UNCRUMPLED: A SCHEME FOR BOUNDING THE PINCH POINTS ON A VARIETY OF PROJECTED SURFACES

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Abstract: Let X be a smooth surface and let $\varphi : X \to \mathbb{P}^N$, with $N \geq 4$, be a finitely ramified map which is birational onto its image $Y = \varphi(X)$, with Y non-degenerate in $\mathbb{P}^N$. In this paper, we produce a lower bound for the length of the pinch scheme of a general linear projection of Y to $\mathbb{P}^3$. We then prove that the lower bound is realized if and only if Y is a rational normal scroll.

We describe an alternative way of viewing this problem, along with a review of the work done on some adjacent problems. We also include a small collection of examples where we compute the number of pinch points contained in a general linear projection to $\mathbb{P}^3$ as evidence for the main theorem. Finally, we conclude with a strengthening of the main result and we describe several future problems stemming from this result.

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CHAPTER I

INTRODUCTION

1.1 Introduction

Algebraic geometry has a long history of using a variety of integers to describe the geometry exhibited by different algebraic varieties (with an emphasis on subvarieties of projective space). Familiar examples of such numbers are degree and (co)dimension, but there are in general many, many more (for example, the genus of a space curve in $\mathbb{P}^N$).

Much of the subject is devoted to understanding these numbers. Consequently, a "golden standard" for theorems in classical algebraic geometry takes the following shape:

- Specify an integer which describes the geometry of a projective variety (and is invariant under some kind of transformation).
- Describe the extremal behavior of the invariant.
- Classify all varieties which realize the extreme values of the invariant.
- (Bonus): Classify varieties which exhibit near-extremal behavior with respect to the invariant.

A highly celebrated result of this form is the Classification of Varieties of Minimal Degree. This result was proven in the case of surfaces by del Pezzo in 1886 [5], and then extended by Bertini in 1907 [2] to include all varieties of minimal degree. A handsome summary can be found in [7]. Stated briefly,

Theorem 1.1.1 (Varieties of Minimal Degree) *Let $X \subset \mathbb{P}^N$ be a non-degenerate variety of degree d with $\dim X = n$. Then*

$$d \geq N - n + 1,$$

with equality holding if and only if X is a rational normal scroll, the Veronese surface $\Phi_2 \subset \mathbb{P}^5$, or a cone over one of these.

It is also well-known that the surfaces of near-minimal degree are del Pezzo surfaces, which have degree N in $\mathbb{P}^N$ (one more than the minimal degree for surfaces which is $N-1$ in $\mathbb{P}^N$).

In this work, we will present a classification theorem of a very similar flavor, along with a corresponding near-minimal classification. This project is primarily concerned with surfaces, rather than varieties in general. The integer with which we concern ourselves is the (necessarily finite) number of pinch points contained in the image of a general linear projection of the surface $X \subset \mathbb{P}^N$ (with $N \geq 4$) to $\mathbb{P}^3$.

We shall state the most general version of the main results here.

Theorem (Surfaces of Minimal Pinch Point Number) *Let X be a smooth projective surface, let $N \geq 4$, and let $\varphi : X \to \mathbb{P}^N$ be a finitely ramified map which is birational onto its non-degenerate image $Y = \varphi(X)$. If $\pi : Y \to \mathbb{P}^3$ is a general linear projection, then the length of the ramification scheme $\mathrm{Ram}(\pi \circ \varphi)$ is at least $2N - 6$, with equality holding if and only if Y is a rational normal scroll.*

Theorem (Surfaces of Near-Minimal Pinch Point Number) *Maintain the assumptions above. For any non-negative integer i, if $N \geq 3 + i \geq 4$, then the ramification scheme $\mathrm{Ram}(\pi \circ \varphi)$ has length exactly $2N - 6 + 2i$ if and only if one of the following holds:*

1. *Y is ruled by lines.*

2. *$N = 5$, $i = 1$, and Y is the Veronese surface.*

3. *$4 \leq N \leq 9$, $i = N - 3$, and Y is a del Pezzo surface of degree N.*

In what remains of this chapter, we will develop the context surrounding these theorems, introduce some notation and conventions to make the statements and proofs more streamlined, and present different lenses through which we can view the length of the pinch scheme of a general linear projection of a smooth surface to $\mathbb{P}^N$. Specifically, we will also introduce the projective characters of Severi, the Gauss class of a surface (and that of a variety in general), and we will explore some of the work that has already been done on the related problem of classifying surfaces of minimal *class*. A summary of notations and conventions can be found at the very end of this chapter.

We now turn our attention to the elementary projective characters of Severi as an avenue towards understanding the pinch point number of a surface.

1.2 The elementary projective characters of Severi

As mentioned in the previous section, algebraic geometry has a long history of describing and classifying varieties, especially curves and surfaces. This is achieved by assigning to them various numbers which describe their geometry, and analyses of these numbers form the backbone of many great chapters in the history of the subject.

Francesco Severi

(Apr 1879 – Dec 1961)

One collection of such descriptive integers consists of Severi's *elementary projective characters* of curves and surfaces. These are discussed at length in [26], particularly in Chapter IX, and several examples are worked out in [22]. These numerical characters are integers which serve to describe the geometry of varieties, and critically, which are *invariant under general projection.* While the nomenclature is slightly dated, the concepts remain relevant even today. We wish to begin this work with a summary of the elementary projective characters for curves and surfaces.

For curves, the word *order*, denoted by the symbol μ_0, was often used to describe the modern notion of degree. For a curve $C \subset \mathbb{P}^N$, this is simply the number of points contained in a general hyperplane section of C.

The *rank* of C, denoted μ_1, is a little more involved. Put simply, μ_1 is the degree of the dual variety $C^* \subset \mathbb{P}^{N*}$. Recall that $\mathbb{P}^{N*}$ is the space of hyperplanes in $\mathbb{P}^N$, and the dual to a smooth variety is the set of hyperplanes containing a projective tangent space to the variety. Then indeed, since the dual variety to the curve C is a hypersurface $C^* \subset \mathbb{P}^{N*}$, it follows that $\mathbb{C}^*$ meets a general line (a complementary-dimensional linear space) in $\mathbb{P}^{N*}$ in a finite set. Translating this notion back to the geometry on C in the original projective space, we conclude that μ_1 counts the number of tangent lines to the curve C which meet a generally chosen codimension 2 plane in $\mathbb{P}^N$.

Surfaces, on the other hand, have four elementary projective characters. Let $S \subset \mathbb{P}^N$ be a surface. The first two characters go by the same names as those in the previous paragraph, namely *order* and *rank*, and are denoted μ_0 and μ_1 respectively. Each is responsible for measuring the so-named characters for a general hyperplane section of S.

The other two characters can be realized as the (finite) number of tangent planes to S which obey certain properties. The *class* of S (not to be confused with $[S] \in A^0(S)$, the intersection theoretic fundamental class of S), is denoted μ_2, and it measures the number of tangent planes which meet a generally chosen codimension 2 plane along a line. Note that this means it is the class of S which reflects the degree of $S^* \subset \mathbb{P}^{N*}$.

Finally, we have the *type* of S, classically denoted ν_2, which is the number of tangent planes to S which meet a general linear space of codimension 4. Alternatively, ν_2 measures the number of pinch points when S (assumed smooth and non-degenerate in $\mathbb{P}^N$) is projected to $\mathbb{P}^3$ from a general codimension 4 linear space.

One unified way to view all four of the projective characters for the surface S is to consider each as the degree of an auxiliary variety to S. That is, the projective characters correspond to the degrees of varieties summarized in Table 1.

Using Severi's projective characters, we can immediately give a coarse description for the objective of this work; we will establish a lower bound for ν_2 in terms of N when $S \subset \mathbb{P}^N$

Projective Character	Auxiliary Variety
$\mu_0 = \deg C$	A hyperplane section
$\mu_1 = \deg C^*$	The dual to a hyperplane section
$\mu_2 = \deg S^*$	The dual variety
$\nu_2 = \deg \operatorname{Tan}(S)$	The tangent variety

Table 1: Projective characters of the surface $S \subset \mathbb{P}^N$ as degrees of auxiliary varieties to S.

is a smooth non-degenerate surface. When referencing a smooth projective surface, we shall continue to use the words *class* and *type*. See Definition 1.6.2 for a more explicit description of the type of a surface.

Next, we will investigate the singularities of general projections to see how pinch points usually arise.

1.3 An introduction to pinch points

Let C be a smooth non-degenerate curve in $\mathbb{P}^N$ with $N \geq 3$, and let $\pi : C \to \mathbb{P}^2$ be a general linear projection with Λ_π the source of projection. Note that

$$\operatorname{codim} \Lambda_\pi = 3.$$

A natural question one might ask is this: How does the image $\pi(C) \subset \mathbb{P}^2$ differ from the original curve C? In particular, what types of singularities does the projection induce on the image $\pi(C)$?

We will give a very brief answer here. Note that if a line tangent to C meets Λ_π, then the image of the contact point of the tangent line to C under π will be a cusp on $\pi(C) \subset \mathbb{P}^2$. Similarly, if a secant line meets Λ_π, then π maps the contact points to the same output in $\mathbb{P}^2$. More generally, any line which is *trisecant* to C and which meets Λ_π corresponds to a triple point, and this notion extends to multisecant lines.

Now, since π is general, Λ_π is a general codimension 3 linear space in $\mathbb{P}^N$, and since the set of lines tangent to C sweep out only a surface, no lines tangent to C meet Λ_π. Similarly, no lines which are trisecant to C are expected to meet a codimension 3 plane in $\mathbb{P}^N$, so the general choice of π guarantees no such intersection. On the other hand, the variety swept out by secant lines is three-dimensional, so it will meet a general $(N-3)$-plane in $\mathbb{P}^N$ in finitely many points.

The conclusion is this: The only singularities in a general projection of a smooth curve C to $\mathbb{P}^2$ are the finitely many simple nodes corresponding to the intersection of the secant variety $\operatorname{Sec} C$ with the source of projection. In this sense, the singularities of such a general projection are "as simple as possible."

We now consider a natural extension: What are the singularities of a general projection of a smooth projective surface to $\mathbb{P}^3$? The answer is given by the classical General Projection Theorem (see [3, Theorem 2.5], [22, Section 2],[23, Section 1],[24, Theorem 1]), which can be phrased as follows:

Theorem 1.3.1 *Let $X \subset \mathbb{P}^N$ be a smooth non-degenerate surface with $N \geq 4$, and let $\pi : X \to \mathbb{P}^3$ be a general linear projection. Then the singular locus of the image surface $\pi(X)$ consists of*

- *A curve Γ consisting of points where the image surface $\pi(X)$ intersects itself transversely. We often refer to the curve Γ as the double curve.*
- *Finitely many triple points. These are points at which the double curve Γ intersects itself in a triple point.*
- *Finitely many pinch points, which occur when the projection map π is ramified.*

The singularities which arise from the General Projection Theorem are called *ordinary singularities.* It is natural to wonder what types of numerical limitations there are on the ordinary singularities of a general projection to $\mathbb{P}^3$. We mentioned at the end of the previous section that this project is concerned with the third type of ordinary singularity, pinch points, which are counted by ν_2, the type of the surface X.

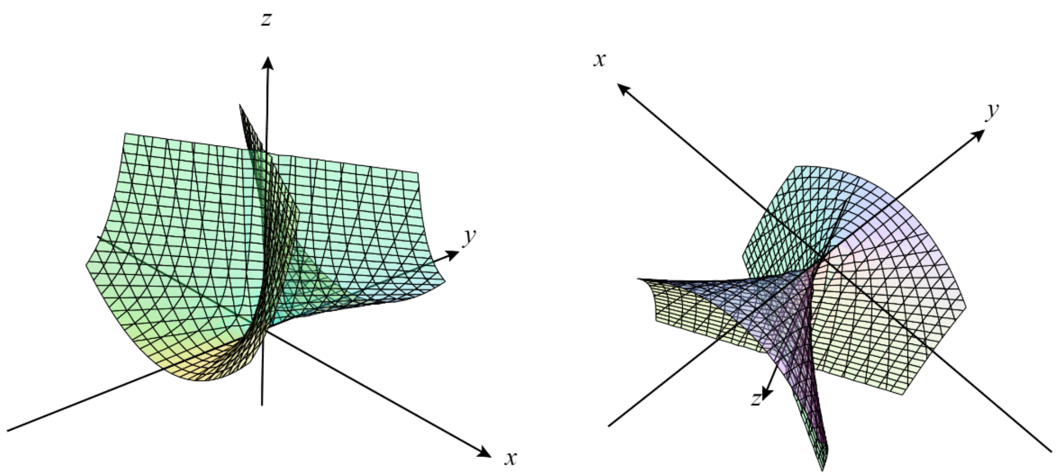


Figure 1: A Whitney Umbrella, the projected image of the cubic rational normal scroll in $\mathbb{P}^4$.

While pinch points arise in this context as the ramification points of the projection map, many times they are encountered in the wild long before the concept of the ramification locus of a map. In the following two examples, we will give the most common examples of surfaces exhibiting pinch points, and we will point to the *smooth* surfaces in higher dimensional spaces whose projections to $\mathbb{P}^3$ yielded these "pinched" surfaces. We shall also revisit these examples (and some others) in Chapter II for some explicit computations.

Our first example is the famous Whitney Umbrella.

Example 1.3.1 Let X be the cubic rational normal scroll $S(1,2)$ in $\mathbb{P}^4$, let $p \in \mathbb{P}^4$ be a general point, and let $\pi_p : \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ be the projection map from p. Then $W = \pi_p(X)$ is the famous Whitney Umbrella. Its equation (after a projective transformation) is often expressed in terms of the affine coordinates of $\mathbb{A}^3_{x,y,z}$ as

$$W = \{(x,y,z) \in \mathbb{A}^3 \mid x^2 = y^2 z\}.$$

In this case, the z-axis is the double curve Γ, and the origin is a pinch point. There are no triple points (since Γ is smooth), but there is a second pinch point on the surface, located at the point at infinity on the z-axis. To see this, one can homogenize the defining equation

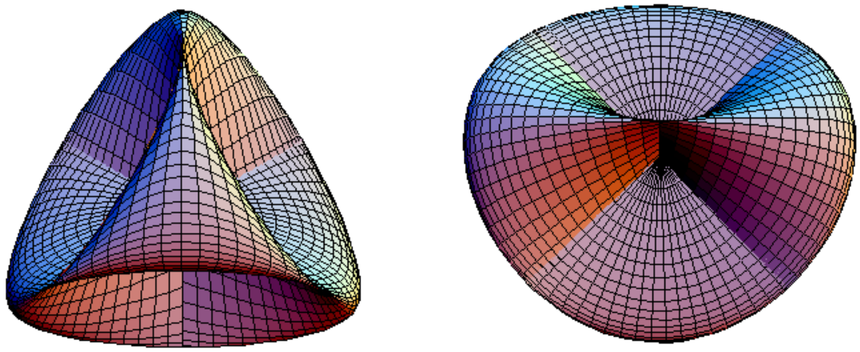


Figure 2: A Roman Surface, the projected image of the Veronese surface in $\mathbb{P}^5$.

for W and then dehomogenize to an affine chart of $\mathbb{P}^3$ containing the point $[0:0:1:0]$. See Figure 1 for a visual.

The second example is the Roman surface, which is often called Steiner's surface, since it was first studied by Jakob Steiner... when he was in Rome.

Jakob Steiner

(Mar 1796 – Apr 1863)

Example 1.3.2 Now, let X be the Veronese surface in $\mathbb{P}^5$. A general projection (whose source is a general line in $\mathbb{P}^5$) of X to $\mathbb{P}^3$ yields the *Roman surface*, which, after a projective transformation and passing to an affine chart, can be expressed as the vanishing locus of the following equation:

$$x^2y^2 + y^2z^2 + z^2x^2 = xyz.$$

Then the double curve Γ is the union of the three coordinate axes, and the unique triple point is their common intersection. There are also three pairs of pinch points, each pair arising on one of the three axes of singularity.

As we continue to study the pinch points of projected surfaces, we will make two important generalizations. First, we shall from now on consider the length of the pinch point *scheme* of a general projection of X to $\mathbb{P}^3$, rather than the literal number of pinch points that arise in the image surface $\pi(X)$. In addition to lending itself nicely to intersection theoretic calculations, this generalization serves to extend the theory to boundary cases where two pinch points coincide, counting points with multiplicity.

Second, instead of a smooth surface *inside* $\mathbb{P}^N$, we will eventually generalize X to be an abstract smooth projective surface which admits a sufficiently "nice" map to $\mathbb{P}^N$. We will call such a map *uncrumpled* (hence, the title of this work). Most notably, an uncrumpled map will have the flexibility of being finitely ramified, so that its image $Y \subset \mathbb{P}^N$ will have finitely many pinch points. Consequently, the pinch point scheme consisting of all pinch points on the image of Y under the general projection π may in fact differ from the scheme of pinch points which were accrued *as a consequence* of the projection.

Before we make the second generalization, we wish to study the pinch point scheme from an entirely different point of view.

1.4 The Gauss class interpretation

For $X \subset \mathbb{P}^N$ a smooth, non-degenerate surface, there is another lens through which one can interpret the type of X: It is one of two coefficients defining the *Gauss class* of $X \subset \mathbb{P}^N$. In fact, we will soon see that the other coefficient is precisely μ_2, the class of X in terms of Severi's projective characters. We begin with a definition of the Gauss class of a smooth variety of any dimension.

Definition 1.4.1 *Let X be any smooth variety of dimension n inside $\mathbb{P}^N$.*

- *The **Gauss map** $\gamma : X \to \mathbb{G}(n, N)$, given by $\gamma(x) = \Lambda_x$, is a regular map from X to the Grassmannian of n-planes in $\mathbb{P}^N$ taking a point $x \in X$ to the projective tangent space to X at x, denoted $\Lambda_x \subset \mathbb{P}^N$. (Note the slight abuse of notation: We refer to both the n-plane Λ_x and its corresponding point in $\mathbb{G}(n, N)$ by the same name.)*

- *The **Gauss class** of X is the intersection theoretic fundamental class of the image $[\gamma(X)]$ inside the Chow ring $A(\mathbb{G}(n,N))$.*

Carl Friedrich Gauss

(Apr 1777 – Feb 1855)

With the tools of intersection theory, we can use the Gauss class of X to answer questions about the geometry of X inside $\mathbb{P}^N$. Now, as a consequence of Zak's Theorem on Tangencies [30, Corollary 2.8], we know that γ is both finite and birational onto its image. But what more can we say about the class $[\gamma(X)] \in A(\mathbb{G}(n,N))$?

Recall that the Chow ring of a variety is graded by (co)dimension, so we can express

$$A(\mathbb{G}(n,N)) = \bigoplus_{j=0}^{(n+1)(N-n)} A_j(\mathbb{G}(n,N)),$$

where $A_j(\mathbb{G}(n,N))$ refers to the abelian group of j-dimensional classes (rational equivalence classes of cycles) in the Chow ring. Note that $\dim \mathbb{G}(n,N) = (n+1)(N-n)$. It is perhaps non-standard to express the grading of $A(\mathbb{G}(n,N))$ by dimension rather than codimension, but since γ is finite and birational onto its image, it follows that $[\gamma(X)] \subset A_n(\mathbb{G}(n,N))$.

Now, it is well known that the Chow group $A^n(\mathbb{G}(n,N))$ is freely generated by a $\mathbb{Z}$-basis of Schubert classes of the form $\sigma_{\mathbf{a}}$, where $\mathbf{a}$ is an integer partition of n. Dual to this fact, the group $A_n(\mathbb{G}(n,N))$ is generated by the dual classes to the Schubert classes $\sigma_{\mathbf{a}}$, denoted $\sigma_{\mathbf{a}}^*$. Formally, we define the dual to a Schubert class as follows:

Definition 1.4.2 *Let $\mathbf{a} = (a_1, a_2, \dots)$ be a partition of n. Then the **dual Schubert class** to the class $\sigma_{\mathbf{a}} \in A(\mathbb{G}(n,N))$ is the class*

$$\sigma_{\mathbf{a}}^* := \sigma_{\mathbf{b}} \in A_n(\mathbb{G}(n,N)),$$

where $a_i + b_{n+1-i} = N - n$ for all $1 \leq i \leq n$. It is the unique Schubert class in the Chow ring $A(\mathbb{G}(n,N))$ whose intersection with $\sigma_{\mathbf{a}}$ is a unique reduced point.

Since the dual Schubert classes generate $A_n(\mathbb{G}(n,N))$, we can write the Gauss class $[\gamma(X)]$ as a $\mathbb{Z}$-linear combination of the form

$$[\gamma(X)] = \sum \gamma_{\mathbf{a}} \cdot \sigma^*_{\mathbf{a}},$$

and we define the *Gauss coefficients* to be the integers $\gamma_{\mathbf{a}}$ in the linear combination. Each of the integers $\gamma_{\mathbf{a}}$ roughly measures something about the "twistedness" of the tangent space of X, and we choose to make them the central objects of study. Each Gauss coefficient is itself some natural expression involving the Chern classes of the bundle of principal parts for $\mathscr{O}_X(1)$, and their study gives rise to a particular bipartite classification problem:

Hermann Cäsar Hannibal Schubert (May 1848 – Jul 1911)

I. For each partition $\mathbf{a}$ of $n = \dim X$, what is the minimum value that the Gauss coefficient $\gamma_{\mathbf{a}}$ can take?

II. Which varieties X realize this minimum?

1.5 Historical work on this project

The questions at the end of the previous section have been answered in some very specific cases. We have sharp lower bounds for the Gauss coefficients when the dimension n of X is either 1 or 2, with the completion of the latter case being the main result of this work. We will address the known cases for $n = 1$ and $n = 2$ separately in what follows in this section.

1.5.1 The $n = 1$ Case

When $n = 1$, X is a degree d curve of genus g, and the analysis is rather straightforward. Note that for any positive integer N, $A_1(\mathbb{G}(1,N))$ is freely generated by σ_1^*. Note that σ_1^* is the class of a line in the Plücker embedding of the Grassmannian; alternatively, we can view $\sigma_1^* = \sigma_{N-1,N-2}$ as the class of lines in $\mathbb{P}^N$ which are contained in a general 2-plane and which contain a fixed general point in that plane. Either way, there is only one Gauss coefficient to consider, since $[\gamma(X)] = \gamma_1 \cdot \sigma_1^*$.

To find γ_1, we compute the intersection product

$$[\gamma(X)] \cdot \sigma_1 = \gamma_1 \cdot \sigma_1^* \cdot \sigma_1 = \gamma_1. \tag{1.5.1}$$

We can view the intersection product in Equation (1.5.1) as computing the intersection of two generally chosen subvarieties in $\mathbb{G}(n, N)$, each belonging to one of the classes σ_1^* and σ_1.

The intersection product of any class with σ_1 amounts to projecting away from a general codimension 2 plane, say Λ, since σ_1 is the class of lines in $\mathbb{P}^N$ which meet Λ nontrivially. Therefore, Equation (1.5.1) is equivalent to taking a general linear projection from X to a generally chosen $\mathbb{P}^1$, which we may call L. Indeed, a general line in $\mathbb{P}^N$ avoids Λ, but each line meeting Λ transversely specifies a hyperplane which meets L transversely, yielding a well-defined output point for the projection map.

Tracing through the computation described in the previous paragraph to find the intersection of σ_1^* and σ_1, we see that γ_1 answers the question: How many lines tangent to X meet a general codimension 2 plane? Note that the answer to such a question is exactly the information contained in μ_1, the rank of the curve X (in the notation of Severi). An equivalent question is this: At how many points of X is the projection map from a generic codimension 2-plane ramified? The answers to both questions can be deduced from a simple application of the Riemann-Hurwitz formula.

Theorem 1.5.1 (The Riemann-Hurwitz Formula) *Let S and T be algebraic curves of genus g_S and g_T respectively, with $\varphi : S \to T$ a (possibly ramified) covering of degree d. For each point $P \in S$, let e_P be the ramification index of φ at P. Then the sum $\gamma_1 = \sum_P (e_P - 1)$ is finite, and*

$$2g_T - 2 = d(2g_S - 2) + \sum_{P \in S} (e_P - 1).$$

Since the target $\mathbb{P}^1$ has genus 0, and since the degree of the projection map is $d = \deg X$, we compute

$$\gamma_1 = 2d + 2g - 2,$$

where g denotes the genus of X.

Bernhard Riemann

(Sep 1826 – Jul 1866)

Observe that since $g \geq 0$, we also have an inequality $\gamma_1 \geq 2d - 2$ with equality holding precisely when X is a rational curve. It is well known that the degree of every non-degenerate curve in $\mathbb{P}^N$ is at least N, so $\gamma_1 \geq 2N - 2$, and equality holds when X is a curve of minimal degree. In other words, the absolute minimal value for γ_1 in $\mathbb{P}^N$ (for any N) is realized by the rational normal curve of degree N.

1.5.2 The $n = 2$ Case

The case where $n = 2$ (so that X is a surface), is already significantly more involved. We begin with the observation that $[\gamma(X)] \in A_2(\mathbb{G}(2, N))$, where the Chow group $A_2(\mathbb{G}(2, N))$ is freely generated by the classes $\sigma^*_{1,1}$ and σ^*_2. Now, computing

$$[\gamma(X)] = \gamma_{1,1} \cdot \sigma^*_{1,1} + \gamma_2 \cdot \sigma^*_2$$

requires two distinct analyses, one for each coefficient. We begin by identifying the following classes in $A^2(\mathbb{G}(2, N))$:

- σ_2 is the class of 2-planes in $\mathbb{P}^N$ meeting a codimension 4 plane nontrivially.
- $\sigma_{1,1}$ is the class of planes in $\mathbb{P}^N$ meeting a codimension 2 plane along a line.

Applying the definition of the dual to a Schubert class to both σ_2 and $\sigma_{1,1}$, we see that

$$\sigma^*_2 = \sigma_{N-2,\ N-2,\ N-4} \qquad \text{and} \qquad \sigma^*_{1,1} = \sigma_{N-2,\ N-3,\ N-3}.$$

Tracing through the definition of γ_2 (in a manner which resembles the discussion of γ_1 in the $n = 1$ case), we see that γ_2 is none

other than the *class* of X, i.e. the degree of its dual variety, classically denoted μ_2. In 1955, E. Marchionna proved that if $d = \deg X$, then $\gamma_2 \geq d - 1$, with equality holding if and only if $X \subset \mathbb{P}^5$ is the Veronese surface [19]. The next year, in 1956, D. Gallarati proved a similar result, namely that for all other smooth projective surfaces, $\gamma_2 \geq d$ with equality holding if and only if X is a scroll over a smooth curve [10]. One year later, (in 1957 [9]), Gallarati was investigating the difference between degree and class, and he proved that if $\gamma_2 > d$, then in fact $\gamma_2 \geq d + 3$. He also classified the surfaces satisfying $3 \leq \gamma_2 - d \leq 10$. A concise statement of this result can be found in [14]. In 1993, [29], C. Turrini and E. Verderio extend the result to classify surfaces whose class is $\gamma_2 - d \leq 16$ and $\gamma_2 \leq 25$. In particular, they show that all surfaces in this classification are either rational or ruled!

Since the minimal degree for a non-degenerate surface in $\mathbb{P}^N$ is $N - 1$, the classification of varieties of minimal degree [7] implies that $\gamma_2 \geq N - 2$ with equality holding if and only if X is the Veronese surface in $\mathbb{P}^5$, and $\gamma_2 = N - 1$ if and only if X is a rational normal scroll.

Although surfaces of small class are still studied to this day ([17], [8], [16]), this completes our summary of the progress made in the analysis of the Gauss coefficients of surfaces. In fact, the results of Gallarati and Marchionna were not originally interpreted in terms of the Gauss class of surfaces at all, and the second Gauss coefficient, $\gamma_{1,1} = \nu_2$, is completely absent from the story so far! In fact, our main result will complete the $\gamma_{1,1}$ case.

1.6 The Main Result

At the end of the previous section, we alluded to the fact that our main result completes what remains of the problem of minimizing the Gauss coefficients of surfaces in $\mathbb{P}^N$. Indeed, we will produce a sharp lower bound for the Gauss coefficient $\gamma_{1,1}$ and classify the surfaces which meet that bound. The techniques we use are very

Dionisio Gallarati

different from those used by Marchionna and Gallarati, in the sense that we rely on tools from projective geometry and intersection theory, the most notable of which is the technique of *generic inner projection.*

Geometrically, when the surface X is smooth, $\gamma_{1,1}$ measures the type of X, denoted ν_2. This can be seen by tracing through the definitions of $\sigma_{1,1}$ and $\sigma^*_{1,1}$. We wish to reiterate that $\gamma_{1,1} = \nu_2$ counts the number of pinch points contained in the image of a general linear projection of X to $\mathbb{P}^3$ (we are still in the case where X is smooth, an assumption we shall soon relax). Indeed, we are asking: How many planes tangent to X meet a generally chosen codimension 4 linear space in $\mathbb{P}^N$?

In order to state the main result in full generality, we first define the following terms.

Definition 1.6.1 *For any finitely ramified map $f : X \to \mathbb{P}^N$, where X is a smooth surface and f is birational onto its image $f(X)$, we say that the* ***ramification scheme*** *or* ***pinch point scheme*** *of f, denoted* $\mathrm{Ram}(f)$*, is the subscheme of X defined locally by the 2×2 minors of a matrix representing df, the map on tangent spaces induced by f. We denote by $\mathfrak{P}(f)$ the length of* $\mathrm{Ram}(f)$.

Definition 1.6.2 *If Y is a smooth projective surface and $\pi : Y \to \mathbb{P}^3$ is a general linear projection, then the* ***type*** *of Y is defined as $\nu_2 = \mathfrak{P}(\pi)$.*

Consider the situation where $\varphi : X \to \mathbb{P}^N$ is an embedding of X to a non-degenerate surface $Y = \varphi(X) \subset \mathbb{P}^N$ with $N \geq 4$, and $\pi : Y \to \mathbb{P}^3$ is a general linear projection. Then Y is smooth, and from [6, Proposition 12.6] we have a formula for $\mathfrak{P}(\pi)$, which is exactly the type ν_2 of Y. We will state this result as a theorem.

Theorem 1.6.1 (The Pinch Point Formula) *Let Y be a smooth non-degenerate surface in $\mathbb{P}^N$, and let $\pi : Y \to \mathbb{P}^3$ be a generally chosen linear projection. Then*

$$\mathfrak{P}(\pi) = \deg\left(6\zeta^2 + 4\zeta K_Y + K_Y^2 - c_2\right), \tag{1.6.1}$$

where ζ represents the class of a hyperplane section of Y, K_Y is the canonical class of Y, and $c_2 = c_2(T_Y)$ is the second Chern class of the tangent bundle of Y.

Theorem 1.6.1 is one of the main tools we use to prove our main result, so we wish to make several important notes here.

- For convenience, we will usually omit the degree map on zero cycles.
- Note that $\deg \zeta^2 = \deg Y$, since the degree of the intersection of two generally chosen hyperplane sections of a surface in $\mathbb{P}^N$ gives exactly the degree of that surface.
- This theorem gives us the exact value for $\gamma_{1,1}$ when dealing with a specific smooth surface in $\mathbb{P}^N$. We devote Chapter II to exploring different examples where this equation is applied.
- Critically, Equation (1.6.1) holds for any finitely ramified map from a smooth surface to $\mathbb{P}^3$. This is because it arises as a specific application of Porteus's formula (see [6, Chapter 12]).

To conclude this chapter, we will state the main theorem in full generality. Note that the last comment above allows us to finally relax the assumption that $\varphi : X \to \mathbb{P}^N$ is an embedding, and instead assume that it is finitely ramified. In such a situation, the map $\pi \circ \varphi$ is indeed a finitely ramified map from a smooth surface to $\mathbb{P}^3$, so we can apply Equation (1.6.1) to it. When this happens, we will write $\mathfrak{P}(\varphi)$ instead of $\mathfrak{P}(\pi \circ \varphi)$, with the latter notation emphasizing the projection map.

Let X be a smooth surface and let $\varphi : X \to \mathbb{P}^N$ with $N \geq 4$ be a finitely ramified map which is birational onto its image $Y = \varphi(X)$, with Y non-degenerate in $\mathbb{P}^N$. We will call such a map *uncrumpled*, see Definition 3.2.1. Then for a general projection $\pi : Y \to \mathbb{P}^3$, Equation (1.6.1) measures

$$\deg(\mathrm{Ram}(\pi \circ \varphi)) = \mathfrak{P}(\varphi) = 6\zeta^2 + 4\zeta K_X + K_X^2 - c_2.$$

Within this context, we use the method of inner projection with a proof by induction on N for the main result:

Theorem 1.6.2 *Let X be a smooth surface, and let $\varphi : X \to \mathbb{P}^N$ be an uncrumpled map to $Y = \varphi(X)$ with $N \geq 4$. Then for a general linear projection $\pi : Y \to \mathbb{P}^3$, we have*

$$\mathfrak{P}(\pi \circ \varphi) \geq 2N - 6,$$

with equality holding if and only if Y is a rational normal scroll.

1.7 Notations and conventions

Throughout this project, we will work over an algebraically closed field $\mathbb{K}$ of characteristic 0. Each *variety* will be assumed to be projective (unless stated otherwise), and the words *curve* and *surface* will refer to varieties of dimension one and two respectively. We adopt the pre-Grothendieck convention for projective spaces, i.e. $\mathbb{P}^N = \mathbb{P}^N_{\mathbb{K}}$ will refer to the set of one dimensional subspaces of $\mathbb{K}^{N+1}$ (as opposed to the one-dimensional quotients of $\mathbb{K}^{N+1}$).

For a set $S \subset \mathbb{P}^N$, we denote by $\langle S \rangle$ the *linear span* of S, the smallest linear space in $\mathbb{P}^N$ which contains S. We say that a variety $X \subset \mathbb{P}^N$ is *non-degenerate* if it is not contained in a hyperplane, i.e. if $\langle X \rangle = \mathbb{P}^N$. For a nonsingular point $x \in X$, we write

$$\Lambda_x := \langle T_x X \rangle$$

for the *projective tangent space*, i.e. the linear space in $\mathbb{P}^N$ tangent to X at x. We denote by $\mathbb{G}(k, N)$ the Grassmannian of k-planes in $\mathbb{P}^N$, so that

$$\mathbb{G}(k, N) = Gr(k + 1, N + 1),$$

where the latter represents the space of $k + 1$ dimensional subspaces of $\mathbb{K}^{N+1}$.

CHAPTER II

SOME WORKED EXAMPLES

2.1 Overview

In this chapter, we will use various techniques in projective geometry and intersection theory to compute the number of pinch points accrued by general linear projections of four distinct (very classical) families of surfaces to $\mathbb{P}^3$. Our strategy will use the Pinch Point Formula very generously (recall that the Pinch Point Formula was stated in Chapter I as Theorem 1.6.1). As such, we begin with a brief exposition of its components before tackling some examples.

Remark 2.1.1 A brief note to the reader: This chapter is more suited to those who have less experience with the tools and characters from projective geometry. The prerequisites needed in this chapter are minimal, and its primary purpose is to expose the reader to some of the classical families of surfaces referenced in the previous paragraph, while still presenting evidence for Theorem 1.6.2.

Throughout this chapter, we will work within the following setting. Let $\varphi : X \to \mathbb{P}^N$ be an embedding of the smooth surface X into $\mathbb{P}^N$, so that we can refer to $X \subset \mathbb{P}^N$ for various surfaces X. Next, let $\pi : \mathbb{P}^N \dashrightarrow \mathbb{P}^3$ be a general linear projection, let $H \subset \mathbb{P}^N$ be a hyperplane, and let $[H]$ be its class in $A(\mathbb{P}^N)$. We define $\zeta \in A(X)$ to be the class of the pullback $\varphi^*[H]$. We also define K_X as the canonical class of X, that is, the first Chern class of the canonical bundle,

$$K_X = c_1(\omega_X) = c_1(\wedge^2 \Omega_X).$$

We will immediately begin simply writing K, omitting the subscript X when there is no risk of ambiguity. Lastly, recall for a smooth surface in characteristic zero the second Chern class of its tangent bundle is the topological Euler characteristic of the surface, $\chi(X)$, when the surface $\varphi(X)$ is viewed as a real 4-manifold. We can now restate Equation (1.6.1) as:

$$\mathfrak{P}(X) = 6\zeta^2 + 4\zeta K + K^2 - \chi(X).$$

In this chapter, we investigate four classic families of surfaces:

- The Veronese surfaces, denoted Φ_d each of which is a d-uple embedding of $\mathbb{P}^2$ into $\mathbb{P}^N$ for some $d \in \mathbb{Z}$ with $d \geq 2$.

- The Segre surfaces, denoted $X_{a,b}$, which are embeddings of $\mathbb{P}^1 \times \mathbb{P}^1$ along the complete linear series $|a, b|$.

- The del Pezzo surfaces of degree N with $3 \leq N \leq 9$ which are isomorphic to the blow-up of $\mathbb{P}^2$ at r general points, with $0 \leq r \leq 6$. We denote by D_N the del Pezzo surface of degree N.

- The rational normal scrolls, denoted $S(a, b)$, each of which is abstractly isomorphic to the Hirzebruch surface $\mathbb{F}_{a-b}$.

For each of the surfaces Y belonging to one of the listed families, we will compute the type ν_2 of Y, given by

$$\mathfrak{P}(\pi : Y \to \mathbb{P}^3),$$

and show that the resulting number obeys the lower bound given in Theorem 1.6.2.

Remark 2.1.2 We wish to highlight the convention established in Chapter I, wherein we omit the projection map in our notation for pinch scheme length. For example, if $v_d : \mathbb{P}^2 \to \mathbb{P}^N$ is the d-th Veronese embedding (covered in the following section), then we will write $\mathfrak{P}(v_d)$ for the pinch scheme length, rather than $\mathfrak{P}(\pi \circ v_d)$.

The pinch scheme lengths we compute in this chapter are consistent with the formulas for ν_2 given in [22] (Note that the author's notation differs slightly from ours in some places).

When appropriate, we will take the opportunity to highlight interesting properties exhibited by specific members of these families; several of these properties hint at some underlying patterns which can be generalized (see Chapter V for more details).

2.2 Family 1: Veronese surfaces

The first class of surfaces we consider is the collection of Veronese surfaces. Recall the definition of the d-th Veronese surface:

Definition 2.2.1 *Let $v_d : \mathbb{P}^2 \to \mathbb{P}^N$ be the d-uple embedding of $\mathbb{P}^2$ into $\mathbb{P}^N$ with $N = \binom{d+2}{d} - 1$ along the complete linear series $|dL|$. Then the **d-th Veronese surface**, denoted Φ_d, is the image of $\mathbb{P}^2$ under v_d, i.e.*

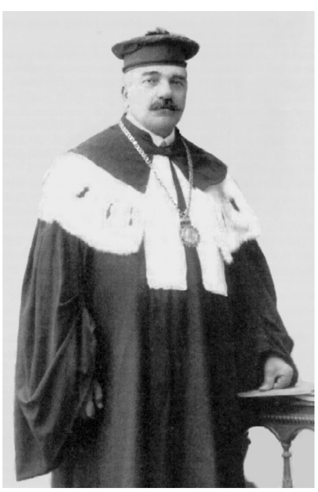
$$\Phi_d = v_d(\mathbb{P}^2).$$

In this section, we will show that

$$\mathfrak{P}(v_d) = 6(d-1)^2. \tag{2.2.1}$$

Our strategy leverages the embedding map to perform the intersection theoretic computation of Equation (1.6.1) in the Chow ring $A(\mathbb{P}^2)$ via pullback along v_d. For this reason, let $L, p \in A(\mathbb{P}^2)$ be the classes of a line and a point in $\mathbb{P}^2$ respectively, and let $[X : Y : Z]$ be the homogeneous coordinates on $\mathbb{P}^2$. By definition, v_d embeds $\mathbb{P}^2$ according to the complete linear series in its homogeneous coordinates. Stated explicitly,

$$v_2([X : Y : Z]) = [X^d : X^{d-1}Y : \cdots : YZ^{d-1} : Z^d].$$



Giuseppe Veronese

(May 1854 – Jul 1917)

As an immediate consequence, we can see that there is a natural bijection between the hyperplane sections of Φ_d and degree d plane curves, since each is described by the vanishing of a k-linear combination of homogeneous degree d monomials in X, Y, Z. Indeed,

$$v_d^* \zeta = dL.$$

Now, two generally chosen degree d plane curves meet in d^2 points, and since this intersection corresponds to the intersection of two generally chosen hyperplane sections of Φ_d, it follows that $\deg \Phi_d = d^2$. Another way of seeing this is by simply computing

$$\deg \Phi_d = \zeta^2 = v_{d*} v_d^* \zeta^2 = v_{d*}(dL)^2 = v_{d*} d^2 p = d^2,$$

via the push-pull formula.

From [6, Example 1.32], the canonical class of projective space $\mathbb{P}^N$ is $K_{\mathbb{P}^N} = -(N+1)H$ where H is the hyperplane class. Since v_d is an isomorphism, it follows that

$$v_d^* K_{\Phi_d} = -3L.$$

To compute $v_d^* c_2(T_{\Phi_d})$, we apply the Whitney Sum formula to the Euler exact sequence for the tangent bundle on $\mathbb{P}^N$ with $N = 2$. For more details, see Appendix A.

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^N} \longrightarrow \mathscr{O}_{\mathbb{P}^N}(1)^{N+1} \longrightarrow T_{\mathbb{P}^N} \longrightarrow 0.$$

Whitney's formula says that the total Chern polynomials of the above vector bundles obey the relationship

$$c\left(\mathscr{O}_{\mathbb{P}^N}(1)^{N+1}\right) = c(\mathscr{O}_{\mathbb{P}^N}) \cdot c(T_{\mathbb{P}^N}).$$

Observe that the left hand side splits as a product of line bundles and that the right hand side reduces to $c(T_{\mathbb{P}^N})$ since $c(\mathscr{O}_{\mathbb{P}^N}) = 1$. Consequently, the total Chern polynomial for the tangent bundle over $\mathbb{P}^N$ is

$$c(T_{\mathbb{P}^N}) = (1+\zeta)^{N+1} = \sum_{k=0}^{N} \binom{N+1}{k} \zeta^k = \sum_{k=0}^{N} c_k(T_{\mathbb{P}^N}).$$

Note that the sum terminates at $k = N$ since $\zeta^{N+1} = 0 \in A(\mathbb{P}^N)$. Applying this formula to $\mathbb{P}^2$, we can compute the second Chern class for the tangent bundle over Φ_d as

$$c_2(T_{\Phi_d}) = \binom{3}{2} \zeta^2 = 3.$$

The last component for the calculation is to apply Theorem 1.6.1:

$$\begin{aligned}\mathfrak{P}(\Phi_d) &= 6\zeta^2 + 4K_{\Phi_d}\zeta + K_{\Phi_d}^2 - c_2(T_{\Phi_d}) \\ &= 6d^2 + v_d^*\left(4K\zeta + K^2 - c_2(T_{\Phi_d})\right) \\ &= 6d^2 + 4(-3L)(dL) + (-3L)^2 - 3 \\ &= 6d^2 - 12d + 9 - 3 \\ &= 6(d-1)^2.\end{aligned}$$

In conclusion, we connect the number $6(d-1)^2$ back to Theorem 1.6.2, which asserts that

$$6(d-1)^2 \geq 2\binom{d+2}{d} - 8.$$

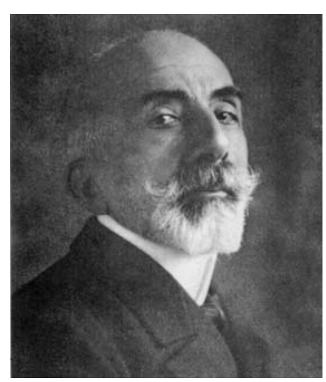
The inequality clearly holds for all $d \geq 2$, giving some evidence for the theorem.

2.3 Family 2: Segre surfaces

We now turn our attention to the second family of surfaces. Let $\sigma_{a,b} : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^N$ with $N = (a+1)(b+1)-1$ be the embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ along the complete linear series of degree $|a, b|$ into $\mathbb{P}^N$, and let $X_{a,b} = \sigma_{a,b}(\mathbb{P}^1 \times \mathbb{P}^1)$. We will show that

$$\mathfrak{P}(\pi : X_{a,b} \to \mathbb{P}^3) = 12ab - 8(a+b) + 4. \tag{2.3.1}$$

As in the previous section, our strategy will be to pull the intersection theoretic computations back to the Chow ring $A(\mathbb{P}^1 \times \mathbb{P}^1)$. In order to apply Theorem 1.6.1, we must collect all terms in the formula.



Corrado Segre

(Aug 1863 – May 1924)

To begin, equip $\mathbb{P}^1 \times \mathbb{P}^1$ with the bihomogeneous coordinates $([s : t], [u : v])$. Then $\sigma_{a,b}$ embeds $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^N$ with $N = (a+1)(b+1) - 1$, and $X_{a,b} = \sigma_{a,b}(\mathbb{P}^1 \times \mathbb{P}^1)$ is given explicitly by the equation

$$\sigma_{a,b}([s : t], [u : v]) = [s^a u^b : s^{a-1}tu^b : \cdots : t^a uv^{b-1} : t^a v^b],$$

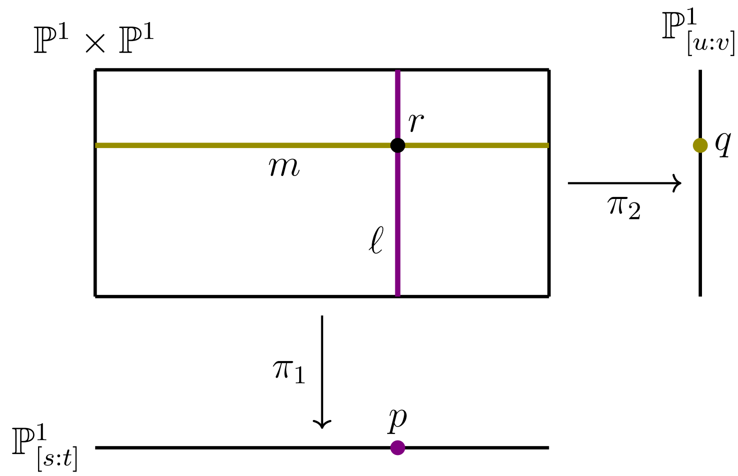


Figure 3: A graphical representation of $\mathbb{P}^1 \times \mathbb{P}^1$, its projection maps, and the pullbacks of general points along those projections.

the complete linear series of bi-degree (a, b). Next, define the projection maps

$$\pi_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1_{[s:t]} \quad \text{and} \quad \pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1_{[u:v]}$$

to the first and second factor. Let p and q be generally chosen points in $\mathbb{P}^1_{[s:t]}$ and $\mathbb{P}^1_{[u:v]}$ respectively, and let ℓ and m be the lines given by the preimages

$$\ell = \pi_1^{-1}(p) \quad \text{and} \quad m = \pi_2^{-1}(q).$$

Finally, let $r = \ell \cap m$. For ease of notation, we will refer to p, q, r, ℓ, and m both as points and lines in $\mathbb{P}^1 \times \mathbb{P}^1$ and as the corresponding classes in the Chow ring $A(\mathbb{P}^1 \times \mathbb{P}^1)$. The setup described in this paragraph is depicted graphically in Figure 3.

We now begin collecting terms for Theorem 1.6.1. Since $\sigma_{a,b}$ is an isomorphism, the total Chern polynomials of $\mathbb{P}^1 \times \mathbb{P}^1$ and $X_{a,b}$ agree. Therefore,

$$K_{X_{a,b}} = K_{\mathbb{P}^1 \times \mathbb{P}^1} \quad \text{and} \quad c_2(T_{X_{a,b}}) = c_2(T_{\mathbb{P}^1 \times \mathbb{P}^1}).$$

Both of these Chern classes can be obtained via the Whitney Sum formula on the Euler sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1} \longrightarrow \mathscr{O}_{\mathbb{P}^1}(1)^2 \oplus \mathscr{O}_{\mathbb{P}^1}(1)^2 \longrightarrow T_{\mathbb{P}^1 \times \mathbb{P}^1} \longrightarrow 0,$$

which, along with [6, Corollary 5.4] gives

$$c(T_{\mathbb{P}^1\times\mathbb{P}^1}) = (1-2\ell)(1-2m).$$

From this, we deduce that

$$K_{\mathbb{P}^1\times\mathbb{P}^1} = -2(\ell+m) \quad \text{and} \quad c_2(T_{\mathbb{P}^1\times\mathbb{P}^1}) = 4r.$$

Next, we wish to analyze ζ, the pullback of a hyperplane along $\sigma_{a,b}$. A hyperplane in $\mathbb{P}^N$, when pulled back to the surface $X_{a,b}$, can be described by the vanishing of a single bi-homogeneous equation of bi-degree (a,b). This can be further pulled back to $\mathbb{P}^1\times\mathbb{P}^1$, to the class of the graph of such a curve, which is linearly equivalent to $a\ell+bm$. The intersections of ℓ and m are described as follows:

$$\ell^2 = m^2 = 0 \quad \text{and} \quad \ell\cdot m = r.$$

Computing the necessary intersection products, we have the following information:

- $\zeta^2 = (a\ell+bm)^2 = a^2\ell^2 + 2ab\ell m + b^2m^2 = 2abr$. Therefore, $\deg X_{a,b} = 2ab$.
- $K_{X_{a,b}} = -2(\ell+m)$, so $K^2 = 4(\ell+m)^2 = 8r$.
- $\zeta\cdot K_{X_{a,b}} = (a\ell+bm)\cdot(-2)(\ell+m) = -2(a+b)r$.
- $c_2(T_{X_{a,b}}) = 4r$.

Thus, by Theorem 1.6.1, we compute

$$\mathfrak{P}(X_{a,b}) = 6\cdot(2ab) + 4\cdot(-2(a+b)) + 8 - 4 = 12ab - 8(a+b) + 4,$$

as desired. Moreover, since, $X_{a,b}\subset\mathbb{P}^N$ with $N = ab+a+b$, to verify Theorem 1.6.2, we ask that

$$\mathfrak{P}(X_{a,b}) = 12ab - 8(a+b) + 4 \geq 2(ab+a+b) - 6.$$

This inequality holds whenever

$$10(a-1)(b-1) \geq 0,$$

which is true for all positive integers a and b. Notice that the *equality* holds only if at least one of a or b is equal to 1, in which case the image of the embedding is identically $S(a, a)$, a rational normal scroll. This observation serves as evidence for the sharpness of the bound, as well as for the classification component of the result.

There is another interesting result that falls out of this computation, one which will serve as evidence for a *strengthening* of the classification result in Chapter V: When $a = b = 2$, the image of $\mathbb{P}^1 \times \mathbb{P}^1$ under $\sigma_{2,2}$ is an anomalous del Pezzo surface of degree 8 in $\mathbb{P}^8$! This is the only class of del Pezzo surfaces whose degree d falls between 3 and 9 which is not abstractly isomorphic to the blow-up of $\mathbb{P}^2$ at $9 - d$ general points. We will cover del Pezzo surfaces in more detail in the next section, and the results will agree with the fact here that

$$\mathfrak{P}(\pi \circ \sigma_{2,2} : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3) = 10.$$

2.4 Family 3: del Pezzo surfaces

The next well known family of surfaces we would like to consider are the del Pezzo surfaces, which arise in the study of varieties of minimal degree.

Pasquale del Pezzo
(May 1859 – Jun 1936)

Recall from Chapter 1 that every n-dimensional variety of minimal degree in $\mathbb{P}^N$ has degree $d \geq N - n + 1.$, and it was a celebrated result of del Pezzo and Bertini [5], [2] that gave the classification of varieties for which the inequality is in fact an equality; these are the so-called varieties of minimal degree.

The del Pezzo surfaces, denoted D_N are not themselves surfaces of minimal degree, as the minimal degree attainable by a nondegenerate surface in $\mathbb{P}^N$ is $N - 1$, attained only by the rational normal scrolls and the Veronese surface in $\mathbb{P}^5$ [7]. Instead, the del Pezzo surfaces have *near* minimal degree. Indeed, the degree N of D_N precisely matches the dimension of $\mathbb{P}^N$ into which D_N is embedded.

Specifically, in this section we will restrict our attention to the del Pezzo surfaces whose degree d satisfies $3 \leq d \leq 9$. As an abstract surface, a del Pezzo surface of degree N, denoted D_N, is isomorphic to the blow-up of $\mathbb{P}^2$ at $r = 9 - N$ general points. There is of course one exception, which we covered at the end of the previous section. The surface D_N is embedded into $\mathbb{P}^N$ via the anticanonical bundle.

Since $N \geq 3$ and $D_N \cong Bl_r\mathbb{P}^2$, (i.e. the blow-up of $\mathbb{P}^2$ at r general points), it is clear that $0 \leq r \leq 6$. In each of these cases, there is some interesting geometry (see for example, [4] and [15]) which we highlight here.

- The surfaces $D_3 \cong Bl_6\mathbb{P}^2$ embedded in $\mathbb{P}^3$ is a cubic surface.

- In $\mathbb{P}^4$, a quartic del Pezzo surface $D_4 \cong Bl_5\mathbb{P}^2$ is the intersection of two quadric hypersurfaces.

- The surface $D_5 \cong Bl_4\mathbb{P}^2$ in $\mathbb{P}^5$ is the intersection of four general hyperplane sections of the image of the Grassmannian $\mathbb{G}(1,4) \subset \mathbb{P}^9$ under its Plüker embedding.

- In $\mathbb{P}^6$, the surface $D_6 \cong Bl_3\mathbb{P}^2$ is one of the following:

  - The intersection of two general hyperplane sections of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ into $\mathbb{P}^8$, given by

$$([X_0 : X_1 : X_2], [Y_0 : Y_1 : Y_2]) \mapsto [X_0Y_0 : X_0Y_1 : \cdots : X_2Y_1 : X_2Y_2].$$

  - A hyperplane section of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^7$, given by

$$([X_0 : X_1], [Y_0 : Y_1], [Z_0 : Z_1]) \mapsto [X_0Y_0Z_0 : X_0Y_0Z_1 : \cdots : X_1Y_1Z_1].$$

- In $\mathbb{P}^7$ and $\mathbb{P}^8$ respectively, the surfaces $D_7 \cong Bl_2\mathbb{P}^2$ and $D_8 \cong Bl_1\mathbb{P}^2$ are best described in this context via the definition of a del Pezzo surface, i.e. the blow-up of $\mathbb{P}^2$ at one or two general points, embedded along the anticanonical embedding. We have already mentioned that there is a del Pezzo surface of degree 8 which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, but we shall not consider it here.

- Finally, in $\mathbb{P}^9$ we have that $D_9 \cong \mathbb{P}^2$ is the 3-uple Veronese embedding of $\mathbb{P}^2$ into $\mathbb{P}^9$ (i.e. $D_9 = \Phi_3$).

Since del Pezzo surfaces are embedded via their anticanonical bundle, we have a relationship between the canonical class K_{D_N} and the section class ζ:

$$-K = \zeta.$$

This relationship introduces a dramatic simplification to the pinch point formula:

$$\mathfrak{P}(\pi : D_N \to \mathbb{P}^3) = 3K^2 - c_2(T_{D_N}).$$

Now, over a field of characteristic zero, $c_2(T_S)$ for a surface S measures the topological Euler characteristic $\chi(S)$ (where S is viewed as a real 4-fold). Therefore, since the blow-up construction at a general point $p \in S$ replaces p with an entire line, we have that

$$\chi(\widetilde{S}) = \chi(S) + 1,$$

and hence

$$c_2(T_{\widetilde{S}}) = c_2(T_S) + 1.$$

Finally, since $D_N \cong Bl_r\mathbb{P}^2$ for $0 \leq r \leq 6$, and since

$$K^2 = \deg D_N = N = 9 - r,$$

we compute

$$\mathfrak{P}(\pi : D_N \to \mathbb{P}^3) = 3(9 - r) - (3 + r) = 4(6 - r) = 4N - 12.$$

One immediately notices that for del Pezzo surfaces, we have $\mathfrak{P}(\pi) = 2(2N - 6)$, and that this equality only agrees with the lower bound proposed in Theorem 1.6.2 when $N = 3$. Perhaps a more subtle, yet interesting, observation is that for all $4 \leq N \leq 9$, the *failure* of the type of D_N to meet the proposed lower bound is two greater than that of D_{N-1}. That is, a general projection of D_4 to $\mathbb{P}^3$ yields two more pinch points than the minimum, then D_5

exceeds the lower bound by four pinch points, and this pattern continues all the way to D_9, which admits 12 pinch points more than the lower bound (which is 12 for a surface in $\mathbb{P}^9$).

Theorem 1.6.2 proposes that $\nu_2 \geq 2N - 6$ with equality holding for surfaces of minimal type. The surfaces of "near-minimal type" are those which have type $\nu_2 = 2N - 4$ (since ν_2 is always even), and indeed, surfaces D_4 are the only del Pezzo surfaces with this property. We saw in a previous section that the Veronese surface Φ_2 also satisfied $\nu_2 = 2N - 4$. In Chapter V, we will show that these are the only non-ruled surfaces of near-minimal type. All other del Pezzo surfaces fail to meet the lower bound in an increasing way (which we shall characterize more clearly, also in Chapter V). This might be somewhat surprising given that every del Pezzo surface is a variety of near-minimal degree, but not all are surfaces of *near-minimal type.*

2.5 Family 4: Rational normal scrolls (Hirzebruch surfaces)

Our last family of surfaces are the rational normal scrolls, which we denote $S(a, b) \subset \mathbb{P}^N$, where $N = a + b + 1$, and $b \geq a$. In this section, we will show with a very detailed analysis that

$$\mathfrak{P}(\pi : S(a, b) \to \mathbb{P}^3) = 2(a + b) - 4.$$

The reader who is familiar with the relative Euler sequence for the cotangent bundle of a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ (see Appendix A) can comfortably skip the discussion in section 2.5.1 where we give transition data for the relevant bundles.

Note that $S(a, b) \cong \mathbb{F}_{b-a}$, where $\mathbb{F}_k$ is the k-th Hirzebruch surface, a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. We will make our computation in the Chow ring $A(S(a, b))$, but to understand the Chern classes of $S(a, b)$, we must first analyze $\mathbb{F}_k$ abstractly. Let $B = \mathbb{P}^1$ be the base curve and let $\rho : \mathbb{F}_k \to B$ be the projective bundle map. The canonical class $K_{\mathbb{F}_k}$ is the first Chern class of the cotangent bundle $\Omega_{\mathbb{F}_k}$, and the second Chern classes of $T_{\mathbb{F}_k}$ and $\Omega_{\mathbb{F}_k}$ are equal. The cotangent bundle arises in the sequence

$$0 \to \rho^*\Omega_B \to \Omega_{\mathbb{F}_k} \to \Omega_\rho \to 0, \tag{2.5.1}$$

which is the sequence used to define the *relative cotangent bundle* Ω_ρ. We will access the total Chern polynomial $c(\Omega_{\mathbb{F}_k})$ via Whitney's formula applied to the sequence in Equation (2.5.1).

2.5.1 The transition data for a collection of bundles

While it is not strictly necessary, we wish to describe each vector bundle in Equation (2.5.1) by its transition data. We begin by investigating the gluing information between some affine charts of $\mathbb{F}_k$ of the form $\mathbb{A}^1 \times \mathbb{A}^1$.

First, note that B has the standard open covering by two affine lines, $\mathbb{A}^1_s$ and $\mathbb{A}^1_t$, with gluing information given by $s = t^{-1}$. This induces two charts on $\mathbb{F}_k$, both isomorphic to $\mathbb{A}^1 \times \mathbb{P}^1$. On the $\mathbb{P}^1$ factor of $\mathbb{A}^1_s \times \mathbb{P}^1$, we write $[X_0 : X_1]$ for the homogeneous coordinates and we denote the affine variables x_{01} and x_{10}, with the understanding that $x_{01} = X_0/X_1$, and $x_{01} = x_{10}^{-1}$. Similarly, on the $\mathbb{P}^1$ factor of $\mathbb{A}^1_t \times \mathbb{P}^1$, we put $[Y_0 : Y_1]$ for the homogeneous coordinates and y_{01} and y_{10} with $y_{01} = y_{10}^{-1}$ for the affine coordinates.

Friedrich Hirzebruch

(Oct 1927 – May 2012)

Using the notation laid out in the previous paragraph, we can state the gluing data for $\mathbb{F}_k$ as follows:

$$s = t^{-1} \quad \text{and} \quad [X_0 : X_1] = [Y_0 : t^k Y_1]. \tag{2.5.2}$$

We will name the four resulting affine charts U_0, U_1, V_0, and V_1; their definitions along with their gluing instructions are given in the following diagram:

$$\begin{tikzcd}
\mathbb{A}^2_{s,x_{10}} \cong U_0 \arrow[rrr, leftrightarrow, "s = t^{-1}", "x_{10} = t^k y_{10}"'] \arrow[d, leftrightarrow, "x_{01} = x_{10}^{-1}"'] & & & V_0 \cong \mathbb{A}^2_{t,y_{10}} \arrow[d, leftrightarrow, "y_{01} = y_{10}^{-1}"] \\
\mathbb{A}^2_{s,x_{01}} \cong U_1 \arrow[rrr, leftrightarrow, "s = t^{-1}", "x_{01} = t^{-k} y_{01}"'] & & & V_1 \cong \mathbb{A}^2_{t,y_{01}}
\end{tikzcd} \tag{2.5.3}$$

We now turn our attention to the cotangent bundle $\Omega_{\mathbb{F}_k}$. We define the trivializing bases of $\Omega_{\mathbb{F}_k}$ on the affine open sets U_0, U_1, V_0, V_1 according to Table 2.

Basis Elements	Affine Chart
$\langle ds, dx_{10}\rangle$	U_0
$\langle ds, dx_{01}\rangle$	U_1
$\langle dt, dy_{10}\rangle$	V_0
$\langle dt, dy_{01}\rangle$	V_1

Table 2: Trivializing bases for the cotangent bundle $\Omega_{\mathbb{F}_k}$ on certain open subsets.

As an example, let $f_{V_0U_0} : \Omega_{\mathbb{F}_k}(V_0) \to \Omega_{\mathbb{F}_k}(U_0)$ be the transition function between the trivializations $\Omega_{\mathbb{F}_k}(V_0)$ and $\Omega_{\mathbb{F}_k}(U_0)$. Taking the exterior derivative of the gluing data in Equation (2.5.2) yields the relationships

$$\begin{aligned} ds &= d(t^{-1}) = -t^{-2} \cdot dt, \\ dx_{10} &= d(t^k y_{10}) = kt^{k-1}y_{10} \cdot dt + t^k \cdot dy_{10}. \end{aligned}$$

In other words, the transition data for $f_{U_0V_0}$ are given by multiplication by the following matrix, which is invertible on the overlap $U_0 \cap V_0$:

$$f_{V_0U_0}\left(\begin{bmatrix} dt \\ dy_{10} \end{bmatrix}\right) = \begin{bmatrix} ds \\ dx_{10} \end{bmatrix} = \begin{bmatrix} -t^{-2} & 0 \\ kt^{k-1}y_{10} & t^k \end{bmatrix} \begin{bmatrix} dt \\ dy_{10} \end{bmatrix}.$$

Transition data for the other pairs of open sets can be obtained in the same manner; these are summarized in Table 3. This concludes our analysis of the transition data for $\Omega_{\mathbb{F}_k}$.

Affine Charts	Transition Matrix	Affine Charts	Transition Matrix
$V_0 \longrightarrow U_0$	$\begin{bmatrix} -t^{-2} & 0 \\ kt^{k-1}y_{10} & t^k \end{bmatrix}$	$U_1 \longrightarrow U_0$	$\begin{bmatrix} 1 & 0 \\ 0 & -x_{10}^{-2} \end{bmatrix}$
$V_1 \longrightarrow U_1$	$\begin{bmatrix} -t^{-2} & 0 \\ -kt^{-k-1}y_{01} & t^k \end{bmatrix}$	$V_1 \longrightarrow V_0$	$\begin{bmatrix} 1 & 0 \\ 0 & -y_{10}^{-2} \end{bmatrix}$

Table 3: Transition matrices between trivializations of $\Omega_{\mathbb{F}_k}$ over selected affine charts.

Next, let Ω_B be the cotangent bundle of the base B. The pullback $\rho^*\Omega_B \subset \Omega_{\mathbb{F}_k}$ sits as a sub-bundle inside $\Omega_{\mathbb{F}_k}$. Indeed, trivializing $\rho^*\Omega_B$ over the affine chart U_0, one can view the corresponding module of differentials $\mathscr{O}_{U_0}\langle ds\rangle$ as a sub-module of $\mathscr{O}_{U_0}\langle ds, dx_{10}\rangle$, the module of differentials for $\Omega_{\mathbb{F}_k}(U_0)$. The same can be said for U_1, V_0, and V_1, which together cover $\mathbb{F}_k$, so the transition data for $\rho^*\Omega_B$ is inherited from that of $\Omega_{\mathbb{F}_k}$.

To exhibit the third bundle in Equation (2.5.1), denoted Ω_ρ, we will examine the quotients of the module of differentials for $\Omega_{\mathbb{F}_k}$. Specifically, for each affine chart U, take the quotient of the module of differentials for $\Omega_{\mathbb{F}_k}(U)$ by the generator of the corresponding module of differentials of $\rho^*\Omega_B(U)$ (e.g. take the quotient by ds over U_0 and U_1, etc.).

We will once more use U_0 and V_0 as an illustration. First, we want to give bases of the quotient modules:

$$\frac{\mathscr{O}_{U_0}\langle ds, dx_{10}\rangle}{\mathscr{O}_{U_0}\langle ds\rangle} = \mathscr{O}_{U_0}\langle \overline{y_s}\rangle \quad \text{and} \quad \frac{\mathscr{O}_{V_0}\langle dt, dy_{10}\rangle}{\mathscr{O}_{V_0}\langle dt\rangle} = \mathscr{O}_{V_0}\langle \overline{y_t}\rangle.$$

Next, we apply the quotient to the transition data for $\Omega_{\mathbb{F}_k}$. The first equation $ds = -t^{-2} \cdot dt$ gives us the trivial relation, and the second factors through the quotient map to become

$$\overline{dx_{10}} = \overline{(kt^{k-1}y_{10}) \cdot dt} + \overline{(t^k) \cdot dy_{10}} = t^k \cdot \overline{dy_{10}}.$$

Similar computations yield:

$$U_0 \longleftrightarrow U_1 : \quad \overline{dx_{10}} = -x_{01}^{-2} \cdot \overline{dx_{01}},$$

$$V_0 \longleftrightarrow V_1 : \quad \overline{dy_{10}} = -y_{10}^{-2} \cdot \overline{dy_{01}},$$

$$U_1 \longleftrightarrow V_1 : \quad \overline{dx_{01}} = t^{-k} \cdot \overline{dy_{01}}.$$

This concludes our analyses of the transition data for the bundles in Equation (2.5.1).

2.5.2 Chern classes of the cotangent bundle

By the Whitney sum formula applied to Equation (2.5.1), if we know the first Chern classes $c_1(\rho^*\Omega_B)$ and $c_1(\Omega_\pi)$, since both bundles have rank 1, we can deduce the Chern classes of $\Omega_{\mathbb{F}_k}$. Since Chern classes respect the pullback operation, we can easily compute

$$c_1(\rho^*\Omega_B) = \rho^* c_1(\Omega_B) = \rho^* c_1(\Omega_{\mathbb{P}^1}) = \rho^*(-2b) = -2F,$$

where $[b]$ is the class of a point in B and $[F]$ is the class of a fiber in $\mathbb{F}_k$ over $b \in B$.

We wish to demonstrate that $c_1(\Omega_\rho)$ can be expressed in terms of:

- The class F of a fiber of ρ, and
- The pullback of either the class of the directrix D or the class of a codirectrix C in the Chow ring of $S(a, b)$ along the embedding map on $\mathbb{F}_k$.

To accomplish the second item, we will actually compute $c_1(\Omega_\rho)$ *twice*. For each computation, we push any rational section through the transition data of Ω_ρ and analyze its degeneracy locus.

We begin with the constant section $\sigma = 1 \cdot \overline{dx_{10}}$ on U_0. Then on V_0 this is the section $t^k \cdot \overline{dy_{10}}$, and on V_1 it becomes $t^k x_t^{-2} \cdot \overline{dy_{01}}$. Thus, the rational section σ has:

- A pole of order 2 along the curve C which is locally defined as $(s, [1:0]) \leftrightarrow (t, [1:0])$, according to the standard gluing instructions. This pole corresponds to $-2C$, which turns out to be the class of a codirectrix on $S(a, b)$.
- A zero of order k along the curve $(0, [Y_0 : Y_1])$, which corresponds to kF (i.e. k times the fiber class F).

Thus, the first Chern class of the relative cotangent bundle is $c_1(\Omega_\rho) = kF - 2C$.

We now compute $c_1(\Omega_\rho)$ *again* by taking the constant rational section $\tau = 1 \cdot \overline{dx_{01}}$ and analyzing its degeneracy locus on the other charts. On U_0 this is the section $-x_{10}^{-2} \cdot \overline{dx_{10}}$, and on V_0 it becomes $-t^{-k}y_{10}^{-2} \cdot \overline{dy_{10}}$. Then τ has:

- A pole of order 2 along the directrix D, locally defined as $(s, [0:1]) \leftrightarrow (t, [0:1])$, with standard gluing instructions. The pole corresponds to $-2D$.
- A **pole** of order k along the curve $(0, [Y_0 : Y_1]$, which corresponds to kF.

Thus, we are forced to conclude that $c_1(\Omega_\rho) = -2D - kF$. Note in particular that the pair of equations we just derived imply the linear equivalence

$$C - D = kF, \tag{2.5.4}$$

with the understanding that $F^2 = C \cdot D = 0$ and $C \cdot F = D \cdot F = 1$. Intersecting both sides of Equation (2.5.4) with D yields

$$D^2 = -k,$$

whereas the same intersection product with C instead of D gives

$$C^2 = k.$$

Since the directrix of a scroll is a more familiar object than the codirectrix, we will choose to express the first Chern class of Ω_ρ as

$$c_1(\Omega_\rho) = -2D - kF.$$

We now compute both Chern classes of $\Omega_{\mathbb{F}_k}$ using the Whitney Sum formula. We will let z serve as our formal variable, and compute:

$$\begin{aligned}
c(\Omega_{\mathbb{F}_k}) &= c(\Omega_{\mathbb{P}^1}) \cdot c(\Omega_\pi) \\
&= (1 - 2Fz)(1 + (-2D - kF)z) \\
&= 1 - 2Fz - 2Dz - kFz + 4DFz^2 + 2kF^2z^2 \\
&= 1 + (-2D - (k+2)F)z + 4pz^2,
\end{aligned}$$

so that the Chern classes of $\Omega_{\mathbb{F}_k}$ are

$$K_{\mathbb{F}_k} = c_1(\Omega_{\mathbb{F}_k}) = -2D - (k+2)F \quad \text{and} \quad c_2(T_{\mathbb{F}_k}) = c_2(\Omega_{\mathbb{F}_k}) = 4p.$$

We now turn our attention to Theorem 1.6.1. Recall that $k = b - a$, and we will show that

Lemma 2.5.1

$$\deg S(a,b) = a + b.$$

Proof. Let D and C be rational normal curves of degrees a and b respectively (with $1 \leq a \leq b$) embedded into disjoint linear subspaces of $\mathbb{P}^N$ with $N = a + b + 1$. Let H and K be hyperplanes in $\mathbb{P}^N$ which are general among hyperplanes containing $\langle D \rangle$ and $\langle C \rangle$. Then $K \cap D$ is a reduced set of a points, and $H \cap C$ is a reduced set of b points. Thus

$$H \cap K \cap S(a,b) = \{a + b \text{ reduced points}\},$$

and the lemma follows from Bertini's theorem since the intersection is finite. ∎

Since D is the class of the directrix, a curve of degree a, it follows that $\zeta \cdot D = a$. Similarly, $\zeta \cdot F = 1$, since projective bundle fibers are mapped one-to-one onto the ruling lines of $S(a,b)$ under the embedding map. These facts, combined with Theorem 1.6.1, allow us to conclude that

$$\begin{aligned}
\mathfrak{P}(S(a,b)) &= 6\deg(S(a,b)) + 4\zeta K_{S(a,b)} + K_{S(a,b)}^2 - c_2(\Omega_{S(a,b)}) \\
&= 6(a+b) + 4\left(-2D\zeta - (b-a+2)\zeta F + D^2 + (b-a+2)DF\right) \\
&\qquad + (b-a+2)^2F^2 - 4 \\
&= 6(a+b) + 4(-2a - (b-a+2) - (b-a) + (b-a+2)) - 4 \\
&= 2(a+b) - 4.
\end{aligned}$$

Since $S(a,b)$ sits inside $\mathbb{P}^{a+b+1}$, we see that rational normal scrolls always meet the lower bound on pinch point scheme length, i.e.

$$\mathfrak{P}(S(a,b)) = 2N - 6 = 2(a+b+1) - 6 = 2(a+b) - 4.$$

Note that this is not sufficient for the classification result; we must yet prove that there are no *other* surfaces which have this property. This is the content of Lemma 4.4.1.

CHAPTER III

THE INNER PROJECTION SETTING

3.1 A brief look ahead

In Chapter II, we presented *evidence* for Theorem 1.6.2, without giving a proof. Here, we set about the task of actually proving it! This will be accomplished over the course of two chapters. In this chapter we will discuss the Inner Projection Setting, a set of definitions and notations designed to give us a language with which we can easily discuss the elements of the proof of Theorem 1.6.2. We also include at the end of this chapter a *sketch* of the proof, since the actual proof is somewhat modular in nature, consisting of several lemmas that assemble rather neatly. In Chapter IV, we systematically state and prove each lemma, making one detour to prove an interesting result: "The ramifications of a map's ramification containing the exceptional divisor." Chapter IV concludes with a full paragraph that makes up a final proof of the main result, stated as Corollary 4.5.1.

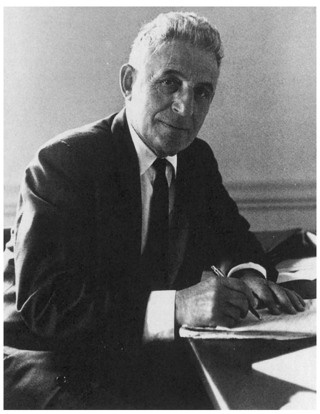
3.2 An important definition: Uncrumpled map

Let X be an irreducible smooth surface. For a map $\varphi : X \to \mathbb{P}^N$, we write $d\varphi : T_X \to \varphi^* T_{\mathbb{P}^N}$ for the map on tangent spaces induced by φ, and we denote by $\mathrm{Ram}(\varphi)$ the ramification scheme of φ, i.e. the subscheme of X for which $\mathrm{rank}(d\varphi) \leq 1$.

The following definition is meant to simplify the statements of most lemmas in Chapter IV. We introduce the notion of an *uncrumpled map*. In Chapters I and II, we worked primarily with smooth surfaces embedded in $\mathbb{P}^N$. Uncrumpled maps on smooth surfaces generalize this situation so that some singular surfaces are admitted.

Definition 3.2.1 *For a smooth surface X, we say that a map $\varphi : X \to \mathbb{P}^N$ is* ***uncrumpled*** *if*

- *$\varphi(X)$ is non-degenerate in $\mathbb{P}^N$,*
- *X is birational onto its image $\varphi(X)$, and*
- $\mathrm{Ram}(\varphi)$ *is finite.*



Oscar Zariski
(Apr 1899 – Jul 1986)

An attractive (though somewhat more sophisticated) property of uncrumpled maps $\varphi : X \to Y$ is that they are necessarily finite. Moreover, since X is assumed smooth (and hence normal), it turns out that X is the normalization of Y, and φ is the normalization map.

We now offer a brief explanation of the preceding claims. That φ is quasi-finite follows from the finite ramification hypothesis, and that φ is proper follows from the fact that X is projective (recall that a map is finite if and only if it is both quasi-finite and proper). A characterization of the normalization of a variety gives a unique isomorphism between the normalization of Y and any other normal surface which admits a finite birational map to Y. The characterization follows from Zariski's Main Theorem, and we refer the reader to Appendix B for a more detailed treatment of the argument presented in this paragraph. We use this characterization to prove Lemma 3.4.2, stated below.

Remark 3.2.1 There is an interesting geometric characterization of surfaces which arise as the image of an uncrumpled map on a smooth surface: They are the surfaces which contain finitely many pinch points and whose normalization is smooth. In particular, they cannot contain any double-points (i.e. points which are locally the vertex of a cone), since double-points are normal and would lift to an impossible singular point in X along the normalization map.

3.3 The Inner Projection Setting

We now define the Inner Projection Setting. This is a sequence of hypotheses, definitions and choices for notation to which we refer whenever we wish to reference a generic inner projection. We have organized the data as an ordered list for the reader's convenience. To emphasize: We refer to this section *frequently* throughout the rest of the text.

a) We begin with the hypotheses of Theorem 1.6.2:

- **i)** Let X be an irreducible smooth surface.
- **ii)** Let $\varphi : X \to \mathbb{P}^N$ with $N \geq 4$ be an uncrumpled map.
- **iii)** Let Y denote the image $Y = \varphi(X)$.

b) Let $x \in X$ be a generally chosen point, and put $y = \varphi(x)$. Note that it is equivalent to let y be a general point in Y, since φ is a birational map between X and Y.

c) Let $\beta_x : \widetilde{X} \to X$ be the blow-up of X at x, let $E = \beta_x^{-1}(x)$ be the exceptional curve, and let $\beta_y : \widetilde{Y} \to Y$ the blow-up of Y at y.

d) The next items deal with certain projection maps and their resolutions to the corresponding blow-ups.

- **i)** Let $\pi_y : Y \dashrightarrow \mathbb{P}^{N-1}$ be the *inner projection from y*. Specifically, π_y is the projection map from $y \in \mathbb{P}^N$ to $\mathbb{P}^{N-1}$ restricted to $Y \setminus \{y\}$.
- **ii)** Define $\pi_x = \pi_y \circ \varphi : X \dashrightarrow \mathbb{P}^{N-1}$ to be the *inner projection from x*. Note that the loci of indeterminacy for π_x and π_y are $\{x\}$ and $\{y\}$ respectively, since y is general in Y.
- **iii)** Let $\widetilde{\varphi} : \widetilde{X} \to \mathbb{P}^{N-1}$ and $\widetilde{\pi}_y : \widetilde{Y} \to \mathbb{P}^{N-1}$ be the resolutions of π_x and π_y respectively to the corresponding blow-ups.
- **iv)** Let $X_x = \widetilde{\varphi}(\widetilde{X})$ and let $Y_y = \widetilde{\pi}_y(\widetilde{Y})$ Note that since $\pi_y(Y \setminus \{y\}) = \pi_x(X \setminus \{x\})$, it follows that $X_x = Y_y$.

e) We use the symbol ζ inside the Chow ring $A(X)$ to denote the pullback of the class of a hyperplane in $\mathbb{P}^N$ along φ. Analogously, let $\widetilde{\zeta} \in A(\widetilde{X})$ to denote the pullback of the class of a hyperplane in $\mathbb{P}^{N-1}$ along $\widetilde{\varphi}$.

f) Lastly, let $\pi : \mathbb{P}^N \dashrightarrow \mathbb{P}^3$ be a general linear projection. Then $\pi \circ \varphi : X \to \mathbb{P}^3$ is, in particular, a map to $\mathbb{P}^3$, and so Theorem 1.6.1 applies. We define the *pinch point scheme length* of the map φ as the output of Equation (1.6.1) when applied to the composition $\pi \circ \varphi$:

$$\mathfrak{P}(\varphi) = \deg\left(6\zeta^2 + 4\zeta K_X + K_X^2 - c_2(T_X)\right).$$

Remark 3.3.1 Since φ is uncrumpled, if $\pi : Y \to \mathbb{P}^3$ is a general linear projection then $\mathfrak{P}(\varphi)$ is precisely the length of $\mathrm{Ram}(\pi \circ \varphi)$. If φ is an embedding, then $\mathfrak{P}(\varphi)$ gives the number of pinch points contained in the surface $\pi(Y)$, and we recover the original notion that $\mathfrak{P}(\varphi)$ is the type ν_2 of Y.

The setting for Section 3.3 can be summarized in the following diagram:

$$\begin{tikzcd}
\widetilde{X} \arrow[d, "\beta_x"] \arrow[ddr, bend right=60, "\widetilde{\varphi}"'] & & \widetilde{Y} \arrow[d, "\beta_y"'] \arrow[ddl, bend left=60, "\widetilde{\pi}_y"] \\
X \arrow[rr, "\varphi"] \arrow[dr, dashed, "\pi_x"] & & Y \arrow[dl, dashed, "\pi_y"'] \\
& X_x = Y_y &
\end{tikzcd} \tag{3.3.1}$$

3.4 A collection of some classical results

As we continue positioning ourselves to where we can prove Theorem 1.6.2, we must present several results which are well-established in the literature, and which will prove to be essential in various proofs in Chapter IV. Each will be stated, named, and equipped with a brief description.

The first result on our list is the General Position Theorem [12], [1]:

Theorem 3.4.1 (General Position Theorem) *Let $C \subset \mathbb{P}^r$, $r \geq 2$, be an irreducible non-degenerate, possibly singular, curve of degree d. Then a general hyperplane meets C in d points any r of which are linearly independent.*

For example, if C is a degree five curve in $\mathbb{P}^3$, then a general 2-plane in $\mathbb{P}^3$ will meet C at exactly 5 reduced points, no three of which are collinear. Similarly, if $C \subset \mathbb{P}^4$ has degree 12, then a general 3-plane meets C at 12 distinct non-coplanar points. Moreover, if one can show that the points of a general hyperplane section of an irreducible curve in $\mathbb{P}^r$ contains r points which are collinear, then that curve must be degenerate in $\mathbb{P}^r$.

Theorem 3.4.2 (Surfaces with a two-dimensional family of plane curves) *Let S be a surface in $\mathbb{P}^N$ with $N \geq 4$. If S contains a 2-dimensional family of plane curves, then S is one of the following:*

- *The Veronese surface in $\mathbb{P}^5$ or its (maybe singular) projection to $\mathbb{P}^4$.*
- *The cubic rational normal scroll $S(1, 2)$.*
- *A cone.*

This is a very classical result due to Segre [20], [27], [25]. If one can exhibit a two parameter family of plane curves in a surface, then there are only a few possibilities for what the surface can actually be.

For completeness, we have elected to include in this section Theorem 3.4.3 and Lemma 3.4.1, both of which are well known results on projective duality. We begin with the popular Reflexivity Theorem [28, Theorem 1.2]:

Theorem 3.4.3 (Reflexivity Theorem) *For any irreducible variety $X \subset \mathbb{P}^N$,*

$$(X^*)^* = X.$$

While the Reflexivity Theorem is very well known, we will use it in a very specific context: If the dimension of the dual variety is "too small," then the original variety must have been ruled. This is the consequence of Lemma 3.4.1 below. When we speak of ruled surfaces, or surfaces which are ruled by lines, we mean that:

Definition 3.4.1 *A surface $X \subset \mathbb{P}^N$ is* ***ruled by lines*** *if its Fano scheme is positive dimensional.*

While it is not necessary for our result, we wish to state the more general version of Lemma 3.4.1. To that end, we note that the concept of a variety being ruled by lines extends to being *ruled by r-planes.* If a variety has this property, then every point on it is contained in an r-plane which is completely contained within the variety. We say that the *defect* of a variety is one less than the codimension of its dual variety. The following lemma states that when the defect of a variety is positive, that variety is ruled [28, Theorem 1.18].

Lemma 3.4.1 *A variety with defect $r \geq 1$ is ruled by r-planes.*

We will use the following version of Lemma 3.4.1: If a surface is 1-defective then it is ruled by lines. This concludes our list of results on projective duality.

The following theorem is a consequence of the Fulton-Hansen Connectedness theorem [18, Theorem 3.4.1].

Theorem 3.4.4 (A Consequence of Fulton-Hansen Connectedness) *Let X be a complete irreducible variety of dimension n, and let $f : X \to \mathbb{P}^r$ be an unramified morphism. If $2n > r$, then f is a closed embedding.*

Our final lemma is more of an observation about the uncrumpled map φ in Section 3.3; the proof can be found in Appendix B.

Lemma 3.4.2 *Maintain the setting of Section 3.3. If Y is ruled by lines, then X is a $\mathbb{P}^1$-bundle over a smooth curve, and φ maps the rulings to lines.*

This concludes our list of preliminary results.

3.5 An outline of the proof of the main result

We wish to conclude this chapter with an overview of the proof of Theorem 1.6.2, which is somewhat modular in nature. In Chapter IV, we will prove many of the statements the reader finds here, and then we will conclude Chapter IV by assembling them together. Rather than lose track of the story arc as we prove the constituent lemmas, we wish to describe the larger picture so that the reader is familiar with each piece, and how it fits in the puzzle.

We begin with the following two equivalent perspectives one can hold regarding a general linear projection $\pi : \mathbb{P}^N \dashrightarrow \mathbb{P}^3$. Let Λ_π be the center of projection of π (i.e. the base locus). Note that $\operatorname{codim} \Lambda_\pi = 4$.

Perspective 1: The map π is a *single* map (to be contrasted with the second perspective) defined by sending a point p to the intersection of the codimension 3 planes containing both p and Λ_π with the target $\mathbb{P}^3$. That is, fix a $\mathbb{P}^3 \subset \mathbb{P}^N \setminus \Lambda_\pi$, denoted H_π, and define

$$\pi(p) = \langle p, \Lambda_\pi \rangle \cap H_\pi.$$

This is usually the first description one sees of a projection. Note that the output is a well-defined point, since the codimensions of the relevant planes satisfy

$$\operatorname{codim}\langle p, \Lambda_\pi \rangle + \operatorname{codim} H_\pi = N.$$

Perspective 2: The map π can be viewed as a composition of $N-3$ projections, each from a general point. Indeed, let $q_1 \in \mathbb{P}^N$ be a generally chosen point, and let

$$\pi_{q_1} : \mathbb{P}^N \dashrightarrow \mathbb{P}^{N-1}$$

be the projection from q_1. More generally, let $q_i \in \mathbb{P}^{N-i+1}$ be a general point such that $1 \leq i \leq N-3$, and let $\pi_{q_i} : \mathbb{P}^{N-i+1} \to \mathbb{P}^{N-i}$ be the corresponding projection map. Note that $H_\pi \subset \mathbb{P}^{N-i}$ for all i, since each q_i is chosen generally. Then

$$\pi = \pi_{q_{N-3}} \circ \pi_{q_{N-4}} \circ \cdots \circ \pi_{q_2} \circ \pi_{q_1}.$$

Indeed, if $N = 4$ the statement is vacuous. Otherwise, one lifts $q_{N-3} \in \mathbb{P}^4$ along $\pi_{q_{N-4}}$ to obtain a *general* line in $\mathbb{P}^5$ whose points all map to the center of projection for the next map. Moreover, the span in $\mathbb{P}^5$ of this line with any other point meets H_π at a single well-defined point, since a general codimension 4 plane fails to meet 3-plane, but a plane of codimension 3 does meet a 3-plane transversely.

Iterating this process of lifting the "cumulative center of projection" reveals that the total composition of the pointwise projection maps is equivalent to projection from the *general* codimension 4 linear space

$$\begin{aligned} \Lambda_\pi &= \left\langle q_1,\ \pi_{q_1}^{-1}\left(\left\langle q_2, \ldots, \pi_{q_{N-4}}^{-1}\left(q_{N-3}\right)\ldots\right\rangle\right)\right\rangle \\ &= \left\langle q_1, \pi_{q_1}^{-1}(q_2), \ldots, \left(\pi_{q_1} \circ \cdots \circ \pi_{q_{N-4}}\right)^{-1}\left(q_{N-3}\right)\right\rangle. \end{aligned}$$

We now take a critical step in our analysis. We seek to bound from below the number of pinch points on $\pi(Y) = (\pi \circ \varphi)(X)$, where $\varphi : X \to \mathbb{P}^N$ is an uncrumpled map, so we make the following specialization. Instead of letting Λ_π be a *general* codimension 4 plane, we choose Λ_π to be *general among those which meet Y at a general point.* The resulting map is fundamentally different, but its analysis will reveal clues about the original setting. Specifically, we are interested in understanding *how this specialization changes the pinch point number.*

The previous paragraph is phrased in terms of Perspective 1, but from Perspective 2 we can describe the specialization thus: Instead of *every* q_i being chosen generally, let the first source of projection q_1 be a general point on Y. If each other q_i is still totally general, then the construction we have here is equivalent to the construction given in the previous paragraph.

In either perspective, the specialization admits another interpretation; rather than thinking about a source of projection which is "slightly more specific" than the general case, we can consider starting with a totally general codimension 4 plane and limiting to one which is general among those which meet Y at a general point. In other words, choose a general point $y \in Y$. As we move Λ_π closer to y, we keep track of the behavior of the pinch points on the image surface $\pi(Y) \subset H_\pi$. We will witness the following behavior: As $\Lambda_\pi \longrightarrow y$, four of the pinch points begin to coalesce, and when $y \in \Lambda_\pi$, those four points mysteriously vanish on $\widetilde{Y}$!

In the next chapter, we will prove this pinch-point-reduction as Lemma 4.3.1 via an intersection theoretic calculation. It is worth noting that the number 4 appears in this context as the answer to an exercise in Semple & Roth [26]. In our case, the context is far more general than that which is presented in the text, and the technique we use is vastly different from the more classical techniques the authors implement.

The strange behavior of pinch points under our specialization of a totally general projection to one which is general among those whose center of projection meets Y at a general point leads us to a conjecture: Is it *always* true that the specialized projection map has four fewer pinch points on the image of Y as compared to the general case? The answer turns out to be yes, with one critical exception: When Y is ruled by lines, the conjectured behavior does not necessarily hold.

Assuming for the moment that Y is not ruled, we now have the ingredients for a proof by induction. If $N = 3$, then Theorem 1.6.2 is true by assumption (aside from the classification result), stating that since $\varphi : X \to \mathbb{P}^3$ is finitely ramified, it follows that $\mathfrak{P}(\varphi) \geq 0 = 2\cdot 3 - 6$. This serves as our base case.

For induction, note that since the pinch point number drops by four when taking a general linear projection of the image of a surface that has been projected once from a general point on it, but the corresponding dimension of the ambient projective space decreases only by 1, it follows that the pinch point number is now closer to the lower bound than it was before. Stated differently, projecting $Y \subset \mathbb{P}^N$ from a general y creates a new surface $Y_y \subset \mathbb{P}^{N-1}$ for which a general linear projection yields four fewer pinch points. But the bound says that

$$\mathfrak{P}(\pi : Y \to \mathbb{P}^3) \geq 2N - 6,$$

while

$$\mathfrak{P}(\pi : Y_y \to \mathbb{P}^3) \geq 2(N-1) - 6 = 2N - 8.$$

But if we also know that the latter pinch point scheme length is four fewer than the former, then we simply proceed by induction to conclude that $\mathfrak{P}(\pi : Y \to \mathbb{P}^3)$ obeys the lower bound.

We call the process of projecting a surface from a general point on it a *general inner projection*, which is described in detail in Section 3.3. We know that either a general inner projection of a surface drops the pinch point number by 4 or that the surface was ruled by lines. It turns out to be rather straightforward to prove the weaker version of the theorem for ruled surfaces only from Theorem 1.6.1; this is the content of Lemma 4.4.1, and it turns every ruled surface into a type of base case for induction. The proof for Lemma 4.4.1 follows the pattern of the examples in Chapter II.

There is one last major obstacle for the induction to parse: How do we know that the inner projection of a surface is again an uncrumpled map from a smooth surface to $\mathbb{P}^{N-1}$? (Recall that this was the hypothesis for Theorem 1.6.2.) It is the answer to this question which occupies the bulk of the proof for Theorem 1.6.2. Here is a summary:

1. The inner projection map has a base point at $x \in X$, and can be extended uniquely to a regular map $\widetilde{\varphi} : \widetilde{X} \to \mathbb{P}^{N-1}$ via the blow-up of X at x, denoted $\beta_x : \widetilde{X} \to X$.

2. The surface $\widetilde{X}$ is smooth.

3. The following three facts are each proved as a lemma or proposition:

   (a) X_x is non-degenerate in $\mathbb{P}^{N-1}$ (Lemma 4.1.1).

   (b) $\widetilde{\varphi}$ is birational onto its image X_x (Lemma 4.1.2).

   (c) $\widetilde{\varphi}$ is finitely ramified (Proposition 4.2.2). The proof for this proposition requires a detour through Proposition 4.2.1.

   Together, these three imply the necessary fact: $\widetilde{\varphi}$ is uncrumpled.

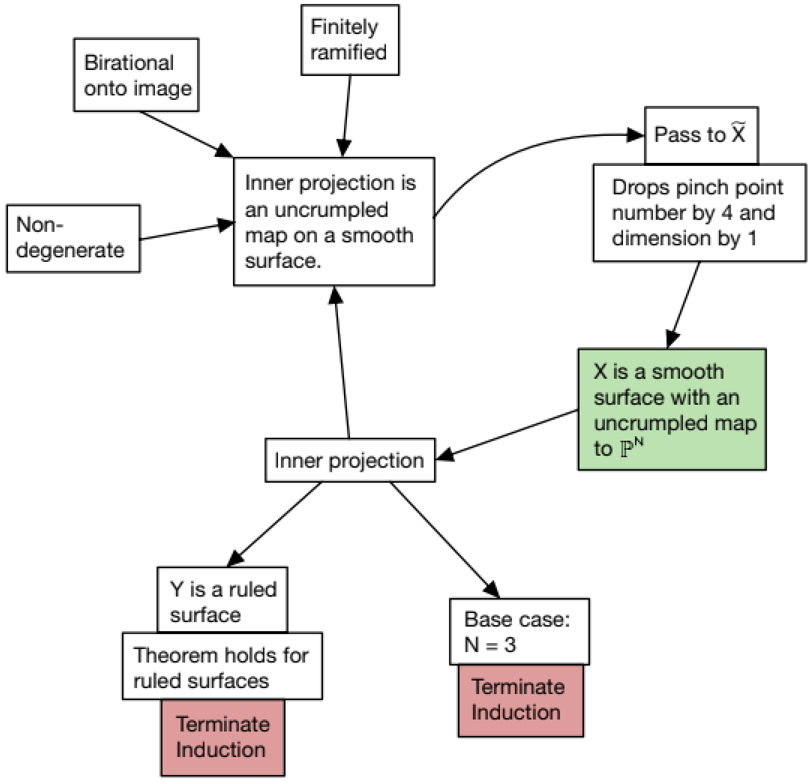


Figure 4: A flow chart depicting the logic of the proof for Theorem 1.6.2.

For the reader's convenience, we have included at the end of this section a flow chart of the logic detailed above, see Figure 4. We encourage the reader to refer back to it as needed throughout Chapter IV.

CHAPTER IV

PROOF OF THE MAIN THEOREM

4.1 Inner projection is non-degenerate and birational onto its image

We now set ourselves to the task of proving Theorem 1.6.2. At the end of Chapter III, we gave an overview of the structure of the proof, while omitting several details. We now establish most of the claims in Section 3.5 as lemmas.

Our first two lemmas serve as observations about the inner projection construction, where we address the corresponding properties of uncrumpled maps. In particular, the fact that $Y_y = X_x$ is a non-degenerate surface in $\mathbb{P}^{N-1}$ is straightforward.

Lemma 4.1.1 *In the context of Section 3.3, the surface Y_y is non-degenerate in $\mathbb{P}^{N-1}$.*

Proof. Suppose for contradiction that Y_y is contained in a hyperplane $H \subset \mathbb{P}^{N-1}$. Then every point in Y is contained in a line meeting H (viewed as a subset of $\mathbb{P}^N$ contained in the target of the map π_y) and y. But then $Y \subset \langle H, y\rangle$, a hyperplane in $\mathbb{P}^N$. This contradicts the hypothesis that Y is non-degenerate in $\mathbb{P}^N$. ■

Lemma 4.1.2 *In the context of Section 3.3, the map $\widetilde{\pi}_y : \widetilde{Y} \to \mathbb{P}^{N-1}$ is birational onto its image Y_y.*

Proof. We begin with the claim that $\dim Y_y = 2$. Indeed, since $\widetilde{Y}$ is an irreducible projective variety and $\widetilde{\pi}_y$ is regular, Y_y is a closed subset of $\mathbb{P}^{N-1}$. If $\dim Y_y = 0$, then Y would be entirely contained in a line (which contains y in particular). This is absurd. If $\dim Y_y = 1$, then Y is a cone over the resulting curve, where y is the vertex of the cone. Since this would be true for a general $y \in Y$, then joining two general points on Y is a line that is completely contained in Y. This condition implies that Y is a degenerate 2-plane, which is contrary to the assumption that Y is non-degenerate in $\mathbb{P}^N$. Thus, Y_y must be two-dimensional.

We now prove that $\widetilde{\pi}_y$ is one-to-one on an open subset of $\widetilde{Y}$. Suppose to the contrary that the degree of $\widetilde{\pi}_y : \widetilde{Y} \to Y_y$ is strictly greater than 1. In other words, by the construction of π_y, if a line is generally chosen among those containing y which are also secant to Y, then that line is in fact at least *trisecant* to Y, i.e. it meets Y at y and at least two other points. Moreover, since y itself was chosen generally, this property can be stated in general:

$$\deg \widetilde{\pi}_y > 1 \implies \text{A general secant line to } Y \text{ is trisecant.}$$

Now, let C be a general hyperplane section of Y. We can make the following observations:

- $Y \subset \mathbb{P}^N$ with $N \geq 4$, so $C \subset \mathbb{P}^r$ with $r \geq 3$.
- Since Y is non-degenerate in $\mathbb{P}^N$, $\deg Y \geq 3$, and so $d = \deg C \geq 3$.
- C is irreducible and non-degenerate in $\mathbb{P}^r$.
- A general secant line to C is trisecant.

By The General Position Theorem (Theorem 3.4.1), a general hyperplane in $\mathbb{P}^r$ meets C in d points, any r of which are linearly independent. Tracing back to what this says about Y, we have that two generally chosen hyperplanes meet Y in at least three points, any three (or more) of which are independent. This is a direct contradiction to the hypothesis that every secant line to Y and hence C is trisecant, since the three points of intersection are collinear. Thus $\deg \widetilde{\pi}_y = 1$. ∎

Remark 4.1.1 Since $Y_y = X_x$, Lemma 4.1.2 implies that $\widetilde{\varphi}$ is birational onto its image. This can be seen by tracing through the diagram in Equation 3.3.1.

We have shown that the resolved inner projection map obeys two of the three properties of generic inner projection, but it remains to show when it is finitely ramified. This is the content of the entire next section.

4.2 A pair of propositions

4.2.1 Ramification of the blowdown map

In this section, we simply wish to prove that the inner projection construction results in a finitely ramified map when resolved to the blow-up. In the proof, we encounter an obstacle wherein the blowdown map could be ramified along the exceptional curve. It turns out that this condition implies that Y is a 2-plane, a result which is interesting in its own right. We will prove it here as Proposition 4.2.1 before proving that inner projection is finitely ramified.

The techniques we will use are somewhat granular; we appeal to the complete local ring attached to our generally chosen point, using the corresponding power series ring to express φ in a form which is compatible with projection. We then trace through the resolution of indeterminacy to the blow-up, and compute the induced map on tangent spaces between affine charts of $\widetilde{X}$. The last step is to find the locus of indeterminacy, and draw a conclusion about the geometry of Y based on the assumption that the resolved projection map is ramified along the exceptional curve.

Finally, we wish to note that the proof makes use of many of the constructions in Section 3.3. Since the blow-up construction is local, Proposition 4.2.1 holds when X is an arbitrary surface; it need not be smooth or irreducible. Moreover, $\varphi : X \to \mathbb{P}^N$ need not be finitely ramified, as long as it is birational onto its image. The relaxed assumptions yield a significantly more general result; nevertheless, we maintain the notation of Section 3.3.

Proposition 4.2.1 *Let X be any surface, and let $\varphi : X \to \mathbb{P}^N$ be birational onto its image $Y = \varphi(X)$. If the inner projection map $\widetilde{\varphi} : \widetilde{X} \to \mathbb{P}^{N-1}$ from a general point $y = \varphi(x) \in Y$ is ramified along the exceptional curve $E \subset \widetilde{X}$, then Y is a 2-plane.*

Proof. We begin our proof by expressing φ in a convenient form. Since X is a surface and x is chosen generally (and is hence smooth), the complete local ring $\widehat{\mathscr{O}_{X,x}}$ is isomorphic to $\mathbb{C}[\![s,t]\!]$, the power series ring with analytic local coordinates s and t, with maximal ideal

$\mathfrak{m}_x = (s,t)$. In terms of s and t, φ has an expression

$$\varphi = [S_0 : S_1 : \ldots : S_N],$$

where the S_j are formal power series in s and t. Since φ is well-defined at x, we can without loss of generality assume that S_0 is a unit in $\mathbb{C}[\![s,t]\!]$, and put

$$\varphi = [1 : f_1 : \ldots : f_n],$$

where $f_j = S_j/S_0$. We can also assume without loss of generality that each f_i vanishes at x, so that

$$y = \varphi(x) = [1 : 0 : \ldots : 0].$$

Once again, because x is general in X, φ is unramified at x, which means that the induced map on tangent spaces is full rank. Consequently, we can find two of the f_i, say f_1 and f_2, which have linearly independent terms of degree one. A different choice of local coordinates would allow us to express $f_1 = s$ and $f_2 = t$.

After applying a suitable automorphism of $\mathbb{P}^N$, we can express φ as

$$\varphi = [1 : s : t : g_3(s,t) : \ldots : g_N(s,t)],$$

where $g_j \in \mathfrak{m}_x^2$ for $j \geq 3$. To clarify, $g_j \in \mathfrak{m}_x^2$ means that all terms of g_j have degree at least 2. The automorphism we apply involves clearing out the linear term of each g_j using linear combinations of the second and third components (remember that there are already no constant terms). This is the convenient form in which we wish to express φ.

Our next task is to compute the resolution of the inner projection map to the blow-up: The inner projection map $\pi_x = \pi_y \circ \varphi$ from $x = \varphi^{-1}(y)$ has a local expression

$$[s : t : g_3(s,t) : \ldots : g_N(s,t)] \in \mathbb{P}^{N-1},$$

which is undefined precisely at x. To resolve this indeterminacy, we pass to the blow-up $\beta_x : \widetilde{X} \to X$ of the surface X at x. From the blow-up construction, we obtain the relation

$sV - tU$, expressed as $s = tu$ on the affine chart of the blow-up given by $V \neq 0$, where we say $u := U/V$. Then the regular map $\widetilde{\varphi} : \widetilde{X} \to \mathbb{P}^{N-1}$, the resolution of π_x, has a local description

$$\begin{aligned}
\widetilde{\varphi}(\widetilde{X}) &= [tu : t : g_3(tu,t) : \ldots : g_N(tu,t)] && \text{(Resolve to the blow-up)} \\
&= [t : tu : g_3(tu,t) : \ldots : g_N(tu,t)] && \text{(Change of coordinates)} \\
&= \left[1 : u : \frac{g_3(tu,t)}{t} : \ldots : \frac{g_N(tu,t)}{t}\right] && \text{(Scale down by } t\text{).}
\end{aligned}$$

At this point, note that since $g_j \in \mathfrak{m}_x^2$ (which means all terms have degree at least 2), it follows that all terms of g_j contain a factor of t^2, so all terms of the power series $\frac{g_j(tu,t)}{t}$ must contain a factor of t. For each j, define

$$h_j(t,u) := \frac{1}{t^2} \cdot g_j(tu,t),$$

with the immediate consequence that on the first canonical affine chart of the target $\mathbb{P}^{N-1}$, we can express the resolution $\widetilde{\varphi}$ as mapping $\widetilde{X}$ locally near E to

$$\big(u, t \cdot h_3(t,u), \ldots, t \cdot h_N(t,u)\big) \in \mathbb{A}^{N-1} \subset \mathbb{P}^{N-1}.$$

Our final task is to compute the locus on which the derivative of $\widetilde{\varphi}$, denoted $d\widetilde{\varphi}$ drops rank. Observe that the terms of $h_j(t,u)$ which are constant with respect to t have degree at most 2 in u, a fact which is relevant when we differentiate and evaluate at $t = 0$. Indeed, since $\widetilde{\varphi}$ is now expressed as a map between affine charts, its derivative $d\widetilde{\varphi}$ evaluated at $t = 0$ is given by the matrix

$$d\widetilde{\varphi}\Big|_{t=0} = \left.\begin{bmatrix} 1 & 0 \\ t \cdot \frac{\partial}{\partial u} h_3(t,u) & h_3(t,u) + t \cdot \frac{\partial}{\partial t} h_3(t,u) \\ \vdots & \vdots \\ t \cdot \frac{\partial}{\partial u} h_N(t,u) & h_N(t,u) + t \cdot \frac{\partial}{\partial t} h_N(t,u) \end{bmatrix}\right|_{t=0} = \begin{bmatrix} 1 & 0 \\ 0 & h_3(0,u) \\ \vdots & \vdots \\ 0 & h_N(0,u) \end{bmatrix}.$$

At $t = 0$, the map $d\widetilde{\varphi}$ fails to be full rank precisely when every term in each $h_j(0,u)$ vanishes. Clearly every term of $h_j(t,u)$ containing a factor of t will vanish; these are precisely the terms in the original power series $g_j(s,t)$ whose degree was at least 3. But all the other terms in $h_j(t,u)$ are constant with respect to t.

The hypothesis that $\widetilde{\varphi}$ is ramified along E imposes the condition that $d\widetilde{\varphi}$ drops rank when $0 = s = t$, which translates into the condition that the corresponding terms in each $h_j(t,u)$ which are constant with respect to t must vanish for all u. But if all quadratic terms in the $g_j(s,t)$ used to express φ are identically 0, then the g_j themselves are originally members of $\mathfrak{m}_x^3 \subset \mathfrak{m}_x^2$. But then the second fundamental form at $y = \varphi(x)$ is identically 0, and since x was chosen generally, Y is a 2-plane. ∎

4.2.2 Inner projection is finitely ramified

Armed with Proposition 4.2.1, we can now address the third property of uncrumpled maps, concerning the circumstances under which $\widetilde{\varphi}$ is finitely ramified. Most of the proof relies on an incidence correspondence and dimension counting argument, and consequently, the logic is somewhat branched.

Proposition 4.2.2 *In the context of Section 3.3, either $\widetilde{\varphi} : \widetilde{X} \to \mathbb{P}^{N-1}$ is finitely ramified, or Y is ruled by lines.*

Proof. We begin by making the claim that $\operatorname{Ram}(\widetilde{\varphi})$ is properly contained inside $\widetilde{X}$ (that is, the ramification scheme is at most one dimensional). Indeed,

$$\dim \operatorname{Ram}(\widetilde{\varphi}) = 2 \implies \operatorname{Ram}(\widetilde{\varphi}) = \widetilde{X},$$

since the ramification scheme of a map is a closed subscheme of the domain. In this case, Y is a cone over a curve (or a point) with vertex $y = \varphi(x)$. But since x is chosen generally, Y is a degenerate 2-plane.

We spend the remainder of this proof establishing that $\operatorname{Ram}\widetilde{\varphi}$ in fact cannot be one-dimensional. Suppose to that end that $\widetilde{\varphi}$ is ramified along a *curve* $R \subset \widetilde{X}$. Proposition 4.2.1 implies that $E \not\subset R$. Then mapping R along the blowdown map, we see that $\beta_x(R) \subset X$ is a curve.

Recall that for any $p \in X \setminus \mathrm{Ram}(\varphi)$, the projective tangent plane is defined as

$$\Lambda_p := \langle d\varphi(T_p X) \rangle \subset \mathbb{P}^N.$$

We define the incidence variety Σ to be the closure of the set

$$\{(p, q) \mid p, q \notin \mathrm{Ram}(\varphi), \text{ and } \varphi(p) \in \Lambda_q \setminus \{\varphi(q)\}\} \subset X \times X.$$

In words, Σ is the set of pairs of points on X at which φ is not ramified, such that the projective tangent plane Λ_q contains $\varphi(p)$, while maintaining that $\varphi(p) \neq \varphi(q)$. Let

$$\pi_1, \pi_2 : \Sigma \to X$$

be the projection maps to the first and second factor respectively.

We now compute $\dim \Sigma$. A generic fiber of π_1 is one-dimensional; this is a restatement of the hypothesis that $\widetilde{\varphi}$ is ramified along the curve R. Indeed, the fiber of π_1 over a point $p \in X$ is the set of points q such that $\varphi(p) \in \Lambda_q \setminus \varphi(q)$, which means that projection from p is ramified at q. The assumption that $\mathrm{Ram}\, \widetilde{\varphi}$ is a curve implies that there are infinitely many such q for a general point p. By dimension counting, we see one-dimensional fibers over a two-dimensional surface, and so we deduce that $\dim \Sigma = 3$.

But now consider the map π_2. The fiber $\pi_2^{-1}(q)$ over a point $q \in X$ is the set of points $p \in X$ such that $\varphi(p)$ is contained in a line tangent to Y at $\varphi(q)$. Equivalently,

$$\pi_2^{-1}(q) = \varphi^{-1}(\Lambda_q \cap Y). \tag{4.2.1}$$

Note that $\dim \pi_2(\Sigma) \geq 1$; this can be seen by dimension counting, since Σ is three-dimensional, but $\dim X = 2$.

We claim that in fact $\pi_2(\Sigma)$ is two-dimensional. To see this, suppose for contradiction that $\pi_2(\Sigma)$ is a curve $C \subset X$. Then the fiber over a general point $c \in C$ will be two-dimensional. Since $\mathrm{Ram}\, \varphi$ is finite and c is general among points in C, we know that Λ_c is a well-defined 2-plane. But by Equation (4.2.1), because the fiber $\pi_2^{-1}(c) = \varphi^{-1}(\Lambda_c \cap Y)$ is a two-dimensional closed subset of X,

$$Y \cap \Lambda_c = Y = \varphi(X) \implies Y \subset \Lambda_c,$$

so Y is again degenerate.

Since $\pi_2(\Sigma)$ is two-dimensional, then in fact, π_2 is surjective, because the image of a regular map on a closed variety is closed. Therefore, a general fiber of π_2 is one-dimensional. Let C_q denote $\Lambda_q \cap Y$ for a general $q \in X$. Then by Equation (4.2.1), $\varphi^{-1}(C_q) = \pi_2^{-1}(q)$ is a general fiber of π_2, and is hence one-dimensional, which implies that C_q is a plane curve.

It is clear that Y contains *some* family of plane curves $\mathscr{F}$. One may ask whether $\mathscr{F}$ is finite, or if the fibers of π_2 sweep out all of X. The latter is indeed the case, since π_1 is surjective, meaning that every point in X is contained in the preimage under φ of some projective tangent plane. But since these fibers sweep out all of X, Y contains a *positive dimensional* family $\mathscr{F}$ of plane curves.

If the plane curve C_q is distinct for each $q \in X$, then $\mathscr{F}$ is two-dimensional (parameterized by X). By Theorem 3.4.2, either Y is the Veronese surface in $\mathbb{P}^5$, Y is a cubic scroll or Y is a cone. The first case is impossible; If Y is the Veronese surface, then $\widetilde{\varphi}$ is an embedding and is hence completely unramified. If Y is a cone or the cubic scroll, then in particular Y is ruled by lines.

Finally, suppose instead that $\dim \mathscr{F} = 1$. Then for a general point $q \in X$, there are infinitely many points in X whose fiber under π_2 is C_q. Let $r \in X$ be general among such points. Then

$$C_q \subset \Lambda_q \cap \Lambda_r.$$

If the planes Λ_q and Λ_r are distinct, then C_q is exactly their line of intersection, and Y is ruled by lines. Otherwise, $\Lambda_q = \Lambda_r$. In this case, let $\pi : \mathbb{P}^N \dashrightarrow \mathbb{P}^3$ be a general linear projection, let $Z = \pi(Y)$, and consider the dual variety $Z^* \subset \mathbb{P}^{3*}$. The condition that $\Lambda_q = \Lambda_r$ implies that infinitely many points on Z share each projective tangent plane. Then Z^* must be at most a curve, so by Lemma 3.4.1, Z is ruled by lines. And since Z is a projection of Y, this property lifts, and we conclude that Y is also ruled by lines. ■

We are now ready to take a large step forward.

4.3 Inner projection usually decreases pinch point number

Assume that Y is *not* ruled by lines.

Proposition 4.2.2 now asserts that $\widetilde{\varphi} : \widetilde{X} \to \mathbb{P}^N$ is finitely ramified. In other words, the assumption that Y is not ruled, taken along with Lemmas 4.1.1 and 4.1.2 and Proposition 4.2.2 (the cumulative results from the previous two sections) imply the following critical fact:

Theorem 4.3.1 *If $\varphi : X \to \mathbb{P}^N$ with $N \geq 4$ is an uncrumpled map on a smooth surface X with $\varphi(X)$ not ruled by lines, then the general inner projection $\widetilde{\varphi} : \widetilde{X} \to \mathbb{P}^{N-1}$ is uncrumpled.*

This fact is critical in the proof of Theorem 1.6.2, since with the running assumption that Y is not ruled by lines, we can present an inductive argument by repeatedly iterating the following lemma, applying it to any surface X which satisfies our hypotheses.

Lemma 4.3.1 *Maintain the context of Section 3.3. If Y is not ruled by lines, then*

$$\mathfrak{P}(\widetilde{\varphi}) = \mathfrak{P}(\varphi) - 4.$$

Proof. Since Y is not ruled, $\widetilde{\varphi}$ is uncrumpled by Theorem 4.3.1. To keep the notation manageable, we write $E = [E]$, $\beta = \beta_x$, and we omit the degree map on zero cycles. We now present nine facts from intersection theory regarding the blow-up, grouped as triplets of equations.

1. Our first group of equations has to do with the fact that pullback along the blowdown map respects intersections involving ζ and K_X in the Chow ring $A(X)$. They follow from [13, Chapter V, Proposition 3.2]. Note that the left hand side takes place in $A(\widetilde{X})$, whereas the right hand product is a computation in $A(X)$.

   (a) $(\beta^*\zeta)^2 = \zeta^2$ (b) $(\beta^* K_X)^2 = K_X^2$ (c) $(\beta^*\zeta)\cdot(\beta^* K_X) = \zeta\cdot K_X$

2. Next, we consider what happens when we intersect various divisors in $A(\widetilde{X})$ with the exceptional curve E. The first two follow as a consequence of the Push-Pull formula, since $\beta(E) = x \in X$, and the intersection of the point class with any other (non-fundamental) class in $A(X)$ is empty. The third point is the famous fact about the self-intersection of the exceptional divisor.

   (a) $(\beta^*\zeta) \cdot E = 0$ (b) $(\beta^* K_X) \cdot E = 0$ (c) $E^2 = -1$

3. The last set of equations relate various classes in $A(\widetilde{X})$ with the corresponding classes in $A(X)$. Note that while both ζ and $\widetilde{\zeta}$ are the pullback of the hyperplane class, the former corresponds to a hyperplane in $\mathbb{P}^N$ while the latter comes from $\mathbb{P}^{N-1}$, and is pulled back along $\widetilde{\varphi}$.

   (a) $\widetilde{\zeta} = \beta^*\zeta - E$ (b) $K_{\widetilde{X}} = \beta^* K_X + E$ (c) $c_2(T_{\widetilde{X}}) = c_2(T_X) + 1$.

Applying Equation (1.6.1) to $\pi \circ \widetilde{\varphi} : \widetilde{X} \to \mathbb{P}^3$, we execute a straightforward computation.

$$\begin{aligned}
\mathfrak{P}(\pi \circ \widetilde{\varphi}) &= 6\widetilde{\zeta}^2 + 4\widetilde{\zeta} \cdot K_{\widetilde{X}} + K_{\widetilde{X}}^2 - c_2(T_{\widetilde{X}}) \\
&= 6(\beta^*\zeta - E)^2 + 4(\beta^*\zeta - E) \cdot (\beta^* K_X + E) + (\beta^* K_X + E)^2 - (c_2(T_X) + 1) \\
&= 6\left((\beta^*\zeta)^2 - 2\beta^*\zeta \cdot E + E^2\right) \\
&\qquad + 4\left(\beta^*\zeta \cdot \beta^* K + \beta^*\zeta \cdot E - \beta^* K \cdot E - E^2\right) \\
&\qquad + \left((\beta^* K)^2 + 2\beta^* K \cdot E + E^2\right) - c_2 - 1 \\
&= 6\left(\zeta^2 - 2 \cdot 0 - 1\right) + 4\left(\zeta \cdot K + 0 - 0 + 1\right) + \left(K^2 + 2 \cdot 0 - 1\right) - c_2 - 1 \\
&= 6\zeta^2 + 4\zeta \cdot K + K^2 - c_2 - 4 \\
&= \mathfrak{P}(\varphi) - 4.
\end{aligned}$$

■

4.4 What happens when the surface is ruled?

In the previous section, we assumed that Y was not ruled; we now adopt the opposite stance. In fact, we will show that Theorem 1.6.2 holds immediately whenever Y is ruled! The proof hinges on the following observation: The reader will note that Equation (1.6.1) has the form

$$6a^2 + 4ab + b^2 - c,$$

which can be expressed as

$$2a^2 + (2a + b)^2 - c.$$

We will show (via the relative Euler sequence on a projective bundle) that the middle term is identically zero for a ruled surface Y, and perform our analysis on the remaining two terms. The strategy in this proof mimics that of the examples in Chapter II.

Lemma 4.4.1 *Maintain the context of Section 3.3, and suppose that Y is ruled by lines. Then*

$$\mathfrak{P}(\varphi) \geq 2N - 6,$$

with equality holding precisely when Y is a rational normal scroll.

Proof. By Lemma 3.4.2, since Y is ruled, X is a projective line bundle over a smooth base curve B. Let $\rho : X \to B$ be the bundle map, and let η represent the class of a section of $\mathscr{O}_X(1)$. Now, from [13, Chapter V, Proposition 2.3], we have that the Picard group $\mathrm{Pic}(X)$ is isomorphic to the direct sum of $\rho^* \mathrm{Pic}(B)$ and the free abelian group generated by η:

$$\mathrm{Pic}(X) \cong \rho^* \mathrm{Pic}(B) \oplus \mathbb{Z} \cdot \langle \eta \rangle.$$

Next, let $F \in A(X)$ denote the class of a ruling line (we will refer to this as the *fiber class*). The Whitney sum formula applied to the relative Euler exact sequence on X yields the canonical class $K_X = -2\eta - \rho^* K_B$, where $\rho^* K_B$ is some integer multiple of the fiber class F. (The canonical class can also be witnessed via adjunction on F.) But since η is the class of a section of $\mathscr{O}_X(1)$, the hyperplane class ζ pulls back to a class in $\mathrm{Pic}(X)$ which is of the form $\eta + aF$, for some integer a. Geometrically, the coefficient on η is representative of the observation that a general line meets a general hyperplane in $\mathbb{P}^N$ transversely. It follows that

$$2\zeta + K_X = 2(\eta + aF) + (-2\eta - \rho^* K_B) \in \rho^* \operatorname{Pic}(B) \oplus \{0\},$$

and so computing $(2\zeta + K_X)^2$ amounts to finding the self intersection of a pure integer multiple of F, which is zero. Hence,

$$(2\zeta + K_X)^2 = 0.$$

Recall also that in characteristic zero, the second Chern class of the surface Y is the topological Euler characteristic of Y, viewed as a four-dimensional real manifold. That is,

$$c_2(T_X) = \chi(X).$$

Moreover, by the fibration property of the Euler characteristic,

$$c_2(T_X) = \chi(\mathbb{P}^1) \cdot \chi(B) = (-2) \cdot (2g_B - 2),$$

where g_B is the genus of B.

Now, we will write Equation (1.6.1) applied to $\pi \circ \widetilde{\varphi}$ as follows:

$$\mathfrak{P}(\pi \circ \widetilde{\varphi}) = 6\zeta^2 + 4\zeta K_X + K_X^2 - c_2(T_X) = (2\zeta + K_X)^2 + 2\zeta^2 - c_2(T_X).$$

Substituting $(2\zeta + K_X)^2 = 0$, $\zeta^2 = \deg Y$, and $c_2(T_X) = 4 - 4g_B$ into the expression above yields

$$\mathfrak{P}(\pi \circ \widetilde{\varphi}) = 2 \deg Y + 4g_B - 4. \tag{4.4.1}$$

Since Y is non-degenerate in $\mathbb{P}^N$, the degree of Y is bounded from below by $\deg Y \geq N - 1$, so

$$\mathfrak{P}(\pi \circ \widetilde{\varphi}) \geq 2N + 4g_B - 6.$$

If Y simultaneously minimizes its degree and the genus g_B (which is necessarily non-negative), then Y is a ruled variety of minimal degree over a rational normal curve; it must be a rational normal scroll of degree $N-1$. In this case, since Y is smooth, $X \cong Y$, φ is an embedding and

$$\mathfrak{P}(\pi \circ \widetilde{\varphi}) = 2N - 6.$$

Thus, the lower bound on the pinch point scheme length is sharp, and we obtain a classification of all ruled surfaces which meet that lower bound. ■

We now have all the ingredients needed to prove the main result!

4.5 A proof of the main theorem

To conclude this chapter, we now prove Theorem 1.6.2. As mentioned at the end of Chapter III, our strategy will be to use induction on N, the dimension of the ambient projective space.

Before we proceed, we must address an interesting discrepancy between the statement and proof of Theorem 1.6.2: The induction uses $N = 3$ as a base case, but the theorem is stated for $N \geq 4$. Indeed, the inequality is in fact trivial for $N = 3$; it asserts the tautology that an uncrumpled map on a smooth surface into $\mathbb{P}^3$ has at least zero pinch points.

Theorem 1.6.2 is stated for $N \geq 4$, not on account of the inequality it claims, but rather because of the classification result, which asserts that the surfaces which *attain* the lower bound are exactly the rational normal scrolls. This is false when $N = 3$, since this conclusion specializes to the claim that the only smooth surface in $\mathbb{P}^3$ is the quadric surface (i.e. the rational normal scroll $S(1,1) \subset \mathbb{P}^3$), which is absurd. Thus, we will use the case where $N = 3$ for induction only, and allow Lemma 4.4.1 to handle the classification result.

Corollary 4.5.1 *In the context of Section 3.3,*

$$\mathfrak{P}(\varphi) \geq 2N - 6,$$

with equality holding if and only if Y is a rational normal scroll.

Proof. We will proceed via induction on N, beginning with the observation that a finitely ramified map $\varphi : X \to \mathbb{P}^3$ has $\mathfrak{P}(\varphi) \geq 0$. By Lemma 4.4.1, if Y is ruled, then we have immediately that $\mathfrak{P}(\varphi) \geq 2N - 6$. We will assume from now on that Y is not ruled. Suppose for induction that

$$\mathfrak{P}(\widetilde{\varphi}) \geq 2(N-1) - 6 = 2N - 8.$$

We know from Theorem 4.3.1 that $\widetilde{\varphi}$ is an uncrumpled map on a smooth surface to $\mathbb{P}^{N-1}$, and so satisfies all necessary hypotheses for induction. Moreover, by Lemma 4.3.1, since $\mathfrak{P}(\widetilde{\varphi}) = \mathfrak{P}(\varphi) - 4$, we have that

$$\mathfrak{P}(\varphi) = \mathfrak{P}(\widetilde{\varphi}) + 4 \geq 2N - 8 + 4 > 2N - 6.$$

The strict inequality forces the classification result to follow from Lemma 4.4.1. ∎

CHAPTER V

A STRENGTHENING AND SOME FUTURE DIRECTIONS

5.1 Surfaces of near-minimal pinch point number

We begin our final chapter with a strengthening of the classification result given in Theorem 1.6.2. It states that the surfaces which exhibit the minimal number of pinch points under general linear projection to $\mathbb{P}^3$ are precisely the rational normal scrolls. By looking deeper into some of the lemmas used to prove Theorem 1.6.2, we can extend this classification result to include surfaces which *nearly* exhibit this minimum.

It is known that the length of the pinch scheme is always even. Indeed, Nöether's theorem implies that 12 divides the sum $c_1^2 + c_2$ (so in particular the sum and hence the difference $c_1^2 - c_2$ is even), and every other term in Equation (1.6.1) is even. Hence, if Y is a surface in $\mathbb{P}^N$, the smallest non-minimal number of pinch points it can exhibit under general projection to $\mathbb{P}^3$ is $2N-4$. This is realized by the Veronese surface in $\mathbb{P}^5$! Indeed, we saw in Example 1.3.2 that a general projection of the Veronese surface to $\mathbb{P}^3$ admits 6 pinch points, namely the pinch points of the Roman surface. We also saw in Chapter II that the quartic del Pezzo surfaces in $\mathbb{P}^4$ have two more pinch points than the minimum. It is natural to wonder, what other surfaces (if indeed there are any) exhibit near-minimal pinch point scheme length?

The question of near-minimal pinch point behavior (see Definition 5.1.1 below) is actually easier to answer for surfaces in $\mathbb{P}^N$ where N is large relative to the difference between $\mathfrak{P}(\pi\circ\varphi)$ and the lower bound $2N-6$. Theorem 5.1.1 below gives a characterization of surfaces $Y=\varphi(X)$, where $\varphi : X \to \mathbb{P}^N$ is an uncrumpled map on a smooth surface X and where N is large enough, such that the difference $\mathfrak{P}(\varphi) - (2N-6)$ is a fixed even number.

Before we state and prove Theorem 5.1.1, we will first establish a pair of Lemmas. Lemma 5.1.1 characterizes surfaces whose inner projection is ruled, and Lemma 5.1.2 shows that the inner projection construction preserves linear normality. We also wish to summarize Theorem 5.1.1 by organizing the results into Table 5.1 before giving a proof of the result.

Lemma 5.1.1 *In the context of Section 3.3, if X_x is ruled by lines, then either Y is ruled by lines or Y is the Veronese surface in $\mathbb{P}^5$.*

Proof. We abbreviate $\beta = \beta_x$. Recall from Theorem 4.3.1 that $\widetilde{\varphi}$ is uncrumpled. Since X_x is ruled, Lemma 3.4.2 asserts that $\widetilde{X}$ is a $\mathbb{P}^1$-bundle over a smooth curve B. Let $\rho : \widetilde{X} \to B$ be the bundle map, and for a general point $b \in B$, let $F = \rho^{-1}(b) \subset \widetilde{X}$ be a general fiber. Then as in the proof of Lemma 4.4.1, $\widetilde{\zeta}$ is a class of the form $\eta + aF$, where η is the class of a section of $\mathscr{O}_{\widetilde{X}}(1)$. But then $\widetilde{\zeta} \cdot F = 1$, since $F^2 = 0$.

By the Push-Pull formula, one has

$$\beta_* F \cdot \zeta = \beta^* \zeta \cdot F. \tag{5.1.1}$$

Recall that $\beta^* \zeta = \widetilde{\zeta} + E$, and when applied to Equation (5.1.1), this fact gives

$$\beta_* F \cdot \zeta \ = \ E \cdot F + \widetilde{\zeta} \cdot F \ = \ E \cdot F + 1. \tag{5.1.2}$$

We now have two cases. Suppose first that $x \notin \beta(F)$, meaning that $E \cdot F = 0$. From Equation (5.1.2), we know that a general hyperplane meets $\beta(F)$ in a single point, and so $(\varphi \circ \beta)(F)$ is a line. Since x is assumed to be general, a general point of $\widetilde{X}$ is contained in a general ruling line, and so we conclude that X (and hence Y) is ruled by lines.

Suppose instead that $x \in \beta(F)$, so that $E \cdot F \geq 1$. Then in fact, $E \cdot F = 1$ identically. Indeed, if $E \cdot F \geq 2$, then there is a subscheme of E of length at least 2 that embeds under $\widetilde{\varphi}$, in which case a general ruling line of X_x is contained in E, a contradiction. Since F and E meet transversely, Equation (5.1.2) implies that $\beta(F)$ is a degree 2 smooth curve containing x. Then $(\varphi \circ \beta)(F) \subset \mathbb{P}^N$ is a smooth conic containing y since φ is birational onto its image. Moreover, because every distinct pair of ruling lines in $\widetilde{X}$ is disjoint, no pair of the corresponding conics intersect at a point in $Y \setminus \{y\}$. By varying $x \in X$, we get a two-dimensional family of plane curves in Y, so by Theorem 3.4.2, Y is the Veronese surface in $\mathbb{P}^5$. ■

Lemma 5.1.2 *If X_x is a smooth linearly normal surface in $\mathbb{P}^{N-1}$, then so is $Y \subset \mathbb{P}^N$.*

Proof. Consider the following diagram of vector spaces:

$$\begin{array}{ccc} H^0(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(1)) & \xrightarrow{f} & H^0(X, \mathscr{O}_X(1)) \\ \uparrow & & \uparrow \\ H^0(\mathbb{P}^{N-1}, \mathscr{O}_{\mathbb{P}^{N-1}}(1)) & \xrightarrow{\widetilde{f}} & H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(1)) \end{array} \tag{5.1.3}$$

By definition, since X_x is assumed to be smooth and linearly normal,

$$H^0(\mathbb{P}^{N-1}, \mathscr{O}_{\mathbb{P}^{N-1}}(1)) \cong H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(1)),$$

where the isomorphism is via $\widetilde{f}$, defined in (5.1.3). Since

$$\mathscr{O}_{\widetilde{X}}(1) = \beta^* \mathscr{O}_X(1)(-E),$$

it follows that

$$H^0(\widetilde{X}, \beta^* \mathscr{O}_X(1)(-E)) \cong H^0(X, \mathscr{I}_x(1)) \subset H^0(X, \mathscr{O}_X(1)),$$

where $\mathscr{I}_x$ is the space of global sections of $\mathscr{O}_X(1)$ vanishing at x. Moreover, the containment $H^0(\mathscr{I}_x) \subset H^0(X, \mathscr{O}_X(1))$ is also codimension 1. Since Y is smooth, f is injective. By rank-nullity, it follows that f is an isomorphism, and so Y is a smooth linearly normal surface in $\mathbb{P}^N$. ■

The lemmas we have just proved will support the conclusions of Theorem 5.1.1, which seeks to classify some other surfaces which exhibit near-minimal pinch point behavior. The concept of *near-minimal pinch point behavior* can be described thus:

Definition 5.1.1 *The surface $Y = \varphi(X) \subset \mathbb{P}^N$ is said to have* ***near-minimal pinch point number*** *δ_Y given by*

$$\delta_Y = \mathfrak{P}(\varphi) - 2N + 6.$$

Although technically the length of the pinch point scheme is a property of the map φ, it is helpful to refer to the near-minimal pinch point number δ_Y as a property of the surface $Y = \varphi(X)$, as long as there is no ambiguity (hence, the choice of notation).

The main result from Theorem 5.1.1 is that if N is sufficiently large, then Y can only be a ruled surface over a smooth curve of bounded genus. In the cases where N is relatively small, more complicated special cases arise. We present two of those cases here: If $\delta_Y = 2$, then Y is the Veronese surface in $\mathbb{P}^5$, and if $\delta_Y = 2N - 6$ for $4 \leq N \leq 9$, then Y is a del Pezzo surface, specifically $Y = D_N$, a del Pezzo surface of degree N. These results are summarized in the following table. Note that the number g_B represents the genus of the base curve for a ruled surface.

<table>
<tr><td>δ_Y</td><td colspan="7">N</td></tr>
<tr><td></td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td><td>$N \geq 10$</td></tr>
<tr><td>0</td><td colspan="7">Theorem 1.6.2: $S(a,b)$ with $a+b+1=N$</td></tr>
<tr><td>2</td><td>D_4</td><td>Φ_2</td><td colspan="5">Ruled surface with $g_B = 0$</td></tr>
<tr><td>4</td><td>-</td><td>D_5</td><td colspan="5">$g_B \in \{0,1\}$</td></tr>
<tr><td>6</td><td>-</td><td>-</td><td>D_6</td><td colspan="4">$g_B \in \{0,1\}$</td></tr>
<tr><td>8</td><td>-</td><td>-</td><td>-</td><td>D_7</td><td colspan="3">$g_B \in \{0,1,2\}$</td></tr>
<tr><td>10</td><td>-</td><td>-</td><td>-</td><td>-</td><td>D_8</td><td colspan="2">$g_B \in \{0,1,2\}$</td></tr>
<tr><td>12</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td><td>$D_9 = \Phi_3$</td><td>$g_B \in \{0,1,2,3\}$</td></tr>
</table>

Table 4: Surfaces Y in $\mathbb{P}^N$ with near minimal pinch point number δ_Y.

In order to make the theorem more concise, we write $\delta_Y = 2i$ for non-negative integers i, and we state and prove Theorem 5.1.1 in terms of i.

Theorem 5.1.1 *Maintain the context of Section 3.3. For any non-negative integer i, if $N \geq 3 + i \geq 4$, then $\delta_Y = 2i$ if and only if one of the following holds:*

1. *Y is ruled by lines.*

2. *$N = 5$, $i = 1$, and Y is the Veronese surface.*

3. *$4 \leq N \leq 9$, $i = N - 3$, and Y is a del Pezzo surface of degree N.*

Proof. Suppose $\mathfrak{P}(\varphi) = 2N - 6 + 2i$. If Y is ruled by lines, the theorem is immediately established, so we assume for the rest of this proof that Y is not ruled. Note in particular that $i = 0$ implies by Theorem 1.6.2 that Y is a rational normal scroll (and is hence ruled), so we can also assume that $i \geq 1$. Now, immediately from Theorem 4.3.1, we have that $\widetilde{\varphi}$ is again uncrumpled, and by Lemma 4.3.1,

$$\mathfrak{P}(\widetilde{\varphi}) = \mathfrak{P}(\varphi) - 4 = 2N - 10 + 2i = 2(N - 1) - 6 + 2(i - 1). \tag{5.1.4}$$

In other words, if Y is not ruled, then the inner projection construction has the following three effects:

1. $X_x \subset \mathbb{P}^{N-1}$ (the dimension of the ambient projective space decreases by one).

2. $\mathfrak{P}(\widetilde{\varphi}) = \mathfrak{P}(\varphi) - 4$.

3. If $\delta_Y = 2i$, then the near minimal pinch point number for X_x is

$$\delta_{X_x} = \delta_Y - 2 = 2(i - 1).$$

We now have two cases.

Case I. Suppose $N \geq 4 + i$. Suppose for induction that if

$$\mathfrak{P}(\varphi) = 2(N - 1) - 6 + 2(i - 1),$$

then either Y is ruled or Y is the Veronese surface $\Phi_2 \subset \mathbb{P}^5$. The base case is trivial; indeed, we have already established that when $i = 0$, Y is a rational normal scroll. We wish to establish the inductive step, which states that if

$$\mathfrak{P}(\varphi) = 2N - 6 + 2i,$$

then either Y is ruled or $Y = \Phi_2$. By Equation (5.1.4), the inductive hypothesis implies that either the surface X_x is ruled or $X_x = \Phi_2$. The latter is impossible since Φ_2 contains no lines of negative self-intersection. But if X_x is ruled, then the inductive step is given by Lemma 5.1.1 with $Y = \Phi_2$ precisely when $N = 5$ and $i = 1$.

Case II. Suppose $N = 3+i$. If $i = 1$, then $N = 4$, and therefore $\mathfrak{P}(\varphi) = 4$. By Equation (5.1.4), $\mathfrak{P}(\widetilde{\varphi}) = 0$, so $\widetilde{\varphi}$ is an unramified map from a smooth surface to $\mathbb{P}^3$. By Theorem 3.4.4, $\widetilde{X}$ embeds into $\mathbb{P}^3$ along $\widetilde{\varphi}$. Moreover, X_x contains a line L whose self intersection is $L^2 = -1$. We can use adjunction and the genus formula (see [6, Section 2.4]) to compute the degree $d = \deg X_x$. Adjunction on $X_x \subset \mathbb{P}^3$ gives $K_{\widetilde{X}} = (d-4)\zeta$. Since the genus of L is 0, the genus formula gives

$$0 = \frac{1}{2}(L^2 + d - 4) + 1.$$

Rearranging, we have

$$-1 = L^2 = 2 - d,$$

so $d = 3$. But then in the case where $i = 1$ and $N = 4$, we have that X_x must be a cubic del Pezzo surface.

Suppose for induction that if $\mathfrak{P}(\varphi) = 2(N-1) - 6 + 2(i-1)$, then Y is a del Pezzo surface of degree $N-1$. As in the previous case, we wish to establish the inductive step which states that if $\mathfrak{P}(\varphi) = 2N - 6 + 2i$ then Y is a del Pezzo surface of degree N.

By Equation (5.1.4) and the inductive hypothesis, X_x is precisely a del Pezzo surface of degree $N-1$. By a characterization of del Pezzo surfaces, X_x is a smooth linearly normal surface in $\mathbb{P}^{N-1}$ of degree $N-1$. But from Lemma 5.1.2, generic inner projection preserves linear normality. Since the degree of the generic inner projection of a surface is also one less than the original surface, we conclude that Y is a degree N linearly normal surface in $\mathbb{P}^N$. Thus, Y is a del Pezzo surface of degree N, with $4 \leq N \leq 9$. ■

Remark 5.1.1 By tracing through the computation in Lemma 4.4.1, we can gain a little more clarity in the cases where Y is ruled. Indeed, one can easily deduce the bound $g \leq \frac{i}{2}$ for the genus g of the base curve over which X is a projective bundle, limiting the complexity of the ruled surface Y.

5.2 Future Work

There are still several curiosities and future directions that arise from our problem, some of which we address here.

5.2.1 An even stronger classification

Our first curiosity comes from Theorem 5.1.1, which strengthens the classification result given in the main theorem. In fact, we now have a classification for surfaces of minimal pinch point number, and surfaces Y of near-minimal pinch point number with $\delta_Y = 2$. In Table 5.1, we see that only one item is missing for us to give a complete classification of surfaces of near-minimal pinch point number $\delta_Y = 4$. Concretely, we need to classify the surfaces in $\mathbb{P}^4$ which admit 4 pinch points under general linear projection to $\mathbb{P}^3$.

The technique used in the proof of Theorem 5.1.1 breaks down in this case. Indeed, since general inner projection reduces both the dimension of the ambient space by 1 and the near minimal pinch point number by 2, we notice that a surface in $\mathbb{P}^4$ with $\delta_Y = 4$ would map via inner projection to a surface in $\mathbb{P}^3$ still containing two pinch points! This presents a challenge, since the author is unaware of a classification of such surfaces in $\mathbb{P}^3$, and it is not clear whether one can carry out a proof by induction even if such a classification exists.

Looking even deeper, we notice that there are only two cases missing for when $\delta_Y = 6$, and it is reasonable to hope that one of these cases can be handled via an inductive argument referenced in the previous paragraph. In this case, we would need to search for surfaces in $\mathbb{P}^4$ with $\delta_Y = 6$, and again this introduces the need for a classification of surfaces in $\mathbb{P}^3$ admitting 4 pinch points. The questions with which we leave the reader are these:

1. For how many (even) numbers δ_Y can we give a complete classification of surfaces Y with near minimal pinch point number δ_Y?

2. For how many classes of surfaces can we fill out the information in Table 5.1?

3. How does the work done on surfaces of small class μ_2 (e.g. the papers by Gallarati and Marchionna, along with other work done since then) relate to the classification of surfaces of small type ν_2?

4. Can we give a lower bound on surface type based on the degree of the surface, in a similar fashion to that of the aforementioned papers?

Rather than extending the result "outward" to include more surfaces or a different type of bound, we can instead consider extending "deeper" to include a broader class of surface.

5.2.2 Relaxing one assumption

Recall that an uncrumpled map $\varphi : X \to Y \subset \mathbb{P}^N$ assumes that Y is non-degenerate and that φ is birational and finitely ramified. One might ask, what happens when we relax the assumption that φ is finitely ramified? If φ is still birational onto its image, then $\dim \mathrm{Ram}(\varphi) \leq 1$, since otherwise Y is a curve. We know what happens if $\mathrm{Ram}(\varphi)$ is finite, so we need only investigate the case where $R = \mathrm{Ram}(\varphi) \subset X$ is a curve.

The notion of $R \subset X$ a curve along which φ is ramified translates to one of two notions on Y:

1. If $\varphi(R) = y$, then $y \in Y$ is a double-point.

2. If $\dim \varphi(R) = 1$, then $\varphi(R)$ is a "ridge" consisting of pinch points in Y.

Note that if we restrict our attention to (non-degenerate) surfaces $Y \subset \mathbb{P}^N$ whose normalization is smooth, this omits the first case, since double-points are normal. Several questions arise.

- What does the number returned by Equation (1.6.1) mean in the context where $\dim \mathrm{Ram}(\varphi) = 1$?

- Can the pinch point formula return a negative number?

- Can every "ridged" surface can be deformed to one in which the ridge is resolved?

- More esoterically, how does our problem change when we relax the finite ramification constraint?

There is one more fun enumerative problem that only occurs for rational normal scrolls.

5.2.3 An interesting enumerative problem

A very different enumerative problem arises when we look closer into the boundary case (that is, general projections of rational normal scrolls). First, observe that the Hilbert scheme of finite sets of $2N-6$ points on X, denoted $\mathrm{Hilb}^{2N-6}(X)$, has dimension $4N-12$. This is precisely the same as

$$\dim \mathbb{G}(N-4, N) = ((N-4)+1)(N+1-(N-4)+1).$$

When X is a rational normal scroll $S(a,b) \subset \mathbb{P}^N$ with $N = a+b+1$, it admits $2N-6$ pinch points under general linear projection from a codimension 4 linear space. Then we get a map

$$\Psi : \mathbb{G}(N-4, N) \dashrightarrow \mathrm{Hilb}^{2N-6}(X),$$

given by $\Psi(\Lambda) = \mathrm{Ram}(\pi_\Lambda)$. A natural question we might ask is: What is the degree of Ψ? In particular, how does the answer depend on the splitting type (a and b) of X? Note that this question only makes sense in the context of rational normal scrolls. Indeed, as soon as we leave the boundary case to consider surfaces whose type exceeds the bound in Theorem 4.5.1, the dimensions of the Hilbert scheme and the corresponding Grassmannian no longer align.

There is one case for which we know the answer: When $N = 4$, $\deg \Psi = 1$, since two generic points on the cubic scroll $S(1, 2)$ uniquely determine a point of projection which realizes them as pinch points. For $N = 5$ however, we see a disparity between the two quartic scrolls $S(2, 2)$ and $S(1, 3)$: The directrix of $S(1, 3)$ is a line in $\mathbb{P}^5$ which meets *every* tangent plane to $S(1, 3)$, but which is nevertheless an invalid source of projection. Investigating higher dimensional spaces reveals even more invalid linear spaces which satisfy the appropriate enumerative properties. The invalid sources of projection cause an excess intersection problem when trying to execute the enumerative computation. This excess gets worse as N or the eccentricity (the difference between a and b) of the scroll increases.

Remark 5.2.1 *In the case where X is a rational normal curve and*

$$\Psi : \mathbb{G}(N-2, N) \to \mathrm{Hilb}^{2N-2},$$

it is well known that $\deg \Psi$ *is given by the N-th Catalan number,*

$$\deg \Psi = \frac{1}{N}\binom{2n-2}{n-1}.$$

Finally, we have a very natural, and impossibly massive generalization to make, inspired by the question: What happens in higher dimensions?

5.2.4 3-folds and beyond

Returning to the context of the larger problem (and returning to the notation of Chapter I), we have seen that for a smooth surface $X \subset \mathbb{P}^N$, its Gauss class can be written as

$$[\gamma(X)] = \gamma_{1,1} \cdot \sigma_{1,1}^* + \gamma_2 \cdot \sigma_2^*,$$

where $A_2(\mathbb{G}(2, N)) = \mathbb{Z}[\sigma_{1,1}^*, \sigma_2^*]$. The integer $\gamma_{1,1} = \nu_2 = \mathfrak{P}(\pi)$, where π is a projection map from a general codimension 4 linear space, and the integer $\gamma_2 = \mu_2$ is the degree of X^*. Theorem 1.6.2 completes the problem of minimizing the Gauss coefficients in the case where X is a surface.

For concreteness, we can consider the next case where $X \subset \mathbb{P}^N$ is a smooth irreducible projective variety of dimension *three*, with the intent to extend into higher dimensions. The setting of the problem now changes to the Chow group $A_3(\mathbb{G}(3,N)) = \mathbb{Z}[\sigma^*_{1,1,1}, \sigma^*_{2,1}, \sigma^*_3]$, wherein the Gauss class of X is given by

$$[\gamma(X)] = \gamma_{1,1,1} \cdot \sigma^*_{1,1,1} + \gamma_{2,1} \cdot \sigma^*_{2,1} + \gamma_3 \cdot \sigma^*_3.$$

This case is still largely unexplored. In Chapter I, we showed that $\gamma_1 = \mu_1$ for a smooth curve. For $n = 2$, the Gauss coefficients represented two of the projective characters of Severi. Here, we know that γ_3 still represents the degree of X^*, and in general, γ_n always represents the degree of X^* for $X^n \subset \mathbb{P}^N$ for $0 < n < N$. Similarly, $\gamma_{1,1,1}$ represents the (finite) number of pinch points in the image of X under general linear projection to $\mathbb{P}^5$, and again the pattern holds for $\gamma_{\mathbf{1}^n}$ representing the pinch point number for a general linear projection to $\mathbb{P}^{2n-1}$. On the other hand, the coefficients in between these extremes, such as $\gamma_{2,1}$ in the case where $\dim X = 3$, have a more complicated geometric meaning.

Extending the problem upwards into higher dimensions, we see that an entire family of classification problems emerges: For each of the generators $\sigma^*_{\mathbf{a}}$ of the Chow group $A_n(\mathbb{G}(n,N))$, where $\mathbf{a}$ is an integer partition of n, we may continue to ask the following questions:

I. What is the minimum value that $\gamma_{\mathbf{a}}$ can take?

II. Which smooth varieties $X \subset \mathbb{P}^N$ realize this minimum?

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APPENDICES

APPENDIX A: Chern Classes and the Euler Exact Sequence

This appendix contains a brief introduction to the Euler exact sequence, as well as some Chern class help.

0.0.1 The Euler exact sequence

We begin with a discussion surrounding the Euler exact sequence for projective space. Let V be an $(N+1)$-dimensional $\mathbb{K}$-vector space, and define as usual

$$\mathbb{P}^N = \mathbb{P}V = \frac{V \setminus \{\mathbf{0}\}}{\mathbb{K}^\times}.$$

Let $\mathcal{S} = \mathscr{O}_{\mathbb{P}^N}(-1)$ be the tautological bundle over $\mathbb{P}^N$, where the fiber over $p \in \mathbb{P}^N$ is the corresponding one-dimensional subspace $\lambda_p \subset V$. Then there is a natural inclusion of vector bundles

$$\mathcal{S} \hookrightarrow \mathscr{O}_{\mathbb{P}^N}^{N+1} = V \otimes \mathscr{O}_{\mathbb{P}^N}.$$

Let $\mathcal{Q}$ denote the quotient bundle, whose fibers over $p \in \mathbb{P}^N$ are V/λ_p. Then we have the following short exact sequence of vector bundles called the *tautological sequence*:

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathscr{O}_{\mathbb{P}^N}^{N+1} \longrightarrow \mathcal{Q} \longrightarrow 0. \tag{A.1}$$

From (A.1), we can define the *tangent bundle* as follows. We first take the tensor product of the sequence (A.1) with the bundle $\mathscr{O}_{\mathbb{P}^N}(1)$. This bundle is the dual to $\mathcal{S}$, often called Serre's twisting sheaf, and the tensor operation is usually called "twisting the sequence up by 1." In this way, we obtain a new sequence, the *Euler exact sequence*:

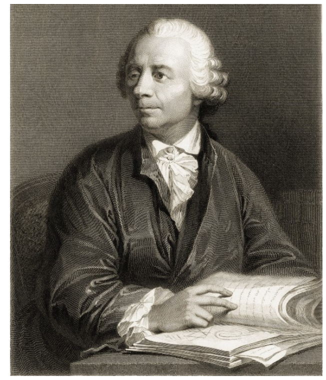
$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^N} \longrightarrow \mathscr{O}_{\mathbb{P}^N}(1)^{N+1} \longrightarrow \mathscr{O}_{\mathbb{P}^N}(1) \otimes \mathcal{Q} \longrightarrow 0.$$

By definition, the third bundle in the sequence is precisely $T_{\mathbb{P}^N}$, typically expressed as

$$T_{\mathbb{P}^N} = \mathrm{Hom}(\mathcal{S}, \mathcal{Q}).$$

Usually, one encounters the tangent bundle defined in terms of its exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^N} \longrightarrow \mathscr{O}_{\mathbb{P}^N}(1)^{N+1} \longrightarrow T_{\mathbb{P}^N} \longrightarrow 0, \qquad \text{(A.2)}$$



Leonhard Euler

(Apr 1707 – Sep 1783)

and if we take the dual of the sequence (A.2), we have the equivalent definition for the cotangent bundle (which is also often referred to as the Euler exact sequence):

$$0 \longrightarrow \Omega_{\mathbb{P}^N} \longrightarrow \mathscr{O}_{\mathbb{P}^N}(-1)^{N+1} \longrightarrow \mathscr{O}_{\mathbb{P}^N} \longrightarrow 0. \qquad \text{(A.3)}$$

0.0.2 Chern classes and Whitney's formula

Let X be a smooth variety, and let $\mathcal{V}$ be a rank r vector bundle on X. In this situation, one can discuss the Chern classes of $\mathcal{V}$. We begin with a definition of the degeneracy locus of a general set of global sections of $\mathcal{V}$.

Definition 0.0.1 *For $1 \leq i \leq r$, we define the* ***degeneracy locus*** *of $r - i$ generally chosen global sections of $\mathcal{V}$ to be the subscheme D of X where they fail to be linearly independent.*

Note that if D is the degeneracy locus of $r - i$ global sections of $\mathcal{V}$, then $\operatorname{codim} D \leq i$, and if all sections belong to a subspace of $H^0(X, \mathcal{V})$ which generates $\mathcal{V}$, then $\operatorname{codim} D = i$ and D is generically reduced. In the event where $\operatorname{codim} D = i$, the *$i$-th Chern class of $\mathcal{V}$*, denoted $c_i(\mathcal{V})$, is exactly $D \in A^i(X)$. The reader may take this to be the definition of the i-th Chern class of $\mathcal{V}$.

Definition 0.0.2 *We define the* ***total Chern class*** *$c(\mathcal{V}) \in A(X)$ to be the unique class given by*

$$c(\mathcal{V}) = 1 + c_1(\mathcal{V}) + c_2(\mathcal{V}) + \cdots + c_r(\mathcal{V}).$$

Note that if $\varphi : Y \to X$ is a regular map on smooth varieties, then the total Chern class of a bundle on the target variety respects the pullback operation:

$$\varphi^* c(\mathcal{V}) = c(\varphi^* \mathcal{V}).$$

Given an exact sequence of vector bundles on X, then the corresponding total Chern classes obey a very satisfying relationship, often called the *Whitney Sum Formula* (or just "Whitney's formula"):

Theorem 0.0.1 (Whitney Sum Formula) *Suppose*

$$0 \longrightarrow \mathcal{V}' \longrightarrow \mathcal{V} \longrightarrow \mathcal{V}'' \longrightarrow 0$$

is a short exact sequence of vector bundles on X. Then

$$c(\mathcal{V}) = c(\mathcal{V}') \cdot c(\mathcal{V}'').$$

The reason for the appearance of the word "Sum" has to do with the specific case where $\mathcal{V}$ splits as the direct sum

$$\mathcal{V} = \mathcal{V}' \oplus \mathcal{V}'',$$

and the bundle maps in the corresponding exact sequence are inclusion into the first factor and projection onto the second.

Definition 0.0.3 *The* ***canonical bundle*** ω_X *on a smooth variety X of dimension n is the top wedge of the cotangent bundle:*

$$\omega_x := \bigwedge^n \Omega_X.$$

The ***canonical class*** K_X *is the first Chern class of the canonical bundle:*

$$K_X := c_1(\omega_X).$$

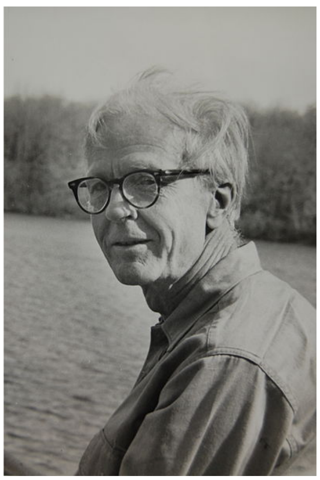
Example 0.0.1 It is well known that the canonical class of projective space is

$$K_{\mathbb{P}^N} = -(N+1)\zeta,$$

where ζ is the hyperplane class, and $\zeta = c_1(\mathscr{O}_{\mathbb{P}^N}(1))$. Applying Whitney's formula to the Euler sequence (A.3), we have

$$c(\Omega_{\mathbb{P}^N}) = c(\mathscr{O}_{\mathbb{P}^N}(1)) = (1-\zeta)^{N+1}.$$

This allows us to conclude that $c_1(\Omega_{\mathbb{P}^N}) = -(N+1)\zeta = K_{\mathbb{P}^N}$. A natural question arises: Is it always true that $c_1(\Omega_X) = K_X$? The answer is yes; we shall justify this below.



Hassler Whitney

(Mar 1907 – May 1989)

One can speak of the *total Chern polynomial* corresponding to $\mathcal{V}$, a tool for simplifying applications of the splitting principle. The total Chern polynomial is defined as the class

$$c_z(\mathcal{V}) := \sum_{i=0}^{r} c_i(\mathcal{V}) z^i \in A(X)[z].$$

Here, z is treated as a formal variable, and $c_z(\mathcal{V})$ is completely determined by $c(\mathcal{V})$. We then consider the *Chern roots* of $\mathcal{V}$; these are classes α_i for which we can write

$$c_z(\mathcal{V}) = \prod_{i=1}^{r}(1+\alpha_i z),$$

essentially allowing the total Chern polynomial to split. In particular, note that the Chern classes of $\mathcal{V}$ can be expressed as

$$c_i(\mathcal{V}) = \sigma_i(\alpha_1, \ldots, \alpha_r),$$

where σ_i is the i-th symmetric function. This is a more literal interpretation of the splitting principle (see [6] for a more rigorous construction), which allows us to easily write down formulas for the Chern classes resulting from applying different operations to vector bundles.

Example 0.0.2 We wish to compute the Chern classes of the k-th wedge power of a rank r bundle $\mathcal{V}$ on X for a fixed integer $1 \leq k \leq r$. Let $\alpha_1, \ldots, \alpha_r$ be the Chern roots of $\mathcal{V}$. The k-th wedge power of a vector space V admits a basis whose elements are in bijection with the size k subsets of a basis for V. The concept extends to the family of vector spaces defined by $\mathcal{V}$, and so the Chern roots of $\bigwedge^k \mathcal{V}$ are

$$\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_k},$$

where $1 \leq i_1 < i_2 < \cdots < i_k \leq r$. In particular, note that for the *determinant line bundle*

$$\det \mathcal{V} := \bigwedge^r \mathcal{V},$$

the first Chern class

$$c_1(\det \mathcal{V}) = \sum_{i=1}^{r} \alpha_i = c_1(\mathcal{V}),$$

by Whitney's formula. This also verifies the identity from Example 0.0.1 that $K_{\mathbb{P}^N} = c_1(\Omega_{\mathbb{P}^N})$.

0.0.3 Relative Euler sequences

We now extend the concept of an Euler sequence to a *family* of projective spaces. Specifically, given a vector bundle, one can investigate its *projectivization*, a construction which is analogous to the construction of the projectivization of a vector space (It is in fact more than an analogy; the projectivization of a vector space can be seen as the projectivization of the rank $N+1$ trivial bundle $\rho : \mathscr{O}^{N+1}_{\operatorname{Spec}\mathbb{K}} \to \operatorname{Spec}\mathbb{K}$).

The construction proceeds as follows. Let $\mathcal{V}$ be a rank $r+1$ vector bundle over a smooth variety X, and let $\rho : \mathbb{P}\mathcal{V} \to X$ be its projectivization, where $\mathbb{P}\mathcal{V} = \operatorname{Proj}(\operatorname{Sym}\mathcal{V}^*)$. Geometrically, we remove the zero section of $\mathcal{V}$ and take the quotient by common scaling, so that the fibers $\mathbb{P}\mathcal{V}_x$ are themselves the projective spaces $\mathbb{P}(\mathcal{V}_x)$. That is, the closed points of $\mathbb{P}\mathcal{V}$ are pairs (x, p) with $x \in X$ and $p \in \mathbb{P}(\mathcal{V}_x)$.

Morally speaking, since the fibers of ρ are projective spaces, each one has its own tautological sequence and corresponding tangent bundle, and these glue together systematically within the tangent bundle $T_{\mathbb{P}\mathcal{V}}$. More formally, for any point $x \in X$, let $\mathcal{V}_x$ be the fiber in $\mathcal{V}$ over x, an $(r+1)$-dimensional vector space, so that $\mathbb{P}\mathcal{V}_x \cong \mathbb{P}^r$. From the tautological sequence (A.1) on each fiber $\mathcal{V}_x$, we have

$$0 \longrightarrow \mathcal{S}_x \longrightarrow \mathscr{O}_{\mathbb{P}\mathcal{V}_x}^{r+1} \longrightarrow \mathcal{Q}_x \longrightarrow 0,$$

where $\mathcal{Q}_x$ is the corresponding quotient bundle. We then have the tangent bundle over the point x given by

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}\mathcal{V}_x} \longrightarrow \mathscr{O}_{\mathbb{P}\mathcal{V}_x}(1)^r \longrightarrow T_{\mathbb{P}\mathcal{V}_x} \longrightarrow 0.$$

As we vary $x \in X$, the tangent bundles vary accordingly, yielding the *relative tangent bundle* $T_{\mathbb{P}\mathcal{V}/X}$. When there is no ambiguity, we will denote the relative tangent bundle T_ρ.

There are two ways through which we can gain access to the bundle T_ρ, both of which involve situating it in an exact sequence. The first strategy is to appeal to the *relative Euler exact sequence*. We begin with the tautological sequence

$$0 \longrightarrow \mathcal{S}_{\mathbb{P}\mathcal{V}} \longrightarrow \rho^*\mathcal{V} \longrightarrow \mathcal{Q}_{\mathbb{P}\mathcal{V}} \longrightarrow 0,$$

where $\mathcal{S}_{\mathbb{P}\mathcal{V}} = \mathscr{O}_{\mathbb{P}\mathcal{V}}(-1)$ is the tautological bundle over the whole $\mathbb{P}\mathcal{V}$ (whose points are pairs of the form (x, p), with $x \in X$ and $\lambda_p \subset \mathcal{V}_x$), and fibers over $\mathcal{Q}_{\mathbb{P}\mathcal{V}}$ are quotients of the form $\mathcal{V}_x/\lambda_p$. We then twist by $\mathscr{O}_{\mathbb{P}\mathcal{V}}(1)$ to obtain

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}\mathcal{V}} \longrightarrow \rho^*\mathcal{V} \otimes \mathscr{O}_{\mathbb{P}\mathcal{V}}(1) \longrightarrow T_{\mathbb{P}\mathcal{V}/X} \longrightarrow 0, \tag{A.4}$$

which is the *relative Euler sequence.* Dually, one can express the relative Euler sequence as the dual to (A.4),

$$0 \longrightarrow \Omega_\rho \longrightarrow \rho^*\mathcal{V} \otimes \mathcal{S} \longrightarrow \mathscr{O}_{\mathbb{P}\mathcal{V}} \longrightarrow 0,$$

where Ω_ρ is the *relative cotangent bundle.*

The second way we access the relative tangent bundle is through its more abstract definition. The bundle map ρ is a smooth map between varieties; therefore, the tangent bundle T_X admits a pullback along ρ to a bundle on $\mathbb{P}\mathcal{V}$. The derivative of ρ, denoted $d\rho : T_{\mathbb{P}\mathcal{V}} \to \rho^*T_X$, is the induced map on vector bundles. The derivative is always surjective, and its kernel is naturally a sub-bundle of $T_{\mathbb{P}\mathcal{V}}$. This is, in fact, the *definition* of the relative tangent bundle,

$$T_\rho = \ker d\rho,$$

and we have just described its defining exact sequence

$$0 \longrightarrow T_\rho \longrightarrow T_{\mathbb{P}\mathcal{V}} \longrightarrow \rho^* T_X \longrightarrow 0. \tag{A.5}$$

As with the Euler sequence (A.3), we can dualize the sequence (A.5) to obtain the defining bundle sequence for the relative cotangent bundle Ω_ρ,

$$0 \longrightarrow \rho^*\Omega_X \longrightarrow \Omega_{\mathbb{P}\mathcal{V}} \longrightarrow \Omega_\rho \longrightarrow 0.$$

One particularly attractive situation to which we can apply Whitney's formula is the relative Euler sequence (A.4) for a projective bundle on the smooth variety X. So doing returns a nice combinatorial formula for the Chern classes of the relative tangent bundle T_ρ. Indeed, we can apply the splitting principle to compute

$$c(\rho^*\mathcal{V} \otimes \mathscr{O}_{\mathbb{P}\mathcal{V}}(1)) = c(T_\rho),$$

and then collect like-terms. This process yields the following result (see [6, Theorem 11.4]):

Theorem 0.0.2 *Let $\rho : \mathbb{P}\mathcal{V} \to X$ be a rank r projective bundle over a smooth variety X. Then the k-th Chern class $c_k(T_\rho)$ is given by*

$$c_k(T_\rho) = \sum_{j=0}^{k} \binom{r+1-j}{k-j} \rho^* c_j(\mathcal{V}) \zeta^{k-j},$$

where $\zeta = c_1(\mathscr{O}_{\mathbb{P}\mathcal{V}}(1))$.

We wish to conclude this section by giving one last characterization of the Chern classes of a vector bundle $\mathcal{V}$ over X through its projectivization $\mathbb{P}\mathcal{V}$. Observe that the dual bundle to $\mathcal{S}_{\mathbb{P}\mathcal{V}}$, namely the line bundle $\mathscr{O}_{\mathbb{P}\mathcal{V}}(1)$, has a non-trivial global section. Therefore, the first Chern class $\zeta = c_1(\mathscr{O}_{\mathbb{P}\mathcal{V}}(1))$ is given by the difference of the classes corresponding to the zero locus and pole locus for a general rational section. This is a construction with which we assume the reader is familiar; note that this locus is invariant under the choice of global section. Symbolically, we write

$$\zeta = c_1(\mathscr{O}_{\mathbb{P}\mathcal{V}}(1)) = [\text{zeros}] - [\text{poles}].$$

Since the construction in the previous paragraph is intrinsic to $\mathcal{V}$, we are presented with an alternative regarding the Chern classes of $\mathcal{V}$:

$\zeta \in A(\mathbb{P}\mathcal{V})$ is integral over $A(X)$, satisfying the unique monic polynomial

$$f(\zeta) = \zeta^r + \rho^* c_1(\mathcal{V})\zeta^{r-1} + \cdots + \rho^* c_r(\mathcal{V}).$$

Indeed, it is true (see [6, Theorem 5.9]) that the pullback $\rho^* : A(X) \to A(\mathbb{P}\mathcal{V})$ is injective, and so we can view $A(X)$ as a subset of $A(\mathbb{P}\mathcal{V})$. The unique coefficients $\rho^* c_i(\mathcal{V})$ in the monic polynomial $f(\zeta)$ translate to unique elements of $A(X)$ which we define to be the Chern classes of $\mathcal{V}$. In fact, the Chow rings $A(\mathbb{P}\mathcal{V})$ and $A(X)$ obey the relationship

$$A(\mathbb{P}\mathcal{V}) = \frac{A(X)[\zeta]}{(f(\zeta))}.$$

In fact, it was a great insight due to Grothendieck to use this construction to *define* the Chern classes of a vector bundle. Not only does it extend the concept of Chern classes to include vector bundles over *singular* varieties X, but also it extends to vector bundles which do not have enough global sections for the degeneracy loci to exhibit the proper codimension.

APPENDIX B: Uncrumpled maps and normalization

One interesting property of uncrumpled maps $\varphi : X \to \mathbb{P}^N$ (see Definition 3.2.1) is that the map φ turns out to be the normalization of the image $Y = \varphi(X)$. In this sense, we relax the constraint that we work with smooth projective surfaces X which admit an *embedding* into $\mathbb{P}^N$ to the constraint that we work with surfaces $Y \subset \mathbb{P}^N$ whose normalization is smooth (with the exception that if the normalization map is not finitely ramified, Y will have an entire curve along which it is pinched).

In this Appendix, we explore these ideas a little more. We will see that the property that X is the normalization of Y will follow from Zariski's Main Theorem, a version of which we begin by stating. Note that this is the version found in [21, Chapter 6], and it is equivalent to the property that the variety X is normal at x.

Theorem 0.0.1 (Zariski's Main Theorem) *Let $f : Z \to X$ be a birational morphism of finite type with $f^{-1}(x)$ finite. Then there exists an open $x \in U \subset X$ such that the restriction*

$$\operatorname{res} f : f^{-1}(U) \to U$$

is an isomorphism.

Next, we present the universal property of normalization, as outlined in [11, Proposition 12.44]. Heuristically, it states that dominant maps from normal varieties are functorial, in the sense that they factor through the normalization of the target. Stated formally:

Lemma 0.0.1 (Universal Property of Normalization) *Let $n : Y' \to Y$ be the normalization of a variety Y. For every integral and normal scheme Z and every dominant morphism $f : Z \to Y$, there exists a unique morphism $g : Z \to Y'$ such that $n \circ g = f$, so the following diagram commutes:*

$$\begin{tikzcd} Z \arrow[rr, "g"] \arrow[dr, "f"'] & & Y' \arrow[dl, "n"] \\ & Y & \end{tikzcd} \tag{B.1}$$

From this property, we can deduce via Zariski's Main Theorem a second characterization of the normalization map. Note that an arrow in the diagram below points in the reverse direction as the corresponding arrow in Lemma 0.0.1.

Lemma 0.0.2 (Universal Property of Normalization) *Let $n : Y' \to Y$ be the normalization of a variety Y. Then for every finite birational morphism $f : Z \to Y$, there is a unique morphism $g : Y' \to Z$ such that the following diagram commutes.*

$$\begin{tikzcd} Z \arrow[dr, "f"'] & & Y' \arrow[ll, "g"'] \arrow[dl, "n"] \\ & Y & \end{tikzcd} \tag{B.2}$$

Proof. Let $f : Z \to Y$ be a finite birational morphism, and let $n_Z : Z' \to Z$ and $n_Y : Y' \to Y$ be the normalizations of Z and Y respectively. Since F is birational, $f \circ n_Z$ is a dominant map from a normal variety to Y. Let $g : Z' \to Y'$ be the unique map induced by the universal property given in Lemma 0.0.1 applied to $f \circ n_Z$. We then have the following commuting diagram,

$$\begin{tikzcd} Z' \arrow[r, "g"] \arrow[d, "n_Z"] & Y' \arrow[d, "n_Y"] \\ Z \arrow[r, "f"] & Y \end{tikzcd} \tag{B.3}$$

with the relationship

$$n_Y \circ g = f \circ n_Z.$$

Now, both normalization maps are finite and birational, and since f is assumed so, it follows that g is also finite and birational. Observe that Y' is normal at each point $y \in Y'$, so by Zariski's Main Theorem, g restricts to an isomorphism on each open set in an open cover of Y'. Therefore,

$$Z' \cong Y'$$

via the unique isomorphism g. The proposed universal property follows with

$$n_Z \circ g^{-1} : Y' \to Z$$

the asserted unique birational morphism. ■

We can use Lemmas 0.0.1 and 0.0.2 to prove the desired proerty of uncrumpled maps.

Proposition 0.0.1 *If $\varphi : X \to Y \subset \mathbb{P}^N$ is an uncrumpled map on a smooth surface X, then X is the normalization of Y, and φ is the normalization map.*

Proof. We first show that φ is finite or equivalently, that φ is both quasi-finite and proper. It is clear that φ is quasi-finite, since $\mathrm{Ram}(\varphi)$ is finite. Indeed, if the fiber over a point in Y were in fact a curve, then the tangent space at every point in the fiber is contracted under $d\varphi$, so $\mathrm{Ram}(\varphi)$ would be infinite. That φ is proper follows from the corresponding property on X.

Hence, φ is a finite morphism, and by assumption, φ is a birational map from X to Y. Now, let $n_Y : Y' \to Y$ be the normalization of Y. We simultaneously apply Lemmas 0.0.1 and 0.0.2 to obtain the following commuting diagram:

$$\begin{tikzcd} X \arrow[rr, bend left, "g"] \arrow[dr, "\varphi"'] & & Y' \arrow[ll, bend left, "f"] \arrow[dl, "n_Y"] \\ & Y & \end{tikzcd} \tag{B.4}$$

Moreover, by Lemma 0.0.1, since φ is dominant and X is normal, $\varphi = n \circ g$, and since φ is finite and birational, Lemma 0.0.2 implies that $n = \varphi \circ f$. But then

$$\varphi = (\varphi \circ f) \circ g \implies f \circ g = id_Y,$$

and

$$n = (n \circ g) \circ f \implies g \circ f = id_X.$$

Thus, since f and g are unique, X is uniquely isomorphic to Y'. ■

Using the fact that X is the normalization of Y with normalization map φ, we can prove Lemma 3.4.2, stated here as

Lemma 0.0.3 *Let $\varphi : X \to \mathbb{P}^N$ be an uncrumpled map on a smooth surface X with $N \geq 4$. If Y is ruled by lines, then X is a $\mathbb{P}^1$-bundle over a smooth curve, and φ maps the rulings to lines.*

Proof. Let $F_1(Y) \subset \mathbb{G}(1, N)$ be the Fano scheme of Y, and choose B to be an irreducible component of $F_1(Y)$. Next, let $n_Y : Y' \to Y$ be the normalization of Y, and let $\alpha : B \to F_1(Y)$ be the embedding of B into the Grassmannian. Let $n_B : B' \to B$ be the normalization of B (since B is not necessarily smooth). Finally, let

$$\mathcal{U} := \mathbb{P}\mathscr{O}_{\mathbb{G}(1,N)}(-1) = \{(p, q) \in \mathbb{P}^N \times \mathbb{G}(1, N) \mid p \in \Lambda_q\}$$

be the universal line in $\mathbb{P}^N$, where Λ_q is the line in $\mathbb{P}^N$ corresponding to the point $q \in \mathbb{G}(1, N)$, and $\mathcal{U}$ has projection maps $\rho_1 : \mathcal{U} \to \mathbb{P}^N$ and $\rho_2 : \mathcal{U} \to \mathbb{G}(1, N)$. This construction is summarized in the diagram (B.5).

$$\begin{tikzcd}
 & & & Y' \arrow[d, "n_Y"'] & \\
n_B^*\alpha^*\mathcal{U} \arrow[r, "\pi_B"] \arrow[d] \arrow[urrr, dashed, bend left, "g"] & \alpha^*\mathcal{U} \arrow[r, "\pi_\alpha"] \arrow[d] & \mathcal{U} \arrow[r, "\rho_1"] \arrow[d, "\rho_2"] & Y & X \arrow[l, "\varphi"] \arrow[ul, "\sim", "f"'] \\
B' \arrow[r, "n_B"] & B \arrow[r, "\alpha"] & F_1(Y) & &
\end{tikzcd} \tag{B.5}$$

The maps π_B and π_α denote the bundle maps induced by the definition of the pullback of a projective bundle, and f is the unique isomorphism given by Proposition 0.0.1 identifying X as the normalization of Y. We will show that there exists a unique isomorphism

$$g : n_B^*\alpha^*\mathcal{U} \to Y'.$$

Since Y is ruled, $\dim B \geq 1$. In fact, $\dim B = 1$, because otherwise either Y contains a double-point or Y is a degenerate 2-plane. The maps α and n_B are both finite and birational by construction, and these properties are inherited by the respective bundle maps π_α and

π_B. When $\mathcal{U}$ is viewed as a projective bundle, it is in fact a $\mathbb{P}^1$-bundle over $\mathbb{G}(1, N)$ with bundle map ρ_2.

Moreover, we claim that $\rho_1 : \rho_2^{-1}(F_1(Y)) \to Y$ is finite and birational onto Y. That ρ_1 is quasi-finite follows from the assumption that through no point in Y pass infinitely many lines, since such a point would be a double-point on Y, contradicting the smoothness assumption on X. Since $\rho_2^{-1}(F_1(Y))$ is projective, ρ_1 is proper and hence finite.

It remains to show that $\deg \rho_1 = 1$. Suppose for contradiction that $\deg \rho_1 \geq 2$. Fix a general point $y \in Y$. Through y there pass at least two lines, say L and M. Since a second line passes through each point in L, there is a family of lines parameterized by L which sweeps out all of Y, and similarly Y is swept out by lines meeting M. Therefore, if we fix a second general point $y' \in Y$, we can find lines L' and M' containing y' and meeting L and M respectively. If either contains the general point y then y is a cone point and Y is a 2-plane. Otherwise, through a general point of M there passes a line through L which is distinct from M, namely the line of intersection of the 2-planes

$$\langle L, M \rangle \cap \langle L', M' \rangle.$$

In this case we have again that a general point of Y is contained in the plane $\langle L, M \rangle$, hence Y is that degenerate 2-plane.

Thus, the composite

$$\rho_1 \circ \pi_\alpha \circ \pi_B : n_B^* \alpha^* \mathcal{U} \to Y$$

is a finite birational morphism from a smooth (and hence normal) surface to Y. By Proposition 0.0.1, there is a unique isomorphism $G : n_B^* \alpha^* \mathcal{U} \to Y'$. Hence, $X \cong n_B^* \alpha^* \mathcal{U}$ via the unique isomorphism $f \circ g^{-1}$, realizing X as a projective bundle over the smooth curve B. Moreover, the composite $\rho_1 \circ \pi_\alpha \circ \pi_B$ maps projective bundle fibers to the rulings of Y. ■

Remark 0.0.2 It may be troubling to see that a doubly-ruled surface does not violate the proposition above. Consider, as a concrete example, Y is the quadric surface in $\mathbb{P}^3$. Because we chose the base curve B over which X is ruled to be an *irreducible component* of $F_1(Y)$, we have specified a particular projective bundle map. The Fano variety of the quadric surface contains two irreducible components, but $\mathbb{P}^1 \times \mathbb{P}^1$ can be realized as a $\mathbb{P}^1$-bundle over either component.

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