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SPECTRAL THEORY OF DIRAC OPERATORS WITH MEASURES

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DEDICATION

to

My parents
Hongyu DU and Shengping Zeng

For

Supporting me to pursue my dream

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Abstract

The purpose of this thesis is to construct the spectral theory of Dirac operators with measures, create a bridge between Dirac operators and canonical systems, and discuss the de Branges spaces of a Dirac operator.

To properly interpret the Dirac equations, we invoke Jan Persson's brilliant work [1], which is relative to linear measure differential equations. The main difficulty we have to face here is discontinuity. Unlike works in [12] by Jonathan Eckhardt and Gerald Teschl, as well as [13] by Christian Remling and Ali Ben Amor, in which the second derivative guarantees the absolute continuity of solutions, this property fails when considering a first-order equation. In all, we will deal with functions of bounded variation rather than absolutely continuous functions.

In Chapter 2, we give a fundamental background, and more details can be found easily in some standard textbooks, for instance, [2,3,8,9,10]. In Chapter 3, we give an explanation of Dirac operators and discuss some properties of such an operator. After that, in chapter 4, we construct boundary conditions, self-adjoint realization, and Weyl theory as well. When assuming the limit point case at infinity, we derive the unique Weyl function which is Herglotz as the limit of Weyl circles. With this function, we obtain the spectral measure of a Dirac operator, and finally, we reach out to the spectral representation theorem. In Chapter 5, we show that Dirac operators are some special canonical systems. There, Volpert's chain rule plays an essential role, and integral should be treated carefully. In Chapter 6, we introduce de Branges spaces generated by Dirac operators, and we try showing that a Paley-Wiener space endowed with a proper inner product gives a Dirac operator with an absolutely continuous measure with respect to Lebesgue measure.

Contents

Chapter 1

Introduction

The Dirac equation is a relativistic wave equation derived by physicist Paul Dirac in 1928 to describe particle physics. In this paper, we want to investigate

$$Jf' - \mu f = g, J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

where μ is a $2{\times}2$ measure on the Borel sets of $[0, \infty)$,and $\mu = \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_2 & -\mu_1 \end{pmatrix}$ satisfying $|\mu_i|([0, N]) < \infty$ for all $N > 0$.

If this measure μ is absolutely continuous with respect to the Lebesgue measure, we just go back to a regular Dirac operator, and in that case, this equation can be interpreted easily as a regular differential equation; however, since singular measures are allowed here, a careful interpretation is indeed needed.

Let's take a Dirac-like measure for instance. Consider $\mu = \begin{pmatrix} 0 & \delta \\ \delta & 0 \end{pmatrix}$ where

δ is Dirac measure at some point x_0. On one hand, we may expect a jump point at x_0 which makes the solution discontinuous at x_0, and this gives a difficulty to determine the value of the solution at x_0; moreover, this even causes the catastrophe: the lack of existence and uniqueness of the solution when talking about initial value problem, which is necessary and automatic in classical theory. On the other hand, unlike in classical theory, we need to interpret the integral carefully: if the measure is absolutely continuous, it doesn't matter how one defines the integral on a closed interval or an open one, but this does matter when the measure is not good enough.

With a compatible interpretation that we come up with in Chapter 3 which can be used for singular measures and continuous measures with respect to the Lebesgue measure, we construct the spectral theory of Dirac operators in Chapter 4: those topics contain the construction of self-adjoint realizations by von Neumann theory and Cayley transformation, general boundary conditions, Weyl theory, and spectral representation theorem.

A canonical system is defined as follows:

$$u'(x) = zJH(x)u(x), J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

on an open interval $x \in (a, b)$, $-\infty \leq a < b \leq \infty$, where z is a complex number, and H satisfies: (1) $H \in \mathbb{R}^{2\times 2}$, (2)$H \in L^1_{loc}(a, b)$, (3) H is Hermitian and positive definite.

The first difficulty of this system comes from the definition: those systems, basically speaking, are not operators in general due to some coefficient matrices $H(x)$ which are not invertible; however, by considering relations rather than graphs in suitable Hilbert spaces, we can still construct self-adjoint realizations, boundary conditions, Weyl circles, etc. For more details, please see [2].

A well-known fact is that Jacobi and Schrödinger equations can be rewritten as canonical systems, also see [2]. We also expect to construct a bridge between canonical systems and Dirac operators, and this is the main result in Chapter 5. Some corollaries will be used in Chapter 6 as well.

People are also interested in the inverse spectral theory. In Chapter 6 We use de Branges theory to investigate this topic: we want to show that de Branges spaces generated by a Dirac operator are Paley-Wiener spaces endowed with some proper inner product, and the inverse is partially true.

Organization of Text:

In Chapter 2, we introduce some definitions, theorems, and conclusions Since those kinds of stuff are classical and can be found in standard textbooks, for example [2,3,5,8,9,10], we just present them without any proofs, and we assume readers can find them easily.

In Chapter 3, we introduce Dirac operators with measures and some basic properties we need to develop our topic. There, we focus on the work by Jan Persson [1], and Jan Persson's work will be introduced in chapter two as well. Jonathan Eckhardt and Gerald Teschl's work is also enlightening, see [12], but we don't invoke this paper there. Christian Remling and Ali Ben Amor's work [13] is also relative to schrödinger operators with measures, but the essential difficulty we need to deal with in this thesis is about discontinuity, which is totally different from the papers we mentioned above.

In Chapter 4, we construct the spectral theory of Dirac operators. In section 4.1, we describe self-adjoint restrictions by von Neuwmann's theory and depict boundary conditions of Dirac operators. In section 4.2, we construct Weyl theory to derive the Weyl function, which gives spectral information of a Dirac operator, and after that, we give the spectral representation theorem.

In Chapter 5, we construct the relation between canonical systems and Dirac operators. We conclude that Dirac operators are special canonical systems with some particular conditions.

In Chapter 6, we want to focus on the inverse problem. We first show that de Branges spaces of a Dirac operator on any intervals are Paley-Wiener spaces endowed with proper inner products, and those inner products are related to a L^1_{loc} function; inversely, any Paley-Wiener space with a proper inner product gives a Dirac operator on an interval. There, reproducing kernels and conjugate kernels of a de Branges space play an important role, and the technical point is to analyze the regularities of two integral equations originating from those kernels, so Fredholm theory and de Branges theory can be applied.

Chapter 2

General Background

In this chapter, we present a fundamental introduction to Herglotz functions, canonical systems, de Branges theory, and integral operators. we have to assume that readers have a basic background (For instance, Hilbert spaces, Lebesgue integral, holomorphic functions, measures, etc.), and we believe that readers have the ability to find out details from standard textbooks, for example, [2,3,5,8,9,10].

2.1 Herglotz Functions

Herglotz functions, sometimes called Nevanlinna functions, play a significant role when it comes to the spectrum of Dirac operators with measures, especially, the Weyl function of a Dirac operator is Herglotz, and this function gives the spectral measure of the operator. We introduce this topic briefly here without any proofs, for readers who are interested in this topic, please see [2],[5],[6].

We call a function (generalized) Herglotz if it is holomorphic from $\mathbb{C}^+$ to ($\overline{\mathbb{C}^+}$) $\mathbb{C}+$. It is well known that a generalized Herglotz function F has the Herglotz representation

$$F(z) = a + bz + \int_{\mathbb{R}} (\frac{1}{t-z} - \frac{t}{t^2+1}) d\rho(t)$$

where a in $\mathbb{R}$, $b \geq 0$, and ρ a positive Borel measure on $\mathbb{R}$ with $\int_{\mathbb{R}} \frac{d\rho(t)}{t^2+1} < \infty$.

Remark. F is a generalized Herglotz function if either it is a Herglotz function or $F = a \in \mathbb{R}_\infty$

Moreover, the triple (a, b, ρ) can be realized from a Herglotz function as follows:

$$a = Re(F(i)), b = \lim_{z \to \infty} \frac{F(z)}{z}$$

with $Im(z) > \epsilon > 0$,

$$d\rho(t) = \frac{1}{\pi} w^* - \lim_{y \to 0+} ImF(t+iy)dt$$

in the sense of weak* convergence.

Sometimes, it is useful to rewrite a Herglotz function as follows:

$$F(z) = a + \int_{\mathbb{R}_\infty} \frac{1+tz}{t-z} dv(t)$$

where $dv(t) = \frac{1}{1+t^2} d\rho(t) + b\delta_\infty$.

The advantage of this form is that v is finite on the compact space $\mathbb{R}_\infty$.

Weyl functions, also known as Weyl-Titchmarsh functions, can be described as Herglotz functions, see [2] and chapter 4 in this thesis. The measure in the Herglotz representation is indeed the spectral measure of the corresponding operator in the spectral representation theorem, also see [2]. Hence, we want to depict supports of the absolutely continuous part and the singular continuous part of the measure in the triple.

Let

$$\Sigma_{ac} = \{t \in \mathbb{R} : 0 < \lim_{y \to 0+} ImF(t + iy) < \infty\}$$

$$\Sigma_s = \{t \in \mathbb{R} : \lim_{y \to 0+} ImF(t + iy) = \infty\}$$

then $\sigma_{ac}(\rho) = \overline{\Sigma_{ac}}^{ess}$, and Σ_s is a support for the singularly continuous part of ρ.

We are also interested in the convergence of a sequence of Herglotz functions. Let $\mathcal{F}$ be the set of all generalized Herglotz functions with the topology of locally uniformly convergence. This space is metrizable, but we don't need this metric here. The following theorems show the connection between the convergence of Herglotz functions and the convergence of those triples (a, b, ρ) or equivalently, (a, v).

Theorem 2.1 $\mathcal{F}$ is compact.

Theorem 2.2 Let $F_n, F \in \mathcal{F} \setminus \{\infty\}$. Then

(1) $F_n \to F$ if and only if $a_n \to a$ and $v_n \to v$ in weak* sense;

(2) $F_n \to \infty$ if and only if $|a_n| + v_n(\mathbb{R}_\infty) \to \infty$

2.2 Canonical Systems

Canonical systems are differential equations that generalize some famous differential equations such as Dirac equations, Jacobi equations, and schrödinger equations.

Precisely, a canonical system is defined as follows:

$$u'(x) = zJH(x)u(x)$$

on an open interval $x \in (a,b)$, $-\infty \leq a < b \leq \infty$, where z is a complex number, and H satisfies: (1) $H \in \mathbb{R}^{2\times 2}$, (2) $H \in L^1_{loc}(a,b)$, (3) H is Hermitian and non-negative definite for (Lebesgue) almost all $x \in (a,b)$.

We denote the collection of all canonical systems on $(0,N)((0,\infty))$ by $C(N)(C)$.

The proper Hilbert space when talking about a canonical system is not $L^2(a,b)$ anymore, but a space called $L^2_H(a,b)$ instead.

Let's define

$$\mathcal{L} = \{f : (a,b) \to \mathbb{C}^2 : f \textit{ (Borel) measurable, } \int_a^b f^* H f < \infty\}$$

with norm $||f|| = (\int_a^b f^* H f)^{\frac{1}{2}}$.

The Hilbert space is defined as the quotient

$$L^2_H(a,b) := \mathcal{L}/\mathcal{N}$$

where $\mathcal{N} = \{f \in \mathcal{L} : ||f|| = 0\}$.

By Weyl theory, it is well known that under the assumption of the limit point case at ∞, if f denotes the (unique, up to a factor) non-trivial $L^2_H(0,\infty)$ solution of $u'(x) = zJH(x)u(x)$ on $[0,\infty)$, then the Weyl function is given by

$$m(z) = \frac{f_1(0,z)}{f_2(0,z)}$$

Moreover, Weyl functions are Herglotz, See [2] for more details.

In the sequel, we always denote by u, v the solution of

$$u'(x) = zJH(x)u(x)$$

satisfying $u(a,z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v(a,z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively when we talk about canonical systems on (a,b).

Analogously as we mentioned above, Weyl functions are important because they contain spectral information about canonical systems. Roughly speaking, if we assume the limit point case at ∞, let

$$Uf = \int_0^\infty u^*(s,t)H(s)f(s)ds, f \in \underset{N>0}{\cup} L_H^2(0,N)$$

$$Uf = \lim_{N\to\infty} U(\chi_{[0,N]}f), f \in L_H^2(0,\infty)$$

define a unitary map $U : L_H^2(0,\infty) \longrightarrow L^2(\mathbb{R},\rho)$ (here, limit is norm limit in $L^2(\mathbb{R},\rho)$), where ρ is the measure from the Herglotz representation of the Weyl function m given above, then this map together with the spectral measure ρ provides a spectral representation.

If the coefficient H in a canonical system is a constant matrix up to a function on $\mathbb{R}$, we then can anticipate solving the equation directly, and we may "delete" those parts from H to simplify the coefficient. We call this scenario singular.

A point $x \in (a,b)$ is called singular if there is $\delta > 0$ and is a vector $v \neq 0 \in \mathbb{R}^2$ such that $H(t)v = 0$ for almost all $|t-x| < \delta$. A non-singular point is called regular.

Obviously, the set of all singular points is open, hence is the union of open intervals, and we call those connected components singular intervals.

Now, we can describe Weyl functions more adequately: theorem 2.3 below says the value of the coefficient b in the Herglotz representation of the Weyl function is relative to the length of the first singular interval of a given type, theorem 2.4 and 2.5 depict the spectral measure when it comes to singular intervals (regular points). See [2] for details.

Theorem 2.3 Consider a canonical system on $[0, \infty)$. Assume the limit point case at ∞, and let m be the Weyl function. The coefficient in the Herglotz representation of m, $b > 0$ if and only if $(0, \infty)$ starts with a singular interval of type e_2.

Theorem 2.4 Under the assumptions of theorem 2.3. Then $\rho(\mathbb{R}) < \infty$ if and only if:
(1) $(0, \infty)$ starts with a singular interval of type $e_\alpha \neq e_2$, or
(2) $(0, \infty)$ starts with a singular interval of type e_2, immediately followed by a second singular interval.

Theorem 2.5 Under the assumptions of theorem 2.3. The spectral measure is compactly supported if and only if the number of regular points on any finite interval is finite.

2.3 De Branges Functions and Spaces

De Branges theory was developed first by de Branges in his four papers [21,22,23,24], this theory can be applied to the inverse spectral theory since the unitary map U in the spectral representation actually provides an isometry between a de Branges space and the Hilbert space, see [2].

A de Branges function is an entire function E such that $|E(z)| > |E^{\#}(z)|$ for $z \in \mathbb{C}^+$. Here, $E^{\#}(z) = \overline{E(\overline{z})}$. The de Branges space of E is defined as

$$B(E) := \{F : F \ entire, \ \frac{F}{E}, \frac{F^{\#}}{E} \in H^2\}$$

where $H^2 = H^2(\mathbb{C}^+)$ is the Hardy space on the upper half plane.

One of the most important observations is that, if E is a de Branges function, then $B(E)$ is a Hilbert space with the inner product

$$[F, G] = \frac{1}{\pi} \int_{\mathbb{R}} \overline{F}(t) G(t) \frac{dt}{|E(t)|^2}$$

Moreover, the reproducing kernels

$$J_w(z) = \frac{\overline{E}(w)E(z) - \overline{E^{\#}}(w)E^{\#}(z)}{2i(\overline{w} - z)}$$

are in $B(E)$, and $[J_w, F] = F(w)$ for all $F \in B(E), w \in \mathbb{C}$.

De Branges functions can be determined by de Branges spaces to some degree, that is, let E_1, E_2 be de Branges functions. $B(E_1) = B(E_2)$ if and only if

$$\begin{pmatrix} ReE_2 \\ ImE_2 \end{pmatrix} = M \begin{pmatrix} ReE_1 \\ ImE_1 \end{pmatrix}$$

for some $M \in SL(2, \mathbb{R})$.

Here, $B(E_1) = B(E_2)$ means they share the same functions and are isometrically equal to one another as Hilbert spaces.

In L. de Branges's four brilliant papers, he came up with some profound theorems, including the characterization of de Branges spaces, the ordering theorem, connection with canonical systems, etc., see [2,16,17] for details, we just state those results without proofs, and readers may omit the rest part of this section on a first reading until Chapter 6.

Theorem 2.6 characterize a Hilbert space as a de Branges space.

Theorem 2.6 Let H be a Hilbert space whose elements are entire functions. Assume that:

1) For every $z \in \mathbb{C}$, point evaluation $z(F) := F(z) \in H^*$;

2) If $F \in H$ with $F(w) = 0$, then $G(z) = \frac{z-\overline{w}}{z-w} F(z) \in H$ and $||F|| = ||G||$;

3) $F \to F^{\#}$ is isometric on H.

Then $H = B(E)$ for some de Branges function E.

conversely, if $B(E)$ is a de Branges space, then it satisfies those assumptions above.

If we give an extra condition, we even have an order among different de Branges spaces (Theorem 2.7). We call a de Branges space $B(E)$ regular, if for all $z_0 \in \mathbb{C}$,

$$F \in B(E) \Rightarrow S_{z_0} F(z) := \frac{F(z) - F(z_0)}{z - z_0} \in B(E)$$

where $S_{z_0}F(z_0) = \lim_{z\to z_0} S_{z_0}F(z)$.

If we have two de Brange spaces $B(E_1), B(E_2)$, then we say $B(E_1) \subset B(E_2)$ if $B(E_1)$ be isometrically contained in $B(E_2)$.

Theorem 2.7(the Ordering Theorem) Let $B(E), B(E_1), B(E_2)$ be regular de Branges spaces and $B(E_1), B(E_2) \subset B(E)$, then either $B(E_1) \subset B(E_2)$ or $B(E_2) \subset B(E_1)$.

There is a natural connection between regular de Branges spaces and canonical systems, i.e., as mentioned above, the spectral representation gives a de Branges space that is regular; and we can recover a canonical system from a de Branges space as follows:

Theorem 2.8 If $B(E)$ is a regular de Branges space, $E(0) = 1$ and $N > 0$, then there is a coefficient $H(x)$ of some canonical system on $(0, N)$ such that $E(z) = u_1(N, z) - iu_2(N, z)$. Moreover, H can be chosen so that trH is a positive constant.

Now, we turn to the type of an entire function. An entire function F is said to be of exponential type if $|F(z)| \leq C(\tau)e^{\tau|z|}, (z \in \mathbb{C})$ for some $\tau > 0$. the infimum of the $\tau > 0$, denoted by $\tau(F)$, is called the type of F. If we consider a canonical system on $[0, N]$ with $H \in L^1(0, N)$, then the type of $E_N(z) = u_1(N, z) - iu_2(N, z)$ is given by

$$\tau_N(E) = \int_0^N \sqrt{detH(x)}dx$$

The next theorem contains information about reproducing kernels and conjugate kernels. Roughly speaking, those kernels can be treated as a basis of a Hilbert space.

Theorem 2.9 Let H be a canonical system on $(0, N)$ defined in the section 2.2, then the space

$$B(E_N) := \{F(z) = \int_0^N u^*(x, \overline{z})H(x)f(x)dx : f \in L^2_H(0, N)\}$$

is a regular de Branges space with $E_N(z) = u_1(N, z) - iu_2(N, z)$.

1) The reproducing kernels are given by

$$J_w(z) = \int_0^N u^*(x, w)H(x)u(x, z)dx$$

2) The conjugate kernels, $K_w(z) := \frac{v^*(N,w)Ju(N,z)-1}{z-\overline{w}}$, are given by

$$K_w(z) = \int_0^N v^*(x, w)H(x)u(x, z)dx = \int_0^N u^*(x, \overline{z})H(x)v(x, \overline{w})dx$$

hence $K_w(z) \in B(E_N)$.

3) Define $\widetilde{F}(z) := [K_z, F]$ for $F(z) = \int_0^N u^*(x, \overline{z})H(x)f(x)dx$, then $\widetilde{F}(z) = \int_0^N v^*(x, \overline{z})H(x)f(x)dx$. Moreover, assume $0 < N_1 < N_2$ and $F \in B(E_{N_1})$, then

$$[K_z^{(N_1)}, F]_{B(E_{N_1})} = [K_z^{(N_2)}, F]_{B(E_{N_2})}$$

4) For all $F, G \in B(E_N)$,we have

$$\widetilde{F}(0)\overline{G(0)} - F(0)\overline{\widetilde{G}(0)} = [S_0G, F] - [G, S_0F]$$

5) The space $B_\tau := \{F \in B(E_N) : \tau(F) \leq \tau_N(E)\}$ is also a de Branges space and $B_\tau = B(E_a)$, where $a = \max\{x \in \mathbb{R} : \tau_x(E) \leq \tau_N(E)\}$.

Paley-Wiener theorem is not a main topic in de Branges theory, but it is strongly related to topics we are interested in in this thesis, so we put it here to emphasize its importance.

Theorem 2.10 (Paley-Wiener theorem) Let F be an entire function, then the following is equivalent:
(1) $F = \int_{\mathbb{R}} f(s)e^{izs}ds$ for some function $f \in L^2(-L, L)$;
(2) $|F(z)| \leq C(L)e^{L|z|}, (z \in \mathbb{C})$ for some constant $C(L) > 0$ and $F(t) \in L^2(\mathbb{R})$.

2.4 Integral Operators

Integral operator theory is a large topic in analysis, we also need to deal with some operators by Fredholm theory.

Young's inequality for integral operators is broadly used. We assume X, Y are measurable spaces, and $K : X \times Y \to \mathbb{R}$ is measurable. Let $p, q, r \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$. If

$$(\int_X |K(x, y)|^p dx)^{\frac{1}{p}} \leq C$$

and

$$(\int_Y |K(x,y)|^p dy)^{\frac{1}{p}} \leq C$$

for all $x \in X$ and all $y \in Y$ respectively,

then

$$(\int_X |\int_Y K(x,y)f(y)dy|^r dx)^{\frac{1}{r}} \leq C(\int_Y |f(y)|^q dy)^{\frac{1}{q}}$$

Especially, if the kernel is given by a function $f(x-y)$, we have Young's Convolution Inequality

$$||f * g||_r \leq ||f||_p ||g||_q$$

Young's inequality can be used for Hilbert-Schmidt integral operators, that is, let M_1, M_2 be measurable subsets of $\mathbb{R}^p, \mathbb{R}^q$ respectively, and $K \in L^2(M_2 \times M_1)$, the operator

$$T : L^2(M_1) \to L^2(M_1)$$

$$Tf(x) = \int_{M_1} K(x,y)f(y)dy$$

One of the most famous results is that Hilbert-Schmidt integral operators are compact. See [10].

We always need to estimate the upper bound of a solution, hence Gronwall's Inequality is helpful. Suppose f, g are non-negative functions on

$[0, L]$, with f continuous and $g \in L^1(0, L)$. If

$$f(x) \leq a + \int_0^x g(t)f(t)dt$$

then

$$f(x) \leq ae^{\int_0^x g(t)dt}$$

Abstractly, the compactness of an operator is significant once we want to know the existence and the uniqueness of solutions, so we invoke Fredholm theory here. Let X, Y be Banach spaces and $T : X \to Y$ a bounded linear operator. T is called Fredholm if its kernel $\ker(T)$ and cokernel $coker(T) = Y \setminus Ran(T)$ are finitely dimensional and $Ran(T)$ is closed (the condition about the range is actually redundant). The index is defined by $ind(T) := dim \ker(T) - dimcoker(T)$.

Theorem 2.11 Let X, Y be Banach spaces, and $T : X \to Y$ compact, then $1 + T$ is Fredholm with $ind(1 + T) = 0$, i.e., $dimker(1 + T) = dim(Y \setminus Ran(1 + K))$.

Chapter 3

Dirac Operators with Measures

In this chapter, we discuss the general definition of a Dirac operator with a measure as the coefficient which is related to the differential equation $Jf' - \mu f = g$, and investigate some basic properties we need in the future.

As we introduced in Chapter 1, the equation above should be interpreted carefully because of the potential discontinuity of the solution (the measure). Due to the form of this equation, we can intuitively say that the solution is of bounded variation because the derivative (in the sense of distribution) is a measure. In section 3.1, we introduce (complex) functions of bounded variation and three theorems by Jan Persson; in section 3.2, we give a compatible definition of a Dirac operator, and we investigate some useful topics such as the transfer matrix, variation of constants and whether the operator is densely defined, etc.

3.1 Basic Concepts

Let N be a positive number. We call a function $f \in L^1[0,N]$ from $\mathbb{R}$ to $\mathbb{C}$ of bounded variation (on $[0,N]$), if the total variation of f, defined by

$$V_0^N(f) := \sup\{\int_0^N |f(t)|\phi'(t)dt : \phi \in C_c^1([0,N],\mathbb{R}), ||\phi||_{L^\infty} \leq 1\}$$

is finite. And we use the notation $BV[0,N]$ to represent the collection of all bounded variation functions on $[0,N]$, i.e.,

$$BV[0,N] := \{f \in L^1[0,N] : V_0^N(f) < \infty\}$$

This definition is equivalent to that the real part and the imaginary part of f are of bounded variation in the sense of real functions.

If $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ is a function from $\mathbb{R}$ to $\mathbb{C} \times \mathbb{C}$, we call f is of bounded variation (on $[0,N]$) if f_1 and f_2 are in $BV[0,N]$, and we also use the same notation $BV[0,N]$, i.e.,

$$BV[0,N] := \{f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in L^1[0,N] : V_0^N(f_i) < \infty, i = 1,2\}$$

We also define the total variation of f by

$$V_0^N(f) := \max_{i=1,2}(V_0^N(f_i))$$

We also need to consider the half-line problem. We define the space of all locally bounded variation functions by

$$BV[0,\infty) := \{f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in L^1_{loc}[0,\infty) : V_0^N(f) < \infty \;\; \textit{for all } N > 0\}$$

Analogously, we say a matrix is in $BV[0,N](BV[0,\infty))$ if all entries are in $BV[0,N](BV[0,\infty))$.

Remark. Even though we use the same notation for different categories, there is no confusion: all components are of (locally) bounded variation. People may be interested in the Tonelli-like, pointwise definition, i.e., the definition containing the sum of differences. The equivalence of those two definitions under some conditions is a difficult topic, and we don't want to discuss it, for readers who want to know more about this topic, see [chapter 7, 15] by Giovanni Leoni.

Let μ be a 2×2 signed Borel measure on $[0,\infty)$ of the form $\begin{pmatrix} \mu_1 & \mu_2 \\ \mu_2 & -\mu_1 \end{pmatrix}$, we define the set of such measures as follows:

$$DS := \{\mu : (1) \max_{i=1,2}(|\mu_i|([0,N]) < \infty \; \textit{for all } N > 0; \;\; (2)\; \mu(\{0\}) = 0\}$$

Here, condition (2) is not essential, and we require this normalization just to avoid discussing the left limit of the solution at 0, as we will see later; but of course, there is no technical difficulty if we remove this condition. On the other hand, condition (1) is essential because it gives a complex measure when considering the cut-off of a measure from DS so that we can apply Jan Persson's theorem; moreover, this condition also implies that there are only countably many points in any compact subset of $[0,\infty)$ such that $\mu\{x\} \neq 0$ at those points. Here, and in the sequel, to avoid too many notations, we will write $\mu\{x\}(\mu(a,b))$ rather than $\mu(\{x\})(\mu((a,b)))$ if there is no confusion.

Given $\mu \in DS$, it is also convenient to define the set of all jump points of μ:

$$S(\mu) := \{x \in (0,\infty) : \mu\{x\} \neq 0\}$$

Assume $\mu \in DS$ and $f \in L^1_{loc}(\mu)$,we interpret integral $\int_a^x d\mu f$ as follows:

$$\int_a^x d\mu f = \begin{cases} \int_{(a,x]} d\mu f & x \geq a \\ -\int_{(x,a]} d\mu f & x < a \end{cases} \tag{3.1}$$

Recall that if f, h are functions of locally bounded variation from $\mathbb{R}$ to $\mathbb{C}$, then integration by parts is given by

$$\int_{[a,b]} f(x+)dh + \int_{[a,b]} df h(x-) = f(b+)h(b+) - f(a-)h(a-)$$

here, d means the relevant Lebesgue–Stieltjes measure associated with the right-continuous representation of the function. See [3,(21.68)] for example. (Even though the formula is not for vector functions in [3], it's not hard to get our version from the original one.)

We still need some notations with respect to matrices. Assume $D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} \in \mathbb{C}^{2\times 2}$, the supremum norm of D is defined by $||D|| = \max_{i=1,2,3,4}(|D_i|)$. This norm is equivalent to the spectral norm $||D||_2$, which is defined as the largest singular value, more precisely, we have $||D|| \leq ||D||_2 \leq 2||D||$. We also observe that the supremum norm is not sub-multiplicative: $||D_1 D_2|| \leq 2||D_1|| \cdot ||D_2||$.

We introduce a function, denoted by g, from $\mathbb{C}^{2\times 2}$ to $\mathbb{C}^{2\times 2}$:

$$g(D) := \sum_{n=1}^{\infty} \frac{D^{n-1}}{n!}$$

Especially, recall an important constant matrix $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Now, we are ready to state Jan Persson's work, see [1] for the general situation.

Theorem 1.[1] Let A be a 2×2 complex Borel measure on the real line and I the identity matrix. Let k be a 2×1 complex Borel measure on the real line.

If $A\{x\} + I$ is invertible for all $x \in \mathbb{R}$, then to each choice of $C \in \mathbb{C}^2$, there is a unique solution f, which is of locally bounded variation and right continuous, of

$$f(x) = C - \int_0^x dA(t)f + \int_0^x dk(t), \ x \geq 0$$

and

$$f(x) = C + \int_0^x dA(t)f - \int_0^x dk(t), \ x < 0$$

Here, the integral should be explained as (3.1).

The approximation below plays a significant role in the sequel.

Let A and k be as in the hypothesis of Theorem 1. Let $\phi \in C(\mathbb{R})$ with $\phi \geq 0$, $\int \phi(t)dt = 1$, and $supp\phi \subseteq [-1, 1]$. Let $\epsilon > 0$ and define $\phi_\epsilon(x) := \frac{\phi(\frac{x}{\epsilon}+1)}{\epsilon}$. Let

$$A_\epsilon(x) := \int_R \phi_\epsilon(x-t)dA(t) \tag{3.2}$$

and

$$k_\epsilon(x) := \int_R \phi_\epsilon(x-t)dk(t) \tag{3.3}$$

Theorem 2.[1] Let A be a 2×2 complex Borel measure on the real line and k a 2×1 complex Borel measure on the real line. Let $\epsilon > 0$, $C \in \mathbb{C}^2$ and let f_ϵ be the solution of

$$f_\epsilon(x) = C - \int_0^x A_\epsilon(t)f dt + \int_0^x k_\epsilon(t)dt \tag{3.4}$$

Then the family $\{f_\epsilon : 0 < \epsilon < 1\}$ is uniformly bounded under the supreme norm of a vector in any compact subset of $\mathbb{R}$.

Theorem 3.[**1**] Under the assumptions of theorem 2. As $\epsilon \to 0$, the family $\{f_\epsilon : 0 < \epsilon < 1\}$ converges pointwisely to the unique solution f of

$$f(x) = C - \int_0^x g(A\{t\})dA(t)f + \int_0^x g(A\{t\})dk(t),\ x \geq 0$$

and

$$f(x) = C + \int_0^x g(A\{t\})dA(t)f - \int_0^x g(A\{t\})dk(t),\ x < 0$$

We need to recall some facts from complex functions. Also see Chapter 1 for more details.

Given a generalized Herglotz function F, we have the Herglotz representation

$$F(z) = a + bz + \int_{\mathbb{R}} (\frac{1}{t-z} - \frac{t}{t^2+1})d\rho(t) \tag{3.5}$$

where a in $\mathbb{R}$, $b \geq 0$, and ρ a positive Borel measure on $\mathbb{R}$ with $\int_{\mathbb{R}} \frac{d\rho(t)}{t^2+1} < \infty$.

we introduce Mobius transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}, z \in \mathbb{C}_\infty$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

The next theorem is useful.

Theorem 4.[2] Let $A \in SL(2, \mathbb{C})$. Then the Mobius transformation $A : z \to Az$ is Herglotz if and only if $i(J - A^*JA) \geq 0$.

3.2 the Properties of Dirac Operators

Let $\mu \in DS$, we want to interpret the differential expression $Jf' - \mu f$ on $[0, \infty)$.

Fix $a > 0$, define

$$Af(x) := Jf(x) - \int_a^x g(\mu\{s\}J)d\mu f$$

if $f \in BV[0, \infty)$ is right continuous.

This definition makes sense. For $f \in BV[0, \infty)$, we know the existence of the left limit and the right limit everywhere, hence, even if this function f is in L^1_{loc}, the right continuous representation is subsistent and unique. Moreover, Af is also right continuous. Now, we introduce an operator T:

$$D(T) := \{f \in L^2[0, \infty) : f \in BV[0, \infty) \textit{ and right continuous,}$$

$$Af \in AC[0, \infty), (Af)' \in L^2[0, \infty)\}$$

$$Tf = -(Af)'$$

As expected, we say $f \in AC[0, \infty)$ if $f \in AC[0, b]$ for all $b > 0$, i.e, f is locally absolutely continuous. Here, even though we have a constant a, the choice of $a > 0$ is irrelevant to the definition of T.

Moreover, let $k \in L^2[0, \infty)$, and we consider the following integral equation:

$$Jf(x) - \int_a^x g(\mu\{s\}J)d\mu f = C - \int_a^x kdt \tag{3.6}$$

where C is a constant.

This equation implies that $f(x)$ is determined by values of f between a and x. For $N > 0$, we define $\widetilde{k}(x) = \chi_{[0,N]}(x)k(x)$ and $\widetilde{\mu} = \mu$ on $[0, N]$ and $\widetilde{\mu} = 0$ outside, then Jan Persson's Theorem 1. gives the existence and uniqueness of the solution of (3.6) when substituting k and μ by $\widetilde{k}$ and the complex measure $\widetilde{\mu}$, it follows that T is indeed an operator. Moreover, since the solution is of locally bounded variation and right continuous, we conclude that f can be extended to 0. This fact also indicates that it is reasonable in the definition of $D(T)$ to assume $f \in BV[0, \infty)$ rather than $f \in BV(0, \infty)$. For convenience, in the sequel, we always assume $a = 0$.

We are also interested in $Jf' - \mu f$ on the interval $[0, N]$. The operator T_N is defined as follows:

$$D(T_N) := \{f \in L^2[0, N] : f \in BV[0, N] \textit{ and right continuous,}$$

$$Af \in AC[0, N], (Af)' \in L^2[0, N]\}$$

$$T_N f = -(Af)'$$

Here, as we mentioned before, k and μ should be treated as $\widetilde{k}$ and the complex measure $\widetilde{\mu}$ in (3.6), and f is the cut-off of the solution of (3.6) on $[0, N]$.

Claim 3.1 Let $f \in D(T_N)(D(T))$, then $f(x-) = e^{J\mu\{x\}}f(x)$. As consequences, f is continuous at $\mu\{x\} = 0$ and $f^*(x-) = f^*(x)e^{-\mu\{x\}J}$.

Proof: As $Af \in AC[0, N]$, we have $\lim_{y \to x^-} Af(y) = Af(x)$, then it follows that $Jf(x-) = Jf(x) + g(\mu\{x\}J)\mu\{x\}f(x) = Je^{J\mu\{x\}}f(x)$ ∎

This observation also gives a way to define $f(0-)$ if necessary, and by choice of our measure, it is natural to assume $f(0-) := f(0)$.

We introduce a useful tool called the transfer matrix of the operator $T_N(T)$. Basically speaking, the transfer matrix T is a 2×2 matrix-valued solution of

$$JT(x) - \int_0^x g(\mu\{s\}J)d\mu T = I$$

where I is the identity.

Obviously, $T(0) = I$. we sometimes need to write T down explicitly as $T(x) = (u(x), v(x)) = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$, where u and v are vector-valued solutions of the integral equation satisfying $u(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively. Of course, u and v are right continuous and of locally bounded variation by definition.

Remark. We can define the transfer matrix T_ϵ analogously for (3.4) if we let $k = 0$ and $A = J\mu$.

Claim 3.2 $det(T(x)) = 1$ for $x \in [0, \infty)$.

Proof: We observe that (3.4) is equivalent to a regular differential equation, and it's easy to show that $det(T_\epsilon(x)) = 1$. It follows from Theorem 3. that $det(T(x)) = \lim_{\epsilon \to 0} det(T_\epsilon(x)) = 1$. ∎

In the sequel, we briefly denote $g(\mu\{s\}J)$ by g in the integral for convenience if there is no confusion, i.e., if there is a "g" in the integral, then the prior recognition is $g(\mu\{s\}J)$.

Claim 3.3(Variation of Constants). Let $f \in D(T_N)(D(T))$ and $T_N f = k(Tf = k)$. Assume $f(0) = C$, then

$$f(x) = T(x)C + T(x)\int_0^x T^{-1}Jkdt \tag{3.7}$$

Proof: Let's define $C(x) := T^{-1}(x)f(x)$ with $C(0) = C$, which is of (locally) bounded variation.

We have

$$JT(x)C(x) - \int_0^x gd\mu T(s)C(s) = JC - \int_0^x kdt \tag{3.8}$$

We claim that $C(x)$ is continuous. Indeed, $T^{-1} = \begin{pmatrix} v_2 & -v_1 \\ -u_2 & u_1 \end{pmatrix}$ is right continuous, and as a product of two right continuous functions, $C(x)$ is also right continuous. On the other hand, by claim 3.1, we have $T(x-) = e^{J\mu\{x\}}T(x)$, then it follows that

$$C(x-) = T^{-1}(x)e^{-J\mu\{x\}}e^{J\mu\{x\}}f(x) = C(x)$$

Hence,

$\int_0^x gd\mu T(s)C(s)$

$$
\begin{aligned}
&= (\int_0^x gd\mu T(s))C(x) - \int_{[0,x]}(\int_0^{s-} gd\mu T)dC(s) \\
&= (\int_0^x gd\mu T)C(x) - \int_{[0,x]}(\int_0^s gd\mu T)dC + \int_{[0,x]} g\mu\{s\}TdC
\end{aligned}
$$

Recall that $\mu\{s\} = 0$ except for countably many points; moreover, the continuity of C implies $C\{\xi\} = C(\xi) - C(\xi-) = 0$ as a measure, it follows that $\int_{[0,x]} g\mu\{s\}TdC = 0$. Hence, we have

$JT(x)C(x) - \int_0^x gd\mu T(s)C(s)$

$$
\begin{aligned}
&= (JT(x) - \int_0^x gd\mu T)C(x) + \int_{[0,x]}(\int_0^s gd\mu T)dC \\
&= JC(x) + \int_{[0,x]}(\int_0^s gd\mu T)dC \\
&= J(\int_0^x dC + C(0)) + \int_0^x(\int_0^s gd\mu T)dC \\
&= JC(0) + \int_0^x(\int_0^s gd\mu T + J)dC \\
&= JC(0) + \int_0^x JTdC
\end{aligned}
$$

Now, it follows from (3.8) that

$$\int_0^x TdC = J\int_0^x kdt$$

Hence, by the approximation of C_c^∞ test functions,

$$C(x) - C = \int_0^x dC = \int_0^x T^{-1}TdC = \int_0^x T^{-1}Jkdt$$

This identity implies (3.7). ■

Our purpose is to construct spectral theory for T, hence we expect this operator to be densely defined so that the adjoint T^* makes sense.

Claim 3.4 $\overline{D(T_N)} = L^2[0,N]$, $\overline{D(T)} = L^2[0,\infty)$

Proof: Let $f \in L^2[0,N]$, we have $T^{-1}f \in L^2[0,N]$ as T^{-1} is bounded on $[0,N]$ under the supremum norm. We can pick up a sequence $\{C_n\}_{n=1}^\infty \subset C_0^\infty(0,N)$ such that $C_n \xrightarrow{L^2} T^{-1}f$, then it follows that $TC_n \xrightarrow{L^2} f$.
We define $k_n(x) := -JT(x)C_n'(x)$, then $k_n \in L^2[0,N]$ since C_n' is also bounded. Moreover, $C_n(x) = \int_0^x T^{-1}Jk_n dt$.
Now, we consider

$$Jf_n(x) - \int_0^x g(\mu\{s\}J)d\mu f_n = -\int_0^x k_n dt$$

Theorem 1. shows that $f_n \in D(T_N)$, and (3.7) shows

$$f_n(x) = T(x) \int_0^x T^{-1} J k_n dt$$

i.e., $f_n(x) = T(x)C_n(x)$. It follows from $f_n \xrightarrow{L^2} f$ that $\overline{D(T_N)} = L^2[0, N]$. To show $\overline{D(T)} = L^2[0, \infty)$, we pick up $f \in L^2[0, \infty)$. Since $\chi_{[0,N]} f \xrightarrow{L^2} f$, we can pick up $\{C_n\}_{n=1}^{\infty} \subset C_0^{\infty}(0, N)$ such that $f_n = TC_n \xrightarrow{L^2} \chi_{[0,N]} f$. Observe that $f_n(N-) = 0$, then claim 1. shows $f_n(N) = 0$, hence the function

$$\widetilde{f_n} = \begin{cases} f_n(x) & x \leq N \\ 0 & x > N \end{cases}$$

is in $D(T)$. it follows from this fact that $\overline{D(T)} = L^2[0, \infty)$. ■

The relation between T_N and T is also interesting.

Claim 3.5 Assume $N > 0$ and $C_1, C_2 \in \mathbb{C}^2$, then there is $f \in D(T_N)$ such that $f(0) = C_1$ and $f(N) = C_2$. Moreover, if $f \in D(T_N)$, then there is a $\widetilde{f} \in D(T)$ such that $f = \chi_{[0,N]} \widetilde{f}$.

Proof: It's clear that there are many $f \in D(T_N)$ satisfying $f(0) = C_1$, then by (3.7), there is a $k \in L^2[0, N]$ such that

$$f(x) = T(x)C_1 + T(x) \int_0^x T^{-1} J k dt$$

hence the value at N is given by

$$f(N) = T(N)C_1 + T(N)\int_0^N T^{-1}Jkdt$$

We define a linear functional

$$l : L^2[0,N] \to \mathbb{C}^2$$

$$l(k) = \int_0^N T^{-1}Jkdt$$

We claim that l is surjective. Indeed, we can write down l as

$$l(k) = \begin{pmatrix} -\int_0^N v^T kdt \\ \int_0^N u^T kdt \end{pmatrix}$$

If $l = 0$, we just pick up $k = 0$.

If $l(k) = \begin{pmatrix} a \\ 0 \end{pmatrix}$ for some $a \neq 0$, we pick up a non-trivial k satisfying $k \in < \overline{u} >^{\perp}$ and $k \notin < \overline{v} >^{\perp}$ and normalize it. Such a k exists, otherwise, $< \overline{v} > \subseteq < \overline{u} >$, then it follows that u and v are linearly dependent, which contradicts with claim 3.2.

If $l(k) = \begin{pmatrix} a \\ b \end{pmatrix}$ for some $a \neq 0$ and $b \neq 0$, we pick up a non-trivial k satisfying $k \in < \overline{au + bv} >^{\perp}$ and $k \notin < \overline{u} >^{\perp}$ and normalize it.

As l is surjective, we conclude that $\exists k \in L^2[0,N]$ such that $l(k) = T^{-1}(N)C_2 - C_1$. And this k gives a unique $f \in D(T_N)$ satisfying $f(0) = C_1$ and $f(N) = C_2$.

To prove the second conclusion, we pick up an arbitrary number M such that $M > N$. $f \in D(T_N)$ gives a $k \in L^2[0, N]$ and $f(N)$. By the same method as above, we can pick up a $\widetilde{k} \in L^2[N, M]$ such that the corresponding $\widetilde{f}$ satisfies $\widetilde{f}(N) = f(N)$ and $\widetilde{f}(M) = 0$. If we glue $f(k)$ and $\widetilde{f}(\widetilde{k})$ together, and set 0 when $x > M$, then we construct an element in $D(T)(R(T))$, and the cut-off on $[0, N]$ of this function is just f. ■

Chapter 4

the Spectral Theory of Dirac Operators

In this chapter, we want to construct the spectral theory of Dirac operators.

In section 4.1, we first investigate the Wronskian of two functions in the domain of a Dirac operator, and this concept will be used to characterize a closed and symmetric operator. Von Neumann theory with Cayley transform there can be applied to construct self-adjoint realizations.

In section 4.2, we discuss Weyl theory: we can show that the transfer matrix is entire as a complex function, then by Mobius transformation generated by this transfer matrix, we construct Weyl circles. The limit point case is defined if those circles converge to a point, and the limit point can be represented by a Herglotz function called the Weyl function which contains the spectral measure of a self-adjoint realization of the operator we are considering.

In section 4.3, by using the Green function, we can construct a Hilbert-Schmidt operator, and Weyl theory allows us to evaluate the spectral measure explicitly. The spectral representation theorem is established there, and we give an alternative proof in Chapter 5.

4.1 Self-adjoint Realizations

In this section, we want to construct self-adjoint extensions of $T_N(T)$. We define the Wronskian of two functions f and h by

$$W_{f,h}(x) := (f^*Jh)(x)$$

If $f, h \in D(T_N)(D(T))$, we have f and h are of (locally) bounded variation and right continuous, then they satisfy claim 3.1, hence we conclude that $W_{f,h}(x-) = W_{f,h}(x) = f^*(x)Jh(x)$, especially, this is true for $x = 0$.

Claim 4.1 Let $a \leq b$ be positive. Suppose $f, h \in BV[0, \infty)$, right continuous, and $Af, Ah \in AC[0, \infty)$, then

$$-\int_a^b ((Af)')^* h dt + \int_a^b f^*(Ah)' dt = W_{f,h}(b) - W_{f,h}(a) \tag{4.1}$$

Proof:

$$\int_a^b f^*(Ah)' dt = f^*(b)Ah(b) - f^*(a-)Ah(a-) - \int_{[a,b]} df^* Ah$$

Notice that

$\int_{[a,b]} df^*Ah$

$$= \int_{[a,b]} df^*Jh - \int_{[a,b]} df^*(\int_0^x gd\mu h)$$
$$= \int_{[a,b]} df^*Jh - f^*(b)\int_0^b gd\mu h + f^*(a-)\int_{(0,a-]} gd\mu h + \int_{[a,b]} f^*(x-)gd\mu h$$

Hence

$$\int_a^b f^*(Ah)'dt = -W_{f,h}(a) + W_{f,h}(b) - \int_{[a,b]} df^*Jh - \int_{[a,b]} f^*(x)e^{-\mu\{x\}J}gd\mu h$$

For $\int_a^b((Af)')^*hdt$, notice that $(Af)^* = -f^*J - \int_0^x f^*d\mu g(-J\mu\{s\})$, then the same calculation shows that :

$\int_a^b((Af)')^*hdt$

$= -W_{f,h}(b) + W_{f,h}(a) + \int_{[a,b]} f^*Jdh - \int_{[a,b]} f^*(x)d\mu g(-J\mu\{x\})e^{J\mu\{x\}}h$

Notice that by integration by parts:

$$\int_{[a,b]} f^*Jdh + \int_{[a,b]} df^*Jh = W_{f,h}(b) - W_{f,h}(a) + \int_{[a,b]} f^*(x)(1-e^{-\mu\{x\}J})Jdh$$

Moreover, since $g(\mu\{x\}J) = 1$ almost everywhere with respect to the Lebesgue measure, we have

$$\int_{[a,b]} f^*(x)e^{-\mu\{x\}J}gd\mu h - \int_{[a,b]} f^*(x)d\mu g(-J\mu\{x\})e^{J\mu\{x\}}h$$

$$= \sum_{x\in S\cap[a,b]} (f^*(x)(e^{-\mu\{x\}J}-1)Jh(x) - f^*(x)J(1-e^{J\mu\{x\}})h(x))$$

$$= \sum_{x\in S\cap[a,b]} f^*(x)(e^{-\mu\{x\}J} + e^{\mu\{x\}J} - 2)Jh(x)$$

We also have

$$\int_{[a,b]} f^*(x)(1-e^{-\mu\{x\}J})Jdh = \sum_{x\in S\cap[a,b]} f^*(x)(1-e^{-\mu\{x\}J})J(h(x)-h(x-))$$

Applying claim 3.1. again, we get

$\int_{[a,b]} f^*(x)(1-e^{-\mu\{x\}J})Jdh$
$= \int_{[a,b]} f^*(x)e^{-\mu\{x\}J}gd\mu h - \int_{[a,b]} f^*(x)d\mu g(-J\mu\{x\})e^{J\mu\{x\}}h$

Assemble all identities we have gotten here, we finally get

$$-\int_a^b ((Af)')^* hdt + \int_a^b f^*(Ah)'dt = W_{f,h}(b) - W_{f,h}(a)$$

$\blacksquare$

Corollary 4.2 Suppose $f, h \in D(T)(D(T_N))$, then

$$\langle T_{(N)}f, h\rangle - \langle f, T_{(N)}h\rangle = \lim_{x\to\infty(N)} W_{f,h}(x) - W_{f,h}(0)$$

as the inner product in L^2.

Proof: Let $b \to \infty(N)$ and $a \to 0$ in (4.1), and recall that f and h can be extended to 0. $\blacksquare$

From now on, we briefly denote $\lim_{x\to\infty} W_{f,h}(x)$ by $W_{f,h}(\infty)$, $\lim_{x\to\infty} f(x)$ by $f(\infty)$ and T by T_∞, hence, the notation, T_N, is indeed T if $N = \infty$. Also, to simplify notations, we simply write $W_{f,h}|_0^N := W_{f,h}(N) - W_{f,h}(0)$.

We denoted the adjoint of T_N by T_o, i.e.,

$$T_o := T_N^*$$

and we define an operator T_{oo} as follows:

$$D(T_{oo}) = \{f \in D(T_N) : f \textit{ has compact support on } (0, N)\}$$

$$T_{oo} = T_N$$

Claim 4.3. $T_{oo}^* \subset T_N$

Proof: Let $f \in D(T_{oo}^*)$, then $\forall k \in D(T_{oo})$, $\exists h \in L^2[0, N](L^2[0, \infty))$ such that

$$\langle f, T_{oo}k \rangle = \langle h, k \rangle$$

Theorem 1.,with a little adaption, gives a right continuous solution $f_1 \in BV[0, \infty)$ of the equation

$$f_1(x) - \int_0^x d\mu f_1 = -\int_0^x h dt$$

But f_1 may fail to be in $L^2[0, N](L^2[0, \infty))$. By (4.1), with the fact that k has compact support, we conclude that

$$\langle h, k \rangle = -\langle (Af_1)', k \rangle = \int_0^N f_1^* T_N k dt$$

Notice that $T_{oo}k = T_N k$, it follows that

$$\int_0^N (f - f_1)^* T_N k dt = 0$$

That is, $\forall p \in R(T_{oo})$, we have $\int_0^N (f - f_1)^* p dt = 0$.

Observe that,by recalling (3.7), $p \in R(T_{oo})$ if and only if

(1) $p \in L^2[0, N](L^2[0, \infty))$;

(2) there are $0 < a < b < N$ such that $p = 0$ out of $[a, b]$;

(3) $\int_0^N T^{-1} J p dt = 0$, where T is the transfer matrix.

Let us denote by K the linear subspace of $L^2[0, N](L^2[0, \infty))$ defined just by condition (2), and consider the following functionals on K:

$$F_1(k) = (1, 0) \int_0^N T^{-1} J k dt, F_2(k) = (0, 1) \int_0^N T^{-1} J k dt$$

$$F(k) = \int_0^N (f - f_1)^* k dt$$

Condition (3) implies if $F_1(k) = F_2(k) = 0$ for a $k \in K$, then $F(k) = 0$. We invoke a lemma which is discussed in [2]:

Lemma. Let $F_1, \ldots, F_n, F : K \to \mathbb{C}$ be linear functionals on a vector space K and assume $\cap N(F_j) \subseteq N(F)$, then F is a linear combination of F_j.

It follows that F is a linear combination of F_1, F_2, i.e., there is a vector $v \in \mathbb{C}^2$ such that

$$\int_0^N (f - f_1 - \overline{T}v)^* k dt = 0$$

for all $k \in K$.

This is true if and only if $f - f_1 - \overline{T}v = 0$ locally, hence globally. It follows that f satisfies

$$f(x) - \int_0^x d\mu f = v - \int_0^x h dt$$

and $f \in L^2[0, N](L^2[0, \infty))$, we conclude that $f \in D(T_N)$, i.e., $T_{oo}^* \subset T_N$.■

Corollary 4.4 (1) T_N is closed. (2) $T_o = \overline{T_{oo}}$, $T_o^* = T_N$; moreover, T_o is closed and symmetric.

Proof: (1) Since T_{oo} is also densely defined and $T_{oo} \subset T_o = T_N^*$ by the definition of T_{oo}, due to claim 4.3, we have $\overline{T_N} \subset T_{oo}^* \subset T_N \subset \overline{T_N}$, i.e., $T_N = \overline{T_N}$.

(2) As the adjoint of T_N, T_o is closed.
$T_{oo}^* = T_N = \overline{T_N}$ implies $\overline{T_{oo}} = T_{oo}^{**} = T_N^* = T_o$. Moreover, $T_N = T_N^{**} = T_o^* \subset T_{oo}^*$, with claim 4.3, it follows that $T_N = T_o^* = T_{oo}^*$. Since $\overline{T_{oo}} \subset T_N$, we conclude that $T_o \subset T_o^*$. ■

Claim 4.5 Let $f \in D(T_N)$. $f \in D(T_o)$ if and only if $\forall h \in D(T_N)$, $W_{f,h}(0) = W_{f,h}(N) = 0$.

Proof: Define an operator $\widetilde{T}$ as follows:

$$D(\widetilde{T}) = \{f \in D(T_N) : W_{f,h}(0) = W_{f,h}(N) = 0, \forall h \in D(T_N)\}$$

$$\widetilde{T} = T_N$$

It follows from Corollary 4.2. that $\widetilde{T} \subset T_o$.

If there is $f \in D(T_N)$ but $f \notin \widetilde{T}$, then without losing generality, let's assume $W_{f,h}(0) \neq 0$ for some $h \in D(T_N)$. Fix $h(0)$, then by claim 3.5., we can choose $\widetilde{h} \in D(T_N)$ such that $\widetilde{h}(0) = h(0)$ and $\widetilde{h}(N) = 0$. This h satisfies $W_{f,\widetilde{h}}(0) = W_{f,h}(0) \neq 0$ and $W_{f,\widetilde{h}}(N) = 0$, hence it follows from corollary 4.2. that $f \notin T_o$. This is equivalent to $\widetilde{T} = T_o$.

For $N = \infty$, we just need a little adaption: we may cut off the half line into a finite interval and an infinite interval, then we vary the part of h on the finite interval. ■

Now, we turn to von Neumann Theory of symmetric relations (see[2]), and introduce the boundary conditions of a Dirac operator with a measure.

We define the null spaces of the operator as follows:

$$N_+ = N(i - T_N), N_- = N(-i - T_N)$$

and defect indices of those two spaces:

$$\gamma_\pm = dim N_\pm$$

By the first formula of von Neumann, we have:

$$D(T_N) = D(T_o) \dot{+} N_+ \dot{+} N_- (direct\ sum)$$

$$T_N(f_0 + g_+ + g_-) = T_o f_0 + ig_+ - ig_-,\ for\ f_o \in D(T_o), g_+ \in N_+, g_- \in N_-$$

Claim 4.6 T_o has equal defect indices $\gamma_- = \gamma_+ = \gamma \leq 2$. As a consequence, the self-adjoint extensions of T_o are exactly the γ-dimensional symmetric extensions of T_o, equivalently, γ-dimensional symmetric restrictions of T_N.

Proof: Let $f \in N_+$, then f satisfies

$$Jf(x) - \int_0^x g(\mu\{s\}J) d\mu f = Jf(0) - i \int_0^x f dt$$

Notice that $g(\mu\{s\}J) = 1$ a.e. with respect to the Lebesgue measure, we actually have

$$Jf(x) - \int_0^x g(\mu\{s\}J)(d\mu - idt) f = Jf(0)$$

Let us define $d\mu_i = d\mu - idt$. It follows from $\mu_i\{x\} = \mu\{x\}$ that

$$Jf(x) - \int_0^x g(\mu_i\{s\}J)(d\mu_i)f = Jf(0) \tag{4.2}$$

By Theorem 1., the dimension of the solution space of the integral equation (4.2) is 2. Since those solutions may fail to be in the $L^2[0,N]$, or $L^2[0,\infty)$, we conclude that $\gamma_+ \leq 2$. Moreover, by taking complex conjugate in (4.2), it follows that $\gamma_- = \gamma_+$.

The rest of the claim is the direct consequence of the Cayley transform, see [10] for instance. ■

Claim 4.7. Suppose $f_i \in D(T_N)$ $(j = 1, \ldots, \gamma)$ are linearly independent modulo $D(T_o)$ and $W_{f_j,f_k}|_0^N = 0$, then the operator S defined by

$$D(S) := \{f \in D(T_N) : W_{f_j,f}|_0^N = 0, j = 1, \ldots, \gamma\} \tag{4.3}$$

$$S = T_N$$

is self-adjoint. conversely, every self-adjoint realization is obtained in this way.

This statement makes sense because $\gamma \neq 0$, otherwise, T_N itself is self-adjoint, and this implies $\forall f \in D(T_N), f(0) = 0$ by claim 4.5, but this does not need to be true.

Proof: Since $T_o \subset S \subset T_N$, and by the construction of S, we conclude that S is at least γ-dimensional extension.

On the other hand, we define functionals

$$F_j : D(T_N) \to \mathbb{C}, F_j(h) = W_{f_j,h}|_0^N$$

Then we can rewrite $D(S)$ as $D(S) = D(T_N) \cap N(F_1) \cdots \cap N(F_\gamma)$.

If $\gamma = 1$, we claim that $N(F_1) \neq D(T_N)$. Indeed, if $h \in N(F_1)$, then $W_{f_1,h}|_0^N = 0$. On the other hand, since f_1 is not in $D(T_o)$, by claim 4.5., there is $h \in D(T_N)$ such that either $W_{f_1,h}(0) = W_{f_1,h}(N) \neq 0$ or $W_{f_1,h}(0) \neq W_{f_1,h}(N)$. Without losing generality, we assume that $W_{f_1,h}(0) \neq 0$, then as we did in the proof of claim 4.5., we can construct $\widetilde{h} \in D(T_N)$ such that $\widetilde{h}(0) = h(0)$ and $\widetilde{h}(N) = 0$, this implies $F_1(\widetilde{h}) \neq 0$. Now, it's safe to say that S is at least a 1-dimensional restriction of T_N.

If $\gamma = 2$, we claim that F_1 and F_2 are linearly independent. Indeed, define $F := \alpha F_1 + \beta F_2 = W_{\alpha f_1 + \beta f_2, \cdot}|_0^N$, and we consider $F = 0$. Since $\alpha f_1 + \beta f_2 \notin D(T_o)$ except for $\alpha = \beta = 0$, hence if $\alpha f_1 + \beta f_2 \notin D(T_o)$, we can construct $\widetilde{h} \in D(T_N)$ as above such that $F(\widetilde{h}) \neq 0$, this contradiction shows that $\alpha = \beta = 0$. Now, if, say, $N(F_1) \subset N(F_2)$, the lemma we used in claim 4.3. implies $F_1 = \beta F_2$ for some β,which contradicts with the linear independence. As $N(F_1) \not\subset N(F_2)$ and $N(F_2) \not\subset N(F_1)$, it's safe to conclude that S is at least 2-dimensional restriction of T_N.

In all, by claim 4.6. and the symmetry of S, It follows that S is self-adjoint.

Conversely, if a self-adjoint extension S of T_o is given, then we can pick up γ elements $f_j \in D(S)$ which are linearly independent modulo $D(T_o)$ and satisfy $W_{f_j,f_k}|_0^N = 0$. We use them to define S_1 by (4.3). Since $S \subset S_1$ and $S,,S_1$ are self-adjoint, it follows that $S = S_1$ as expected. ■

4.2 Weyl Theory

In this section, we want to construct Weyl theory for Dirac operators with measures, then spectral theory as well. We are interested in the separated boundary condition, i.e., in the statement of claim 4.7, we have $W_{f_j,f}(0) = W_{f_j,f}(N) = 0$ separately. As usual, a calculation shows that this is equivalent to $e^*_{\alpha_1}Jf(0) = e^*_{\alpha_2}Jf(N) = 0$ where $e_{\alpha_j} = \begin{pmatrix} \cos\alpha_j \\ \sin\alpha_j \end{pmatrix}$ for some $\alpha_j \in \mathbb{R}$, see [2].

Given $N < \infty$, we consider the self-adjoint restriction of T_N with separated boundary condition: $f_2(0) = 0, e^*_\beta Jf(N) = 0$ for some $\beta \in [0,\pi)$. The Titchmarsh-Weyl m function, $m_N^\beta : \mathbb{C}^+ \to C_\infty$, with $\mathbb{C}^+ = \{z \in \mathbb{C} : Imz > 0\}$ and C_∞ the Riemann sphere, is defined as usual:

take a non-trivial solution f of $T_N f = zf$ that satisfies $e^*_\beta Jf(N) = 0$, then

$$m_N^\beta(z) := \frac{f_1(0,z)}{f_2(0,z)} \tag{4.4}$$

We also generalize the transfer matrix we introduced in Chapter 3. We call a 2×2 matrix $T(x, z)$ the transfer matrix if it is the solution of

$$JT(x,z) - \int_0^x g(\mu\{s\}J) d\mu T(t,z) = J - z \int_0^x T(t,z)dt$$

where I is the identity.

Also, we write T down explicitly as $T(x, z) = (u(x, z), v(x, z)) = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$, and it is obvious that the previous transfer matrix $T(x)$ we defined is $T(x) = T(x, 0)$.

Using Mobius transformation, we can rewrite m function as follows:

$$m_N^\beta(z) = T^{-1}(N, z) \cot\beta \tag{4.5}$$

The approximation we introduced in Chapter 3 is super important here, so we state it more clearly.

Let $\mu_z := \chi_{[0,N]}(t)(d\mu - zdt)$ be a complex measure, and define

$$A_\epsilon(x, z) := J \int_R \phi_\epsilon(x - t) d\mu_z(t)$$

where ϕ_ϵ is defined as in (3.2).

Consider the differential equation:

$$f_\epsilon' + A_\epsilon(x, z) f_\epsilon = 0$$

and we call $T_\epsilon(x, z) = (u_\epsilon(x, z), v_\epsilon(x, z))$ the transfer matrix of this equation if

$$T_\epsilon'(x, z) + A_\epsilon(x, z) T_\epsilon(x, z) = 0, T_\epsilon(0, z) = I \tag{4.6}$$

Observe that $T(x, z)$ satisfies

$$JT(x, z) - \int_0^x g(\mu_z\{s\} J) d\mu_z T(t, z) = J$$

By Theorem 2. and Theorem 3., for a fixed z, it follows that $\{T_\epsilon(x, z) : 0 < \epsilon < 1\}$ is uniformly bounded under the supreme norm of matrices and that $\lim_{\epsilon \to 0} T_\epsilon(x, z) = T(x, z)$ pointwisely with respect to x.

Claim 4.8 (1) $detT(x, z) = 1$ and (2) for $z \in \mathbb{C}^+$, $i(T^*(x, z) JT(x, z) - J) \geq 0$

Proof: It is easy to show that, as the transfer matrices of regular differential equations, $detT_\epsilon(x, z) = 1$, hence $detT(x, z) = \lim_{\epsilon \to 0} detT_\epsilon(x, z) = 1$.

Notice that

$$\frac{d}{dx}(T_\epsilon^*(x, z) JT_\epsilon(x, z)) = (\overline{z} - z) T_\epsilon^*(x, z) T_\epsilon(x, z) \int_0^N \phi_\epsilon(x - t) dt$$

Hence we have

$$T_\epsilon^*(x,z)JT_\epsilon(x,z) - J = -2iImz \int_0^x T_\epsilon^*(s,z)T_\epsilon(s,z)(\int_0^N \phi_\epsilon(s-t)dt)ds$$

i.e., $i(T_\epsilon^*(x,z)JT_\epsilon(x,z) - J) \geq 0$.

As $T(x,z)$ is the pointwise limit of $T_\epsilon(x,z)$, we conclude that

$$i(T^*(x,z)JT(x,z) - J) \geq 0$$

■

Claim 4.9 For any fixed x, $T(x,z)$ is entire.

Proof: Since (4.6) is a regular differential equation, then the standard theory of differential equations shows that $T_\epsilon(x,z)$ is entire for any fixed x. For example, see [4].

The integral form of the differential equation (4.6) is :

$$JT_\epsilon(x,z)$$

$$= J + \int_0^x ds(\int_0^N \phi_\epsilon(s-t)d\mu(t))T_\epsilon(s,z) - z\int_0^x ds(\int_0^N \phi_\epsilon(s-t)dt)T_\epsilon(s,z)$$

Fix a x and let K be a compact subset of $\mathbb{C}$, we consider the family $F := \{T_\epsilon(x,z) : 0 < \epsilon < 1, z \in K\}$.

Since the supreme matrix norm of the kernel is given by

$|| \int_0^N \phi_\epsilon(s-t)d\mu(t) - z \int_0^N \phi_\epsilon(s-t)dt||$

$$\leq || \int_0^N \phi_\epsilon(s-t)d\mu(t)|| + |z| \int_0^N \phi_\epsilon(s-t)dt$$

Moreover,

$|| \int_0^N \phi_\epsilon(s-t)d\mu(t)|| = \max_{i=1,2,3,4} (\int_0^N \phi_\epsilon(s-t)d|\mu_i|(t))$

$$\leq \int_0^N \phi_\epsilon(s-t) \sum_{i=1}^{4} d|\mu_i|(t)$$

Hence we have

$||T_\epsilon(x,z)||$

$$\leq 1 + 2\int_0^x ds(\int_0^N \phi_\epsilon(s-t) \sum_{i=1}^{4} d|\mu_i|(t)) + |z| \int_0^N \phi_\epsilon(s-t)dt)||T_\epsilon(s,z)||$$

Since $\int_0^N \phi_\epsilon(s-t) \sum_{i=1}^4 d|\mu_i|(t) + |z| \int_0^N \phi_\epsilon(s-t)dt$ is in $L^1(R)$, By Gronwall's inequality, we have

$$||T_\epsilon(x,z)|| \leq e^{2\int_0^x (\int_0^N \phi_\epsilon(s-t)\sum_{i=1}^4 d|\mu_i|(t) + 2|z|\int_0^N \phi_\epsilon(s-t)dt)} \leq e^{2\sum_{i=1}^4 |\mu_i|[0,N] + 4|z|N}$$

It follows from this inequality that F is uniformly bounded on K.

On the other hand, let us consider the differential equation

$$u'(x,z) + A_\epsilon(x,z)u(x,z) = k(x,z),\ u(0,z) = C$$

By variation of constants, it follows that

$$u(x,z) = T_\epsilon(x,z)(C + \int_0^x T_\epsilon^{-1}(s,z)k(s,z)ds)$$

If we pick up $z, z_0 \in K$, then we have two equations in the form of (4.6), hence

$$(T_\epsilon(x,z) - T_\epsilon(x,z_0)) + A_\epsilon(x,z)(T_\epsilon(x,z) - T_\epsilon(x,z_0))$$

$$= (z - z_0)J(\int_0^N \phi_\epsilon(x-t)dt)T_\epsilon(x,z_0)$$

It follows from a variation of constants that

$$T_\epsilon(x,z) - T_\epsilon(x,z_0) = (z - z_0)T_\epsilon(x,z)\int_0^x T_\epsilon^{-1}(s,z)JT_\epsilon(s,z_0)(\int_0^N \phi_\epsilon(s-t)dt)$$

This implies that

$$||T_\epsilon(x,z) - T_\epsilon(x,z_0)|| \leq 4|z - z_0| \cdot ||T_\epsilon(x,z)|| \int_0^x ||T_\epsilon^{-1}(s,z)|| \cdot ||T_\epsilon(s,z_0)||$$

Since F is uniformly bounded on K, we conclude that there is a constant M which is irrelevant to ϵ and x such that

$$||T_\epsilon(x,z) - T_\epsilon(x,z_0)|| \leq M|z - z_0|$$

It follows that F is equicontinuous.

Now, Arzela-Ascoli theorem works: there exists a subsequence of F which converges uniformly. Since $T(x,z)$ is the pointwise limit, then it is the uniform limit. For K is arbitrary, as the uniform limit of holomorphic functions on any compact set, $T(x,z)$ is entire. ■

Claim 4.10 $m_N^\beta(z)$ is a Herglotz function.

Proof: We first show that $m_N^\beta(z) \neq a \in \mathbb{R}_\infty$. Indeed, if $m_N^\beta(z) = a$, then the solution f that we used in the definition of (4.4) satisfies some boundary condition at 0; however, this is impossible because a complex number z cannot be an eigenvalue of any self-adjoint restriction of T_N.

Claim 4.9. implies, in (4.4), that $f_2(0,z)$ is entire, hence either $f_2(0,z) \equiv 0$ or zeros of $f_2(0,z)$ have no accumulation points. Because $m_N^\beta(z) \neq \infty$ as we gained above, it follows that $m_N^\beta(z)$ has a meromorphic extension. Moreover, since $f(0,z)$ is real on the real line, the non-real poles of $m_N^\beta(z)$ come in complex conjugate pairs. If we write the Mobius transformation explicitly, we have

$$m_N^\beta(z) = \frac{v_2(N,z)\cot\beta - v_1(N,z)}{-u_2(N,z)\cot\beta + u_1(N,z)}$$

If $m_N^\beta(z)$ has a pole at $z \in \mathbb{C}^+$, we conclude from $u_2(N,z)\cot\beta - u_1(N,z) = 0$ that u is in the domain of the self-adjoint restriction we defined at the beginning of the section, but this implies that z is an eigenvalue of that self-adjoint restriction, which is impossible. As a consequence, $m_N^\beta(z)$ is holomorphic on $\mathbb{C}^+$.

By Theorem 4 in Chapter 3, claim 4.8, and (4.5), we conclude that $m_N^\beta(z)$ maps $\mathbb{C}^+$ to $\overline{\mathbb{C}^+}$, hence is a generalized Herglotz function, and Herglotz. Moreover, we have that all poles are on the real line. ■

Now, we are ready to define the Weyl circle $C(N,z)$ and the Weyl disk $D(N,z)$ as follows:

$$C(N,z) := \{T^{-1}(N,z)q : q \in \mathbb{R}_\infty\}$$

$$D(N,z) := \{T^{-1}(N,z)q : q \in \overline{\mathbb{C}^+}\}$$

Obviously, if $N_1 \leq N_2$, then $D(N_2,z) \subseteq D(N_1,z)$.

Claim 4.11 The radius R of $D(N,z)$ is given by

$$\frac{1}{R} = 2Imz \cdot ||u(\cdot,z)||^2_{L^2[0,N]}$$

Proof: Since $T^{-1}(N,z) = \begin{pmatrix} v_2(N,z) & -v_1(N,z) \\ -u_2(N,z) & u_1(N,z) \end{pmatrix}$, by a standard calculation based on Mobius transformation (see [2]), we have

$$\frac{1}{R} = 2|Im(-\overline{u_2(N,z)}u_1(N,z)|$$

Moreover,

$$2iIm(-\overline{u_2(N,z)}u_1(N,z))$$

$$= -u^*(N,z)Ju(N,z) = -\lim_{\epsilon\to 0}\int_0^N (u_\epsilon^*(\cdot,z)Ju_\epsilon(\cdot,z))'(t)dt$$

Recall the calculation in claim 4.8, we have

$$-\lim_{\epsilon\to 0}\int_0^N (u_\epsilon^*(\cdot,z)Ju_\epsilon(\cdot,z))'(t)dt$$

$$= 2iImz\lim_{\epsilon\to 0}\int_0^N u_\epsilon^*(t,z)u_\epsilon(t,z)(\int_0^N \phi_\epsilon(t-s)ds)dt$$

Recall that $\{u_\epsilon(x,z) : 0 < \epsilon < 1\}$ is uniformly bounded on $[0,N]$ by Theorem 2.,it follows that

$$2iIm(-\overline{u_2(N,z)}u_1(N,z)) = 2iImz||u(\cdot,z)||^2_{L^2[0,N]}$$

This gives us the identity we need. ■

Since $D(N,z)$ are nested and compact, we define a non-empty set as follows:

$$D(z) := \bigcap_{N>0} D(N,z)$$

We call the limit point case at ∞ if for any z, there is just one non-trivial solution up to a factor (in $BV[0,\infty)$ and right continuous) of $(Af)' = -zf$ which is in $L^2[0,\infty)$, and the limit circle case if for all z, all solutions (in $BV[0,\infty)$ and right continuous) of $(Af)' = -zf$ are in $L^2[0,\infty)$.

Remark. Since the spectrum of T is nonempty as the space is a complex Hilbert space, there must be $z \in \sigma(T)$ so that $Tf = zf$ for some $0 \neq f \in D(T)$, hence we just have those two scenarios.

Claim 4.12

1) Assume the limit point case at ∞, then $D(N,z)$ is a point;

2) Assume the limit circle case at ∞, then $D(N,z)$ is a circle.

Proof: Let $M \in \mathbb{C}$, and we define $f_M(x,z) = TM = T \cdot \begin{pmatrix} M \\ 1 \end{pmatrix} = v(x,z) + Mu(x,z)$. We claim that

$$M \in D(N,z) \Longleftrightarrow Imz \int_0^N f_M^* f_M dt \leq ImM$$

$$M \in C(N,z) \Longleftrightarrow Imz \int_0^N f_M^* f_M dt = ImM$$

Indeed, in the proof of claim 4.8, we actually conclude that

$$T^*(x,z)JT(x,z) - J = -2iImz \int_0^x T^*T(s,z)ds$$

This implies

$$f_M^*(N)Jf_M(N) - 2iImM = -2iImz\int_0^N f_M^* f_M dt$$

Hence

$$Imz\int_0^N f_M^* f_M dt = ImM - Imf_M(N,z)$$

Thus

$$Imz\int_0^N f_M^* f_M dt \leq ImM \Leftrightarrow Imf_M(N,z) \geq 0 \Leftrightarrow TM \in \overline{\mathbb{C}^+}$$

$$Imz\int_0^N f_M^* f_M dt = ImM \Leftrightarrow Imf_M(N,z) = 0 \Leftrightarrow TM \in \mathbb{R}_\infty$$

To prove 1), we assume $D(z)$ contains at least 2 points, say, M_1 and M_2. It is easy to show that f_{M_1} and f_{M_2} are linearly independent and in $L^2[0,\infty)$, but this contradicts the definition of limit point case.

To prove 2), notice that $D(L,z) = \bigcap_{L>N>0} D(N,z)$, hence we have $D(Z) = \lim_{L\to\infty} D(L.z)$. By checking the center and radius, we can reach this conclusion. ■

Claim 4.13 Assume the limit point case at ∞, then $m(z) \in D(z)$ is Herglotz. If f is the non-trivial solution up to a factor (in $BV[0,\infty)$ and right continuous) of $(Af)' = -zf$ which is in $L^2[0,\infty)$, then $m(z) = \frac{f_1(0,z)}{f_2(0,z)}$.

Proof: $m(z) = \lim_{N\to\infty} m_N^\beta(z)$ is a locally uniform limit of Herglotz functions, hence Herglotz.

Let $f_m(x,z) := Tm$, then $f_m \in L^2[0,\infty)$ and $m(z) = \frac{f_1(0,z)}{f_2(0,z)}$. $\blacksquare$

4.3 Spectral Representation Theorem

Now, we want to investigate the spectral representation theorem, and there is no need to restrict our self-adjoint realization so strictly, hence we consider T_N with general separated boundary conditions, i.e., we define the self-adjoint restriction $S_{\alpha,\beta}$ of T_N as follows:

$$D(S_{\alpha,\beta}) = \{f \in D(T_N) : \sin\alpha f_1(0) - \cos\alpha f_2(0) = 0,$$

$$\sin\beta f_1(N) - \cos\beta f_2(N) = 0\}$$

$$S_{\alpha,\beta} = T_N$$

for some $\alpha, \beta \in [0,\pi)$

Let u_0, u_N be non-trivial solutions of $T_N f = zf$ satisfying $\sin\alpha u_{0,1}(0) - \cos\alpha u_{0,2}(0) = 0$ and $\sin\beta u_{N,1}(N) - \cos\beta u_{N,2}(N) = 0$ respectively and we normalize them such that $detM(0) = 1$ where $M(x) = (u_0(x), u_N(x))$. Notice that

$$JM(x,z) - \int_0^x g(\mu_z\{s\}J)d\mu_z M(t,z) = JM(0)$$

We have the following claim with respect to the Green function.

Claim 4.15 Let $z \in \mathbb{C} \setminus \mathbb{R}$, then $\forall k \in L^2[0, N]$, we have

$$(S_{\alpha,\beta} - z)^{-1} k = \int_0^N G(x, t, z) k(t) dt$$

where

$$G(x, t, z) = \begin{cases} u_N(x, z) \cdot u_0^\top(t, z) & t \leq x \\ u_0(x, z) \cdot u_N^\top(t, z) & x < t \end{cases}.$$

Proof: Since $S_{\alpha,\beta}$ is self-adjoint, then $\forall z \in \mathbb{C} \setminus \mathbb{R}$, we have $(S_{\alpha,\beta} - z)^{-1} \in B(L^2[0, N])$. Let $k \in L^2[0, N]$, then $\exists f \in D(S_{\alpha,\beta} - z)$ such that $(S_{\alpha,\beta} - z)f = k$. By variation of constants again, we get an analogous conclusion

$$f(x) = M(x)C + M(x) \int_0^x M^{-1} Jkdt$$

where $C = M(0)^{-1} f(0)$.

Moreover, we have $f(N) = M(N)C + M(N) \int_0^N M^{-1} Jkdt$.

Define $Q := \int_0^N M^{-1} Jkdt$, then $Q = M^{-1}(N)f(N) - M^{-1}(0)f(0)$. Suppose $f(0) = m \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$, $f(N) = n \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix}$ for some real numbers m, n, and $u_0(0) = p \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$, $u_N(N) = q \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix}$ for $p, q \neq 0$. then we have

$$Q = n \begin{pmatrix} q\sin\beta & -q\cos\beta \\ -u_{0,2}(N) & u_{0,1}(N) \end{pmatrix} \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix} - m \begin{pmatrix} u_{N,2}(0) & -u_{N,1}(0) \\ -p\sin\alpha & p\cos\alpha \end{pmatrix} \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

Observe that $detM = u_{0,1}(x)u_{N,2}(x) - u_{0,2}(x)u_{N,1}(x) = 1$ since M is a rotation of the transfer matrix, hence we have

$$Q = \begin{pmatrix} -\frac{m}{p} \\ \frac{n}{q} \end{pmatrix} := \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$$

Since $C = M(0)^{-1}f(0) = \begin{pmatrix} \frac{m}{p} \\ 0 \end{pmatrix} = \begin{pmatrix} -Q_1 \\ 0 \end{pmatrix}$, we actually have

$$f(x) = M(x)(\begin{pmatrix} -Q_1 \\ 0 \end{pmatrix} + \int_0^x M^{-1}Jkdt) = \int_0^N G(x,t,z)k(t)dt$$

And here,

$$G(x,t,z) = M(x) \cdot \begin{pmatrix} \chi_{(x,N)}(t)u_{N,1}(t,z) & \chi_{(x,N)}(t)u_{N,2}(t,z) \\ \chi_{(0,x)}(t)u_{0,1}(t,z) & \chi_{(0,x)}(t)u_{0,2}(t,z) \end{pmatrix}$$

If we calculate this product, it gives us that

$$G(x,t,z) = \begin{cases} u_N(x,z) \cdot u_0^\top(t,z) & t \leq x \\ u_0(x,z) \cdot u_N^\top(t,z) & x < t \end{cases}$$

■

Claim 4.16 $(S_{\alpha,\beta} - i)^{-1}$ is a Hibert-Schmidt operator. As a consequence, $\sigma(S_{\alpha,\beta}) = \{E_n\}$ is purely discrete and $\sum \frac{1}{1+E_n^2} < \infty$.

Proof: Notice that $G(x,t,z)$ is square integrable with respect to the Lebesgue measure, and $(S_{\alpha,\beta} - i)^{-1}$ is normal. ■

We turn back to the case that $\alpha = 0$ and β keeps free, and the self-adjoint restriction is denoted by S_β.

By Weyl theory, we have a Herglotz function $m_N^\beta(z)$ with the integral expression

$$m_N^\beta(z) = a + bz + \int_{\mathbb{R}} (\frac{1}{t-z} - \frac{t}{t^2+1}) d\rho(t)$$

Since $m_N^\beta(z)$ is meromorphic and real on the real line, it follows that $\rho(t)$ is discrete with atoms precisely at the poles of $m_N^\beta(z)$, more precisely, those poles are eigenvalues of S_β. In fact, we can describe S_β as a spectral representation.

Claim 4.17 The measure ρ associated with $m_N^\beta(z)$ is given by

$$\rho = \sum_{E \in \sigma(S_\beta)} \frac{\delta_E}{||u(\cdot, E)||^2_{L^2[0,N]}}$$

where δ_E is the Dirac measure at E.

Moreover, ρ and the map:

$$U : L^2[0, N] \longrightarrow L^2(\mathbb{R}, \rho)$$

$$Uf(t) = \int_0^N u^*(s, t) f(s) ds$$

set up a spectral representation of S_β.

Proof: We want to calculate $\rho\{t\}$ for a fixed t. Since we know $\rho\{t\} = \lim_{y\to 0+}(-iym_N^\beta(t+iy))$ and $Imz\int_0^N f^*_{m_N^\beta} f_{m_N^\beta}dt = Im(m_N^\beta(z))$ from claim 4.12, we have

$$\begin{aligned}\rho\{t\} &= \lim_{y\to 0+} y^2 \int_0^N f^*_{m_N^\beta} f_{m_N^\beta}dt \\ &= \lim_{y\to 0+} y^2(||v(\cdot,t+iy)||^2 + |m_N^\beta(t+iy)|^2||u(\cdot,t+iy)||^2 \\ &\qquad + \int_0^N (v^* m_N^\beta u + u^*(m_N^\beta)^* v))\end{aligned}$$

We first claim that $\lim_{y\to 0+}||u(\cdot,t+iy)||^2 = ||u(\cdot,t)||^2$ and $\lim_{y\to 0+}||v(\cdot,t+iy)||^2 = ||v(\cdot,t)||^2$ for a fixed t.

Indeed, let us consider

$$JT(x,t+iy) - \int_0^x g(\mu\{s\}J)d\mu T(s,t+iy) = J-(t+iy)\int_0^x T(s,t+iy)dt$$

Recall $d\mu_t = \chi_{[0,N]}(s)(d\mu - tds)$, then

$$JT(x,t+iy) - \int_0^x g(\mu_t\{s\}J)d\mu_t T(s,t+iy) = J - iy\int_0^x T(s,t+iy)dt$$

Hence, by variation of constants, we get

$$T(x,t+iy) = T(x,t)(I + iy\int_0^x T^{-1}(s,t)JT(s,t+iy)ds) \qquad (4.7)$$

As a transfer matrix, $\sup_{x\in[0,N]} ||T(x,t)|| < M$ for some $M \in \mathbb{R}$, thus we have

$$\sup_{x\in[0,N]} ||T(x,t+iy)|| \leq 2M(1+2yNM \sup_{x\in[0,N]} ||T(x,t+iy)||)$$

If y is small enough, say, $y < \frac{1}{8NM^2}$, we have

$$\sup_{x\in[0,N]} ||T(x,t+iy)|| < 4M$$

From (4.7) we also have

$$T(x,t+iy) - T(x,t) = iyT(x,t)\int_0^x T^{-1}(s,t)JT(s,t+iy)ds)$$

Hence

$$\sup_{x\in[0,N]} ||T(x,t+iy) - T(x,t)|| \leq 16NM^3y$$

for small enough y.

This implies $\lim_{y\to 0+} T(x,t+iy) = T(x,t)$ uniformly with respect to x in $[0,N]$, and Lebesgue dominated convergence theorem gives the identities we desire.

Notice that $\lim_{y\to 0+} y^2 \int_0^N v^* m_N^\beta u = \lim_{y\to 0+} y \int_0^N v^*(ym_N^\beta)u$, we conclude that

$$\rho\{t\} = \rho^2\{t\}||u(\cdot,t)||^2$$

This implies

$$\rho\{t\} = \frac{1}{||u(\cdot,t)||^2}$$

if $t \in \sigma(S_\beta)$, hence we have

$$\rho = \sum_{E \in \sigma(S_\beta)} \frac{\delta_E}{||u(\cdot, E)||^2_{L^2[0,N]}}$$

As usual, $\{\frac{u(\cdot,E_n)}{||u(\cdot,E_n)||} : E_n \in \sigma(S_\beta)\}$ forms an ONB of $\overline{D(S_\beta)}$, the map given with this spectral measure sets up a spectral representation of S_β.■

For the half-line problem, if we have the limit circle case at ∞, then there is nothing new, and we just need to give a boundary condition at ∞, and everything is crystal. If we assume the limit point case at ∞, then we get a unique m function and its measure ρ as well.

Claim 4.18 Assume limit point case at ∞, let

$$Uf = \int_0^\infty u^*(s,t)f(s)ds, f \in \bigcup_{N>0} L^2[0,N]$$

$$Uf = \lim_{N \to \infty} U(\chi_{[0,N]}f), f \in L^2[0,\infty)$$

define a unitary map $U : L^2[0,\infty) \longrightarrow L^2(\mathbb{R},\rho)$ (here, limit is norm limit in $L^2(\mathbb{R},\rho)$).

Then this map together with measure ρ provides a spectral representation.

The proof is classical, and we just skip it here, see [2] for more details.

We also provide an alternative proof in Chapter 5 after we understand the relation between canonical systems and Dirac operators.

Chapter 5

Dirac Operators as Canonical Systems

In this chapter, we want to study the relationship between Dirac operators with measures and canonical systems.

In section 5.1, we give some notations needed in this chapter, and some consequences used in section 5.2 are given. In section 5.2, we construct a mapping between Dirac operators with measures and a subset of canonical systems. We also prove that this mapping is bijective. In section 5.3, we give some corollaries which may be used in Chapter 7 based on our main result. Some of those corollaries can be proved easily when considering absolutely continuous measures. Also, a depiction of Weyl functions is given at the end of this chapter.

5.1 Some Notations and Preparations

A canonical system is defined as follows:

$$u'(x) = zJH(x)u(x)$$

on an open interval $x \in (a,b)$, $-\infty \leq a < b \leq \infty$, where z is a complex number, and H satisfies: (1) $H \in \mathbb{R}^{2\times 2}$, (2) $H \in L^1_{loc}(a,b)$, (3) H is Hermitian and non-negative definite for (Lebesgue) almost all $x \in (a,b)$.

Here, the coefficient H can represent this system uniquely up to a normalization(for example, trace normed), hence we sometimes simply say a canonical system H, see [2] for more details.

From now on, we temporarily fix the interval $[0,N]$ for $N < \infty$ and consider measures in DS. As we mentioned in Chapter 3, it is not essential that the part of a measure in DS on (N,∞) when we consider $[0,N]$, hence it is safe to assume the part of a measure on (N,∞) is 0, i.e., we consider

$$DS(N) := \{\mu \in DS : \mu(N,\infty) = 0\}$$

We also define a subset of canonical systems in $[0,N]$ as follows:

$$CD(N) := \{H \in C(N) : (1) H \in BV[0,N] \textit{ and right continuous;}$$

$$(2) det H = 1;\ (3) H(0) = 1\}$$

As expected, condition (1) means all entries of H are of bounded variation and right continuous; conditions (2) and (3) are the normalization that we desire.

We also need to consider the chain rule in this thesis which is due to Volpert. For our purpose, we don't need to describe the derivative globally, hence we just need the following weak version:

Volpert's chain rule: Let $I \subset \mathbb{R}$ be an open interval, $f : \mathbb{R}^m \to \mathbb{R}^n$ continuously differentiable , $u : I \to \mathbb{R}^m$ is of bounded variation, and S the set of all jump points of u defined as the set of all $x \in I$ where the approximate limit $\tilde{u}$ does not exist at x. Then

$$d(f(u)) = du \cdot df(\tilde{u})$$

in the sense of measures on $I \setminus S$, where d is the distributional derivative. See [11] for more details.

The description on S is much more intricate, we just ignore that. Moreover, this theorem was generalized by L.Ambrosio and G.Dal Maso in [11].

The following conclusion is important here.

Claim 5.1 $A = e^B$ for some $B \in \mathbb{R}^{2\times 2}$ satisfying $B = B^\top, trB = 0$ if and only if $A \in \mathbb{R}^{2\times 2}$ satisfies $detA = 1$, $A = A^\top$ and $A > 0$.

Proof: If $A = e^B$, then $\overline{A} = e^{\overline{B}} = e^B = A$, hence $A \in \mathbb{R}^{2\times 2}$.

Also, we have $A^\top = e^{B^\top} = e^B = A$ and $detA = e^{trB} = 1$.

Notice that eigenvalues of A are of the form e^λ, hence $A > 0$.

Conversely, $A^* = A$ and $A > 0$ give a matrix $B = \ln A$ by spectral theorem when we consider the spectral norm, i.e., $A = e^B$. The same calculation as above gives properties of B. ■

Pick up $H \in CD(N)$. Since $H^\top = H, H \geq 0$ and $detH(x) = \prod_{\lambda_i \in \sigma(H(x))} \lambda_i = 1$, we actually have $H > 0$ for all $x \in [0, N]$ and

$$H = \begin{pmatrix} R_1^2 & R_1 R_2 \cos\delta \\ R_1 R_2 \cos\delta & R_2^2 \end{pmatrix}$$

for some real function $R_1, R_2 > 0$ and δ such that $R_1 R_2 \sin\delta = 1$.

It is possible to find out the square root of H, denoted by $H^{\frac{1}{2}}$, as follows:

$$H^{\frac{1}{2}} = \begin{pmatrix} \frac{R_1^2+1}{\sqrt{R_1^2+R_2^2+2}} & \frac{R_1 R_2 \cos\delta}{\sqrt{R_1^2+R_2^2+2}} \\ \frac{R_1 R_2 \cos\delta}{\sqrt{R_1^2+R_2^2+2}} & \frac{R_2^2+1}{\sqrt{R_1^2+R_2^2+2}} \end{pmatrix} \tag{5.1}$$

Since $H \in BV[0, N]$ and right continuous; moreover, $\sqrt{R_1^2 + R_2^2 + 2} > 2$, it follows that $H^{\frac{1}{2}} \in BV[0, N]$ and right continuous.

We also denote the inverse of $H^{\frac{1}{2}}$ by $H^{-\frac{1}{2}}$ and want to consider when

$x > 0$ that $H^{-\frac{1}{2}}(x)H(x-)H^{-\frac{1}{2}}(x)$. Moreover, the collection of all jump points of H is defined by

$$S_N(H) := \{x \in (0, N] : H(x-) \neq H(x)\}$$

Obviously, this set contains countably many points, hence sometimes it is convenient to write $S_N(H) = \{x_1, x_2, x_3, \cdots\}$.

Claim 5.2 Assume $H \in CD(N)$, then there is a matrix $M(x) \in \mathbb{R}^{2\times 2}$ on $(0, N]$ satisfying $M(x) = M(x)^\top, trM(x) = 0$ for all $x \in (0, N]$ such that on $(0, N]$

$$H^{-\frac{1}{2}}(x)H(x-)H^{-\frac{1}{2}}(x) = e^{2JM(x)} \tag{5.2}$$

Moreover, $\sum\limits_{x \in S_N(H)} ||M(x)|| < \infty$

Proof: Since $detH(x-) = \lim\limits_{y \to x-} detH(y) = 1$, it follows that

$$det(H^{-\frac{1}{2}}(x)H(x-)H^{-\frac{1}{2}}(x)) = 1$$

Also, $H^{-\frac{1}{2}}(x)H(x-)H^{-\frac{1}{2}}(x) \in \mathbb{R}^{2\times 2}$ and symmetric. We want to show that $H^{-\frac{1}{2}}(x)H(x-)H^{-\frac{1}{2}}(x) > 0$. Indeed, since we have $H > 0$, it follows that $H^{-\frac{1}{2}} > 0$. Moreover, as $v^*H(x)v > 0$ for all $v \in \mathbb{C}^2$, we have $v^*H(x-)v \geq 0$. Since $detH(x-) = \prod\limits_{\lambda_i \in \sigma(H(x-))} \lambda_i = 1$, this implies that $\lambda_i > 0$, which is equivalent to $H(x-) > 0$. Thus, as a product of positive definite matrices, we have $H^{-\frac{1}{2}}(x)H(x-)H^{-\frac{1}{2}}(x) > 0$.

By claim 5.1, we have the existence of M. We also observe that $M(x) = 0$ if and only if H is continuous at x.

Since $H^{-\frac{1}{2}}, H \in BV[0, N]$, we have $\sup_{x\in[0,N]} ||H^{-\frac{1}{2}}(x)|| < \infty$, and from comparing with total variation of H, $\sum_{x\in S_N(H)} ||H(x) - H(x-)|| < \infty$ it follows

$$\sum_{x\in S_N(H)} ||H^{-\frac{1}{2}}(x)H(x-)H^{-\frac{1}{2}}(x) - I||$$

$$\leq 4 \sum_{x\in S_N(H)} ||H^{-\frac{1}{2}}(x)||^2 \cdot ||H(x) - H(x-)|| < \infty$$

This estimation demonstrates that we can pick up a number $L > 0$ such that

$$\sum_{i=L}^{\infty} ||H^{-\frac{1}{2}}(x_i)H(x_i-)H^{-\frac{1}{2}}(x_i) - I|| < \frac{1}{4}$$

If $i \geq L$, We have

$$\ln\left(H^{-\frac{1}{2}}(x_i)H(x_i-)H^{-\frac{1}{2}}(x_i)\right)$$

$$= \sum_{k=1}^{\infty}(-1)^{k+1}\frac{(H^{-\frac{1}{2}}(x_i)H(x_i-)H^{-\frac{1}{2}}(x_i) - I)^k}{k}$$

Hence

$$\begin{aligned}
||2JM(x_i)|| &= 2||M(x_i)|| \\
&\leq \sum_{k=1}^{\infty} \frac{2^{k-1}||H^{-\frac{1}{2}}(x_i)H(x_i-)H^{-\frac{1}{2}}(x_i) - I||^k}{k} \\
&\leq \frac{1}{2}\sum_{k=1}^{\infty}(2||H^{-\frac{1}{2}}(x_i)H(x_i-)H^{-\frac{1}{2}}(x_i) - I||)^k
\end{aligned}$$

This tells that

$$
\begin{aligned}
&\sum_{i=L}^{\infty} ||M(x_i)|| \\
&\qquad \leq \frac{1}{4}\sum_{k=1}^{\infty}(\sum_{i=L}^{\infty}(2||H^{-\frac{1}{2}}(x_i)H(x_i-)H^{-\frac{1}{2}}(x_i) - I||)^k) \\
&\qquad \leq \frac{1}{4}\sum_{k=1}^{\infty} 2^k(\sum_{i=L}^{\infty} ||H^{-\frac{1}{2}}(x_i)H(x_i-)H^{-\frac{1}{2}}(x_i) - I||)^k \\
&\qquad \leq \frac{1}{4}\sum_{k=1}^{\infty} \frac{1}{2^k} \\
&\qquad = \frac{1}{4}
\end{aligned}
$$

Which implies $\sum_{x\in S_N(H)} ||M(x)|| < \infty$. ■

5.2 the Main Result

We pick up $\mu \in DS(N)$, then the transfer matrix $T(x)$, or equivalently $T(x,0)$, with respect to this μ is unique due to Theorem 1. If we define $H(x) := T^{\top}(x)T(x)$, then it is trivial to show that $H \in CD(N)$. In fact, this observation gives a mapping between $DS(N)$ and $CD(N)$:

$$F : DS(N) \longrightarrow CD(N), \mu \mapsto H = T^{\top}T$$

Our main result in this chapter is the following claim:

Claim 5.3 F is bijective.

Proof: First of all, we discuss the relationship between $\mu \in DS(N)$ and its transfer matrix T. As we already mentioned, if μ is given, we just have a unique transfer matrix. On the other hand, if T is the transfer matrix of $\mu_1, \mu_2 \in DS(N)$, then we must have

$$\int_0^x (g(\mu_1\{s\}J)d\mu_1 - g(\mu_2\{s\}J)d\mu_2)T = 0$$

The left-hand side of the above equation is of bounded variation, and the distributional derivative is just 0 as a function, and this implies, by the approximation of C_c^∞ test functions, that

$\int_0^x (g(\mu_1\{s\}J)d\mu_1 - g(\mu_2\{s\}J)d\mu_2)$

$$= \int_0^x (g(\mu_1\{s\}J)d\mu_1 - g(\mu_2\{s\}J)d\mu_2)T \cdot T^{-1} = 0$$

Since $\int_{\{x\}} (g(\mu_1\{s\}J)d\mu_1 = \int_{\{x\}} (g(\mu_2\{s\}J)d\mu_2$, we conclude that

$$e^{\mu_1\{x\}}J = e^{\mu_2\{x\}}J$$

or equivalently, $\mu_1\{x\} = \mu_2\{x\}$. If A is a Borel set of $(0, N]$ out of the support of jump part of $\mu_1\{x\}(\mu_1\{x\})$, then we have

$$\int_A (g(\mu_1\{s\}J)d\mu_1 = \int_A (g(\mu_2\{s\}J)d\mu_2 = \int_A d\mu_1 = \int_A d\mu_2$$

by approximating χ_A by C_c^∞ test functions.

Combining those two facts together, we conclude that for any Borel set of

$(0, N]$, we have $\int_A d\mu_1 = \int_A d\mu_2$, i.e., $\mu_1 = \mu_2$.

Now, the statement above shows that, in order to complete the proof, we just need to find out a unique transfer matrix T such that $H = T^\top T$ for any $H \in CD(N)$.

Next, we assume the existence of such a transfer matrix, and we want to show that, under this assumption, this decomposition of H is unique, in other words, there is exactly one matrix satisfying this decomposition and is the transfer matrix for some $\mu \in DS(N)$. After that, we will show the existence.

By the assumption, we have $T^\top T = H^{\frac{1}{2}} H^{\frac{1}{2}}$, this gives $(TH^{-\frac{1}{2}})^\top (TH^{-\frac{1}{2}}) = I$. Since we also know that $det(TH^{-\frac{1}{2}}) = 1$, it follows from those two facts that

$$TH^{-\frac{1}{2}} = R_\theta$$

for some function $\theta(x)$ and $R_\theta(x) = \begin{pmatrix} \cos\theta(x) & -\sin\theta(x) \\ \sin\theta(x) & \cos\theta(x) \end{pmatrix}$ such that $R_\theta(0) = 1$. Moreover, $R_\theta \in BV[0, N]$ is right continuous.

As the transfer matrix of some Dirac operator $\sigma \in DS(N)$, we have $T(x-) = e^{J\sigma\{x\}} T(x)$, thus it follows that

$$H^{\frac{1}{2}}(x-) = R_\theta^{-1}(x-) T(x-) = R_\theta^{-1}(x-) R_\theta(x) e^{J R_\theta^{-1}(x) \sigma\{x\} R_\theta(x)} H^{\frac{1}{2}}(x)$$

Moreover,

$$H(x-) = (H^{\frac{1}{2}}(x-))^\top H^{\frac{1}{2}}(x-) = H^{\frac{1}{2}}(x)e^{2JR_\theta^{-1}(x)\sigma\{x\}R_\theta(x)}H^{\frac{1}{2}}(x)$$

Thus by claim 5.2, we conclude that

$$\sigma\{x\} = R_\theta(x)M(x)R_\theta^{-1}(x)$$

and

$$R_\theta(x-) = R_\theta(x)e^{JM(x)}H^{\frac{1}{2}}(x)H^{-\frac{1}{2}}(x-) \tag{5.3}$$

Those two identities imply that, even if we had two different transfer matrices (even though this is impossible as we will see later), the jump points of those corresponding rotations are the same. Moreover, the first identity also implies that x is a jump point of T if and only if x is a jump point of H.

Also recall that $T, H, H^{\frac{1}{2}}, R_\theta$ are continuous at x_0 if and only if $\mu\{x_0\} = 0$, and the continuity on $(0, N] \setminus S_N(H)$ implies the equivalence between a function u and its approximate limit $\tilde{u}$ on $(0, N] \setminus S_N(H)$, in other words, we can refine Volpert's chain rule (Theorem 2.34) by substituting $\tilde{u}$ by u directly in this case.

We can say more about $e^{JM(x)}H^{\frac{1}{2}}(x)H^{-\frac{1}{2}}(x-)$. By claim 5.2 and $e^{JM} =$

e^{-MJ} we have

$$(e^{JM(x)}H^{\frac{1}{2}}(x)H^{-\frac{1}{2}}(x-))^{\top}e^{JM(x)}H^{\frac{1}{2}}(x)H^{-\frac{1}{2}}(x-) = 1$$

Also observe that $det(e^{JM(x)}H^{\frac{1}{2}}(x)H^{-\frac{1}{2}}(x-)) = 1$, hence there is a function $\beta(x)$ such that

$$e^{JM(x)}H^{\frac{1}{2}}(x)H^{-\frac{1}{2}}(x-) = R_{\beta}(x)$$

We define a signed Borel measure $\widetilde{\mu}$ on $[0, N]$ by

$$\widetilde{\mu}(A) := \begin{cases} \int_{A\setminus\{0\}} JdT \cdot T^{-1}, & A \neq \{0\} \\ 0, & A = \{0\} \end{cases}$$

and using this measure, we define μ as follows:

(I)

$$\mu\{x\} := \begin{cases} R_{\theta}(x)M(x)R_{\theta}^{-1}(x), & x \in (0, N] \\ 0, & x = 0 \end{cases}$$

(II) For any Borel set A on $(0, N]$ that doesn't contain jump points of H, $\mu(A) := \widetilde{\mu}(A)$.

It's easy to check that this μ is a signed Borel measure.

Moreover, if $x \in (0, N]$, then

$$\widetilde{\mu}\{x\} = \int_{\{x\}} JdT \cdot T^{-1} = J(T(x) - T(x-))T^{-1}(x) = J(1 - e^{JR_{\theta}(x)M(x)R_{\theta}^{-1}(x)})$$

Notice that $g(\mu\{x\}J)\mu\{x\} = J(1 - e^{JR_\theta(x)M(x)R_\theta^{-1}(x)})$, we get

$$\widetilde{\mu}\{x\} = g(\mu\{x\}J)\mu\{x\}$$

This identity, with (II), shows that for any Borel set A on $(0, N]$, we have

$$\int_A d\widetilde{\mu} = \int_A g(\mu\{s\}J)d\mu$$

Since $\int_0^x d\widetilde{\mu}T = JT(x) - J$, it follows that

$$JT(x) - \int_0^x g(\mu\{s\}J)d\mu T = J$$

Notice that T is the transfer matrix of σ, then the same argument we use at the beginning of this proof shows that $\mu = \sigma \in DS(N)$.

We already proved that, if T is such a transfer matrix, then the corresponding measure is given by (I) and (II). Now, let us focus on $\widetilde{\mu}$ and write down μ explicitly to check how many measures given by (I) and (II) are indeed in $DS(N)$.

Recall (5.1), (5.2), and (5.3), it follows that if H is continuous at x, then so are $H^{\frac{1}{2}}(x)$ and R_θ, and this shows that jump points of $H^{\frac{1}{2}}(x)$ and R_θ are in $S_N(H)$. Thus, on $(0, N] \setminus S_N(H)$, by Volpert's chain rule, we have

$$d\widetilde{\mu} = J(dR_\theta H^{\frac{1}{2}} + R_\theta dH^{\frac{1}{2}})T^{-1} = R_\theta J dH^{\frac{1}{2}} H^{-\frac{1}{2}} R_\theta^{-1} + J dR_\theta R_\theta^{-1}$$

To avoid too many notations, let us simply write $H^{\frac{1}{2}} = \begin{pmatrix} h_1 & h_3 \\ h_3 & h_2 \end{pmatrix}$, then we have

$$JdH^{\frac{1}{2}}H^{-\frac{1}{2}} = \begin{pmatrix} h_3dh_2 - h_2dh_3 & h_3dh_3 - h_1dh_2 \\ h_2dh_1 - h_3dh_3 & h_1dh_3 - h_3dh_1 \end{pmatrix}$$

As $detH^{\frac{1}{2}} = 1$, we have $h_2dh_1 - h_3dh_3 = h_3dh_3 - h_1dh_2$, hence in fact, $JdH^{\frac{1}{2}}H^{-\frac{1}{2}}$ is symmetric. Once again, we denote this measure by

$$JdH^{\frac{1}{2}}H^{-\frac{1}{2}} = \begin{pmatrix} F_1 & F_3 \\ F_3 & F_2 \end{pmatrix}$$

Also, notice that this measure is determined only by H.

Now, we can calculate $d\widetilde{\mu}$:

$R_\theta JdH^{\frac{1}{2}}H^{-\frac{1}{2}}R_\theta^{-1}$

$$= \begin{pmatrix} F_1\cos^2\theta + F_2\sin^2\theta - \sin 2\theta F_3 & \frac{1}{2}\sin 2\theta(F_1 - F_2) + \cos 2\theta F_3 \\ \frac{1}{2}\sin 2\theta(F_1 - F_2) + \cos 2\theta F_3 & F_2\cos^2\theta + F_1\sin^2\theta + \sin 2\theta F_3 \end{pmatrix}$$

and

$$JdR_\theta R_\theta^{-1} = \begin{pmatrix} \sin\theta d\cos\theta - \cos\theta d\sin\theta & -(\sin\theta d\sin\theta + \cos\theta d\cos\theta) \\ \sin\theta d\sin\theta + \cos\theta d\cos\theta & \sin\theta d\cos\theta - \cos\theta d\sin\theta \end{pmatrix}$$

By Volpert's chain rule again (or integration by parts),

$$d(\cos^2\theta + \sin^2\theta) = 2(\sin\theta d\sin\theta + \cos\theta d\cos\theta) = 0$$

Moreover, by the construction of μ and the fact that $\mu = \sigma \in DS(N)$, it follows

$$\sin\theta d\cos\theta - \cos\theta d\sin\theta = -\frac{1}{2}(F_1 + F_2)$$

With the help of the fact that $\sin\theta d\sin\theta + \cos\theta d\cos\theta = 0$, we conclude

$$d\begin{pmatrix}\cos\theta \\ \\ \sin\theta\end{pmatrix} = \begin{pmatrix} 0 & -\frac{F_1+F_2}{2} \\ \\ \frac{F_1+F_2}{2} & 0 \end{pmatrix}\begin{pmatrix}\cos\theta \\ \\ \sin\theta\end{pmatrix}$$

If $x \in S_N(H)$, recall (5.3), then we have $R_\theta(x-) = R_\theta(x)R_\beta(x)$, and it implies that

$$\begin{pmatrix}\cos\theta(x-) \\ \\ \sin\theta(x-)\end{pmatrix} = R_\beta\begin{pmatrix}\cos\theta(x) \\ \\ \sin\theta(x)\end{pmatrix}$$

Also recall the fact $\int_{\{x\}} d\begin{pmatrix}\cos\theta \\ \\ \sin\theta\end{pmatrix} = \begin{pmatrix}\cos\theta(x) \\ \\ \sin\theta(x)\end{pmatrix} - \begin{pmatrix}\cos\theta(x-) \\ \\ \sin\theta(x-)\end{pmatrix}$, then it follows that

$$\int_{\{x\}} d\begin{pmatrix}\cos\theta \\ \\ \sin\theta\end{pmatrix} = \int_R (1 - R_\beta)d\delta_x\begin{pmatrix}\cos\theta \\ \\ \sin\theta\end{pmatrix}$$

where δ_x represents the Dirac measure at x.

We define a measure ω on $\mathbb{R}$ by

$$d\omega = \chi_{(0,N]\setminus S_N(H)} \begin{pmatrix} 0 & \frac{F_1+F_2}{2} \\ -\frac{F_1+F_2}{2} & 0 \end{pmatrix} + \sum_{x\in S_N(H)} (R_\beta(x)-1)d\delta_x$$

where, as usual, $\chi_{(0,N]\setminus S_N(H)}$ is the indicator of $(0,N]\setminus S_N(H)$.

What we have discussed above reveals the fact that $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ satisfies the following equation:

$$\begin{pmatrix} \cos\theta(x) \\ \sin\theta(x) \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \int_0^x d\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = -\int_0^x d\omega \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

Let us consider this integral equation

$$f(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \int_0^x d\omega f(s)$$

First of all, we claim that the measure ω is complex.

Indeed, the first part is fine since H is just of bounded variation on $[0,N]$, hence we just need to show that

$$\sum_{x\in S_N(H)} ||R_\beta(x)-1|| < \infty$$

or equivalently,

$$\sum_{i=1}^{\infty} ||e^{JM(x_i)} H^{\frac{1}{2}}(x_i) H^{-\frac{1}{2}}(x_i-) - 1|| < \infty$$

Notice that

$$||e^{JM(x_i)} H^{\frac{1}{2}}(x_i) H^{-\frac{1}{2}}(x_i-) - 1||$$

$$\leq ||(\sum_{k=1}^{\infty} \frac{(JM(x_i))^k}{k!}) H^{\frac{1}{2}}(x_i) H^{-\frac{1}{2}}(x_i-)|| + ||H^{\frac{1}{2}}(x_i) H^{-\frac{1}{2}}(x_i-) - 1||$$

and

$$||H^{\frac{1}{2}}(x_i) H^{-\frac{1}{2}}(x_i-) - 1|| \leq 2||H^{\frac{1}{2}}(x_i)|| \cdot ||H^{-\frac{1}{2}}(x_i-) - H^{-\frac{1}{2}}(x_i)||$$

$$||(\sum_{k=1}^{\infty} \tfrac{(JM(x_i))^k}{k!}) H^{\frac{1}{2}}(x_i) H^{-\frac{1}{2}}(x_i-)||$$

$$\leq 4||H^{\frac{1}{2}}(x_i)|| \cdot ||H^{-\frac{1}{2}}(x_i-)|| \cdot (\frac{1}{2} \sum_{k=1}^{\infty} \frac{2^k ||M(x_i)||^k}{k!})$$

Since $H^{\frac{1}{2}} \in BV[0, N]$, and recall $e^{JM(x)} H^{\frac{1}{2}}(x) H^{-\frac{1}{2}}(x-) = R_\beta(x)$, we conclude that there is a constant, denoted by C, such that

$$\max\{\sup_{x \in (0,N]} ||H^{\frac{1}{2}}(x)||, \sup_{x \in (0,N]} ||H^{\frac{1}{2}}(x-)||\} < C$$

Hence

$$\sum_{i=1}^{\infty} ||H^{\frac{1}{2}}(x_i) H^{-\frac{1}{2}}(x_i-) - 1|| \leq 2C \cdot V_0^N(H^{-\frac{1}{2}}) < \infty$$

and

$$\begin{aligned}\sum_{i=1}^{\infty}||(\sum_{k=1}^{\infty}\frac{(JM(x_i))^k}{k!})H^{\frac{1}{2}}(x_i)H^{-\frac{1}{2}}(x_i-)|| &\leq 2C^2\sum_{k=1}^{\infty}\frac{2^k\sum_{i=1}^{\infty}||M(x_i)||^k}{k!} \\ &\leq 2C^2\sum_{k=1}^{\infty}\frac{2^k(\sum_{i=1}^{\infty}||M(x_i)||)^k}{k!} \\ &= 2C^2(e^{2\sum_{i=1}^{\infty}||M(x_i)||}-1) \\ &< \infty\end{aligned}$$

we combine those estimations together, then we can reach our conclusion that the measure ω is complex.

On the other hand, it is easy to check that

$$\omega\{x\}+I=\begin{cases} I & x\notin S_N(H) \\ R_\beta(x) & x\in S_N(H)\end{cases}$$

Hence, by Jan Persson's Theorem 1, the solution of the integral equation is unique. This unique solution gives a unique R_θ, which means that the transfer matrix $T=R_\theta H^{\frac{1}{2}}$ is unique.

The only issue left is the existence of such a transfer matrix. In fact, this part also comes from the integral equation above.

The measure ω which is determined only by H that we introduced above gives a unique transfer matrix T_0 (which is of bounded variation) satisfying

$$T_0(x) = I - \int_0^x d\omega T_0$$

If we write down T_0 explicitly as $T_0 = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$, we have

$$\begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \int_0^x d\omega \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Observe that $d\omega$ is of the form $\begin{pmatrix} d\omega_1 & d\omega_2 \\ -d\omega_2 & d\omega_1 \end{pmatrix}$, hence we can rewrite this equation as follows:

$$\begin{pmatrix} -u_2(x) \\ u_1(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \int_0^x d\omega \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix}$$

From the uniqueness of the solution of the equation, it follows that

$$\begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix} = \begin{pmatrix} -u_2(x) \\ u_1(x) \end{pmatrix}$$

We want to show that $detT_0 = 1$. Notice that, since here we are interested in the value at a point, so integration by parts works, and we don't need to invoke the chain rule.

By integration by parts, we have the following two equations:

$$\int_0^x u_1(s)du_1 = u_1^2(x) - u_1^2(0) - \int_0^x u_1(s-)du_1$$

$$\int_0^x u_2(s)du_2 = u_2^2(x) - u_2^2(0) - \int_0^x u_2(s-)du_2$$

Recall that $u_i\{x\} = u_i(x) - u_i(x-)$, hence it follows that

$$2\int_0^x (u_1(s)du_1 + u_2(s)du_2) = detT_0 - 1 + \sum_{s\in(0,x]\cap S_N(H)} (u_1^2\{s\} + u_2^2\{s\})$$

the right-hand side is from the fact that

$$\begin{pmatrix} u_1\{x\} \\ u_2\{x\} \end{pmatrix} = \omega\{x\} \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}$$

or equivalently

$$\begin{pmatrix} u_1(x-) \\ u_2(x-) \end{pmatrix} = -(\omega\{x\} + 1) \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}$$

Because we have

$$u_1^2\{s\} + u_2^2\{s\} = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}^\top (R_\beta^\top - I)(R_\beta - I) \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}$$

we actually get

$$\sum_{s\in(0,x]\cap S_N(H)} (u_1^2\{s\} + u_2^2\{s\}) = 2\sum_{s\in(0,x]\cap S_N(H)} (1 - \cos\beta(s))(u_1^2(s) + u_2^2(s))$$

On the other hand, the distributional derivatives of u_1, u_2 can be obtained directly from the integral equation, hence it follows that

$$\int_0^x (u_1(s)du_1 + u_2(s)du_2) = \int_0^x (u_1^2(s) + u_2^2(s)) \sum_{x \in S_N(H)} (1 - \cos_\beta(x)) d\delta_x(s)$$

If we combine all observations above together, we conclude this important result that $detT_0(x) = 1$.

Now, it follows that

$$T_0 = R_\theta$$

for some θ such that R_θ is of bounded variation on $[0, N]$ and right continuous.

We claim that $T := R_\theta H^{\frac{1}{2}}$ is the transfer matrix of some measure $\mu \in DS(N)$. Notice that $T^\top T = H$ since $detT_0(x) = 1$.

Indeed, notice that T is of bounded variation, hence dT is a Borel measure. We define a signed Borel measure on $[0, N]$ by

$$\widetilde{\mu}(A) := \begin{cases} \int_{A \setminus \{0\}} JdT \cdot T^{-1}, & A \neq \{0\} \\ 0, & A = \{0\} \end{cases}$$

And measure μ by

(I)

$$\mu\{x\} := \begin{cases} R_\theta(x)M(x)R_\theta^{-1}(x), & x \in (0,N] \\ 0, & x = 0 \end{cases}$$

(II) For any Borel set A on $(0,N]$ that doesn't contain jump points of H, $\mu(A) := \widetilde{\mu}(A)$.

We still have, if $x \in (0,N]$, that

$$\widetilde{\mu}\{x\} = \int_{\{x\}} JdT \cdot T^{-1}$$

$$= J(T(x) - T(x-))T^{-1}(x) = J(1 - R_\theta(x-)H^{\frac{1}{2}}(x-)H^{-\frac{1}{2}}(x)R_\theta^{-1}(x))$$

Since R_θ satisfies $R_\theta(x) = I - \int_0^x d\omega R_\theta$, we conclude that

$$R_\theta(x-) = (\omega\{x\} + I)R_\theta(x)$$

Notice that

$$(\omega\{x\} + I)R_\theta(x) = R_\theta(x)(\omega\{x\} + I)$$

and

$$\omega\{x\} + I = e^{JM(x)}H^{\frac{1}{2}}(x)H^{-\frac{1}{2}}(x-)$$

it follows that

$$\widetilde{\mu}\{x\} = J(1 - e^{JR_\theta(x)M(x)R_\theta^{-1}(x)}) = J(1 - e^{J\mu\{x\}}) = g(\mu\{x\}J)\mu\{x\}$$

Thus, we have

$$\int_0^x d\widetilde{\mu} = \int_0^x g(\mu\{s\}J)d\mu$$

Since $\int_0^x d\widetilde{\mu}T = JT(x) - J$ by the definition of $\widetilde{\mu}$, it follows that

$$JT(x) - \int_0^x g(\mu\{s\}J)d\mu T = J$$

Next, we want to show that $\mu \in DS(N)$.

From $R_\theta(x-) = (\omega\{x\} + I)R_\theta(x)$, we conclude that $R_\theta(x)$ is continuous if and only if $\omega\{x\} = 0$, and this is always true if $x \notin S_N(H)$. If a Borel set A on $(0, N]$ that doesn't contain jump points of H is given, then we have, by Volpert's chain rule again, that

$$\mu(A) = \int_A R_\theta J dH^{\frac{1}{2}} H^{-\frac{1}{2}} R_\theta^{-1} + J \int_A dR_\theta R_\theta^{-1}$$

Moreover, by the definition of ω, we have that

$$dR_\theta = \begin{pmatrix} 0 & \frac{F_1+F_2}{2} \\ -\frac{F_1+F_2}{2} & 0 \end{pmatrix} R_\theta$$

in the sense of distribution on A, hence,

$$J\int_A dR_\theta R_\theta^{-1} = -\int_A \begin{pmatrix} \frac{F_1+F_2}{2} & 0 \\ 0 & \frac{F_1+F_2}{2} \end{pmatrix}$$

Recall that $JdH^{\frac{1}{2}}H^{-\frac{1}{2}} = \begin{pmatrix} F_1 & F_3 \\ F_3 & F_2 \end{pmatrix}$ and (I), it follows that μ has the correct form.

Define $||\mu||(A) := \max_{i=1,2}(|\mu_i|(A))$ for a Borel set A on $[0, N]$, then

$$||\mu||([0, N]) \leq ||\mu||(S_N(H)) + ||\mu||([0, N] \setminus S_N(H))$$

Since we have $\sum_{x \in S_N(H)} ||M(x)|| < \infty$, $||\mu||([0, N] \setminus S_N(H)) = ||\widetilde{\mu}||([0, N] \setminus S_N(H))$ and T is of bounded variation, it follows that $||\mu||([0, N]) < \infty$, i.e., $\mu \in DS(N)$.

We completed the proof. ■

5.3 Some Corollaries

We have some interesting corollaries.

Let us define a subset of $DS(N)$ as follows:

$$DS_{ac}(N) := \{\mu \in DS(N) : d\mu \ll dt\}$$

where dt is the Lebesgue measure on $\mathbb{R}$.

Obviously, a measure from $DS_{ac}(N)$ gives a regular Dirac equation. We define a subset of $CD(N)$ as well:

$$CD_{ac}(N) := \{H \in C(N) : (1) H \in AC[0,N];\ (2) detH = 1;$$

$$(3) H(0) = 1\}$$

The restriction of F on $DS_{ac}(N)$, denoted by $F|_{DS_{ac}(N)}$, maps $DS_{ac}(N)$ into $CD_{ac}(N)$. To obtain this conclusion, we just need to observe that the transfer matrix T satisfies $JT' = fT$ for some $f \in L^1[0,N]$, which means T is absolutely continuous. Moreover, we have the following claim:

Corollary 5.4 $F|_{DS_{ac}(N)}$ is bijective.

The proof of this claim 5.4 is much easier, and we don't want to repeat this tedious calculation here, so we just state the sketch of the proof.

The sketch of the proof: Pick up $H \in CD_{ac}(N)$, then we still have

$$H = \begin{pmatrix} R_1^2 & R_1 R_2 \cos\delta \\ R_1 R_2 \cos\delta & R_2^2 \end{pmatrix}$$

for some real function $R_1, R_2 > 0$ and δ such that $R_1 R_2 \sin\delta = 1$. Notice that $R_i^2 \in AC[0,N]$ and they are not 0, then we conclude that $R_i, \delta = \arcsin \frac{1}{R_1 R_2} \in AC[0,N]$.

We claim that, as in the proof of claim 5.3, the unique θ given by

$$\theta = \frac{1}{2}R_1R_2\cos\delta - \int_0^x R_1'R_2\cos\delta dt$$

derives the unique rotation R_θ.

Moreover, as a function rather than a measure, the coefficient f can be deduced easily from a differential equation:

$$f = JT'T^{-1}$$

This gives a measure in $DS_{ac}(N)$. ■

One can expect that if we just delete the discrete part of a measure, then we may have an analogous conclusion. So we also define a subset of $DS(N)$ as follows:

$$DS_c(N) := \{\mu \in DS(N) : \mu\{x\} = 0,\ \forall x \in [0, N]\}$$

Also, the subset of $CD(N)$ is defined as:

$$CD_c(N) := \{H \in C(N) : (1)H \in C[0, N];\ (2)detH = 1;$$

$$(3)H(0) = 1\}$$

Here, that H is continuous at boundary points means the value of H is the same as the left (right) limit at the corresponding boundary point.

Moreover, the continuity on a compact set is essential since this gives a bounded function, hence is of bounded variation.

Once again, The restriction of F on $DS_c(N)$, denoted by $F|_{DS_c(N)}$, maps $DS_c(N)$ into $CD_c(N)$. We have the following conclusion:

Corollary 5.5 $F|_{DS_c(N)}$ is bijective.

The proof is absolutely the same as the proof of claim 5.3, the only improvement is $R_\beta(x) = 1$, hence the measure μ defined satisfies $\mu\{x\} = 0$. We just skip the proof.

Now we are ready to prove claim 4.18.

Proof of the claim 4.18:

With a little adaption, we conclude from claim 5.3 that $\mu \in DS$ if and only if $H = T^\top T \in C$, $H \in BV[0,\infty)$, $detH = 1$ and $H(0) = 1$. Here, $T(x) = T(x,0)$ is the transfer matrix of the Dirac operator with respect to μ.

We claim that, $f \in L^2[0,\infty)$ is a solution of

$$Jf(x) - \int_0^x g(\mu\{s\}J)d\mu f = Jf(0) - z\int_0^x f dt$$

if and only if $g := T^{-1}f \in L^2_H[0,\infty)$ is a solution of

$$g' = zJHg = zJT^\top Tg, g(0) = f(0)$$

Indeed, notice that by the definition of the norm in $L^2_H[0,\infty)$, we have $||g||_{L^2_H} = ||f||_{L^2}$. Moreover, claim 3.3 characterizes a Dirac equation: since

$$g(x) = g(0) + z\int_0^x JT^\top Tg dt$$

and observe that $JT^\top = T^{-1}J$, then we have

$$f(x) = T(x)f(0) + T(x)\int_0^x T^{-1}J(zf)dt$$

Comparing with (3.7), we conclude that f is indeed the solution of the Dirac equation with $k = zf$, and this equation is the desired one.

This observation, with the definition of Weyl functions of Dirac operators and of canonical systems, implies that the given Dirac operator and the corresponding canonical system share the same Weyl function, and hence spectral measure.

Now, claim 4.18 is a direct conclusion of the spectral representation theorem of canonical systems, see[2]. ■

We turn to a depiction of Weyl functions to end this chapter.

Claim 5.6 Assume $\mu \in DS$ and the limit point case at ∞. Then the

Weyl function is given by

$$m(z) = a + \int_{\mathbb{R}} (\frac{1}{t-z} - \frac{t}{t^2+1}) d\rho(t)$$

where $\rho(\mathbb{R}) = \infty$ and ρ is not compact supported.

Proof: Notice that the corresponding canonical system has 1 as the determinant, which implies there is no singular point. This claim follows from theorem 2.3, 2.4 and 2.5. $\blacksquare$

Chapter 6

De Branges Spaces of Dirac Operators

In this chapter, we want to discuss the inverse spectral theory. Here, de Branges theory plays a significant role: given a Dirac operator with a measure, we can define a de Branges function originating from a solution of eigenvalue problems, and the spectral representation theorem from Chapter 4 in fact gives an isometry between L^2 and the de Branges space generated by the de Branges function. On the other hand, the spectral representation theorem also implies that the spectral measure gives some information about the inner product of the de Branges space.

In section 6.1, we give a transformation that can transfer the integral equation with respect to a Dirac operator with a measure to another integral equation which can be solved by iteration. In section 6.2, we introduce some background about measures and Fourier transform. In section 6.3, we discuss de Branges spaces generated by a Dirac operator and characterize those spaces as Paley-Wiener spaces with some proper inner products. In section 6.4, with a stronger assumption, we show that a Paley-Wiener space with a proper inner product gives a regular de Branges space, hence a canonical system can be found from this de Branges space. In section 6.6, we discuss the regularities of two integral equations so that we may write down this canonical system explicitly.

6.1 A Transformation

In this section, we will extend a well-known transformation. For convenience, we restrict ourselves on the interval $[0, N]$, but as we have done many times, there is no technical difficulty to extend to the half line $[0, \infty)$.

Consider $[0, N]$ and let $\mu \in DS$. Analogously, we define

$$S_N(\mu) := \{x \in [0, N] : x \in S(\mu)\} = \{x_1, x_2, x_3, \cdots\}$$

and a function on $(0, \infty)$ by

$$t(x) = (\mu_1^2\{x\} + \mu_2^2\{x\})^{\frac{1}{2}}$$

obviously, $t(x) \neq 0$ if and only if $x \in S(\mu)$. This function gives a measure ω which is defined as follows:

$$\omega := \sum_{x \in S_N(\mu)} \frac{e^{t(x)} + e^{-t(x)} - 2}{2} \delta_x$$

where δ_x is the Dirac measure at x.

Claim 6.1 There is a unique function k in $BV[0,\infty)$ which is right continuous such that

$$k(x) = 1 - \int_0^x d\omega k$$

As a consequence, the collection of all jump points of k is exactly $S_N(\mu)$.

Proof: First of all, we claim that ω is complex. Indeed, as we have done in Chapter 5, we have

$$|\omega|(\mathbb{R}) = \tfrac{1}{2} \sum_{x \in S_N(\mu)} (e^{t(x)} + e^{-t(x)} - 2) = \tfrac{1}{2} \sum_{x \in S_N(\mu)} (\textstyle\sum_{n=1}^{\infty} \frac{t^{2n}(x)}{(2n)!})$$

$$= \tfrac{1}{2} \textstyle\sum_{n=1}^{\infty} \frac{\sum\limits_{x \in S_N(\mu)} t^{2n}(x)}{(2n)!}$$

$$\leq \tfrac{1}{2} \textstyle\sum_{n=1}^{\infty} \frac{(\sum\limits_{x \in S_N(\mu)} t(x))^{2n}}{(2n)!}$$

$$= \tfrac{1}{2}(e^{\sum\limits_{x \in S_N(\mu)} t(x)} + e^{-\sum\limits_{x \in S_N(\mu)} t(x)} - 2) < \infty$$

The last inequality comes from $\sum\limits_{x \in S_N(\mu)} t(x) < \infty$.

On the other hand, we have for $x \in S_N(\mu)$,

$$\omega\{x\} + 1 = \frac{e^{t(x)} + e^{-t(x)}}{2} \neq 0$$

Now, we apply Jan Persson's theorem (see [1] for $n = 1$) to get this conclusion. If one wants to use theorem 2 that we just introduced in Chapter 3, then a diagonal 2×2 matrix with entries k works. ■

The following property is essential when we define the transformation.

Claim 6.2 There is $\epsilon, M > 0$ such that $\epsilon \leq |k| \leq M$.

Proof: Since $k \in BV[0, \infty)$ is right continuous, hence the existence of M is trivial. We just need to show the existence of ϵ.

First, we claim that $k(x) \neq 0$. Indeed, if $k(x_0) = 0$ for some x_0, then we have $0 = 1 - \int_0^{x_0} d\omega k$, hence the equation

$$k(x) = (1 - \int_0^{x_0} d\omega k) - \int_{x_0}^{x} d\omega k = - \int_{x_0}^{x} d\omega k, x \geq x_0$$

$$k(x) = \int_{x}^{x_0} d\omega k, x < x_0$$

has a unique solution which is still $k(x)$; however, 0 is the solution of this equation, which means $k = 0$. This contradicts with $k(0) = 1$.

Second, as before, we have $k(x-) = (1 + \omega\{x\})k(x)$, or equivalently, $k(x) = \frac{k(x-)}{1+\omega\{x\}}$. Moreover, we also have that $k(x) = 1$ for $x < 0$ and $k(x) = k(N)$ for $x > N$.

If $\inf\limits_{x\in[0,N]} |k| = 0$, then there is a sequence $\{x_n\}_{n=1}^{\infty} \subset [0, N]$ such that for any $n \in \mathbb{N}^+$,

$$|k(x_n)| < \frac{1}{n}$$

Since $[0, N]$ is compact, there is a convergent subsequence of $\{x_n\}_{n=1}^{\infty}$ which we still denote by $\{x_n\}_{n=1}^{\infty}$. In other words, $\lim\limits_{n\to\infty} x_n = x_0 \in [0, N]$.
If x_n approximates to x_0 from the right-hand side, i.e., there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $x_0 < x_{n_k}$ and $\lim\limits_{k\to\infty} x_{n_k} = x_0$, then by the existence of the right limit of a function of bounded variation, it follows that

$$|k(x_0)| = |k(x_0+)| = \lim_{k\to\infty} |k(x_{n_k})| \leq \lim_{k\to\infty} \frac{1}{n_k} = 0$$

If x_n can approximate to x_0 from the left-hand side, then we still have such a subsequence, hence

$$|k(x_0)| = \frac{|k(x_0-)|}{1+\omega\{x_0\}} = \frac{\lim\limits_{k\to\infty} |k(x_{n_k})|}{1+\omega\{x_0\}} \leq \lim_{k\to\infty} \frac{1}{n_k(1+\omega\{x_0\})} = 0$$

In all, we conclude that $k(x_0) = 0$, which is impossible. Since $\inf\limits_{x\in[0,N]} |k| > 0$, we simply let $\epsilon = \inf\limits_{x\in[0,N]} |k|$. ■

It's time to introduce the transformation.

Assume y is the solution of the equation

$$Jy(x) - \int_0^x g(\mu\{s\}J)d\mu y = C - z\int_0^x y dt$$

We define a function

$$f := \frac{1}{k} Q^{-1} y$$

where $Q(x,z) = \begin{pmatrix} e^{izx} & e^{-izx} \\ -ie^{izx} & ie^{-izx} \end{pmatrix}$.

Due to claim 6.2, f is well-defined everywhere. Moreover, we have the following conclusion:

Claim 6.3 On the interval $[0, N]$, f satisfies

$$f(x) = \frac{1}{2} \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} C + \int_0^x \begin{pmatrix} 0 & e^{-2izs} dP(s) \\ e^{2izs} d\overline{P}(s) & 0 \end{pmatrix} f(s)$$

where $dP(s) = \frac{e^{t(s)} - e^{-t(s)}}{t(s)(e^{t(s)} + e^{-t(s)})} (d\mu_2(s) - id\mu_1(s))$

Proof: First of all, we claim that

$$Jg(\mu\{x\}J) = \frac{e^{t(x)} - e^{-t(x)}}{2t(x)} J + \frac{e^{t(x)} + e^{-t(x)} - 2}{2t^2(x)} \mu\{x\}$$

here, coefficients should be interpreted as limits, i.e., when $t(x) = 0$, then $\frac{e^{t(x)} - e^{-t(x)}}{2t(x)} = 1$ and $\frac{e^{t(x)} + e^{-t(x)} - 2}{2t^2(x)} = \frac{1}{2}$.

Indeed, if $t(x) = 0$, then $\mu\{x\} = 0$, which implies $Jg(0) = J$.

If $t(x) \neq 0$, and notice that $(\mu\{x\}J)^2 = t^2(x)I$, we have

$$\begin{aligned} g(\mu J) &= \sum_{n=1}^{\infty} \frac{(\mu J)^{n-1}}{n!} \\ &= \frac{e^t - e^{-t}}{2t} I + \frac{e^t + e^{-t} - 2}{2t^2} \mu J \end{aligned}$$

Observe that $J\mu J = \mu$, thus we get the desired identity.

Now, we fix $z \in \mathbb{C}$. Thanks to claim 6.2, it follows that f is of bounded variation on $[0, N]$ and right continuous. Also notice that $y(0-) = y(0)$ as $\mu\{0\} = 0$, we have

$$\int_{[0,x]} d(kQ)f = k(x)Q(x)f(x) - y(0) - \int_{[0,x]} k(s-)Q(s)df(s) \tag{6.1}$$

by integration by parts on $[0, N]$.

By Volpert's chain rule, we have

$$\begin{aligned} \int_{[0,x]} d(kQ)f &= \int_{[0,x]\setminus S_N(\mu)} d(kQ)f + \int_{[0,x]\cap S_N(\mu)} d(kQ)f \\ &= \int_{[0,x]\setminus S_N(\mu)} d(k)Qf + \int_{[0,x]\setminus S_N(\mu)} k(dQ)f \\ &+ \int_{[0,x]\cap S_N(\mu)} d(kQ)f \end{aligned}$$

Observe that

$$\int_{[0,x]\setminus S_N(\mu)} k(dQ)f = zJ \int_{[0,x]\setminus S_N(\mu)} kQf ds = zJ \int_0^x kQf ds$$

and

$$\int_{[0,x]\cap S_N(\mu)} d(kQ)f = \sum_{s\in[0,x]\cap S_N(\mu)} (k(s)-k(s-))Q(s)f(s)$$
$$= \int_{[0,x]\cap S_N(\mu)} d(k)Qf$$

Hence, it follows that

$$\int_{[0,x]} d(kQ)f = \int_{[0,x]} d(k)Qf + zJ\int_0^x kQf ds \tag{6.2}$$

Moreover, by the definition of y, we get

$$k(x)Q(x)f(x) - y(0) = -J(\int_0^x kg(\mu J)d\mu Qf - z\int_0^x kQf dt) \tag{6.3}$$

Combining (6.1), (6.2) and (6.3) together, we get

$$\begin{aligned}
&-\int_0^x kJg(\mu J)d\mu Qf \\
&= \int_{[0,x]} d(k)Qf + \int_{[0,x]} k(s-)Q(s)df(s) \\
&= -\int_0^x \frac{e^{t(s)}-e^{-t(s)}}{2t(s)}kJd\mu Qf - \int_0^x \frac{e^{t(s)}+e^{-t(s)}-2}{2t^2(s)}k\mu\{s\}d\mu Qf
\end{aligned}$$

Let's investigate the measure $dk\cdot I + \frac{e^{t(s)}+e^{-t(s)}-2}{2t^2(s)}k\mu\{s\}d\mu$.

We have

$$\begin{aligned}
&\int_0^x (dk \cdot I + \frac{e^{t(s)} + e^{-t(s)} - 2}{2t^2(s)} k\mu\{s\} d\mu) \\
&= (k(x) - 1)I + \sum_{s \in [0,x] \cap S_N(\mu)} k(s) \frac{e^{t(s)} + e^{-t(s)} - 2}{2} I \\
&= (k(x) - 1 + \int_0^x d\omega k)I = 0
\end{aligned}$$

Also notice that k and f are continuous at 0, hence we have

$$\int_{[0,x]} d(k)Qf = -\int_0^x \frac{e^{t(s)} + e^{-t(s)} - 2}{2t^2(s)} k\mu\{s\} d\mu Qf$$

Thus

$$\int_{[0,x]} k(s-)Q(s)df(s) = -\int_0^x \frac{e^{t(s)} - e^{-t(s)}}{2t(s)} kJ d\mu Qf$$

Recall $k(x-) = (1 + \omega\{x\})k(x)$, we get

$$f(x) - f(0) = \int_0^x df = -\int_0^x \frac{e^{t(s)} - e^{-t(s)}}{t(s)(e^{t(s)} + e^{-t(s)})} (Q^{-1} J d\mu Q) f$$

This one, with $f(0) = \frac{1}{2} \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} C$, gives the desired equation. ■

6.2 Some Notations about Measures

In this section, we introduce some notations used in the sequel.

Let $\mu \in \mathcal{M}^b(\mathbb{R})$ the complex Borel measures space on $\mathbb{R}$. We do have many equivalent ways to define the Fourier transform of μ, for example, via distribution theory or via Fourier analysis of locally compact Abelian groups [18]. Here, we just simply use the most intuitive way, that is, the Fourier transform of μ is defined by

$$\widehat{\mu}(t) = \int_{\mathbb{R}} e^{its} d\mu(s)$$

From this form, one can immediately know that $\widehat{\mu}(t)$ is bounded by the total variation of μ, and moreover, thanks to Lebesgue's dominated convergence theorem, $\widehat{\mu}(t)$ is continuous on $\mathbb{R}$. We want to mention that Lebesgue's dominated convergence theorem doesn't work in the sense of nets, so we actually use the sequence version, then deal with sequences in complex numbers. If μ is absolutely continuous with respect to the Lebesgue measure, then the classical Riemann-Lebesgue lemma shows that $\widehat{\mu}(t)$ in fact is continuous on the Riemann sphere $\mathbb{R}_\infty$; however, this is no need to be true for a measure containing discrete part. the easiest example could be δ_1.

Given measurable spaces (X_1, M_1) and (X_1, M_1), a measurable function $f : X_1 \to X_2$, and a (positive, complex, signed, etc.) measure μ on M_1, the pushforward measure of μ under f is defined to be the measure $\mu_*(A) := \mu(f^{-1}(A))$ for $A \in M_2$. μ_* plays as a change of variable.

Let $\mu, \rho \in \mathcal{M}^b(\mathbb{R})$, the convolution $\mu * \rho$ is the pushforward measure of the product measure $\mu \times \rho$ under the addition map $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, (x, y) \to$

$x+y$, i.e., $\mu * \rho(A) = \int_{\mathbb{R}\times\mathbb{R}} 1_A(x+y)d(\mu(x)\times\rho(y))$.

Claim 6.4 Let $\mu \in \mathcal{M}^b(\mathbb{R})$, and $\mu_d = \sum_{n=1}^{\infty} c_n\delta_{x_n}$ be discrete part of μ. Then we have

$$\lim_{R\to\infty}\frac{1}{2R}\int_{-R}^{R}|\widehat{\mu}(t)|^2dt = \sum_{n=1}^{\infty}|c_n|^2$$

where $\widehat{\mu}(t) = \int_{\mathbb{R}} e^{its}d\mu(s)$ is the Fourier transform of μ.

Proof: Let us define μ_r to be the pushforward measure of $\overline{\mu}$ under $f(x) = -x$, i.e., $\mu_r(A) = \overline{\mu}(f^{-1}(A))$ for any Borel set $A \subset \mathbb{R}$. Obviously, μ_r is a complex measure; moreover, we have $\widehat{\mu_r}(t) = \overline{\widehat{\mu}(t)}$.

We consider $\mu * \mu_r$. The Fourier transform of this convolution is

$$\widehat{\mu * \mu_r}(t) = \int_{\mathbb{R}\times\mathbb{R}} e^{it(x+y)}d\mu(x)\times\mu_r(y) = \widehat{\mu_r}(t)\widehat{\mu}(t) = |\widehat{\mu}(t)|^2$$

On the other hand, as the convolution of two complex measures, it is also complex. It hence follows from Fubini's theorem that

$$\frac{1}{2R}\int_{-R}^{R}|\widehat{\mu}(t)|^2dt = \frac{1}{2R}\int_{-R}^{R}\widehat{\mu * \mu_r}(t)dt = \int_{\mathbb{R}} f_R(s)d\mu * \mu_r(s)$$

where

$$f_R(s) = \frac{1}{2R}\int_{-R}^{R} e^{its}dt$$

Since $|f_R| \leq 1$, $f_R(0) = 1$ and $f_R(s) = \frac{e^{iRs} - e^{-iRs}}{2iRs}$ if $s \neq 0$, we conclude that $\lim_{R \to \infty} f_R(s) = 0$ except for $s = 0$. By Lebesgue's dominated convergence theorem for sequences, and treat $\frac{1}{2R} \int_{-R}^{R} |\widehat{\mu}(t)|^2 dt$ as a function on $\mathbb{R}$, we have

$$\lim_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} |\widehat{\mu}(t)|^2 dt = \mu * \mu_r\{0\}$$

By the definition of convolution again,

$$\mu * \mu_r\{0\} = \int_{\mathbb{R} \times \mathbb{R}} 1_{\{0\}}(x+y) d\mu(x) \times \mu_r(y) = \int_{\mathbb{R}} \mu_r\{-x\} d\mu(x) = \sum_{n=1}^{\infty} |c_n|^2$$

Combining all together, we finally get

$$\lim_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} |\widehat{\mu}(t)|^2 dt = \sum_{n=1}^{\infty} |c_n|^2$$

■

An intriguing corollary of this claim 6.4 gives an intuition about some measures called Rajchman measures.

Corollary 6.5 Assume $\mu \in \mathcal{M}^b(\mathbb{R})$ is continuous (with respect to the Lebesgue measure) on $\mathbb{R}$. then those two conditions are equivalent:

(1) $\lim_{t \to \pm\infty} |\widehat{\mu}(t)|$ exists;

(2) $\lim_{t \to \pm\infty} |\widehat{\mu}(t)| = 0$.

Proof: We just need to show (1) $\Rightarrow$ (2). Assume (2) is not true, then from claim 6.4, it follows that

$$\lim_{R\to\infty}\frac{1}{2R}\int_{-R}^{R}|\widehat{\mu}(t)|^2dt = 0$$

As μ is complex, we know that $\widehat{\mu}(t)$ is continuous and bounded by the total variation of μ. We pick up a $M \in \mathbb{R}^+$, if we have $|\widehat{\mu}(t)| > \epsilon$ for some $\epsilon > 0$ when $t > M$, then when R is large enough, we have

$$\begin{aligned}\frac{1}{2R}\int_{-R}^{R}|\widehat{\mu}(t)|^2dt &= \frac{1}{2R}\int_{-R}^{M}|\widehat{\mu}(t)|^2dt + \frac{1}{2R}\int_{M}^{R}|\widehat{\mu}(t)|^2dt \\ &> \frac{R-M}{2R}\epsilon^2\end{aligned}$$

When $R \to +\infty$, we have $\lim\limits_{R\to\infty}\frac{1}{2R}\int_{-R}^{R}|\widehat{\mu}(t)|^2dt \geq \frac{\epsilon^2}{2}$. This contradiction implies that $\lim\limits_{t\to+\infty}|\widehat{\mu}(t)| = 0$. The same conclusion for $t \to -\infty$ can be achieved once we consider $\frac{1}{2R}\int_{-R}^{-M}|\widehat{\mu}(t)|^2dt$. ■

Corollary 6.6 Assume $\mu \in \mathcal{M}^b(\mathbb{R})$. If $\lim\limits_{t\to\pm\infty}|\widehat{\mu}(t)| = 0$, then μ is continuous.

Proof: We still pick up a $M \in \mathbb{R}^+$ such that $|\widehat{\mu}(t)| < \epsilon$ for some $\epsilon > 0$ when $|t| > M$, then for large R,

$$\begin{aligned}\frac{1}{2R}\int_{-R}^{R}|\widehat{\mu}(t)|^2dt &= \frac{1}{2R}(\int_{-M}^{M}|\widehat{\mu}(t)|^2dt + \int_{M}^{R}|\widehat{\mu}(t)|^2dt + \int_{-R}^{-M}|\widehat{\mu}(t)|^2dt) \\ &< \frac{R-M}{R}\epsilon^2 + \frac{1}{2R}\int_{-M}^{M}|\widehat{\mu}(t)|^2dt\end{aligned}$$

Thus we have

$$\lim_{R\to\infty} \frac{1}{2R}\int_{-R}^{R} |\widehat{\mu}(t)|^2 dt \le \epsilon^2$$

As ϵ is arbitrary, and with claim 6.4, we conclude that the discrete part of μ is empty. ■

We denote continuous complex measures by $\mathcal{M}_c^b(\mathbb{R})$, i.e.,

$$\mathcal{M}_c^b(\mathbb{R}) := \{\mu \in \mathcal{M}^b(\mathbb{R}) : \mu \textit{ is continuous}\}$$

Rajchman measures are special measures in $\mathcal{M}^b(\mathbb{R})$ defined as follows:

$$\mathcal{M}_0^b(\mathbb{R}) := \{\mu \in \mathcal{M}^b(\mathbb{R}) : \lim_{t\to\pm\infty} |\widehat{\mu}(t)| = 0\}$$

Absolutely, by Riemann-Lebesgue lemma and corollary 6.6 above, we conclude that

$$L^1(\mathbb{R}) \subset \mathcal{M}_0^b(\mathbb{R}) \subset \mathcal{M}_c^b(\mathbb{R})$$

We want to mention that Menshov constructed a singular continuous Rajchman measure, and moreover, the Cantor-Lebesgue measure is a continuous measure that is not a Rajchman measure. See [19,20] for more details. Notice that the Fourier transform of a Rajchman measure is defined on Riemann sphere $\mathbb{R}_\infty$.

Now, we are ready to present Wiener's lemma.

Claim 6.7 (Wiener's Lemma) Assume $\mu \in \mathcal{M}_0^b(\mathbb{R})$ and $d\mu \ll dt$. If $1 + \widehat{\mu}(t) \neq 0$ on $\mathbb{R}$, then there is a complex measure $\rho \in \mathcal{M}_0^b(\mathbb{R})$ and and $d\rho \ll dt$ such that

$$\frac{1}{1+\widehat{\mu}} = 1 + \widehat{\rho}$$

where dt is the Lebesgue measure.

We present a closed property without proof due to Rajchman and Milicer-Gruzewska.

Claim 6.8 Let $\rho \in \mathcal{M}_0^b(\mathbb{R})$. If a complex measure μ satisfies $\mu \ll |\rho|$, then $\mu \in \mathcal{M}_0^b(\mathbb{R})$.

$\mathcal{M}_0^b(\mathbb{R})$ is sometimes said to be a L-space or a band because of claim 6.8.

We have discussed $\mu * \rho$ for some $\mu, \rho \in \mathcal{M}^b(\mathbb{R})$, and we already know that $\mu * \rho \in \mathcal{M}^b(\mathbb{R})$. Next, we will discuss one special case: $d\rho \ll dt$. By Lebesgue-Radon-Nikodym theorem, this is equivalent to $d\rho = f dt$ for some $f \in L^1(\mathbb{R}, dt)$.

Let A be a Borel set of $\mathbb{R}$, then the value of the convolution at A is given by

$$\mu * \rho(A) = \int_{\mathbb{R}} \int_{\mathbb{R}} 1_A(x+y) d\rho(x) d\mu(y)$$
$$= \int_{\mathbb{R}} (\int_A f(x-y) dx) d\mu(y)$$
$$= \int_A (\int_{\mathbb{R}} f(x-y) d\mu(y)) dx$$

Here, the third equality is from the fact that μ is complex and Fubini's theorem.

Since we have

$$\int_{\mathbb{R}} | \int_{\mathbb{R}} f(x-y) d\mu(y)| dx \leq \int_{\mathbb{R}} (\int_{\mathbb{R}} |f(x-y)| dx) d|\mu|(y) < \infty$$

it follows that $\int_{\mathbb{R}} f(x-y) d\mu(y) \in L^1(\mathbb{R}, dt)$. In fact, this is the Radon derivative of measure $\mu * \rho$. We denote this function by

$$\mu * f(x) := \int_{\mathbb{R}} f(x-y) d\mu(y)$$

Claim 6.9 Let $\mu \in \mathcal{M}^b(\mathbb{R})$ and $f \in L^1(\mathbb{R}, dt)$. Then $\mu * f(x) \in L^1(\mathbb{R}, dt)$ and

$$||\mu * f||_{L^1(\mathbb{R})} \leq |\mu|(\mathbb{R}) \cdot ||f||_{L^1(\mathbb{R})}$$

6.3 De Branges Spaces

Let $N < \infty$, $\mu \in DS$ and we consider T_N. As usual, let $u(x, z)$ be the solution satisfying $T_N u = zu$ with $u(0, z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. we define

$$E_N(z) = u_1(N, z) - iu_2(N, z)$$

Claim 6.10 $E_N(z)$ is a de Branges function. The reproducing kernels of $B(E_N)$ are given by

$$J_w(z) = \int_0^N u^*(x, w)u(x, z)dx$$

Proof: $E_N(z)$ is entire because $T(N, z)$ is entire.

Let's calculate

$$\begin{aligned} J_w(z) &= \frac{\overline{E_N}(w)E_N(z) - \overline{E_N^{\#}}(w)E_N^{\#}(z)}{2i(\overline{w} - z)} \\ &= \frac{1}{\overline{w} - z} u^*(x, w)Ju(x, z) \\ &= \int_0^N u^*(x, w)u(x, z)dx \end{aligned}$$

The third equality comes from the same approximation used in claim 4.8.

In particular, taking $w = z$, we obtain that

$$\frac{|E_N(z)|^2 - |E_N^{\#}(z)|^2}{4Imz} = ||u(\cdot, z)||^2 > 0$$

Hence, $E_N(z)$ is a de Branges function. ■

Claim 6.11 The formula

$$Uf(z) = \int_0^N u^*(x, \overline{z}) f(x) dx$$

defines an isometry $U : L^2[0, N] \to B(E_N)$.

With claim 6.10, the proof is irrelevant to the type of operators, hence claim 5.3 guarantees this claim since we have the same version for canonical systems, see [2]. We don't work explicitly on this proof. Moreover, this theorem implies that if $N_1 < N_2$, then $B(E_{N_1}) \subset B(E_{N_2})$.

Now, it is time to invoke a theorem by Louis de Branges. The version we need here is a little bit different from de Branges's original one, but it is more explicit for our purpose. For readers who are interested in the original version, please see [24].

Theorem (De Branges) Let $N_1 < N_2 < \infty$, $\mu \in DS$. If $F \in B(E_{N_1})$ and $|h| \leq \tau_{N_2} - \tau_{N_1}$ where τ_{N_i} is the type of E_{N_i}, then $e^{ihz} F \in E_{N_2}$.

Let $\rho \in \mathcal{M}^b(\mathbb{R})$ and compactly supported, we define a complex function $\widehat{\mu}(z)$ by

$$\widehat{\rho}(z) = \int_{\mathbb{R}} e^{izs} d\rho(s)$$

This function is entire by Morera's theorem, and moreover, the restriction of this function on $\mathbb{R}$ is the Fourier transform of ρ, see [9]. This property also indicates the usage of the notation $\widehat{\rho}$.

We denote the pushforward measure of $\overline{\mu}$ under $f(x) = -x$ by μ_r, and recall the property $\widehat{\mu_r}(t) = \overline{\widehat{\mu}(t)}$. In the sequel, we sometimes use $d\mu \in \mathcal{M}^b(\mathbb{R})$ rather than the formal notation $\mu \in \mathcal{M}^b(\mathbb{R})$ just for convenience.

Claim 6.12 Assume $\mu \in DS$. Then for any arbitrary $N \in (0, \infty)$, $B(E_N)$ is a Paley-Wiener space as sets, i.e.,

$$B(E_N) = PW_N := \{F(z) = \widehat{f}(z) : f \in L^2(-N, N)\}$$

Proof: For convenience, we still use μ to denote the cut-off of μ on $[0, N]$, i.e., $\mu \in DS(N)$.

Conclusions from section 6.1 show that

$$E_\delta(z) = 2k(\delta)e^{-iz\delta} f_2(\delta, z)$$

where f is from claim 6.3, $f(0, z) = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\delta \in (0, N]$.

We first focus on a small interval, i.e., let us consider the interval $[0, \delta]$ with $||\mu||((0, \delta]) = \max\limits_{i=1,2}(|\mu_i|((0, \delta])) < \frac{1}{8}$. we apply iteration on $[0, \delta]$ by putting

$$f_0 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; f_{n+1} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \int_0^x \begin{pmatrix} 0 & e^{-2izs} dP(s) \\ e^{2izs} d\overline{P}(s) & 0 \end{pmatrix} f_n(s)$$

where $dP(s) = \frac{e^{t(s)} - e^{-t(s)}}{t(x)(e^{t(s)} + e^{-t(s)})}(d\mu_2(s) - id\mu_1(s))$.

We claim that the solution can be written as

$$f(x, z) = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \sum_{n \geq 1} v_n(x, z)$$

where

$$v_n(x,z) = \begin{cases} \frac{1}{2}\int_0^x \int_0^{t_1} \cdots \int_0^{t_{n-1}} \begin{pmatrix} dP(t_1) d\overline{P}(t_2) \cdots dP(t_n) e^{-2iz(t_1 - t_2 + t_3 - \cdots + t_n)} \\ d\overline{P}(t_1) dP(t_2) \cdots d\overline{P}(t_n) e^{2iz(t_1 - t_2 + t_3 - \cdots + t_n)} \end{pmatrix} \\ \frac{1}{2}\int_0^x \int_0^{t_1} \cdots \int_0^{t_{n-1}} \begin{pmatrix} d\overline{P}(t_1) dP(t_2) \cdots d\overline{P}(t_n) e^{-2iz(t_1 - t_2 + t_3 - \cdots - t_n)} \\ dP(t_1) d\overline{P}(t_2) \cdots dP(t_n) e^{2iz(t_1 - t_2 + t_3 - \cdots - t_n)} \end{pmatrix} \end{cases},$$

and the first (second) branch is for odd (even) n.

Indeed, we denote v_n by $\begin{pmatrix} v_{n,1} \\ v_{n,2} \end{pmatrix}$. Since $v_{n,2} = v_{n,1}^{\#}$, to check the convergence of $\sum\limits_{n \geq 1} v_n(x, z)$, we just need to check $\sum\limits_{n \geq 1} v_{n,2}(x, z)$. Notice that $0 < t_1 - t_2 + t_3 - \cdots \pm t_n \leq \delta$ and $\frac{e^{t(s)} - e^{-t(s)}}{t(x)(e^{t(s)} + e^{-t(s)})} \leq 1$ for $t \geq 0$, it follows that

$$\begin{aligned}
|v_{n,2}| &\leq \frac{e^{2|z|N}}{2} \int_0^{\delta} \int_0^{t_1} \cdots \int_0^{t_{n-1}} d|P|(t_n) d|P|(t_{n-1}) \cdots d|P|(t_1) \\
&\leq \frac{e^{2|z|N}}{2} (\int_0^{\delta} d|P|(t))^n \\
&\leq \frac{e^{2|z|N}}{2} \cdot \frac{1}{4^n}
\end{aligned}$$

hence we conclude that $\sum\limits_{n\geq 1} v_n(x,z)$ converges uniformly on $[0,\delta]$ and on a compact set of $\mathbb{C}$.

Let $s = 2(t_1 - t_2 + t_3 - \cdots \pm t_n)$. We just consider $v_{n,2}(x,z)$ when n is odd. For any even n, we have an analogous result.

Write

$$\Delta = \{(t_1, t_2, \cdots, t_n) : 0 < t_n \leq t_{n-1} \leq \cdots \leq t_1 \leq \delta\}$$

The transform

$$T : \mathbb{R}^n \to \mathbb{R}^n, (t_1, t_2, \cdots, t_n) \mapsto (s, t_2, \cdots, t_n)$$

satisfies

$$T' = \begin{pmatrix} 2 & -2 & 2 & \cdots & 2 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

hence, T is invertible.

We have

$$v_{n,2}(\delta, z) = \frac{1}{2}\int_{\mathbb{R}^n} d(\overline{P} \times P \cdots \times \overline{P})(t) e^{2iz(t_1 - t_2 + t_3 - \cdots + t_n)} \chi_\Delta(t)$$

where $t = (t_1, t_2, \cdots, t_n)$.

We denote the pushforward of $\overline{P} \times P \cdots \times \overline{P}$ under T by $T_*(\overline{P} \times P \cdots \times \overline{P})$, then we actually have

$$v_{n,2}(\delta, z) = \frac{1}{2}\int_{\mathbb{R}^n} d(T_*(\overline{P} \times P \cdots \times \overline{P}))(x) e^{izs} \chi_{T(\Delta)}(x)$$

where $x = (s, t_2, \cdots, t_n)$.

Let $\partial T(\Delta)(s)$ be the cut-off of $T(\Delta)$ at s, i.e., the collection of all points in $T(\Delta)$ such that the first coordinate is s. We define a measure on $\mathbb{R}$ by

$$\mu_n(A) := \int_A \chi_{[0,2\delta]}(s) \int_{\partial T(\Delta)(s)} d(T_*(\overline{P} \times P \cdots \times \overline{P}))(s, t_2, \cdots, t_n)$$

Obviously, this one is compactly supported; moreover, by the definition of a pushforward measure and T, we have

$$\begin{aligned} |\mu_n|(\mathbb{R}) &\leq \int_0^\delta \int_0^{t_1} \cdots \int_0^{t_{n-1}} d|P|(t_n) d|P|(t_{n-1}) \cdots d|P|(t_1) \\ &\leq \frac{1}{4^n} \end{aligned}$$

Thus, $\mu_n \in \mathcal{M}^b(\mathbb{R})$ and compactly supported by $[0, 2\delta]$, and

$$v_{n,2}(\delta, z) = \frac{1}{2} \int_{\mathbb{R}} e^{izs} d\mu_n(s)$$

Notice that this conclusion is also true if n is even. The total variation of μ_n implies that the sequence of complex measures $\{\sum_{i=1}^{n} \mu_i\}_{n=1}^{\infty}$ converges in $\mathcal{M}^b(\mathbb{R})$, and moreover, the limit, which is denoted by ρ, is compactly supported by $[0, 2\delta]$ with $|\rho|(\mathbb{R}) \leq \frac{1}{2}$.

Since measures in the sequence above are compactly supported by $[0, 2\delta]$, we conclude that

$$\sum_{n=1}^{\infty} v_{n,2}(\delta, z) = \frac{1}{2} \lim_{k \to \infty} \int_{\mathbb{R}} e^{izs} d \sum_{n=1}^{k} \mu_n(s) = \frac{1}{2} \int_{\mathbb{R}} e^{izs} d\rho(s) = \frac{\widehat{\rho}(z)}{2}$$

and this gives the desired result:

$$E_\delta(z) = k(\delta) e^{-iz\delta}(1 + \widehat{\rho}(z)) \tag{6.4}$$

for some $\rho \in \mathcal{M}^b(\mathbb{R})$ and is supported by $[0, 2\delta]$.

On the closed upper half plane, i.e., $Imz \geq 0$, we have

$$|\widehat{\rho}(z)| \leq |\rho|(\mathbb{R}) \leq \frac{1}{2}$$

hence, it follows that

$$\frac{|k(\delta)|}{2}|e^{-iz\delta}| \leq |E_\delta(z)| \leq \frac{3|k(\delta)|}{2}|e^{-iz\delta}|$$

on $\overline{\mathbb{C}+}$.

This estimation, with the fact that $\frac{|k(\delta)|}{2} > 0$, gives $B(E_\delta(z)) = B(e^{-iz\delta})$ as sets. With the famous result saying that $B(e^{-iz\delta}) = PW_\delta$ as sets (see[2]), we get the conclusion that $B(E_\delta(z)) = PW_\delta$ as sets if δ is small enough.

Next, we want to show $PW_N \subset B(E_N)$ as sets.

We first observe that, because of claim 5.3 and the exponential type formula, the type of E_L, denoted by τ_L, can be determined by $\tau_L = L$ for $L \in [0, \infty)$. Let $|h| \leq N - \delta$ and $F = \hat{f} \in PW_\delta$. Clearly, $e^{ihz}F = \widehat{\delta_h * f} \in B(E_N)$ due to the theorem (de Branges) above. On the other hand, since f runs in $L^2(-\delta, \delta)$, then $\delta_h * f$ runs in $L^2(-\delta - h, \delta - h)$. If we pick up some proper h, $L^2(-N, N)$ can be decomposed into the direct sum of finitely many such subspaces. As $B(E_N)$ is Hilbert, we conclude that $PW_N \subset B(E_N)$.

Moreover, we claim that $B(E_N) \subset PW_N$ as sets.

First, we want to show that $|E_N(t)| < C$ on $\mathbb{R}$ for some constant C, or equivalently, $f_2(N, t)$ is bounded.

Since $\mu \in DS$, we can pick up finitely many point masses so that the rest of point masses is small enough, precisely, for $\epsilon > 0$, there exists an integer m such that for $x_n \in S_N(\mu)$ we have $\sum_{n=m}^{\infty} |\mu|(x_n) < \epsilon$. Moreover, as $[0, N]$ is compact, we can find out finitely many points, denoted by $\delta_0 = 0 < \delta_1 < \delta_2 < \cdots < \delta_{M-1} < \delta_M = N$ so that $||\mu||((\delta_n, \delta_{n+1})) = \max_{i=1,2}(|\mu_i|((\delta_n, \delta_{n+1}))) < \frac{1}{8}$. We need to be aware of that δ_n can be a point mass which has a large weight, and this is why we just consider open intervals rather than closed ones.

Now we consider the integral equation in claim 6.3 on intervals $I_n := (\delta_n, \delta_{n+1})$ separately. On I_0, this is indeed what we did at the beginning of the proof, and the conclusion is that $f(\delta_1-, t)$ is bounded. To evaluate $f(\delta_1, t)$, notice that the point δ_1 will update $f(\delta_1-, t)$ to $f(\delta_1, t)$ by a constant matrix (which is not the identity if δ_1 is a point mass), thus $f(\delta_1, t)$ is still bounded. On I_n, we consider the iteration as above by putting:

$$f_0 = f(\delta_n, z), \ f_{n+1} = f(\delta_n, z) + \int_{\delta_n}^{x} \begin{pmatrix} 0 & e^{-2izs} dP(s) \\ e^{2izs} d\overline{P}(s) & 0 \end{pmatrix} f_n(s)$$

An analogous calculation as above shows that $f(\delta_{n+1}-, t)$ is bounded if $f(\delta_n, t)$ is so, and as expected, $f(\delta_{n+1}, t)$ is also bounded since δ_{n+1} updates $f(\delta_{n+1}-, t)$ by a constant matrix relative to the mass of this point. In summary, it follows from induction that $|E_N(t)| < C$ on $\mathbb{R}$ for some constant C.

Let $F \in B(E_N)$. the exponential type formula, theorem 2.9, claim 5.3, and claim 6.11 together show that $\tau(F) \leq N$. Moreover,

$$\int_{\mathbb{R}} |F(t)|^2 dt = \int_{\mathbb{R}} |\frac{F(t)}{E_N(t)}|^2 |E_N(t)|^2 dt < C^2 || \frac{F(z)}{E_N(z)} ||_{H^2}$$

Since $\frac{F}{E_N} \in H^2$ by the definition of $B(E_N)$, it follows that $F(t) \in L^2(\mathbb{R})$. Now, the Paley-Wiener theorem implies that $B(E_N) \subset PW_N$. ■

Claim 6.13 Assume $\mu \in DS$ and $d\mu \ll dt$. Then for any arbitrary $N \in (0, \infty)$, $B(E_N)$ is a Paley-Wiener space as sets, i.e.,

$$B(E_N) := PW_N = \{F(z) = \widehat{f}(z) : f \in L^2(-N, N)\}$$

Moreover, $\frac{1}{|E_N(t)|^2} = 1 + \widehat{\phi}(t)$ on $\mathbb{R}$ for some $\phi \in L^1(\mathbb{R})$ such that $\phi(x) = \overline{\phi}(-x)$. Thus, for $F = \widehat{f} \in B(E_N)$, the norm can be written as

$$||F||^2_{B(E_N)} = 2\langle f, f + \phi * f \rangle$$

Proof: The first part about Paley-Wiener spaces is from claim 6.12 directly. To show the second part, we just need to observe, due to Volpert's chain rule, that the estimation of $|v_{n,2}|$ from the proof of claim 6.12 can be improved on $[0, N]$ as

$$
\begin{aligned}
|v_{n,2}| &\leq \frac{e^{2|z|N}}{2} \int_0^N \int_0^{t_1} \cdots \int_0^{t_{n-1}} d|P|(t_n) d|P|(t_{n-1}) \cdots d|P|(t_1) \\
&= \frac{e^{2|z|N}}{2} \int_0^N \int_0^{t_1} \cdots (\int_0^{t_{n-2}} \int_0^{t_{n-1}} d|P|(t_n) d|P|(t_{n-1})) \cdots d|P|(t_1) \\
&= \frac{e^{2|z|N}}{2} \int_0^N \int_0^{t_1} \cdots (\int_0^{t_{n-3}} \frac{1}{2} (\int_0^{t_{n-2}} d|P|(t_n))^2 d|P|(t_{n-2})) \cdots d|P|(t_1) \\
&= \frac{e^{2|z|N} (|P|[0, N])^n}{2n!}
\end{aligned}
$$

And moreover, we have $d\mu_n \ll dt$ since $dP \ll dt$, thus we get the conclusion that:

$$
E_N(z) = e^{-izN} (1 + \widehat{\rho}(z))
$$

for some $\rho \in L^1(0, 2N)$ (formally, ρ is absolutely continuous, but we still use this notation).

Recall that $E_N(z) \neq 0$, otherwise $u(N, z) = 0$.

A calculation shows, for $t \in \mathbb{R}$, that

$$
\frac{1}{|E_N(t)|^2} = \frac{1}{1 + \hat{g}(t)}
$$

where $g := \rho + \rho_r + \rho * \rho_r$ with $\rho_r(t) = \overline{\rho}(-t)$.

By Young's convolution inequality, we conclude that $g \in L^1(\mathbb{R})$. Now, Wiener's lemma (claim 6.7) implies that there is a $\phi \in L^1(\mathbb{R})$ such that $\frac{1}{1+\hat{g}(t)} = 1 + \hat{\phi}$, this gives

$$
\frac{1}{|E_N(t)|^2} = 1 + \hat{\phi}
$$

Moreover, if we take complex conjugate on both sides, we have $\hat{\phi} = \overline{\hat{\phi}}$. Recall that $\overline{\hat{\phi}} = \hat{\phi}_r$, and this gives $\phi = \phi_r$. By the definition, we have

$$||F||^2_{B(E_N)} = \frac{1}{\pi}\int_{\mathbb{R}} |F|^2 \frac{dt}{|E_N|^2} = \frac{1}{\pi}\int_{\mathbb{R}} |\hat{f}|^2 dt + \frac{1}{\pi}\int_{\mathbb{R}} \overline{\hat{f}} \cdot \widehat{\phi * f} dt$$

Parseval identity implies that

$$||F||^2_{B(E_N)} = 2||f||^2_{L^2} + 2\langle f, \phi * f\rangle$$

which is indeed the desired result. ■

Remark. Claim 6.13 implies that, even though the convolution operator $\phi * \cdot$ with $\phi \in L^1(\mathbb{R})$ maps $L^2(-N, N)$ into $L^2(\mathbb{R})$, $||F||^2_{B(E_N)}$ is determined by $\phi * f$ on $(-N, N)$, that is, by the restriction of ϕ to $(-2N, 2N)$. Without loss of generality, we can assume that $\phi \in L^1(-2N, 2N)$. Moreover, as a norm, we require that $\langle f, f + \phi * f\rangle > 0$ for non-zero $f \in L^2(-N, N)$.

6.4 PW_x as a Regular De Branges Space

Fix N, let $\phi \in L^1(-2N, 2N)$ and $x \in (0, N]$. We define an operator, denoted by T^x_ϕ, as follows:

$$T^x_\phi : L^2[-x, x] \to L^2[-x, x], T^x_\phi f = \int_{-x}^{x} \phi(t - s) f(s) ds$$

This definition makes sense because of Young's inequality for integral operators. Notice that this operator is essentially different from the convolution operator on $L^2[-x,x]$: the convolution operator is not compact unless $\phi = 0$, whereas T_ϕ^x is compact as we will see later. However, it is useful to point out the relation between those two types of operators once we consider different x: if we identify $\phi \in L^1(-2N, 2N)$ and $f \in L^2[-x,x]$ with their extensions (set 0 out of intervals) in $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ respectively, then the convolution $\phi * f \in L^2(\mathbb{R})$, and $T_\phi^x f = \phi * f$ on $[-x,x]$.

We define a subset of $L^1(\mathbb{R})$,denoted by Φ_N, as follows:

$$\Phi_N := \{\phi \in L^1(-2N, 2N) : 1)\phi(x) = \overline{\phi}(-x); 2)1 + T_\phi^N > 0\}$$

where $1 + T_\phi^N > 0$ means $\langle f, (1 + T_\phi^N)f\rangle > 0$ for non-zero $f \in L^2[-N,N]$. Notice that $\langle f, (1 + T_\phi^N)f\rangle = \langle f, f + \phi * f\rangle$ if ϕ is treated as its extension (set 0 out of the $(-2N, 2N)$) in $L^1(\mathbb{R})$.

Let $\phi \in \Phi_N$, and $F = \hat{f}, H = \hat{h} \in PW_x$, we define an inner product (which is easy to verify) as follows:

$$[F, H]_{\phi,x} := 2\langle f, (1 + T_\phi^x)h\rangle$$

The norm is given by

$$||F||_{\phi,x}^2 = [F, F]_{\phi,x}$$

Claim 6.14 T_ϕ^x is self-adjoint and compact.

Proof: Let $f, g \in L^2[-x,x]$, then it follows from Fubini theorem that

$$\langle f, T_\phi^x g\rangle = \int_{-x}^{x}\int_{-x}^{x} \overline{f}(t)\phi(t-s)dtg(s)ds$$

Recall that $\phi(x) = \overline{\phi}(-x)$, then the left-hand side is indeed $\langle T_\phi^x f, g\rangle$, and this shows that T_ϕ^x is self-adjoint.

Pick up $\phi_n \in C_c^\infty(-2N,2N)$ so that $\phi_n \to \phi$ in $L^1(-2N,2N)$. Since $T_{\phi_n}^x$ are Hilbert-Schmidt integral operators, hence are compact as well. Young's inequality implies that

$$||T_{\phi_n}^x f - T_\phi^x f||_{L^2[-x,x]} = ||T_{\phi_n-\phi}^x f||_{L^2[-x,x]} \leq ||\phi_n - \phi||_{L^1(-2N,2N)}||f||_{L^2[-x,x]}$$

and this estimation shows that $T_{\phi_n}^x \to T_\phi^x$ in $B(L^2[-x,x])$. As a result, T_ϕ^x is also compact. ■

Claim 6.15 PW_x with the norm $||\cdot||_{\phi,x}$, denoted by $(PW_x, ||\cdot||_{\phi,x})$, is a Hilbert space.

Proof: Consider $f \in L^2[-x,x]$, and the extension of f (set 0 out of $[-x,x]$) in $L^2[-N,N]$,denoted by f for convenience.

We have for all $f \in L^2[-x,x]$,

$$0 < \langle f, (1+T_\phi^N)f\rangle = \langle f, (1+T_\phi^x)f\rangle$$

Since T_ϕ^x is self-adjoint and compact, we have $\sigma(T_\phi^x) = \sigma_p(T_\phi^x) \cup \{0\}$ and 0 is the only accumulation point; moreover,

$$\sigma(1 + T_\phi^x) = \sigma_p(T_\phi^x + 1) \cup \{1\} = (\sigma_p(T_\phi^x) + 1) \cup \{1\}$$

From the inequality above, we conclude that if $\lambda \in \sigma_p(T_\phi^x + 1)$, then $\lambda > 0$,otherwise there must be a $f \in L^2[-x, x]$ so that

$$\langle f, (1 + T_\phi^x) f \rangle = \lambda ||f||^2 \leq 0$$

Since 1 is the only accumulation point of $\sigma_p(1 + T_\phi^x)$, then there are at most finitely many eigenvalues in $[0, 1 - \epsilon]$ for any $\epsilon > 0$. Let's denote the smallest eigenvalue by $\lambda_{min} > 0$, the spectral theorem shows that

$$\langle f, (1 + T_\phi^x) f \rangle = \int_{\sigma(1+T_\phi^x)} td||E_{\phi,x}(t) f||^2 \geq \lambda_{min} ||f||^2$$

where $E_{\phi,x}(t)$ is the spectral family of $1 + T_\phi^x$.
On the other hand, we have $\langle f, (1 + T_\phi^x) f \rangle \leq ||1 + T_\phi^x|| \cdot ||f||^2$. Those two inequalities together imply that

$$\lambda_{min} ||f||^2 \leq ||F||^2_{\phi,x} \leq ||1 + T_\phi^x|| \cdot ||f||^2$$

That is, $(PW_x, || \cdot ||_{\phi,x})$ is Hilbert. ■

Claim 6.16 $(PW_x, || \cdot ||_{\phi,x})$ is a de Branges space.

Proof: Theorem 2.6 characterizes a de Branges space, so we check 1) to 3) one by one.

1) For $z \in \mathbb{C}$, we have

$$|z(F)| = |\int f e^{izt} dt| \leq e^{|z|x} \int_{-x}^{x} |f| dt \leq \sqrt{2x} e^{|z|x} ||f||_{L^2[-x,x]}$$

Recall $||f||_{L^2[-x,x]} \leq \frac{1}{\sqrt{\lambda_{min}}} ||F||_{\phi,x}$, we have

$$|z(F)| \leq \sqrt{\frac{2x}{\lambda_{min}}} e^{|z|x} ||F||_{\phi,x}$$

It follows from the inequality above that the linear functional $z \in PW_x^*$.

2) Recall the Paley-Wiener theorem

$$PW_x = \{F : F is\ entire, \int_{\mathbb{R}} |F|^2 < \infty, |F(z)| < C_F e^{x|z|}\}$$

Let's pick up $F \in PW_x$ with $F(w) = 0$, and write $G(z) = \frac{z-\overline{w}}{z-w} F(z)$. Consider the power series of F at w, then it's easy to see that G is also entire.

Also, we have

$$\int_{\mathbb{R}} |G|^2 = \int_{\mathbb{R}} |\frac{t-\overline{w}}{t-w}|^2 |F|^2 = \int_{\mathbb{R}} |F|^2 < \infty$$

To show that $|G(z)| < C_G e^{x|z|}$ for some constant C_G, we consider G in the closed unit ball $\overline{B_1}(w)$. since $G(z)e^{-xz}$ is entire, then is bounded in compact set $\overline{B_1}(w)$. We choose a large number M s.t. $|G(z)e^{-xz}| < M$ in $\overline{B_1}(w)$, i.e., $|G(z)| < M|e^{xz}| \leq Me^{x|z|}$ in $\overline{B_1}(w)$. On the other hand, in $\mathbb{C} \setminus B_1(w)$, since $\lim\limits_{z\to\infty} |\frac{z-\overline{w}}{z-w}| = 1$, we conclude that $|\frac{z-\overline{w}}{z-w}| < N$ for some large

N, i.e., $|G(z)| < NC_F e^{x|z|}$. In summary, $|G(z)| < max\{M, NC_F\}e^{x|z|}$.

Now, it follows from the Paley-Wiener theorem that $G \in PW_x$, that is, there is $g \in L^2[-x, x]$ s.t. $G = \hat{g}$.

Let's evaluate $||G||^2_{\phi,x}$ (with some obviously simplified notations).

$$||G||^2_{\phi,x} = 2||g||^2 + 2\langle g, \phi * g\rangle = \frac{1}{\pi}||G||^2_{L^2} + \frac{1}{\pi}\langle G, \hat{\phi} \cdot G\rangle$$

Notice that $|G(t)|^2 = |F(t)|^2$, we get

$$||G||^2_{\phi,x} = \frac{1}{\pi}\langle F, (1 + \hat{\phi}) \cdot F\rangle = 2\langle f, (1 + \phi) * f\rangle = ||F||^2_{\phi,x}$$

3) Recall again that $F^{\#}(z) = \widehat{\overline{f_r}}(z)$ where $f_r(t) = f(-t)$. We have

$$||F^{\#}||^2_{\phi,x} = 2\langle \overline{f_r}, (1 + T^x_\phi)\overline{f_r}\rangle = 2\langle \overline{f_r}, (1 + \phi) * \overline{f_r}\rangle = ||F||^2_{\phi,x}$$

This gives the isometry. ■

Claim 6.17 $(PW_x, || \cdot ||_{\phi,x})$ is regular.

Proof: Let $F \in PW_x$, we want to show that $(S_{z_0}F)(z) = \frac{F(z)-F(z_0)}{z-z_0} \in PW_x$.

By checking the power series of $F(z)$ at z_0, it's easy to see that $S_{z_0}F$ is entire.

To evaluate $\int_{\mathbb{R}} |(S_{z_0}F)(t)|^2 dt$, we need to consider two cases: a) $z_0 \notin \mathbb{R}$; b) $z_0 \in \mathbb{R}$.

a)

$$\int_{\mathbb{R}} |\frac{F(t)-F(z_0)}{t-z_0}|^2 dt \leq \frac{2}{|Imz_0|^2}\int_{\mathbb{R}} |F|^2 + 2|F(z_0)|^2 \int_{\mathbb{R}} \frac{dt}{|t-z_0|^2} < \infty$$

b) We can find a small ball centered at z_0, denoted by $B_\epsilon(z_0)$, and split the integral into two parts:

$$\int_{\mathbb{R}} |(S_{z_0}F)(t)|^2 dt = \int_{\mathbb{R}\setminus B_\epsilon(z_0)} |\frac{F(t)-F(z_0)}{t-z_0}|^2 + \int_{B_\epsilon(z_0)\cap\mathbb{R}} |\frac{F(t)-F(z_0)}{t-z_0}|^2$$

by the same reason in a), we have $\int_{\mathbb{R}\setminus B_\epsilon(z_0)} |\frac{F(t)-F(z_0)}{t-z_0}|^2 dt < \infty$.
Since $(S_{z_0}F)(z)$ is continuous, $|(S_{z_0}F)(z)|$ reaches out to its maximum, denoted by M, in $\overline{B_\epsilon(z_0)}$, i.e., $\int_{B_\epsilon(z_0)} |\frac{F(t)-F(z_0)}{t-z_0}|^2 dt \leq 2\epsilon M^2$. If we assemble those two parts together, we always have $\int_{\mathbb{R}} |(S_{z_0}F)(t)|^2 dt < \infty$.
We have $|(S_{z_0}F)(z)| \leq M \leq Me^{x|z|}$ in $\overline{B_\epsilon(z_0)}$; moreover, in $\mathbb{C}\setminus\overline{B_\epsilon(z_0)}$, we also have

$$|\frac{F(z)-F(z_0)}{z-z_0}| \leq \frac{1}{\epsilon}(|F(z)|+|F(z_0)|) \leq \frac{C_F}{\epsilon}(e^{x|z|}+e^{x|z_0|})$$

hence, for an adequately large number C, it's clear that

$$\frac{C_F}{\epsilon}(e^{x|z|}+e^{x|z_0|}) \leq Ce^{x|z|}$$

In summary, it follows that

$$|\frac{F(z)-F(z_0)}{z-z_0}| \leq max\{C, M\}e^{x|z|}$$

Now, the Paley-wiener theorem gives the desired conclusion. ■

6.5 the Canonical System Given by PW_N

We have proved that $(PW_N, ||\cdot||_{\phi,N})$ is a regular de Branges space, i.e., there is a de Branges function, denoted by E_N, so that $(PW_N, ||\cdot||_{\phi,N}) = B(E_N)$. Without the loss of the generality, we can normalize E_N s.t. $E_N(0) = 1$. Now, by theorem 2.8, there is a canonical system H with $trH = 1$ on $(0, N)$ such that

$$E_N(z) = u_1(N, z) - iu_2(N, z)$$

and the corresponding reproducing kernels J_w are given by

$$J_w(z) = \int_0^N u^*(t, w)H(t)u(t, z)dt$$

where $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is the solution of $u' = zJHu$ satisfying $u(0, z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
In the sequel, we simply denote $(PW_x, ||\cdot||_{\phi,x})$ by PW_x, and B_x the de Branges space of $u' = zJHu$ on $(0, x)$. Also, for convenience, we define $PW_0 = B_0 = \{0\}$.
For PW_x, we have $PW_{x_1} \subset PW_{x_2}$ if $x_1 \leq x_2$.
For B_x, we still have $B_{x_1} \subset B_{x_2}$ if $x_1 \leq x_2$ and x_1 is regular. See [17] for details.
We denote all regular values of the canonical system on $[0, N]$ by R. Since

PW_x and B_x are regular due to claim 6.17 and theorem 2.9, it follows from the ordering theorem (theorem 2.7) and the fact $PW_N = B_N$ that either $PW_x \subset B_t$ or $B_t \subset PW_x$ for $t \in R$. Define for $t \in R$ a function $x(t)$ by

$$x(t) = \inf\{x \in [0, N] : B_t \subset PW_x\}$$

By the definition, we have that $t = 0, N$ are regular (see Corollary 10.11 in [17]), i.e., $x(0) = 0, x(N) = N$. It is also clear that $x(t)$ is increasing. We apply a modification: if $(0, N)$ starts with a singular interval $(0, a)$ and $E_a(z) = 1$, we delete this initial interval and rescale the rest so that we still have a canonical system on $(0, N)$. Clearly, this modification does not change B_N.

Claim 6.18 $\forall x \in (0, N)$, $PW_x = \bigcap_{y>x} PW_y = \overline{\bigcup_{y<x} PW_y}$. This closure is taken in PW_N.

Proof: Since $PW_x \subset PW_y$ if $x \leq y$, it's clear that $PW_x \subset \bigcap_{y>x} PW_y$.
On the other hand, let $F \in \bigcap_{y>x} PW_y$, then $F \in PW_{x+\frac{1}{n}}$ for all positive integers n, i.e., $F(z) = \int f_n(t)e^{izt}dt$ where $f_n \in L^2(-(x+\frac{1}{n}), x+\frac{1}{n})$. By the uniqueness of the inverse Fourier transform, we get that all f_n are the same. This is true only if f_n are supported by $[-x, x]$, hence $F \in PW_x$.
Since $PW_y \subset PW_x$ if $y \leq x$, hence $\overline{\bigcup_{y<x} PW_y} \subset PW_x$ as PW_y are closed in PW_x.
Let $F \in PW_x$, then $F = \hat{f}$ for some $f \in L^2(-x, x)$. Define $f_n = \chi_{(-(x-\frac{1}{n}), x-\frac{1}{n})} f$, then $F_n \to F$ in $(PW_x, ||\cdot||_{\phi,x})$, hence it follows $F \in$

$\overline{\bigcup_{y<x} PW_y}$. ■

Claim 6.19 The (modified) canonical system $u' = zJHu$ has no singular points. Moreover, $PW_{x(t)} = B_t$ for all $t \in [0, N]$.

Proof: If $t \in R$, then either $PW_x \subset B_t$ or $B_t \subset PW_x$ for $t \in R$, hence it follows that $\overline{\bigcup_{y<x(t)} PW_y} \subset B_t \subset \bigcap_{y>x(t)} PW_y$. By claim 6.18, we have $PW_{x(t)} = B_t$.

If (a, b) is a singular interval, then for regular values a, b, we have $PW_{x(a)} = B_a, PW_{x(b)} = B_b$, i.e.,

$$PW_{x(b)} \ominus PW_{x(a)} = B_b \ominus B_a$$

Corollary 10.11 in [17] again implies that the right-hand side is one-dimensional; however, the left-hand side cannot be one-dimensional, this contradiction shows that there are no singular points. As a result, $PW_{x(t)} = B_t$ for all $t \in [0, N]$. ■

6.6 Two Integral Equations

1) Reproducing kernels for $w = 0$ in PW_x

Let's denote reproducing kernels for $w = 0$ in PW_x by $J_0(x, z)$, then there exist $j(x, t) \in L^2(-x, x)$ such that $J_0(x, z) = \int_{-x}^{x} j(x, t) e^{izt} dt$.

$\forall F = \int f e^{izt} dt \in PW_x$, we have $[J_0, F]_{\phi,x} = F(0)$. Hence it follows

$$F(0) = \int_{-x}^{x} f dt = \langle 1, f \rangle_{L^2(-x,x)} = 2\langle j, (1 + T_\phi^x) f \rangle$$

As the operator $1 + T_\phi^x$ is self-adjoint, we actually have

$$\langle 1, f\rangle_{L^2(-x,x)} = \langle 2(1 + T_\phi^x)j, f\rangle$$

Since f is arbitrary in $L^2(-x, x)$, we conclude that

$$(1 + T_\phi^x)j = \frac{1}{2}$$

or equivalently,

$$j(x,t) + \int_{-x}^{x} \phi(t-s)j(x,s)ds = \frac{1}{2} \tag{6.5}$$

on $t \in [-x, x]$.

2) Conjugate kernels for $w = 0$ in PW_x

We introduce some notations first.

The signal function, denoted by $sgn(x)$, is the function defined by

$$sgn(x) := \begin{cases} 1, & x \in [0, \infty) \\ -1, & x \in (\infty, 0) \end{cases}$$

For $\phi \in \Phi_N$, we define a function $\Phi(x)$ by

$$\Phi(x) = \int_0^x \phi(s)ds$$

The usage of the notation Φ will not bring any confusion, and we also follow the convention: if $x < 0$, then $\int_0^x \phi(s)ds = -\int_x^0 \phi(s)ds$. Also notice that we have the property $\overline{\Phi}(x) = -\Phi(-x)$.

We define a function by $\psi(s) = (2\Phi(s) + sgn(s))i$, then define a bounded linear functional for $F = \widehat{f} \in PW_x$ as follows:

$$\widehat{F}(0) = \int_{-x}^{x} \overline{\psi}(t) f(t) dt$$

We stop to insert the following property of $\widehat{F}(0)$ which will be used later when we construct the connection between PW_x and B_x.

Claim 6.20 For all $F = \widehat{f}, G = \widehat{g} \in PW_x$,we have

$$\widehat{F}(0)\overline{G(0)} - F(0)\overline{\widehat{G}(0)} = [S_0 G, F]_{\phi,x} - [G, S_0 F]_{\phi,x}$$

Proof: Let's write $s_0 F(t) := I_f(t) - \chi_{(0,x)}(t) F(0)$, where $I_f(t) = \int_{-x}^{t} f(t)dt$. Notice that $I_f(x) = F(0)$, then it is easy to show that $S_0 F(z) = -i\widehat{s_0 F}(z)$ as Fourier transform.

Thus we get

$$[S_0 G, F]_{\phi,x} = 2i\langle s_0 G, (1 + T_\phi^x) f\rangle$$

$$[G, S_0 F]_{\phi,x} = -2i\langle (1 + T_\phi^x) g, s_0 F\rangle$$

Notice that

$\langle I_g, (1 + T_\phi^x) f\rangle = \langle (1 + T_\phi^x) I_g, f\rangle$

$= \overline{(1 + T_\phi^x) I_g} \cdot I_f|_{-x}^{x} - \langle (1 + T_\phi^x) g, I_f\rangle + \overline{G(0)} \int_{-x}^{x} \phi(x - t) I_f(t) dt$

i.e.,

$\langle I_g, (1 + T_\phi^x) f\rangle + \langle (1 + T_\phi^x) g, I_f\rangle$

$= F(0)\overline{(1 + T_\phi^x) I_g}(x) + \overline{G(0)} \int_{-x}^{x} \phi(x - t) I_f(t) dt$

On the other hand, since $\langle(1+T_\phi^x)g, I_f\rangle = \langle g, (1+T_\phi^x)I_f\rangle$, the same computation shows that

$\langle I_g, (1+T_\phi^x)f\rangle + \langle(1+T_\phi^x)g, I_f\rangle$

$= \overline{G(0)}(1+T_\phi^x)I_f(x) + F(0)\int_{-x}^x \phi(t-x)\overline{I_g(t)}dt$

Hence we have

$$F(0)\overline{(1+T_\phi^x)I_g}(x) + \overline{G(0)}\int_{-x}^x \phi(x-t)I_f(t)dt$$
$$= \overline{G(0)}(1+T_\phi^x)I_f(x) + F(0)\int_{-x}^x \phi(t-x)\overline{I_g(t)}dt$$

We also know from the equation above that

$[S_0G, F]_{\phi,x} - [G, S_0F]_{\phi,x}$

$= 2i(\langle I_g, (1+T_\phi^x)f\rangle + \langle(1+T_\phi^x)g, I_f\rangle - \overline{G(0)}\langle(1+T_\phi^x)\chi_{(0,x)}, f\rangle - F(0)\langle g, (1+T_\phi^x)\chi_{(0,x)}\rangle$

$= \overline{G(0)}\cdot i(\int_{-x}^x \phi(x-t)I_f(t)dt + (1+T_\phi^x)I_f(x) - 2\langle(1+T_\phi^x)\chi_{(0,x)}, f\rangle)$

$-F(0)\cdot(-i)(\int_{-x}^x \phi(t-x)\overline{I_g(t)}dt + \overline{(1+T_\phi^x)I_g}(x) - 2\langle g, (1+T_\phi^x)\chi_{(0,x)}\rangle)$

Notice that

$$T_\phi^x I_f(x) = \int_{-x}^x \phi(x-t)I_f(t)dt = \int_{-x}^x \Phi(x-t)f(t)dt$$

and

$$\langle(1+T_\phi^x)\chi_{(0,x)}, f\rangle = \langle\chi_{(0,x)}, f\rangle + \int_{-x}^x (\Phi(x-t) - \Phi(-t))f(t)dt$$

we finally get

$$[S_0G, F]_{\phi,x} - [G, S_0F]_{\phi,x} = \widehat{F}(0)\overline{G(0)} - F(0)\overline{\widehat{G}(0)}$$

$\blacksquare$

Now, let's resume our discussion about $\widehat{F}(0)$. As a bounded linear functional on PW_x, the Rieze representation theorem then guarantees that there must be a unique $K_0(x,z) = \int_{-x}^{x} k(x,t)e^{izt}dt \in PW_x$ so that $[K_0, F]_{\phi,x} = \widehat{F}(0)$ for all $F = \widehat{f} \in PW_x$. In other words, we get

$$\langle 2(1+T_\phi^x)k, f\rangle = \int_{-x}^{x} \overline{\psi}(t)f(t)dt$$

i.e., $(1+T_\phi^x)k = \frac{1}{2}\psi$, or equivalently,

$$k(x,t) + \int_{-x}^{x} \phi(t-s)k(x,s)ds = \frac{1}{2}\psi(t) \tag{6.6}$$

on $t \in [-x, x]$.

6.7 the Regularities of the Integral Equations

In this section, we always assume $\phi \in \Phi_N$, $x \in (0, N]$. We want to consider the following differential equation:

$$p(x,t) + \int_{-x}^{x} \phi(t-s)p(x,s)ds = g(x,t)$$

for $t \in [-x, x]$ in some proper spaces.

We can also extend this equation to 0, i.e., if $x = 0$, we define $p(0,0) = g(0,0)$. To simplify our notations, let's define a triangle with respect to a number $m \in (0, N]$ by

$$\Delta_m := \{(x,t) \in \mathbb{R}^2 : 0 < |t| < x < m\}$$

It is clear that if $g(x,t) = \frac{1}{2}$ and $g(x,t) = \frac{1}{2}\psi(t)$, then this differential equation gives (6.5), (6.6) respectively.

For our purpose, we do need to define some operators and analyze those operators carefully. Recall that

$$T_\phi^x : L^2[-x,x] \to L^2[-x,x], T_\phi^x f = \int_{-x}^{x} \phi(t-s)f(s)ds$$

By Young's inequality for integral operators, we can define

$$L_\phi^x : L^1[-x,x] \to L^1[-x,x], L_\phi^x f = \int_{-x}^{x} \phi(t-s)f(s)ds$$

and

$$C_\phi^x : C[-x,x] \to C[-x,x], C_\phi^x f = \int_{-x}^{x} \phi(t-s)f(s)ds$$

Here, $C[-x,x]$ is the Banach space of all continuous functions endowed with the supreme norm.

Of course, since $C[-x,x] \subset L^2[-x,x] \subset L^1[-x,x]$ as sets, if we pick up some function $f \in C[-x,x]$ for instance, then we do have $T_\phi^x f = L_\phi^x f = C_\phi^x f$; however, the advantage of using different notations for "the same" operator is that we don't need to always emphasize spaces when jumping back and forth among those different spaces.

Let's define

$$H[-x,x] := C[-x,x] \oplus L\{\chi[0,x]\}$$

where L means all linear combinations.

Of course, we can treat $H[-x,x]$ as the direct sum of two Banach spaces, then the scalar multiplication and vector addition can be defined as usual so that $H[-x,x]$ becomes a vector space.

For $(f, a\chi[0,x]) \in H[-x,x]$, we define the norm (which is easy to check) as follows:

$$||(f, a\chi[0,x])||_{H[-x,x]} := ||f||_{C[-x,x]} + |a|$$

It's easy to see that $H[-x,x]$ with the norm above is a Banach space; moreover, $C[-x,x]$ can be embedded into $H[-x,x]$ isometrically. In all, we have the chain:

$$C[-x,x] \subset H[-x,x] \subset L^2[-x,x] \subset L^1[-x,x]$$

and the second $\subset$ may be interpreted as $f + a\chi[0,x]$ for $(f, a\chi[0,x]) \in H[-x,x]$.

Let $f \in C[-x,x]$. We have

$$\int_{-x}^{x} \phi(t-s)(f(s) + a\chi[0,x](s))ds = \int_{-x}^{x} \phi(t-s)f(s)ds + a\int_{t-x}^{t} \phi(s)ds$$

It is clear that the right-hand side is continuous, hence we can define an operator

$$H_\phi^x : H[-x,x] \to H[-x,x]$$

$$H_\phi^x(f, a\chi[0,x]) = (\int_{-x}^{x} \phi(t-s)(f(s) + a\chi[0,x](s))ds, 0)$$

It is helpful to keep in mind that the range of H^x_ϕ is actually in $C[-x,x]$ and that $\frac{1}{2}\psi(t)$ can be treated uniquely as $(i(\Phi(s)-\frac{1}{2}), i\chi[0,x](s)) \in H[-x,x]$.

Claim 6.21 $C^x_\phi, H^x_\phi, L^x_\phi$ are compact.

Proof: Pick up $\phi_n \in C^\infty_c(-2N,2N)$ so that $\phi_n \to \phi$ in $L^1(-2N,2N)$ and $||\phi_n - \phi||_{L^1} < 1$.

We first claim that $C^x_{\phi_n}$ is compact.

Let $\{f_m\} \subset C[-x,x]$ s.t. $||f_m||_{C[-x,x]} \leq 1$. Since $||C^x_{\phi_n} f_m||_{C[-x,x]} \leq ||\phi||_{L^1} + 1$, we get that $\{C^x_{\phi_n} f_m\}$ is uniformly bounded. On the other hand, since

$$|C^x_{\phi_n} f_m(t) - C^x_{\phi_n} f_m(t_0)| \leq \int_{-x}^{x} |\phi_n(t-s) - \phi_n(t_0 - s)| ds$$

and ϕ_n is continuous and compactly supported, hence is uniformly continuous, thus it follows that $\{C^x_{\phi_n} f_m\}$ is equicontinuous. Now Arzela-Ascoli theorem tells that $C^x_{\phi_n}$ is compact.

Also notice that $C^x_{\phi_n} \to C^x_\phi$ in $B(C[-x,x])$, we get that C^x_ϕ is compact as well.

The same idea can be applied to H^x_ϕ, hence H^x_ϕ is also compact.

For L^x_ϕ, we just need to notice that the range of $L^x_{\phi_n}$ is in $C[-x,x]$, then the same idea also shows that L^x_ϕ is compact. ■

Claim 6.22 Let X, Y be Banach spaces. An operator $T : X \to Y$ has a continuous inverse if and only if

$$\gamma_T := \inf\{||Tx|| : x \in D(T), ||x|| \geq 1\} > 0.$$

We have that $||T^{-1}|| = \gamma_T^{-1}$.

Proof: Let's assume $\gamma_T > 0$ first.

Then it follows that $\ker(T) = \{0\}$, hence the inverse exists. We have

$$||T^{-1}|| = \sup_{y \in D(T^{-1})} ||\frac{T^{-1}y}{||y||}|| = \sup_{x \in D(T)} ||\frac{x}{||Tx||}|| = \frac{1}{\inf\limits_{x \in D(T)} ||T\frac{x}{||x||}||}$$

This show that T^{-1} is continuous and $||T^{-1}|| = \gamma_T^{-1}$.

On the other hand, if $\gamma_T = 0$, then either there is $x \neq 0$ such that $Tx = 0$, i.e., T has no inverse; or T has the inverse but there is a sequence $\{x_n\}$ such that $||x_n|| = 1$ and $||y_n|| := ||Tx_n|| \to 0$; however, this implies that $||\frac{T^{-1}y_n}{||y_n||}|| = \frac{1}{||y_n||} \to \infty$, i.e., T^{-1} is not continuous. ■

Claim 6.23 $1 + T_\phi^x$, $1 + C_\phi^x$, $1 + H_\phi^x$, and $1 + L_\phi^x$ are bijections.

Proof: Since all operators listed are Fredholm with index 0, hence we just need to show that their kernels are $\{0\}$.

For $1 + T_\phi^x$, if $f \in L^2[-x, x]$, then as what is got in the proof of the claim 6.15 (we still denote by f the extension of f by setting 0 out of $[-x, x]$), we have

$$\langle f, (1 + T_\phi^x)f\rangle = \langle f, (1 + T_\phi^N)f\rangle \geq \lambda ||f||^2_{L^2[-N,N]} = \lambda ||f||^2_{L^2[-x,x]}$$

where $\lambda > 0$ is the smallest eigenvalue of $1 + T_\phi^N$.

If γ is an eigenvalue of $1 + T_\phi^x$, then for a corresponding eigenvector f_γ, we must have

$$\langle f_\gamma, (1 + T_\phi^x) f_\gamma \rangle = \gamma ||f_\gamma||^2 \geq \lambda ||f_\gamma||^2$$

Hence, it follows that all eigenvalues of $1 + T_\phi^x$ are not less than λ. By spectral theorem again,

$$||(1 + T_\phi^x) f||^2 = \int_{\sigma(1+T_\phi^x)} t^2 d||E_{\phi,x}(t) f||^2 \geq \lambda^2 ||f||^2$$

where $E_{\phi,x}(t)$ is the spectral family of $1 + T_\phi^x$.

Now, claim 6.22 implies that $1 + T_\phi^x$ is invertable with $||(1 + T_\phi^x)^{-1}|| \leq \lambda^{-1}$, specially, $\ker(1 + T_\phi^x) = \{0\}$.

For $1 + C_\phi^x$, since any eigenvalue of $1 + C_\phi^x$ must be an eigenvalue of $1 + T_\phi^x$, hence all eigenvalues of $1 + C_\phi^x$ are not less than λ, i.e., 0 cannot be an eigenvalue, thus $\ker(1 + C_\phi^x) = \{0\}$.

For $1 + H_\phi^x$, the desired conclusion is from the conclusion of $1 + C_\phi^x$ directly.

For $1 + L_\phi^x$, we change our strategy: we will show that $Ran(1 + L_\phi^x) = L^1[-x, x]$, hence $dim \ker(1 + L_\phi^x) = 0$. Let $g \in L^1[-x, x]$, then we have $\{g_n\} \subset C[-x, x]$ so that $g_n \to g$ in $L^1[-x, x]$. Since we already proved that $1 + C_\phi^x$ is a bijection, then there must be $\{f_n\} \subset C[-x, x]$ so that $(1 + C_\phi^x) f_n = g_n$; moreover, this fact implies $(1 + L_\phi^x) f_n = g_n$, i.e., $\{g_n\} \subset$

$Ran(1+L_\phi^x)$. Since we know that $Ran(1+L_\phi^x)$ is closed, it follows that $g \in Ran(1+L_\phi^x)$ as the limit of $\{g_n\}$. ■

Let's turn to $1+H_\phi^x$ again.

Suppose $g = (g_c, a\chi[0,x]) \in H[-x,x]$, then claim 6.23 implies there is just one $f = (f_c, a\chi[0,x]) \in H[-x,x]$ such that $(1+H_\phi^x)f = g$, i.e.,

$$f_c(x,t) + \int_{-x}^{x} \phi(t-s) f_c(x,s)ds = g_c(x,t) - a\int_{t-x}^{t} \phi(s)ds$$

This gives the unique solution $f_c + a\chi[0,x] \in L^2[-x,x]$ of the equation $(1+T_\phi^x)(f_c + a\chi[0,x]) = g_c + a\chi[0,x]$.

Specially, from (6.6), we have $k(x,t) = k_c(x,t) + i\chi[0,x]$ where $k_c \in C[-x,x]$ is the solution of

$$k_c(x,t) + \int_{-x}^{x} \phi(t-s) k_c(x,s)ds = i\Phi(t-x) - \frac{1}{2}i$$

Claim 6.24 Assume $g(x,t) \in C(\overline{\Delta_N})$. For any $x \in [0,N]$, there is a unique $p(x,t) \in C[-x,x]$ which is the solution of

$$p(x,t) + \int_{-x}^{x} \phi(t-s)p(x,s)ds = g(x,t)$$

Moreover, we have

1) $p(x,t) \in C(\overline{\Delta_N})$.

2) Under an extra assumption that for all fixed $x \in [0,N]$, $g(x,t) \in AC[-x,x]$, we have that $p(x,t) \in AC[-x,x]$ with respect to t. More-

over, the partial derivative $p_t(x,t)$ satisfies

$$p_t(x,t) + \int_{-x}^{x} \phi(t-s)p_t(x,s) = g_t(x,t) + \phi(t-x)p(x,x) - \phi(t+x)p(x,-x)$$

where $g_t(x,t)$ is the partial derivative of g with respect to t.

Proof: The existence of $p(x,t) \in C[-x,x]$ for a fixed $x \in [0,N]$ is from claim 6.23 directly.

1) We first show that $p(x,t)$ is continuous at 0. The purpose here is to avoid $x = 0$ in the sequel.

Let $1 > \epsilon > 0$. Recall that $||C_\phi^x f|| \le ||f||_{C[-x,x]} \sup\limits_{t\in[-x,x]} \int_{t-x}^{t+x} |\phi(s)|ds$, we can pick up a small N_1 by absolute continuity so that if $b - a \le 2N_1$, then $\int_a^b |\phi(s)|ds < \epsilon$. Thus, for all $x \in [0, N_1]$, we have $||C_\phi^x|| < \epsilon$.

It is clear that $(1 + C_\phi^x)^{-1} = \sum\limits_{n=0}^{\infty}(-C_\phi^x)^n$ and $||(1 + C_\phi^x)^{-1}|| \le \frac{1}{1-\epsilon}$ for all $x \in [0, N_1]$. Let $(x,t) \in \overline{\Delta_{N_1}}$, then it follows from $p(0,0) = g(0,0)$ that

$$p(x,t) - p(0,0) = (1 + C_\phi^x)^{-1}(g(x,t) - g(0,0) - C_\phi^x g(0,0))$$

Since $g(x,t) \in C(\overline{\Delta_N})$, then N_1 can be chosen small enough so that $|g(x,t) - g(0,0)| < \epsilon$, hence the equality above gives

$$|p(x,t) - p(0,0)| \le \frac{\epsilon(1 + |g(0,0)|)}{1-\epsilon}$$

This implies that $p(x,t)$ is continuous at $(0,0)$.

The rest part is the continuity on $\Theta_{N_1} := \overline{\Delta_N} \setminus \overline{\Delta_{N_1}} \cup \{(N_1,t) : |t| \le N_1\}$ for any $N_1 > 0$.

Let's fix a N_1. We first introduce a method to "annihilate" the variable x, see [4] chapter 6 for example. Let h be a continuous function as follows:

$$h : \Theta_{N_1} \to [N_1, N] \times [-1, 1], h(x, t) = (x, \frac{t}{x})$$

Notice that the construction of the function makes sense because we already avoid $x = 0$. Moreover, on Θ_{N_1}, p solves the equation

$$p(x, t) + \int_{-x}^{x} \phi(t - s)p(x, s)ds = g(x, t)$$

if and only if on $[N_1, N] \times [-1, 1]$, $q := p \circ h^{-1}$ solves

$$q(x, t) + \int_{-1}^{1} x\phi(xt - xs)q(x, s)ds = g \circ h^{-1}(x, t)$$

We denote the second operator from $C[-1, 1]$ to $C[-1, 1]$ for a fixed $x \in [N_1, N]$ by $1 + K_\phi^x$, with the help of C_ϕ^x, we know that $1 + K_\phi^x$ is boundedly invertible. The advantage of this operator is that the second coordinate is irrelevant to x, hence we can plug $q(x_0, t)$ into $1 + K_\phi^x$ as needed and don't need to worry about whether they are corresponding.

Let $1 > \epsilon > 0$, if we fix $x_0 \in [N_1, N]$, and pick up $x \in [N_1, N]$ such that $|x - x_0| < \delta$ for some proper $\delta > 0$, we want to analyze $||(K_\phi^x - K_\phi^{x_0})f|| \leq ||f|| \sup\limits_{t \in [-1,1]} \int_{-1}^{1} |x\phi(xt - xs) - x_0\phi(x_0 t - x_0 s)|ds$ for $f \in C[-1, 1]$.
Let $\{\phi_n\} \subset C_c^\infty(-2N, 2N)$ satisfying $\phi_n(x) = \overline{\phi_n}(-x)$ and $\phi_n \to \phi$ in $L^1(-2N, 2N)$.

Since we have for $f \in L^2[-N, N]$,

$$|\langle f, (1 + T_\phi^N) f\rangle - \langle f, (1 + T_{\phi_n}^N) f\rangle| \le ||f||^2 ||\phi - \phi_n||_{L^1}$$

we conclude that for all large n, $\phi_n \in \Phi_N$, hence without loss of generality, we assume all $\phi_n \in \Phi_N$ in the sequel.

Let's pick up a large enough n such that $||\phi - \phi_n|| < \frac{\epsilon}{8||(1+K_\phi^{x_0})^{-1}||}$, we have
$\int_{-1}^{1} |x\phi(xt - xs) - x_0\phi(x_0 t - x_0 s)| ds$
$\le \int_{-1}^{1} |x\phi(xt - xs) - x\phi_n(xt - xs)| ds + \int_{-1}^{1} |x_0\phi_n(x_0 t - x_0 s) - x_0\phi(x_0 t - x_0 s)| ds + \int_{-1}^{1} |x\phi_n(xt - xs) - x_0\phi_n(x_0 t - x_0 s)| ds$
The first and the second terms are not greater than $||\phi - \phi_n||$, moreover, since
$\int_{-1}^{1} |x\phi_n(xt - xs) - x_0\phi_n(x_0 t - x_0 s)| ds$
$\le \int_{-1}^{1} x|\phi_n(xt - xs) - \phi_n(x_0 t - x_0 s)| ds + |x - x_0| \int_{-1}^{1} |\phi_n(x_0 t - x_0 s)| ds$
and ϕ_n uniformly continuous, if δ is chosen properly small enough such that $|\phi_n(xt - xs) - \phi_n(x_0 t - x_0 s)| \le \frac{\epsilon}{8N||(1+K_\phi^{x_0})^{-1}||}$, and that the second term is also small, then we can get

$$||K_\phi^x - K_\phi^{x_0}|| < \frac{\epsilon}{||(1 + K_\phi^{x_0})^{-1}||}$$

.

Consider the following equation about K_ϕ^x:

$$(1 + K_\phi^{x_0})^{-1}(g \circ h^{-1}(x, t)) = q(x, t) + (1 + K_\phi^{x_0})^{-1}(K_\phi^x - K_\phi^{x_0})(q(x, t))$$

Since we have $|g| \leq M$ for some M, then

$$||q(x,\cdot)||_{C[-1,1]} \leq M||(1+K_\phi^{x_0})^{-1}|| + \epsilon||q(x,\cdot)||_{C[-1,1]}$$

equivalently,

$$\sup_{|x-x_0|<\delta} ||q(x,\cdot)||_{C[-1,1]} \leq \frac{M||(1+K_\phi^{x_0})^{-1}||}{1-\epsilon}$$

We also have $(1+K_\phi^{x_0})^{-1}(g \circ h^{-1}(x_0,t)) = q(x_0,t)$. If δ is chosen smaller enough, then we have $||g \circ h^{-1}(x,\cdot) - g \circ h^{-1}(x_0,\cdot)||_{C[-1,1]} < \epsilon$, hence in fact, we have the following estimate:

$$||q(x_0,\cdot) - q(x,\cdot)||_{C[-1,1]} \leq ||(1+K_\phi^{x_0})^{-1}||\epsilon + \frac{M||(1+K_\phi^{x_0})^{-1}||}{1-\epsilon}\epsilon$$

On the other hand, if we consider x and t simultaneously, we have

$$|q(x,t) - q(x_0,t_0)| \leq |q(x,t) - q(x_0,t)| + |q(x_0,t) - q(x_0,t_0)|$$

This implies $q(x,t) \in C([N_1,N] \times [-1,1])$, that is, as the composition of two continuous functions, $p(x,t) \in C(\Theta_{N_1})$.

If we change N_1 in Θ_{N_1}, and with the fact that p is continuous at 0, we conclude that $p(x,t) \in C(\overline{\Delta_N})$.

2) Fix $x \in (0,x]$. Let $\{\phi_n\}, \{(g_n)_t(x,t)\} \subset C_c^\infty(-2N,2N)$ and $\phi_n \to \phi, (g_n)_t(x,t) \to g_t(x,t)$ in $L^1(-2N,2N)$. We write $g_n(x,t) = g(x,0) + \int_0^t (g_n)_t(x,s)ds$, then $g_n \to g$ in $C[-x,x]$, and we have solutions p_n satisfy-

ing

$$(1 + C^x_{\phi_n})p_n = g_n(x,t)$$

or equivalently,

$$p_n(x,t) = -\int_{-x}^{x} \phi_n(t-s)p_n(x,s)ds + g_n(x,t)$$

Since ϕ_n is differentiable and its derivative is bounded, then by applying the mean value theorem and dominated convergence theorem to ϕ_n, we conclude that $\int_{-x}^{x} \phi_n(t-s)p_n(x,s)ds$ is differentiable as a function of t, and the derivative is $\int_{-x}^{x} \phi_n'(t-s)p_n(x,s)ds$. Since $p_n(x,t) \in C(\overline{\Delta_N})$, hence it is bounded, and ϕ_n' is also bounded, then we actually conclude that by mean value theorem again, $\int_{-x}^{x} \phi_n(t-s)p_n(x,s)ds$ is Lipschitz, hence absolutely continuous with respect to t, thus it follows that p_n is absolutely continuous. We denote those derivatives by $(p_n)_t$, then we have

$$(1 + L^x_{\phi_n})(p_n)_t = (g_n)_t(x,t) + \phi_n(t-x)p_n(x,x) - \phi_n(t+x)p_n(x,-x)$$

Here, we have to change the space we are in (the operator notation we are using) because the right-hand side converges in L^1 rather than $C[-x,x]$. Since $(1 + C^x_{\phi_n})^{-1} = (1 - (1 + C^x_{\phi})^{-1}C^x_{\phi-\phi_n})^{-1}(1 + C^x_{\phi})^{-1}$, and for all large n, we have $||C^x_{\phi-\phi_n}|| \leq ||\phi - \phi_n||_{L^1} \leq \frac{\epsilon}{||(1+C^x_{\phi})^{-1}||}$ for small $0 < \epsilon < 1$, hence it follows that

$$||(1 + C^x_{\phi_n})^{-1} - (1 + C^x_{\phi})^{-1}|| \leq \frac{||(1 + C^x_{\phi})^{-1}||}{1-\epsilon}\epsilon$$

For $1 + L^x_{\phi_n}$, we have an analogous conclusion.

From $p_n = (1 + C^x_{\phi_n})^{-1} g_n$, we conclude that $\{p_n\}$ is a Cauchy sequence in $C[-x,x]$ that converges to $p(x,t)$, hence $\{p_n\}$ is uniformly bounded, as a consequence, $(g_n)_t(x,t) + \phi_n(t-x)p_n(x,x) - \phi_n(t+x)p_n(x,-x)$ is convergent to $g_t(x,t) + \phi(t-x)p(x,x) - \phi(t+x)p(x,-x)$ in $L^1[-x,x]$. It follows as above that $\{(p_n)_t\}$ is Cauchy in $L^1[-x,x]$, i.e., there is a function in $L^1[-x,x]$, denoted by p_t, such that $\lim_{n\to\infty}(p_n)_t \to p_t$. If we take limit for $(1 + L^x_{\phi_n})(p_n)_t$, then we can get the equation we need; moreover, we have

$$p_n(x,t) = p_n(x,0) + \int_0^t (p_n)_t(x,s)ds$$

once we take limit on both sides, we get

$$p(x,t) = p(x,0) + \int_0^t p_t(x,s)ds$$

■

Since our purpose is just to analyze two equations obtained in section 6.6, we can summarize the claim 6.24 as follows:

Claim 6.25 Assume that $g(x,t) = \frac{1}{2}$ or $g(x,t) = i\Phi(t-x) - \frac{1}{2}i$.

1) For any $x \in [0,N]$, there is a unique $p(x,t) \in C[-x,x]$ that is the solution of

$$p(x,t) + \int_{-x}^{x} \phi(t-s)p(x,s)ds = g(x,t)$$

Moreover, we have $p(x,t) \in C(\overline{\Delta_N})$, and $p(x,t) \in AC[-x,x]$ with respect to t. The partial derivative $p_t(x,t)$ satisfies

$$p_t(x,t) + \int_{-x}^{x} \phi(t-s)p_t(x,s) = g_t(x,t) + \phi(t-x)p(x,x) - \phi(t+x)p(x,-x)$$

where $g_t(x,t)$ is the partial derivative of g with respect to t.

2) For any $x \in [0,N]$,we denote by $p_x(x,t)$ the solution of

$$p_x(x,t) + \int_{-x}^{x} \phi(t-s)p_x(x,s) = g_x(x,t) - \phi(t-x)p(x,x) - \phi(t+x)p(x,-x)$$

where $g_x(x,t)$ is the partial derivative of g with respect to x.

Then for any $x \in [0,N]$, we have

$$\int_{-x}^{x} p(x,s)ds = \int_{-x}^{x} p(|s|,s)ds + \int_{-x}^{x} ds \int_{|s|}^{x} p_x(m,s)dm$$

Proof: 1) This is just a restatement of claim 6.24.

2) We first discuss the uniform upper bound of operators $(1+C_\phi^x)^{-1}$ with respect to x.

As in claim 6.22, we define on $[0,N]$ a function for $\phi \in \Phi_N$:

$$\gamma_{C_\phi}(x) := \inf\{||(1+C_\phi^x)f|| : f \in C[-x,x], ||f|| \geq 1\}$$

We claim that $\gamma_{C_\phi}(x)$ is continuous on $[0,N]$.

Indeed, let's recall the operator $1+K_\phi^x$. In the proof of part 1), we discussed $||K_\phi^x - K_\phi^{x_0}||$, and the conclusion there can be summarized as K_ϕ^x is continuous as a mapping about x.

We pick up $f \in C[-x,x]$ such that $||f|| = 1$, then We have $(1 + C_\phi^x)f = (1 + K_\phi^x)k$ where $k(t) = f(xt)$, and $(1 + C_\phi^{x_0})w = (1 + K_\phi^{x_0})k$ where $w(t) = f(\frac{x}{x_0}t)$. Because

$$||(1 + K_\phi^x)k|| \le ||(1 + K_\phi^{x_0})k|| + ||K_\phi^x - K_\phi^{x_0}|| \cdot ||k||_{C[-1,1]}$$

we get

$$||(1 + C_\phi^x)f|| \le ||(1 + C_\phi^{x_0})w|| + ||K_\phi^x - K_\phi^{x_0}||$$

If we take infimum on the left-hand side first, then take infimum again on the right-hand side, we conclude that $\gamma_{C_\phi}(x) \le \gamma_{C_\phi}(x_0) + ||K_\phi^x - K_\phi^{x_0}||$. Once we switch K_ϕ^x and $K_\phi^{x_0}$, then we can get the symmetric one: $\gamma_{C_\phi}(x_0) \le \gamma_{C_\phi}(x) + ||K_\phi^x - K_\phi^{x_0}||$. The inequality

$$|\gamma_{C_\phi}(x_0) - \gamma_{C_\phi}(x)| \le ||K_\phi^x - K_\phi^{x_0}||$$

implies, by taking $x \to x_0$, that $\gamma_{C_\phi}(x)$ is continuous on $(0, N]$. If $x_0 = 0$, then $\lim_{x \to x_0} ||C_\phi^x|| = 0$, this gives the continuity at 0.
Now it follows that $\gamma_{C_\phi}(x) \ge \inf_{x \in [0,N]} \gamma_{C_\phi}(x) = \gamma_{C_\phi}(x_0)$ for some $x_0 \in [0, N]$. Moreover, by claim 6.22, claim 6.23 and closed graph theorem, we know that $\gamma_{C_\phi}(x_0) > 0$, hence it follows from this fact that $\gamma_{C_\phi}(x) \ge c(\phi)$ for some number only related to ϕ and $c(\phi) > 0$, i.e.,

$$\sup_{x \in [0,N]} ||(1 + C_\phi^x)^{-1}|| \le \frac{1}{c(\phi)}$$

. For $1 + L_\phi^x$, we have an analogous conclusion saying that

$$\sup_{x\in[0,N]} ||(1 + L_\phi^x)^{-1}|| \leq \frac{1}{l(\phi)}$$

for a number only related to ϕ and $l(\phi) > 0$. To see this, we just need to notice that as above $||f||_{L^1[-x,x]} = x||k||_{L^1[-1,1]}$, $||f||_{L^1[-x,x]} = \frac{x}{x_0}||w||_{L^1[-x_0,x_0]}$, and apply Young's inequality for integral operators again for $||K_\phi^x - K_\phi^{x_0}||$.

We just need to deal with $g = \frac{1}{2}$, the other scenario can be done as the same (the only nuance is that we need to consider g_n obtained by substituting ϕ by ϕ_n in g).

As in the proof of claim 6.24, Let $\{\phi_n\} \subset C_c^\infty(-2N, 2N)$ and $\phi_n \to \phi$ in $L^1(-2N, 2N)$, and we denote the difference quotient of a given function f by

$$S_\epsilon f(x,t) = \frac{f(x+\epsilon, t) - f(x,t)}{\epsilon}$$

Again, we have

$$||p_n - p||_{C[-x,x]} \leq \frac{1}{2}||(1 + C_{\phi_n}^x)^{-1} - (1 + C_\phi^x)^{-1}|| \leq \frac{\epsilon}{2c(\phi)(1-\epsilon)}$$

i.e., p_n is convergent to p uniformly not only about t but about x, as a result, we conclude that

$$\sup_{(x,t)\in\overline{\Delta_N}} |p_n(x,t)| < M$$

for all n and a large number M.

On the other hand, we also have for $\epsilon > 0$

$$(1+C^{x}_{\phi_n})S_\epsilon p_n = \frac{1}{\epsilon}(-\int_x^{x+\epsilon} \phi_n(t-s)p_n(x+\epsilon, s) - \int_{-(x+\epsilon)}^{-x} \phi_n(t-s)p_n(x+\epsilon, s))$$

and for $\epsilon < 0$

$$(1+C^{x+\epsilon}_{\phi_n})S_\epsilon p_n = \frac{1}{\epsilon}(-\int_x^{x+\epsilon} \phi_n(t-s)p_n(x, s) - \int_{-(x+\epsilon)}^{-x} \phi_n(t-s)p_n(x, s))$$

Let's deal with the first equation.

Since the right-hand side converges to $-\phi_n(t-x)p_n(x,x) - \phi_n(t+x)p_n(x,-x)$ in $C[-x,x]$, hence $S_\epsilon p_n \to (p_n)_x$ in $C[-x,x]$ when $\epsilon \to 0$ from the right-hand side.

To deal with the second equation, consider $(1 + K^{x+\epsilon}_{\phi_n})(S_\epsilon p_n \circ h^{-1})$ and $(1 + K^{x}_{\phi_n})((p_n)_x \circ h^{-1})$, moreover, for a small enough $\epsilon > 0$,

$$(1 + K^{x+\epsilon}_{\phi_n})^{-1} = (1 - (1 + K^{x}_{\phi_n})^{-1}(K^{x}_{\phi_n} - K^{x+\epsilon}_{\phi_n}))^{-1}(1 + K^{x}_{\phi_n})^{-1}$$

it follows that $S_\epsilon p_n \to (p_n)_x$ in $C[-x,x]$ when $\epsilon \to 0$ from the left-hand side.

In all, p_n is differentiable everywhere in term of x, and the (partial) derivative is $(p_n)_x$, and

$$p_n(x,t) = p_n(|t|, t) + \int_{|t|}^{x} (p_n)_x(s,t)ds$$

Once again, since $\{p_n\}$ is uniformly bounded in $\overline{\Delta_N}$, it follows that $-\phi_n(t-$

$x)p_n(x,x) - \phi_n(t+x)p_n(x,-x) \to -\phi(t-x)p(x,x) - \phi(t+x)p(x,-x)$ in $L^1[-x,x]$ uniformly in $x \in [0,N]$, as a consequence, it follows from the uniform bound of $||(1+L_\phi^x)^{-1}||$ in term of x that

$$\lim_{n\to\infty} ||(p_n)_x - p_x||_{L^1[-x,x]} = 0$$

uniformly in x.

This limit implies that Fubini theorem works, and

$$\lim_{n\to\infty} \int_0^x dm \int_{-m}^m ((p_n)_x(m,s) - p_x(m,s))ds = 0$$

Moreover, if we pick up a large n, then we have

$\int_0^x dm \int_{-m}^m |p_x(m,s)|ds$
$\leq \int_0^x dm \int_{-m}^m |(p_n)_x(m,s) - p_x(m,s)|ds + \int_0^x dm \int_{-m}^m |(p_n)_x(m,s)|ds$

The first term of the second line is small enough, and $(p_n)_x$ is bounded, hence $\int_{-x}^x |p_x(x,s)|ds$ is integrable in term of x.

By Fubini theorem, we have

$0 = \lim_{n\to\infty} \int_0^x dm \int_{-m}^m ((p_n)_x(m,s) - p_x(m,s))ds$
$= \lim_{n\to\infty} \int_{-x}^x ds \int_{|s|}^x ((p_n)_x(m,s) - p_x(m,s))dm$
$= \lim_{n\to\infty} \int_{-x}^x (p_n(x,s) - p_n(|s|,s) - \int_{|s|}^x p_x(m,s)dm)ds$
$= \int_{-x}^x (p(x,s) - p(|s|,s) - \int_{|s|}^x p_x(m,s)dm)ds$

This gives

$$\int_{-x}^{x} p(x,s)ds = \int_{-x}^{x} p(|s|,s)ds + \int_{-x}^{x} ds \int_{|s|}^{x} p_x(m,s)dm$$

■

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