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DIAGRAMMATIC CATEGORIES WHICH ARISE FROM DIRECTED GRAPHS

A DISSERTATION APPROVED FOR THE
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Dedicated to Meghan, Jude, Huck, Dwight, and Roberta

Acknowledgments

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Contents

Abstract

There is a categorical equivalence between the Temperley–Lieb category $TL(2)$ and the full subcategory of $SU(2)$-**mod** with objects given by $V^{\otimes k}$ where V is the tautological $SU(2)$-module and k is a non-negative integer. The first results in this dissertation develop new diagrammatic categories which are shown to be equivalent to similarly defined full subcategories of G-**mod** for certain finite subgroups G of $SU(2)$. The diagrams which generate the Temperley–Lieb category are shown to be linear combinations of the generating diagrams for these newly defined diagrammatic categories. The main result of this paper utilizes the representation graph of a group G, $R(V,G)$, and gives a general construction of a diagrammatic category $\mathbf{Dgrams}_{R(V,G)}$. The proof of the main theorem shows that, given explicit criteria, there is an equivalence of categories between a quotient category of $\mathbf{Dgrams}_{R(V,G)}$ and a full subcategory of $G-$ **mod** with objects being the tensor products of finitely many irreducible G-modules.

Chapter 1

Introduction

The main subject of this dissertation is to develop a class of diagrammatic categories which arises from the study of the representation theory of certain groups. Our motivating examples are the finite subgroups of $SU(2)$. The main theorem, however, goes beyond these motivating examples.

In order to give context, we must first pay homage to the giants whose shoulders these results rest upon. Felix Klein classified the finite subgroups of the special unitary group, $SU(2)$. There are two infinite families of finite subgroups along with 3 exceptional subgroups: the cyclic groups of order n, $\mathbf{C}_n$; the binary dihedral groups, $\mathbf{D}_n$, of order $4n$; the binary tetrahedral group $\mathbf{T}$; the binary octahedral group $\mathbf{O}$; and the binary icosahedral group $\mathbf{I}$. Around 1980, McKay made the observation that certain affine Dynkin diagrams and the representation graphs associated with these finite subgroups are identical [1]. See Section 2.2 for details.

In a different direction, for $k \in \mathbb{Z}_{\geq 0}$, $\delta \in \mathbb{C}$, the diagrammatic Temperley–Lieb algebras, $TL_k(\delta)$, were developed by the authors of the same name in [2]. For $k \in \mathbb{Z}_{\geq 0}$, there are isomorphisms between the endomorphism algebra

$$Z_k(SU(2)) := \mathrm{End}_{SU(2)}(V^{\otimes k})$$

of the natural module V for $SU(2)$ and the Temperley–Lieb algebra $TL_k(2)$. In [3], Barnes, Benkart, and Halverson combined the work of McKay and Temperley–Lieb by describing the endomorphism algebras of the finite subgroups of $SU(2)$ and presenting diagrammatics for the $\mathbf{C}_n$ and $\mathbf{D}_n$ cases.

The study of endomorphism algebras like $Z_k(SU(2))$ can be generalized to more general homomorphism spaces. For example, we can study $\mathrm{Hom}_{SU(2)}\left(V^{\otimes k}, V^{\otimes \ell}\right)$ for all k, ℓ. This gives us new tools, new perspective, and a richer understanding of the representation theory. With this generalization in mind, the diagrammatic Temperley–Lieb *category* was developed, see [4] and [5]. This category admits a fully faithful monoidal functor to the category whose objects are tensor products of V and the morphisms are all $SU(2)$-linear maps. In particular, the Temperley–Lieb algebras appear as endomorphism algebras in the Temperley–Lieb category.

Surprisingly, entire categories can be easier to derive than individual endomorphism algebras. In particular, the Temperley–Lieb category has generating diagrams known as the cup, cap, and identity strand, and there is a diagrammatic basis for each space of homomorphisms, $\mathrm{Hom}_{SU(2)}(V^{\otimes k}, V^{\otimes \ell})$, which can be described as all non-crossing diagrams with k nodes on the bottom of the diagram and ℓ nodes on the top.

This introduction provides a small roadmap for the dissertation at large. In Chapter 2, the reader will find some pertinent background and motivation for the development of diagrammatic categories. We also discuss representation graphs as they will be a key tool in this thesis. In Chapter 3 comes the definition of a diagrammatic category $C_n^\star$ and of the fully faithful and essentially surjective functor onto the category of $\mathbf{C}_n$-modules of the form $V^{\otimes k}$ for some $k \geq 0$. As the groups $\mathbf{C}_n$ and $\mathbf{D}_n$

are closely related, so too are the representation theories. Thus in Chapter 4, we may realize a new diagrammatic category, $\mathcal{D}_n^\star$ in terms of the diagrams used to define $C_n^\star$. We also prove an equivalence of categories between the diagrammatic category $\mathcal{D}_n^\star$ and the category of $\mathbf{D}_n$-modules of the form $V^{\otimes k}$ for some $k \geq 0$. In Chapter 5, we expand the set of objects we are considering. We define a diagrammatic category with multiple generating objects which correspond to all of the simple $\mathbf{C}_n$-modules and provide explicit relations giving a diagrammatic description of the monoidal full subcategory generated by the irreducible $\mathbf{C}_n$-modules, $\mathbf{C}_n$-$\mathbf{mod}_{\text{irr}}$. In Chapter 6, we utilize the representation graph of a group G, $R(V, G)$, and give a general construction of a diagrammatic category $\mathbf{Dgrams}_{R(V,G)}$. The proof of the main theorem shows that, given explicit criteria, there is an equivalence of categories between a quotient category of $\mathbf{Dgrams}_{R(V,G)}$ and G-$\mathbf{mod}_{\text{irr}}$. In the final chapter, we give a few final remarks regarding generalization to directed graphs and give a few examples which show that these results apply outside of the context of $SU(2)$ and its finite subgroups.

We shall close out the introduction with a discussion of certain directions in which this work might extend. For the constructions in this dissertation, the functor to the category of G-modules can be thought of as a functor to the category of $\mathbb{C}$-vector spaces once we forget the G-action. In other words, we have a representation of each of these diagrammatic categories. Just as one group can have many representations, one category can have many interesting representations. This is an active area of research.

For example, Sam and Snowden explore the representation theory of the Brauer category in [6]. They specifically mention that much of the theory they develop could be transferable to other categories, like the Temperley–Lieb categories and its variants. In particular, we expect it applies to the categories introduced in this thesis.

Similarly, Brundan and Vargas give a concrete diagrammatic definition of the affine partition category, and use it to study the representation theory of the partition category [7]. It is with these two papers in mind that we may ask the following questions.

Question 1.1. *Can we classify and study the representations of the diagrammatic categories associated to the finite subgroups of $SU(2)$? In particular, what is the categorical representation theory of these diagrammatic categories, and can we extend some notions such as highest weight module, semi-simplicity, irreducible modules, etc. to these categories?*

In addition to the above questions, there are other directions one might consider exploring. The hands-on combinatorial nature of this area makes it easy to compute interesting examples and special cases. Another direction could be to explore how these categories react to changes in certain parameters. For example, the Temperley–Lieb category, when not considering the connection to $SU(2)$, can be defined with a parameter $\delta \in \mathbb{C}$ where $\delta = 2$ is the Temperley–Lieb category for $SU(2)$. What would introducing such a parameter to these diagrammatic categories change about the combinatorics or representation theory? For example, one might explore how these categories decategorify. Still another: we consider these categories over the complex numbers; what happens if we consider them over other fields?

Chapter 2

Preliminaries

This dissertation will assume general knowledge in the areas of group theory, representation theory, and category theory. However, beyond basic definitions and some fundamental theorems, this dissertation should be self-contained. To that end, this chapter discusses some of the background material this dissertation utilizes.

2.1. A Discussion centered around $SU(2)$

Throughout this dissertation, the special unitary group $SU(2)$ and its finite subgroups are used as a source of study, motivation, and examples. In an effort to settle on a starting point, we can define $SU(2)$ in the following way:

$$\begin{aligned} SU(2) &= \{A \in GL(2,\mathbb{C}) |\ A^* = A^{-1} \text{and } \det A = 1\} \\ &= \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} |\ \alpha, \beta \in \mathbb{C}, \alpha\bar{\alpha} + \beta\bar{\beta} = 1 \right\} \end{aligned} \tag{2.1}$$

where $\bar{\alpha}$ is the complex conjugate of α.

Let us explore some of the general facts about the finite-dimensional representation theory of $SU(2)$. We will refer to the category of all finite-dimensional $SU(2)$-

modules and all $SU(2)$-module homomorphisms as $SU(2)$-**mod**. First, all the finite-dimensional $SU(2)$-modules are semisimple; that is, any finite-dimensional $SU(2)$-module is isomorphic to a direct sum of simple modules. Second, the simple modules can be categorized, up to isomorphism, by their dimension. Let $\mathbb{N}$ the set of all non-negative integers (i.e., $0 \in \mathbb{N}$). For $r \in \mathbb{N}$, we will let $V(r)$ be the $(r+1)$st dimensional simple $SU(2)$-module. We can fix a particular representative for $V(1)$, namely the natural or tautological module for $SU(2)$, $\mathbb{C}^2$, the space of column vectors of height 2 with entries in $\mathbb{C}$. The action of $SU(2)$ is given by matrix multiplication.

In general, let G be a group, and let V be a G-module. The k-fold tensor product of V, $V^{\otimes k} = V \otimes V \otimes \cdots \otimes V$ can be realized as a G-module using the diagonal action; i.e., for $g \in G$ and $v_i \in V$ for all $0 \leq i \leq k$,

$$g.\left(v_1 \otimes v_2 \otimes \cdots \otimes v_k\right) = \left(g.v_1 \otimes g.v_2 \otimes \cdots \otimes g.v_k\right).$$

Furthermore, if V and W are G-modules, the diagonal action works in a similar way for $V \otimes W$.

Using the so-called diagonal action for tensor products we have the following consequence of the Clebsch-Gordon formula:

$$V(1) \otimes V(r) \cong V(r-1) \oplus V(r+1) \tag{2.2}$$

for $r \geq 1$ and $V(1) \otimes V(0) \cong V(1)$. From this formula we see that for all $r \in \mathbb{N}$, there is a $k \in \mathbb{N}$ such that $V(r)$ is a submodule of $V^{\otimes k}$. In particular, there is a minimal k which admits $V(r)$ as a direct summand precisely once, namely, $k = r$. Thus, there are, up to scaling, canonical projections from $V^{\otimes r}$ onto $V(r)$ for all $r \in \mathbb{N}$. More generally, given any finite-dimensional $SU(2)$-module M, there exists a $k \in \mathbb{N}$ such that M is a direct summand of $V^{\otimes k}$.

We can now define a category of representations using this idea.

Definition 2.1. *We denote by $SU(2)$-**mod**$_V$ the full monoidal subcategory of $SU(2)$-**mod** with generating object V.*

In this category, objects are $V^{\otimes k}$ for $k \in \mathbb{N}$, and morphisms are elements of the vector spaces $\mathrm{Hom}_{SU(2)}\left(V^{\otimes k}, V^{\otimes \ell}\right)$ where $k, \ell \in \mathbb{N}$. This category is monoidal and $\mathbb{C}$-linear. We define these and other categorical notions in Section 2.3 when we discuss diagrammatic categories.

This realization of a portion of the representation theory of $SU(2)$ leads to the notion of a representation graph which we explore in the next section.

2.2. Representation Graphs and the McKay Correspondence

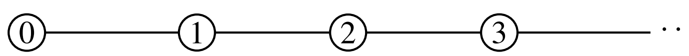
This section is a summary of the work in [3], which covers this material more comprehensively. Their work provided important motivating ideas for the constructions in this dissertation.

Let us set some notation. Let $\{G^{(a)}\}_{a \in A}$ be a set of isomorphism class representatives for the simple G-modules. Let V be some G-module, not necessarily simple.

Definition 2.2. *The representation graph $R(V, G)$ is a directed graph with nodes* *labeled by $a \in A$, and if $V \otimes G^{(a)} \cong \bigoplus_b \left(G^{(b)}\right)^{m_b}$ where m_b is the multiplicity of $G^{(b)}$ in $V \otimes G^{(a)}$, $R(V, G)$ has m_b directed edges from node a to node b. In the event that there is a pair of directed edges, one from a to b and one from b to a, we will represent this by a single undirected edge between a to b.*

To illustrate the definition, let us construct an example explicitly.

Example 2.3. *Let $G = SU(2)$ and let $V = \mathbb{C}^2$. Let $G^{(a)} = V(a)$ for $a \in A := \mathbb{N}$. Notice that $V = V(1) = G^{(1)}$ is simple, and in fact for each $a \in \mathbb{N}$, there is one irreducible G-module of dimension $a + 1$. From the Clebsch–Gordon formula 2.2, $V \otimes G^{(a)} = G^{(a-1)} \oplus G^{(a+1)}$ for all $a \in \mathbb{N}$. Thus, the representation graph is the undirected graph*



where the node a corresponds to $G^{(a)} = V(a)$.

In the 19th century, Felix Klien classified all the finite subgroups of $SU(2)$. There are two families indexed by $n \in \mathbb{N}$: the cyclic groups $\mathbf{C}_n$ and the binary dihedral groups $\mathbf{D}_n$; along with three exceptional groups: the binary tetrahedral group, **T**; binary octahedral group, **O**; and binary icosahedral group, **I**. In 1980, McKay made his rather beautiful observation that the representation graphs $R(G, V)$ of these groups using the natural module V for $SU(2)$ as the defining module are in one-to-one correspondence with the affine Dynkin diagrams of certain types. The following example makes explicit the correspondence when considering the binary tetrahedral group.

Example 2.4. *The binary tetrahedral group* $\mathbf{T}$ *is generated by* X*,* Y*, and* A *where*

$$X = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad A = \frac{1}{2}\begin{pmatrix} 1+i & 1+i \\ -1+i & 1-i \end{pmatrix}$$

and $i = \sqrt{-1}$*. Furthermore, the simple* $\mathbf{T}$*-modules can be characterized as follows: there are three* 1*-dimensional simple* $\mathbf{T}$*-modules which we will call* $T^{(0)}$*,* $T^{(4)}$*, and* $T^{(4')}$*, three* 2*-dimensional simple* $\mathbf{T}$*-modules which we will call* $T^{(1)}$*,* $T^{(3)}$*, and* $T^{(3')}$*, and one* 3*-dimensional simple* $\mathbf{T}$*-modules which we will call* $T^{(2)}$*. To make this construction explicit, we fix an isomorphism class representative for each simple* $\mathbf{T}$*-module.*

$T^{(0)}$ *is the trivial module.*

$T^{(1)} = \mathbb{C}$*-span*$\{w_{-1}, w_1\}$ *where*

$Xw_{-1} = iw_{-1}$, $Xw_1 = -iw_1$, $Yw_{-1} = -w_1$, $Yw_1 = w_{-1}$,

$Aw_{-1} = \frac{1}{2}(1+i)w_{-1} + \frac{1}{2}(i-1)w_1$, $Aw_1 = \frac{1}{2}(1+i)w_{-1} - \frac{1}{2}(i-1)w_1$.

$T^{(2)} = \mathbb{C}$*-span*$\{w_{-2}, w_2, w_{0'}\}$ *where*

$Xw_{-2} = -w_{-2}$, $Xw_2 = -w_2$, $Xw_{0'} = -w_{0'}$,

$Yw_{-2} = w_2$, $Yw_2 = w_{-2}$, $Yw_{0'} = -w_{0'}$,

$Aw_{-2} = \frac{1}{2}iw_{-2} - \frac{1}{2}iw_2 - \frac{1}{2}w_{0'}$, $Aw_2 = \frac{1}{2}iw_{-2} - \frac{1}{2}iw_2 + \frac{1}{2}w_{0'}$, *and* $Aw_{0'} = iw_{-2} + iw_2$.

$T^{(3)} = \mathbb{C}$*-span*$\{w_{-3}, w_3\}$ *where*

$Xw_{-3} = iw_{-3}$, $Xw_3 = -iw_3$, $Yw_{-3} = -w_3$, $Yw_3 = w_{-3}$,

$Aw_{-3} = \frac{1}{4}(\sqrt{3} - 1 - i(1+\sqrt{3}))w_{-3} + \frac{1}{4}(\sqrt{3} + 1 + i(-1+\sqrt{3}))w_3$,

$Aw_3 = \frac{1}{4}(\sqrt{3} - 1 - i(1+\sqrt{3}))w_{-3} - \frac{1}{4}(\sqrt{3} + 1 + i(-1+\sqrt{3}))w_3$.

$T^{(3')} = \mathbb{C}$*-span*$\{w_{-3'}, w_{3'}\}$ *where*

$Xw_{-3'} = iw_{-3'}$, $Xw_{3'} = -iw_{3'}$, $Yw_{-3'} = -w_{3'}$, $Yw_{3'} = w_{-3'}$,

$Aw_{-3'} = \frac{1}{4}(-\sqrt{3} - 1 + i(-1 + \sqrt{3}))w_{-3'} + \frac{1}{4}(-\sqrt{3} + 1 - i(1 + \sqrt{3}))w_{3'}$,

$Aw_{3'} = \frac{1}{4}(-\sqrt{3} - 1 + i(-1 + \sqrt{3}))w_{-3'} - \frac{1}{4}(-\sqrt{3} + 1 - i(1 + \sqrt{3}))w_{3'}$.

$T^{(4)} = \mathbb{C}$-*span*$\{w_4\}$ *where*

$Xw_4 = w_4$, $Yw_4 = w_4$, $Aw_4 = \frac{1}{2}(-i\sqrt{3} - 1)w_4$.

$T^{(4')} = \mathbb{C}$-*span*$\{w_4\}$ *where*

$Xw_{4'} = w_{4'}$, $Yw_{4'} = w_{4'}$, $Aw_{4'} = \frac{1}{2}(i\sqrt{3} - 1)w_{4'}$.

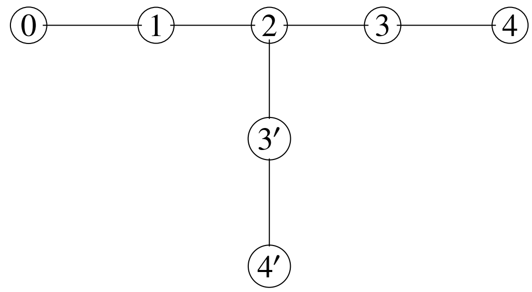
Notice that $T^{(1)} \cong V$ *where* V *is the natural module for* $SU(2)$. *Now, we are ready to build the representation graph* $R(V, \mathbf{T})$. *Firstly, the ith node of* $R(V, \mathbf{T})$ *corresponds to the simple* $\mathbf{T}$*-module* $T^{(i)}$. *Using the definition of the simples above, we can compute explicitly the direct sum decompositions of certain modules. In particular,*

$$T^{(1)} \otimes T^{(0)} \cong T^{(1)}, \qquad T^{(1)} \otimes T^{(1)} \cong T^{(0)} \oplus T^{(2)}, \qquad T^{(1)} \otimes T^{(4')} \cong T^{(3')} \tag{2.3}$$

$$T^{(1)} \otimes T^{(4)} \cong T^{(3)}, \qquad T^{(1)} \otimes T^{(3)} \cong T^{(2)} \oplus T^{(4)}, \qquad T^{(1)} \otimes T^{(3')} \cong T^{(2)} \oplus T^{(4')}, \tag{2.4}$$

$$\text{and } T^{(1)} \otimes T^{(2)} \cong T^{(1)} \oplus T^{(3)} \oplus T^{(3')} \tag{2.5}$$

Thus,



(2.6)

is the realization of the representation graph $R(V, \mathbf{T})$. *Observe that this is the affine Dynkin diagram* $\hat{E}_6$.

In a similar manner, the representation graphs for the other finite subgroups of $SU(2)$, $\mathbf{C}_n$, $\mathbf{D}_n$, $\mathbf{O}$, and $\mathbf{I}$, respectively correspond to the Dynkin diagram $\hat{A}_{n-1}$, $\hat{D}_{n+2}$, $\hat{E}_7$, and $\hat{E}_8$.

It is advantageous for this thesis to establish some notation. Given a representation graph $R(V, G)$, we let $P(a, b)$ be the set of all paths from a to b. We let $P(a, b)_k$ be the subset of $P(a, b)$ consisting of all paths of length k. A path $\mathbf{p} \in P(a, b)_k$ can be identified with a k-tuple $\mathbf{p} = (a, b_1, b_2, \ldots, b_{k-1}, b)$ which traverses the nodes $b_i \in I_G$ for $i \in \{1, 2, \ldots, k-1\}$.

Example 2.5. *Considering the representation graph of* $\mathbf{T}$*,* $R\left(T^{(1)}, \mathbf{T}\right)$ *from (2.6). There are* 5 *paths of length* 4 *from the node labeled by* 1 *to the node labeled by* 3*. Thus,* $P(1, 3)_4$ *has* 5 *elements, namely* $(1, 2, 3, 4, 3)$*,* $(1, 2, 3, 2, 3)$*,* $(1, 2, 1, 2, 3)$*,* $(1, 2, 3', 2, 3)$*, and* $(1, 0, 1, 2, 3)$.

Using (2.3), we know that $T^{(3)}$ *is a direct summand of* $\left(T^{(1)}\right)^{\otimes 5}$ *and has multiplicity* 5*. Consider the path* $\mathbf{p} = (1, 2, 3, 4, 3)$*. Each path corresponds to a unique isomorphic copy of* $T^{(3)}$ *as a submodule of* $\left(T^{(1)}\right)^{\otimes 5}$ *in a canonical way. This* *construction is given in general for a group G in 6.1.*

2.3. Diagrammatic Categories

As the main goal of this dissertation is to develop diagrammatic categories which describe certain categories of representations, let us begin with a few categorical notions. In order to define a category, one must give a collection of objects and a collection of morphisms which contains the identity morphism for each object, are closed under composition, and satisfy associativity. The diagrammatic categories in this dissertation will all be strict, monoidal, and $\mathbb{C}$-linear. The following are the necessary definitions from [8] with some of the technical details suppressed.

Definition 2.6. *A* monoidal category *is a quintuple $(C, \otimes, a, \mathbf{1}, \iota)$, where C is a category, $\otimes : C \otimes C \longrightarrow C$ is a bifunctor called the tensor product bifunctor, $a : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ is the* associator *and a natural isomorphism for all objects X, Y, and Z in C, $\mathbf{1}$ is an object of C, and $\iota : \mathbf{1} \otimes \mathbf{1} \xrightarrow{\sim} \mathbf{1}$ is the* unitor *and an isomorphism, all subject to the pentagon axiom and the unit axiom.*

Essentially, a monoidal category allows for tensor products of objects and morphisms in which there is an associator and a unit object. A $\mathbb{C}$-linear category asserts that the class of morphisms are in fact vector spaces over the field $\mathbb{C}$ and with composition acting linearly.

In a similar way to group or monoid presentation, we can define a $\mathbb{C}$-linear monoidal category using generators and relations. For a technical discussion of this, see [8, 9]. Let C be a monoidal category. A collection S of objects in C

generates the objects of $\mathcal{C}$ if every object can be realized as the tensor product of elements of S. Furthermore, a collection M of morphisms in $\mathcal{C}$ generates the morphisms of $\mathcal{C}$ if every morphism can be realized using linear combinations, compositions, and tensor products of elements of M. On the other hand, given a set of objects S and a set of morphisms M, we can construct the free monoidal category on these sets. One can also impose *relations* on morphisms between objects. Let R be a collection of relations for morphisms in $\mathcal{C}$, and let $\mathcal{I}_R$ be the tensor ideal generated by R. If $\mathcal{C}$ is generated by S and M, then the quotient category $\mathcal{C}/\mathcal{I}_R$ is said to be generated by S and M subject to the relations R.

Definition 2.7. *A* strict *monoidal category is a monoidal category in which the associator and the unitor are identity morphisms.*

There is a subtle issue with the functors in this dissertation. All of our diagrammatic categories are strict, yet the target categories are from representation theory, and the unitor of the category of G-modules for a group G is not the identity morphism. However, this is not really an issue since we have Mac Lane's Strictness Theorem from [8, 9].

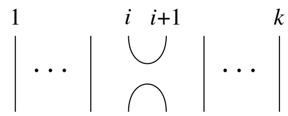
Theorem 2.8. *Any monoidal category is monoidally equivalent to a strict monoidal category.*

In order to give an example of the above definitions, let us first discuss some motivation. Much of our discussion will be centered around defining diagrammatic algebras and categories which are specifically designed to mirror the workings of a category coming from representation theory.

For example, consider the well-known Temperley–Lieb algebra $TL_k(\delta)$ which can be defined by generators $e_1, \ldots, e_{k-1}$ and subject to the relations

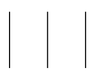
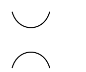
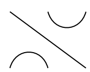
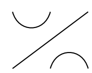
$$e_i^2 = \delta e_i, \; e_i e_{i\pm 1} e_i = e_i, \text{ and } e_i e_j = e_j e_i \text{ for } |i-j| > 1.$$

The algebra $TL_k(\delta)$ can be viewed diagrammatically as well where

$e_i :=$ 

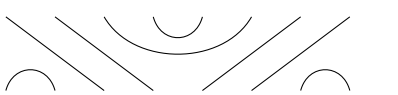
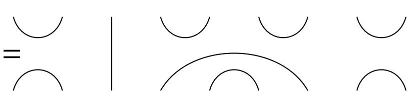
.

Then the Temperley–Lieb algebra $TL_3(\delta)$ has a basis given by the following diagrams:

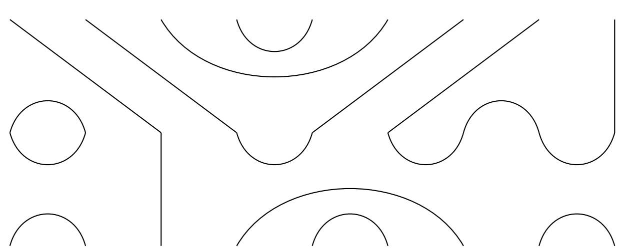
 ,  ,  ,  , and  .

The composition product is given by vertically stacking diagrams as shown in the next example. Furthermore, whenever there is a closed connected component, we delete it and multiply the resulting diagram by a factor of δ.

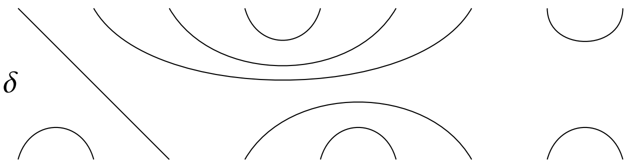
Example 2.9. *Let*

$d_1 =$  *and* $d_2 =$  .

We connect the diagrams in the obvious way:



and we use isotopies to straighten out connected components, as well as delete any connected components contained completely in the middle of the diagram to get

$d_1 \circ d_2 := \delta$ 

By setting $\delta = 2$, we get the following theorem.

Theorem 2.10. *[10] For all $k \geq 0$, there is an isomorphism of algebras*

$$TL_k(2) \xrightarrow{\cong} End_{SU(2)}\left(V^{\otimes k}\right).$$

Thus, we have a diagrammatic presentation for the endomorphism algebra $\mathrm{End}_{SU(2)}(V^{\otimes k})$.

We are now ready to give an example of a monoidal $\mathbb{C}$-linear category given by generators and relations. In particular, we can generalize this description and obtain the Temperley–Lieb category $TL(\delta)$ by allowing the number of vertices on top and bottom to vary. Thus, $TL(\delta)$ can be defined as the monoidal $\mathbb{C}$-linear category generated by one object $\bullet$ and the morphisms

 ,  , and  .

Composition is given by vertical concatenation, when this is possible. The monoidal product is given by horizontal concatenation. These operations are subject to the same relations as above, namely isotopy equivalence and a factor of δ gets multiplied for each closed connected component deleted. There is then a fully faithful functor

$$TL(2) \xrightarrow{\cong} SU(2)\text{-}\mathbf{mod}$$

given on objects by $\bullet^{\otimes k} \mapsto V^{\otimes k}$. This functor defines an equivalence into $SU(2)$-$\mathbf{mod}_V$. From this equivalence, we have a diagrammatic basis for the spaces of $SU(2)$-invariant homomorphisms, $\mathrm{Hom}_{SU(2)}\left(V^{\otimes k}, V^{\otimes \ell}\right)$ for all $k, \ell \in \mathbb{Z}$. In particular, the non-crossing diagrams with k nodes on bottom and ℓ nodes on top and where each node has valence precisely 1 form this basis.

Chapter 3

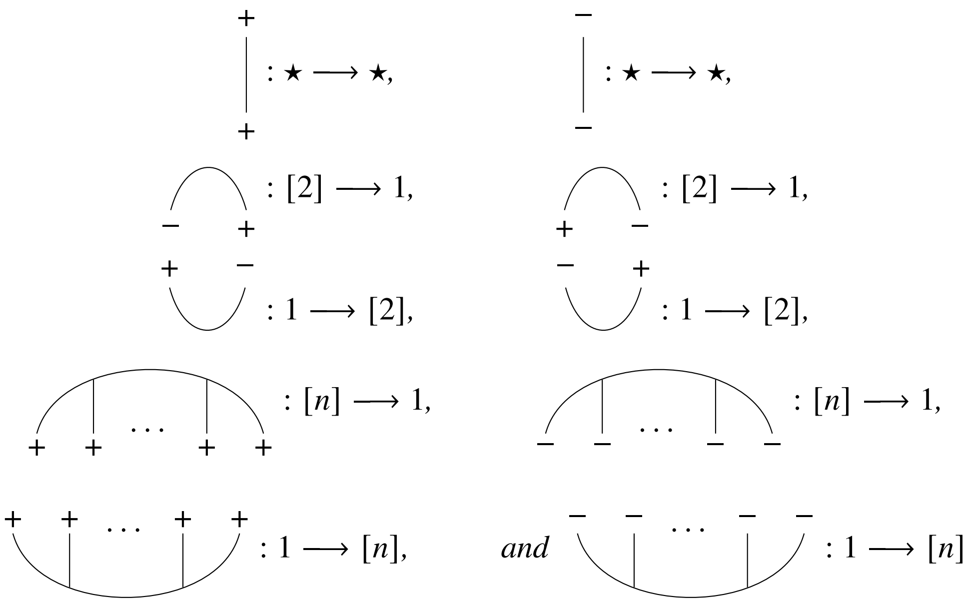
Categorical Equivalence of $C_n^\star$

We are now ready to introduce some diagrammatic categories. We begin by developing a diagrammatic category with the cyclic group, $\mathbf{C}_n$, in mind. Of all the groups that we will consider, these are by far the nicest behaved. The representation theory is well-known and established in many contexts.

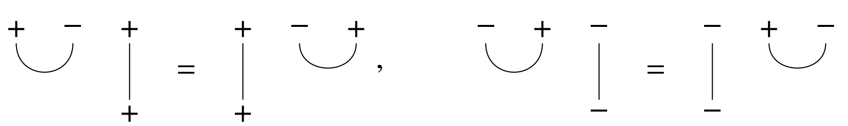
3.1. The Category $C_n^\star$

First, we establish some notation: Let $I_k = \{\epsilon := (\epsilon_1, \epsilon_2, \ldots, \epsilon_k) | \epsilon_i \in \{+, -\}\}$. Fix an $\epsilon \in I_k$. Let $|\epsilon_+|$ be the number of $+$ components in ϵ and $|\epsilon_-|$ be the number of $-$ components in ϵ, and let $|\epsilon| := ||\epsilon_+| - |\epsilon_-||$ be the absolute value of the difference of $|\epsilon_+|$ and $|\epsilon_-|$. Given a fixed $\star$, we use the convention that $[k] := \star \cdots \star$ is the concatenation of k $\star$'s, and $[0]$ corresponds to the empty concatenation; furthermore, we will follow the convention that the empty diagram is the identity morphism from $[0]$ to $[0]$.

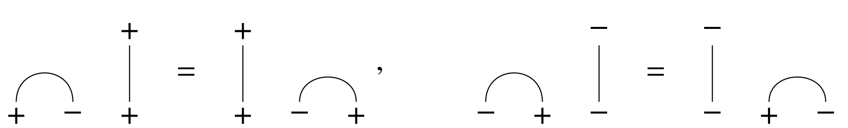
Definition 3.1. *We let $C_n^\star$ be the $\mathbb{C}$-linear monoidal category generated by the unique object $\star$ with the tensor product being defined as horizontal concatenation* *and eight morphisms:*



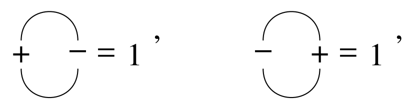
where the composition of diagrams is vertical stacking from bottom to top. By convention, if a − and a + are matched anywhere as a result of stacking, the result is the 0 *morphism. The tensor product of diagrams is horizontal concatenation. The generators are subject to the following relations:*



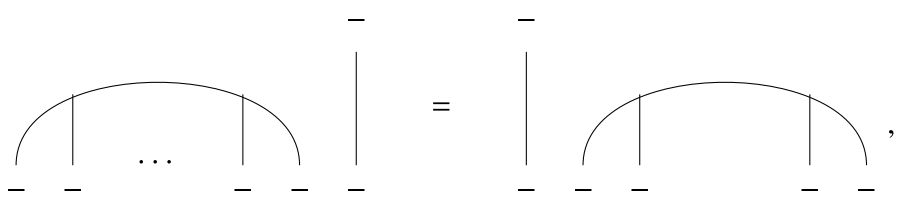
(3.1)



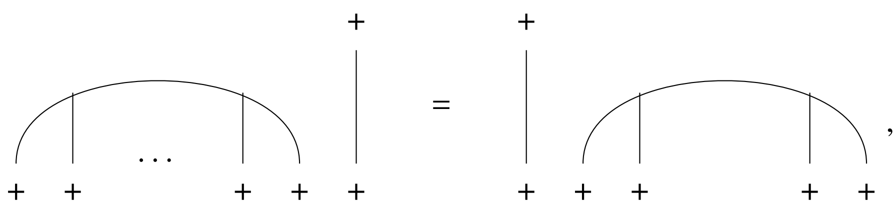
(3.2)



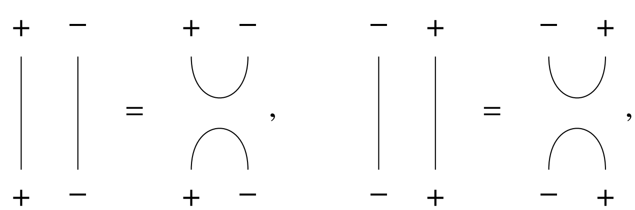
(3.3)



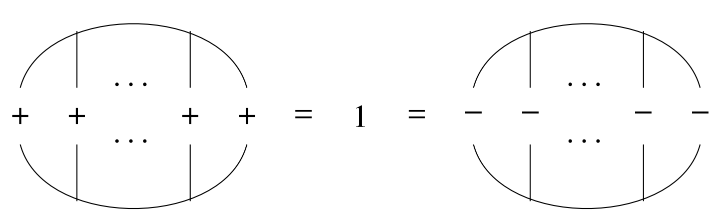
(3.4)



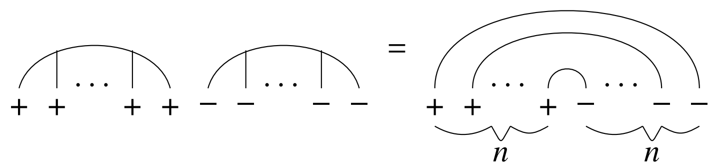
(3.5)



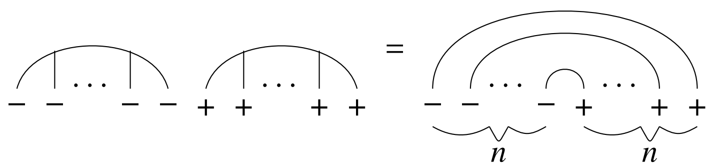
(3.6)



(3.7)



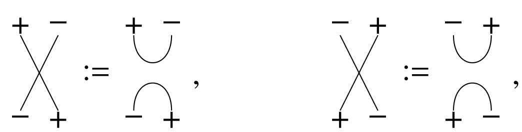
, *and* (3.8)



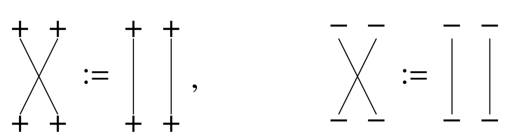
. (3.9)

Objects from this category are $[k] = \star \cdots \star$ for all $k \in \mathbb{N}$, and the morphisms of this category are elements of $\mathrm{Hom}_{C_n^\star}([k],[l])$ and are $\mathbb{C}$-linear combinations of diagrams with an element from I_k on bottom and an element from I_l on top.

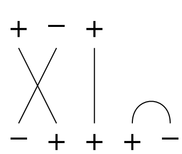
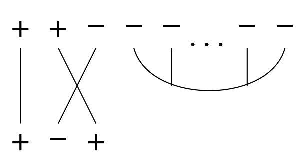
It is convenient to introduce the following crossing diagrams:

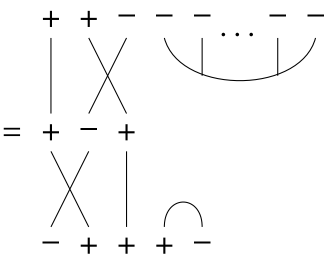


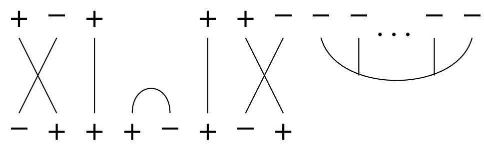
(3.10)



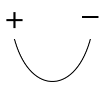
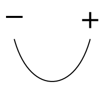
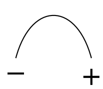
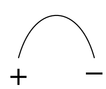
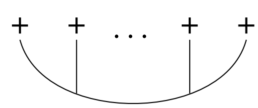
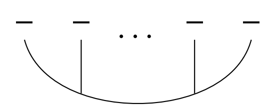
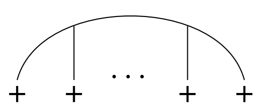
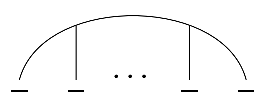
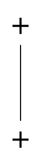
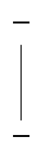
(3.11)

Example 3.2. *Let* $d_1 =$  *and* $d_2 =$  .

Then $d_1 \circ d_2 =$  ,

and $d_1 \otimes d_2 =$ 

Lemma 3.3. *Let* $\epsilon \in I_k$ *and* $\delta \in I_l$. *If there is a diagram with labeling* ϵ *on bottom and* δ *on top, then we must have* $|\epsilon| \equiv |\delta| \mod n$.

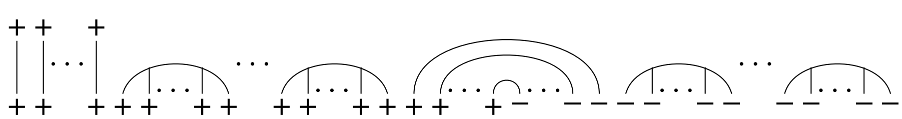
Proof. Considering how the generating diagrams tensor together, any of the diagrams  ,  ,  , or  will have a net 0 addition to either the $|\epsilon|$ or $|\delta|$. Any of the diagrams  ,  ,  , or  will either add or subtract n to either the $|\epsilon|$ or $|\delta|$, and the diagrams  or  will add or subtract 1 from both the $|\epsilon|$ and $|\delta|$ simultaneously. Thus $|\epsilon| \equiv |\delta| \mod n$. □

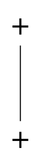
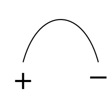
Lemma 3.4. *Any diagram generated by the above diagrams can be reflected across its horizontal axis to create a new diagram which is also generated by the above diagrams.*

Proof. The claim is true for the generating diagrams by inspection. Considering any generating diagram, there exists a diagram which is the exact vertical reflection. For the identity strands, they are the vertical reflection of themselves. The cup and cap diagrams are vertical reflections of each other. Likewise, the grouping and ungrouping diagrams are vertical reflections of each other. Therefore, for each diagram constructed by the above generators the vertical reflection can also be generated. □

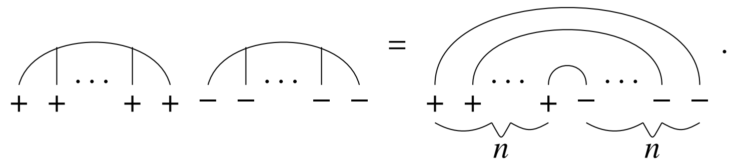
Theorem 3.5. *Any two diagrams with the same labelings are equal as morphisms in $C_n^\star$.*

Proof. Using 3.11, it is easy to see that any crossings in a diagram can be uncrossed, and thus, any crossing diagram is equivalent to some non-crossing diagram. Hence, we need only show that all non-crossing diagrams are equivalent. Let $\underline{i} = (+, \ldots, +, -, \ldots, -)$ with $|\underline{i}| \geq 0$ and $\underline{j} = (+, \ldots, +)$ where $0 \leq |\underline{j}| < n$ and $|\underline{i}| \equiv |\underline{j}| \mod n$. If $|\underline{i}| < 0$, the argument is completely analogous to the following with the + and − components exchanged. We show that any non-crossing diagram from $\underline{i}$ to $\underline{j}$ is equivalent to the following diagram which we will call d_0:



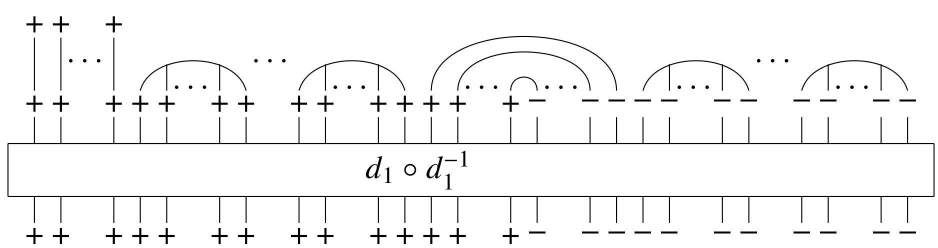
where there are precisely $|\underline{j}|$ , and the number of  is less than n.

Given any diagram d, we may use the relation (3.4) to move all of the n-pairings of minuses to the right. As $\underline{j} = (+, \ldots, +)$, there are no minus identity strands, which means every minus is either paired with $n - 1$ other minuses or with a +. Now we may use the relation (3.5) to move any n-pairing of pluses to the left of any plus paired with a minus. Now we may reduce the number of plus-minus pairings modulo n by using the relation



Finally, by using relation (3.5) again, we can move any plus identity strands to the far left, and we are left with the desired diagram. Thus all diagrams from $\underline{i}$ to $\underline{j}$ are equivalent.

Without loss of generality, we may assume that $\underline{i}$ consists of s + components and t − components. We construct a diagram d_1 in the following way: for any configuration $\underline{x}$ of s + components and t − components with $s \geq t$, we may find the first + in $\underline{x}$, reading the tuple from left to right. We may send $\underline{i}$ to $\underline{x}$ by sending this first + to the first + in $\underline{i}$, and then iterating this process for the remaining + components in $\underline{x}$. Now we may repeat this process for the − components in $\underline{x}$. The resulting diagram is a permutation of the components, and thus invertible by reversing this process. We may now stack diagrams in the following way:



which is a diagram from $\underline{i}$ to $\underline{j}$. Now we observe two things. The first is that any diagram from $\underline{x}$ to $\underline{i}$ is invertible or the 0 map, and the second is that, since all diagrams from $\underline{i}$ to $\underline{j}$ are equivalent, all diagrams from $\underline{x}$ to $\underline{j}$ are also equivalent. Therefore, as we have the vertical symmetry of diagrams, this shows that any two diagrams with the same labelings are equivalent. □

Given $\epsilon \in I_k$ and $\delta \in I_l$ such that $|\epsilon| \equiv |\delta| \mod n$, by 3.3 and 3.5 there exists a diagram d_ϵ^δ in $\mathrm{Hom}_{C_n^\star}([k],[l])$, and any such diagrams are all equal as morphisms.

Corollary 3.6. *$Hom_{C_n^\star}([k],[l])$ is spanned by $\{d_\epsilon^\delta \in Hom_{C_n^\star}([k],[l]) | \epsilon \in I_k, \delta \in I_l\}$.*

3.2. The Functor $\mathcal{F}_n$

Let $\mathbf{C}_n$ be the subgroup of $SU(2)$ generated by $g = \begin{pmatrix} \xi_n & 0 \\ 0 & \xi_n^{-1} \end{pmatrix}$ where ξ_n is a fixed primitive nth root of unity. We let $\mathbf{C}_n$**-mod** be the category of finite-dimensional $\mathbf{C}_n$-modules. Recall, $V = \mathbb{C}^2$ is the natural module for $SU(2)$.

Definition 3.7. *Let $\mathbf{C}_n$**-mod**$_V$ be the $\mathbb{C}$-linear monoidal full-subcategory generated by the object V and the class of morphisms is the collection of sets $Hom_{\mathbf{C}_n}(V^{\otimes k}, V^{\otimes l})$ where we let $k,\ l \in \mathbb{N}$.*

Let $v_{-1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ be the standard basis for V.

Definition 3.8. *For $n \geq 1$, define a functor $\mathcal{F}_n : C_n^\star \longrightarrow \mathbf{C}_n$**-mod**$_V$ sending object to object by the rule $\star \mapsto V$, so $[k] \mapsto V^{\otimes k}$. On the generating morphisms, $\mathcal{F}_n$ is given by the rules given in figure 3.1.*

Theorem 3.9. *The functors $\mathcal{F}_n$ are well defined.*

Proof. Recall from definition 3.1 the relations imposed on the diagrams. We check

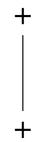
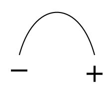
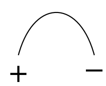
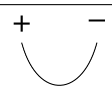
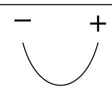
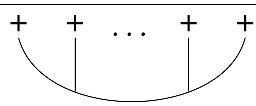
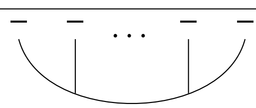
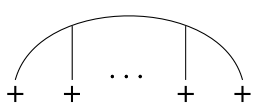
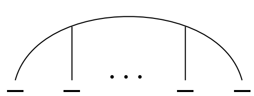
Definition of the Functor $\mathcal{F}_n$ for morphisms		
Diagram D	Domain and Codomain of $\mathcal{F}_n(D)$	Definition of $\mathcal{F}_n(D)$
	$V \longrightarrow V$	$\begin{cases} v_{-1} \mapsto 0 \\ v_1 \mapsto v_1 \end{cases}$
	$V \longrightarrow V$	$\begin{cases} v_{-1} \mapsto v_{-1} \\ v_1 \mapsto 0 \end{cases}$
	$V^{\otimes 2} \longrightarrow V^{\otimes 0}$	$\begin{cases} v_{-1} \otimes v_1 \mapsto 1 \\ \text{otherwise} \mapsto 0 \end{cases}$
	$V^{\otimes 2} \longrightarrow V^{\otimes 0}$	$\begin{cases} v_1 \otimes v_{-1} \mapsto 1 \\ \text{otherwise} \mapsto 0 \end{cases},$
	$V^{\otimes 0} \longrightarrow V^{\otimes 2}$	$1 \mapsto v_1 \otimes v_{-1}$
	$V^{\otimes 0} \longrightarrow V^{\otimes 2}$	$1 \mapsto v_{-1} \otimes v_1$
	$V^{\otimes 0} \longrightarrow V^{\otimes n}$	$1 \mapsto v_1 \otimes \cdots \otimes v_1$
	$V^{\otimes 0} \longrightarrow V^{\otimes n}$	$1 \mapsto v_{-1} \otimes \cdots \otimes v_{-1}$
	$V^{\otimes n} \longrightarrow V^{\otimes 0}$	$v_1 \otimes \cdots \otimes v_1 \mapsto 1$
	$V^{\otimes n} \longrightarrow V^{\otimes 0}$	$v_{-1} \otimes \cdots \otimes v_{-1} \mapsto 1$

Figure 3.1: Defining the functor $\mathcal{F}_n$ on morphisms

that these relations are satisfied after the application of $\mathcal{F}_n$. We will abbreviate our calculations somewhat by defining the left hand side of an equation d_1 and the right hand side of the equation d_2. Also, we will consider one equation from each line in definition 3.1 as any other equations in the same line are analogous by switching -1 and 1.

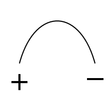
In (3.1), we consider the first equality, and apply $\mathcal{F}_n$ to each side:

$$\mathcal{F}_n(d_1)(v_{-1}) = 0 = \mathcal{F}_n(d_2)(v_{-1}),$$
$$\mathcal{F}_n(d_1)(v_1) = v_1 \otimes v_{-1} \otimes v_1,$$
$$\text{and } \mathcal{F}_n(d_2)(v_1) = v_1 \otimes v_{-1} \otimes v_1.$$

Considering (3.2) and applying $\mathcal{F}_n$:

$$\mathcal{F}_n(d_1)(v_1 \otimes v_{-1} \otimes v_1) = 1 \otimes v_1 = v_1$$
$$\text{and } \mathcal{F}_n(d_2)(v_1 \otimes v_{-1} \otimes v_1) = v_1 \otimes 1 = v_1,$$

and $\mathcal{F}_n(d_1)$ and $\mathcal{F}_n(d_2)$ applied to any other basis element of $V^{\otimes 3}$ is 0.
Considering (3.3) and applying $\mathcal{F}_n$:

$$\mathcal{F}_n(d_1)(1) = \mathcal{F}_n\left(\right.$$


$$\left.\right)(v_1 \otimes v_{-1}) = 1$$

Considering (3.4) and applying $\mathcal{F}_n$:

$$\mathcal{F}_n(d_1)(v_{-1} \otimes v_{-1} \otimes \cdots \otimes v_{-1} \otimes v_{-1} \otimes v_{-1}) = 1 \otimes v_{-1} = v_{-1}$$

$$\text{and } \mathcal{F}_n(d_2)(v_{-1} \otimes v_{-1} \otimes v_{-1} \otimes \cdots \otimes v_{-1} \otimes v_{-1}) = v_{-1} \otimes 1 = v_{-1},$$

where in both cases, the basis element

$$v_{-1} \otimes v_{-1} \otimes \cdots \otimes v_{-1} \otimes v_{-1} \otimes v_{-1} = v_{-1} \otimes v_{-1} \otimes v_{-1} \otimes \cdots \otimes v_{-1} \otimes v_{-1}$$

is the tensor of v_{-1} $n+1$-many times. Also, $\mathcal{F}_n(d_1)$ and $\mathcal{F}_n(d_2)$ applied to any other basis element of $V^{\otimes(n+1)}$ is 0.

Note that for (3.5), the calculation is analogous to the case of (3.4) but applied to the tensor product of v_1 $n+1$-many times.

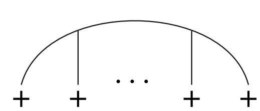
Considering (3.6) and applying $\mathcal{F}_n$:

$$\mathcal{F}_n(d_1)(v_1 \otimes v_{-1}) = v_1 \otimes v_{-1}$$
$$\text{and } \mathcal{F}_n(d_2)(v_1 \otimes v_{-1}) = v_1 \otimes v_{-1}.$$

and $\mathcal{F}_n(d_1)$ and $\mathcal{F}_n(d_2)$ applied to any other basis element of $V^{\otimes 2}$ is 0.

Considering (3.7) and applying $\mathcal{F}_n$:

$$\mathcal{F}_n(d_1)(1) = \mathcal{F}_n\left(\quad \right)(v_1 \otimes v_1 \otimes \cdots \otimes v_1 \otimes v_1) = 1.$$



Considering (3.7) and applying $\mathcal{F}_n$:

$$\mathcal{F}_n(d_1)(v_1 \otimes v_1 \otimes \cdots \otimes v_1 \otimes v_1 \otimes v_{-1} \otimes v_{-1} \otimes \cdots \otimes v_{-1} \otimes v_{-1}) = 1,$$

where the basis element is the tensor product of v_1 n-many times followed by

v_{-1} n-many times, and $\mathcal{F}_n(d_1)$ and $\mathcal{F}_n(d_2)$ applied to any other basis element of $V^{\otimes 2n}$ is 0.

Note that for (3.8), the calculation is analogous to the case of (3.7) but applied to the tensor product of v_{-1} n-many times followed by v_1 n-many times.

These calculations show that the functors $\mathcal{F}_n$ are well-defined.

□

Theorem 3.10. *The functors $\mathcal{F}_n$ are faithful.*

Proof. Fix $n \in \mathbb{N}$. Suppose that $\mathcal{F}_n\left(\sum_{\substack{\delta \in I_k, \\ \epsilon \in I_\ell \\ |\epsilon| \equiv |\delta| \mod n}} A^\epsilon_\delta d^\epsilon_\delta\right) = 0$, where d^ϵ_δ is a diagram in $\mathrm{Hom}_{C_n^\star}([k], [\ell])$ and $A^\epsilon_\delta \in \mathbb{C}$.

Let $v_{\underline{i}} := v_{i_1} \otimes \cdots \otimes v_{i_k} \in V^{\otimes k}$. By linearity, we have

$$\mathcal{F}_n\left(\sum_{\substack{\delta,\epsilon \\ |\epsilon| \equiv |\delta| \mod n}} A^\epsilon_\delta d^\epsilon_\delta\right) v_i = \sum_{\substack{\delta,\epsilon \\ |\epsilon| \equiv |\delta| \mod n}} A^\epsilon_\delta \mathcal{F}_n\left(d^\epsilon_\delta\right) v_i = 0$$

for each $v_{\underline{i}} \in V^{\otimes k}$.

By definition, $\mathcal{F}_n\left(d^\epsilon_\delta\right) v_{\underline{i}} = \begin{cases} v_\epsilon & \text{for } \underline{i} = \delta \\ 0 & \text{otherwise} \end{cases}$, and thus we must have

$$\sum_{\substack{\delta,\epsilon \\ |\epsilon| \equiv |\delta| \mod n}} A^\epsilon_\delta \mathcal{F}_n\left(d^\epsilon_\delta\right) v_i = \sum_{\substack{\epsilon \\ |\epsilon| \equiv |\underline{i}| \mod n}} A^\epsilon_{\underline{i}} v_\epsilon = 0.$$

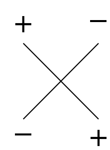
As the v_ϵ are basis elements, we must have that $A^\epsilon_{\underline{i}} = 0$ for all ϵ. As $\underline{i}$ was arbitrarily chosen, we must have that $A^\epsilon_\delta = 0$ for all δ and ϵ. Therefore, $\mathcal{F}_n$ is faithful. □

Corollary 3.11. *The spanning set for $Hom_{C_n^\star}([k],[l])$, $\{d_\epsilon^\delta \in Hom_{C_n^\star}([k],[l])\}$, is linearly independent, and thus a basis.*

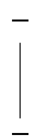
Theorem 3.12. *The functor $\mathcal{F}_n$ is full.*

Proof. As the maps $\mathcal{F}_n$ are faithful, we will abuse notation and identify a diagram in $C_n^\star$ with that of the corresponding morphism in $\mathrm{Hom}_{\mathbf{C}_n\text{-}\mathbf{mod}_V}(V^{\otimes k}, V^{\otimes l})$. As the vertical reflection of each generator is another generator, it suffices to show that the image under $\mathcal{F}_n$ of the generating diagrams generate $\mathrm{Hom}_{\mathbf{C}_n}(V^{\otimes k}, V^{\otimes r})$ where $0 \le r < n$. We consider the basis of $\mathrm{Hom}_{\mathbf{C}_n}(V^{\otimes k}, V^{\otimes r})$ which sends a basis element $v_{\underline{i}}$ of $V^{\otimes k}$ where $v_{\underline{i}} = v_{i_1} \otimes \cdots \otimes v_{i_k}$, $i_1 \dots i_k \in \{1, -1\}$, to some basis element, $v_{\underline{i}'}$, of $V^{\otimes r}$. Furthermore, as $\sum_{j=1}^{k} i_j =: |\underline{i}| \mod n$ determines the irreducible $\mathbf{C}_n$-module which contains $v_{\underline{i}}$, by Schur's Lemma we must have $|\underline{i}| \equiv |\underline{i}'| \mod n$.

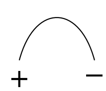
Thus, let $|\underline{i}| \equiv i' \mod n$ where $0 \le i' < n$, then we give an algorithm for finding diagrams which take $v_{\underline{i}}$ to $v_{\underline{i}'}$ where $v_{\underline{i}'} = v_1 \otimes \cdots \otimes v_1$.

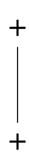
First, by considering the composition of a sequence of 

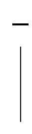
tensored with the required 

and 

diagrams, we may send $v_{\underline{i}}$ to $v_1 \otimes \cdots \otimes v_1 \otimes v_{-1} \otimes \cdots \otimes v_{-1}$ where there are t v_1 tensor factors and $k - t$ v_{-1} tensor factors.

Now we may use 

on the tth and $(t+1)$st positions tensored with the image of the required 

and 

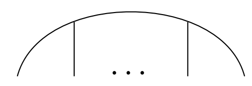
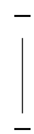
to get

$$v_1 \otimes \cdots \otimes v_1 \otimes 1 \otimes v_{-1} \otimes \cdots \otimes v_{-1}$$

$$= v_1 \otimes \cdots \otimes v_1 \otimes v_{-1} \otimes \cdots \otimes v_{-1} := v_{\underline{i''}}$$

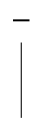
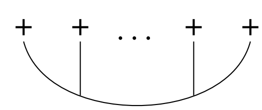
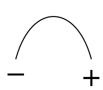
with $t-1$ v_1 tensor factors and $k-t-1$ v_{-1} tensor factors.

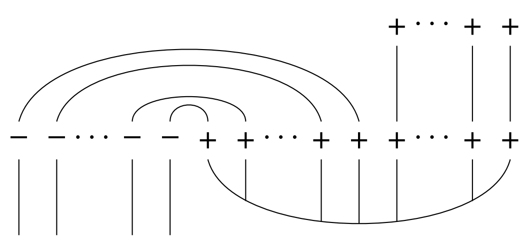
Iterating this process finitely many times will eventually leave us with either $v_{\underline{a}} = v_{-1} \otimes \cdots \otimes v_{-1}$ or $v_{\underline{b}} = v_1 \otimes \cdots \otimes v_1$. Furthermore, $|\underline{a}| \equiv r \equiv |\underline{b}| \mod n$.

If we are left with $v_{\underline{a}}$, then $|\underline{a}| = (-a')n + i'$. Now we may apply the tensor product of $a'-1$  and $i'-n$ . The resulting vector is

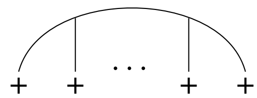
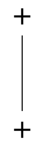
$$v_{\underline{s}} = v_{-1} \otimes \cdots \otimes v_{-1}$$

where $|\underline{s}| = i' - n$.

We may now use the following combination of , , and  to form the morphism which sends $v_{\underline{s}}$ to $v_{\underline{i'}}$:

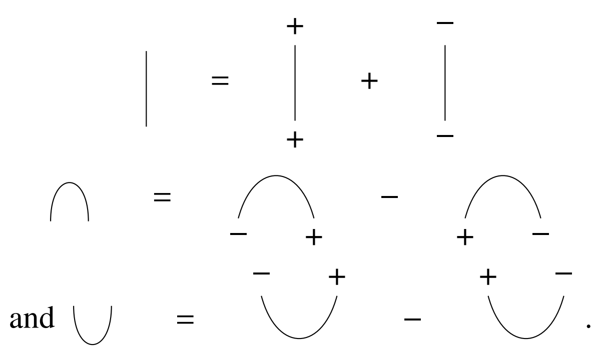


If, however, we are left with $v_{\underline{b}}$, then $|\underline{b}| = b'n + i'$. Now by applying the tensor

product of b'  and i' . The resulting vector is precisely $v_{\underline{i'}}$.
Therefore, the functor $\mathcal{F}_n$ is full. □

Corollary 3.13. *The categories $C_n^\star$ and $\mathbf{C}_n$-$\boldsymbol{mod}_V$ are equivalent as monoidal $\mathbb{C}$-linear categories.*

Recall that, in general, given a subgroup H of a group G and G-modules M and N, $\mathrm{Hom}_G(M, N)$ is a subset of $\mathrm{Hom}_H(M, N)$. With this in mind, one can ask whether the diagrams developed in this chapter combine to become the Temperley–Lieb diagrams in $TL(2)$. The objects of each category are indeed the same. Furthermore, observe the following relationship:



After checking that the relations are satisfied, this gives us a presentation for $TL(2)$ in terms of the diagrams from $C_n^\star$.

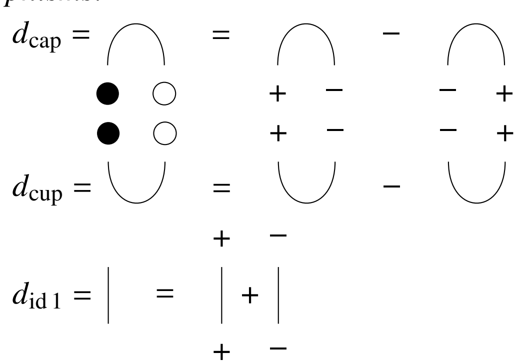
Chapter 4

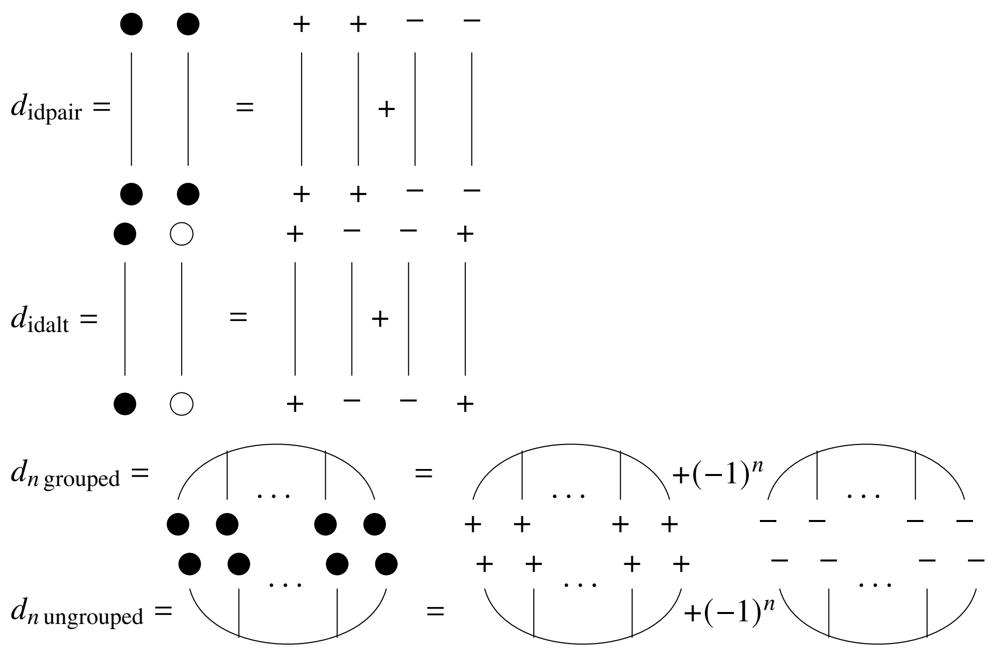
The D_n case

Continuing on with the motivation of constructing these McKay diagrammatic categories, we will move on to the next family of finite subgroups of $SU(2)$. The definitions below benefit from viewing the morphism spaces of $\mathbf{D}_n$-$\mathbf{mod}_V$ as subspaces of categories that we studied in the previous chapter.

We define two families of categories. One family of categories will be defined as subcategories of $C^{\star}_{2n}$, and the other will be defined as subcategories of $\mathbf{C}_{2n}$-$\mathbf{mod}_V$.

Definition 4.1. *We let $\mathcal{D}^{\star}_n$ be the $\mathbb{C}$-linear monoidal subcategory of $C^{\star}_{2n}$ generated by the unique object $\star$ with the tensor product being defined as horizontal concatenation and $[k] := \star \cdots \star$ is the concatenation of k $\star$'s and the following morphisms:*





Consider the family of subgroups $\mathbf{D}_n$ of SU_2 generated by g and h where

$$g = \begin{pmatrix} \xi_{2n}^{-1} & 0 \\ 0 & \xi_{2n} \end{pmatrix}, \qquad h = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{4.1}$$

where ξ_{2n} is a fixed primitive $2n$th root of unity and $i^2 = -1$. We can compute the following relations on the generators g and h: $g^{2n} = 1$, $g^n = h^2$, and $h^{-1}gh = g^{-1}$. Given that V is the SU_2-module described in section 2 and $\mathbf{C}_{2n} \subset \mathbf{D}_n$, we have that $\mathrm{Hom}_{\mathbf{D}_n}(V^{\otimes k}, V^{\otimes l}) \subset \mathrm{Hom}_{\mathbf{C}_{2n}}(V^{\otimes k}, V^{\otimes l})$. From this we will define the subcategory of $\mathbf{C}_{2n}$-$\mathbf{mod}_V$, $\mathbf{D}_n$-$\mathbf{mod}_V$.

Definition 4.2. *Let $\mathbf{D}_n$-**mod**$_V$ be the full $\mathbb{C}$-linear monoidal subcategory of $\mathbf{D}_n$-**mod** generated by the object V. The morphism spaces are precisely* $\mathrm{Hom}_{\mathbf{D}_n}(V^{\otimes k}, V^{\otimes l})$ *for $k, \ell \in \mathbb{N}$.*

Lemma 4.3. *The category $\mathbf{D}_n$-**mod**$_V$ is equal to a monoidal subcategory of $\mathbf{C}_{2n}$-**mod**$_V$.*

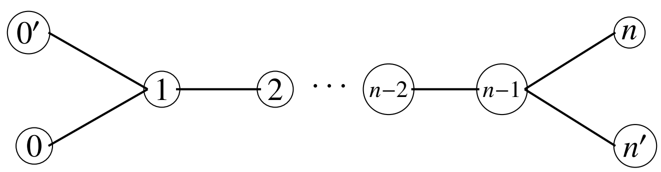
Proof. Let $\mathbf{C}_{2n}^h$ the subcategory of $\mathbf{C}_{2n}$-**mod** with the same defining module, V, and the morphisms $d \in \mathrm{Hom}_{\mathbf{C}_{2n}}\left(V^{\otimes k}, V^{\otimes \ell}\right)$ such that $dh = hd$. By definition, $\mathbf{C}_{2n}^h$ and $\mathbf{D}_n$-**mod**$_V$ have the same defining object.

Furthermore, as $\mathbf{C}_{2n}$ is a subgroup of $\mathbf{D}_n$, morphisms of $\mathbf{D}_n$-**mod**$_V$ belong to the sets $\mathrm{Hom}_{\mathbf{C}_{2n}}(V^{\otimes k}, V^{\otimes l})$. By definition, the morphisms of $\mathbf{D}_n$-**mod**$_V$ commute with the elements of $\mathbf{D}_n$. By 4.1, $\mathbf{D}_n$ is generated by g, which is an element of $\mathbf{C}_{2n}$, and h. Thus, the morphisms of $\mathbf{D}_n$-**mod**$_V$ are precisely the morphisms in $\mathrm{Hom}_{\mathbf{C}_{2n}}(V^{\otimes k}, V^{\otimes l})$ which commute with $h \in \mathbf{D}_n$. Therefore, $\mathbf{C}_{2n}^h = \mathbf{D}_n$-**mod**$_V$. □

As these two categories are equal, we will no longer distinguish between the two, instead denoting them both as $\mathbf{D}_n$-**mod**$_V$.

Now, define the functor $\mathcal{G}_n$ to be the restriction of $\mathcal{F}_{2n}$ to $\mathbf{D}_n$-**mod**$_V$, and thus $\mathcal{G}_n : \mathcal{D}_n^\star \longrightarrow \mathbf{C}_{2n}$-**mod**$_V$ is a faithful functor. We will prove in theorem 4.5 that the functor $\mathcal{G}_n$ is full onto $\mathbf{D}_n$-**mod**$_V$. First, we will need to prove a lemma. To that end, it will be advantageous to describe the irreducible $\mathbf{D}_n$-modules and how we may view these irreducible modules as they appear as submodules of $V^{\otimes k}$.

We will do this by following the construction in [3]. By the McKay correspondence, the irreducible $\mathbf{D}_n$ modules can be enumerated by the nodes in the affine Dynkin diagram of type $\hat{D}_n$, $i = 1, \ldots, n-1, 0, 0', n, n'$.



There are four 1-dimensional irreducible modules, which we will denote by $\mathbf{D}_n^{(0)}, \mathbf{D}_n^{(0')}, \mathbf{D}_n^{(n)}$, and $\mathbf{D}_n^{(n')}$, and $n-1$ 2-dimensional irreducible modules, which we will denote by $\mathbf{D}_n^{(j)}$ where $j = 1, \ldots, n-1$, and $V := \mathbf{D}_n^{(1)}$. From Proposition 3.4 in [3], we have the following:

- $\mathbf{D}_n^{(j)} \otimes V \cong \mathbf{D}_n^{(j-1)} \oplus \mathbf{D}_n^{(j+1)}$ for $1 < j < n-1$;
- $\mathbf{D}_n^{(1)} \otimes V \cong \mathbf{D}_n^{(0')} \oplus \mathbf{D}_n^{(0)} \oplus \mathbf{D}_n^{(2)}$;
- $\mathbf{D}_n^{(n-1)} \otimes V \cong \mathbf{D}_n^{(n')} \oplus \mathbf{D}_n^{(n)} \oplus \mathbf{D}_n^{(n-2)}$;
- $\mathbf{D}_n^{(0)} \otimes V \cong \mathbf{D}_n^{(1)} \cong V,\ \mathbf{D}_n^{(0')} \otimes V \cong \mathbf{D}_n^{(1)} \cong V$;
- $\mathbf{D}_n^{(n)} \otimes V \cong \mathbf{D}_n^{(n-1)},\ \mathbf{D}_n^{(n')} \otimes V \cong \mathbf{D}_n^{(n-1)}$.

Lemma 4.4. *If $Hom_{\mathbf{D}_n}(V^{\otimes k}, V^{\otimes l})$ is nontrivial, then $2 \mid (k-l)$.*

Proof. We will consider 0 and 0′ as even and we consider n' as even if n is even and odd if n is odd, then it suffices to show that the irreducible $\mathbf{D}_n$-modules which show up in the direct sum decomposition of $V^{\otimes k}$ are indexed by even numbers if k is even and indexed by odd numbers if k is even.

Thus, let $k = 2k'$ for some $k' \in \mathbb{Z}$. If $k' = 0$ then $k = 0$ and $V^{\otimes k} \cong \mathbb{C} \cong \mathbf{D}_n^{(0)}$, and if $k' = 1$ then $k = 2$ and $V^{\otimes k} \cong \mathbf{D}_n^{(1)} \otimes V \cong \mathbf{D}_n^{(0)} \oplus \mathbf{D}_n^{(0')} \oplus \mathbf{D}_n^{(2)}$.

Let n be even. Now suppose that only even indexed irreducible $\mathbf{D}_n$-modules show up in the decomposition of $V^{\otimes 2k'}$. That is,

$$V^{\otimes 2k'} \cong \left(\mathbf{D}_n^{(0)}\right)^{k_0} \oplus \left(\mathbf{D}_n^{(0')}\right)^{k_{0'}} \oplus \bigoplus_{i=1}^{\frac{n-2}{2}} \left(\mathbf{D}_n^{(2i)}\right)^{k_{2i}} \oplus \left(\mathbf{D}_n^{(n)}\right)^{k_n} \oplus \left(\mathbf{D}_n^{(n')}\right)^{k_{n'}}.$$

Thus we can compute

$$
\begin{aligned}
V^{\otimes(2k'+2)} &\cong \left(\bigoplus_{j\in\{0,0',n,n'\}} \left(\mathbf{D}_n^{(j)}\right)^{k_j} \oplus \bigoplus_{i=1}^{\frac{n-2}{2}} \left(\mathbf{D}_n^{(2i)}\right)^{k_{2i}} \right) \otimes V \otimes V \\
&\cong \left(\bigoplus_{j\in\{0,0'\}} \left(\mathbf{D}_n^{(1)}\right)^{k_j} \oplus \bigoplus_{j\in\{n,n'\}} \left(\mathbf{D}_n^{(n-1)}\right)^{k_j} \oplus \bigoplus_{i=1}^{\frac{n-2}{2}} \left(\left(\mathbf{D}_n^{(2i-1)}\right)^{k_{2i}} \oplus \left(\mathbf{D}_n^{(2i+1)}\right)^{k_{2i}} \right) \right) \otimes V \\
&\cong \left(\mathbf{D}_n^{(0)}\right)^{k_0} \oplus \left(\mathbf{D}_n^{(0')}\right)^{k_{0'}} \oplus \left(\mathbf{D}_n^{(0)}\right)^{k_2} \oplus \left(\mathbf{D}_n^{(0')}\right)^{k_2} \\
&\qquad \oplus \bigoplus_{i=2}^{\frac{n-2}{2}} \left(\left(\mathbf{D}_n^{(2i-2)}\right)^{k_{2i}} \oplus \left(\mathbf{D}_n^{(2i)}\right)^{2(k_{2i})} \oplus \left(\mathbf{D}_n^{(2i+2)}\right)^{k_{2i}} \right) \\
&\qquad\quad \oplus \left(\mathbf{D}_n^{(n-2)}\right)^{k_n} \oplus \left(\mathbf{D}_n^{(n-2)}\right)^{k_{n'}} \oplus \left(\mathbf{D}_n^{(n)}\right)^{k_n} \oplus \left(\mathbf{D}_n^{(n')}\right)^{k_{n'}} .
\end{aligned}
$$

A similar computation can be done for the case where n is odd.

Furthermore, an analogous induction shows that if k is odd, then the irreducible $\mathbf{D}_n$-modules that show up in the direct sum decomposition of $V^{\otimes k}$ are indexed by odd numbers. Therefore, by Schur's Lemma, if $2 \nmid (k-l)$, then $\mathrm{Hom}_{\mathbf{D}_n}(V^{\otimes k}, V^{\otimes l})$ is trivial.

□

Now we are ready for the following theorem.

Theorem 4.5. *The functor $\mathcal{G}_n$ is full onto $\mathbf{D}_n$-**mod**$_V$.*

Proof. As $\mathcal{F}_{2n}$ is faithful, $\mathcal{G}_n$ is faithful as well, and thus we will identify diagrams with their images. Notice that $\mathrm{Hom}_{\mathbf{D}_n}(V^{\otimes k}, V^{\otimes l}) \subset \mathrm{Hom}_{\mathbf{C}_{2n}}(V^{\otimes k}, V^{\otimes l})$. Consider an element $d \in \mathrm{Hom}_{\mathbf{D}_n}(V^{\otimes k}, V^{\otimes l})$ then d is an element of $\mathrm{Hom}_{\mathbf{C}_{2n}}(V^{\otimes k}, V^{\otimes l})$ which satisfies $hd = dh$ where $h = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ given $i = \sqrt{-1}$. That is to say, we can let $d = \sum_{\underline{s},\underline{t}} A_{\underline{s}}^{\underline{t}} d_{\underline{s}}^{\underline{t}}$ where $d_{\underline{s}}^{\underline{t}}$ is a simple diagram with the labeling $\underline{s}$ on bottom and $\underline{t}$ on top with coefficient $A_{\underline{s}}^{\underline{t}}$. Now we apply each side of $hd = dh$ to a basis vector, $v_{\underline{r}}$ to get

$$hd(v_{\underline{r}}) = \left(\sum_{\underline{s},\underline{t}} A_{\underline{s}}^{\underline{t}} h d_{\underline{s}}^{\underline{t}} v_{\underline{r}}\right) = \left(\sum_{|\underline{t}|\equiv|\underline{r}| \mod n} A_{\underline{r}}^{\underline{t}} h v_{\underline{t}}\right) = i^l \left(\sum_{|\underline{r}|\equiv|\underline{t}| \mod n} A_{\underline{r}}^{\underline{t}} v_{-\underline{t}}\right).$$

On the other hand,

$$dh(v_{\underline{r}}) = \left(\sum_{\underline{s},\underline{t}} A_{\underline{s}}^{\underline{t}} d_{\underline{s}}^{\underline{t}}\right) h v_{\underline{r}} = \left(\sum_{\underline{s},\underline{t}} A_{\underline{s}}^{\underline{t}} d_{\underline{s}}^{\underline{t}}\right) i^k v_{-\underline{r}} = i^k \left(\sum_{|\underline{t}|\equiv|-\underline{r}| \mod n} A_{-\underline{r}}^{\underline{t}} v_{\underline{t}}\right).$$

Thus, we have $i^l \left(\sum_{|\underline{r}|\equiv|\underline{t}| \mod n} A_{\underline{r}}^{\underline{t}} v_{-\underline{t}}\right) = i^k \left(\sum_{|-\underline{r}|\equiv|\underline{t}| \mod n} A_{-\underline{r}}^{\underline{t}} v_{\underline{t}}\right)$. By reindexing we get that $A_{\underline{r}}^{\underline{t}} = i^{k-l} A_{-\underline{r}}^{-\underline{t}}$ for all $|\underline{r}| \equiv |\underline{t}| \mod n$, and so

$$d = \sum_{|\underline{s}|\equiv|\underline{t}| \mod n} A_{\underline{s}}^{\underline{t}} (d_{\underline{s}}^{\underline{t}} + i^{k-l} d_{-\underline{s}}^{-\underline{t}}).$$

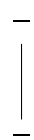
By lemma 4.4, we have that $2 \mid k - l$, and so $i^{k-l} = (-1)^{\frac{k-l}{2}}$, giving us

$$d = \sum_{|\underline{s}|\equiv|\underline{t}| \mod n} A_{\underline{s}}^{\underline{t}} (d_{\underline{s}}^{\underline{t}} + (-1)^{\frac{k-l}{2}} d_{-\underline{s}}^{-\underline{t}}).$$

Therefore, it is enough to show that the list of diagrams generate

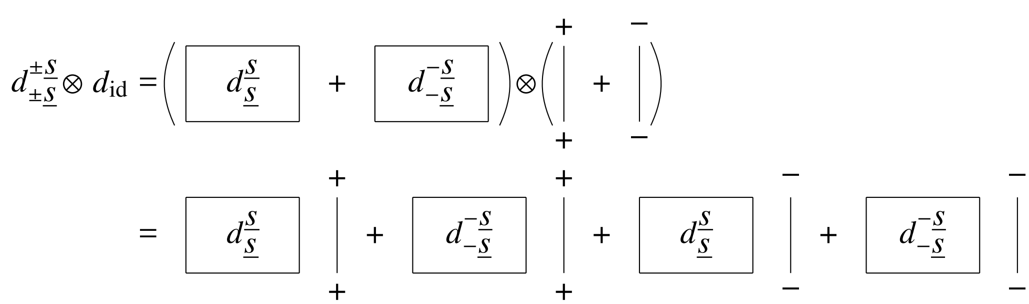
$$d^{\pm\underline{t}}_{\pm\underline{s}} = d^{\underline{t}}_{\underline{s}} + i^{k-l} d^{-\underline{t}}_{-\underline{s}}$$

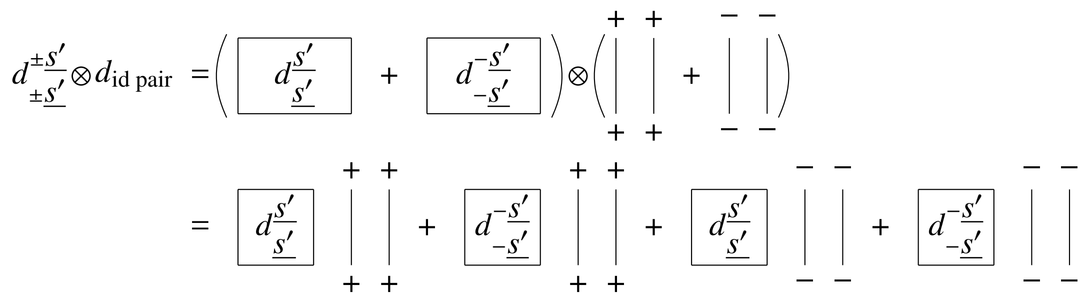
for any labelings $\underline{s}$ and $\underline{t}$ such that $|\underline{s}| \equiv |\underline{t}| \mod n$.

First, we will show that the diagrams generate the set $B_k = \{d^{\pm\underline{s}}_{\pm\underline{s}} = d^{\underline{s}}_{\underline{s}} + d^{-\underline{s}}_{-\underline{s}}\}$ for all $k \geq 1$. As all diagrams in $\mathrm{Hom}_{\mathbf{C}_{2n}}(V^{\otimes k}, V^{\otimes l})$ with the same labeling are equivalent, we can view $d^{\underline{s}}_{\underline{s}}$ and $d^{-\underline{s}}_{-\underline{s}}$ as the diagram which consists only of  or  in which $d^{\underline{s}}_{\underline{s}}$ sends $v_{\underline{s}}$ to $v_{\underline{s}}$ and $d^{-\underline{s}}_{-\underline{s}}$ sends $v_{-\underline{s}}$ to $v_{-\underline{s}}$. In particular, it will be shown that B_k is generated by d_{id}, $d_{\mathrm{id\ pair}}$, and $d_{\mathrm{id\ alt}}$.

To prove this, we induct on k. Indeed, if $k = 1$, then $B_1 = \{d_{\mathrm{id}}\}$, and if $k = 2$, we have $B_2 = \{d_{\mathrm{id\ pair}}, d_{\mathrm{id\ alt}}\}$. Thus the base case is satisfied.

Now we assume that B_k and B_{k-1} are generated by d_{id}, $d_{\mathrm{id\ pair}}$, and $d_{\mathrm{id\ alt}}$. We notice that $\underline{s}$is a k-tuple of +'s and −'s, and let $\underline{s'}$ be the $k-1$-tuple which is identical to $\underline{s}$ in the first $k-1$ entries.





The claim now is $\left(d_{\pm\underline{s}}^{\pm\underline{s}} \otimes d_{\text{id}}\right) \circ \left(d_{\pm\underline{s}'}^{\pm\underline{s}'} \otimes d_{\text{id pair}}\right)$ generates all elements of B_{k+1}. There are two possibilities: either the kth component of $\underline{s}$ is + or it is −. If the kth component of $\underline{s}$ is a +, then obviously the kth component of $-\underline{s}$ is a −. Thus, in this situation $\left(d_{\pm\underline{s}}^{\pm\underline{s}} \otimes d_{\text{id}}\right) \circ \left(d_{\pm\underline{s}'}^{\pm\underline{s}'} \otimes d_{\text{id pair}}\right)$ accounts for all diagrams in B_{k+1} in which the kth and $(k+1)$st labels match. On the other hand, if the kth component of $\underline{s}$ is a −, then obviously the kth component of $-\underline{s}$ is a +. Thus, in this situation $\left(d_{\pm\underline{s}}^{\pm\underline{s}} \otimes d_{\text{id}}\right) \circ \left(d_{\pm\underline{s}'}^{\pm\underline{s}'} \otimes d_{\text{id pair}}\right)$ accounts for all diagrams in B_{k+1} in which the k-th and $k+1$-st labels are opposite. The set of these two situations give us the complete list of elements in B_{k+1}.

As we can generate B_k and B_l, we can always precompose and post compose with elements from B_k and B_l respectively. This controls the input and output labelings respectively of our diagrams in the desired way.

In order to show that the morphisms in definition 4.1 generate all of the $\mathbb{C}$-vector spaces $\text{Hom}_{\mathbf{D}_n}(V^{\otimes k}, V^{\otimes l})$, we give explicitly an algorithm for constructing each $d_{\pm\underline{s}}^{\pm\underline{t}}$ where $|\underline{s}| \equiv |\underline{t}| \mod 2n$.

Given such an element $d_{\pm\underline{s}}^{\pm\underline{t}}$, we consider the first diagram in the sum. This element is in $\text{Hom}_{\mathbf{C}_{2n}}(V^{\otimes k}, V^{\otimes l})$ and sends $v_{\underline{s}}$ to $v_{\underline{t}}$, call it $d_{\underline{s}}^{\underline{t}}$. Thus this diagram is some element of $\text{Hom}_{\mathbf{C}_{2n}}(V^{\otimes k}, V^{\otimes l})$. We now consider the generators used to build $d_{\underline{s}}^{\underline{t}}$. As every generator for $\text{Hom}_{\mathbf{C}_{2n}}(V^{\otimes k}, V^{\otimes l})$ appears as a term in one of the diagrams we are claiming to generate $\mathrm{Hom}_{\mathbf{D}_n}(V^{\otimes k}, V^{\otimes l})$, we can construct an element d' of $\mathrm{Hom}_{\mathbf{D}_n}(V^{\otimes k}, V^{\otimes l})$ where $d_{\underline{s}}^{\underline{t}}$ is one of the terms. Furthermore, we know that each term in d' is unique as the labeling of each term will be unique. That is to say, each term of d' will have a coefficient of 1 or -1.

Next, we observe that since each contributive diagram in the tensor product for $d_{\underline{s}}^{\underline{t}}$ has a partner which has exactly the opposite labeling, thus $d_{-\underline{s}}^{-\underline{t}}$ has a nonzero coefficient in d', and that coefficient is either a 1 or a -1.

We may now precompose with $d_{\pm\underline{s}}^{\pm\underline{s}}$, an element of B_k, and post-compose with $d_{\pm\underline{t}}^{\pm\underline{t}}$, an element of B_l. Thus we have

$$d_{\pm\underline{t}}^{\pm\underline{t}} \circ d' \circ d_{\pm\underline{s}}^{\pm\underline{s}} = \pm d_{\underline{s}}^{\underline{t}} \pm d_{-\underline{s}}^{-\underline{t}}.$$

The last thing to check is that the coefficients are consistent with $d_{\pm\underline{s}}^{\pm\underline{t}}$.

Recall that $d_{\pm\underline{s}}^{\pm\underline{t}} = d_{\underline{s}}^{\underline{t}} + (-1)^{\frac{k-l}{2}} d_{-\underline{s}}^{-\underline{t}}$, and we consider 2 different cases, when n is odd and when n is even.

When n is odd we have the diagrams which would contribute to having an opposite sign are d_{cap}, d_{cup}, $d_{n\text{ grouped}}$, and $d_{n\text{ ungrouped}}$. As we can scale by -1, we need only consider $d_{\underline{s}}^{\underline{t}} \pm d_{-\underline{s}}^{-\underline{t}}$. If we have $d_{\underline{s}}^{\underline{t}} + d_{-\underline{s}}^{-\underline{t}}$, then then the number of generators with a -1 as a coefficient on the second term must be even, i.e. the number of d_{cap}, d_{cup}, $d_{n\text{ grouped}}$, and $d_{n\text{ ungrouped}}$ used is even. This must mean that $2 \mid \dfrac{k-l}{2}$. Thus, we must have that $d_{\pm\underline{s}}^{\pm\underline{t}} = d_{\underline{s}}^{\underline{t}} + d_{-\underline{s}}^{-\underline{t}}$.

If we have $d_{\underline{s}}^{\underline{t}} - d_{-\underline{s}}^{-\underline{t}}$, then the number of d_{cap}, d_{cup}, $d_{n\text{ grouped}}$, and $d_{n\text{ ungrouped}}$ used is odd. This means $2 \nmid \dfrac{k-l}{2}$, thus $d_{\pm\underline{s}}^{\pm\underline{t}} = d_{\underline{s}}^{\underline{t}} - d_{-\underline{s}}^{-\underline{t}}$. Thus we have that the coefficients are consistent if n is odd.

If n is even, the only contributors to a positive or negative second term are d_{cap}

and d_{cup}, which an analgous arguement can be applied. Therefore, we have that our list of 8 diagrams generate $\text{Hom}_{\mathbf{D}_n}(V^{\otimes k}, V^{\otimes l})$. □

It would be interesting to develop similar diagrammatic categories for G-**mod**$_V$ when G is one of these exceptional subgroups. However, preliminary calculations suggest that the diagrammatics will be rather intricate. We leave this for future work.

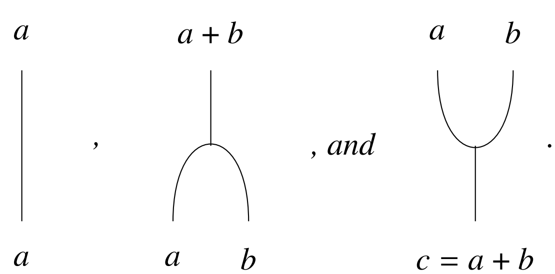
Instead, we choose to develop more refined diagrammatics which works equally well for all of the finite subgroups of $SU(2)$.

Chapter 5

Categories with Irreducible $\mathbf{C}_n$-modules as Objects

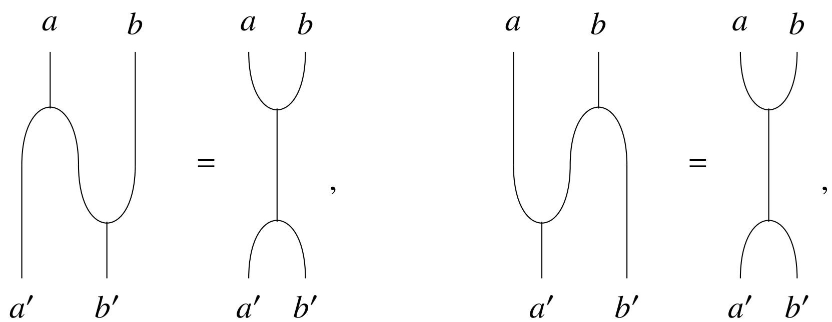
Let us explore some new families of categories. We will again use the convention that the empty diagram is the morphism from (0) to (0) which represents multiplication by 1 where $(0) = \bar{0}$ is the identity object.

Definition 5.1. *Let C_n^{irr} be the $\mathbb{C}$-linear monoidal category with objects generated by $a \in {\mathbb{Z}}/{n\mathbb{Z}}$ with the tensor product being defined by concatenation. Denote the concatenation of the integers $a_1, a_2, \ldots, a_k$ as $[a_1, a_2, \ldots, a_k]$. The morphisms are generated by the following diagrams:*

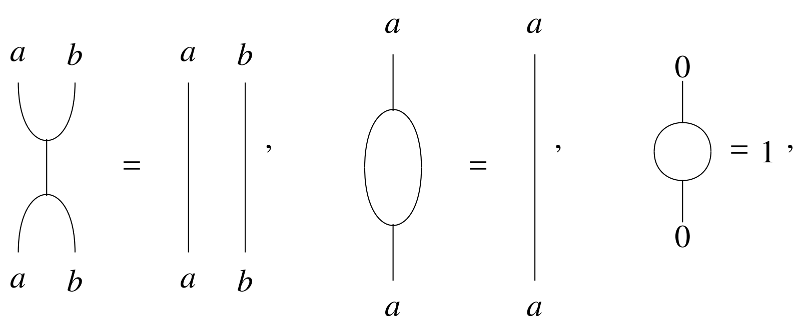


where a, b, and $c \in {\mathbb{Z}}/{n\mathbb{Z}}$. We will sometimes refer to these as the identity diagram, the merge diagram, and the split diagram respectively.

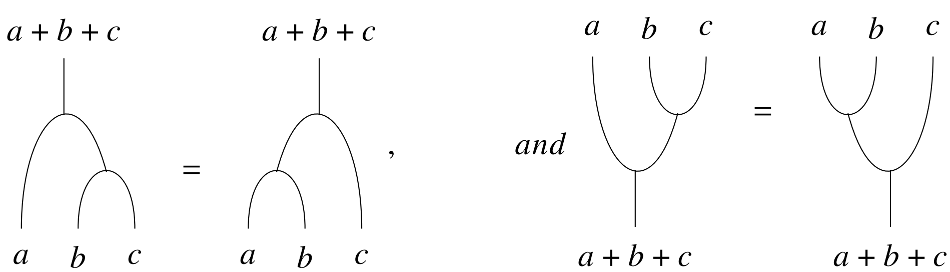
We impose the following relations:



(5.1)

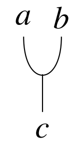


(5.2)



(5.3)

where $a,\ b,\ c,\ a',$ *and* $b' \in \mathbb{Z}/_{n\mathbb{Z}}$, *and* $a + b \equiv a' + b' \mod n$.

Remark 5.2. *It is worth noting that the when using the split map on an integer mod* *n, we must specify which two integers are the target. For example,* 

and 

are equal if and only if $a' \equiv a \mod n$ *and* $b' \equiv b \mod n$.

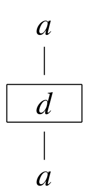
Given a diagram d, we will denote s_d as the number of split diagrams used in the construction of d and m_d as the number of merge diagrams used in d.

Lemma 5.3. *The difference* $s_d - m_d$ *is precisely the difference between the number of tensor factors in the target and the number of tensor factors in the source.*

Proof. Fix a diagram $d \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{irr}}}\left(\bigotimes_{i=1}^{k} a_i, \bigotimes_{j=1}^{\ell} b_j\right)$ with $k, \ell \in \mathbb{N}$. Notice, there are ℓ tensor factors corresponding to ℓ strings on the bottom of d. Reading the d from bottom to top, observe that a merge diagram will subtract one from the number of tensor factors of the top of d, and a split diagram will add one to the number of tensor factors of the top of the d. Furthermore, an identity strand will not change the number of tensor factors. Thus, $s_d - m_d = k - \ell$. □

Lemma 5.4. *Any diagram in* $\mathrm{Hom}_{\mathcal{C}^{irr}}(a, a)$ *is equal to* 

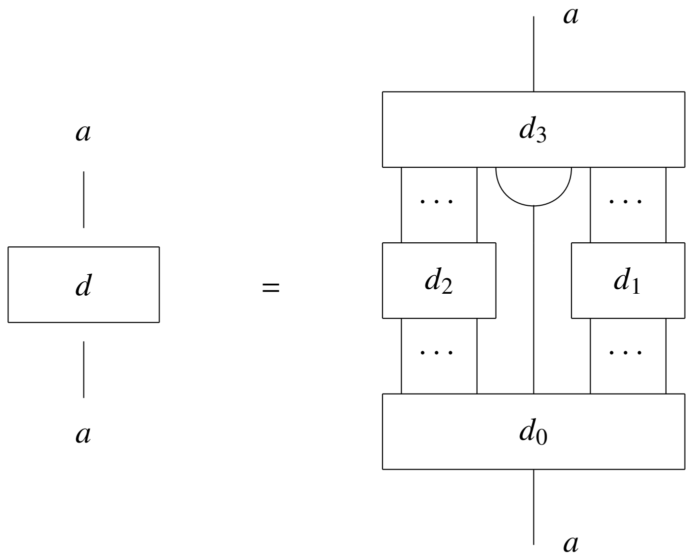
as morphisms in $\mathcal{C}_n^{irr}$.

Proof. Let 

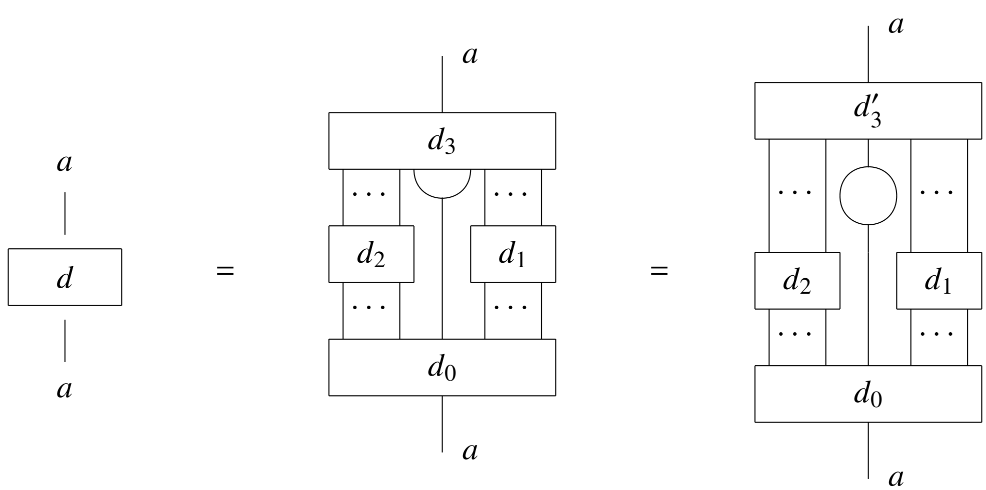
be a diagram in $\mathrm{Hom}_{\mathcal{C}^{\mathrm{irr}}}(a, a)$. From Lemma 5.3, the number of merge diagrams in d is equal to the number of split diagrams in d. So, we induct on the number of split diagrams in d, s_d. If $s_d = 0$, then $m_d = 0$ as well, and d must be the identity strand on a.

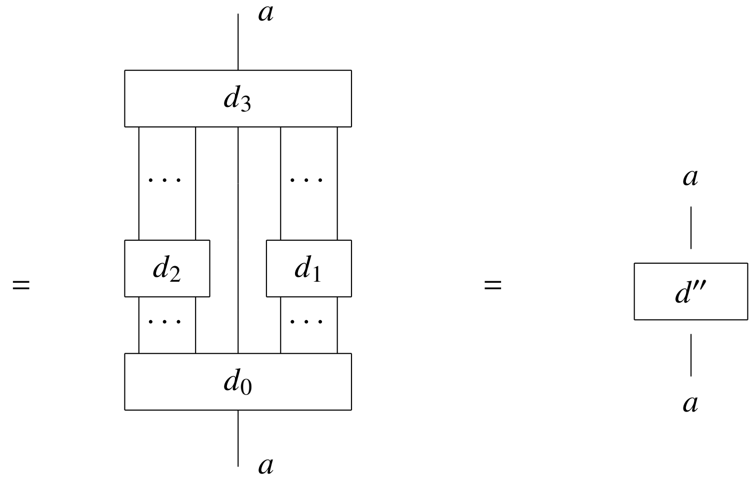
Now, assume we have that any diagram d' is equal to the identity strand on a for

$s_{d'} < k$ for some k. Suppose $s_{d'} = k$. Then, we can isolate a highest split diagram in d'. Thus we have

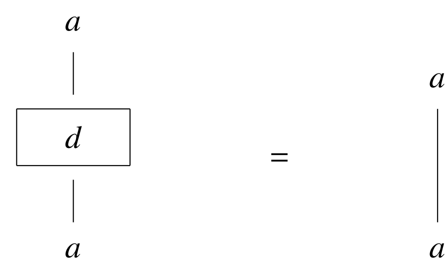


where d_3 only contains identity strands and merge diagrams. In particular, since d_3 has at least two tensor factors in the domain and only one tensor factor in the codomain, by Lemma 5.3, d_3 must contain at least one merge diagram. Using the associativity relation for merge diagrams in (5.3) iteratively, we can position a merge diagram directly above the split diagram. Hence,





where d'' has $k-1$ splits, i.e. $s_{d''} < k$. Therefore, using the induction hypothesis



□

The above lemmas will be helpful in proving Theorem 5.7. Now let us explore a particular category from representation theory.

We denote $\mathbf{C}_n$**-mod**$_{\text{irr}}$ as the full $\mathbb{C}$-linear monoidal subcategory of $\mathbf{C}_n$**-mod** where the generating objects are the irreducible $\mathbf{C}_n$-modules $\mathbf{C}_n^{(a)}$ where $a \in \mathbb{Z}/n\mathbb{Z}$. As all irreducible $\mathbf{C}_n$-modules are 1-dimensional, and

$$\dim(M \otimes N) = \dim(M) \cdot \dim(N),$$

then by Schur's Lemma, $\mathrm{Hom}_{\mathbf{C}_n}(M, N)$ is either 0-dimensional or 1-dimensional. If it is 1-dimensional, then $M \cong \mathbf{C}_n^{(a)} \cong N$ for some $a \in \mathbb{Z}/n\mathbb{Z}$. We pick bases for these irreducible $\mathbf{C}_n$-modules and let v_a denote our chosen basis vector of $\mathbf{C}_n^{(a)}$.

We can choose $\mathbf{C}_n$-module homomorphisms,

$$m_{a,b}^c : \mathbf{C}_n^{(a)} \otimes \mathbf{C}_n^{(b)} \longrightarrow \mathbf{C}_n^{(c)}$$

$$v_a \otimes v_b \mapsto v_c$$

where $c \equiv a + b \mod n$,

$$s_c^{a,b} : \mathbf{C}_n^{(c)} \longrightarrow \mathbf{C}_n^{(a)} \otimes \mathbf{C}_n^{(b)}$$

$$v_c \mapsto v_a \otimes v_b$$

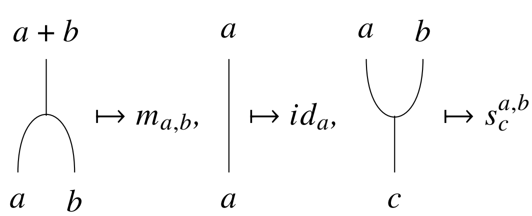
where $c \equiv a + b \mod n$, and

$$id_a : \mathbf{C}_n^{(a)} \longrightarrow \mathbf{C}_n^{(a)}$$

is the identity map. Notice that $m_{a,b}^c \circ s_c^{a,b} = id_c$ for all $a, b, c \in \mathbb{Z}/n\mathbb{Z}$.

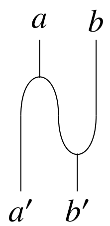
Theorem 5.5. *There exists a well-defined functor of monoidal $\mathbb{C}$-linear categories $\mathcal{F}_n^{irr} : C_n^{irr} \longrightarrow \mathbf{C}_n$-**mod**$_{irr}$ determined by the following rules:*

$$a \mapsto \mathbf{C}_n^{(a)}$$

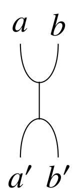


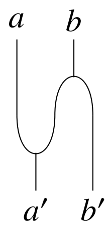
for each $a, b, c \in \mathbb{Z}/n\mathbb{Z}$.

Proof. We check that the above relations are preserved by the functor $\mathcal{F}_n^{\text{irr}}$.

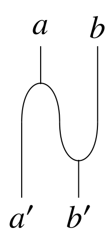
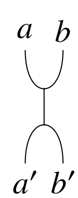
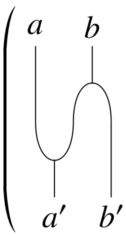
$\mathcal{F}_n^{\text{irr}}\Bigg($  $\Bigg)(v_{a'} \otimes v_{b'}) = (m^a_{a',c} \otimes id_b \circ id_{a'} \otimes s^{c,b}_{b'})(v_{a'} \otimes v_{b'})$

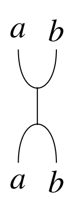
$$= (m^a_{a',c} \otimes id_b)(v_{a'} \otimes v_c \otimes v_b) = v_a \otimes v_b;$$

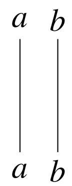
$\mathcal{F}_n^{\text{irr}}\Bigg($  $\Bigg)(v_{a'} \otimes v_{b'}) = s^{a,b}_c \circ m^c_{a',b'}(v_{a'} \otimes v_{b'}) = s^{a,b}_c(v_c) = v_a \otimes v_b;$

$\mathcal{F}_n^{\text{irr}}\Bigg($  $\Bigg)(v_{a'} \otimes v_{b'}) = (id_a \otimes m^a_{c,b'} \circ s^{a,c}_{a'} \otimes id_{b'})(v_{a'} \otimes v_{b'})$

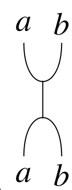
$$= (id_a \otimes m^a_{c,b'})(v_a \otimes v_c \otimes v_{b'}) = v_a \otimes v_b.$$

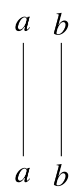
Thus, $\mathcal{F}_n^{\text{irr}}\Bigg($  $\Bigg) = \mathcal{F}_n^{\text{irr}}\Bigg($  $\Bigg) = \mathcal{F}_n^{\text{irr}}\Bigg($  $\Bigg)$.

$\mathcal{F}_n^{\text{irr}}\Bigg($  $\Bigg)(v_a \otimes v_b) = s^{a,b}_c \circ m^c_{a,b}(v_a \otimes v_b) = s^{a,b}_c(v_c) = v_a \otimes v_b;$

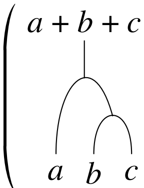
$$\mathcal{F}_n^{\text{irr}}\left(\right.$$ 

$$\left. \right)(v_a \otimes v_b) = id_a \otimes id_b(v_a \otimes v_b) = v_a \otimes v_b.$$

Thus, $\mathcal{F}_n^{\text{irr}}\left(\right.$ 

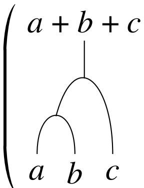
$\left. \right) = \mathcal{F}_n^{\text{irr}}\left(\right.$ 

$\left. \right).$

$$\mathcal{F}_n^{\text{irr}}\left(\right.$$ 

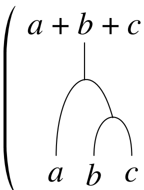
$$\left. \right)(v_a \otimes v_b \otimes v_c) = m_{a,b+c}^{a+b+c} \circ id_a \otimes m_{b,c}^{b+c}(v_a \otimes v_b \otimes v_c)$$

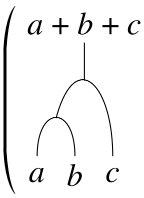
$$= m_{a,b+c}^{a+b+c}(v_a \otimes v_{b+c}) = v_{a+b+c};$$

$$\mathcal{F}_n^{\text{irr}}\left(\right.$$ 

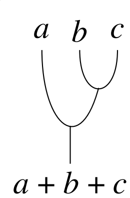
$$\left. \right)(v_a \otimes v_b \otimes v_c) = m_{a+b,c}^{a+b+c} \circ m_{a,b}^{a+b} \otimes id_c(v_a \otimes v_b \otimes v_c)$$

$$= m_{a+b,c}^{a+b+c}(v_{a+b} \otimes v_c) = v_{a+b+c}.$$

Thus, $\mathcal{F}_n^{\text{irr}}\left(\right.$ 

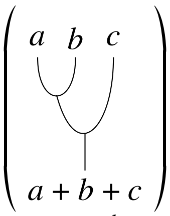
$\left. \right) = \mathcal{F}_n^{\text{irr}}\left(\right.$ 

$\left. \right).$

$$\mathcal{F}_n^{\text{irr}}\left(\right.$$ 

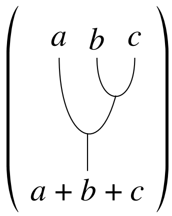
$$\left. \right)(v_{a+b+c}) = id_a \otimes s_{b+c}^{b,c} \circ s_{a+b+c}^{a,b+c}(v_{a+b+c})$$

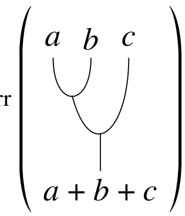
$$= id_a \otimes s_{b+c}^{b,c}(v_a \otimes v_{b+c}) = v_a \otimes v_b \otimes v_c;$$

$\mathcal{F}_n^{\mathrm{irr}}$ 

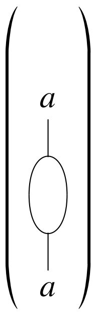
$(v_{a+b+c}) = s_{a+b}^{a,b} \otimes id_c \circ s_{a+b+c}^{a+b,c}(v_{a+b+c})$

$= s_{a+b}^{a,b} \otimes id_c(v_{a+b} \otimes v_c) = v_a \otimes v_b \otimes v_c.$

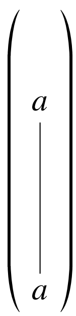
Thus, $\mathcal{F}_n^{\mathrm{irr}}$ 

$= \mathcal{F}_n^{\mathrm{irr}}$ 

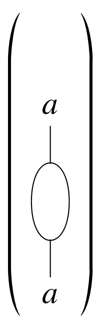
.

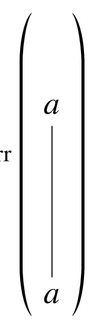
$\mathcal{F}_n^{\mathrm{irr}}$ 

$(v_a) = m_{b,c}^a \circ s_a^{b,c}(v_a) = m_{b,c}^a(v_b \otimes v_c) = v_a$

$\mathcal{F}_n^{\mathrm{irr}}$ 

$(v_a) = id_a(v_a) = v_a.$

Thus, $\mathcal{F}_n^{\mathrm{irr}}$ 

$= \mathcal{F}_n^{\mathrm{irr}}$ 

.

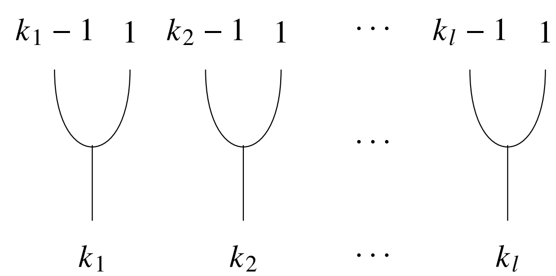
Therefore, the functor $\mathcal{F}_n^{\mathrm{irr}}$ is well defined.

□

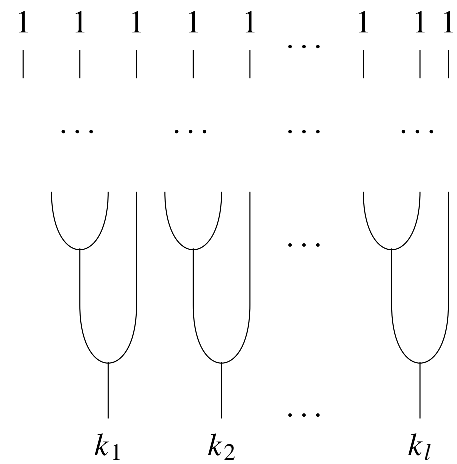
Theorem 5.6. *The functors $\mathcal{F}_n^{irr} : \mathcal{C}_n^{irr} \longrightarrow \mathbf{C}_n$-**mod**$_{irr}$ are full.*

Proof. It suffices to show that given $[k_1, k_2, \ldots, k_l] \in (\mathbb{Z}/n\mathbb{Z})^l$ and a morphism $f \in \mathbf{C}_n$-$\mathbf{mod}_{\text{irr}}$ where $f : \bigotimes_{i=1}^{l} \mathbf{C}_n^{(k_i)} \longrightarrow \bigotimes_{j=1}^{k_1+k_2+\cdots+k_l} \mathbf{C}_n^{(1)}$, there exists a morphism $d \in \mathcal{C}_n^{\text{irr}}$ such that $\mathcal{F}_n^{\text{irr}}(d) = f$. As $\bigotimes_{i=1}^{l} \mathbf{C}_n^{(k_i)}$ and $\bigotimes_{j=1}^{k_1+k_2+\cdots+k_l} \mathbf{C}_n^{(1)}$ are one dimensional $\mathbf{C}_n$-modules, they are irreducible as $\mathbf{C}_n$-modules, and thus, up to scaling, there is only one non-zero morphism. Said another way, we need only show there exists a $d \in \mathcal{C}_n^{\text{irr}}$ such that $\mathcal{F}_n^{\text{irr}}(d) : \bigotimes_{i=1}^{l} \mathbf{C}_n^{(k_i)} \longrightarrow \bigotimes_{j=1}^{k_1+k_2+\cdots+k_l} \mathbf{C}_n^{(1)}$.

We construct a diagram from $[k_1, k_2, \ldots, k_l]$ to $[1, 1, \ldots, 1]$ where there are $k_1 + k_2 + \cdots + k_l = 1 + 1 + \cdots + 1$. Consider the following diagram:



for $k_i \not\equiv 1 \mod n$. If $k_i \equiv 1 \mod n$, we replace the split diagram with the identity strand. We continue to stack the split diagram until $k_i - j \equiv 1 \mod n$ and tensor with the identity strand where needed. This process is finite, and thus, we get the resulting diagram:

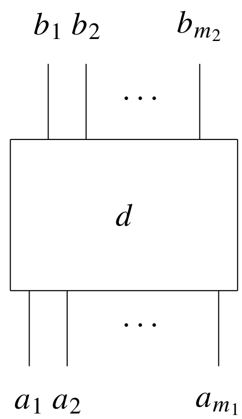


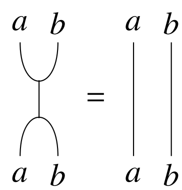
It is clear that the image of this diagram under the functor $\mathcal{F}_n^{\text{irr}}$ is a non-zero homomorphism which sends the vector $v_{k_1} \otimes v_{k_2} \otimes \cdots \otimes v_{k_l}$ to the vector $v_1 \otimes v_1 \otimes \cdots \otimes v_1$, and therefore, $\mathcal{F}_n^{\text{irr}}$ is full. □

Theorem 5.7. *The functor $\mathcal{F}_n^{irr} : C_n^{irr} \longrightarrow$ $\mathbf{C}_n$-**mod**$_{irr}$ is faithful.*

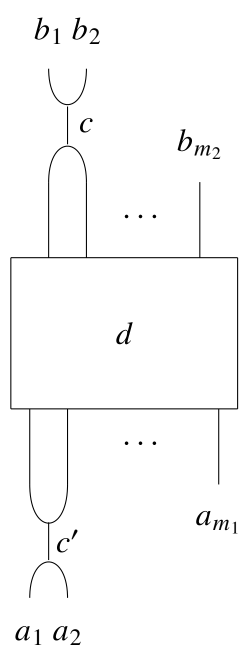
Proof. Let $[a_1, a_2, \ldots, a_{m_1}] \in \left(\mathbb{Z}/_{n\mathbb{Z}}\right)^{m_1}$ and $[b_1, b_2, \ldots, b_{m_2}] \in \left(\mathbb{Z}/_{n\mathbb{Z}}\right)^{m_2}$. We have that $\text{Hom}_{\mathbf{C}_n}\left(\bigotimes_{i=1}^{m_2} \mathbf{C}_n^{(a_i)}, \bigotimes_{j=1}^{m_2} \mathbf{C}_n^{(b_j)}\right)$ has dimension 1 iff $\sum_{i=1}^{m_1} a_i \equiv \sum_{j=1}^{m_2} b_j \mod n$ and 0 otherwise. Thus, it suffices to show that there is one morphism up to scaling by $\mathbb{C}$ in C_n^{irr} between $[a_1, a_2, \ldots, a_{m_1}]$ and $[b_1, b_2, \ldots, b_{m_2}]$ when $\sum_{i=1}^{m_1} a_i \equiv \sum_{j=1}^{m_2} b_j \mod n$.

Consider a diagram $d \in \text{Hom}_{C_n^{\text{irr}}}\left(\bigotimes_{i=1}^{m_1} a_i, \bigotimes_{j=1}^{m_2} b_j\right)$ where $\sum_{i=1}^{m_1} a_i \equiv \sum_{j=1}^{m_2} b_j \mod n$:



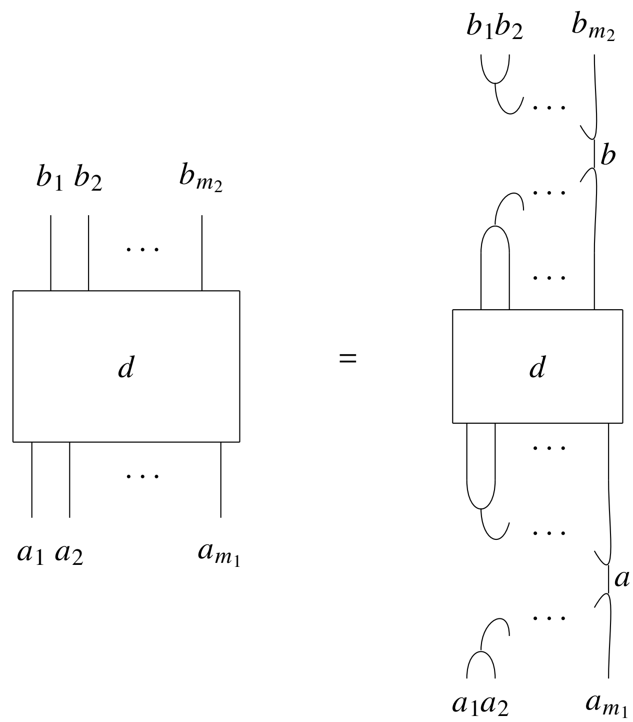
Using the relation 

from 5.2, we can rewrite d as

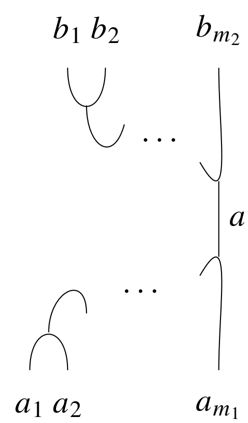


.

Now, consider using the relation iteratively to get the following equality:



where $a \equiv \sum_{i=1}^{m_1} a_i \mod n$ and $b \equiv \sum_{j=1}^{m_2} b_j \mod n$, and since $\sum_{i=1}^{m_1} a_i \equiv \sum_{j=1}^{m_2} b_j \mod n$, then $a \equiv b \mod n$. Thus by Lemma 5.4 the diagram on the right hand side of the equation is equal to



where $a \equiv \sum_{i=1}^{m_1} a_i \equiv \sum_{j=1}^{m_2} b_j \mod n$. Thus, there is one diagram up to scaling in

$\mathrm{Hom}_{\mathcal{C}_n^{\mathrm{irr}}}\left(\bigotimes_{i=1}^{m_1} a_i, \bigotimes_{j=1}^{m_2} b_j\right)$. Therefore, the functor $\mathcal{F}_n^{\mathrm{irr}}$ is faithful.

□

Now that we are familiar with a specific example of the type of diagrammatic category we would like to construct, the next chapter develops diagrammatic categories which utilize the representation graphs given a group G and a module V.

Chapter 6

The Categories G-$\mathbf{mod}_{\text{irr}}$ and $\mathbf{Dgrams}_{R(V,G)}$

As advertised, this section develops the construction of a diagrammatic category based on a given representation graph. However, we can be even more general. Let Γ be a directed graph with no multiple parallel edges, that is, no two nodes have two or more directed edges with the same direction between them, and with the set of vertices indexed by the set I_Γ. The constructions in this chapter can be used to define a diagrammatic category associated to the graph Γ. Furthermore, as we have done in this dissertation, one can start with a semisimple symmetric monoidal k-linear category over some field k and consider the full subcategory $\mathcal{C}$ where the objects are monoidally generated by the simple objects. Regardless of whether or not this category comes from representation theory, we may construct a representation graph. That is, we may fix a simple object x and construct the graph with nodes corresponding to the simple objects and a directed edge from vertex v to vertex u if the simple object corresponding to u is a direct summand of $x \otimes v$. If this directed graph has no multiple edges, then using the ideas in this chapter, one can diagrammatically define a category which is categorically equivalent to C.

6.1. The Category G-$\mathbf{mod}_{\mathrm{irr}}$

We first consider when Γ is the representation graph for a group G. Let G be a group, not necessarily finite, and let V be a G-module such that the resulting representation graph $R(V,G)$ is a connected graph with no multiple parallel edges. Furthermore, we will assume that V corresponds to a node in $R(V,G)$. That is, we will assume V is a simple G-module. It is worth mentioning here that $SU(2)$ and the finite subgroups of $SU(2)$ are examples of such G, but there are others as well. See [11–14].

First, let us set some notation. Let $\{G^{(a)}\}_{a\in I_G}$ be a set of fixed isomorphism class representatives of simple G-modules with I_G being an indexing set for the finite-dimensional simple G-modules. Furthermore, as V is a simple G-module and I_G is an indexing set for the simple, G-modules, one of the elements of I_G corresponds to V. For notational convenience, we let this index be the symbol 1. In particular, we will use V and $G^{(1)}$ interchangeably. Furthermore, for $a, b \in I_G$, we will also use $b \to a$ to denote that b is adjacent to a in $R(V,G)$. Note that in an undirected graph $b \to a$ implies $a \to b$.

Definition 6.1. *We let G-$\mathbf{mod}_{irr}$ be the full monoidal subcategory of G-$\mathbf{mod}$ with objects generated by $G^{(a)}$ where $a \in I_G$.*

Notice, the morphisms of this category are elements of the $\mathbb{C}$-vector spaces

$$\mathrm{Hom}_G\left(\bigotimes_{i=1}^{n} G^{(a_i)}, \bigotimes_{j=1}^{m} G^{(b_j)}\right)$$

where $a_i, b_j \in I_G$ and $n, m \in \mathbb{N}$.

We define certain G-module homomorphisms concretely. Since $R(V,G)$ has no multiple edges by assumption, the space $\mathrm{Hom}_G\left(V \otimes G^{(a)}, G^{(b)}\right)$ is 1-dimensional for each b adjacent to a in $R(V,G)$ and 0-dimensional otherwise. We can fix such maps for each b adjacent to a and name them ${m_1}^{b}{}_{a}$. Furthermore, with the ${m_1}^{b}{}_{a}$ fixed, for each b which is adjacent to a in $R(V,G)$ there are unique non-zero G-module homomorphisms, which we name ${s^1}_{b}{}^{a}$, which span $\mathrm{Hom}_G\left(G^{(b)}, V \otimes G^{(a)}\right)$ such that the following is satisfied:

$$\sum_{b \to a} {s^1}_{b}{}^{a} \circ {m_1}^{b}{}_{a} = \mathrm{id}_{G^{(1)} \otimes G^{(a)}} \,. \tag{6.1}$$

Let us consider an example: let $\mathbf{T}$ be the binary tetrahedral group. We will use notation consistent with Example 2.4.

Example 6.2. *We let* $\mathbf{T}$-***mod***$_{irr}$ *be the full monoidal subcategory of* $\mathbf{T}$-***mod*** *with objects generated by* $T^{(a)}$ *with* $a \in I_{\mathbf{T}} = \{1, 2, 3, 4, 3', 4'\}$. *Notice, the morphisms of this category are in* $\mathrm{Hom}_{\mathbf{T}}\left(\bigotimes_{k=1}^{n} T^{(a_k)}, \bigotimes_{\ell=1}^{m} T^{(b_\ell)}\right)$.

We will consider the following $\mathbf{T}$*-module homomorphisms:*

$${m_1}^{1}{}_{0} : T^{(1)} \otimes T^{(0)} \longrightarrow T^{(1)} \qquad {m_1}^{2}{}_{1} : T^{(1)} \otimes T^{(1)} \longrightarrow T^{(2)}$$

$$v_{-1} \otimes 1 \mapsto v_{-1} \qquad v_{-1} \otimes v_{-1} \mapsto v_{-2}$$

$$v_1 \otimes 1 \mapsto v_1 \qquad v_1 \otimes v_1 \mapsto v_2$$

$$v_{-1} \otimes v_1 + v_1 \otimes v_{-1} \mapsto v_{0'}$$

$$m_{1\,1}^{\,0} : T^{(1)} \otimes T^{(1)} \longrightarrow T^{(0)}$$

$$v_{-1} \otimes v_1 - v_1 \otimes v_{-1} \mapsto 1$$

$$m_{1\,2}^{\,1} : T^{(1)} \otimes T^{(2)} \longrightarrow T^{(1)}$$

$$-\frac{1}{2} v_{-1} \otimes v_{0'} + v_1 \otimes v_{-2} \mapsto v_{-1}$$

$$\frac{1}{2} v_1 \otimes v_{0'} - v_{-1} \otimes v_2 \mapsto v_1$$

$$m_{1\,3}^{\,2} : T^{(1)} \otimes T^{(3)} \longrightarrow T^{(2)}$$

$$v_1 \otimes v_3 \mapsto v_2 - i\sqrt{3} v_{-2}$$

$$v_{-1} \otimes v_{-3} \mapsto v_{-2} - i\sqrt{3} v_2$$

$$v_{-1} \otimes v_3 + v_1 \otimes v_{-3} \mapsto -2 v_{0'}$$

$$m_{1\,2}^{\,3} : T^{(1)} \otimes T^{(2)} \longrightarrow T^{(3)}$$

$$-v_{-1} \otimes v_{0'} - v_1 \otimes v_{-2} + i\sqrt{3} v_1 \otimes v_2 \mapsto v_{-3}$$

$$v_1 \otimes v_{0'} + v_{-1} \otimes v_2 - i\sqrt{3} v_{-1} \otimes v_{-2} \mapsto v_3$$

$$m_{1\,3'}^{\;2} : T^{(1)} \otimes T^{(3')} \longrightarrow T^{(2)}$$

$$v_1 \otimes v_{3'} \mapsto v_{-2} + i\sqrt{3}v_2$$

$$v_{-1} \otimes v_{-3'} \mapsto v_2 + i\sqrt{3}v_{-2}$$

$$v_{-1} \otimes v_{3'} + v_1 \otimes v_{-3'} \mapsto -2v_{0'}$$

$$m_{1\,2}^{\;3'} : T^{(1)} \otimes T^{(2)} \longrightarrow T^{(3')}$$

$$-v_{-1} \otimes v_{0'} - v_1 \otimes v_{-2} - i\sqrt{3}v_1 \otimes v_2 \mapsto v_{-3'}$$

$$v_1 \otimes v_{0'} + v_{-1} \otimes v_2 + i\sqrt{3}v_{-1} \otimes v_{-2} \mapsto v_{3'}$$

$$m_{1\,3}^{\;4} : T^{(1)} \otimes T^{(3)} \longrightarrow T^{(4)}$$

$$v_{-1} \otimes v_3 - v_1 \otimes v_{-3} \mapsto v_4$$

$$m_{1\,3'}^{\;4'} : T^{(1)} \otimes T^{(3')} \longrightarrow T^{(4')}$$

$$v_{-1} \otimes v_{3'} - v_1 \otimes v_{-3'} \mapsto v_{4'}$$

$$m_{1\,4}^{\;3} : T^{(1)} \otimes T^{(4)} \longrightarrow T^{(3)}$$

$$v_{-1} \otimes v_4 \mapsto v_{-3}$$

$$v_1 \otimes v_4 \mapsto v_3$$

$$m_{1\,4'}^{\;3'} : T^{(1)} \otimes T^{(4')} \longrightarrow T^{(3')}$$

$$v_{-1} \otimes v_{4'} \mapsto v_{-3'}$$

$$v_1 \otimes v_{4'} \mapsto v_{3'}$$

We will then define $s^{1\;a}_{\;b} : T^{(b)} \longrightarrow T^{(1)} \otimes T^{(a)}$ *to be the map satisfying the relation* $\sum_{b \to a} s^{1\;a}_{\;b} \circ m_{1\;a}^{\;b} = \mathrm{id}_{T^{(1)} \otimes T^{(b)}}$

Recall from Section 2.2 that $P(a,b)_k$ is the set paths from a to b of length k and is subset of $P(a,b)$. Let $\mathbf{p} = \{b_0, b_1, \ldots, b_k\} \in P(a,b)_k$ where $b_0 = 1$ and $b_k = b$.

We fix $\pi_{\mathbf{p}}$ to be the map from $\left(G^{(1)}\right)^{\otimes k}$ onto the irreducible submodule $G^{(b)}$ using the previously fixed maps, $m_{1}{}^{b}{}_{a}$ and $s^{1}{}_{b}{}^{a}$, in the following way:

$$\pi_{\mathbf{p}} := \left(m_{1}{}^{b}{}_{b_{k-1}}\right) \circ \left(\mathrm{id}_V \otimes m_{1}{}^{b_{k-1}}{}_{b_{k-2}}\right) \circ \cdots \circ \left(\left(\mathrm{id}_{G^{(1)}}\right)^{\otimes(k-3)} \otimes m_{1}{}^{b_2}{}_{b_1}\right) \circ \left(\left(\mathrm{id}_V\right)^{\otimes(k-2)} \otimes m_{1}{}^{b_1}{}_{1}\right)$$

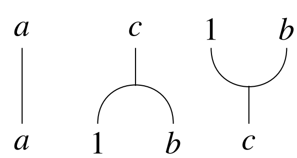
Since the identity maps and the $m_{1}{}^{b}{}_{a}$ are canonical (up to scaling), so then is $\pi_{\mathbf{p}}$.

Similarly, we let $\iota_{\mathbf{p}}$ be the map from $G^{(b)}$ into $\left(G^{(1)}\right)^{\otimes k}$ such that $\pi_{\mathbf{p}} \circ \iota_{\mathbf{p}} = \mathrm{id}_{G^{(b)}}$. For each irreducible G-module $G^{(b)}$, there is a minimal k_b such that $G^{(b)} \subset \left(G^{(1)}\right)^{\otimes k_b}$, and since the representation graph has no multiple edges, this corresponds to a single path $\mathbf{q} \in P(a,b)_{k_b}$. Thus, $G^{(b)}$ shows up exactly once in $\left(G^{(1)}\right)^{\otimes k_b}$, and thus we let $\pi_{\mathbf{q}}$ and $\iota_{\mathbf{q}}$ be the corresponding projection and inclusion maps.

6.2. The Category $\mathbf{Dgrams}_{R(V,G)}$

Now we turn to a diagrammatic category which needs only the data of the representation graphs $R(V,G)$ presented in the previous section to construct.

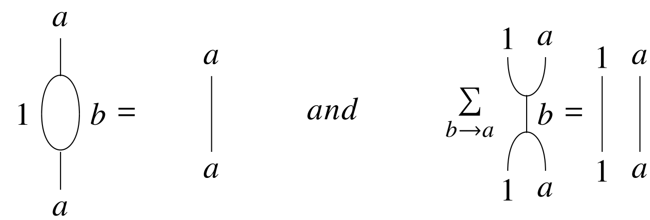
Definition 6.3. *We let* $\mathbf{Dgrams}_{R(V,G)}$ *be the* $\mathbb{C}$*-linear monoidal category with objects generated by* $k \in I_G$ *and morphisms generated by the following diagrams:*



where $a, b,$ *and* $c \in I_G$ *such that* c *is adjacent to* b *in the representation graph,*

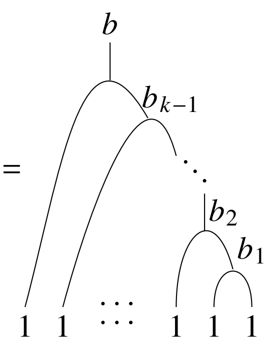
$R(V, G)$.

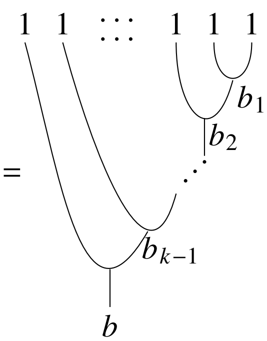
The generators are subject to the following relations:



(6.2)

Let us set some notation for some morphisms in the category $\mathbf{Dgrams}_{R(V,G)}$. Recall the notation we introduced in Section 2.2: for $\mathbf{p} = (1, b_1, \ldots, b_{k-1}, b) \in P(1, b)_k$, we let

$u_{\mathbf{p}} :=$ 

, and $d_{\mathbf{p}} :=$ 

Lemma 6.4. *As morphisms in* $\mathbf{Dgrams}_{R(V,G)}$

$$\sum_{b \in I_G} \sum_{\mathbf{p} \in P(1,b)_k} d_{\mathbf{p}} \circ u_{\mathbf{p}} = \mathrm{id}_{1^{\otimes k}}$$

for all $k \in \mathbb{N}_{\geq 2}$.

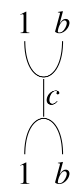
Proof. We proceed by induction on k. For $k = 2$, the statement is precisely the second relation in Definition 6.3. Now let us suppose that

$$\sum_{b \in I_G} \sum_{\mathbf{p} \in P(1,b)_k} d_{\mathbf{p}} \circ u_{\mathbf{p}} = \mathrm{id}_{1^{\otimes k}}$$

for some $k \geq 2$. Using this hypothesis, we have

$$\mathrm{id}_{1^{\otimes(k+1)}} = \sum_{b \in I_G} \sum_{\mathbf{p} \in P(1,b)_k} \mathrm{id}_1 \otimes (d_{\mathbf{p}} \circ u_{\mathbf{p}})$$

$$= \sum_{b \in I_G} \sum_{\mathbf{p} \in P(1,b)_k} \left(\sum_{c \to b} \mathrm{id}_1 \otimes d_{\mathbf{p}} \circ \right.$$



$$\left. \circ\, \mathrm{id}_1 \otimes u_{\mathbf{p}} \right)$$

$$= \sum_{c \in I_G} \sum_{\mathbf{p} \in P(1,c)_{k+1}} d_{\mathbf{p}} \circ u_{\mathbf{p}},$$

which was to be shown. □

The following gives an example of the construction of a diagrammatic category in this way.

Example 6.5. *Recall the representation graph $R\left(T^{(1)}, \mathbf{T}\right)$ from (2.6). Then we can construct the $\mathbb{C}$-linear monoidal category $\mathbf{Dgrams}_{R\left(T^{(1)}, \mathbf{T}\right)}$ be with objects generated by $k \in I_{\mathbf{T}}$ and morphisms generated and related in the same manner as in Definition 6.3.*

6.3. The Functor $\mathcal{H}_{R(V,G)}$

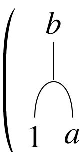
The following definitions and theorems show that there is a full functor from $\mathbf{Dgrams}_{R(V,G)}$ onto G-$\mathbf{mod}_{\mathrm{irr}}$. Recall the maps $m_1{}^b{}_a$ and $s^1{}_b{}^a$ given before (6.1).

Definition 6.6. *We let $\mathcal{H}_{R(V,G)} : \mathbf{Dgrams}_{R(V,G)} \longrightarrow G$-$\boldsymbol{mod}_{irr}$ be the monoidal*

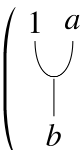
$\mathbb{C}$*-linear functor determined by the following rules:*

$$\mathcal{H}_{R(V,G)}(a) = G^{(a)} \text{ for } a \in I_G,$$

$$\mathcal{H}_{R(V,G)}\left(\quad \right) = m_{1\ a}^{\ b},$$



$$\text{and } \mathcal{H}_{R(V,G)}\left(\quad \right) = s_{\ a}^{1\ b}.$$



Note that by the way $m_{1\ a}^{\ b}$ and $s_{\ a}^{1\ b}$ were chosen, the relations in (6.2) are automatically satisfied.

As there will be no confusion as to which representation graph, for the rest of this chapter we will suppress the $R(V, G)$ in the notation of Definitions 6.3 and 6.6 and say that **Dgrams** := **Dgrams**$_{R(V,G)}$ and $\mathcal{H} := \mathcal{H}_{R(V,G)}$.

Lemma 6.7. *The functor $\mathcal{H}$ is full onto* $\mathrm{Hom}_G\left(\left(G^{(1)}\right)^{\otimes k}, G^{(b)}\right)$ *and* $\mathrm{Hom}_G\left(G^{(b)}, \left(G^{(1)}\right)^{\otimes k}\right)$ *for any $k \in \mathbb{N}$ and any $b \in I_G$.*

Proof. To prove that $\mathcal{H}$ is full onto $\mathrm{Hom}_G\left(\left(G^{(1)}\right)^{\otimes k}, G^{(b)}\right)$, it suffices to show that $\mathrm{Hom}_G\left((G^{(1)})^{\otimes k}, G^{(b)}\right)$ is spanned by

$$B_k^b := \left\{ \mathcal{H}\left(u_{\mathbf{p}}\right) =: \pi_{\mathbf{p}} \;\middle|\; \mathbf{p} \in P(1, b)_k \right\},$$

which we will show by inducting on k.

Since the representation graph of G does not contain any multiple edges, then up to scaling $m_{1\ b}^{\ c} : G^{(1)} \otimes G^{(b)} \longrightarrow G^{(c)}$ is canonical for all $G^{(c)} \subset G^{(1)} \otimes G^{(b)}$. Thus each $\mathrm{Hom}_G\left(G^{(1)} \otimes G^{(b)}, G^{(c)}\right)$ is either 0 or spanned by $m_{1\ b}^{\ c}$. Furthermore, since $G^{(1)}$ is simple, and $\mathcal{H}\left(\begin{array}{c} a \\ | \\ a \end{array}\right) = \mathrm{id}_{G^{(1)}}$, the base case is trivial.

Now suppose that $\mathrm{Hom}_G\left((G^{(1)})^{\otimes k}, G^{(b)}\right)$ is spanned by D_k^b for some k and for all b. Then we consider $\mathrm{Hom}_G\left((G^{(1)})^{\otimes(k+1)}, G^{(c)}\right)$.

Since $G^{\otimes(k+1)} = G^{(1)} \otimes (G^{(1)})^{\otimes k}$, we can construct $\pi_{\mathbf{p}} = m_{1\ b}^{\ c} \circ \left(\mathrm{id}_{G^{(1)}} \otimes \pi_{\mathbf{q}}\right)$, where $\mathbf{p} \in P(1,c)_{k+1}$ and $\mathbf{q} \in P(1,b)_k$. So up to scaling, we have morphisms

$$(G^{(1)})^{\otimes(k+1)} \xrightarrow{\mathrm{id}_{G^{(1)}} \otimes \pi_{\mathbf{q}}} G^{(1)} \otimes G^{(b)} \xrightarrow{m_{1\ b}^{\ c}} G^{(c)}$$

which are canonically based on the path in the representation graph. Thus for each $G^{(c)} \subset (G^{(1)})^{\otimes(k+1)}$, there is a canonical projection $\pi_{\mathbf{p}}$. Therefore, $\mathcal{H}$ is full on $\mathrm{Hom}_G\left((G^{(1)})^{\otimes k}, G^{(b)}\right)$ for all $k \in \mathbb{N}$ and $b \in I$.

It is analogously shown using $d_{\mathbf{p}}$, $\iota_{\mathbf{p}}$, and $s^{1\ b}_{\ c}$ where $\mathbf{p} \in P(1,b)_k$ that $\mathcal{H}$ is full onto $\mathrm{Hom}_G\left(G^{(b)}, \left(G^{(1)}\right)^{\otimes k}\right)$.

□

Lemma 6.8. *The functor $\mathcal{H}$ is full onto*

$$\mathrm{Hom}_G\left(\bigotimes_{i=1}^{n} G^{(a_i)}, G^{(b)}\right) \textit{ and } \mathrm{Hom}_G\left(G^b, \bigotimes_{i=1}^{n} G^{(a_i)}\right)$$

for $a_i, b \in I$.

Proof. By the previous lemma and the fact that $\mathcal{H}$ is a monoidal, $\mathbb{C}$-linear functor, it suffices to show that any morphism in $\mathrm{Hom}_G\left(\bigotimes_{i=1}^{n} G^{(a_i)}, G^{(b)}\right)$ can be realized through $\mathbb{C}$-linearity, composition, and tensor products of morphisms in $\mathrm{Hom}_G\left(\left(G^{(1)}\right)^{\otimes k}, G^{(b)}\right)$ and $\mathrm{Hom}_G\left(G^{(b)}, \left(G^{(1)}\right)^{\otimes k}\right)$.

Let $f \in \mathrm{Hom}_G\left(\bigotimes_{i=1}^{n} G^{(a_i)}, G^{(b)}\right)$. For each a_i, there exists a minimal k_i such that $G^{(a_i)} \subset (G^{(1)})^{\otimes k_i}$, and thus there are morphisms, $\pi^{a_i}_{k_{a_i}} \in \mathrm{Hom}_G\left(\left(G^{(1)}\right)^{\otimes k_{a_i}}, G^{(a_i)}\right)$ and $\iota^{k_{a_i}}_{a_i} \in \mathrm{Hom}_G\left(G^{(b)}, \left(G^{(1)}\right)^{\otimes k_{a_i}}\right)$ such that $\pi^{a_i}_{k_{a_i}} \circ \iota^{k_{a_i}}_{a_i} = \mathrm{id}_{G^{(a_i)}}$. Thus,

$$f = f \circ \left(\bigotimes_{i=1}^{n} \pi^{a_i}_{k_{a_i}} \circ \bigotimes_{i=1}^{n} \iota^{k_{a_i}}_{a_i}\right) = \left(f \circ \bigotimes_{i=1}^{n} \pi^{a_i}_{k_{a_i}}\right) \circ \bigotimes_{i=1}^{n} \iota^{k_{a_i}}_{a_i},$$

and since

$$\left(f \circ \bigotimes_{i=1}^{n} \pi^{a_i}_{k_{a_i}}\right) \in \mathrm{Hom}_G\left(\left(G^{(1)}\right)^{\otimes k_{a_i}}, G^{(b)}\right) \text{ and } \iota^{k_{a_i}}_{a_i} \in \mathrm{Hom}_G\left(G^{(a_i)}, \left(G^{(1)}\right)^{\otimes k_{a_i}}\right),$$

then G^{irr} is full onto $\mathrm{Hom}_G\left(\bigotimes_{i=1}^{n} G^{(a_i)}, G^{(b)}\right)$ for $a_i, b \in I$.

An analogous argument shows $\mathcal{H}$ is full onto $\mathrm{Hom}_G\left(G^{b}, \bigotimes_{i=1}^{n} G^{(a_i)}\right)$. □

Lemma 6.9. *The functor $\mathcal{H}$ is full onto $Hom_G\left(\left(G^{(1)}\right)^{\otimes k}, \left(G^{(1)}\right)^{\otimes \ell}\right)$ for any $k, \ell \in \mathbb{N}$.*

Proof. Given a morphism, $f \in \mathrm{Hom}_G\left(\left(G^{(1)}\right)^{\otimes k}, \left(G^{(1)}\right)^{\otimes \ell}\right)$ and a path $\mathbf{p} \in P(1,b)_k$, we have in the image of $\mathcal{H}$ canonical projections, $\pi_{\mathbf{p}}$, and inclusions, $\iota_{\mathbf{p}}$, onto and from $G^{(b)}$ such that $\pi_{\mathbf{p}} \circ \iota_{\mathbf{p}} = \mathrm{id}_{G^{(b)}}$ and $\sum_{\mathbf{p} \in P(1,b)_k} \iota_{\mathbf{p}} \circ \pi_{\mathbf{p}} = \mathrm{id}_{\left(G^{\otimes k}\right)}$. Thus, we have

$$
\begin{aligned}
f &= \left(\sum_{\mathbf{p}\in P(1,b)_\ell} \iota_{\mathbf{p}} \circ \pi_{\mathbf{p}}\right) \circ f \circ \left(\sum_{\mathbf{q}\in P(1,b)_k} \iota_{\mathbf{q}} \circ \pi_{\mathbf{q}}\right) \\
&= \left(\sum_{\mathbf{p}\in P(1,b)_\ell} \sum_{\mathbf{q}\in P(1,b)_k} \iota_{\mathbf{p}} \circ \left(\pi_{\mathbf{p}} \circ f \circ \iota_{\mathbf{q}} \circ \pi_{\mathbf{q}}\right)\right)
\end{aligned}
$$

where the sums are taken over all paths of length k and ℓ from 1 to b. Since $\pi_{\mathbf{p}} \circ f \circ \iota_{\mathbf{q}} \circ \pi_{\mathbf{q}} \in \mathrm{Hom}_G\left(\left(G^{(1)}\right)^{\otimes k}, G^{(b)}\right)$ and $\iota_{\mathbf{p}} \in \mathrm{Hom}_G\left(G^{(b)}, \left(G^{(1)}\right)^{\otimes \ell}\right)$, $\mathcal{H}$ is full onto $\mathrm{Hom}_G\left(\left(G^{(1)}\right)^{\otimes k}, \left(G^{(1)}\right)^{\otimes \ell}\right)$ for any $k, \ell \in \mathbb{N}$. □

Theorem 6.10. *The functor $\mathcal{H}$ is full.*

Proof. Consider a morphism $f \in \mathrm{Hom}_G\left(\bigotimes_{i=1}^{n} G^{(a_i)}, \bigotimes_{j=1}^{m} G^{(b_j)}\right)$. Using notation from the proofs above, we have

$$
\begin{aligned}
f &= \bigotimes_{j=1}^{m}\left(\pi^{b_j}_{\ell_{b_j}} \circ \iota^{\ell_{b_j}}_{b_j}\right) \circ f \circ \bigotimes_{i=1}^{n}\left(\pi^{a_i}_{k_{a_i}} \circ \iota^{k_{a_i}}_{a_i}\right) \\
&= \bigotimes_{j=1}^{m} \pi^{b_j}_{\ell_{b_j}} \circ \left(\bigotimes_{j=1}^{m} \iota^{\ell_{b_j}}_{b_j} \circ f \circ \bigotimes_{i=1}^{n} \pi^{a_i}_{k_{a_i}}\right) \circ \bigotimes_{i=1}^{n} \iota^{k_{a_i}}_{a_i},
\end{aligned}
$$

and by setting $k = \sum_{i=1}^{n} k_{a_i}$ and $\ell = \sum_{j=1}^{m} \ell_{b_j}$, we have that

$$
\left(\bigotimes_{j=1}^{m} \iota^{\ell_{b_j}}_{b_j} \circ f \circ \bigotimes_{i=1}^{n} \pi^{a_i}_{k_{a_i}}\right) \in \mathrm{Hom}_G\left(\left(G^{(1)}\right)^{\otimes k}, \left(G^{(1)}\right)^{\otimes \ell}\right),
$$

$$\pi^{b_j}_{\ell_{b_j}} \in \mathrm{Hom}_G\left(\left(G^{(1)}\right)^{\otimes \ell_{b_j}}, G^{(b_j)}\right),$$

and

$$\iota^{k_{a_i}}_{a_i} \in \mathrm{Hom}_G\left(G^{(a_i)}, \left(G^{(1)}\right)^{\otimes k_{a_i}}\right).$$

Therefore the functor $\mathcal{H}$ is full. $\square$

6.4. The Induced Functor $\overline{\mathcal{H}}_{R(V,G)}$

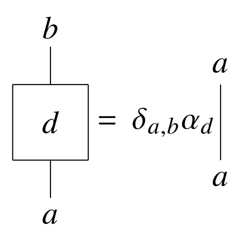
We now explore the kernel of $\mathcal{H}$. Assume $\mathcal{I}$ is a tensor ideal of **Dgrams** such that for all objects X, Y in **Dgrams**, $\mathcal{H}(f) = 0$ for every morphism $f \in \mathcal{I}(X, Y)$. Let $\overline{\textbf{Dgrams}} := {}^{\textbf{Dgrams}}/_{\mathcal{I}}$. Then there is an induced functor

$$\overline{\mathcal{H}} : \overline{\textbf{Dgrams}} \longrightarrow G\text{-}\textbf{mod}.$$

Let us assume that

$$\mathrm{Hom}_{\overline{\textbf{Dgrams}}}(a, b) = \begin{cases} \mathbb{C} \cdot \mathrm{id}_a & a = b \\ 0 & a \neq b \end{cases}.$$

That is, in $\overline{\textbf{Dgrams}}$ we have for all $a \in I_G$



(6.3)

where $\delta_{a,b}$ is the Kronecker delta and $\alpha_d \in \mathbb{C}$.

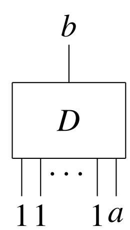
Lemma 6.11. *Suppose the equality in 6.3 is satisfied. The functor $\overline{\mathcal{H}}$ is faithful on $Hom_{\overline{\mathbf{Dgrams}}}\left(1^{\otimes k}, b\right)$ and $Hom_{\overline{\mathbf{Dgrams}}}\left(b, 1^{\otimes k}\right)$ for all $b \in I_G$ and $k \in \mathbb{N}$.*

Proof. Recall that $\mathcal{H}$ is full, the set $\left\{\pi_{{}^1p_k^b} | {}^1p_k^b \in {}^1 P_k^b\right\}$ forms a basis for

$$\mathrm{Hom}_G\left(\left(G^{(1)}\right)^{\otimes k}, \left(G^{(1)}\right)^{\otimes \ell}\right),$$

and $\mathcal{H}\left(u_{{}^1p_k^b}\right) = \pi_{{}^1p_k^b}$. Thus, the $u_{{}^1p_k^b}$ are linearly independent.

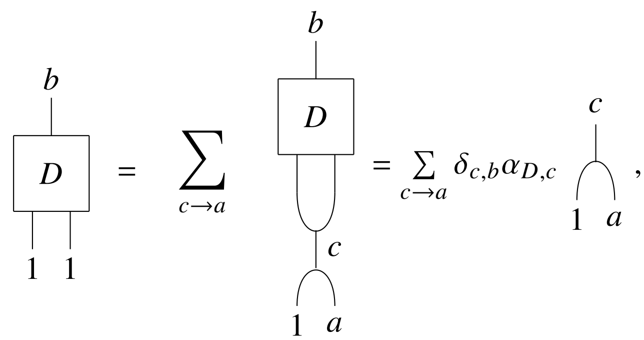
It then suffices to show that any diagram in $\mathrm{Hom}_{\overline{\mathbf{Dgrams}}}\left(1^{\otimes k}, b\right)$ can be written as a linear combination of the diagrams $u_{\mathbf{p}}$ where $\mathbf{p} \in P(1,b)_k$. It will be convenient to instead show the following equality:



$$= \sum_{\mathbf{p} \in P(a,b)_k} \alpha_{\mathbf{p}} u_{\mathbf{p}}$$

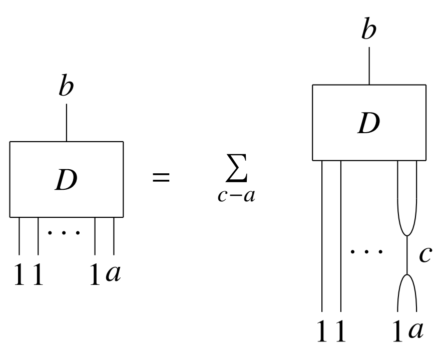
where D is a diagram in $\mathrm{Hom}_{\overline{\mathbf{Dgrams}}}\left(1^{\otimes k} \otimes a, b\right)$.

We induct on k. For $k = 0$, an immediate consequence of the relation (6.3) is that any diagram is either the identity on a, or it is 0. For $k = 1$, the second relation in (6.2) results in the following:

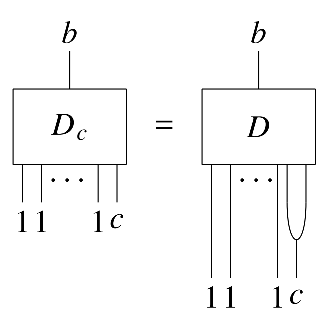


which was to be shown.

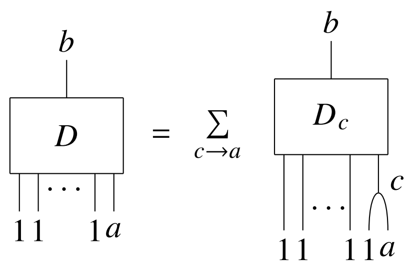
Now, suppose that any diagram in $\mathrm{Hom}_{\overline{\mathbf{Dgrams}}}\left(1^{\otimes k} \otimes a, b\right)$ can be written as a linear combination of the diagrams $u_{\mathbf{p}}$ where $\mathbf{p} \in P(a,b)_k$, and let the following diagram be a diagram in $\mathrm{Hom}_{\overline{\mathbf{Dgrams}}}\left(1^{\otimes (k+1)} \otimes a, b\right)$ for some $b \in I_G$. Using the second relation in (6.2), we get that



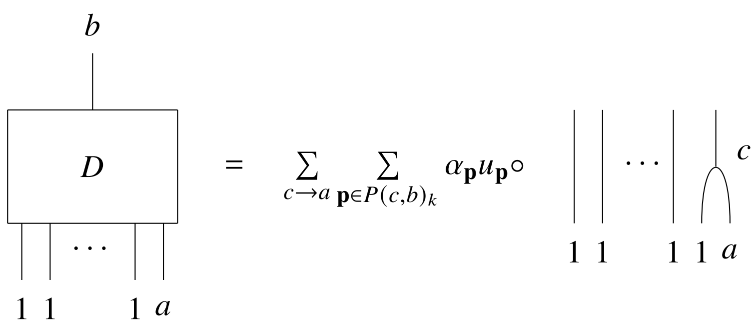
Now we set



resulting in



and thus, by the induction hypothesis,



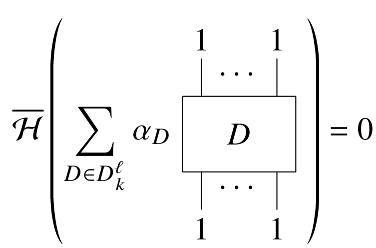
which shows the desired result. Therefore, the $\overline{\mathcal{H}}$ is faithful on $\mathrm{Hom}_{\overline{\mathbf{Dgrams}}}\left(1^{\otimes k}, b\right)$ for all $b \in I_G$.

By considering the vertical reflection of each diagram, the analogous argument shows that $\overline{\mathcal{H}}$ is faithful from $\mathrm{Hom}_{\overline{\mathbf{Dgrams}}}\left(b, 1^{\otimes k}\right)$.

□

Lemma 6.12. *The functor $\overline{\mathcal{H}}$ is faithful on $\mathrm{Hom}_{\overline{\mathbf{Dgrams}}}\left(1^{\otimes k}, 1^{\otimes \ell}\right)$ for all $k, \ell \in \mathbb{N}$.*

Proof. Let D_k^ℓ be the set of all diagrams in $\mathrm{Hom}_{\overline{\mathbf{Dgrams}}}\left(1^{\otimes k}, 1^{\otimes \ell}\right)$. Now suppose that



where only finitely many of the α_D are non-zero.

Let $\mathbf{q} \in P(1,b)_\ell$. Then

$$0 = \overline{\mathcal{H}}\left(u_{\mathbf{q}}\right) \circ \overline{\mathcal{H}}\left(\sum_{D \in D_k^\ell} \alpha_D \begin{array}{c} 1 \; \cdots \; 1 \\ \boxed{\;D\;} \\ 1 \; \cdots \; 1 \end{array} \right)$$

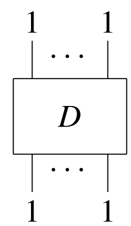
$$= \overline{\mathcal{H}}\left(\sum_{D \in D_k^\ell} \alpha_D \left(u_{\mathbf{q}} \circ \begin{array}{c} 1 \; \cdots \; 1 \\ \boxed{\;D\;} \\ 1 \; \cdots \; 1 \end{array} \right)\right).$$

Now notice that for any $b \in I_G$,

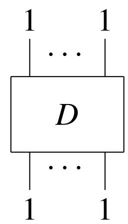
$$u_{\mathbf{q}} \circ \begin{array}{c} 1 \; \cdots \; 1 \\ \boxed{\;D\;} \\ 1 \; \cdots \; 1 \end{array} \in \mathrm{Hom}_{\overline{\mathbf{Dgrams}}}\left(1^{\otimes k}, b\right),$$

and thus by the previous lemma, we have the following equalities:

$$0 = \sum_{D \in D_k^\ell} \alpha_D \left(u_{\mathbf{q}} \circ \begin{array}{c} 1 \; \cdots \; 1 \\ \boxed{\;D\;} \\ 1 \; \cdots \; 1 \end{array} \right) = \sum_{D \in D_k^\ell} \alpha_D \left(d_{\mathbf{q}} \circ u_{\mathbf{q}} \circ \begin{array}{c} 1 \; \cdots \; 1 \\ \boxed{\;D\;} \\ 1 \; \cdots \; 1 \end{array} \right)$$

$$= \sum_{b \in I_G} \sum_{\mathbf{p} \in P(1,b)_\ell} \sum_{D \in D_k^\ell} \alpha_D \left(d_{\mathbf{p}} \circ u_{\mathbf{p}} \circ \right.$$  $$\left. \right),$$

and since $\sum_{\mathbf{p} \in P(1,b)_\ell} d_{\mathbf{p}} \circ u_{\mathbf{p}} = \mathrm{id}_{1^{\otimes \ell}}$, we have

$$0 = \sum_{D \in D_k^\ell} \alpha_D$$  .

Therefore, $\overline{\mathcal{H}}$ is faithful from $\mathrm{Hom}_{\overline{\mathbf{Dgrams}}}\left(1^{\otimes k}, 1^{\otimes \ell}\right)$ for all $k, \ell \in \mathbb{N}$. □

Theorem 6.13. *The functor $\overline{\mathcal{H}}$ is faithful on $\overline{\mathbf{Dgrams}}$.*

Proof. Let $D_{a_n}^{b_m}$ be the set of all diagrams in $\mathrm{Hom}_{\overline{\mathbf{Dgrams}}}\left(\bigotimes_{i=1}^{n} a_i, \bigotimes_{j=1}^{m} b_j \right)$ where $a_i, b_j \in \mathrm{Obj}\left(\overline{\mathbf{Dgrams}}\right)$ for all $1 \leq i \leq n$ and $1 \leq j \leq n$.

Suppose $\overline{\mathcal{H}}\left(\sum_{d \in D_{a_n}^{b_m}} \alpha_d d \right) = 0$. Then we also have

$$\left(\iota_{\ell_{b_1}}^{b_1} \otimes \cdots \otimes \iota_{\ell_{b_m}}^{b_m} \right) \circ \overline{\mathcal{H}} \left(\sum_{d \in D_{a_n}^{b_m}} \alpha_d d \right) \circ \left(\pi_{k_{a_1}}^{a_1} \otimes \cdots \otimes \pi_{k_{a_n}}^{a_n} \right) = 0,$$

and using the fact that $\overline{\mathcal{H}}$ is a monoidal $\mathbb{C}$-linear functor along side the lemmas above, we have the following string of equalities:

$$\left(\iota^{b_1}_{\ell_{b_1}} \otimes \cdots \otimes \iota^{b_m}_{\ell_{b_m}}\right) \circ \overline{\mathcal{H}}\left(\sum_{d \in D^{b_m}_{a_n}} \alpha_d d\right) \circ \left(\pi^{a_1}_{k_{a_1}} \otimes \cdots \otimes \pi^{a_n}_{k_{a_n}}\right)$$

$$= \overline{\mathcal{H}}\left(\left(d^{b_1}_{\ell_{b_1}} \otimes \cdots \otimes d^{b_m}_{\ell_{b_m}}\right) \circ \sum_{d \in D^{b_m}_{a_n}} \alpha_d d \circ \left(u^{a_1}_{k_{a_1}} \otimes \cdots \otimes u^{a_n}_{k_{a_n}}\right)\right)$$

$$= \overline{\mathcal{H}}\left(\sum_{d \in D^{b_m}_{a_n}} \left(\alpha_d \left(d^{b_1}_{\ell_{b_1}} \otimes \cdots \otimes d^{b_m}_{\ell_{b_m}}\right) \circ d \circ \left(u^{a_1}_{k_{a_1}} \otimes \cdots \otimes u^{a_n}_{k_{a_n}}\right)\right)\right) = 0.$$

Since $\overline{\mathcal{H}}$ is faithful on $\mathrm{Hom}_{\overline{\mathbf{Dgrams}}}\left(1^{\otimes k}, 1^{\otimes \ell}\right)$, then

$$\sum_{d \in D^{b_m}_{a_n}} \left(\alpha_d \left(d^{b_1}_{\ell_{b_1}} \otimes \cdots \otimes d^{b_m}_{\ell_{b_m}}\right) \circ d \circ \left(u^{a_1}_{k_{a_1}} \otimes \cdots \otimes u^{a_n}_{k_{a_n}}\right)\right) = 0.$$

This implies that

$$\left(u^{b_1}_{\ell_{b_1}} \otimes \cdots \otimes u^{b_m}_{\ell_{b_m}}\right) \cdot \cdot \sum_{d \in D^{b_m}_{a_n}} \left(\alpha_d \left(d^{b_1}_{\ell_{b_1}} \otimes \cdots \otimes d^{b_m}_{\ell_{b_m}}\right) \circ d \circ \left(u^{a_1}_{k_{a_1}} \otimes \cdots \otimes u^{a_n}_{k_{a_n}}\right)\right) \cdot \cdot \left(d^{a_1}_{k_{a_1}} \otimes \cdots \otimes d^{a_n}_{k_{a_n}}\right) = 0,$$

and thus

$$\sum_{d \in D^{b_m}_{a_n}} \alpha_d \left(u^{b_1}_{\ell_{b_1}} \circ d^{b_1}_{\ell_{b_1}} \otimes \cdots \otimes u^{b_m}_{\ell_{b_m}} \circ d^{b_m}_{\ell_{b_m}}\right) \circ d \circ \left(u^{a_1}_{k_{a_1}} \circ d^{a_1}_{k_{a_1}} \otimes \cdots \otimes u^{a_n}_{k_{a_n}} \circ d^{a_n}_{k_{a_n}}\right) = 0.$$

Now notice, since $u^a_{k_a} \circ d^a_{k_a} = \alpha_{a,k_a} \operatorname{id}_a$ with α_{a,k_a} not 0, we finally have

$$\alpha \sum_{d \in D^{bm}_{an}} \alpha_d d = 0$$

where α is the non-zero scalar $\alpha = \prod_{i=1}^{n} \prod_{j=1}^{m} \alpha_{a_i,k_{a_i}} \alpha_{b_j,\ell_{b_j}}$. Therefore, $\overline{\mathcal{H}}$ is faithful. □

Combining the fullness result given in Theorem 6.10 and the faithfulness results in Theorem 6.13 yields the following result.

Theorem 6.14. *Let $R(V, G)$ be a representation graph which is connected and contains no multiple parallel edges. Let $\mathcal{I}$ be a tensor ideal of* **Dgrams** *which satisfies (6.3). Then there is an equivalence of categories*

$$\overline{H} : \mathbf{Dgrams} \big/ _{\mathcal{I}} \longrightarrow G - \mathbf{mod}_{irr}.$$

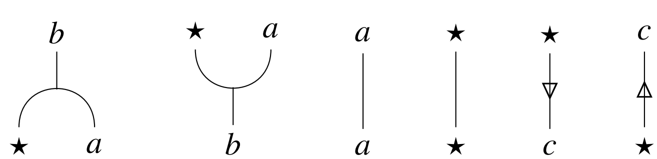
Of course, it remains to determine $\mathcal{I}$, for example, by giving a set of relations. This will presumably depend on the specifics of the representation theory of G and would need to be determined on a case-by-case basis.

Chapter 7

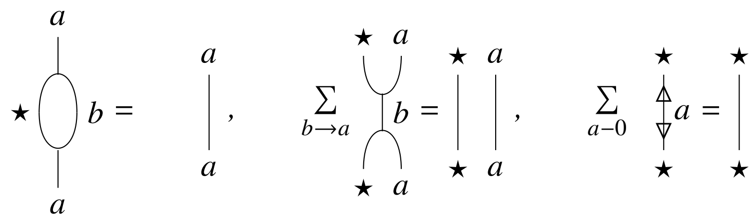
Final Remarks

It is worth noting that we can be even more general in our set up with much the same result. Suppose instead that we begin with a semi-simple, monoidal, $\mathbb{C}$-linear category $\mathcal{M}$ and restrict to the full subcategory monoidally generated by the simple objects, which we can denote as $\mathcal{M}_{\text{irr}}$, the objects of which can be indexed by $I_{\mathcal{M}}$. By fixing an object V of $\mathcal{M}$, we may construct a directed graph $\Gamma_{\mathcal{M},V}$ in an analogous way to a representation graph. Assume $\Gamma := \Gamma_{\mathcal{M},V}$ is a directed, connected graph which does not contain any multiple parallel edges between vertices, we may form the following definition. For convenience, we will identify the unit object of $\mathcal{M}$, $\mathbb{1}$, with the vertex of Γ corresponding to $\mathbb{1}$.

Definition 7.1. *Let* **Dgrams**$_{\Gamma,\star}$ *be defined as the monoidal $\mathbb{C}$-linear category with objects generated by $\star$ and $a \in I_{\mathbb{M}}$ and morphisms generated by*



for all $a, b \in I_{\mathcal{M}}$ and $c \in I_{\mathcal{M}}$ adjacent to $\mathbb{1}$ in the directed graph Γ. The generating diagrams are subjected to the following relations:



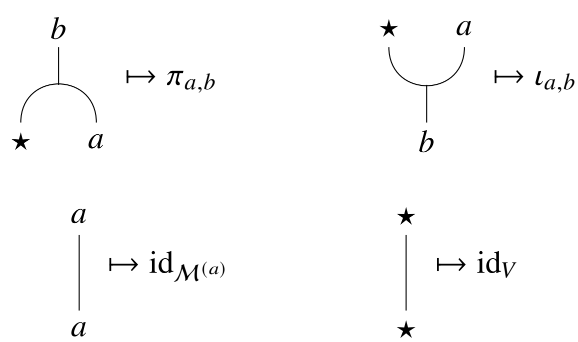
Denote by $\mathcal{M}^{(a)}$ the simple object of $\mathcal{M}$ corresponding to the index $a \in I_{\mathcal{M}}$. Let $\pi_{a,b}$ be a map in $\mathrm{Hom}_{\mathcal{M}}\left(V \otimes \mathcal{M}^{(a)}, \mathcal{M}^{(b)}\right)$. As Γ has no multiple edges and $\mathcal{M}^{(b)}$ is simple, $\pi_{a,b}$ is unique, up to scaling. Let $\iota_{a,b}$ be in $\mathrm{Hom}_{\mathcal{M}}\left(\mathcal{M}^{(b)}, V \otimes \mathcal{M}^{(a)}\right)$, such that $\pi_{a,b} \circ \iota_{a,b} = \mathrm{id}_{\mathcal{M}^{(b)}}$. Furthermore, we can define a unique, up to scaling, map, $\pi_{V,c}$ in $\mathrm{Hom}_{\mathcal{M}}\left(V, \mathcal{M}^{(c)}\right)$ when $\mathcal{M}^{(c)}$ is a direct summand of V, and let $\iota_{V,c}$ in $\mathrm{Hom}_{\mathcal{M}}\left(\mathcal{M}^{(c)}, V\right)$ such that $\pi_{V,c} \circ \iota_{V,c} = \mathrm{id}_{\mathcal{M}^{(c)}}$.

Now, we can define a monoidal, $\mathbb{C}$-linear functor

$$\mathcal{H} : \mathbf{Dgrams}_{\Gamma,\star} \longrightarrow \mathcal{M}_{\mathrm{irr}}$$

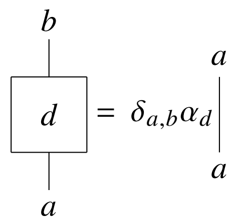
which is given on the generating objects and morphisms as follows:

$$a \mapsto \mathcal{M}^{(a)} \qquad \star \mapsto V$$

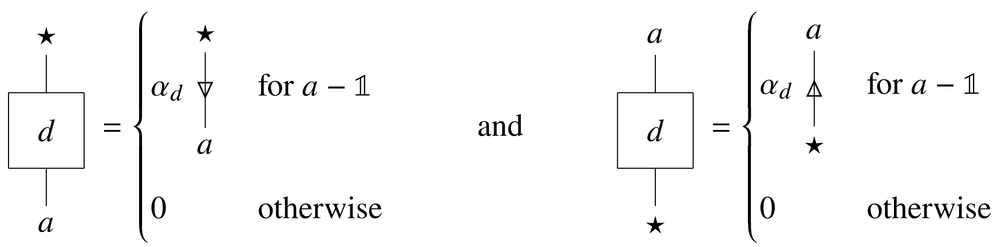


and extend monoidally and $\mathbb{C}$-linearly. The proofs are analogous to show $\mathcal{H}$ is a full functor from $\mathbf{Dgrams}_{\Gamma,\star}$ to $\mathcal{M}_{\text{irr}}$.

Furthermore, we can define an induced faithful functor from $\mathbf{Dgrams}_{\Gamma,\star} / \mathcal{I}$ where we let $\mathcal{I}$ be the tensor ideal generated by the following relations:

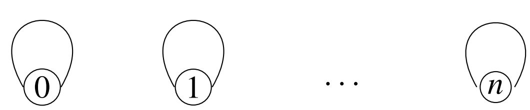


, where $\delta_{a,b}$ is the Kronecker delta, and $\alpha_d \in \mathbb{C}$,



The proofs are analogous to show this construction admits of a fully faithful functor.

Let us now explore some limitations on these constructions. First, we need connectedness in our representation graph as the following will illuminate. Consider a finite group G with the set $\{G^{(i)}\}_{0 \le i \le n}$ an exhaustive list of irreducible G-modules, up to isomorphism, and let the defining module, V, for our representation graph be the trivial module for G. Then we have that $R(V,G)$ is



which would result in a diagrammatic category which does not recover any homomorphisms in $\mathrm{Hom}_G\left(G^{(a)} \otimes G^{(b)}, G^{(c)}\right)$ even when $G^{(c)}$ shows up in the direct sum decomposition of $G^{(a)} \otimes G^{(b)}$.

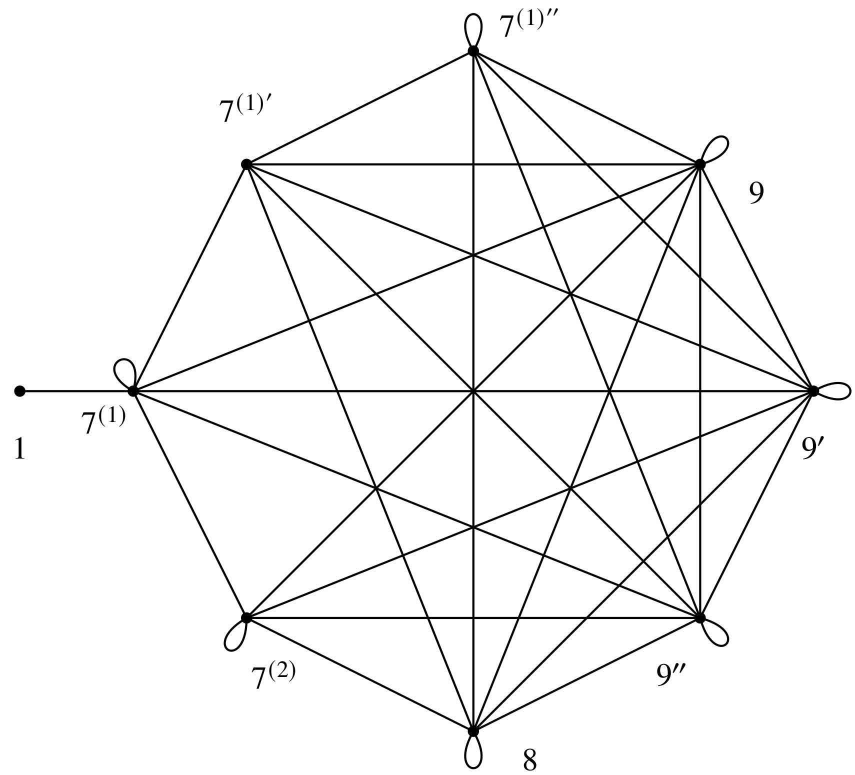
Another limitation of our construction is that we assumed the representation graph has no multiple edges between two vertices. This corresponds to a multiplicity free condition on the direct sum decomposition of the tensor product of the defining module V and each simple module. Without this condition, there is not a canonical way to decompose this tensor product into simples, and we must make non-trivial choices.

On the other hand, please note that we make no assumption that these graphs be finite. In particular, the representation graphs $R(SU(2), V)$, $R(\mathbf{C}_\infty, V)$, and $R(\mathbf{D}_\infty, V)$ where V is the natural module for $SU(2)$, have infinitely many nodes. In this situation, our approach still applies and we can construct a diagrammatic category which encodes the corresponding representation theory.

Beyond the subgroups of $SU(2)$, there are numerous representation graphs in the literature. They often are aptly named McKay graphs. Here are just a few places the reader can explore: [13], [14], [11], and [12]. In particular, the following example uses Evans' and Pugh's explicit computation of a representation graph $R(PSL(2;8), \Sigma_7^{(1)})$ to construct a diagrammatic category for a setting unlike any other in this thesis.

Example 7.2. *Let $G = PSL(2;8)$ denote the projective special linear group of degree 2 over the finite field of order 8. This is an irreducible primitive group of order 504. There are nine irreducible G-modules. We will follow the notation from Evans and Pugh and let the irreducible G-modules be indexed in the following way:*

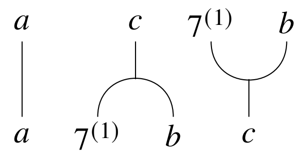
let the trivial module of dimension 1 *be denoted* Σ_1*; there are four* 7*-dimensional irreducible modules denoted* $\Sigma_7^{(1)}$, $\Sigma_7^{(1)'}$, $\Sigma_7^{(1)''}$, *and* $\Sigma_7^{(2)}$*; there is one* 8*-dimensional irreducible module denoted* Σ_8*; finally, there are three* 9*-dimensional irreducible modules* Σ_9, Σ_9', *and* Σ_9''. *For consistency with the notation in this dissertation, let* $\Sigma_7^{(1)} = V$. *Thus, Evans and Pugh compute the representation graph* $R(V, G)$ *to be the following undirected graph:*



Now we can construct the monoidal $\mathbb{C}$*-linear diagrammatic category* $\mathbf{Dgrams}_{R(V,PSL(2;8))}$. *Keeping consistent with the notation from Definition 6.3, let*

$$I_{PSL(2;8)} = \{1, 7^{(1)}, 7^{(1)'}, 7^{(1)''}, 7^{(2)}, 8, 9, 9', 9''\}.$$

The set of objects are generated by $a \in I_{PSL(2;8)}$ *and the morphisms are generated* *by*



*where a, b, and $c \in I_{PSL(2;8)}$. Then by Theorem 6.10, there is an essentially surjective and full functor from $\textbf{Dgrams}_{R(V,PSL(2;8))}$ onto the full subcategory of $PSL(2;8)$-mod with objects generated by the irreducible $PSL(2;8)$-modules. By Theorem 6.13, if relations are imposed to ensure (6.3), then the resulting category $\overline{\textbf{Dgrams}}_{R(V,PSL(2;8))}$ is equivalent to $PSL(2;8)$-**mod**$_{irr}$.*

We shall finish this thesis with another example. The following example shows that the constructions in this dissertation apply to situations outside of representation theory. In particular, the Fibonacci category has objects which have non-integer dimensions. For a more comprehensive understanding of this setting, see [15].

Example 7.3. *Let $\mathcal{F}ib$ be the semi-simple, monoidal, $\mathbb{C}$-linear category in which* Obj $(\mathcal{F}ib)$ *are generated by two objects, I and X, which follow the following tensor product decomposition rules:*

$$I \otimes I \cong I, \quad I \otimes X \cong X \otimes I \cong X, \quad \text{and} \quad X \otimes X \cong I \oplus X.$$

Now, define $\mathcal{F}ib_{irr}$ as the full subcategory monoidally generated by X and I. Thus, we can construct the following graph $\Gamma := \Gamma_{\mathcal{F}ib_{irr}}$ with generating object X:

.

From the graph Γ*, we can construct* $\mathbf{Dgrams}_\Gamma$ *and functor*

$$\mathcal{H} : \mathbf{Dgrams}_\Gamma \longrightarrow \mathcal{F}ib_{irr}$$

consistent with the other constructions in this thesis.

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