_{m=-\infty}^{0} (c_1(\mathcal{L}_i) + \lambda'_i - c_1(\mathcal{L}_j) - \lambda'_j + m\hbar)}{\prod_{m=-\infty}^{d-\langle \beta, c_1(\mathcal{L}_j)\rangle_X} (c_1(\mathcal{L}_i) + \lambda'_i - c_1(\mathcal{L}_j) - \lambda'_j + m\hbar)} \pi^* J_\beta, \quad (131)$$

the restriction of S to the section X_i. For any $k > 0$ let

$$h_{ijk} = \frac{c_1(\mathcal{L}_j) + \lambda'_j - c_1(\mathcal{L}_i) - \lambda'_i}{k}.$$

Lemma 8.5.1 *The hypergeometric series S_i satisfies the following recursion relations:*

$$S_i = 1 + \sum_{(d,\beta) \neq (0,0)} q_1^d q_2^\beta P_{id\beta}(\hbar)$$

$$+ \sum_{k>0} \sum_{j \neq i} q_1^k \frac{C_{ij}(k)}{(c_1(\mathcal{L}_i) - c_1(\mathcal{L}_j) + \lambda'_i - \lambda'_j + k\hbar)} S_j(\hbar = h_{ijk}) \quad (132)$$

where $P_{id\beta}$ are Laurent polynomials in $\hbar$ and $C_{ij}(k)$ are a set of $H^*X \otimes \mathbb{Q}(\lambda')$-valued coefficients.

Proof. The hypergeometric series S_i has poles at $\hbar = 0$ or $\hbar = h_{ijk}$ for some $j \neq i$ and some $k > 0$. The latter poles only arise from the coefficients of $q_1^d q_2^\beta$ with $d - \int_\beta c_1(\mathcal{L}_j) \geq k > 0$.

We compute

$$k\cdot \mathrm{Res}_{h_{ijk}} S_i = \sum_{d-\int_\beta c_1(\mathcal{L}_j)\geq k} q_1^d {q_2}^\beta \overline{S}_{i,j,k}^{d,\beta}(\hbar) = \sum_{d-\int_\beta c_1(\mathcal{L}_j)\geq k} q_1^d {q_2}^\beta \frac{\prod_{m=-\infty}^0 mh_{ijk}}{\prod_{m=-\infty}^{d-\int_\beta c_1(\mathcal{L}_i)} mh_{ijk}}$$

$$\prod_{a\neq i,(m,a)\neq(k,j)} \frac{\prod_{m=-\infty}^0 (c_1(\mathcal{L}_i)+\lambda_i'-c_1(\mathcal{L}_a)-\lambda_a'+mh_{ijk})}{\prod_{m=-\infty}^{d-\langle\beta,c_1(\mathcal{L}_a)\rangle x}(c_1(\mathcal{L}_i)+\lambda_i'-c_1(\mathcal{L}_a)-\lambda_a'+mh_{ijk})}\pi^* J_\beta. \tag{133}$$

The goal is to find a presentation

$$S_i = \sum_{j\neq i}\sum_{k=1}^{\infty} q_1^k \frac{C_{ij}(k)S_j(\hbar=h_{ijk})}{(c_1(\mathcal{L}_i)-c_1(\mathcal{L}_j)+\lambda_i'-\lambda_j'+k\hbar)} + \sum_{d,\beta} q_1^d q_2^\beta P_{id\beta}(\hbar) \tag{134}$$

where $P_{id\beta}(\hbar)$ is a Laurent polynomial in $\hbar$ and $C_{ij}(k)$ are some suitable coeficients. We will do this by substracting the polar parts of S_i at h_{ijk}'s. Equation (134) is equivalent to

$$k\cdot \mathrm{Res}_{h_{ijk}} S_i = q_1^k C_{ij}(k) S_j(\hbar = h_{ijk}). \tag{135}$$

Let $S_j(\hbar)=\sum_{d,\beta} q_1^d q_2^\beta S_{j,d,\beta}(\hbar)$. Then to prove (135) we must show that

$$\overline{S}_{i,j,k}^{d,\beta} = C_{ij}(k) S_{j,d-k,\beta} \tag{136}$$

or

$$C_{ij}(k)\prod_a \frac{\prod_{m=-\infty}^0 (c_1(\mathcal{L}_j)+\lambda_j'-c_1(\mathcal{L}_a)-\lambda_a'+mh_{ijk})}{\prod_{m=-\infty}^{d-k-\int_\beta c_1(\mathcal{L}_a)}(c_1(\mathcal{L}_j)+\lambda_j'-c_1(\mathcal{L}_a)-\lambda_a'+mh_{ijk})} =$$

$$\frac{\prod_{m=-\infty}^0 mh_{ijk}}{\prod_{m=-\infty}^{d-\int_\beta c_1(\mathcal{L}_i)} mh_{ijk}} \prod_{a\neq i,(m,a)\neq(k,j)} \frac{\prod_{m=-\infty}^0 (c_1(\mathcal{L}_i)+\lambda_i'-c_1(\mathcal{L}_a)-\lambda_a'+mh_{ijk})}{\prod_{m=-\infty}^{d-\int_\beta c_1(\mathcal{L}_a)}(c_1(\mathcal{L}_i)+\lambda_i'-c_1(\mathcal{L}_a)-\lambda_a'+mh_{ijk})}. \tag{137}$$

A careful investigation of S_i reveals that the coeficient of $q_1^d q_2^\beta$ in S_i is zero unless $d-\int_\beta c_1(\mathcal{L}_i)\geq 0$. Also recall that $d-k-\int_\beta c_1(\mathcal{L}_j)\geq 0$. It follows that both sides of the equation (137) are nonzero. We can then find the coefficients $C_{ij}(k)$. For $a \neq j$ we use the following identity

$$c_1(\mathcal{L}_j) - c_1(\mathcal{L}_a) + \lambda'_j - \lambda'_a + mh_{ijk} = c_1(\mathcal{L}_i) - c_1(\mathcal{L}_a) + \lambda'_i - \lambda'_a + (m+k)h_{ijk}. \quad (138)$$

Substituting this identity in the left side of (137) after some algebraic transformations we obtain:

$$C_{ij}(k) = \frac{1}{\prod_a \prod_{m=1,(a,m)\neq(j,k)}^{k} (c_1(\mathcal{L}_i) + \lambda'_i - c_1(\mathcal{L}_a) - \lambda'_a)}. \quad (139)$$

The lemma is proven.†

8.6 What is left to complete the proof of this conjecture

Three more things are needed to complete the proof of this conjecture.

First, the double polynomiality condition for the hypergeometric series $I_{\mathbb{P}(V)}$. This needs an understanding of the relations between the linear sigma models of X and that of $\mathbb{P}(V)$.

Second, the recursion relations for the series $J_{\mathbb{P}(V)}$.

Third, a uniqueness result which determines $J_{\mathbb{P}(V)}$ uniquely from the three properties displayed here. This should not be much different from the case of a toric variety.

These will be left for future work.

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