

Domain wall dynamics in classical spin chains: free propagation, subdiffusive spreading, and topological soliton emission

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The non-equilibrium dynamics of domain wall initial states in a classical anisotropic Heisenberg chain exhibits a striking coexistence of apparently linear and non-linear behaviours: the propagation and spreading of the domain wall can be captured quantitatively by *linear*, i.e. non-interacting, spin wave theory absent its usual justifications; while, simultaneously, for a wide range of easy-plane anisotropies, emission can take place of stable topological solitons—a process and objects intrinsically associated with interactions and non-linearities. The easy-axis domain wall only has transient dynamics, the isotropic one broadens diffusively, while the easy-plane one yields a pair of ballistically counter-propagating domain walls which, unusually, broaden *subdiffusively*, their width scaling as $t^{1/3}$.

Introduction— A prototypical setting for non-equilibrium dynamics is an initial state with two neighbouring regions in different stationary states of the same Hamiltonian. A single sharp domain wall between distinct stationary states can move and spread, carrying energy – and, possibly, other conserved quantities. This type of dynamics has been studied in many contexts, including the spin-1/2 quantum XXZ chain [1–33], other quantum spin chains [30, 34–38], quantum field theories [39–41], the continuum Landau-Lifshitz model of classical spin densities [30, 42], two-dimensional quantum systems [43, 44], and the simple exclusion process [45].

Here, we examine this setting in a *classical* one-dimensional anisotropic (XXZ) Heisenberg chain. This is, arguably, the simplest incarnation of this problem. As we show below, it is highly tractable, and opens a complementary perspective on the much studied related problems of quantum Heisenberg chains, where signatures of interesting phenomena such as KPZ scaling [25, 46] have been experimentally observed [47, 48].

One surprising finding is that we can describe the domain wall dynamics *quantitatively* using *linear* spin wave theory across a wide range of parameters. At the same time, we also find emission of (possibly multiple) topological solitons, which can only be generated by non-linear scattering processes, and which must be stabilised by non-linear interactions.

We find qualitatively distinct behaviour in the easy-axis, isotropic, and easy-plane cases. Our results and setup are summarised in Fig. 1. For easy-plane anisotropy, the domain wall splits into two ballistically counter-propagating ones (Fig. 1(c)). Since the Hamiltonian is non-integrable and intrinsically non-linear, and since the propagating domain walls have high energy compared to the background, they can, in principle, emit or decay into other excitations — giving the non-equilibrium setup an inherent non-linear flavour. It is therefore all the more surprising that, over the entire range of easy-plane

anisotropy $\Delta \in [0, 1)$, domain walls propagate ballistically. This is reminiscent of the behaviour of quasiparticles in integrable systems [14, 30, 49–59], or that of operator spreading [60–62]. For the latter, ballistic behaviour is accompanied by diffusive broadening [60–62]. More generally, broadening in interacting many-body systems is typically diffusive, with exceptions usually associated with integrability or a lack of interactions.

In sharp contrast to this expectation, we show that the propagating domain walls broaden subdiffusively, as $\sim t^{1/3}$, in the entire easy-plane regime $\Delta \in [0, 1)$. We find that the propagation speed, profile, and $t^{1/3}$ scaling can be quantitatively obtained from *linear* spin wave theory. At the same time, above a critical angle $\varphi_c(\Delta)$ between the domains separated by the propagating domain wall (Fig. 1(b)), the linear behaviour of the propagating domain walls coexists with the aforementioned, inherently *non-linear* feature of the emission of topological solitons. We provide a heuristic picture for all of these processes.

At the isotropic Heisenberg point $\Delta = 1$, the domain walls can no longer propagate, and the subdiffusive spreading gives way to a diffusive melting of the original domain wall (Fig. 1(c)). Nor can the domain walls propagate in the easy-axis case ($\Delta > 1$), where the melting is fully arrested and a static soliton is approached asymptotically (Fig. 1(c)).

The behaviour for $\Delta \geq 1$ is analogous to that known for quantum spin- $\frac{1}{2}$ chains [3, 42] – a classical-quantum analogy which is, in itself, remarkable. By contrast, the $t^{1/3}$ broadening we find over the entire range $0 \leq \Delta < 1$ appears, in the quantum spin- $\frac{1}{2}$ case, only at the $\Delta = 0$ point [2, 9, 10, 12], being associated with the non-interacting (free-fermion) nature of this case. The existence and emission of the solitons have, to the best of our knowledge, not been previously observed – either in the quantum model or in a corresponding continuum Landau-Lifshitz model.

In the following, we provide details for these claims,

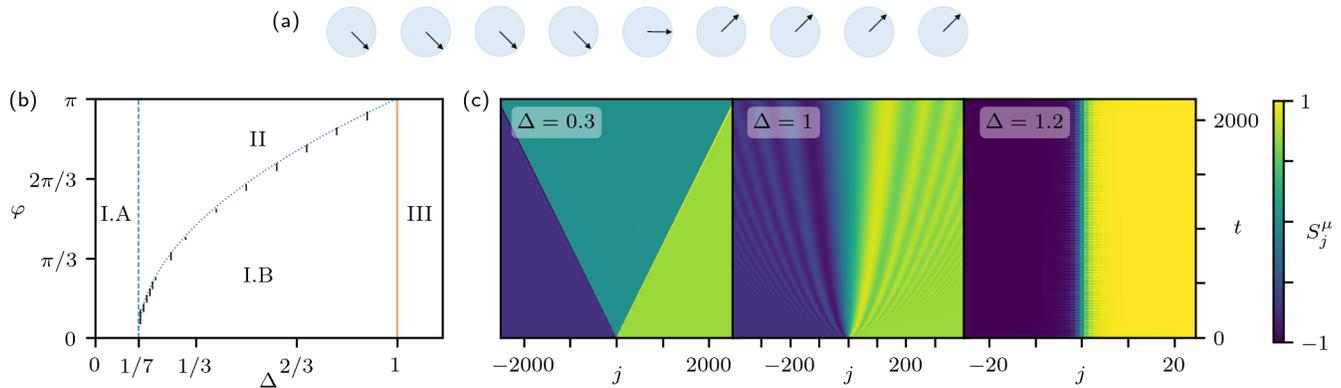


FIG. 1. (a) Schematic of the initial conditions (3), in the xy -plane, shown for $\varphi = \pi/2$. (b) Boundaries between the different dynamical regimes as a function of the anisotropy Δ and amplitude φ . I.A and I.B are the two linear regimes (distinguished by whether the oscillations are behind or ahead of the counter-propagating domain walls, respectively), where all of the dynamical features are well-described by linear spin wave theory; II is the easy-plane non-linear regime, where topological solitons coexist with the spreading domain walls. III is the easy-axis regime, with a single, static domain wall. The isotropic point $\Delta = 1$ corresponds to the transition between I.B and III, and is well-described by linear spin wave theory with a single, diffusively broadening domain wall. The vertical bars denote the uncertainty in determining the transition between I.B and II from the simulations. (c) Overview of the domain wall dynamics shown for the easy-plane $\Delta = 0.3$, the isotropic point $\Delta = 1$, and the easy-axis $\Delta = 1.2$, respectively. Note the different ranges of the time axis. $\varphi = \pi/2$ for the easy-plane and the isotropic case, where S^y is plotted. S^z is plotted for the easy-axis case, with $\varphi = \pi$. Ballistic counter-propagation is observed in the easy-plane, diffusive melting of the original domain wall is seen at the isotropic point, whilst the easy-axis approaches a very narrow static soliton.

and conclude with a discussion of the broader import of this work.

Model— We consider the classical XYZ spin chain,

$$H = - \sum_i (J_x S_i^x S_{i+1}^x + J_y S_i^y S_{i+1}^y + J_z S_i^z S_{i+1}^z). \quad (1)$$

where the $\mathbf{S}_i \in S^2$ are classical $O(3)$ vectors at sites i of a chain. We assume the coupling constants $J_\mu \geq 0$ are all ferromagnetic. The dynamics is given by the classical equations of motion,

$$\dot{S}_i^\mu = -\epsilon^{\mu\nu\lambda} J_\nu (S_{i+1}^\nu + S_{i-1}^\nu) S_i^\lambda, \quad (2)$$

which follow from the fundamental Poisson brackets $\{S_i^\mu, S_j^\nu\} = \delta_{ij} \epsilon^{\mu\nu\lambda} S_j^\lambda$.

We set $J_x = J_y = 1$ (which implicitly defines all units), and $0 \leq J_z = \Delta$. The XY-point $\Delta = 0$ corresponds to the free-fermion limit of the quantum spin- $\frac{1}{2}$ chain, but is, in the classical case, an interacting model.

Easy-plane, $\Delta < 1$ — We consider a sharp domain wall in the in-plane components as the initial condition,

$$\mathbf{S}_i = \cos(\varphi/2) \hat{\mathbf{x}} + \text{sgn}(i) \sin(\varphi/2) \hat{\mathbf{y}} \quad (3)$$

for some amplitude φ that sets the magnetisation jump across the domain wall as illustrated in Fig. 1(a). The $O(2)$ isotropy implies that any choice of φ connects two easy-plane ground states. We set the spin at $i = 0$ to lie halfway between the two domains, $\mathbf{S}_{i=0} = \hat{\mathbf{x}}$, to ensure that there is some dynamical evolution, even if $\varphi = \pi$.

Numerically integrating the equations of motion (2) with initial conditions (3) and open boundaries reveals that two counter-propagating domain walls immediately emerge from $i = 0$: a left-moving one connecting the $(-)$ -domain to the expanding $\hat{\mathbf{x}}$ -domain; and a right-moving one connecting the $\hat{\mathbf{x}}$ -domain to the $(+)$ -domain as seen in Fig. 1(c).

The size of the $\hat{\mathbf{x}}$ -domain grows linearly with time, implying ballistic domain wall motion. The underlying velocity $c = \sqrt{2(1 - \Delta)}$ does not differ measurably from the group velocity of long-wavelength spin waves, despite the non-linearity of the equations of motion.

To investigate the long-time dynamics of the domain wall numerically, we switch to its co-moving frame [63]. We then find, numerically, that this easy-plane dynamics exhibits three qualitatively distinct regimes, cf. Fig. 1(b): two linear regimes, I.A and I.B, so-called because they are well-described by *linear* spin wave theory in their entirety; and a non-linear regime II characterised by an instability to the emission of topological solitons.

Within the linear regime we find, in addition to the ballistic motion of the domain walls, a sub-diffusive spreading, with their width scaling as $t^{1/3}$. We demonstrate this scaling collapse of the full domain wall profiles in Fig. 2(a,b). In the non-linear regime we observe the emission of a soliton during domain wall propagation shown in Fig. 2(c) which moves ballistically at a slower speed than the domain wall. We show with a purely ballistic scaling collapse in Fig. 2(d) that, indeed, this soliton does not disperse.

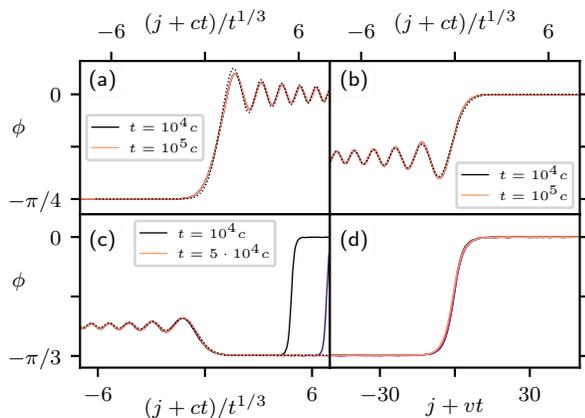


FIG. 2. Dynamics in the easy-plane case. Dotted lines show spin wave predictions, where relevant. Only the left-moving domain walls are shown. (a) Linear regime I.A ($\Delta = 0$, $\varphi = \pi/2$), showing the ballistic propagation and subdiffusive spreading of the domain wall, with the oscillations trailing. (b) Linear regime I.B ($\Delta = 0.3$, $\varphi = 3\pi/10$), where the oscillations are now ahead of the domain wall. (c) Nonlinear regime II ($\Delta = 0.25$, $\varphi = \pi/2$), showing that the domain wall decays by emitting a topological soliton, though its speed and width-scaling are unaffected. (d) Same parameters and times as (c), but in the co-moving frame of the soliton.

Spin-wave theory— We next demonstrate that the spin-wave description of the easy-plane dynamics, remarkably, captures *all* of the relevant features in what we call the linear regimes, and correctly predicts velocity and width-scaling of the domain walls even in the non-linear regime.

We expand each spin about the \hat{x} -domain,

$$\mathbf{S}_i = \hat{x}\sqrt{1 - l_i^2} + \mathbf{l}_i, \quad (4)$$

and retain only terms linear in \mathbf{l}_i in the equations of motion (which is, a priori, not controlled, as φ is large!).

The analytical solution of the resulting problem is presented in the Suppl. Mat.[63]. The central asymptotic result is readily stated: at long times, the left-moving domain wall is a function \mathcal{D} of the variable $(j + ct)/(3\alpha t)^{1/3}$:

$$S_j^y(t) \sim \sin\left(\frac{\varphi}{2}\right) \mathcal{D}\left(\frac{j + ct}{(3\alpha t)^{1/3}}\right). \quad (5)$$

The linear spin wave calculation thus correctly predicts, asymptotically, two ballistically counter-propagating domain walls, each with a width scaling as $w(t) \propto t^{1/3}$. We also observe good quantitative agreement of the spin-wave prediction (dotted) with the profiles obtained in the full numerical simulation (solid curves) in Fig. 2(a,b). The integral form is different from that appearing in the quantum free-fermion case [1, 12] but is similar to those appearing in recent studies of caustics and catastrophes at light-cones [64–66].

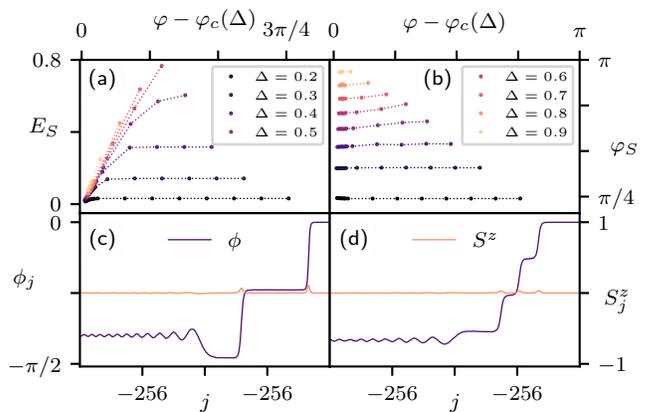


FIG. 3. Soliton emission in the easy-plane dynamics. (a), (b) Dependence of the soliton energy E_S and soliton amplitude φ_S , respectively, on the initial amplitude φ . We observe that the amplitude of the emitted solitons is almost constant. (c), (d) Two- and three-soliton emission, respectively, when $\varphi \gg \varphi_S$.

Soliton emission in the non-linear regime— We next discuss the emission of topological solitons in the easy-plane regime. As observed in Fig. 2(c, d) the moving domain wall can emit a stable (non-dispersing) ballistically-propagating soliton connecting two ground states. We find that this emission only takes place above a critical, anisotropy-dependent amplitude $\varphi_c(\Delta)$, and, in particular, only in the regime $1/7 < \Delta < 1$, as shown in Fig. 1(b). The energy carried by these solitons is seen to depend both on the anisotropy Δ and the initial amplitude of the domain wall φ . Importantly, we observe that, at fixed Δ , the angular amplitude φ_S of the emitted soliton (the in-plane angle between the two ground states the soliton connects) does not depend on φ (see Fig. 3(b)), and, in fact, is equal to the critical value $\varphi_c = \varphi_S$. Finally, we observe n -soliton emission if $\varphi > (2n - 1)\varphi_S$, as shown in Figs. 3(c) & 3(d).

To explain this phenomenology we begin with an observation on the kinematics of magnons. When $\Delta < 1/7$, the spin-wave dispersion has *negative* curvature at small q ; this ensures that two-magnon scattering is elastic. In contrast, inelastic scattering is possible for $\Delta > 1/7$, allowing the dynamic instability towards soliton emission [63]. Taking, for now, the fixed soliton amplitude as given, we provide an energetic argument for the stability regions. The domain wall energy depends monotonically on its amplitude, which must, therefore, decrease if soliton emission is to occur. The initial amplitude of the domain wall is $\varphi/2$, and after the emission the new amplitude is $|\varphi_S - \varphi/2|$. Thus, $\varphi_c = \varphi_S$. This also shows that n -soliton emission is possible if $\varphi > (2n - 1)\varphi_S$, as observed in Figs. 3(c) & 3(d).

To explain the uniqueness of the created solitons we propose the following heuristic model of soliton pro-

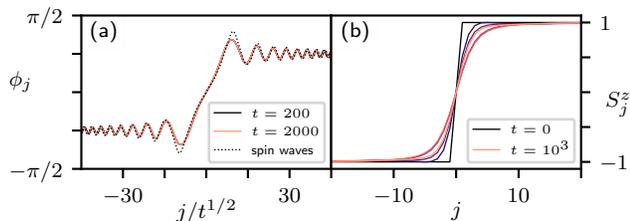


FIG. 4. Dynamics (a) at the isotropic point, and (b) in the easy-axis regime ($\Delta = 1.2$). There are no propagating domain walls in either case: the initial state spreads diffusively at the isotropic point, whilst it approaches the static soliton in the easy-axis case.

duction. We assume that the model supports a two-parameter family of soliton solutions characterised by their energy E_S and amplitude φ_S , which determine the other physical properties—in particular, their speed and width. As the interactions are local, and the soliton is observed to be created at the ballistically-moving centre of the domain wall over an extended time, the speed of the soliton must initially be matched to the Δ -dependent domain wall speed so that energy can be efficiently transferred. Further, since the soliton is seeded by the domain wall, it must begin with zero energy. The condition that the velocities match, combined with the restriction to low-energy solitons $E_S \rightarrow 0$, selects a unique soliton from the two-parameter family, with some fixed amplitude $\varphi_S(\Delta)$. As the dynamics proceeds, energy is transferred from the domain wall to the soliton, slowing down the latter and leading to its separation from the domain wall. The amplitude φ_S , however, is topological (the soliton connects different boundary conditions), and cannot be changed by local processes.

Easy-axis and isotropic dynamics— We briefly remark on the domain wall dynamics at the isotropic point ($\Delta = 1$) and in the easy-axis case ($\Delta > 1$).

At the isotropic point, there can be no propagating domain walls, because all components of the magnetisation are conserved; instead, we observe that the initially sharp domain wall spreads diffusively (Fig. 4(a)). This can be understood within the linear spin-wave picture. At $\Delta = 1$, the dispersion switches from an odd-power expansion to the even expansion [63]

$$\omega_q = 2(1 - \cos(q)) \sim q^2 + \dots, \quad q \sim 0. \quad (6)$$

There are no linear (dispersionless) terms—so the centre of the domain wall does not move—and the width is now controlled by the quadratic, not cubic, term. Details of the calculation are presented in [63]. The final, asymptotic answer can be conveniently expressed in terms of the normalised Fresnel integrals,

$$S_j^y(t) \sim \sin\left(\frac{\varphi}{2}\right) \left[\mathcal{C}\left(\frac{j}{\sqrt{2\pi t}}\right) + \mathcal{S}\left(\frac{j}{\sqrt{2\pi t}}\right) \right], \quad (7)$$

which shows good quantitative agreement with the full solution as seen in Fig. 4(a).

For the easy-axis, we change the initial conditions so that the domain wall occurs in the z -components, ensuring that the state has finite energy. Specifically,

$$\mathbf{S}_{i<0} = -\hat{z}, \quad \mathbf{S}_{i=0} = \hat{x}, \quad \mathbf{S}_{i>0} = +\hat{z}. \quad (8)$$

Now, since the z -magnetisation is conserved, there can be no propagating domain wall solutions; some dissipative spin-wave radiation escapes, before the state settles down, in an oscillatory manner (Fig. 4(b)), to the static soliton

$$S_j^z = \tanh(j \cosh^{-1}(\Delta)). \quad (9)$$

Conclusions and outlook— We have investigated the non-equilibrium dynamics of a classical anisotropic spin chain with a sharp domain wall as the initial conditions, revealing a phenomenology which is both remarkably rich—particularly in the easy-plane case, with its ballistic propagation and subdiffusive spreading of the domain wall, alongside the existence and emission of solitons—and remarkably tractable, as we have shown that many of these features can be understood within the framework of linear spin-wave theory.

This work thus sheds some light on the question of when, and to what extent, classical treatments can account for a priori complex quantum dynamics, by providing closely related instances of where this appears to be (im)possible: while the $\Delta \geq 1$ regimes and the XY point ($\Delta = 0$) appear to be entirely analogous both classically and quantum-mechanically [15, 22, 32, 33, 42], the $0 < \Delta < 1$ regime is qualitatively distinct in the classical case. While reflecting some properties of the quantum $\Delta = 0$ case [9, 10], it exhibits an entirely new phenomenon of soliton emission not found in previous studies of either the $S = 1/2$ quantum case or the continuum Landau-Lifshitz model.

This makes a connection to the broader question of under what conditions non-linear settings – e.g. a priori beyond the linear response regime – may still be described by simplified linear theories. This issue has appeared prominently in the study of KPZ dynamics [47, 48, 59] expected for small jumps in the initial condition, but in fact observed for larger ones. How a description of the ‘doubly non-linear’ phenomenon of the *emission* of (single or even multiple) *stable solitons* can co-exist with a linear description of the propagation of the emitting domain wall is a tantalising open question for future theoretical work.

Overall, it has become clear that spin chains, not just quantum but also classical, host many unexplored features. The dynamics of the classical Heisenberg spin chain (1) is now being examined from various perspectives [67–79], and is presumably good for many surprises in future studies.

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Supplementary Material

In the Supplementary Material, we describe in more detail the numerical procedure for switching to the co-moving frame of the ballistically propagating domain walls in the easy-plane dynamics; give an expanded account of the spin-wave calculations for the easy-plane and isotropic domain walls; and expatiate on the kinematics of two-magnon scattering and the implications regarding topological soliton emission.

S-I. CO-MOVING FRAME

In this section we describe the precise numerical procedure for switching to the co-moving frame of the (left-moving) domain wall in the easy-plane, which allows us to investigate the dynamics to much longer times than could be achieved on a fixed finite-size system.

We begin with the sharp domain wall initial conditions (3) on a finite chain of length L (with sites $i \in [-\frac{L}{2}, \frac{L}{2}) \cap \mathbb{Z}$) and open boundaries. After a reset time $t_R = L/4c$ (i.e., the time at which the left domain wall reaches site $i = -L/4$, where c is the domain wall speed), we shift the state rightward by $L/4$, i.e., we map $\mathbf{S}_i \mapsto \mathbf{S}_{i-L/4}$, the rightmost quarter of the state is discarded, and the leftmost quarter is reset to the $(-)$ -domain. We store a snapshot of this reset state, and then continue with the numerical time evolution, resetting periodically.

Clearly, resetting the state in this manner induces extra sources of error, viz., any information in the rightmost quarter is lost, and any information which reaches the finite left-boundary will be reflected. The (approximate) locality of the domain walls, however, ameliorates any potential problems here—since their width $w \sim t^{1/3}$ scales slowly with time, only modest system sizes $L \sim 10^4$ are needed to reach long times $t = 10^6$.

S-II. SPIN-WAVE CALCULATION IN THE EASY-PLANE CASE

Let us now present the spin-wave calculation for the easy-plane domain wall dynamics in more detail.

To ensure that the Fourier-modes are well-defined, we will work, as an intermediate step, with periodic boundary conditions for a system of size L , and initial conditions

$$\begin{aligned} \mathbf{S}_{i=0} &= \hat{\mathbf{x}} \\ \mathbf{S}_{i=1, \dots, L/2-1} &= \cos(\varphi/2)\hat{\mathbf{x}} + \sin(\varphi/2)\hat{\mathbf{y}} \\ \mathbf{S}_{i=L/2} &= \hat{\mathbf{x}} \\ \mathbf{S}_{L/2+1, \dots, L-1} &= \cos(\varphi/2)\hat{\mathbf{x}} - \sin(\varphi/2)\hat{\mathbf{y}}. \end{aligned} \quad (\text{S1})$$

We will take the thermodynamic limit when it is explicitly safe to do so. Now, following the usual procedure of classical spin wave theory, we expand each spin about the $\hat{\mathbf{x}}$ -domain,

$$\mathbf{S}_i = \hat{\mathbf{x}} \sqrt{1 - l_i^2} + \mathbf{l}_i, \quad (\text{S2})$$

and retain only terms linear in \mathbf{l}_i in the equations of motion (2). We obtain

$$\begin{aligned} \dot{l}_i^x &= 0, \\ \dot{l}_i^y &= 2J_x l_i^z - J_z(l_{i+1}^z + l_{i-1}^z), \\ \dot{l}_i^z &= -2J_x l_i^y + J_y(l_{i+1}^y + l_{i-1}^y). \end{aligned} \quad (\text{S3})$$

These equations are readily solved by Fourier transformation. Since $l_q^z(0) = 0$, we have

$$l_q^y(t) = S_q^y(t) = \cos(\omega_q t) S_q^y(0), \quad (\text{S4})$$

where the dispersion is given by

$$\omega_q = 2\sqrt{(1 - \cos q)(1 - \Delta \cos q)}. \quad (\text{S5})$$

The procedure now is to Fourier transform the initial conditions, apply the time-evolution to each q -mode, and then invert the transform. After some algebra, we obtain

$$S_j^y(t) = \frac{-2i}{L} \sum_{n=1,3,\dots}^{L-1} e^{2\pi i n j/L} \cot\left(\frac{n\pi}{L}\right) \cos(\omega_q t), \quad (\text{S6})$$

where $q = 2\pi n/L$. We may now safely pass to the thermodynamic limit $L \rightarrow \infty$, which yields

$$S_j^y(t) = -2i \sin\left(\frac{\varphi}{2}\right) \int_{-\pi}^{\pi} \frac{dq}{4\pi} e^{iqj} \cos(\omega_q t) \cot \frac{q}{2}. \quad (\text{S7})$$

To make further analytic progress, we expand the dispersion in powers of q :

$$\omega_q \sim c|q| - \alpha|q|^3 + \gamma|q|^5 + \dots, \quad |q| \sim 0, \quad (\text{S8})$$

where the coefficients are given as functions of Δ by

$$c = \sqrt{2(1-\Delta)}, \quad \alpha = \frac{1-7\Delta}{12c}, \quad \gamma = \frac{1-62\Delta+\Delta^2}{960(1-\Delta)c}. \quad (\text{S9})$$

Let us assume, for now, that $\alpha > 0$ (i.e., $0 < \Delta < 1/7$). We insert the expansion of the dispersion at this order, and note that the modulus signs can be ignored because the cosine is even. Then, splitting the cosine to make clear how this separates into left- and right-moving domain walls, we have

$$S_j^y(t) \sim -2i \sin\left(\frac{\varphi}{2}\right) \sum_{\sigma=\pm} \int_{-\pi}^{\pi} \frac{dq}{8\pi} e^{iq(j+\sigma ct) - i\sigma\alpha q^3 t} \cot \frac{q}{2}. \quad (\text{S10})$$

We make the substitution

$$k^3 = 3\alpha q^3 t \Rightarrow q = \frac{k}{(3\alpha t)^{1/3}}, \quad (\text{S11})$$

whereupon the integral becomes

$$S_j^y(t) \sim \sum_{\sigma=\pm} \int_{-(3\alpha t)^{1/3}\pi}^{(3\alpha t)^{1/3}\pi} \frac{dk}{4\pi} \exp\left(ik \frac{j+\sigma ct}{(3\alpha t)^{1/3}}\right) \left[\frac{-2i \sin\left(\frac{\varphi}{2}\right) e^{-i\sigma k^3/3}}{2(3\alpha t)^{1/3}} \cot\left(\frac{k}{2(3\alpha t)^{1/3}}\right) \right]. \quad (\text{S12})$$

We now consider the $t \rightarrow \infty$ asymptotics. We send the limits of the integrals to $\pm\infty$, and use the asymptotic equivalence

$$\frac{1}{X} \cot\left(\frac{k}{X}\right) \sim \frac{1}{k}, \quad X \rightarrow \infty, \quad (\text{S13})$$

which yields

$$S_j^y(t) \sim \sin\left(\frac{\varphi}{2}\right) \sum_{\sigma=\pm} \mathcal{F}_{[f\sigma]}^{-1}\left(\frac{j+\sigma ct}{(3\alpha t)^{1/3}}\right), \quad (\text{S14})$$

where

$$\mathcal{F}_{[f]}^{-1}(X) := \int_{\mathbb{R}} \frac{dk}{2\pi} e^{ikX} f(k), \quad f_{\sigma}(k) = \frac{-ie^{-i\sigma k^3/3}}{k}. \quad (\text{S15})$$

We note that this integral has been previously considered in the context of undular tidal bores [64], and more generally in the context of catastrophes in waves at horizons [66], and related integrals generically describe light cones in quenches in quantum spin chains [65, 66].

The calculation is the same for $\alpha < 0$ ($\Delta > 1/7$), except that an overall minus sign is attached to Eq. (S14), since we have to flip over the integration limits in Eq. (S12). The functions $\mathcal{F}_{[f\sigma]}^{-1}$ can be explicitly evaluated in terms of the generalised hypergeometric functions ${}_pF_q$, though the full expressions are somewhat unwieldy; it is apparent, however, that the spin-wave calculation correctly predicts two ballistically counter-propagating domain walls which broaden subdiffusively, and, as shown in Fig. 2 of the main text, reproduces the full profile reasonably well. The spin wave calculation, cannot, of course, predict the emission of topological solitons, since they are intrinsically non-linear objects.

Precisely at the point $\Delta = 1/7$, the cubic term vanishes and an analogous calculation would predict that the width should scale as $t^{1/5}$. We do not observe this in the numerical simulations (which use the full equations of motion), which may be due to non-linear interactions between the spin wave modes renormalising the dispersion.

S-III. SPIN-WAVE CALCULATION AT THE ISOTROPIC POINT

The spin-wave calculation at the isotropic point proceeds with exactly the same steps (initial conditions, equations of motion, Fourier transform, and thermodynamic limit) as the easy-plane case—up to the insertion of the dispersion relation,

$$\omega_q = 2(1 - \cos(q)) \sim q^2 + \dots, \quad q \sim 0, \quad (\text{S16})$$

into Eq. (S7). From this point, we have

$$S_j^y(t) = -2i \sin\left(\frac{\varphi}{2}\right) \int_{-\pi}^{\pi} \frac{dq}{4\pi} e^{iqj} \cos(q^2 t) \cot \frac{q}{2}, \quad (\text{S17})$$

and we now use the substitution

$$k^2 = 2q^2 t, \quad \Rightarrow \quad q = \frac{k}{(2t)^{1/2}}. \quad (\text{S18})$$

Together with the asymptotic relation (S13), this yields

$$S_j^y(t) = \sin\left(\frac{\varphi}{2}\right) \mathcal{F}_{[g]}^{-1} \left(\frac{j}{(2t)^{1/2}} \right), \quad (\text{S19})$$

in the notation of Eq. (S15), with

$$g(k) = \frac{-2i}{k} \cos\left(\frac{k^2}{2}\right); \quad (\text{S20})$$

or, in terms of the normalised Fresnel integrals,

$$S_j^y(t) = \sin\left(\frac{\varphi}{2}\right) \left[\mathcal{C}\left(\frac{j}{\sqrt{2\pi t}}\right) + \mathcal{S}\left(\frac{j}{\sqrt{2\pi t}}\right) \right]. \quad (\text{S21})$$

S-IV. MAGNON KINEMATICS

We close with some observations on the kinematics of two-magnon scattering, and the implications for soliton emission observed numerically. We consider two-magnon scattering, since these are the lowest-order processes which could lead to some dynamic instability.

For two magnons with momenta q_1, q_2 , which scatter to two magnons with momenta q_3, q_4 , conservation of momentum implies that we may write these as:

$$\begin{aligned} q_1 &= p & q_2 &= p' \\ q_3 &= p' - q & q_4 &= p + q \end{aligned} \quad (\text{S22})$$

for some momentum transfer q . On the other hand, energy conservation imposes a further constraint:

$$\omega_{p+q} + \omega_{p'-q} = \omega_{p'} + \omega_p. \quad (\text{S23})$$

Let us assume that all of the magnon momenta have the same sign; for soliton formation we do not consider back-scattering, since any scattering processes must be local, and left and right-movers quickly separate in the sharp domain wall set-up we consider, and would therefore not contribute to the dynamical formation of a soliton.

Now, we use the fifth-order expansion of ω_q ,

$$\omega_q \sim cq - \alpha q^3 + \gamma q^5. \quad (\text{S24})$$

Inserting this into Eq. (S23) yields a quartic equation for the allowed momentum transfers q in terms of p, p', α , and γ (the latter two being functions of Δ). There are two trivial (elastic) solutions,

$$q = 0, \quad q = p' - p, \quad (\text{S25})$$

and two non-trivial, but potentially complex solutions,

$$q = \frac{p' - p}{2} \pm \frac{\sqrt{5}\sqrt{12\alpha\gamma - 5\gamma^2(3p^2 + 2pp' + 3p'^2)}}{10\gamma}. \quad (\text{S26})$$

However, since we are assuming that p, p' have the same sign, these inelastic solutions are never real for $\Delta < 1/7$, where $\alpha > 0, \gamma < 0$. For $\Delta > 1/7$ ($\alpha < 0, \gamma < 0$), however, there is a range of momenta for which inelastic scattering *is* possible, given by the condition

$$\frac{12}{5} \frac{|\alpha|}{|\gamma|} > 3p^2 + 2pp' + 3p'^2. \quad (\text{S27})$$

We conjecture that the fact that two-magnon scattering is always elastic for $\Delta < 1/7$ explains why regime I.A (cf. Fig. 1(b)) is never observed to emit solitons, for any initial domain wall amplitude φ , since the main dynamical processes that would manifest such an instability are forbidden.