POLYNOMIALS HAVING SMALL HEIGHTS ON LEMNISCATES

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Abstract: The Mahler measure of a complex polynomial is the geometric mean of that polynomial over the unit circle. By a result of Kronecker, for nonconstant integer polynomials, the Mahler measure is equal to 1 if and only if all roots of the polynomial are 0 or roots of unity. In 1933, Lehmer asked if the Mahler measure for all other nonconstant integer polynomials had a lower bound greater than 1. In fact, Lehmer noted that the smallest such measure he had found belonged to a polynomial having degree 10 and to this day, no polynomial has been found which lowers this bound. We explore a generalization of the Mahler measure to lemniscates and investigate which properties of the classical Mahler measure are preserved by this generalization. In particular, we are interested in the analogues of Lehmer's question, and we investigate this matter both analytically and computationally. Our work is largely restricted to the classical Bernoulli lemniscate and its variations, but many of our results have applicability to a broad range of lemniscates.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

INTRODUCTION

1.1 Outline

The Mahler measure of a complex polynomial $P(z) \in \mathbb{C}[z]$, denoted $M(P)$, is the geometric mean of $|P|$ over the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. This measure has been extensively studied in the preceding decades. For example, D. H. Lehmer used the Mahler measure in the 1930s to help in his search of what were then large primes. In the 1960s, Mahler revisited the measure in investigating polynomial height functions, and the measure now bears his name.

As part of his search for primes, Lehmer was interested in monic irreducible integer polynomials $P(z) \in \mathbb{Z}[z]$ where $M(P)$ was small. For such polynomials, it is known that when $M(P) = 1$, $P(z) = z$ or P is a cyclotomic polynomial. For $M(P) > 1$, Lehmer noted the smallest measure which he could find was degree 10 polynomial $\mathcal{L}(z)$ with $M(\mathcal{L}) \approx 1.1762$, and it is conjectured that this is the smallest such value.

Since the time of Lehmer, much work has gone into proving or disproving the conjecture. The conjecture has been proven for large subsets of the integer polynomials. For example, the conjecture is known to be true for nonreciprocal integer polynomials, totally real integer polynomials, and integer polynomials whose coefficients are all odd. However, the conjecture still remains unproven nearly nine decades later.

We study a generalization of the Mahler measure over lemniscates L. In this study, we focus on analogous results to the classical Mahler measure. In particular, we investigate an analogue of the classical Lehmer conjecture and which classical results pertaining to this conjecture are preserved or fail in this generalized setting.

1.2 Historical Background

1.2.1 The Mahler Measure

In a 1933 paper entitled 'Factorization of certain cyclotomic functions' [12], D. H. Lehmer outlined a method for finding primes, which were large by contemporary standards. His method was to take a monic, integer polynomial

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 \in \mathbb{Z}[z].$$

By the fundamental theorem of algebra, $P(z)$ factors over $\mathbb{C}$,

$$P(z) = \prod_{k=1}^{n}(z - \alpha_k).$$

For each $m \in \mathbb{N}$, define

$$\Delta_m(P) := \prod_{k=1}^{n}(\alpha_k^m - 1). \tag{1.2.1}$$

It is always the case that $\Delta_m(P)$ is an integer. Moreover, if α_k is an N-th root of unity, then $\Delta_m(P) = 0$ for N dividing m. Thus, it is often assumed that no α_k is a root of unity. By computing $\Delta_m(P)$ for various polynomials, Lehmer was able to produce primes which were large by contemporary standards.

In his paper, Lehmer showed that the sequence $\{\Delta_m(P)\}_{m\in\mathbb{N}}$ is more likely to produce primes if it does not grow too quickly, where the rate of growth was measured as the ratio of successive terms:

$$\left|\frac{\Delta_{m+1}(P)}{\Delta_m(P)}\right|.$$

The limit of this measure of growth gives rise to an expression which shall be of great importance:

Proposition 1.2.1 *Provided that no root α_k of P has $|\alpha_k| = 1$,*

$$\lim_{m\to\infty}\left|\frac{\Delta_{m+1}(P)}{\Delta_m(P)}\right| = \prod_{k=1}^{n}\max\{1, |\alpha_k|\}. \tag{1.2.2}$$

In 1960 and 1961, Mahler published 'An application of Jensen's formula to polynomials' [13] and 'On the zeros of the derivative of a polynomial' [14], respectively. In these papers, Mahler utilized a generalization of (1.2.2) to any polynomial in $\mathbb{C}[z]$. Contrasting to Lehmer, Mahler's interest in (1.2.2) lay in its use as a height function for polynomials. Broadly speaking, height functions are used to measure the complexity or size of a mathematical object, and Mahler's work focused on comparing an assortment of such height functions for polynomials.

Definition 1.2.1 *For any nonzero polynomial*

$$P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = a_n\prod_{k=1}^{n}(z - \alpha_k)$$

in $\mathbb{C}[z]$, define the Mahler measure of P to be

$$M(P) := \exp\left(\frac{1}{2\pi}\int_0^{2\pi}\log\left|P(e^{it})\right| dt\right). \tag{1.2.3}$$

The Mahler measure is the geometric mean of $|P(z)|$ for z on the unit circle $\mathbb{T}$. However,

$M(P)$ is often given in the more familiar form:

$$M(P) := |a_n| \prod_{k=1}^{n} \max\{1, |\alpha_k|\}, \tag{1.2.4}$$

with an empty product assumed to be 1. The equivalence of (1.2.3) and (1.2.4) follows as an easy consequence of Jensen's formula [1]:

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|e^{it} - \alpha\right| dt = \max\{0, \log|\alpha|\}.$$

It is clear that in the case of P being a monic polynomial in $\mathbb{Z}[z]$, (1.2.4) reduces to (1.2.2).

Additionally, if α is an algebraic integer whose complete set of conjugates is denoted by $\{\alpha_k\}_{k=1}^n$, then the minimal polynomial of α is

$$P(z) = \prod_{k=1}^{n}(z - \alpha_k) \in \mathbb{Z}[z],$$

and we define the Mahler measure of α to be

$$M(\alpha) := M(P) = \prod_{k=1}^{n} \max\{1, |\alpha_k|\}.$$

We make note of the fact that M is multiplicative. That is, for $P(z), Q(z) \in \mathbb{C}[z]$, $M(PQ) = M(P)M(Q)$ and $M(0) = 0$, where $0 \in \mathbb{C}[z]$ is the zero polynomial. Additionally, for $P(z) \in \mathbb{C}[z]$ with $P(0) \neq 0$, let

$$Q(z) := z^n P\left(\frac{1}{z}\right) = a_0 \prod_{k=1}^{n}\left(z - \frac{1}{\alpha_k}\right). \tag{1.2.5}$$

Then,

$$M(Q) = |a_0| \prod_{k=1}^{n} \max\left\{1, \frac{1}{|\alpha_k|}\right\}.$$

Definition 1.2.2 *A polynomial $P(z) \in \mathbb{C}[z]$ of degree n is said to be reciprocal if it satisfies $z^n P(1/z) = P(z)$. Furthermore, an algebraic integer α is said to be reciprocal if the minimal polynomial of α is reciprocal.*

It follows by (1.2.5) that if $P(z)$ is reciprocal, we must have

$$a_n \prod_{k=1}^{n}(z - \alpha_k) = a_0 \prod_{k=1}^{n}\left(z - \frac{1}{\alpha_k}\right).$$

Since the zeros of each product must coincide up to reordering, then a polynomial $P(z) \in \mathbb{C}[z]$ is reciprocal if and only if $P(\alpha_k) = P(1/\alpha_k) = 0$ for each root α_k of P. This implies that the product of two reciprocal polynomials is reciprocal and furthermore, an algebraic integer α is reciprocal if and only if α is conjugate to $1/\alpha$.

On the other hand, if we consider $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, then if $P(z)$ is reciprocal, we must have

$$\begin{aligned} P(z) &= z^n P(1/z) \\ &= a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 \\ &= a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n. \end{aligned}$$

Equating coefficients, we must then have $a_k = a_{n-k}$ for $k = 0, \ldots, n$. Hence, the coefficients of $P(z)$ are palindromic. Finally, reciprocal polynomials with real coefficients have the following useful result:

Proposition 1.2.2 *If $P(z) \in \mathbb{R}[z]$ is reciprocal with odd degree, then $P(-1) = 0$. Moreover, if the degree of $P(z) \in \mathbb{Z}[z]$ is odd and greater than 1, then P is reducible over $\mathbb{Z}$.*

Proof. Since $P(z)$ has odd degree and is reciprocal, then $P(\alpha) = P(1/\alpha) = 0$ for some $\alpha \in \mathbb{R}$. If $|\alpha| \neq 1$, then $\alpha \neq 1/\alpha$, so $\frac{P(z)}{z^2-(\alpha+1/\alpha)z+1}$ is a real polynomial and is reciprocal. Repeating this process until only one real zero remains, we must have that $\alpha = 1/\alpha$ and we conclude that $\alpha = -1$ since $(z-1)Q(z)$ is not reciprocal for $Q(z)$ reciprocal.[1] In particular, if $P(z) \in \mathbb{Z}[z]$ is of odd degree, then $P(-1) = 0$, so $(z+1) \mid P(z)$. Thus, if the degree of P is greater than 1, then P is reducible over $\mathbb{Z}$. ■

In addition, Mahler showed that for a polynomial $P(z) = a_n z^n + \cdots + a_0 \in \mathbb{C}[z]$,

$$|a_k| \leq \binom{n}{k} M(P) \quad (k = 0, \ldots, n) \quad \text{and} \quad M(P') \leq nM(P). \tag{1.2.6}$$

Letting the height of P, denoted $H(P)$, and the length of P, denoted $L(P)$, be defined by

$$H(P) := \max_{0 \leq k \leq n} |a_k| \quad \text{and} \quad L(P) := \sum_{k=0}^{n} |a_k|,$$

it is clear that $H(cP) = |c|\,H(P)$ and $L(cP) = |c|\,L(P)$ for $c \in \mathbb{C}$. With (1.2.6), we then have [11, p. 3-4]

$$H(P) \leq \binom{n}{\lfloor n/2 \rfloor} M(P) \quad \text{and} \quad 2^{-n} L(P) \leq M(P). \tag{1.2.7}$$

From which, we then have that if $M(P)$ is bounded, $H(P)$ is bounded. It then follows that the Mahler measure satisfies the Northcott property as a height function for integer polynomials.

Proposition 1.2.3 (Northcott property of the Mahler measure) *For $n \in \mathbb{N}$ and $d > 0$, there exist only finitely many $P(z) \in \mathbb{Z}[z]$ with $\deg P \leq n$ such that $M(P) < d$.*

[1] If $Q(z) = a_n z^n + \cdots + a_0$ is reciprocal, then $a_n = a_0$, so $(z-1)Q(z) = a_n z^n + \cdots - a_0$ is not reciprocal.

Another known inequality of the Mahler measure is given by [15, cor. 1.13]:

$$M(P) \le \sqrt{\frac{1}{2\pi}\int_0^{2\pi} |P(e^{it})|^2\, dt} = \sqrt{\sum_{k=0}^{n} |a_k|^2}. \tag{1.2.8}$$

The properties we have listed of the Mahler measure are but a small sampling. In the decades since Lehmer and Mahler published their celebrated papers, several articles, books and surveys have been published focused wholly or in part on the properties of the Mahler measure. Among them are Borwein [2], Boyd [5], Everest and Ward [11], Smyth [26], and most recently, McKee and Smyth [15].

1.2.2 The Multivariable Mahler Measure

The definition of the Mahler measure can be extended to any nonzero multivariable polynomial $P(z_1, \ldots, z_n) \in \mathbb{C}[z_1, \ldots, z_n]$ as follows:

Definition 1.2.3 *For any nonzero polynomial $P(z_1, \ldots, z_n) \in \mathbb{C}[z_1, \ldots, z_n]$, define the (multivariable) Mahler measure of P to be*

$$M(P) := \exp\left(\int_0^{2\pi} \cdots \int_0^{2\pi} \log\left|P(e^{it_1}, \ldots, e^{it_n})\right| \frac{dt_1}{2\pi} \cdots \frac{dt_n}{2\pi}\right).$$

It is clear that when P is a single-variable polynomial, the above definition is equivalent to (1.2.4). By a result of Boyd [6], the single-variable Mahler measure and the multivariable Mahler measure are related in the following manner:

Theorem 1.2.1 (Boyd, 1981) *Let $P(z_1, \ldots, z_n) \in \mathbb{C}[z_1, \ldots, z_n]$, then*

$$M(P(z_1, \ldots, z_n)) = \lim_{r_2 \to \infty} \cdots \lim_{r_n \to \infty} M(P(z, z^{r_2}, \ldots, z^{r_n})).$$

This result tells us that for any $P(z_1, \ldots, z_n) \in \mathbb{C}[z_1, \ldots, z_n]$, there exists a sequence of polynomials $\{P_n\}_{n=1}^{\infty} \subset \mathbb{Z}[z]$ such that $\lim_{n\to\infty} M(P_n) = M(P)$. Perhaps the most well known of such sequences being

$$\lim_{n\to\infty} M(z^n + z + 1) = M(z_2 + z_1 + 1) = 1.38135\ldots. \tag{1.2.9}$$

1.2.3 Lehmer's Problem

As we noted, Lehmer was interested in finding monic $P(z) \in \mathbb{Z}[z]$ such that $M(P)$ was small. In the case where $M(P) = 1$, we rely upon the following result of Kronecker [10] :

Theorem 1.2.2 (Kronecker, 1857) *Let α be a nonzero algebraic integer and denote its complete set of conjugates by $\{\alpha_k\}_{k=1}^{n}$. If $|\alpha_k| \le 1$ for all $k = 1, \ldots, n$, then α is a root of unity.*

Note that for α a nonzero algebraic integer whose complete set of conjugates is denoted by $\{\alpha_k\}_{k=1}^{n}$, we have that $M(\alpha) = 1$ if and only if $|\alpha_k| \le 1$ for $k = 1, 2, \ldots, n$. Thus, we may alternatively restate the previous theorem as

Theorem 1.2.3 *Let $P(z) \in \mathbb{Z}[z]$ be irreducible and monic. Then $M(P) = 1$ if and only if P is a cyclotomic polynomial or $P(z) = z$. Similarly, for α an algebraic integer, $M(\alpha) = 1$ if and only if α is a root of unity or $\alpha = 0$.*

Therefore, for $P(z) = a_n z^n + \cdots + a_1 z + a_0 \in \mathbb{Z}[z]$, P is completely understood when $M(P) \leq 1$. We turn our attention to $M(P) > 1$. By (1.2.4), $M(P) \geq |a_n|$, so for $|a_n| \geq 2$, we have that $M(P) \geq 2$. This, along with the fact that M is multiplicative, mean that for $P(z) \in \mathbb{Z}[z]$, polynomials which are monic and irreducible only need to be considered when trying to minimize $M(P) > 1$.

In his paper, Lehmer posed the problem of whether monic polynomials $P(z) \in \mathbb{Z}[z]$ such that $M(P) > 1$ can be chosen with $M(P)$ arbitrarily close to 1. Lehmer noted that the smallest $M(P) > 1$ he could find was

$$M(\alpha_{\mathcal{L}}) = \alpha_{\mathcal{L}} = 1.1762\ldots$$

where $\alpha_{\mathcal{L}}$ is the only root lying outside the closed unit disc of the polynomial

$$\mathcal{L}(z) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1. \tag{1.2.10}$$

While Lehmer only posed the problem as whether such polynomials could be arbitrarily found, the following conjecture still bears his name:

Conjecture 1.2.1 (The Classical Lehmer Conjecture) *For all α such that $M(\alpha) > 1$, we have*

$$\inf M(\alpha) > 1.$$

To this day, no polynomial $P(z) \in \mathbb{Z}[z]$ has been found such that $1 < M(P) < \alpha_L$. Consequently, a stronger conjecture sometimes stated as the Lehmer conjecture is:

Conjecture 1.2.2 (The (Stronger) Lehmer Conjecture) *For all α such that $M(\alpha) > 1$, we have*

$$\inf M(\alpha) = \alpha_L.$$

Much work has gone into trying to prove (or disprove) the Lehmer conjecture. While the conjecture has never been proven in its entirety, it has been proven for many subsets of $\mathbb{Z}[z]$. For nonreciprocal polynomials in $\mathbb{Z}[z]$, the Lehmer Conjecture was proven by Breusch [7].

Theorem 1.2.4 (Breusch, 1951) *Let $P(z) \in \mathbb{Z}[z]$ with $P(z)$ nonreciprocal and $M(P) > 1$, then*

$$M(P) \geq M\left(z^3 - z^2 - \frac{1}{4}\right) = 1.1796\ldots.$$

While this proved the Lehmer conjecture was true for nonreciprocal polynomials, it was not an optimal bound because $z^3 - z^2 - \frac{1}{4} \notin \mathbb{Z}[z]$. Later, Smyth [25] would improve upon the lower bound given by Breusch.

Theorem 1.2.5 (Smyth, 1971) *Let $P(z) \in \mathbb{Z}[z]$ with $P(z)$ nonreciprocal and $M(P) > 1$, then*

$$M(P) \geq M\left(z^3 - z - 1\right) = 1.3247\ldots.$$

Since $z^3 - z - 1$ is a nonreciprocal integer polynomial, this lower bound for nonreciprocal integer polynomials cannot be improved. However, for nonreciprocal integer polynomials whose coefficients are all odd, P. Borwein, Mossinghoff and Hare [4] proved that:

Theorem 1.2.6 (P. Borwein, Mossinghoff and Hare, 2004) *Let $P(z) \in \mathbb{Z}[z]$ be a nonreciprocal polynomial whose coefficients are all odd, then*

$$M(P) \geq M(z^2 - z - 1) = \phi = \frac{1+\sqrt{5}}{2} = 1.618\ldots.$$

As $z^2 - z - 1$ is a nonreciprocal polynomial whose coefficients are all odd, this lower bound cannot be improved. Further, Borwein, Dobrowolski and Mossinghoff [3] proved a similar result which did not require the polynomial in question to be nonreciprocal.

Theorem 1.2.7 (Borwein, Dobrowolski and Mossinghoff, 2007) *Let $P(z) \in \mathbb{Z}[z]$ be a noncyclotomic irreducible polynomial whose coefficients are all odd, then*

$$M(P) \geq 5^{1/4} = 1.4953\ldots.$$

Another notable lower bound proven by Schinzel [23], who showed that for those nonzero algebraic integers $\alpha \neq \pm 1$ which are totally real, i.e. whose conjugates are all real, a lower bound depending upon the degree of the minimal polynomial of α exists:

Theorem 1.2.8 (Schinzel) *Let $\alpha \neq \pm 1$ be a nonzero totally real algebraic integer having a minimal polynomial of degree n. Then,*

$$M(\alpha) \geq \phi^{n/2} \quad \textit{where } \phi = \frac{1+\sqrt{5}}{2}.$$

Finally, when considering all nonzero algebraic integers α which are not roots of unity, there exist lower bounds which depend upon the degree of n. In particularly, Dobrowolski [9] proved that:

Theorem 1.2.9 (Dobrowolski, 1979) *If α is a nonzero algebraic integer of degree n that is not a root of unity, then*

$$M(\alpha) > 1 + \frac{1}{1200}\left(\frac{\log\log n}{\log n}\right)^3. \tag{1.2.11}$$

Additionally, in 1996, Voutier [28] improved the constant $\frac{1}{1200}$ seen in (1.2.11) to $\frac{1}{4}$. His refinement has yet to be further improved.

Related to the Mahler and of interest to our discussion of lower bounds is the house function, which we define as:

Definition 1.2.4 *Let α be an algebraic integer and denote its complete set of conjugates by $\{\alpha_k\}_{k=1}^n$. The house of α is defined as*

$$\overline{|\alpha|} := \max_{1\leq k\leq n} |\alpha_k|. \tag{1.2.12}$$

It follows immediately by (1.2.2) and (1.2.12) that

$$M(\alpha)^{1/n} \leq \overline{|\alpha|} \leq M(\alpha). \tag{1.2.13}$$

Furthermore, as with the Mahler measure, $\overline{|\alpha|} > 1$ for nonzero algebraic integers which are not roots of unity. In particular, Schinzel and Zassenhaus [24] showed

Theorem 1.2.10 (Schinzel and Zassenhaus, 1965) *If α is a nonzero algebraic integer of degree n that is not a root of unity and if $2s$ of its conjugates are nonreal, then*

$$M(\alpha) \geq \overline{|\alpha|} > 1 + 4^{-s-2}.$$

This result provides a lower bound for $M(\alpha)$ that depends only on how many non-real conjugates α has. Additionally, Schinzel and Zassenhaus conjectured the following lower bound for the house function:

Conjecture 1.2.3 (Schinzel and Zassenhaus, 1965) *If α is a nonzero algebraic integer of degree n that is not a root of unity, then*

$$\overline{|\alpha|} \geq 1 + c/n \tag{1.2.14}$$

for some absolute constant $c > 0$.

The Lehmer conjecture implies the Schinzel and Zassenhaus conjecture since

$$\overline{|\alpha|} \geq M(\alpha)^{1/n} > 1 + \frac{\log M(\alpha)}{n}.$$

However, the Schinzel and Zassenhaus conjecture does not imply the Lehmer conjecture. In fact, the Schinzel and Zassenhaus conjecture is no longer a conjecture, as it was recently proven true by Dimitrov [8].

Theorem 1.2.11 (Dimitrov, 2019) *If α is a nonzero algebraic integer of degree n that is not a root of unity, then*

$$M(\alpha) \geq \overline{|\alpha|} \geq 2^{\frac{1}{4n}} > 1 + \frac{\log 2}{4n}. \tag{1.2.15}$$

Like the theorem of Dobrowolski, Dimitrov's result provides a lower bound for the Mahler measure of nonzero algebraic integer α which is not a root of unity. In both cases, this lower bound depends upon the degree n of the minimal polynomial of α and both lower bounds converge to 1 as n grows to infinity. Hence, the Lehmer conjecture remains unresolved.

CHAPTER II

GENERALIZATIONS OF THE MAHLER MEASURE

2.1 Brief History and Foundational Concepts

In 2011 and 2021, Pritsker published the papers entitled 'Distribution of algebraic numbers' [18] and 'Heights of polynomials over lemniscates' [19], respectively. In the former paper, Pritsker outlined a generalization of the Mahler measure to an arbitrary compact set $E \subset \mathbb{C}$ having capacity equal to 1. His work improved upon a generalization of the Mahler measure first published by Rumely in his paper 'On Bilu's equidistribution theorem' [21]. In the latter paper, building off his earlier work, Pritsker defined a generalization of the Mahler measure to an arbitrary lemniscate. For lemniscates which have capacity equal to 1, the two generalizations are equivalent. In any case, both generalizations rely upon results from potential theory, so we begin with an overview of this subject.

2.1.1 Potential Theory

Definition 2.1.1 *Let μ be a Borel measure on $\mathbb{C}$. The support of μ, denoted* $\operatorname{supp}\mu$*, is the set of $x \in \mathbb{C}$ such that $\mu(U) > 0$ for each open neighborhood U of x.*

Additionally, for a set $X \subset \mathbb{C}$, we denote by $\mathcal{P}(X)$ the collection of probability measures μ on $\mathbb{C}$ such that $\operatorname{supp}\mu \subset X$.

Definition 2.1.2 *A sequence $(\mu_n)_{n\in\mathbb{N}}$ of probability measures on $\mathbb{C}$ is weak*-convergent to $\mu \in \mathcal{P}(X)$, denoted $\mu_n \xrightarrow{*} \mu$, if*

$$\lim_{n\to\infty} \int \phi \, d\mu_n = \int \phi \, d\mu$$

for each $\phi \in C_c(\mathbb{C})$, where $C_c(\mathbb{C})$ is the space of continuous functions of compact support equipped with the sup-norm.

Additionally, in the case where $X \subset \mathbb{C}$ is a compact set and $\mu_n \in \mathcal{P}(X)$ for each $n \in \mathbb{N}$, then $\mu_n \xrightarrow{*} \mu$ for $\mu \in \mathcal{P}(X)$ if and only if

$$\lim_{n\to\infty} \int \phi \, d\mu_n = \int \phi \, d\mu$$

for each $\phi \in C(X)$, where $C(X)$ is the collection of continuous functions on X.

Definition 2.1.3 *Let μ be a finite Borel measure on $\mathbb{C}$ with compact support. The (loga-*

rithmic) potential for μ is the function $p_\mu : \mathbb{C} \to [-\infty, \infty)$ defined by

$$p_\mu(z) := \int \log|z-w|\, d\mu(w).$$

Using the logarithmic potential, we can then define the related notion of energy.

Definition 2.1.4 *Let μ be a finite Borel measure on $\mathbb{C}$ with compact support. Its (logarithmic) energy $I(\mu)$ is defined as*

$$I(\mu) := \iint \log|z-w|\, d\mu(z)d\mu(w) = \int p_\mu(z)d\mu(z).$$

In measure theory, sets having measure zero are notable because integration over such sets are always zero, and an analogous concept arises from energy in potential theory.

Definition 2.1.5 *(a) A subset $E \subset \mathbb{C}$ is called polar if $I(\mu) = -\infty$ for every nonzero finite Borel measure μ having* $\operatorname{supp}\mu \subset E$.

(b) A property is said to hold quasi everywhere (q.e.) on $S \subset \mathbb{C}$ if it holds everywhere on $S \setminus E$, for a Borel polar set E.

The logarithmic energy is essential to many of the results of potential theory. The following lemma makes use of the logarithmic energy to provide a necessary and sufficient condition for equality of measures [22, lem. 1.8].

Lemma 2.1.1 *Let μ_1 and μ_2 be positive Borel measures such that $I(\mu_1)$ and $I(\mu_2)$ are both finite, and let $\mu = \mu_1 - \mu_2$ be a signed Borel measure with compact support such that $\mu(\mathbb{C}) = 0$. Then*

$$I(\mu) = \int p_\mu(z)d\mu = \iint \log|z-w|\, d\mu(z)d\mu(w) \leq 0. \tag{2.1.1}$$

Further, $I(\mu) = 0$ if and only if $\mu_1 \equiv \mu_2$.

Definition 2.1.6 *Let $E \subset \mathbb{C}$ be compact. If there exists $\mu_E \in \mathcal{P}(E)$ such that*

$$I(\mu_E) = \sup_{\mu \in \mathcal{P}(E)} I(\mu),$$

then μ_E is called an equilibrium measure for E.

The existence of an equilibrium measure can be guaranteed for suitable $E \subset \mathbb{C}$. Further, by Lemma 1, this equilibrium measure is unique under certain conditions [20, thm. 3.3.2, 3.7.6] [22, thm. 1.3].

Theorem 2.1.1 *Every compact $E \subset \mathbb{C}$ has an equilibrium measure μ_E. Further, if E is non-polar, then μ_E is unique with* $\operatorname{supp}\mu_E \subset \partial D$, *where D is the unbounded connected component of $\mathbb{C} \setminus E$.*

The following theorem, which is sometimes called the Fundamental Theorem of Potential Theory, provides many useful inequalities about potential and energy with respect to an equilibrium measure [20, thm. 3.3.4]:

Theorem 2.1.2 (Frostman's Theorem) *Let E be a non-polar compact subset of $\mathbb{C}$, and let μ_E be the equilibrium measure of E. Then*

1. $p_{\mu_E} \geq I(\mu_E)$ *on* $\mathbb{C}$*;*

2. $p_{\mu_E} = I(\mu_E)$ *on* $E \setminus P$*, where P is a polar subset of ∂E.*

We now turn our focus to Green's functions, which we will show have useful relationships with potential and energy [20, def. 4.4.1].

Definition 2.1.7 *Let D be a proper subdomain of $\overline{\mathbb{C}}$ (the Riemann sphere). A Green's function for D is a map $g_D : D \times D \to (-\infty, \infty]$, such that for each $w \in D$:*

(a) $g_D(\cdot, w)$ *is harmonic on $D \setminus \{w\}$, and bounded outside each neighborhood of w;*

(b) $g_D(w, w) = \infty$*, and as $z \to w$,*

$$g_D(z, w) = \begin{cases} \log|z| + O(1), & w = \infty, \\ -\log|z - w| + O(1), & w \neq \infty; \end{cases}$$

(c) $g_D(z, w) \to 0$ *as $z \to \zeta$ quasi everywhere for $\zeta \in \partial D$.*

As with the equilibrium measure, in some cases, we are able to guarantee the existence and uniqueness of the Green's function for D [20, thm. 4.4.2].

Theorem 2.1.3 *If D is a domain $\overline{\mathbb{C}}$ such that ∂D is non-polar, then there exists a unique Green's function g_D.*

We now define the logarithmic capacity, which will allow us to relate the equilibrium measure to Green's functions, as we shall later demonstrate.

Definition 2.1.8 *The (logarithmic) capacity of $E \subset \mathbb{C}$ is defined as*

$$\operatorname{cap}(E) := \sup_{F \subset E} e^{I(\mu_F)}$$

where F is compact and μ_F is the equilibrium measure of F. In particular, if E is compact with the equilibrium measure μ_E, then

$$\operatorname{cap}(E) = e^{I(\mu_E)}.$$

When D is a proper subdomain of $\overline{\mathbb{C}}$, with ∂D non-polar and $\infty \in D$, then $E = \overline{\mathbb{C}} \setminus D \subset \mathbb{C}$ is compact and non-polar. Letting μ_E be the equilibrium measure of E, we have [20, p. 107]

$$g_D(z, \infty) = \begin{cases} p_{\mu_E}(z) - I(\mu_E), & z \in D \setminus \{\infty\}, \\ \infty, & z = \infty. \end{cases}$$

Thus, for $z \in \mathbb{C} \setminus E$,

$$g_D(z, \infty) = \int \log|z - w| \, d\mu_E(w) - \log \operatorname{cap}(E). \tag{2.1.2}$$

Moreover, if $\operatorname{cap}(E) = 1$, then $\log \operatorname{cap}(E) = 0$ and the previous expression reduces to

$$g_D(z,\infty) = \int \log|z-w|\, d\mu_E(w) = p_{\mu_E}(z). \tag{2.1.3}$$

Hence, for $\operatorname{cap}(E) = 1$, (2.1.3) extends $g_D(z,\infty)$ to all of $\overline{\mathbb{C}}$.

We define the notion of a regular boundary point of a domain and the regularity of a domain.

Definition 2.1.9 *Let D be a proper subdomain of $\overline{\mathbb{C}}$, and let $z_0 \in \partial D$. A barrier at z_0 is a subharmonic function b defined on $D \cap N$, where N is an open neighborhood of z_0, satisfying*

$$b < 0 \text{ on } D \cap N \quad \text{and} \quad \lim_{z \to z_0} b(z) = 0.$$

A boundary point at which a barrier exists is called regular, and is otherwise irregular. If every $z \in \partial D$ is regular, then D is called a regular domain.

The following proposition shall prove useful when later relating the two generalizations of the Mahler measure which we shall introduce shortly.

Proposition 2.1.1 *Let $E \subset \mathbb{C}$ be compact with $\operatorname{cap}(E) = 1$, and let D be the unbounded connected component of $\overline{\mathbb{C}} \setminus E$. If D is regular, then*

$$g_D(z,\infty) = 0 \quad (z \in \overline{\mathbb{C}} \setminus D) \tag{2.1.4}$$

where $g_D(z,\infty)$ has been extended to $\overline{\mathbb{C}}$ by (2.1.3).

Proof. By (2.1.3), $g_D(z,\infty) = p_\mu(z)$ for $z \neq \infty$ where μ is the equilibrium measure of E and $\operatorname{supp}\mu \subset \partial D$. Since D is regular, then $p_\mu(z) = \log \operatorname{cap}(E) = 0$ for all $z \in \partial D$ (see [20, thm. 4.2.4]). Moreover, by Frostman's theorem, $p_\mu = \log \operatorname{cap}(E) = 0$ on $\overline{\mathbb{C}} \setminus \overline{D}$. Hence, $p_\mu = 0$ on $\overline{\mathbb{C}} \setminus D$. ■

2.1.2 Lemniscates of a Polynomial

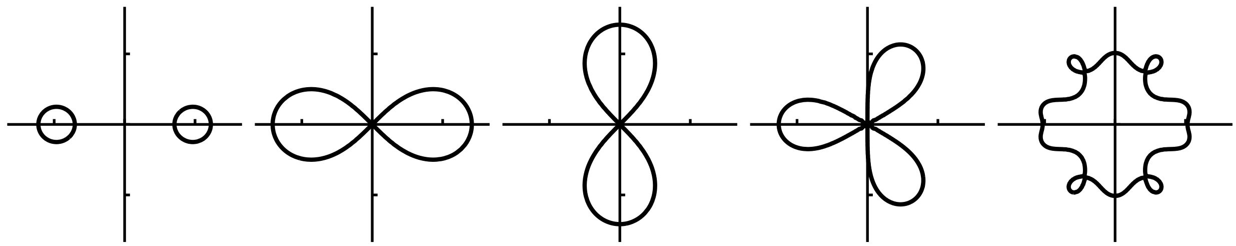
Definition 2.1.10 *For a polynomial $V(z) = a_m \prod_{k=1}^{m}(z-\zeta_k) \in \mathbb{C}[z]$ with $a_m \neq 0$, and for any $r > 0$, define its lemniscate as*

$$L := \{z \in \mathbb{C} : |V(z)| = r\} \tag{2.1.5}$$

and define the filled-in lemniscate E as the union of the lemniscate L and its interior in $\mathbb{C}$

$$E := \{z \in \mathbb{C} : |V(z)| \leq r\}. \tag{2.1.6}$$

Were it allowed that $r = 0$, then both L and E would be the set of zeros of V. If $r > 0$, but r is sufficiently small, then L consists of closed curves each enclosing a zero of V, and if V has only simple zeros, then there are m such closed curves. As r increases, these closed curves eventually meet at critical points of V and for sufficiently large r, L is a single closed curve which encloses all the zeros of V.



$\left|z^2-1\right|=\frac{1}{2}$ $\left|z^2-1\right|=1$ $\left|z^2+1\right|=1$ $\left|z^3-1\right|=1$ $\left|z^{10}-z^4+1\right|=\frac{6}{5}$

Figure 1: Examples of lemniscates and the polynomials used to generate them.

In addition, many properties of lemniscates with respect to potential theory are well known and can be explicitly stated. In particular, for E as described in (2.1.6), the equilibrium measure of E [27, p. 350], capacity of E [20, p. 134-135], and Green's function of $\mathbb{C}\setminus E$ [20, p. 133-134] are known explicitly. Further, as stated previously, lemniscates consist of at most finitely many connected components, so regularity is also well understood [20, thm. 3.8.3, 4.2.4].

Proposition 2.1.2 *Let $V(z) = a_m z^m + \cdots + a_0 \in \mathbb{C}[z]$ with $a_m \neq 0$ and $r > 0$, and define L and E as in (2.1.5) and (2.1.6). Then the equilibrium measure μ_E of E (and L) is given by*

$$d\mu_E(z) = \frac{|V'(z)|}{2\pi m r}\,|dz| \tag{2.1.7}$$

restricted to L. In addition, the capacity of E is given by

$$\operatorname{cap}(E) = \left(\frac{r}{|a_m|}\right)^{1/m}. \tag{2.1.8}$$

Finally, letting D be the unbounded connected component of $\overline{\mathbb{C}}\setminus E$, we have that D is regular with the Green's function for D given by

$$g_D(z,\infty) = \frac{1}{m}\log\frac{|V(z)|}{r} \quad (z \in \mathbb{C}\setminus E). \tag{2.1.9}$$

2.2 Generalizations of the Mahler Measure

The following generalization of the Mahler measure appeared in 2021 in a paper published by Pritsker [19], and it allows for the generalization of the Mahler measure to an arbitrary lemniscate. This generalization produces many results which are analogous to the classical Mahler measure.

Definition 2.2.1 *Let $V(z) = a_m\prod_{k=1}^m (z-\zeta_k) \in \mathbb{C}[z]$ with $a_m \neq 0$, $r > 0$, and $L = \{z \in \mathbb{C} : |V(z)| = r\}$ be a lemniscate as defined in (2.1.5). If $P(z) = c_n\prod_{k=1}^n (z-z_k) \in \mathbb{C}[z]$ is any polynomial with $c_n \neq 0$, then the generalized Mahler measure $M_L(P)$ of L is defined as*

$$M_L(P) := \exp\left(\int_L \log|P(z)|\frac{|V'(z)|}{2\pi m r}\,|dz|\right). \tag{2.2.1}$$

Analogous to the classical Mahler measure $M(P)$ being the geometric mean of $|P(z)|$ for z on the unit circle, $M_L(P)$ serves as the geometric mean of $|P(z)|$ for z on L. In addition, as was the case with the classical Mahler measure, this generalization may be restated as a closed form expression.

Proposition 2.2.1 (Pritsker, 2021) *Let L be a lemniscate $\{z \in \mathbb{C} : |V(z)| = r\}$ defined as in (2.1.5). If $P(z) = c_n \prod_{k=1}^n (z - z_k) \in \mathbb{C}[z]$ is any polynomial with $c_n \neq 0$, then*

$$M_L(P) = |c_n|\,|a_m|^{-n/m} \left(\prod_{k=1}^{n} \max\{r, |V(z_k)|\} \right)^{1/m}. \tag{2.2.2}$$

Furthermore, we have

$$M_L(P) \geq |a_m|^{-n/m} \, |\mathrm{Res}(P, V)|^{1/m}, \tag{2.2.3}$$

where $\mathrm{Res}(P, V)$ *is the resultant of P and V.*

Proof. Letting μ be the equilibrium measure of L, we have by (2.1.7) that $d\mu = \dfrac{|V'(z)|}{2\pi m r}\,|dz|$ and $\operatorname{supp}\mu = L$. Hence, by (2.2.1), $M_L(P) = \exp\left(\int \log|P|\, d\mu\right)$, so

$$\log M_L(P) = \int \log|P(z)|\, d\mu(z) = \log|c_n| + \sum_{k=1}^{n} \int \log|z - z_k|\, d\mu(z). \tag{2.2.4}$$

For each z_k, $z_k \in E$ or $z_k \in \mathbb{C} \setminus E$. In the former case, by Frostman's theorem and (2.1.8)

$$\int \log|z - z_k|\, d\mu(z) - \log \operatorname{cap}(E) = \frac{1}{m} \log \frac{r}{|a_m|} \quad (z_k \in E).$$

In the latter case, by (2.1.2) and (2.1.9), we have

$$\int \log|z - z_k|\, d\mu(z) = \frac{1}{m} \log \frac{|V(z_k)|}{|a_m|} \quad (z_k \in \mathbb{C} \setminus E).$$

Thus, in either case

$$\int \log|z - z_k|\, d\mu(z) = \frac{1}{m} \log \frac{\max\{r, |V(z_k)|\}}{|a_m|}.$$

We then have that (2.2.2) follows by the above and (2.2.4). The definition of the resultant and (2.2.2) immediately imply (2.2.3). ■

Letting $V(z) = z$, $r = 1$, and L defined as in (2.1.5), we see that that (2.2.1) and (2.2.2) reduce to (1.2.3) and (1.2.4), respectively. Hence, the classical Mahler measure is a special case of the generalization. In the classical case, Kronecker's theorem proved very useful when classifying polynomials, and an analogous statement exists for the generalization of the Mahler measure in some cases.

Theorem 2.2.1 (Pritsker, 2021) *Let $V(z) = z^m + \cdots + a_1 z + a_0 \in \mathbb{Z}[z]$ be monic, let $r = 1$, and let L be as defined by (2.1.5). The generalized Mahler measure (2.2.2) satisfies*

$$M_L(P) \geq 1, \quad P \in \mathbb{Z}[z], P \not\equiv 0. \tag{2.2.5}$$

Equality is attained above if and only if P has leading coefficient ± 1, and all roots of P are located in E as defined by (2.1.6).

More precisely, $M_L(P) = 1$ for a monic irreducible $P(z) \in \mathbb{Z}[z]$ if and only if either $P \mid V$ or $P \mid \Phi \circ V$ for some cyclotomic polynomial Φ. Thus, if α is an algebraic integer contained in E with all its conjugates, then

(i) α is a root of V, when $\alpha \in E \setminus L$,

(ii) α is a root of $\Phi \circ V$, for a cyclotomic polynomial Φ, when $\alpha \in L$.

This generalization of the Mahler measure leads to an analogous statement for a generalization of the Lehmer conjecture. Consider the greatest lower bound for the generalized Mahler measure of all polynomials $P(z) \in \mathbb{Z}[z]$ satisfying $M_L(P) > 1$:

$$B_L := \inf \{M_L(P) : P \in \mathbb{Z}[z], M_L(P) > 1\}. \tag{2.2.6}$$

Conjecture 2.2.1 (The Lehmer Conjecture on Lemniscates) *Let $V(z) = z^m + \cdots + a_1 z + a_0 \in \mathbb{Z}[z]$ be monic, let $r = 1$, and define L as in (2.1.5). Then $B_L > 1$.*

Letting L be as described above, for $P(z) = c_n \prod_{k=1}^{n}(z - z_k) \in \mathbb{Z}[z]$ a nonzero polynomial, we have

$$P(V(z)) = \prod_{k=1}^{n} (V(z) - z_k) = c_n \prod_{\substack{1 \leq k \leq n \\ 1 \leq j \leq m}} (z - \beta_{k,j}),$$

where $\beta_{k,1}, \ldots, \beta_{k,m}$ are the m roots of $V(z) = z_k$. It then follows that

$$M_L(P \circ V) = c_n \prod_{\substack{1 \leq k \leq n \\ 1 \leq j \leq m}} \max\{1, |V(\beta_{k,j})|\}^{1/m} = c_n \prod_{k=1}^{n} \max\{1, |z_k|\} = M(P).$$

Thus, $B_L \leq B_{\mathbb{T}}$ where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, which in part gives rise to the following result:

Theorem 2.2.2 (Pritsker, 2021) *Let L be defined by (2.1.5), with $V(z) = z^m + \cdots + a_1 z + a_0 \in \mathbb{Z}[z]$ and $r = 1$. Then*

$$(B_{\mathbb{T}})^{1/m} \leq B_L \leq B_{\mathbb{T}}. \tag{2.2.7}$$

An immediate result follows from (2.2.7), that is

Corollary 2.2.1 *The classical Lehmer conjecture is true if and only if the generalized Lehmer conjecture is true.*

As stated at the start of this chapter, there have been multiple other generalizations of the Mahler measure. In addition to the generalizations outlined in (2.2.1) and (2.2.2), another generalization of the Mahler measure was published by Rumely [21] in 1999, and further explored by Pritsker [18] in 2011. Rather than generalizing the Mahler measure to lemniscates, this alternative measure generalizes the Mahler measure to arbitrary compact sets $E \subset \mathbb{C}$ with $\operatorname{cap}(E) = 1$. We shall later show that for lemniscates L having $\operatorname{cap}(L) = 1$, the two generalizations are identical.

Definition 2.2.2 *Let $E \subset \mathbb{C}$ be compact with $\operatorname{cap}(E) = 1$ and let D be the unbounded connected component of $\overline{\mathbb{C}} \setminus E$. For any polynomial $P(z) = c_n \prod_{k=1}^{n}(z - z_k) \in \mathbb{C}[z]$ with $c_n \neq 0$, define the Mahler measure of P on E as*

$$M_E(P) := |c_n| \exp\left(\sum_{z_k \in D} g_D(z_k, \infty)\right), \tag{2.2.8}$$

where $g_D(z,\infty)$ is the Green's function on D extended to $\overline{\mathbb{C}}$ by (2.1.3).

To avoid confusion, when working with a lemniscate with L and E as defined in (2.1.5) and (2.1.6), respectively, we shall use $M_L(P)$ to mean (2.2.2) and $M_E(P)$ to mean (2.2.8). However, any confusion is purely notational, because when a lemniscate has capacity equal to one, the following proposition gives that the two measures are identically equal.

Proposition 2.2.2 *Let $V(z) = a_m z^m + \cdots + a_0 \in \mathbb{C}[z]$ and $r > 0$ with $|a_m| = r$, define L and E as in (2.1.5) and (2.1.6), respectively. For any polynomial $P(z) = c_n \prod_{k=1}^{n}(z - z_k)$ with $c_n \neq 0$, we have*

$$M_E(P) = M_L(P).$$

Proof. Let D be the unbounded connected component of $\overline{\mathbb{C}} \setminus E$, then D is regular. Since $|r/a_m| = 1$, then $\operatorname{cap}(E) = 1$ by (2.1.8). Hence, $g_D(z,\infty) = 0$ for all $z \notin D$ by (2.1.4). It follows that

$$\log M_L(P) = \log|c_n| + \sum_{k=1}^{n} g_D(z_k,\infty) = \log|c_n| + \sum_{z_k \in D} g_D(z_k,\infty) = \log M_E(P).$$

■

The usefulness of the generalization in (2.2.8) is made apparent by the following theorem.

Theorem 2.2.3 (Pritsker, 2011) *Let $P_n(z) = c_n \prod_{k=1}^{m_n}(z - \alpha_{k,n})$ with $m_n \to \infty$ as $n \to \infty$ be a sequence of polynomials with integer coefficients and simple zeros. Suppose that $E \subset \mathbb{C}$ is a compact set having $\operatorname{cap}(E) = 1$ with μ_E the equilibrium measure of E. Then*

$$\lim_{n\to\infty} (M_E(P_n))^{1/m_n} = 1 \tag{2.2.9}$$

if and only if

$$\begin{cases} \text{(i)} \displaystyle\lim_{n\to\infty} |c_n|^{1/m_n} = 1, \\ \text{(ii)} \displaystyle\lim_{R\to\infty}\lim_{n\to\infty} \left(\prod\nolimits_{|\alpha_{k,n}|\geq R} |\alpha_{k,n}|\right)^{1/m_n} = 1, \quad \text{and} \\ \text{(iii)} \ \tau_n = \displaystyle\frac{1}{m_n}\sum_{k=1}^{m_n} \delta_{\alpha_{k,n}} \xrightarrow{*} \mu_E \quad \text{as } n \to \infty. \end{cases} \tag{2.2.10}$$

The following theorem, which features in the proof for Theorem 2.2.3, provides a lower bound for the energy of a measure arising as the limit of a sequence of measures for the roots of integer polynomials.

Theorem 2.2.4 *Let $P_n(z) = a_n \prod_{k=1}^{n}(z - z_k) \in \mathbb{Z}[z]$ with $P_n(z)$ having simple zeroes and $\lim_{n\to\infty} |a_n|^{1/n} = 1$. Assume further that there exists a bounded closed disc D such that the roots of P_n lie in D for all $n \in \mathbb{N}$. If $\tau_n = \frac{1}{n}\sum_{k=1}^{n} \delta_{n,k} \xrightarrow{*} \tau$ with τ having compact support, then $I(\tau) \geq 0$.*

Proof. We may assume that $\operatorname{supp} \tau \subset D$ by making D sufficiently large. Let $\{f_M\}_{M=1}^{\infty}$ be defined by $f_M(z, w) = \min\{M, -\log|z - w|\}$ for $z, w \in D$, so that $f_M : D \times D \to \mathbb{R}$ is continuous for each $M \in \mathbb{N}$ and f_M increases to $-\log|z - w|$. By the Monotone Convergence Theorem since $\tau_n \times \tau_n \xrightarrow{*} \tau \times \tau$, we then have

$$
\begin{aligned}
I(\tau) &= \iint \log|z - w|\, \tau(z)\tau(w) \\
&= -\lim_{M\to\infty} \iint_{D\times D} \min\{M, -\log|z - w|\}\, \tau(z)\tau(w) \\
&= -\lim_{M\to\infty} \left(\lim_{n\to\infty} \iint \min\{M, -\log|z - w|\}\, \tau_n(z)\tau_n(w) \right) \\
&= -\lim_{M\to\infty} \lim_{n\to\infty} \left(-\frac{M}{n} + \frac{2}{n^2} \sum_{1\leq k<j\leq n} \log|z_k - z_j| \right).
\end{aligned}
$$

As M is fixed in the inner limit, we then have that

$$
\begin{aligned}
I(\tau) &= \lim_{n\to\infty} \left(\frac{2}{n^2} \sum_{1\leq k<j\leq n} \log|z_k - z_j| \right) \\
&= \lim_{n\to\infty} \left(\frac{2}{n} \log|a_n|^{1/n} - 2\log|a_n|^{1/n} + \frac{1}{n^2} \log|\mathrm{Disc}(P_n)| \right).
\end{aligned}
$$

Where $\mathrm{Disc}(P_n)$ is the discriminant of P_n. Since each $P_n \in \mathbb{Z}[n]$ has simple zeros, then $|\mathrm{Disc}(P_n)| \in \mathbb{N}$. By assumption, $\lim_{n\to\infty} |a_n|^{1/n} = 1$, so $\lim_{n\to\infty} \log|a_n|^{1/n} = 0$. It follows that $I(\tau) \geq 0$. ■

We now introduce the notion of mutual energy between two measures. The provides a name for an expression often arising when comparing the energies of two measures.

Definition 2.2.3 *For μ and τ finite Borel measures on $\mathbb{C}$ with compact support, let the mutual energy of μ and τ, denoted $I(\mu, \tau)$, be defined as*

$$
I(\mu, \tau) := \iint \log|z - w|\, d\mu(z) d\tau(w).
$$

The following theorem provides a lower bound for the mutual energy of measures which arise as the limit measures for roots of integer polynomials.

Theorem 2.2.5 *Let $\{P_n\}_{n=1}^{\infty}$ and $\{Q_m\}_{m=1}^{\infty}$ be sequences of monic irreducible integer polynomials such that $\deg P_n = n$, $\deg Q_m = m$ and whose roots all lie in some bounded disc. Further, assume that roots of P_n and Q_m are disjoint for all $m, n \in \mathbb{N}$. Define*

$$\mu_n := \frac{1}{n} \sum_{P_n(z)=0} \delta_z, \quad \textit{and} \quad \tau_m := \frac{1}{m} \sum_{Q_m(z)=0} \delta_z.$$

If $\mu_n \xrightarrow{*} \mu$ and $\tau_m \xrightarrow{*} \tau$, then $I(\mu, \tau) \geq 0$.

Proof. Since the roots of P_n and Q_m are bounded, and μ, τ have compact supports, then for some closed disc D with sufficiently large radius, the roots of P_n and Q_m are contained in D for each $m, n \in \mathbb{N}$ and $\operatorname{supp} \mu, \operatorname{supp} \tau \subset D$. Let $\{f_M\}_{M=1}^{\infty}$ be defined by $f_M(z, w) = \min\{M, -\log|z - w|\}$ for $z, w \in D$, so that $f_M : D \times D \to \mathbb{R}$ is continuous for each $M \in \mathbb{N}$ and f_M increases to $-\log|z - w|$. By the Monotone Convergence Theorem, and since $\mu_n \xrightarrow{*} \mu$ and $\tau_n \xrightarrow{*} \tau$, then $\mu_n \times \tau_n \xrightarrow{*} \mu \times \tau$, we then have by the compactness of D that

$$\begin{aligned}
I(\mu, \tau) &= \iint \log|z - w| \, d\mu(z) d\tau(w) \\
&= \iint_{D \times D} \log|z - w| \, d\mu(z) d\tau(w) \\
&= -\lim_{M \to \infty} \iint_{D \times D} \min\{M, -\log|z - w|\} \, d\mu(z) d\tau(w) \\
&= -\lim_{M \to \infty} \left(\lim_{n \to \infty} \iint \min\{M, -\log|z - w|\} \, d\mu_n(z) d\tau_n(w) \right) \\
&= -\lim_{n \to \infty} \left(\frac{1}{n^2} \sum_{P_n(z)=0} \sum_{Q_n(w)=0} -\log|z - w| \right) \\
&= \lim_{n \to \infty} \frac{1}{n^2} \log|\operatorname{Res}(P_n, Q_n)| .
\end{aligned}$$

The resultant of two integer polynomials is always an integer, and since the roots of P_n and Q_m are disjoint for all $m, n \in \mathbb{N}$, then $|\operatorname{Res}(P_n, Q_m)| \geq 1$ for all $m, n \in \mathbb{N}$. It follows that $I(\mu, \tau) \geq 0$. ■

CHAPTER III

LEMNISCATES WITH CAPACITY ONE

In the previous sections, we defined the Mahler measure and generalized it to arbitrary lemniscates. Many of the important results for this generalization depended upon the capacity of the lemniscate being one. Thus, we shall focus largely on such lemniscates going forward.

3.1 The Mahler Measure over Lemniscates with Capacity One

We begin our focus on lemniscates having capacity one by stating and proving several useful results which apply to such lemniscates. Recall that the classical Mahler measure acts as a height function for integer polynomials with respect to degree and measure. It is also the case that for any lemniscate having capacity one, the generalized Mahler measure for this lemniscate acts as a height function for integer polynomials.

Proposition 3.1.1 (Northcott property for the Mahler measure over lemniscates) *Let $V(z) = a_m z^m + \cdots + a_0 \in \mathbb{C}[z]$, $r > 0$, and $|a_m| = r$. Define L and E as in (2.1.5) and (2.1.6), respectively. For $N \in \mathbb{N}$ and $d > 0$, there exist only finitely many $P(z) \in \mathbb{Z}[z]$ with $\deg P \leq N$ such that $M_L(P) < d$.*

Proof. Since $|a_m| = r$, then by (2.2.2), for $P(z) = c_n z^n + \cdots + c_0 = c_n \prod_{k=1}^{n}(z - z_k) \in \mathbb{Z}[z]$ with $c_n \neq 0$ and $n \leq N$, we have

$$M_L(P) = |c_n| \left(\prod_{k=1}^{n} \max\{1, |V(z_k)/r|\} \right)^{1/m}.$$

It follows that if $M_L(P) < d$, then $|c_n| < d$ and $z_k \in K = \{z \in \mathbb{C} : |V(z)| \leq d^m r\}$ for $k = 1, \ldots, n$. Since K is compact, then there exists $R \geq 1$ such that $|z| < R$ for all $z \in K$. Hence, symmetric polynomials in $z_1, \ldots, z_n$ are bounded. As the coefficients of P are the products of c_n and these symmetric polynomials, then the coefficients of P are bounded. Thus, $H(P) = \max_{0 \leq k \leq n} |c_k|$ is bounded, and so only finitely many P with $\deg P \leq n$ and $M_L(P) < d$ exist. ∎

The following result provides a necessary condition for a sequence $\{\mu_n\}_{n \in \mathbb{N}}$ of probability measures to converge weak* to a probability measure μ.

Lemma 3.1.1 *Let $\mu \in \mathcal{P}(\mathbb{C})$ with compact support and let $(\mu_n)_{n \in \mathbb{N}} \subset \mathcal{P}(\mathbb{C})$ with $\mu_n \xrightarrow{*} \mu$. Then for any $N \in \mathbb{N}$,*

$$\operatorname{supp} \mu \subset \overline{\textstyle\bigcup_{n=N}^{\infty} \operatorname{supp} \mu_n}. \tag{3.1.1}$$

Proof. Suppose, for contradiction, that there exists $N \in \mathbb{N}$ such that $\operatorname{supp}\mu \not\subset \overline{\bigcup_{n=N}^{\infty} \operatorname{supp}\mu_n}$, then there exists $z_0 \in \operatorname{supp}\mu$ and $r > 0$ such that $B(z_0, r) \cap (\bigcup_{n=N}^{\infty} \operatorname{supp}\mu_n) = \emptyset$. Consider the function $f : \mathbb{C} \to [0, 1]$ defined by

$$f(z) := \begin{cases} 1 & \text{if } |z - z_0| \leq r/2, \\ \frac{2r-2|z-z_0|}{r} & \text{if } r/2 < |z - z_0| < r, \\ 0 & \text{if } |z - z_0| \geq r. \end{cases}$$

It is clear that $f \in C_c(\mathbb{C})$ and $\int f d\mu_n = 0$ for all $n \in \mathbb{N}$. By weak*-convergence, we then have

$$\int f d\mu = \lim_{n\to\infty} \int f d\mu_n = 0.$$

However, by definition of $\operatorname{supp}\mu$, $\int f d\mu \geq \mu(B(z_0, r/2)) > 0$. Hence, no such $z_0 \in \operatorname{supp}\mu$ exists, so no such $N \in \mathbb{N}$ exists, and $\operatorname{supp}\mu \subset \overline{\bigcup_{n=N}^{\infty} \operatorname{supp}\mu_n}$ for all $N \in \mathbb{N}$. ■

Theorem 3.1.1 *Let $E \subset \mathbb{C}$ be compact having* $\operatorname{cap}(E) = 1$ *and the equilibrium measure μ_E. For $F \subset \mathbb{C}$ such that* $\operatorname{supp}\mu_E \not\subset \overline{F}$*, denote by $\mathbb{Z}_n(F)$ the collection of polynomials with integer coefficients of exact degree n whose roots lie entirely in F. Then there exists $C > 1$ such that*

$$\inf \{M_E(P) : P \in Z_n(F), M_E(P) > 1\} \geq C^n. \tag{3.1.2}$$

Proof. Suppose, for contradiction, that no such d exists. Then for each $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$\inf \{M_E(P) : P \in Z_n(F), M_E(P) > 1\} < (1 + \varepsilon)^n.$$

It follows that there exists a sequence $P_n = c_n \prod_{k=1}^{m_n}(z - z_{k,n})$ such that $M_E(P_n)^{1/m_n} \to 1$ as $n \to \infty$, and we may assume that each such P_n has simple zeros since M_E is multiplicative. Moreover, as the generalized Mahler measure is a height function, then for each $\varepsilon > 0$ and $n \in \mathbb{N}$ there exist only finitely many $P(z) \in \mathbb{Z}[z]$ with $\deg P \leq n$ and $M_E(P) < (1 + \varepsilon)^n$, so $m_n \to \infty$ as $n \to \infty$. Let $\tau_n = \frac{1}{m_n} \sum_{k=1}^{n} \delta_{k,n}$, then by (2.2.10), $\tau_n \xrightarrow{*} \mu_E$ and by (3.1.1),

$$\operatorname{supp}\mu_E \subset \overline{\bigcup_{n=1}^{\infty} P_n^{-1}(0)} \subset \overline{F},$$

which contradicts the assumption that $\operatorname{supp}\mu_E \not\subset \overline{F}$. By contradiction, there must exist $\varepsilon > 0$ such that for all $n \in \mathbb{N}$,

$$\inf \{M_E(P) \in Z_n(F) \mid M_E(P) > 1\} \geq (1 + \varepsilon)^n.$$

Letting $C = 1 + \varepsilon$, we complete our proof. ■

Corollary 3.1.1 *Let $V(z) = a_m z^m + \cdots + a_0 \in \mathbb{C}[z]$, $r > 0$ with $|a_m| = r$. Define L and E be as in (2.1.5) and (2.1.6), respectively. Then there exists $C > 1$ such that for all totally real $P(z) \in \mathbb{Z}[z]$ of degree n having simple zeros with $M_L(P) > 1$, $M_L(P) > C^n$.*

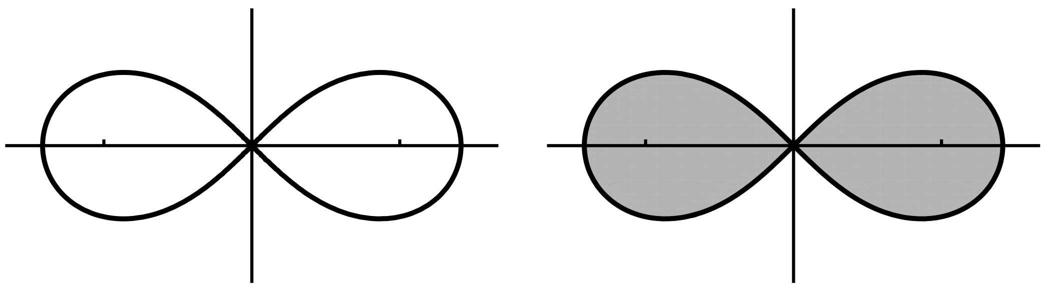
Proof. By Proposition 2.1.2, the equilibrium measure μ_E of E and L has $\operatorname{supp}\mu_E = L$. Since $r > 0$, then E has a non-empty interior, so L contains non-real numbers and $L \not\subset \mathbb{R}$. By Theorem 3.1.1, the result follows. ■

3.2 The Bernoulli Lemniscate

We now turn out attention to particular lemniscates which have capacity one. The unit circle is such a lemniscate, but the classical Mahler measure has been extensively studied for decades, so we shall focus on other lemniscates. With the exception of the unit circle, perhaps the best known lemniscate having capacity one is the Bernoulli lemniscate.

Definition 3.2.1 *Let $V(z) = z^2 - 1 \in \mathbb{Z}[z]$ and let L and E be as described by (2.1.5) and (2.1.6), respectively. That is,*

$$L := \left\{ z \in \mathbb{C} : \left| z^2 - 1 \right| = 1 \right\} \quad \textit{and} \quad E := \left\{ z \in \mathbb{C} : \left| z^2 - 1 \right| \leq 1 \right\}. \tag{3.2.1}$$



$L = \left\{ z \in \mathbb{C} : \left| z^2 - 1 \right| = 1 \right\}$ $E = \left\{ z \in \mathbb{C} : \left| z^2 - 1 \right| \leq 1 \right\}$

Figure 2: The unfilled and filled Bernoulli lemniscate, respectively, depicted in the complex plane.

For the remainder of this section, we assume L and E are as described above unless otherwise stated. In particular, for $P(z) = c_n \prod_{k=1}^{n} (z - z_k) \in \mathbb{Z}[z]$, we define $M_L(P)$ as in (2.2.1) and (2.2.2). That is,

$$M_L(P) := \exp\left(\frac{1}{2\pi} \int_L \log |P(z)| \, |z| \, |dz| \right) = |c_n| \sqrt{\prod_{k=1}^{n} \max\{1, |z_k^2 - 1|\}}. \tag{3.2.2}$$

As in (2.2.3), we then have

$$M_L(P) \geq |\mathrm{Res}(P, V)|^{1/2}. \tag{3.2.3}$$

Additionally, we define L_-, L_+, G_-, G_+ respectively as

$$L_- := \{ z \in L : \mathrm{Re}\, z \leq 0 \} \quad \text{and} \quad L_+ := \{ z \in L : \mathrm{Re}\, z \geq 0 \},$$
$$G_- := \left\{ z \in \mathbb{C} : \mathrm{Re}\, z < 0, \left| z^2 - 1 \right| < 1 \right\} \quad \text{and} \quad G_+ := \left\{ z \in L : \mathrm{Re}\, z > 0, \left| z^2 - 1 \right| < 1 \right\}.$$

and we define V_-^{-1} and V_+^{-1} respectively as

$$z = V_-^{-1}(w) := -\sqrt{w+1} \quad \text{and} \quad z = V_+^{-1}(w) := \sqrt{w+1},$$

where the branch of the square root function is the principal branch. Then V_-^{-1} and V_+^{-1} are single valued branches of V^{-1}. When restricted to $\overline{\mathbb{D}}$, we have that V_-^{-1} and V_+^{-1} form bijections from $\overline{\mathbb{D}}$ to $L_- \cup G_-$ and $L_+ \cup G_+$, respectively. To verify, note that if $\sqrt{w_1+1} = \sqrt{w_2+1}$, then $w_1+1 = w_2+1$ because V_+^{-1} is a single valued branch of V^{-1}. On the other hand, if $z \in L_+ \cup G_+$, then $|z^2-1| \leq 1$, so there exists $|w| \leq 1$ such that $z^2 - 1 = w$, so $V_+^{-1}(w) = z$. We verify V_-^{-1} is a bijection in a similar manner. Moreover, V_-^{-1} and V_+^{-1} are both analytic for $w \neq -1$, so they are conformal maps for all points $w \neq -1$. In particular, they are conformal maps from $\mathbb{D}$ onto G_- and G_+, respectively.

Theorem 3.2.1 (Looney, 2022) *Let $P(z) = \prod_{k=1}^n (z - z_k) \in \mathbb{Z}[z]$ be totally real. If $M_L(P) > 1$, then*

$$M_L(P) \geq \phi^{n/4} \quad \text{where} \quad \phi = \frac{1+\sqrt{5}}{2}.$$

Further, for $P(z) = (z^2 - z - 1)^m$ with $m \in \mathbb{N}$, equality is attained.

Proof. Let $Q(z) := \prod_{k=1}^n (z - z_k^2)$, so that $Q(z) \in \mathbb{Z}[z]$ is totally real. Letting $T(z) := Q(z+1)$, we have that $T(z) = \prod_{k=1}^n (z + 1 - z_k^2) = \prod_{k=1}^n (z - (z_k^2 - 1)) \in \mathbb{Z}[z]$. It follows that

$$M(T) = \prod_{k=1}^n \max\left\{1, \left|z_k^2 - 1\right|\right\} = M_L^2(P) > 1.$$

As $T(z) \in \mathbb{Z}[z]$ is totally real with $\deg T = n$ and $M(T) > 1$, then by Theorem 1.2.8, we have that $M_L^2(P) = M(T) \geq \phi^{n/2}$. It follows that $M_L(P) \geq \phi^{n/4}$.

Letting $P(z) = (z^2 - z - 1)^m$, we have that

$$M_L(P) = M_L^m(z^2 - z - 1) = \phi^{m/2}.$$

Thus equality is attained and this bound cannot be further improved. ∎

An immediate and useful inequality follows from our choosing the Bernoulli lemniscate.

Theorem 3.2.2 (Looney, 2022) *Let $P(z) = z^n + \cdots + c_1 z + c_0 = \prod_{k=1}^n (z - z_k) \in \mathbb{Z}[z]$ be irreducible with $\deg P \geq 2$. If $M_L(P) < \sqrt{2}$, then*

$$|P(1)| = |P(-1)| = 1.$$

Proof. Let $Q(z) := \prod_{k=1}^n (z - z_k^2)$, so that $Q(z) \in \mathbb{Z}[z]$. Writing P as the sum of its odd and even parts, we have

$$P(z) = P_0(z^2) + zP_1(z^2)$$

where $P_0(z) = c_0 + c_2 z + c_4 z^2 + \ldots$ and $P_1(z) = c_1 + c_3 z + c_5 z^2 + \ldots$. By Graeffe's root-squaring method [16, Ch. 8], we then have

$$Q(z) = (-1)^n (P_0^2(z) - zP_1^2(z)).$$

Now, let $T(z) := Q(z+1) \in \mathbb{Z}[z]$, then

$$T(z) = \prod_{k=1}^n (z - (z_k^2 - 1)) = (-1)^n (P_0^2(z+1) - (z+1)P_1^2(z+1)).$$

It follows that $M(T) = M_L^2(P)$, and so $1 \le M(T) < 2$. Thus, $|T(0)| = 1$, and we then have

$$1 = |T(0)| = \left|P_0^2(1) - P_1^2(1)\right| = |P_0(1) + P_1(1)|\,|P_0(1) - P_1(1)| = |P(1)|\,|P(-1)|\,.$$

Since $P(z) \in \mathbb{Z}[z]$, $P(1), P(-1) \in \mathbb{Z}$, we must then have that $|P(z)| = |P(-1)| = 1$. ■

We may also prove this result using the submean value property for subharmonic functions.

Alternate proof. Since V_-^{-1} and V_+^{-1} are conformal maps for $w \ne -1$, then $P \circ V_-^{-1}$ and $P \circ V_+^{-1}$ are analytic for $w \ne -1$. Hence, $\log\left|P(V_-^{-1})\right|$ and $\log\left|P(V_+^{-1})\right|$ are subharmonic functions for $w \ne -1$ and in particular for $w \in \mathbb{D}$. We then have,

$$\begin{aligned}
\log M_L(P) &= \int_L \log|P(z)|\,\frac{|V'(z)|}{4\pi}\,|dz| \\
&= \frac{1}{4\pi}\int_{L_-} \log|P(z)|\,|d(V(z))| + \frac{1}{4\pi}\int_{L_+} \log|P(z)|\,|d(V(z))| \\
&= \frac{1}{4\pi}\int_{|w|=1} \log\left|P(V_-^{-1}(w))\right|\,|dw| + \frac{1}{4\pi}\int_{|w|=1} \log\left|P(V_+^{-1}(w))\right|\,|dw| \\
&\ge \frac{1}{2}\log\left|P(V_-^{-1}(0))\right| + \frac{1}{2}\log\left|P(V_+^{-1}(0))\right| \quad \text{(By submean value property.)} \\
&= \log\sqrt{|P(-1)|\,|P(1)|}.
\end{aligned}$$

Thus, $M_L(P) \ge \sqrt{|P(-1)|\,|P(1)|}$. It follows that if $|P(-1)| \ge 2$ or $|P(1)| \ge 2$, then $M_L(P) \ge \sqrt{2}$. By contraposition, if $M_L(P) < \sqrt{2}$, then $|P(-1)| = |P(1)| = 1$. ■

As an immediate consequence of Theorem 3.2.2, for most polynomials, we are able to quickly determine if most polynomials have relatively large measures.

Corollary 3.2.1 *For any polynomial $P(z) = c_n z^n + \cdots + c_0 \in \mathbb{Z}[z]$ with $\deg P \ge 2$, if $M_L(P) < \sqrt{2}$, then*

$$\left|\sum_{k=0}^{n} c_k\right| = \left|\sum_{k=0}^{n} (-1)^k c_k\right| = 1. \tag{3.2.4}$$

An important question when it comes to trying to prove or disprove Conjecture 2.2.1 is whether there exist infinitely many polynomials all having measures below some fixed bound. The following proposition provides a sequence of polynomials whose sequence of measures converges.

Proposition 3.2.1 *Let $P_n(z) = z(z^2-1)^n - 1$, then $M_L(P_n) \le 3^{1/4}$ for all $n \in \mathbb{N}$. Further,*

$$\lim_{n\to\infty} M_L(P_n) = \sqrt{M(z_2 + z_1 + 1)} = 1.17530\ldots$$

Proof. To begin, for all $n \in \mathbb{N}$, we may write $P_n(z) = \prod_{k=1}^{2n+1}(z - z_{k,n})$ as the sum of its odd and even parts as

$$P_n(z) = P_{0,n}(z^2) + zP_{1,n}(z^2)$$

where $P_{0,n}(z) = -1$ and $P_{1,n}(z) = (z-1)^n$. Letting $Q_n(z) := \prod_{k=1}^{2n+1}(z - z_{k,n}^2)$, by Graeffe's root-squaring method, we have

$$Q_n(z) = (-1)^{2n+1}(P_{0,n}^2(z) - zP_{1,n}^2(z)) = z(z-1)^{2n} - 1.$$

Then letting $T_n(z) := Q_n(z+1)$, we then have that

$$T_n(z) = \prod_{k=1}^{2n+1}(z - (z_{k,n}^2 - 1)) = z^{2n+1} + z^{2n} - 1.$$

By (1.2.8), it then follows that

$$M_L^2(P_n) = M(T_n) = M(z^{2n+1} + z^{2n} - 1) \leq \sqrt{3}.$$

Thus, $M_L(P_n) \leq 3^{1/4}$ for all $n \in \mathbb{N}$. Now, letting $P(z_1, z_2) = z_2 z_1 + z_2 - 1 \in \mathbb{Z}[z_1, z_2]$, we have by Theorem 1.2.1, that

$$\lim_{n\to\infty} M(T_n) = \lim_{n\to\infty} M(P(z^{2n}, z)) = M(P) = 1.38135\ldots.$$

Hence, $\lim_{n\to\infty} M_L(P_n) = M(P)^{1/2} = 1.17530\ldots.$ ■

3.2.1 Notable Measures on the Bernoulli Lemniscate

Using equation (3.2.4), we are able to search for polynomials $P(z) \in \mathbb{Z}[z]$ for which $M_L(P)$ by the following algorithm:

1. Generate a polynomial $P(z) = z^n + \cdots + a_1 z + a_0 \in \mathbb{Z}[z] = \prod_{k=1}^{n}(z - z_k)$ satisfying $|P(1)| = |P(-1)| = 1$.

2. Using Graeffe's root-squaring method, compute $T(z) = \prod_{k=1}^{n}(z - (z_k^2 - 1))$ and verify that $2^{-n}L(T) < M(T)$. This step can be repeated by applying Graeffe's method to T.

3. Compute $M_L(P)$ and verify that $M_L(P) < \sqrt{2}$. As $M_L^2(P) = M(T)$, $M_L(P)$ may be computable by looking up the value of $M(T)$ in an existing database, such as Mossinghoff's [17].

4. If $P(z)$ is irreducible, then note its value.

Using the algorithm described above, we are able to generate a large amount of polynomials P_n with measure $1 < M_L(P_n) < \sqrt{2}$.

n	$\deg P_n$	$a_0, a_1, \ldots$	$M_L(a_0 + a_1 z + \ldots)$
1	3	$-1, -1, 0, 1$	$1.15096\ldots$
2	5	$1, 3, -1, -3, 0, 1$	$1.16177\ldots$
3	6	$-1, -1, 3, 0, -3, 0, 1$	$1.17087\ldots$
4	7	$-1, -1, 0, 3, 0, -3, 0, 1$	$1.17446\ldots$
5	4	$1, -1, -2, 0, 1$	$1.17485\ldots$
6	5	$-1, 1, 0, -2, 0, 1$	$1.18738\ldots$
7	5	$-1, 2, 0, -3, 0, 1$	$1.20136\ldots$
8	4	$2, 1, -2, -1, 1$	$1.24740\ldots$
9	2	$-1, -1, 1$	$1.27201\ldots$
10	5	$1, 2, -1, -2, 0, 1$	$1.28375\ldots$
11	6	$-1, 2, 3, -1, -3, 0, 1$	$1.29042\ldots$
12	4	$1, 1, -1, -1, 1$	$1.31228\ldots$
13	6	$-1, 0, 2, 0, -3, 0, 1$	$1.32471\ldots$
14	6	$-1, -1, 1, 2, -2, -1, 1$	$1.33284\ldots$
15	4	$1, 2, -2, -1, 1$	$1.33538\ldots$
16	7	$-1, -1, 1, 3, -1, -3, 0, 1$	$1.34541\ldots$
17	7	$1, -1, -1, 3, 2, -3, -1, 1$	$1.34614\ldots$
18	7	$-1, 0, 2, 3, -1, -3, 0, 1$	$1.35727\ldots$
19	8	$1, 2, -2, -3, 3, 3, -3, -1, 1$	$1.35760\ldots$
20	6	$-1, -3, 3, 3, -3, -1, 1$	$1.36212\ldots$
21	8	$-1, 0, -1, 0, 3, 0, -3, 0, 1$	$1.38027\ldots$
22	5	$-1, 3, 2, -3, -1, 1$	$1.38158\ldots$
23	8	$1, 0, -1, 0, 3, 0, -3, 0, 1$	$1.40126\ldots$
24	6	$-1, -2, 2, 2, -2, -1, 1$	$1.40133\ldots$
25	8	$1, 1, -1, -3, 3, 3, -3, -1, 1$	$1.40183\ldots$

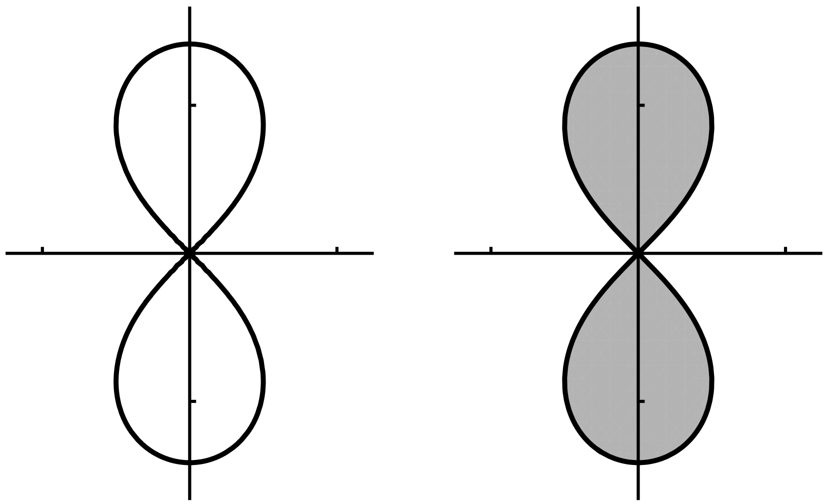
Table 1: Polynomials P_n such that $1 < M_L(P_n) < \sqrt{2}$ where L is the Bernoulli lemniscate.

3.3 The Rotated Bernoulli Lemniscate

We define the rotated Bernoulli lemniscate in the same vein as we defined the classical Bernoulli lemniscate.

Definition 3.3.1 *Let $V(z) = z^2 + 1 \in \mathbb{Z}[z]$ and let L and E be defined as in (2.1.5) and (2.1.6), respectively. That is,*

$$L := \left\{z \in \mathbb{C} : \left|z^2 + 1\right| = 1\right\} \quad \textit{and} \quad E := \left\{z \in \mathbb{C} : \left|z^2 + 1\right| \leq 1\right\}.$$



$L = \left\{z \in \mathbb{C} : \left|z^2 + 1\right| = 1\right\}$ $E = \left\{z \in \mathbb{C} : \left|z^2 - 1\right| \leq 1\right\}$

Figure 3: The unfilled and filled rotated Bernoulli lemniscate, respectively, depicted in the complex plane.

Proceeding as before, for $P(z) = c_n \prod_{k=1}^{n}(z - z_k) \in \mathbb{Z}[z]$, we define $M_L(P)$ as in (2.2.1) and (2.2.2) by

$$M_L(P) := \exp\left(\frac{1}{2\pi}\int_L \log|P(z)|\,|z|\,|dz|\right) = |c_n|\sqrt{\prod_{k=1}^{n}\max\{1, |z_k^2 + 1|\}}.$$

By Corollary 3.1.1, there exists a constant $C > 1$ such that for all totally real monic $P(z) \in \mathbb{Z}[z]$ with degree n and $M_L(P) > 1$, $M_L(P) > C^n$. One such value of C is known for the rotated Bernoulli lemniscate.

Proposition 3.3.1 *Let $P(z) \in \mathbb{Z}[z]$ be monic and totally real with degree n having simple zeros. If $M_L(P) > 1$ then*

$$M_L(P) \geq \phi^{n/2} \quad \textit{where } \phi = \frac{1+\sqrt{5}}{2}.$$

Proof. We may assume that $P(z) = \prod_{k=1}^{n}(z - z_k)$ is irreducible since $M_L(P)$ is multiplicative and $M_L(P) > 1$. By Theorem 1.2.8, if $P(\pm 1) \neq 0$, then

$$M(P) \geq \phi^{n/2}.$$

By our choice of L and the fact that $z_k \in \mathbb{R}$ for $k = 1, \ldots, n$, we have

$$\sqrt{z_k^2 + 1} \geq \max\{1, |z_k|\}$$

for $k = 1, \ldots, n$. It follows that

$$M_L(P) = \sqrt{\prod_{k=1}^{n} \max\{1, |z_k^2 + 1|\}} = \sqrt{\prod_{k=1}^{n} |z_k^2 + 1|} \geq \prod_{k=1}^{n} \max\{1, |z_k|\} = M(P) \geq \phi^{n/2}.$$

On the other hand, if $P(\pm 1) = 0$, then $P = z \pm 1$, and $M_L(P) = \sqrt{2} \geq \phi^{1/2}$. Thus, in any case, $M_L(P) \geq \phi^{n/2}$. ■

Proposition 3.3.2 *Let $P(z) \in \mathbb{Z}[z]$ be an irreducible monic polynomial with degree ≥ 1. If $1 < M_L(P) < \sqrt{2}$, then*

$$|P(i)| = |P(-i)| = 1. \tag{3.3.1}$$

Proof. Suppose that $M_L(P) < \sqrt{2}$, then

$$\sqrt{|P(i)P(-i)|} < M_L(P) < \sqrt{2}.$$

Hence,

$$2 > |P(i)P(-i)| = \left|P(i)\overline{P(i)}\right| = \left|P(-i)\overline{P(-i)}\right| = |P(i)|^2 = |P(-i)|^2.$$

Since $M_L(P) > 1$ and P is monic and irreducible, then $P \nmid (z^2 + 1)$ by Theorem 2.2.1. Thus,

$$0 < |P(i)|^2 = |P(-i)|^2 < 2.$$

Moreover, since $P(i), P(-i) \in \mathbb{Z}[i]$, then $|P(i)|^2, |P(-i)|^2 \in \mathbb{Z}$, so $|P(i)|^2 = |P(-i)|^2 = 1$. ■

3.3.1 Notable Measures on the Rotated Bernoulli Lemniscate

Similarly to the classical Bernoulli lemniscate, we are able to make use of (3.3.1) to search for polynomials $P(z) \in \mathbb{Z}[z]$ for which $1 < M_L(P) < \sqrt{2}$.

1. Generate a polynomial $P(z) = z^n + \cdots + a_1 z + a_0 \in \mathbb{Z}[z] = \prod_{k=1}^{n}(z - z_k)$ satisfying $|P(i)| = |P(-i)| = 1$.

2. Using Graeffe's root-squaring method, compute $T(z) = \prod_{k=1}^{n}(z - (z_k^2 + 1))$ and verify that $2^{-n}L(T) < M(T)$. This step can be repeated by applying Graeffe's method to T.

3. Compute $M_L(P)$ and verify that $M_L(P) < \sqrt{2}$. As with the Bernoulli lemniscate, $M_L(P)$ may be computable by looking up the value of $M(T)$ in an existing database.

4. If $P(z)$ is irreducible, then note its value.

n	$\deg P_n$	$a_0, a_1, \ldots$	$M_L(a_0 + a_1 z + \ldots)$
1	3	$-1, 2, -1, 1$	$1.15096\ldots$
2	6	$1, -2, 5, -3, 4, -1, 1$	$1.16615\ldots$
3	7	$-1, 1, 0, 3, 0, 3, 0, 1$	$1.17844\ldots$
4	6	$1, -1, 3, 0, 3, 0, 1$	$1.18224\ldots$
5	4	$1, -1, 2, 0, 1$	$1.18375\ldots$
6	3	$-1, 1, 0, 1$	$1.21060\ldots$
7	6	$2, -1, 3, -1, 3, 0, 1$	$1.24515\ldots$
8	6	$1, -2, 3, -1, 3, 0, 1$	$1.29257\ldots$
9	7	$-1, 2, -2, 3, -1, 3, 0, 1$	$1.29406\ldots$
10	6	$1, -2, 3, -2, 3, -1, 1$	$1.29620\ldots$
11	7	$-1, 2, -3, 3, -3, 3, -1, 1$	$1.29748\ldots$
12	4	$1, -2, 2, -1, 1$	$1.31228\ldots$
13	8	$1, 2, 2, 3, 5, 1, 4, 0, 1$	$1.32256\ldots$
14	6	$1, 0, 1, 0, 2, 0, 1$	$1.32471\ldots$
15	8	$1, 0, 2, -2, 5, -3, 4, -1, 1$	$1.33265\ldots$
16	8	$1, -1, 3, -1, 5, -2, 4, -1, 1$	$1.33871\ldots$
17	7	$-1, 0, -1, 2, -2, 3, -1, 1$	$1.34027\ldots$
18	5	$-1, 3, -2, 3, -1, 1$	$1.34858\ldots$
19	5	$-2, 3, -3, 3, -1, 1$	$1.35015\ldots$
20	6	$1, 0, 4, -2, 4, -1, 1$	$1.35568\ldots$
21	8	$1, -2, 3, -3, 5, -3, 4, -1, 1$	$1.35620\ldots$
22	6	$2, -3, 5, -3, 4, -1, 1$	$1.35863\ldots$
23	7	$-1, 2, 0, 5, 0, 4, 0, 1$	$1.35962\ldots$
24	5	$-1, 0, -1, 2, 0, 1$	$1.36479\ldots$
25	6	$1, -1, 2, 0, 2, 0, 1$	$1.37646\ldots$
26	8	$-1, 0, 1, 0, 3, 0, 3, 0, 1$	$1.38027\ldots$
27	8	$1, 0, 1, 0, 3, 0, 3, 0, 1$	$1.40126\ldots$

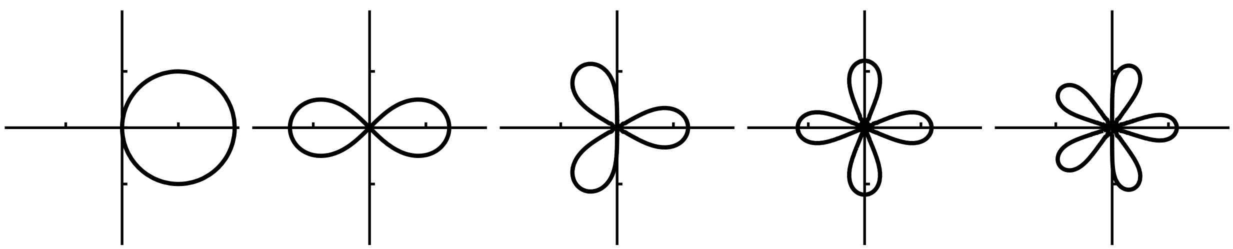
Table 2: Polynomials P_n such that $1 < M_L(P_n) < \sqrt{2}$ where L is the rotated Bernoulli lemniscate.

3.4 Asymptotic Behavior of Measures Over Sequences of Lemniscates

To this point, our study of Mahler measures on lemniscates have been concerned with measuring many polynomials on fixed lemniscates. Our attention now turns to the study of measuring a single polynomial over several lemniscates.

Definition 3.4.1 *Let $V_m(z) := z^m - 1 \in \mathbb{Z}[z]$ for $m = 1, 2, \ldots,$ and define both L_m and E_m as in (2.1.5) and (2.1.6) with $r = 1$ for $m = 1, 2, \ldots.$ That is,*

$$L_m := \{z \in \mathbb{C} : |z^m - 1| = 1\} \quad \textit{and} \quad E_m := \{z \in \mathbb{C} : |z^m - 1| \le 1\}.$$



$|z-1| = 1$ $\quad$ $\left|z^2-1\right| = 1$ $\quad$ $\left|z^3-1\right| = 1$ $\quad$ $\left|z^4-1\right| = 1$ $\quad$ $\left|z^5-1\right| = 1$

Figure 4: The first five terms of L_m.

Using the terminology of (2.2.1) and (2.2.2), for a polynomial $P(z) = c_n \prod_{k=1}^n (z - z_k) \in \mathbb{C}[z]$ we define the sequence of Mahler measures M_m as

$$M_m(P) := \exp\left(\frac{1}{2\pi}\int_{L_m} \log|P(z)| \left|z^{m-1}\right| |dz|\right) = |c_n| \left(\prod_{k=1}^n \max\{1, |z_k^m - 1|\}\right)^{1/m}.$$

Theorem 3.4.1 *For $P(z) = c_n \prod_{k=1}^n (z - z_k)$, we have*

$$\lim_{m\to\infty} M_m(P) = M(P).$$

Proof. Let $P(z) = c_n \prod_{k=1}^n (z - z_k) \in \mathbb{C}[z]$, then for each z_k, we have two possible cases:

(i) If $|z_k| \le 1$, then $0 \le |z_k^m - 1| \le 2$ and $1 \le \max\{1, |z_k^m - 1|\} \le 2$. Thus,

$$1 \le \max\{1, |z_k^m - 1|\}^{1/m} \le 2^{1/m}.$$

It then follows that $\lim_{m\to\infty} \max\{1, |z_k^m - 1|\}^{1/m} = 1$.

(ii) If $|z_k| > 1$, then $\lim_{m\to\infty} |z_k^m - 1| = \infty$. Hence, there exists $N \in \mathbb{N}$ such that for all $m \ge N$, we have $|z_k^m - 1| > 1$. It then follows that

$$\lim_{m\to\infty} \max\{1, |z_k^m - 1|\}^{1/m} = \lim_{m\to\infty} |z_k^m - 1|^{1/m} = \lim_{m\to\infty} |z_k| \left|1 - z_k^{-m}\right|^{1/m} = |z_k|.$$

Thus, $\lim_{m\to\infty} \max\{1, |z_k^m - 1|\}^{1/m} = \max\{1, |z_k|\}$, and consequently

$$\begin{aligned}\lim_{m\to\infty} M_m(P) &= |c_n| \lim_{m\to\infty} \left(\prod_{k=1}^{n} \max\{1, |z_k^m - 1|\}\right)^{1/m} \\ &= |c_n| \prod_{k=1}^{n} \lim_{m\to\infty} \left(\max\{1, |z_k^m - 1|\}^{1/m}\right) \\ &= |c_n| \prod_{k=1}^{n} \max\{1, |z_k|\} = M(P).\end{aligned}$$

■

Theorem 3.4.2 *Let $P_m(z) = z^m - z - 1 \in \mathbb{Z}[z]$, then $M_m(P_m) > 1$ for all $m \geq 2$ and*

$$\lim_{m\to\infty} M_m(P_m) = 1. \tag{3.4.1}$$

Proof. For $m \geq 2$, let $\alpha_1, \ldots, \alpha_m$ be the roots of P_m, then $\alpha_k^m - 1 = \alpha_k$ for $k = 1, \ldots, m$. It follows that

$$M_m(P_m) = \prod_{k=1}^{m} \max\{1, |\alpha_k^m - 1|\}^{1/m} = \prod_{k=1}^{m} \max\{1, |\alpha_k|\}^{1/m} = M(P_m)^{1/m}.$$

Restricting P_m to $\mathbb{R}$, we have that $P_m(1) = -1$ and $P_m(x) \to \infty$ as $x \to \infty$. By the intermediate value theorem for real functions, P_m has a real root $\alpha > 1$, so

$$M_m(P_m) \geq |\alpha|^{1/m} > 1.$$

By (1.2.8), we also have that

$$M(P_m) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log\left|P_m(e^{it})\right| dt\right) \leq \sqrt{\frac{1}{2\pi}\int_0^{2\pi} \left|P_m(e^{it})\right|^2 dt} = \sqrt{3}.$$

Hence, for each $m \geq 2$, we have

$$1 < M_m(P_m) = M(P_m)^{1/m} \leq \sqrt[2m]{3}.$$

By the squeeze theorem, it follows that $\lim_{m\to\infty} M_m(P_m) = 1$.

■

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