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ON THE PRESCRIBED RICCI CURVATURE OF NONCOMPACT HOMOGENEOUS SPACES WITH TWO ISOTROPY SUMMANDS

A DISSERTATION APPROVED FOR THE
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Dedicated to

Jesus, the $\lambda\acute{o}\gamma o\sigma$, *who created and sustains all; rescuing and vindicating – even from the dead.*

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Contents

Abstract

The current dissertation works within the setting of noncompact homogeneous spaces G/H in which G is semi-simple. In particular, we frequently work with a decomposition of the Lie algebra $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$, where $\mathfrak{h} \oplus \mathfrak{p}''$ is the maximal compact in $\mathfrak{g}$ and $\mathfrak{p}'$ is the negative one eigenspace from the Cartan decomposition. In such a setting we primarily set out to understand G invariant metrics and Ricci curvature, and the relationship these are in with Lie theoretic conditions. There are three basic components to this work with the second holding most of our attention. The first component is an investigation into spaces, G/H, in which we can always obtain some decomposition with $(\mathfrak{p}'', \mathfrak{p}') = 0$ (what we call a Cartan orthogonal pair), building out results indicating that there are many examples of such spaces. The second component is an investigation into simply connected G/H with two isotropy irreducible summands. Here, we classify such spaces and solve the so-called Prescribed Ricci Curvature problem for all such G/H. The third component is an investigation into a particularly nice setting of G/H with G simple and having three irreducible summands in which $[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}$ for each irreducible isotropy representation, $\mathfrak{p}_i$. Here, we provide Lie theoretic conditions for obtaining diagonal ric, begin an investigation into the signature of such spaces, and work through an example, $SO(n,2)/SO(n)$. A final consequence of these three components is a description of the signature of all spaces G/H in which G is simple and G/H has negative scalar curvature for all metrics.

Introduction

A classic question in the field of Riemannian geometry is *what is the relationship between Ricci curvature and geometric or topological objects?* In other words, given a set of conditions on the Ricci curvature, what can be said about the possible spaces with said curvature, and, vice-versa, given a space or a collection of spaces, what can be said about the possible Ricci curvature values? This old and important question comes up in examples such as Myer's theorem which says that if a complete Riemannian manifold has Ricci curvature bounded below by a positive constant then the space is compact (see [dC92]). Moreover, restricting ourselves to the homogeneous setting (the setting addressed in the present work), we have several similar results and questions both new and old.

Going back to the late 1970's we have a result of B. Bergery found in Theorem 2 of [BB78] which shows the logical equivalence between a homogeneous space G/H having a universal cover diffeomorphic to Euclidean space and admitting only negative scalar curvature (the trace of Ricci curvature). Further, through the investigation of Einstein metrics (i.e metrics g such that $ric_g(.,.) = \lambda g(.,.)$) on homogeneous spaces, more results can be seen such as Bochner's result in [Boc46] (confer with [Bes87, 7.4]) which indicates that if the scalar curvature is negative then G/H is necessarily noncompact.

In the noncompact homogeneous setting (the primary emphasis for us), there has recently been a resolution given to a decades old question known as Alekseevskii's Conjecture.

Alekseevskii's Conjecture: A connected homogeneous Einstein space with negative scalar curvature is diffeomorphic to a Euclidean space.

With a proof put forth in [BL23], we now have yet another relationship between the Ricci curvature and geometric spaces. Furthermore, In 7.5 of [Bes87] the formulation of the conjecture informs us that H must be a maximal compact subgroup of G, providing us with Lie theoretic implications, and in Theorem A of [BL22] another Lie theoretic implication is provided that G/H must also be a solvmanifold (i.e diffeomorphic to a solvable Lie group).

Most of these results then, many stemming from the Einstein problem, provide us with one direction of the relationship of interest, namely, provided a condition on the Ricci curvature, we get certain kinds of spaces. The other direction of interest is what we turn to now.

The Einstein problem, can from one perspective, be thought of as a special case of a problem that has received considerable attention lately known as the *Prescribe Ricci Curvature Problem* (PRP). This problem has two questions wrapped into one as it seeks to get a full description of the Ricci curvature for a given manifold or a given collection of manifolds. The first question of interest is *what conditions on* $T(.,.)$ *are necessary and sufficient to there existing a metric* g *with* $ric_g(.,.) = T(.,.)$*?* This question can, geometrically, be thought of as seeking to understand the image of $ric_g(.,.)$ for our space G/H. The second question is similar, and it seeks to understand the image of $ric_g(.,.)$ up to scaling as it asks *what conditions on* $T(.,.)$ *are necessary and sufficient to there existing a* $c > 0$ *such that* $ric_g(.,.) = cT(.,.)$ *for some metric* g*?*

Staying within the homogeneous setting, there are some natural choices of spaces to choose from to make things simple. If G/H is isotropy irreducible, the answer is immediate since by Schur's Lemma $ric_g(.,.) = \lambda g(.,.)$ and with λ being positive, negative, or zero based upon G/H being compact, noncompact, or admitting only flat metrics, respectively. Relaxing the condition on G/H some, we can ask about when H is maximal in G. In the compact setting, recently in Theorem 1.1 of [Pul16], a sufficient condition on $T(.,.)$ was provided to the question for $ric(.,.) = cT(.,.)$ when H is connected, showing that if $T(.,.)$ is positive-semidefinite, but not identically 0 then there is a solution. If G is noncompact, then we consider G/H where H is a maximal compact subgroup of G, and if G is semi-simple then G/H is a noncompact symmetric space (see [Hel01]) and is necessarily the Riemannian product of irreducible symmetric spaces. Thus, $ric_g = ric_{g_1} + ... + ric_{g_n}$ with each g_i unique up to scaling by Schur's Lemma, and with $ric_{g_i} = \lambda_i g_i(.,.)$ by virtue of being defined on an isotropy irreducible space.

Restricting ourselves now to the setting G/H in which G is semi-simple and H is connected (where we will stay for most of the paper), we have other natural settings to consider. In both the compact and noncompact settings, the PRP for simple Lie groups G with the so-called *naturally reductive metrics* are addressed in [APZ21] and [AGP20]. Moreover, in the compact setting, spaces G/H with two inequivalent isotropy summands are addressed with a complete solution to $ric(.,.) = cT(.,.)$ provided in [GP17] and [BP20]. However, in noncompact setting, much is left unaddressed, and it is to this case and others that we turn our attention.

In the survey of the PRP, [BP20], more results can be found, and, as is noted there, little work has been done in the noncompact setting we are interested in. It is for this reason that we consider our work to be worthy of attention. In the current work, beyond addressing the PRP in the two isotropy irreducible summand setting, we also provide work in the direction of the PRP indicating possible diagonal values for the Ricci tensor (i.e, studying the signature of $ric(.,.)$) in the setting of three isotropy irreducible summands where $\mathfrak{g}$ is simple and each irreducible subrepresentation, $\mathfrak{p}_i$, has the property that $[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}$. This setting serves as a kind of "dual" to the generalized Wallach spaces classified by authors such as Nikonorov in [Nik21] and other works cited therein. Our work in this three irreducible summand setting is assisted by Nikornorov's work [Nik00] which provides a formula for the diagonal values of the Ricci tensor (viewed as a (1,1) tensor) for metrics $(.,.)$ where the compact and noncompact pieces of the isotropy represention for G/H are orthogonal. The conclusions of Nikonorov's paper sparks some other interesting questions that we find worthy of attention.

In [Nik00], Nikonorov provides us with an expression for the diagonal of the Ricci tensor that is far less formidable than we otherwise have (see Corollary 7.38 of [Bes87]). This, in combination with the desire to understand the relationship between geometric objects and Ricci curvature, also sparks an interest in a better understanding of which spaces G/H admit only metrics $(.,.)$ where the compact and noncompact pieces of the isotropy representation are orthogonal (a property for $(\mathfrak{g}, \mathfrak{h})$ we call being a Cartan orthogonal pair). Knowing which spaces admit such metrics would then allow us to have a nice correspondence between Ricci tensor possibilities and geometric possibilities. Not all spaces admit this condition (as we will show), but there are also many spaces which do (as we will show).

Taking now this backdrop of results and questions in the field, we ask the following questions.

Question 1: What is the solution to the PRP for G/H where G is noncompact semisimple, H is connected, and G/H has two isotropy irreducible summands?

Question 2: What is the signature of the Ricci curvature tensor for the noncompact setting that is "dual" to generalized Wallach spaces? (See Chapter 4 for more precision.)

Regarding **Question 1**, we provide a complete solution with three different filtrations of solutions based upon the compact part of the isotropy representation, $\mathfrak{p}''$, being trivial, having $[\mathfrak{p}'', \mathfrak{p}''] \subset \mathfrak{h}$, or having $[\mathfrak{p}'', \mathfrak{p}''] \subset \mathfrak{h} \oplus \mathfrak{p}''$. We remark here that our approach to this problem is unlike previous works done for the PRP such as [Pul16] and [AGP20] which use variation techniques to solve when $ric = cT$, and is also unlike the two isotropy summand setting addressed in [GP17] and [BP20] in that we do not restrict ourselves to inequivalent summands.

Regarding **Question 2**, we begin an investigation, but a complete solution was out of reach. We do, however, provide some motivating examples for clarity and completeness.

It is with these questions in mind that we turn our attention to the setting of G/H in which G is noncompact semi-simple. The subsequent material is according to the following program, and most results pertain to those noncompact G/H in which G is semi-simple (we are clear when we are working more broadly). In Chapter 1 we have our preliminaries in which we introduce the necessary objects and tools for our study (also containing our precise definition of Cartan orthogonal pairs). In Chapter 2 we present both new and old results regarding the consequences of being a Cartan orthogonal pair along with new sufficient conditions to be an orthogonal pair. These results will prove to be helpful in Chapter 4. Moreover, in Chapter 2 we also devote our attention here to new examples of Cartan orthogonal pairs. There we also discuss one G/H that is not a Cartan orthogonal pair (found in Example 3 of Section 2.2) which serves as a correction to Example 4 in [Nik00] and indicates a gap in Theorem B of [AL17] (but we hasten to note that the proposed results in [BL23] resolve the gap). In Chapter 3 we turn our attention to **Question 1**, providing a classification of the homogeneous spaces with two isotropy irreducible summands. In Chapter 4 we address **Question 2**, providing a step in the direction of a complete result, and, in part, extending the methods employed by Nikonorov in [Nik00] to find a nice expression for the Ricci curvature tensor in the given three irreducible summand setting. A nice consequence of our results found at the end of Chapter 4 is a description of the signature of G/H in which G is simple and G/H admits only metrics with negative scalar curvature.

Remark 0.1. As was stated, there is much more to be said about the PRP, but we specifically wish to mention two other works that served to be inspiring for the current work. The first is the work of Lauret and Will in [LW22] which investigates a property called Ricci local invertibility (see Definition 1.2 therein). This work is in the direction of the PRP in both the compact and noncompact setting. The second work that has not been mentioned but served as inspiration for its work in the direction of the PRP is that of Arroyo and Lafuente [AL22] which investigates the signature of the Ricci curvature tensor in the nilpotent setting.

Chapter 1

Preliminaries

1.1. The Ricci Tensors and Homogeneous Spaces

The following basic content regarding homogeneous spaces and curvature can be found in [dC92], [Pet16], [Bes87], and [NRS06].

Any given Riemannian manifold (M, g) that is a *homogeneous space* has a presentation in terms of Lie groups as a coset space, G/H, in which G acts transitively and by isometry on M and H is compact and acts as isotropy. Transitively meaning that, for a point $p \in M$ and a point $q \in M$ there is a $k \in G$ such that $k.p = q$, isometry meaning that (by an abuse of notation for the action) if $X_p, Y_p \in T_pM$, $g(k.X_p, k.Y_p)_{k.p} = g(X_p, Y_p)_p$ (i.e g is a left G-invariant metric), and isotropy meaning that for some $p \in M$ we have $H = \{g \in G : g.p = p\}$. This allows for an identification $M \cong G/H$, and we will henceforth consider homogeneous spaces with this presentation.

Remark 1.1. We will, by a common abuse of notation, drop the point $p \in M$ in the description of vectors and metrics unless needed for clarity.

In Riemannian Geometry, a question of interest pertains to the curvature of a given space (homogeneous or not), M. To study curvature, we use the language of tensors as in [Pet16], which allows us to easily think about changing tensor type as follows:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \quad \text{is the curvature tensor of type} \quad (1,3)$$
$$R(X,Y,Z,W) = g(R(X,Y)Z,W) \quad \text{is the curvature tensor of type} \quad (0,4)$$

From this curvature tensor, we may consider other measurements of curvature such as Sectional, Ricci, and Scalar. In this paper we will primarily be concerned with Ricci curvature with the occasional Scalar curvature tensor appearing. *Ricci curvature* is defined as follows, where $\{e_i\}$ is an orthonormal basis for our metric $g(.,.)$ in T_pM and $X, Y \in T_pM$.

$$ric(X,Y) = \sum_{i=1}^{n} g(R(X,e_i)e_i, Y) \quad \text{is the curvature tensor of type} \quad (0,2)$$
$$Ric(X) = \sum_{i=1}^{n} R(X,e_i)e_i \quad \text{is the curvature tensor of type} \quad (1,1)$$
$$ric(X,Y) = g(Ric(X),Y)$$

The *scalar curvature* S is defined at a point $p \in M$ as the trace of Ric at that point, or equivalently:

$$S(p) = \sum_{i=1}^{n} g(Ric_p(e_i), e_i)_p.$$

On a given homogeneous space G/H, there is a convenient way to describe the G-invariant metrics on the space using Lie algebra data. First, we observe that if we wish to think about the metric on the space globally, since G is a Lie group that acts by isometry on G/H, it suffices to study the metric and the curvature at a single point.

Indeed, on a general manifold M, if $\phi : M \to M$ is a diffeomorphism, then $g(d\phi_p X_p, d\phi_p Y_p)_{\phi(p)} = \phi^* g(X_p, Y_p)_p$ where ϕ^* is the pull-back of ϕ and $d\phi_p$ is the derivative of ϕ at $p \in M$. From this, we get that $ric_g(d\phi_p(X_p), d\phi_p(Y_p))_{\phi(p)} = ric_{\phi^* g}(X_p, Y_p)_p$ as well. If we furthermore assume ϕ to be an isometry of M, then these equalities become $g(d\phi_p X_p, d\phi_p Y_p)_{\phi(p)} = g(X_p, Y_p)_p$ and $ric_g(d\phi_p(X_p), d\phi_p(Y_p))_{\phi(p)} = ric_g(X_p, Y_p)_p$.

Therefore, in our homogeneous setting, since G acts by isometry on G/H, if $X_{eH}, Y_{eH} \in T_{eH}(G/H)$ and $k \in G$, then $g(k.X_{eH}, k.Y_{eH})_{kH} = g(X_{eH}, Y_{eH})_{eH}$. Thus, $ric_g(X_{kH}, Y_{kH})_{kH} = ric_g(X_{eH}, Y_{eH})_{eH}$ as well. In the homogeneous setting then, we can allow ourselves to restrict our study of Ricci curvature to studying the curvature at a single point. This is where the Lie algebra data comes in. For greater understanding of this dynamic, we turn our attention to some of the needed basics of Lie theory and Representation theory.

1.2. Lie Theory, Representation Theory, and Ricci Curvature

The following information can be found in [Kna02], [Hel01], [Hal15], and [FH91].

Recall that for a Lie group G, its Lie algebra $\mathfrak{g}$ is naturally identified as the tangent space to G at $e \in G$, and is also often identified as the vector fields invariant under the left group action on G. We denote a representation (using the notation of Chapter 4 in [Hal15]) (Π, V) on G by $\Pi : G \to GL_n(V)$ and a representation (π, V) on $\mathfrak{g}$ by $\pi : \mathfrak{g} \to \mathfrak{gl}_n(V)$, which by use of the exponential map $exp : \mathfrak{g} \to G$, we have the following relation:

$$\pi(x) = \frac{d}{dt}|_0 \Pi(exp(tX)) \tag{1.1}$$

We call $g(.,.)$ on V a Π *invariant metric of* G if for $v, w \in V$, $g(\Pi(k)v, \Pi(k)w) = g(v, w)$ for all $k \in G$. Moreover, we call $g(.,.)$ on V a π *invariant metric of* $\mathfrak{g}$ if for $v, w \in V$, $g(\pi(x)v, w) = -g(v, \pi(x)w)$ for all $x \in \mathfrak{g}$. It follows from use of (1.1) that if $g(.,.)$ is Π invariant then $g(.,.)$ is also π invariant.

A representation of G or $\mathfrak{g}$ on V is *irreducible* if there is no invariant subspace of V aside from $\{0\}$ and V. That is, in the case of Lie groups (Lie algebras are defined similarly), if $\Pi : G \to GL_n(V)$, if $W \subset V$ such that $\Pi(k)w \in W$ for all $k \in G$ and $w \in W$, then $W = \{0\}$ or V. We call a representation V of G (and so also $\mathfrak{g}$) *completely reducible* if V can be decomposed into a sum of irreducible representations like so: $V = \bigoplus_{i=1}^{n} V_i$. The following is well-known and can be found in the resources listed above:

Proposition 1.2. If G is a compact Lie group with Lie algebra $\mathfrak{g}$, then any representation (Π, V) or (π, V) is completely reducible.

A common representation of interest in Lie theory which will be the primary representation of interest in the current work is the *adjoint representation* denoted by $Ad : G \to GL(\mathfrak{g})$ and $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. Ad is properly thought of as the derivative of conjugation, and ad the derivative of Ad. However, ad is equivalently defined as $ad(x)y = [x, y]$ where $[.,.]$ is the Lie bracket, and it is through this lens that we will primarily consider ad. In the current work, we will primarily consider the representation $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ and restrictions of that representation as we study homogeneous spaces.

With the above Lie theoretic and Representation theoretic tools at hand, we can now turn our attention back to the question of metrics and curvature on a homogeneous space, G/H, where H is a compact subgroup of G acting as isotropy. If we consider the representation of $\mathfrak{h}$, $ad|_{\mathfrak{h}}: \mathfrak{g} \to \mathfrak{g}$, we know by complete reducibility, that we can get a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ in which $\mathfrak{p}$ is an invariant complement of $\mathfrak{h}$ in $\mathfrak{g}$. Here, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ is called a *reductive decomposition*. There are two helpful examples of how to derive this decomposition.

Example 1.3. $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ where $\mathfrak{p} = \{x \in \mathfrak{g} : (x, y) = 0 \text{ for all } y \in \mathfrak{h}\}$ and $(.,.)$ is an Ad_H invariant definite bilinear form on $\mathfrak{g}$.

Example 1.4. $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ where $\mathfrak{p} = \{x \in \mathfrak{g} : B(x, y) = 0 \text{ for all } y \in \mathfrak{h}\}$ and $B(.,.)$ is the Killing form for $\mathfrak{g}$ where G/H is a homogeneous space in which G acts almost effectively and is semi-simple.

In general, $\mathfrak{p}$ is not necessarily irreducible but is a sum of irreducible representations, but when $\mathfrak{p}$ is irreducible, we call G/H an *isotropy irreducible* homogeneous space. Regardless, from this decomposition, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, and the identification of $\mathfrak{g}$ with T_eG, there is a natural identification of $\mathfrak{p}$ with $T_{eH}G/H$. The identification is defined as follows:

$$x \mapsto X_x = \frac{d}{dt}|_0 exp(tx).eH$$

The above map takes $x \in \mathfrak{g}$ to $X_x \in T_{eH}G/H$, and one can observe that if $x \in \mathfrak{h}$, since $H.eH = eH$, $X_x = 0$.

From this identification, we can get a one-to-one correspondence between the G-invariant metrics on G/H and the Ad_H invariant metrics on $\mathfrak{p}$. The identification of metrics is done as follows:

$$g(X_x, Y_y)_{eH} := (x, y)_e$$

Here, the identification of G-invariant metrics on the left with Ad_H invariant metrics on the right is achievable because the representation formed by the G action is equivalent to the Ad_H representation. Therefore, since we can study curvature on G/H at a point, and since we can identify $\mathfrak{p}$ with $T_{eH}G/H$, we can study the Ricci tensor in terms of Ad_H invariant metrics on $\mathfrak{p}$ in $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$. Consequently, we can study $ad_{\mathfrak{h}}$ invariant metrics on $\mathfrak{p}$ to understand the Ricci curvature of G-invariant metrics on G/H. Having this all put together, a remark on nation is warranted.

Remark 1.5. On a homogeneous space G/H, $ad_{\mathfrak{h}}$ will be used to denote $ad|_{\mathfrak{h}}\colon \mathfrak{h} \to \mathfrak{gl}(\mathfrak{p})$ where $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ is a reductive decomposition. That is, $ad_{\mathfrak{h}}$ will denote the representation defined by $ad_{\mathfrak{h}}(x)(y) = [x, y]$ where $x \in \mathfrak{h}$ and $y \in \mathfrak{p}$.
When $\mathfrak{h}$ is clear from the context, we will instead use the notation $ad_x(y) = [x, y]$ for specific elements of the Lie algebra.

Restricting ourselves to the case when G is unimodular (i.e $trace(ad_x) = 0$ for $x \in \mathfrak{g}$), we have the following formula for the Ricci curvature (chapter 7 of [Bes87]) where $\{e_i\}$ is an orthonormal basis for an $ad_{\mathfrak{h}}$ invariant metric $(.,.)$ on $\mathfrak{p}$ in $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ and $B(.,.)$ is the Killing form on $\mathfrak{g}$ defined by $B(x, y) := tr(ad_x \circ ad_y)$.

$$ric(x, y) = -\frac{1}{2} \sum_i ([x, e_i]_{\mathfrak{p}}, [y, e_i]_{\mathfrak{p}}) - \frac{1}{2} B(x, y) + \frac{1}{4} \sum_{i,j} ([e_i, e_j]_{\mathfrak{p}}, x)([e_i, e_j]_{\mathfrak{p}}, y) \quad (1.2)$$

1.3. Schur's Lemma and Some Consequences

When working with metrics, inner products, or bilinear forms invariant under some Lie algebra representation, Schur's Lemma is an essential tool to many proofs and examples. Given that this is the case, we have this section dedicated to Schur's Lemma and many of the immediate consequences of Schur's Lemma that get used in Lie theory and Reimannian geometry. Many of the following results and statements can be found in resources such as [Hal15], [FH91], [Oni04], and [BtD85]. Some proofs are included for completeness, especially for results hard to come by in the references just mentioned.

Definition 1.6. Given two representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ and $\tau : \mathfrak{g} \longrightarrow \mathfrak{gl}(W)$, we say that $\phi : V \longrightarrow W$ is an *intertwining map* if $\phi \circ \rho(x) = \tau(x) \circ \phi$ for all $x \in \mathfrak{g}$. If $V = W$, then we use the term *equivariant map*.

Definition 1.7. We say that $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ and $\tau : \mathfrak{g} \longrightarrow \mathfrak{gl}(W)$ are *isomorphic representations* of $\mathfrak{g}$ if there is an intertwining map, $\phi : V \longrightarrow W$, for ρ and τ such that ϕ is an isomorphism of vector spaces.

Schur's Lemma Let $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ and $\tau : \mathfrak{g} \longrightarrow \mathfrak{gl}(W)$ be representations of $\mathfrak{g}$ such that V and W are both either real or complex. If V is irreducible then any intertwining map $\phi : V \longrightarrow W$ is either an isomorphism or 0. Moreover, if $\psi : V \longrightarrow V$ is an equivariant map and V is complex then $\phi = \lambda Id_V$ or is 0 ($\lambda \in \mathbb{C}$), and if V is real and ψ is self-adjoint then $\psi = \lambda Id_V$ ($\lambda \in \mathbb{R}$).

An immediate consequence of Schur's Lemma as stated above on irreducible representations is that if V and W are any completely reducible representations (see Section 1.2), then the existence of a non-trivial intetwining map $\phi : V \longrightarrow W$ implies that there is some $V_0 \subset V$ and $W_0 \subset W$ such that $V_0 \simeq W_0$ as representations.

The following two lemmas are well-known, and a proof for each is included for completeness.

Lemma 1.8. If $\langle ., . \rangle$ and $(., .)$ are two inner products invariant under the representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ (where $\mathfrak{g}$ is the Lie algebra of a compact Lie group), then there is a symmetric (with respect to $\langle ., . \rangle$) positive definite $L : V \to V$ such that L is an intertwinining map for ρ and $(v, w) = \langle Lv, w \rangle$ for all $v, w \in V$.

Proof: By properties of inner products on vector spaces, we have that $(v, w) = \langle Lv, w \rangle$ for all $v, w \in V$ where L is symmetric with respect to $\langle ., . \rangle$ and L is positive definite. What must be shown is that L is an equivariant map for ρ.

$$\begin{aligned}
(\rho(x)(v), w) &= \langle L(\rho(x)(v)), w \rangle \\
&= \langle \rho(x)(v), L(w) \rangle \\
&= -\langle v, \rho(x)(L(w)) \rangle \\
&= -(L^{-1}(\rho(x)(L(w))), v)
\end{aligned}$$

Therefore, by ρ invariance of $(., .)$ used on the left hand side above,

$$-(v, \rho(x)(w)) = -(L^{-1}(\rho(x)(L(w))), v).$$

Thus, we have that $\rho(x)(w) = L^{-1}(\rho(x)(L(w)))$, so $L(\rho(x)(w)) = \rho(x)(L(w))$, making L a ρ equivariant map. ∎

Lemma 1.9. If $\rho : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{p} \oplus \mathfrak{q})$ is a representation for $\mathfrak{g}$ where G is a compact Lie group, then if there is no irreducible representation in $\mathfrak{p}$ isomorphic to an irreducible representation in $\mathfrak{q}$ then all ρ- invariant inner products on $\mathfrak{p} \oplus \mathfrak{q}$ are such that $\mathfrak{p}$ and $\mathfrak{q}$ are orthogonal.

Proof: Assume to the contrary and there is such a inner product $(.,.)$ where $(x, y) \neq 0$ for some $x \in \mathfrak{p}$ and $y \in \mathfrak{q}$. Well, since $\mathfrak{p}, \mathfrak{q}$ are invariant representations, there is a ρ-invariant inner product $\langle .,. \rangle$ on $\mathfrak{p} \oplus \mathfrak{q}$ such that $\langle v, w \rangle = 0$ for all $v \in \mathfrak{p}$ and $w \in \mathfrak{q}$. Moreover, there is a ρ equivariant positive definite $L : \mathfrak{p} \oplus \mathfrak{q} \longrightarrow \mathfrak{p} \oplus \mathfrak{q}$ such that $(x, y) = \langle Lx, y \rangle$. Let $x \in \mathfrak{p}$ and $y \in \mathfrak{q}$. If $\langle Lx, y \rangle \neq 0$ then Lx projects onto $\mathfrak{q}$ from $\mathfrak{p} \oplus \mathfrak{q}$ non-trivially. However, $\text{proj} : \mathfrak{p} \oplus \mathfrak{q} \longrightarrow \mathfrak{q}$ is a ρ intertwining map. Indeed, for all $v \in \mathfrak{p}$ and $w \in \mathfrak{q}$,

$$\begin{aligned}
\text{proj}(\rho(x)(v + w)) &= \text{proj}((\rho(x)(v + w))_{\mathfrak{p}} + (\rho(x)(v + w))_{\mathfrak{q}}) \\
&= (\rho(x)(v + w))_{\mathfrak{q}} \\
&= (\rho(x)w)\mathfrak{q} \\
&= \rho(x)w \\
&= \rho(x)(\text{proj}(v + w)).
\end{aligned}$$

Thus, we have a non-trivial intertwining map $\phi = \text{proj} \circ L|_{\mathfrak{p}} : \mathfrak{p} \longrightarrow \mathfrak{q}$, a contradiction to the assumption that $\mathfrak{p}$ and $\mathfrak{q}$ share no equivalent irreducible subrepresentations. ■

An important consequence of Schur's Lemma as found in resources such as [Bes87] and [Hel01] is that if G/H is an isotropy irreducible space (or more specificially, an irreducible symmetric space), then $ric(x, y) = \lambda(x, y)$ for any $ad_{\mathfrak{h}}$ inner product, $(.,.)$, on $\mathfrak{p}$ in $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$. This follows from our statement regarding inner products being described by $(v, w) = \langle Lv, w \rangle$ having an analagous statement being true for arbitrary ρ invariant bilinear forms (allowing for L to not necessarily be positive definite or invertible in general).

1.4. Semi-Simple Lie Groups and Lie Algebras

The theory of semi-simple Lie groups and Lie algebras is well known, well-studied, and the semi-simple Lie algebras are all classified. Moreover, semi-simple Lie groups are well-known to be unimodular (Corollary 8.31 of Section 3 of Chapter VIII in [Kna02]). Since we will be focused on spaces G/H where $\mathfrak{g}$ is semi-simple, we spend some time reminding the reader of some important results for semi-simple Lie algebras and Lie groups. The following information can be found in [Hel01], [Kna02], and some in [Oni04].

If $\mathfrak{g}$ is a *semi-simple* Lie algebra then $\mathfrak{g} = \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_n$ where each $\mathfrak{g}_i$ is a simple Lie algebra that is an ideal of $\mathfrak{g}$. A *simple* Lie algebra is a non-abelian Lie algebra in which there are no non-trivial ideals (this definition provides an important consequence of being semi-simple, namely, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$). Recalling that $\mathfrak{g}$ is a vector space with a Lie bracket, we are reminded that the vector space for $\mathfrak{g}$ can be $\mathbb{R}$ or $\mathbb{C}$, called a real or complex Lie algebra, respectively. For each real semi-simple Lie algebra, $\mathfrak{g}$, there is a complex semi-simple Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g}$.

For each complex semi-simple Lie algebra, there are two different ways to obtain an associated real Lie algebra: finding a real form or by realification. It is well known that every complex semi-simple Lie algebra has a real semi-simple Lie algebra associated to it called the *real form*, meaning, if $\mathfrak{g}$ is a complex semi-simple Lie algebra, the real form $\mathfrak{g}_0$ is one such that $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$. On the other hand, a if $\mathfrak{g}$ is a complex semi-simple Lie algebra the *realification of* $\mathfrak{g}$ is done by by restricting the scalars of $\mathfrak{g}$. Using that any complex semi-simple Lie algebra $\mathfrak{g}$ is such that $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$ for a real form $\mathfrak{g}_0$, we may consider the realification of $\mathfrak{g}$, $\mathfrak{g}_{\mathbb{R}} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$ with restricted scalars.

Types of Real Simple Lie Algebras

Using the relationship between complex simple Lie algebras and real simple Lie algebras, there are two disctinct types of real simple Lie algebras $\mathfrak{g}$:

Type 1 $\mathfrak{g}^{\mathbb{C}}$ is simple (i.e $\mathfrak{g}$ is a real form of a complex simple Lie algebra)

Type 2 $\mathfrak{g}^{\mathbb{C}}$ is semi-simple (i.e $\mathfrak{g}$ is a realification of a complex simple Lie algebra)

These two different types of real simple Lie algebras will prove to be helpful as we turn to understanding decompositions of semi-simple Lie algebras and (in the coming sections) symmetric spaces.

Cartan Decompositions The context this work is most dedicated to is that of homogeneous spaces G/H in which G is a noncompact real semi-simple Lie group. Noncompact semi-simple Lie groups and algebras have a nice structure theory that is well-studied. A principle structure of interest in this paper is the *Cartan decomposition* which we describe in detail below.

If G is a noncompact semi-simple Lie group (G is noncompact and $\mathfrak{g}$ is semi-simple) then there is an involution $\theta : \mathfrak{g} \to \mathfrak{g}$ such that $B_\theta(x, y) = -B(x, \theta(y))$ is positive-definite. Such an involution is called a *Cartan involution*. Corresponding to the Cartan involution is the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ which $\theta(x) = x$ for $x \in \mathfrak{k}$, $\theta(x) = -x$ for $x \in \mathfrak{p}$. This decomposition has the following properties:

$$
\begin{aligned}
B(x,x) &< 0 \text{ for all } x \in \mathfrak{k} \\
B(x,x) &> 0 \text{ for all } x \in \mathfrak{p} \\
B(\mathfrak{k}, \mathfrak{p}) &= 0 \\
[\mathfrak{k}, \mathfrak{p}] &\subset \mathfrak{p}
\end{aligned}
\tag{1.3}
$$

$$[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

For a given Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, K is the maximal compact of G (unique up to conjugation) and G/K is a noncompact symmetric space. In the case in which $\mathfrak{g}$ is simple, $\mathfrak{p}$ is an irreducible $ad_{\mathfrak{k}}$ representation, G/K is (up to isometry) an irreducible symmetric space, and $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$. Irreducible symmetric spaces are classified, a list of which can be found in Chapter X of [Hel01] and Chapter 7 of [Bes87]. Any symmetric space that is not irreducible can be decomposed into a Riemannian product (i.e $(M, g) = (M_1 \times \ldots \times M_n, g_1 + \ldots + g_n)$) of irreducible symmetric spaces called a *DeRham irreducible decomposition*.

Compact Real Forms

Among the simple real forms, there are compact real forms and noncompact real forms. There are three important and well known facts regarding compact real forms that we will need later.

1. There is a one to one relationship between compact simple Lie algebras and complex simple Lie algebras

2. $B(x, x) < 0$ for $x \in \mathfrak{g}_0$ where $\mathfrak{g}_0$ is a compact real form and $B(.,.)$ is the Killing form on $\mathfrak{g}_0$. In general, $B(x, x) < 0$ is equivalent to being compact semi-simple.

3. If $\mathfrak{g}$ is complex simple and $\mathfrak{g}_{\mathbb{R}} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$ where $\mathfrak{g}_0$ is the compact real form, then $\mathfrak{g}_{\mathbb{R}} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$ is the Cartan decomposition of $\mathfrak{g}_{\mathbb{R}}$ with Cartan involution given by complex conjugation with restricted scalars.

The Killing Form

The following Lemma is useful, and can be found in Lemma 6.1 of Chapter III in [Hel01].

Lemma 1.10. Let $B_0(x, y)$ be the Killing form for $\mathfrak{g}_0$, a simple Lie algebra of Type 1 above, and let $B(x, y)$ be the Killing form of $\mathfrak{g}$, the complexification of $\mathfrak{g}_0$, and let $B_{\mathbb{R}}(x, y)$ be the Killing form of $\mathfrak{g}_{\mathbb{R}}$, the realification of $\mathfrak{g}$.

1. $B_0(x, y) = B(x, y)$ for all $x, y \in \mathfrak{g}_0$

2. $B_{\mathbb{R}}(x, y) = 2Re(B(x, y))$ for all $x, y \in \mathfrak{g}_{\mathbb{R}}$

1.5. Dual Symmetric Spaces

Coming from the relationship between complex semi-simple Lie algebras and real semi-simple Lie algebras, we get a relationship between compact and noncompact irreducible symmetric spaces. The content here can be found in Chapter V Section 2 and Chapter VIII Section 5 of [Hel01], with a list of noncompact symmetric spaces with their duals listed in Chapter X. For a more detailed explanation, we recommend this resource. Here, we start with a description of the two dual types of irreducible symmetric spaces, then we provide two examples, and then become more precise.

Proposition 1.11. If G/K is a noncompact irreducible symmetric space then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition of a simple Lie algebra $\mathfrak{g}$. If $\mathfrak{g}^{\mathbb{C}}$ is simple then $\mathfrak{k}$ in $\mathfrak{g}$ can be simple, semi-simple, or reductive with dimension 1 center, and all such cases do arise. If $\mathfrak{g}$ is the realification of a complex simple then we have Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$ where $\mathfrak{k}$ is a compact simple. Conversely, all such $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ are noncompact irreducible spaces.

Proposition 1.12. If G/K is a compact irreducible symmetric space then $\mathfrak{g}$ is either simple or $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}$ where $\mathfrak{k}$ is compact simple. If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is simple then $\mathfrak{k}$ can be simple, semi-simple, or reductive with dimension 1 center, and all such cases do arise. If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}$ then the reductive decomposition is $\mathfrak{g} = \Delta(\mathfrak{k}) \oplus \mathfrak{p}$ where $\Delta(\mathfrak{k}) = \{x + x : x \in \mathfrak{k}\}$ and $\mathfrak{p} = \{x - x : x \in \mathfrak{k}\}$. Conversely, all such $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ are compact irreducible symmetric spaces.

Remark 1.13. For G/K, a symmetric space of noncompact or compact type, our decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $B(\mathfrak{k}, \mathfrak{p}) = 0$ has $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ (See Section 1 of Chapter V in [Hel01]).

Example 1.14. $SL(n,\mathbb{R})/SO(n)$ is a noncompact symmetric space. The Cartan decomposition $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R}) = \mathfrak{so}(n) \oplus \mathfrak{p}$ where $\mathfrak{p}$ is the real vector spaces formed by $n \times n$ symmetric traceless matrices (which we will later denote by $symm(n)$). $\mathfrak{sl}(n,\mathbb{R})^{\mathbb{C}} = \mathfrak{sl}(n,\mathbb{C}) = \mathfrak{sl}(n,\mathbb{R}) \oplus i\mathfrak{sl}(n,\mathbb{R})$, and by the Cartan decomposition $\mathfrak{sl}(n,\mathbb{C}) = \mathfrak{so}(n) \oplus \mathfrak{p} \oplus i\mathfrak{so}(n) \oplus i\mathfrak{p}$. Since by the Cartan properties (1.3) $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$, implying $[i\mathfrak{p}, i\mathfrak{p}] \subset \mathfrak{k}$ as well. Thus, $\mathfrak{so}(n) \oplus i\mathfrak{p}$ is a subset of $\mathfrak{sl}(n,\mathbb{C})$ closed under the bracket. Moreover, letting $\mathfrak{g}^* = \mathfrak{so}(n) \oplus i\mathfrak{p}$ with restricted scalars, one can see that $\mathfrak{sl}(n,\mathbb{C}) = \mathfrak{g}^* \oplus i\mathfrak{g}^*$, and by Lemma 1.3 $B(x,x) < 0$ for $x \in \mathfrak{g}^*$. Therefore, $\mathfrak{g}^*$ is a compact real form of $\mathfrak{sl}(n,\mathbb{C})$, and one can check that $\mathfrak{g}^* = \mathfrak{su}(n)$. From this, then, we get a compact symmetric space $SU(n)/SO(n)$ that is dual to $SL(n,\mathbb{R})/SO(n)$.

Example 1.15. $SL(n,\mathbb{C})_{\mathbb{R}}/SU(n)$ is a noncompact irreducible symmetric space. The Cartan decomposition is $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})_{\mathbb{R}} = \mathfrak{su}(n) \oplus i\mathfrak{su}(n)$. Note that in this case we have $\mathfrak{g}^{\mathbb{C}} = \mathfrak{su}(n) \oplus i\mathfrak{su}(n) \oplus i\mathfrak{su}(n) \oplus \mathfrak{su}(n)$ which is semi-simple and we can let $\mathfrak{g}^* = \mathfrak{su}(n) \oplus \mathfrak{su}(n)$ which corresponds to the compact irreducible symmetric space $SU(n)SU(n)/\Delta(SU(n))$ which is the compact dual to $SL(n,\mathbb{C})_{\mathbb{R}}/SU(n)$. At the Lie algebra level, we have $\mathfrak{su}(n) \oplus \mathfrak{su}(n) = \Delta(\mathfrak{su}(n)) \oplus \{x - x : x \in \mathfrak{su}(n)\}$, and we can observe that $\mathfrak{su}(n) \simeq \Delta(\mathfrak{su}(n))$.

Now let us be more precise. If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a reductive decomposition (not necessarily noncompact) for an irreducible symmetric space then we consider $\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{p}$ in $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k} \oplus \mathfrak{p} \oplus i\mathfrak{k} \oplus i\mathfrak{p}$ with restricted scalars, and we have G^*/K as the *dual symmetric space* to G/K. Using Lemma 1.3, we can see that if $B_{\mathfrak{g}}(x,x) < 0$ for all $x \in \mathfrak{g}$ then $B_{\mathfrak{g}^*}(x,x) > 0$ for $x \in i\mathfrak{p}$. The converse is true since $B(x,x) < 0$ for $x \in \mathfrak{k}$ (see(1.3)). Therefore, if $\mathfrak{g}$ is compact simple then $\mathfrak{g}^*$ is noncompact simple. Conversely, if $\mathfrak{g}$ is noncompact simple then $\mathfrak{g}^*$ is compact simple. In the present work, we will be interested obtaining a dual of an irreducible symmetric space. We here describe how to find the dual, considering the two different types of real simple Lie algebras and the different situations arising from each. In the second setting below, there is an identification that occurs allowing this simpler presentation. For more on this, we direct the reader's attention to Proposition 2.3 in Chapter V of [Hel01].

In the following $\mathfrak{g}$ is noncompact and $\mathfrak{g}^*$ is compact and we give the reductive decompositions for the symmetric spaces.

1. $\mathfrak{g}^{\mathbb{C}}$ is simple:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \leftrightarrow \mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{p} \text{ and both are simple}$$

2. $\mathfrak{g}^{\mathbb{C}}$ is semi-simple:

$$\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k} \leftrightarrow \mathfrak{g}^* = \Delta(\mathfrak{k}) \oplus \{x - x : x \in \mathfrak{k}\}$$

$$\text{and } \mathfrak{g} \text{ is simple while } \mathfrak{g}^* = \mathfrak{k} \oplus \mathfrak{k} \text{ is not since } \mathfrak{k} \text{ is simple}$$

The process of finding the dual at the level of irreducible symmetric spaces can be extended to the broader symmetric space setting simply by dualizing each factor like so:
If $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_n$ with $\mathfrak{k} = \mathfrak{k}_1 \oplus \ldots \oplus \mathfrak{k}_n$ as the maximal compact, then the dual $\mathfrak{g}^*$ will be formed by the direct sum of the duals $(\mathfrak{g}_i^*, \mathfrak{k}_i)$ coming from $(\mathfrak{g}_i, \mathfrak{k}_i)$.

1.6. Representations of Real, Complex, and Quaternionic Types

The following information is well-known by many and can be found in [FH91], [BtD85], or [Oni04], unless stated otherwise. Since we will be frequently interested in knowing about all the types of intertwining or equivariant maps to say things about the Ricci curvature or the types of metrics that appear, this section will prove to be essential for us.

Following the terminology in [BtD85], if V is a complex irreducible representation of a compact Lie group H (or the representation of the associated Lie algebra, $\mathfrak{h}$), then V is precisely one of the following types:

1. *real type* if there is a conjugate linear equivariant map $J : V \to V$ such that $J^2 = Id$

2. *quaternionic type* if there is a conjugate linear equivariant map $J : V \to V$ such that $J^2 = -Id$

3. *complex type* if $\overline{V} \not\simeq V$.

Taking a real irreducible representation V, we say that V is of real, complex, or quaternionic type by considering the complexification $V^{\mathbb{C}}$ and if it is real, complex, or quaternionic type. Moreover, (see Thoerem 6.7 of [BtD85]) the space of equivariant maps for the representation of $\mathfrak{h}$, $Hom_{\mathfrak{h}}(V, V)$, is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ depending on if V is of real, complex, or quaternionic type respectively. Consequently, an irreducible real reprsentation V of $\mathfrak{h}$ where H is compact must have even dimension if V is complex type and must have dimension equal to a multiple of 4 if quaternionic type. The significance of these types for our purposes manifests in our study of $\mathfrak{p}$, an $ad_{\mathfrak{h}}$ representation coming from the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$. In this setting $\mathfrak{p}$ is a real irreducible representation of $\mathfrak{h}$, where H is a compact Lie group.

It turns out (see Chapter 2 of [Wol84]) that for an irreducible symmetric space G/K, there are different types depending on the isotropy action of $ad_{\mathfrak{k}}$ on $\mathfrak{p}$ in $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

1. G/K is a *Hermitian irreducible symmetric space* if $\mathfrak{p}$ is of complex type (see Chapter X of [Hel01] and the Appendices of [Kna02] for a list of such spaces)

2. G/K is a *quaternionic irrecucible symmetric space* if $\mathfrak{p}$ is of quaternionic type (see Table 1 in [Wol65] and the Appendices of [Kna02] for a list of such spaces)

Remark 1.16. The importance of these types of representations becomes apparent in Chapter 2 when we consider intertwining maps in Condition i. and ii. in 2.1 for being our so-called Cartan orthogonal pairs, as well as examples where these conditions are used such as in Section 2.2. However, there is another important implication regarding these types of representations in the effect they have upon the metrics (or, in the Lie algebra seeting, inner products) than occur on a space G/H in which $\mathfrak{p}$ in $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ has equivalent irreducible representations occuring within. That is, when our $\mathfrak{p}$ with decomposition into irreducibles given by $\mathfrak{p} = \mathfrak{p}_1 \oplus \dots \mathfrak{p}_n$ some $\mathfrak{p}_i$ and $\mathfrak{p}_j$ are equivalent. When an equivalence occurs, if one wants to understand all inner products $(.,.)$ in terms of a fixed metric $\langle .,. \rangle$, then one must take the type of representation into consideration. Indeed, given $(.,.)$ and $\langle .,. \rangle$, we can understand $(x, y) = \langle \Phi x, y \rangle$ where $\Phi : \mathfrak{p} \to \mathfrak{p}$ is a positive definite equivariant map. Any such Φ has a square root ϕ (i.e., $\phi^2 = \Phi$), so $(x, y) = \langle \phi x, \phi y \rangle$ is an equivalent understanding of $(.,.)$. Moreover, ϕ is equivariant. Now, $\phi : \mathfrak{p} \to \mathfrak{p}$ may be restricted to $\mathfrak{p}_i$, call it ϕ_i, and by Schur's Lemma $\phi_i : \mathfrak{p}_i \to \mathfrak{p}$ will have 0 image one any irreducible component of $\mathfrak{p}$ that is not equivalent ot $\mathfrak{p}_j$. However, for a $\mathfrak{p}_j \simeq \mathfrak{p}_i$, an arbitrary $\mathrm{proj}_{\mathfrak{p}_j} \circ \phi_i : \mathfrak{p}_i \to \mathfrak{p}_j$, will belong to either $GL(n, \mathbb{R})$, $GL(n, \mathbb{C})$, or $GL(n, \mathbb{H})$, depending on if $\mathfrak{p}_i$ is of real, complex, or quaternionic type, respectively. Therefore, the type of irreducible representations directly impacts the number of parameters the inner product is dependent upon. This fact will soon prove to be quite important.

1.7. Orthogonal Irreducible Decompositions

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}'$ be a Cartan decomposition of a noncompact semi-simple Lie algebra. We may consider $\mathfrak{h} \subset \mathfrak{k}$, a compact subalgebra and the noncompact homogeneous space G/H with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$ where $\mathfrak{h} \oplus \mathfrak{p}'' = \mathfrak{k}$. Fixing an $ad_{\mathfrak{h}}$ invariant inner product $(.,.)$, we may consider an orthogonal decomposition which is unique up to isomorphism. We have the following as a consequence of the Spectral theorem for real linear maps. We include a proof here for completion as we were unable to find a reference with this worked out.

Definition 1.17. We say that two bilinear forms $T(.,.)$ and $L(.,.)$ are *simultaneously diagonalized* on a vector space V if for some basis $\{e_i\}$ of V, $T(e_i, e_j) = L(e_i, e_j) = 0$.

Lemma 1.18. For $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ and $(.,.)$ as described above, $(.,.)$ may be simultaneously diagonalized with the Killing form $B(.,.)$ on $\mathfrak{p}$.

<u>Proof:</u> By the $ad_{\mathfrak{h}}$ invariance of $B(.,.)$ and by $B(.,.)$ being definite (since $\mathfrak{g}$ is semi-simple), we know that there is an isomorphism $\Phi : \mathfrak{p} \to \mathfrak{p}$ such that $(x, y) = B(\Phi x, y)$ where Φ is $ad_{\mathfrak{h}}$ invariant and self-adjoint with respect to $B(.,.)$. Using the self-adjointness of Φ and the Spectral theorem for real linear maps, let $\{e_k\}$ be a $B(.,.)$ orthonormal basis of Φ eigenvectors on $\mathfrak{p}$. This provides us with an irreducible decomposition $\mathfrak{p} = \mathfrak{p}_1 \oplus ...\mathfrak{p}_n$ since if $\mathfrak{p}_i = Span\{e_i : \Phi(e_i) = \lambda_i e_i\}$, then by Φ being an equivariant map we have that $\mathfrak{p}_i$ is an invariant subspace of $\mathfrak{p}$. Now, let $\{e_i^\alpha\}$ being an orthonormal basis for $\mathfrak{p}_i$ of eigenvectors associated with the eigenvalue λ_i. We can then see that $(e_i^\alpha, e_j^\beta) = B(\Phi(e_i^\alpha), e_j^\beta) = B(\lambda_i e_i^\alpha, e_j^\beta) = 0$. Thus, $(\mathfrak{p}_i, \mathfrak{p}_j) = B(\mathfrak{p}_i, \mathfrak{p}_j) = 0$. Schur's Lemma gives us that on an irreducible representation $\mathfrak{p}_i$, $(.,.) = \lambda_i B(.,.)$, which proves the result. ■

Note that if there are no equivalent representations, our decomposition of $\mathfrak{p} = \mathfrak{p}_1 \oplus ... \oplus \mathfrak{p}_n$ into $ad_{\mathfrak{h}}$ irreducible representations is unique up to scaling as a consequence of Schur's Lemma. However, if there are equivalent representations, then the given decomposition is dependent upon the choice of the inner product (but still isomorphic) as our choice of basis was depended upon Φ.

In the current work, we will be interested in studying those $ad_{\mathfrak{h}}$ invariant inner products for which $(\mathfrak{p}'', \mathfrak{p}') = 0$. This is automatically the case if there are no $\mathfrak{p}_i \subset \mathfrak{p}''$ isomorphic to any $\mathfrak{p}_j \subset \mathfrak{p}'$ by Schur's Lemma as we showed in a more general setting in Lemma 1.9. However, it may be the case that there is a $\mathfrak{p}_i \subset \mathfrak{p}''$ isomorphic to some $\mathfrak{p}_j \subset \mathfrak{p}'$. In this case, we will not necessarily have $(\mathfrak{p}'', \mathfrak{p}') = 0$ for the same decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$, but we are not without hope of a similar orthogonality. Indeed, since the maximal compact K in G is unique up to conjugacy, there still may exist a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}'' \oplus \mathfrak{q}'$ in which $(\mathfrak{q}'', \mathfrak{q}') = 0$, providing us still with an irreducible decomposition in which the compact and noncompact piece of the isotropy are orthogonal. To sum this up succinctly, it may be the case that even if $\mathfrak{p}''$ and $\mathfrak{p}'$ have isomorphic irreducible representations, there is some other (isomorphic) Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}'$ in which $(\mathfrak{q}'', \mathfrak{q}') = 0$. In [Nik00], this situation was investigated and a nice formula for $ric(.,.)$ was determined for a metric in which the compact and noncompact piece of the isotropy are orthogonal. See (1.4) below.

Consider $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$ to be decomposed into irreducible $ad_{\mathfrak{h}}$ representations $\mathfrak{p}' = \mathfrak{p}_1 \oplus ... \oplus \mathfrak{p}_u$ and $\mathfrak{p}'' = \mathfrak{p}_{u+1} \oplus ... \oplus \mathfrak{p}_v$ where $(.,.)$ and $\langle .,. \rangle = B(.,.)_{\mathfrak{p}'} - B(.,.)_{\mathfrak{p}''}$ are simultaneously diagonalized on $\mathfrak{p} = \mathfrak{p}' \oplus \mathfrak{p}''$. Let the ordering $\mathfrak{p}_1, ..., \mathfrak{p}_n$ be given by $x_1 \leq ... \leq x_u$ where $(.,.)_{\mathfrak{p}_i} = x_i \langle .,. \rangle_{\mathfrak{p}_i}$. Similarly, order the $\mathfrak{p}_{n+1}, ..., \mathfrak{p}_v$ and define r_i to be $ric(.,.)_{\mathfrak{p}_i} = r_i(.,.)_{\mathfrak{p}_i}$. Let b_i be 1 when $\mathfrak{p}_i \subset \mathfrak{p}''$ and -1 when $\mathfrak{p}_i \subset \mathfrak{p}'$, let d_i be the dimension of $\mathfrak{p}_i$, and let $\{e_i^\alpha\}$ be an orthonormal basis of $\mathfrak{p}_i$ with respect to $\langle .,. \rangle$. By Lemma 2 in

[Nik00] we have the following:

$$r_i = \frac{b_i}{2x_i} + \frac{1}{4d_i} \sum_{1 \le j,k \le v} (\sum_{\alpha,\beta,\gamma} \langle [e_i^\alpha, e_j^\beta], e_k^\gamma \rangle^2)(\frac{x_i}{x_j x_k} - \frac{x_k}{x_i x_j} - \frac{x_j}{x_i x_k}). \qquad (1.4)$$

It is important to observe that the r_i describe the values of the $(1,1)$ tensor. Indeed, the $(0,2)$ tensor is scale invariant, meaning $ric_{\lambda g}(x,y) = ric_g(x,y)$, while the $(1,1)$ tensor is not with $Ric_{\lambda g}(x) = \frac{1}{\lambda} Ric_g(x)$. The above formula is not scale invariant as can be shown by observing that r_i for the inner product described by $(\lambda x_1, ..., \lambda x_n)$ is $\frac{1}{\lambda} r_i'$ where r_i' is corresponds to the inner product given by $(x_1, ..., x_n)$.

The provision of such a formula for the diagonal entries of $ric(.,.)$ becomes immensely helpful as we begin to investigate the Ricci curvature of various spaces G/H in which $\mathfrak{g}$ is noncompact semi-simple. We could, of course, utilize this formula to describe the Ricci curvature under the constraint that there are no isomorphic isotropy irreducible summands, since then $ric(\mathfrak{p}_i, \mathfrak{p}_j) = 0$ by Schur's Lemma and we have $(\mathfrak{p}'', \mathfrak{p}') = 0$. Or, we could describe the diagonal of the Ricci curvature tensor in all G/H with $\mathfrak{g}$ noncompact semi-simple spaces, but restrict our metrics to those for which $(\mathfrak{p}'', \mathfrak{p}') = 0$. Another option, though, is to investigate the question of *which spaces can we always apply this formula regardless of the metric chosen?* It is in the pursuit of this question that we build the following definition which will be used in Chapter 2.

Recall that for $\mathfrak{g}$, a noncompact semi-simple Lie algebra, and $\mathfrak{h}$ corresponding to a compact subgroup $H \subset G$, we have $\mathfrak{h} \subset \mathfrak{k}$ where $\mathfrak{k}$ is our maximal compact in $\mathfrak{g}$. Moreover, with a choice (unique up to conjugation) of $\mathfrak{k}$, we have an associated Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}'$. Now, recall that for some $ad_{\mathfrak{h}}$ invariant $\mathfrak{p}$ in $\mathfrak{g}$, we have a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ that we call a reductive decomposition. Due to the frequency of use of the following decomposition in the current work and the relation between the Cartan and reductive decompositions therein, we supply the following definition to simplify matters.

Definition 1.19. With the above setup, we refer to $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$ as a *reductive Cartan Decomposition* for the pair $(\mathfrak{g}, \mathfrak{h})$ when $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{p}''$ with $B(\mathfrak{h}, \mathfrak{p}'') = B(\mathfrak{p}'', \mathfrak{p}') = 0$ and $\mathfrak{p} = \mathfrak{p}'' \oplus \mathfrak{p}'$.

As for the existence of such a decomposition, we can be assured that there exists such a decomposition for any $\mathfrak{h} \subset \mathfrak{k}$ where $\mathfrak{k}$ is the (unique up to conjugation) maximal compact of a noncompact semi-simple $\mathfrak{g}$. Indeed, if $\mathfrak{g}$ is semi-simple then we may choose a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}'$ which provides that $B(\mathfrak{k}, \mathfrak{p}') = 0$ (See the Cartan decomposition properties in 1.3). Moreover, since $\mathfrak{h} \subset \mathfrak{k}$, we can get a $\mathfrak{p}'' = \{x \in \mathfrak{k} : B(x, y) = 0 \text{ for all } y \in \mathfrak{h}\}$ using that $B(.,.)$ is non-dgenerate by $\mathfrak{g}$ being semi-simple. Thus, we have a reductive Cartan decomposition for $(\mathfrak{g}, \mathfrak{h})$. It is worth noting that our reductive Cartan decomposition is dependent upon a choice of Cartan decomposition.

A principle reason of interest for such a decomposition is because it plays an integral role in our development and understanding of the following kinds of pairs $(\mathfrak{g}, \mathfrak{h})$ in which $\mathfrak{g}$ is noncompact semi-simple and $\mathfrak{h} \subset \mathfrak{k}$ where $\mathfrak{k}$ is the maximal compact in $\mathfrak{g}$. The following pairs are precisely the pairs that the question asked above seeks to investigate.

Definition 1.20. We say $(\mathfrak{g}, \mathfrak{h})$ is a *Cartan orthogonal pair* if the following condition is satisfied:
Given a G-invariant metric on G/H, g, there exists a Cartan decomposition, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}'$ such that our reductive Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$ has $(\mathfrak{p}'', \mathfrak{p}') = 0$ for $(.,.)$, the $ad_{\mathfrak{h}}$ invariant inner product on $\mathfrak{p} = \mathfrak{p}'' \oplus \mathfrak{p}'$ associated with g (See Section 1.2).

Remark 1.21. Unless stated otherwise, going forward, $\mathfrak{p}''$ will be used for an $ad_{\mathfrak{h}}$ invariant complement to $\mathfrak{h}$ in $\mathfrak{k}$, the maximal compact of a noncompact semi-simple $\mathfrak{g}$, and $\mathfrak{p}'$ will refer to the -1 eigenspace from the Cartan decomposition.

Remark 1.22. One abuse of notation to be used will be $(.,.)_{\mathfrak{p}_i}$ for $(.,.)_{\mathfrak{p}_i \times \mathfrak{p}_i}$ in which the $\mathfrak{p}_i$ component is taken for any x, y in $(.,.)$. Similarly, we will frequently be writing $(.,.)_i$ for $(.,.)_{\mathfrak{p}_i \times \mathfrak{p}_i}$.

Remark 1.23. Unless stated otherwise, henceforth for $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$, $\langle .,. \rangle = B(.,.)_{\mathfrak{p}'} - B(.,.)_{\mathfrak{p}''}$ where $B(.,.)$ is the Killing form.

The following lemma will prove to be useful for us later on. A short explanation is given, but for a more detailed look, we recommend [Kna02] and [Hel01].

Lemma 1.24. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$ with $\mathfrak{p} = \mathfrak{p}'' \oplus \mathfrak{p}'$. The map $\text{proj}_{\mathfrak{p}} \circ \text{ad}_x : \mathfrak{p} \to \mathfrak{p}$ is symmetric for $x \in \mathfrak{p}'$ and skew symmetric for $x \in \mathfrak{p}''$, both relative to the metric $\langle .,. \rangle$.

Proof: This proof follows from the definition of $\langle .,. \rangle = B_{\mathfrak{p}'} - B_{\mathfrak{p}''}$ and the $ad_{\mathfrak{g}}$ invariance of $B(.,.)$. ■

Remark 1.25. Unless unclear from the context, by an abuse of notation, we will say that ad_x is (skew) symmetric with respect to $\langle .,. \rangle$ for x in $(\mathfrak{p}'')$ $\mathfrak{p}'$.

Chapter 2

Cartan Orthogonal Pairs

Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$ be noncompact semi-simple with reductive Cartan decomposition for $(\mathfrak{g}, \mathfrak{h})$ (see Definition 1.19). In this chapter, we wish to discuss Cartan orthogonal pairs, but first, we motivate the definition (Definition 1.20) and provide some more information.

This definition is in-part motivated by [Nik00] and Theorem 2 found within. There, Nikonorov showed that for a reductive Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$, $(\mathfrak{g}, \mathfrak{h})$ is a Cartan orthogonal pair if and only if the following two conditions are met for any given intertwining map $\phi : \mathfrak{p}'' \to \mathfrak{p}'$:

$$\text{Condtion i. } [\phi(x), \phi(y)] \subset \mathfrak{h} \oplus ker\phi \text{ for all } x, y \in \mathfrak{p}'' \tag{2.1}$$

$$\text{Condition ii. } \phi([x, y]_{\mathfrak{p}''}) = [x, \phi(y)] + [\phi(x), y] \text{ for all } x, y \in \mathfrak{p}''$$

This provides a representation theoretic approach to determining if $(\mathfrak{g}, \mathfrak{h})$ is a Cartan orthogonal pair in the presence of isomorphic representations $\mathfrak{p}_i \subset \mathfrak{p}''$ and $\mathfrak{p}_j \subset \mathfrak{p}'$. If there are no such isomorphisms, then by the use of Lemma 1.9 we have in our case that $(\mathfrak{p}'', \mathfrak{p}') = 0$ for any $ad_{\mathfrak{h}}$ invariant inner product $(.,.)$ (which shows that this condition is much stronger than being a Cartan orthogonal pair since any $ad_{\mathfrak{h}}$ invariant inner product works for the one decomposition). Another use of Schur's Lemma to see that $(\mathfrak{g}, \mathfrak{h})$ is a Cartan orthogonal pair in such a case is the proof of Nikonorov's in the Corollary of [Nik00]. There it is shown that the above two conditions are met since, by Schur's Lemma (see Section 1.3), the only intertwining maps are the trivial maps which immediately satisfy both conditions. We summarize this in the following Lemma.

Lemma 2.1. If $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$ is a reductive Cartan decomposition for $(\mathfrak{g}, \mathfrak{h})$ in which there is no $\mathfrak{p}_i \subset \mathfrak{p}''$ and $\mathfrak{p}_j \subset \mathfrak{p}'$ such that $\mathfrak{p}_i \simeq \mathfrak{p}_j$, then $(\mathfrak{g}, \mathfrak{h})$ is a Cartan orthogonal pair in which $(\mathfrak{p}'', \mathfrak{p}') = 0$ for all $ad_{\mathfrak{h}}$ invariant inner products on $\mathfrak{p}'' \oplus \mathfrak{p}'$.

Example 2.2. Consider $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$ with $\mathfrak{h} = \mathfrak{so}(n-1)$. In this case, $(\mathfrak{g}, \mathfrak{h})$ is not a Cartan orthogonal pair. Here, $\mathfrak{so}(n, \mathbb{C}) = \mathfrak{so}(n) \oplus i\mathfrak{so}(n)$, and taking $\mathfrak{so}(n) = \mathfrak{so}(n-1) \oplus \mathfrak{p}$ from the irreducible symmetric space Cartan decomposition (1.3), we have $\mathfrak{so}(n, \mathbb{C}) = \mathfrak{so}(n-1) \oplus \mathfrak{p} \oplus i\mathfrak{so}(n-1) \oplus i\mathfrak{p}$. One can observe that $\mathfrak{p}$ and $i\mathfrak{p}$ are isomorphic as $\mathfrak{so}(n-1)$ representations, and one can see that condition ii above fails. Indeed, since $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h}$ by $\mathfrak{so}(n) = \mathfrak{so}(n-1) \oplus \mathfrak{p}$ being a Cartan decomposition, condition ii requires $[\phi(x), y] + [x, \phi(y)] = [ix, y] + [x, iy] = 2i[x, y] = 0$ which is not the case as $[x, y] = 0$ only if y is parallel to x.

Example 2.3. Consider $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ and $\mathfrak{h} = \mathfrak{so}(n-1)$ with $n > 3$. Looking to Example 4 of [Nik00], we can see that $(\mathfrak{g}, \mathfrak{h})$ is in fact a Cartan orthogonal pair.

Spaces described by Cartan orthogonal pairs are of geometric interest because of what can be said when you have an $ad_{\mathfrak{h}}$ invariant inner product with $(\mathfrak{p}'', \mathfrak{p}') = 0$. In Theorem 1 of [Nik00], it was shown by using 1.4 that if $r_1 \geq r_u$ then $r_u > 0$ which implies that such metrics are not Einstein (see Theorem 7.4 in [Bes87]). Furthermore, this result shows how certain metrics cannot produce non-positive Ricci curvature (which we will see more of in

Chapter 2). Therefore, if we have a space G/H such that $(\mathfrak{g}, \mathfrak{h})$ is a Cartan orthogonal pair, we know that G/H has no metrics that are Einstein, we have a nice formula for the diagonal entries of the $(1,1)$ Ricci tensor (See Section 1.1), Ric, and we have a relationship between what happens to Ric on $\mathfrak{p}''$ and $\mathfrak{p}'$.

In the current chapter, we explore more properties of Cartan orthogonal pairs, determining another geometric consequence of being a Cartan orthogonal pair, as well as a variety of Lie theoretic conditions to be one. We finish with some new examples of Cartan orthogonal pairs along with a non-trivial non-example that serves as a correction to Example 4 in [Nik00], as mentioned in the Introduction.

2.1. Properties of Cartan Orthogonal Pairs

We begin with a simple, but nice geometric consequence to being a Cartan orthogonal pair that has not yet been demonstrated (to the author's knowledge). From there, we move to some propositions and lemmas that are aimed at determining what kinds of spaces are Cartan orthogonal pairs and how we might be able to build new ones.

Proposition 2.4. For $\mathfrak{g}$ noncompact semi-simple, choose a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}'$ and reductive Cartan decomposotion $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$ for $(\mathfrak{g}, \mathfrak{h})$ (see Definition 1.19). Furthermore, we choose an $ad_{\mathfrak{h}}$ invariant inner product $(.,.)$ such that $\mathfrak{p} = \mathfrak{p}'' \oplus \mathfrak{p}'$ is orthogonal. Under these conditions, $ric(\mathfrak{p}'', \mathfrak{p}') = 0$.

Proof: Much of this proof follows from the Cartan decomposition properties (see 1.3). Let $x \in \mathfrak{p}''$ and $y \in \mathfrak{p}'$. Also let $\{x_i\}$ be an orthonormal basis for $(.,.)$ such that each $x_i \in \mathfrak{p}''$

or $\mathfrak{p}'$. From 1.2 we have:

$$ric(x, y) = \frac{-1}{2}\sum_i([x, x_i]_\mathfrak{p}, [y, x_i]_\mathfrak{p}) - \frac{1}{2}B(x, y) + \frac{1}{4}\sum_{i,j}([x_i, x_j]_\mathfrak{p}, x)([x_i, x_j]_\mathfrak{p}, y)$$

Observe that $B(x, y) = 0$ by properties of the Cartan decomposition, so the second term goes away. For the first and third terms, we first recall the following property of a Cartan decomposition:

$$[\mathfrak{k}, \mathfrak{p}'] \subset \mathfrak{p}',$$
$$[\mathfrak{p}', \mathfrak{p}'] \subset \mathfrak{k}.$$

Since $x \in \mathfrak{p}'' \subset \mathfrak{k}$ and $y \in \mathfrak{p}'$, $([x, x_i]_\mathfrak{p}, [y, x_i]_\mathfrak{p}) = 0$ because x_i is either in $\mathfrak{p}'$ or $\mathfrak{p}''$. Thus, the first term is 0.

Similarly with the third term, since $x_i \in \mathfrak{p}'$ or $\mathfrak{p}'' \subset \mathfrak{k}$,

$$[x_i, x_j] \in \mathfrak{p}' \text{ or } \mathfrak{k}$$

which implies that either $([x_i, x_j], x) = 0$ or $([x_i, x_j], y) = 0$ for each x_i, x_j pair. Therefore, the third term is also 0. ■

Lemma 2.5. If $\mathfrak{g}$ is noncompact simple and $\mathfrak{k}$ is not simple, then $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2 \oplus \mathfrak{k}_3 \oplus \mathfrak{k}_4$ where $\mathfrak{k}_i$ is an ideal of $\mathfrak{k}$ which is either 0, central, or simple. Further, $\mathfrak{k}$ has the following possibilities:

- $\mathfrak{k} = \mathfrak{so}(4) \oplus \mathfrak{so}(4) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ (only when $\mathfrak{g} = \mathfrak{so}(4, 4)$)
- $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2 \oplus \mathbb{R}$ with $\mathfrak{k}_1, \mathfrak{k}_2$ simple
- $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ with $\mathfrak{k}_1, \mathfrak{k}_2$ simple

- $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathbb{R}$ with $\mathfrak{k}_1$ simple.

Proof: The proof of this can be determined by looking at the list of irreducible symmetric spaces in Chapter 7, Section H of [Bes87] and Chapter X, Section 6 of [Hel01] with the incidental isomorphisms found therein. ■

Lemma 2.6. Let $\mathfrak{g}$ be noncompact simple with $\mathfrak{k}$ not simple and $N_{\mathfrak{g}}(\mathfrak{k}_i) = \{z \in \mathfrak{g} : [z, x] \in \mathfrak{k}_i$ for all $x \in \mathfrak{k}_i\}$ where $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2 \oplus \mathfrak{k}_3 \oplus \mathfrak{k}_4$ as in Lemma 2.4. $N_{\mathfrak{g}}(\mathfrak{k}_i) = \mathfrak{k}$ for all i.

Proof: First, observe that $\mathfrak{k} \subset N_{\mathfrak{g}}(\mathfrak{k}_i)$ since each $\mathfrak{k}_j$ is an ideal in $\mathfrak{k}$. Now, we want to show that $N_{\mathfrak{g}}(\mathfrak{k}_i) \subset \mathfrak{k}$ and we will do that by showing that $N_{\mathfrak{g}}(\mathfrak{k}_i) \cap \mathfrak{p}' = \{0\}$. Since $\mathfrak{k}$ is a subalgebra of $N_{\mathfrak{g}}(\mathfrak{k}_i)$ and $[\mathfrak{k}, \mathfrak{p}'] \subset \mathfrak{p}'$ by the Cartan properties (1.3), we have that $[\mathfrak{k}, N_{\mathfrak{g}}(\mathfrak{k}_i) \cap \mathfrak{p}'] \subset N_{\mathfrak{g}}(\mathfrak{k}_i) \cap \mathfrak{p}'$. By the irreducibility of $\mathfrak{p}'$ under the $ad_{\mathfrak{k}}$ action (see Proposition 1.11 in the preliminaries), we know that $N_{\mathfrak{g}}(\mathfrak{k}_i) \cap \mathfrak{p}' = \{0\}$ or $\mathfrak{p}'$. If $\{0\}$, then we are done. If $N_{\mathfrak{g}}(\mathfrak{k}_i) \cap \mathfrak{p}' = \mathfrak{p}'$ then $N_{\mathfrak{g}}(\mathfrak{k}_i) = \mathfrak{k} \oplus \mathfrak{p}' = \mathfrak{g}$, but this implies $\mathfrak{k}_i$ is an ideal of $\mathfrak{g}$ contradicting $\mathfrak{g}$ being simple. ■

Proposition 2.7. Suppose $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$ is a reductive Cartan decomposition for $(\mathfrak{g}, \mathfrak{h})$ with $\mathfrak{g}$ noncompact simple. Further, suppose that $\mathfrak{k}$ is not simple. If $\mathfrak{h}$ contains an ideal of $\mathfrak{k}$ then $(\mathfrak{g}, \mathfrak{h})$ is a Cartan orthogonal pair.

Proof: Let $\mathfrak{h} = \mathfrak{k}' \oplus \mathfrak{h}_0$ where $\mathfrak{k}'$ is the ideal of $\mathfrak{k}$ assumed to be in $\mathfrak{h}$. In what follows, using Lemma 2.6 and Lemma 2.1, we will show that for any reductive Cartan decomposition, $(\mathfrak{p}'', \mathfrak{p}') = 0$ for all $ad_{\mathfrak{h}}$ invariant inner products on $\mathfrak{p} = \mathfrak{p}'' \oplus \mathfrak{p}'$. We will do this in two steps. First, we will show that $\mathfrak{p}'' \subset \ker ad_{\mathfrak{k}'} = \bigcap_{x \in \mathfrak{k}'} \ker ad_x$, but $\ker ad_{\mathfrak{k}'} \bigcap \mathfrak{p}' = \{0\}$. Then we will show how this implies that we are in the setting of Lemma 2.1, giving us the desired result.

Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$ be a reductive Cartan decomposition with $\mathfrak{h} = \mathfrak{k}' \oplus \mathfrak{h}_0$, as before. Since

$\mathfrak{k} = \mathfrak{k}' \oplus \mathfrak{k}''$ where $\mathfrak{k}''$ is a complementary ideal to $\mathfrak{k}'$ in $\mathfrak{k}$, we know that $\mathfrak{p}'' \subset \mathfrak{k}''$, implying that $\mathfrak{p}'' \subset \ker ad_{\mathfrak{k}'}$. Now, by Lemma 2.6, we know that $\ker ad_{\mathfrak{k}'} \subset \mathfrak{k}$, so $\ker ad_{\mathfrak{k}'} \cap \mathfrak{p}' = \{0\}$, which completes our first step.

Now, if there is a non-trivial $ad_{\mathfrak{h}}$ intertwining map $\phi : \mathfrak{p}'' \to \mathfrak{p}'$, then for $x \in \mathfrak{k}' \subset \mathfrak{h}$, $ad_x(\phi(v)) = \phi(ad_x(v)) = \phi(0) = 0$ for $v \in \mathfrak{p}'' \subset \ker ad_{\mathfrak{k}'}$. Since x and v were arbitrary, $ad_{\mathfrak{k}'}$ must have kernel in $\mathfrak{p}'$, $\{\phi(v)\}$, a contradiction. Therefore, there can be no non-trivial $ad_{\mathfrak{h}}$ intertwining maps $\phi : \mathfrak{p}'' \to \mathfrak{p}'$ which implies by Schur's Lemma that there is no $\mathfrak{p}_i \subset \mathfrak{p}''$ and $\mathfrak{p}_j \subset \mathfrak{p}'$ in which $\mathfrak{p}_i \simeq \mathfrak{p}_j$ as $ad_{\mathfrak{h}}$ representations. This places us in the setting of Lemma 2.1 which implies that $(\mathfrak{p}'', \mathfrak{p}') = 0$ for every $ad_{\mathfrak{h}}$ invariant inner product on $\mathfrak{p} = \mathfrak{p}'' \oplus \mathfrak{p}'$. ■

The preceding proposition, in light of Lemma 2.5, provides us with ample examples of Cartan orthogonal pairs in the setting of $\mathfrak{g}$ being noncompact simple. However, what about the case of semi-simple? Can we construct semi-simple Cartan orthogonal pairs from simple Cartan orthogonal pairs? More generally, can we take two semi-simple Cartan orthogonal pairs, and construct a new semi-simple Cartan orthogonal pair by direct product?

One approach is to use Proposition 3.4 from [AL17] to say that if $(\mathfrak{g}_1, \mathfrak{h}_1)$ and $(\mathfrak{g}_2, \mathfrak{h}_2)$ are Cartan orthogonal pairs in which $ad_{\mathfrak{h}_1}$ does not act trivially on any subspace of $\mathfrak{p}_1$ in $\mathfrak{g}_1 = \mathfrak{h}_1 \oplus \mathfrak{p}_1$, then $G_1G_2/H_1H_2 = G_1/H_1 \times G_2/H_2$ is a Riemannian product. That is, any $ad_{\mathfrak{h}_1 \oplus \mathfrak{h}_2}$ invariant inner products on $\mathfrak{p}_1 \oplus \mathfrak{p}_2$, $(.,.)$, can be written as as sum of $ad_{\mathfrak{h}_i}$ inner products like so: $(.,.) = (.,.)_1 + (.,.)_2$ where $(.,.)_i$ is an $ad_{\mathfrak{h}_i}$ invariant metric on $\mathfrak{p}_i$ for $i = 1, 2$. From this we clearly get that $(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{h}_1 \oplus \mathfrak{h}_2)$ is a Cartan orthogonal pair. However, there are examples of Cartan orthogonal pairs that do not match this criterion as seen above in Lemma 2.7 (e.g. $(\mathfrak{so}(n,m), \mathfrak{so}(n))$). Moreover, it is not the case in such instances that the only inner products come from product metrics as above. Thus, we wish to find a less restrictive condition to build Cartan orthogonal pairs from Cartan orthogonal pairs.

Allow us to simplify matters and restate what we want. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \subset \mathfrak{g}$ in which $\mathfrak{g}_i = \mathfrak{h}_i \oplus \mathfrak{p}_i'' \oplus \mathfrak{p}_i'$ is a reductive Cartan decomposition, $\mathfrak{g}_i$ is noncompact semi-simple, and $(\mathfrak{g}_i, \mathfrak{h}_i)$ is a Cartan orthogonal pair for $i = 1, 2$. We want to find a condition for $(\mathfrak{g}, \mathfrak{h})$ to be a Cartan orthogonal pair even in the presence of trivial representations in $\mathfrak{p}_i = \mathfrak{p}_i'' \oplus \mathfrak{p}_i'$ for $i = 1$ or $i = 2$. To do this, we recognize that the strength of the assumption (from [AL17, Prop. 3.4]) that $\mathfrak{p}_1$ contains no trivial representations is that it forces there to be no nontrivial $ad_{\mathfrak{h}}$ intertwining maps $\mathfrak{p}_1 \to \mathfrak{p}_2$, causing $(\mathfrak{p}_1, \mathfrak{p}_2) = 0$ for any $ad_{\mathfrak{h}}$ invariant inner product $(.,.)$ on $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$. What we want to do is to find a condition that allows for a trivial representation in $\mathfrak{p}_1$ or $\mathfrak{p}_2$ but forces $(\mathfrak{p}'', \mathfrak{p}') = 0$ for some reductive Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$. Let us consider what is required more precisely.

Recall (See Definition 1.20) that to show something is a Cartan orthogonal pair in general, we begin with an arbitrary G invariant metric, g, for our G/H. In this case, $G/H = (G_1G_2)/(H_1H_2)$, so g is both G_1 and G_2 invariant. Thus, by restriction, we may consider g a G_i invariant metric on G_i/H_i, denoting the restriction of g by g_i for both $i = 1, 2$. Now, $(\mathfrak{g}_1, \mathfrak{h}_1)$ and $(\mathfrak{g}_2, \mathfrak{h}_2)$ are Cartan orthogonal pairs, so taking $(\mathfrak{g}_1, \mathfrak{h}_1)$, we can find a reductive Cartan decomposition $\mathfrak{g}_1 = \mathfrak{h}_1 \oplus \mathfrak{p}_1'' \oplus \mathfrak{p}_1'$ such that $(\mathfrak{p}_1'', \mathfrak{p}_1')_1 = 0$ where $(.,.)_1$ is our unique $ad_{\mathfrak{h}_1}$ invariant inner product coming from g_1 on G_1/H_1 (See Section 1.2 or 7.24 in [Bes87] for this uniquely associated inner product). Similarly, since $(\mathfrak{g}_2, \mathfrak{h}_2)$ is a Cartan orthogonal pair we can find a reductive Cartan decomposition $\mathfrak{g}_2 = \mathfrak{h}_2 \oplus \mathfrak{p}_2'' \oplus \mathfrak{p}_2'$ such that $(\mathfrak{p}_2'', \mathfrak{p}_2')_2 = 0$ where $(.,.)_2$ is our unique $ad_{\mathfrak{h}_2}$ invariant inner product coming from g_2 on G_2/H_2. This provides us with a reductive Cartan decomposition for G/H given by

$\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{p}_1'' \oplus \mathfrak{p}_2'' \oplus \mathfrak{p}_1' \oplus \mathfrak{p}_2'$ where $\mathfrak{p}'' = \mathfrak{p}_1'' \oplus \mathfrak{p}_2''$ and $\mathfrak{p}' = \mathfrak{p}_1' \oplus \mathfrak{p}_2'$. Moreover, since G invariant metrics for G/H are uniquely associated with $ad_{\mathfrak{h}}$ invariant inner products on $\mathfrak{p} = \mathfrak{p}_1'' \oplus \mathfrak{p}_2'' \oplus \mathfrak{p}_1' \oplus \mathfrak{p}_2'$, we know our unique $ad_{\mathfrak{h}}$ invariant $(.,.)$ on $\mathfrak{p}$ associated with g restricts to $(.,.)_1$ on $\mathfrak{p}_1'' \oplus \mathfrak{p}_1'$ and $(.,.)_2$ on $\mathfrak{p}_2'' \oplus \mathfrak{p}_2'$. Thus, we have $(\mathfrak{p}_1'', \mathfrak{p}_1') = (\mathfrak{p}_2'', \mathfrak{p}_2') = 0$.

If we want $(\mathfrak{g}, \mathfrak{h})$ to be a Cartan orthogonal pair, it is now sufficient to find a condition guaranteeing $(\mathfrak{p}_1'', \mathfrak{p}_2') = (\mathfrak{p}_2'', \mathfrak{p}_1') = 0$. Thus, we seek a condition to ensure that any $ad_{\mathfrak{h}}$ intertwining maps $\mathfrak{p}_1'' \to \mathfrak{p}_2'$ and $\mathfrak{p}_2'' \to \mathfrak{p}_1'$ are trivial, implying that $(\mathfrak{p}_1'', \mathfrak{p}_2') = (\mathfrak{p}_2'', \mathfrak{p}_1') = 0$ for any $ad_{\mathfrak{h}}$ invariant $(.,.)$ on $\mathfrak{p}$. With this desired condition in mind, in the following lemma we utilize a similar (but different) technique to the one from [AL17, Prop. 3.4] regarding the nonexistence of trivial representations. In doing so, we provide a sufficient condition to take two Cartan orthogonal pairs and construct a new one through direct sum.

Lemma 2.8. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be semi-simple and $(\mathfrak{g}_1, \mathfrak{h}_1)$, $(\mathfrak{g}_2, \mathfrak{h}_2)$ be Cartan orthogonal pairs. If each $ad_{\mathfrak{h}_i}$ does not act trivially on any invariant subspace of $\mathfrak{p}_i'$ for any reductive Cartan decomposition, $\mathfrak{g}_i = \mathfrak{h}_i \oplus \mathfrak{p}_i'' \oplus \mathfrak{p}_i'$, then $(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{h}_1 \oplus \mathfrak{h}_2)$ is a Cartan orthogonal pair.

Proof: Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$. Following the above commentary, we have already established that given a G invariant metric on G/H, g, we can get a reductive Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$ with $\mathfrak{p} = \mathfrak{p}_1'' \oplus \mathfrak{p}_2'' \oplus \mathfrak{p}_1' \oplus \mathfrak{p}_2'$ such that $(\mathfrak{p}_1'', \mathfrak{p}_1') = (\mathfrak{p}_2'', \mathfrak{p}_2') = 0$ for the $ad_{\mathfrak{h}}$ invariant inner product $(.,.)$ on $\mathfrak{p}$ associated with g. Having established this, we finish our proof by showing that $(\mathfrak{p}_1'', \mathfrak{p}_2') = (\mathfrak{p}_2'', \mathfrak{p}_1') = 0$, and we do this by proving the stronger statement that $(\mathfrak{p}_1'', \mathfrak{p}_2') = (\mathfrak{p}_2'', \mathfrak{p}_1') = 0$ for an $ad_{\mathfrak{h}}$ invariant metric on $\mathfrak{p}$.

We will show that $(\mathfrak{p}_1'', \mathfrak{p}_2') = 0$ and the other case is exactly the same with subscripts changed. For this proof, we will use a similar argument as in Proposition 2.7, using kernels of $ad_{\mathfrak{h}_i}$ to show the nonexistence of any nontrivial $ad_{\mathfrak{h}}$ intertwining map $\phi : \mathfrak{p}_1'' \to \mathfrak{p}_2'$. This will imply, by Lemma 1.9, that $(\mathfrak{p}_1'', \mathfrak{p}_2') = 0$, providing us with the desired result.

Recall that in $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are ideals which means that $[\mathfrak{g}_1, \mathfrak{g}_2] = 0$. Therefore, $\mathfrak{p}_1'' \subset \mathfrak{g}_1$ is necessarily in the kernel of $ad_{\mathfrak{g}_2}$ and therefore in the kernel of $ad_{\mathfrak{h}_2}$. So, if there was a non-trivial $ad_{\mathfrak{h}}$ intertwining map $\phi : \mathfrak{p}_1'' \to \mathfrak{p}_2'$, then for $x \in \mathfrak{h}_2 \subset \mathfrak{h}$ and $v \in \mathfrak{p}_1''$, $ad_x(\phi(v)) = \phi(ad_x(v)) = \phi(0) = 0$. This, however, is a contradiction to the assumption regarding $ad_{\mathfrak{h}_2}$ having no trivial representations in $\mathfrak{p}_2'$. Therefore, there are no non-trivial $ad_{\mathfrak{h}}$ intertwining maps $\phi : \mathfrak{p}_1'' \to \mathfrak{p}_2'$, implying by Schur's Lemma (see 1.3) that there are no irreducible subrepresentations of $\mathfrak{p}_1''$ isomorphic to any irreducible subrepresenation of $\mathfrak{p}_2'$. Moreover, by Lemma 1.9, we have that $(\mathfrak{p}_1'', \mathfrak{p}_2') = 0$ for every $ad_{\mathfrak{h}}$ invariant inner product on $\mathfrak{p}$. ■

Remark 2.9. An example showing that $(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{h}_1 \oplus \mathfrak{h}_2)$ is not a Cartan orthogonal pair when $(\mathfrak{g}_1, \mathfrak{h}_1)$ and $(\mathfrak{g}_2, \mathfrak{h}_2)$ are Cartan orthogonal pairs is saved until after the next couple of results for simplicity (See Example 2.16 in the next section).

Remark 2.10. Lemma 2.8 addresses the combining of Cartan orthogonal pairs to get another Cartan orthogonal pair. It is natural to ask about going the opposite direction: starting with a Cartan orthogonal pair for semi-simple $\mathfrak{g}$ and decomposing into separate Cartan orthogonal pairs. The ability to do this successfully is clear immediately. If $(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{h}_1 \oplus \mathfrak{h}_2)$ is a Cartan orthogonal pair then by restricting $ad_{\mathfrak{h}_1 \oplus \mathfrak{h}_2}$ inner products, we get that $(\mathfrak{g}_1, \mathfrak{h}_1)$ and $(\mathfrak{g}_2, \mathfrak{h}_2)$ are Cartan orthogonal pairs.

Proposition 2.11. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_n$ be a noncompact semi-simple Lie algbebra with each $\mathfrak{g}_i$ a simple ideal in $\mathfrak{g}$. Moreover, assume that the maximal compact $\mathfrak{k}_i$ of $\mathfrak{g}_i$ is not simple for each i. If $\mathfrak{h} = \mathfrak{h}_1 \oplus ... \oplus \mathfrak{h}_n$ is such that $\mathfrak{h}_i \subset \mathfrak{k}_i$ for all i and $\mathfrak{h}_i$ contains an ideal of

$\mathfrak{k}_i$ for all i then $(\mathfrak{g}, \mathfrak{h})$ is a Cartan orthogonal pair.

Proof: As before, let $\mathfrak{g}_i = \mathfrak{h}_i \oplus \mathfrak{p}_i'' \oplus \mathfrak{p}_i'$ be a reductive Cartan decomposition for $(\mathfrak{g}_i, \mathfrak{h}_i)$, and note by Proposition 2.7 that all $(\mathfrak{g}_i, \mathfrak{h}_i)$ are Cartan orthogonal pairs. Moreover, by Lemma 2.6, one can see that $ad_{\mathfrak{h}_i}$ does not act trivially on any invariant subspace of $\mathfrak{p}_i'$, so by applying Lemma 2.8, we have that $(\mathfrak{g}, \mathfrak{h})$ is a Cartan orthogonal pair. ■.

Thus far, we have provided results that indicate that we could, if called for, produce numerous examples of Cartan orthogonal pairs, and we have a method for constructing more Cartan orthogonal pairs. In all of the above results, though, we were consistently proving that these spaces satisfy a stronger condition than what is required for a Cartan orthogonal pair (recall the equivalent conditions in 2.1) as we were proving that no non-trivial intertwining maps from $\mathfrak{p}'' \to \mathfrak{p}'$ existed. What we would now like to do is determine a sufficient (Lie theoretic) condition for $(\mathfrak{g}, \mathfrak{h})$ to be a Cartan orthogonal pair even in the presence of nontrivial $ad_{\mathfrak{h}}$ intertwining maps $\mathfrak{p}'' \to \mathfrak{p}'$. In the following proposition, we not only find a sufficient condition, but we are able to say what the intertwining map is in such cases. To see the usefulness of the following proposition, consider the Example 2.13 and Example 2.14 in Section 2.2, and also compare our result in Example 2.15 with Example 1 in [Nik00].

Proposition 2.12. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$ be a reductive Cartan decomposition for $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{g}$ is noncompact simple. Assume there is a nonzero $x \in \mathfrak{p}'$ be such that $[\mathfrak{h}, x] = 0$. Fix an $x \in \mathfrak{p}'$ such that $[\mathfrak{h}, x] = 0$.

1. There is a nonzero $ad_{\mathfrak{h}}$ intertwining map $\phi : \mathfrak{p}'' \to \mathfrak{p}'$, namely, $ad_x|_{\mathfrak{p}''} : \mathfrak{p}'' \to \mathfrak{p}'$.

2. If $\mathfrak{p}''$ is irreducible and of real type (see Section 1.6), then for some irreducible $\mathfrak{p}_i \subset \mathfrak{p}'$, we have $ad_x(\mathfrak{p}'') = \mathfrak{p}_i$, $ad_x : \mathfrak{p}'' \to \mathfrak{p}_i$ as an isomorphism, and $ad_x^2 = \lambda Id$.

3. If $\mathfrak{p}''$ is irreducible of real type and $[\mathfrak{p}'', \mathfrak{p}''] \subset \mathfrak{h}$ then by 2 there is some $\mathfrak{p}_i \subset \mathfrak{p}'$ isomorphic to $\mathfrak{p}''$. Assume that no other $\mathfrak{p}_j \subset \mathfrak{p}'$ is isomorphic to $\mathfrak{p}''$. In this case, $(\mathfrak{g}, \mathfrak{h})$ is a Cartan orthogonal pair.

Proof: Since x is such that $[\mathfrak{h}, x] = 0$, we have that $ad_{\mathfrak{h}} \circ ad_x = ad_x \circ ad_{\mathfrak{h}}$ by the Jacobian identity. By the Cartan decomposition properties (1.3), if $x \in \mathfrak{p}'$, we know that $ad_x(y) \in \mathfrak{p}'$ for any $y \in \mathfrak{k}$. Since $ad_x(y) = 0$ for $y \in \mathfrak{h}$, there must exist a $y \in \mathfrak{p}''$ such that $ad_x(y) \neq 0$ else $\mathbb{R}\{x\}$ be a trivial representation of $ad_{\mathfrak{k}}$, a contradiction to $\mathfrak{p}'$ being an irreducible $ad_{\mathfrak{k}}$ representation. Thus, $ad_x|_{\mathfrak{p}''} : \mathfrak{p}'' \to \mathfrak{p}'$ is a non-trivial $ad_{\mathfrak{h}}$ intertwining map. This proves 1.

To prove 2, we first remark that by 1 we already have a non-zero $ad_{\mathfrak{h}}$ intertwining map, $ad_x|_{\mathfrak{p}''} : \mathfrak{p}'' \to \mathfrak{p}'$. Since $\mathfrak{p}''$ is irreducible, we can say by Schur's Lemma that there is an irreducible $\mathfrak{p}_i \subset \mathfrak{p}'$ such that $ad_x(\mathfrak{p}'') = \mathfrak{p}_i$ with $ad_x : \mathfrak{p}'' \to \mathfrak{p}_i$ being an isomorphism. This provides us with the first two parts of our desired result. To see that $ad_x^2 = \lambda Id$, we will first show that $ad_x(\mathfrak{p}_i) = \mathfrak{p}''$ and then utilize our assumptions on $\mathfrak{p}''$.

We know from the Cartan decomposition properties that $ad_x(\mathfrak{p}_i) \subset \mathfrak{h} \oplus \mathfrak{p}''$. More than that, though, we have $ad_x(\mathfrak{p}_i) \subset \mathfrak{p}''$. Indeed, if $v \in \mathfrak{p}_i$ and $w \in \mathfrak{h}$ then $B(ad_x(v), w) = -B(v, ad_x(w)) = 0$ with the final equality being true by the assumption that $ad_x(\mathfrak{h}) = 0$. Having that $ad_x(\mathfrak{p}_i) \subset \mathfrak{p}''$, we consider the map $ad_x : \mathfrak{p}_i \to \mathfrak{p}''$ which is also an $ad_{\mathfrak{h}}$ intertwining map by $ad_x(\mathfrak{h}) = 0$. By being an intertwining map, we know that either $ad_x(\mathfrak{p}_i) = \{0\}$ or $ad_x(\mathfrak{p}_i) = \mathfrak{p}''$ by Schur's Lemma. To see that $ad_x(\mathfrak{p}_i) \neq \{0\}$, let $w \in \mathfrak{p}''$ and observe that $B(ad_x(ad_x(w)), w) = -B(ad_x(w), ad_x(w))$. Since $ad_x(w) \in \mathfrak{p}_i \subset \mathfrak{p}'$ and $B(.,.) < 0$ on $\mathfrak{p}'$ by the Cartan decomposition properties, we have that $B(ad_x(w), ad_x(w)) \neq 0$. Therefore, we have $ad_x(ad_x(w)) \neq 0$ for non-zero $w \in \mathfrak{p}''$, implying that $ad_x(\mathfrak{p}_i) \neq \{0\}$. Thus, $ad_x(\mathfrak{p}_i) = \mathfrak{p}''$.

Since we now have $ad_x(\mathfrak{p}_i) = \mathfrak{p}''$ and $ad_x(\mathfrak{p}'') = \mathfrak{p}_i$, we know $ad_x \circ ad_x : \mathfrak{p}'' \to \mathfrak{p}''$ is an non-trivial $ad_{\mathfrak{h}}$ equivariant map on $\mathfrak{p}''$. Utilizing the assumption that $\mathfrak{p}''$ is irreducible of real type now, we have that $ad_x^2 = \lambda Id$ for $\lambda \in \mathbb{R}$ as desired.

Now we prove 3. Recall that to prove $(\mathfrak{g}, \mathfrak{h})$ is a Cartan orthogonal pair, we must satisfy the two conditions found in 2.1. Looking to those two conditions, and provided our assumptions regarding $\mathfrak{p}''$ being irreducible with $[\mathfrak{p}'', \mathfrak{p}''] \subset \mathfrak{h}$, we must prove the following for any $ad_{\mathfrak{h}}$ intertwining map, $\phi : \mathfrak{p}'' \to \mathfrak{p}'$:

$$[\phi(x), \phi(y)] \subset \mathfrak{h}$$

$$0 = [x, \phi(y)] + [\phi(x), y] \text{for all } x, y \in \mathfrak{p}''.$$

Since $\mathfrak{p}''$ is irreducible of real type and there is only one $\mathfrak{p}_i \subset \mathfrak{p}'$ isomorphic to $\mathfrak{p}''$ by assumption, we can conclude by 2 that all such ϕ are obtained by $\lambda ad_x|_{\mathfrak{p}''} : \mathfrak{p}'' \to \mathfrak{p}_i$ for $\lambda \in \mathbb{R}$. Thus, it suffices to prove that $[ad_x(y), ad_x(z)] = [[x, y], [x, z]] \subset \mathfrak{h}$ and $0 = [y, ad_x(z)] + [ad_x(y), z]$ for $y, z \in \mathfrak{p}''$. The second condition is obvious by the Jacobi identity implying $[y, ad_x(z)] + [ad_x(y), z] = ad_x([y, z]) = 0$. (We note here that $[\mathfrak{p}'', \mathfrak{p}''] \subset \mathfrak{h}$ is not needed for the second condition of a Carton orthogonal pair to be satisfied.) We now check that for $y, z \in \mathfrak{p}''$, $[[x, y], [x, z]] \subset \mathfrak{h}$ by using the Jacobi identity, skew-symmetry of brackets, and result 2.

$$\begin{aligned}
[[x, y], [x, z]] &= [[[x, y], x], z] + [x, [[x, y], z]] \\
&= [[[x, y], x], z] + [x, -[[z, x], y] - [x, [z, y]]] \\
&= [-ad_x^2(y), z] + [x, [[x, z], y]] + [x, [x, [y, z]]]
\end{aligned}$$

$$
\begin{aligned}
&= [-ad_x^2(y), z] + [[x, [x, z]], y] + [[x, z], [x, y]] + ad_x^2([y, z]) \\
2[[x, y], [x, z]] &= [-ad_x^2(y), z] + [[x, [x, z]], y] + ad_x^2([y, z]) \\
&= [-ad_x^2(y), z] + [ad_x^2(z), y] + ad_x^2([y, z]) \\
&= -\lambda[y, z] + \lambda[z, y] + 0 \\
&= -2\lambda[y, z] \in \mathfrak{h}
\end{aligned}
$$

Thus, $[[x, y], [x, z]] = -\lambda[y, z] \subset \mathfrak{h}$. ■

2.2. Examples of Cartan Orthogonal Pairs

We now turn our attention to some examples of Cartan orthogonal pairs and some examples of $(\mathfrak{g}, \mathfrak{h})$ that are not Cartan orthogonal pairs. The goal in many of these examples is to demonstrate the usefulness of Proposition 2.12 for showing that particular pairs are in fact Cartan orthogonal pairs. In all examples below, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$ is our reductive Cartan decomposition for $(\mathfrak{g}, \mathfrak{h})$. In many of the examples below, we also spend a good deal of time justifying the usage of particular dimensions of spaces and not others. Much of this is reduced to the issue of wanting $\mathfrak{p}''$ to be irreducible of real type with $[\mathfrak{p}'', \mathfrak{p}''] \subset \mathfrak{h}$ which we are able to conclude by our K/H being irreducible symmetric spaces of real type (See Remark 1.13).

Example 2.13. Let $\mathfrak{g} = \mathfrak{so}(n,n)$ and $\mathfrak{h} = \Delta(\mathfrak{so}(n)) = \{x + x : x \in \mathfrak{so}(n)\} \subset \mathfrak{so}(n) \oplus \mathfrak{so}(n) = \mathfrak{k}$. For $n \geq 3$ and $n \neq 4$, $(\mathfrak{g}, \mathfrak{h})$ is a Cartan orthogonal pair.

Using Section 2 of Chapter X in [Hel01], we get the following subspaces of $\mathfrak{g}$ that are helpful for understanding our decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$ where $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{p}''$:

$$\mathfrak{g} = \left\{ \left[\begin{array}{c|c} M & N \\ \hline N^t & Q \end{array} \right] : M, Q \in \mathfrak{so}(n) \text{ , } N \in M(n, \mathbb{R}) \right\}.$$

$$\mathfrak{k} = \left\{ \left[\begin{array}{c|c} M & 0 \\ \hline 0 & Q \end{array} \right] : M, Q \in \mathfrak{so}(n) \right\}.$$

$$\mathfrak{p}' = \left\{ \left[\begin{array}{c|c} 0 & N \\ \hline N^t & 0 \end{array} \right] : N \in M(n, \mathbb{R}) \right\}$$

$$\mathfrak{h} = \left\{ \left[\begin{array}{c|c} M & 0 \\ \hline 0 & M \end{array} \right] : M \in \mathfrak{so}(n) \right\}.$$

$$\mathfrak{p}'' = \left\{ \left[\begin{array}{c|c} Q & 0 \\ \hline 0 & -Q \end{array} \right] : Q \in \mathfrak{so}(n) \right\}$$

Our goal is to utilize Proposition 2.12 to show that $(\mathfrak{g}, \mathfrak{h})$ is a Cartan orthogonal pair, so we need that $\mathfrak{g}$ is simple and $\mathfrak{p}''$ is irreducible, of real type, and $[\mathfrak{p}'', \mathfrak{p}''] \subset \mathfrak{h}$. Thus, we consider when $n \geq 3$ but not 4. We ignore $n = 2$ and $n = 4$ for separate reasons, so allow us to explain.

If $n = 4$ then $K/H = SO(n)SO(n)/\Delta(SO(n))$ is not an irreducible symmetric space as $\mathfrak{so}(4) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ (see the incidental isomorphisms following Table V in Section 6 of Chapter X in [Hel01]) is not simple. Thus, we don't have $\mathfrak{p}''$ irreducible (see 1.5 or Section 6 of Chapter X in [Hel01]).

If $n = 2$, then by the incidental isomorphisms just referenced from [Hel01, Ch. X, Sec. 6], $\mathfrak{so}(2,2) \simeq \mathfrak{sl}(n,\mathbb{R}) \oplus \mathfrak{sl}(n,\mathbb{R})$, so $\mathfrak{so}(2,2)$ is not simple.

To justify our using $n \geq 3$ but not $n = 4$, consider Type BDI in Section 2 of Chapter X in [Hel01] and Table IV of chapter X in [Hel01]. In Type BDI, we find that for all such n, $SO(n,n)/SO(n)SO(n)$ is an irreducible symmetric space, so all such $\mathfrak{so}(n,n)$ are simple. Thus, we have $\mathfrak{g}$ is simple.

In Table IV all such n except for $n = 3$ and $n = 6$ are provided as irreducible symmetric spaces for $SO(n)SO(n)/\Delta(SO(n))$. However, when $n = 3$ and $n = 6$, by considering the incidental isomorphisms $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$ and $\mathfrak{so}(6) \simeq \mathfrak{su}(4)$ in [Hel01, Ch. X, Sec. 6] along side Table IV in Chapter X of [Hel01] again, we see that these cases of $SO(n)SO(n)/\Delta(SO(n))$ are in fact irreducible symmetric spaces. Thus, for all $n \geq 3$ and $n \neq 4$, we have that $\mathfrak{p}''$ is irreducible and $[\mathfrak{p}'', \mathfrak{p}''] \subset \mathfrak{h}$ (See Remark 1.13).

Lastly, for all such n, $SO(n)SO(n)/\Delta(SO(n))$ is not a Hermitian or Quaternionic symmetric space (see Section 6 of Chapter X in [Hel01] and Table 1 of [Wol65]), so $\mathfrak{p}''$ is irreducible of real type (see 1.6).

To summarize, we now have that $\mathfrak{g}$ is simple and $\mathfrak{p}''$ is irreducible of real type with $[\mathfrak{p}'', \mathfrak{p}''] \subset \mathfrak{h}$. Our focus not turns to getting a decomposition of $\mathfrak{p}'$ into irreducibles to establish that all the assumptions of Proposition 2.12 are met and then we conclude our result.

Let $H = \left[\begin{array}{c|c} M & 0 \\ \hline 0 & M \end{array}\right] \in \mathfrak{h}$ and $P = \left[\begin{array}{c|c} 0 & N \\ \hline N^t & 0 \end{array}\right] \in \mathfrak{p}'$ and observe that

$$\begin{aligned} ad_H(P) &= \left[\begin{array}{c|c} 0 & MN - NM \\ \hline MN^t - N^tM & 0 \end{array}\right] \\ &= \left[\begin{array}{c|c} 0 & MN - NM \\ \hline (MN - NM)^t & 0 \end{array}\right]. \end{aligned}$$

From this, we can observe that the $ad_{\mathfrak{h}}$ representation on $\mathfrak{p}'$ is isomorphic to the $ad_{\mathfrak{so}(n)}$ representation on $M(n, \mathbb{R})$.

The Lie algebra, $M(n,\mathbb{R})$, has the decomposition $M(n,\mathbb{R}) = \mathfrak{sl}(n,\mathbb{R}) \oplus \{\lambda I\}$ where $\{\lambda I\}$ are matrices that are multiples of the identity, a trivial irreducible representation of $ad_{\mathfrak{so}(n)}$. Moreover, $\mathfrak{sl}(n,\mathbb{R})$ has the following decomposition into irreducibles under $ad_{\mathfrak{so}(n)}$ following from the irreducible symmetric space $SL(n,\mathbb{R})/SO(n)$ (see Section 2, Chapter X of [Hel01]) and from the fact that $\mathfrak{so}(n)$ is simple for $n \geq 3$ and $n \neq 4$:

$$\mathfrak{sl}(n,\mathbb{R}) = \mathfrak{so}(n) \oplus \mathrm{symm}(n) \text{ where } \mathrm{symm}(n) = \{B \in \mathfrak{sl}(n,\mathbb{R}) : B^t = B\}$$

Thus, we have the following decomposition of $\mathfrak{p}'$ into irreducibles:

$$\mathfrak{p}' = \left\{\left[\begin{array}{c|c} 0 & \lambda I \\ \hline \lambda I & 0 \end{array}\right] : \lambda \in \mathbb{R}\right\} \oplus \left\{\left[\begin{array}{c|c} 0 & N \\ \hline -N & 0 \end{array}\right] : N \in \mathfrak{so}(n)\right\} \oplus \left\{\left[\begin{array}{c|c} 0 & B \\ \hline B & 0 \end{array}\right] : \begin{array}{r} trB = 0, \\ B = B^t \end{array}\right\},$$

and we will denote the irreducible representations in the decomposition by $\mathfrak{p}_1$, $\mathfrak{p}_2$, and $\mathfrak{p}_3$, respectively.

$\dim\mathfrak{p}_1 = 1$, $\dim\mathfrak{p}_3 = n(n+1)/2 - 1$, and $\dim\mathfrak{p}'' = n(n-1)/2$. Thus, $\mathfrak{p}_1$ is not isomorphic to $\mathfrak{p}''$ unless (possibly) when $n = 2$ (a case ignored here), and $\mathfrak{p}_3$ is never isomorphic to $\mathfrak{p}''$ since the dimensions are inequivalent.

By Proposition 2.12, we know there is an isomorphism $\mathfrak{p}'' \simeq \mathfrak{p}_i$ for some i, and by dimensionality, it must be with $\mathfrak{p}_2$. Moreover, there is no other isomorphism with $\mathfrak{p}''$ in $\mathfrak{p}'$. Thus, Proposition 2.12 gives us that $(\mathfrak{g}, \mathfrak{h})$ is a Cartan orthogonal pair.

Example 2.14. Let $\mathfrak{g} = \mathfrak{sp}(n,n)$ with $\mathfrak{h} = \{x + x : x \in \mathfrak{sp}(n)\} \subset \mathfrak{sp}(n) \oplus \mathfrak{sp}(n) = \mathfrak{k}$. For

$n \geq 1$, $(\mathfrak{g}, \mathfrak{h})$ is a Cartan orthogonal pair.

Using Section 2 of Chapter X in [Hel01] again we get the following subspaces of $\mathfrak{g}$ that are helpful for understanding our decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$ where $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{p}''$:

$$\mathfrak{g} = \left\{ \left[\begin{array}{c|c|c|c} z_{11} & z_{12} & z_{13} & z_{14} \\ \hline \bar{z}_{12}^t & z_{22} & z_{14}^t & z_{24} \\ \hline -\bar{z}_{13} & \bar{z}_{14} & \bar{z}_{11} & -\bar{z}_{12} \\ \hline \bar{z}_{14}^t & -\bar{z}_{24} & -\bar{z}_{12}^t & \bar{z}_{22} \end{array} \right] : z_{ij} \in M(p, \mathbb{C}) \,, \begin{array}{l} z_{11}, z_{22} \text{ skew-Hermitian and} \\ z_{13}, z_{24} \text{ symmetric} \end{array} \right\}$$

$$\mathfrak{p}' = \left\{ \left[\begin{array}{c|c|c|c} 0 & z_{12} & 0 & z_{14} \\ \hline \bar{z}_{12}^t & 0 & z_{14}^t & 0 \\ \hline 0 & \bar{z}_{14} & 0 & -\bar{z}_{12} \\ \hline \bar{z}_{14}^t & 0 & -z_{12}^t & 0 \end{array} \right] : z_{ij} \in M(p, \mathbb{C}) \right\}$$

$$\mathfrak{k} = \left\{ \left[\begin{array}{c|c|c|c} z_{11} & 0 & z_{13} & 0 \\ \hline 0 & z_{22} & 0 & z_{24} \\ \hline -\bar{z}_{13} & 0 & \bar{z}_{11} & 0 \\ \hline 0 & -\bar{z}_{24} & 0 & \bar{z}_{22} \end{array} \right] : z_{ij} \in M(p, \mathbb{C}) \,, \begin{array}{l} z_{11}, z_{22} \text{ skew-Hermitian and} \\ z_{13}, z_{24} \text{ symmetric} \end{array} \right\}$$

$$\mathfrak{p}'' = \left\{ \left[\begin{array}{c|c|c|c} z_{11} & 0 & z_{13} & 0 \\ \hline 0 & -z_{11} & 0 & -z_{13} \\ \hline -\bar{z}_{13} & 0 & \bar{z}_{11} & 0 \\ \hline 0 & \bar{z}_{13} & 0 & -\bar{z}_{11} \end{array} \right] : z_{ij} \in M(p, \mathbb{C}) \,, \begin{array}{l} z_{11} \text{ skew-Hermitian and} \\ z_{13} \text{ symmetric} \end{array} \right\}$$

$$\mathfrak{h} = \left\{ \left[\begin{array}{c|c|c|c} z_{11} & 0 & z_{13} & 0 \\ \hline 0 & z_{11} & 0 & z_{13} \\ \hline -\bar{z}_{13} & 0 & \bar{z}_{11} & 0 \\ \hline 0 & -\bar{z}_{13} & 0 & \bar{z}_{11} \end{array} \right] : z_{ij} \in M(p, \mathbb{C}) \, , \begin{array}{l} z_{11} \text{ skew-Hermitian and} \\ z_{13} \text{ symmetric} \end{array} \right\}.$$

Again, our goal is to utilize Proposition 2.12 to show that $(\mathfrak{g}, \mathfrak{h})$ is a Cartan orthogonal pair, so we need that $\mathfrak{g}$ is simple and $\mathfrak{p}''$ is irreducible, of real type, and $[\mathfrak{p}'', \mathfrak{p}''] \subset \mathfrak{h}$ along with a trivial $ad_{\mathfrak{h}}$ representation. In this case, we can use $n \geq 1$, as we will now demonstrate.

$K/H = Sp(n)Sp(n)/\Delta Sp(n)$ is an irreducible symmetric space of real type, not Hermitian or Quaternionic (see Chapter X of [Hel01] and Table 1 of [Wol65]). Since Table IV of Chapter X in [Hel01] only lists $n \geq 3$, to see that $n = 1$ and $n = 2$ are still irreducible symmetric spaces, consider the incidental isomorphisms $\mathfrak{sp}(1) \simeq \mathfrak{su}(2)$ and $\mathfrak{sp}(2) \simeq \mathfrak{so}(5)$, and Table IV will again confirm that these are irreducible symmetric spaces. Thus, $\mathfrak{p}''$ is irreducible of real type and $[\mathfrak{p}'', \mathfrak{p}''] \subset \mathfrak{h}$ (See Remark 1.13). Lastly, Type C II in Section 2 of Chapter X in [Hel01] will confirm that $\mathfrak{sp}(n, n)$ is simple for all such n as $Sp(n, n)/Sp(n)Sp(n)$ is an irreducible symmetric space.

To utilize Proposition 2.12, we now seek to understand the $ad_{\mathfrak{h}}$ representation on $\mathfrak{p}'$ and a decomposition of $\mathfrak{p}'$ into irreducible factors. To do this, we will first find an equivalent representation of $ad_{\mathfrak{sp}(n)}$ on 2×2 block matrices which we will call $\mathfrak{q}'$ and then find a decomposition of $\mathfrak{q}'$ into irreducibles, $\mathfrak{q}' = \mathfrak{q}_1 \oplus \mathfrak{q}_2 \oplus \mathfrak{q}_3$. In our decomposition of $\mathfrak{q}'$, we will find that $\mathfrak{q}_1$ is isomorphic to the isotropy representation for the irreducible symmetic space $Sp(n, \mathbb{C})_{\mathbb{R}}/Sp(n)$, $\mathfrak{q}_2$ is isomorphic to the isotropy representation from the irreducible symmetric space $SU(2n)/Sp(n)$, and $\mathfrak{q}_3$ is a one-dimensional trivial representation.

Consider the following definition of $\mathfrak{sp}(n)$ from Section 2 of Chapter X in [Hel01]

$$\mathfrak{sp}(n) = \left\{ \left[\begin{array}{c|c} z_{11} & z_{13} \\ \hline -\bar{z}_{13} & \bar{z}_{11} \end{array} \right] : z_{11}, z_{13} \in M(n, \mathbb{C}) \text{ , } z_{11} \text{skew-Hermitian , and } z_{13} \text{ symmetric} \right\},$$

and let $\mathfrak{q}'$ be defined by

$$\mathfrak{q}' = \left\{ \left[\begin{array}{c|c} z_{12} & z_{14} \\ \hline \bar{z}_{14} & -\bar{z}_{12} \end{array} \right] : z_{ij} \in M(n, \mathbb{C}) \right\}.$$

<u>Claim:</u> The $ad_{\mathfrak{h}}$ representation on $\mathfrak{p}'$ is isomorphic to the $ad_{\mathfrak{sp}(n)}$ representation on $\mathfrak{q}'$.

<u>Proof of Claim:</u>

$$\text{Let } X = \left[\begin{array}{c|c|c|c} x_{11} & 0 & x_{13} & 0 \\ \hline 0 & x_{11} & 0 & x_{13} \\ \hline -\bar{x}_{13} & 0 & \bar{x}_{11} & 0 \\ \hline 0 & -\bar{x}_{13} & 0 & \bar{x}_{11} \end{array} \right] \in \mathfrak{h} \quad \text{and } V = \left[\begin{array}{c|c|c|c} 0 & z_{12} & 0 & z_{14} \\ \hline \bar{z}^t_{12} & 0 & z^t_{14} & 0 \\ \hline 0 & \bar{z}_{14} & 0 & -\bar{z}_{12} \\ \hline \bar{z}^t_{14} & 0 & -z^t_{12} & 0 \end{array} \right] \in \mathfrak{p}'.$$

$$\text{Let } x = \left[\begin{array}{c|c} x_{11} & x_{13} \\ \hline -\bar{x}_{13} & \bar{x}_{11} \end{array} \right] \in \mathfrak{sp}(n) \text{ and } v = \left[\begin{array}{c|c} z_{12} & z_{14} \\ \hline \bar{z}_{14} & -\bar{z}_{12} \end{array} \right] \in \mathfrak{q}'.$$

To see how X is mapped to x in the isomorphism $\mathfrak{h} \to \mathfrak{sp}(n)$, we have provided a red and blue box in our matrix to help one see the diagonal copies of $\mathfrak{sp}(n)$ in $\mathfrak{sp}(n, n)$ above. In adddtion, one can see the isomorphism by looking at the embedding of $\mathfrak{sp}(n) \oplus \mathfrak{sp}(n) \to \mathfrak{sp}(n,n)$ found in Type CII in Section 2 of Chapter X in [Hel01]. We will show that there is an intertwining map (intertwining $ad_{\mathfrak{h}}$ with $ad_{\mathfrak{sp}(n)}$) $\phi : \mathfrak{p}' \to \mathfrak{q}'$ determined by z_{12} in the $V_{1,2}$ position being mapped to z_{12} in the $v_{1,1}$ position, and z_{14} in the $V_{1,4}$ position being mapped to z_{14} in the $v_{1,2}$ position.

To show this, since $\mathfrak{p}'$ is already known to be invariant under $ad_{\mathfrak{h}}$, it suffices to show that $[x,v] \in \mathfrak{q}'$ (to verify invariance of $\mathfrak{q}'$ under $ad_{\mathfrak{sp}(n)}$) and that $\phi([X,V]) = [x,v]$ which is done by checking that $[X,V]_{1,2} = [x,v]_{1,1}$ and $[X,V]_{1,4} = [x,v]_{1,2}$.

First, we show that $\mathfrak{q}'$ is invariant. Observe,

$$[x,v] = \left[\begin{array}{c|c} x_{11}z_{12} + x_{13}\bar{z}_{14} - (z_{12}x_{11} + z_{14}(-\bar{x}_{13})) & x_{11}z_{14} + x_{13}(-\bar{z}_{12}) - (z_{12}x_{13} + z_{14}\bar{x}_{11}) \\ \hline -\bar{x}_{13}z_{12} + \bar{x}_{11}\bar{z}_{14} - (\bar{z}_{14}x_{11} + \bar{z}_{12}\bar{x}_{13}) & -\bar{x}_{13}z_{14} + \bar{x}_{11}(-\bar{z}_{12}) - (\bar{z}_{14}x_{13} - \bar{z}_{12}\bar{x}_{11}) \end{array}\right].$$

To see that $[x,v]_{2,2} = -\overline{[x,v]}_{1,1}$ and $[x,v]_{2,1} = \overline{[x,v]}_{1,2}$, we rearrange the terms in $[x,v]_{2,1}$ and $[x,v]_{2,2}$ from above and check for equivalence. Observe,

$$[x,v] = \left[\begin{array}{c|c} x_{11}z_{12} + x_{13}\bar{z}_{14} - (z_{12}x_{11} + z_{14}(-\bar{x}_{13})) & x_{11}z_{14} + x_{13}(-\bar{z}_{12}) - (z_{12}x_{13} + z_{14}\bar{x}_{11}) \\ \hline \bar{x}_{11}\bar{z}_{14} - \bar{x}_{13}z_{12} - (\bar{z}_{12}\bar{x}_{13} + \bar{z}_{14}x_{11}) & -\bar{x}_{11}\bar{z}_{12} - \bar{x}_{13}z_{14} - (-\bar{z}_{12}\bar{x}_{11} + \bar{z}_{14}x_{13}) \end{array}\right].$$

Now that we have that $\mathfrak{q}'$ is invariant under $ad_{\mathfrak{sp}(n)}$, we check that under ϕ as described above, $[X,V]_{1,2}$ is mapped to $[x,v]_{1,1}$ and $[X,V]_{1,4}$ is mapped to $[x,v]_{1,2}$ by checking that $[X,V]_{1,2} = [x,v]_{1,1}$ and $[X,V]_{1,4} = [x,v]_{1,2}$.

$$\begin{aligned}[X,V]_{1,2} &= x_{11}z_{12} + x_{13}\bar{z}_{14} - (z_{12}x_{11} + z_{14}(\bar{x}_{13})) \\ &= [x,v]_{1,1}\end{aligned}$$

$$
\begin{aligned}
[X,V]_{1,4} &= x_{11}z_{14} + x_{13}(-\bar{z}_{12}) - (z_{12}x_{13} + z_{14}\bar{x}_{11}) \\
&= [x,v]_{1,2}
\end{aligned}
$$

Thus, we have proven our claim and we can understand our $ad_{\mathfrak{h}}$ representation on $\mathfrak{p}'$ by understanding the simpler, isomorphic representation of $ad_{\mathfrak{sp}(n)}$ on $\mathfrak{q}'$.

Observe that we have the following vector space decomposition for our

$$
\mathfrak{q}' = \left\{ \left[\begin{array}{c|c} z_{12} & z_{14} \\ \hline \bar{z}_{14} & -\bar{z}_{12} \end{array} \right] : z_{ij} \in M(n,\mathbb{C}) \right\} :
$$

$\mathfrak{q}' = \mathfrak{q}_1 \oplus \mathfrak{q}_2 \oplus \mathfrak{q}_3$ where

$$
\mathfrak{q}_1 = \left\{ \left[\begin{array}{c|c} z_{12} & z_{14} \\ \hline \bar{z}_{14} & -\bar{z}_{12} \end{array} \right] : z_{ij} \in M(n,\mathbb{C}) \text{ , } z_{12} \text{ Hermitian, } z_{14} \text{ symmetric} \right\}
$$

$$
\mathfrak{q}_2 = \left\{ \left[\begin{array}{c|c} z_{12} & z_{14} \\ \hline \bar{z}_{14} & -\bar{z}_{12} \end{array} \right] : z_{ij} \in M(n,\mathbb{C}) \text{ , } z_{12} \in \mathfrak{su}(n) \text{ , } z_{14} \in \mathfrak{so}(n,\mathbb{C}) \right\}
$$

$$
\mathfrak{q}_3 = \left\{ \left[\begin{array}{c|c} z_{12} & 0 \\ \hline 0 & -\bar{z}_{12} \end{array} \right] : z_{ij} \in M(n,\mathbb{C}) \text{ , } z_{12} = \lambda I \text{ , } \bar{\lambda} = -\lambda \right\}.
$$

We can see that $\mathfrak{q}_3$ is a trivial representation of dimension 1. We will now show, as mentioned before, that $\mathfrak{q}_1$ is isomorphic to the isotoropy representation for the irreducible symmetic space $Sp(n,\mathbb{C})_{\mathbb{R}}/Sp(n)$ and $\mathfrak{q}_2$ is isomorphic to the isotropy representation from the irreducible symmetric space $SU(2n)/Sp(n)$.

First, we show that $\mathfrak{q}_1 = i\mathfrak{sp}(n)$, recalling that $\mathfrak{sp}(n,\mathbb{C})_{\mathbb{R}} = \mathfrak{sp}(n) \oplus i\mathfrak{sp}(n)$ is the Cartan decomposition of $\mathfrak{sp}(n,\mathbb{C})_{\mathbb{R}}$ (see Section 1.5).

Consider $\mathfrak{sp}(n) = \left\{ \left[\begin{array}{c|c} x & y \\ \hline -\bar{y} & -x^t \end{array} \right] : x, y \in M(n,\mathbb{C}) \text{ , } x \text{ skew-Hermitian, and } y \text{ symmetric} \right\}$, which implies that we have

$i\mathfrak{sp}(n) = \left\{ \left[\begin{array}{c|c} ix & iy \\ \hline \overline{iy} & -ix^t \end{array} \right] : x, y \in M(n,\mathbb{C}) \text{ , } x \text{ skew-Hermitian, and } y \text{ symmetric} \right\}$. Thus, if we let $z_{12} = ix$ and $z_{14} = iy$, then we get $\mathfrak{q}_1$, so $\mathfrak{q}_1 = i\mathfrak{sp}(n)$ as a vector space, with isomorphism at the representation level following from both representations being defined by the matrix multiplication bracket.

To see that $\mathfrak{q}_2$ is the irreducible isotropy representation from $SU(2n)/Sp(n)$, one simply has to compare Type AII in Section 2 of Chapter X in [Hel01] with $\mathfrak{q}_2$ and note that in both cases, the $ad_{\mathfrak{sp}(n)}$ action is the matrix multiplication defined bracket.

Therefore, we have $\mathfrak{q}' = \mathfrak{q}_1 \oplus \mathfrak{q}_2 \oplus \mathfrak{q}_3$ as a decomposition into irreducibles. Now, observe that $\dim\mathfrak{p}'' = n(2n+1)$, $\dim\mathfrak{q}_1 = n(2n+1)$, and $\dim\mathfrak{q}_2 = (n-1)(2n+1)$, and one can check by comparing dimensions that $\mathfrak{q}_2$ cannot be isomorphic to $\mathfrak{p}''$. Moreover, we have $\mathfrak{q}_3$ as a trivial representation with dimension 1. Therefore, we have all the assumptions met for Proposition 2.12, and we can conclude that $\mathfrak{p}'' \simeq \mathfrak{q}_1$ and $(\mathfrak{g}, \mathfrak{h})$ is a Cartan orthogonal pair.

Example 2.15. Let $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{R})$ with $\mathfrak{h} = \mathfrak{so}(n) \subset \mathfrak{so}(n+1) = \mathfrak{k}$. For $n \geq 3$, $(\mathfrak{g}, \mathfrak{h})$ is a Cartan orthogonal pair, but for $n = 2$, $(\mathfrak{g}, \mathfrak{h})$ is not a Cartan orthogonal pair.

Using Section 2 in Chapter X of [Hel01] again, we have $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$ where $\mathfrak{h}$, $\mathfrak{p}''$, and $\mathfrak{p}'$ are defined as follows:

$$\mathfrak{h} = \left\{ \left[\begin{array}{c|c} A & 0 \\ \hline 0 & 0 \end{array} \right] : A \in \mathfrak{so}(n) \right\}$$

$$\mathfrak{p}'' = \left\{ \left[\begin{array}{c|c} 0 & x^t \\ \hline -x^t & 0 \end{array} \right] : x \in \mathbb{R}^n \right\}$$

$$\mathfrak{p}' = \left\{ \left[\begin{array}{c|c} C & x \\ \hline x & -tr(C) \end{array} \right] : x \in \mathbb{R}^n \text{ and } C^t = C \right\}.$$

Here, we have that $\mathfrak{g}$ is simple for all $n > 1$, and $K/H = SO(n+1)/SO(n)$ is an irreducible symmetric space of real type (see Chapter X of [Hel01] and Table 1 of [Wol65]) except when $n = 2$. When $n = 2$ we have $SO(3)/SO(2)$ which is of Hermitian type, so in this case, the irreducible representation $\mathfrak{p}''$ is of complex type.

Nikonorov, in Example 4 of [Nik00], shows that $(\mathfrak{g}, \mathfrak{h})$ is a Cartan orthogonal pair with the irreducible decomposition of $\mathfrak{p}'$ given by $\mathfrak{p}' = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$ with $\mathfrak{p}_1$, $\mathfrak{p}_2$, and $\mathfrak{p}_3$ as follows:

$$\mathfrak{p}_1 = \left\{ \left[\begin{array}{c|c} B & 0 \\ \hline 0 & 0 \end{array} \right] : B^t = B \text{ and } trB = 0 \right\}$$

$$\mathfrak{p}_2 = \left\{ \left[\begin{array}{c|c} 0 & x \\ \hline x^t & 0 \end{array} \right] : x \in \mathbb{R}^n \right\}$$

$$\mathfrak{p}_3 = \left\{ \left[\begin{array}{c|c} \lambda I & 0 \\ \hline 0 & -n\lambda \end{array} \right] : \lambda \in \mathbb{R} \right\}.$$

In that example, Nikonorov constructs the intertwining map; however, by utilizing our

Proposition 2.12, we can see that since $\mathfrak{p}_3$ is trivial, $\mathfrak{p}_1 \not\simeq \mathfrak{p}''$ by dimensionality, and $[\mathfrak{p}'', \mathfrak{p}''] \subset \mathfrak{h}$ by $SO(n+1)/SO(n)$ being an irreducible symmetric space (see Remark 1.13), so long as $\mathfrak{p}''$ is irreducible of real type, $(\mathfrak{g}, \mathfrak{h})$ is a Cartan orthogonal pair. Thus, for $n > 2$ $(\mathfrak{g}, \mathfrak{h})$ is a Cartan orthogonal pair.

When $n = 2$, $\mathfrak{p}''$ is of complex type, a fact overlooked in [Nik00] and [AL17] as well. Their conclusion that $SL(3, \mathbb{R})/SO(2)$ is not Einstein, it turns out, has a gap in it. The solution to that gap is resolved in [BL23] in which the Alekseevski conjecture is given in the positive. For our purposes though, we simply wish to show that $(\mathfrak{sl}(3, \mathbb{R}), \mathfrak{so}(2))$ is not a Cartan orthogonal pair. This will give us a non-trivial example that is not a Cartan orthogonal pair, and will also show the necessity of having a real representation for part 3 of Proposition 2.12 to be true.

Claim: The pair $(\mathfrak{sl}(3, \mathbb{R}), \mathfrak{so}(2))$ is not a Cartan orthogonal pair.

Proof of Claim: Consider then $\mathfrak{sl}(3, \mathbb{R}) = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$. Let $\phi : \mathfrak{p}'' \to \mathfrak{p}_2$ be defined by

$$\begin{bmatrix} 0 & 0 & x_1 \\ 0 & 0 & x_2 \\ -x_1 & -x_2 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & 0 & x_2 \\ 0 & 0 & -x_1 \\ x_2 & -x_1 & 0 \end{bmatrix}.$$

Let $Z = \begin{bmatrix} 0 & z & 0 \\ -z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{h}$, $X = \begin{bmatrix} 0 & 0 & x_1 \\ 0 & 0 & x_2 \\ -x_1 & -x_2 & 0 \end{bmatrix} \in \mathfrak{p}''$, and $Y = \begin{bmatrix} 0 & 0 & y_1 \\ 0 & 0 & y_2 \\ -y_1 & -y_2 & 0 \end{bmatrix} \in \mathfrak{p}''$.

We will show that ϕ is an intertwining map and then prove that condition ii for $(\mathfrak{sl}(3, \mathbb{R}), \mathfrak{so}(2))$ to be a Cartan orthogonal pair (see 2.1) is not satisfied (observe that condition i is automatically satisfied since the bracket structure of $\mathfrak{p}_2$ is not changing).

Observe, $[Z, X] = \begin{bmatrix} 0 & 0 & zx_2 \\ 0 & 0 & -zx_1 \\ -zx_2 & zx_1 & 0 \end{bmatrix}$ and $\phi([Z, X]) = \begin{bmatrix} 0 & 0 & -zx_1 \\ 0 & 0 & -zx_2 \\ -zx_1 & -zx_2 & 0 \end{bmatrix}$.

Moreover, we have that $\phi(X) = \begin{bmatrix} 0 & 0 & x_2 \\ 0 & 0 & -x_1 \\ x_2 & -x_1 & 0 \end{bmatrix}$ and $[Z, \phi(X)] = \begin{bmatrix} 0 & 0 & -zx_1 \\ 0 & 0 & -zx_2 \\ -zx_1 & -zx_2 & 0 \end{bmatrix}$.

Thus, we have, $\phi([Z, X]) = [Z, \phi(X)]$, meaning that ϕ is an intertwining map.

To show condition ii is not satisfied, first recall that condition ii states that for any $ad_{\mathfrak{h}}$ intertwining map $\phi : \mathfrak{p}'' \to \mathfrak{p}'$,

$$\phi([X, Y]_{\mathfrak{p}''}) = [X, \phi(Y)] + [\phi(X), Y] \text{ for all } X, Y \in \mathfrak{p}''.$$

Since $[\mathfrak{p}'', \mathfrak{p}''] \subset \mathfrak{h}$, we have that the left hand side is 0. Therefore, to conclude our claim, we need to show that there is an $X, Y \in \mathfrak{p}''$ such that $[\phi(X), Y] + [X, \phi(Y)] \neq 0$.

$$[\phi(X), Y] = \begin{bmatrix} -2x_2y_1 & x_1y_1 - x_2y_2 & 0 \\ x_1y_1 - x_2y_2 & 2x_1y_2 & 0 \\ 0 & 0 & 2x_2y_1 - 2x_1y_2 \end{bmatrix}$$
$$[X, \phi(Y)] = \begin{bmatrix} 2x_1y_2 & -x_1y_1 + x_2y_2 & 0 \\ -x_1y_1 + x_2y_2 & -2x_2y_1 & 0 \\ 0 & 0 & 2x_2y_1 - 2x_1y_2 \end{bmatrix}$$

Observing that $-2x_2y_1 + 2x_1y_2 \neq 0$ for arbitrary x_1, x_2, y_1, y_2, we have proven our claim that $(\mathfrak{sl}(3,\mathbb{R}), \mathfrak{so}(2))$ is not a Cartan orthogonal pair.

Example 2.16. For such a $\mathfrak{g}$ and $\mathfrak{h}$ as described below, $(\mathfrak{g}, \mathfrak{h})$ is not a Cartan orthogonal pair.

Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ where $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{h}_1, \mathfrak{h}_2$ are defined as follows:

$\mathfrak{g}_1 = \mathfrak{so}(3,3)$ with $\mathfrak{h}_1 = \mathfrak{so}(3) \subset \mathfrak{so}(3) \oplus \mathfrak{so}(3) = \mathfrak{k}_1$ where $\mathfrak{h}_1 = \mathfrak{so}(3)$ is one of the ideals in $\mathfrak{k}_1$;

$\mathfrak{g}_2 = \mathfrak{so}(3,3)$ with $\mathfrak{h}_2 = \Delta(\mathfrak{so}(3)) \subset \mathfrak{so}(3) \oplus \mathfrak{so}(3) = \mathfrak{k}_2$.

In this setting, by Proposition 2.7 we know that $(\mathfrak{g}_1, \mathfrak{h}_1)$ is a Cartan orthogonal pair. Moreover, by Example 2.13, we know that $(\mathfrak{g}_2, \mathfrak{h}_2)$ is a Cartan orthogonal pair as well. Consider $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{p}_1'' \oplus \mathfrak{p}_2'' \oplus \mathfrak{p}_1' \oplus \mathfrak{p}_2'$ a reductive Cartan decomposition for $(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{h}_1 \oplus \mathfrak{h}_2)$ where $\mathfrak{p}_1'' = \mathfrak{so}(3)$, the complementary ideal to $\mathfrak{h}_1$ in $\mathfrak{k}_1$.

Claim: $\mathfrak{p}_1'' = \mathfrak{so}(3)$ is a three dimensional trivial $ad_{\mathfrak{h}_1 \oplus \mathfrak{h}_2}$ representation.

Proof of Claim: We will show that $ad_{\mathfrak{h}_1}$ acts trivially on $\mathfrak{p}_1''$ and then show that $ad_{\mathfrak{h}_2}$ acts trivially on $\mathfrak{p}_1''$. First, $ad_{\mathfrak{h}_1}$ acts trivially on $\mathfrak{p}_1''$ because $\mathfrak{p}_1''$ is an ideal of $\mathfrak{h}_1$ in $\mathfrak{k}_1 = \mathfrak{h}_1 \oplus \mathfrak{p}_1''$. Next, to see that $ad_{\mathfrak{h}_2}$ acts trivially on $\mathfrak{p}_1''$, we observe that this is true by $\mathfrak{g}_1$ and $\mathfrak{g}_2$ being ideals in $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\mathfrak{h}_2 \subset \mathfrak{g}_2$ and $\mathfrak{p}_1'' \subset \mathfrak{g}_1$. Thus, we have proven our claim.

Claim: There is a trivial one-dimensional $ad_{\mathfrak{h}_1 \oplus \mathfrak{h}_2}$ representation in $\mathfrak{p}_2' \subset \mathfrak{g}_2$.

Proof of Claim: Looking to Example 2.13, we can see that, inside $\mathfrak{p}_2'$ is a one dimensional trivial $ad_{\mathfrak{h}_2}$ representation. Moreover, by $\mathfrak{g}_1$ and $\mathfrak{g}_2$ being ideals in $\mathfrak{g}$, $ad_{\mathfrak{h}_1}$ acts trivially on all of $\mathfrak{p}'_2$. Thus, we have a one dimensional trivial representation of $ad_{\mathfrak{h}_1 \oplus \mathfrak{h}_2}$ in $\mathfrak{p}'_2$. This proves our claim, and for simplicity, we denote this one dimensional trivial representation by $\mathfrak{q}_0$.

Using the above two claims, we now show that $(\mathfrak{g}, \mathfrak{h})$ is indeed not a Cartan orthogonal pair. First, let $\phi : \mathfrak{p}''_1 \to \mathfrak{q}_0 \subset \mathfrak{p}'_2$ be a non-zero linear map. Since our representations are trivial under $ad_{\mathfrak{h}_1 \oplus \mathfrak{h}_2}$, ϕ is an $ad_{\mathfrak{h}_1 \oplus \mathfrak{h}_2}$ intertwining map. We will check that condition ii in 2.1 fails, causing $(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{h}_1 \oplus \mathfrak{h}_2)$ to not be a Cartan orthogonal pair.

If $x, y \in \mathfrak{p}''_1 \subset \mathfrak{g}_1$ then $\phi(x), \phi(y) \in \mathfrak{g}_2$. Moreover, by $\mathfrak{g}_1, \mathfrak{g}_2$ being ideals in $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, we have $[x, \phi(y)] + [\phi(x), y] = 0$. Since $[\mathfrak{p}''_1, \mathfrak{p}''_1] = \mathfrak{p}''_1$ by $\mathfrak{p}''_1$ being simple (see Section 1.4), we know there are $x, y \in \mathfrak{p}''_1$ such that $\phi([x, y]_{\mathfrak{p}''}) \neq 0$; thus, condition ii of being a Cartan orthogonal pair is not satisfied and $(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{h}_1 \oplus \mathfrak{h}_2)$ is not a Cartan orthogonal pair.

Remark 2.17. It is worth noting that the argument above relies upon the fact that in $\mathfrak{p}''_1$ there is at least a two dimensional trivial representation and that there is a $[x, y]_{\mathfrak{p}''}$ not in the kernel of ϕ. This indicates that some improvement upon Lemma 2.8 is not far out of reach.

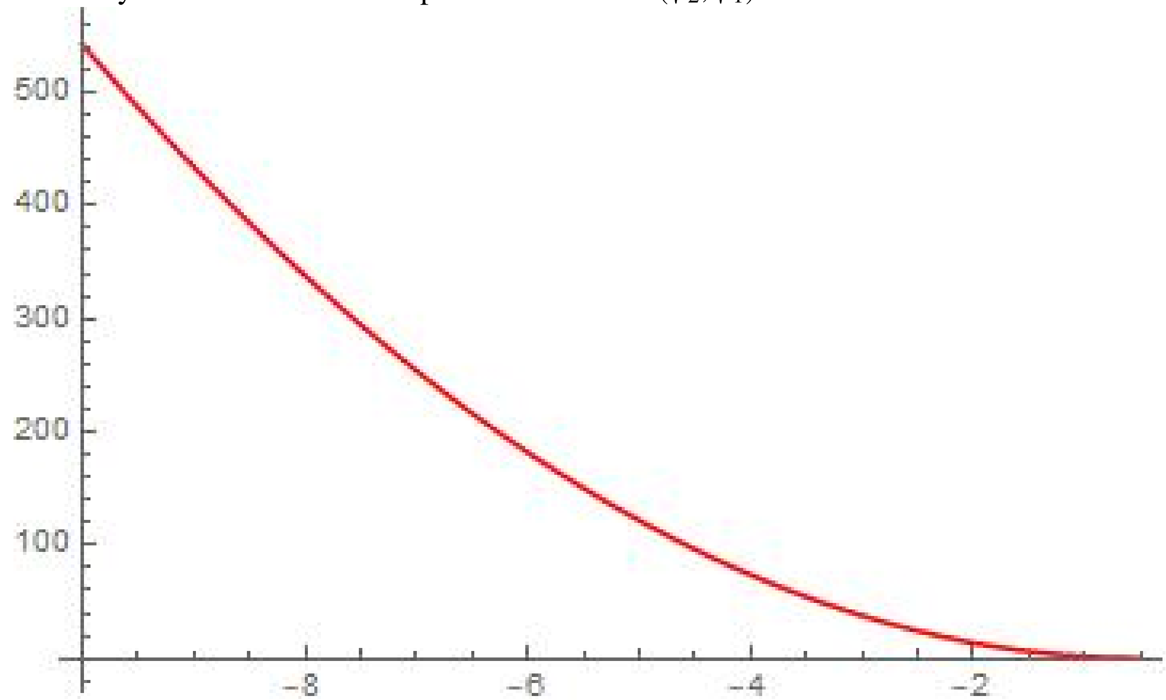
Chapter 3

Two Irreducible Summands

In [DK08] (with corrections in Theorem A.1 of [He12] and Remark 6.1 of [LL22]), there is an investigation into the classification and Ricci curvature for the connected, simply connect, compact, G/H in which G is simple, H is connected, and G/H has two irreducible summands. That is (see Section 1.1), G/H in which the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ has an irreducible decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2$. In [DK08] (and aforementioned corrections), a classification was given, and information regarding the Ricci tensor and Einstein metrics was provided. In the present work, we are in the noncompact setting, and we wish to ask a similar set of questions about the classification and the Ricci tensor.

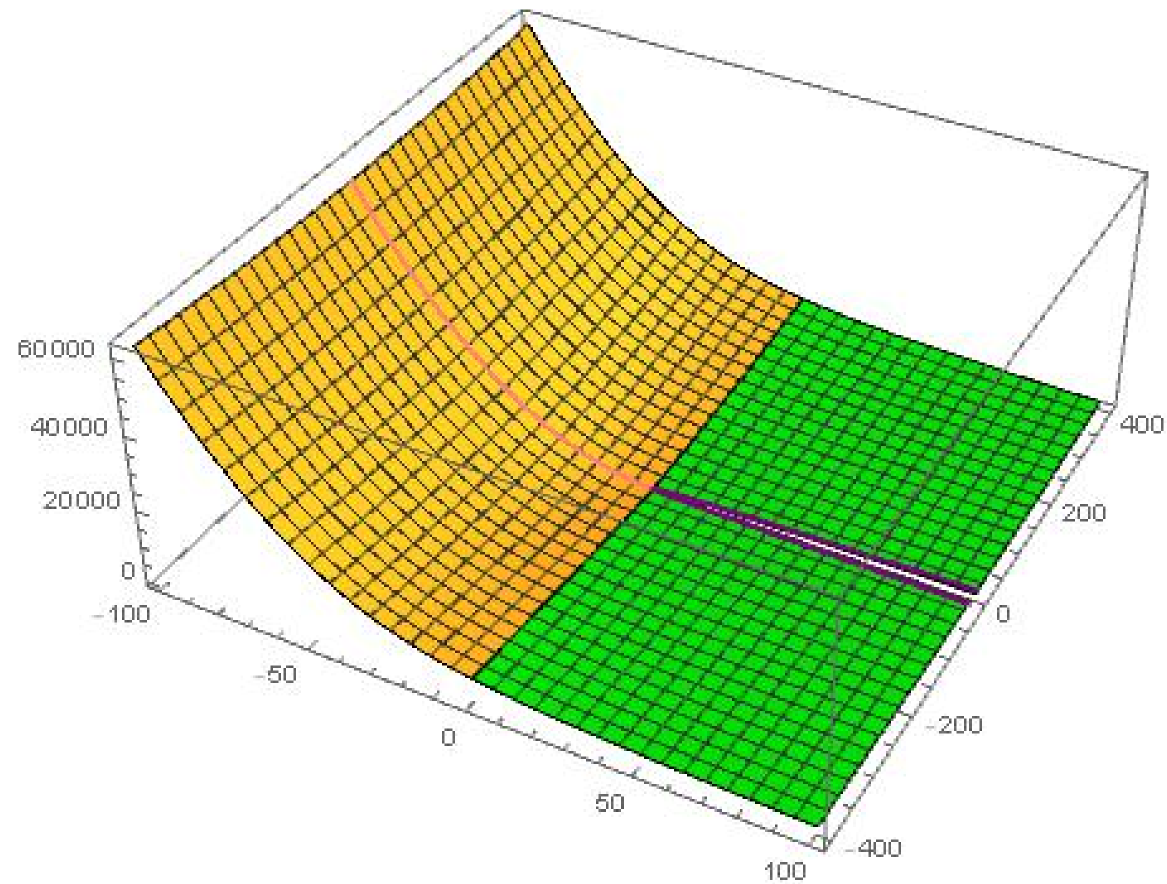
In the first section (3.1), we investigate the classification of connected, noncompact G/H with two isotropy irreducible summands in which $\mathfrak{g}$ is semi-simple (not just simple).

In the second section (3.2), we investigate the Ricci tensor and solve the so-called prescribed Ricci curvature problem (PRP) in the setting of two isotropy irreducible summands in which $\mathfrak{g}$ is noncompact semi-simple with the exception of one space, $SO_0(1,7)/G_2$. For this exceptional case, our results in this section only provide a partial solution since our solution only addresses those inner products in which $(\mathfrak{p}_2, \mathfrak{p}_1) = 0$.



Produced in Mathematica ([Inc]), the above picture provides an example graph of the solutions to $ric = T$ in solving the PRP for inner products in which $(\mathfrak{p}_2, \mathfrak{p}_1) = 0$. The above example is $SO_0(1,7)/G_2$ when we restrict our inner products to those that satisfy $(\mathfrak{p}_2, \mathfrak{p}_1) = 0$

In the third section (3.3), we investigate our exceptional case, $SO_0(1,7)/G_2$, which is unique in that it is the one case in which the two irreducible isotropy summands are isomorphic (see the remark at the end of section 2 of [DK08] and then one can check the dimensions of the corrected spaces from [He12] and [LL22] to see that no isomorphism can exist). Here, we completely solve the prescribed Ricci curvature problem for $SO_0(1,7)/G_2$, and with the results in Section 3.2 we have a complete solution to the prescribed Ricci curvature problem for simply connected, noncompact G/H with G semi-simple, H connected, and G/H having two isotropy irreducible summands.



Produced in Mathematica ([Inc]), the above picture provides a graph of the solutions to $ric = T$ in solving the PRP for $SO_0(1,7)/G_2$.

Regarding the compact case with two irreducible summands, we direct the reader's attention to [GP17] and [BP20]. There, the complete solution to the PRP is not found as they address all but those spaces in which the two irreducible isotropy representations are isomorphic. They do, however, solve the PRP (specifically for solutions to $ric = cT$ which implies the case where $c = 1$) for those spaces in which the two irreducible isotropy representations are not isomorphic and without the restriction of G being semi-simple.

3.1. Classification

The following lemma is included for completion as it is used in Theorem 3.2 regarding the classification of two isotropy irreducible summand spaces.

Lemma 3.1. If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}'$ is a Cartan decomposition of a noncompact simple Lie algebra and $\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{p}'$ is the dual to $\mathfrak{g}$ (see Section 1.5), then $\mathfrak{p}$ and $i\mathfrak{p}$ are isomorphic irreducible $ad_{\mathfrak{k}}$ representations.

Proof: Consider $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k} \oplus \mathfrak{p}' \oplus i\mathfrak{k} \oplus i\mathfrak{p}'$ (see Section 1.4) and restrict the scalars. We may consider the real linear map $i : \mathfrak{p}' \to i\mathfrak{p}'$ and by definition of $\mathfrak{g}^{\mathbb{C}}$, $[x, iv] = i[x, v]$, so we are done. ∎

Using the classification from [DK08] in the compact setting with two isotropy summands, we will classify spaces with two isotropy summands in the noncompact setting with the restriction that G in G/H is semi-simple. In the following, we work with simply connected G/H with G simple and H connected.

Theorem 3.2. Let G/H be simply connected with G a connected semi-simple Lie group with no compact factors and $H \subset G$, a compact, connected subgroup. If G/H has exactly two irreducible representations then G/H is described by one of the following:

1. G and H have Lie algebras $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $\mathfrak{h} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ where $\mathfrak{g}_i$ is noncompact simple and $\mathfrak{k}_i$ is the maximal compact in $\mathfrak{g}_i$. In this case, G/H is a symmetric space.

2. G has a simple Lie algebra $\mathfrak{g}$, and H has Lie algebra $\mathfrak{h} \subsetneq \mathfrak{k}$ where $\mathfrak{k}$ is the maximal compact in $\mathfrak{g}$. A classification of such G/H is determined by the $(\mathfrak{g}, \mathfrak{k}, \mathfrak{h})$ triple belonging to Tables 3.1 through 3.4. In this case, G/H is not a symmetric space.

Proof: Let $\mathfrak{g}$ be noncompact semi-simple and $\mathfrak{k}$ the maximal compact subalgebra of $\mathfrak{g}$. If $\mathfrak{g}$ is simple, then, as discussed in Proposition 1.11, G/K is an irreducible symmetric space, and the reductive decomposition of $\mathfrak{g}$ is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}'$ with $\mathfrak{p}'$ being an irreducible $ad_{\mathfrak{k}}$ invariant complement to $\mathfrak{k}$ in $\mathfrak{g}$. Therefore, if $\mathfrak{g}$ is simple, we will have to consider an $\mathfrak{h} \subsetneq \mathfrak{k}$, a case we will turn to after resolving the case in which $\mathfrak{g}$ is semi-simple and not simple.

If $\mathfrak{g}$ is not simple, then G/K has a decomposition into irreducible factors coming from the DeRham decomposition discussed in Section 1.2. Thus, the reductive decomposition of $\mathfrak{g}$ is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{k}_1 \oplus ... \oplus \mathfrak{k}_n \oplus \mathfrak{p}_1 \oplus ... \oplus \mathfrak{p}_n$ in which $\mathfrak{p}_i$ is the irreducible $ad_{\mathfrak{k}_i}$ invariant complement of $\mathfrak{k}_i$ in $\mathfrak{g}_i$ and $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$. Therefore, if $\mathfrak{g}$ is semi-simple and not simple and if we consider the spaces G/H in which the isotropy representation has exactly two irreducible summands, we must restrict to $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ where $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are simple ideals in $\mathfrak{g}$. Moreover, we must have $\mathfrak{h} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ where $\mathfrak{k}_1, \mathfrak{k}_2$ are the maximal compacts in $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively. Indeed, if $\mathfrak{h} \subsetneq \mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$, the maximal compact in $\mathfrak{g}$, then the isotropy representation would have a (not necessarily irreducible) decomposition, $\mathfrak{p} = \mathfrak{q} \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2$ in which $\mathfrak{q}$ is an $ad_{\mathfrak{h}}$ invariant complement of $\mathfrak{h}$ inside $\mathfrak{k}$, giving us more than two irreducible summands. Thus, a complete description of spaces G/H in which G is semi-simple but not simple is given by $G_1/K_1 \times G_2/K_2$ in which $\mathfrak{g}_i$ are noncompact simple and $\mathfrak{k}_i$ is the maximal compact inside $\mathfrak{g}_i$. Such spaces are symmetric spaces.

Now, let $\mathfrak{g}$ be noncompact simple and consider the spaces G/H in which there are exactly two irreducible summands in the isotropy representation. As shown above, we must choose $\mathfrak{h} \subsetneq \mathfrak{k}$ to obtain more than one irreducible summand. Since our reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ has a decomposition of $\mathfrak{p}$ into irreducibles that is unique up to isomorphism, we may choose a decomposition of $\mathfrak{p}$ that is covenient for us. Therefore, for $\mathfrak{h} \subsetneq \mathfrak{k}$, we choose a reductive Cartan decomposition for $(\mathfrak{g}, \mathfrak{h})$ given by $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$ where $\mathfrak{h} \oplus \mathfrak{p}'' = \mathfrak{k}$

and $\mathfrak{p}'$ is the irreducible $ad_{\mathfrak{k}}$ representation from the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}'$ (See Definition 1.19). In this case, $\mathfrak{p} = \mathfrak{p}'' \oplus \mathfrak{p}'$, so to have exactly two irreducible summands, the $ad_{\mathfrak{h}}$ representations $\mathfrak{p}'$ and $\mathfrak{p}''$ must be irreducible. This then allows us to restrict ourselves to the case when $\mathfrak{g}$ is not the realification of a complex simple Lie algebra (see Section 1.4) because otherwise, $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$ (see Section 1.5) and we get at least three irreducible summands for the isotropy action, as seen from the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus i\mathfrak{h} \oplus i\mathfrak{p}''$.

We now use the duality of symmetric spaces as discussed in Section 1.5. We know that if we have a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}'$ then $\mathfrak{g}$ has a dual, $\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{p}'$ and $\mathfrak{g}^*$ is compact with G^*/K a compact irreducible symmetric space. Moreover, we know that $\mathfrak{p}'$ and $i\mathfrak{p}'$ are isomorphic $ad_{\mathfrak{k}}$ representations by Proposition 3.1. Using this isomorphism of representations, we can pass from $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$ to $\mathfrak{g}^* = \mathfrak{h} \oplus \mathfrak{p}'' \oplus i\mathfrak{p}'$ and vice versa. Therefore, for $\mathfrak{g}$ simple, if G/H is noncompact with two irreducible summands, we have $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$ and we will find a corresponding compact G^*/H with two irreducible summands $\mathfrak{g}^* = \mathfrak{h} \oplus \mathfrak{p}'' \oplus i\mathfrak{p}'$ with $\mathfrak{g}^*$ compact simple. Using the complete list of compact G^*/H with two irreducible summands in which G^* is connected and simple given by Dickenson and Kerr in [DK08] (with corrections in Appendix A of [He12] and Remark 6.1 of [LL22]), we can then get a complete list of noncompact G/H with two irreducible summands. We now turn to how we can get that list.

By observing the work in [DK08], we note that the list in the noncompact setting will be smaller since in the compact setting one can have G^*/H with H maximal in G^* but still have two irreducible summands. In this case, H will not be inside a $K \subset G^*$ such that G^*/K is a compact irreducible symmetric space, so we could not obtain G^*/H from any noncompact G/H with two irreducible summands using the duality. Instead, we must restrict ourselves in the compact setting to the case in which there is an intermediate subgroup $H \subset K \subset G^*$. Furthermore, since there are $H \subset K \subset G$ such that K is maximal in G^* but G^*/K is isotropy irreducible and not symmetric (see tables 5 and 6 in Chapter 7 of [Bes87] for these), we must further restrict ourselves to those $H \subset K \subset G^*$ in which G^*/K are irreducible symmetric spaces (which can be checked by the help of Chapter X of [Hel01]). We can then achieve our own list in the noncompact setting by the following procedure similar to the procedure used in Section 3 of [AL17]:

a. In the compact setting, select $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}^*$ in which $(\mathfrak{g}^*, \mathfrak{k})$ is a pair associated with a compact irreducible symmetric space and G^*/H has two irreducible summands.

b. Use the duality to achieve $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$ such that G/H has exactly two irreducible summands in the noncompact setting.

A check of the lists in [DK08] (and corrections in [He12] and [LL22]) shows that there are compact G^*/H in which $H \subset K \subset G^*$ and G^*/K is a compact irreducible symmetric space, but there are also some spaces in which G^*/K is isotropy irreducible, but not symmetric. We wish to keep the former to use our duality, and ignore the latter. Thus, we can get our complete list in the noncompact setting by dualizing (in the sense of **b** above) each G^*/H in the lists given in [DK08], [He12], and [LL22] while ignoring the following items in the list: I.20, I.21, I.22, I.23, I.29, III.9, III.10, III.11, IV.3, IV.6, IV.13, IV.18, IV.30, IV.31, IV.32, IV.33, IV. 41, IV.42, IV.43, IV.44. ∎

Table 3.1

$(\mathfrak{g}, \mathfrak{k}, \mathfrak{h})$	Constraint	Label in [DK08]
$(\mathfrak{su}(\frac{n(n-1)}{2}, m), \mathfrak{su}(\frac{n(n-1)}{2}) \oplus \mathfrak{su}(m) \oplus \mathbb{R}, \mathfrak{su}(n) \oplus \mathfrak{su}(m) \oplus \mathbb{R})$	$n \geq 5$	II.1
$(\mathfrak{su}(\frac{n(n+1)}{2}, m), \mathfrak{su}(\frac{n(n+1)}{2}) \oplus \mathfrak{su}(m) \oplus \mathbb{R}, \mathfrak{su}(n) \oplus \mathfrak{su}(m) \oplus \mathbb{R})$	$n \geq 2$	II.2
$(\mathfrak{su}(27, m), \mathfrak{su}(27) \oplus \mathfrak{su}(m) \oplus \mathbb{R}, \mathfrak{e}_6 \oplus \mathfrak{su}(m) \oplus \mathbb{R})$	NA	II.3
$(\mathfrak{su}(16, m), \mathfrak{su}(16) \oplus \mathfrak{su}(m) \oplus \mathbb{R}, \mathfrak{so}(10) \oplus \mathfrak{su}(m) \oplus \mathbb{R})$	NA	II.4
$(\mathfrak{su}(pq, m), \mathfrak{su}(pq) \oplus \mathfrak{su}(m) \oplus \mathbb{R}, \mathfrak{su}(p) \oplus \mathfrak{su}(q) \oplus \mathfrak{u}(1) \oplus \mathfrak{su}(m))$	$p, q \geq 2$	II.5
$(\mathfrak{su}(n, m), \mathfrak{su}(n) \oplus \mathfrak{su}(m) \oplus \mathbb{R}, \mathfrak{so}(n) \oplus \mathfrak{u}(1) \oplus \mathfrak{su}(m))$	$n \geq 3$	II.6
$(\mathfrak{su}(n, m), \mathfrak{su}(n) \oplus \mathfrak{su}(m) \oplus \mathbb{R}, \mathfrak{su}(n) \oplus \mathfrak{su}(m))$	$m, n \geq 2$	II.7
$(\mathfrak{su}(2n, m), \mathfrak{su}(2n) \oplus \mathfrak{su}(m) \oplus \mathbb{R}, \mathfrak{sp}(n) \oplus \mathfrak{su}(m) \oplus \mathbb{R})$	$n \geq 2$	II.8
$(\mathfrak{su}^*(14), \mathfrak{sp}(7), \mathfrak{sp}(3))$	NA	II.9
$(\mathfrak{su}^*(32), \mathfrak{sp}(16), \mathfrak{so}(12))$	NA	II.10
$(\mathfrak{su}^*(56), \mathfrak{sp}(28), \mathfrak{e}_7)$	NA	II.11
$(\mathfrak{su}^*(4), \mathfrak{sp}(2), \mathfrak{su}(2))$	$n \geq 5$	II.1
$(\mathfrak{su}^*(14), \mathfrak{sp}(7), \mathfrak{sp}(3))$	NA	II.9
$(\mathfrak{su}^*(32), \mathfrak{sp}(16), \mathfrak{so}(12))$	NA	II.10
$(\mathfrak{su}^*(56), \mathfrak{sp}(28), \mathfrak{e}_7)$	NA	II.11
$(\mathfrak{su}^*(4), \mathfrak{sp}(2), \mathfrak{su}(2)))$	NA	II.12
$(\mathfrak{su}^*(20), \mathfrak{sp}(10), \mathfrak{su}(6))$	NA	[He12] II.15
$(\mathfrak{sp}(m, 2), \mathfrak{sp}(m) \oplus \mathfrak{sp}(2), \mathfrak{sp}(m) \oplus \mathfrak{su}(2))$	NA	III.1
$(\mathfrak{sp}(m, 7), \mathfrak{sp}(m) \oplus \mathfrak{sp}(7), \mathfrak{sp}(m) \oplus \mathfrak{sp}(3))$	NA	III.2
$(\mathfrak{sp}(m, 10), \mathfrak{sp}(m) \oplus \mathfrak{sp}(10), \mathfrak{sp}(m) \oplus \mathfrak{su}(6))$	NA	III.3
$(\mathfrak{sp}(m, 16), \mathfrak{sp}(m) \oplus \mathfrak{sp}(16), \mathfrak{sp}(m) \oplus \mathfrak{so}(12))$	NA	III.4
$(\mathfrak{sp}(m, 28), \mathfrak{sp}(m) \oplus \mathfrak{sp}(28), \mathfrak{sp}(m) \oplus \mathfrak{e}_7)$	NA	III.5
$(\mathfrak{sp}(m, n), \mathfrak{sp}(m) \oplus \mathfrak{sp}(n), \mathfrak{sp}(m) \oplus \mathfrak{su}(n) \oplus \mathbb{R})$	NA	III.6
$(\mathfrak{sp}(m, n), \mathfrak{sp}(m) \oplus \mathfrak{sp}(n), \mathfrak{sp}(m) \oplus \mathfrak{so}(n)\mathfrak{sp}(1))$	$n \geq 3$	III.7

Table 3.2

$(\mathfrak{g},\mathfrak{k},\mathfrak{h})$	Constraint	Label in [DK08]
$(\mathfrak{sp}(n,\mathbb{R}),\mathfrak{su}(n)\oplus\mathbb{R},\mathfrak{su}(n))$	$k\geq 3$	III.8
$(\mathfrak{sp}(2m,\mathbb{R}),\mathfrak{su}(2m)\oplus\mathbb{R},\mathfrak{sp}(m)\oplus\mathfrak{u}(1))$	$m\geq 2$	[He12] III.12
$(\mathfrak{g}_2(2),\mathfrak{so}(4),\mathfrak{u}(2))$	$U(2)_3\not\subset SU(3)$	IV.1
$(\mathfrak{f}_4^{(4)},\mathfrak{sp}(3)\oplus\mathfrak{sp}(1),\mathfrak{sp}(3)\oplus\mathfrak{u}(1))$	NA	IV.2
$(\mathfrak{f}_4^{(-20)},\mathfrak{so}(9),\mathfrak{so}(7)\oplus\mathfrak{so}(2))$	NA	IV.4
$(\mathfrak{f}_4^{(-20)},\mathfrak{so}(9),\mathfrak{so}(6)\oplus\mathfrak{so}(3))$	NA	IV.5
$(\mathfrak{e}_6^{(-14)},\mathfrak{so}(10)\oplus\mathfrak{so}(2),\mathfrak{so}(10))$	NA	IV.7
$(\mathfrak{e}_6^{(-14)},\mathfrak{so}(10)\oplus\mathfrak{so}(2),\mathfrak{so}(9)\oplus\mathfrak{so}(2))$	NA	IV.8
$(\mathfrak{e}_6^{(-14)},\mathfrak{so}(10)\oplus\mathfrak{so}(2),\mathfrak{so}(7)\oplus\mathfrak{so}(3)\oplus\mathfrak{so}(2))$	NA	IV.9
$(\mathfrak{e}_6^{(-14)},\mathfrak{so}(10)\oplus\mathfrak{so}(2),\mathfrak{so}(5)\oplus\mathfrak{so}(5)\oplus\mathfrak{so}(2))$	NA	IV.11
$(\mathfrak{e}_6^{(-14)},\mathfrak{so}(10)\oplus\mathfrak{so}(2),\mathfrak{sp}(2)\oplus\mathfrak{so}(2))$	NA	IV.12
$(\mathfrak{e}_6^{(2)},\mathfrak{su}(6)\oplus\mathfrak{su}(2),\mathfrak{su}(6)\oplus\mathfrak{u}(1))$	NA	IV.14
$(\mathfrak{e}_6^{(2)},\mathfrak{su}(6)\oplus\mathfrak{su}(2),\mathfrak{su}(5)\oplus\mathbb{R}\oplus\mathfrak{su}(2))$	NA	IV.15
$(\mathfrak{e}_6^{(2)},\mathfrak{su}(6)\oplus\mathfrak{su}(2),\mathfrak{so}(6)\oplus\mathfrak{su}(2))$	NA	IV.16
$(\mathfrak{e}_6^{(2)},\mathfrak{su}(6)\oplus\mathfrak{su}(2),\mathfrak{su}(3)\oplus\mathfrak{su}(2))$	NA	IV.17
$(\mathfrak{e}_7^{(-25)},\mathfrak{e}_6\oplus\mathfrak{so}(2),\mathfrak{e}_6)$	NA	IV.19
$(\mathfrak{e}_7^{(-25)},\mathfrak{e}_6\oplus\mathfrak{so}(2),\mathfrak{sp}(4)\oplus\mathfrak{so}(2))$	NA	IV.20
$(\mathfrak{e}_7^{(-25)},\mathfrak{e}_6\oplus\mathfrak{so}(2),\mathfrak{g}_2\oplus\mathfrak{so}(2))$	NA	IV.21
$(\mathfrak{e}_7^{(-25)},\mathfrak{e}_6\oplus\mathfrak{so}(2),su(3)\oplus\mathfrak{so}(2))$	NA	IV.22
$(\mathfrak{e}_7^{(7)},\mathfrak{su}(8),\mathfrak{su}(7)\oplus\mathbb{R})$	NA	IV.23
$(\mathfrak{e}_7^{(-5)},\mathfrak{so}(12)\oplus\mathfrak{sp}(1),\mathfrak{so}(12)\oplus\mathfrak{u}(1))$	NA	IV.24
$(\mathfrak{e}_7^{(-5)},\mathfrak{so}(12)\oplus\mathfrak{sp}(1),\mathfrak{so}(11)\oplus\mathfrak{sp}(1))$	NA	IV.25
$(\mathfrak{e}_7^{(-5)},\mathfrak{so}(12)\oplus\mathfrak{sp}(1),\mathfrak{so}(10)\oplus\mathfrak{so}(2)\oplus\mathfrak{sp}(1))$	NA	IV.26
$(\mathfrak{e}_7^{(-5)},\mathfrak{so}(12)\oplus\mathfrak{sp}(1),\mathfrak{so}(9)\oplus\mathfrak{so}(3)\oplus\mathfrak{sp}(1))$	NA	IV.27
$(\mathfrak{e}_7^{(-5)},\mathfrak{so}(12)\oplus\mathfrak{sp}(1),\mathfrak{sp}(7)\oplus\mathfrak{sp}(5)\oplus\mathfrak{sp}(1))$	NA	IV. 28
$(\mathfrak{e}_7^{(-5)},\mathfrak{so}(12)\oplus\mathfrak{sp}(1),\mathfrak{so}(6)\oplus\mathfrak{so}(6)\oplus\mathfrak{sp}(1))$	NA	IV.29

Table 3.3

$(\mathfrak{g}, \mathfrak{k}, \mathfrak{h})$	Constraint	Label in [DK08]
$(\mathfrak{e}_8^{(8)}, \mathfrak{so}(16), \mathfrak{so}(15))$	NA	IV.34
$(\mathfrak{e}_8^{(8)}, \mathfrak{so}(16), \mathfrak{so}(15) \oplus \mathfrak{so}(2))$	NA	IV.35
$(\mathfrak{e}_8^{(8)}, \mathfrak{so}(16), \mathfrak{so}(13) \oplus \mathfrak{so}(3))$	NA	IV.36
$(\mathfrak{e}_8^{(8)}, \mathfrak{so}(16), \mathfrak{so}(11) \oplus \mathfrak{so}(5))$	NA	IV.38
$(\mathfrak{e}_8^{(8)}, \mathfrak{so}(16), \mathfrak{so}(10) \oplus \mathfrak{so}(6))$	NA	IV.39
$(\mathfrak{e}_8^{(8)}, \mathfrak{so}(16), \mathfrak{so}(9) \oplus \mathfrak{so}(7))$	NA	IV.40
$(\mathfrak{e}_8^{(8)}, \mathfrak{so}(16), \mathfrak{so}(9))$	NA	[LL22]
$(\mathfrak{e}_8^{(-24)}, \mathfrak{sp}(1) \oplus \mathfrak{e}_7, \mathfrak{u}(1) \oplus \mathfrak{e}_7)$	NA	IV.45
$(\mathfrak{e}_8^{(-24)}, \mathfrak{sp}(1) \oplus \mathfrak{e}_7, \mathfrak{sp}(1) \oplus \mathfrak{su}(8))$	NA	IV.46
$(\mathfrak{e}_8^{(-24)}, \mathfrak{sp}(1) \oplus \mathfrak{e}_7, \mathfrak{sp}(1) \oplus \mathfrak{su}(3))$	NA	IV.47

Remark 3.3. As remarked at the end of section 2 of [DK08] (and can be easily be checked with the corrections in [He12] and [LL22]), the only G^*/H with $\mathfrak{g}^* = \mathfrak{h} \oplus \mathfrak{p}'' \oplus i\mathfrak{p}'$ in which $i\mathfrak{p}' \simeq \mathfrak{p}''$ is $SO(8)/G_2$. Thus, in the noncompact setting, we similarly have that the only $(\mathfrak{g}, \mathfrak{k}, \mathfrak{h})$ triple in which $\mathfrak{p}'' \simeq \mathfrak{p}'$ is $(\mathfrak{so}(1,7), \mathfrak{so}(7), \mathfrak{g}_2)$. Thus, if we want to restrict ourselves to the spaces in which G/H has two irreducible summands and $(\mathfrak{g}, \mathfrak{h})$ is a Cartan orthogonal pair (See Defintion 1.20), the only space potentially giving us problems is $SO_0(1,7)/G_2$ where $SO_0(1,7)$ is the connected component containing the identity in $SO(1,7)$. Inequivalence of representations is not necessary for $(\mathfrak{so}(1,7), \mathfrak{g}_2)$ to be a Cartan orthogonal pair. However, if an equivalence is present, then since $\mathfrak{p}''$ is irreducible, condition ii for being a Cartan orthogonal pair (See 2.1) requires any intertwining map $\phi : \mathfrak{p}'' \to \mathfrak{p}'$ to have $[\phi(x), \phi(y)] \subset \mathfrak{h}$. This condition cannot be met since $[\mathfrak{p}', \mathfrak{p}'] = \mathfrak{k}$ in this case by the Cartan decomposition properties when $\mathfrak{g}$ is simple (See 1.3). Therefore, if we wish to only consider inner prodcuts in which $(\mathfrak{p}', \mathfrak{p}'') = 0$ for some Cartan decomposition, we must necessarily restrict the space of inner products in the case of $SO_0(1,7)/G_2$, but not for any other G/H coming from the triple $(\mathfrak{g}, \mathfrak{k}, \mathfrak{h})$ in Tables 3.1 through 3.4. In Section 3.3 of this chapter, we consider the case of $SO_0(1,7)/G_2$ more thoroughly without this assumption, but the results in Section 3.2 below apply to all the other spaces in Table 3.1 through 3.4. However, we may include $SO_0(1,7)/G_2$ in the results in Section 3.2 if we restrict our inner products to those in which $(\mathfrak{p}'', \mathfrak{p}') = 0$ for some Cartan decomposition.

3.2. Prescribed Ricci Curvature

The prescribed Ricci curvature problem (PRP) on a homogeneous space G/H is an investigation into a complete description of the $(0, 2)$ Ricci curvature tensor (See Section 1.1 for comments on the tensor type of Ricci curvature) $ric_g(., .)$ for a G-invariant metric g on T_eG/H. There are two components to this: one in which we ask about the image of $ric_g(., .)$ and the other in which we ask about the image of $ric_g(., .)$ up to scaling. As we have consistently done (See Section 1.2), the PRP investigation is done at the Lie algebra level where we consider the $(0, 2)$ Ricci tensor on $\mathfrak{p}$ which comes from a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ for G/H. At the Lie algebra level then, the PRP is asking the two following questions:

1. For what $(0, 2)$ $ad_{\mathfrak{h}}$ invariant tensors $T(., .)$ on $\mathfrak{p}$ is there a an $ad_{\mathfrak{h}}$ invariant inner product $(., .)$ such that $ric_{(.,.)}(., .) = T(., .)$?

2. For what $(0, 2)$ tensors $T(., .)$ on $\mathfrak{p}$ is there a $c > 0$ such that we have an $ad_{\mathfrak{h}}$ invariant inner product $(., .)$ satisfying $ric_{(.,.)}(., .) = cT$?

In particular, the main goal is to find sufficient and necessary conditions on $T(., .)$ such that $ric_{(.,.)}(., .) = T(.,)$ for some $ad_{\mathfrak{h}}$ invariant $(., .)$ for the first question, and sufficient and necessary conditions on $T(., .)$ such that $ric_{(.,.)}(., .) = cT$ for some $c > 0$ and some $ad_{\mathfrak{h}}$ invariant $(., .)$ for the second question. In this section, we turn our attention to this problem in our context of interest: noncompact G/H in which $\mathfrak{g}$ is semi-simple and there are two irreducible summands.

Remark 3.4. We restrict ourselves to $c > 0$ since if $c < 0$, all we must do is consider the solutions to $ric(., .) = cT(., .)$ for $c > 0$ and negate $T(., .)$.

Remark 3.5. For this problem, it will simplify matters if we change notation for $\mathfrak{p}''$ and

$\mathfrak{p}'$. Thus, we let $\mathfrak{p}_1 = \mathfrak{p}'$ and $\mathfrak{p}_2 = \mathfrak{p}''$. Moreover, when we use $\langle .,. \rangle$, we are using the fixed inner product $\langle .,. \rangle = B_{\mathfrak{p}'} - B_{\mathfrak{p}''}$ as mentioned in Remark 1.23.

Remark 3.6. Unless greater specificity is required, we will drop the $(.,.)$ subscript on the Ricci tensor and just write $ric(.,.)$ or ric to refer to the general $(0,2)$ Ricci tensor. Similarly, we will frequently write just T instead of $T(.,.)$.

Lemma 3.7. For G/H with G semi-simple noncompact and G/H having two irreducible summands, every $(0,2)$ $ad_{\mathfrak{h}}$ invariant tensor T on the isotropy representation $\mathfrak{p} = \mathfrak{p}_2 \oplus \mathfrak{p}_1$ is of the form $T = t_1\langle .,. \rangle_1 + t_2\langle .,. \rangle_2$ with the exception of $SO_0(1,7)/G_2$.

Proof: Since $T(.,.)$ is $ad_{\mathfrak{h}}$ invariant, and since $\langle .,. \rangle$ is $ad_{\mathfrak{h}}$ invariant, we know that in general, $T(x,y) = \langle \Phi x, y \rangle$ where Φ is symmetric and an $ad_{\mathfrak{h}}$ equivariant map on $\mathfrak{p}$. To prove the desired result, all we need to show is that Φ is necessarily diagonal. That is, we need to show that any $ad_{\mathfrak{h}}$ intertwining map $\phi : \mathfrak{p}_1 \to \mathfrak{p}_2$ is 0.
If $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_1$ is simple, then, as noted in Remark 3.3, since $\mathfrak{p}_2 \not\simeq \mathfrak{p}_1$ we have, by Schur's Lemma (see Section 1.3), that all $ad_{\mathfrak{h}}$ intertwining maps $\mathfrak{p}_1 \to \mathfrak{p}_2$ are 0, providing the desired result. Similarly, when we have $\mathfrak{g}$ semi-simple, by Theorem 3.2, we know $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\mathfrak{h} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $G/H = G_1/K_1 \times G_2/K_2$. In this case, $ad_{\mathfrak{k}_i}$ acts irreducibly on $\mathfrak{p}_i$ and trivially on $\mathfrak{p}_j$ $(j \neq i)$ as seen in Section 1.4. This implies that no non-trivial, $ad_{\mathfrak{k}_1 \oplus \mathfrak{k}_2}$ intertwining maps $\mathfrak{p}_2 \to \mathfrak{p}_1$ exists, giving us the desired result. ■

Remark 3.8. In the following theorems and corollaries regarding solutions to $ric = T$ and $ric = cT$, we assume that $\mathfrak{g}$ is noncompact semi-simple and G/H has two irreducible summands. By Lemma 3.7 we have $T = t_1\langle .,. \rangle_1 + t_2\langle .,. \rangle_2$ and an arbitrary $ad_{\mathfrak{h}}$ invariant inner product is of the form $(.,.) = x_1\langle .,. \rangle_1 + x_2\langle .,. \rangle_2$ with $x_1, x_2 > 0$. If $\mathfrak{g}$ is not simple, then recall by Theorem 3.2 that $G/H = G_1/K_1 \times G_2/K_2$, a product of two irreducible symmetric spaces. If $\mathfrak{g}$ is simple, then we work with $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_1$ where the provided decomposition is a reductive Cartan decomposition.

Theorem 3.9. If $\mathfrak{g}$ is not simple then the only T such that $ric = T$ is $T = ric_{\langle .,. \rangle}$ where $\langle .,. \rangle$ is our fixed inner product. Moreover, our $T < 0$ in this case.

Proof: Here, by the deRahm decomposition for symmetric spaces (See Section 1.4) we have $ric = ric_1 + ric_2$ where ric_1 , ric_2 are the Ricci tensors for G_1/K_1 , G_2/K_2, respectively. Now, by G_i/K_i being a noncompact irreducible symmetric space, we have that $ric_{\langle .,. \rangle_i} = \lambda_i \langle .,. \rangle_i$ for some $\lambda_i < 0$ for $i = 1, 2$ (See Section 1.3). Moreover, $ric_{c\langle .,. \rangle_i} = ric_{\langle .,. \rangle_i}$ (See Section 1.1), and by Schur's Lemma (See Section 1.3), $\alpha\langle .,. \rangle_i$ exhausts all $ad_{\mathfrak{k}_i}$ inner products on $\mathfrak{p}_i$ where $\alpha > 0$. Thus, for an arbitrary $ad_{\mathfrak{h}}$ invariant inner product $(.,.)$, $ric_{(.,.)} = \lambda_1\langle .,. \rangle_1 + \lambda_2\langle .,. \rangle_2 = ric_{\langle .,. \rangle}$. Therefore, $ric = T$ if and only if $T = ric_{\langle .,. \rangle}$, and so $T < 0$ by $\lambda_1, \lambda_2 < 0$. ■

Theorem 3.10. If $\mathfrak{g}$ is not simple then $ric = cT$ for $c > 0$ if and only if T is a scalar multiple of $ric_{\langle .,. \rangle}$ where $\langle .,. \rangle$ is our fixed inner product, and $c = \frac{T(x,x)}{ric_{\langle .,. \rangle}(x,x)}$ for some $x \in \mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$. Moreover, our $T < 0$ in this case.

Proof: The proof of Theorem 3.9 allows us to easily see the solutions to $ric = cT$. Since ric is given by $ric_{(.,.)} = ric_{\langle .,. \rangle} < 0$ for all $ad_{\mathfrak{h}}$ invariant inner products, $(.,.)$, we have a solution to $ric = cT$ for $c > 0$ if and only if T has $T < 0$ and is a scalar multiple of $ric_{\langle .,. \rangle}$ with $c = \frac{T(x,x)}{ric_{\langle .,. \rangle}(x,x)}$ for some $x \in \mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$. ■

Considering the PRP in the case where $\mathfrak{g}$ is simple, the situation becomes more complex. One complexity is (as has been mentioned) that $\mathfrak{g} = \mathfrak{so}(1, 7)$ must be handled differently in order to solve the PRP in its entirety (and will be handled in Section 3.3). Another complexity is how our formula for ric can vary rather drastically with the bracket relation on $\mathfrak{p}_2 = \mathfrak{p}''$. Due to these variations, we consider the solutions to $ric = T$ and $ric = cT$ in three different settings: first with $\mathfrak{p}_2$ such that $[\mathfrak{p}_2, \mathfrak{p}_2] \not\subset \mathfrak{h}$, second with $\mathfrak{p}_2$ such that $[\mathfrak{p}_2, \mathfrak{p}_2] \subset \mathfrak{h}$, and then third with $\mathfrak{p}_2$ being a trivial $ad_{\mathfrak{h}}$ representation. It turns out that the solutions in the second and third situations follow easily from the first and can be thought of as specialized situations of the first one. Thus, we provide results for $ric = T$ and $ric = cT$ in the setting of $[\mathfrak{p}_2, \mathfrak{p}_2] \not\subset \mathfrak{h}$, in Theorem 3.13 and Theorem 3.14, respectively, and then we provide corollaries describing the $[\mathfrak{p}_2, \mathfrak{p}_2] \subset \mathfrak{h}$ and trivial representation settings.

Before providing solutions, though, we first provide formulas describing an arbitrary $(0, 2)$ Ricci tensor in terms of an arbitrary $ad_{\mathfrak{h}}$ inner product and Lie algebra data.

Lemma 3.11. Let G/H being a noncompact space with two irreducible summands in which $\mathfrak{g}$ is simple with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_1$, as mentioned in Remark 3.8. In such a setting we have the following formulas for R_1, R_2 defining $ric(.,.) = R_1\langle ., .\rangle_1 + R_2\langle ., .\rangle_2$:

$$R_1 = \frac{-1}{2} - \frac{p_1}{2d_1}\frac{x_2}{x_1} < 0$$
$$R_2 = \frac{2d_2 - p_2 - 2p_1}{4d_2} + \frac{p_1}{4d_2}\left(\frac{x_2}{x_1}\right)^2 > 0.$$

Here, $d_1 = dim(\mathfrak{p}_1)$, $d_2 = dim(\mathfrak{p}_2)$, $p_1 = \sum_{\alpha,\beta,\gamma}\langle[e_1^\alpha, e_2^\beta], e_1^\gamma\rangle^2$, and $p_2 = \sum_{\alpha,\beta,\gamma}\langle[e_2^\alpha, e_2^\beta], e_2^\gamma\rangle^2$ where $\{e_i^\alpha\}$ is the notation coming from Eqn.1.4 for an orthonormal basis with respect to $\langle ., .\rangle$, our fixed metric on $\mathfrak{p} = \mathfrak{p}_2 \oplus \mathfrak{p}_1$. Recall that $\{e_1^\alpha\}$ and $\{e_2^\alpha\}$ are understood to be orthonormal bases on $\mathfrak{p}_1$ and $\mathfrak{p}_2$ respectively.

Proof: Following the set up by Nikonorov in [Nik00] as discussed in Section 1.7, we have an arbitrary $ad_{\mathfrak{h}}$ invariant inner product $(., .) = x_1\langle ., .\rangle_1 + x_2\langle ., .\rangle_2$ for $x_i > 0$ and the Ricci tensor for that inner product, $ric = r_1(., .)_1 + r_2(., .)_2$ where $r_1, r_2 \in \mathbb{R}$. From Eqn.1.4

we have:

$$r_1 = \frac{-1}{2x_1} + \frac{1}{4d_1} \sum_{1 \le j,k \le 2} (\sum_{\alpha,\beta,\gamma} \langle [e_1^\alpha, e_j^\beta], e_k^\gamma \rangle^2)(\frac{x_1}{x_j x_k} - \frac{x_k}{x_1 x_j} - \frac{x_j}{x_1 x_k})$$

$$r_2 = \frac{1}{2x_2} + \frac{1}{4d_2} \sum_{1 \le j,k \le 2} (\sum_{\alpha,\beta,\gamma} \langle [e_2^\alpha, e_j^\beta], e_k^\gamma \rangle^2)(\frac{x_2}{x_j x_k} - \frac{x_k}{x_2 x_j} - \frac{x_j}{x_2 x_k}).$$

The first step will be to generate simplified formulas for r_1 and r_2 in our setting dependent upon x_1 and x_2 (along with terms p_1 and p_2 dependent upon the bracket of $\mathfrak{g}$ on $\mathfrak{p}$). From there, we will determine our R_1 and R_2.

By the Cartan decomposition properties (1.3) and the (skew) symmetry of ad_{e_i} (Lemma 1.24), we get that in r_1, our term $\langle [e_1^\alpha, e_j^\beta], e_k^\gamma \rangle^2$ has:

$$\langle [e_1^\alpha, e_2^\beta], e_1^\gamma \rangle^2 = \langle [e_1^\alpha, e_1^\beta], e_2^\gamma \rangle^2$$
$$\langle [e_1^\alpha, e_1^\beta], e_1^\gamma \rangle^2 = \langle [e_1^\alpha, e_2^\beta], e_2^\gamma \rangle^2 = 0$$

Similarly, in r_2 our $\langle [e_2^\alpha, e_j^\beta], e_k^\gamma \rangle^2$ term has:

$$\langle [e_2^\alpha, e_2^\beta], e_1^\gamma \rangle^2 = \langle [e_2^\alpha, e_1^\beta], e_2^\gamma \rangle^2 = 0$$

Letting $p_1 = \sum_{\alpha,\beta,\gamma} \langle [e_1^\alpha, e_1^\beta], e_2^\gamma \rangle^2 = \sum_{\alpha,\beta,\gamma} \langle [e_1^\alpha, e_2^\beta], e_1^\gamma \rangle^2 = \sum_{\alpha,\beta,\gamma} \langle [e_2^\alpha, e_1^\beta], e_1^\gamma \rangle^2$ and

$p_2 = \sum_{\alpha,\beta,\gamma}\langle[e_2^\alpha, e_2^\beta], e_2^\gamma\rangle^2$, we get the following formulas for r_1 and r_2:

$$r_1 = \frac{-1}{2x_1} + \frac{p_1}{4d_1}\left[\left(\frac{x_1}{x_1x_2} - \frac{x_2}{x_1x_1} - \frac{x_1}{x_1x_2}\right) + \left(\frac{x_1}{x_2x_1} - \frac{x_1}{x_1x_2} - \frac{x_2}{x_1x_1}\right)\right]$$

$$r_2 = \frac{1}{2x_2} + \frac{1}{4d_2}\left[p_1\left(\frac{x_2}{x_1x_1} - \frac{x_1}{x_2x_1} - \frac{x_1}{x_2x_1}\right) + p_2\left(\frac{x_2}{x_1x_2} - \frac{x_2}{x_2x_2} - \frac{x_2}{x_2x_2}\right)\right].$$

Simplifying both terms and getting a common denominator we get the following:

$$r_1 = \frac{-d_1x_1 - p_1x_2}{2d_1x_1^2}$$

$$r_2 = \frac{p_1(x_2^2 - 2x_1) + (2d_2 - p_2)x_1^2}{4d_2x_1^2x_2}$$

$$= \frac{(2d_2 - p_2 - 2p_1)x_1^2 + p_1x_2^2}{4d_2x_1^2x_2}.$$

By Lemma 1 in [Nik00], we have $2p_1 \leq d_1$ and $p_1 + p_2 \leq d_2$, with equality only when $\mathfrak{p}_1$ and $\mathfrak{p}_2$, respectively, are trivial representations for $ad_{\mathfrak{h}}$. Moreover, since $p_1 = \sum_{\alpha,\beta,\gamma}\langle[e_1^\alpha, e_1^\beta], e_2^\gamma\rangle^2$, we know that $p_1 > 0$ since by the Cartan decomposition properties $[\mathfrak{p}_1, \mathfrak{p}_1] = \mathfrak{k} \supset \mathfrak{p}_2$. We can thus observe that $r_1 < 0$ and $r_2 > 0$.

We now relate ric to the background inner product, placing ourselves in the $(0, 2)$ tensor setting. Since $(.,.) = x_1\langle .,.\rangle_1 + x_2\langle .,.\rangle_2$ and $ric(.,.) = r_1(.,.)_1 + (.,.)_2$, we can write $ric(.,.) = x_1r_1\langle .,.\rangle_1 + x_2r_2\langle .,.\rangle$. Define $R_1 = x_1r_1$ and $R_2 = x_2r_2$ and we have the following:

$$R_1 = x_1\frac{-d_1x_1 - p_1x_2}{2d_1x_1^2}$$

$$= \frac{-1}{2} - \frac{p_1}{2d_1}\frac{x_2}{x_1} \tag{3.1}$$

$$\begin{aligned} R_2 &= x_2 \frac{(2d_2 - p_2 - 2p_1)x_1^2 + p_1 x_2^2}{4d_2 x_1^2 x_2} \\ &= \frac{2d_2 - p_2 - 2p_1}{4d_2} + \frac{p_1}{4d_2}\left(\frac{x_2}{x_1}\right)^2 \end{aligned}$$

Note that $R_1 = x_1 r_1 < 0$ since $x_1 > 0$ and $r_1 < 0$; likewise, $R_2 = x_2 r_2 > 0$ since $x_2 > 0$ and $r_2 > 0$. Thus, we have our desired result. ■

Lemma 3.12. Let p_1, p_2, d_1, d_2, and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_1$ be defined as in Lemma 3.11. In general, we have

$$\frac{2d_2 - p_2 - 2p_1}{4d_2} \geq 0.$$

If we assume that $[\mathfrak{p}_2, \mathfrak{p}_2] \not\subset \mathfrak{h}$, then we have

$$\frac{2d_2 - p_2 - 2p_1}{4d_2} > 0.$$

If we assume that $[\mathfrak{p}_2, \mathfrak{p}_2] \subset \mathfrak{h}$, then we have

$$\frac{2d_2 - p_2 - 2p_1}{4d_2} = \frac{d_2 - p_1}{2d_2} \geq 0$$

with equality if and only if $\mathfrak{p}_2$ is a trivial $ad_{\mathfrak{h}}$ representation.

Proof: As was mentioned in the proof of Lemma 3.11, by Lemma 1 in [Nik00], $p_1 + p_2 \leq d_2$ with equality if and only if $\mathfrak{p}_1$ and $\mathfrak{p}_2$, respectively, are trivial representations. By $p_1 + p_2 \leq d_2$ we are able to conclude that $\frac{2d_2 - p_2 - 2p_1}{4d_2} \geq 0$ for general $\mathfrak{p}_2$, proving the first claim.

If $[\mathfrak{p}_2, \mathfrak{p}_2] \not\subset \mathfrak{h}$ then we know that $p_2 = \sum_{\alpha,\beta,\gamma} \langle [e_2^\alpha, e_2^\beta], e_2^\gamma \rangle^2 > 0$. Thus, we have that

$$2d_2 - p_2 - 2p_1 > 2d_2 - 2p_2 - 2p_1 \geq 0$$

and we are able to conclude that $\dfrac{2d_2 - p_2 - 2p_1}{4d_2} > 0$, proving the second claim.

If $[\mathfrak{p}_2, \mathfrak{p}_2] \subset \mathfrak{h}$ then we know that $p_2 = 0$ and we get $\dfrac{2d_2 - p_2 - 2p_1}{4d_2} = \dfrac{d_2 - p_1}{2d_2} \geq 0$ with $p_1 = d_2$ if and only if $\mathfrak{p}_2$ is trivial, as mentioned above. ■

Theorem 3.13. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_1$ be noncompact simple with $[\mathfrak{p}_2, \mathfrak{p}_2] \not\subset \mathfrak{h}$. For G/H in this case, $ric = T$ has a solution T if and only if

$$t_2 = \frac{d_1^2}{d_2 p_1} t_1^2 + \frac{d_1^2}{d_2 p_1} t_1 + \frac{p_1(2d_2 - p_2 - 2p_1) + d_1^2}{4 d_2 p_1} \text{ with } t_1 \in (-\infty, \frac{-1}{2}).$$

Proof: The goal of this proof is to determine sufficient and necessary conditions on $T(.,.) = t_1\langle .,. \rangle_1 + t_2 \langle .,. \rangle_2$ such that $ric = T$ for some $ad_{\mathfrak{h}}$ invariant inner product. Since Lemma 3.11 provides us with $ric(.,.)$ in terms R_1 and R_2 which are dependent only upon the pair (x_1, x_2) defining our inner product and p_1, p_2, d_1, d_2 which are determined by the Lie data, what we seek are the solutions to the following system of equations:

$$\begin{cases} R_1 &= t_1 \\ R_2 &= t_2. \end{cases}$$

Note that by $R_1 < 0$ and $R_2 > 0$, we know that $t_1 < 0$ and $t_2 > 0$. Now, plugging into R_1

and R_2 we have the following:

$$\frac{-1}{2} - \frac{p_1}{2d_1}\frac{x_2}{x_1} = t_1$$
$$\frac{2d_2 - p_2 - 2p_1}{4d_2} + \frac{p_1}{4d_2}\left(\frac{x_2}{x_1}\right)^2 = t_2.$$

Let $\lambda = \frac{x_1}{x_2}$ and observe that λ can take on any positive value, implying that t_1 can take on any value in $(-\infty, \frac{-1}{2})$. Our approach is as follows. We use the equation with t_1 to solve for λ and then we substitute λ into the equation with t_2, providing an equation of t_2 in terms of t_1. Once we have that, we will know that for any $t_1 \in (-\infty, \frac{-1}{2})$, we can get a t_2 such that $ric = T$ has a solution, providing sufficient and necessary conditions as desired.

$$\frac{-1}{2} - \frac{p_1}{2d_1}\lambda = t_1$$
$$\lambda = \frac{-2d_1}{p_1}(t_1 + \frac{1}{2})$$

$$\frac{2d_2 - p_2 - 2p_1}{4d_2} + \frac{p_1}{4d_2}(\lambda)^2 = t_2$$
$$\frac{2d_2 - p_2 - 2p_1}{4d_2} + \frac{p_1}{4d_2}\left(\frac{-2d_1}{p_1}(t_1 + \frac{1}{2})\right)^2 = t_2$$
$$\frac{2d_2 - p_2 - 2p_1}{4d_2} + \frac{p_1}{4d_2}\left(\frac{4d_1^2}{p_1^2}(t_1^2 + t_1 + \frac{1}{4})\right) = t_2$$
$$\frac{2d_2 - p_2 - 2p_1}{4d_2} + \frac{d_1^2}{d_2 p_1}\left(t_1^2 + t_1 + \frac{1}{4}\right) = t_2$$
$$\frac{d_1^2}{d_2 p_1}t_1^2 + \frac{d_1^2}{d_2 p_1}t_1 + \frac{p_1(2d_2 - p_2 - 2p_1) + d_1^2}{4d_2 p_1} = t_2$$

Thus, the solutions to $ric = T$ are given by

$$t_2 = \frac{d_1^2}{d_2 p_1} t_1^2 + \frac{d_1^2}{d_2 p_1} t_1 + \frac{p_1(2d_2 - p_2 - 2p_1) + d_1^2}{4d_2 p_1} \text{ for any } t_1 \in (-\infty, \frac{-1}{2}).$$

■

Theorem 3.14. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_1$ be noncompact simple with $[\mathfrak{p}_2, \mathfrak{p}_2] \not\subset \mathfrak{h}$. Recall our setting and the definition of some notation from Remark 3.8. For G/H in this case, the equation $ric = cT$ for $c > 0$ has a solution if and only if (t_1, t_2) is a pair satisfying

$$\frac{t_2}{t_1} \leq \frac{d_1^2}{d_2 p_1} - \frac{d_1}{d_2}\sqrt{\frac{d_1^2}{p_1^2} + \frac{2d_2 - p_2 - 2p_1}{p_1}} \leq 0$$

where $t_1 > 0$, $t_2 < 0$, and equality on the right only occurs when $\mathfrak{p}_2$ is trivial (See Corollary 3.19 for that setting).

When the above inequality is satisfied, there is always one solution, namely c_+ (where c_+ takes the + in 3.2 below) and (x_1, x_2), the pair unique up to scaling given by $\frac{x_2}{x_1} = \frac{-2d_1}{p_1}(c_+ t_1 + \frac{1}{2})$.

In addition to this one solution, there is a second solution if and only if our pair (t_1, t_2) with $t_1 > 0$ and $t_2 < 0$ satisfies

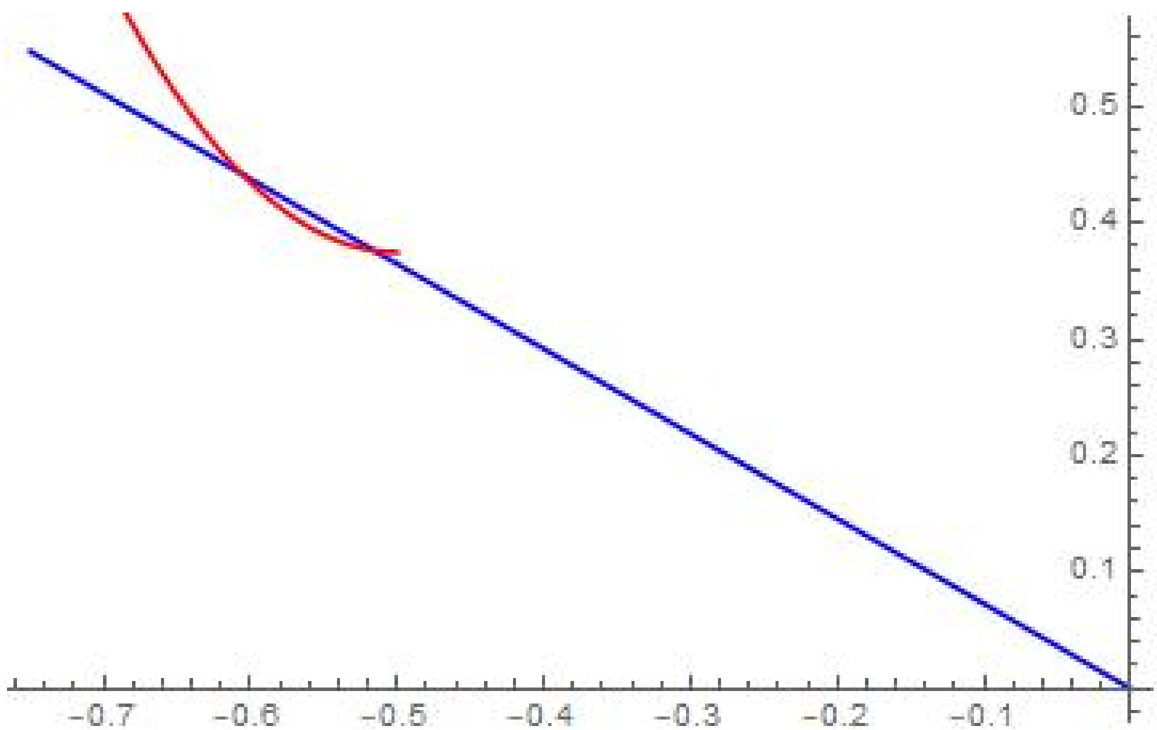
$$-\frac{2d_2 - p_2 - 2p_1}{2d_2} < \frac{t_2}{t_1} < \frac{d_1^2}{d_2 p_1} - \frac{d_1}{d_2}\sqrt{\frac{d_1^2}{p_1^2} + \frac{2d_2 - p_2 - 2p_1}{p_1}}.$$

The second solution is c_- (where c_- takes the $-$ in 3.2 below) and (x_1, x_2), the pair unique up to scaling given by $\frac{x_2}{x_1} = \frac{-2d_1}{p_1}(c_- t_1 + \frac{1}{2})$.

$$c = \frac{-\left(\frac{d_1^2}{d_2 p_1} t_1 - t_2\right) \pm \sqrt{\left(\frac{d_1^2}{d_2 p_1} t_1 - t_2\right)^2 - 4\left(\frac{d_1^2}{d_2 p_1} t_1^2\right)\left(\frac{d_1^2}{4 p_1 d_2} + \frac{2d_2 - p_2 - 2p_1}{4d_2}\right)}}{2\frac{d_1^2}{d_2 p_1} t_1^2} \tag{3.2}$$

Moreover, $c_+ = c_-$ when the discriminant in c is zero, and this happens precisely when

$$\frac{t_2}{t_1} = \frac{d_1^2}{d_2 p_1} - \frac{d_1}{d_2}\sqrt{\frac{d_1^2}{p_1^2} + \frac{2d_2 - p_2 - 2p_1}{p_1}}.$$



The above image was produced in Mathematica ([Inc]) for the image of ric (in red) with $G/H = SO_0(1,7)/G_2$ when we restrict our $ad_{\mathfrak{g}_2}$ inner products to those in which $(\mathfrak{p}_2, \mathfrak{p}_1) = 0$. In this case $d_1 = d_2 = 7$ and $p_1 = p_2 = \frac{7}{6}$ (See Remark 3.23). Since $ric = cT$ is looking for T produced by taking the image of ric and multiplying by some constant, all such T can be understood by what is called the *cone* of the image of ric (See Exercise 16 in Chapter 8 of Section 2 in [CLO15]). The cone of the image of ric is the collection of lines through the origin that intersect the image of ric. Since we concern ourselves with $c > 0$ specifically, we ignore one half of the line through the origin depending on the point on the image of ric under consideration. In our case $t_1 < 0$ and $t_2 > 0$, so we would always ignore the half of the line at the origin and in the fourth quadrant if we orient our plane with t_1 being the horizontal axis and t_2 the vertical. The line provided above helps to illustrate a subset of points describing T in $ric = cT$. Additionally, the line illustrates the need for more than one c value depending on how the line intersects T in certain cases (here we are in the case of $[\mathfrak{p}_2, \mathfrak{p}_2] \not\subset \mathfrak{h}$).

Remark 3.15. A remark is warranted before we begin our proof. In the following, we will be using Lemma 3.12 to come to conclusions regarding the existence of solutions to $ric = cT$. For the purposes of proving the corollaries to follow in a simpler fashion, we use the general $\mathfrak{p}_2$ setting of the lemma unless forced to assume $[\mathfrak{p}_2, \mathfrak{p}_2] \not\subset \mathfrak{h}$. We will see that the only setting that really changes the solutions in a significant way (i.e. it is more than just a formula change for the c and the (t_1, t_2)) is the setting in which $\mathfrak{p}_2$ is a trivial representation, as this is the only setting in which $\dfrac{2d_2 - p_2 - 2p_1}{p_1} = 0$. When the need for strict inequality occurs, we are sure to make note of it and point the reader to Corollary 3.19.

Proof: The goal of this proof is to find necessary and sufficient conditions on $T = t_1\langle ., .\rangle_1 + t_2\langle ., .\rangle_2$ such there there are solutions to $ric = cT$ and then provide what $c > 0$ and $ad_{\mathfrak{h}}$ inner product (determined by the pair (x_1, x_2)) to expect for a given (t_1, t_2) for which there is a solution. As before, since Lemma 3.11 provides ric in terms of R_1 and R_2 which are dependent upon the pair (x_1, x_2) and p_1, p_2, d_1, d_2 which come from the Lie data, we seek to find solutions to the following system:

$$\begin{cases} R_1 &= ct_1 \\ R_2 &= ct_2. \end{cases}$$

Plugging in for R_1, R_2, we get the following equations:

$$\begin{aligned} \frac{-1}{2} - \frac{p_1}{2d_1}\frac{x_2}{x_1} &= ct_1 \\ \frac{2d_2 - p_2 - 2p_1}{4d_2} + \frac{p_1}{4d_2}\left(\frac{x_2}{x_1}\right)^2 &= ct_2. \end{aligned}$$

Note that once again, by $R_1 < 0$ and $R_2, c > 0$ we have $t_1 < 0$ and $t_2 > 0$. Our approach is similar to that of Theorem 3.13, but since we need more information than just what t_1, t_2 satisfy the equation, there are some differences.

In this case we are looking for conditions on the pair (t_1, t_2) that are sufficient and necessary to the existence of a $c > 0$ and (x_1, x_2) such that we have a solution to the given system of equations. To do so, we again set $\lambda = \frac{x_2}{x_1}$, and solve for λ in terms of c and t_1. We then get an equation for c in terms of t_1 and t_2, and using the condition that $c > 0$, we obtain sufficient and necessary conditions on (t_1, t_2) such that we get a $c > 0$. Then, we investigate what subset of those (t_1, t_2) giving us $c > 0$ satisfy $\lambda > 0$. This will provide us with solutions to the given system of equations, providing us with the desired result.

$$\begin{aligned} \frac{-1}{2} - \frac{p_1}{2d_1}\frac{x_2}{x_1} &= ct_1 \\ \frac{-1}{2} - \frac{p_1}{2d_1}\lambda &= ct_1 \\ \lambda &= \frac{-2d_1}{p_1}\left(ct_1 + \frac{1}{2}\right) \qquad (\star) \end{aligned}$$

$$
\begin{aligned}
\frac{2d_2 - p_2 - 2p_1}{4d_2} + \frac{p_1}{4d_2}\left(\frac{x_2}{x_1}\right)^2 &= ct_2 \\
\frac{2d_2 - p_2 - 2p_1}{4d_2} + \frac{p_1}{4d_2}\lambda^2 &= ct_2 \\
\frac{2d_2 - p_2 - 2p_1}{4d_2} + \frac{p_1}{4d_2}\left(\frac{-2d_1}{p_1}\left(ct_1 + \frac{1}{2}\right)\right)^2 &= ct_2 \\
\frac{2d_2 - p_2 - 2p_1}{4d_2} + \frac{p_1}{4d_2}\left(\frac{4d_1^2}{p_1^2}\left(c^2t_1^2 + ct_1 + \frac{1}{4}\right)\right) &= ct_2 \\
\left(\frac{d_1^2}{d_2p_1}t_1^2\right)c^2 + \left(\frac{d_1^2}{d_2p_1}t_1 - t_2\right)c + \frac{d_1^2}{4p_1d_2} + \frac{2d_2 - p_2 - 2p_1}{4d_2} &= 0 \qquad (\star\star)
\end{aligned}
$$

Observing that the above equation is quadratic in c, so we solve for c using the quadratic formula.

$$
c = \frac{-\left(\frac{d_1^2}{d_2p_1}t_1 - t_2\right) \pm \sqrt{\left(\frac{d_1^2}{d_2p_1}t_1 - t_2\right)^2 - 4\left(\frac{d_1^2}{d_2p_1}t_1^2\right)\left(\frac{d_1^2}{4p_1d_2} + \frac{2d_2 - p_2 - 2p_1}{4d_2}\right)}}{2\frac{d_1^2}{d_2p_1}t_1^2} \tag{3.3}
$$

So long as $c > 0$ and the resulting $\lambda > 0$, any t_1, t_2, c satisfying $(\star\star)$ above provides a solution to $ric = cT$. Thus, we use c above to find what conditions on t_1 and t_2 are necessary and sufficient for $c > 0$ and then we use $(\star)$ to determine what conditions are necessary and sufficient for $\lambda > 0$. Once we have those, we will have all the solutions to $(\star\star)$ and thus all the solutions to $ric = cT$. There are multiple steps here with (in the end) more than one possible solution in certain settings. For this reason, we finish the proof with a set of claims:

Claim 1: $c > 0$ if and only if c is real

Claim 2: c is real if and only if (t_1, t_2) satisfies 3.4

$$\frac{t_2}{t_1} \leq \frac{d_1^2}{d_2 p_1} - \frac{d_1}{d_2}\sqrt{\frac{d_1^2}{p_1^2} + \frac{2d_2 - p_2 - 2p_1}{p_1}} \tag{3.4}$$

Claim 3: For any (t_1, t_2) satisfying 3.4, there is one solution given by c_+ and $\lambda = \frac{-2d_1}{p_1}\left(c_+ t_1 + \frac{1}{2}\right)$ where c_+ takes the + in 3.3

Claim 4: For any (t_1, t_2) satisfying 3.5, there is a second solution given by c_- and $\lambda = \frac{-2d_1}{p_1}\left(c_- t_1 + \frac{1}{2}\right)$ where c_- takes the $-$ in 3.3.

$$-\frac{2d_2 - p_2 - 2p_1}{2d_2} < \frac{t_2}{t_1} < \frac{d_1^2}{d_2 p_1} - \frac{d_1}{d_2}\sqrt{\frac{d_1^2}{p_1^2} + \frac{2d_2 - p_2 - 2p_1}{p_1}} \tag{3.5}$$

<u>**Claim 1:**</u> $c > 0$ if and only if c is real

<u>Proof of **Claim 1**</u>: First recall that $t_1 < 0$, $t_2 > 0$, and $\dfrac{2d_2 - p_2 - 2p_1}{4d_2} \geq 0$ by the general setting in Lemma 3.12. These inequalities allows us to also conclude that

$$4\left(\frac{d_1^2}{d_2 p_1} t_1^2\right)\left(\frac{d_1^2}{4p_1 d_2} + \frac{2d_2 - p_2 - 2p_1}{4d_2}\right) > 0.$$

These inequalities allow us to see that $c > 0$ if and only if c is real as the numerator and denominator in c will always be positive. Indeed, the denominator is always positive since every term is positive, and the numerator being positive requires some more detailed checking. First, recall the numerator of c:

$$-\left(\frac{d_1^2}{d_2p_1}t_1 - t_2\right) \pm \sqrt{\left(\frac{d_1^2}{d_2p_1}t_1 - t_2\right)^2 - 4\left(\frac{d_1^2}{d_2p_1}t_1^2\right)\left(\frac{d_1^2}{4p_1d_2} + \frac{2d_2 - p_2 - 2p_1}{4d_2}\right)}$$

which is always positive if we take the plus sign and c is real since $t_1 < 0$ and $t_2 > 0$. If we take the minus sign, then we need the term on the right to have smaller magnitude than the term on the left, which is true by $4\left(\frac{d_1^2}{d_2p_1}t_1^2\right)\left(\frac{d_1^2}{4p_1d_2} + \frac{2d_2 - p_2 - 2p_1}{4d_2}\right) > 0$ implying that

$$\sqrt{\left(\frac{d_1^2}{d_2p_1}t_1 - t_2\right)^2 - 4\left(\frac{d_1^2}{d_2p_1}t_1^2\right)\left(\frac{d_1^2}{4p_1d_2} + \frac{2d_2 - p_2 - 2p_1}{4d_2}\right)} < \left|\left(\frac{d_1^2}{d_2p_1}t_1 - t_2\right)\right|.$$

Therefore, we have a $c > 0$ just by finding when c is real. This concludes the proof of **Claim 1**.

By **Claim 1**, we can get sufficient and necessary conditions on (t_1, t_2) for solutions to $ric = cT$ by finding the conditions for which c is real and $\lambda = \frac{-2d_1}{p_1}\left(ct_1 + \frac{1}{2}\right) > 0$. We hasten to remark that since c has the possibility of a + or a −, there is a possibility of more than one set of solutions to $ric = cT$, one solution with c_+ and $\lambda = \frac{-2d_1}{p_1}\left(c_+t_1 + \frac{1}{2}\right)$ and another with c_- and $\lambda = \frac{-2d_1}{p_1}\left(c_-t_1 + \frac{1}{2}\right)$. Here, c_+ and c_- are the c in 3.3 taking the + and the −, respectively.

We now focus on finding conditions on (t_1, t_2) when c is real.

<u>**Claim 2:**</u> c is real if and only if (t_1, t_2) satisfies 3.4

<u>Proof of **Claim 2**</u>: To prove this claim, we look for t_1, t_2 such that the discriminant in c

is non-negative:

$$\left(\frac{d_1^2}{d_2 p_1} t_1 - t_2\right)^2 - 4\left(\frac{d_1^2}{d_2 p_1} t_1^2\right)\left(\frac{d_1^2}{4 p_1 d_2} + \frac{2d_2 - p_2 - 2p_1}{4d_2}\right) \geq 0.$$

If we divide by t_1^2 and distribute the 4 we have

$$\left(\frac{d_1^2}{d_2 p_1} - \frac{t_2}{t_1}\right)^2 - \frac{d_1^2}{d_2 p_1}\left(\frac{d_1^2}{p_1 d_2} + \frac{2d_2 - p_2 - 2p_1}{d_2}\right) \geq 0.$$

We now let $t = \frac{t_2}{t_1}$ and we determine when

$$f(t) = \left(\frac{d_1^2}{d_2 p_1} - t\right)^2 - \frac{d_1^2}{d_2 p_1}\left(\frac{d_1^2}{p_1 d_2} + \frac{2d_2 - p_2 - 2p_1}{d_2}\right) \geq 0$$

since this will provide us with equivalent conditions to c being real.

Our $f(t)$ is a parabola in t that is concave up with vertex

$$\left(\frac{d_1^2}{d_2 p_1}\,,\ -\frac{d_1^2}{d_2 p_1}\left(\frac{d_1^2}{p_1 d_2} + \frac{2d_2 - p_2 - 2p_1}{d_2}\right)\right).$$

Thus, the minimum of $f(t)$ is negative with one zero of $f(t)$ being positive and the other unknown. We thus solve $f(t) = 0$ to determine the conditions for $f(t) \geq 0$, recalling that we really only care about $t < 0$ since $t_1 < 0$ and $t_2 > 0$.

$$f(t) = 0$$

$$\left(\frac{d_1^2}{d_2p_1} - t\right)^2 - \frac{d_1^2}{d_2p_1}\left(\frac{d_1^2}{p_1d_2} + \frac{2d_2 - p_2 - 2p_1}{d_2}\right) = 0$$

$$\left(\frac{d_1^2}{d_2p_1} - t\right)^2 = \frac{d_1^2}{d_2p_1}\left(\frac{d_1^2}{p_1d_2} + \frac{2d_2 - p_2 - 2p_1}{d_2}\right)$$

$$\frac{d_1^2}{d_2p_1} - t = \pm\sqrt{\frac{d_1^2}{d_2p_1}\left(\frac{d_1^2}{p_1d_2} + \frac{2d_2 - p_2 - 2p_1}{d_2}\right)}$$

$$t = \frac{d_1^2}{d_2p_1} \pm \sqrt{\frac{d_1^2}{d_2p_1}\left(\frac{d_1^2}{p_1d_2} + \frac{2d_2 - p_2 - 2p_1}{d_2}\right)}$$

This provides us with t such that $f(t) = 0$, but we drop the case with + since we only want $t < 0$ and we observe when we have $t < 0$.

$$t = \frac{d_1^2}{d_2p_1} - \sqrt{\frac{d_1^2}{d_2p_1}\left(\frac{d_1^2}{p_1d_2} + \frac{2d_2 - p_2 - 2p_1}{d_2}\right)}$$

$$t = \frac{d_1^2}{d_2p_1} - \frac{d_1}{d_2}\sqrt{\frac{d_1^2}{p_1^2} + \frac{2d_2 - p_2 - 2p_1}{p_1}} \leq 0$$

where the final inequality is true by $\dfrac{2d_2 - p_2 - 2p_1}{p_1} \geq 0$. Observe that $t = 0$ only when $\dfrac{2d_2 - p_2 - 2p_1}{p_1} = 0$, and by Lemma 3.12 we know this only happens when $\mathfrak{p}_2$ is a trivial representation. For that setting, please see Corollary 3.19. In the current setting of $[\mathfrak{p}_2, \mathfrak{p}_2] \not\subset \mathfrak{h}$, though, we may conclude that our t for $f(t) = 0$ is negative.

Since the t found above where $f(t) = 0$ is negative and $f(t)$ is a parabola concave up, we can conclude that for $t \leq \frac{d_1^2}{d_2p_1} - \frac{d_1}{d_2}\sqrt{\frac{d_1^2}{p_1^2} + \dfrac{2d_2 - p_2 - 2p_1}{p_1}}$ we have $f(t) \geq 0$. Thus, having (t_1, t_2) such that

$$\frac{t_2}{t_1} \leq \frac{d_1^2}{d_2 p_1} - \frac{d_1}{d_2}\sqrt{\frac{d_1^2}{p_1^2} + \frac{2d_2 - p_2 - 2p_1}{p_1}} \tag{3.6}$$

is equivalent to having c real and therefore $c > 0$ as well. This concludes our proof of **Claim 2**.

It will be helpful when we reach the end of the proof of **Claim 4** to go ahead and note that $\frac{t_2}{t_1} = t = \frac{d_1^2}{d_2 p_1} - \frac{d_1}{d_2}\sqrt{\frac{d_1^2}{p_1^2} + \frac{2d_2 - p_2 - 2p_1}{p_1}}$ happens when $f(t) = 0$ which is when the discriminant of c is 0 and $c_+ = c_-$.

Now, we want to verify conditions for which not only $c > 0$, but $\lambda = \frac{-2d_1}{p_1}\left(ct_1 + \frac{1}{2}\right) > 0$.

<u>**Claim 3:**</u> For any (t_1, t_2) satisfying 3.4, there is one solution given by c_+ and $\lambda = \frac{-2d_1}{p_1}\left(c_+ t_1 + \frac{1}{2}\right)$ where c_+ takes the + in 3.3

<u>Proof of **Claim 3**</u>: Observe that $\lambda > 0$ occurs if and only if $ct_1 < \frac{-1}{2}$. Let us examine that inequality more closely:

$$ct_1 = \frac{-\left(\frac{d_1^2}{d_2 p_1}t_1 - t_2\right) \pm \sqrt{\left(\frac{d_1^2}{d_2 p_1}t_1 - t_2\right)^2 - 4\left(\frac{d_1^2}{d_2 p_1}t_1^2\right)\left(\frac{d_1^2}{4p_1 d_2} + \frac{2d_2 - p_2 - 2p_1}{4d_2}\right)}}{2\frac{d_1^2}{d_2 p_1}t_1} < \frac{-1}{2}$$

$$-\left(\frac{d_1^2}{d_2 p_1}t_1 - t_2\right) \pm \sqrt{\left(\frac{d_1^2}{d_2 p_1}t_1 - t_2\right)^2 - 4\left(\frac{d_1^2}{d_2 p_1}t_1^2\right)\left(\frac{d_1^2}{4p_1 d_2} + \frac{2d_2 - p_2 - 2p_1}{4d_2}\right)} > \frac{-d_1^2 t_1}{d_2 p_1}$$

$$t_2 \pm \sqrt{\left(\frac{d_1^2}{d_2 p_1}t_1 - t_2\right)^2 - 4\left(\frac{d_1^2}{d_2 p_1}t_1^2\right)\left(\frac{d_1^2}{4p_1 d_2} + \frac{2d_2 - p_2 - 2p_1}{4d_2}\right)} > 0 \tag{$*$}$$

This creates two cases for determining what conditions on (t_1, t_2) provide $ct_1 < \frac{-1}{2}$, one in which we have the + above in $(*)$ and another in which we have the − above. Considering the case with the +, we recall that $t_2 > 0$, so as long as c is real (and therefore positive), we have at least one solution. This completes the proof of **Claim 3**.

We now turn our attention to the possibility of a second set of solutions determined by taking the − in $(*)$.

Claim 4: For any (t_1, t_2) satisfying 3.5, there is a second solution given by c_- and $\lambda = \frac{-2d_1}{p_1}\left(c_- t_1 + \frac{1}{2}\right)$ where c_- takes the − in 3.3

Proof of **Claim 4**: In this case, we are looking for when the discriminant above in $(*)$ is less than t_2^2. Allow us to investigate:

$$\begin{aligned}
\left(\frac{d_1^2}{d_2 p_1} t_1 - t_2\right)^2 - 4\left(\frac{d_1^2}{d_2 p_1} t_1^2\right)\left(\frac{d_1^2}{4p_1 d_2} + \frac{2d_2 - p_2 - 2p_1}{4d_2}\right) &< t_2^2 \\
\frac{d_1^4}{d_2^2 p_1^2} t_1^2 + t_2^2 - 2\frac{d_1^2}{d_2 p_1} t_1 t_2 - \frac{d_1^4}{d_2^2 p_1^2} t_1^2 - \frac{d_1^2 t_1^2 (2d_2 - p_2 - 2p_1)}{d_2^2 p_1} &< t_2^2 \\
-2\frac{d_1^2}{d_2 p_1} t_1 t_2 - \frac{d_1^2 t_1^2 (2d_2 - p_2 - 2p_1)}{d_2^2 p_1} &< 0 \\
\text{now divide by } -t_1 \frac{d_1^2}{d_2 p_1} & \\
2t_2 + t_1 \frac{2d_2 - p_2 - 2p_1}{d_2} &< 0 \\
\frac{t_2}{t_1} &> -\frac{2d_2 - p_2 - 2p_1}{2d_2}
\end{aligned}$$

Also note that having a (t_1, t_2) satisfy $(*)$ is equivalent to $ct_1 < \frac{-1}{2} < 0$, so we do not have to worry about if such a (t_1, t_2) will not produce a $c > 0$ since we have $t_1 < 0$. Thus, for

$$-\frac{2d_2 - p_2 - 2p_1}{2d_2} < \frac{t_2}{t_1} < \frac{d_1^2}{d_2 p_1} - \frac{d_1}{d_2}\sqrt{\frac{d_1^2}{p_1^2} + \frac{2d_2 - p_2 - 2p_1}{p_1}} \tag{3.7}$$

we have a second solution. Again, we remark that by Lemma 3.12, when $\mathfrak{p}_2$ is a trivial representation (the setting considered in Corollary 3.19), we have the lower and upper bound being 0, and this is the only setting in which this can happen. We also exclude equality on the right as this happens if an only if the discriminant in c is 0 and in this case $c_+ = c_-$, so there is only one solution. This completes our proof of **Claim 4**.

With the completion of **Claim 4**, we have our desired result providing necessary and sufficient conditions on (t_1, t_2) for solutions to $ric = cT$, providing the c values and the (x_1, x_2) that determine our metric for the given (t_1, t_2) as well. ∎

Corollary 3.16. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_1$ be noncompact simple with $[\mathfrak{p}_2, \mathfrak{p}_2] \subset \mathfrak{h}$ and $\mathfrak{p}_2$ not a trivial representation. For G/H in this case, $ric = T$ has a solution T if and only if

$$t_2 = \frac{d_1^2}{d_2 p_1} t_1^2 + \frac{d_1^2}{d_2 p_1} t_1 + \frac{p_1(2d_2 - 2p_1) + d_1^2}{4 d_2 p_1} \text{ with } t_1 \in (-\infty, \frac{-1}{2}).$$

<u>Proof:</u> If $[\mathfrak{p}_2, \mathfrak{p}_2] \subset \mathfrak{h}$, then we have $p_2 = 0$ in our R_1 and R_2 defined in Eqn.3.1. Thus, the solutions to $ric = T$ can be determined from Theorem 3.13 to be any (t_1, t_2) such that $t_2 = \frac{d_1^2}{d_2 p_1} t_1^2 + \frac{d_1^2}{d_2 p_1} t_1 + \frac{p_1(2d_2 - 2p_1) + d_1^2}{4 d_2 p_1}$ with $t_1 \in (-\infty, \frac{-1}{2})$. ∎

Corollary 3.17. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_1$ be noncompact simple with $[\mathfrak{p}_2, \mathfrak{p}_2] \subset \mathfrak{h}$ and $\mathfrak{p}_2$ not a trivial representation. Recall our setting and the definition of some notation from Remark 3.8. For G/H in this case, the equation $ric = cT$ has a solution if and only if (t_1, t_2) is a pair satisfying

$$\frac{t_2}{t_1} \leq \frac{d_1^2}{d_2 p_1} - \frac{d_1}{d_2}\sqrt{\frac{d_1^2}{p_1^2} + \frac{2d_2 - 2p_1}{p_1}}$$

where $t_1 > 0$ and $t_2 < 0$.

When the above inequality is satisfied, there is always one solution, namely c_+ (where c_+ takes the $+$ in 3.8 below) and (x_1, x_2), the pair unique up to scaling given by $\frac{x_2}{x_1} = \frac{-2d_1}{p_1}(c_+ t_1 + \frac{1}{2})$.

In addition to this one solution, there is a second solution if and only if our pair (t_1, t_2) with $t_1 > 0$ and $t_2 < 0$ satisfies

$$-\frac{d_2 - p_1}{d_2} < \frac{t_2}{t_1} < \frac{d_1^2}{d_2 p_1} - \frac{d_1}{d_2}\sqrt{\frac{d_1^2}{p_1^2} + \frac{2d_2 - 2p_1}{p_1}}.$$

The second solution is c_- (where c_- takes the $-$ in 3.8 below) and (x_1, x_2) the pair unique up to scaling given by $\frac{x_2}{x_1} = \frac{-2d_1}{p_1}(c_- t_1 + \frac{1}{2})$.

$$c = \frac{-\left(\frac{d_1^2}{d_2 p_1} t_1 - t_2\right) \pm \sqrt{\left(\frac{d_1^2}{d_2 p_1} t_1 - t_2\right)^2 - 4\left(\frac{d_1^2}{d_2 p_1} t_1^2\right)\left(\frac{d_1^2}{4 p_1 d_2} + \frac{2d_2 - 2p_1}{4d_2}\right)}}{2\frac{d_1^2}{d_2 p_1} t_1^2} \tag{3.8}$$

Moreover, $c_+ = c_-$ when the discriminant in c is zero, and this happens precisely when

$$\frac{t_2}{t_1} = \frac{d_1^2}{d_2 p_1} - \frac{d_1}{d_2}\sqrt{\frac{d_1^2}{p_1^2} + \frac{2d_2 - 2p_1}{p_1}}.$$

Proof: If $[\mathfrak{p}_2, \mathfrak{p}_2] \subset \mathfrak{h}$ and $\mathfrak{p}_2$ is not a trivial representation, then we have $p_2 = 0$ in our R_1 and R_2 defined in Eqn.3.1. Moreover, by Lemma 3.12 we have $\dfrac{2d_2 - p_2 - 2p_1}{4d_2} = \dfrac{d_2 - p_1}{2d_2} > 0$. Using Theorem 3.14 then we can say that (t_1, t_2) satisfying

$$\frac{t_2}{t_1} \leq \frac{d_1^2}{d_2 p_1} - \frac{d_1}{d_2}\sqrt{\frac{d_1^2}{p_1^2} + \frac{2d_2 - 2p_1}{p_1}} \tag{3.9}$$

is a sufficient and necessary condition to having a solution to $ric = cT$, namely c_+ from 3.10 below and (x_1, x_2) satisfying $\frac{x_2}{x_1} = \frac{-2d_1}{p_1}\left(c_+ t_1 + \frac{1}{2}\right)$.

Moreover, having (t_1, t_2) satisfy

$$-\frac{d_2 - p_1}{d_2} < \frac{t_2}{t_1} < \frac{d_1^2}{d_2 p_1} - \frac{d_1}{d_2}\sqrt{\frac{d_1^2}{p_1^2} + \frac{2d_2 - 2p_1}{p_1}} \tag{**}$$

is a sufficient and necessary condition to have a second solution to, namely c_- from 3.10 below and (x_1, x_2) satisfying $\frac{x_2}{x_1} = \frac{-2d_1}{p_1}(c_- t_1 + \frac{1}{2})$.

$$c = \frac{-\left(\frac{d_1^2}{d_2 p_1} t_1 - t_2\right) \pm \sqrt{\left(\frac{d_1^2}{d_2 p_1} t_1 - t_2\right)^2 - 4\left(\frac{d_1^2}{d_2 p_1} t_1^2\right)\left(\frac{d_1^2}{4 p_1 d_2} + \dfrac{2d_2 - 2p_1}{4d_2}\right)}}{2\frac{d_1^2}{d_2 p_1} t_1^2} \tag{3.10}$$

Again, we do not have a second solution for

$$\frac{t_2}{t_1} = \frac{d_1^2}{d_2 p_1} - \frac{d_1}{d_2}\sqrt{\frac{d_1^2}{p_1^2} + \frac{2d_2 - 2p_1}{p_1}}$$

as this is precisely when $c_+ = c_-$ since the discriminant of c is 0. ∎

Corollary 3.18. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_1$ be noncompact simple with $\mathfrak{p}_2$ trivial. For G/H in this case, $ric = T$ has a solution T if and only if $t_2 = d_1^2 t_1^2 + d_1^2 t_1 + \frac{d_1^2}{4}$ for $t_1 \in (-\infty, \frac{-1}{2})$.

Proof: If $\mathfrak{p}_2$ is a trivial representation then $p_2 = 0$ and $p_1 = d_2 = 1$ (See Lemma 3.12) in our R_1 and R_2 defined in Eqn.3.1. Therefore, just as before we use the result in Theorem 3.13 to determine the solutions to $ric = T$. In this case, we have solutions of the form $t_2 = d_1^2 t_1^2 + d_1^2 t_1 + \frac{d_1^2}{4}$ with $t_1 \in (-\infty, \frac{-1}{2})$. ■

Corollary 3.19. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_1$ be noncompact simple with $\mathfrak{p}_2$ trivial. Recall our setting and the definition of some notation from Remark 3.8. For G/H in this case, the equation $ric = cT$ has a solution for any given (t_1, t_2) with $t_1 > 0$ and $t_2 < 0$. In this case, c is defined by 3.11 below and our inner product is defined by the (x_1, x_2) pair unique up to scaling satisfying $\frac{x_2}{x_1} = -2d_1(ct_1 + \frac{1}{2})$.

$$c = \frac{-(d_1^2 t_1 - t_2) + \sqrt{t_2^2 - 2d_1^2 t_1 t_2}}{2d_1^2 t_1^2} \tag{3.11}$$

Proof: Once again we have $p_2 = 0$ and $p_1 = d_2 = 1$ (See Lemma 3.12) in our R_1 and R_2 defined in Eqn.3.1. Moreover, by Lemma 3.12 we know that $\dfrac{2d_2 - p_2 - 2p_1}{4d_2} = 0$, which we discussed beforehand in Remark 3.15 as being a situation which would present some changes in the types of solutions that arise in the proof of Theorem 3.14. We discuss those changes now.

Note that if we simply plug into the first solution to $ric = cT$ from Theorem 3.14, then we have any (t_1, t_2) such that $\frac{t_2}{t_1} \leq 0$ as solutions with the c value being c_+ from

$$c = \frac{-(d_1^2 t_1 - t_2) \pm \sqrt{t_2^2 - 2d_1^2 t_1 t_2}}{2d_1^2 t_1^2}. \tag{3.12}$$

Recalling the proof of **Claim 2** in Theorem 3.14, we saw that the term $\dfrac{2d_2 - p_2 - 2p_1}{p_1} = 0$ causes us to have c values that are real if and only if such that $\frac{t_2}{t_1} \leq 0$, and we saw in the proof of **Claim 3** that as long as c was real, we had a solution with c_+ as the c value. However, as was mentioned in the proof of **Claim 2**, we are interested only in $t_2 < 0$ and $t_1 > 0$ by necessity of $c, R_1 > 0$ and $R_2 < 0$. Thus, we must exclude the $\frac{t_2}{t_1} = 0$ from our solution set, implying that the sufficient and necessary condition to having a solution is $\frac{t_2}{t_1} < 0$. Therefore, for any (t_1, t_2) with $t_1 < 0$ and $t_2 > 0$ gives a solution to $ric = cT$.

Moreover, as the proof of **Claim 4** in Theorem 3.14 shows, due to $\dfrac{2d_2 - p_2 - 2p_1}{p_1} = 0$ in this case, more than one solution does not exist because it would require $0 < \frac{t_2}{t_1} < 0$, which cannot happen. Thus, we do not have c_- in this case. This gives us the desired result. ■

3.3. $SO_0(1,7)/G_2$

In this section we consider $\mathfrak{so}(1,7) = \mathfrak{g}_2 \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$ in which $\mathfrak{p}'' \simeq \mathfrak{p}'$ with $\dim \mathfrak{p}'' = \dim \mathfrak{p}' = 7$. The goal of this section will be to determine the $ad_{\mathfrak{g}_2}$ invariant $T(.,.)$ for which we have solutions to $ric = T$ and $ric = cT$, also supplying a way to find $c > 0$ in the second equation.

In the subsequent material, we will work with an orthonormal basis with respect to $\langle .,. \rangle$ on $\mathfrak{p} = \mathfrak{p}'' \oplus \mathfrak{p}'$, $\{x_1, ..., x_{14}\}$, with $\{x_1, ..., x_7\}$ a basis for $\mathfrak{p}''$ and $\{x_8, ..., x_{14}\}$ a basis for $\mathfrak{p}'$. Consequently, our indexing here is opposite from the previous section. As will be discussed in Lemma 3.28 and Corollary 3.30, we are interested in working with a particular nice choice of an $\langle .,. \rangle$ orthonormal basis that will make our $ad_{\mathfrak{g}_2}$ equivariant maps have matrices where the blocks are diagonal. Corollary 3.30, in particular, show us that our T is determined by (t_1, t_2, t_3) such that $t_1 = T(x_1, x_1)$, $t_2 = T(x_8, x_8)$, and $t_3 = T(x_1, x_8)$ for our choice of basis. Therefore, we also have that ric is determined by (r_1, r_2, r_3) such that $r_1 = ric(x_1, x_1)$, $r_2 = ric(x_8, x_8)$, and $r_3 = ric(x_1, x_8)$.

Remark 3.20. Our basis of choice is determined later in Appendix A.1 where we used Sympy in Python ([MSP+17]) to determine our basis and compute ric. To get the diagonal blocks we wanted for our equivariant maps once we had an orthonormal basis with respect to $\langle .,. \rangle$, to our elated surprise, all that was required of us was to re-order it.

Theorem 3.21. For $SO_0(1,7)/G_2$ with (t_1, t_2, t_3) being defined as above, there is a T such that $ric = T$ if and only if (t_1, t_2, t_3) is such that:

Case 1 If $t_3 = 0$ then $(t_1, t_2, 0)$ is such that $t_1 = 6t_2^2 + 6t_2 + \frac{15}{8}$ with $t_2 < \frac{-1}{2}$.

Case 2 If $t_3 \neq 0$ then (t_1, t_2, t_3) is such that

$$t_1 = \begin{cases} f_1(t_2, t_3)\,, & t_2 \leq \frac{-3}{4} \text{ and } |t_3| > 0 \\ f_1(t_2, t_3)\,, & -\frac{3}{4} < t_2 \leq \frac{-1}{2} \text{ and } |t_3| > 0 \text{ and } |t_3| \neq \frac{1}{2}\sqrt{\frac{3}{2}}\sqrt{4t_2+3} \\ f_1(t_2, t_3)\,, & t_2 > \frac{-1}{2} \text{ and } |t_3| > \frac{\sqrt{3}}{4}\sqrt{2t_2+1} \text{ and } |t_3| \neq \frac{1}{2}\sqrt{\frac{3}{2}}\sqrt{4t_2+3} \\ \frac{3}{4}\,, & t_2 > \frac{-3}{4} \text{ and } |t_3| = \frac{1}{2}\sqrt{\frac{3}{2}}\sqrt{4t_2+3} \end{cases}$$

where f_1 is described by Eqns.3.13 below.

Theorem 3.22. For $SO_0(1,7)/G_2$ with (t_1, t_2, t_3) being defined as above, there is a T such that $ric = cT$ for some $c > 0$ if and only if T is determined by (t_1, t_2, t_3) such that:

Case 1 If $t_3 = 0$ then (t_1, t_2, t_3) is such that $\frac{1}{3}\left(-\sqrt{5}-2\right) \leq \frac{t_2}{t_1} < 0$. In this case for a given solution (t_1, t_2, t_3), $c = \frac{c_0}{t_1}$ where c_0 is the solution(s) to the implicit equation $c_0 = 6(c_0\frac{t_2}{t_1})^2 + 6c_0\frac{t_2}{t_1} + \frac{15}{8}$. When $\frac{t_2}{t_1}$ has the conditions above, we always have real solutions for c_0.

Case 2 If $t_3 \neq 0$ then (t_1, t_2, t_3) is such that $\frac{t_2}{t_1} = l$, and $\frac{|t_3|}{t_1} = m$ in the 14 regions defined in Step 6 (3.3) below. For a given solution (t_1, t_2, t_3), we have $c = \frac{c_0}{t_1}$ where c_0 is the solution(s) to the implicit equation $c_0 = f_1(c_0\dfrac{t_2}{t_1}, c_0\dfrac{t_3}{t_1})$ and f_1 is as defined in Theorem 3.21, unless (t_1, t_2, t_3) is a multiple of $(\frac{3}{4}, r_2, r_3)$ with $r_2 > \frac{-3}{4}$ and $|r_3| = \frac{1}{2}\sqrt{\frac{3}{2}}\sqrt{4r_2+3}$. In this case, $c = \frac{c_1}{t_1}$ where $c_1 = \frac{3}{4}$. For $\frac{t_2}{t_1}$ and $\frac{t_3}{t_1}$ in the regions above, we always have real solutions for c_0.

Remark 3.23. Comparing our solution in **Case 1** of Theorem 3.21 with our solution from Theorem 3.13, and knowing that $d_1 = d_2 = 7$ in this case, we can see that $p_1 = p_2 = \frac{7}{6}$.

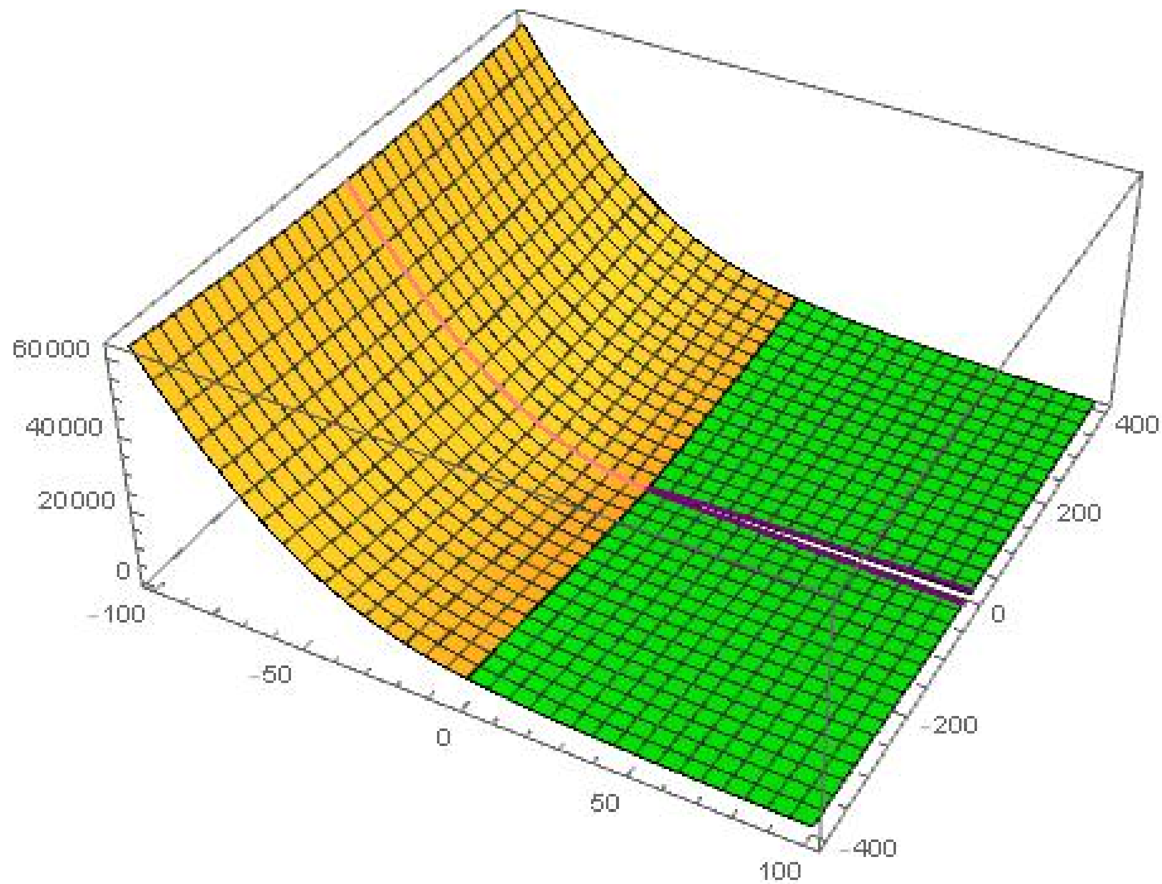
Remark 3.24. When we get to the end of Step 5 (3.3) with the Appendix references provided, we have a proof for Theorem 3.21 above. When we get to the end of Step 6 (3.3) with the Appendix references provided, we have a proof of Theorem 3.22 above.

Let $f_1(t_2, t_3)$ be the first root in the set of roots (in increasing order) to the following polynomial in t_1

$$128t_1^3 + t_1^2\left(-768t_2^2 - 768t_2 - 432\right) + t_1\left(1152t_2^2 + 1536t_2t_3^2 + 1152t_2 + 1536t_3^2 + 432\right) - 432t_2^2 - 432t_2 - 768t_3^4 - 288t_3^2 - 135 \tag{3.13}$$

This root has the following radical form, but we caution that radical forms involving parameters and not just constant values can lose solutions.

$$\begin{aligned}
&\frac{1}{8}\left(16t_2^2 + 16t_2 + 9\right) \\
&+ \frac{1}{8}\left(\sqrt[3]{\left(16t_2^2 + 16t_2 + 3\right)^3 - 192(4t_2 + 3)(2t_2(4t_2 + 5) + 5)t_3^2 + 8\sqrt{2}\sqrt{-\left((4t_2 + 1)^3 - 72t_3^2\right)\left(3(4t_2 + 3)^2t_3 + 16t_3^3\right)^2} + 1536t_3^4}\right) \\
&+ \frac{1}{8}\left(\frac{(4t_2 + 1)^2(4t_2 + 3)^2}{\sqrt[3]{\left(16t_2^2 + 16t_2 + 3\right)^3 - 192(4t_2 + 3)(2t_2(4t_2 + 5) + 5)t_3^2 + 8\sqrt{2}\sqrt{-\left((4t_2 + 1)^3 - 72t_3^2\right)\left(3(4t_2 + 3)^2t_3 + 16t_3^3\right)^2} + 1536t_3^4}}\right) \\
&- \frac{1}{8}\left(\frac{256(t_2 + 1)t_3^2}{\sqrt[3]{\left(16t_2^2 + 16t_2 + 3\right)^3 - 192(4t_2 + 3)(2t_2(4t_2 + 5) + 5)t_3^2 + 8\sqrt{2}\sqrt{-\left((4t_2 + 1)^3 - 72t_3^2\right)\left(3(4t_2 + 3)^2t_3 + 16t_3^3\right)^2} + 1536t_3^4}}\right)
\end{aligned}$$

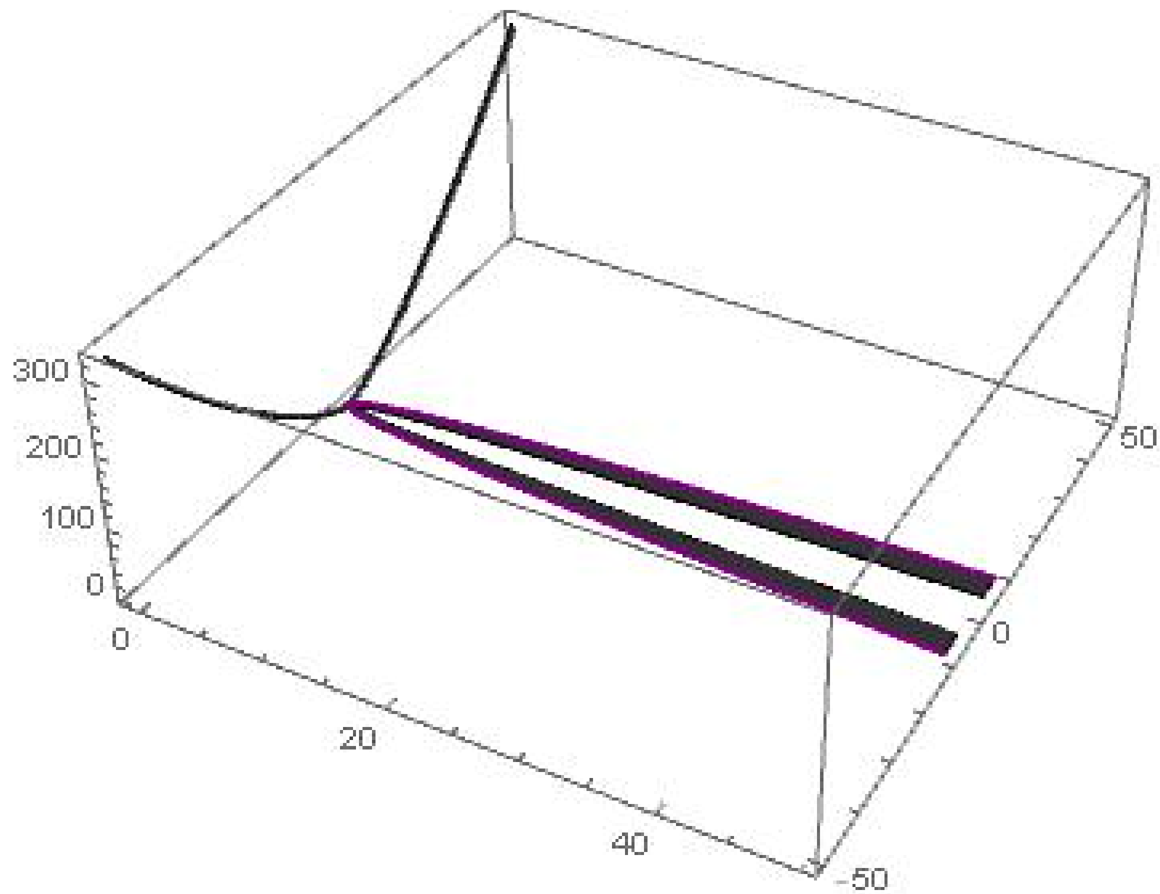


The above image was produced in Mathematica for the image of ric in Theorem 3.21.

The gold sheet curving up has a pink strip that can be faintly seen in the middle. The gold is the $z, t_3 \neq 0$ with $t_2 \leq \frac{-3}{4}$ solution but only graphed out to $-100 \leq t_2$, and the pink strip is $z, t_3 = 0$ which has $t_2 \leq \frac{-1}{2}$, also only graphed out to $-100 \leq t_2$. The gold and the pink were graphed with $0 < |t_3| \leq 400$.

The green is the image where $z \neq 0$, $t_2 > \frac{-1}{2}$, and $|t_3| > \frac{1}{2}\sqrt{\frac{3}{2}}\sqrt{3 + 4t_2}$. Here, we only graphed out to $t_2 \leq 100$ with $|t_3| \leq 400$.

There are other colors that can be seen in the image above describing the other parts of our solution to $ric = T$. To help show them somewhat more clearly, we have the following image which has more restrictive bounds on the parameters ($t_2 \leq 50$ and $|t_3| < 50$ instead of the values used above of 100 and 400, respectively). Still, in this image, there are some strips that are hard to see (such as the solutions where $t_1 = \frac{3}{4}$). The piece going upward is the image where $-\frac{3}{4} < t_2 \leq -\frac{1}{2}$ and the the rest is a combination of the image where $-\frac{1}{2} < t_2$ and $\frac{\sqrt{3}}{4}\sqrt{2t_2+1} < |t_3| < \frac{1}{2}\sqrt{\frac{3}{2}}\sqrt{4t_2+3}$ and the image where $t_1 = \frac{3}{4}$.



An Overview of our Approach:

In the case of $SO_0(1,7)/G_2$, we know from I.16 in [DK08] and our dual process in Theorem 3.2 that $\mathfrak{so}(1,7) = \mathfrak{g}_2 \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$ is such that $\mathfrak{p}'' \simeq \mathfrak{p}'$ of dimension 7. Since $ric_g(.,.)$ is dependent upon the metric g (see Section 1.1), if we want to solve $ric_g = T$ and $ric_g = cT$, then we will need to understand ric_g for any possible metric. Any $SO_0(1,7)$-invariant metric on $SO_0(1,7)/G_2$ can be understood as an $ad_{\mathfrak{g}_2}$ invariant inner product on $\mathfrak{p} = \mathfrak{p}'' \oplus \mathfrak{p}'$ (see Section 1.2), so we use all such inner products and our equation for $ric(.,.)$ in Eqn.1.2 to determine the possible values of $ric(.,.)$. Working out the solutions to $ric = T$ and $ric = cT$ turns out to be an involved problem even in this simple example, so we first provide a step-by-step overview of the process involved in reaching our solution before getting into the details.

Step 1 Determine a description of an arbitrary $ad_{\mathfrak{g}_2}$ invariant inner product $(.,.) = \langle \Phi .,. \rangle = \langle \phi .,\phi . \rangle$ where $\Phi : \mathfrak{p} \to \mathfrak{p}$ is a positive definite $ad_{\mathfrak{g}_2}$ equivariant map and $\phi : \mathfrak{p} \to \mathfrak{p}$ is such that $\phi^2 = \Phi$ and is also an $ad_{\mathfrak{g}_2}$ equivariant map. In this step, we determine that ϕ is dependent upon three variables (a,b,c). We also determine that Φ is determined by three variables which we label (x,y,z) and we note the polynomial relationship between (a,b,c) and (x,y,z).

Step 2 Using the arbitrary inner product description of $(.,.) = \langle \phi .,\phi . \rangle$ (since defining an orthonormal basis on $\mathfrak{p}$ for $(.,.)$ is easiest with this description), we find a formula for $ric(.,.)$ for an arbitrary $ad_{\mathfrak{g}_2}$ inner prodcut. The formula obtained depends only upon ϕ, $\langle .,. \rangle$, and $\{x_i\}$ where $\{x_i\}$ is a $\langle .,. \rangle$-orthonormal basis on $\mathfrak{p}$.

Step 3 Compute $ric(.,.)$ to obtain a function dependent only upon our (a,b,c) defining ϕ using the description found in Step 2. For this, we use SymPy in Python ([MSP+17]) and acquire three terms r_1, r_2, and r_3 which are scale-invariant rational functions.

Step 4 Using polynomial relationships between (a, b, c) for ϕ and (x, y, z) for Φ, we use an algebraic geometry tool known as elimination ideal to get our r_i in terms of (x, y, z). This step was made possible by the built-in elimination ideal function in Mathmematica ([Inc]), and cuts the degree of the polynomials (in both the numerator and denominator of our r_i) in half.

Step 5 Using Mathematica ([Inc]), we use built-in functions and utilize the scale-invariance of our r_i to find all (t_1, t_2, t_3) such that $(r_1, r_2, r_3) = (t_1, t_2, t_3)$ for some (x, y, z) defining our $ad_{\mathfrak{g}_2}$ invariant inner product, thus solving the problem of $ric = T$.

Step 6 Using built-in functions from Mathematica ([Inc]), and by projecting (r_1, r_2, r_3) onto a plane, we find all (t_1, t_2, t_3) such that there is a c and an (x, y, z) with $(r_1, r_2, r_3) = c(t_1, t_2, t_3)$. We also provide a description of the c values needed for a given (t_1, t_2, t_3).

As we work through these six steps, we provide the mathematics here, and we explain some of the methods used in our code. However, we save the code from Python and Mathematica for Appendix A.

Remark 3.25. Step 4 turned out to be a necessary step for us. The following two steps were attempted using the r_i in terms of (a, b, c), but the Mathematica functions utilized did not finish processing, even after more than 24 hours of run-time.

Remark 3.26. In Step 5, using the scale invariance also turned out to be necessary as the functions in Mathematica spent hours running with no output without using the scale invariance.

Remark 3.27. In Mathematica, we used ***AbsoluteTiming*** to find the run times for obtaining solutions to $ric = T$ and $ric = cT$ using the methodology discussed in Step 5 and Step 6, respectively. The run time for solving $ric = T$ was 126.031 seconds (so just over 2 minutes), and the run time for solving $ric = cT$ was 346.735 seconds (so just under 6 minutes).

Step 1

Any $ad_{\mathfrak{g}_2}$ invariant inner product $(.,.)$ on $\mathfrak{p}$ can be determined by $(v, w) = \langle \Phi v, w \rangle = \langle \phi v, \phi w \rangle$ where $\Phi = \phi^2$ is a positive definite $ad_{\mathfrak{g}_2}$ equivariant map and ϕ is an invertible symmetric map. More than that, ϕ is also an $ad_{\mathfrak{g}_2}$ equivariant map since Φ has a matrix representation that is diagonal for a basis of eigenvectors of $\mathfrak{p}$ which implies that ϕ also has a matrix representation that is diagonal for the same basis of eigenvectors. Thus, $\Phi \circ ad_{\mathfrak{g}_2} = ad_{\mathfrak{g}_2} \circ \Phi$ will imply $\phi \circ ad_{\mathfrak{g}_2} = ad_{\mathfrak{g}_2} \circ \phi$. In this first step, we seek to understand the form of Φ and ϕ so that we can ultimately understand any $ad_{\mathfrak{g}_2}$ invariant inner product in terms of Φ or ϕ, and so that we can understand the algebraic relationship between Φ and ϕ.

Lemma 3.28. For $\mathfrak{so}(1,7) = \mathfrak{g}_2 \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$, let $\mathfrak{p} = \mathfrak{p}'' \oplus \mathfrak{p}'$. If $M : \mathfrak{p} \to \mathfrak{p}$ is a symmetric (with respect to $\langle .,. \rangle$) $ad_{\mathfrak{g}_2}$ equivariant map then for a nice choice of basis, M has a matrix of the form:

$$M = \left[\begin{array}{c:c} aId_{\mathfrak{p}''} & cId \\ \hdashline cId & bId_{\mathfrak{p}'} \end{array}\right]$$

where $a, b, c \in \mathbb{R}$. Here, unless $c = 0$, the nice basis of choice is $\{x_1, ..., x_7, Lx_1, ...Lx_7\}$ where $\{x_i\}$ is an orthonormal basis with respect to $\langle .,. \rangle$ on $\mathfrak{p}''$ and $L : \mathfrak{p}'' \to \mathfrak{p}'$ is an $ad_{\mathfrak{g}_2}$ intertwining map defined by $L = \text{proj}_{\mathfrak{p}'} \circ M|_{\mathfrak{p}''}$. If $c = 0$ then the basis on $\mathfrak{p}$ can be any basis orthonormal with respect to $\langle .,. \rangle$.

Proof: First, if M is a symmetric (with respect to $\langle .,. \rangle$) $ad_{\mathfrak{g}_2}$ equivariant map then we know that in general,

$$M = \left[\begin{array}{c|c} A & L^t \\ \hline L & B \end{array}\right]$$

where $A : \mathfrak{p}'' \to \mathfrak{p}''$ is symmetric, $B : \mathfrak{p}' \to \mathfrak{p}'$ is symmetric, L is defined as in the the statement of our Lemma, and L^t is transposed with respect to $\langle .,. \rangle$ (along with the symmetry of A and B). Since $\mathfrak{p}''$ and $\mathfrak{p}'$ are irreducible, and since A and B are symmetric, we know by Schur's Lemma (See Section 1.3) that $A = aId_{\mathfrak{p}''}$ and $B = bId_{\mathfrak{p}'}$. Moreover, since $\dim\mathfrak{p}'' = \dim\mathfrak{p}' = 7$, we know that $\mathfrak{p}''$ and $\mathfrak{p}'$ are irreducible representations of real type (See Section 1.6), meaning that $a, b \in \mathbb{R}$.

Now, $L : \mathfrak{p}'' \to \mathfrak{p}'$ is (by Schur's Lemma) an isomorphism or 0. If 0, then we are done and $c = 0$ in the statement of the Lemma. If L is an isomorphism then we know again by Schur's Lemma that $LL^t = \lambda Id_{\mathfrak{p}''}$ with $\lambda \in \mathbb{R}$. Moreover, by choosing $\{x_1, ..., x_7, Lx_1, ..., Lx_7\}$ as a basis for $\mathfrak{p}$ where $\{x_1, ..., x_7\}$ is an orthonormal basis for $\mathfrak{p}''$, we know that L becomes a diagonal matrix. Thus, we know that $L = cId$ with $c \in \mathbb{R}$ by $\mathfrak{p}''$ and $\mathfrak{p}'$ being irreducible of real type. ■

Remark 3.29. It is worth noting that in the case that $L : \mathfrak{p}'' \to \mathfrak{p}'$ is not the 0 map we have that any other $ad_{\mathfrak{g}_2}$ intertwining map $\mathfrak{p}'' \to \mathfrak{p}'$ is a multiple of L. Indeed, if $N : \mathfrak{p}'' \to \mathfrak{p}'$ was another $ad_{\mathfrak{g}_2}$ intertwining map, then $N^{-1}L = \lambda Id_{\mathfrak{p}''}$ with $\lambda \in \mathbb{R}$ by $\mathfrak{p}''$ being irreducible of real type. Thus, $N = \lambda L$. This is worth noting since our basis was dependent upon L, but there is really only one choice for L up to scaling.

Corollary 3.30. For $\mathfrak{so}(1,7) = \mathfrak{g}_2 \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$, let $\mathfrak{p} = \mathfrak{p}'' \oplus \mathfrak{p}'$. If $T(.,.)$ is an $ad_{\mathfrak{g}_2}$ invariant bilinear form then

$$T(v,w) = \begin{cases} t_1\langle v,w\rangle \ , \ v,w \in \mathfrak{p}'' \\ t_2\langle v,w\rangle \ , \ v,w \in \mathfrak{p}' \\ t_3\langle v,w\rangle \ , \ v \in \mathfrak{p}'' \ , \ w \in \mathfrak{p}' \end{cases}$$

where $(t_1, t_2, t_3) \in \mathbb{R}^3$.

Proof: If $T(.,.)$ is an $ad_{\mathfrak{g}_2}$ bilinear form then, as discussed in Section 1.3, $T(v,w) = \langle Mv, w\rangle$ where $v, w \in \mathfrak{p}$ and M is symmetric with respect to $\langle .,.\rangle$ and an $ad_{\mathfrak{g}_2}$ equivariant map. By Lemma 3.28 we know that for a nice choice of basis, M can be described by:

$$M = \left[\begin{array}{c|c} t_1 Id_{\mathfrak{p}''} & t_3 Id \\ \hline t_3 Id & t_2 Id_{\mathfrak{p}'} \end{array}\right]$$

for $t_1, t_2, t_3 \in \mathbb{R}$. Thus, we have that

$$T(v,w) = \langle Mv, w\rangle = \begin{cases} t_1\langle v,w\rangle \ , \ v,w \in \mathfrak{p}'' \\ t_2\langle v,w\rangle \ , \ v,w \in \mathfrak{p}' \\ t_3\langle v,w\rangle \ , \ v \in \mathfrak{p}'' \ , \ w \in \mathfrak{p}', \end{cases}$$

as desired. ■

Lemma 3.31. For $\mathfrak{so}(1,7) = \mathfrak{g}_2 \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$, let $\mathfrak{p} = \mathfrak{p}'' \oplus \mathfrak{p}'$. Any $ad_{\mathfrak{g}_2}$ invariant inner product $(.,.)$ can be written as $(.,.) = \langle \Phi.,.\rangle = \langle \phi.,\phi.\rangle$ where Φ is positive definite, ϕ is symmetric and invertible, and both Φ and ϕ are $ad_{\mathfrak{g}_2}$ equivariant maps. Moreover, for a nice choice of basis (See Lemma 3.28) we have the following descriptions of Φ and ϕ:

$$\Phi = \left[\begin{array}{c|c} xId_{\mathfrak{p}''} & zId \\ \hline zId & yId_{\mathfrak{p}'} \end{array}\right] \text{ with } x, y > 0 \text{ and } xy - z^2 > 0$$

$$\phi = \left[\begin{array}{c|c} aId_{\mathfrak{p}''} & cId \\ \hline cId & bId_{\mathfrak{p}'} \end{array}\right] \text{ with } ab - c^2 \neq 0.$$

Proof: In the following proof, we first prove the statement regarding Φ. Then, using the positive definiteness of Φ and properties of square roots, we then prove the statement regarding ϕ. When proving the statement regarding ϕ, we prove the statement in the 2×2 matrice setting and then show that the statement holds in the 14×14 setting we are in.

As discussed in Section 1.3, we know that any $ad_{\mathfrak{g}_2}$ invariant metric on $\mathfrak{p} = \mathfrak{p}'' \oplus \mathfrak{p}'$ is of the form $(.,.) = \langle \Phi .,. \rangle$ where $\Phi : \mathfrak{p} \to \mathfrak{p}$ is a symmetric (with respect to $\langle .,. \rangle$) positive definite $ad_{\mathfrak{g}_2}$ equivariant map. Moreover, since Φ is symmetric with respect to $\langle .,. \rangle$, we know by by Lemma 3.28 that for a nice choice of basis we have Φ of the form

$$\Phi = \left[\begin{array}{c|c} xId_{\mathfrak{p}''} & zId \\ \hline zId & yId_{\mathfrak{p}'} \end{array}\right].$$

Since Φ is positive definite, $tr\Phi = 7(x+y) > 0$, and $det\Phi = (xy - z^2)^7 > 0$, we can conclude that $x, y > 0$ and $xy - z^2 > 0$. Thus,

$$P = \left\{ \left[\begin{array}{c|c} xId_{\mathfrak{p}''} & zId \\ \hline zId & yId_{\mathfrak{p}'} \end{array}\right] : x, y > 0 \text{ and } xy - z^2 > 0 \right\}$$

is the set describing all positive definite matrices on $\mathfrak{p}$. This concludes the proof for the first statement regarding Φ.

By being positive definite, Φ has a square root matrix ϕ such that $\phi^2 = \Phi$. To complete the proof, we consider the 2×2 matrix setting, determining what the collection of ϕ is, and then we show that understanding the 2×2 matrix setting is enough to determine the

14×14 matrix setting our problem is placed within.

In the 2×2 setting, we consider positive definite $\Phi = \begin{bmatrix} x & z \\ z & y \end{bmatrix}$. By being positive definite, there is a C such that $L = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = C\Phi C^{-1}$ where λ_1, λ_2 are the eigenvalues of Φ. Since $trL = tr\Phi$ and $detL = det\Phi$, we know that Φ is positive definite if and only if $tr\Phi > 0$ and $det\Phi > 0$. Therefore, Φ is positive definite if and only if $x, y > 0$ and $xy - z^2 > 0$.

Now L has square root matrices described by, $l = \begin{bmatrix} \pm\sqrt{\lambda_1} & 0 \\ 0 & \pm\sqrt{\lambda_2} \end{bmatrix}$, and one can observe that all symmetric ϕ such that $\phi^2 = \Phi$ are such that $l = C\phi C^{-1}$ for some l. Since l can be any diagonal matrix with nonzero determinant (because the λ_i can be anything positive), it is then the case that ϕ can be any symmetric matrix with nonzero determinant.

Now we show that the collection of invertible, symmetric 2×2 matrices generates all positive definite 2×2 matrices by squaring, and then show how the 14×14 matrice setting has the same multiplication structure, completing our proof.

Let $\phi = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$ with $ab - c^2 \neq 0$. In this case, $\phi^2 = \begin{bmatrix} a^2 + c^2 & c(a+b) \\ c(a+b) & b^2 + c^2 \end{bmatrix}$, which is positive definite if and only if $tr\phi^2 > 0$ and $det\phi^2 > 0$, and this is the case if and only if

$$(a^2 + c^2)(b^2 + c^2) - c^2(a+b)^2 > 0.$$

$$\text{Now, } (a^2 + c^2)(b^2 + c^2) - c^2(a+b)^2 = a^2b^2 + c^4 - c^2(2ab)$$

$$= (ab - c^2)^2,$$

and $(ab - c^2)^2 > 0$ if and only if $ab - c^2 \neq 0$. Thus, ϕ^2 is clearly positive definite. Lastly, observe that $a^2 + c^2$ and $b^2 + c^2$ can be any positive number while $c(a + b)$ can be any nonzero number. Thus, in the 2×2 setting, the invertible, symmetric $\{\phi\}$ generates the positive definite $\{\Phi\}$ by squaring each ϕ.

To extend this to the 14×14 matrix case is now a trivial observation that if, in the 2×2 setting, $\{\phi\}$ generates all possible positive definite matrices $\{\Phi\}$ by squaring each ϕ, then by having the same multiplication structure:

$$\left[\begin{array}{c:c} aId_{\mathfrak{p}''} & cId \\ \hdashline cId & bId_{\mathfrak{p}'} \end{array}\right]\left[\begin{array}{c:c} aId_{\mathfrak{p}''} & cId \\ \hdashline cId & bId_{\mathfrak{p}'} \end{array}\right] = \left[\begin{array}{c:c} (a^2 + c^2)Id_{\mathfrak{p}''} & c(a + b)Id \\ \hdashline c(a + b)Id & (b^2 + c^2)Id_{\mathfrak{p}'} \end{array}\right]$$

we can see that the analogous statement is true in the 14×14 case. That is, P as defined above, is generated by

$$\left\{ \left[\begin{array}{c:c} aId_{\mathfrak{p}''} & cId \\ \hdashline cId & bId_{\mathfrak{p}'} \end{array} \right] : ab - c^2 \neq 0 \right\}$$

by squaring the matrices. Thus, any inner product $(.,.) = \langle \Phi.,.\rangle = \langle \phi^2.,.\rangle = \langle \phi.,\phi.\rangle$, with Φ and ϕ having the form desired. Moreover, since ϕ has the same block form as Φ, it is clear that Φ being $ad_{\mathfrak{g}_2}$ equivariant implies that ϕ is $ad_{\mathfrak{g}_2}$ equivariant. ■

Observe from how our $\{\phi\}$ generates our $\{\Phi\}$ in the proof above, that we also have a polynomial relationship between the (a, b, c) defining ϕ and the (x, y, z) defining Φ:

$$\begin{aligned} a^2 + c^2 &= x \\ b^2 + c^2 &= y \\ c(a + b) &= z. \end{aligned} \tag{3.14}$$

Step 2

In the present step, we will be finding a formula for $ric(.,.)$ that is dependent only upon ϕ, our fixed inner product $\langle .,. \rangle$, and a $\langle .,. \rangle$ orthonormal basis, $\{x_i\}$. In the end, we provide a formula for $ric(x, y)$ in which one can determine how to work with matrix representations of ϕ and $ad_{\mathfrak{p}}$ with respect to the basis $\{x_i\}$ (except for the Killing form term which we intentionally leave as is).

Lemma 3.32. Consider G/H to be any homogeneous space for which G is unimodular with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$. For an arbitrary $ad_{\mathfrak{h}}$ invariant inner product $(.,.) = \langle \phi.,\phi. \rangle$, we have the following formula for $ric(.,.)$ in terms of the base inner product $\langle .,. \rangle$, a $\langle .,. \rangle$ orthonormal basis $\{x_i\}$, and ϕ.

$$ric(x, y) = \frac{-1}{2}\sum_{x_i}\langle \phi(\sum_{k}(\phi^{-1})_{ki}ad_{\mathfrak{p}}((x_k))(x)), \phi(\sum_{k}(\phi^{-1})_{ki}ad_{\mathfrak{p}}((x_k))(y))\rangle - \frac{1}{2}B(x, y)$$

$$+\frac{1}{4}\sum_{x_i,x_j}\langle \phi(\sum_{k}(\phi^{-1})_{ki}ad_{\mathfrak{p}}((x_k))(\phi^{-1}(x_j))), \phi(x)\rangle\langle \phi(\sum_{k}(\phi^{-1})_{ki}ad_{\mathfrak{p}}((x_k))(\phi^{-1}(x_j))), \phi(y)\rangle$$

Proof: Using Eqn.1.2, we have the following in which, by an abuse of notation, we consider $ad_{\mathfrak{p}}(x)$ to be defined by $[x,.]_{\mathfrak{p}}$ where $x \in \mathfrak{p}$:

$$\begin{aligned}
ric(x, y) &= \frac{-1}{2}\sum_{x_i}([\phi^{-1}(x_i), x]_{\mathfrak{p}}, [\phi^{-1}(x_i), y]_{\mathfrak{p}}) - \frac{1}{2}B(x, y) \\
&+ \frac{1}{4}\sum_{x_i,x_j}([\phi^{-1}(x_i), \phi^{-1}(x_j)], x)([\phi^{-1}(x_i), \phi^{-1}(x_j)], y) \\
&= \frac{-1}{2}\sum_{x_i}(ad_{\mathfrak{p}}(\phi^{-1}(x_i)(x)), ad_{\mathfrak{p}}(\phi^{-1}(x_i)(y))) - \frac{1}{2}B(x, y) \\
&+ \frac{1}{4}\sum_{x_i,x_j}(ad_{\mathfrak{p}}(\phi^{-1}(x_i))(\phi^{-1}(x_j)), x)(ad_{\mathfrak{p}}(\phi^{-1}(x_i))(\phi^{-1}(x_j)), y) \\
&= \frac{-1}{2}\sum_{x_i}\langle \phi \circ ad_{\mathfrak{p}}(\phi^{-1}(x_i)(x)), \phi \circ ad_{\mathfrak{p}}(\phi^{-1}(x_i)(y))\rangle - \frac{1}{2}B(x, y)
\end{aligned}$$

$$+\frac{1}{4}\sum_{x_i,x_j}\langle\phi\circ ad_{\mathfrak{p}}(\phi^{-1}(x_i))(\phi^{-1}(x_j)),\phi(x)\rangle\langle\phi\circ ad_{\mathfrak{p}}(\phi^{-1}(x_i))(\phi^{-1}(x_j)),\phi(y)\rangle.$$

Using the linearity of ad and the fact that $\phi^{-1}(x_i) = (\phi^{-1})_{1i}x_1 + ... + (\phi^{-1})_{ni}x_n$ where $(\phi^{-1})_{ij} = \langle\phi^{-1}x_j, x_i\rangle$ are the matrix entries given by a matrix representation of ϕ^{-1} with respect to the basis $\{x_i\}$, have our result:

$$ric(x,y) = \frac{-1}{2}\sum_{x_i}\langle\phi(\sum_k(\phi^{-1})_{ki}ad_{\mathfrak{p}}((x_k))(x)),\phi(\sum_k(\phi^{-1})_{ki}ad_{\mathfrak{p}}((x_k))(y))\rangle - \frac{1}{2}B(x,y)+$$
$$\frac{1}{4}\sum_{x_i,x_j}\langle\phi(\sum_k(\phi^{-1})_{ki}ad_{\mathfrak{p}}((x_k))(\phi^{-1}(x_j))),\phi(x)\rangle\langle\phi(\sum_k(\phi^{-1})_{ki}ad_{\mathfrak{p}}((x_k))(\phi^{-1}(x_j))),\phi(y)\rangle \tag{3.15}$$

■

Step 3

Having our description of $ric(.,.)$ for an arbitrary inner product, we can see that it is described totally in terms of the Killing form $B(.,.)$, the background inner product $\langle .,. \rangle$, a basis $\{x_i\}$ on $\mathfrak{p}$ for which $\langle .,. \rangle$ is orthonormal, the $ad_{\mathfrak{p}}(x_k)$ maps, and an equivariant map ϕ. We now turn our attention to finding $ric(x, y)$ for an arbitrary inner product on $\mathfrak{p} = \mathfrak{p}'' \oplus \mathfrak{p}'$ in $\mathfrak{so}(1, 7) = \mathfrak{g}_2 \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$. Using Eqn.3.15, our process is as follows:

i. Fix a background inner product and get an orthonormal basis for $\mathfrak{p} = \mathfrak{p}'' \oplus \mathfrak{p}'$.

ii. Find the equivariant maps ϕ described in Lemma 3.31 by finding an appropriately ordered basis.

iii. Find the matrices $ad_{\mathfrak{p}}(x_k)$ for each x_k coming from the basis in i.

iv. Determine $ric(.,.)$ in terms of the a, b, c defining ϕ (which describes an arbitrary $ad_{\mathfrak{g}_2}$ invariant inner product).

To do this, we use Python ([MSP+17]) and Maple ([Map]) with the code found in Appendix A.1.

First, we discuss part i which is seeking to fix a background inner product and get an orthonormal basis for $\mathfrak{p} = \mathfrak{p}'' \oplus \mathfrak{p}'$. In this step we used Maple to produce a collection of matrices representing a basis of $\mathfrak{g}_2$ as a subalgebra of $\mathfrak{so}(7)$ as well as matrices representing a basis for $\mathfrak{so}(7)$, the maximal compact in $\mathfrak{so}(1, 7)$. From there, we were able to copy and paste the matrices into Python, and in Python we built out the rest of $\mathfrak{so}(1, 7)$ using Type BD I in Section 2 of Chapter X in [Hel01] to ensure that we appropriately constructed $\mathfrak{so}(1, 7)$. This process involved finding a basis appropriate for $\mathfrak{p}''$ in $\mathfrak{so}(7)$ so that $\mathfrak{g}_2 \oplus \mathfrak{p}'' = \mathfrak{so}(7)$. We describe this process below with the code provided in Appendix A.1.

Using the matrices for bases of $\mathfrak{g}_2$ and $\mathfrak{so}(7)$, we constructed a 7 dimensional complement to $\mathfrak{g}_2$ in $\mathfrak{so}(7)$ which we call $\mathfrak{q}$. We wish for this to not be just any complement $\mathfrak{q}_0$ where $\mathfrak{g}_2 \oplus \mathfrak{q}_0 = \mathfrak{so}(7)$, rather, we want a $\mathfrak{p}''$ such that $\mathfrak{g}_2 \oplus \mathfrak{p}'' = \mathfrak{so}(7)$ and $\mathfrak{p}''$ is an $ad_{\mathfrak{h}}$ invariant complement to $\mathfrak{g}_2$ with $\langle \mathfrak{g}_2, \mathfrak{p}'' \rangle = 0$. Thus, we begin to construct our $\langle ., . \rangle$ by defining it to be $\langle u, v \rangle = -6tr(uv)$ for $u, v \in \mathfrak{so}(7)$ which is $-B(u, v)$ for $\mathfrak{so}(1, 7)$ restricted to $\mathfrak{so}(7)$ (See Section 8 in Chapter III of [Hel01] and Lemma 1.10). Using this $ad_{\mathfrak{g}_2}$ invariant inner product on $\mathfrak{so}(7)$, our matrices describing bases for $\mathfrak{g}_2$ and $\mathfrak{q}$, and the Gram-Schmidt process, we are able to get an orthonormal basis for $\mathfrak{p}''$ with the desired trait that $\langle \mathfrak{g}_2, \mathfrak{p}'' \rangle = 0$.

Having matrices defining a basis for $\mathfrak{g}_2 \oplus \mathfrak{p}'' = \mathfrak{so}(7)$, we then embed those matrices into $\mathfrak{so}(1, 7)$ by adding a row of 0's above and a column of 0's to the left. Lastly, we get a basis of matrices for $\mathfrak{p}'$, using Type BD I from [Hel01] to define them. We then rescale them so that they will be orthonormal with respect to our inner product on $\mathfrak{p}'$ which is defined by $\langle u, v \rangle = 6tr(uv) = B(u, v)$ for $u, v \in \mathfrak{p}'$. (So we are working with the same fixed inner product on $\mathfrak{p} = \mathfrak{p}'' \oplus \mathfrak{p}'$ that we have used throughout this document, $\langle ., . \rangle = B_{\mathfrak{p}'} - B_{\mathfrak{p}''}$, as mentioned in Remark 1.23.)

One last thing is done at this stage, and that is to reorder the basis of $\mathfrak{p}''$ which will ensure that our ϕ we will soon define has the correct form as decribed by Lemma 3.31.

After we constructed our basis of $\mathfrak{so}(1, 7) = \mathfrak{g}_2 \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$ as described above, we ran a series of checks to ensure that $\langle \mathfrak{g}_2, \mathfrak{p}'' \rangle = \langle \mathfrak{p}'', \mathfrak{p}' \rangle = 0$ and that our basis on $\mathfrak{p} = \mathfrak{p}'' \oplus \mathfrak{p}'$ is is orthonormal with respect to $\langle ., . \rangle$. These checks have their code provided in Appendix A.1.

We now turn to part ii which was concerned with finding the equivariant maps ϕ described in Lemma 3.31 by finding an appropriately ordered basis. The code for this and the checks is provided in Appendix A.1. The list of matrices describing our basis is provided in Appendix A.1.

In this process, we constructed matrices representing $ad_{\mathfrak{p}}(e_k) : \mathfrak{p} \to \mathfrak{p}$ where e_k is a basis element in $\mathfrak{g}_2$ from part i. We do this by using $(ad_{\mathfrak{p}}(e_k))_{i,j} = \langle ad_{\mathfrak{p}}(e_k)(x_j), x_i \rangle$ where $x_i, x_j \in \mathfrak{p}$ come from the orthonormal basis given in part i. Once we have that, we construct a matrix representing our intertwining map, ϕ, as described previously in Lemma 3.31. We then check that $ad_{e_k}\phi - \phi ad_{e_k} = 0$ for all e_k. This concludes part ii.

Part iii was concerned with finding the matrices $ad_{\mathfrak{p}}(x_k)$ for each x_k coming from the basis of $\mathfrak{p}$ in part i. In this part, we construct matrices representing $ad_{\mathfrak{p}}(x_k) : \mathfrak{p} \to \mathfrak{p}$ using the Cartan decomposition properties discussed in 1.3 and $(ad_{\mathfrak{p}}(x_k))_{i,j} = \langle ad_{\mathfrak{p}}(x_k)x_j, x_i \rangle$. The code for both part iii and iv is provided in Apppendix A.1, and it is to part iv that we turn our attention to now.

Part iv is quite involved as it is concerned with determining $ric(.,.)$ in terms of a, b, c in ϕ. In the code, we construct three different functions, one for each term appearing in our formula found in Lemma 3.15. Since we have constructed 14×14 matrices for ϕ and our collection $\{ad_{\mathfrak{p}}(x_i)\}$, our basis elements $\{x_i\}$ in $\mathfrak{p}$ are represented as the standard $\mathbb{R}^{14}$ basis elements with our innerproduct being the dot product.

The idea for the two terms that are not the Killing form is, for each x_i, to create a single matrix representing $\sum_k (\phi^{-1})_{ki} ad_{\mathfrak{p}}((x_k))$, and to apply that matrix to x, y, or $\phi^{-1}(x_j)$ (which are just vectors in $\mathbb{R}^{14}$) depending on the term. Then we take that result and multiply by ϕ, and then apply the dot product. Once that has happened, you simply loop through the correct index summing such terms. We provide our process with the first term below.

As a reminder, we provide the first term here:

$$\frac{-1}{2}\sum_{x_i}\langle\phi(\sum_k(\phi^{-1})_{ki}ad_{\mathfrak{p}}((x_k))(x)),\phi(\sum_k(\phi^{-1})_{ki}ad_{\mathfrak{p}}((x_k))(y))\rangle.$$

Using a loop through the index k in Python and our already achieved ϕ and $\{ad_p(x_k)\}$, we construct $M_i = \sum_k(\phi^{-1})_{ki}ad_{\mathfrak{p}}((x_k))$ which turns the term above into:

$$\frac{-1}{2}\sum_{x_i}\langle\phi(M_i x),\phi(M_i y)\rangle.$$

Now, ϕ and M_i are 14×14 matrices with $x, y \in \mathbb{R}^{14}$. Thus, we loop through the index i in Python applying the dot product of $\phi M_i x$ with $\phi M_i y$ for each i, summing over the index of i. This will provide us with all but the $\frac{-1}{2}$ factor which we apply at the end.

For the Killing form, we take in x and y as vectors in $\mathbb{R}^{14}$ and construct the appropriate 14×14 matrix for x and y using the basis elements found in part i. That is, if $x \in \mathbb{R}^{14}$ and represents an element of $\mathfrak{p}$, and if $\{e_i\}$ is our basis of 14×14 matrices for $\mathfrak{p}$ constructed in part i, then $x = (x_1, ..., x_{14})$ has a 14×14 matrix representation given by $x_1e_1 + ... + x_{14}e_{14}$. Once we have the correct matrix for x and the correct matrix for y, we use that $B(x, y) = 6tr(xy)$ for $\mathfrak{so}(1, 7)$ to calculate $B(x, y)$.

Once we run our code to produce $ric(x, y)$ (by using ϕ^{-1} instead of ϕ as our equivariant map to make terms simpler), we ran checks to ensure that we have what we expected. From Remark 3.30, we expect for $ric(x_i, x_i) \neq 0$ for $x \leq 14$ and for $ric(x_{x_i}, x_{i+7}) = ric(x_{x_{i+7}}, x_{x_i}) \neq 0$

for $i \leq 7$. We further expect for all other terms to be zero. From more general properties of ric, we expect for $ric(x_i, x_j) = ric(x_j, x_i)$, and for the expressions defining $ric(x_i, x_j)$ to be scale invariant with respect to a multiple of ϕ as this describes our metric. Lastly, again from Remark 3.30, we expect for $ric(x_1, x_1) = ... = ric(x_7, x_7)$, $ric(x_8, x_8) = ... = ric(x_{14}, x_{14})$, and $ric(x_1, x_8) = ... = ric(x_8, x_{14})$. In the end, we label $r_1 = ric(x_1, x_1)$, $r_2 = ric(x_8, x_8)$ and $r_3 = ric(x_1, x_8)$. The code for these checks is provided in Appendix A.1. We provide the r_1, r_2, and r_3 in terms of (a, b, c) below.

$$r_1 = \frac{9a^4\left(b^4+c^4\right)-36a^3bc^2\left(b^2-c^2\right)+6a^2c^2\left(b^4+20b^2c^2+c^4\right)+12abc^2\left(b^4+5b^2c^2-2c^4\right)+b^8+10b^6c^2+27b^4c^4+10b^2c^6+10c^8}{24\left(c^2-ab\right)^4}$$

$$r_2 = \frac{-2\left(\left(a^2+c^2\right)^2+2c^2(a+b)^2+\left(b^2+c^2\right)^2\right)\left(c^2-ab\right)^2+c^2\left(2a^2b+a\left(b^2+c^2\right)+b^3+3bc^2\right)^2+c^2(3a+b)^2\left(a^2+c^2\right)^2+2\left(a^3b+2a^2c^2+3abc^2+b^2c^2+c^4\right)^2-12\left(c^2-ab\right)^4}{24\left(c^2-ab\right)^4}$$

$$r_3 = -\frac{c(a+b)\left(b^2+c^2\right)\left(3a^2+b^2+4c^2\right)^2}{24\left(c^2-ab\right)^4}$$

Step 4

As we observed after the proof of Lemma 3.31 with Eqns.3.14, the relationship between ϕ and Φ provides us with polynomial relationships between the (a, b, c) defining ϕ and the (x, y, z) defining Φ:

$$a^2 + c^2 = x$$
$$b^2 + c^2 = y$$
$$c(a + b) = z.$$

Moreover, we have that $det\Phi = (det\phi)^2$. Using the polynomial relationships and recognizing that the denominator of each r_i is a constant multiple of $(det\phi)^4$, we are able to use a function in Mathematica, ***Eliminate***, that finds elimination ideals (see the first three chapters of [CLO15] for more information on elimination ideals) to determine what the numerator of each r_i is in terms of (x, y, z). Checking that our newfound r_i terms are correct is done by substituting back in for x, y, and z. The Mathematica code for the r_1, r_2, and r_3 in terms of (a, b, c), the process of finding the r_1, r_2, and r_3 in terms of (x, y, z), and the checks to ensure that our new r_i terms are correct can be found in Appendix A.2.

Below we provide our rational functions describing r_1, r_2, and r_3 in terms of (x, y, z).

$$r_1 = \frac{9x^2y^2 - 18xyz^2 + y^4 + 6y^2z^2 + 18z^4}{24\left(z^2 - xy\right)^2}$$
$$r_2 = \frac{-3x^2\left(4y^2 - 3z^2\right) - 2x\left(y^3 - 12yz^2\right) + 3y^2z^2 - 6z^4}{24\left(z^2 - xy\right)^2}$$
$$r_3 = -\frac{yz(3x + y)^2}{24\left(z^2 - xy\right)^2}$$

Step 5

The goal now is to find the solutions to $ric = T$ which we do by finding equations describing the (t_1, t_2, t_3) such that there exists an (x, y, z) with $x, y > 0$ and $xy - z^2 > 0$ with $(r_1, r_2, r_3) = (t_1, t_2, t_3)$. As can be seen in the r_i provided in Step 4, if $(x, y, z) \mapsto (r_1, r_2, r_3)$, then $(x, y, -z) \mapsto (r_1, r_2, -r_3)$. Moreover, $r_3 = 0$ if and only if $z = 0$. This means that if we find solutions in the case of $z > 0$ then we have solutions to case when $z < 0$ given by $(t_1, t_2, -t_3)$ where (t_1, t_2, t_3) is a solution to the $z > 0$ case. Therefore, we may consider solutions for when $t_3 > 0$ and obtain solutions for when $t_3 \neq 0$ by taking using $|t_3|$ in our conditions on t_3.

Moreover, if $z = 0$ then we already have our solution since this is the case provided by Theorem 3.13 (but with t_1 and t_2 swapped!). However, to show how our methods of using Mathematica cohere with that solution, we also provide in Appendix A.2 the solution produced by Mathematica in the z = 0 setting. We also (in the same subsection of the Appendix) provide the solution to $ric = cT$ in the setting with $z = 0$ to show how the Mathematica code coheres with Theorem 3.14 as well. We save the $z = 0$ setting for last in the Appendix since we are applying the methods from the $z \neq 0$ setting.

Now we focus our attention to looking for $(r_1, r_2, r_3) = c(t_1, t_2, t_3)$, but with $t_3, r_3, z > 0$. Finding the solutions to $ric = T$ in this case requires us to use a combination of functions in Mathematica, ***Resolve*** and ***Exists***. We sought to utilize these functions with r_1, r_2, and r_3 as is without using scale invariance; however, after hours of running, there was no output. Thus, we utilized the scale invariance and set $det\Phi = xy - z^2 = 1$. With $det\Phi = 1$, r_1 and r_2 became polynomials in terms of (x, y) with rational coefficients, but we could not eliminate the z in r_3, so r_3 became a polynomial in terms of (x, y, z) with rational coefficients. Since

z was not eliminated completely, when using the combination of ***Resolve*** and ***Exists*** in Mathematica, we had to ensure that the polynomial relationship $xy - z^2 = 1$ was provided as a constraint. The code for this is provided in Appendix A.2. We provide the new r_1, r_2, and r_3 below:

$$r_1 = \frac{1}{24}\left(9x^2y^2 + 6x\left(y^2 - 3\right)y + y^4 - 6y^2 + 18\right)$$

$$r_2 = \frac{1}{24}(9x^3y + xy(-12 + y^2) - 3(2 + y^2) + x^2(-9 + 6y^2))$$

$$r_3 = -\frac{1}{24}y(3x + y)^2 z \text{ and } xy - z^2 = 1$$

Remark 3.33. One might think that eliminating z could be done using elimination theory from Algebraic Geometry (See Chapters 1-3 of [CLO15] for this) as we did to get r_1, r_2, and r_3 in terms of (x, y, z). However, the ***Eliminate*** function, through its use of Grobner bases, only guarantees (without additional assumptions and algorithms being taken into consideration) an algebraic closure of the image of the function in consideration. Thus, when seeking to eliminate z in r_3 by using $xy - z^2 = 1$, the resulting equation for r_3 in terms of (x, y) need not have the same graph as r_3 in terms of (x, y, z). Instead, we expect the algebraic closure of the graph of r_3. To find the image itself is called (in [CLO15]) the *implicitization problem* for functions that are rational or polynomial, and is much more involved than just elimination. Thus, in some sense, it appears that we may have just been lucky that we were able to get r_1, r_2, and r_3 completely in terms of (x, y, z) using this elimination trick.

The output that Mathematica provides from ***Resolve*** and ***Exists*** describes the region that is the image of ric strictly in terms of (t_1, t_2, t_3) (but in Mathematica, we use (k, l, m) for simplicity in the code). The output was given by multiple regions with roots to polynomials being used to describe the region. However, we were able to simplify the regions and we were able to use the ***ToRadicals*** function to get the roots of polynomials as functions written explicitly in terms of two variables. To simplify the regions, one observation we made was that the roots of the polynomials given turned out to be described by the same function in each region. Once we realized that, we were able to simplify the presentation of the regions, and the simplified form is given below, describing our (t_1, t_2, t_3) such that $(r_1, r_2, r_3) = (t_1, t_2, t_3)$ for some (x, y, z) with $z > 0$:

$$t_1 = \begin{cases} f_1(t_2, t_3)\,, & t_2 \leq \frac{-3}{4} \text{ and } t_3 > 0 \\ f_1(t_2, t_3)\,, & -\frac{3}{4} < t_2 \leq \frac{-1}{2} \text{ and } t_3 > 0 \text{ and } t_3 \neq \frac{1}{2}\sqrt{\frac{3}{2}}\sqrt{4t_2 + 3} \\ f_1(t_2, t_3)\,, & t_2 > \frac{-1}{2} \text{ and } t_3 > \frac{\sqrt{3}}{4}\sqrt{2t_2 + 1} \text{ and } t_3 \neq \frac{1}{2}\sqrt{\frac{3}{2}}\sqrt{4t_2 + 3} \\ \frac{3}{4}\,, & t_2 > \frac{-3}{4} \text{ and } t_3 = \frac{1}{2}\sqrt{\frac{3}{2}}\sqrt{4t_2 + 3} \end{cases}$$

where $f_1(t_2, t_3)$ is described Eqns.3.13. The Mathematica for this step is provided in Appendix A.2.

This provides us with the solution desired for $z, t_3 > 0$. To finish **Case 2** of Theorem 3.21 regarding $t_3 \neq 0$, one must only make our conditions on t_3 become conditions on $|t_3|$.

As mentioned before, we provide the Mathematica code for **Case 1** of Theorem 3.21 in Appendix A.2. The solution in this case follows the same program as **Case 2**, but with $z = r_3 = 0$. With those conditions, we use ***Resolve*** and ***Exists*** to find the following solution:

$$t_1 = \frac{3}{8}(5 + 16t_2 + 16t_2^2) \text{ for } t_2 < -\frac{1}{2}$$

Remark 3.34. Our f_1 comes from $func1$ in our code from Mathematica, and we remark again that in the code provided, $(t_1, t_2, t_3) = (k, l, m)$.

Proof of Theorem 3.21: Steps 1 through 5 above along with the work provided in Appendix A.1 and Appendix A.2 complete the proof. ■

Step 6

We want to find the solutions to $ric = cT$, and we have shown that this amounts to a description of (t_1, t_2, t_3) such that there exists a c where $(r_1, r_2, r_3) = c(t_1, t_2, t_3)$ for some (x, y, z) with $x, y > 0$ an $xy - z^2 > 0$. In the end, it is nicest to not just have a description of the (t_1, t_2, t_3) without reference to the metric described by (x, y, z), but also a description of the c without reference to the (x, y, z).

As a reminder, we provide our r_1, r_2, r_3 without using any scale invariance below:

$$r_1 = \frac{9x^2y^2 - 18xyz^2 + y^4 + 6y^2z^2 + 18z^4}{24\left(z^2 - xy\right)^2}$$
$$r_2 = \frac{-3x^2\left(4y^2 - 3z^2\right) - 2x\left(y^3 - 12yz^2\right) + 3y^2z^2 - 6z^4}{24\left(z^2 - xy\right)^2}$$
$$r_3 = -\frac{yz(3x + y)^2}{24\left(z^2 - xy\right)^2}.$$

Our approach to solving this is as follows. First, using the function ***FunctionRange*** in Mathematica we determined that $r_1 > \frac{3}{8} > 0$. Using this, $(r_1, r_2, r_3) = c(t_1, t_2, t_3)$ is true if and only if $(1, \frac{r_2}{r_1}, \frac{r_3}{r_1}) = (1, \frac{t_2}{t_1}, \frac{t_3}{t_1})$. Thus, we get a description of the (t_1, t_2, t_3) desired if we can describe $R = \{(1, l, m) : (1, l, m) = (1, \frac{r_2}{r_1}, \frac{r_3}{r_1})\}$. That is, we know that (t_1, t_2, t_3) is a solution to $(r_1, r_2, r_3) = c(t_1, t_2, t_3)$ if and only if (t_1, t_2, t_3) is such that $(1, \frac{t_2}{t_1}, \frac{t_3}{t_1}) \in R$. In addition to finding a description of R, we would like to find the c such that $(r_1, r_2, r_3) = c(t_1, t_2, t_3)$.

Claim: For $(r_1, r_2, r_3) = c(t_1, t_2, t_3)$, $c = \frac{c_0}{t_1}$ where c_0 is given by the implicit equation $c_0 = f_1(c_0\frac{t_2}{t_1}, c_0\frac{t_3}{t_1})$ where f_1 is defined by Eqn.3.13, unless (t_1, t_2, t_3) is a multiple of $(\frac{3}{4}, r_2, r_3)$ with $r_2 > \frac{-3}{4}$ and $|r_3| = \frac{1}{2}\sqrt{\frac{3}{2}}\sqrt{4r_2 + 3}$. In this case, $c = \frac{c_1}{t_1}$ where $c_1 = \frac{3}{4}$.

Moreover, our c values so described are real so long as (t_1, t_2, t_3) are solutions to $ric = cT$.

<u>Proof of **Claim**</u>: For a solution (t_1, t_2, t_3), $(1, \frac{t_2}{t_1}, \frac{t_3}{t_1}) = (1, l, m)$ from R, so $\frac{1}{t_1}(t_1, t_2, t_3) = (1, l, m)$. Letting $c = \frac{1}{t_1}$ then gets us to a corresponding element of R, but not the (r_1, r_2, r_3) in the image of ric. The corresponding (r_1, r_2, r_3) is one that satisfies the equation $(1, \frac{r_2}{r_1}, \frac{r_3}{r_1}) = (1, l, m) = \frac{1}{t_1}(t_1, t_2, t_3)$. Observe that we then could have $(r_1, r_2, r_3) = \frac{r_1}{t_1}(t_1, t_2, t_3)$, but that would cause our $c = \frac{r_1}{t_1}$ which makes our c dependent upon our (x, y, z) which we wish to avoid. Allow us to analyze this equality more:

$$\begin{aligned}(r_1, r_2, r_3) &= \frac{r_1}{t_1}(t_1, t_2, t_3)\\ &= (r_1, \frac{r_1 t_2}{t_1}, \frac{r_1 t_3}{t_1})\\ &\text{which implies that}\\ r_2 &= \frac{r_1 t_2}{t_1}\\ r_3 &= \frac{r_1 t_3}{t_1}.\end{aligned}$$

Now, from Step 5, we know that if (r_1, r_2, r_3) is a point on the image of ric, then if we have r_2, r_3, then we can determine r_1 by $r_1 = f_1(r_2, r_3) = f_1(\frac{r_1 t_2}{t_1}, \frac{r_1 t_3}{t_1})$ unless $r_1 = \frac{3}{4}$, $r_2 > \frac{-3}{4}$, and $|r_3| = \frac{1}{2}\sqrt{\frac{3}{2}}\sqrt{4r_2 + 3}$. We save this exception for after the rest of the solutions.

Since (t_1, t_2, t_3) is provided as a solution, we can thus understand $r_1 = f_1(\frac{r_1 t_2}{t_1}, \frac{r_1 t_3}{t_1})$ as an implicit equation describing r_1 in terms of a solution (t_1, t_2, t_3). Thus, we have our c value in terms of (t_1, t_2, t_3), namely, $\frac{c_0}{t_1}$ where c_0 is a solution in terms of (t_1, t_2, t_3) determined by solving for c_0 in the implicit equation $c_0 = f_1(\frac{c_0 t_2}{t_1}, \frac{c_0 t_3}{t_1})$.

Now for the exceptional situation. In this case, following the above procedure, we could consider $r_1 = g_1(r_2, r_3) = \frac{3}{4}$ with c values given by $c = \frac{c_1}{t_1}$ where c_1 is a solution to $c_1 = g_1(\frac{c_1 t_2}{t_1}, \frac{c_1 t_3}{t_1}) = \frac{3}{4}$.

To see that the c values are real, consider that we were working with (t_1, t_2, t_3) were solutions to $ric = cT$, implying that for a (t_1, t_2, t_3) that is a solution, there is a real c with the description provided. This ends the proof our **Claim**.

Following the above procedure, we can see that in **Case 1** of Theorem 3.22, when $z = r_3 = t_3 = 0$, we can obtain our c value in a similar way. First, recall that the image of ric in this case is described by $r_1 = \frac{3}{8}(5 + 16r_2 + 16r_2^2)$. Now, if $(r_1, r_2) = c(t_1, t_2)$ then, for a solution (t_1, t_2) we can get $c = \frac{c_0}{t_1}$ where c_0 is a solution in terms of (t_1, t_2) determined by solving for c_0 in the implicit equation $c_0 = \frac{3}{8}(5 + 16(\frac{c_0 t_2}{t_1}) + 16(\frac{c_0 t_2}{t_1})^2)$.

Now, we discuss how we determine the region R which describes the (t_1, t_2, t_3) with a solution to $ric = cT$.

As in Step 5, we can restrict ourselves to the setting in which $z > 0$ since we are looking for $c > 0$, so we need only consider when $r_3 > 0$. As was previously mentioned, we provide in Appendix A.2 our solution to the case in which $z = r_3 = 0$ as well.

Just as before in Step 5, we us the ***Resolve*** and ***Exists*** functions in Mathematica to determine the image of the map described by $(1, \frac{r_2}{r_1}, \frac{r_3}{r_1}) = (1, l, m)$. The code for this is provided in Appendix A.2.

The output that Mathematica provided was, as in Step 5, capable of being described more simply and is described as a collection of regions described in a piece-wise fashion.

However, the description of the region R is still quite messy, being described by the union of 14 different regions involving 14 different functions. These functions are, again, described implicitly as the roots of polynomials, but this time only a couple were capable of being expressed explicitly. We provide the regions below and then the polynomials needed, with the correct root being specified (note that the nth root, or Root n, is the nth root in the set of roots placed in increasing order).

The region R are those $(1, l, m)$ described by the following regions:

$$\text{Region 1:} \begin{cases} m > 0 & \frac{1}{2}\left(m^2 - 2\right) \le l < m^2 \end{cases}$$

$$\text{Region 2:} \begin{cases} 0 < m < \frac{1}{\sqrt{3}} & f_{275m1} \le l < \frac{1}{3}\left(3m^2 - 4\right) \end{cases}$$

$$\text{Region 3:} \begin{cases} 0 < m \le \sqrt{\frac{2}{3}} & f_{2213m1} \le l \le \frac{1}{2}\left(m^2 - 2\right) \\ m > \sqrt{\frac{2}{3}} & f_{2213m1} \le l < \frac{1}{3}\left(3m^2 - 4\right) \end{cases}$$

$$\text{Region 4:} \begin{cases} 0 < m \le .625... & l = f_{4507m1} \\ \frac{1}{\sqrt{3}} < m < .625... & l = f_{4507m2} \end{cases}$$

$$\text{Region 5:} \begin{cases} m > \frac{1}{\sqrt{3}} & f_{275m1} \le l < m^2 \end{cases}$$

$$\text{Region 6:} \begin{cases} \frac{1}{\sqrt{3}} < m \le .986... & f_{1168m1} < l < \frac{1}{3}\left(3m^2 - 4\right) \\ \frac{1}{\sqrt{3}} < m \le 1.11... & f_{1168m2} < l < f_{25m1} \\ 1.11... < m < 1.40... & f_{2213m1} \le l < f_{25m1} \\ m \ge 2.22... & f_{25m1} < l < \frac{1}{3}\left(3m^2 - 4\right) \end{cases}$$

$$\text{Region 7:} \begin{cases} \frac{1}{\sqrt{3}} < m \le 1.11... & f_{1168m2} < l \le f_{275m1} \\ 1.11... < m \le 1.56... & f_{2213m1} \le l \le f_{275m1} \end{cases}$$

$$\text{Region 8:} \begin{cases} \frac{1}{\sqrt{3}} < m \le 2.22... & f_{25m1} < l \le f_{275m1} \\ m > 2.22... & f_{2213m1} \le l \le f_{275m1} \end{cases}$$

$$\text{Region 9: } \begin{cases} .281\ldots < m \le \frac{1}{\sqrt{3}} & f_{166360m2} < l < m^2 \\ \frac{1}{\sqrt{3}} < m \le \sqrt{\frac{2}{3}} & f_{166360m1} < l < \frac{1}{3}\left(3m^2 - 4\right) \\ \frac{1}{\sqrt{3}} < m \le \sqrt{\frac{2}{3}} & f_{166360m2} < l < f_{25m1} \\ \sqrt{\frac{2}{3}} < m \le .875\ldots & f_{166360m1} < l < f_{25m1} \end{cases}$$

$$\text{Region 10: } \begin{cases} 0 < m \le .372\ldots & f_{4507m1} < l < m^2 \\ .372\ldots < m \le .556\ldots & f_{4507m1} < l < f_{150m2} \\ .556\ldots < m < 1.52\ldots & f_{2213m1} \le l < f_{150m2} \\ 1.52\ldots \le m < 1.52\ldots & f_{150m1} < l < f_{150m2} \end{cases}$$

$$\text{Region 11: } \begin{cases} .372\ldots < m \le \frac{1}{\sqrt{3}} & f_{150m2} < l < m^2 \\ \frac{1}{\sqrt{3}} < m \le 1.17\ldots & f_{150m2} < l < f_{25m1} \\ 1.17\ldots < m < 1.40\ldots & f_{150m1} < l < f_{25m1} \\ 1.40\ldots \le m \le 1.52\ldots & f_{150m1} < l < \frac{1}{3}\left(3m^2 - 4\right) \end{cases}$$

$$\text{Region 12: } \begin{cases} \frac{3\sqrt{3}}{5} < m \le 1.17\ldots & f_{150m2} < l \le f_{275m1} \\ 1.17\ldots < m \le 1.52\ldots & f_{150m1} < l \le f_{275m1} \end{cases}$$

$$\text{Region 13: } \begin{cases} \frac{3\sqrt{3}}{5} < m \le 1.13\ldots & f_{1168m2} < l < f_{150m2} \\ 1.09\ldots < m \le 1.13\ldots & f_{2213m1} \le l < f_{150m1} \end{cases}$$

$$\text{Region 14: } \begin{cases} 0 < m < .423... & f_{2213m1} \leq l < f_{4507m1} \\ .423... \leq m < .556... & f_{2213m1} \leq l \leq f_{275m1} \\ .556... \leq m < \frac{1}{\sqrt{3}} & f_{4507m1} < l \leq f_{275m1} \\ \frac{1}{\sqrt{3}} \leq m < .625... & f_{4507m1} < l < f_{4507m2} \end{cases}$$

In the above regions, the numerical values bounding m are approximations provided by Mathematica. For the exact expression, we reference our Mathematica code in Appendix A.2. Note that the exact expressions are given as a roots of polynomials. The bounds for l above can be found below and are the indicated roots to the given polynomials in l where m is treated as a constant.

$$f_{27m1} = \text{Root 1 of the following polynomial in } l \tag{3.16}$$

$$-2 - 75m^2 + (6 - 180m^2)l + (180 - 378m^2)l^2 + (756 - 324m^2)l^3 + (1134 - 243m^2)l^4 + 486l^5$$

$$f_{2213m1} = \text{Root 1 of the following polynomial in } l$$

$$486l^5 + l^4\left(1134 - 243m^2\right) + l^3\left(972m^2 + 756\right) + l^2\left(-648m^4 - 1242m^2 + 180\right) + l\left(5616m^4 - 3780m^2 + 6\right) - 2160m^6 + 5112m^4 + 213m^2 - 2$$

$$f_{4507m1} = \text{Root 1 of the following polynomial in } l$$

$$4 - 507m^2 + (51 - 1404m^2)l + (252 - 1674m^2)l^2 + (594 - 972m^2)l^3 + (648 - 243m^2)l^4 + 243l^5$$

$$f_{4507m2} = \text{Root 2 of the following polynomial in } l$$

$$4 - 507m^2 + (51 - 1404m^2)l + (252 - 1674m^2)l^2 + (594 - 972m^2)l^3$$
$$+ (648 - 243m^2)l^4 + 243l^5$$

f_{25m1} = Root 1 of the following polynomial in l

$$- 25m^2 + (1 + 60m^2)l + (12 - 126m^2)l^2 + (54 + 108m^2)l^3 + (108 - 81m^2)l^4 + 81l^5$$

f_{1168m1} = Root 1 of the following polynomial in l

$$1 - 168m^2 + 144m^4 + (12 + 144m^2)l + (54 + 216m^2)l^2 + 108l^3 + 81l^4$$

f_{1168m2} = Root 2 of the following polynomial in l

$$1 - 168m^2 + 144m^4 + (12 + 144m^2)l + (54 + 216m^2)l^2 + 108l^3 + 81l^4$$

$f_{1166360m1}$ = Root 1 of the following polynomial in l

$$16 - 6360m^2 + 47961m^4 + (216 + 8478m^2 + 149796m^4)l$$
$$+ (1161 + 54432m^2 + 176094m^4)l^2 + (3132 + 64476m^2 + 92340m^4)l^3$$
$$+ (4374 + 29160m^2 + 18225m^4)l^4 + (2916 + 4374m^2)l^5 + 729l^6$$

$f_{166360m2}$ = Root 2of the following polynomial in l

$$16 - 6360m^2 + 47961m^4 + (216 + 8478m^2 + 149796m^4)l$$
$$+ (1161 + 54432m^2 + 176094m^4)l^2 + (3132 + 64476m^2 + 92340m^4)l^3$$
$$+ (4374 + 29160m^2 + 18225m^4)l^4 + (2916 + 4374m^2)l^5 + 729l^6$$

f_{150m1} = Root 1 of the following polynomial in l

$$- 150m^2 + 75m^4 + (2 - 231m^2 + 540m^4)l + (42 + 4806m^2 + 3402m^4)l^2$$

$$+ (378 + 14742m^2 + 8748m^4)l^3 + (1890 - 6966m^2 + 19683m^4)l^4$$

$$+ (5670 - 45927m^2)l^5 + (10206 - 13122m^2)l^6 + 10206l^7 + 4374l^8$$

f_{150m2} = Root 2 of the following polynomial in l

$$- 150m^2 + 75m^4 + (2 - 231m^2 + 540m^4)l + (42 + 4806m^2 + 3402m^4)l^2$$

$$+ (378 + 14742m^2 + 8748m^4)l^3 + (1890 - 6966m^2 + 19683m^4)l^4$$

$$+ (5670 - 45927m^2)l^5 + (10206 - 13122m^2)l^6 + 10206l^7 + 4374l^8$$

Of the above roots, there are only two for which Mathematica could produce a radical description:

$$f_{1168m1} = \frac{1}{3}\left(-2\sqrt{-3m^2 - 2\sqrt{3}m} - 1\right)$$

$$f_{1168m2} = \frac{1}{3}\left(2\sqrt{-3m^2 - 2\sqrt{3}m} - 1\right).$$

Having the above regions and functions, we have the solution to $ric = cT$ for $z, t_3 > 0$. To find solutions for $t_3 \neq 0$, as before, one must exchange conditions on t_3 to be conditions on $|t_3|$, meaning that for $t_3 \neq 0$, we want (t_1, t_2, t_3) to be such that $\frac{t_2}{t_1} = l$ and $\frac{|t_3|}{t_1} = m$ in the 14 regions above. Thus, we have solution to **Case 2** of Theorem 3.22.

As mentioned before, we provide the Mathematica code for **Case 1** of Theorem 3.22 in Appendix A.2. The proof for this case follows the same program as **Case 2**, but with $z = r_3 = 0$ which greatly simplifies things. Indeed, if one looks to our Mathematica work in the $z = 0$ setting, one can notice that finding the description of $R = \{(1, l) : (1, l) = (1, \frac{r_2}{r_1})\}$

using ***Exists*** and ***Resolve*** amounts to one simple region described by:

$$\frac{1}{3}\left(-\sqrt{5}-2\right) \leq l < 0.$$

Therefore, (t_1, t_2) providing solutions in the case of $z = 0$ are those which satisfy

$$\frac{1}{3}\left(-\sqrt{5}-2\right) \leq \frac{t_2}{t_1} < 0.$$

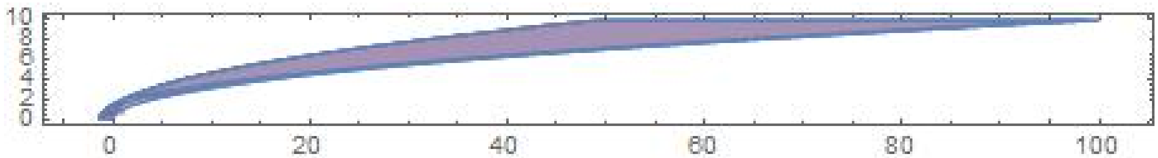
This provides us with the desired solution of **Case 1** in Theorem 3.22.

Having solutions provided above and the proof of our **Claim** above, we have a description of our (t_1, t_2, t_3) and $c > 0$ such that $(r_1, r_2, r_3) = c(t_1, t_2, t_3)$ for some $ad_{\mathfrak{h}}\ (x, y, z)$ with $xy - z^2 > 0$ and $x, y > 0$.

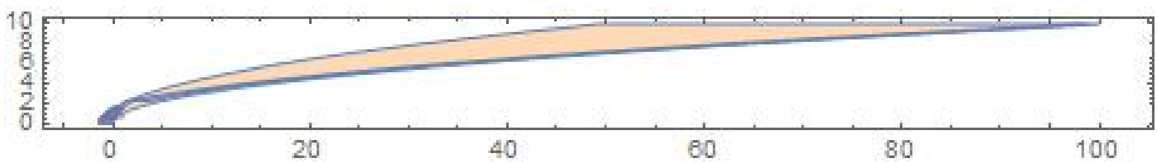
Proof of Theorem 3.22: Steps 1 through 6 above along with the work provided in Mathematica in Appendix A.1 and Appendix A.2 complete the proof. ■

Remark 3.35. In the following pages, we provide images depicting R. The images indicate that there is still some simplification that could happen. We remind the reader that R describes the solutions for $z > 0$ which in turn gives us solutions for $z \neq 0$. The $z < 0$ solutions would just be reflections of the graphs seen about the horizontal axes.

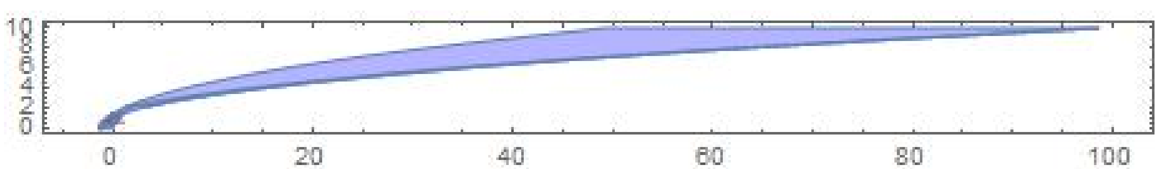
In the following images, we depict the region R with the vertical axis being the m axis and the horizontal axis the l axis. The first image is all of the regions shown at the same time. We remove various regions to show how there appears to be great overlap between regions, indicating that further simplification of the description of R could be made.



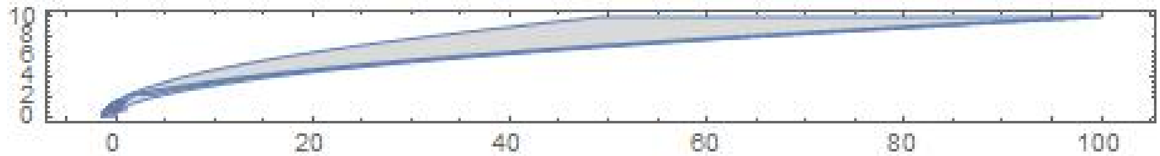
Below we have R with all of the regions except for the second part of Region 3 and Region 5. The larger portion that is orange is Region 1.



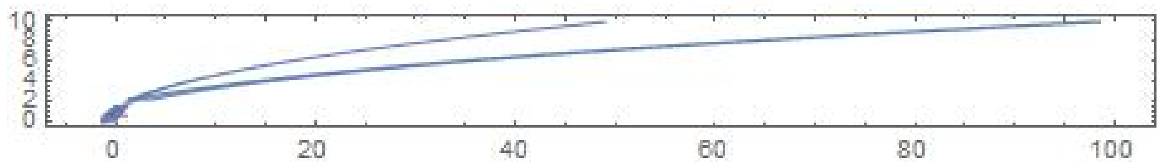
Below we have R with all of the regions except for Region 1 and Region 5. The larger portion in blue is the second part of Region 3.



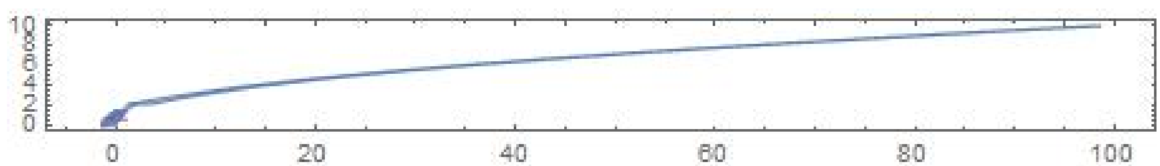
Below we have R with all of the regions except for Region 1 and the second part of Region 3. The larger grey part is Region 5.



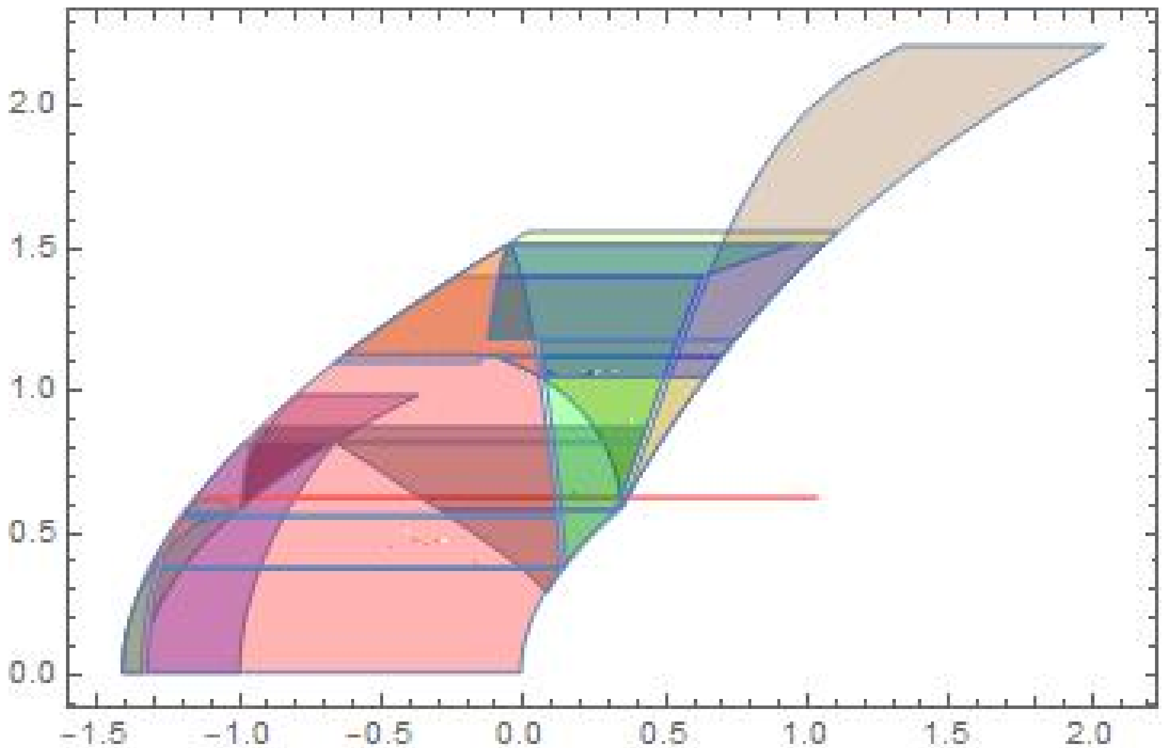
Below we have R with all of the regions except for Region 1, the second part of Region 3, and Region 5. The thin part going up with greater m values is the second part of Region 8, and the thin part going up with lesser m values is the last region in Region 6.



Below we have R with all of the regions except for Region 1, the second part of Region 3, Region 5, and the second part of Region 8. The thin part going up is the last region in Region 6.



Below we have R with all of the regions except for Region 1, the second part of Region 3, Region 5, the last part of Region 6, and the second part of Region 8. Notice by observing the bounds that the amount of R seen has decreased significantly, but there is still much overlap between regions.



Chapter 4

Three Irreducible Summands

In the pursuit of understanding the image of ric as before in Chapter 3, a partial understanding of the image can be found by determing the signature of ric, that is, the number of positive, negative, or zero values along the diagonal of $Ric(.)$, the $(1,1)$ Ricci tensor (See Section 1.1). In papers such as [AL22] and the papers cited therein, the signature of ric is precisely the geometric property of interest. In [AL22], the objects of interest are nilmanifolds with left invariant metrics. In the present chapter, we concern ourselves with the setting of G/H with G noncompact simple and with reductive Cartan decomposition (See Definition 1.19) $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$ such that $\mathfrak{p}' = \mathfrak{p}_1 \oplus \mathfrak{p}_2$, $[\mathfrak{p}'', \mathfrak{p}'']$, $[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}$, and $\mathfrak{p}''$, $\mathfrak{p}_1$, $\mathfrak{p}_2$ are irreducible. In this setting, we investigate the signature of ric, particularly for metrics whose associated inner product is diagonalized over such a decomposition (See Section 4.2). The G/H we restrict ourselves to serve as a kind of noncompact "dual" to the generalized Wallach spaces classified in [Nik21] (See Definition 4.1 below), and in Section 4.3 we address a specific example, $SO(n,2)/SO(2)$, which also gets attention in [BB78] and [Nik00].

In understanding the signature of ric or the image of ric, something that can be helpful is having a presentation of ric that is itself diagonal. Since ric is symmetric, there always exists a basis in which ric is diagonal; however, one may want to know is if there are nice Lie-theoretic conditions that cause ric to be diagonal in such a way that the bracket conditions on the isotropy representation and the inner product play nicely together. In fact, a question of interest in papers such as [LW13] and [Kri21] is *what kind of bracket conditions can be placed on a basis for which ric is diagonal for that basis?* We also concern ourselves with a question like this (specifically in Section 4.1) where we find sufficient conditions placed on the bracket relations in our isotropy representation for our ric to be diagonal. In addition to finding a sufficient condition to diagonalize ric, our results in Section 4.1 are later used in Section 4.2 to indicate what conditions on our isotropy representation sufficiently place us within the setting of interest for our results regarding the signature of ric (See Section 4.2 for more precision on the setting and how the conditions get used).

Since our objects of interest (the ones with $[\mathfrak{p}'', \mathfrak{p}''], [\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}$ with noncompact simple $\mathfrak{g}$) are highly motivated by generalized Wallach spaces in the compact setting, allow us to explain what generalized Wallach spaces are, why we say "dual", and how to obtain noncompact examples of the desired spaces from generalized Wallach spaces.

Definition 4.1. A *generalized Wallach space* is a compact homogeneous space G/H in which the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ has $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$ as a decomposition into irreducibles that are pairwise orthogonal with respect to $-B(.,.)$ such that $[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}$ for all i.

Remark 4.2. The above definition and classifiction of generalized Wallach spaces comes from Nikonorov's [Nik21] which is a correction to Nikonorov's [Nik16]. When needed, we will cite the corrected version which has the citations for the former version within.

Remark 4.3. We say *"dual" generalized Wallach space* for multiple reasons, and the next two examples show why. One reason is because you can take a generalized Wallach space, G/H and use the dual process used in Theorem 3.2 to find noncompact examples of spaces with three irreducible isotropy representations, $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ in which $[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}$. In fact, as we will show, one generalized Wallach space, by use of this dual process, can produce more than one corresponding noncompact spaces with the desired qualities. The second reason is because you cannot necessarily take a noncompact space with the desired properties and use the dual process to get a generalized Wallach space. Thus, these spaces are "dual" in the sense that you can get some of them through this process using dual symmetric spaces, but you can't get all of them. Moreover, unlike dual symmetric spaces (See Section 1.5), there is not a single noncompact space corresponding to a given generalized Wallach space through the dual process.

Before we show these two examples that help clarify our remark above, we first remind the reader of the dual process used in Theorem 3.2 (with a couple of modifications) to see how this process can be used in the setting we are in. The following dual process goes from the compact setting to the noncompact setting; however, as is noted in the proof of the Theorem 3.2, this dual process can be used to go from noncompact to compact as well, as we are using the duality of symmetric spaces (See Section 1.5) which allows one to go from the compact setting to the noncompact setting and vice versa. Also, we remind the reader that this process is used in Section 3 of [AL17] to go from the compact setting to the noncompact setting.

a. In the compact setting, select $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}^*$ in which $(\mathfrak{g}^*, \mathfrak{k})$ is a pair associated with a compact irreducible symmetric space and G^*/H is a compact homogeneous space.

b. Use the duality of symmetric spaces to achieve $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$ such that G/H is a noncompact homogeneous space.

Note: To go from noncompact to compact, use this same process but exchange compact with noncompact and vice versa in the description.

Example 4.4. Let $G/H = SO(n,\mathbb{C})_{\mathbb{R}}/SO(n-1)$ where $SO(n,\mathbb{C})_{\mathbb{R}}$ is the realifiction (See Section 1.4) of $SO(n,\mathbb{C})$. This space has the desired property of three irreducible summands $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ such that $[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h} = \mathfrak{so}(n-1)$ for all i, but there is no generalized Wallach space obtained by using dual symmetric spaces to pass from the noncompact setting to the compact setting.

Here, $\mathfrak{so}(n,\mathbb{C})_{\mathbb{R}} = \mathfrak{so}(n-1) \oplus \mathfrak{p} \oplus i\mathfrak{so}(n-1) \oplus i\mathfrak{p}$ where $\mathfrak{so}(n) = \mathfrak{so}(n-1) \oplus \mathfrak{p}$ is the reductive decomposition for the compact irreducible symmetric space $SO(n)/SO(n-1)$. With this decomposition, we can see that $[\mathfrak{p},\mathfrak{p}]$, $[i\mathfrak{so}(n-1), i\mathfrak{so}(n-1)]$, $[i\mathfrak{p}, i\mathfrak{p}] \subset \mathfrak{h}$. However, the corresponding compact space using the duality process mentioned above would be $SO(n) \times SO(n)/\Delta(SO(n-1))$ which is not a generalized Wallach space according to the description of such spaces given in Theorem 1 of [Nik21].

Example 4.5. Let $G/H = SO(n+1+1)/SO(n)$ which is a generalized Wallach space from Theorem 1 in [Nik21] (citing Table 1 in [Nik16]). By using the daul process, we have have two spaces, $SO(n+1,1)/SO(n)$ and $SO(n,2)/SO(n)$, which are noncompact spaces with three irreducible summands $\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3$ such that $[\mathfrak{q}_i, \mathfrak{q}_i] \subset \mathfrak{so}(n)$.

Here we have $\mathfrak{g} = \mathfrak{so}(n+1+1)$ and $\mathfrak{h} = \mathfrak{so}(n) \oplus \mathfrak{so}(1) \oplus \mathfrak{so}(1) \simeq \mathfrak{so}(n)$. Since $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$ and $(\mathfrak{h} \oplus \mathfrak{p}_i, \mathfrak{h})$ is a symmetric pair (see Lemma 6 in [Nik21]), we get $\mathfrak{k}_1 = \mathfrak{h} \oplus \mathfrak{p}_1 = \mathfrak{so}(n+1)$, $\mathfrak{k}_2 = \mathfrak{h} \oplus \mathfrak{p}_2 = \mathfrak{so}(n+1)$ and $\mathfrak{k}_3 = \mathfrak{h} \oplus \mathfrak{p}_3 = \mathfrak{so}(n) \oplus \mathfrak{so}(2)$. Thus, by taking $(\mathfrak{g}, \mathfrak{k}_1, \mathfrak{h})$ and $(\mathfrak{g}, \mathfrak{k}_2, \mathfrak{h})$ and using our dual process, we will in this case produce the same noncompact space described at the Lie algebra level by the pair: $(\mathfrak{so}(n+1,1), \mathfrak{so}(n))$.

In this case, we have $\mathfrak{so}(n+1,1) = \mathfrak{h}\oplus\mathfrak{p}_1\oplus i\mathfrak{p}_2\oplus i\mathfrak{p}_3$ and $\mathfrak{so}(n+1,1) = \mathfrak{h}\oplus\mathfrak{p}_2\oplus i\mathfrak{p}_1\oplus i\mathfrak{p}_3$ where $\mathfrak{so}(n+1) = \mathfrak{h}\oplus\mathfrak{p}_1 = \mathfrak{h}\oplus\mathfrak{p}_2$. The corresponding homogeneous space is $SO(n+1,1)/SO(n)$ and since $[i\mathfrak{p}_i, i\mathfrak{p}_i] \subset \mathfrak{so}(n)$ for $i = 1,2,3$, we have a noncompact space with the desired qualities.

Taking the triple $(\mathfrak{g}, \mathfrak{k}_3, \mathfrak{h}) = (\mathfrak{so}(n+1+1), \mathfrak{so}(n)\oplus\mathfrak{so}(2), \mathfrak{so}(n))$ and using the dual process, we obtain the homogeneous space described by the pair $(\mathfrak{so}(n,2), \mathfrak{so}(n))$ where $\mathfrak{so}(n,2) = \mathfrak{so}(n)\oplus\mathfrak{p}_3\oplus i\mathfrak{p}_1\oplus i\mathfrak{p}_2$ with $\mathfrak{so}(n)\oplus\mathfrak{so}(2) = \mathfrak{so}(n)\oplus\mathfrak{p}_3$. The corresponding noncompact homogeneous spaces is $SO(n,2)/SO(n)$ which also has the desired qualities.

With the above questions, properties, and objects of interest, we have the following program for the present chapter. In Section 4.1 we find a sufficient condition placed on the irreducible representations so that $ric(.,.)$ is diagonal, and more than that, simultaneously diagonalized with $B(.,.)$ (See Section 1.7). In Section 4.2 we turn to finding expressions for the diagonal of $Ric(.)$ and, using the results from Section 4.1, we investigate the signature of our spaces of interest in which $[\mathfrak{p}'', \mathfrak{p}''], [\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}$. Finally, in Section 4.3 we provide an example of interest involving the results from the previous two sections.

4.1. Diagonalizing *ric*

The two following lemmas and then the theorem are notably similar to Nikonorov's Theorem 2 and the lemmas used therein (see [Nik00]). The following results serve as a kind of extension of Nikonorov's methods. These results are ultimately interested in understanding what happens to inner products and ric in the presence of isomorphic $ad_{\mathfrak{h}}$ representations inside $\mathfrak{p}'$ in a reductive Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$ (See Definition 1.19). More specifically, we aim at discovering how one can obtain bracket conditions such that ric is diagonal even in the presence of isomorphisms in the isotropy representation.

Remark 4.6. It is worth noting that we do not use that $\mathfrak{g}$ is semi-simple in the following two lemmas, and it is for that reason that we do not assume that $\mathfrak{g}$ is semi-simple.

Lemma 4.7. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ be a reductive decomposition such that there exist $\mathfrak{p}_1$ and $\mathfrak{p}_2$ irreducible in $\mathfrak{p}$ with $\mathfrak{p}_1 \simeq \mathfrak{p}_2$. If $\mathfrak{q}$ is an irreducible representation in $\mathfrak{p}_1 \oplus \mathfrak{p}_2$ then $\mathfrak{q} = \{av + b\phi(v) : v \in \mathfrak{p}_1$ and $\phi : \mathfrak{p}_1 \to \mathfrak{p}_2$ is an $ad_{\mathfrak{h}}$ intertwining map$\}$.

Proof: First observe in $\mathfrak{q}$ defined above that if $b = 0$ then $\mathfrak{q} = \mathfrak{p}_1$ and if $a = 0$ then $\mathfrak{q} = \mathfrak{p}_2$. For this reason, we assume that $\mathfrak{q} \neq \mathfrak{p}_1, \mathfrak{p}_2$ and we fix $\mathfrak{q}$ as some irreducible $ad_{\mathfrak{h}}$ representation. We will show that for some intertwining $\phi : \mathfrak{p}_1 \to \mathfrak{p}_2$, $\mathfrak{q} = \{v + \phi(v) : v \in \mathfrak{p}_1\}$, proving our claim.

Let $v = v_1 + v_2$ and $w = v_1 + w_2$ both be in $\mathfrak{q}$ with v_1 being the $\mathfrak{p}_1$ component for v and w and with v_2, w_2 the $\mathfrak{p}_2$ components for v and w, respectively. Note that $v - w = v_2 - w_2 \in \mathfrak{q}$, and since $\mathfrak{q} \neq \mathfrak{p}_2$, by $\mathfrak{q}$ and $\mathfrak{p}_2$ being irreducible we know that $\mathfrak{q} \cap \mathfrak{p}_2 = \{0\}$. Therefore, we can conclude that $v_2 - w_2 = 0$, implying that $v_2 = w_2$. This implies that for each $v \in \mathfrak{q}$, the $\mathfrak{p}_1$ component of v uniquely determines the $\mathfrak{p}_2$ component. Therefore, we may consider the linear map $\phi : \mathfrak{p}_1 \to \mathfrak{p}_2$ given by $\phi = \text{proj}_{\mathfrak{p}_2} \circ \text{proj}_{\mathfrak{p}_1}^{-1}$ and check that ϕ is in fact an intertwining map. This is seen by checking that $\text{proj}_{\mathfrak{p}_1}$ and $\text{proj}_{\mathfrak{p}_2}$ are both $ad_{\mathfrak{h}}$ intertwining maps and are therefore (by Schur's Lemma in 1.3) vector space isomorphisms as well. We check $\text{proj}_{\mathfrak{p}_1} : \mathfrak{q} \to \mathfrak{p}_1$ since $\text{proj}_{\mathfrak{p}_2}$ is the same argument with different subscripts. Let $x \in \mathfrak{h}$ and $v = v_1 + v_2 \in \mathfrak{q}$.

$$\begin{aligned} \text{proj}_{\mathfrak{p}_1}(ad_x(v)) &= \text{proj}_{\mathfrak{p}_1}([x, v_1 + v_2]) \\ &= \text{proj}_{\mathfrak{p}_1}([x, v_1] + [x, v_2]) \\ &= [x, v_1] \text{ since } \mathfrak{p}_1 \text{ is } ad_{\mathfrak{h}} \text{ invariant} \\ &= ad_x(\text{proj}_{\mathfrak{p}_1}(v)) \end{aligned}$$

Thus, we have $\text{proj}_{\mathfrak{p}_1}$ as an $ad_{\mathfrak{h}}$ intertwining map, as desired, and in this case, $\mathfrak{q} = \{v + \phi(v) : v \in \mathfrak{p}_1\}$. ■

Remark 4.8. It is worth noting that in the above proof, we fixed $\mathfrak{q}$ and built ϕ from that fixed $\mathfrak{q}$. However, one could also go in the reverse order in which we choose a fixed $ad_{\mathfrak{h}}$ intertwining $\psi : \mathfrak{p}_1 \to \mathfrak{p}_2$ and build an associated $\mathfrak{q} = \{av + b\psi(v)\}$ which is an $ad_{\mathfrak{h}}$ irreducible representation. This is noteworthy since the proof in the above Lemma helps one in finding an intertwining map $\phi : \mathfrak{p}_1 \to \mathfrak{p}_2$ given an irreducible representation $\mathfrak{q}$, but only the statement of the lemma is helpful in constructing an irreducible representation $\mathfrak{q}$ given an intertwining map. Moreover, one can check that such a $\mathfrak{q}$ is in fact and irreducible representation by using that $\text{proj}_{\mathfrak{p}_1} : \mathfrak{q} \to \mathfrak{p}_1$ is an $ad_{\mathfrak{h}}$ intertwining map. Indeed, if $V \subset \mathfrak{q}$ was an invariant subspace, $\text{proj}_{\mathfrak{p}_1}(V) \subset \mathfrak{p}_1$ would be invariant in $\mathfrak{p}_1$, but $\mathfrak{p}_1$ is irreducible and the same dimension as $\mathfrak{q}$. Thus, $\text{proj}_{\mathfrak{p}_1}(V) = \{0\}$ which implies $V = \{0\}$, or $\text{proj}_{\mathfrak{p}_1}(V) = \mathfrak{p}_1$ which implies $V = \mathfrak{q}$.

Lemma 4.9. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ be a reductive decomposition such that there exist $\mathfrak{p}_1$ and $\mathfrak{p}_2$ irreducible in $\mathfrak{p}$ with $\mathfrak{p}_1 \simeq \mathfrak{p}_2$. Assume that $[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}$ for $i = 1, 2$.
If for all $ad_{\mathfrak{h}}$ intertwining maps $\phi : \mathfrak{p}_1 \to \mathfrak{p}_2$, $[v, \phi(w)]_{\mathfrak{p}} + [\phi(v), w]_{\mathfrak{p}} = 0$ for all $v, w \in \mathfrak{p}_1$, then $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{h}$ for all $ad_{\mathfrak{h}}$ irreducible representations $\mathfrak{q}$ in $\mathfrak{p}_1 \oplus \mathfrak{p}_2$. The converse is true when $\mathfrak{q} \neq \mathfrak{p}_1$ or $\mathfrak{p}_2$.

Proof: We first prove the forward direction. Let $\mathfrak{q}$ be an $ad_{\mathfrak{h}}$ irreducible representation in $\mathfrak{p}_1 \oplus \mathfrak{p}_2$. If $\mathfrak{p}_1 \not\simeq \mathfrak{p}_2$ then $\mathfrak{q}$ is $\mathfrak{p}_1$ or $\mathfrak{p}_2$ and we are done by assumption. If $\mathfrak{p}_1 \simeq \mathfrak{p}_2$ then $\mathfrak{q} = \{av + b\phi(v) : v \in \mathfrak{p}_1 \text{ and } \phi \text{ is an intertwining map}\}$ by Lemma 4.7. Now,

$$[av+b\phi(v), aw+b\phi(w)]_{\mathfrak{p}} = a^2[v,w]_{\mathfrak{p}} + b^2[\phi(v),\phi(w)]_{\mathfrak{p}} + ab[v,\phi(w)]_{\mathfrak{p}} + ab[\phi(v),w]_{\mathfrak{p}}$$

and by assumption, each term is 0, so $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{h}$.

To see that the converse is true, observe by Lemma 4.7 and our assumption, that $0 = [v + \phi(v), w + \phi(w)]_{\mathfrak{p}} = [v, \phi(w)]_{\mathfrak{p}} + [\phi(v), w]_{\mathfrak{p}}$ for any $ad_{\mathfrak{h}}$ intertwining $\phi : \mathfrak{p}_1 \to \mathfrak{p}_2$. ■

In the following theorem, we are extending the results of Nikonorov in [Nik00] and our definition of Cartan orthogonal pairs in Definition 1.20. In addition to extending these results and definition, we address the consequences these conditions have on *ric*.

Remark 4.10. Observe that in the following theorem, we do not assume that $[\mathfrak{p}'', \mathfrak{p}''] \subset \mathfrak{h}$ except for in claim **2** as it is not needed for the other results. This particular result will be used later in Section 4.2.

Theorem 4.11. Consider the homogeneous space G/H such that $\mathfrak{g}$ is noncompact semi-simple and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$ is a reductive Cartan decomposition. Assume $\mathfrak{p}''$ is irreducible and that $\mathfrak{p}' = \mathfrak{p}_1 \oplus \mathfrak{p}_2$ is a decomposition into irreducibles in which $B(\mathfrak{p}_1, \mathfrak{p}_2) = 0$. Set $\mathfrak{p} = \mathfrak{p}'' \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2$. The following is true:

$$[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h} \text{ for } i = 1, 2 \iff [\mathfrak{p}'', \mathfrak{p}_i] \subset \mathfrak{p}_j \text{ with } i \neq j. \tag{4.1}$$

Moreover, we have the following claims for $\mathfrak{g}$.

1. Provided $[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}$ for $i = 1, 2$, if $(.,.)$ is an $ad_{\mathfrak{h}}$ invariant inner product on $\mathfrak{p}'' \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2$ for which this decomposition is orthogonal, $ric(.,.)$ is diagonal.

2. Provided $[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}$ for $i = 1, 2$, further suppose that at most two irreducible representations are equivalent in $\mathfrak{p}'' \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2$ and $[\mathfrak{p}'', \mathfrak{p}''] \subset \mathfrak{h}$. We have the following:

For any G invariant metric, there is a reductive Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}'' \oplus \mathfrak{q}'$ (with $\mathfrak{q}' = \mathfrak{q}_1 \oplus \mathfrak{q}_2$) such that the unique corresponding $ad_{\mathfrak{h}}$ invariant inner product, $(.,.)$, on $\mathfrak{q} = \mathfrak{q}'' \oplus \mathfrak{q}_1 \oplus \mathfrak{q}_2$ is simultaneously diagonalized with $B(.,.)$ and $[\mathfrak{q}'', \mathfrak{q}''], [\mathfrak{q}_i, \mathfrak{q}_i] \subset \mathfrak{h}$ if and only if the following condition is satisfied:

If $\mathfrak{m}$ and $\mathfrak{n}$ are the two equivalent irreducible representations among $\mathfrak{p}'', \mathfrak{p}_1, \mathfrak{p}_2$, then for any isomorphism $\phi : \mathfrak{m} \to \mathfrak{n}$, $0 = [x, \phi(y)]_{\mathfrak{p}} + [\phi(x), y]_{\mathfrak{p}}$ for all $x, y \in \mathfrak{m}$.

Consequently, ric is diagonal with respect to the given decomposition, $\mathfrak{q} = \mathfrak{q}'' \oplus \mathfrak{q}_1 \oplus \mathfrak{q}_2$.

Proof: First we prove the equivalence.

Let $x, y \in \mathfrak{p}_i$ and $z \in \mathfrak{p}''$. Given the $ad_{\mathfrak{g}}$ invariance of the Killing form B, we have

$$B([y, z], x) = -B(z, [y, x]).$$

If $[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}$, the right hand side is 0 as $B(\mathfrak{h}, \mathfrak{p}'') = 0$ for a reductive Cartan decomposition, and since by the Cartan properties (See 1.3) $[y, z] \in \mathfrak{p}'$, we have $[y, z] \in \mathfrak{p}_j$ for $j \neq i$. This proves the forwards direction.

If $[\mathfrak{p}'', \mathfrak{p}_i] \subset \mathfrak{p}_j$ for $i \neq j$ then the left hand side is 0, and since (again by the Cartan properties) $[y, x] \in \mathfrak{k} = \mathfrak{h} \oplus \mathfrak{p}''$ but $z \in \mathfrak{p}''$, we know $[y, x] \in \mathfrak{h}$. This proves the equivalence (4.1).

Now we prove **1.**

By Proposition 2.4 we are able to reduce the problem to showing that $ric(\mathfrak{p}_1, \mathfrak{p}_2) = 0$. Now let $x \in \mathfrak{p}_1$ and $y \in \mathfrak{p}_2$. We know that $B(x, y) = 0$, so by Eqn.1.2 we have that

$$ric(x, y) = \frac{-1}{2} \sum_i ([x, x_i]_{\mathfrak{p}}, [y, x_i]_{\mathfrak{p}}) + \frac{1}{4} \sum_{i,j} ([x_i, x_j]_{\mathfrak{p}}, x)([x_i, x_j]_{\mathfrak{p}}, y).$$

Now, we can see that if $x_i \in \mathfrak{p}_1$ or $x_i \in \mathfrak{p}_2$ then by $[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}$, $([x, x_i]_{\mathfrak{p}}, [y, x_i]_{\mathfrak{p}}) = 0$. Furthermore, if $x_i \in \mathfrak{p}''$ then $[x, x_i] \in \mathfrak{p}_2$ and $[y, x_i] \in \mathfrak{p}_1$ by equivalence 4.1. Since $(\mathfrak{p}_1, \mathfrak{p}_2) = 0$ by assumption, this gives us that the first term is 0.

To prove the last term is zero, we look at the three different cases based on where x_i, x_j live in $\mathfrak{p} = \mathfrak{p}'' \oplus \mathfrak{p}'$. Now, if $x_i, x_j \in \mathfrak{p}'$ then $[x_i, x_j] \in \mathfrak{k}$ by Cartan decomposition properties. Thus, by $(\mathfrak{p}'', \mathfrak{p}') = 0$ we know that $([x_i, x_j]_{\mathfrak{p}}, x) = 0$. If $x_i, x_j \in \mathfrak{p}''$ then again $[x_i, x_j] \in \mathfrak{k}$, so $([x_i, x_j]_{\mathfrak{p}}, x) = 0$. Lastly, if $x_i \in \mathfrak{p}''$ and $x_j \in \mathfrak{p}'$ then by $[\mathfrak{p}'', \mathfrak{p}_i] \subset \mathfrak{p}_j$ (due to equivalence 4.1), $([x_i, x_j]_{\mathfrak{p}}, x)([x_i, x_j]_{\mathfrak{p}}, y) = 0$. Thus, we have that $ric(x, y) = 0$.

Now we prove **2.**

If there are no isomorphisms, then we are done by Schur's Lemma (See Section 1.3). As for the presence of isomorphisms, we prove this in two cases, first where $\mathfrak{p}_1 \simeq \mathfrak{p}_2$, and second where $\mathfrak{p}'' \simeq \mathfrak{p}_1$. In either case, by Lemma 4.9 and our assumptions, we know that $0 = [\phi(x), y]_{\mathfrak{p}} + [x, \phi(y)]_{\mathfrak{p}}$ for any intertwining may $\phi : \mathfrak{m} \to \mathfrak{n}$ and all $x, y \in \mathfrak{m}$ and if and only if any decomposition into irreducibles has $[\mathfrak{q}'', \mathfrak{q}''], [\mathfrak{q}_1, \mathfrak{q}_1], [\mathfrak{q}_2, \mathfrak{q}_2] \subset \mathfrak{h}$ (depending on the case, $\mathfrak{q}'' = \mathfrak{p}''$ or $\mathfrak{q}_i = \mathfrak{p}_i$ for some i since there is one representation not isomorphic to any other and therefore unique up to scaling). What remains to be checked is the claim regarding our decomposition being simultaneously diagonalized with $B(.,.)$.

If $\mathfrak{p}_1 \simeq \mathfrak{p}_2$, then since we can always choose a $(.,.)$ orthogonal decomposition into irreducibles simultaneously diagonalized with $B(.,.)$ on $\mathfrak{p}'$ (See Lemma 1.18 for how this is always possible), we have a desired decomposition. (Note that this part did not require the condition on the intertwining maps.)

If $\mathfrak{p}'' \simeq \mathfrak{p}_1$ (the case with $\mathfrak{p}_2$ is the same with subscripts changed), then by the two conditions of being a Cartan orthogonal pair (See 2.1), we know that for any metric, we have a reductive Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}'' \oplus \mathfrak{q}'$ with $\mathfrak{q}' = \mathfrak{q}_1 \oplus \mathfrak{q}_2$ such that $(\mathfrak{q}'', \mathfrak{q}') = 0$ if and only if $[\mathfrak{p}_1, \mathfrak{p}_1] \subset \mathfrak{h}$ and $\phi([x, y]_{\mathfrak{p}''}) = [\phi(x), y] + [x, \phi(y)]$ for any $ad_{\mathfrak{h}}$ intertwining map $\phi : \mathfrak{p}'' \to \mathfrak{p}'$ and any $x, y, \in \mathfrak{p}''$. Moreover, we know that $0 = \phi([x, y]_{\mathfrak{p}''})$ by assumption, and by the Cartan decomposition properties, $[\phi(x), y]_{\mathfrak{p}} + [x, \phi(y)]_{\mathfrak{p}} = [\phi(x), y] + [x, \phi(y)]$. Also, we know that $\mathfrak{q}_2 = \mathfrak{p}_2$ since $\mathfrak{p}_2$ is unique up to scaling.

Therefore, given our assumptions, $0 = [\phi(x), y]_{\mathfrak{p}} + [x, \phi(y)]_{\mathfrak{p}}$ if and only if there is a reductive Cartan decomposition such that $(\mathfrak{q}'', \mathfrak{q}') = 0$ for every metric. Moreover, for any reductive Cartan decompositon, we have $B(\mathfrak{q}'', \mathfrak{q}') = 0$ by the Cartan decomposition properties, and we always have $B(\mathfrak{q}_1, \mathfrak{q}_2) = 0$ by $\mathfrak{p}''$, $\mathfrak{p}_1 \not\simeq \mathfrak{p}_2$.

To prove our final statement regarding ric being diagonal on our decomposition, one must only apply our claim **1** to our reductive Cartan decomposition in which $(.,.)$ and $B(.,.)$ are simultaneously diagonalized with $[\mathfrak{q}_i, \mathfrak{q}_i] \subset \mathfrak{h}$ for $i = 1, 2$. ■

4.2. The Signature

In this section, we begin to investigate the signature of our spaces and metrics, G/H and g, for which $\mathfrak{g}$ is noncompact simple and there is some reductive Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$ such that $\mathfrak{p}' = \mathfrak{p}_1 \oplus \mathfrak{p}_2$, $[\mathfrak{p}_i, \mathfrak{p}_i]$, $[\mathfrak{p}'', \mathfrak{p}''] \subset \mathfrak{h}$, $\mathfrak{p}''$, $\mathfrak{p}_1$, $\mathfrak{p}_2$ are irreducible, and our decomposition is orthogonal with respect to, $(.,.)$, the inner product associated with g (See 1.2). Our results are incomplete regarding a complete solution to the signature of such spaces; however, some first steps of approaching a complete solution are made.

Remark 4.12. As our following example indicates, just because there exist metrics for which we have $\mathfrak{p}'' \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2$ orthogonal with respect to $(.,.)$ and $[\mathfrak{p}''\mathfrak{p}'']$, $[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}$, that does not necessarily mean that all metrics will have such a condition.

Example 4.13. We showed in Example 2.2 that $(\mathfrak{so}(n, \mathbb{C})_{\mathbb{R}}, \mathfrak{so}(n-1))$ is not a Cartan orthogonal pair where $\mathfrak{p} \simeq i\mathfrak{p}$ in $\mathfrak{so}(n, \mathbb{C})_{\mathbb{R}} = \mathfrak{so}(n-1) \oplus \mathfrak{p} \oplus i\mathfrak{so}(n-1) \oplus i\mathfrak{p}$. Moreover, we have $[\mathfrak{p}, \mathfrak{p}], [i\mathfrak{so}(n-1), i\mathfrak{so}(n-1)], [i\mathfrak{p}, i\mathfrak{p}] \subset \mathfrak{so}(n-1)$ by choosing $\mathfrak{p}$ such that $\mathfrak{so}(n) = \mathfrak{so}(n-1) \oplus \mathfrak{p}$ is our reductive decomposition for the symmetric space $SO(n)/SO(n-1)$.

Since $(\mathfrak{so}(n, \mathbb{C})_{\mathbb{R}}, \mathfrak{so}(n-1))$ is not a Cartan orthogonal pair, we know that there is a metric such that there is no reductive Cartan decomposition $\mathfrak{so}(n, \mathbb{C})_{\mathbb{R}} = \mathfrak{so}(n-1) \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$ with $(\mathfrak{p}'', \mathfrak{p}') = 0$ where $(.,.)$ is the associated inner product. Now, by Lemma 4.7, for such a metric, any $(.,.)$ orthogonal decomposition, $\mathfrak{so}(n, \mathbb{C})_{\mathbb{R}} = \mathfrak{so}(n-1) \oplus i\mathfrak{so}(n-1) \oplus \mathfrak{q}_1 \oplus \mathfrak{q}_2$, would have $\mathfrak{q}_1 = \{ax + ibx : x \in \mathfrak{p}\}$ with $a, b \neq 0$. Thus, for $x, y \in \mathfrak{p}$ such that $[x, y] \neq 0$ (i.e x and y are not parallel, in this case),

$$[ax + ibx, ay + iby] = a^2[x, y] + b^2[ix, iy] + ab([ix, y] + [x, iy])$$

$$= a^2[x,y] - b^2[x,y] + 2abi[x,y] \notin \mathfrak{h}.$$

Therefore, $[\mathfrak{q}_1\mathfrak{q}_1] \not\subset \mathfrak{h}$.

With the above remark and example in mind, we provide the following definition.

Definition 4.14. For G/H with $\mathfrak{g}$ noncompact simple, define $\mathcal{M}^{adm}_{G/H}$ to be the set of *admissible* G invariant metrics on G/H. That is, the set of G invariant metrics such that there is a reductive Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$ with $\mathfrak{p}' = \mathfrak{p}_1 \oplus \mathfrak{p}_2$ such that our unique $ad_{\mathfrak{h}}$ inner product $(.,.)$ is diagonal on $\mathfrak{p}'' \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2$ (a decomposition into irreducibles) and $[\mathfrak{p}'', \mathfrak{p}''], [\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}$ for all i.

Remark 4.15. As was mentioned in the proof of **2** in Theorem 4.11, we may assume that for any inner product $(.,.)$ coming from a metric in $\mathcal{M}^{adm}_{G/H}$, $(.,.)$ is simultaneously diagonalized with $B(.,.)$. This follows from $B(\mathfrak{p}'', \mathfrak{p}') = 0$ and the ability to simultaneously diagonalize $B(.,.)$ with $(.,.)$ on $\mathfrak{p}'$ by Lemma 1.18. Moreover, by working with metrics in $\mathcal{M}^{adm}_{G/H}$, we are working with inner products that have $(\mathfrak{p}'', \mathfrak{p}') = 0$ for some reductive Cartan decomposition. Being simultaneously diagonalized with $B(.,.)$ and having $(\mathfrak{p}'', \mathfrak{p}') = 0$, we are therefore able to use Eqn.1.4 to determine the diagonal values on the $(1,1)$ Ricci tensor.

Proposition 4.16. Suppose G/H is noncompact with simple $\mathfrak{g}$. For every metric in $\mathcal{M}^{adm}_{G/H}$, ric is diagonal on the given decomposition $\mathfrak{p}'' \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2$ with the following formulas describing the diagonal of the $(1,1)$ tensor $Ric(.)$ (here we set $\mathfrak{p}_3 = \mathfrak{p}''$):

$$r_1 = \frac{-d_1 x_2 x_3 + p(x_1^2 - x_2^2 - x_3^2)}{2 d_1 x_2 x_2 x_3}$$

$$r_2 = \frac{-d_2x_1x_3 + p(x_2^2 - x_1^2 - x_3^2)}{2d_2x_1x_2x_3}$$
$$r_3 = \frac{d_3x_1x_2 + p(x_3^2 - x_1^2 - x_2^2)}{2d_3x_1x_2x_3}.$$

Here, $x_1, x_2, x_3 > 0$, $p = \sum_{\alpha,\beta,\gamma}\langle[e_1^\alpha, e_2^\beta], e_3^\gamma\rangle^2$, $\{e_i^\alpha\}$ is an orthonormal basis for $\mathfrak{p}_i$ with respect to $\langle.,.\rangle$, and $d_i = \dim\mathfrak{p}_i$.

Moreover, our scalar curvature formula S is as follows:

$$S = \frac{-d_1x_2x_3 - d_2x_1x_3 + d_3x_1x_2 - p(x_1^2 + x_2^2 + x_3^2)}{2x_1x_2x_3}$$

Proof: For a metric in $\mathcal{M}_{G/H}^{adm}$, we know there is some decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}_3 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2$ such that $[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}$ for all i with the corresponding inner product $(.,.)$ simultaneously diagonalized with $B(.,.)$. By Theorem 4.11, we know ric is diagonal, and we want to describe the diagonal values. To do this, we use Eqn.1.4 which is greatly simplified due to the bracket conditions we have assumed.

Since $[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}$ for $i = 1, 2$, we know by Theorem 4.11 that $[\mathfrak{p}_3, \mathfrak{p}_i] \subset \mathfrak{p}_j$ for $i \neq j$ and $i, j \neq 3$. By $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{p}_3$ being maximal in $\mathfrak{g}$, $[\mathfrak{p}_1, \mathfrak{p}_1], [\mathfrak{p}_2, \mathfrak{p}_2] \subset \mathfrak{h}$, and $[\mathfrak{p}', \mathfrak{p}'] = \mathfrak{k}$ (from the Cartan decomposition properties), we have $[\mathfrak{p}_1, \mathfrak{p}_2]_{\mathfrak{p}} \subset \mathfrak{p}_3$. Moreover, by (skew) symmetry of $ad_{e_l} : \mathfrak{p} \to \mathfrak{p}$ with respect to $\langle.,.\rangle = B_{\mathfrak{p}'} - B_{\mathfrak{p}''}$ (See Lemma 1.24), we have

$$\langle[e_i^\alpha, e_j^\beta], e_k^\gamma\rangle^2 = \langle[e_j^\alpha, e_i^\beta], e_k^\gamma\rangle^2 = \langle[e_i^\alpha, e_k^\beta], e_i^\gamma\rangle^2.$$

Therefore, by $[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}$ for all i, we can conclude that, unless i, j, k are pairwise distinct, we have

$$\sum_{\alpha,\beta,\gamma} \langle [e_i^\alpha, e_j^\beta], e_k^\gamma \rangle^2 = 0.$$

Now, let $p = \sum_{\alpha,\beta,\gamma} \langle [e_1^\alpha, e_2^\beta], e_3^\gamma \rangle^2$, and from our formula in Eqn.1.4 for the $(1,1)$ Ricci tensor, we get the following r_i (where each $x_i > 0$) along the diagonal:

$$\begin{aligned}
r_1 &= \frac{-1}{2x_1} + \frac{p}{4d_1}\left[\left(\frac{x_1}{x_2x_3} - \frac{x_3}{x_1x_2} - \frac{x_2}{x_1x_3}\right) + \left(\frac{x_1}{x_3x_2} - \frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2}\right)\right]\\
r_2 &= \frac{-1}{2x_2} + \frac{p}{4d_2}\left[\left(\frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2} - \frac{x_1}{x_2x_3}\right) + \left(\frac{x_2}{x_3x_1} - \frac{x_1}{x_2x_3} - \frac{x_3}{x_1x_2}\right)\right]\\
r_3 &= \frac{1}{2x_3} + \frac{p}{4d_3}\left[\left(\frac{x_3}{x_2x_1} - \frac{x_2}{x_1x_3} - \frac{x_1}{x_2x_3}\right) + \left(\frac{x_3}{x_1x_2} - \frac{x_1}{x_2x_3} - \frac{x_2}{x_3x_1}\right)\right].
\end{aligned}$$

Simplifying each and combining our terms we get:

$$\begin{aligned}
r_1 &= \frac{-d_1x_2x_3 + p(x_1^2 - x_2^2 - x_3^2)}{2d_1x_1x_2x_3}\\
r_2 &= \frac{-d_2x_1x_3 + p(x_2^2 - x_1^2 - x_3^2)}{2d_2x_1x_2x_3}\\
r_3 &= \frac{d_3x_1x_2 + p(x_3^2 - x_1^2 - x_2^2)}{2d_3x_1x_2x_3}.
\end{aligned}$$

This provides us with the desired result for the diagonal values of $Ric(.)$. Now, for the scalar curvature, since each r_i is the multiple of the identity for the ith block of the $(1,1)$ Ricci tensor, we get that $S = d_1r_1 + d_2r_2 + d_3r_3$. Thus, we have

$$S = \frac{-d_1x_2x_3 - d_2x_1x_3 + d_3x_1x_2 - p(x_1^2 + x_2^2 + x_3^2)}{2x_1x_2x_3},$$

as desired. ■

Remark 4.17. We observe that each r_i has all of $\mathbb{R}$ as a possible value (when considered one at a time) since $x_1, x_2, x_3 > 0$ can be chosen to get arbitrarily large independently of one another. We also observe that S can get infinitely negative; however, from Sections 9 and 10 of [BB78], we know that S is only always negative in the case when $\mathfrak{g} = \mathfrak{so}(n, 2)$ and $\mathfrak{h} = \mathfrak{so}(n)$ (here $d_3 = 1$).

Remark 4.18. Looking to the definition of $\mathcal{M}^{adm}_{G/H}$, we can see from Theorem 4.11 that $\mathcal{M}^{adm}_{G/H}$ is the set of all G invariant metrics if the two following conditions are met.

a. There exists one such decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}_3 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2$ with $[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}$ for all i.

b. There is at most one pair, $\mathfrak{p}_i, \mathfrak{p}_j$ isomorphic with $0 = [\phi(x), y]_{\mathfrak{p}} + [x, \phi(y)]_{\mathfrak{p}}$ for any isomorphism $\phi : \mathfrak{p}_i \longrightarrow \mathfrak{p}_j$ and for any $x, y \in \mathfrak{p}_i$.

The case when $\mathfrak{p}_1 \simeq \mathfrak{p}_2 \simeq \mathfrak{p}_3$ is one yet to be investigated.

Proposition 4.19. Under the conditions of Proposition 4.16, $r_1 + r_2 < 0$. That is, there can be at most one non-negative r_i in $\mathfrak{p}' = \mathfrak{p}_1 \oplus \mathfrak{p}_2$ from our reductive Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$.

Proof: From Proposition 4.16, we have the following equations for the r_i.

$$\begin{aligned} r_1 &= \frac{-d_1 x_2 x_3 + p(x_1^2 - x_2^2 - x_3^2)}{2 d_1 x_1 x_2 x_3} \\ r_2 &= \frac{-d_2 x_1 x_3 + p(x_2^2 - x_1^2 - x_3^2)}{2 d_2 x_1 x_2 x_3} \\ r_3 &= \frac{d_3 x_1 x_2 + p(x_3^2 - x_1^2 - x_2^2)}{2 d_3 x_1 x_2 x_3}. \end{aligned}$$

Since the denominators are all positive, it suffices to show that the numerators of r_1 and r_2

cannot be simultaneously non positive. To show this, we take the sum of the numerators:

$$\begin{aligned}&-d_1x_2x_3 + p(x_1^2 - x_2^2 - x_3^2) + -d_2x_1x_3 + p(x_2^2 - x_1^2 - x_3^2)\\&= -d_1x_2x_3 - d_2x_1x_3 - 2px_3^2 < 0.\end{aligned}$$

Since the sum of the numerators is negative, the numerators cannot be simultaneously non-negative implying that if $r_1 \geq 0$ then $r_2 < 0$ and vice versa. ∎

Example 4.20. Suppose we are under the conditions of Proposition 4.16 except that $\mathfrak{h}$ is trivial. In this case, $G/H = G$, so all irreducible isotropy representations are trivial. Thus, we are working with the case in which G is a noncompact semi-simple Lie group of dimension 3 which must be $SL_2(\mathbb{R})$. Milnor, in Corollary 4.7 of [Mil76], showed that the signature of $SL(2, \mathbb{R})$ is (using Milnor's notation), $(+, -, -)$ or $(0, 0, -)$. That is, if one r_i is positive, the other two are negative, or if two r_i are zero, then the third is negative.

From the above result and example, we know that mixed signature is possible. Great effort was put into determining conditions for there to exist metrics in with $r_i < 0$ for all i; however, for now, this is left as an open area to continue exploration. We leave the following section, exploring primarily $SO(n, 2)/SO(n)$, as motivation for future endeavors to determine the set of signatures for spaces with metrics in $\mathcal{M}^{adm}_{G/H}$.

4.3. $SO(n,2)/SO(n)$

In this section, we consider a specific example of interest for applying our results from the previous two sections, particularly, Theorem 4.11. One reason for highlighting this example is because it is previously discussed in [Nik00], but also because the scalar curvature is negative for all metrics, as noted in [BB78]. The fruit of these results, joined with Milnor's work regarding $SL(2,\mathbb{R})$ in [Mil76] and our results from Chapter 3, is a complete description of the signature of spaces G/H in which we have strictly negative scalar curvature for all metrics and $\mathfrak{g}$ simple (excluding the irreducible symmetric spaces as these are well-known).

Proposition 4.21. Consider $SO(n,2)/SO(n)$ with decomposition $\mathfrak{so}(n,2) = \mathfrak{so}(n) \oplus \mathfrak{so}(2) \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2$ where $\mathfrak{p}_1 \simeq \mathfrak{p}_2$ and $n \geq 3$. Let $\mathfrak{g} = \mathfrak{so}(n,2)$, $\mathfrak{h} = \mathfrak{so}(n)$, and $\mathfrak{p}_3 = \mathfrak{so}(2)$. Given any G invariant metric on G/H, there is a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}_3 \oplus \mathfrak{q}_1 \oplus \mathfrak{q}_2$ such that $[\mathfrak{q}_i, \mathfrak{q}_i] \subset \mathfrak{h}$ for all i and the $ad_{\mathfrak{h}}$ invariant inner product, $(.,.)$, is simultaneously diagonalized with $B(.,.)$ and ric is diagonal.
Consequently, all G invariant metrics are in $\mathcal{M}^{adm}_{G/H}$.

Proof: Using Chapter X of [Hel01] and that $\mathfrak{g} = \mathfrak{so}(n,2)$, $\mathfrak{k} = \mathfrak{so}(n) \oplus \mathfrak{so}(2)$, and $\mathfrak{h} = \mathfrak{so}(n)$, we can get the following decomposition in which $A \in \mathfrak{so}(n)$, $z \in \mathbb{R}$, and $x, y \in \mathbb{R}^n$.

$$\mathfrak{g} = \left[\begin{array}{c|c|c} A & x & y \\ \hline x^t & 0 & 0 \\ \hline y^t & 0 & 0 \end{array}\right] \quad \mathfrak{k} = \left[\begin{array}{c|c|c} A & 0 & 0 \\ \hline 0 & 0 & z \\ \hline 0 & -z & 0 \end{array}\right]$$

$$\mathfrak{h} = \left[\begin{array}{c:c:c} A & 0 & 0 \\ \hdashline 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 \end{array}\right] \quad \mathfrak{p}'' = \left[\begin{array}{c:c:c} 0 & 0 & 0 \\ \hdashline 0 & 0 & z \\ \hdashline 0 & -z & 0 \end{array}\right]$$

$$\mathfrak{p}' = \left[\begin{array}{c:c:c} 0 & x & y \\ \hdashline x^t & 0 & 0 \\ \hdashline y^t & 0 & 0 \end{array}\right]$$

First, we seek to simultaneously diagonalize $(.,.)$ on $\mathfrak{p}$ with $B(.,.)$. Nikonorov in [Nik00] showed that $(\mathfrak{p}'', \mathfrak{p}') = 0$ for all $ad_{\mathfrak{h}}$ invariant innerproducts, observing that $\mathfrak{p}''$ is irreducible and dimension 1 where as $\mathfrak{p}'$ decomposes into two irreducible respresentations of dimension n:

$$\mathfrak{p}_1 = \left[\begin{array}{c:c:c} 0 & x & 0 \\ \hdashline x^t & 0 & 0 \\ \hdashline 0 & 0 & 0 \end{array}\right] \text{ and } \mathfrak{p}_2 = \left[\begin{array}{c:c:c} 0 & 0 & y \\ \hdashline 0 & 0 & 0 \\ \hdashline y^t & 0 & 0 \end{array}\right].$$

Notice that $\mathfrak{p}_1 \simeq \mathfrak{p}_2$ as $ad_{\mathfrak{h}}$ representations by the intertwining map ψ defined below:

$$\psi\left(\left[\begin{array}{c:c:c} 0 & x & 0 \\ \hdashline x^t & 0 & 0 \\ \hdashline 0 & 0 & 0 \end{array}\right]\right) = \left[\begin{array}{c:c:c} 0 & 0 & x \\ \hdashline 0 & 0 & 0 \\ \hdashline x^t & 0 & 0 \end{array}\right].$$

Moreover, observe that $[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}$ for $i = 1, 2$, and since $\mathfrak{p}''$ is one dimensional, $[\mathfrak{p}'', \mathfrak{p}''] = \{0\} \subset \mathfrak{h}$ as well.

We now wish to use the results in Theorem 4.11, so we need to understand the $ad_{\mathfrak{h}}$ representation on $\mathfrak{p}_1$ given below:

$$\left[\left[\begin{array}{c:c:c} A & 0 & 0 \\ \hdashline 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 \end{array}\right], \left[\begin{array}{c:c:c} 0 & x & 0 \\ \hdashline x^t & 0 & 0 \\ \hdashline 0 & 0 & 0 \end{array}\right]\right] =$$

$$\left[\begin{array}{c|c|c} 0 & Ax & 0 \\ \hline -x^t A & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right].$$

Using Type BD I in Section 2 of Chapter X in [Hel01], one can observe that this representation is the same as the $ad_{\mathfrak{so}(n)}$ representation in $SO(n,1)/SO(n)$, an irreducible representation of real type (see Section 1.6 and the references therein). Therefore, any $ad_{\mathfrak{h}}$ intertwining $\phi : \mathfrak{p}_1 \to \mathfrak{p}_2$ is of the form $\lambda\psi$ for $\lambda \in \mathbb{R}$. This allows us now to see that $[v, \phi(w)] + [\phi(v), w] = \lambda([v, \psi(w)] + [\psi(v), w])$. Moreover, it is easy to check that $[v, \psi(w)] + [\psi(v), w] = 0$. Therefore, by an application of claim **2** in Theorem 4.11, we get that for any G invariant metric on G/H, there is a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{q}_1 \oplus \mathfrak{q}_2$ such that $[\mathfrak{q}_i, \mathfrak{q}_i] \subset \mathfrak{h}$ for all i and our inner product, $(.,.)$, is simultaneously diagonalized with $B(.,.)$ on $\mathfrak{p}'' \oplus \mathfrak{q}_1 \oplus \mathfrak{q}_2$. Furthermore, we know that $ric(.,.)$ is diagonal, as desired. ■

Remark 4.22. Nikonorov showed in Example 1 of [Nik00] that there are metrics for which every r_i is negative for $SO(n,2)/SO(n)$. Our next result goes further to say that there are also metrics in which (r_1, r_2, r_3) has exactly one non-negative term and that there can be no more.

Proposition 4.23. For $SO(n,2)/SO(n)$ with $n \geq 3$ there can be at most one r_i such that r_i is non-negative where r_i is as in Proposition 4.16.

Proof: By Proposition 4.21, we know that every left invariant metric for $SO(n,2)/SO(n)$

is in $\mathcal{M}^{adm}_{G/H}$. Thus, we know that every metric has ric described by the r_i in Proposition 4.16.

Since $d_3 = 1$, by Lemma 1 in [Nik00], $2p = 1$. Since we have $d_1 = d_2 = n$, we have the following formulas for the r_i coming from Proposition 4.16 (we multiplied the top and bottom by 2):

$$
\begin{aligned}
r_1 &= \frac{-2nx_2x_3 + x_1^2 - x_2^2 - x_3^2}{4nx_1x_2x_3} \\
r_2 &= \frac{-2nx_1x_3 + x_2^2 - x_1^2 - x_3^2}{4nx_1x_2x_3} \\
r_3 &= \frac{2x_1x_2 + x_3^2 - x_2^2 - x_1^2}{4x_1x_2x_3}
\end{aligned}
$$

From Proposition 4.19, we know that $r_1 + r_2 < 0$.

Thus, if $r_1 \geq 0$, $r_2 < 0$ and vice versa. To finish, we show that $r_3 \geq 0$ implies $r_1 < 0$ and $r_2 < 0$.

$r_3 \geq 0$ if and only if $x_3^2 \geq x_2^2 + x_1^2 - 2x_1x_2 = (x_2 - x_1)^2$, so $x_3 \geq |x_2 - x_1|$. If $x_1 = x_2$ then $r_1, r_2 < 0$ is easy to see from the r_i found above, so assume that $x_2 > x_1$ and observe:

$$
\begin{aligned}
r_1 &\leq \frac{-2nx_2(x_2 - x_1) + x_1^2 - x_2^2 - (x_2 - x_1)^2}{4nx_1x_2x_3} \\
&= \frac{-2nx_2 - x_1 - x_2 - (x_2 - x_1)}{4nx_1x_2x_3}(x_2 - x_1) \\
&= \frac{-nx_2 - 2x_2}{4nx_1x_2x_3}(x_2 - x_1) \\
&< 0 \text{ since } x_1 < x_2
\end{aligned}
$$

$$
\begin{aligned}
r_2 &\leq \frac{-2nx_1(x_2 - x_1) + x_2^2 - x_1^2 - (x_2 - x_1)^2}{4nx_1x_2x_3} \\
&= \frac{-2nx_2 + x_2 + x_1 - (x_2 - x_1)}{4nx_1x_2x_3}(x_2 - x_1)
\end{aligned}
$$

$$= \frac{-2nx_2 + 2x_1}{4nx_1x_2x_3}(x_2 - x_1)$$
$$< 0 \text{ since } x_1 < x_2 \text{ and } n > 1.$$

By the symmetry of r_1 and r_2 in relation to x_1 and x_2, it is easy to see the same is true if we assume $x_2 < x_1$. ■

In [AL22] while studying the signature of spaces in the nilpotent setting, Arroyo and Lafuente determined $\sigma_{Ric}(N)$, the set of signatures of the Ricci curvature for left invariant metrics on a connected nilpotent Lie group N. Moreover, they described the set of signatures completely in terms of Lie-theoretic data. In the following theorem, we summarize our results above, adopting the same notation used by Arroyo and Lafuente in which

$$\sigma_{Ric}(G/H) = \{\sigma(Ric(g)) : g, \text{ a } G\text{invariant Riemannian metric on } G/H\}$$

and $\sigma(Ric(g)) = (s^-, s^0, s^+) \in \mathbb{Z}^3_{\geq 0}$. Here, s^- indicates the number of negative values in the signature, s^0 the 0 values, and s^+ the positive values.

Corollary 4.24. The set of signatures for Ric on $SO(n,2)/SO(n)$ with $n \geq 3$ is

$$\{(2+2n,0,0),(2+n,n,0),(2n,2,0),(2+n,0,n)(2n,0,2)\}$$

where the elements in the set are describing (s^-, s^0, s^+).

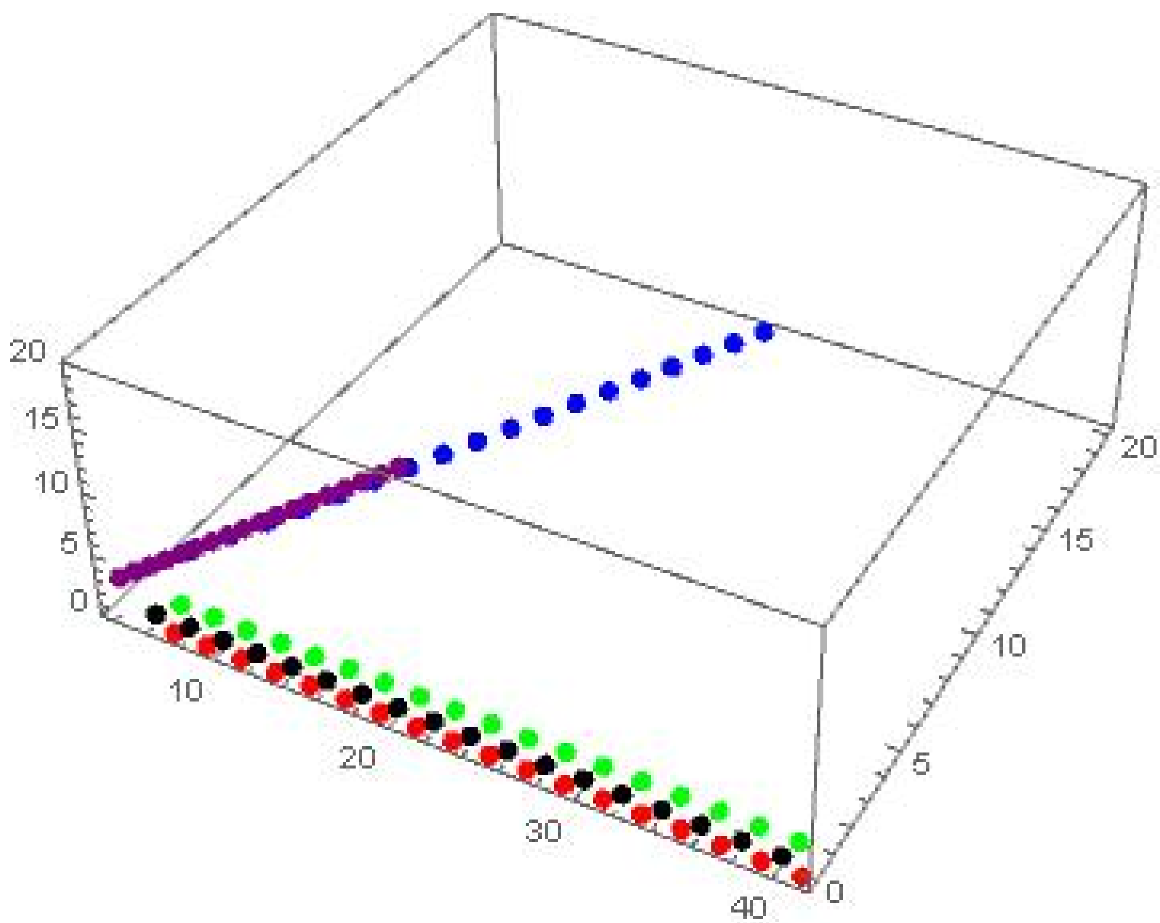
<u>Proof:</u> By Proposition 4.21, we know that every metric is in $\mathcal{M}^{adm}_{G/H}$. Moreover, by Example 1 in [Nik00] we know there exist metrics in which $r_i < 0$ for all i, and by Proposition 4.23, we know there is at most one non-negative r_i. This allows us to see that the signature of $SO(n,2)/SO(n)$ is given by the following. (Recall $\dim\mathfrak{p}'' = 1$, and $\dim\mathfrak{p}_1 =$

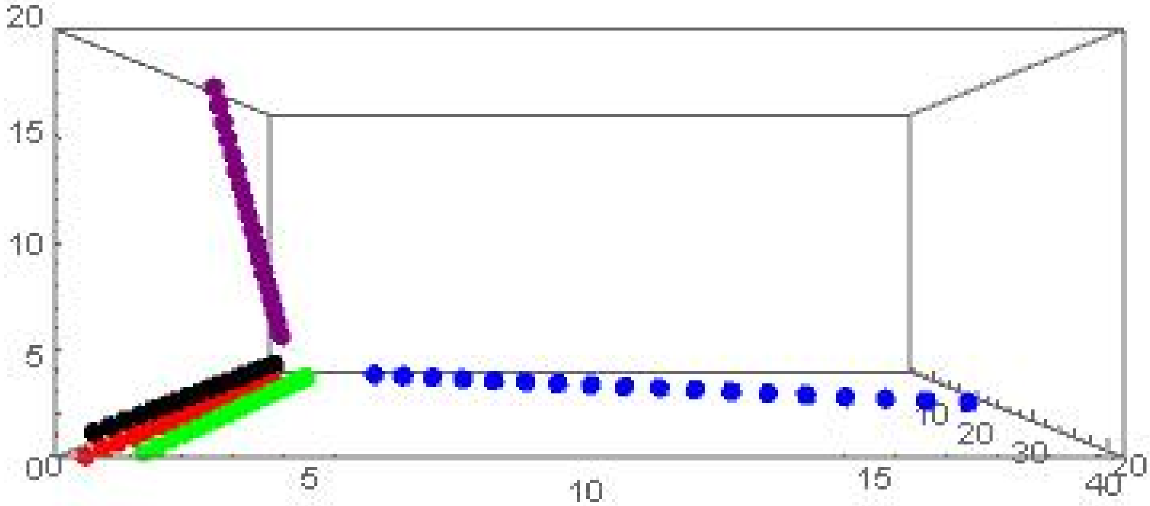
$\dim \mathfrak{p}_2 = n$.)

$$\{(1+2n,0,0),(1+n,n,0),(1+n,n,0)(2n,1,0),(1+n,0,n),(1+n,0,n),(2n,0,1)\}$$

By removing the duplicates, we have our desired result. ■

We provide two graphs of the signature from $n = 3$ to $n = 20$ (two graphs so that different perspectives are provided). The red points are representative the first element in the set from the corollary above, the blue points for the second element, the green points for the third, the purple points for the fourth, and the black points for the fifth.

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20
40

With the above results for the signature of ric for $SO(n,2)/SO(n)$ and Example 4.20 regarding $SL(2,\mathbb{R})$ from [Mil76], we can now describe completely the signature of G/H in which $\mathfrak{g}$ is simple and the scalar curvature is always negative. If G/H is an irreducible symmetric space, then the signature for each space is described by $\{(d,0,0)\}$ where d is the dimension of the given irreducible symmetric space. We ignore this case in the following as it is well-known.

From Sections 9 and 10 of [BB78] (Section 10 has the table we are referencing), we know that if, for connected H and simple G, G/H has strictly negative scalar curvature for all G invariant metrics (and is not an irreducible symmetric space), then $(\mathfrak{g},\mathfrak{h})$ is described by:

$(\mathfrak{sl}(2,\mathbb{R}),\{0\})$

$(\mathfrak{so}(n,2),\mathfrak{so}(n))$ with $n \geq 3$

$(\mathfrak{su}(m,n),\mathfrak{su}(m)\oplus\mathfrak{su}(n))$ with $m \geq n \geq 1$ and $(n,m) \neq (1,1),(2,2)$

$(\mathfrak{so}^*(2n),\mathfrak{su}(n))$ with $n \geq 5$

$(\mathfrak{sp}(n,\mathbb{R}),\mathfrak{su}(n))$ with $n \geq 3$

$(\mathfrak{e}_6^{-14},\mathfrak{so}(10))$

$(\mathfrak{e}_7^{-25},\mathfrak{e}_6)$.

The third through seventh items on the list, can be found in Tables 3.1 through 3.4 from Theorem 3.2 as spaces with two isotropy summands. Furthermore, for each of the spaces with two irreducible summands, the reductive Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$ has trivial $\mathfrak{p}''$. Thus, by a simple application of Corollary 3.18 and looking at the dimensions of $\mathfrak{p}'$ from the classification of symmetric spaces in Chapter X of [Hel01], we describe the set of signatures using the notation from [AL22].

Corollary 4.25. If G/H has negative scalar curvature for all metrics (and is not an irreducible symmetric space), G is simple, and H is connected, then the set of signatures of ric is according to the following descriptions (with the appropriate conditions on the n, m from above):

$$\begin{aligned}
(\mathfrak{sl}(2,\mathbb{R}), \{0\}) &: \{(2,0,1), (1,2,0)\} \\
(\mathfrak{so}(n,2), \mathfrak{so}(n)) &: \{(2+2n,0,0), (2+n,n,0), (2n,2,0), (2+n,0,n), (2n,0,2)\} \\
(\mathfrak{su}(m,n), \mathfrak{su}(m) \oplus \mathfrak{su}(n)) &: \{(2mn,0,1)\} \\
(\mathfrak{so}^*(2n), \mathfrak{su}(n)) &: \{(n(n-1),0,1)\} \\
(\mathfrak{sp}(n,\mathbb{R}), \mathfrak{su}(n)) &: \{(n(n+1),0,1)\} \\
(\mathfrak{e}_6^{-14}, \mathfrak{so}(10)) &: \{(32,0,1)\} \\
(\mathfrak{e}_7^{-25}, \mathfrak{e}_6) &: \{(54,0,1)\}
\end{aligned}$$

Proof: From Corollary 3.18 we know that $r_1 < 0$ on $\mathfrak{p}'$ and $r_2 > 0$ on $\mathfrak{p}''$. Thus, for the spaces with two irreducible summands, we have our result simply by checking the dimensions of $\mathfrak{p}'$ since $\mathfrak{p}''$ has dimension 1. ■

Appendix A

Programming Usage

A.1. Python Usage

In this section, we provide the Python code that uses sympy ([MSP+17]) to determine an orthonormal basis for $\mathfrak{p} = \mathfrak{p}'' \oplus \mathfrak{p}'$ in $\mathfrak{so}(1,7) = \mathfrak{g}_2 \oplus \mathfrak{p}'' \oplus \mathfrak{p}'$ with respect to the metric $\langle .,. \rangle = B_{\mathfrak{p}'}(.,.) - B_{\mathfrak{p}''}(.,.)$.

Finding a Basis for $\mathfrak{so}(1,7)$

```
from sympy import *
from sympy import Matrix
from sympy.abc import a, b,c,d

def bracket(a,b):
return (a*b - b*a)

def printbasistolatex(Basis):
i = 1
for x in Basis:
print(i)
print_latex(x)
print("\\" "\\")
i = i+1

#This is G2 as given by Maple a subalgebra of so(7). I was able to tell Maple to copy to python code and then I pasted it here
G2 = [Matrix([[0,1,0,0,0,0,0],[-1,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],
[0,0,0,0,0,1,0],[0,0,0,0,-1,0,0],[0,0,0,0,0,0,0]]),
Matrix([[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,1],
[0,0,0,0,0,1,0],[0,0,0,0,-1,0,0],[0,0,0,-1,0,0,0]]),
Matrix([[0,0,1,0,0,0,0],[0,0,0,0,0,0,0],[-1,0,0,0,0,0,0],[0,0,0,0,0,0,0],
[0,0,0,0,0,0,1],[0,0,0,0,0,0,0],[0,0,0,0,-1,0,0]]),
Matrix([[0,0,0,0,0,0,0],[0,0,1,0,0,0,0],[0,-1,0,0,0,0,0],[0,0,0,0,0,0,0],
[0,0,0,0,0,0,0],[0,0,0,0,0,0,1],[0,0,0,0,0,-1,0]]),
```

```
Matrix([[0,0,0,0,0,0,1],[0,0,0,0,0,0,0],[0,0,0,0,1,0,0],
[0,0,0,0,0,0,0],[0,0,-1,0,0,0,0],[0,0,0,0,0,0,0],[-1,0,0,0,0,0,0]]),
Matrix([[0,0,0,0,0,0,0],[0,0,0,0,0,0,1],[0,0,0,0,0,1,0],[0,0,0,0,0,0,0],
[0,0,0,0,0,0,0],[0,0,-1,0,0,0,0],[0,-1,0,0,0,0,0]]),
Matrix([[0,0,0,1,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,-1,0],[-1,0,0,0,0,0,0],
[0,0,0,0,0,0,0],[0,0,1,0,0,0,0],[0,0,0,0,0,0,0]]),
Matrix([[0,0,0,0,0,0,0],[0,0,0,1,0,0,0],[0,0,0,0,1,0,0],
[0,-1,0,0,0,0,0],[0,0,-1,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0]]),
Matrix([[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,1,0,0],
[0,0,0,-1,0,0,0],[0,0,0,0,0,0,1],[0,0,0,0,0,-1,0]]),
Matrix([[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,1,0],
[0,0,0,0,0,0,-1],[0,0,0,-1,0,0,0],[0,0,0,0,1,0,0]]),
Matrix([[0,0,0,0,1,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,-1],
[0,0,0,0,0,0,0],[-1,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,1,0,0,0,0]]),
Matrix([[0,0,0,0,0,0,0],[0,0,0,0,1,0,0],[0,0,0,-1,0,0,0],[0,0,1,0,0,0,0],
[0,-1,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0]]),
Matrix([[0,0,0,0,0,1,0],[0,0,0,0,0,0,0],[0,0,0,1,0,0,0],[0,0,-1,0,0,0,0],
[0,0,0,0,0,0,0],[-1,0,0,0,0,0,0],[0,0,0,0,0,0,0]]),
Matrix([[0,0,0,0,0,0,0],[0,0,0,0,0,1,0],[0,0,0,0,0,0,-1],
[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,-1,0,0,0,0,0],[0,0,1,0,0,0,0]])]

#This is so(7) as given by Maple. I copied as python code again and pasted here. I don't  like its presentation, and I want each element's negative
```

```
so7 = [Matrix
  ([[0,-1,0,0,0,0,0],[1,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],
[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0]]),
Matrix
  ([[0,0,-1,0,0,0,0],[0,0,0,0,0,0,0],[1,0,0,0,0,0,0],[0,0,0,0,0,0,0],
[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0]]),
Matrix
  ([[0,0,0,-1,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[1,0,0,0,0,0,0],
[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0]]),
Matrix
  ([[0,0,0,0,-1,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],
[1,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0]]),
Matrix
  ([[0,0,0,0,0,-1,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],
[0,0,0,0,0,0,0],[1,0,0,0,0,0,0],[0,0,0,0,0,0,0]]),
Matrix
  ([[0,0,0,0,0,0,-1],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],
[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[1,0,0,0,0,0,0]]),
Matrix
  ([[0,0,0,0,0,0,0],[0,0,-1,0,0,0,0],[0,1,0,0,0,0,0],[0,0,0,0,0,0,0],
[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0]]),
Matrix
  ([[0,0,0,0,0,0,0],[0,0,0,-1,0,0,0],[0,0,0,0,0,0,0],[0,1,0,0,0,0,0],
[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0]]),
Matrix
  ([[0,0,0,0,0,0,0],[0,0,0,0,-1,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],
[0,1,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0]]),
Matrix
  ([[0,0,0,0,0,0,0],[0,0,0,0,0,-1,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],
[0,0,0,0,0,0,0],[0,1,0,0,0,0,0],[0,0,0,0,0,0,0]]),
```

```
Matrix([[0,0,0,0,0,0,0],[0,0,0,0,0,0,-1],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],
[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,1,0,0,0,0,0]]),
Matrix([[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,-1,0,0,0],[0,0,1,0,0,0,0],
[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0]]),
Matrix([[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,-1,0,0],[0,0,0,0,0,0,0],
[0,0,1,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0]]),
Matrix([[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,-1,0],[0,0,0,0,0,0,0],
[0,0,0,0,0,0,0],[0,0,1,0,0,0,0],[0,0,0,0,0,0,0]]),
Matrix([[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,-1],[0,0,0,0,0,0,0],
[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,1,0,0,0,0]]),
Matrix([[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,-1,0,0],
[0,0,0,1,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0]]),
Matrix([[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,-1,0],
[0,0,0,0,0,0,0],[0,0,0,1,0,0,0],[0,0,0,0,0,0,0]]),
Matrix([[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,-1],
[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,1,0,0,0]]),
Matrix([[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],
[0,0,0,0,0,-1,0],[0,0,0,0,1,0,0],[0,0,0,0,0,0,0]]),
Matrix([[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],
[0,0,0,0,0,0,-1],[0,0,0,0,0,0,0],[0,0,0,0,1,0,0]]),
```

```
Matrix([[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],[0,0,0,0,0,0,0],
[0,0,0,0,0,0,0],[0,0,0,0,0,0,-1],[0,0,0,0,0,1,0]])]
Newso7 = []
for x in so7:
Newso7 = Newso7 + [-x]



#The G2 basis we have is not orthogonal under trace pairing, and we ultimately want it's orthogonal complement
#The approach is to us gram-schmidt on G2 with the trace pairing
#We then get a linearly independent complement of G2 in so(7) and gram-schmidt the orthogonal basis for G2 with the lin. ind. complement

#First we defined a metric of so(7), the scalar is chosen to make things look nicer
def metric(u,v):
return(-S(6)*trace(u*v))

#This defines the projection of v onto u using the metric we defined. This is needed for gram-schmidt
def proj(u,v):
return((S(metric(u,v))/metric(u,u))*u)

#This defines the gram-schmidt procedure function. It needs a basis, the dimension of the basis, and the size the sqaure matrices
#This function returns an orthonormal basis
def gramschmidt(Basis, Dim, n):
NewBasis = [S(1)/(sqrt(metric(Basis[0], Basis[0])))*Basis[0]]
i = 1
```

```
j = 0
Sub = zeros(n, n)
while i < Dim:
while j < i:
Sub = Sub + proj(NewBasis[j], Basis[i])
j = j + 1
E = Basis[i] - Sub
E = (S(1)/(sqrt(metric(E,E))))*E
i = i+1
NewBasis = NewBasis + [E]
j = 0
Sub = zeros(n,n)
return(NewBasis)




#This is to get my orthonormal G2
NewG2 = gramschmidt(G2, 14, 7)


#Below is a Linearly independent complement to G2 in so(7)
#The process to achieve this was to look at the original G2 write out
  the elements as basis elements of so(7)
#There was an obvious choice once you write out the origial G2
  elements
K1 = Newso7[0] + Newso7[17]
K2 = Newso7[2] - Newso7[10]
K3 = Newso7[4] - Newso7[8]
K4 = Newso7[1] - Newso7[16]
K5 = Newso7[3] + Newso7[9]
```

```
K6 = Newso7[5] + Newso7[7]
K7 = Newso7[6] + Newso7[15]
LIC = [K1, K2, K3, K4, K5, K6, K7]

TheBasis = NewG2 + LIC

#This will provide a basis for the p'' in so(7) and should keep the
 basis for G2 the same
TheNewBasis = gramschmidt(TheBasis, 21, 7)


#We now have an orthonormal g2 and we have an orthonormal basis for an
  invariant complement


#We now want to place our basis for so(7) inside so(1,7)
#This will only require adding a column and row of zeros to the left
 and top


so17Basis1 = []
for x in TheNewBasis:
row = zeros(1, 7)
col = zeros(8, 1)
R = x.row_insert(0, row)
C = R.col_insert(0, col)
so17Basis1 = so17Basis1 + [C]


#I now want to create a basis that uses the original basis given for
 g2 to avoid any concern about the new g2 basis not being g2
```

```
#Placing the original g2 basis in so(1,7) which doesn't change bracket relations or dimension
G28 = []
for x in G2:
row = zeros(1, 7)
col = zeros(8, 1)
R = x.row_insert(0, row)
C = R.col_insert(0, col)
G28 = G28 + [C]


#Now we need our p' and to complete our so17Basis that is orthonormal
#There is an obvious choice for a basis of p' to make
#We will have to rescale our basis in p' piece to ensure that this basis is not just orthogonal but orthonormal for our metric

#These are matrices with ones along the first row except the diagonal and 0's elsewhere
e1 = Matrix([[0,1, 0,0,0,0,0,0], zeros(7,8)])
e2 = Matrix([[0,0, 1,0,0,0,0,0], zeros(7,8)])
e3 = Matrix([[0,0, 0,1,0,0,0,0], zeros(7,8)])
e4 = Matrix([[0,0, 0,0,1,0,0,0], zeros(7,8)])
e5 = Matrix([[0,0, 0,0,0, 1,0,0], zeros(7,8)])
e6 = Matrix([[0,0, 0,0,0, 0,1,0], zeros(7,8)])
e7 = Matrix([[0,0, 0,0,0, 0,0, 1], zeros(7,8)])


p1 = e1 + transpose(e1)
p2 = e2 + transpose(e2)
p3 = e3 + transpose(e3)
```

```
p4 = e4 + transpose(e4)
p5 = e5 + transpose(e5)
p6 = e6 + transpose(e6)
p7 = e7 + transpose(e7)


P = [p1, p2, p3, p4, p5, p6, p7]







so17Basis = G28 + so17Basis1[14:21] + P
#printbasistolatex(so17Basis)

#I now want to reorder the basis in p'' to get the intertwing map to
  be diagonal in all four blocks, and in addition to this
#We will rescale the basis in p' so that it is orthonormal

A = so17Basis[0:14]
B = [so17Basis[20], -so17Basis[17], so17Basis[14], -so17Basis[18],
  so17Basis[15], so17Basis[19], -so17Basis[16]]
C = []
for x in so17Basis[21:28]:
y = S(1)/sqrt(12)*x
C = C + [y]
so17Basis = A + B + C
```

```
#The following assignments are in some sense redundant, but it puts a nice name on them

#G2 in so(1,7)
G28 = so17Basis[0:14]

#p''
P2 = so17Basis[14:21]

#p'
P1 = so17Basis[21:28]


#Here is the metric to make so17Basis orthonormal. P1 is the only not orthonormal piece left
#It is important to recognize that this metric works fine on P1 so long as it is an elemnt of P1, the collection of basis elements
#However, if you try to use the metric on a linear combination of basis elements in P1, it will not work
#The purpose of defining the metric this way is to make it simple for checking orthonormality of the basis
def metric1(u,v):
if u in P1 and v in P1:
return(S(6)*trace(u*v))
else:
return(metric(u,v))
```

Code Checks for $\mathfrak{so}(1,7)$ Basis

In the following, we provide the code we use to check that our basis is what we want. That is, we check that our basis is orthonormal with respect to $\langle .,. \rangle$, we check that we span $\mathfrak{p}$, and we ensure that $\mathfrak{p}''$ is an $ad_{\mathfrak{g}_2}$ invariant complement by checking that it is a $\langle .,. \rangle$ orthogonal complement to $\mathfrak{g}_2$.

```python
from sympy import *
from sympy import Matrix
from sympy.abc import a, b,c,d
from so17orthonormalbasis import *

#This file checks that I have a decomposition for so(1,7) = g2 + p'' + p'
#The one thing that is not checked is that when I take the g2 I get from Maple and place it in so(1,7) that it is still g2.
#One can eye-ball it if needed, but all I am doing is placing a row of 0's and a column of 0's above and to the left of the matrices in g2
#This does not change bracket relations or dimension, so I am still in g2.


#The ordering of the checks:
#First, we check that g2 + p'' is in so(7) and an orthogonal decomposition
#Second, we check that so(7) + p' is an orthogonal decomposition and that p' matches Helgason's definition in Ch. X
#Third, we check that p'' has an orthonormal decomposition
#Fourth, we check that p' has an orthonormal decomposition
#Last, we check that so(1,7) has the right dimension of 28
```

```
#The only other thing one might want to check is that the so(7) is in
 the bottom right block and the p' consists of E_1i + E_i1, this is
 an easy check by the eye or you can look at the source code


#This is the so(1,7) basis whose first 14 entries are the G2 above but
  placed in so(1,7) instead of just so(7)
#The next 14 entries make up p'' and p' (7 dim each)
so17Basis

#G2 in so(1,7)
G28

#p''
P2

#p'
P1





print("Check 1: G28 + P2, are they orthogonal commplements in so(7)
 and are there enough elements to span?")
#Check that G28 and p'' are orthogonal under trace(uv) and are in so
 (7) inside so(1,7)
#If you get an "uh oh for G28 and p2" then you are not in so(7)
#If you get something besides [0,21], you don't have orthogonality (
 the 0) or enough to span (the 21)
sum = 0
```

```
for x in G28:
for y in P2:
if transpose(x) == -x and transpose(y) == -y:
sum = sum + trace(x*y)
else:
print("uh oh for G28 and p2")
print([x,y])
print("no uh oh message? Success, your elements are skew symmetric!")
print("If there is an uh oh message, then the pair [x,y] that comes
 with it are the elements in G28 and P2 respectively that are not in
 so(7)")
print([sum, len(G28 + P2)])
print("If the pair above is [0, 21] then you have orthogonality
 between elements in G28 and P2 and there are 21 elements in G28 + P2
 ")
print("If the first number is not 0, you don't have an orthogonal
 complement")
print("If the second number is not 21 then you don't have enough
 elements to span so(7)")



K = G28 + P2


print("Check 2: Is P1 orthogonal to the maximal compact so(7) = K and
 are there enough elements to span so(1,7)?")
#Checking that P1 is orthogonal to the maximal compact so(7) = K
#Also checking that P1 is as defined in Helgason
#If you get an "uh oh for p1" then it is not defined as in Helgason
```

```
#If you get a number other than [0, 7] then you don't have
 orthogonality (the 0) or enough to span (the 7)
sum = 0
for x in K:
for y in P1:
if transpose(y) == y:
sum = sum + trace(x*y)
else:
print("uh oh for P1")
print(y)
print("no uh oh message? Success! Your elements are symmetric!")
print("If there is an uh oh message then the element that comes with
 it is not in p'")
print([sum, len(P1)])
print("If the above [x,y] is [0,7] then you have P1 as an orthogonal
 complement to K and K + P1 has enough elements to span so(1,7)")
print("If not x in [x,y] is not 0 then you don't have an orthogonal
 complement and if y is not 28 then you don't have enough elements to
  span so(1,7)")


print("Check 3: Are P1 and P2 elements orthonormal for their
 respective metrics?")
#Checking orthonormality of P2 using the metric defined in called file
#If the sum is anything other than 0 orthogonality is a problem
#If "uh oh P2 not orthonormal" appears then the element is not norm 1
sum = 0
for x in P2:
for y in P2:
if x == y:
if metric(x,y) == 1:
```

```
pass
else:
print("uh oh P2 not orthonormal")
print(x)
else:
sum = sum + metric(x,y)
print("If no uh oh message for P2 then your norms are 1! Otherwise,
 the element given is not normalized for the metric")
print(sum)
print("If the number above is not 0 then you don't have orthogonality
 in P2!")



#Checking orthonormality of P1 using the metric defined in called file
#If the sum is anything other than 0 orthogonality is a problem
#If "uh oh P1 not orthonormal" appears then the element is not norm 1
sum = 0
for x in P1:
for y in P1:
if x == y:
if metric1(x,y) == 1:
pass
else:
print("uh oh P1 not orthonormal")
print(x)
else:
sum = sum + metric(x,y)
print("If no uh oh message for P1 then your norms are 1! Otherwise,
 the element given is not normalized for the metric")
print(sum)
```

```
print("If the number above is not 0 then you don't have orthogonality
 in P1!")

#This checks how many elements are in so17, I expect 28
print(len(so17Basis))

```

Below is what we get when we run the above piece of code.

```
Check 1: G28 + P2, are they orthogonal commplements in so(7) and are
 there enough elements to span?
no uh oh message? Success, your elements are skew symmetric!
If there is an uh oh message, then the pair [x,y] that comes with it
 are the elements in G28 and P2 respectively that are not in so(7)
[0, 21]
If the pair above is [0, 21] then you have orthogonality between
 elements in G28 and P2 and there are 21 elements in G28 + P2
If the first number is not 0, you don't have an orthogonal complement
If the second number is not 21 then you don't have enough elements to
 span so(7)
Check 2: Is P1 orthogonal to the maximal compact so(7) = K and are
 there enough elements to span so(1,7)?
no uh oh message? Success! Your elements are symmetric!
If there is an uh oh message then the element that comes with it is
 not in p'
[0, 7]
If the above [x,y] is [0,7] then you have P1 as an orthogonal
 complement to K and K + P1 has enough elements to span so(1,7)
If not x in [x,y] is not 0 then you don't have an orthogonal
 complement and if y is not 28 then you don't have enough elements to
  span so(1,7)
Check 3: Are P1 and P2 elements orthonormal for their respective
```

```
 metrics?
If no uh oh message for P2 then your norms are 1! Otherwise, the
 element given is not normalized for the metric
0
If the number above is not 0 then you don't have orthogonality in P2!
If no uh oh message for P1 then your norms are 1! Otherwise, the
 element given is not normalized for the metric
0
If the number above is not 0 then you don't have orthogonality in P1!
28

```

Our Orthonormal basis on $\mathfrak{p}'' \oplus \mathfrak{p}'$

Here we provide the basis that we ultimately use to compute *ric*. This basis is orthonormal with respect to $\langle ., . \rangle$.

Our basis for $\mathfrak{p}''$:

$$1\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&\frac{1}{6}&0&0&0&0\\ 0&0&-\frac{1}{6}&0&0&0&0&0\\ 0&0&0&0&0&\frac{1}{6}&0&0\\ 0&0&0&0&-\frac{1}{6}&0&0&0\\ 0&0&0&0&0&0&0&-\frac{1}{6}\\ 0&0&0&0&0&0&\frac{1}{6}&0 \end{bmatrix} \qquad 2\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&-\frac{1}{6}&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&\frac{1}{6}&0&0&0&0&0&0\\ 0&0&0&0&0&0&\frac{1}{6}&0\\ 0&0&0&0&0&0&0&\frac{1}{6}\\ 0&0&0&0&-\frac{1}{6}&0&0&0\\ 0&0&0&0&0&-\frac{1}{6}&0&0 \end{bmatrix}$$

$$3\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&\frac{1}{6}&0&0&0&0&0\\ 0&-\frac{1}{6}&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&\frac{1}{6}\\ 0&0&0&0&0&0&-\frac{1}{6}&0\\ 0&0&0&0&0&\frac{1}{6}&0&0\\ 0&0&0&0&-\frac{1}{6}&0&0&0 \end{bmatrix} \qquad 4\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&-\frac{1}{6}&0&0\\ 0&0&0&0&0&0&-\frac{1}{6}&0\\ 0&0&0&0&0&0&0&-\frac{1}{6}\\ 0&0&0&0&0&0&0&0\\ 0&\frac{1}{6}&0&0&0&0&0&0\\ 0&0&\frac{1}{6}&0&0&0&0&0\\ 0&0&0&\frac{1}{6}&0&0&0&0 \end{bmatrix}$$

$$5\begin{bmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&\frac{1}{6}&0&0&0\\0&0&0&0&0&0&0&-\frac{1}{6}\\0&0&0&0&0&0&\frac{1}{6}&0\\0&-\frac{1}{6}&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&-\frac{1}{6}&0&0&0&0\\0&0&\frac{1}{6}&0&0&0&0&0\end{bmatrix}\qquad 6\begin{bmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&\frac{1}{6}\\0&0&0&0&\frac{1}{6}&0&0&0\\0&0&0&0&0&-\frac{1}{6}&0&0\\0&0&-\frac{1}{6}&0&0&0&0&0\\0&0&0&\frac{1}{6}&0&0&0&0\\0&0&0&0&0&0&0&0\\0&-\frac{1}{6}&0&0&0&0&0&0\end{bmatrix}$$

$$7\begin{bmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&-\frac{1}{6}&0\\0&0&0&0&0&\frac{1}{6}&0&0\\0&0&0&0&\frac{1}{6}&0&0&0\\0&0&0&-\frac{1}{6}&0&0&0&0\\0&0&-\frac{1}{6}&0&0&0&0&0\\0&\frac{1}{6}&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{bmatrix}$$

Our basis for $\mathfrak{p}'$

$$8\begin{bmatrix}0&\frac{\sqrt{3}}{6}&0&0&0&0&0&0\\\frac{\sqrt{3}}{6}&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{bmatrix}\qquad 9\begin{bmatrix}0&0&\frac{\sqrt{3}}{6}&0&0&0&0&0\\0&0&0&0&0&0&0&0\\\frac{\sqrt{3}}{6}&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{bmatrix}$$

$$10\begin{bmatrix} 0 & 0 & 0 & \frac{\sqrt{3}}{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$11\begin{bmatrix} 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$12\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$13\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$14\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Finding the Equivariant maps ϕ

We now provide the code that builds our equivariant maps, ϕ, and our $ad_{\mathfrak{g}_2}$ matrices for a given basis element of $\mathfrak{g}_2$ acting on a basis element of $\mathfrak{p}$ as provided above. That is, if x_i is a basis element of $\mathfrak{p}$ from above and e_j is an element of $\mathfrak{g}_2$ from above, we here build the maps $ad_{e_j}(x_j) : \mathfrak{p} \to \mathfrak{p}$.

```
from so17orthonormalbasis import *

#This is the so(1,7) basis whose first 14 entries are the G2 above but
  placed in so(1,7) instead of just so(7)
#The next 14 entries make up p'' and p' (7 dim each)
so17Basis

#G2 in so(1,7)
G28

#p''
P2

#p'
P1



#First we build our adg2 matrices and we do so in one big loop
#We build our matrix by using the inner product. That is M_ij = <Mej,
 e_i> for a matrix M and an orthonormal basis with respect to <.,.>
#The result of this process is a list of matrices that are 14x14 with
 the top left 7x7 going to p'' and the bottom right going to p'
admat = []
```

```
k = 0
while k < 14: #This loop is selecting basis elements from G28 to enter ino the bracket
adg = zeros(14,14)
i = 14
while 13 < i < 21: #This loop is selecting basis elements from p'' for the bracket
BR = bracket(so17Basis[k], so17Basis[i]) #bracket is called from so17orthonormalbasis.py
j = 14
while 13 < j < 21:
if metric(BR, so17Basis[j]) == 0: #metric(.,.) is called from so17orthonormalbasis.py
pass
else:
adg[j-14, i-14] = metric(BR, so17Basis[j])
j = j+1
i = i + 1

i = 21 #This loop is selecting basis elements from p' for the bracket
while 20 < i < 28:
BR = bracket(so17Basis[k], so17Basis[i])
j = 21
while 20 < j < 28:
if S(6) * trace(BR * so17Basis[j]) == 0: # Here we can't call metric1 because BR will be a scalar multiple of a basis element. See comment in the file so17orthonormalbasis.py where metric1(.,.) is defined. This is at the end of the code.
pass
else:
adg[j - 14, i - 14] = S(6) * trace(BR * so17Basis[j])
```

```
j = j +1
i = i +1
admat = admat + [adg]
k = k +1





Psi = zeros(14, 14)
Psi[7,0] = c
Psi[8, 1] = c
Psi[9, 2] = c
Psi[10, 3] = c
Psi[11, 4] = c
Psi[12, 5] = c
Psi[13, 6] = c

#We make it symmetric since we ultimately want that
Psi = Psi + transpose(Psi)

#Here we get our multiple of the identity for the top left and top
 right
#We do this separately because sympy wants to treat my Psi above as
 immutatable on the diagonal for some reason
#So it is not letting me go through and redefine the diagonal entries,
  but it will let me add something to them
T = zeros(14,14)
i =0
while i < 7:
T[i,i] = a
```

```
T[i+7, i+7] = b
i = i +1

Phi = Psi + T


```

Now, we provide the code used to check that $ad_{e_i}\phi - \phi ad_{e_i} = 0$ for each e_i in the provided basis of $\mathfrak{g}_2$. We remind the reader that *Phi* in the code currently corresponds to ϕ from Lemma 3.31.

```
from so17orthonormalbasis import *
from IntertwiningMap import *

#This is the collection of 14x14 matrices for adg2 acting on p'' + p'
#There are 14 entries as is checked here
admat
print("The number of adg2 maps is ", len(admat))
print("Was it 14? If so, success! If not... oops")


#This is the intertwining map on p'' + p'
Phi


#This checks to make sure that the Phi intertwines.
#It fails if you get "uh oh" and it tells you which element it fails
  for
i = 0
for x in admat:
if Phi*x - x*Phi == zeros(14,14):
pass
```

```
else:
print("uh oh")
print(x)
print("if no uh oh, then success! Phi is an intertwining map!")

```

Here we provide the output when the code to check the intertwining maps is run.

```
The number of adg2 maps is  14
Was it 14? If so, success! If not... oops
if no uh oh, then success! Phi is an intertwining map!

```

Finding the $ad_{\mathfrak{p}}$ maps and $ric(.,.)$

In the following, we provide the code that demonstrates how we built our Ricci tensor values using the basis, inner product, and equivariant maps achieved prior. We note that we here also provide the way we construct our $ad_{\mathfrak{p}}(x_i) : \mathfrak{p} \longrightarrow \mathfrak{p}$ maps where x_i comes from the provided orthonormal basis for $\mathfrak{p}$.

```
from so17orthonormalbasis import *
from IntertwiningMap import *

#Now to create the ad maps for the p'' and p' parts
#Since ad will now have some g_2 in it, we will need to just take the inner product value for the matrix values
#Other than that, it is the same process for the adp'' as it was for the adg2
#For adp', by Cartan decomposition properties, it will be top right and bottom left blocks with values, so the metric usages must change
adpmat = []
k = 14
while 13 < k < 21: #Selecting an element from p'' to calculate ad for that element
adp = zeros(14, 14)
i = 14
while 13 < i < 21:
BR = bracket(so17Basis[k], so17Basis[i]) #if x in p'' then [x,y] has a component in p'' for y in p'' so we use the metric on p'' here
j = 14
while 13 < j < 21:
if trace(BR*so17Basis[j]) == 0:
pass
else:
```

```
adp[j-14, i-14] = metric(BR, so17Basis[j])
j = j+1
i = i + 1


i = 21
while 20 < i < 28:
BR = bracket(so17Basis[k], so17Basis[i]) #if x in p'' then [x,y] is in p' for y in p' so we use the metric on p' here
j = 21
while 20 < j < 28:
if trace(BR*so17Basis[j]) == 0:
pass
else:
adp[j - 14, i - 14] = S(6)*trace(BR*so17Basis[j])
j = j +1
i = i +1
adpmat = adpmat + [adp]
k = k +1

#For the second 7, by the Cartan properties, the ad maps will send p'' to p' and p' to p''.
k = 21
while 20 < k < 28:
adp = zeros(14, 14)
i = 14
while 13 < i < 21:
BR = bracket(so17Basis[k], so17Basis[i])
j = 21
while 20 < j < 28:
if trace(BR*so17Basis[j]) == 0:
```

```
pass
else:
adp[j - 14, i - 14] = S(6)*trace(BR*so17Basis[j])
j = j + 1
i = i +1

i = 21
while 20 < i < 28:
BR = bracket(so17Basis[k], so17Basis[i])
j = 14
while 13 < j < 21:
if trace(BR*so17Basis[j]) == 0:
pass
else:
adp[j-14, i-14] = metric(BR, so17Basis[j])
j = j+1
i = i + 1
adpmat = adpmat + [adp]
k = k + 1


Phi_inv = Phi.inv()




#The next three functions are described in the LaTex file "SO17_G2"

#x and y are vectors of length 14
#phi is any 14 by 14 matrix but we want it to be an intertwining map
#admaps is all the ad_p maps
```

```
def Term1(x,y, phi, admaps):
sum = 0
i = 0
phi_inv = phi.inv()
while i < 14:
ad = zeros(14,14)
k = 0
while k < 14:
ad = ad + phi_inv[k,i]*admaps[k]
k = k +1
first = phi*ad*x
second = phi*ad*y
sum = sum + first.dot(second)
i = i + 1
return(sum)

#x and y are vectors of lenght 14
#phi is any 14 by 14 matrix but we want it to be an intertwining map
#admaps is all the ad_p maps
#PBasis1 is the standard orthonormal basis in R^14

def Term3(x, y, phi, admaps, PBasis1):
sum = 0
j = 0
phi_inv = phi.inv()
while j < 14:
i = 0
while i < 14:
ad = zeros(14,14)
k = 0
while k < 14:
```

```
ad = ad + phi_inv[k, i]*admaps[k]
k = k +1
first = phi*ad*phi_inv*PBasis1[j] #The left part of the inner product in both pieces
second = phi*x #The right part of the inner product in the first inner product
third = phi*y #The right part of the inner product in the second inner product
sum = sum + (first.dot(second))*(first.dot(third))
i = i +1
j = j +1
return(sum)

#Killing here will take x and y the same as the other terms, but will first produce the matrices in so(1,7) corresponding to x and y
#Using the matrices, we utilize that B(.,.) = 6tr(..)
def Killing(x,y, PBasis2):
i = 0
mx = zeros(8,8)
my = zeros(8,8)
while i < 14:
mx = mx + x[i]*PBasis2[i]
my = my + y[i]*PBasis2[i]
i = i + 1
tr = trace(mx*my)
return(6*tr)

def ric(x,y, phi, admaps, PBasis1, PBasis2):
rxy = S(-1)/2*Term1(x,y, phi, admaps) + S(1)/4*Term3(x, y, phi, admaps, PBasis1) - S(1)/2*Killing(x,y, PBasis2)
return(rxy)
```

```

M = eye(14,14)
E_1 = M.col(0)
E_2 = M.col(1)
E_3 = M.col(2)
E_4 = M.col(3)
E_5 = M.col(4)
E_6 = M.col(5)
E_7 = M.col(6)
E_8 = M.col(7)
E_9 = M.col(8)
E_10 = M.col(9)
E_11 = M.col(10)
E_12 = M.col(11)
E_13 = M.col(12)
E_14 = M.col(13)

#This creates our standard basis with a 1 in the ith spot
PBasis1 = [E_1, E_2, E_3, E_4, E_5, E_6, E_7, E_8, E_9, E_10, E_11,
 E_12, E_13, E_14]

#This selects our orthonormal basis for p = p'' + p'
PBasis2 = so17Basis[14:28]

```

Code for Checking $ric(.,.)$

Here, we provide the code for the various checks we ran to make sure that our code is producing what we expected. Due to shape of our equivariant maps and the symmetric form of $ric(.,.)$, we expect a certain shape for our $ric(.,.)$ corresponding to what is discussed in Remark 3.30. The code makes sure that we get zeros where expected, equal terms where expected, and that our $ric(.,.)$ is scale invariant.

```
from so17orthonormalbasis import *
from IntertwiningMap import *
from so17Ricci import *

#ric(.,.) should have a shape that matches the Phi. I should get 0's everwhere except along "diagonal" terms in the "four blocks"
#In terms of a (0,2) tensor this means that ric(e_i e_i) is non zero and ric(e_i e_i+7) = ric(e_i+7 e_i) is nonzero but zero elsewhere
#We also check bilinearity along the "off diagonal blocks"
#We also check scale invariance



#I ran the following code to check to make sure that it gave zeros where it should
i = 0
while i < 14:
j = 0
while j < 14:
if i == j:
pass
else:
if ric(PBasis1[i],PBasis1[j], Phi_inv, adpmat, PBasis1, PBasis2) == 0:
pass
```

```
else:
print([i+1,j+1])
j = j+1
i = i +1
print("This checks out if you see [1, 8] , ..., [7, 14] and the same
  pairs swapped around since those correspond to the diagonal entries
  of the top right and bottom left blocks")

#Checking if the "top left block" is a multiple of the identity
i = 0
while i < 6:
x = ric(PBasis1[i], PBasis1[i], Phi_inv, adpmat, PBasis1, PBasis2)
y = ric(PBasis1[i+1], PBasis1[i+1], Phi_inv, adpmat, PBasis1, PBasis2)
if x == y:
pass
else:
print("uh oh")
print([i, i+1])
i = i +1
print("No uh oh? Success! The values representing the top left block
  entries are a multiple of the identity")
print("See an uh oh message? The pair [x,y] tells you which basis
  elements are the problem.")


#Checking if the "bottom right block" is a multiple of the identity
i = 7
while 6 < i < 13:
x = ric(PBasis1[i], PBasis1[i], Phi_inv, adpmat, PBasis1, PBasis2)
y = ric(PBasis1[i+1], PBasis1[i+1], Phi_inv, adpmat, PBasis1, PBasis2)
if x == y:
```

```
pass
else:
print("uh oh")
print([i, i+1])
i = i +1
print("No uh oh? Success! The values representing the bottom right
  block entries are a multiple of the identity")
print("See an uh oh message? The pair [x,y] tells you which basis
  elements are the problem.")

#Checking the "top right block" is a multiple of the idenity
i = 0
while i < 6:
x = ric(PBasis1[i], PBasis1[i+7], Phi_inv, adpmat, PBasis1, PBasis2)
y = ric(PBasis1[i+1], PBasis1[i+8], Phi_inv, adpmat, PBasis1, PBasis2)
if x == y:
pass
else:
print("uh oh")
print([i, i+7])
i = i +7
print("No uh oh? Success! The values representing the top right block
  entries are a multiple of the identity")
print("See an uh oh message? The pair [x,y] tells you which basis
  elements are the problem.")

#Check the "bottom left block" is a multiple of the identity
i = 0
while i < 6:
x = ric(PBasis1[i+7], PBasis1[i], Phi_inv, adpmat, PBasis1, PBasis2)
y = ric(PBasis1[i+8],PBasis1[i+1],  Phi_inv, adpmat, PBasis1, PBasis2)
```

```
if x == y:
pass
else:
print("uh oh")
print([i, i+7])
i = i +7
print("No uh oh? Success? The values representing the bottom left
 block entries are a multiple of the identity")
print("See an uh oh message? The pair [x,y] tells you which basis
 elements are the problem.")

#Checking symmetry of the "off diagonal blocks"
i = 0
while i < 6:
x = ric(PBasis1[i], PBasis1[i+7], Phi_inv, adpmat, PBasis1, PBasis2)
y = ric(PBasis1[i+7], PBasis1[i], Phi_inv, adpmat, PBasis1, PBasis2)
if x == y:
pass
else:
print("uh oh")
print([i, i+1])
i = i +1
print("No uh oh? Success! The top left block and bottom right block
 have the same diagonal entries")
print("See an uh oh message? The pair [x,y] tells you which basis
 elements are the problem.")


#Checking scale invariance
#Expect to get 3 zeros here
```

```
r1 = ric(PBasis1[0],PBasis1[0], Phi_inv, adpmat, PBasis1, PBasis2) -
  ric(PBasis1[0],PBasis1[0], d*Phi_inv, adpmat, PBasis1, PBasis2)
r2 = ric(PBasis1[8],PBasis1[8], Phi_inv, adpmat, PBasis1, PBasis2) -
  ric(PBasis1[8],PBasis1[8], d*Phi_inv, adpmat, PBasis1, PBasis2)
r3 = ric(PBasis1[0],PBasis1[7], Phi_inv, adpmat, PBasis1, PBasis2) -
  ric(PBasis1[0],PBasis1[7], d*Phi_inv, adpmat, PBasis1, PBasis2)

print("Do you see three zeros below? If so, success! ric is scale
  invariant. Otherwise, the nonzero term corresponds to a not scale
  invariant value")
r1 = simplify(r1)
print_latex(r1)
r2 = simplify(r2)
print_latex(r2)
r3 = simplify(r3)
print_latex(r3)

```

Below is what the above code prints when run.

```
[1, 8]
[2, 9]
[3, 10]
[4, 11]
[5, 12]
[6, 13]
[7, 14]
[8, 1]
[9, 2]
[10, 3]
[11, 4]
[12, 5]
```

```
[13, 6]
[14, 7]
This checks out if you see [1, 8] , ..., [7, 14] and the same pairs
  swapped around since those correspond to the diagonal entries of the
   top right and bottom left blocks
No uh oh? Success! The values representing the top left block entries
  are a multiple of the identity
See an uh oh message? The pair [x,y] tells you which basis elements
  are the problem.
No uh oh? Success! The values representing the bottom right block
  entries are a multiple of the identity
See an uh oh message? The pair [x,y] tells you which basis elements
  are the problem.
No uh oh? Success! The values representing the top right block entries
   are a multiple of the identity
See an uh oh message? The pair [x,y] tells you which basis elements
  are the problem.
No uh oh? Success? The values representing the bottom left block
  entries are a multiple of the identity
See an uh oh message? The pair [x,y] tells you which basis elements
  are the problem.
No uh oh? Success! The top left block and bottom right block have the
  same diagonal entries
See an uh oh message? The pair [x,y] tells you which basis elements
  are the problem.
Do you see three zeros below? If so, success! ric is scale invariant.
  Otherwise, the nonzero term corresponds to a not scale invariant
  value
0
0
0
```

The last code we provide from Python is the actual expression for r_1, r_2, and r_3 in terms of (a, b, c). We provide the code with its Mathematica output as this is what we use in the next section when we begin to work in Mathematica.

```
r1 = ric(PBasis1[0],PBasis1[0], Phi_inv, adpmat, PBasis1, PBasis2)
r2 = ric(PBasis1[8],PBasis1[8], Phi_inv, adpmat, PBasis1, PBasis2)
r3 = ric(PBasis1[0],PBasis1[7], Phi_inv, adpmat, PBasis1, PBasis2)

r1 = simplify(r1)
r1 = simplify(r1)
print(mathematica_code(r1))
r2 = simplify(r2)
r2 = simplify(r2)
print(mathematica_code(r2))
r3 = simplify(r3)
r3 = simplify(r3)
print(mathematica_code(r3))


(1/24)*(9*a^4*b^4 + 9*a^4*c^4 - 36*a^3*b^3*c^2 + 36*a^3*b*c^4 + 6*a^2*
  b^4*c^2 + 120*a^2*b^2*c^4 + 6*a^2*c^6 + 12*a*b^5*c^2 + 60*a*b^3*c^4
  - 24*a*b*c^6 + b^8 + 10*b^6*c^2 + 27*b^4*c^4 + 10*b^2*c^6 + 10*c^8)
  /(a^4*b^4 - 4*a^3*b^3*c^2 + 6*a^2*b^2*c^4 - 4*a*b*c^6 + c^8)

(1/24)*(c^2*(3*a^3 + a^2*b + 3*a*c^2 + b*c^2)^2 + c^2*(2*a^2*b + a*b^2
   + a*c^2 + b^3 + 3*b*c^2)^2 - 12*(a*b - c^2)^4 - 2*(a*b - c^2)^2*(2*
  c^2*(a + b)^2 + (a^2 + c^2)^2 + (b^2 + c^2)^2) + 2*(a^3*b + 2*a^2*c
  ^2 + 3*a*b*c^2 + b^2*c^2 + c^4)^2)/(a*b - c^2)^4

-c*(9*a^5*b^2 + 9*a^5*c^2 + 9*a^4*b^3 + 9*a^4*b*c^2 + 6*a^3*b^4 + 30*a
```

```
^3*b^2*c^2 + 24*a^3*c^4 + 6*a^2*b^5 + 30*a^2*b^3*c^2 + 24*a^2*b*c^4
+ a*b^6 + 9*a*b^4*c^2 + 24*a*b^2*c^4 + 16*a*c^6 + b^7 + 9*b^5*c^2 +
24*b^3*c^4 + 16*b*c^6)/(24*a^4*b^4 - 96*a^3*b^3*c^2 + 144*a^2*b^2*c
^4 - 96*a*b*c^6 + 24*c^8)


```

A.2. Mathematica Usage

First, we paste what we got from Python in the previous section into Mathematica so that we can begin to do computations using built in functions Mathematica provides. At the end of this piece of coding in Mathematica, we also provide where we checked the range of our r_1 (called $r31$ here). This is important for Step 6 (3.3) when we utilize the fact that $r_1 > 0$.

```
In[37]:= r31 =
Simplify[-c*(9*a^5*b^2 + 9*a^5*c^2 + 9*a^4*b^3 + 9*a^4*b*c^2 +
6*a^3*b^4 + 30*a^3*b^2*c^2 + 24*a^3*c^4 + 6*a^2*b^5 +
30*a^2*b^3*c^2 + 24*a^2*b*c^4 + a*b^6 + 9*a*b^4*c^2 +
24*a*b^2*c^4 + 16*a*c^6 + b^7 + 9*b^5*c^2 + 24*b^3*c^4 +
16*b*c^6)/(24*a^4*b^4 - 96*a^3*b^3*c^2 + 144*a^2*b^2*c^4 -
96*a*b*c^6 + 24*c^8)]

r21 = Simplify[(1/
24)*(c^2*(3*a^3 + a^2*b + 3*a*c^2 + b*c^2)^2 +
c^2*(2*a^2*b + a*b^2 + a*c^2 + b^3 + 3*b*c^2)^2 -
12*(a*b - c^2)^4 -
2*(a*b - c^2)^2*(2*
c^2*(a + b)^2 + (a^2 + c^2)^2 + (b^2 + c^2)^2) +
2*(a^3*b + 2*a^2*c^2 + 3*a*b*c^2 + b^2*c^2 + c^4)^2)/(a*b -
c^2)^4]
r11 = Simplify[(1/
24)*(9*a^4*b^4 + 9*a^4*c^4 - 36*a^3*b^3*c^2 + 36*a^3*b*c^4 +
6*a^2*b^4*c^2 + 120*a^2*b^2*c^4 + 6*a^2*c^6 + 12*a*b^5*c^2 +
60*a*b^3*c^4 - 24*a*b*c^6 + b^8 + 10*b^6*c^2 + 27*b^4*c^4 +
10*b^2*c^6 + 10*c^8)/(a^4*b^4 - 4*a^3*b^3*c^2 + 6*a^2*b^2*c^4 -
4*a*b*c^6 + c^8)]
```

```
Out[37]= -(((a + b) c (b^2 + c^2) (3 a^2 + b^2 + 4 c^2)^2)/(
24 (-a b + c^2)^4))

Out[38]= ((3 a + b)^2 c^2 (a^2 + c^2)^2 - 12 (-a b + c^2)^4 +
2 (a^3 b + 2 a^2 c^2 + 3 a b c^2 + b^2 c^2 + c^4)^2 +
c^2 (2 a^2 b + b^3 + 3 b c^2 + a (b^2 + c^2))^2 -
2 (-a b +
c^2)^2 (2 (a + b)^2 c^2 + (a^2 + c^2)^2 + (b^2 +
c^2)^2))/(24 (-a b + c^2)^4)

Out[39]= (b^8 + 10 b^6 c^2 + 27 b^4 c^4 + 10 b^2 c^6 + 10 c^8 -
36 a^3 b c^2 (b^2 - c^2) + 12 a b c^2 (b^4 + 5 b^2 c^2 - 2 c^4) +
9 a^4 (b^4 + c^4) +
6 a^2 c^2 (b^4 + 20 b^2 c^2 + c^4))/(24 (-a b + c^2)^4)

In[40]:= FunctionRange[r11, {a, b, c}, t1]

Out[40]= t1 > 3/8
```

Now, we wish to use the relations between the (a, b, c) defining ϕ and the (x, y, z) defining Φ in Lemma 3.31 to determine r_1, r_2, and r_3 in terms of (x, y, z). The relations are $x = a^2 + c^2$, $y = b^2 + c^2$, and $z = c(a + b)$. To accomplish this, we use the ***Eliminate*** function in Mathematica on the numerators of each r_i and observe that the denominator is $det\phi^4 = det\Phi^2$.

```
In[31]:= Eliminate[{
0 ==  (a + b) *c* (b^2 + c^2) *(3 *a^2 + b^2 + 4 *c^2)^2,
x == a^2 + c^2 , y == b^2 + c^2 , z == c*(a + b)}, {a, b, c}]

Out[31]= 9 x^2 y z + 6 x y^2 z == -y^3 z
```

```
In[3]:= num3 = Factor[9 x^2 y z + 6 x y^2 z + y^3 z]

Out[3]= y (3 x + y)^2 z

In[6]:= Eliminate[{0 == (3 a + b)^2 c^2 (a^2 + c^2)^2 -
12 (-a b + c^2)^4 +
2 (a^3 b + 2 a^2 c^2 + 3 a b c^2 + b^2 c^2 + c^4)^2 +
c^2 (2 a^2 b + b^3 + 3 b c^2 + a (b^2 + c^2))^2 -
2 (-a b +
c^2)^2 (2 (a + b)^2 c^2 + (a^2 + c^2)^2 + (b^2 + c^2)^2),
x == a^2 + c^2 , y == b^2 + c^2 , z == c*(a + b)}, {a, b, c}]

Out[6]= x y (2 y^2 - 24 z^2) + x^2 (12 y^2 - 9 z^2) ==
z^2 (3 y^2 - 6 z^2)

In[4]:= num2 =
Simplify[x y (2 y^2 - 24 z^2) + x^2 (12 y^2 - 9 z^2) -
z^2 (3 y^2 - 6 z^2)]

Out[4]= -3 y^2 z^2 + 6 z^4 + 3 x^2 (4 y^2 - 3 z^2) +
2 x (y^3 - 12 y z^2)

In[8]:= Eliminate[{0 ==
b^8 + 10 b^6 c^2 + 27 b^4 c^4 + 10 b^2 c^6 + 10 c^8 -
36 a^3 b c^2 (b^2 - c^2) + 12 a b c^2 (b^4 + 5 b^2 c^2 - 2 c^4) +
9 a^4 (b^4 + c^4) + 6 a^2 c^2 (b^4 + 20 b^2 c^2 + c^4),
x == a^2 + c^2 , y == b^2 + c^2 , z == c*(a + b)}, {a, b, c}]

Out[8]= -y^4 - 6 y^2 z^2 - 18 z^4 == 9 x^2 y^2 - 18 x y z^2
```

```
In[5]:= num1 =
Simplify[-y^4 - 6 y^2 z^2 - 18 z^4 - 9 x^2 y^2 + 18 x y z^2]

Out[5]= -9 x^2 y^2 - y^4 + 18 x y z^2 - 6 y^2 z^2 - 18 z^4

In[6]:= den = 24*(-x*y  + z^2)^2

Out[6]= 24 (-x y + z^2)^2

In[7]:= r1 = -num1/den
r2 = -num2/den
r3 = -num3/den

Out[7]= (9 x^2 y^2 + y^4 - 18 x y z^2 + 6 y^2 z^2 +
18 z^4)/(24 (-x y + z^2)^2)

Out[8]= (3 y^2 z^2 - 6 z^4 - 3 x^2 (4 y^2 - 3 z^2) -
2 x (y^3 - 12 y z^2))/(24 (-x y + z^2)^2)

Out[9]= -((y (3 x + y)^2 z)/(24 (-x y + z^2)^2))
```

We now check that the r_i we have in terms of (x, y, z) are correct by substituting back in for x, y, and z in terms of (a, b, c) and seeing if our result is equivalent to our original.

```
In[47]:= checkr1 =
Simplify[ReplaceAll[
ReplaceAll[ReplaceAll[r1, x -> a^2 + c^2], y -> b^2 + c^2],
z -> c*(a + b)]]
checkr2 =
Simplify[ReplaceAll[
ReplaceAll[ReplaceAll[r2, x -> a^2 + c^2], y -> b^2 + c^2],
```

```
z -> c*(a + b)]]
checkr3 =
Simplify[ReplaceAll[
ReplaceAll[ReplaceAll[r3, x -> a^2 + c^2], y -> b^2 + c^2],
z -> c*(a + b)]]

Out[47]= (
18 (a + b)^4 c^4 - 18 (a + b)^2 c^2 (a^2 + c^2) (b^2 + c^2) +
6 (a + b)^2 c^2 (b^2 + c^2)^2 +
9 (a^2 + c^2)^2 (b^2 + c^2)^2 + (b^2 + c^2)^4)/(24 (-a b + c^2)^4)

Out[48]= -((
6 (a + b)^4 c^4 - 3 (a + b)^2 c^2 (b^2 + c^2)^2 +
3 (a^2 + c^2)^2 (-3 (a + b)^2 c^2 + 4 (b^2 + c^2)^2) +
2 (a^2 + c^2) (-12 (a + b)^2 c^2 (b^2 + c^2) + (b^2 + c^2)^3))/(
24 (-a b + c^2)^4))

Out[49]= -(((a + b) c (b^2 + c^2) (3 a^2 + b^2 + 4 c^2)^2)/(
24 (-a b + c^2)^4))

In[50]:= Simplify[r11 - checkr1]
Simplify[r21 - checkr2]
Simplify[r31 - checkr3]

Out[50]= 0

Out[51]= 0

Out[52]= 0

```

$ric = T$ with $z > 0$

As is noted in Step 5 (3.3), we can get solutions to $ric = T$ by looking at when $z > 0$ and then changing the sign on r_3 in order to get the $z < 0$ case solutions. Here, we provide the solutions with $z > 0$. To do this, we use a combination of the Mathematica functions ***Resolve*** and ***Exists*** and in the code we have the following identification: $(t_1, t_2, t_3) = (k, l, m)$. This combination of ***Resolve*** and ***Exists*** seeks to find algebraic conditions in terms of the variables k, l, m based off the conditions provided in terms of x, y, z, k, l, m. That is, by using ***Resolve*** and ***Exists*** as we do below, we are able to find the conditions on k, l, and m such that $r_1(x, y, z) = k$, $r_2(x, y, z) = l$ and $r_3(x, y, z) = m$. When we do this, we specify that we want real conditions, and we specify necessary constraints such as our usage of scale invariance and $x, y > 0$.

Our approach is as follows. First, we find r_1 and r_2 where $det\Phi = 1$, but not r_3 as an explicit formulation could not be achieved (only an implicit equation), although it is clear that the denominator becomes the constant 24 under this condition. We then use ***Resolve*** and ***Exists*** in Mathematica, providing the polynomial condition that $det\Phi = xy - z^2 = 1$. The output provided by Mathematica was a long list of equations (which we provide below). However, we were able to determine that all these functions are the same with the exception of the constant $\frac{3}{4}$ that appears. We call said function *func1*.

The different expressions that Mathematica provides for k in terms of l and m are ***Root*** expressions which provide solutions in terms of roots of polynomials where the variable that for the polynomial is the #1 term which Mathematica refers to as *pure functions*. To give an example, $Root[-1+\#1^2\&, 1]$ is -1 and $Root[-1+\#1^2\&, 2]$ is 1 because the first one is saying to take the first root of $-1+x^2$ and the second one is saying to take the second root of $-1+x^2$.

Some roots of polynomials have a radical form while others do not. To get the radical form (if it exists), one uses ***ToRadicals***. We remark that in the code, ***&&*** means *and* and **||** means *or*.

```
In[63]:= det1r1 =
Simplify[ReplaceAll[ReplaceAll[1/24*Numerator[r1], z^2 -> x*y - 1],
z^4 -> (x*y - 1)^2]]

Out[63]= 1/24 (18 - 6 y^2 + 9 x^2 y^2 + y^4 + 6 x y (-3 + y^2))

In[64]:= det1r2 =
Simplify[ReplaceAll[ReplaceAll[1/24*Numerator[r2], z^2 -> x*y - 1],
z^4 -> (x*y - 1)^2]]

Out[64]= 1/24 (9 x^3 y + x y (-12 + y^2) - 3 (2 + y^2) +
x^2 (-9 + 6 y^2))

Resolve[Exists[{x, y, z}, det1r1 - k == 0 && det1r2 - l == 0 && (1/24)
  *Numerator[r3] - m == 0 && x*y - z^2 == 1 && x > 0 && y > 0 && m >
  0], Reals]

(l <= -(3/4) && m > 0 &&
k == Root[-135 - 432 l - 432 l^2 - 288 m^2 -
768 m^4 + (432 + 1152 l + 1152 l^2 + 1536 m^2 +
1536 l m^2) #1 + (-432 - 768 l - 768 l^2) #1^2 + 128 #1^3 &,
1]) || (-(3/4) <
l <= -(1/
2) && ((0 < m < 1/2 Sqrt[3/2] Sqrt[3 + 4 l] &&
k == Root[-135 - 432 l - 432 l^2 - 288 m^2 -
768 m^4 + (432 + 1152 l + 1152 l^2 + 1536 m^2 +
```

```
1536 l m^2) #1 + (-432 - 768 l - 768 l^2) #1^2 +
128 #1^3 &, 1]) || (m == 1/2 Sqrt[3/2] Sqrt[3 + 4 l] &&
k == 3/4) || (m > 1/2 Sqrt[3/2] Sqrt[3 + 4 l] &&
k == Root[-135 - 432 l - 432 l^2 - 288 m^2 -
768 m^4 + (432 + 1152 l + 1152 l^2 + 1536 m^2 +
1536 l m^2) #1 + (-432 - 768 l - 768 l^2) #1^2 +
128 #1^3 &, 1]))) || (-(1/2) <
l < -(1/4) && ((1/4 Sqrt[3] Sqrt[1 + 2 l] < m <
1/2 Sqrt[3/2] Sqrt[3 + 4 l] &&
k == Root[-135 - 432 l - 432 l^2 - 288 m^2 -
768 m^4 + (432 + 1152 l + 1152 l^2 + 1536 m^2 +
1536 l m^2) #1 + (-432 - 768 l - 768 l^2) #1^2 +
128 #1^3 &, 1]) || (m == 1/2 Sqrt[3/2] Sqrt[3 + 4 l] &&
k == 3/4) || (m > 1/2 Sqrt[3/2] Sqrt[3 + 4 l] &&
k == Root[-135 - 432 l - 432 l^2 - 288 m^2 -
768 m^4 + (432 + 1152 l + 1152 l^2 + 1536 m^2 +
1536 l m^2) #1 + (-432 - 768 l - 768 l^2) #1^2 +
128 #1^3 &, 1]))) || (l == -(1/
4) && ((Sqrt[3/2]/4 < m < Sqrt[3]/2 &&
k == Root[-27 - 144 m^2 - 384 m^4 + (108 + 576 m^2) #1 -
144 #1^2 + 64 #1^3 &, 1]) || (m == Sqrt[3]/2 &&
k == 3/4) || (m > Sqrt[3]/2 &&
k == Root[-27 - 144 m^2 - 384 m^4 + (108 + 576 m^2) #1 -
144 #1^2 + 64 #1^3 &, 1]))) || (-(1/4) < l <
0 && ((1/4 Sqrt[3] Sqrt[1 + 2 l] < m <
1/2 Sqrt[3/2] Sqrt[3 + 4 l] &&
k == Root[-135 - 432 l - 432 l^2 - 288 m^2 -
768 m^4 + (432 + 1152 l + 1152 l^2 + 1536 m^2 +
1536 l m^2) #1 + (-432 - 768 l - 768 l^2) #1^2 +
128 #1^3 &, 1]) || (m == 1/2 Sqrt[3/2] Sqrt[3 + 4 l] &&
k == 3/4) || (m > 1/2 Sqrt[3/2] Sqrt[3 + 4 l] &&
```

```
k == Root[-135 - 432 l - 432 l^2 - 288 m^2 -
768 m^4 + (432 + 1152 l + 1152 l^2 + 1536 m^2 +
1536 l m^2) #1 + (-432 - 768 l - 768 l^2) #1^2 +
128 #1^3 &, 1]))) || (l ==
0 && ((Sqrt[3]/4 < m < 3/(2 Sqrt[2]) &&
k == Root[-135 - 288 m^2 - 768 m^4 + (432 + 1536 m^2) #1 -
432 #1^2 + 128 #1^3 &, 1]) || (m == 3/(2 Sqrt[2]) &&
k == 3/4) || (m > 3/(2 Sqrt[2]) &&
k == Root[-135 - 288 m^2 - 768 m^4 + (432 + 1536 m^2) #1 -
432 #1^2 + 128 #1^3 &, 1]))) || (l >
0 && ((1/4 Sqrt[3] Sqrt[1 + 2 l] < m <
1/2 Sqrt[3/2] Sqrt[3 + 4 l] &&
k == Root[-135 - 432 l - 432 l^2 - 288 m^2 -
768 m^4 + (432 + 1152 l + 1152 l^2 + 1536 m^2 +
1536 l m^2) #1 + (-432 - 768 l - 768 l^2) #1^2 +
128 #1^3 &, 1]) || (m == 1/2 Sqrt[3/2] Sqrt[3 + 4 l] &&
k == 3/4) || (m > 1/2 Sqrt[3/2] Sqrt[3 + 4 l] &&
k == Root[-135 - 432 l - 432 l^2 - 288 m^2 -
768 m^4 + (432 + 1152 l + 1152 l^2 + 1536 m^2 +
1536 l m^2) #1 + (-432 - 768 l - 768 l^2) #1^2 +
128 #1^3 &, 1])))

fl1 = Root[-135 - 432 l - 432 l^2 - 288 m^2 -
768 m^4 + (432 + 1152 l + 1152 l^2 + 1536 m^2 +
1536 l m^2) #1 + (-432 - 768 l - 768 l^2) #1^2 + 128 #1^3 &, 1]

In[23]:= func1 = FullSimplify[ToRadicals[fl1]]

Out[23]= 1/8 (9 + 16 l +
16 l^2 + ((1 + 4 l)^2 (3 + 4 l)^2)/((3 + 16 l + 16 l^2)^3 -
192 (3 + 4 l) (5 + 2 l (5 + 4 l)) m^2 + 1536 m^4 +
```

```
8 Sqrt[2]
Sqrt[-((1 + 4 l)^3 - 72 m^2) (3 (3 + 4 l)^2 m + 16 m^3)^2])^(
1/3) - (256 (1 + l) m^2)/((3 + 16 l + 16 l^2)^3 -
192 (3 + 4 l) (5 + 2 l (5 + 4 l)) m^2 + 1536 m^4 +
8 Sqrt[2]
Sqrt[-((1 + 4 l)^3 - 72 m^2) (3 (3 + 4 l)^2 m + 16 m^3)^2])^(
1/3) + ((3 + 16 l + 16 l^2)^3 -
192 (3 + 4 l) (5 + 2 l (5 + 4 l)) m^2 + 1536 m^4 +
8 Sqrt[2]
Sqrt[-((1 + 4 l)^3 - 72 m^2) (3 (3 + 4 l)^2 m + 16 m^3)^2])^(
1/3))

fl1 = Root[-135 - 432 l - 432 l^2 - 288 m^2 -
768 m^4 + (432 + 1152 l + 1152 l^2 + 1536 m^2 +
1536 l m^2) #1 + (-432 - 768 l - 768 l^2) #1^2 + 128 #1^3 &, 1]

In[23]:= func1 = FullSimplify[ToRadicals[fl1]]

Out[23]= 1/8 (9 + 16 l +
16 l^2 + ((1 + 4 l)^2 (3 + 4 l)^2)/((3 + 16 l + 16 l^2)^3 -
192 (3 + 4 l) (5 + 2 l (5 + 4 l)) m^2 + 1536 m^4 +
8 Sqrt[2]
Sqrt[-((1 + 4 l)^3 - 72 m^2) (3 (3 + 4 l)^2 m + 16 m^3)^2])^(
1/3) - (256 (1 + l) m^2)/((3 + 16 l + 16 l^2)^3 -
192 (3 + 4 l) (5 + 2 l (5 + 4 l)) m^2 + 1536 m^4 +
8 Sqrt[2]
Sqrt[-((1 + 4 l)^3 - 72 m^2) (3 (3 + 4 l)^2 m + 16 m^3)^2])^(
1/3) + ((3 + 16 l + 16 l^2)^3 -
192 (3 + 4 l) (5 + 2 l (5 + 4 l)) m^2 + 1536 m^4 +
8 Sqrt[2]
Sqrt[-((1 + 4 l)^3 - 72 m^2) (3 (3 + 4 l)^2 m + 16 m^3)^2])^(
```

```
1/3))

```

As mentioned, the above values for k are determined by l and m, and over each region described in terms of l and m, it turns out that the provided ***Root*** functions describing k all come from the same function in terms of l and m. To check this, we utilized the ***Simplify*** function in mathematica, subtracting different Root functions describing k from the first output for k provided which we call $fl1$. Whenever the subtraction was not 0, we were able to determine that m or l was constant in those cases where as $fl1$ was in terms of both l and m. What we then saw was that if we plugged in the specific l or m values into $fl1$ and used ***Simplify*** with subtraction again, we got 0 (except for when the function was the constant $\frac{3}{4}$). Thus, we were able to see that the output for k was actually described by the same function in almost every region described by the l and m parameters. This led to the simplification provided in Step 5 (3.3). Below we provide an easier to see version of $fl1$ as it is presented in Mathematica and then the function this root function is describing.

fl1:

$$\mathrm{Root}\left[\begin{array}{r}128\#1^3+\#1^2\left(-768l^2-768l-432\right)+\#1\left(1152l^2+1536lm^2+1152l+1536m^2+432\right)\\ -432l^2-432l-768m^4-288m^2-135\end{array}\&,1\right]$$

The root of this polynomial is given by:

$$\begin{aligned}&\frac{1}{8}\left(16l^2+16l+9\right)\\&+\frac{1}{8}\left(\sqrt[3]{\left(16l^2+16l+3\right)^3-192(4l+3)(2l(4l+5)+5)m^2+8\sqrt{2}\sqrt{-\left((4l+1)^3-72m^2\right)\left(3(4l+3)^2m+16m^3\right)^2}+1536m^4}\right)\\&+\frac{1}{8}\left(\frac{(4l+1)^2(4l+3)^2}{\sqrt[3]{\left(16l^2+16l+3\right)^3-192(4l+3)(2l(4l+5)+5)m^2+8\sqrt{2}\sqrt{-\left((4l+1)^3-72m^2\right)\left(3(4l+3)^2m+16m^3\right)^2}+1536m^4}}\right)\\&-\frac{1}{8}\left(\frac{256(l+1)m^2}{\sqrt[3]{\left(16l^2+16l+3\right)^3-192(4l+3)(2l(4l+5)+5)m^2+8\sqrt{2}\sqrt{-\left((4l+1)^3-72m^2\right)\left(3(4l+3)^2m+16m^3\right)^2}+1536m^4}}\right)\end{aligned}\tag{A.1}$$

ric= cT with $z > 0$

Just as before, we are able to restrict ourselves to $r_3, z > 0$ to get solutions to the $r_3, z \neq 0$ setting. The solution is found in a similar manner except we are now looking to use ***Resolve*** and ***Exists*** on $\frac{r_2}{r_1}$ and $\frac{r_3}{r_1}$. We make this choice because $(r_1, r_2, r_3) = c(t_1, t_2, t_3)$ if and only if $(1, \frac{r_2}{r_1}, \frac{r_3}{r_1}) = (1, \frac{t_2}{t_1}, \frac{t_3}{t_1})$ since $r_1 > 0$. In the following, $l = \frac{t_2}{t_1}$ and $m = \frac{t_3}{t_1}$, and we are seeking to describe the region $R = \{(1, l, m) : (1, l, m) = (1, \frac{r_2}{r_1}, \frac{r_3}{r_1})\}$. Moreover, one can see that the description of R as an output in Mathematica is quite long. The output can be thought of as 15 regions being described in a piece-wise fashion, and after we provide the initial output in Mathematica, we break the output up into the 15 sections. We break the sections up based on the m values that Mathematica is using to distinguish between varying solutions.

After we provide the 15 regions in different sections, we provide the several different functions involved which are again described as roots of polynomials as before. When we list these, we use a naming system for the functions based upon the first few numbers that appear in the polynomial as well as an indication of what root is being taken. The number indicating the root is the number that occurs after the m, so that if you were to see $f123m1$ and $f123m2$ you would know that these are using the same polynomial, but the first one is using the first root and the second one is using the second root. After providing the regions and the roots of the polynomials, we also indicate which of these roots of polynomials can be determined using radicals.

We then show that the regions that occur in which m is a constant number can be included inside other regions. As in the $ric = T$ case, the ***Resolve*** and *Exist* combination is not giving us simplified results, this time in the sense that some parts of the 15 regions overlap with one another. This was checked by hand using the notation f_{number} to refer to *fnumber* in the Mathematica code. We provide some of the steps in the simplification after the code to see how we got from the 15 regions provided in Mathematica to the considerably simpler solution provided.

```
In[10]:= Resolve[
Exists[{x, y, z},
r2/r1 - l == 0 && r3/r1 - m == 0 && x*y - z^2 > 0 && x > 0 &&
y > 0 && m > 0 ], Reals]

Out[10]= (m > 0 && 1/2 (-2 + m^2) <= l < m^2) || (0 < m < 1/Sqrt[3] &&
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &, 1] <=
l < 1/3 (-4 + 3 m^2)) || (0 < m <= Sqrt[2/3] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <=
1/2 (-2 + m^2)) || (m > Sqrt[2/3] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
1/3 (-4 + 3 m^2)) || (0 < m <= 1/Sqrt[3] &&
l == Root[
4 - 507 m^2 + (51 - 1404 m^2) #1 + (252 -
1674 m^2) #1^2 + (594 - 972 m^2) #1^3 + (648 -
243 m^2) #1^4 + 243 #1^5 &, 1]) || (1/Sqrt[3] < m <
Root[-324 + 692 #^2 + 336 #^4 + 45 #^6& , 2,
0] && (l ==
Root[4 -
```

```
507 m^2 + (51 - 1404 m^2) #1 + (252 -
1674 m^2) #1^2 + (594 - 972 m^2) #1^3 + (648 -
243 m^2) #1^4 + 243 #1^5 &, 1] ||
l == Root[
4 - 507 m^2 + (51 - 1404 m^2) #1 + (252 -
1674 m^2) #1^2 + (594 - 972 m^2) #1^3 + (648 -
243 m^2) #1^4 + 243 #1^5 &, 2])) || (m ==
Root[-324 + 692 #^2 + 336 #^4 + 45 #^6& , 2, 0] &&
l == Root[
4 - 507 m^2 + (51 - 1404 m^2) #1 + (252 -
1674 m^2) #1^2 + (594 - 972 m^2) #1^3 + (648 -
243 m^2) #1^4 + 243 #1^5 &, 1]) || (0 < m < 1/Sqrt[3] &&
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &, 1] <=
l < 1/3 (-4 + 3 m^2)) || (1/Sqrt[3] < m <
Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& , 2, 0] &&
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &, 1] <=
l < m^2) || (m ==
Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& , 2, 0] &&
1/3 (-4 + 3 m^2) <= l <
m^2) || (Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& , 2, 0] <
m < Root[
32768 - 2020986 #^2 - 10826991 #^4 - 2570157 #^6 - 220725 #\
^8 + 151875 #^10& , 4, 0] &&
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &, 1] <=
l < m^2) || (m == Root[
```

```
32768 - 2020986 #^2 - 10826991 #^4 - 2570157 #^6 - 220725 #\
^8 + 151875 #^10& , 4, 0] &&
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1] <= l <
m^2) || (m > Root[
32768 - 2020986 #^2 - 10826991 #^4 - 2570157 #^6 - 220725 #\
^8 + 151875 #^10& , 4, 0] &&
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &, 1] <=
l < m^2) || (1/Sqrt[3] < m <= Root[
1 - 36 #^2 + 36 #^4& , 4,
0] && (Root[
1 - 168 m^2 +
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 +
108 #1^3 + 81 #1^4 &, 1] < l < 1/3 (-4 + 3 m^2) ||
Root[1 - 168 m^2 +
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 +
108 #1^3 + 81 #1^4 &, 2] < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1])) || (Root[1 - 36 #^2 + 36 #^4& , 4, 0] < m <
Root[-27 + 19 #^2 - 9 #^4 + 9 #^6& , 2,
0] && (Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l < 1/3 (-4 + 3 m^2) ||
Root[1 - 168 m^2 +
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 +
108 #1^3 + 81 #1^4 &, 2] < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
```

```
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1])) || (m ==
Root[-27 + 19 #^2 - 9 #^4 + 9 #^6& , 2,
0] && (Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l < 1/3 (-4 + 3 m^2) ||
1/3 (-4 + 3 m^2) < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1])) || (Root[-27 + 19 #^2 - 9 #^4 + 9 #^6& , 2, 0] < m <
Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1]) || (Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2,
0] <= m < Root[1 - 27 #^2 - 657 #^4 + 135 #^6& , 4, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
1/3 (-4 + 3 m^2)) || (m >= Root[
1 - 27 #^2 - 657 #^4 + 135 #^6& , 4, 0] &&
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1] < l <
1/3 (-4 + 3 m^2)) || (1/Sqrt[3] < m <= Root[
1 - 36 #^2 + 36 #^4& , 4,
0] && (Root[
```

```
1 - 168 m^2 +
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 +
108 #1^3 + 81 #1^4 &, 1] < l < 1/3 (-4 + 3 m^2) ||
Root[1 - 168 m^2 +
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 +
108 #1^3 + 81 #1^4 &, 2] < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1])) || (Root[1 - 36 #^2 + 36 #^4& , 4, 0] < m <
Root[-27 + 19 #^2 - 9 #^4 + 9 #^6& , 2,
0] && (Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l < 1/3 (-4 + 3 m^2) ||
Root[1 - 168 m^2 +
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 +
108 #1^3 + 81 #1^4 &, 2] < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1])) || (m ==
Root[-27 + 19 #^2 - 9 #^4 + 9 #^6& , 2,
0] && (Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l < 1/3 (-4 + 3 m^2) ||
1/3 (-4 + 3 m^2) < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
```

```
1])) || (Root[-27 + 19 #^2 - 9 #^4 + 9 #^6& , 2, 0] < m <
Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& , 2, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || (m ==
Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& , 2, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <=
1/3 (-4 + 3 m^2)) || (m >
Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& , 2, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
1/3 (-4 + 3 m^2)) || (1/Sqrt[3] < m <
Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2, 0] &&
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1] < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || (m ==
Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2, 0] &&
1/3 (-4 + 3 m^2) < l <=
```

```
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || (Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2, 0] <
m < Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& , 2, 0] &&
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1] < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || (m ==
Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& , 2, 0] &&
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1] < l <=
1/3 (-4 +
3 m^2)) || (Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& ,
2, 0] < m <= Root[1 - 27 #^2 - 657 #^4 + 135 #^6& , 4, 0] &&
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1] < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || (Root[1 - 27 #^2 - 657 #^4 + 135 #^6& , 4, 0] < m < Root[
32768 - 2020986 #^2 - 10826991 #^4 - 2570157 #^6 - 220725 #\
^8 + 151875 #^10& , 4, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
```

```
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || (m == Root[
32768 - 2020986 #^2 - 10826991 #^4 - 2570157 #^6 - 220725 #\
^8 + 151875 #^10& , 4, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <=
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1]) || (m >
Root[32768 - 2020986 #^2 - 10826991 #^4 - 2570157 #^6 - 220725 #\
^8 + 151875 #^10& , 4, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1]) || (Root[-1 + 12 #^2 + 9 #^4& , 2, 0] < m <= 1/Sqrt[3] &&
Root[16 - 6360 m^2 +
47961 m^4 + (216 + 8478 m^2 + 149796 m^4) #1 + (1161 +
54432 m^2 + 176094 m^4) #1^2 + (3132 + 64476 m^2 +
92340 m^4) #1^3 + (4374 + 29160 m^2 +
18225 m^4) #1^4 + (2916 + 4374 m^2) #1^5 + 729 #1^6 &, 2] <
l < m^2) || (1/Sqrt[3] < m < Sqrt[2/
3] && (Root[
16 - 6360 m^2 +
47961 m^4 + (216 + 8478 m^2 + 149796 m^4) #1 + (1161 +
54432 m^2 + 176094 m^4) #1^2 + (3132 + 64476 m^2 +
92340 m^4) #1^3 + (4374 + 29160 m^2 +
18225 m^4) #1^4 + (2916 + 4374 m^2) #1^5 + 729 #1^6 &,
```

```
1] < l < 1/3 (-4 + 3 m^2) ||
Root[16 - 6360 m^2 +
47961 m^4 + (216 + 8478 m^2 + 149796 m^4) #1 + (1161 +
54432 m^2 + 176094 m^4) #1^2 + (3132 + 64476 m^2 +
92340 m^4) #1^3 + (4374 + 29160 m^2 +
18225 m^4) #1^4 + (2916 + 4374 m^2) #1^5 + 729 #1^6 &,
2] < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1])) || (m == Sqrt[2/
3] && (Root[
16 - 6360 m^2 +
47961 m^4 + (216 + 8478 m^2 + 149796 m^4) #1 + (1161 +
54432 m^2 + 176094 m^4) #1^2 + (3132 + 64476 m^2 +
92340 m^4) #1^3 + (4374 + 29160 m^2 +
18225 m^4) #1^4 + (2916 + 4374 m^2) #1^5 + 729 #1^6 &,
1] < l < 1/3 (-4 + 3 m^2) ||
1/3 (-4 + 3 m^2) < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1])) || (Sqrt[2/3] < m < Root[
18 - 965 #^2 - 4029 #^4 + 4293 #^6 + 3375 #^8& , 4, 0] &&
Root[16 - 6360 m^2 +
47961 m^4 + (216 + 8478 m^2 + 149796 m^4) #1 + (1161 +
54432 m^2 + 176094 m^4) #1^2 + (3132 + 64476 m^2 +
92340 m^4) #1^3 + (4374 + 29160 m^2 +
18225 m^4) #1^4 + (2916 + 4374 m^2) #1^5 + 729 #1^6 &, 1] <
l < Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1]) || (m == Root[
18 - 965 #^2 - 4029 #^4 + 4293 #^6 + 3375 #^8& , 4, 0] &&
```

```
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1]) || (Root[
18 - 965 #^2 - 4029 #^4 + 4293 #^6 + 3375 #^8& , 4, 0] < m <
Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1]) || (Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2,
0] <= m < Root[1 - 27 #^2 - 657 #^4 + 135 #^6& , 4, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
1/3 (-4 + 3 m^2)) || (m >= Root[
1 - 27 #^2 - 657 #^4 + 135 #^6& , 4, 0] &&
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1] < l <
1/3 (-4 + 3 m^2)) || (0 < m <= Root[-1 + 6 #^2 + 9 #^4& , 2, 0] &&
Root[4 -
507 m^2 + (51 - 1404 m^2) #1 + (252 - 1674 m^2) #1^2 + (594 -
972 m^2) #1^3 + (648 - 243 m^2) #1^4 + 243 #1^5 &, 1] < l <
m^2) || (Root[-1 + 6 #^2 + 9 #^4& , 2, 0] < m < Root[
32768 - 1855650 #^2 + 10014057 #^4 - 3642597 #^6 - 204525 #\
```

```
^8 + 151875 #^10& , 4, 0] &&
Root[4 -
507 m^2 + (51 - 1404 m^2) #1 + (252 - 1674 m^2) #1^2 + (594 -
972 m^2) #1^3 + (648 - 243 m^2) #1^4 + 243 #1^5 &, 1] < l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2]) || (m ==
Root[32768 - 1855650 #^2 + 10014057 #^4 - 3642597 #^6 - 204525 #\
^8 + 151875 #^10& , 4, 0] &&
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &, 1] < l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2]) || (Root[
32768 - 1855650 #^2 + 10014057 #^4 - 3642597 #^6 - 204525 #\
^8 + 151875 #^10& , 4, 0] < m <= Root[
1 - 147 #^2 + 423 #^4 + 135 #^6& , 4, 0] &&
Root[4 -
507 m^2 + (51 - 1404 m^2) #1 + (252 - 1674 m^2) #1^2 + (594 -
972 m^2) #1^3 + (648 - 243 m^2) #1^4 + 243 #1^5 &, 1] < l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
```

```
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2]) || (Root[
1 - 147 #^2 + 423 #^4 + 135 #^6& , 4, 0] < m <
Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #\
^10 + 81 #^12& , 2, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2]) || (m ==
Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #\
^10 + 81 #^12& , 2, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
1/3 (-4 +
3 m^2)) || \
(Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #^10 + 81 #\
^12& , 2, 0] < m < Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
Root[-150 m^2 +
```

```
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2]) || (m ==
Root[686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4,
0] && Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] < l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2]) || (Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4, 0] <
m < Root[-2812500 - 11259427868 #^2 - 6144055152 #^4 + 1352551761 #\
^6 + 692176752 #^8 + 340122240 #^10& , 2, 0] &&
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] < l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &,
```

```
2]) || (Root[-1 + 6 #^2 + 9 #^4& , 2, 0] < m <= 1/Sqrt[3] &&
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2] < l <
m^2) || (1/Sqrt[3] < m <= Root[-2 - 9 #^2 + 9 #^4& , 2, 0] &&
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2] < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1]) || (Root[-2 - 9 #^2 + 9 #^4& , 2, 0] < m <
Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #\
^10 + 81 #^12& , 2,
0] && (Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] < l <
1/3 (-4 + 3 m^2) ||
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
```

```
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2] < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1])) || (m ==
Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #\
^10 + 81 #^12& , 2,
0] && (Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] < l <
1/3 (-4 + 3 m^2) ||
1/3 (-4 + 3 m^2) < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1])) || (Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #\
^8 - 837 #^10 + 81 #^12& , 2, 0] < m <
Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2, 0] &&
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1]) || (Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2,
0] <= m < Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4, 0] &&
Root[-150 m^2 +
```

```
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] < l <
1/3 (-4 + 3 m^2)) || (m == Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] < l <
1/3 (-4 + 3 m^2)) || (Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4, 0] <
m < Root[1 - 27 #^2 - 657 #^4 + 135 #^6& , 4, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
1/3 (-4 + 3 m^2)) || (m >= Root[
1 - 27 #^2 - 657 #^4 + 135 #^6& , 4, 0] &&
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1] < l <
1/3 (-4 + 3 m^2)) || (1/Sqrt[3] < m <= (3 Sqrt[3])/5 &&
Root[1 - 168 m^2 +
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 + 108 #1^3 +
81 #1^4 &, 2] < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1]) || ((
3 Sqrt[3])/5 < m <= Root[-2 - 9 #^2 + 9 #^4& , 2, 0] &&
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
```

```
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2] < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1]) || (Root[-2 - 9 #^2 + 9 #^4& , 2, 0] < m < Root[
17179869184 - 758241767424 #^2 + 738811183104 #^4 - 1873773558144 #\
^6 + 2178248345280 #^8 + 730684767357 #^10 - 1168450231581 #\
^12 - 312846367473 #^14 + 307409258025 #^16& , 4,
0] && (Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] < l <
1/3 (-4 + 3 m^2) ||
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2] < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1])) || (m == Root[
17179869184 - 758241767424 #^2 + 738811183104 #^4 - 1873773558144 #\
^6 + 2178248345280 #^8 + 730684767357 #^10 - 1168450231581 #\
^12 - 312846367473 #^14 + 307409258025 #^16& , 4,
0] && (Root[
1 - 168 m^2 +
```

```
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 +
108 #1^3 + 81 #1^4 &, 2] < l < 1/3 (-4 + 3 m^2) ||
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2] < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1])) || (Root[
17179869184 - 758241767424 #^2 + 738811183104 #^4 - 1873773558144 #\
^6 + 2178248345280 #^8 + 730684767357 #^10 - 1168450231581 #\
^12 - 312846367473 #^14 + 307409258025 #^16& , 4, 0] < m <
Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #\
^10 + 81 #^12& , 2,
0] && (Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] < l <
1/3 (-4 + 3 m^2) ||
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2] < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
```

```
1])) || (m ==
Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #\
^10 + 81 #^12& , 2,
0] && (Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] < l <
1/3 (-4 + 3 m^2) ||
1/3 (-4 + 3 m^2) < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1])) || (Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #\
^8 - 837 #^10 + 81 #^12& , 2, 0] < m <
Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2, 0] &&
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1]) || (Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2,
0] <= m < Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4, 0] &&
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
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19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] < l <
1/3 (-4 + 3 m^2)) || (m == Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] < l <
1/3 (-4 + 3 m^2)) || (Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4, 0] <
m < Root[1 - 27 #^2 - 657 #^4 + 135 #^6& , 4, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
1/3 (-4 + 3 m^2)) || (m >= Root[
1 - 27 #^2 - 657 #^4 + 135 #^6& , 4, 0] &&
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1] < l <
1/3 (-4 + 3 m^2)) || (1/Sqrt[3] < m <= (3 Sqrt[3])/5 &&
Root[1 - 168 m^2 +
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 + 108 #1^3 +
81 #1^4 &, 2] < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || ((3 Sqrt[3])/5 < m <= Root[-2 - 9 #^2 + 9 #^4& , 2, 0] &&
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
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19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2] < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || (Root[-2 - 9 #^2 + 9 #^4& , 2, 0] < m < Root[
17179869184 - 758241767424 #^2 + 738811183104 #^4 - 1873773558144 #\
^6 + 2178248345280 #^8 + 730684767357 #^10 - 1168450231581 #\
^12 - 312846367473 #^14 + 307409258025 #^16& , 4,
0] && (Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] < l <
1/3 (-4 + 3 m^2) ||
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2] < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1])) || (m == Root[
17179869184 - 758241767424 #^2 + 738811183104 #^4 - 1873773558144 #\
^6 + 2178248345280 #^8 + 730684767357 #^10 - 1168450231581 #\
^12 - 312846367473 #^14 + 307409258025 #^16& , 4,
0] && (Root[
1 - 168 m^2 +
```

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144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 +
108 #1^3 + 81 #1^4 &, 2] < l < 1/3 (-4 + 3 m^2) ||
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2] < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1])) || (Root[
17179869184 - 758241767424 #^2 + 738811183104 #^4 - 1873773558144 #\
^6 + 2178248345280 #^8 + 730684767357 #^10 - 1168450231581 #\
^12 - 312846367473 #^14 + 307409258025 #^16& , 4, 0] < m <
Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #\
^10 + 81 #^12& , 2,
0] && (Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] < l <
1/3 (-4 + 3 m^2) ||
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2] < l <=
Root[-2 -
```

```
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1])) || (m ==
Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #\
^10 + 81 #^12& , 2,
0] && (Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] < l <
1/3 (-4 + 3 m^2) ||
1/3 (-4 + 3 m^2) < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1])) || (Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #\
^8 - 837 #^10 + 81 #^12& , 2, 0] < m < Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4, 0] &&
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || (m == Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
```

```
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || (Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4, 0] <
m < Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& , 2, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || (m ==
Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& , 2, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <=
1/3 (-4 + 3 m^2)) || (m >
Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& , 2, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
1/3 (-4 + 3 m^2)) || (1/Sqrt[3] < m <= Root[
1 - 36 #^2 + 36 #^4& , 4, 0] &&
Root[1 - 168 m^2 +
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```
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 + 108 #1^3 +
81 #1^4 &, 1] < l < 1/3 (-4 + 3 m^2)) || (Root[
1 - 36 #^2 + 36 #^4& , 4, 0] < m <= (3 Sqrt[3])/5 &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
1/3 (-4 + 3 m^2)) || ((3 Sqrt[3])/5 < m <=
Root[-2 - 9 #^2 + 9 #^4& , 2,
0] && (Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l < 1/3 (-4 + 3 m^2) ||
Root[1 - 168 m^2 +
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 +
108 #1^3 + 81 #1^4 &, 2] < l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &,
2])) || (Root[-2 - 9 #^2 + 9 #^4& , 2, 0] < m <
Root[-27 + 19 #^2 - 9 #^4 + 9 #^6& , 2,
0] && (Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
```

```
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] ||
Root[1 - 168 m^2 +
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 +
108 #1^3 + 81 #1^4 &, 2] < l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2])) || (m ==
Root[-27 + 19 #^2 - 9 #^4 + 9 #^6& , 2,
0] && (Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] ||
1/3 (-4 + 3 m^2) < l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &,
2])) || (Root[-27 + 19 #^2 - 9 #^4 + 9 #^6& , 2, 0] < m < Root[
```

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17179869184 - 758241767424 #^2 + 738811183104 #^4 - 1873773558144 #\
^6 + 2178248345280 #^8 + 730684767357 #^10 - 1168450231581 #\
^12 - 312846367473 #^14 + 307409258025 #^16& , 4,
0] && (Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] ||
Root[1 - 168 m^2 +
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 +
108 #1^3 + 81 #1^4 &, 2] < l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2])) || (m ==
Root[17179869184 - 758241767424 #^2 + 738811183104 #\
^4 - 1873773558144 #^6 + 2178248345280 #^8 + 730684767357 #\
^10 - 1168450231581 #^12 - 312846367473 #^14 + 307409258025 #^16& , 4,
0] && (Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
Root[
1 - 168 m^2 +
```

```
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 +
108 #1^3 + 81 #1^4 &, 2] ||
Root[1 - 168 m^2 +
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 +
108 #1^3 + 81 #1^4 &, 2] < l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2])) || (Root[
17179869184 - 758241767424 #^2 + 738811183104 #^4 - 1873773558144 #\
^6 + 2178248345280 #^8 + 730684767357 #^10 - 1168450231581 #\
^12 - 312846367473 #^14 + 307409258025 #^16& , 4, 0] < m <
Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #\
^10 + 81 #^12& , 2, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2]) || (m ==
Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #\
^10 + 81 #^12& , 2, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
```

```
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
1/3 (-4 +
3 m^2)) || \
(Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #^10 + 81 #\
^12& , 2, 0] < m < Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2]) || (m ==
Root[686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4,
0] && Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] < l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2]) || (Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4, 0] <
m < Root[-2812500 - 11259427868 #^2 - 6144055152 #^4 + 1352551761 #\
^6 + 692176752 #^8 + 340122240 #^10& , 2, 0] &&
Root[-150 m^2 +
```

```
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] < l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2]) || (0 < m <
Root[32768 - 1855650 #^2 + 10014057 #^4 - 3642597 #^6 - 204525 #\
^8 + 151875 #^10& , 4, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
Root[4 -
507 m^2 + (51 - 1404 m^2) #1 + (252 - 1674 m^2) #1^2 + (594 -
972 m^2) #1^3 + (648 - 243 m^2) #1^4 + 243 #1^5 &,
1]) || (Root[
32768 - 1855650 #^2 + 10014057 #^4 - 3642597 #^6 - 204525 #\
^8 + 151875 #^10& , 4, 0] <= m < Root[
1 - 147 #^2 + 423 #^4 + 135 #^6& , 4, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
```

```
1]) || (Root[1 - 147 #^2 + 423 #^4 + 135 #^6& , 4, 0] <= m < 1/
Sqrt[3] &&
Root[4 -
507 m^2 + (51 - 1404 m^2) #1 + (252 - 1674 m^2) #1^2 + (594 -
972 m^2) #1^3 + (648 - 243 m^2) #1^4 + 243 #1^5 &, 1] < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || (m == 1/Sqrt[3] &&
Root[4 -
507 m^2 + (51 - 1404 m^2) #1 + (252 - 1674 m^2) #1^2 + (594 -
972 m^2) #1^3 + (648 - 243 m^2) #1^4 + 243 #1^5 &, 1] < l <
1/3 (-4 + 3 m^2)) || (1/Sqrt[3] < m <
Root[-324 + 692 #^2 + 336 #^4 + 45 #^6& , 2,
0] && (Root[
4 - 507 m^2 + (51 - 1404 m^2) #1 + (252 -
1674 m^2) #1^2 + (594 - 972 m^2) #1^3 + (648 -
243 m^2) #1^4 + 243 #1^5 &, 1] < l <
Root[4 -
507 m^2 + (51 - 1404 m^2) #1 + (252 -
1674 m^2) #1^2 + (594 - 972 m^2) #1^3 + (648 -
243 m^2) #1^4 + 243 #1^5 &, 2] ||
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1] < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1])) || (Root[-324 + 692 #^2 + 336 #^4 + 45 #^6& , 2, 0] <= m <
Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2, 0] &&
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1] < l <=
```

```
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || (m ==
Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2, 0] &&
1/3 (-4 + 3 m^2) < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || (Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2, 0] <
m < Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& , 2, 0] &&
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1] < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || (m ==
Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& , 2, 0] &&
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1] < l <=
1/3 (-4 +
3 m^2)) || (Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& ,
2, 0] < m <= Root[1 - 27 #^2 - 657 #^4 + 135 #^6& , 4, 0] &&
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1] < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || (Root[1 - 27 #^2 - 657 #^4 + 135 #^6& , 4, 0] < m < Root[
32768 - 2020986 #^2 - 10826991 #^4 - 2570157 #^6 - 220725 #\
^8 + 151875 #^10& , 4, 0] &&
```

```
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || (m == Root[
32768 - 2020986 #^2 - 10826991 #^4 - 2570157 #^6 - 220725 #\
^8 + 151875 #^10& , 4, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <=
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1]) || (m >
Root[32768 - 2020986 #^2 - 10826991 #^4 - 2570157 #^6 - 220725 #\
^8 + 151875 #^10& , 4, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1])
```

In the following, we are taking the output provided above that describes R and we splitting them up into 15 different regions. Each region is specified based on the bounds for the m values provided since the entire region R is a described in piece-wise fashion. We remark that in the code, ***&&*** means *and* and || means *or*.

```
1

(m > 0 && 1/2 (-2 + m^2) <= l < m^2)


2

0 < m < 1/Sqrt[3] &&
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &, 1] <=
l < 1/3 (-4 + 3 m^2)) ||


3

(0 < m <= Sqrt[2/3] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <=
1/2 (-2 + m^2)) || (m > Sqrt[2/3] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l < 1/3 (-4 + 3 m^2)
```

```

4


(0 < m <= 1/Sqrt[3] &&
l == Root[
4 - 507 m^2 + (51 - 1404 m^2) #1 + (252 -
1674 m^2) #1^2 + (594 - 972 m^2) #1^3 + (648 -
243 m^2) #1^4 + 243 #1^5 &, 1]) || (1/Sqrt[3] < m <
Root[-324 + 692 #^2 + 336 #^4 + 45 #^6& , 2,
0] && (l ==
Root[
4 - 507 m^2 + (51 - 1404 m^2) #1 + (252 -
1674 m^2) #1^2 + (594 - 972 m^2) #1^3 + (648 -
243 m^2) #1^4 + 243 #1^5 &, 1] ||
l == Root[
4 - 507 m^2 + (51 - 1404 m^2) #1 + (252 -
1674 m^2) #1^2 + (594 - 972 m^2) #1^3 + (648 -
243 m^2) #1^4 + 243 #1^5 &, 2])) || (m ==
Root[-324 + 692 #^2 + 336 #^4 + 45 #^6& , 2, 0] &&
l == Root[
4 - 507 m^2 + (51 - 1404 m^2) #1 + (252 -
1674 m^2) #1^2 + (594 - 972 m^2) #1^3 + (648 -
243 m^2) #1^4 + 243 #1^5 &, 1])

5

(0 < m < 1/Sqrt[3] &&
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &, 1] <=
```

```
l < 1/3 (-4 + 3 m^2)) || (1/Sqrt[3] < m <
Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& , 2, 0] &&
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &, 1] <=
l < m^2) || (m ==
Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& , 2, 0] &&
1/3 (-4 + 3 m^2) <= l <
m^2) || (Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& , 2, 0] <
m < Root[
32768 - 2020986 #^2 - 10826991 #^4 - 2570157 #^6 - 220725 #\
^8 + 151875 #^10& , 4, 0] &&
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &, 1] <=
l < m^2) || (m == Root[
32768 - 2020986 #^2 - 10826991 #^4 - 2570157 #^6 - 220725 #\
^8 + 151875 #^10& , 4, 0] &&
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1] <= l <
m^2) || (m > Root[
32768 - 2020986 #^2 - 10826991 #^4 - 2570157 #^6 - 220725 #\
^8 + 151875 #^10& , 4, 0] &&
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &, 1] <=
l < m^2)

6

(1/Sqrt[3] < m <= Root[
```

```
1 - 36 #^2 + 36 #^4& , 4,
0] && (Root[
1 - 168 m^2 +
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 +
108 #1^3 + 81 #1^4 &, 1] < l < 1/3 (-4 + 3 m^2) ||
Root[1 - 168 m^2 +
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 +
108 #1^3 + 81 #1^4 &, 2] < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1])) || (Root[1 - 36 #^2 + 36 #^4& , 4, 0] < m <
Root[-27 + 19 #^2 - 9 #^4 + 9 #^6& , 2,
0] && (Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l < 1/3 (-4 + 3 m^2) ||
Root[1 - 168 m^2 +
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 +
108 #1^3 + 81 #1^4 &, 2] < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1])) || (m ==
Root[-27 + 19 #^2 - 9 #^4 + 9 #^6& , 2,
0] && (Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l < 1/3 (-4 + 3 m^2) ||
1/3 (-4 + 3 m^2) < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1])) || (Root[-27 + 19 #^2 - 9 #^4 + 9 #^6& , 2, 0] < m <
```

```
Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1]) || (Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2,
0] <= m < Root[1 - 27 #^2 - 657 #^4 + 135 #^6& , 4, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
1/3 (-4 + 3 m^2)) || (m >= Root[
1 - 27 #^2 - 657 #^4 + 135 #^6& , 4, 0] &&
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1] < l <
1/3 (-4 + 3 m^2))

7

(1/Sqrt[3] < m <= Root[
1 - 36 #^2 + 36 #^4& , 4,
0] && (Root[
1 - 168 m^2 +
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 +
108 #1^3 + 81 #1^4 &, 1] < l < 1/3 (-4 + 3 m^2) ||
Root[1 - 168 m^2 +
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 +
108 #1^3 + 81 #1^4 &, 2] < l <=
Root[-2 -
```

```
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1])) || (Root[1 - 36 #^2 + 36 #^4& , 4, 0] < m <
Root[-27 + 19 #^2 - 9 #^4 + 9 #^6& , 2,
0] && (Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l < 1/3 (-4 + 3 m^2) ||
Root[
1 - 168 m^2 +
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 +
108 #1^3 + 81 #1^4 &, 2] < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1])) || (m ==
Root[-27 + 19 #^2 - 9 #^4 + 9 #^6& , 2,
0] && (Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l < 1/3 (-4 + 3 m^2) ||
1/3 (-4 + 3 m^2) < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1])) || (Root[-27 + 19 #^2 - 9 #^4 + 9 #^6& , 2, 0] < m <
Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& , 2, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <=
```

```
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || (m ==
Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& , 2, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <=
1/3 (-4 + 3 m^2)) || (m >
Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& , 2, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l < 1/3 (-4 + 3 m^2))

8

(1/Sqrt[3] < m <
Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2, 0] &&
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1] < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || (m ==
Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2, 0] &&
1/3 (-4 + 3 m^2) < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
```

```
1]) || (Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2, 0] <
m < Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& , 2, 0] &&
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1] < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || (m ==
Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& , 2, 0] &&
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1] < l <=
1/3 (-4 +
3 m^2)) || (Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& ,
2, 0] < m <= Root[1 - 27 #^2 - 657 #^4 + 135 #^6& , 4, 0] &&
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1] < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || (Root[1 - 27 #^2 - 657 #^4 + 135 #^6& , 4, 0] < m < Root[
32768 - 2020986 #^2 - 10826991 #^4 - 2570157 #^6 - 220725 #\
^8 + 151875 #^10& , 4, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || (m == Root[
32768 - 2020986 #^2 - 10826991 #^4 - 2570157 #^6 - 220725 #\
```

```
^8 + 151875 #^10& , 4, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <=
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1]) || (m >
Root[32768 - 2020986 #^2 - 10826991 #^4 - 2570157 #^6 - 220725 #\
^8 + 151875 #^10& , 4, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1])

9

(Root[-1 + 12 #^2 + 9 #^4& , 2, 0] < m <= 1/Sqrt[3] &&
Root[16 - 6360 m^2 +
47961 m^4 + (216 + 8478 m^2 + 149796 m^4) #1 + (1161 +
54432 m^2 + 176094 m^4) #1^2 + (3132 + 64476 m^2 +
92340 m^4) #1^3 + (4374 + 29160 m^2 +
18225 m^4) #1^4 + (2916 + 4374 m^2) #1^5 + 729 #1^6 &, 2] <
l < m^2) || (1/Sqrt[3] < m < Sqrt[2/
3] && (Root[
16 - 6360 m^2 +
47961 m^4 + (216 + 8478 m^2 + 149796 m^4) #1 + (1161 +
54432 m^2 + 176094 m^4) #1^2 + (3132 + 64476 m^2 +
92340 m^4) #1^3 + (4374 + 29160 m^2 +
18225 m^4) #1^4 + (2916 + 4374 m^2) #1^5 + 729 #1^6 &,
```

```
1] < l < 1/3 (-4 + 3 m^2) ||
Root[16 - 6360 m^2 +
47961 m^4 + (216 + 8478 m^2 + 149796 m^4) #1 + (1161 +
54432 m^2 + 176094 m^4) #1^2 + (3132 + 64476 m^2 +
92340 m^4) #1^3 + (4374 + 29160 m^2 +
18225 m^4) #1^4 + (2916 + 4374 m^2) #1^5 + 729 #1^6 &,
2] < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1])) || (m == Sqrt[2/
3] && (Root[
16 - 6360 m^2 +
47961 m^4 + (216 + 8478 m^2 + 149796 m^4) #1 + (1161 +
54432 m^2 + 176094 m^4) #1^2 + (3132 + 64476 m^2 +
92340 m^4) #1^3 + (4374 + 29160 m^2 +
18225 m^4) #1^4 + (2916 + 4374 m^2) #1^5 + 729 #1^6 &,
1] < l < 1/3 (-4 + 3 m^2) ||
1/3 (-4 + 3 m^2) < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1])) || (Sqrt[2/3] < m < Root[
18 - 965 #^2 - 4029 #^4 + 4293 #^6 + 3375 #^8& , 4, 0] &&
Root[16 - 6360 m^2 +
47961 m^4 + (216 + 8478 m^2 + 149796 m^4) #1 + (1161 +
54432 m^2 + 176094 m^4) #1^2 + (3132 + 64476 m^2 +
92340 m^4) #1^3 + (4374 + 29160 m^2 +
18225 m^4) #1^4 + (2916 + 4374 m^2) #1^5 + 729 #1^6 &, 1] <
l < Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1]) || (m == Root[
18 - 965 #^2 - 4029 #^4 + 4293 #^6 + 3375 #^8& , 4, 0] &&
```

```
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1]) || (Root[
18 - 965 #^2 - 4029 #^4 + 4293 #^6 + 3375 #^8& , 4, 0] < m <
Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1]) || (Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2,
0] <= m < Root[1 - 27 #^2 - 657 #^4 + 135 #^6& , 4, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
1/3 (-4 + 3 m^2)) || (m >= Root[
1 - 27 #^2 - 657 #^4 + 135 #^6& , 4, 0] &&
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1] < l <
1/3 (-4 + 3 m^2))

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(0 < m <= Root[-1 + 6 #^2 + 9 #^4& , 2, 0] &&
Root[4 -
```

```
507 m^2 + (51 - 1404 m^2) #1 + (252 - 1674 m^2) #1^2 + (594 -
972 m^2) #1^3 + (648 - 243 m^2) #1^4 + 243 #1^5 &, 1] < l <
m^2) || (Root[-1 + 6 #^2 + 9 #^4& , 2, 0] < m < Root[
32768 - 1855650 #^2 + 10014057 #^4 - 3642597 #^6 - 204525 #\
^8 + 151875 #^10& , 4, 0] &&
Root[4 -
507 m^2 + (51 - 1404 m^2) #1 + (252 - 1674 m^2) #1^2 + (594 -
972 m^2) #1^3 + (648 - 243 m^2) #1^4 + 243 #1^5 &, 1] < l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2]) || (m ==
Root[32768 - 1855650 #^2 + 10014057 #^4 - 3642597 #^6 - 204525 #\
^8 + 151875 #^10& , 4, 0] &&
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &, 1] < l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2]) || (Root[
32768 - 1855650 #^2 + 10014057 #^4 - 3642597 #^6 - 204525 #\
^8 + 151875 #^10& , 4, 0] < m <= Root[
1 - 147 #^2 + 423 #^4 + 135 #^6& , 4, 0] &&
Root[4 -
507 m^2 + (51 - 1404 m^2) #1 + (252 - 1674 m^2) #1^2 + (594 -
972 m^2) #1^3 + (648 - 243 m^2) #1^4 + 243 #1^5 &, 1] < l <
```

```
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2]) || (Root[
1 - 147 #^2 + 423 #^4 + 135 #^6& , 4, 0] < m <
Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #\
^10 + 81 #^12& , 2, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2]) || (m ==
Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #\
^10 + 81 #^12& , 2, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
1/3 (-4 +
3 m^2)) || \
(Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #^10 + 81 #\
^12& , 2, 0] < m < Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
```

```
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2]) || (m ==
Root[686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4,
0] && Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] < l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2]) || (Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4, 0] <
m < Root[-2812500 - 11259427868 #^2 - 6144055152 #^4 + 1352551761 #\
^6 + 692176752 #^8 + 340122240 #^10& , 2, 0] &&
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] < l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
```

```
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2])

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(Root[-1 + 6 #^2 + 9 #^4& , 2, 0] < m <= 1/Sqrt[3] &&
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2] < l <
m^2) || (1/Sqrt[3] < m <= Root[-2 - 9 #^2 + 9 #^4& , 2, 0] &&
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2] < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1]) || (Root[-2 - 9 #^2 + 9 #^4& , 2, 0] < m <
Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #\
^10 + 81 #^12& , 2,
0] && (Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
```

```
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] < l <
1/3 (-4 + 3 m^2) ||
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2] < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1])) || (m ==
Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #\
^10 + 81 #^12& , 2,
0] && (Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] < l <
1/3 (-4 + 3 m^2) ||
1/3 (-4 + 3 m^2) < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1])) || (Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #\
^8 - 837 #^10 + 81 #^12& , 2, 0] < m <
Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2, 0] &&
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
```

```
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1]) || (Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2,
0] <= m < Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4, 0] &&
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] < l <
1/3 (-4 + 3 m^2)) || (m == Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] < l <
1/3 (-4 + 3 m^2)) || (Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4, 0] <
m < Root[1 - 27 #^2 - 657 #^4 + 135 #^6& , 4, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
1/3 (-4 + 3 m^2)) || (m >= Root[
1 - 27 #^2 - 657 #^4 + 135 #^6& , 4, 0] &&
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1] < l <
1/3 (-4 + 3 m^2))
```

```
12

(1/Sqrt[3] < m <= (3 Sqrt[3])/5 &&
Root[1 - 168 m^2 +
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 + 108 #1^3 +
81 #1^4 &, 2] < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1]) || ((
3 Sqrt[3])/5 < m <= Root[-2 - 9 #^2 + 9 #^4& , 2, 0] &&
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2] < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1]) || (Root[-2 - 9 #^2 + 9 #^4& , 2, 0] < m < Root[
17179869184 - 758241767424 #^2 + 738811183104 #^4 - 1873773558144 #\
^6 + 2178248345280 #^8 + 730684767357 #^10 - 1168450231581 #\
^12 - 312846367473 #^14 + 307409258025 #^16& , 4,
0] && (Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] < l <
1/3 (-4 + 3 m^2) ||
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
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8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2] < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1])) || (m == Root[
17179869184 - 758241767424 #^2 + 738811183104 #^4 - 1873773558144 #\
^6 + 2178248345280 #^8 + 730684767357 #^10 - 1168450231581 #\
^12 - 312846367473 #^14 + 307409258025 #^16& , 4,
0] && (Root[
1 - 168 m^2 +
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 +
108 #1^3 + 81 #1^4 &, 2] < l < 1/3 (-4 + 3 m^2) ||
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2] < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1])) || (Root[
17179869184 - 758241767424 #^2 + 738811183104 #^4 - 1873773558144 #\
^6 + 2178248345280 #^8 + 730684767357 #^10 - 1168450231581 #\
^12 - 312846367473 #^14 + 307409258025 #^16& , 4, 0] < m <
Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #\
^10 + 81 #^12& , 2,
0] && (Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
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19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] < l <
1/3 (-4 + 3 m^2) ||
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2] < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1])) || (m ==
Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #\
^10 + 81 #^12& , 2,
0] && (Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] < l <
1/3 (-4 + 3 m^2) ||
1/3 (-4 + 3 m^2) < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1])) || (Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #\
^8 - 837 #^10 + 81 #^12& , 2, 0] < m <
Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2, 0] &&
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
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19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] < l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &,
1]) || (Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2,
0] <= m < Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4, 0] &&
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] < l <
1/3 (-4 + 3 m^2)) || (m == Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] < l <
1/3 (-4 + 3 m^2)) || (Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4, 0] <
m < Root[1 - 27 #^2 - 657 #^4 + 135 #^6& , 4, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
1/3 (-4 + 3 m^2)) || (m >= Root[
1 - 27 #^2 - 657 #^4 + 135 #^6& , 4, 0] &&
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1] < l <
1/3 (-4 + 3 m^2))
```

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13

(1/Sqrt[3] < m <= (3 Sqrt[3])/5 &&
Root[1 - 168 m^2 +
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 + 108 #1^3 +
81 #1^4 &, 2] < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || ((3 Sqrt[3])/5 < m <= Root[-2 - 9 #^2 + 9 #^4& , 2, 0] &&
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2] < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || (Root[-2 - 9 #^2 + 9 #^4& , 2, 0] < m < Root[
17179869184 - 758241767424 #^2 + 738811183104 #^4 - 1873773558144 #\
^6 + 2178248345280 #^8 + 730684767357 #^10 - 1168450231581 #\
^12 - 312846367473 #^14 + 307409258025 #^16& , 4,
0] && (Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] < l <
1/3 (-4 + 3 m^2) ||
```

```
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2] < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1])) || (m == Root[
17179869184 - 758241767424 #^2 + 738811183104 #^4 - 1873773558144 #\
^6 + 2178248345280 #^8 + 730684767357 #^10 - 1168450231581 #\
^12 - 312846367473 #^14 + 307409258025 #^16& , 4,
0] && (Root[
1 - 168 m^2 +
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 +
108 #1^3 + 81 #1^4 &, 2] < l < 1/3 (-4 + 3 m^2) ||
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2] < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1])) || (Root[
17179869184 - 758241767424 #^2 + 738811183104 #^4 - 1873773558144 #\
^6 + 2178248345280 #^8 + 730684767357 #^10 - 1168450231581 #\
^12 - 312846367473 #^14 + 307409258025 #^16& , 4, 0] < m <
Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #\
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^10 + 81 #^12& , 2,
0] && (Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] < l <
1/3 (-4 + 3 m^2) ||
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2] < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1])) || (m ==
Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #\
^10 + 81 #^12& , 2,
0] && (Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] < l <
1/3 (-4 + 3 m^2) ||
1/3 (-4 + 3 m^2) < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
```

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1])) || (Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #\
^8 - 837 #^10 + 81 #^12& , 2, 0] < m < Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4, 0] &&
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || (m == Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || (Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4, 0] <
m < Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& , 2, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
```

```
1]) || (m ==
Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& , 2, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <=
1/3 (-4 + 3 m^2)) || (m >
Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& , 2, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l < 1/3 (-4 + 3 m^2))

14

(1/Sqrt[3] < m <= Root[1 - 36 #^2 + 36 #^4& , 4, 0] &&
Root[1 - 168 m^2 +
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 + 108 #1^3 +
81 #1^4 &, 1] < l < 1/3 (-4 + 3 m^2)) || (Root[
1 - 36 #^2 + 36 #^4& , 4, 0] < m <= (3 Sqrt[3])/5 &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
1/3 (-4 + 3 m^2)) || ((3 Sqrt[3])/5 < m <=
Root[-2 - 9 #^2 + 9 #^4& , 2,
0] && (Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l < 1/3 (-4 + 3 m^2) ||
Root[1 - 168 m^2 +
```

```
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 +
108 #1^3 + 81 #1^4 &, 2] < l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &,
2])) || (Root[-2 - 9 #^2 + 9 #^4& , 2, 0] < m <
Root[-27 + 19 #^2 - 9 #^4 + 9 #^6& , 2,
0] && (Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] ||
Root[1 - 168 m^2 +
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 +
108 #1^3 + 81 #1^4 &, 2] < l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2])) || (m ==
Root[-27 + 19 #^2 - 9 #^4 + 9 #^6& , 2,
0] && (Root[-2 + 213 m^2 + 5112 m^4 -
```

```
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] ||
1/3 (-4 + 3 m^2) < l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &,
2])) || (Root[-27 + 19 #^2 - 9 #^4 + 9 #^6& , 2, 0] < m < Root[
17179869184 - 758241767424 #^2 + 738811183104 #^4 - 1873773558144 #\
^6 + 2178248345280 #^8 + 730684767357 #^10 - 1168450231581 #\
^12 - 312846367473 #^14 + 307409258025 #^16& , 4,
0] && (Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] ||
Root[1 - 168 m^2 +
```

```
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 +
108 #1^3 + 81 #1^4 &, 2] < l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2])) || (m ==
Root[17179869184 - 758241767424 #^2 + 738811183104 #\
^4 - 1873773558144 #^6 + 2178248345280 #^8 + 730684767357 #\
^10 - 1168450231581 #^12 - 312846367473 #^14 + 307409258025 #^16& , 4,
0] && (Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
Root[
1 - 168 m^2 +
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 +
108 #1^3 + 81 #1^4 &, 2] ||
Root[1 - 168 m^2 +
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 +
108 #1^3 + 81 #1^4 &, 2] < l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2])) || (Root[
17179869184 - 758241767424 #^2 + 738811183104 #^4 - 1873773558144 #\
^6 + 2178248345280 #^8 + 730684767357 #^10 - 1168450231581 #\
^12 - 312846367473 #^14 + 307409258025 #^16& , 4, 0] < m <
```

```
Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #\
^10 + 81 #^12& , 2, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2]) || (m ==
Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #\
^10 + 81 #^12& , 2, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
1/3 (-4 +
3 m^2)) || \
(Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #^10 + 81 #\
^12& , 2, 0] < m < Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
```

```
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2]) || (m ==
Root[686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4,
0] && Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] < l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2]) || (Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4, 0] <
m < Root[-2812500 - 11259427868 #^2 - 6144055152 #^4 + 1352551761 #\
^6 + 692176752 #^8 + 340122240 #^10& , 2, 0] &&
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1] < l <
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 +
8748 m^4) #1^3 + (1890 - 6966 m^2 +
19683 m^4) #1^4 + (5670 - 45927 m^2) #1^5 + (10206 -
13122 m^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 2])

15
```

```
(0 < m < Root[
32768 - 1855650 #^2 + 10014057 #^4 - 3642597 #^6 - 204525 #\
^8 + 151875 #^10& , 4, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
Root[4 -
507 m^2 + (51 - 1404 m^2) #1 + (252 - 1674 m^2) #1^2 + (594 -
972 m^2) #1^3 + (648 - 243 m^2) #1^4 + 243 #1^5 &,
1]) || (Root[
32768 - 1855650 #^2 + 10014057 #^4 - 3642597 #^6 - 204525 #\
^8 + 151875 #^10& , 4, 0] <= m < Root[
1 - 147 #^2 + 423 #^4 + 135 #^6& , 4, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || (Root[1 - 147 #^2 + 423 #^4 + 135 #^6& , 4, 0] <= m < 1/
Sqrt[3] &&
Root[4 -
507 m^2 + (51 - 1404 m^2) #1 + (252 - 1674 m^2) #1^2 + (594 -
972 m^2) #1^3 + (648 - 243 m^2) #1^4 + 243 #1^5 &, 1] < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || (m == 1/Sqrt[3] &&
Root[4 -
```

```
507 m^2 + (51 - 1404 m^2) #1 + (252 - 1674 m^2) #1^2 + (594 -
972 m^2) #1^3 + (648 - 243 m^2) #1^4 + 243 #1^5 &, 1] < l <
1/3 (-4 + 3 m^2)) || (1/Sqrt[3] < m <
Root[-324 + 692 #^2 + 336 #^4 + 45 #^6& , 2,
0] && (Root[
4 - 507 m^2 + (51 - 1404 m^2) #1 + (252 -
1674 m^2) #1^2 + (594 - 972 m^2) #1^3 + (648 -
243 m^2) #1^4 + 243 #1^5 &, 1] < l <
Root[4 -
507 m^2 + (51 - 1404 m^2) #1 + (252 -
1674 m^2) #1^2 + (594 - 972 m^2) #1^3 + (648 -
243 m^2) #1^4 + 243 #1^5 &, 2] ||
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1] < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1])) || (Root[-324 + 692 #^2 + 336 #^4 + 45 #^6& , 2, 0] <= m <
Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2, 0] &&
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1] < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || (m ==
Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2, 0] &&
1/3 (-4 + 3 m^2) < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || (Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2, 0] <
```

```
m < Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& , 2, 0] &&
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1] < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || (m ==
Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& , 2, 0] &&
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1] < l <=
1/3 (-4 +
3 m^2)) || (Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& ,
2, 0] < m <= Root[1 - 27 #^2 - 657 #^4 + 135 #^6& , 4, 0] &&
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1] < l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || (Root[1 - 27 #^2 - 657 #^4 + 135 #^6& , 4, 0] < m < Root[
32768 - 2020986 #^2 - 10826991 #^4 - 2570157 #^6 - 220725 #\
^8 + 151875 #^10& , 4, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <=
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &,
1]) || (m == Root[
32768 - 2020986 #^2 - 10826991 #^4 - 2570157 #^6 - 220725 #\
^8 + 151875 #^10& , 4, 0] &&
```

```
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <=
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1]) || (m >
Root[32768 - 2020986 #^2 - 10826991 #^4 - 2570157 #^6 - 220725 #\
^8 + 151875 #^10& , 4, 0] &&
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 -
243 m^2) #1^4 + 486 #1^5 &, 1] <= l <
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1])

```

In the 15 regions provided above, we have the following functions appearing. These functions are provided as roots of polynomials and only two of these functions had radical forms.

```
In[2]:= f275m1 =
Root[-2 -
75 m^2 + (6 - 180 m^2) #1 + (180 - 378 m^2) #1^2 + (756 -
324 m^2) #1^3 + (1134 - 243 m^2) #1^4 + 486 #1^5 &, 1]

In[3]:= f2213m1 =
Root[-2 + 213 m^2 + 5112 m^4 -
2160 m^6 + (6 - 3780 m^2 + 5616 m^4) #1 + (180 - 1242 m^2 -
648 m^4) #1^2 + (756 + 972 m^2) #1^3 + (1134 - 243 m^2) #1^4 +
486 #1^5 &, 1]

In[4]:= f4507m1 =
Root[4 - 507 m^2 + (51 - 1404 m^2) #1 + (252 -
1674 m^2) #1^2 + (594 - 972 m^2) #1^3 + (648 - 243 m^2) #1^4 +
243 #1^5 &, 1]

In[5]:= f4507m2 =
Root[4 - 507 m^2 + (51 - 1404 m^2) #1 + (252 -
1674 m^2) #1^2 + (594 - 972 m^2) #1^3 + (648 - 243 m^2) #1^4 +
243 #1^5 &, 2]

In[6]:= f25m1 =
Root[-25 m^2 + (1 + 60 m^2) #1 + (12 - 126 m^2) #1^2 + (54 +
108 m^2) #1^3 + (108 - 81 m^2) #1^4 + 81 #1^5 &, 1]

In[7]:= f1168m1 =
ToRadicals[
```

```
Root[1 - 168 m^2 +
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 + 108 #1^3 +
81 #1^4 &, 1]]

Out[7]= 1/3 (-1 - 2 Sqrt[-2 Sqrt[3] m - 3 m^2])

In[8]:= f1168m2 =
ToRadicals[
Root[1 - 168 m^2 +
144 m^4 + (12 + 144 m^2) #1 + (54 + 216 m^2) #1^2 + 108 #1^3 +
81 #1^4 &, 2]]

Out[8]= 1/3 (-1 + 2 Sqrt[-2 Sqrt[3] m - 3 m^2])

In[9]:= f166360m2 =
Root[16 - 6360 m^2 +
47961 m^4 + (216 + 8478 m^2 + 149796 m^4) #1 + (1161 +
54432 m^2 + 176094 m^4) #1^2 + (3132 + 64476 m^2 +
92340 m^4) #1^3 + (4374 + 29160 m^2 +
18225 m^4) #1^4 + (2916 + 4374 m^2) #1^5 + 729 #1^6 &, 2]

In[10]:= f166360m1 =
Root[16 - 6360 m^2 +
47961 m^4 + (216 + 8478 m^2 + 149796 m^4) #1 + (1161 +
54432 m^2 + 176094 m^4) #1^2 + (3132 + 64476 m^2 +
92340 m^4) #1^3 + (4374 + 29160 m^2 +
18225 m^4) #1^4 + (2916 + 4374 m^2) #1^5 + 729 #1^6 &, 1]

In[11]:= f150m2 =
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
```

```
3402 m^4) #1^2 + (378 + 14742 m^2 + 8748 m^4) #1^3 + (1890 -
6966 m^2 + 19683 m^4) #1^4 + (5670 -
45927 m^2) #1^5 + (10206 - 13122 m^2) #1^6 + 10206 #1^7 +
4374 #1^8 &, 2]

In[12]:= f150m1 =
Root[-150 m^2 +
75 m^4 + (2 - 231 m^2 + 540 m^4) #1 + (42 + 4806 m^2 +
3402 m^4) #1^2 + (378 + 14742 m^2 + 8748 m^4) #1^3 + (1890 -
6966 m^2 + 19683 m^4) #1^4 + (5670 -
45927 m^2) #1^5 + (10206 - 13122 m^2) #1^6 + 10206 #1^7 +
4374 #1^8 &, 1]
```

We now provide the 15 regions with our notation described above and show a few steps of the simplifications that took place. When we describe the regions, we have some places where the m value is described as a constant instead of on an interval. In the code provided above, one sees the ***Root*** function used to describe that constant, but the Mathematica Notebook presents these as approximations. For simplifying the description of our regions, we provide the approximations for these m values instead of the ***Root*** descriptions.

There are two big steps in our simplification process. The first simplification is to look for obvious overlaps between the regions. For instance, one will see that most of the twelfth region ends up disappearing in this first simplification, and this is due to most of the region overlapping with other regions, making majority of the twelfth region a collection of redundancies.

The second simplification is to consolidate those regions described by constant values of m into the interval they belong. We will describe this process more precisely when we get there. In this second step of simplifying, we were also able to further simplify our regions and we ended up reducing the number of regions describing R by finding more overlaps between regions.

$$1: \begin{cases} m > 0 & \frac{1}{2}\left(m^2 - 2\right) \leq l < m^2 \end{cases}$$

$$2: \begin{cases} 0 < m < \frac{1}{\sqrt{3}} & f_{275m1} \leq l < \frac{1}{3}\left(3m^2 - 4\right) \end{cases}$$

$$3: \begin{cases} 0 < m \leq \sqrt{\frac{2}{3}} & f_{2213m1} \leq l \leq \frac{1}{2}\left(m^2 - 2\right) \\ m > \sqrt{\frac{2}{3}} & f_{2213m1} \leq l < \frac{1}{3}\left(3m^2 - 4\right) \end{cases}$$

$$4: \begin{cases} 0 < m \leq \frac{1}{\sqrt{3}} & l = f_{4507m1} \\ \frac{1}{\sqrt{3}} < m < .625... & l = f_{4507m1} \\ \frac{1}{\sqrt{3}} < m < .625... & l = f_{4507m2} \\ m = .625... & l = f_{4507m1} \end{cases}$$

$$5: \begin{cases} 0 < m < \frac{1}{\sqrt{3}} & f_{275m1} \leq l < \frac{1}{3}\left(3m^2 - 4\right) \\ \frac{1}{\sqrt{3}} < m < 1.56... & f_{275m1} \leq l < m^2 \\ m = 1.56... & \frac{1}{3}\left(3m^2 - 4\right) \leq l < m^2 \\ 1.56... < m < 2.48... & f_{275m1} \leq l < m^2 \\ m = 2.48... & f_{25m1} \leq l < m^2 \\ m > 2.48... & f_{275m1} \leq l < m^2 \end{cases}$$

$$6: \begin{cases} \frac{1}{\sqrt{3}} < m \leq .986... & f_{1168m1} < l < \frac{1}{3}\left(3m^2 - 4\right) \\ \frac{1}{\sqrt{3}} < m \leq .986... & f_{1168m2} < l < f_{25m1} \\ .986... < m < 1.11... & f_{2213m1} \leq l < \frac{1}{3}\left(3m^2 - 4\right) \\ .986... < m < 1.11... & f_{1168m2} < l < f_{25m1} \\ m = 1.11... & f_{2213m1} \leq l < \frac{1}{3}\left(3m^2 - 4\right) \\ m = 1.11... & 1/3(-4 + 3m^2) < l < f_{25m1} \\ 1.11... < m < 1.40... & f_{2213m1} \leq l < f_{25m1} \\ 1.40... \leq m < 2.22... & f_{2213m1} \leq l < \frac{1}{3}\left(3m^2 - 4\right) \\ m \geq 2.22... & f_{25m1} < l < \frac{1}{3}\left(3m^2 - 4\right) \end{cases}$$

$$
7:\begin{cases}
\frac{1}{\sqrt{3}} < m \leq .986\ldots & f_{1168m1} < l < \frac{1}{3}\left(3m^2-4\right) \\
\frac{1}{\sqrt{3}} < m \leq .986\ldots & f_{1168m2} < l \leq f_{275m1} \\
.986\ldots < m < 1.11\ldots & f_{2213m1} \leq l < \frac{1}{3}\left(3m^2-4\right) \\
.986\ldots < m < 1.11\ldots & f_{1168m2} < l \leq f_{275m1} \\
m = 1.11\ldots & f_{2213m1} \leq l < \frac{1}{3}\left(3m^2-4\right) \\
m = 1.11\ldots & \frac{1}{3}\left(3m^2-4\right) < l \leq f_{275m1} \\
1.11\ldots < m < 1.56\ldots & f_{2213m1} \leq l \leq f_{275m1} \\
m = 1.56\ldots & f_{2213m1} \leq l \leq \frac{1}{3}\left(3m^2-4\right) \\
m > 1.56\ldots & f_{2213m1} \leq l < \frac{1}{3}\left(3m^2-4\right)
\end{cases}
$$

$$
8:\begin{cases}
\frac{1}{\sqrt{3}} < m < 1.40\ldots & f_{25m1} < l \leq f_{275m1} \\
m = 1.40\ldots & \frac{1}{3}\left(3m^2-4\right) < l \leq f_{275m1} \\
1.40\ldots < m < 1.56\ldots & f_{25m1} < l \leq f_{275m1} \\
m = 1.56\ldots & f_{25m1} < l \leq \frac{1}{3}\left(3m^2-4\right) \\
1.56\ldots < m \leq 2.22\ldots & f_{25m1} < l \leq f_{275m1} \\
2.22\ldots < m < 2.48\ldots & f_{2213m1} \leq l \leq f_{275m1} \\
m = 2.48\ldots & f_{2213m1} \leq l \leq f_{25m1} \\
m > 2.48\ldots & f_{2213m1} \leq l < f_{25m1}
\end{cases}
$$

$$9:\begin{cases} .281... < m \leq \frac{1}{\sqrt{3}} & f_{166360m2} < l < m^2 \\ \frac{1}{\sqrt{3}} < m < \sqrt{\frac{2}{3}} & f_{166360m1} < l < \frac{1}{3}\left(3m^2-4\right) \\ \frac{1}{\sqrt{3}} < m < \sqrt{\frac{2}{3}} & f_{166360m2} < l < f_{25m1} \\ m = \sqrt{\frac{2}{3}} & f_{166360m1} < l < \frac{1}{3}\left(3m^2-4\right) \\ m = \sqrt{\frac{2}{3}} & \frac{1}{3}\left(3m^2-4\right) < l < f_{25m1} \\ \sqrt{\frac{2}{3}} < m < .875... & f_{166360m1} < l < f_{25m1} \\ m = .875... & f_{2213m1} < l < f_{25m1} \\ .875... < m < 1.40... & f_{2213m1} \leq l < f_{25m1} \\ 1.40 \leq m < 2.22... & f_{2213m1} \leq l < \frac{1}{3}\left(3m^2-4\right) \\ m \geq 2.22... & f_{25m1} < l < \frac{1}{3}\left(3m^2-4\right) \end{cases}$$

$$10:\begin{cases} 0 < m \leq .372... & f_{4507m1} < l < m^2 \\ .372... < m < .423... & f_{4507m1} < l < f_{150m2} \\ m = .423... & f_{275m1} < l < f_{150m2} \\ .423... < m \leq .556... & f_{4507m1} < l < f_{150m2} \\ .556... < m < 1.17... & f_{2213m1} \leq l < f_{150m2} \\ m = 1.17... & f_{2213m1} \leq l < \frac{1}{3}\left(3m^2-4\right) \\ 1.17... < m < 1.52... & f_{2213m1} \leq l < f_{150m2} \\ m = 1.52... & f_{2213m1} < l < f_{150m2} \\ 1.52... < m < 1.52... & f_{150m1} < l < f_{150m2} \end{cases}$$

$$
11 : \begin{cases}
.372... < m \leq \frac{1}{\sqrt{3}} & f_{150m2} < l < m^2 \\
\frac{1}{\sqrt{3}} < m \leq 1.09... & f_{150m2} < l < f_{25m1} \\
1.09... < m < 1.17... & f_{150m1} < l < \frac{1}{3}\left(3m^2 - 4\right) \\
1.09... < m < 1.17... & f_{150m2} < l < f_{25m1} \\
m = 1.17... & f_{150m1} < l < \frac{1}{3}\left(3m^2 - 4\right) \\
m = 1.17... & \frac{1}{3}\left(3m^2 - 4\right) < l < f_{25m1} \\
1.17... < m < 1.40... & f_{150m1} < l < f_{25m1} \\
1.40... \leq m < 1.52... & f_{150m1} < l < \frac{1}{3}\left(3m^2 - 4\right) \\
m = 1.52... & f_{2213m1} < l < \frac{1}{3}\left(3m^2 - 4\right) \\
1.52... < m < 2.22... & f_{2213m1} \leq l < \frac{1}{3}\left(3m^2 - 4\right) \\
m \geq 2.22... & f_{25m1} < l < \frac{1}{3}\left(3m^2 - 4\right)
\end{cases}
$$

$$12 : \begin{cases} \frac{1}{\sqrt{3}} < m \leq \frac{3\sqrt{3}}{5} & f_{1168m2} < l < f_{25m1} \\ \frac{3\sqrt{3}}{5} < m \leq 1.09... & f_{150m2} < l < f_{25m1} \\ 1.09... < m < 1.13... & f_{150m1} < l < \frac{1}{3}\left(3m^2 - 4\right) \\ 1.09... < m < 1.13... & f_{150m2} < l < f_{25m1} \\ m = 1.13... & f_{1168m2} < l < \frac{1}{3}\left(3m^2 - 4\right) \\ m = 1.13... & f_{150m2} < l < f_{25m1} \\ 1.13... < m < 1.17... & f_{150m1} < l < \frac{1}{3}\left(3m^2 - 4\right) \\ 1.13... < m < 1.17... & f_{150m2} < l < f_{25m1} \\ m = 1.17... & f_{150m1} < l < \frac{1}{3}\left(3m^2 - 4\right) \\ m = 1.17... & \frac{1}{3}\left(3m^2 - 4\right) < l < f_{25m1} \\ 1.17... < m < 1.40... & f_{150m1} < m < f_{25m1} \\ 1.40... \leq m < 1.52... & f_{150m1} < l < \frac{1}{3}\left(3m^2 - 4\right) \\ m = 1.52... & f_{2213m1} < l < \frac{1}{3}\left(3m^2 - 4\right) \\ 1.52... < m < 2.22... & f_{2213m1} \leq l < \frac{1}{3}\left(3m^2 - 4\right) \\ m \geq 2.22... & f_{25m1} < l < \frac{1}{3}\left(3m^2 - 4\right) \end{cases}$$

$$13 : \begin{cases} \frac{1}{\sqrt{3}} < m \leq \frac{3\sqrt{3}}{5} & f_{1168m2} < l \leq f_{275m1} \\ \frac{3\sqrt{3}}{5} < m \leq 1.09... & f_{150m2} < l \leq f_{275m1} \\ 1.09... < m < 1.13... & f_{150m1} < l < \frac{1}{3}\left(3m^2 - 4\right) \\ 1.09... < m < 1.13... & f_{150m2} < l \leq f_{275m1} \\ m = 1.13... & f_{1168m2} < l < \frac{1}{3}\left(3m^2 - 4\right) \\ m = 1.13... & f_{150m2} < l \leq f_{275m1} \\ 1.13... < m < 1.17... & f_{150m1} < l < \frac{1}{3}\left(3m^2 - 4\right) \\ 1.13... < m < 1.17... & f_{150m2} < l \leq f_{275m1} \\ m = 1.17... & f_{150m1} < l < \frac{1}{3}\left(3m^2 - 4\right) \\ m = 1.17... & \frac{1}{3}\left(3m^2 - 4\right) < l \leq f_{275m1} \\ 1.17... < m < 1.52... & f_{150m1} < l \leq f_{275m1} \\ m = 1.52... & f_{2213m1} < l \leq f_{275m1} \\ 1.52 < m < 1.56... & f_{2213m1} \leq l \leq f_{275m1} \\ m = 1.56... & f_{2213m1} \leq l \leq \frac{1}{3}\left(3m^2 - 4\right) \\ m > 1.56... & f_{2213m1} \leq l < \frac{1}{3}\left(3m^2 - 4\right) \end{cases}$$

$$
14 : \begin{cases}
\frac{1}{\sqrt{3}} < m \leq .986\ldots & f_{1168m1} < l < \frac{1}{3}\left(3m^2 - 4\right) \\
.986\ldots < m \leq \frac{3\sqrt{3}}{5} & f_{2213m1} \leq l < \frac{1}{3}\left(3m^2 - 4\right) \\
\frac{3\sqrt{3}}{5} < m \leq 1.09\ldots & f_{2213m1} \leq l < \frac{1}{3}\left(3m^2 - 4\right) \\
\frac{3\sqrt{3}}{5} < m \leq 1.09\ldots & f_{1168m2} < l < f_{150m2} \\
1.09\ldots < m < 1.11\ldots & f_{2213m1} \leq l < f_{150m1} \\
1.09\ldots < m < 1.11\ldots & f_{1168m2} < l < f_{150m2} \\
m = 1.11\ldots & f_{2213m1} \leq l < f_{150m1} \\
m = 1.11\ldots & \frac{1}{3}\left(3m^2 - 4\right) < l < f_{150m2} \\
1.11\ldots < m < 1.13\ldots & f_{2213m1} \leq l < f_{150m1} \\
1.11\ldots < m < 1.13\ldots & f_{1168m2} < l < f_{150m2} \\
m = 1.13\ldots & f_{2213m1} \leq l < f_{1168m2} \\
m = 1.13\ldots & f_{1168m2} < l < f_{150m2} \\
1.13\ldots < m < 1.17\ldots & f_{2213m1} \leq l < f_{150m2} \\
m = 1.17\ldots & f_{2213m1} \leq l < \frac{1}{3}\left(3m^2 - 4\right) \\
1.17\ldots < m < 1.52\ldots & f_{2213m1} \leq l < f_{150m2} \\
m = 1.52\ldots & f_{2213m1} < l < f_{150m2} \\
1.52\ldots < m < 1.52\ldots & f_{150m1} < m < f_{150m2}
\end{cases}
$$

$$
15 : \begin{cases}
0 < m < .423\ldots & f_{2213m1} \le l < f_{4507m1} \\
.423\ldots \le m < .556\ldots & f_{2213m1} \le l \le f_{275m1} \\
.556\ldots \le m < \frac{1}{\sqrt{3}} & f_{4507m1} < l \le f_{275m1} \\
m = \frac{1}{\sqrt{3}} & f_{4507m1} < l < \frac{1}{3}\left(3m^2 - 4\right) \\
\frac{1}{\sqrt{3}} < m < .625\ldots & f_{4507m1} < l < f_{4507m2} \\
\frac{1}{\sqrt{3}} < m < .625\ldots & f_{25m1} < l \le f_{275m1} \\
.625\ldots \le m < 1.40\ldots & f_{25m1} < l \le f_{275m1} \\
m = 1.40\ldots & \frac{1}{3}\left(3m^2 - 4\right) < l \le f_{275m1} \\
1.40\ldots < m < 1.56\ldots & f_{25m1} < l \le f_{275m1} \\
m = 1.56 & f_{25m1} < l \le \frac{1}{3}\left(3m^2 - 4\right) \\
1.56\ldots < m \le 2.22\ldots & f_{25m1} < l \le f_{275m1} \\
2.22 < m < 248 & f_{2213m1} \le l \le f_{275m1} \\
m = 2.48\ldots & f_{2213m1} \le l \le f_{25m1} \\
m > 2.48\ldots & f_{2213m1} \le l < f_{25m1}
\end{cases}
$$

By looking for overlaps between the 15 regions provided above, we are able simplify those 15 regions into the following:

$$1 : \begin{cases} m > 0 & \frac{1}{2}\left(m^2 - 2\right) \leq l < m^2 \end{cases}$$

$$2 : \begin{cases} 0 < m < \frac{1}{\sqrt{3}} & f_{275m1} \leq l < \frac{1}{3}\left(3m^2 - 4\right) \end{cases}$$

$$3 : \begin{cases} 0 < m \leq \sqrt{\frac{2}{3}} & f_{2213m1} \leq l \leq \frac{1}{2}\left(m^2 - 2\right) \\ m > \sqrt{\frac{2}{3}} & f_{2213m1} \leq l < \frac{1}{3}\left(3m^2 - 4\right) \end{cases}$$

$$4 : \begin{cases} 0 < m \leq \frac{1}{\sqrt{3}} & l = f_{4507m1} \\ \frac{1}{\sqrt{3}} < m < .625... & l = f_{4507m1} \\ \frac{1}{\sqrt{3}} < m < .625... & l = f_{4507m2} \\ m = .625... & l = f_{4507m1} \end{cases}$$

$$5 : \begin{cases} \frac{1}{\sqrt{3}} < m < 1.56... & f_{275m1} \leq l < m^2 \\ m = 1.56... & \frac{1}{3}\left(3m^2 - 4\right) \leq l < m^2 \\ 1.56... < m < 2.48... & f_{275m1} \leq l < m^2 \\ m = 2.48... & f_{25m1} \leq l < m^2 \\ m > 2.48... & f_{275m1} \leq l < m^2 \end{cases}$$

$$6:\begin{cases} \frac{1}{\sqrt{3}} < m \leq .986... & f_{1168m1} < l < \frac{1}{3}\left(3m^2-4\right) \\ \frac{1}{\sqrt{3}} < m \leq .986... & f_{1168m2} < l < f_{25m1} \\ .986... < m < 1.11... & f_{1168m2} < l < f_{25m1} \\ m = 1.11... & 1/3(-4+3m^2) < l < f_{25m1} \\ 1.11... < m < 1.40... & f_{2213m1} \leq l < f_{25m1} \\ m \geq 2.22... & f_{25m1} < l < \frac{1}{3}\left(3m^2-4\right) \end{cases}$$

$$7:\begin{cases} \frac{1}{\sqrt{3}} < m \leq .986... & f_{1168m2} < l \leq f_{275m1} \\ .986... < m < 1.11... & f_{1168m2} < l \leq f_{275m1} \\ m = 1.11... & \frac{1}{3}\left(3m^2-4\right) < l \leq f_{275m1} \\ 1.11... < m < 1.56... & f_{2213m1} \leq l \leq f_{275m1} \\ m = 1.56... & f_{2213m1} \leq l \leq \frac{1}{3}\left(3m^2-4\right) \end{cases}$$

$$8:\begin{cases} \frac{1}{\sqrt{3}} < m < 1.40... & f_{25m1} < l \leq f_{275m1} \\ m = 1.40... & \frac{1}{3}\left(3m^2-4\right) < l \leq f_{275m1} \\ 1.40... < m < 1.56... & f_{25m1} < l \leq f_{275m1} \\ m = 1.56... & f_{25m1} < l \leq \frac{1}{3}\left(3m^2-4\right) \\ 1.56... < m \leq 2.22... & f_{25m1} < l \leq f_{275m1} \\ 2.22... < m < 2.48... & f_{2213m1} \leq l \leq f_{275m1} \\ m = 2.48... & f_{2213m1} \leq l \leq f_{25m1} \\ m > 2.48... & f_{2213m1} \leq l < f_{25m1} \end{cases}$$

$$
9:\begin{cases}
.281\ldots < m \le \frac{1}{\sqrt{3}} & f_{166360m2} < l < m^2 \\
\frac{1}{\sqrt{3}} < m < \sqrt{\frac{2}{3}} & f_{166360m1} < l < \frac{1}{3}\left(3m^2-4\right) \\
\frac{1}{\sqrt{3}} < m < \sqrt{\frac{2}{3}} & f_{166360m2} < l < f_{25m1} \\
m = \sqrt{\frac{2}{3}} & f_{166360m1} < l < \frac{1}{3}\left(3m^2-4\right) \\
m = \sqrt{\frac{2}{3}} & \frac{1}{3}\left(3m^2-4\right) < l < f_{25m1} \\
\sqrt{\frac{2}{3}} < m < .875\ldots & f_{166360m1} < l < f_{25m1} \\
m = .875\ldots & f_{2213m1} < l < f_{25m1}
\end{cases}
$$

$$
10:\begin{cases}
0 < m \le .372\ldots & f_{4507m1} < l < m^2 \\
.372\ldots < m < .423\ldots & f_{4507m1} < l < f_{150m2} \\
m = .423\ldots & f_{275m1} < l < f_{150m2} \\
.423\ldots < m \le .556\ldots & f_{4507m1} < l < f_{150m2} \\
.556\ldots < m < 1.17\ldots & f_{2213m1} \le l < f_{150m2} \\
1.17\ldots < m < 1.52\ldots & f_{2213m1} \le l < f_{150m2} \\
m = 1.52\ldots & f_{2213m1} < l < f_{150m2} \\
1.52\ldots < m < 1.52\ldots & f_{150m1} < l < f_{150m2}
\end{cases}
$$

$$11 : \begin{cases} .372... < m \leq \frac{1}{\sqrt{3}} & f_{150m2} < l < m^2 \\ \frac{1}{\sqrt{3}} < m \leq 1.09... & f_{150m2} < l < f_{25m1} \\ 1.09... < m < 1.17... & f_{150m1} < l < \frac{1}{3}\left(3m^2 - 4\right) \\ 1.09... < m < 1.17... & f_{150m2} < l < f_{25m1} \\ m = 1.17... & f_{150m1} < l < \frac{1}{3}\left(3m^2 - 4\right) \\ m = 1.17... & \frac{1}{3}\left(3m^2 - 4\right) < l < f_{25m1} \\ 1.17... < m < 1.40... & f_{150m1} < l < f_{25m1} \\ 1.40... \leq m < 1.52... & f_{150m1} < l < \frac{1}{3}\left(3m^2 - 4\right) \\ m = 1.52... & f_{2213m1} < l < \frac{1}{3}\left(3m^2 - 4\right) \end{cases}$$

$$12 : \begin{cases} m = 1.13... & f_{1168m2} < l < \frac{1}{3}\left(3m^2 - 4\right) \end{cases}$$

$$13 : \begin{cases} \frac{3\sqrt{3}}{5} < m \leq 1.09... & f_{150m2} < l \leq f_{275m1} \\ 1.09... < m < 1.13... & f_{150m2} < l \leq f_{275m1} \\ m = 1.13... & f_{150m2} < l \leq f_{275m1} \\ 1.13... < m < 1.17... & f_{150m2} < l \leq f_{275m1} \\ m = 1.17... & \frac{1}{3}\left(3m^2 - 4\right) < l \leq f_{275m1} \\ 1.17... < m < 1.52... & f_{150m1} < l \leq f_{275m1} \\ m = 1.52... & f_{2213m1} < l \leq f_{275m1} \end{cases}$$

$$14:\begin{cases} \frac{3\sqrt{3}}{5} < m \leq 1.09... & f_{1168m2} < l < f_{150m2} \\ 1.09... < m < 1.11... & f_{2213m1} \leq l < f_{150m1} \\ 1.09... < m < 1.11... & f_{1168m2} < l < f_{150m2} \\ m = 1.11... & f_{2213m1} \leq l < f_{150m1} \\ m = 1.11... & \frac{1}{3}\left(3m^2-4\right) < l < f_{150m2} \\ 1.11... < m < 1.13... & f_{2213m1} \leq l < f_{150m1} \\ 1.11... < m < 1.13... & f_{1168m2} < l < f_{150m2} \\ m = 1.13... & f_{2213m1} \leq l < f_{1168m2} \\ m = 1.13... & f_{1168m2} < l < f_{150m2} \end{cases}$$

$$15:\begin{cases} 0 < m < .423... & f_{2213m1} \leq l < f_{4507m1} \\ .423... \leq m < .556... & f_{2213m1} \leq l \leq f_{275m1} \\ .556... \leq m < \frac{1}{\sqrt{3}} & f_{4507m1} < l \leq f_{275m1} \\ m = \frac{1}{\sqrt{3}} & f_{4507m1} < l < \frac{1}{3}\left(3m^2-4\right) \\ \frac{1}{\sqrt{3}} < m < .625... & f_{4507m1} < l < f_{4507m2} \end{cases}$$

In the regions and functions provided above, there are times when m is equal to a constant value. Like in the fifth region, we have $m = 1.56....$ Sometimes in instances like this, there is no reason to have this region separated from the subregion just above and/or below where the m value provided is a strict upper or lower bound. For instance, in the fifth region when $m = 1.56...$, it appears in that subregion that we have different bounds on l than for the bounds on l on in the subregion just before and just after. However, it turns out that when $m = 1.56...$ that $f_{275m1} = \frac{1}{3}(3m^2 - 4)$, so we can actually consolidate those three subregions into one subregion in the fifth region. Simplifications like these happened many times, and we provide the calculations done in Mathematica to check this below. To do this, we used the function ***RootReduce*** in Mathematica, subtracting the ***Root*** functions with the appropriate m values plugged in. Getting a 0 as an output confirmed that we could simplify by consolidating regions in a manner just described.

```
In[7]:= RootReduce[
Root[4 - 507 Root[-324 + 692 #^2 + 336 #^4 + 45 #^6& , 2,
0]^2 + (51 -
1404 Root[-324 + 692 #^2 + 336 #^4 + 45 #^6& , 2,
0]^2) #1 + (252 -
1674 Root[-324 + 692 #^2 + 336 #^4 + 45 #^6& , 2,
0]^2) #1^2 + (594 -
972 Root[-324 + 692 #^2 + 336 #^4 + 45 #^6& , 2,
0]^2) #1^3 + (648 -
243 Root[-324 + 692 #^2 + 336 #^4 + 45 #^6& , 2, 0]^2) #1^4 +
243 #1^5 &, 2] -
Root[4 - 507 Root[-324 + 692 #^2 + 336 #^4 + 45 #^6& , 2,
0]^2 + (51 -
1404 Root[-324 + 692 #^2 + 336 #^4 + 45 #^6& , 2,
```

```
0]^2) #1 + (252 -
1674 Root[-324 + 692 #^2 + 336 #^4 + 45 #^6& , 2,
0]^2) #1^2 + (594 -
972 Root[-324 + 692 #^2 + 336 #^4 + 45 #^6& , 2,
0]^2) #1^3 + (648 -
243 Root[-324 + 692 #^2 + 336 #^4 + 45 #^6& , 2, 0]^2) #1^4 +
243 #1^5 &, 1]]

Out[7]= 0

In[8]:=
RootReduce[
Root[-2 -
75 Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& , 2,
0]^2 + (6 -
180 Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& , 2,
0]^2) #1 + (180 -
378 Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& , 2,
0]^2) #1^2 + (756 -
324 Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& , 2,
0]^2) #1^3 + (1134 -
243 Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& , 2,
0]^2) #1^4 + 486 #1^5 &, 1] -
1/3 (-4 +
3 Root[-18 - 47 #^2 + 147 #^4 - 117 #^6 + 27 #^8& , 2, 0]^2)]

Out[8]= 0

In[3]:= RootReduce[
Root[-25 Root[
32768 - 2020986 #^2 - 10826991 #^4 - 2570157 #^6 - 220725 #\
```

```
^8 + 151875 #^10& , 4,
0]^2 + (1 +
60 Root[32768 - 2020986 #^2 - 10826991 #^4 - 2570157 #\
^6 - 220725 #^8 + 151875 #^10& , 4, 0]^2) #1 + (12 -
126 Root[
32768 - 2020986 #^2 - 10826991 #^4 - 2570157 #^6 - 220725 #\
^8 + 151875 #^10& , 4, 0]^2) #1^2 + (54 +
108 Root[
32768 - 2020986 #^2 - 10826991 #^4 - 2570157 #^6 - 220725 #\
^8 + 151875 #^10& , 4, 0]^2) #1^3 + (108 -
81 Root[32768 - 2020986 #^2 - 10826991 #^4 - 2570157 #\
^6 - 220725 #^8 + 151875 #^10& , 4, 0]^2) #1^4 + 81 #1^5 &, 1] -
Root[-2 -
75 Root[32768 - 2020986 #^2 - 10826991 #^4 - 2570157 #\
^6 - 220725 #^8 + 151875 #^10& , 4,
0]^2 + (6 -
180 Root[
32768 - 2020986 #^2 - 10826991 #^4 - 2570157 #^6 - 220725 #\
^8 + 151875 #^10& , 4, 0]^2) #1 + (180 -
378 Root[
32768 - 2020986 #^2 - 10826991 #^4 - 2570157 #^6 - 220725 #\
^8 + 151875 #^10& , 4, 0]^2) #1^2 + (756 -
324 Root[
32768 - 2020986 #^2 - 10826991 #^4 - 2570157 #^6 - 220725 #\
^8 + 151875 #^10& , 4, 0]^2) #1^3 + (1134 -
243 Root[
32768 - 2020986 #^2 - 10826991 #^4 - 2570157 #^6 - 220725 #\
^8 + 151875 #^10& , 4, 0]^2) #1^4 + 486 #1^5 &, 1]]

Out[3]= 0
```

```
In[13]:= RootReduce[
Root[1 - 168 Root[-27 + 19 #^2 - 9 #^4 + 9 #^6& , 2, 0]^2 +
144 Root[-27 + 19 #^2 - 9 #^4 + 9 #^6& , 2,
0]^4 + (12 +
144 Root[-27 + 19 #^2 - 9 #^4 + 9 #^6& , 2, 0]^2) #1 + (54 +
216 Root[-27 + 19 #^2 - 9 #^4 + 9 #^6& , 2, 0]^2) #1^2 +
108 #1^3 + 81 #1^4 &, 2] -
1/3 (-4 + 3 Root[-27 + 19 #^2 - 9 #^4 + 9 #^6& , 2, 0]^2)]

Out[13]= 0



In[14]:= RootReduce[
Root[-25 Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2,
0]^2 + (1 +
60 Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2,
0]^2) #1 + (12 -
126 Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2,
0]^2) #1^2 + (54 +
108 Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2,
0]^2) #1^3 + (108 -
81 Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2,
0]^2) #1^4 + 81 #1^5 &, 1] -
1/3 (-4 +
3 Root[-27 - 82 #^2 + 192 #^4 - 126 #^6 + 27 #^8& , 2, 0]^2)]

Out[14]= 0
```

```
In[16]:= RootReduce[
Root[16 - 6360 Sqrt[2/3]^2 +
47961 Sqrt[
2/3]^4 + (216 + 8478 Sqrt[2/3]^2 +
149796 Sqrt[2/3]^4) #1 + (1161 + 54432 Sqrt[2/3]^2 +
176094 Sqrt[2/3]^4) #1^2 + (3132 + 64476 Sqrt[2/3]^2 +
92340 Sqrt[2/3]^4) #1^3 + (4374 + 29160 Sqrt[2/3]^2 +
18225 Sqrt[2/3]^4) #1^4 + (2916 + 4374 Sqrt[2/3]^2) #1^5 +
729 #1^6 &, 2] - 1/3 (-4 + 3 Sqrt[2/3]^2)]

Out[16]= 0



In[18]:= RootReduce[
Root[16 -
6360 Root[
18 - 965 #^2 - 4029 #^4 + 4293 #^6 + 3375 #^8& , 4, 0]^2 +
47961 Root[
18 - 965 #^2 - 4029 #^4 + 4293 #^6 + 3375 #^8& , 4,
0]^4 + (216 +
8478 Root[
18 - 965 #^2 - 4029 #^4 + 4293 #^6 + 3375 #^8& , 4, 0]^2 +
149796 Root[
18 - 965 #^2 - 4029 #^4 + 4293 #^6 + 3375 #^8& , 4,
0]^4) #1 + (1161 +
54432 Root[
18 - 965 #^2 - 4029 #^4 + 4293 #^6 + 3375 #^8& , 4, 0]^2 +
176094 Root[
```

```
18 - 965 #^2 - 4029 #^4 + 4293 #^6 + 3375 #^8& , 4,
0]^4) #1^2 + (3132 +
64476 Root[
18 - 965 #^2 - 4029 #^4 + 4293 #^6 + 3375 #^8& , 4, 0]^2 +
92340 Root[
18 - 965 #^2 - 4029 #^4 + 4293 #^6 + 3375 #^8& , 4,
0]^4) #1^3 + (4374 +
29160 Root[
18 - 965 #^2 - 4029 #^4 + 4293 #^6 + 3375 #^8& , 4, 0]^2 +
18225 Root[
18 - 965 #^2 - 4029 #^4 + 4293 #^6 + 3375 #^8& , 4,
0]^4) #1^4 + (2916 +
4374 Root[
18 - 965 #^2 - 4029 #^4 + 4293 #^6 + 3375 #^8& , 4,
0]^2) #1^5 + 729 #1^6 &, 1] -
Root[-2 +
213 Root[
18 - 965 #^2 - 4029 #^4 + 4293 #^6 + 3375 #^8& , 4, 0]^2 +
5112 Root[
18 - 965 #^2 - 4029 #^4 + 4293 #^6 + 3375 #^8& , 4, 0]^4 -
2160 Root[
18 - 965 #^2 - 4029 #^4 + 4293 #^6 + 3375 #^8& , 4,
0]^6 + (6 -
3780 Root[
18 - 965 #^2 - 4029 #^4 + 4293 #^6 + 3375 #^8& , 4, 0]^2 +
5616 Root[
18 - 965 #^2 - 4029 #^4 + 4293 #^6 + 3375 #^8& , 4,
0]^4) #1 + (180 -
1242 Root[
18 - 965 #^2 - 4029 #^4 + 4293 #^6 + 3375 #^8& , 4, 0]^2 -
648 Root[
```

```
18 - 965 #^2 - 4029 #^4 + 4293 #^6 + 3375 #^8& , 4,
0]^4) #1^2 + (756 +
972 Root[
18 - 965 #^2 - 4029 #^4 + 4293 #^6 + 3375 #^8& , 4,
0]^2) #1^3 + (1134 -
243 Root[
18 - 965 #^2 - 4029 #^4 + 4293 #^6 + 3375 #^8& , 4,
0]^2) #1^4 + 486 #1^5 &, 1]]

Out[18]= 0

In[19]:= RootReduce[
Root[4 - 507 Root[
32768 - 1855650 #^2 + 10014057 #^4 - 3642597 #^6 - 204525 #\
^8 + 151875 #^10& , 4,
0]^2 + (51 -
1404 Root[
32768 - 1855650 #^2 + 10014057 #^4 - 3642597 #^6 - 204525 #\
^8 + 151875 #^10& , 4, 0]^2) #1 + (252 -
1674 Root[
32768 - 1855650 #^2 + 10014057 #^4 - 3642597 #^6 - 204525 #\
^8 + 151875 #^10& , 4, 0]^2) #1^2 + (594 -
972 Root[
32768 - 1855650 #^2 + 10014057 #^4 - 3642597 #^6 - 204525 #\
^8 + 151875 #^10& , 4, 0]^2) #1^3 + (648 -
243 Root[
32768 - 1855650 #^2 + 10014057 #^4 - 3642597 #^6 - 204525 #\
^8 + 151875 #^10& , 4, 0]^2) #1^4 + 243 #1^5 &, 1] -
Root[-2 -
75 Root[32768 - 1855650 #^2 + 10014057 #^4 - 3642597 #\
^6 - 204525 #^8 + 151875 #^10& , 4,
```

```
0]^2 + (6 -
180 Root[
32768 - 1855650 #^2 + 10014057 #^4 - 3642597 #^6 - 204525 #\
^8 + 151875 #^10& , 4, 0]^2) #1 + (180 -
378 Root[
32768 - 1855650 #^2 + 10014057 #^4 - 3642597 #^6 - 204525 #\
^8 + 151875 #^10& , 4, 0]^2) #1^2 + (756 -
324 Root[
32768 - 1855650 #^2 + 10014057 #^4 - 3642597 #^6 - 204525 #\
^8 + 151875 #^10& , 4, 0]^2) #1^3 + (1134 -
243 Root[
32768 - 1855650 #^2 + 10014057 #^4 - 3642597 #^6 - 204525 #\
^8 + 151875 #^10& , 4, 0]^2) #1^4 + 486 #1^5 &, 1]]

Out[19]= 0

In[21]:=
RootReduce[
Root[-150 \
Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #^10 + 81 #\
^12& , 2, 0]^2 +
75 Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #\
^10 + 81 #^12& , 2,
0]^4 + (2 -
231 Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #\
^10 + 81 #^12& , 2, 0]^2 +
540 Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #\
^10 + 81 #^12& , 2, 0]^4) #1 + (42 +
4806 Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #\
^10 + 81 #^12& , 2, 0]^2 +
3402 Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #\
```

```
^10 + 81 #^12& , 2, 0]^4) #1^2 + (378 +
14742 Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #\
^8 - 837 #^10 + 81 #^12& , 2, 0]^2 +
8748 Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #\
^10 + 81 #^12& , 2, 0]^4) #1^3 + (1890 -
6966 Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #\
^10 + 81 #^12& , 2, 0]^2 +
19683 Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #\
^8 - 837 #^10 + 81 #^12& , 2, 0]^4) #1^4 + (5670 -
45927 Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #\
^8 - 837 #^10 + 81 #^12& , 2, 0]^2) #1^5 + (10206 -
13122 Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #\
^8 - 837 #^10 + 81 #^12& , 2, 0]^2) #1^6 + 10206 #1^7 + 4374 #1^8 &,
2] -  1/3 (-4 +
3 Root[-486 - 9 #^2 + 3053 #^4 - 4914 #^6 + 3096 #^8 - 837 #\
^10 + 81 #^12& , 2, 0]^2)]

Out[21]= 0

In[23]:= RootReduce[
Root[-2 +
213 Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4,
0]^2 + 5112 Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4,
0]^4 - 2160 Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4,
0]^6 + (6 -
3780 Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4,
0]^2 + 5616 Root[
```

```
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4,
0]^4) #1 + (180 -
1242 Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4,
0]^2 - 648 Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4,
0]^4) #1^2 + (756 +
972 Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4,
0]^2) #1^3 + (1134 -
243 Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4,
0]^2) #1^4 + 486 #1^5 &, 1] -
Root[-150 Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4,
0]^2 + 75 Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4,
0]^4 + (2 -
231 Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4,
0]^2 + 540 Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4,
0]^4) #1 + (42 +
4806 Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4,
0]^2 + 3402 Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4,
0]^4) #1^2 + (378 +
14742 Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4,
0]^2 +
```

```
8748 Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4,
0]^4) #1^3 + (1890 -
6966 Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4,
0]^2 + 19683 Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4,
0]^4) #1^4 + (5670 -
45927 Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4,
0]^2) #1^5 + (10206 -
13122 Root[
686 - 49551 #^2 - 203679 #^4 - 60129 #^6 + 68445 #^8& , 4,
0]^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1]]

Out[23]= 0



In[1]:= RootReduce[
Root[1 - 168 Root[
17179869184 - 758241767424 #^2 + 738811183104 #\
^4 - 1873773558144 #^6 + 2178248345280 #^8 + 730684767357 #\
^10 - 1168450231581 #^12 - 312846367473 #^14 + 307409258025 #^16& , 4,
0]^2 + 144 Root[
17179869184 - 758241767424 #^2 + 738811183104 #\
^4 - 1873773558144 #^6 + 2178248345280 #^8 + 730684767357 #\
^10 - 1168450231581 #^12 - 312846367473 #^14 + 307409258025 #^16& , 4,
0]^4 + (12 +
144 Root[
17179869184 - 758241767424 #^2 + 738811183104 #\
```

```
^4 - 1873773558144 #^6 + 2178248345280 #^8 + 730684767357 #\
^10 - 1168450231581 #^12 - 312846367473 #^14 + 307409258025 #^16& , 4,
0]^2) #1 + (54 +
216 Root[
17179869184 - 758241767424 #^2 + 738811183104 #\
^4 - 1873773558144 #^6 + 2178248345280 #^8 + 730684767357 #\
^10 - 1168450231581 #^12 - 312846367473 #^14 + 307409258025 #^16& , 4,
0]^2) #1^2 + 108 #1^3 + 81 #1^4 &, 2] -
Root[-150 Root[
17179869184 - 758241767424 #^2 + 738811183104 #\
^4 - 1873773558144 #^6 + 2178248345280 #^8 + 730684767357 #\
^10 - 1168450231581 #^12 - 312846367473 #^14 + 307409258025 #^16& , 4,
0]^2 + 75 Root[
17179869184 - 758241767424 #^2 + 738811183104 #\
^4 - 1873773558144 #^6 + 2178248345280 #^8 + 730684767357 #\
^10 - 1168450231581 #^12 - 312846367473 #^14 + 307409258025 #^16& , 4,
0]^4 + (2 -
231 Root[
17179869184 - 758241767424 #^2 + 738811183104 #\
^4 - 1873773558144 #^6 + 2178248345280 #^8 + 730684767357 #\
^10 - 1168450231581 #^12 - 312846367473 #^14 + 307409258025 #^16& , 4,
0]^2 + 540 Root[
17179869184 - 758241767424 #^2 + 738811183104 #\
^4 - 1873773558144 #^6 + 2178248345280 #^8 + 730684767357 #\
^10 - 1168450231581 #^12 - 312846367473 #^14 + 307409258025 #^16& , 4,
0]^4) #1 + (42 +
4806 Root[
17179869184 - 758241767424 #^2 + 738811183104 #\
^4 - 1873773558144 #^6 + 2178248345280 #^8 + 730684767357 #\
^10 - 1168450231581 #^12 - 312846367473 #^14 + 307409258025 #^16& , 4,
0]^2 + 3402 Root[
```

```
17179869184 - 758241767424 #^2 + 738811183104 #\
^4 - 1873773558144 #^6 + 2178248345280 #^8 + 730684767357 #\
^10 - 1168450231581 #^12 - 312846367473 #^14 + 307409258025 #^16& , 4,
0]^4) #1^2 + (378 +
14742 Root[
17179869184 - 758241767424 #^2 + 738811183104 #\
^4 - 1873773558144 #^6 + 2178248345280 #^8 + 730684767357 #\
^10 - 1168450231581 #^12 - 312846367473 #^14 + 307409258025 #^16& , 4,
0]^2 +
8748 Root[
17179869184 - 758241767424 #^2 + 738811183104 #\
^4 - 1873773558144 #^6 + 2178248345280 #^8 + 730684767357 #\
^10 - 1168450231581 #^12 - 312846367473 #^14 + 307409258025 #^16& , 4,
0]^4) #1^3 + (1890 -
6966 Root[
17179869184 - 758241767424 #^2 + 738811183104 #\
^4 - 1873773558144 #^6 + 2178248345280 #^8 + 730684767357 #\
^10 - 1168450231581 #^12 - 312846367473 #^14 + 307409258025 #^16& , 4,
0]^2 + 19683 Root[
17179869184 - 758241767424 #^2 + 738811183104 #\
^4 - 1873773558144 #^6 + 2178248345280 #^8 + 730684767357 #\
^10 - 1168450231581 #^12 - 312846367473 #^14 + 307409258025 #^16& , 4,
0]^4) #1^4 + (5670 -
45927 Root[
17179869184 - 758241767424 #^2 + 738811183104 #\
^4 - 1873773558144 #^6 + 2178248345280 #^8 + 730684767357 #\
^10 - 1168450231581 #^12 - 312846367473 #^14 + 307409258025 #^16& , 4,
0]^2) #1^5 + (10206 -
13122 Root[
17179869184 - 758241767424 #^2 + 738811183104 #\
^4 - 1873773558144 #^6 + 2178248345280 #^8 + 730684767357 #\
```

```
^10 - 1168450231581 #^12 - 312846367473 #^14 + 307409258025 #^16& , 4,
0]^2) #1^6 + 10206 #1^7 + 4374 #1^8 &, 1]]

Out[1]= 0



In[24]:= RootReduce[
Root[4 - 507 Sqrt[
1/3]^2 + (51 - 1404 Sqrt[1/3]^2) #1 + (252 -
1674 Sqrt[1/3]^2) #1^2 + (594 -
972 Sqrt[1/3]^2) #1^3 + (648 - 243 Sqrt[1/3]^2) #1^4 +
243 #1^5 &, 2] - 1/3 (-4 + 3 Sqrt[1/3]^2)]

Out[24]= 0

```

Using the above checks and looking for more regions that overlap among the 15 regions provided before, we have the following set of regions as our final simplification of the description of R provided in the initial output in Mathematica.

$$\begin{cases} m > 0 & \frac{1}{2}\left(m^2-2\right) \leq l < m^2 \end{cases}$$

$$\begin{cases} 0 < m < \frac{1}{\sqrt{3}} & f_{275m1} \leq l < \frac{1}{3}\left(3m^2-4\right) \end{cases}$$

$$\begin{cases} 0 < m \leq \sqrt{\frac{2}{3}} & f_{2213m1} \leq l \leq \frac{1}{2}\left(m^2-2\right) \\ m > \sqrt{\frac{2}{3}} & f_{2213m1} \leq l < \frac{1}{3}\left(3m^2-4\right) \end{cases}$$

$$\begin{cases} 0 < m \le .625\ldots & l = f_{4507m1} \\ \frac{1}{\sqrt{3}} < m < .625\ldots & l = f_{4507m2} \end{cases}$$

$$\begin{cases} m > \frac{1}{\sqrt{3}} & f_{275m1} \le l < m^2 \end{cases}$$

$$\begin{cases} \frac{1}{\sqrt{3}} < m \le .986\ldots & f_{1168m1} < l < \frac{1}{3}\left(3m^2 - 4\right) \\ \frac{1}{\sqrt{3}} < m \le 1.11\ldots & f_{1168m2} < l < f_{25m1} \\ 1.11\ldots < m < 1.40\ldots & f_{2213m1} \le l < f_{25m1} \\ m \ge 2.22\ldots & f_{25m1} < l < \frac{1}{3}\left(3m^2 - 4\right) \end{cases}$$

$$\begin{cases} \frac{1}{\sqrt{3}} < m \le 1.11\ldots & f_{1168m2} < l \le f_{275m1} \\ 1.11\ldots < m \le 1.56\ldots & f_{2213m1} \le l \le f_{275m1} \end{cases}$$

$$\begin{cases} \frac{1}{\sqrt{3}} < m \le 2.22\ldots & f_{25m1} < l \le f_{275m1} \\ m > 2.22\ldots & f_{2213m1} \le l \le f_{275m1} \end{cases}$$

$$\begin{cases} .281\ldots < m \le \frac{1}{\sqrt{3}} & f_{166360m2} < l < m^2 \\ \frac{1}{\sqrt{3}} < m \le \sqrt{\frac{2}{3}} & f_{166360m1} < l < \frac{1}{3}\left(3m^2 - 4\right) \\ \frac{1}{\sqrt{3}} < m \le \sqrt{\frac{2}{3}} & f_{166360m2} < l < f_{25m1} \\ \sqrt{\frac{2}{3}} < m \le .875\ldots & f_{166360m1} < l < f_{25m1} \end{cases}$$

$$\begin{cases} 0 < m \le .372\ldots & f_{4507m1} < l < m^2 \\ .372\ldots < m \le .556\ldots & f_{4507m1} < l < f_{150m2} \\ .556\ldots < m < 1.52\ldots & f_{2213m1} \le l < f_{150m2} \\ 1.52\ldots \le m < 1.52\ldots & f_{150m1} < l < f_{150m2} \end{cases}$$

$$\begin{cases} .372... < m \le \frac{1}{\sqrt{3}} & f_{150m2} < l < m^2 \\ \frac{1}{\sqrt{3}} < m \le 1.17... & f_{150m2} < l < f_{25m1} \\ 1.17... < m < 1.40... & f_{150m1} < l < f_{25m1} \\ 1.40... \le m \le 1.52... & f_{150m1} < l < \frac{1}{3}\left(3m^2 - 4\right) \end{cases}$$

$$\begin{cases} \frac{3\sqrt{3}}{5} < m \le 1.17... & f_{150m2} < l \le f_{275m1} \\ 1.17... < m \le 1.52... & f_{150m1} < l \le f_{275m1} \end{cases}$$

$$\begin{cases} \frac{3\sqrt{3}}{5} < m \le 1.13... & f_{1168m2} < l < f_{150m2} \\ 1.09... < m \le 1.13... & f_{2213m1} \le l < f_{150m1} \end{cases}$$

$$\begin{cases} 0 < m < .423... & f_{2213m1} \le l < f_{4507m1} \\ .423... \le m < .556... & f_{2213m1} \le l \le f_{275m1} \\ .556... \le m < \frac{1}{\sqrt{3}} & f_{4507m1} < l \le f_{275m1} \\ \frac{1}{\sqrt{3}} \le m < .625... & f_{4507m1} < l < f_{4507m2} \end{cases}$$

$ric = T$ and $ric = cT$ with $z = 0$

In the following, we provide the solution to $ric = T$ and $ric = cT$ provided that $z = 0$ (Recall that z describes the off-block diagonals in our $ad_{\mathfrak{g}_2}$ equivariant map Φ that describes our metrics). To do this, we use a combination of the Mathematica functions ***Resolve*** and ***Exists*** again. This combination seeks to find algebraic conditions in terms of the variables k and l based off the conditions provided in terms of x, y, k, l. We specify real solutions only as well.

First, we provide the solution for $ric = T$ and then we provide the solution for $ric = cT$ as discussed in Step 5 (3.3) and Step 6 (3.3), respectively. When we provide solutions to $ric = cT$, we provide two different solutions for completeness. The first solution we provide is the solution with the $z \neq 0$ case in which we use the image of $(1, \frac{r_2}{r_1}, 0)$ to find solutions and c. We use this ratio because these solutions will be using the same ratio we used when we solved $ric = cT$ with $z > 0$ (Recall that there we used $(1, \frac{r_2}{r_1}, \frac{r_3}{r_1})$). After we find these solutions, we provide the code to find the values of c.

After finding those solutions, we find a second solution which is done to cohere with the result in Theorem 3.14, recognizing that our r_1 and r_2 here are in opposite order from said result. We provide the solution from Mathematica here to help show that our approach with Mathematica coheres with the approach we used by hand, noticing that the solution we have has the same form.

```
In[14]:= t1 = Simplify[ReplaceAll[r1, z -> 0]]

Out[14]= 1/24 (9 + y^2/x^2)

In[16]:= t2 = Simplify[ReplaceAll[r2, z -> 0]]

```

```
Out[16]= -((6 x + y)/(12 x))

In[17]:= Resolve[
Exists[{x, y}, t1 - k == 0 && t2 - l == 0 && x > 0 && y > 0], Reals]

Out[17]= 1/2 + l < 0 && -(15/8) + k - 6 l - 6 l^2 == 0

In[18]:= Solve[-(15/8) + k - 6 l - 6 l^2 == 0, k]

Out[18]= {{k -> 3/8 (5 + 16 l + 16 l^2)}}


In[55]:= Resolve[
Exists[{x, y}, t2/t1 - l == 0 && x > 0 && y > 0], Reals]

Out[55]= 1/3 (-2 - Sqrt[5]) <= l < 0
```

In the following, I am finding the c value for $(r_1, r_2, 0) = c(t_1, t_2, 0)$ which we simplify into thinking of as $(r_1, r_2) = c(t_1, t_2)$. Recall from Step 6 (3.3) that the c value is $c_0\frac{1}{t_1}$ where c_0 is a solution to the implicit equation $r_1 = f_1(\frac{r_1 t_2}{t_1}, \frac{r_1 t_3}{t_1})$ where f_1 is the function that described the image of ric. Since our function describing the image of ric in the $z = 0$ setting was given above as $k = \frac{3}{8}(5 + 16l + 16l^2)$ for a point $(r_1, r_2) = (k, l)$ on the image of ric, we use k instead of f_1 here (although f_1 turns out to be the same when we let $t_3 = 0$). Thus, we are interested in getting the c values by solving for r_1 in $r_1 = \frac{3}{8}(5 + 16\frac{r_1 t_2}{t_1} + 16(\frac{r_1 t_2}{t_1})^2)$. Below this is done with $c2$ being used for r_1 and $z2$ being used for $\frac{t_2}{t_1}$. Thus, we have a description of our c_0.

```
In[56]:= Simplify[ReplaceAll[3/8 (5 + 16 l + 16 l^2), l -> c2*z2]]
```

```
Out[56]= 15/8 + 6 c2 z2 + 6 c2^2 z2^2

In[58]:= Reduce[
c2 == 15/8 + 6 c2 z2 + 6 c2^2 z2^2  && 1/3 (-2 - Sqrt[5]) <= z2 &&
z2 < 0, c2, Reals]

Out[58]= (z2 == 1/3 (-2 - Sqrt[5]) &&
c2 == (3 (1 - 2 (-2 - Sqrt[5])))/(4 (-2 - Sqrt[5])^2) - (
3 Sqrt[1 - 4 (-2 - Sqrt[5]) - (-2 - Sqrt[5])^2])/(
4 (2 + Sqrt[5])^2)) || (1/3 (-2 - Sqrt[5]) < z2 <
0 && (c2 == (1 - 6 z2)/(12 z2^2) -
1/12 Sqrt[(1 - 12 z2 - 9 z2^2)/z2^4] ||
c2 == (1 - 6 z2)/(12 z2^2) +
1/12 Sqrt[(1 - 12 z2 - 9 z2^2)/z2^4]))
```

Similarly, we get the solution utilizing the same ratio as Theorem 3.14, $\frac{t_1}{t_2}$, as opposed to $\frac{t_2}{t_1}$ since the indexing got flipped. To see how to solve for c, we use the same approach as in Step 6 (3.3) and we see that we end up with $c = c_0\frac{1}{t_2}$ with c_0 being described by $r_2\frac{t_1}{t_2}$ and $r_1 = r_2\frac{t_1}{t_2}$. Thus, if we want to get c for a given solution (t_1, t_2), then we want to solve for r_2 using the description of r_1 in terms of r_2 as before. That is, by using $k = \frac{3}{8}(5 + 16l + 16l^2)$ again, we solve for r_2 in $r_2\frac{t_1}{t_2} = \frac{3}{8}(5 + 16r_2 + 16r_2^2)$. We do this below with $c2$ used for r_2 and $z1$ used for $\frac{t_1}{t_2}$. This provides us with a description of c_0.

```
In[59]:= Resolve[
Exists[{x, y}, t1/t2 - k == 0 && x > 0 && y > 0], Reals]

Out[59]= k <= 3 (2 - Sqrt[5])

In[60]:= Reduce[
```

```
c2*z1 == 3/8 (5 + 16 c2 + 16 c2^2) && z1 <= 3 (2 - Sqrt[5]) &&
c2 < -1/2, c2, Reals]

Out[60]= (z1 <= -(3/4) &&
c2 == 1/12 (-6 + z1) - 1/12 Sqrt[-9 - 12 z1 + z1^2]) || (-(3/4) <
z1 < 6 -
3 Sqrt[5] && (c2 ==
1/12 (-6 + z1) - 1/12 Sqrt[-9 - 12 z1 + z1^2] ||
c2 == 1/12 (-6 + z1) + 1/12 Sqrt[-9 - 12 z1 + z1^2])) || (z1 ==
6 - 3 Sqrt[5] &&
c2 == -(Sqrt[5]/4) -
1/12 Sqrt[-9 - 12 (6 - 3 Sqrt[5]) + (6 - 3 Sqrt[5])^2])
```

Taking the above description of c_0 and that $c = \frac{c_0}{t_2}$, one can take the c from Theorem 3.14 and see that we have the same c. Indeed, this is done by swapping t_1 for t_2 from Theorem 3.14, letting $z1 = \frac{t_1}{t_2}$ here, and setting $d_1 = d_2 = 7$ and $p_1 = p_2 = \frac{7}{6}$ (See Remark 3.23). From there, one can see that we have the same c values.

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