DIVISOR LABELLING OF STAIRCASE DIAGRAMS AND FIBER BUNDLE STRUCTURES ON SCHUBERT VARIETIES

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Abstract: Let $\mathrm{Gr}(r, n)$ denote the Grassmannian of r-dimensional subspaces of the n-dimensional vector space $\mathbb{C}^n$ over the field of complex numbers. Each $\mathrm{Gr}(r, n)$ contains a unique codimension 1 Schubert subvariety called the Schubert divisor of the Grassmannian. In this project, we will discuss the correspondence between the set of permutations avoiding the patterns 3412, 52341, 52431, and 53241, and the set of Schubert varieties in the complete flag variety which are iterated fiber bundles of Grassmannians or Grassmannian Schubert divisors. Using this geometrical structure, we calculate the generating function that enumerates the permutations avoiding these patterns.

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CHAPTER I

PRELIMINARIES

1.1 Introduction

Let G be a complex semisimple algebraic group with the Weyl group W and B a Borel subgroup of G. G can be expressed as a disjoint union of double cosets of B parameterized by the elements of W and the decomposition of $G = \bigsqcup_{w \in W} BwB$ is called the Bruhat decomposition of G. For any $w \in W$, the variety

$$X_w := \overline{BwB/B}$$

is called a Schubert variety in the complete flag variety $X := G/B$. If $G = \mathtt{SL}_n(\mathbb{C})$, then W is the symmetric group $\mathfrak{S}_n$, and in this case, we write $X = \mathcal{F}\ell(n)$, which is called the complete flag variety of type A. In 1987, Ryan [18] proved that the smooth Schubert varieties of type A are iterated fiber bundles of Grassmannian. In 1989, Wolper [22] generalized Rayn's result for any algebraically closed field of characteristic zero.

In 1990, Lakshmibai and Sandhya [15] gave a permutation pattern avoidance criterion to identify a smooth Schubert variety of type A and showed that a Schubert variety X_w is smooth if and only if w avoids the patterns 3412 and 4231. Since then permutation pattern avoidance has been used as an important tool to characterize the geometric properties of Schubert varieties in $\mathcal{F}\ell(n)$. In 1990, by using the fiber bundle structures of smooth Schubert varieties, Haiman [13] was able to calculate the generating function for the number of permutations that avoid 3412 and 4231. In 1998, Billey [5] gave a pattern avoidance criterion for rationally smooth Schubert varieties in types B and C. There are many other characterizations of (rationally) smooth Schubert varieties. For example, in 1994, Carrell and Peterson [9] proved that a Schubert variety X_w is rationally smooth if and only if the Poincaré polynomial of X_w is palindromic. A survey of these results and more can be found in [1, 3]. A theorem of Deodhar, Peterson, and Carrel-Kuttler shows that in types A, D, and E, a Schubert variety X_w is smooth if and only if it is rationally smooth. Recently, Richmond and Slofstra [16] showed that every rationally smooth Schubert variety in any finite classical type is an iterated fiber bundle of Grassmannian.

Pattern avoidance criteria also have been used to characterize other geometric properties of Schubert varieties. In 2013, Úlfarsson and Woo [12] proved that a Schubert variety X_w is a local complete intersection if and only if w avoids 53241, 52341, 52431, 35142, 42513, and 426153. In 2001, Billey and Warrington [6] showed that the Bott-Samelson resolution of a Schubert variety X_w is small if and only if w is 321-hexagon-avoiding. In 2002, Gasharov and Reiner [11] showed that a Schubert variety X_w is defined by inclusions if and only if w avoids 4231, 35142, 42513, and 351624. In 2005, Woo and Yong [24] gave a pattern avoidance criterion of the Gorenstein Schubert variety. In 2007, Bousquet-Mélou and Butler [8] showed that a Schubert variety X_w is factorial if and only if w is 4231 and $3\underline{41}2$ avoiding. In 2007, Tenner [20] showed that the principal order ideal below w in Bruhat order is isomorphic to a Boolean lattice if and only if w avoids 321 and 3412. This is equivalent to saying that the Bott-Samelson resolution of X_w is isomorphic to X_w if and only if w is 321 and 3412 avoiding. In 2009, Woo, Billey, and Weed [23] proved that a permutation w is 653421, 632541, 463152, 526413, 546213, and 465132 avoiding and the singular locus of X_w has exactly 1 component if and only if the Kazhdan-Lusztig polynomial $P_{id,w}(1) = 2$. More results relating to permutation pattern avoidance criterion can be found in Tenner's Database [19].

One of the main goals of this project is to calculate the number of Schubert varieties in type A which are iterated fiber bundles of Grassmannian Schubert varieties of codimension at most 1. In [17], Richmond and Slofstra showed that the fibers in the fiber bundle structure of a Schubert variety can be combinatorially encoded using blocks in a labelled staircase diagram denoted by $(\mathcal{D}, \lambda)$, where $\mathcal{D}$ is the staircase diagram and λ the labelling of $\mathcal{D}$. More specifically, each labelled staircase diagram $(\mathcal{D}, \lambda)$ corresponds to a unique permutation $\Lambda(\mathcal{D}, \lambda)$. In this context, they showed that the set of "nearly maximally labelled" staircase diagrams is in bijection with the set of Schubert varieties that are iterated fiber bundles of Grassmannian Schubert varieties, and the bijection is given by

$$(\mathcal{D}, \lambda) \longleftrightarrow X_{\Lambda(\mathcal{D}, \lambda)}.$$

In particular, they showed that $X_{\Lambda(\mathcal{D},\lambda)}$ is smooth if and only if λ is the unique "maximal labelling" of the staircase diagram $\mathcal{D}$ of type A. Consequently, by counting the number of all possible type A staircase diagrams, they reproduced Haiman's generating function for counting the number of smooth varieties in type A and computed the smooth and rationally smooth Schubert varieties in the finite classical types B, C, and D. In Chapter II, we give a brief overview of staircase diagrams of type A. In Chapter III, we define a particular labelling called "divisor labelling" of a staircase diagram of type A. We prove that $(\mathcal{D}, \lambda)$ is a divisor-labelled staircase diagram of type A if and only if $\Lambda(\mathcal{D}, \lambda)$ avoids the patterns 3412, 52341, 52431, and 53241. We calculate the number of permutations avoiding these patterns by studying iterated fiber bundle structures on Schubert varieties in $\mathcal{F}\ell(n)$ whose fibers are Grassmannian Schubert varieties of codimension at most 1. In Chapter IV, we calculate the generating function of this type of Schubert variety. To state our main result, here we give a brief overview of Schubert varieties in type A.

Let s_i denote the simple transposition $(i, i+1)$ in $\mathfrak{S}_n$. It is well known that $\mathfrak{S}_n$ is generated by the set $\mathcal{S} := \{s_1, s_2, \cdots, s_{n-1}\}$. We elaborate this property in Section 2.1. Let $[1, n] := \{1, 2, \cdots, n\}$. For any subset $\mathbf{a} = \{a_1 < \cdots < a_k < n\}$ of $[1, n]$, define the **partial flag variety**

$$\mathcal{F}\ell(\mathbf{a}, n) := \{V_\bullet^{\mathbf{a}} := (V_{a_1} \subset V_{a_2} \subset \cdots \subset V_{a_k} \subset \mathbb{C}^n) \mid \dim(V_{a_i}) = a_i\}.$$

In particular, if $\mathbf{a} = [1, n]$, then $\mathcal{F}\ell(\mathbf{a}, n)$ is the complete flag variety $\mathcal{F}\ell(n)$.

Fix a basis $\{e_1, e_2, \cdots, e_n\}$ of $\mathbb{C}^n$. For $1 \leq i \leq n$, let E_i be the subspace generated by the set $\{e_1, e_2, \cdots, e_i\}$. For any subset $J \subset \mathcal{S}$, let $\mathbf{a}_J = \{i : s_i \notin J\}$. Then each $w \in \mathfrak{S}_n$ defines a **Schubert variety** X_w^J in the partial flag variety $\mathcal{F}\ell(\mathbf{a}_J, n)$, where

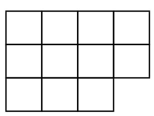
$$X_w^J := \{V_\bullet^{\mathbf{a}_J} \in \mathcal{F}\ell(\mathbf{a}_J,\, n) \,|\, \dim(E_i \cap V_j) \geq r_w[i,j] \,\forall\, i,j\},\ \text{where } r_w[i,j] = \big|[1,i] \cap w([1,j])\big|.$$

In particular, we have the Schubert variety $X_w := X_w^{\emptyset}$ in the complete flag vcariety $\mathcal{F}\ell(n)$.

If $\mathbf{a} = \{r < n\}$, then $\mathcal{F}\ell(\mathbf{a}, n)$ is the Grassmannian $\mathrm{Gr}(r,n)$ of r-dimensional subspaces of $\mathbb{C}^n$. If X_w^J is a Schubert variety in the Grassmannian $\mathrm{Gr}(r,n)$, then the permutation w corresponds to a unique partition λ_w such that the Young diagram of λ_w is contained in an $r \times (n-r)$ rectangle. For any such partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$, the Grassmannian Schubert variety X_λ is defined as follows.

$$X_\lambda = \{V \in \mathrm{Gr}(r,n) \,|\, \dim\big(V \cap E_{i+\lambda_{r+1-i}}\big) \geq i \text{ for all } 1 \leq i \leq r\}.$$

The dimension of X_λ is given by the number of boxes in the Young diagram of λ. This implies that $\mathrm{Gr}(r,n)$ contains exactly one codimension-1 Schubert subvariety given by the partition $\lambda = (r, \ldots, r, r-1)$, where r is repeated $n-r-1$ times. We call this Schubert variety the **Schubert divisor** of the Grassmannian $\mathrm{Gr}(r,n)$. For example,

$(4,4,3) \longleftrightarrow$  corresponds to the unique Schubert divisor of the Grassmannian $\mathrm{Gr}(4,7)$.

If $\mathbf{b} \subseteq \mathbf{a} \subseteq [1,n]$, then there is a projection map $\pi_{\mathbf{b}}^{\mathbf{a}} : \mathcal{F}\ell(\mathbf{a}, n) \twoheadrightarrow \mathcal{F}\ell(\mathbf{b}, n)$ given by

$$\pi_{\mathbf{b}}^{\mathbf{a}}(V_\bullet^{\mathbf{a}}) = V_\bullet^{\mathbf{b}}.$$

This map is naturally a fiber bundle on $\mathcal{F}\ell(\mathbf{a}, n)$ and hence given any collection of nested subsets

$$[1,n] = \sigma_n \supset \cdots \sigma_2 \supset \sigma_1 = \{n\},$$

where $|\sigma_j| = j$, there is an iterated fiber bundle structure on the complete flag variety

$$\mathcal{F}\ell(n) \overset{\pi_{\sigma_{n-1}}^{\sigma_n}}{\twoheadrightarrow} \mathcal{F}\ell(\sigma_{n-1}, n) \overset{\pi_{\sigma_{n-2}}^{\sigma_{n-1}}}{\twoheadrightarrow} \cdots \overset{\pi_{\sigma_2}^{\sigma_3}}{\twoheadrightarrow} \mathcal{F}\ell(\sigma_2, n) \overset{\pi_{\sigma_1}^{\sigma_2}}{\twoheadrightarrow} \mathcal{F}\ell(\sigma_1, n) \simeq \{pt\}.$$

For $w \in \mathfrak{S}_n$, we say that the Schubert variety X_w has a **complete parabolic bundle structure** if there exists a nested collection of subsets $[1,n] = \sigma_n \supset \cdots \supset \sigma_2 \supset \sigma_1 = \{n\}$

such that the projection maps $\pi_{\sigma_i}^{\sigma_{i+1}}$ induce an iterated fiber bundle structure on X_w,

$$X_w = X_n \overset{\pi_{\sigma_{n-1}}^{\sigma_n}}{\twoheadrightarrow} X_{n-1} \overset{\pi_{\sigma_{n-2}}^{\sigma_{n-1}}}{\twoheadrightarrow} \cdots \overset{\pi_{\sigma_2}^{\sigma_3}}{\twoheadrightarrow} X_2 \overset{\pi_{\sigma_1}^{\sigma_2}}{\twoheadrightarrow} X_1, \text{ where } X_i := \pi_{\sigma_i}^{[1,n]}(X_w) \subset \mathcal{F}\ell(\sigma_i, n).$$

In this case, the fibers of each induced projection map are isomorphic to Grassmannian Schubert varieties (see example 3.1.11). It is not true that all Schubert varieties have complete parabolic bundle structures. For example, if $w = s_2s_3s_1s_2$, then X_w does not have a complete parabolic bundle structure (see Example 4.3 in [2]). In [2], Alland and Richmond showed that a permutation $w \in \mathfrak{S}_n$ avoids the patterns 3412, 52341, and 635241 if and only if X_w has a complete parabolic bundle structure. As we mentioned earlier, Ryan showed in [18] that smooth Schubert varieties are iterated fiber bundles of Grassmanian fibers. The following theorem is the main result of this thesis:

Theorem 1.1.1 *The Schubert variety $X_w \subseteq \mathcal{F}\ell(n)$ has a complete parabolic bundle structure with fibers isomorphic to Grassmannian or Grassmannian Schubert divisors if and only if w avoids the patterns* 3412, 52341, 52431, *and* 53241.

Moreover, if $Z(x) = \sum_{n\geq 0} z_n x^n$, where z_n is the number of $w \in \mathfrak{S}_{n+1}$ avoiding such patterns, then

$$Z(x) = \frac{-4x^6 + 24x^5 - 58x^4 + 73x^3 - 49x^2 + 17x - 2 - x\sqrt{x^4 - 2x^3 + 7x^2 - 6x + 1}}{2(x-1)(2x^6 - 14x^5 + 37x^4 - 46x^3 + 28x^2 - 9x + 1)}.$$

We know by [10, Theorem IV.7] that the growth of the coefficients of a generating series is controlled by the singularity of the smallest modulus. For the generating function $Z(x)$ in Theorem 1.1.1, the singularity with the smallest modulus is $\alpha \approx 0.203086$, which is a zero of the polynomial $2x^6 - 14x^5 + 37x^4 - 46x^3 + 28x^2 - 9x + 1$ in the denominator of $Z(x)$.

Define the constant Z_α by

$$Z_\alpha := \lim_{x\to\alpha} (x - \alpha)Z(x).$$

The following is an immediate corollary of Theorem 1.1.1.

Corollary 1.1.2 *Let z_n be the number of $w \in \mathfrak{S}_n$ avoiding the patterns 3412, 52341, 52431,* *and 53241. Then*

$$z_n \sim \frac{Z_\alpha}{\alpha^{n+1}},$$

and in particular, the asymptotic rate at which z_n is growing is equal to the limit

$$\lim_{n\to\infty} \frac{z_{n+1}}{z_n} = \frac{1}{\alpha} \approx 4.92402.$$

Note that the asymptotic growth rate of the number of smooth Schubert varieties in $\mathcal{F}\ell(n)$ is approximately 4.382985 [17]. Here we provide some values of z_n.

n	1	2	3	4	5	6	7	8	9
z_n	1	2	6	23	100	460	2172	10397	50173

1.2 A summary of the findings in this work

The following is a summary of the findings of this project.

Let P_n be the set of permutations in $\mathfrak{S}_{n+1}$ avoiding the patterns 3412, 52341, 52431, and 53241,

Q_n the set of Schubert varieties in $\mathcal{F}\ell(n+1)$ which are fiber bundles of Grassmannians or Grassmannian Schubert divisors, and

R_n the set of divisor-labelled staricase diagrams of support contained in $[s_1, s_n]$.

Then $|P_n| = |Q_n| = |R_n|$. Moreover, $|P_n|$ is the coefficient of x^n in the formal power series expansion of

$$Z(x) = \frac{-4x^6 + 24x^5 - 58x^4 + 73x^3 - 49x^2 + 17x - 2 - x\sqrt{x^4 - 2x^3 + 7x^2 - 6x + 1}}{2(x-1)(2x^6 - 14x^5 + 37x^4 - 46x^3 + 28x^2 - 9x + 1)}.$$

CHAPTER II

STAIRCASE DIAGRAM AND SCHUBERT VARIETY

In this chapter, we recall several results from [17] concerning staircase diagrams, labelled staircase diagrams, and the Schubert variety corresponding to a labelled staircase diagram. In general, a staircase diagram is a collection of connected subsets (blocks) of vertices of a graph Γ where the blocks are allowed to overlap each other in a particular way such that $\mathcal{D}$ resembles a staircase with steps of irregular length. If the underlying graph Γ is a simple path, then we label the blocks of $\mathcal{D}$ by elements of $\mathfrak{S}_n$ and obtain a unique element $\Lambda(\mathcal{D}) \in \mathfrak{S}_n$ such that the Schubert variety $X_{\Lambda(\mathcal{D})} \in \mathcal{F}\ell(n)$ has iterated fiber bundle structure. In this case, Γ is called a Dynkin diagram of type A which corresponds to a Coxeter group of type A. In the following section, we give a brief overview of the Coxeter group of type A.

2.1 Coxeter group in type A

Definition 2.1.1 *Let X be a non-empty set. A binary relation $\prec$ on X is called a **partial order** if the following three conditions are satisfied:*

1. *$\prec$ is reflexive, i.e. for evry $x \in X$, $x \prec x$.*

2. *$\prec$ is anti-symmetric, i.e. for all $x, y \in X$ if $x \prec y$ and $y \prec x$, then $x = y$.*

3. *$\prec$ is transitive, i.e. for all $x, y, z \in X$, if $x \prec y$ and $y \prec z$, then $x \prec z$.*

*If $\prec$ is a partial order relation on X, then we call $(X, \prec)$ a **partially ordered set** (poset).*

Sometimes we write X instead of $(X, \prec)$ when $\prec$ is clear. If $x, y \in X$ such that either $x \prec y$ or $y \prec x$, then we say that x and y are **comparable**. If $x \in X$ and $y \in X$ are not comparable, then we say that they are **incomparable**. If x and y are two elements of the poset X, then we say that y **covers** x, if $x \prec y$, but there is no $z \in X \setminus \{x, y\}$ such that $x \prec z \prec y$. A subset Y of the poset X is called a **chain** if every pair of elements in Y are comparable. If Y is a subset of X such that there is no $z \notin Y$ such that $x \prec z \prec y$ whenever $x, y \in Y$, then Y is called a **saturated subset** of X.

Definition 2.1.2 *Suppose that $X = \{x_1, x_2, \cdots, x_n\}$ is a partially ordered set such that for all $1 \leq i < j \leq n$, either $x_i \prec x_j$ or they are incomparable. Then $x_1 \prec x_2 \cdots \prec x_n$ is called a **linear extension** of the partial ordering of X. Thus, a linear extension of a partial order is a total order that is compatible with the partial order.*

Definition 2.1.3 *A **Coxeter group** is a group W with a set of generators $S = \{r_1, r_2, \cdots, r_n\}$ with relations $(r_i r_j)^{m_{ij}} = 1$, where $m_{ii} = 1$ for all i and $m_{ij} > 1$ whenever $i \neq j$. We use the convention that $m_{ij} = \infty$ when there exists no m such that $(r_i r_j)^m = 1$. The pair (W, S) is called a **Coxeter system**. The Coxeter system is associated with the $n \times n$ symmetric matrix (m_{ij}) called the **Coxeter matrix** of (W, S) with the (i, j)-th entry being m_{ij}.*

Every symmetric matrix (m_{ij}) such that $m_{ii} = 1$ for all i and $m_{ij} \in \{2, 3, \cdots\} \cup \{\infty\}$ for all $i \neq j$ is a Coxeter matrix. It is a fact that up to isomorphism there is a one-to-one correspondence between Coxeter matrices and Coxeter systems. A Coxeter matrix can be encoded by a graph called the **Coxeter-Dynkin diagram** of the Coxeter matrix (or, equivalently, of the Coxeter system) where the graph satisfies the following conditions:

- The vertices of the graph are labelled by generator subscripts.
- Two vertices i and j are adjacent if and only if $m_{ij} > 2$.
- An edge is labelled with the value of m_{ij} whenever $m_{ij} > 3$.
- Two vertices are not connected by an edge if and only if the corresponding generators commute, i.e. $m_{ij} = 2$.

If the graph has two or more connected components, then the associated group is the direct product of the groups associated with the individual components. Thus the disjoint union of the Coxeter diagrams yields a direct product of Coxeter groups.

Definition 2.1.4 *The **symmetric group** $\mathfrak{S}_n$ is the set of all bijections from the set $[1, n] = \{1, 2, \cdots, n\}$ to itself. These bijections are called **permutations**. The binary group operation in $\mathfrak{S}_n$ is the composition of permutations. The compositions of permutations are read from right to left.*

There are several standard notations to represent a permutation. One of them is cycle notation. For example, the cycle $(3, 5, 6)$ stands for the permutation which sends 3 to 5, 5 to 6, 6 to 3, and keeps all other elements of $[1, n]$ fixed.

It is customary to start with the smallest entry when we write a cycle in $\mathfrak{S}_n$. If there exists no common element in two cycles, then we say that the cycles are disjoint. Any two disjoint cycles are commutative. Every permutation can be written (uniquely) as a product (composition) of disjoint cycles.

A permutation $w \in \mathfrak{S}_n$ can also be expressed in one-line notation as

$$w = w(1)w(2)\cdots w(n),$$

and in two-line notation as

$$w = \begin{pmatrix} 1 & 2 & \cdots & n \\ w(1) & w(2) & \cdots & w(n) \end{pmatrix}.$$

We can also consider the permutation w as a sequence of n distinct numbers in $[1, n]$. For $0 < t_1 < t_2 < \cdots < t_m < n + 1$, we will call

$$w(t_1)w(t_2)\cdots w(t_m) = \begin{pmatrix} t_1 & t_2 & \cdots & t_m \\ w(t_1) & w(t_2) & \cdots & w(t_m) \end{pmatrix}$$

a sub-sequence of w. Moreover, if $t_{i+1} = t_i + 1$ for all $1 < i < m$, then we say that the subsequence is saturated.

If $S = s_1 s_2 \cdots s_k$ and $T = t_1 t_2 \cdots t_k$ are two finite sequences such that for all i, j,

$$s_i < s_j \Longleftrightarrow t_i < t_j,$$

then we say that S and T are **order isomorphic**, and we write

$$S \sim T.$$

For example,

$$2719 \sim 4628.$$

Sometimes we call a permutation a pattern when expressed in one-line notation. For $n \geq m$, $w \in \mathfrak{S}_n$, and $p \in \mathfrak{S}_m$, we say that w **contains** the pattern p if w has a subsequence which is order isomorphic to p. If w does not contain p, then we say that w avoids p.

Example 2.1.5 *Let $w = 362541 \in \mathfrak{S}_6$. Observe that*

- *w contains $132 \sim 254$.*
- *w contains $231 \sim 362$.*
- *w avoids 123 since w does not contain an increasing subsequence of more than two entries.*

A cycle of the type (i, j) is called a **transposition**. Moreover, if $j = i + 1$, then $s_i := (i, i+1)$ is called a **simple (elementary) transposition**.

If (i, j) is a cycle, then it is easy to see that

$$s_j (i, j) s_j = (i, j+1),$$

which implies that every cycle can be expressed as a product of simple transpositions. Consequently, we can express a permutation as a product of simple transpositions. Thus we see that $\mathcal{S} := \{s_1, s_2, \cdots, s_{n-1}\}$ generates the symmetric group $\mathfrak{S}_n$.

Example 2.1.6 *Let $w = s_1 s_2 s_4$. Then*

$$w(1) = s_1 s_2 s_4(1) = s_1 s_2(1) = s_1(1) = 2,$$

$$w(2) = s_1 s_2 s_4(2) = s_1 s_2(2) = s_1(3) = 3,$$

$$w(3) = s_1 s_2 s_4(3) = s_1 s_2(3) = s_1(2) = 1,$$

$$w(4) = s_1 s_2 s_4(4) = s_1 s_2(5) = s_1(5) = 5,$$

$$w(5) = s_1 s_2 s_4(5) = s_1 s_2(4) = s_1(4) = 4,$$

and

$$w(k) = s_1 s_2 s_4(k) = s_1 s_2(k) = s_1(k) = k, \text{ for } k > 5.$$

Thus in one-line notation $w = 23154$, in two-line notation $w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$, and in disjoint cycle notations, $w = (1, 2, 3)(4, 5)$.

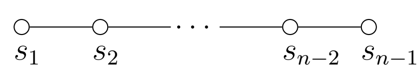
Notice that for all distinct i and j,

$$s_i^2 = s_i s_i = 1,$$

$$s_i s_j = s_j s_i \iff |i - j| > 1, \text{ and}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}.$$

It follows that the symmetric group $\mathfrak{S}_n$ is a Coxeter group with the set of generators $\mathcal{S} = \{s_1, s_2, \cdots, s_{n-1}\}$ and the Coxeter-Dynkin diagram of $\mathfrak{S}_n$ is a simple path Γ with $n - 1$ vertices.



The graph Γ is called the Coxeter-Dynkin diagram of type A_{n-1}.

Definition 2.1.7 *Let (W, S) be a Coxeter system and $w \in W$. The least number of generators in S needed to express w is called the **length** of w which we denote by $\ell(w)$.*

Definition 2.1.8 *If $w = s_{j_1} s_{j_2} \cdots s_{j_k}$ such that $\ell(w) = k$, then we say that $s_{j_1} s_{j_2} \cdots s_{j_k}$ is a **reduced expression** for w.*

Example 2.1.9 *Let* $w = s_3s_2s_3s_2s_1$*. Then*

$$w = s_2s_3s_2s_2s_1 = s_2s_3s_1 = s_2s_1s_3,$$

which implies that $\ell(w) = 3$.

Definition 2.1.10 *Suppose that* (W, S) *is a Coxeter system. There is a natural partial order called* ***Bruhat order*** *on* W*. The name was given by Verma in [21]. Let* $u, v \in W$*.*

- *We say that* $u \leq v$ *in (strong) Bruhat order if there exists a reduced expression of* u *which is a subsequence of some reduced expressions of* v*. If* $u \leq v$ *in the Bruhat order and* $\ell(v) = 1 + \ell(u)$*, then we say that* v *covers* u *in the Bruhat order.*

- *If* $v = su$ *for some* $s \in S$ *and* $\ell(v) = 1 + \ell(u)$*, then we say that* v *covers* u *in the weak left Bruhat order.*

- *If* $v = us$ *for some* $s \in S$ *and* $\ell(v) = 1 + \ell(u)$*, then we say that* v *covers* u *in the weak right Bruhat order.*

We will denote the strong, weak left, and weak right Bruhat order relations by the symbols $\leq$*,* $\leq_L$ *and* $\leq_R$*, respectively.*

Definition 2.1.11 *Suppose that* $w = s_{t_1}s_{t_2}\cdots s_{t_\ell}$ *is a reduced expression. Let* $[k, k+m] \subset [1, \ell]$*. Then* $w' = s_{t_k}s_{t_{k+1}}\cdots s_{t_{k+m}}$ *is also a reduced expression. In this case, we call* w' *a* ***factor*** *of* w*.*

Definition 2.1.12 *A permutation* w *is called* ***fully commutative*** *if no reduced expression of* w *contains a factor of the type* $s_is_{i+1}s_i$*.*

Example 2.1.13 $w = s_1s_2s_3s_5s_6$ *is fully commutative.*

Lemma 2.1.14 *(Humphreys [14]) There is a unique epimorphism*

$$\epsilon : \mathfrak{S}_n \to \{-1, 1\}$$

given by

$$\epsilon(w) = (-1)^{\ell(w)}.$$

As a result, for any simple transposition s,

$$\ell(sw) = \ell(w) \pm 1$$

and

$$\ell(ws) = \ell(w) \pm 1.$$

Definition 2.1.15 *Let $\mathcal{S} = \{s_1, s_2, \cdots, s_{n-1}\}$ and $\ell : \mathfrak{S}_n \to \mathbb{Z}_{\geq 0}$ denote the length function. Define*

$$D_L(w) := \{s \in \mathcal{S} \,\big|\, \ell(sw) = \ell(w) - 1\} \text{ and}$$
$$D_R(w) := \{s \in \mathcal{S} \,\big|\, \ell(ws) = \ell(w) - 1\}.$$

- *$D_L(w)$ is called the set of left descents of w.*
- *$D_R(w)$ is called the set of right descents of w.*

Remark 2.1.16 *It follows that $s \in D_L(w) \iff$ some reduced expression of w begins with s. Similarly, $s \in D_R(w) \iff$ some reduced expression of w ends with s.*

Definition 2.1.17 *For $u \in \mathfrak{S}_n$, let*

$$\mathtt{Supp}(u) := \{s \in \mathcal{S} \,|\, s \text{ lies in a reduced expression of } u\}.$$

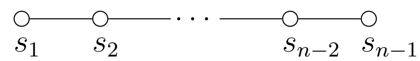
Consequently,

$$\mathtt{Supp}(u) = \{s \in \mathcal{S} \,|\, s \leq u \text{ in the Bruhat order }\}.$$

Example 2.1.18 *Let $u = s_1 s_2 s_1$. Then $\mathtt{Supp}(u) = \{s_1, s_2\}$.*

2.2 Staircase diagram of type A

As we mentioned in the previous section that if $W = \mathfrak{S}_n$ and $\mathcal{S} = \{s_1, s_2, \cdots, s_{n-1}\}$, where each s_i is the simple transposition $(i, i+1)$, then $(W, \mathcal{S})$ is a Coxeter system, and the Coxeter-Dynkin diagram Γ of W is of type A_{n-1} that is a simple path with $n-1$ vertices.



We call an interval $B = [s_i,\, s_j] := \{s_i, s_{i+1}, \ldots, s_j\}$ in Γ a **block**, where $i \leq j$. In other words, if B is a block, then the induced subgraph corresponding to B in the Coxeter-Dynkin diagram Γ is connected. For a block $B = [s_i, s_j]$, we define $L(B) = i$.

Definition 2.2.1 *A* ***staircase diagram*** *of type A is a partially ordered set* $(\mathcal{D}, \preceq)$, *where* $\mathcal{D} = \{B_1, B_2, \cdots, B_k\}$ *is a set of blocks in* Γ *such that for all* $i \neq j$,

1. $B_i \not\subset B_j$,

2. B_i *and* B_j *are comparable whenever* $B_i \cup B_j$ *is a block,*

3. $B_i \cup B_j$ *is a block whenever* B_i *covers* B_j, *and*

4. *if* $B_{j_1} \prec B_{j_2} \prec \cdots \prec B_{j_l}$ *is a chain in* $\mathcal{D}$, *then either* $L(B_{j_1}) < L(B_{j_2}) < \cdots < L(B_{j_l})$ *or* $L(B_{j_1}) > L(B_{j_2}) > \cdots > L(B_{j_l})$.

Note that staircase diagrams can be defined over an arbitrary graph. For a general definition of a staircase diagram, we refer to [17]. In this project, we restrict to staircase diagrams of type A as defined above.

Definition 2.2.2 *If* $\mathcal{D}$ *is a staircase diagram, then by* $\texttt{Supp}(\mathcal{D})$, *we denote the* ***support*** *of* $\mathcal{D}$, *which is defined as follows:*

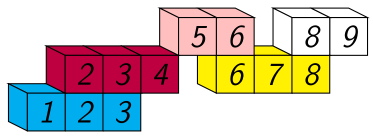
$$\texttt{Supp}(\mathcal{D}) := \bigcup_{B \in \mathcal{D}} B.$$

If $\texttt{Supp}(\mathcal{D})$ is a block, then we say that $\mathcal{D}$ is a **connected** staircase diagram. If $\mathcal{D}$ is a staircase diagram and $\mathcal{D}' \subset \mathcal{D}$ is a saturated subset, then we call $\mathcal{D}'$ a subdiagram of

$\mathcal{D}$. We see that a subdiagram of a staircase diagram is itself a staircase diagram with the induced partial order. Note that every staircase diagram is a disjoint union of its maximally connected subdiagrams.

Staircase diagrams can be represented by pictures that resemble staircases with steps of irregular length as shown in the example below.

Example 2.2.3 *The picture*

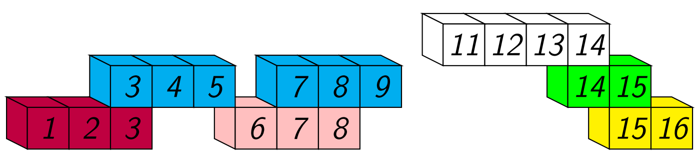


represents a connected staircase diagram

$$\mathcal{D} = \{[s_1, s_3] \prec [s_2,\, s_4] \prec [s_5, s_6] \succ [s_6, s_8] \prec [s_8, s_9]\}$$

with support $[s_1,\, s_9]$*, where for notational simplicity, we pictorially label* s_i *by* i*, and in the picture the covering relations for* $\mathcal{D}$ *are given by vertical adjacencies.*

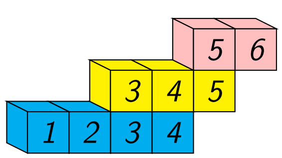
Example 2.2.4 *The support of the staircase diagram*

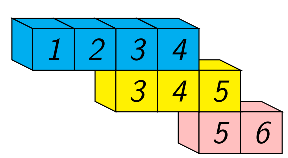


is $[s_1, s_9] \sqcup [s_{11}, s_{16}]$ *and the diagram has two connected subdiagrams.*

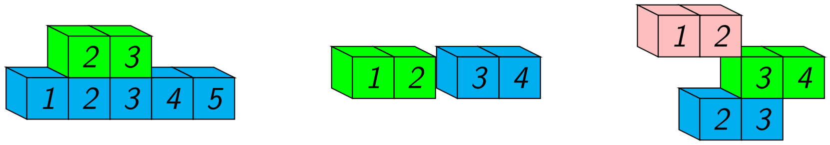
Notice that by flipping a staircase diagram $\mathcal{D}$, we get another staircase diagram with the reverse partial order which we denote by $\texttt{flip}(\mathcal{D})$.

Example 2.2.5 *Suppose that* $\mathcal{D} = \{[s_1, s_4] \prec [s_3, s_5] \prec [s_5, s_6]\}$*. Then* $\texttt{flip}(\mathcal{D}) = \{[s_5, s_6] \prec [s_3, s_5] \prec [s_1, s_4]\}$.

$\mathcal{D} =$ 

$\texttt{flip}(\mathcal{D}) =$ 

Example 2.2.6 *The following three diagrams are not valid staircase diagrams.*



- *The first diagram violates the first condition of a staircase diagram.*
- *The second diagram violates the second condition of a staircase diagram.*
- *The third diagram violates the fourth condition of a staircase diagram.*

2.3 The Schubert variety associated with a labelled staircase diagram

Definition 2.3.1 *Given a staircase diagram $\mathcal{D}$, define $J_R : \mathcal{D} \to \mathcal{S}$ and $J_L : \mathcal{D} \to \mathcal{S}$ by*

$$J_R(B) := \{s \in B \,|\, s \in B' \text{ for some } B' \in \mathcal{D} \text{ covered by } B\}$$

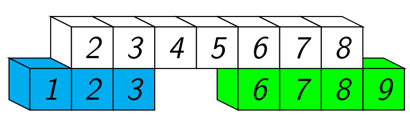
and

$$J_L(B) := \{s \in B \,|\, s \in B' \text{ for some } B' \in \mathcal{D} \text{ that covers } B\}.$$

Note that if $\{B_1 \prec B_2 \prec \cdots \prec B_k\}$ is a linear extension of $\mathcal{D}$, then for each i,

$$J_R(B_i) = (B_1 \cup \cdots \cup B_{i-1}) \cap B_i.$$

Example 2.3.2 *In the following staircase diagram*



$$J_R([s_2, s_8]) = \{s_2, s_3, s_6, s_7, s_8\}\,,$$

$$J_L[s_1,\, s_3] = \{s_2,\, s_3\},$$

$$J_L[s_6,\, s_9] = \{s_6,\, s_7,\, s_8\}, \text{ and}$$

$$J_L([s_2,\, s_8]) = J_R([s_1,\, s_3]) = J_R([s_6,\, s_9]) = \emptyset.$$

Definition 2.3.3 *For any subset $J \subset \mathcal{S}$, let u_J denote the unique maximal element (with respect to length) in the subgroup generated by J. An element $w \in \mathfrak{S}_n$ is called a **maximal** element if $w = u_{\mathtt{Supp}(w)}$; otherwise we call w a **non-maximal** element.*

Definition 2.3.4 *Suppose that $\mathcal{D}$ is a staircase diagram. A **labelling** of $\mathcal{D}$ is a mapping $\lambda : \mathcal{D} \to \mathfrak{S}_n$ such that*

1. $J_R(B) \subseteq D_R(\lambda(B))$,
2. $J_L(B) \subseteq D_L(\lambda(B))$, *and*
3. $\mathtt{Supp}(\lambda(B)u_{J_R(B)}) = \mathtt{Supp}(u_{J_L(B)}\lambda(B)) = B$.

Since the definition of a labelling of a staircase diagram is symmetric, $\lambda : \mathcal{D} \to \mathfrak{S}_n$ is a labelling of $\mathcal{D}$ if and only if $\lambda^{-1} : \mathtt{flip}(\mathcal{D}) \to \mathfrak{S}_n$ given by

$$\lambda^{-1}(B) = (\lambda(B))^{-1}$$

is a labelling of flip($\mathcal{D}$).

For a labelling λ and a linear extension $\{B_1, B_2, \cdots, B_k\}$ of a staircase diagram $\mathcal{D}$, let

$$\overline{\lambda}(B) := \lambda(B)u_{J_R(B)}$$

and

$$\Lambda(\mathcal{D}, \lambda) := \overline{\lambda}(B_k) \cdots \overline{\lambda}(B_2)\overline{\lambda}(B_1).$$

We will write $\Lambda(\mathcal{D})$ in place of $\Lambda(\mathcal{D}, \lambda)$ when λ is clear. Moreover, if $\mathcal{D}'$ is a subdiagram of $\mathcal{D}$, then we will write $\Lambda(\mathcal{D}')$ in place of $\Lambda(\mathcal{D}, \lambda|_{\mathcal{D}'})$. If λ is a labelling of a staircase diagram $\mathcal{D}$, and B and B' are two incomparable blocks in $\mathcal{D}$, then $\mathtt{Supp}(\overline{\lambda}(B)) = B$, $\mathtt{Supp}(\overline{\lambda}(B')) = B'$, and $\overline{\lambda}(B)$ and $\overline{\lambda}(B')$ commute. Thus we see that $\Lambda(\mathcal{D})$ does not depend on the choice of a linear extension of $\mathcal{D}$.

Lemma 2.3.5 *[17] Let λ be a labelling of a staircase diagram $\mathcal{D}$. Then*

$$(\Lambda(\mathcal{D}, \lambda))^{-1} = \Lambda(\mathtt{flip}(\mathcal{D}), \lambda^{-1}).$$

Example 2.3.6 *Let $\mathcal{D}$ be a staircase diagram, and $\lambda : \mathcal{D} \to \mathfrak{S}_n$ a mapping such that $\forall\, B \in \mathcal{D}$, $\lambda(B) = u_B$. Then λ is a labelling of $\mathcal{D}$ and it is called the **maximal labelling** of $\mathcal{D}$.*

From now on, we will denote the maximal labeling by λ_{max}.

Remark 2.3.7 *If $\mathcal{D} = \{[s_1, s_n]\}$, then*

$$\Lambda(\mathcal{D}, \lambda_{max}) = \begin{pmatrix} 1 & 2 & \cdots & n+1 \\ n+1 & n & \cdots & 1 \end{pmatrix} = (n+1)n \cdots 21.$$

2.4 Billey-Postnikov decomposition

Let $W = \mathfrak{S}_n$. For any $J \subset \mathcal{S}$, let $W_J = \langle J \rangle$ denote the subgroup generated by J and W^J the set of minimum length coset representatives of W/W_J. Each element $w \in W$ has a unique parabolic decomposition $w = vu$, where $v \in W^J$ and $u \in W_J$[7].

Definition 2.4.1 *Let $w = vu$ be the parabolic decomposition of $w \in \mathfrak{S}_n$ with respect to a subset $J \subset \mathfrak{S}_n$. Then $w = vu$ is called a **Billey-Postnikov (BP) decomposition** if*

$$\mathtt{Supp}(v) \cap J \subset D_L(u).$$

*Furthermore, if $|\mathtt{Supp}(w)| = |J| + 1$, then we say that $w = vu$ is a **Grassmannian BP decomposition** with respect to J. For more details, we refer to [16, 4].*

Proposition 2.4.2 *([17]) Let $\{B_1, B_2, \cdots B_k\}$ be a linear extension of a staircase diagram $\mathcal{D}$. Consider the subdiagram $\mathcal{D}^i := \{B_1, B_2, \cdots B_{i-1}\}$, $i \in [2, k]$. If λ is a labelling of $\mathcal{D}$, then*

$$\Lambda(\mathcal{D}^{i+1}) = \overline{\lambda}(B_i).\Lambda(\mathcal{D}^i)$$

is a BP decomposition with respect to $\mathtt{Supp}(\mathcal{D}^i)$ for every $i \in [2, k]$.

If w is non-maximal and it has a Grassmannian BP decomposition $w = vu$ such that $\mathtt{Supp}(u) \subset \mathtt{Supp}(v)$, then w is called a **nearly maximal** element.

Definition 2.4.3 *A labelling* λ *of a staircase diagram* $\mathcal{D}$ *is called a* ***nearly maximal*** *labelling of* $\mathcal{D}$*, if* $\forall\, B \in \mathcal{D}$, $\lambda(B)$ *is either maximal or nearly maximal. For any* $B \in \mathcal{D}$*, if* $\lambda(B) = vu$ *is the Grassmannian BP decomposition with respect to some subset* $J \in \mathfrak{S}_n$ *such that* $\mathtt{Supp}(u) \subset \mathtt{Supp}(v)$*, then we say that* $\lambda(B)$ *is a nearly maximal labelling of* B *with respect to* J*.*

2.5 Nearly maximal labelling of a block in a staircase diagram

In this section, we will classify the nearly maximal labelling of a staircase diagram $\mathcal{D} = \{[s_1, s_n]\}$ consisting of a single block. For any $s_m \in [s_1, s_n]$, let u_m denote the maximal element of the subgroup generated by $J_m := [s_1, s_n] \setminus \{s_m\}$. For $1 \leq k \leq n - m$, let

$$\delta_k := \begin{cases} s_{k+1}s_{k+2}\cdots s_{k+m-1} & ;\ \text{if } m > 1 \\ e & ;\ \text{otherwise} \end{cases}$$

and

$$w_m := \begin{cases} \delta_{n-m}\delta_{n-m-1}\cdots\delta_2\delta_1 & ;\ \text{if } m < n \\ e & ;\ \text{otherwise} \end{cases}$$

Definition 2.5.1 *For* $1 \leq m \leq n$*, we define*

$$v_m := \begin{cases} (s_n s_{n-1}\cdots s_{m+1})(s_1 s_2 \cdots s_m) & ;\ \textit{if } m < n \\ s_1 s_2 \cdots s_m & ;\ \textit{otherwise} \end{cases}$$

$$\Delta_1 := \Delta_{(1,m,n)} = w_m v_m,$$

$$\Delta_2 := \Delta_{(2,m,n)} = \begin{cases} u_{\{s_1,s_2,\cdots,s_{m-1}\}} & ;\ \textit{if } m \neq 1 \\ e & ;\ \textit{if } m = 1 \end{cases},$$

$$\Delta_3 := \Delta_{(3,m,n)} = \begin{cases} u_{\{s_{m+1},s_{m+2},\cdots,s_n\}} & ;\ \textit{if } m \neq n \\ e & ;\ \textit{if } m = n \end{cases},\ \textit{and}$$

$$\Delta := \Delta_1\Delta_2\Delta_3.$$

Lemma 2.5.2 *In two-line notation,*

$$\Delta_1 = \begin{pmatrix} 1 & 2 & \cdots & m & m+1 & m+2 & \cdots & n+1 \\ n-m+2 & n-m+3 & \cdots & n+1 & 1 & 2 & \cdots & n-m+1 \end{pmatrix}$$

and Δ *is the unique maximal element of* $\mathfrak{S}_{n+1}$.

Proof. Let $x \in \{1, 2, \cdots, n+1\}$. Now we consider four cases.

Case 1: Let $0 < x < m$. We have for all y, $r > 0$, $s_{y+r}(y) = y$, $s_y(y+r+1) = y+r+1$, and $s_y(y) = y+1$. Therefore,

$$\begin{aligned} v_m(x) &= (s_n s_{n-1} \cdots s_{m+1})(s_1 s_2 \cdots s_m)(x) \\ &= (s_n s_{n-1} \cdots s_{m+1})(s_1 s_2 \cdots s_x)(x) \\ &= (s_n s_{n-1} \cdots s_{m+1})(x+1) \\ &= x+1. \end{aligned}$$

Now,

$$\begin{aligned} \delta_1(x+1) &= s_2 s_3 \cdots s_m(x+1) \\ &= s_2 s_3 \cdots s_{x+1}(x+1) \\ &= s_2 s_3 \cdots s_x(x+2) \\ &= x+2. \end{aligned}$$

Similarly $\delta_2(x+2) = x+3$. In general, for $0 < x < m$,

$$\delta_{n-m}(x+n-m) = x+n-m+1.$$

Therefore, $\Delta_1(x) = x+n-m+1$ for $0 < x < m$.

Case 2: Let $x = m$. We have

$$\begin{aligned} v_m(m) &= (s_n s_{n-1} \cdots s_{m+1})(s_1 s_2 \cdots s_m)(m) \\ &= (s_n s_{n-1} \cdots s_{m+1})(m+1) \\ &= n+1. \end{aligned}$$

Moreover, $w_m(n+1) = n+1$. Hence $\Delta_1(m) = n+1$.

Case 3: Let $x = m+1$. Now $v_m(m+1) = (s_n s_{n-1} \cdots s_{m+1})(s_1 s_2 \cdots s_m)(m+1) = 1$ and $w_m(1) = 1$. Therefore, $\Delta_1(m+1) = 1$

Case 4: Let $x > m+1$. Then $x = m+r+1$ for some $r > 0$. Now

$$v_m(x) = v_m(m+r+1) = (s_n s_{n-1} \cdots s_{m+1})(s_1 s_2 \cdots s_m)(m+r+1) = m+r.$$

Moreover,

$$\begin{aligned} \delta_{r-1}\delta_{r-2}\cdots\delta_1(m+r) &= m+r, \\ \delta_r(m+r) &= r+1, \text{ and} \\ \delta_{n-m}\delta_{n-m-1}\cdots\delta_{r+1}(r+1) &= r+1. \end{aligned}$$

Thus $w_m(m+r) = r+1$. Hence

$$\Delta_1(m+r+1) = r+1.$$

Thus, we see that $\Delta_1(x) = x-m$, if $x > m+1$. Hence

$$\Delta_1(x) = \begin{cases} x - m + (n+1) & \text{if } 1 \le x \le m \\ x - m & \text{if } (m+1) \le x \le (n+1) \end{cases}.$$

This completes the first part of the proof.

Since Δ_2 and Δ_3 are maximal elements,

$$\Delta_2 = \begin{pmatrix} 1 & 2 & \cdots & m & m+1 & m+2 & \cdots & n+1 \\ m & m-1 & \cdots & 1 & m+1 & m+2 & \cdots & n+1 \end{pmatrix}$$

and

$$\Delta_3 = \begin{pmatrix} 1 & 2 & \cdots & m & m+1 & m+2 & \cdots & n+1 \\ 1 & 2 & \cdots & m & n+1 & n & \cdots & m+1 \end{pmatrix}.$$

i.e.,

$$\Delta_2(x) = \begin{cases} m+1-x & \text{if } x = 1, 2, \cdots, m \\ x & \text{if } x = (m+1), (m+2), \cdots, (n+1) \end{cases}$$

and

$$\Delta_3(x) = \begin{cases} x & \text{if } x = 1, 2, \cdots, m \\ m+n+2-x & \text{if } x = (m+1), (m+2), \cdots, (n+1) \end{cases}.$$

Now one can check that if $1 \leq x \leq n+1$, then

$$\Delta(x) = \Delta_1\Delta_2\Delta_3(x) = n+2-x.$$

Hence

$$\Delta = \begin{pmatrix} 1 & 2 & \cdots & n+1 \\ n+1 & n & \cdots & 1 \end{pmatrix}.$$

This completes the proof. ■

Lemma 2.5.3 λ_{min} *defined by*

$$\lambda_{min}([s_1, s_n]) = v_m(\Delta_2\Delta_3)$$

is the unique nearly maximal labelling of $\{[s_1, s_n]\}$ *with respect to* $J_m := [s_1, s_n] \setminus \{s_m\}$ *such that*

$$\ell(v_m) = n.$$

Proof. It follows from the definition of nearly maximal labelling that λ defined by the parabolic decomposition $\lambda([s_1, s_n]) = vu$ with respect to J_m is a nearly maximal labelling of $\{[s_1, s_n]\}$ whenever the following three conditions are satisfied:

1. $\texttt{Supp}(v) = [s_1, s_n]$.

2. $u = u_m$.

3. m is the unique right descents in v.

Observe that $\texttt{Supp}(v_m) = [s_1, s_n]$, $\Delta_2\Delta_3 = u_m$ and m is the unique right descent in v_m. Moreover, if $v' \in \mathfrak{S}_{n+1}$ such that $\ell(v') = n$, $\texttt{Supp}(v') = [s_1, s_n]$, and m is the unique right descents in v' then $v' = s_n s_{n-1} \cdots s_{m+1} s_1 s_2 \cdots s_m = v_m$. This completes the proof. ■

Remark 2.5.4 *If $v \in \mathfrak{S}_{n+1}$ such that* $\mathtt{Supp}(v) = [s_1, s_n]$ *and m is the unique right descent in v, then $v_m \leq v$ in left weak Bruhat order. Thus if λ is a nearly maximal labelling of $\{[s_1, s_n]\}$ with respect to J_m, then*

$$\lambda_{min}([s_1, s_n]) \leq_L \lambda([s_1, s_n]) \leq_L \lambda_{max}([s_1, s_n]).$$

It is easy to see that each δ_k is fully commutative. In fact, each δ_k has a unique reduced expression. Now with the assumption that, in Definition 2.5.1, the expressions used for Δ_2 and Δ_3 are reduced, the number of generators s_i used in the expression of Δ is

$$= (n-m)(m-1) + n + \frac{m(m-1)}{2} + \frac{(n-m)(n-m+1)}{2} = \frac{n(n+1)}{2},$$

which is equal to the length of Δ. Therefore,

$$(s_{k+1}s_{k+2}\cdots s_{k+m-1})(s_k s_{k+1}\cdots s_{k+m-2})\cdots(s_2 s_3 \cdots s_m)$$

is a reduced expression of $\delta_k\delta_{k-1}\cdots\delta_1$.

Lemma 2.5.5 *$\delta_k\delta_{k-1}$ is fully commutative.*

Proof. Let $w = \delta_k\delta_{k-1} = (s_{k+1}s_{k+2}\cdots s_{k+m-1})(s_k s_{k+1}\cdots s_{k+m-2})$, where each δ_j has a unique reduced expression. Also for all i and j, the i^{th} entry of δ_j is s_{j+i}.

Since s_i and s_{i+1} do not commute, hence in the expression of w, if we move the i^{th} entry of δ_k to the right, then we also have to move the $(i+1)^{\text{th}}$ entry of δ_k to the right. Similarly, if we move the i^{th} entry of δ_{k-1} to the left, then we also have to move the $(i-1)^{\text{th}}$ entry of δ_{k-1} to the left. Moreover, we cannot move the i^{th} entry (which is s_{k+i}) of δ_k to the right of the i^{th} entry (which is s_k) of δ_{k-1}.

The reduced expression of δ_k contains exactly one $s_a = s_{k+i}$, which is the i^{th} entry in δ_k. Similarly, the reduced expression of δ_{k-1} contains exactly one $s_b = s_{k+i}$, which is the $(i+1)^{th}$ entry in δ_{k-1}. Hence, in w, if we move s_a to the right and s_b to the left so that the distance between s_a and s_b is minimum, then we will be ended up with the factor that is either $s_{k+i}s_{k+i+1}s_{k+i-1}s_{k+i}$ or $s_{k+i}s_{k+i-1}s_{k+i+1}s_{k+i}$, each of which is fully commutative. This completes the proof. ■

Repeatedly applying the argument used in Lemma 2.5.5, we can prove the following corollary.

Corollary 2.5.1 *For all $k > 0$, $w_m := \delta_k \delta_{k-1} \cdots \delta_1$ is fully commutative. Moreover, s_{k+1} is the unique left descent and m is the unique right descent in $\delta_k \delta_{k-1} \cdots \delta_1$.*

We see that we can represent w_m by an $(n-m) \times (m-1)$ Young Tableau, where we use French notation with the diagonal entries being m, the i-th upper diagonals with entries $m-i$, and i-th lower diagonals with $m+i$. For example, if $n = 7$ and $m = 4$, then w_m is given by the following tableau.

4	3	2
5	4	3
6	5	4

It follows that λ_{max} defined by $\lambda_{max}([s_1, s_n]) = \Delta$ is the maximal labelling of $\{[s_1, s_n]\}$. We call a nearly maximal labelling λ of $\{[s_1, s_n]\}$ **non-trivial** if λ is neither λ_{max} nor λ_{min}. Note that $\lambda_{max}([s_1, s_n]) = \lambda_{min}([s_1, s_n])$, if $m \in \{s_1, s_n\}$. Since every non-trivial nearly maximal labelling of $[s_1, s_n]$ can be extended to the maximal labelling of $[s_1, s_n]$, hence every non-trivial nearly maximal labelling of $\{[s_1, s_n]\}$ is obtained from Δ by removing some entries from w_m. Moreover from the proof of the previous lemma, we see that the i^{th} entry of δ_j in w_m can be removed if the following two conditions are met.

1. The k^{th} entry of δ_j is removed for all $k < i$.

2. The i^{th} entry of δ_k is removed for all $k > j$.

Thus we obtain the following lemma.

Lemma 2.5.6 *The permutation $\lambda([s_1, s_n])$ is a nearly maximal labelling of $\mathcal{D} = \{[s_1, s_n]\}$ if and only if there exists $s_m \in [s_1, s_n]$ and a partition μ whose Young diagram is contained in a $(n-m) \times (m-1)$ rectangle such that*

$$\lambda([s_1, s_n]) = w_\mu (s_n s_{n-1} \cdots s_{m+1})(s_1 s_2 \cdots s_m) u_m,$$

where w_μ is defined in the example below.

Example 2.5.7 *Let $n = 7$, $m = 4$, and $\mu = (3, 2, 2)$. Consider the Young tableaux of μ given by*

$$\begin{array}{|c|c|c|}\hline 4 & 3 & 2 \\ \hline 5 & 4 \\ \cline{1-2} 6 & 5 \\ \cline{1-2}\end{array}$$

Here we use French notation with the diagonal entries $m = 4$, the i-th upper diagonals with entries $m-i$, and i-th lower diagonals with $m+i$. Then w_μ is the associated permutation corresponding to the reverse row word of the tableaux. In this case,

$$w_\mu = (s_5 s_6)(s_4 s_5)(s_2 s_3 s_4)$$

and

$$\lambda([s_1, s_7]) = w_\mu (s_7 s_6 s_5)(s_1 s_2 s_3 s_4) u_4.$$

Corollary 2.5.8 *If λ is a nearly maximal labelling of $\mathcal{D} = \{[s_k, s_n]\}$, then*

$$\Lambda(\mathcal{D}, \lambda) = \lambda([s_k, s_n]) = \begin{pmatrix} 1 & 2 & \cdots & k-1 & k & k+1 & \cdots & n & n+1 \\ 1 & 2 & \cdots & k-1 & n+1 & \alpha_{k+1} & \cdots & \alpha_n & k \end{pmatrix},$$

for some $\alpha_{k+1}, \alpha_{k+2}, \cdots, \alpha_n \in [k+1, n]$, and

$$\lambda_{max}([s_k, s_n]) = \begin{pmatrix} 1 & 2 & \cdots & k-1 & k & k+1 & \cdots & n & n+1 \\ 1 & 2 & \cdots & k-1 & n+1 & n & \cdots & k+1 & k \end{pmatrix},$$

where λ_{max} is the maximal labelling.

CHAPTER III

Divisor-labelled staircase diagram

The goal of this chapter is to provide a pattern avoidance criterion of a permutation $w \in \mathfrak{S}_n$ such that the corresponding Schubert variety $X_w \in \mathcal{F}\ell(n)$ has a complete parabolic bundle structure with fibers that are isomorphic to Grassmannians or Grassmannian Schubert divisors.

3.1 Fiber bundle structure of a Schubert variety

Definition 3.1.1 *A map $\pi : X \to Y$ between algebraic varieties is called a **fiber bundle** with fiber F if for each point $y \in Y$, the fiber $\pi^{-1}(y)$ is isomorphic to F and there is a Zariski open neighborhood U of y such that $\pi^{-1}(U) \cong U \times F$.*

For any subset $J \subset \mathcal{S} = [s_1, s_{n-1}]$, we have the natural projection map

$$\pi : \mathcal{F}\ell(n) \to \mathcal{F}\ell(\mathbf{a}_J, n).$$

If $w = vu$ is the parabolic decomposition of w with respect to J, then the restriction of π to X_w gives the projection

$$\pi : X_w \to X_v^J.$$

The following theorem makes the connection between the geometry of Schubert varieties and BP decompositions.

Theorem 3.1.1 ([16]) *The parabolic decomposition $w = vu$ is a BP decomposition if and only if the restriction $\pi : X_w \to X_v^J$ is a fiber bundle with fibers isomorphic to X_u.*

Definition 3.1.2 ([17]) *A* ***complete BP decomposition*** *of an element $w \in \mathfrak{S}_n$ is a factorization $w = v_n \cdots v_2 v_1$ such that $v_i(v_{i-1} \cdots v_2 v_1)$ is a Grassmannian BP decomposion for every $i \in [2, n]$.*

Lemma 3.1.3 ([17]) *An element $w \in \mathfrak{S}_n$ is either maximal or nearly maximal if only if w has a complete BP decomposition $w = v_n \cdots v_2 v_1$ such that $\mathtt{Supp}(v_{i-1}) \subset \mathtt{Supp}(v_i)$ for all $i \in [2, n]$.*

For every labelled staircase diagram $(\mathcal{D}, \lambda)$, we obtain a unique Schubert variety $X_{\Lambda(\mathcal{D})}$, since $\Lambda(\mathcal{D})$ does not depend on the choice of the linear extension of $\mathcal{D}$. By Proposition 2.4.2, we see that the blocks of $\mathcal{D}$ determine the fibers of the fiber bundle structure of the Schubert variety $X_{\Lambda(\mathcal{D})}$ and the partial order in the blocks of $\mathcal{D}$ determines the sequence of the fibers. In [17], Richmond and Slofstra showed that the following three conditions are equivalent.

1. $(\mathcal{D}, \lambda)$ is a nearly maximal labelled staircase diagram.

2. $\Lambda(\mathcal{D})$ is a maximal or nearly maximal element.

3. The Schubert variety $X_{\Lambda(\mathcal{D})}$ has a complete parabolic bundle structure.

In [2], Alland and Richmond showed that a permutation w avoids the patterns 3412, 52341, and 635241 if and only if the Schubert variety X_w has a complete parabolic bundle structure. Thus we have the following theorem.

Theorem 3.1.4 ([2, 17]) *There is a bijection between any two of the following three sets,*

1. *the set of permutations avoiding the patterns $3412, 52341,$ and 635241 in $\mathfrak{S}_n$,*

2. *the set of Schubert varieties in $\mathcal{F}\ell(n)$ with complete parabolic bundle structures,*

3. *the set of nearly maximal labelled staircase diagrams of support contained in $[s_1,\, s_{n-1}]$,*

and the bijections are giev by

$$w \leftrightarrow X_w \leftrightarrow w = \Lambda(\mathcal{D}, \lambda).$$

Remark 3.1.5 *Suppose that $k < m < n$ and $\mathcal{D} = \{B\} = \{[s_k, s_n]\}$ is a staircase diagram consisting of a single block. Let λ be a nearly maximal labelling of B with respect to the set $\{[s_k, s_n]\} \setminus \{s_m\}$. Then any sequence*

$$[n] = \sigma_n \supset \cdots \supset \sigma_2 \supset \sigma_1 = \{n\},$$

where $|\sigma_j| = j$ and $\sigma_2 = \{n + k - m,\, n\}$, induces a complete parabolic bundle structure on $X_{\Lambda(\mathcal{D})}$,

$$X_{\Lambda(\mathcal{D})} = X_n \overset{\pi^{\sigma_n}_{\sigma_{n-1}}}{\twoheadrightarrow} X_{n-1} \overset{\pi^{\sigma_{n-1}}_{\sigma_{n-2}}}{\twoheadrightarrow} \cdots \overset{\pi^{\sigma_3}_{\sigma_2}}{\twoheadrightarrow} X_2 \overset{\pi^{\sigma_2}_{\sigma_1}}{\twoheadrightarrow} X_1, \text{ where}$$

$$X_i := \pi^{[1,n]}_{\sigma_i}(X_{\Lambda(\mathcal{D})}) \subset \mathcal{F}\ell(\sigma_i, n).$$

Remark 3.1.6 *Let $\{B_1, B_2, \cdots, B_\ell\}$ be a linear extension of a staircase diagram $\mathcal{D}$ of type A_{n-1}. For $i \in [2, \ell]$, let*

$$\mathcal{D}^i = \{B_1, B_2, \cdots, B_{i-1}\}$$

and

$$\boldsymbol{b}_i = \{j \,\big|\, s_j \in [s_1, s_n] \setminus \mathtt{Supp}(\mathcal{D}^i\}).$$

Assume that $\boldsymbol{b}_1 = [1, n]$ and $\boldsymbol{b}_{\ell+1} = \{n\}$. Observe that

$$[1, n] = \boldsymbol{b}_1 \supset \boldsymbol{b}_2 \supset \cdots \supset \boldsymbol{b}_{\ell+1} = \{n\}.$$

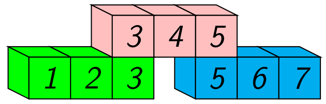
By Proposition 2.4.2 and Theorem 3.1.1, the sequence of projection maps

$$V_\bullet^{\boldsymbol{b}_1} \twoheadrightarrow V_\bullet^{\boldsymbol{b}_2} \twoheadrightarrow \cdots \twoheadrightarrow V_\bullet^{\boldsymbol{b}_\ell} \twoheadrightarrow (\mathbb{C}^n) \cong \{pt\} \tag{3.1.1}$$

induces a sequence of fiber bundle structures on the corresponding sequence of Schubert varieties in the partial flag varieties. Now by Remark 3.1.5 together with the sequence (3.1.1), we get a sequence of projection maps that induces a complete parabolic bundle structure on $X_{\Lambda(\mathcal{D})}$ which is illustrated by the following example.

Example 3.1.7 *Consider the following staircase diagram with the linear extension*

$$\mathcal{D} = \{\{1,2,3\},\{5,6,7\},\{3,4,5\}\}.$$



Let λ be a nearly maximal labelling of $\mathcal{D}$ such that the restrictions of λ on the blocks $[s_1, s_3]$ and $[s_3, s_5]$ are maximal and the restriction of λ on the block $[s_5, s_7]$ are non-maximal. Thus the restriction $\lambda_{[s_5,s_7]}$ is a non-maximal nearly maximal labelling of the block $[s_5, s_7]$ with respect to $\{s_5, s_7\}$. By (3.1.1), we get the sequence of projection maps

$$(V_1 \subset V_2 \subset \cdots \subset V_8 = \mathbb{C}^8) \twoheadrightarrow (V_4 \subset V_5 \subset \cdots \subset \mathbb{C}^8) \twoheadrightarrow (V_4 \subset \mathbb{C}^8) \twoheadrightarrow (\mathbb{C}^8). \tag{3.1.2}$$

Notice that in the sequence (3.1.2), we started with the complete flag $(V_1 \subset V_2 \subset \cdots \subset \mathbb{C}^8)$. Then we ignored all V_i's such that s_i belongs to the first block in the linear extension of $\mathcal{D}$. Then we ignored all V_i's such that s_i belongs to the second block. Then we ignored all V_i's such that s_i belongs to the third block.

Now, by Remark 3.1.5, we can reach from $(V_1 \subset V_2 \subset \cdots \subset \mathbb{C}^8)$ to $(V_4 \subset V_5 \subset \cdots \subset \mathbb{C}^8)$ by ignoring one V_i each time from $V_1, V_2,$ and V_3 in that order, since the first block in $\mathcal{D}$ has the maximal labelling. In a similar way, we can reach from $(V_4 \subset V_6 \subset \cdots \subset \mathbb{C}^8)$ to $(V_4 \subset \mathbb{C}^8)$ by ignoring one V_i each time with the exception that we need to ignore V_6 at the end since the second block has the nearly maximal labelling with respect to the complement of $\{s_6\}$. Continuing in this way, we finally get the sequence of projection maps given by

$$\begin{aligned} &(V_1 \subset V_2 \subset \cdots \subset \mathbb{C}^8) \twoheadrightarrow (V_2 \subset V_3 \subset \cdots \subset \mathbb{C}^8) \twoheadrightarrow (V_3 \subset V_4 \subset \cdots \subset \mathbb{C}^8) \\ &\twoheadrightarrow (V_4 \subset V_5 \subset \cdots \subset \mathbb{C}^8) \twoheadrightarrow (V_4 \subset V_6 \subset V_7 \subset \mathbb{C}^8) \twoheadrightarrow (V_4 \subset V_6 \subset \mathbb{C}^8) \\ &\twoheadrightarrow (V_4 \subset \mathbb{C}^8) \twoheadrightarrow (\mathbb{C}^8), \end{aligned} \tag{3.1.3}$$

which induces a complete parabolic bundle structure on $X_{\Lambda(\mathcal{D})}$.

Definition 3.1.8 *We call a nearly maximal labelling λ of a staircase diagam $\mathcal{D}$ a* ***divisor labelling*** *of $\mathcal{D}$ if for each $B \in \mathcal{D}$, $\ell(\lambda(B)) = \ell(u_B)$ or $\ell(u_B) - 1$, where ℓ denotes the length function.*

The following lemma follows from Lemma 2.5.6.

Lemma 3.1.9 *If λ is a divisor labeling of $\mathcal{D} = \{B\} = \{[s_k, s_n]\}$, then*

$$\lambda(B) = \lambda_{max}(B)s_i = s_{n+k-i}\lambda_{max}(B),$$

for some i such that $k < i < n$.

Let $(\mathcal{D}, \lambda)$ be a nearly maximal labelled staircase diagram with support in $[s_1, s_{n-1}]$. Let $w = \Lambda(\mathcal{D})$. By Lemma 3.1.3, w has a complete BP decomposition $w = v_n \cdots v_2 v_1$. For $i \in [2, n]$, let $w_i = v_{i-1} \cdots v_2 v_1$. Then each X_{w_i} is a Grassmannian Schubert variety. If λ is the maximal labelling, then each X_{w_i} is smooth, and so of codimension 0. If λ is a divisor labelling then each X_{w_i} is of codimension 0 or 1. Thus we have the following lemma.

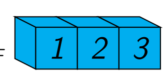
Lemma 3.1.10 *The following two sets are in bijection,*

1. *the set of divisor-labelled staircase diagrams of support contained in $[s_1,\, s_{n-1}]$,*

2. *the set of Schubert varieties in $\mathcal{F}\ell(n)$ having complete parabolic bundle structures with fibers isomorphic to Grassmannians or Grassmannian Schubert divisors,*

and the bijection is given by $(\mathcal{D},\, \lambda) \leftrightarrow X_{\Lambda(\mathcal{D},\, \lambda)}$.

Example 3.1.11 *Let $\mathcal{D}$ be the staircase diagram*

$\mathcal{D} =$ 

consisting of a single block $[s_1, s_3]$ and λ a non-maximal divisor labelling of $\mathcal{D}$. Then $\Lambda(\mathcal{D}) =$ 4231. Let $w =$ 4231. Fix a complete flag $(E_1 \subset E_2 \subset E_3 \subset \mathbb{C}^4) \in \mathcal{F}\ell(4)$. Now we compute the Schubert variety

$$X_w = \{(V_1 \subset V_2 \subset V_3 \subset V_4 = \mathbb{C}^4) : \mathtt{dim}(E_i \cap V_j) \geq r_w[i, j]\, \forall\, i, j\}.$$

We have

$r_w[1, 1] = |\{1\} \cap \{4\}| = 0 \Rightarrow \mathtt{dim}(E_1 \cap V_1) \geq 0,$

$r_w[1,2] = |\{1\} \cap \{4,2\}| = 0 \Rightarrow \mathtt{dim}(E_1 \cap V_2) \geq 0,$

$r_w[1,3] = |\{1\} \cap \{4,2,3\}| = 0 \Rightarrow \mathtt{dim}(E_1 \cap V_3) \geq 0,$

$r_w[1,4] = |\{1\} \cap \{4,2,3,1\}| = 1 \Rightarrow \mathtt{dim}(E_1 \cap V_4) \geq 1,$

$r_w[2,1] = |\{1,2\} \cap \{4\}| = 0 \Rightarrow \mathtt{dim}(E_2 \cap V_1) \geq 0,$

$r_w[2,2] = |\{1,2\} \cap \{4,2\}| = 1 \Rightarrow \mathtt{dim}(E_2 \cap V_2) \geq 1,$

$r_w[2,3] = |\{1,2\} \cap \{4,2,3\}| = 1 \Rightarrow \mathtt{dim}(E_2 \cap V_3) \geq 1,$

$r_w[2,4] = |\{1,2\} \cap \{4,2,3,1\}| = 2 \Rightarrow \mathtt{dim}(E_2 \cap V_4) \geq 2,$

$r_w[3,1] = |\{1,2,3\} \cap \{4\}| = 0 \Rightarrow \mathtt{dim}(E_3 \cap V_1) \geq 0,$

$r_w[3,2] = |\{1,2,3\} \cap \{4,2\}| = 1 \Rightarrow \mathtt{dim}(E_3 \cap V_2) \geq 1,$

$r_w[3,3] = |\{1,2,3\} \cap \{4,2,3\}| = 2 \Rightarrow \mathtt{dim}(E_3 \cap V_3) \geq 2,$

$r_w[3,4] = |\{1,2,3\} \cap \{4,2,3,1\}| = 3 \Rightarrow \mathtt{dim}(E_3 \cap V_4) \geq 3,$

$r_w[4,1] = |\{1,2,3,4\} \cap \{4\}| = 1 \Rightarrow \mathtt{dim}(E_4 \cap V_1) \geq 1,$

$r_w[4,2] = |\{1,2,3,4\} \cap \{4,2\}| = 2 \Rightarrow \mathtt{dim}(E_4 \cap V_2) \geq 2,$

$r_w[4,3] = |\{1,2,3,4\} \cap \{4,2,3\}| = 3 \Rightarrow \mathtt{dim}(E_4 \cap V_3) \geq 3,$ *and*

$r_w[4,4] = |\{1,2,3,4\} \cap \{4,2,3,1\}| = 4 \Rightarrow \mathtt{dim}(E_4 \cap V_4) \geq 4.$

Here all "$\mathtt{dim}(E_i \cap V_j) \geq r_w[i,j]$*" type conditions are redundant except that* $\mathtt{dim}(E_2 \cap V_2) \geq 1$*. Therefore*

$$X_{4231} = \{(V_1 \subset V_2 \subset V_3 \subset V_4) : \mathtt{dim}(E_2 \cap V_2) \geq 1\}.$$

Now one can check that X_{4231} *has an iterated fiber bundle structure via the following sequence of projection maps:*

$$(V_1 \subset V_2 \subset V_3 \subset \mathbb{C}_4) \to (V_1 \subset V_2 \subset \mathbb{C}_4) \to (V_2 \subset \mathbb{C}_4) \to (\mathbb{C}_4),$$

where the fibers of the first two projection maps are Grassmannians and the fibers of the last project map are isomorphic to a Grassmannian Schubert divisor.

3.2 Pattern avoidance and divisor-labelled staircase diagram

We noticed in Section 2.5 that all nearly maximal labellings of $[s_1, s_n]$ are obtained by choosing $s_m \in [s_1, s_n]$ and a partition $\mu \subseteq (n-m) \times (m-1)$. In the following lemmas, it will be convenient to consider the dual partition to μ in $(n-m) \times (m-1)$ which we will denote by $\mu^\vee$.

Let $\lambda(n, m, \mu)$ denote the nearly maximal labelling corresponding to the data $m \leq n$ and partition μ. Also, let

$$\mathtt{Supp}(u^\vee) := \{j : s_j \in \mathtt{Supp}\big(\lambda_{max}([s_1, s_n])(\lambda(n,m,\mu))^{-1}\big)\}.$$

Example 3.2.1 *Let $n = 8$, $m = 4$, and $\mu = (3, 2, 1, 1)$. Consider the following Young tableau of μ.*

4	3	2
5	4	
6		
7		

Here, $\mu^\vee = (2, 2, 1)$ and $\mathtt{Supp}(\mu^\vee) = \{3, 4, 5, 6\}$.

Note that $\lambda(n, m, \mu)$ is a divisor labelling if $|\mathtt{Supp}(\mu^\vee)|$ is at most 1. Moreover, if $\mu^\vee$ is empty, then we obtain the maximal labelling in which case the labelling is given by the permutation $(n+1)n \cdots 21$.

Let $p_1 = 3412$, $p_2 = 52341$, $p_3 = 52431$ $p_4 = 53241$, and $p_5 = 635241$. Notice that the pattern p_5 contains both p_3 and p_4.

Lemma 3.2.2 *Let $i > 0$, $j \geq 0$, and $\alpha = s_{i+j}s_{i+j-1} \cdots s_i$. Then*

$$\alpha = \begin{pmatrix} 1 & 2 & \cdots & i-1 & i & i+1 & \cdots & i+j & i+j+1 \\ 1 & 2 & \cdots & i-1 & i+j+1 & i & \cdots & i+j-1 & i+j \end{pmatrix}.$$

Proof. We consider three cases.

Case 1: Let $x < i$. Then $\alpha(x) = x$, since $s_k(x) = x$ for all $k > x$.

Case 2: Let $x = i$. Then $\alpha(x) = i + j + 1$, since $s_x(x) = x + 1$.

Case 3: Let $x > i$. Then

$$\begin{aligned}\alpha(x) &= s_{i+j}s_{i+j-1}\cdots s_i(x)\\ &= s_{i+j}s_{i+j-1}\cdots s_x s_{x-1}(x)\\ &= s_{i+j}s_{i+j-1}\cdots s_x(x-1)\\ &= x-1.\end{aligned}$$

Hence the lemma. ■

Lemma 3.2.3 *Let $\alpha^\vee = (\ell_1 \geq \ell_2 \geq \cdots \geq \ell_s)$ and $\beta^\vee = (\ell_2 \geq \ell_3 \geq \cdots \geq \ell_s)$. Then for large enough n, $\lambda(n, m, \alpha)$ contains $\lambda(n-1, m, \beta)$.*

Proof. Consider $\lambda(n, m, \mu)$ and let $\mu^\vee = \ell_1$. Suppose that the largest entry in $\texttt{Supp}(\mu^\vee)$ is v. With the help of Lemma 3.2.2, one can check that $\lambda(n, m, \mu) = (n+1)\cdots(v+1)\cdots 1$, where the $(m+1)^{th}$ entry is $v+1$, and all other entries are decreasing. This implies that $\lambda(n, m, \mu)$ contains the maximal pattern $n(n-1)\cdots 1$. Consider $\lambda(n-1, m, \beta)$ and notice that all entries in $\texttt{Supp}(\beta^\vee)$ are less than v. Hence $\lambda(n, m, \alpha)$ is obtained from $\lambda(n, m, \mu) = (n+1)\cdots(v+1)\cdots 1$ by a sequence of swaps of the entries in $\lambda(n, m, \mu)$ that are less than v. By the same sequence of swaps of these entries applied on the maximal pattern $n(n-1)\cdots 1$, we get the permutation $\lambda(n-1, m, \beta)$. However, swapping a pair of entries $\leq v$ in $\lambda(n, m, \mu)$ preserves the position of the entries $\geq v$ in $\lambda(n, m, \mu)$. Hence, $\lambda(n, m, \alpha)$ contains $\lambda(n-1, m, \beta)$. ■

Lemma 3.2.4 *Let $\lambda(n, m, \mu)$ be a nearly maximal labelling of $\mathcal{D} = \{[s_1, s_n]\}$. If the size of the partition $|\mu^\vee| > 1$ (i.e $\lambda(n, m, \mu)$ is not a divisor labelling), then the permutation $\lambda(n, m, \mu)$ contains p_3 or p_4.*

Proof. We prove the lemma by the induction on n.

Special Case: Assume that $\mu^\vee$ is a single row such that $|\mu^\vee| > 1$. Then

$$\texttt{Supp}(\mu^\vee) = \{n-m+1, n-m+2, \cdots, n-m+\ell\},$$

for some $m > 2$, $\ell > 1$. Let $\alpha = s_{n-m+\ell}s_{n-m+\ell-1}\cdots s_{n-m+1}$. Then $\lambda(n,m,\mu) = \alpha\Delta$. By Lemma 3.2.2,

$$\alpha = \begin{pmatrix} 1 & \cdots & n-m & n-m+1 & n-m+2 & \cdots & n-m+\ell+1 \\ 1 & \cdots & n-m & n-m+\ell+1 & n-m+1 & \cdots & n-m+\ell \end{pmatrix}.$$

Now

$$\begin{aligned} &\alpha\Delta(1) = \alpha(n+1) = n+1, \\ &\alpha\Delta(m-1) = \alpha(n-m+3) = n-m+2, \\ &\alpha\Delta(m) = \alpha(n-m+2) = n-m+1, \\ &\alpha\Delta(m+1) = \alpha(n-m+1) = n-m+\ell+1, \text{ and} \\ &\alpha\Delta(n+1) = \alpha(1) = 1. \end{aligned} \tag{3.2.1}$$

Thus we see that $\lambda(n,m,\mu)$ contains a subsequence

$$(n+1)(n-m+2)(n-m+1)(n-m+\ell+1)1 \sim 52431,$$

and hence, it contains p_4. Similarly, we can show that if $\mu^\vee$ is a single column such that $|\mu^\vee| \geq 2$, then $\lambda(n,m,\mu)$ contains $p_3 = 52431$. Thus we see that the lemma is true for $n = 4$.

Now for the sake of induction, assume that for any permutation $\lambda(n-1,m,\mu)$, if $|\mu^\vee| > 1$, then the permutation $\lambda(n-1,m,\mu)$ contains p_3 or p_4. We will show that the same property holds for $\lambda(n,m,\mu)$, where $|\mu^\vee| > 1$. Let $\mu^\vee = (\ell_1 \geq \ell_2 \geq \cdots \geq \ell_s)$ and $\beta^\vee = (\ell_2 \geq \ell_3 \geq \cdots \geq \ell_s)$. By the previous lemma, $\lambda(n,m,\mu)$ contains $\lambda(n-1,m,\beta)$. If $|\beta^\vee| > 1$ then, by the induction hypothesis, $\lambda(n,m,\mu)$ contains p_3 or p_4,.

Now assume that $|\beta^\vee| = 1$. Then $\mu^\vee$ is a hook with two parts and $m < n-1$. One can check that the smallest and the largest entries in $\texttt{Supp}(\mu^\vee)$ are s_{n-m} and $s_{n-m+\ell}$, respectively, for some $\ell > 0$, and

$$\lambda(n,m,\mu) = s_{n-m}(s_{n-m+\ell}s_{n-m+\ell-1}\cdots s_{n-m+1})\Delta = s_{n-m}\alpha\Delta, \text{ where}$$

$$\alpha = s_{n-m+\ell}s_{n-m+\ell-1}\cdots s_{n-m+1}.$$

Now by Equation (3.2.1),

$$\lambda(n,m,\mu)(1) = s_{n-m}(n+1) = n+1,$$

$$\lambda(n,m,\mu)(m) = s_{n-m}(n-m+1) = n-m,$$

$$\lambda(n,m,\mu)(m+1) = s_{n-m}(n-m+\ell+1) = n-m+\ell+1,$$

$$\lambda(n,m,\mu)(m+2) = s_{n-m}(n-m) = n-m+1, \text{ and}$$

$$\lambda(n,m,\mu)(n+1) = s_{n-m}(1) = 1.$$

The sequence $(n+1)(n-m)(n-m+\ell+1)(n-m+1)1$ is order isomorphic to p_3. Therefore, $\lambda(n,m,\mu)$ contains p_3. This completes the proof. ∎

Remark 3.2.5 *Let $\mathcal{D} = \{[s_k, s_n]\}$ be a staircase diagram consisting of a single block. If λ is a non divisor nearly maximal labelling of $\mathcal{D}$, then*

$$\Lambda(\mathcal{D}) = \begin{pmatrix} 1 & 2 & \cdots & k-1 & k & k+1 & \cdots & n & n+1 \\ 1 & 2 & \cdots & k-1 & n+1 & w_{k+1} & \cdots & w_n & k \end{pmatrix}$$

such that the subsequence $\begin{pmatrix} k & k+1 & \cdots & n & n+1 \\ n+1 & w_{k+1} & \cdots & w_n & k \end{pmatrix}$ *contains p_3 or p_4.*

We now show that Lemma 3.2.4 extends to connected staircase diagrams.

Lemma 3.2.6 *Let $\mathcal{D}$ be a (connected) staircase diagram with support $[s_1,\ s_n]$ such that $[s_k,\ s_n] \in \mathcal{D}$. If λ is a nearly maximal labelling of $\mathcal{D}$, then the $(n+1)^{th}$ entry in $\Lambda(\mathcal{D})$ is at most k.*

Proof. We will prove the lemma by the induction on n. Clearly the lemma is true when $\mathcal{D}$ consists of a single block.

Suppose that the lemma is true for any staircase diagram whose support is a proper subset of $[s_1,\ s_n]$.

Let $\mathcal{D}' = \mathcal{D} \setminus \{[s_k,\ s_n]\}$, $\mathtt{Supp}(\mathcal{D}') = [s_1, s_\ell]$, and

$$\Lambda(\mathcal{D}') = \begin{pmatrix} 1 & 2 & \cdots & \ell & \ell+1 \\ w_1 & w_2 & \cdots & w_\ell & w_{\ell+1}. \end{pmatrix}.$$

Then by our induction hypothesis $w_{\ell+1} < k$. Now,

$$u_{[s_k,s_\ell]} = \begin{pmatrix} 1 & \cdots & k-1 & k & k+1 & \cdots & \ell & \ell+1 \\ 1 & \cdots & k-1 & \ell+1 & \ell & \cdots & k+1 & k \end{pmatrix}.$$

By Corollary 2.5.8

$$\lambda([s_k, s_n]) = \begin{pmatrix} 1 & \cdots & k-1 & k & k+1 & \cdots & n & n+1 \\ 1 & \cdots & k-1 & n+1 & * & \cdots & * & k \end{pmatrix}.$$

Therefore

$$\lambda([s_k, s_n])u_{[s_k,s_\ell]} = \begin{pmatrix} 1 & 2 & \cdots & n & n+1 \\ \eta_1 & \eta_2 & \cdots & \eta_n & k \end{pmatrix},$$

for some η_i and

$$u_{[s_k,s_\ell]}\lambda([s_k, s_n]) = \begin{pmatrix} 1 & 2 & \cdots & n & n+1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n & \ell+1 \end{pmatrix}$$

for some α_i. Now we consider two cases.

Case 1: Let $[s_k,\ s_n]$ is a maximum element of $\mathcal{D}$. In this case,

$$\begin{aligned} \Lambda(\mathcal{D}) &= \lambda([s_k, s_n])u_{[s_k,s_l]}\Lambda(\mathcal{D}') \\ &= \begin{pmatrix} 1 & 2 & \cdots & n & n+1 \\ \eta_1 & \eta_2 & \cdots & \eta_n & k \end{pmatrix} \begin{pmatrix} 1 & 2 & \cdots & \ell & \ell+1 \\ w_1 & w_2 & \cdots & w_\ell & w_{\ell+1} \end{pmatrix}. \end{aligned}$$

Case 2: Let $[s_k,\ s_n]$ is a minimum element of $\mathcal{D}$. In this case,

$$\begin{aligned} \Lambda(\mathcal{D}) &= \Lambda(\mathcal{D}')u_{[s_k,s_l]}\lambda([s_k, s_n]) \\ &= \begin{pmatrix} 1 & 2 & \cdots & \ell & \ell+1 \\ w_1 & w_2 & \cdots & w_\ell & w_{\ell+1} \end{pmatrix} \begin{pmatrix} 1 & 2 & \cdots & n & n+1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n & \ell+1 \end{pmatrix}. \end{aligned}$$

In both cases, we see that the $(n+1)$-th entry in $\Lambda(\mathcal{D}, \lambda)$ is atmost k. ■

Remark 3.2.7 *Let λ be a nearly maximal labelling of a staircase diagram $\mathcal{D}$ such that $\Lambda(\mathcal{D}, \lambda)$ avoids p_3 or p_4. Then λ^{-1} is also a nearly maximal labelling of* `flip`$(\mathcal{D})$*, and*

$\Lambda(\texttt{flip}(\mathcal{D}), \lambda^{-1})$ *avoids* p_3 *and* p_4. *To see this, suppose that* λ^{-1} *is not a nearly maximal labelling of* $\texttt{flip}(\mathcal{D})$. *Then* $\Lambda(\texttt{flip}(\mathcal{D}), \lambda^{-1})$ *contains* 635241. *Therefore* $\Lambda(\mathcal{D}, \lambda) = \Lambda(\texttt{flip}(\texttt{flip}(\mathcal{D})), (\lambda^{-1})^{-1})$ *contains* 642531. *Hence* $\Lambda(\mathcal{D}, \lambda)$ *contains* 642531. *However* 642531 *contains* p_3 *and* p_4, *which is a contradiction.*

Lemma 3.2.8 *Let* $\mathcal{D}$ *be a connected staircase diagram with support* $[s_1, s_n]$, λ *a nearly maximal labelling of* $\mathcal{D}$, *and* β_n *the unique block in* $\mathcal{D}$ *containing* s_n. *Suppose that* β_n *is maximal in* $\mathcal{D}$ *and* $\mathcal{D}' = \mathcal{D} \setminus \{\beta_n\}$.

1. *If* λ *restricted to* β_n *is a divisor labelling, then* $\Lambda(\mathcal{D})$ *contains* $\Lambda(\mathcal{D}')$.

2. *If* λ *restricted to* β_n *is a non-divisor labelling, then* $\Lambda(\mathcal{D})$ *contains* p_3 *or* p_4.

Proof. Let $\beta_n = [s_k, s_n]$, $\texttt{Supp}(\mathcal{D}') = [s_1, s_\ell]$, and $\Lambda(\mathcal{D}') = w_1 w_2 \cdots w_{\ell+1}$. Now, $\Lambda(D) = \overline{\lambda}(\beta_n)\Lambda(\mathcal{D}')$.

Let S_k^n be the subsequence $(n+1)n \cdots k$. Since $\mathcal{D}$ is connected, $k \leq \ell+1$. Now we consider the following two cases.

$$\begin{aligned} &\text{Case } A: \quad k = \ell + 1, \text{ and so, } \beta_n = [s_{\ell+1}, s_n]. \\ &\text{Case } B: \quad k < \ell + 1. \end{aligned}$$

Case A : In this case, $\overline{\lambda}_{max}(\beta_n)\Lambda(\mathcal{D}') = u_{[s_k,\ s_n]}\ \Lambda(\mathcal{D}')$, where

$$u_{[s_k,\ s_n]} = \begin{pmatrix} 1 & \cdots & k-1 & k & \cdots & n+1 \\ 1 & \cdots & k-1 & n+1 & \cdots & k \end{pmatrix}.$$

Thus, in one-line notation,

$$\overline{\lambda}_{max}(\beta_n)\Lambda(\mathcal{D}') = \eta_1\eta_2 \cdots \eta_{\ell+1} n(n-1) \cdots (\ell+1), \text{ where}$$

$$\eta_i = \begin{cases} n+1 & : \text{ if } w_i = \ell + 1 \\ w_i & : \text{ otherwise} \end{cases}.$$

We see that $\overline{\lambda}_{max}(\beta_n)\Lambda(\mathcal{D}')$ contains the pattern $\Lambda(\mathcal{D}')$, since $\ell+1$ is the largest entry in the one-line notation of $\Lambda(\mathcal{D}')$. We also see that $s_d\overline{\lambda}_{max}(\beta_n)\Lambda(\mathcal{D}')$ contains the pattern $\Lambda(\mathcal{D}')$, if $\ell+1 < d < n$. Thus we see that if λ restricted to β_n is a divisor labelling, then $\Lambda(\mathcal{D})$ contains $\Lambda(\mathcal{D}')$.

Case B : In this case, $\overline{\lambda}_{max}(\beta_n)\Lambda(\mathcal{D}') = u_{[s_k,\ s_n]}\ u_{[s_k,\ s_\ell]}\ \Lambda(\mathcal{D}')$, where

$$u_{[s_k,\ s_n]} = \begin{pmatrix} 1 & \cdots & k-1 & k & \cdots & n+1 \\ 1 & \cdots & k-1 & n+1 & \cdots & k \end{pmatrix} \text{ and}$$
$$u_{[s_k,\ s_\ell]} = \begin{pmatrix} 1 & \cdots & k-1 & k & \cdots & \ell+1 \\ 1 & \cdots & k-1 & \ell+1 & \cdots & k \end{pmatrix}.$$

Thus, in one-line notation,

$$\overline{\lambda}_{max}(\beta_n)\Lambda(\mathcal{D}') = \eta_1\eta_2\cdots\eta_\ell w_{\ell+1}(n+k-\ell-1)(n+k-\ell-2)\cdots(k+1)k, \text{ where}$$
$$\eta_i = \begin{cases} w_i & : \text{ if } w_i < k \\ w_i + n - \ell & : \text{ otherwise} \end{cases}.$$

Notice that in this case, $\overline{\lambda}_{max}(\beta_n)\Lambda(\mathcal{D}')$ also contains the pattern $\Lambda(\mathcal{D}')$. Moreover, by the previous lemma $w_{\ell+1} < k$, and so, $\overline{\lambda}_{max}(\beta_n)\Lambda(\mathcal{D}')$ contains the sub-sequence S^n_k . Also since $k < d < n$, $s_d\overline{\lambda}_{max}(\beta_n)\Lambda(\mathcal{D}')$ contains the pattern $\Lambda(\mathcal{D}')$ too. This completes the first part of the proof.

In the previous part, we see that $\overline{\lambda}_{max}(\beta_n)$ contains the sub-sequence $(n+1)n\cdots k$. Now suppose that λ restricted to β_n is non-divisor. Then

$$\Lambda(\mathcal{D}) = s_{d_1}s_{d_2}\cdots s_{d_t}\overline{\lambda}_{max}(\beta_n)\Lambda(\mathcal{D}'),$$

for some d_i such that

$$s_{d_1}s_{d_2}\cdots s_{d_t}\lambda_{max}(\beta_n) = \begin{pmatrix} 1 & 2 & \cdots & k-1 & k & k+1 & \cdots & n & n+1 \\ 1 & 2 & \cdots & k-1 & n+1 & \eta_{k+1} & \cdots & \eta_n & k \end{pmatrix}$$

for some η_i such that$s_{d_1}s_{d_2}\cdots s_{d_t}\lambda_{max}(\beta_n)$ contains p_3 or p_4. Cosequently, $\Lambda(\mathcal{D})$ contains a subsequence

$$\left(n+1, \eta_{k+1}, \cdots, \eta_n, k\right),$$

which contains p_3 or p_4. This completes the proof. ■

Lemma 3.2.9 *Let $\mathcal{D}$ be a connected staircase diagram and λ a non-divisor nearly maximal labelling of $\mathcal{D}$. Then $\Lambda(\mathcal{D})$ contains p_3 or p_4.*

Proof. Without any loss of generality, suppose that $\mathtt{Supp}(\mathcal{D}) = [s_1, s_n]$. We will prove the lemma by the induction on n. The lemma is true for $n < 5$. Suppose that the lemma is true for any staircase diagram with $|\mathtt{Supp}(\mathcal{D})| < n$. Let β_n be the unique block containing s_n in $\mathcal{D}$. Now by Remark 3.2.7, we can assume without any loss of generality that β_n is maximum in $\mathcal{D}$. Let $\mathcal{D}'$ be the sub-diagram defined in Lemma 3.2.8. Now we consider two cases.

Case 1: Let λ restricted to β_n be a divisor labelling. Then by Lemma 3.2.8, $\Lambda(\mathcal{D})$ contains $\Lambda(\mathcal{D}')$. Hence by our induction hypothesis $\Lambda(\mathcal{D})$ contains p_3 or p_4.

Case 2: Let λ restricted to β_n be a non divisor labelling. Then by 3.2.8, $\Lambda(\mathcal{D})$ contains p_3 or p_4. This completes the proof. ■

Lemma 3.2.10 *Let λ be a divisor labelling of a (connected) staircase diagram $\mathcal{D}$, then the permutation pattern $\Lambda(\mathcal{D})$ avoids p_3 and p_4.*

Proof. We will prove the lemma by the induction on the number of blocks in $\mathcal{D}$. First note that if $\mathcal{D} = \{B\}$ is a single block, then it can be verified that any divisor labelling of $\mathcal{D}$ corresponds to a permutation avoiding p_3 and p_4. In fact, any such labelling will avoid 132 and 213 all of which are contained in p_2 and p_3 respectively.

Now let $\mathcal{D}$ be a staircase diagram such that $|\mathcal{D}| \geq 2$. Without any loss of generality, suppose that $\mathtt{Supp}(\mathcal{D}) = [s_1, s_n]$, and as in the previous lemma, let $\beta_n = [s_k, s_n]$ denote the unique block containing s_n, β_n is maximal, $\mathcal{D}' = \mathcal{D} \setminus [s_k,\ s_n]$, $\mathtt{Supp}(\mathcal{D}') = \{s_1, s_\ell\}$, and

$$\Lambda(\mathcal{D}') = \begin{pmatrix} 1 & 2 & \cdots & \ell & \ell+1 \\ w_1 & w_2 & \cdots & w_\ell & w_{\ell+1} \end{pmatrix},$$

where $w_{\ell+1} < k$. By our induction hypothesis, $\Lambda(\mathcal{D}')$ avoids p_3 and p_4. Now we consider two cases.

Case 1: Let $\lambda|_{\{\beta_n\}}$ be maximal. In this case,

$$\Lambda(\mathcal{D}) = \begin{pmatrix} 1 & \cdots & \ell & \ell+1 & \ell+2 & \ell+3 & \cdots & n & n+1 \\ \eta_1 & \cdots & \eta_\ell & w_{\ell+1} & n+k-\ell-1 & n+k-\ell-2 & \cdots & k+1 & k \end{pmatrix},$$

where

$$\eta_i = \begin{cases} w_i & \text{if } w_i < k \\ w_i + n - l & \text{otherwise} \end{cases}.$$

Case 2: Let $\lambda|_{\{\beta_n\}}$ be divisor but not maximal. In this case, $\Lambda(\mathcal{D})$ is almost the same as it is in Case 1, except there is a swap between two consecutive entries that are bigger than $n+k-\ell-1$.

In both cases, if $\Lambda(\mathcal{D})$ contains p_3 or p_4, then $\Lambda(\mathcal{D}')$ also contains the same, since $w_{\ell+1} < k$. This completes the proof. ■

If a staircase diagram has disconnected support, then the permutation corresponding to any nearly maximal labelling is an element of the parabolic subgroup $W_{J_1} \times \cdots W_{J_k}$ where J_i's denote the connected components of $\mathtt{Supp}(\mathcal{D})$. Hence if the corresponding permutation contains one of 3412, 52341, 52431, and 53241, it must contain the pattern in one of the connected components. This leads to the following corollary:

Corollary 3.2.11 *There is a bijection between the following sets:*

1. *The set of permutations of* $[1, n]$ *avoiding the patterns* 3412*,* 52341, 52431, *and* 53241,
2. *The set of Schubert varieties in* $\mathcal{F}\ell(n)$ *which have complete parabolic bundle structures where the fibers are isomorphic to Grassmannians or Grassmannian Schubert divisors,*
3. *The set of divisor-labelled staircase diagrams of support contained in* $[s_1, s_{n-1}]$,

and the bijections are given by $w \leftrightarrow X_w \leftrightarrow w = \Lambda(D, \lambda)$*, where* λ *is a divisor labelling of* $\mathcal{D}$.

Note that if λ is a nearly maximal labelling of $\mathcal{D}$, then λ^{-1} may or may not be a nearly maximal labelling of $\texttt{flip}(\mathcal{D})$. Likewise, if w is a nearly maximal element in $\mathfrak{S}_n$, then w^{-1} is not necessarily a nearly maximal element in $\mathfrak{S}_n$. If both w and w^{-1} are nearly maximal, then we say that w is **almost maximal**. It follows from Remark 3.2.7 that λ is a divisor labelling of $\mathcal{D}$ if and only if λ^{-1} is a divisor labelling of $\texttt{flip}(\mathcal{D})$. Thus if $(\lambda, \mathcal{D})$ is a divisor-labelled staircase diagram, then $\Lambda(D, \lambda)$ is an almost maximal element in $\mathfrak{S}_n$.

CHAPTER IV

GENERATING FUNCTION

In this chapter, we will find the generating function

$$Z(x) = \sum_{n \geq 0} z_n x^n,$$

where $z_0 = 1$, and for $n > 0$, z_n is the number of divisor-labelled staircase diagrams of support contained in $[s_1, s_n]$. This will in turn give us the number of Schubert varieties in $\mathcal{F}\ell(n+1)$ that are iterated fiber bundles of Grassmannians or Grassmannian Schubert divisors.

Definition 4.0.1 *Suppose that $\mathfrak{D}$ is a set of staircase diagrams, $\mathcal{D} \in \mathfrak{D}$, and $B \in \mathcal{D}$. Then we define the following:*

$N_{\mathfrak{D}} = |\{(\mathcal{D}, \lambda) : \mathcal{D} \in \mathfrak{D}$ *and* λ *is a divisor labelling of* $\mathcal{D}\}|$,

$N_{\mathcal{D}} = N_{\{\mathcal{D}\}}$, *and*

$N_B = |\{(B, \lambda) : \lambda$ *is a divisor labelling of* B *such that* λ *can be extended to a labelling of* $\mathcal{D}\}|$.

If $\mathcal{D}$ is a staircase diagram and $\mathfrak{D}$ is a set of staircase diagrams, then it follows from the definition that

$$N_{\mathcal{D}} = \prod_{B_i \in \mathcal{D}} N_{B_i}$$

and

$$N_{\mathfrak{D}} = \sum_{\mathcal{D} \in \mathfrak{D}} N_{\mathcal{D}}.$$

The following is a technical lemma that describes the values N_{B_i} in a given staircase diagram.

Lemma 4.0.2 *Let B be a block in a staircase diagram $\mathcal{D}$ such that $|\mathtt{Supp}(B)| = \ell$. Let $k_1 = min\{|J_R(B)|, |J_L(B)|\}$ and $k_2 = max\{|J_R(B)|, |J_L(B)|\}$. If $\ell = 1$, then $N_B = 1$. Otherwise, we have*

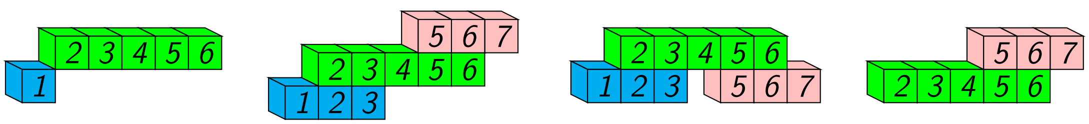
(a) $N_B = \ell - 1 \quad$ *if* $k_2 = 0$,

(b) $N_B = \ell - k_2 \quad$ *if* $k_1 > 0$,

(c) $N_B = \ell + 1 - k_2 \quad$ *if* $k_1 = 0$ *and B has overlapping boxes on both sides, and*

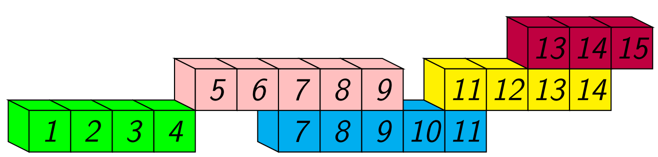
(d) $N_B = \ell - k_2 \quad$ *if* $k_1 = 0$ *and B has overlapping boxes on exactly one side.*

Example 4.0.3 *Here we list four diagrams where the blocks with support $B = [2, 6]$ have properties* (a), (b), (c), *and* (d), *respectively.*



Proof. Without any loss of generality, let $B = [s_1, s_\ell]$. Note that a non-maximal divisor-labelling of B is of the form $s_i \lambda_{max}(B) = \lambda_{max}(B) s_{\ell+1-i}$, for some $1 < i < \ell$. This together with part (1) of Definition 2.3.4 implies that there are $\ell - 2$ choices of s_i in part (a), $\ell - 1 - k_2$ choices of s_i in parts (b) and (d), and $\ell - k_2$ choices of s_i in part (c). This completes the proof. ■

Example 4.0.4 *In the staircase diagram*

$\mathcal{D} =$  ,

$$N_{[s_1,\ s_4]} = 4 - 1 = 3,$$

$$N_{[s_5,\ s_9]} = 5 - 3 = 2,$$

$$N_{[s_7,\ s_{11}]} = 5 + 1 - 4 = 2,$$

$$N_{[s_{11},\ s_{14}]} = 4 - 2 = 2,$$

and

$$N_{[s_{13},\ s_{15}]} = 3 - 2 = 1.$$

Hence

$$N_{\mathcal{D}} = (3)(2)(2)(2)(1) = 24.$$

The following lemma follows immediately from Lemma 4.0.2.

Lemma 4.0.5 *Suppose that* $b_n = N_{[s_1, s_n]}$, *the number of divisor labellings of the staircase diagram* $\{[s_1,\ s_n]\}$. *Then*

$$\begin{aligned} \mathtt{F}_0(x) &:= \sum_{n>0} b_n x^n \\ &= x + \sum_{n=2}^{\infty} (n-1)x^n \\ &= x + \left(\frac{x}{1-x}\right)^2. \end{aligned}$$

Definition 4.0.6 *We call a staircase diagram* $\mathcal{D}$ ***strongly connected*** *if* $\mathcal{D}$ *is connected and for each pair of adjacent blocks* B_1, B_2 *in* $\mathcal{D}$, $|B_1 \cap B_2| > 0$.

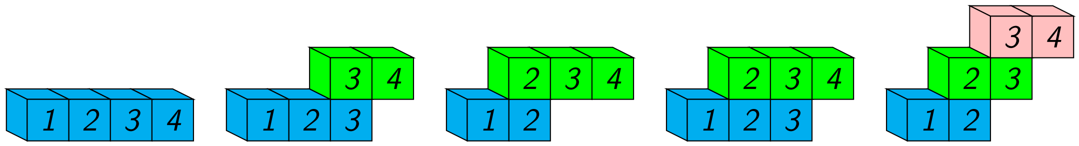
Let

$$\mathtt{C}(n) := \text{the set of connected staircase diagrams of support } [s_1, s_n],$$

$$\mathtt{SC}(n) := \{\mathcal{D} \in \mathtt{C}(n)\ : \mathcal{D} \text{ is strongly connected}\},\ \text{and}$$

$$\mathtt{SCI}(n) := \{\mathcal{D} \in \mathtt{SC}(n) : \ \mathcal{D} \text{ is increasing}\}.$$

Example 4.0.7 $\mathtt{SCI}(4)$ *consists of the following five staircase diagrams.*



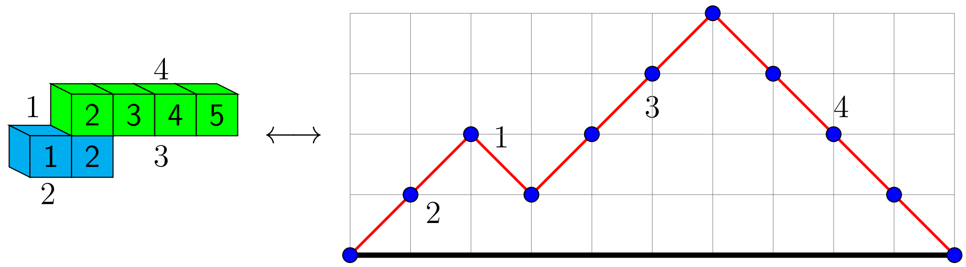
Thus, $N_{\mathtt{SCI}(4)} = 3 + (2)(1) + (1)(2) + (1)(1) + (1)(1)(1) = 9.$

Observe that a staircase diagram $\mathcal{D}$ in $\mathtt{SCI}(n)$ corresponds to a unique Dyck path P such that

1. no valley of P lies on the x-axis,

2. the k-th mount of P corresponds to the k-th block in $\mathcal{D}$, and

3. the length of the left (resp. right) side of a mount in P equals the number of exposed boxes of the corresponding block from the bottom (resp. top).



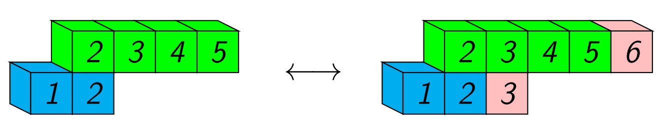
Definition 4.0.8 *If $B = [s_i, s_j]$ is a block, $\mathcal{D} = \{B_1 \prec B_2 \prec \cdots \prec B_k\}$ is an increasing staircase diagram, and $\mathfrak{D}$ is a set of increasing staircase diagrams, then we define the following:*

$$\overline{B} = [s_i,\, s_{j+1}],$$

$$\overline{\mathcal{D}} = \{\overline{B_1} \prec \overline{B_2} \prec \cdots \prec \overline{B_k}\}, \text{ and}$$

$$\overline{\mathfrak{D}} = \{\overline{\mathcal{D}} : \mathcal{D} \in \mathfrak{D}\}.$$

Remark 4.0.9 *It is easy to check that if $\mathcal{D}$ is an increasing staircase diagram then so is $\overline{\mathcal{D}}$. Moreover, there is a natural bijection between the sets $\mathfrak{D}$ and $\overline{\mathfrak{D}}$ given by $\mathcal{D} \mapsto \overline{\mathcal{D}}$.*



Lemma 4.0.10

$$N_{\overline{\mathtt{SCI}(n)}} = \begin{cases} N_{\mathtt{SCI}(n)} & \text{if } n = 1 \\ N_{\mathtt{SCI}(n)} + 1 & \text{if } n > 1 \end{cases}.$$

Proof. It follows from Lemma 4.0.2 and Remark 4.0.9 that if $\mathcal{D}$ is an increasing staircase diagram, then

$$N_{\overline{\mathcal{D}}} = \begin{cases} N_{\mathcal{D}} + 1 & ; \text{ if } \mathcal{D} \text{ consists of a single block of length greater than } 1 \\ N_{\mathcal{D}} & ; \text{ otherwise} \end{cases}.$$

Moreover, if $n > 1$, then $\texttt{SCI}(n)$ contains exactly one staircase diagram consisting of a single block of length greater than 1. This completes the proof. ∎

Lemma 4.0.11 *There is a bijection between* $\texttt{SCI}(n)$ *and* $\cup_{k=1}^{n-1}\big(\overline{\texttt{SCI}(k)} \times \texttt{SCI}(n-k)\big)$ *which preserves the number of divisor labelling.*

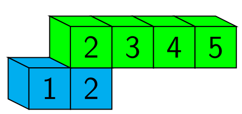
Proof. Recall that each diagram in $\overline{\texttt{SCI}(k)}$ is of the type $\overline{D_l}$ for some $\mathcal{D}_l \in \texttt{SCI}(k)$. For $\mathcal{D}_l = \{B_1 \prec B_2 \prec \cdots \prec B_m\} \in \texttt{SCI}(k)$ and $\mathcal{D}_r = \{B_{m+1} \prec B_{m+2} \prec \cdots \prec B_s\} \in \texttt{SCI}(n-k)$, define

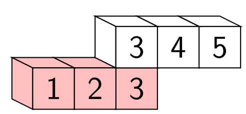
$$\mathcal{D}_l^r = \{\overline{B_1} \prec \overline{B_2} \prec \cdots \overline{B_m} \prec B_{m+1}(+k) \prec B_{m+2}(+k) \prec \cdots \prec B_s(+k)\} \in \texttt{SCI}(n),$$

where for all j,

$$B_j(+k) = \{s_{i+k} \,|\, s_i \in B_j\}.$$

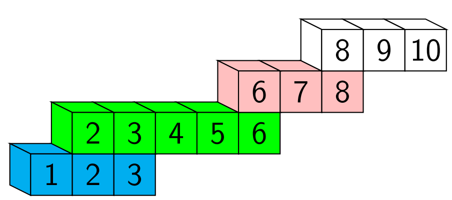
For example, let

$\mathcal{D}_l =$ 

and $\mathcal{D}_r =$ 

.

Then

$\mathcal{D}_l^r =$ 

.

It is easy to see that

$$(N_{\overline{\mathcal{D}_\ell}})(N_{\mathcal{D}_r}) = (N_{\mathcal{D}_\ell^r}).$$

Notice that D_l^r is uniquely determined by D_l and D_r. Thus the function

$$F : \cup_{k=1}^{n-1}\big(\overline{\texttt{SCI}(k)} \times \texttt{SCI}(n-k)\big) \to \texttt{SCI}(n)$$

defined by

$$\big(\overline{D_l},\, D_r\big) \mapsto \begin{cases} D_l^r & : \text{ if } |\texttt{Supp}(D_r)| > 1 \\ \\ \overline{D_l} & : \text{ if } |\texttt{Supp}(D_r)| = 1 \end{cases}$$

is one-one.

Let $\mathcal{D} = \{B_1 \prec B_2 \prec \cdots B_m \prec B_{m+1} \prec \cdots \prec B_s\} \in \texttt{SCI}(n) \setminus \overline{\texttt{SCI}(n-1)}$, where m is the least positive integer such that

$$B_m \cap B_{m+1} = \{s_k\}$$

for some k. For each $i \leq m$, let B_i' be the block such that

$$\overline{B_i'} = B_i,$$

and for $i > m$, let B_i' be the block defined by

$$B_i' = \{s_{j-k+1} \,|\, s_j \in B_i\}.$$

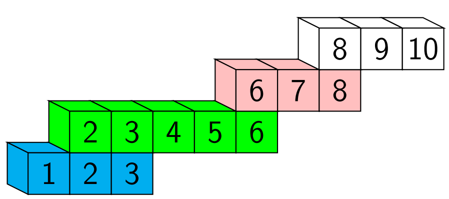
Let $\mathcal{D}_l = \{B_1' \prec B_2' \prec \cdots B_m'\}$ and $\mathcal{D}_r = \{B_{m+1}' \prec B_{m+2}' \prec \cdots B_s'\}$. Now one can check that

$$F(\overline{\mathcal{D}_l},\, \mathcal{D}_r) = \mathcal{D}.$$

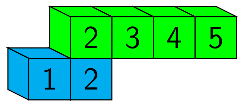
For example, if

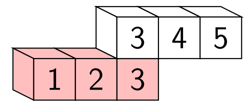
$$\mathcal{D} = \{B_1 = [s_1, s_3] \prec B_2 = [s_2, s_6] \prec B_3 = [s_6, s_8] \prec B_4 = [s_8, s_{10}]\},$$

then $B_1' = [s_1, s_2]$, $B_2' = [s_2, s_5]$, $B_3' = [s_1, s_3]$, $B_4' = [s_3, s_5]$, $\mathcal{D}_l = \{B_1' \prec B_2'\}$, and $\mathcal{D}_r = \{B_3' \prec B_4'\}$.

$\mathcal{D} =$ 

,

$\mathcal{D}_l =$ 

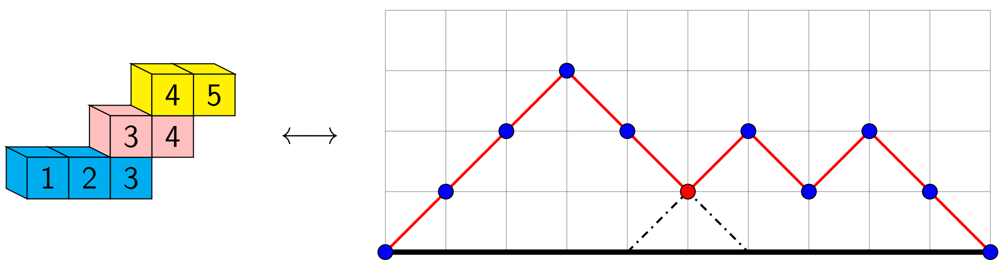
and $\mathcal{D}_r =$ 

.

Thus we see that F is onto, and hence F is a bijection. Moreover,

$$(N_{\overline{D_l}})(N_{D_r}) = N_{D_l^r}.$$

To visualize the map F, observe that if the first valley of a Dyck path of a staircase diagram in $\mathtt{SCI}(n)$ stays at the position $(2k-1, 1)$, then the diagram decomposes into two diagrams; one is in $\overline{\mathtt{SCI}(k)}$, and the other is in $\mathtt{SCI}(n-k)$.



$\blacksquare$

Corollary 4.0.12 *$N_{\mathtt{SCI}(n)}$ satisfies the following Catalan type equation:*

$$N_{\mathtt{SCI}(n)} = \sum_{k=1}^{n-1} N_{\overline{\mathtt{SCI}(k)}} \times N_{\mathtt{SCI}(n-k)}.$$

Lemma 4.0.13 *Define the generating series*

$$a(x) := \sum_{n>0} N_{\mathtt{SCI}(n)} x^n.$$

Then

$$a(x) = \frac{1 - x - x^2 - \sqrt{x^4 - 2x^3 + 7x^2 - 6x + 1}}{2(1-x)}.$$

Proof. We have

$$\begin{aligned}
a(x) &= \sum_{n>0} N_{\mathtt{SCI}(n)} x^n \\
&= x + \sum_{n>1} \Big(\sum_{k=1}^{n-1} N_{\overline{\mathtt{SCI}(k)}} \times N_{\mathtt{SCI}(n-k)} \Big) x^n \\
&= x + \sum_{k=1}^{\infty} \Big(\sum_{n=k+1}^{\infty} N_{\overline{\mathtt{SCI}(k)}} \times N_{\mathtt{SCI}(n-k)} \Big) x^n \\
&= x + \sum_{k=1}^{\infty} \Big(\sum_{n=1}^{\infty} N_{\overline{\mathtt{SCI}(k)}} \times N_{\mathtt{SCI}(n)} \Big) x^{n+k} \\
&= x + \Big(\sum_{k=1}^{\infty} N_{\overline{\mathtt{SCI}(k)}} x^k \Big) \Big(\sum_{n=1}^{\infty} N_{\mathtt{SCI}(n)} x^n \Big).
\end{aligned}$$

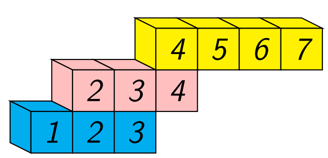
By Lemma 4.0.10, $\displaystyle\sum_{k=1}^{\infty} N_{\overline{\mathtt{SCI}(k)}} x^k = \frac{x^2}{1-x} + a(x)$. Hence,

$$a(x) = x + \Big(\frac{x^2}{1-x} + a(x) \Big) a(x),$$

which is a quadratic equation in $a(x)$. Since the coefficients in the formal power series expansion of $a(x)$ are positive, $a(x) = \dfrac{1 - x - x^2 - \sqrt{x^4 - 2x^3 + 7x^2 - 6x + 1}}{2(1-x)}$. ∎

Let $\mathtt{SCI}(k,n) \subset \mathtt{SCI}(n)$ be the subset of staircase diagrams such that the block containing s_1 of a diagram in $\mathtt{SCI}(k,n)$ equals $[s_1, s_{k+1}]$ and the covering block contains s_2 (i.e. these two blocks have k overlapping boxes).

Example 4.0.14 *The follwing staircase diagram is contained in* $\mathtt{SCI}(2,7)$.



Lemma 4.0.15 *Define the generating series*

$$G(x,k) := \sum_{n=k+2}^{\infty} N_{\mathtt{SCI}(k,n)} \; x^n.$$

Suppose that

$$a := a(x)$$

and

$$d := ax - x^2.$$

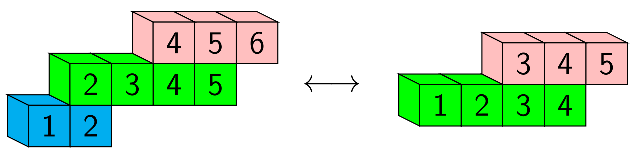
Then $G(x, k) = da^{k-1}$.

Proof. Notice that there is a bijection from $\mathtt{SCI}(1, n)$ to $\mathtt{SCI}(n-1)$ given by

$$\mathcal{D} \mapsto \mathcal{D} \setminus \{[s_1, s_2]\}$$

and

$$N_{\mathcal{D}} = N_{\mathcal{D} \setminus \{[s_1, s_2]\}}.$$



Thus

$$\begin{aligned}
G(x, 1) &= \sum_{n=3}^{\infty} N_{\mathtt{SCI}(1,n)} x^n \\
&= \sum_{n=3}^{\infty} N_{\mathtt{SCI}(n-1)} x^n \\
&= \sum_{n=2}^{\infty} N_{\mathtt{SCI}(n)} x^{n+1} \\
&= -x^2 + \sum_{n=1}^{\infty} N_{\mathtt{SCI}(n)} x^{n+1} \\
&= -x^2 + xa.
\end{aligned}$$

Hence

$$G(x, 1) = d. \tag{4.0.1}$$

As in Corollary 4.0.12, we can show that if $k > 1$, then

$$N_{\mathtt{SCI}(k,\,n)} = \sum_{i=k+1}^{n-1} N_{\overline{\mathtt{SCI}(k-1,\,i)}} N_{\mathtt{SCI}(n-i)}. \tag{4.0.2}$$

Moreover, for $k > 1$, it follows from the definition of $\overline{\texttt{SCI}(k-1, i)}$ that

$$N_{\overline{\texttt{SCI}(k-1,\,i)}} = N_{\texttt{SCI}(k-1,\,i)}. \tag{4.0.3}$$

Thus, combining Equations (4.0.2) and (4.0.3), we get that if $k > 1$, then

$$N_{\texttt{SCI}(k,\,n)} = \sum_{i=k+1}^{n-1} N_{\texttt{SCI}(k-1,\,i)} N_{\texttt{SCI}(n-i)}. \tag{4.0.4}$$

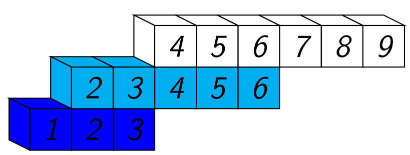
This implies that if $k > 1$, then

$$\begin{aligned} G(x, k) &= \sum_{n=k+2}^{\infty} N_{\texttt{SCI}(k,n)} x^n \\ &= \Big(\sum_{i=k+1}^{\infty} N_{\texttt{SCI}(k-1,i)} x^i \Big) \Big(\sum_{n=1}^{\infty} N_{\texttt{SCI}(n)} x^n \Big) \\ &= G(x, k-1) a. \end{aligned} \tag{4.0.5}$$

Now the lemma follows from Equations (4.0.1) and (4.0.5). ■

Let $\texttt{SCI}_\ell(k,\, n) \subset \texttt{SCI}(k, n)$ be the subset of diagrams such that the block containing s_n has ℓ overlapping boxes.

Example 4.0.16 $\texttt{SCI}_3(2, 9)$ *contains the following diagram.*



Observe that

$$\texttt{SCI}(k,\, n) = \bigcup_{\ell > 0} \texttt{SCI}_\ell(k,\, n),$$

and so,

$$N_{\texttt{SCI}(k,\,n)} = \sum_{\ell > 0} N_{\texttt{SCI}_\ell(k,\,n)}.$$

Let $\texttt{SCI}(k, \ell, n) \subset \texttt{SCI}(k, n)$ denote the subset of diagrams such that the block containing s_n contains $\ell + 1$ boxes, where ℓ boxes are overlapping. Let $\texttt{L}(x, k, \ell)$ be the generating series of the number of divisor-labelled diagrams of this type. i.e.,

$$\mathtt{L}(x,k,\ell) = \sum_{n>0} N_{\mathtt{SCI}(k,\ell,n)} x^n.$$

It is clear that

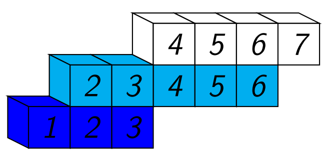
$$\mathtt{L}(x,k,\ell) = \mathtt{L}(x,\ell,k).$$

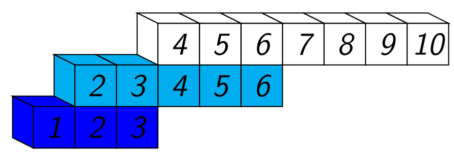
Definition 4.0.17 *Let $\mathcal{D}$ be a staircase diagram with support $[s_1, s_n]$. We define $\mathcal{D}^{+m}$ to be the staircase diagram obtained from $\mathcal{D}$ by adding $m \geq 0$ boxes to the unique block of $\mathcal{D}$ containing s_n. In particular, $\mathcal{D}^{+0} = \mathcal{D}$.*

It is easy to see that

$$N_{\mathcal{D}^{+m}} = (m+1)N_{\mathcal{D}}.$$

Example 4.0.18

Let $\mathcal{D} =$ 

. *Then* $\mathcal{D}^{+3} =$ 

.

Lemma 4.0.19 *For any positive integer k,*

$$G(x,k) = \sum_{\ell>0} \frac{\mathtt{L}(x,k,\ell)}{(1-x)^2}.$$

Proof. For $\ell > 0$ and $\mathcal{D} \in \mathtt{SCI}_\ell(k,n)$, if the unique block in $\mathcal{D}$ containing s_n has $m+1$ exposed boxes for some $m \geq 0$, then

$$\mathcal{D} = (\mathcal{D}')^{+m}$$

for some $\mathcal{D}' \in \mathtt{SCI}(k, \ell, n-m)$. Therefore,

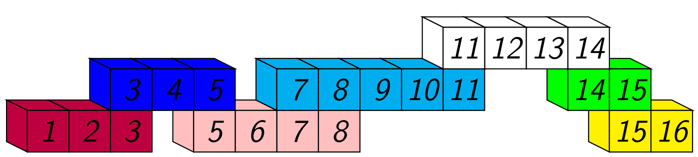
$$\begin{aligned}
\sum_{n>0} N_{\mathtt{SCI}_\ell(k,n)} x^n &= \sum_{n>0} \left(\sum_{\mathcal{D} \in \mathtt{SCI}_\ell(k,n)} N_{\mathcal{D}} \right) x^n \\
&= \sum_{n>0} \left(\sum_{m=0}^{n-1} \sum_{\mathcal{D}' \in \mathtt{SCI}(k,\ell,n-m)} (m+1) N_{\mathcal{D}'} x^{n-m} \right) x^m \\
&= \sum_{m \geq 0} (m+1) \left(\sum_{n>m} \sum_{\mathcal{D}' \in \mathtt{SCI}(k,\ell,n-m)} N_{\mathcal{D}'} x^{n-m} \right) x^m \\
&= \sum_{m \geq 0} (m+1) \mathtt{L}(x,k,\ell) x^m \\
&= \frac{\mathtt{L}(x,k,\ell)}{(1-x)^2},
\end{aligned}$$

and so, $G(x,k) = \sum_{n>0} N_{\mathtt{SCI}(k,n)} x^n = \sum_{n>0} \sum_{l>0} N_{\mathtt{SCI}_\ell(k,n)} x^n = \sum_{\ell=1}^{\infty} \frac{\mathtt{L}(x,k,\ell)}{(1-x)^2}$. ■

Definition 4.0.20 *Let $\mathtt{T}_t(\ell, n) \subset \mathtt{SC}(n)$ denote the subset of staircase diagrams such that*

1. *each diagram in $\mathtt{T}_t(\ell, n)$ has exactly $t \geq 0$ local extremas (i.e local maximum or minimum blocks),*
2. *the block containing s_n contains $\ell + 1$ boxes where ℓ boxes are overlapping, and*
3. *the block containing s_1 is a minimum.*

Example 4.0.21 $\mathtt{T}_3(1, 16)$ *contains the following staircase diagram.*



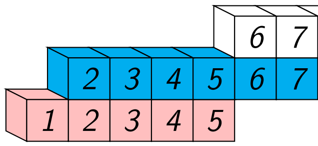
Here the local extremas of the diagram are $[s_3, s_5]$, $[s_5, s_8]$ and $[s_{11}, s_{14}]$.

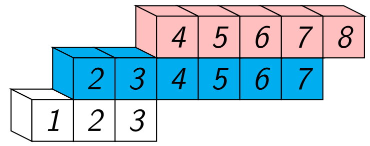
Let

$$\mathtt{Turn}_t(x,k) := \sum_{n>0} N_{\mathtt{T}_t(k,n)} x^n.$$

Definition 4.0.22 *If $\mathcal{D}$ is a connected staircase diagram, then* $\texttt{rev}(\mathcal{D})$ *is the diagram obtained from $\mathcal{D}$ by filliping $\mathcal{D}$ and then re-labelling the boxes in the reverse order.*

Example 4.0.23

Let $\mathcal{D} =$ 

. *Then* $\texttt{rev}(\mathcal{D}) =$ 

.

Lemma 4.0.24 *For any positive integer k,* $\texttt{Turn}_0(x,k) = G(x,k)$.

Proof. Observe that $\texttt{T}_0(k,n) \subset \texttt{SCI}(n)$ is the subset of staircase diagrams such that the block containing s_n in a staircase diagram in $\texttt{T}_0(k,n)$ contains $k+1$ boxes where k boxes are overlapping. Thus, the sets $\texttt{T}_0(k,n)$ and $\texttt{SCI}(k,n)$ are in bijection and the bijection is given by

$$\mathcal{D} \mapsto \texttt{rev}(\mathcal{D}).$$

Therefore

$$N_{\texttt{T}_0(k,n)} = N_{\texttt{SCI}(k,n)},$$

and so,

$$\sum_{n>0} N_{\texttt{T}_0(k,n)}\, x^n = G(x,k).$$

Hence $\texttt{Turn}_0(x,k) = G(x,k)$. ■

Let $\Lambda_{\texttt{type}}$ be the set of staircase diagrams of type $\{B_1 \prec B_2 \succ B_3\}$ or $\{B_1 \succ B_2 \prec B_3\}$ such that $N_{B_1} = 1 = N_{B_3}$. Thus if $\mathcal{D} \in \Lambda_{\texttt{type}}$ such that $\mathcal{D} = \{B_1 \prec B_2 \succ B_3\}$ or $\{B_1 \succ B_2 \prec B_3\}$, then by Lemma 4.0.2,

$$N_{\mathcal{D}} = 1 + \text{the number of exposed boxes in } B_2.$$

Remark 4.0.25 *Note that for $t > 0$, each staircase diagram $\mathcal{D} \in \texttt{T}_t(k,n)$ decomposes into three diagrams $\mathcal{D}_1, \mathcal{D}_2$, and $\mathcal{D}_3$ such that*

$$N_{\mathcal{D}} = N_{\mathcal{D}_1} N_{\mathcal{D}_2} N_{\mathcal{D}_3},$$

where

$$\mathcal{D}_1 \in \mathtt{T}_{t-1}(j, m_1),$$

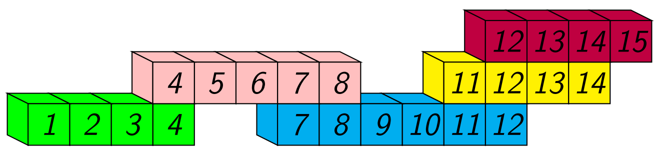
$$\mathcal{D}_2 \in \Lambda_{\mathtt{type}}, \ \textit{and}$$

$$\mathcal{D}_3 \ \textit{or}\ \mathtt{flip}(\mathcal{D}_3) \in \mathtt{SCI}(\ell, k, m_3),$$

for some j, m_1, ℓ, m_3. *Moreover, if* m_2 *is the number of exposed boxes in the local extrema of* $\mathcal{D}_2$, *then*

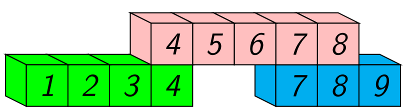
$$m_1 + m_2 + m_3 = n + 2.$$

Example 4.0.26 *The diagram*

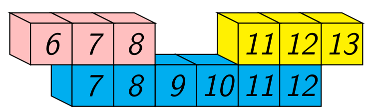


$\in \mathtt{T}_2(3, 15)$

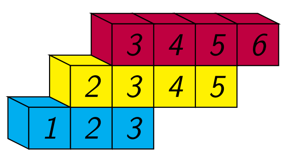
decomposes into the following three diagrams:



$\in \mathtt{T}_1(2, 9)$,



$\in \Lambda_{\mathtt{type}}$, *and*



$\in \mathtt{SCI}(2, 3, 6)$.

Lemma 4.0.27 *For any* $t \geq 0$, $\mathtt{Turn}_t(x, k) = M^t G(x, k)$, *where* $M = \dfrac{(a-x)(2-x)}{1-a}$.

Proof. We will prove the lemma by the induction on t. Clearly the lemma is true for $t = 0$. Let

$$\mathtt{Turn}_{t-1}(x, k) = M^{t-1} G(x, k).$$

If $t > 0$, then by Remark 4.0.25,

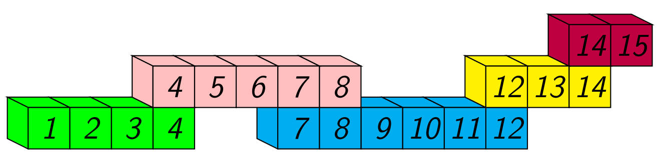
$$\begin{aligned}
\mathtt{Turn}_t(x,k) &= \sum_{n>0} N_{\mathtt{T}_t(k,n)} x^n \\
&= \sum_{n>0} \Bigg(\sum_{m_1+m_2+m_3=n+2} \Bigg(\sum_{j>1,\,\ell>1} N_{T_{t-1}(j,m_1)} (1+m_2) N_{\mathtt{SCI}(\ell,k,m_3)} \Bigg) \Bigg) x^n \\
&= \sum_{j=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{1}{x^2} \mathtt{Turn}_{t-1}(x,j)(2x+3x^2+4x^3+\cdots)\mathtt{L}(x,\ell,k) \\
&= \sum_{j=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{(2-x)}{x(1-x)^2} \mathtt{Turn}_{t-1}(x,j)\mathtt{L}(x,\ell,k) \\
&= M^{t-1} \sum_{j=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{(2-x)}{x(1-x)^2} G(x,j)\mathtt{L}(x,\ell,k) \\
&= M^{t-1} \sum_{j=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{(2-x)}{x(1-x)^2} d a^{j-1}\mathtt{L}(x,\ell,k) \\
&= M^{t-1}(a-x)(2-x)\Bigg(\sum_{j=1}^{\infty} a^{j-1} \Bigg) \Bigg(\sum_{\ell=1}^{\infty} \frac{\mathtt{L}(x,\ell,k)}{(1-x)^2} \Bigg)
\end{aligned}$$

Therefore, by Lemma 4.0.19,

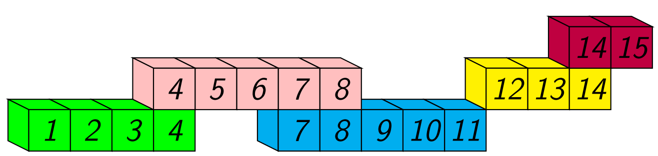
$$\mathtt{Turn}_t(x,k) = M^{t-1}\frac{(a-x)(2-x)}{(1-a)}G(x,k) = M^{t-1}MG(x,k) = M^t G(x,k). \qquad \blacksquare$$

Recall that $\mathtt{SC}(n)$ is the set of connected staircase diagrams with support $[s_1,\, s_n]$, where the intersection of every pair of adjacent blocks in each staircase diagram is non-empty.

Example 4.0.28 $\mathtt{SC}(15)$ *contains the diagram*



but it does not contain the diagram



.

Lemma 4.0.29 *For any positive integer n,*

$$F(x) := \sum_{n=1}^{\infty} N_{\mathtt{SC}(n)} x^n = \mathtt{F}_0(x) + \frac{2d}{(1-M)(1-a)(1-x)^2}.$$

Proof. We have

$$\sum_{n>0} N_{[s_1,\, s_n]} x^n = \mathtt{F}_0(x).$$

For $t \geq 0, m > 0$, and $k > 0$, let

$$\mathtt{A}_{(t,k,m)} := \{\mathcal{D} : \mathcal{D} \in \mathtt{T}_t(k,n) \text{ or } \mathcal{D} = (\mathcal{D}')^{+m}, \text{ where } \mathcal{D}' \in \mathtt{T}_t(k, n-m)\}$$

and

$$\mathtt{flip}(\mathtt{A}_{(t,k,m)}) := \{\mathtt{flip}(\mathcal{D}) : \mathcal{D} \in \mathtt{A}_{(t,k,m)}\}.$$

Therefore,

$$\begin{aligned}
\sum_{n>0} \sum_{\mathcal{D} \in \mathtt{A}_{(t,k,m)}} N_{\mathcal{D}} x^n &= \sum_{n>0} \sum_{m=0}^{n-1} \sum_{\mathcal{D}' \in \mathtt{T}_t(k,n-m)} N_{\mathcal{D}'} (m+1) x^n \\
&= \sum_{m=0}^{\infty} \sum_{n=m+1}^{\infty} \sum_{\mathcal{D}' \in \mathtt{T}_t(k,n-m)} N_{\mathcal{D}'} (m+1) x^{n-m} x^m \\
&= \sum_{m=0}^{\infty} \mathtt{Turn}_t(x,k)(m+1) x^m \\
&= \frac{\mathtt{Turn}_t(x,k)}{(1-x)^2}.
\end{aligned}$$

Observe that if $\mathcal{D} \in \mathtt{C}(n) \setminus \{[s_1, s_n]\}$, then $\mathcal{D} \in \mathtt{A}_{(t,k,m)} \cup \mathtt{flip}(\mathtt{A}_{(t,k,m)})$, for some (t,k,m). Moreover, $N_{\mathtt{flip}(\mathtt{A}_{(t,k,m)})} = N_{\mathtt{A}_{(t,k,m)}}$. Hence,

$$\begin{aligned}
F(x) &= \mathtt{F}_0(x) + 2 \sum_{k=1}^{\infty} \sum_{t=0}^{\infty} \frac{\mathtt{Turn}_t(x,k)}{(1-x)^2} \\
&= \mathtt{F}_0(x) + \frac{2}{(1-x)^2} \sum_{k=1}^{\infty} \sum_{t=0}^{\infty} M^t G(x,k) \\
&= \mathtt{F}_0(x) + \frac{2d}{(1-M)(1-a)(1-x)^2},
\end{aligned}$$

which completes the proof. ∎

Suppose that $t_n := N_{\mathtt{C}(n)}$ is the number of divisor-labelled connected staircase diagrams with support $[s_1,\, s_n]$ and z_n is the number of divisor-labelled staircase diagrams with support contained in $[s_1,\, s_n]$. Then

$$z_n = z_{n-1} + t_n + \sum_{k=2}^{n} z_{n-k} t_{k-1},$$

since every staircase diagram is a disjoint union of staircase diagrams with connected support. Let

$$\mathtt{F}_T(x) := \sum_{n>0} t_n x^n$$

and

$$Z(x) := \sum_{n\geq 0} z_n x^n.$$

Therefore,

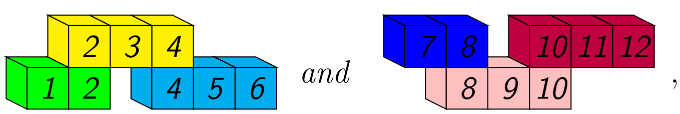
$$\begin{aligned} Z(x) &= 1 + \sum_{n=1}^{\infty} z_{n-1} x^n + \sum_{n=1}^{\infty} t_n x^n + \sum_{n=2}^{\infty}\sum_{k=2}^{n} z_{n-k} t_{k-1} x^n \\ &= 1 + xZ(x) + \mathtt{F}_T(x) + x\mathtt{F}_T(x)Z(x). \end{aligned}$$

Hence,

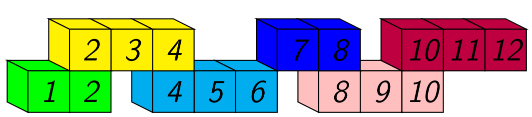
$$Z(x) = \frac{1 + \mathtt{F}_T(x)}{1 - x - x\mathtt{F}_T(x)}.$$

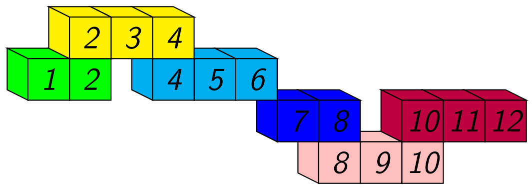
Observe that if $\mathcal{D}$ is a connected staircase diagram having two non-overlapping adjacent blocks, then $\mathcal{D}$ decomposes into two diagrams, say $\mathcal{D}_1$ and $\mathcal{D}_2$, where $\mathcal{D}_1 \in \mathtt{SC}(m_1)$ for some m_1 and $\mathcal{D}_2 \in \mathtt{C}(m_2)$ for some m_2. In other words, we obtain $\mathcal{D}$ by "gluing" $\mathcal{D}_1$ and $\mathcal{D}_2$.

Example 4.0.30 *By gluing the staircase diagrams*



we obtain the following two diagrams:





Let $f_n := N_{\mathtt{SC}(n)}$. Thus we see that

$$t_n = f_n + 2\sum_{k>0} f_{n-k}t_k.$$

Hence

$$\begin{aligned}\mathtt{F}_T(x) &= \sum_{n=1}^{\infty} t_n x^n \\ &= \sum_{n=1}^{\infty} f_n x^n + 2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \mathtt{f}_{n-k}t_k x^n \\ &= F(x) + 2F(x)F_T(x).\end{aligned}$$

Therefore,

$$\mathtt{F}_T(x) = \frac{F(x)}{1-2F(x)},$$

and so,

$$Z(x) = \frac{1-F(x)}{1-x+(x-2)F(x)}.$$

By simplifying the right-hand side of $Z(x)$, we get the following theorem.

Theorem 4.0.31 *Let z_n be the number divisor-labelled staircase diagrams of support contained in $[s_1, s_n]$, $z_0 = 1$, and $Z(x) := \sum_{n\geq 0} z_n x^n$. Then*

$$Z(x) = \frac{-4x^6+24x^5-58x^4+73x^3-49x^2+17x-2-x\sqrt{x^4-2x^3+7x^2-6x+1}}{2(x-1)(2x^6-14x^5+37x^4-46x^3+28x^2-9x+1)}.$$

Now Theorem 1.1.1 follows from Corollary 3.2.11 and Theorem 4.0.31.

REFERENCES

[1] H. Abe and S. Billey, *Consequences of the Lakshmibai-Sandhya theorem: the ubiquity of permutation patterns in Schubert calculus and related geometry*, Adv. Stud. Pure Math. **71** (2016), 1–52.

[2] T. Alland and E. Richmond, *Staircase diagrams and enumeration of Schubert varieties*, Journal of Combinatorial Theory, Series A **154** (2018), 533–550.

[3] S. Billey and V. Lakshmibai, *Singular loci of Schubert varieties*, Progress in Mathematics, vol. 182, Birkhäuser Boston, Inc., Boston, MA, 2000.

[4] S. Billey and A. Postnikov, *Smoothness of Schubert varieties via patterns in root subsystems*, Adv. Appl. Math. **34(3)** (2005), 447 – 466.

[5] S. C. Billey, *Pattern avoidance and rational smoothness of Schubert varieties*, Adv. Math. **139(1)** (1998), 141 – 156.

[6] S. C. Billey and G. Warrington, *Kazdan-Lusztig polynomials for 321-hexagon-avoiding permutations*, J. Algebraic Combin. **13** (2001), 111 – 136.

[7] A. Björner and F. Brenti, *Combinatorics of coxeter groups*, Graduate Texts in Mathematics, Springer-Verlag, Berlin, Heidelberg, 2005.

[8] M. Bousquet-Mélou and S. Butler, *Forest-like permutations*, Annals of Combinatorics **11** (2007), 335 – 354.

[9] J. B. Carrel, *The bruhat graph of the Coxeter group, a conjecture of Deodhar, and rational smoothness of Schubert varieties*, Algebraic groups and their generalizations: classical methods (University Park, PA, 1991), Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, RI **56** (1994), 53 – 61.

[10] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, Cambridge **4** (2009).

[11] V. Gasharov and V. Reiner, *Cohomology of smooth Schubert varieties in partial flag manifolds*, J. London Math. Soc. **66(2)** (2002), 550 – 562.

[12] Ulfarsson H. and A. Woo, *Which Schubert varieties are local complete intersections?*, Proc. London Math. Soc. **107(5)** (2013), 1004 – 1052.

[13] M. Haiman, *Enumeration of smooth Schubert varieties*, Unpublished (1990).

[14] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge University Press, Cambridge (1990).

[15] V. Lakshmibai and B. Sandhya, *Criterion for smoothness of Schubert varieties in* $SL(n, \mathbb{C})/B$, Proc. Indian Acad. Sci. Math. Sci. **100** (1990), 334–338.

[16] E. Richmond and W. Slofstra, *Baily-Postnikov decompositions and the fibre bundle structure of Schubert varieties*, Math. Ann. **366** (2016), 31–55.

[17] ______, *Staircase diagrams and enumeration of smooth Schubert varieties*, Journal of Combinatorial Theory, Series A **150** (2017), 328–376.

[18] K. M. Ryan, *On Schubert varieties in the flag manifold of* $SL(n, \mathbb{C})$, Mathematische Annalen **276** (1987), 205–224.

[19] B. Tenner, *Database of Permutation Pattern Aavoidance.*, http://math.depaul.edu/bridget/patterns.html.

[20] ———, *Pattern avoidance and the Bruhat order*, J. Comb. Theory Series A **114** (2007), 888–905.

[21] D. N. Verma, *Mobius inversion for the Bruhat ordering on a Weyl group*, Ann. Sci. Ecole Norm. Sup. **4** (1971), 393 – 398.

[22] J. S. Wolper, *A combinatorial approach to the singularities of Schubert varieties*, Adv. Math. **76(2)** (1989), 184 – 193.

[23] A. Woo, *Permutations with kazhdan-lusztig polynomial* $p_{id,w}(q) = 1+q^h$, The Electronic Journal of Combinatorics **16(2)** (2009).

[24] A. Woo and A. Yong, *When is a Schubert variety gorenstein?*, Advances in Mathematics **207** (2005), 205 – 220.

APPENDICES

0.1 Notations

$N_{\texttt{SCI}(n)}$ is the number of divisor-labelled strongly connected increasing staircase diagram of support $[s_1, s_n]$.

$N_{\texttt{SC}(n)}$ is the number of divisor-labelled strongly connected staircase diagrams of support $[s_1, s_n]$.

$N_{\texttt{C}(n)}$ is the number of divisor-labelled connected staircase diagrams of support $[s_1, s_n]$.

z_n is the number of divisor-labelled staircase diagrams with support contained in $[s_1, s_n]$, which is equal to the number of permutations in $\mathfrak{S}_{n+1}$ avoiding 3412, 52341, 52431, and 53241.

0.2 Recursive relations

$$t_n = f_n + 2\sum_{k=1}^{n-1} f_{n-k}t_k$$

and

$$z_n = z_{n-1} + t_n + \sum_{k=2}^{n} z_{n-k}t_{k-1},$$

where $f_n := N_{\texttt{SC}(n)}$ and $t_n := N_{\texttt{C}(n)}$.

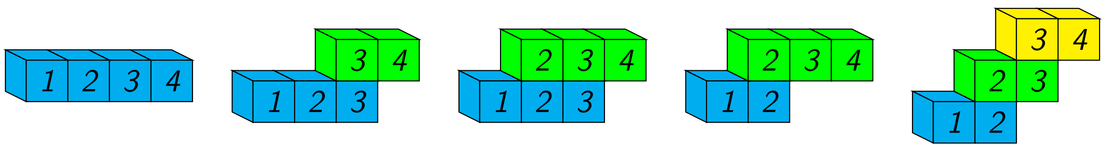
0.3 Generating functions

$$a(x) = \sum_{n>0} N_{\texttt{SCI}(n)}x^n = \frac{1 - x - x^2 - \sqrt{x^4 - 2x^3 + 7x^2 - 6x + 1}}{2(1-x)}$$

$$= x + x^2 + 3x^3 + 9x^4 + 29x^5 + 99x^6 + \cdots,$$

where the coefficient of x^n is the number of divisor labelled strictly increasing staircase diagrams of support $[s_1, s_n]$.

Example 0.3.1 *By* `SCI(n)`*, we denote the the set of strongly connected staircase diagrams of support* $[s_1, s_n]$*. Note that* $|\texttt{SCI(n)}| = C_n$*, the Catalan number.* `SCI(5)` *contains the following 5 staircase diagrams.*



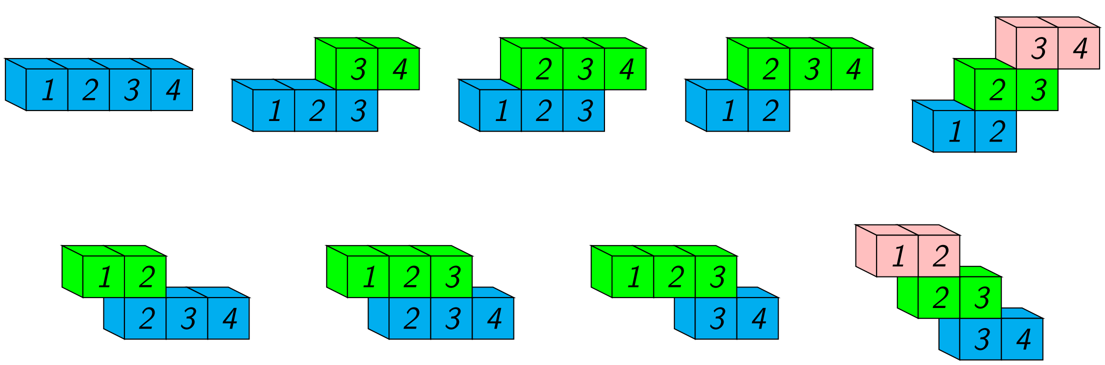
The first diagram is a single block of 4 *boxes. Therefore the diagram has* $4-1=3$ *divisor labellings.*

In the second diagram, the first block $[s_1, s_3]$ *has* 3 *boxes with* 1 *overlapping boxes and so, the block has* $3-1=2$ *divisor labellings. Similarly, the second block* $[s_3, s_4]$ *has* 1 *divisor labelling. Thus the second diagram has a total of* $2 \times 1 = 2$ *divisor labellings.*

Similarly, the third digram has 1 *divisor labelling, the fourth diagram has* 2 *divisor labellings and the fifth diagram has* 1 *divisor labelling.*

Therefore $N_{\texttt{SCI}}(4) = 3+2+1+2+1 = 9$*, which is the coefficient of* x^4 *is the formal power series of* $a(x)$*.*

Example 0.3.2 *The set of strongly connected staircase diagrams of support* $[s_1, s_n]$ *is denoted by* $\texttt{SC}(n)$*.* $\texttt{SC}(4)$ *consists of 9 staircase diagrams which are listed below.*



Thus $N_{\mathsf{SC}(4)} = 3+2+1+2+1+2+1+2+1 = 15$, *which is the coefficient of* x^4 *in the formal power series of* $F(x)$, *where*

$$F(x) = \sum_{n>0} N_{\mathsf{SC}(n)} x^n = x + \left(\frac{x}{1-x}\right)^2 + \frac{2xa(x) - 2x^2}{(1-x)^2(1+2x-x^2+(x-3)a(x))}$$
$$= x + x^2 + 4x^3 + 15x^4 + 58x^5 + 231x^6 + 940x^7 + \cdots.$$

The number of Schubert varieties in $\mathcal{F}\ell(n)$ having complete fiber bundle structures with fibers isomorphic to Grassmannians or Grassmannian Schubert divisors is the coefficient of x^n in the formal power series of $Z(x)$, where

$$Z(x) = \sum_{n\geq 0} z_n x^n = \frac{1-F(x)}{1-x+(x-2)F(x)}$$
$$= \frac{-4x^6+24x^5-58x^4+73x^3-49x^2+17x-2-x\sqrt{x^4-2x^3+7x^2-6x+1}}{2(x-1)(2x^6-14x^5+37x^4-46x^3+28x^2-9x+1)}$$
$$= 1 + 2x + 6x^2 + 23x^3 + 100x^4 + 460x^5 + 2172x^6 + 10397x^7 + 50173x^8 + \cdots$$

z_n is also equal to the number of permutations in $\mathfrak{S}_{n+1}$ avoiding the patterns 3412, 52341, 52431 and 53241.

0.4 Some computations using Sagemath

```
import sage.combinat.permutation as permutation
W = WeylGroup("A9",prefix="s")
[s1,s2,s3, s4,s5,s6,s7,s8,s9] = W.simple_reflections()
p1=[3,4,1,2]
p2=[5,2,3,4,1]
p3=[5,2,4,3,1]
p4=[5,3,2,4,1]

def divisor_pattern(n):
    return Permutations(n, avoiding=[p1,p2,p3,p4]).cardinality()
[divisor_pattern(i+1) for i in range(9)]
```

```
[1, 2, 6, 23, 100, 460, 2172, 10397, 50173]
```

Figure 1: Number of permutations avoiding 3412, 52341, 52431, and 53241

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