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ABSTRACT

We introduce a generalized notion of combinatorial species called A-species, where A is a Hopf algebra. The role played by the symmetric group, S_n, in the classical theory of species is now replaced with the wreath product $A \wr S_n$. We show that category of A-species admits a monoidal structure under the Cauchy and Hadamard product. Under certain choices of A, we recover the notion of Joyal's species, $\mathcal{H}$-species defined by Choquette and Bergeron, and B_r-species defined by Henderson. We define many bilax monoidal functors involving the category of A-species. The first functor we construct, S^A, goes from the category of species to A-species. This functor sends the regular representation of the symmetric group S_n to the regular representation of $A \wr S_n$, and we use this functor to construct the appropriate definitions of A-Hopf monoids built from common Hopf monoids. When $A = \mathbb{K}C_r$, we recover the functor defined by Choquette and Bergeron. We then define A-Fock functors, which are bilax functors between the category of A-species and the category of vector spaces. We analyze the images of certain A-Hopf monoids under them and show that they are all isomorphic as Hopf algebras to the Hopf algebra of polynomials invariant under the hyperoctrahedral group, $\mathbb{C}\langle\langle x\rangle\rangle^{B_r}$. We end by showing a sub Hopf algebra of a quotient of the ring of C_r-colored set partitions functions surjects onto $\mathbb{C}\langle\langle x\rangle\rangle^{B_r}$ and has a one-sided inverse.

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CHAPTER 0

Introduction

Hopf Algebras were first encountered in the work of Heinz Hopf in 1941, where he introduced the notion of Hopf algebras in connection with the homology of Lie groups; however, a formal definition wasn't coined until 1956 by Pierre Cartier. For a complete history, see [5] and [14]. Hopf Algebras are a particularly interesting object to study as they turn up in many areas–algebraic topology, representation theory, combinatorics, and applications in physics. Informally, Hopf algebras are vector spaces over a field $\mathbb{K}$, with an algebra and coalgebra structure that is compatible along with a unique antihomomorphism, called the antipode (see Subsection 3.1.1).

The idea of using Hopf algebras to study combinatorial structures goes back to the work of Joni and Rota in 1979 in [24]. These Hopf algebras tend to have bases that are naturally labeled by combinatorial structures (set partitions, linear orders, trees, etc.), and their algebraic structures encode basic operations of these combinatorial objects. Often, in these combinatorial Hopf algebras [1], the product and coproduct encode how to combine and split combinatorial objects.

A motivating example is the ring of symmetric functions ([20], [29], [13]), as these are well-studied and have many applications. There have been many generalizations of the ring of symmetric functions, many of which are combinatorial Hopf algebras: the quasisymmetric functions, $QSym$ ([36], [19], [28]), the noncommutative symmetric functions, $NSym$ ([18]), the ring of symmetric functions in noncommuting variables, $NCSym$ or Π ([35], [11]), and the ring of odd symmetric functions, $O\Lambda$ ([26], [17]).

One place to find combinatorial Hopf algebras is within the theory of combinatorial species. The theory of combinatorial species originated in the work of Joyal ([25]) in 1981 as a way to understand generating functions. Since then, species have been used in different areas of mathematics. They have striking similarities to the theory of algebraic data types in functional programming languages such as Haskel ([38]), they are used in Euler integration as a new way to interpret the Euler integral which also extends to magnitude homology and configuration spaces ([30]), and even have applications in physics ([33]).

Loosely speaking, a species is a way to take a set (labels) and convert it to a family of structures labelled by said set; for example, if we take the set $S = \{1, 2, 3\}$, we can consider all the graphs whose vertices are labelled by S. More formally, a species, $\mathbf{p}$, is a functor between the category of finite sets and set bijections and the category of vector spaces. That is, for each finite set I, we get a vector space $\mathbf{p}[I]$, along with linear maps, $\mathbf{p}[I] \to \mathbf{p}[J]$, induced from the underlying set bijections $I \to J$, see Chapter 4. An alternative way to view

[1] There doesn't seem to be a universal definition of a combinatorial Hopf algebra, but what seems to be agreed upon by authors is that there should be a basis labelled by combinatorial objects, the grading should come from the "size" of the objects, and the structure constants from the product and coproduct should count the number of ways one could combine or split elements.

a species is as thinking of each of the components, $\mathbf{p}[I]$, as a module for the symmetric group S_n where $|I| = n$.

We can consider the category of species. It turns out this category admits various operations; in particular, the category of species admits a monoidal structure. With respect to these operations, there are analogous structures akin to algebras (monoids), coalgebras (comonoids), bialgebras (bimonoids), and Hopf algebras (Hopf monoids). In recent years, Aguair and Mahajan, in [3], did extensive work involving the category of species as a monoidal category. We also encourage the reader to see [4] and [6]. They introduced the notion of a bilax monoidal functor, which is a functor between two monoidal categories that preserves bimonoids. Within this, they determined what conditions were necessary for when a bilax monoidal functor preserves Hopf monoids and call this a bistrong monoidal functor. They primarily focus on the category of species. They defined four important Fock functors, $K, \overline{K}$, $K^{\vee}$, and $\overline{K}^{\vee}$, which correspond to the S_n-coinvariants and the S_n-invariants. All of which are bilax monoidal functors, with both $\overline{K}$ and $\overline{K}^{\vee}$ having the additional bistrong property; hence, they preserve Hopf monoids. They used these functors as a way to construct Hopf algebras from Hopf monoids. Many well-studied Hopf algebras can be recovered in this fashion. This gives an indication that this is a natural combinatorial setting to work in. We also see that a single Hopf monoid can have many different associated Hopf algebras; for example, consider the Hopf monoid of set partitions, $\mathbf{\Pi}$, as defined in Section 3.5. Applying the functors yields

$$K(\mathbf{\Pi}) \cong \Pi = NCSym \cong K^{\vee}(\mathbf{\Pi}) \text{ and } \overline{K}(\mathbf{\Pi}) \cong \Lambda = Sym \cong \overline{K}^{\vee}(\mathbf{\Pi}).$$

Many generalizations of species have been studied, see [10], [21], and [3]. In Chapter 19 of [3], Aguiar and Mahajan define the notions of decorated species and colored species. In [10], Choquette and Bergeron defined the notion of $\mathcal{H}$-species, which give modules for the hyperoctrahedral group, $C_2 \wr S_n$. They construct a functor, $\mathcal{S}$, that constructs an $\mathcal{H}$-species from any species. One thing to note, is that under this functor the regular representation of S_n is sent to the regular representation of $C_2 \wr S_n$, allowing them to define the notion of $\mathcal{H}$ linear orders appropriately. In [21] and [22], Henderson defined a generalization of Joyal's species which he called B_r-modules. Here, rather than the components being modules for the symmetric group, one has modules for $C_r \wr S_n$, i.e., the wreath product of the cyclic group of order with the symmetric group. In fact, he remarks that this holds for any group G instead of restricting to only C_r. Henderson's definition encompasses both Joyal's species and $\mathcal{H}$-species.

The goal of this thesis is to generalize the category of species even further. Let $\mathbb{K}$ be a field and A be a Hopf algebra over a field $\mathbb{K}$. We define a notion of A-species. We define our underlying category $\mathbf{Set}^A$ to consist of all finite sets decorated with A and all morphisms between them. Informally, an A-species is a functor, $\mathbf{p}$, between $\mathbf{Set}^A$ and $\mathbf{Vec}_{\mathbb{K}}$, which consists of a family of vector spaces $\mathbf{p}[I_A]$, one for each object $I_A \in \mathbf{Set}^A$. In particular, each $\mathbf{p}[I_A]$ can be viewed as a module for $A \wr S_n$. In other words, the role of the symmetric group in classical species is now replaced with $A \wr S_n$. We can consider the category formed by all such A-species and morphisms between them, call this $\mathbf{Sp}^A$. Our category encompasses some of the generalizations above. When $A = \mathbb{K}$, we recover the classical notion of species ([25]). When $A = \mathbb{K}C_2$, we recover $\mathcal{H}$-species, and when $A = \mathbb{K}G$ ([10]), we recover Henderson's notion of species corresponding to $G \wr S_n$-modules ([21], [22]).

As done by Aguiar and Mahajan ([3]), we explore analogous properties of this category $\mathbf{Sp}^A$ to that of $\mathbf{Sp}$. Under two operations similar to ones used in classical species, this category becomes a monoidal category. Hence, we can apply ideas/tools used within monoidal categories to our category. We define what it means to be a monoid, comonoid, bimonoid, and Hopf monoid in $\mathbf{Sp}^A$ and construct natural analogues of bilax monoidal functors which preserve the notion of (Hopf/bi/co)monoids. One such functor, $\mathcal{S}^A : \mathbf{Sp} \to \mathbf{Sp}^A$, gives a way to define an A-species from any species. For any A, this functor sends the regular representation of S_n to the regular representation of $A \wr S_n$ for every $n \geq 0$. When $A = \mathbb{K}C_2$, we recover the functor defined by Choquette and Bergeron. From this functor, we can define the correct version of what the A-species of linear orders, set partitions, etc. should be. Other functors that we define are similar to the Fock functors defined by Aguiar and Mahajan, which we call A-Fock functors, and give a way of constructing graded Hopf algebras from Hopf monoids in $\mathbf{Sp}^A$. Following notation of Aguiar and Mahjan, these are denoted by $K_A, \tilde{K}_A, K_A^\vee$, and $\widetilde{K_A^\vee}$. We end by showing a string of relationships by applying various A-Fock functors to Hopf monoids formed by operations on the A-Hopf monoid of linear orders $(\mathbf{L}_A)$, set partitions $(\mathbf{\Pi}_A)$, and superclass functions on unitriangular groups $(\mathbf{scf}_A(U))$. The relationships are as follows:

$$\overline{K}_A(\mathbf{scf}_A(U)) \cong \overline{K}_A((\mathbf{L} \times \mathbf{\Pi})_A) \cong K_A(\mathbf{\Pi}_A) \cong \widetilde{K}_A(\mathbf{L}_A \times \mathbf{\Pi}_A) \cong \tilde{\Pi}^{(B)},$$

where $\tilde{\Pi}^{(B)}$ is a colored version of the ring of symmetric functions in noncommutative variables (see [1] and [2]).

There are many other well-studied species which should have interesting generalizations to A-species. In [34], Proudfoot constructed a category FS_B whose objects are finite sets along with an involution with exactly one fixed point and bijections that respect this involution; one natural question would be to see if there exists a bilax monoidal functor from $\mathbf{Sp}^A \to FS_B$. It also remains an open question to compute the Hopf algebras associated to these A-species. Conversely, there are interesting combinatorial Hopf algebras that have not yet had a description in terms of species. For example, it would be interesting to determine what suitable choice of A and A-species would be needed to recover the ring of odd symmetric functions of Ellis and Khovanov, [17].

The content of this thesis is organized as follows: Chapters 1 through 6 discuss the necessary background information. Chapter 7 and its sections introduces our notion of A-species, where A is a Hopf algebra. Here, we start by describing the building blocks of A-species, i.e., the category of A-sets, and define A-species– a functor $\mathbf{p} : \mathbf{Set}^A \to \mathbf{Vec}_{\mathbb{K}}$. We consider the category of A-species, and show this is monoidal with respect to two different products. We end by showing how A-species generalizes classical species, $\mathcal{H}$-species, and B_r-species. In Chapter 8, we construct A-species by decorating with A-modules in various ways. In Chapter 9 and its sections, we construct a bilax monoidal functor, $\mathcal{S}^A$, from the category of species to the category of A-species. This functor sends the regular representation of the symmetric group, S_n, to the regular representation of $A \wr S_n$ and generalizes the notion of the functor defined by Choquette and Bergeron. In Chapter 10 and its sections, we define A-versions of the Fock functors defined by Aguiar and Mahajn ([3]). Within, we describe morphisms between these functors and end by making explicit the structure of the Hopf algebra obtained by applying one of these functors to a A-Hopf monoid. In Chapter 11, we look at three examples of A-Hopf monoids. Chapter 12 and its sections explores the relationships between A-Hopf monoids constructed via examples from Chapter 11 and their images under certain A-Fock functors described in Chapter 10. In Chapter 13, we end by defining a projection map from quotient of the Hopf algebra of C_r-colored set partitions to the Hopf algebra of B_r-invariant functions.

CHAPTER 1

Combinatorial Preliminaries

In this first chapter, we will discuss the basics of the necessary combinatorial background that will be used throughout later sections. In this section, we primarily follow the notation used by Rosas, see [11] and [35].

1.1. Combinatorics of Set Partitions

Let I be a finite set. When I is of the form $\{1, ..., n\}$ for some positive integer n, we simply write $[n]$. For positive integer m, let $m + [n] = \{m+1, ..., m+n\}$.

Definition 1.1.1. A *(set) partition*, X, of I, denoted $X \vdash I$, is a family of disjoint nonempty sets, called *blocks*, $X_1, ..., X_t$ whose union is I. We write

$$X = X_1|X_2|\cdots|X_t.$$

When an order on I is given, we usually order the blocks in increasing order of their minimal elements. The *length* of X is the number of blocks of X, denoted $\ell(X)$.

We say a *decomposition* of I, is an ordered sequence of disjoint subsets of I, say $S = (S_1, S_2, ..., S_k)$ such that $I = \bigsqcup_{i=1}^{k} S_i$. Note, that these subsets, unlike set partitions, can be empty, because we want to be able to consider the following decompositions $I = \emptyset \sqcup I$ and $I = I \sqcup \emptyset$.

Remark 1.1.2. If following Aguiar and Mahajan, our definition of set partition corresponds their notion of a *linear set composition*, please reference [3].

1.1.1. Partial Order on Set Partitions

The partial ordering on the set of partitions of I is given by refinement: Let π and σ be partitions of I. We write $\pi \leq \sigma$ if each block of π is a subset of a block of σ. For example

$$1|25|346|7 \leq 1346|257 \text{ but } 1257|346 \not\leq 1346|257.$$

1.2. Constructing New Set Partitions

Given set partitions, $\pi \vdash [n]$ and $\sigma \vdash [m]$ such that $\ell(\pi) = s$ and $\ell(\sigma) = r$, we are interested in constructing new set partitions in the following ways:

(1) We let $\wedge$ denote the *greatest lower bound* with respect to the partial order $\leq$ given above. For example,

$$1|23 \wedge 123 = 1|23 \ , \ 1|23 \wedge 1|2|3 = 1|2|3 \ , \ 1|23 \wedge 12|3 = 1|2|3.$$

(2) Let $\pi \sqcup \sigma \vdash [n+m]$ denote the following set partition:

$$\pi \sqcup \sigma := \pi_1|\cdots|\pi_s|st(\sigma_1)|\cdots|st(\sigma_r),$$

where st denotes the *standardization map*, the unique order preserving map from $[m] \to n + [m]$. This is also sometimes denoted $\pi|\sigma$.

(3) For any set partition, X, of a set I with $|I| = n$, we can consider the operator $(-)^{\downarrow}$. This maps the set partition X to the appropriate set partition of $[n]$ along the pullback of the unique order preserving bijection $[n] \to I$. For example $(18|2|37)^{\downarrow} = 15|2|34$.

(4) Given $S \subseteq I$, the restriction of $X|_S$ is the partition of S whose blocks are the nonempty intersections of the blocks of X with S. For example, if $X = 1|25|346|7$ and $S = \{1, 3, 4\}$, then $X|_{\{1,3,4\}} = 1|34$.

1.3. Integer Partitions

Let $n \geq 0$. We say $\lambda = (\lambda_1, ..., \lambda_t)$ is an *integer partition* of n if $\sum_{i=1}^{t} \lambda_i = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t$. There is a natural mapping from set partitions to integer partitions given by

$$\lambda(\pi) = \lambda(X_1|X_2|\cdots|X_t) = (|X_{i_1}|, .., |X_{i_t}|), \tag{1}$$

where $(|X_{i_1}|, |X_{i_2}|, ..., |X_{i_t}|)$ is the partition obtained by listing $|X_1|, ..., |X_t|$ in weakly decreasing order. We say that $\lambda(\pi)$ is the *integer partition type* of π. We will also need the notation $\lambda! = \lambda_1! \cdots \lambda_t!$ and can extend to set partitions via $\pi! = \lambda(\pi)!$.

CHAPTER 2

Category Theory

In this chapter, we will discuss the required background information regarding category theory; we are interested in looking at monoidal categories. For a more extensive treatment, please refer to [3] and [27]. Throughout this chapter, we follow the exposition of Aguiar and Mahajan, see [3]. The reader may wish to skip to later chapters and refer back to this section as needed.

2.1. Basics of Category Theory

We begin with some basic definitions.

DEFINITION 2.1.1. A *category*, $\mathcal{C}$, consists of the following data:

- A class of objects denoted $Ob(\mathcal{C})$.

- For each pair of objects $X, Y \in Ob(\mathcal{C})$, a set $\mathrm{Hom}_{\mathcal{C}}(X, Y) = \mathrm{Hom}(X, Y)$ of morphisms between X and Y,

such that the morphisms satisfy the following conditions:

- For all $X, Y, Z \in Ob(\mathcal{C})$, and $f \in \mathrm{Hom}(X, Y)$, $g \in \mathrm{Hom}(Y, Z)$ there is a composition operation
$$\circ : \mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z)$$
$$(g, f) \mapsto g \circ f$$
such that $g \circ f$ is a morphism and $\circ$ is associative.

- For all $X \in Ob(\mathcal{C})$, there is an identity morphism $1_X \in \mathrm{Hom}(X, X)$. For all $X, Y \in Ob(\mathcal{C})$ and $f \in \mathrm{Hom}(X, Y)$, we have $f \circ 1_X = f$ and $1_Y \circ f = f$.

DEFINITION 2.1.2. A *subcategory* of $\mathcal{C}$, is a category $\mathcal{S}$ whose objects are a subcollection of objects of $\mathcal{C}$ and morphisms are a subcollection of the collection of morphisms of $\mathcal{C}$ such that:

- If $f : X \to Y$ is a morphism in $\mathcal{S}$, then $X, Y \in \mathcal{S}$.

- If $f : X \to Y$ and $g : Y \to Z$ are morphisms in $\mathcal{S}$, then $g \circ f$ is a morphism in $\mathcal{S}$.

- If $X \in \mathcal{S}$, then so is 1_X.

We further say that $\mathcal{S}$ is a *full subcategory* if for any $X, Y \in \mathcal{S}$, every morphism $f : X \to Y$ in $\mathcal{C}$ is also a morphism in $\mathcal{S}$, in other words the inclusion functor $\iota : \mathcal{S} \hookrightarrow \mathcal{C}$ is full.

The following are the some of the underlying categories we will be interested in:

Example 2.1.3.

- The category of sets, **Set**, consists of sets for objects and the morphisms are all set maps between them.

- The category, $\mathbf{Set}^{\times}$, consists of finite sets for objects and the morphisms are bijections between them.

- The category of vector spaces over a field, $\mathbb{K}$, $\mathbf{Vec}_{\mathbb{K}}$, consists of vector spaces for objects and the morphisms are linear maps between them.

Definition 2.1.4. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor*, $F : \mathcal{C} \to \mathcal{D}$ is a mapping that:

- assigns an object $F(X) \in \mathcal{D}$ for each object $X \in \mathcal{C}$,

- associates each morphism $f : X \to Y \in \mathcal{C}$ to a morphism $F(f) : F(X) \to F(Y) \in \mathcal{D}$ such that the following conditions hold:

  - $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for every object $X \in \mathcal{C}$,

  - $F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$.

We say a functor is said to be *full* if F is surjective on morphisms, i.e., for every pair of objects X and Y in $\mathcal{C}$ and every morphism $g : F(X) \to F(Y)$ in $\mathcal{D}$, there is a morphism $f : X \to Y$ in $\mathcal{C}$ such that $F(f) = g$.

Definition 2.1.5. Given two functors F and G between categories $\mathcal{C}$ and $\mathcal{D}$, a *natural transformation* $\alpha : F \to G$ assigns a morphism (section maps) $\alpha_X : F(X) \to G(X)$ for each object X in $\mathcal{C}$ and these section maps such that for each morphism $f : X \to Y$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\alpha_Y"] & G(Y)
\end{tikzcd}$$

We say that α_X is natural in X. We denote a natural transformation by $\alpha : F \to G$ or $\alpha : F \Rightarrow G$. When all section maps are invertible, we say that α is a *natural isomorphism*, and write $F \cong G$.

Definition 2.1.6. We say that two categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there exists functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $FG \cong 1_{\mathcal{C}}$ and $GF \cong 1_{\mathcal{D}}$.

2.2. Monoidal Categories

Now we focus on a specific type of category and properties involving it. Throughout this thesis, we will be working with monoidal categories. We start by giving the definition of a monodial category.

DEFINITION 2.2.1. A *monoidal category* $(\mathcal{C}, \cdot)$ is a a category $\mathcal{C}$ with a functor

$$\cdot : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

together with

1. a natural isomorphism

$$\alpha_{A,B,C} : (A \cdot B) \cdot C \xrightarrow{\simeq} A \cdot (B \cdot C)$$

   such that the following diagram commutes

$$\begin{tikzcd} & (A \cdot B) \cdot (C \cdot D) \arrow[dr, "\alpha_{A,B,C \cdot D}"] & \\ ((A \cdot B) \cdot C) \cdot D \arrow[ur, "\alpha_{A \cdot B,C,D}"] \arrow[d, "\alpha_{A,B,C} \cdot \mathrm{id}_D"'] & & A \cdot (B \cdot (C \cdot D)) \\ (A \cdot (B \cdot C)) \cdot D \arrow[rr, "\alpha_{A,B \cdot C,D}"'] & & A \cdot ((B \cdot C) \cdot D). \arrow[u, "\mathrm{id}_A \cdot \alpha_{B,C,D}"'] \end{tikzcd} \tag{2}$$

2. $\mathcal{C}$ has a distinguished object e with natural isomorphisms

$$\lambda_A : A \to e \cdot \quad \text{and} \quad \rho_A : A \to A \cdot e$$

   such that the following diagram commutes

$$\begin{tikzcd} A \cdot (e \cdot B) \arrow[rr, "\alpha_{A,e,B}"] & & (A \cdot e) \cdot B \\ & A \cdot B. \arrow[ul, "\mathrm{id}_A \cdot \lambda_B"] \arrow[ur, "\rho_A \cdot \mathrm{id}_B"'] & \end{tikzcd} \tag{3}$$

This object is called the *unit object*.

We say a monoidal category is *strict* if the above natural isomorphisms are identities.

DEFINITION 2.2.2. A *braided monoidal category* is a monoidal category $(\mathcal{C}, \cdot)$ together with a natural isomorphism

$$\beta_{A,B} : A \cdot B \to B \cdot A$$

such that the following diagrams commute

$$\begin{tikzcd} & B \cdot A \cdot C \arrow[dr, "\mathrm{id}_B \cdot \beta_{A,C}"] & & & A \cdot C \cdot B \arrow[dr, "\beta_{A,C} \cdot \mathrm{id}_B"] & \\ A \cdot B \cdot C \arrow[ur, "\beta_{A,B} \cdot \mathrm{id}_C"] \arrow[rr, "\beta_{A,B \cdot C}"'] & & B \cdot C \cdot A & A \cdot B \cdot C \arrow[ur, "\mathrm{id}_A \cdot \beta_{B,C}"] \arrow[rr, "\beta_{A \cdot B,C}"'] & & C \cdot A \cdot B. \end{tikzcd} \tag{4}$$

We further say that the category $\mathcal{C}$ is *symmetric* if $\beta^2 = \mathrm{id}$.

Example 2.2.3. The category $\mathbf{Vec}_{\mathbb{K}}$ is an example of a symmetric monoidal category, with $\cdot$ being the usual tensor product of vector spaces and the unit of $\mathbf{Vec}$ being $\mathbb{K}$. The symmetric braiding is given by

$$\begin{aligned} \beta_{A,B} : A \otimes B &\to B \otimes A \\ a \otimes b &\mapsto b \otimes a. \end{aligned}$$

2.3. Hopf Monoids

Definition 2.3.1. A *monoid* in a monoidal category $(\mathcal{C}, \cdot, e)$ is a triple (A, μ, ι) where $A \in \mathcal{C}$,

$$\mu : A \cdot A \to A \text{ and } \iota : e \to A$$

satisfy the associativity and unit axioms, i.e., the following diagrams must commute:

$$\begin{tikzcd} A \cdot A \cdot A \arrow[r, "\mathrm{id} \cdot \mu"] \arrow[d, "\mu \cdot \mathrm{id}"'] & A \cdot A \arrow[d, "\mu"] \\ A \cdot A \arrow[r, "\mu"'] & A \end{tikzcd} \qquad \begin{tikzcd} e \cdot A \arrow[r, "\iota \cdot \mathrm{id}"] \arrow[dr, "\lambda"'] & A \cdot A \arrow[d, "\mu"] & A \cdot e \arrow[l, "\mathrm{id} \cdot \iota"'] \\ & A \arrow[ur, "\rho"'] & \end{tikzcd} . \tag{5}$$

A *morphism* of monoids $f : (A, \mu, \iota) \to (A', \mu', \iota')$ in a monoidal category $\mathcal{C}$, is a map $A \to A'$ such that the following diagrams commute:

$$\begin{tikzcd} A \cdot A \arrow[r, "f \cdot f"] \arrow[d, "\mu"'] & A' \cdot A' \arrow[d, "\mu'"] \\ A \arrow[r, "f"'] & A \end{tikzcd} \quad , \quad \begin{tikzcd} e \arrow[r, "\iota"] \arrow[dr, "\iota'"'] & A \arrow[d, "f"] \\ & A' \end{tikzcd} . \tag{6}$$

We say that a monoid (A, μ, ι) in a braided monoidal category is *commutative* if $\mu \circ \beta = \mu$; in terms of diagrams, we need the following to commute:

$$\begin{tikzcd} A \cdot A \arrow[rr, "\beta"] \arrow[dr, "\mu"'] & & A \cdot A \arrow[dl, "\mu"] \\ & A & \end{tikzcd} \tag{7}$$

Definition 2.3.2. A *comonoid* in a monoidal category $(\mathcal{C}, \cdot, e)$ is a triple (C, Δ, ε) where $C \in \mathcal{C}$,

$$\Delta : C \to C \cdot C \quad \text{and} \quad \varepsilon : C \to e$$

satisfy the coassociativity and counital properties. In terms of diagrams commuting, reverse the arrows in the monoid diagrams and replace μ with Δ and ι with ε.
A *morphism* of comonoids $(C, \Delta, \varepsilon) \to (C', \Delta', \varepsilon')$ is a map $C \to C'$ such that the diagrams in (**??**) commute when arrows are reversed and μ, μ' are replaced with Δ, Δ', and ι, ι' are replaced with ε and ε'. We say that a comonoid (C, Δ, ε) in a braided monoidal category is cocommutative if $\beta \circ \Delta = \Delta$. The diagram that corresponds to this is the same as Diagram (5) above, with arrows reversed and Δ replacing μ.

DEFINITION 2.3.3. A *bimonoid* in a braided monoidal category $(\mathcal{C}, \cdot, e)$ is a quintuple $(H, \mu, \iota, \Delta, \varepsilon)$ where $H \in \mathcal{C}$ such that (H, μ, ι) is a monoid, (H, Δ, ε) is a comonoid, and the two structures are compatible in the sense that the following diagrams commute:

$$\begin{tikzcd} H \cdot H \cdot H \cdot H \arrow[rr, "\mathrm{id} \cdot \beta \cdot \mathrm{id}"] & & H \cdot H \cdot H \cdot H \arrow[d, "\mu \cdot \mu"] \\ H \cdot H \arrow[u, "\Delta \cdot \Delta"] \arrow[r, "\mu"'] & H \arrow[r, "\Delta"'] & H \cdot H \end{tikzcd} \tag{8}$$

$$\begin{tikzcd} H \cdot H \arrow[r, "\varepsilon \cdot \varepsilon"] \arrow[d, "\mu"'] & e \cdot e \arrow[d, "\lambda_e^{-1}"] \\ H \arrow[r, "\varepsilon"'] & e \end{tikzcd} \qquad \begin{tikzcd} e \arrow[r, "\iota"] \arrow[d, "\lambda_I"] & H \arrow[d, "\Delta"] \\ e \cdot e \arrow[r, "\iota \cdot \iota"'] & H \cdot H \end{tikzcd} \tag{9}$$

$$\begin{tikzcd} & H \arrow[dr, "\varepsilon"] & \\ e \arrow[ur, "\iota"] \arrow[rr, leftrightarrow, "\sim"'] & & e. \end{tikzcd} \tag{10}$$

A *morphism of bimonoids* is a morphism of the underlying monoid and comonoids.

An equivalent way to characterize a bimonoid H, is to say that H is both a monoid and comonoid, such that Δ and ε are morphisms of monoids (or equivalently, μ and ι are morphisms of comonoids).

DEFINITION 2.3.4. Given a monoid (A, μ, ι) and comonoid (C, Δ, ε) in a braided monoidal category $(\mathcal{C}, \cdot, e)$, one can form the *convolution monoid*, $M(C, A)$, which is the set of all morphisms from C to A. In other words, $M(C, A) = \mathrm{Hom}_{\mathcal{C}}(C, A)$ with the *convolution product*:
Let $f, g \in \mathrm{Hom}_{\mathcal{C}}(C, A)$, the product $f * g$ is formed from the following composite:

$$C \xrightarrow{\Delta} C \cdot C \xrightarrow{f \cdot g} A \cdot A \xrightarrow{\mu} A \tag{11}$$

In $M(C, A)$, the map $\iota \circ \varepsilon$ is the identity.

DEFINITION 2.3.5. A *Hopf monoid* in a braided monoidal category $(\mathcal{C}, \cdot, e, \beta)$ is a bimonoid H for which the identity map $id : H \to H$ is invertible in the convolution monoid $\mathrm{End}(H)$. In other words, there exists a map $s : H \to H$, called the *antipode*, such that the following two diagrams commute:

$$\begin{tikzcd} H \cdot H \arrow[rr, "\mathrm{id} \cdot s"] & & H \cdot H \arrow[d, "\mu"] \\ H \arrow[u, "\Delta"] \arrow[r, "\varepsilon"'] & e \arrow[r, "\iota"'] & H \end{tikzcd} \quad \begin{tikzcd} H \cdot H \arrow[rr, "s \cdot \mathrm{id}"] & & H \cdot H \arrow[d, "\mu"] \\ H \arrow[u, "\Delta"] \arrow[r, "\varepsilon"'] & e \arrow[r, "\iota"'] & H. \end{tikzcd} \tag{12}$$

REMARK 2.3.6. If the antipode exists, then it must be unique since it is the inverse to the $\mathrm{id} : H \to H \in \mathrm{End}(H)$ with respect to the convolution product.

2.4. Monoidal Functors

Again, we follow the notation of Aguair and Mahajan [3]. We let $(\mathcal{C}, \cdot, e)$ and $(\mathcal{D}, \star, e')$ be two monoidal categories and F be a functor from $\mathcal{C}$ to $\mathcal{D}$. We write $\mathcal{M}$ to denote the tensor product functors[1], i.e., $\mathcal{M} : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ via $D \times D \mapsto D \star D$ for all $D \in \mathcal{D}$, or $\mathcal{M} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ via $X \times X \mapsto X \cdot X$ for all $X \in \mathcal{C}$. We write $F^2 := \mathcal{M} \circ (F \times F)$ and $F_2 := F \circ \mathcal{M}$. These are both functors from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{D}$. Let $\mathcal{I}$ denote the category with a single object and the only morphism being the identity, and let $F^0 : \mathcal{I} \to \mathcal{D}$ and $F_0 : \mathcal{I} \to \mathcal{D}$ be the functors that send the unique object of $\mathcal{I}$ to e' and $F(e)$ respectively.

Definition 2.4.1. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is *lax monoidal* if there is a natural transformation $\varphi : F^2 \implies F_2$, where for every pair of $A, B \in \mathcal{C}$ we have

$$F(A) \star F(B) \xrightarrow{\varphi_{A,B}} F(A \cdot B),$$

and a morphism $\varphi_0 : e' \to F(e)$ such that the following conditions are satisfied:

(1) **Associativity:** φ is associative, i.e., for all $A, B, C \in \mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd} F(A) \star F(B) \star F(C) \arrow[r, "\mathrm{id} \star \varphi_{B,C}"] \arrow[d, "\varphi_{A,B} \star \mathrm{id}"'] & F(A) \star F(B \cdot C) \arrow[d, "\varphi_{A,B\cdot C}"] \\ F(A \cdot B) \star F(C) \arrow[r, "\varphi_{A\cdot B, C}"'] & F(A \cdot B \cdot C). \end{tikzcd} \tag{13}$$

(2) **Unitality:** φ is left and right unital, i.e., for all $A \in \mathcal{C}$, the following diagrams commute:

$$\begin{tikzcd} e' \star F(A) \arrow[d, "\varphi_0 \star \mathrm{id}"'] & F(A) \arrow[l, "\lambda_{F(A)}"'] \arrow[d, "F(\lambda_A)"] \\ F(e) \star F(A) \arrow[r, "\varphi_{e,A}"'] & F(e \cdot A) \end{tikzcd} \qquad \begin{tikzcd} F(A) \star e' \arrow[d, "\mathrm{id} \star \varphi_0"'] & F(A) \arrow[l, "\rho_{F(A)}"'] \arrow[d, "F(\rho_A)"] \\ F(A) \star F(e) \arrow[r, "\varphi_{A,e}"'] & F(A \cdot e). \end{tikzcd} \tag{14}$$

Remark 2.4.2. We can view φ_0 as a natural transformation between F^0 and F_0.

Definition 2.4.3. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is *colax monoidal* if there is a natural transformation $\psi : F_2 \implies F^2$, where for every pair of $A, B \in \mathcal{C}$ we have

$$F(A \cdot B) \xrightarrow{\varphi_{A,B}} F(A) \star F(B),$$

and a morphism $\psi_0 : F(e) \to e'$ such that the following conditions are satisfied:

(1) **Coassociativity:** ψ is coassociative, i.e., we need the diagrams formed by reversing the arrows in Diagram (13) and replacing φ with ψ to commute.
(2) **Conitality:** ψ is left and right counital, i.e., we need the diagrams formed by reversing the arrows in Diagram (14) and replacing φ with ψ to commute.

Definition 2.4.4. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is *bilax monoidal* if there exists natural transformations φ and ψ:

[1] Here we follow the terminology of Aguair and Mahajan, although it would make sense to call these functors monoidal product functors because they directly use the monoidal product of the respective category

$$F(A) \star F(B) \underset{\psi_{A,B}}{\overset{\varphi_{A,B}}{\rightleftarrows}} F(A \cdot B)$$

and morphisms $\varphi_0 : e' \to F(e)$ and $\psi_0 : F(e) \to e'$ in $\mathcal{D}$ such that F is lax and colax and the conditions below are satisfied:

(1) **Braiding:** The following diagram commutes for all $A, B, C, D \in \mathcal{C}$:

$$\begin{tikzcd}
& F(A \cdot B) \star F(C \cdot D) \arrow[dl, "\varphi_{A\cdot B, C\cdot D}"'] \arrow[dr, "\psi_{A,B} \star \psi_{C,D}"] & \\
F(A \cdot B \cdot C \cdot D) \arrow[d, "F(\mathrm{id} \cdot \beta \cdot \mathrm{id})"'] & & F(A) \star F(B) \star F(C) \star F(D) \arrow[d, "\mathrm{id} \star \beta \star \mathrm{id}"] \\
F(A \cdot C \cdot B \cdot D) \arrow[dr, "\psi_{A\cdot C, B\cdot D}"'] & & F(A) \star F(C) \star F(B) \star F(D) \arrow[dl, "\varphi_{A,C} \star \varphi_{B,D}"] \\
& F(A \cdot C) \star F(B \cdot D) &
\end{tikzcd} \tag{15}$$

where β is used for the braiding in either $\mathcal{C}$ or $\mathcal{D}$.

(2) **Unitality:** The following diagrams must commute:

$$\begin{tikzcd}
e' \arrow[r, "\varphi_0"] \arrow[d, "\lambda_{e'}"'] & F(e) \arrow[r, "F(\lambda_I)"] & F(e \cdot e) \arrow[d, "\psi_{e,e}"] \\
e' \star e' \arrow[rr, "\varphi_0 \star \varphi_0"'] & & F(e) \star F(e)
\end{tikzcd}
\quad
\begin{tikzcd}
e' & F(e') \arrow[l, "\psi_0"'] & F(e \cdot e) \arrow[l, "F(\lambda_e^{-1})"'] \\
e' \star e' \arrow[u, "\lambda_{e'}^{-1}"] & & F(e) \star F(e) \arrow[ll, "\psi_0 \star \psi_0"] \arrow[u, "\varphi_{e,e}"']
\end{tikzcd} \tag{16}$$

$$\begin{tikzcd}
& F(e) \arrow[dr, "\psi_0"] & \\
e' \arrow[ur, "\varphi_0"] \arrow[rr, leftrightarrow, "\simeq"'] & & e'
\end{tikzcd} \tag{17}$$

Definition 2.4.5. A lax (colax) monoidal functor (F, φ, φ_0) (resp. (F, ψ, ψ_0)) between two monoidal categories $(\mathcal{C}, \cdot, e, \beta)$ and $(\mathcal{D}, \star, e', \beta)$ is *braided* if the right-hand (resp. left-hand) digram below commutes:

$$\begin{tikzcd}
F(A \cdot B) \arrow[r, "\psi_{A,B}"] \arrow[d, "F(\beta)"'] & F(A) \star F(B) \arrow[r, "\varphi_{A,B}"] \arrow[d, "\beta"] & F(A \cdot B) \arrow[d, "F(\beta)"] \\
F(B \cdot A) \arrow[r, "\psi_{B,A}"'] & F(B) \star F(A) \arrow[r, "\varphi_{B,A}"'] & F(B \cdot A)
\end{tikzcd} \tag{18}$$

We say that a lax monoidal functor (F, φ, φ_0) is *strong* if φ and φ_0 are invertible. We say that a colax monoidal functor (F, ψ, ψ_0) is *costrong* if ψ and ψ_0 are invertible. A bilax monoidal functor $(F, \varphi, \varphi_0, \psi, \psi_0)$ is *bistrong* if it is both strong and costrong.

Proposition 2.4.6. (Aguiar and Mahajan, 3.45 [3]) *If* $(F, \varphi, \varphi_0, \psi, \psi_0)$ *is a bilax monoidal functor with* $\varphi_0\psi_0 = 1$ *and* $\varphi\psi = 1$ *then* F *is a bistrong monoidal functor.*

Let (F, φ, φ_0) and (G, ξ, ξ_0) be lax monoidal functors between monoidal categories, where $F : (\mathcal{C}, \cdot, e) \to (\mathcal{D}, \star, e')$ and $G : (\mathcal{D}, \star, e') \to (\mathcal{E}, \Box, e'')$. Define the composition of lax monoidal functors to be

$$(GF, \varphi\xi, \varphi_0\xi_0) : \mathcal{C} \to \mathcal{E}$$

where the functor $GF : \mathcal{C} \to \mathcal{E}$ is the composite of F and G, and the natural transformations

$$\varphi\xi : (GF)^2 \to (GF)_2 \ \text{ and } \ \varphi_0\xi_0 : e'' \to GF(e)$$

are defined by the following diagrams:

$$\begin{tikzcd}
GF(A)\Box GF(B) \arrow[rr, dashed, "(\varphi\xi)_{A,B}"] \arrow[dr, "\xi_{F(A),F(B)}"'] & & GF(A\cdot B) & e'' \arrow[rr, dashed, "\varphi_0\xi_0"] \arrow[dr, "\xi_0"'] & & GF(e) \\
& G(F(A)\star F(B)) \arrow[ur, "G(\varphi_{A,B})"'] & & & G(e') \arrow[ur, "G(\varphi_0)"'] &
\end{tikzcd} \tag{19}$$

The composition of colax functors is defined similarly, with arrows reversed and the appropriate maps replaced with the colax maps.

Theorem 2.4.7. (Aguair, Mahajan, 3.21[3]) *If* $F : \mathcal{C} \to \mathcal{D}$ *and* $G : \mathcal{D} \to \mathcal{E}$ *are lax (colax, bilax) monoidal, then the functor* $GF : \mathcal{C} \to \mathcal{E}$ *is lax (colax, bilax) monoidal.*

2.5. Morphisms of Monoidal Functors

Definition 2.5.1. Let $(\mathcal{C}, \cdot)$ and $(\mathcal{D}, \star)$ be two monoidal categories. Let (F, φ, φ_0) and (G, ξ, ξ_0) be two lax monoidal functors from $\mathcal{C}$ to $\mathcal{D}$. A *morphism* from F to G of lax monoidal functors is a natural transformation $\alpha : F \implies G$ such that the following diagrams commute for all $A, B \in \mathcal{C}$:

$$\begin{tikzcd}
F(A)\star F(D) \arrow[r, "\varphi_{A,B}"] \arrow[d, "\alpha_A\star\alpha_B"'] & F(A\cdot B) \arrow[d, "\alpha_{A\cdot B}"] & & F(e) \arrow[dd, "\alpha_I"] \\
G(A)\star G(B) \arrow[r, "\xi_{A,B}"'] & G(A\cdot B) & e' \arrow[ur, "\varphi_0"] \arrow[dr, "\xi_0"'] & \\
& & & G(e)
\end{tikzcd} \tag{20}$$

Now, let (F, ψ, ψ_0) and (G, δ, δ_0) be two colax monoidal functors from $\mathcal{C}$ to $\mathcal{D}$. A *morphism* from F to G of colax monoidal functors is a natural transformation $\alpha : F \implies G$ such that the following diagrams commute for all $A, B \in \mathcal{C}$:

$$
\begin{tikzcd}
F(A \cdot B) \arrow[r, "\psi_{A,B}"] \arrow[d, "\alpha_A \star \alpha_B"'] & F(A) \star F(B) \arrow[d, "\alpha_{A \cdot B}"] \\
G(A \cdot B) \arrow[r, "\delta_{A,B}"'] & G(A) \star G(B)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(e) \arrow[dd, "\alpha_I"'] \arrow[dr, "\psi_0"] & \\
& e' \\
G(e) \arrow[ur, "\delta_0"'] &
\end{tikzcd}
\tag{21}
$$

A *morphism of bilax functors* is a morphism is such that Diagrams (20) and (21) commute, i.e., a morphism of lax and colax functors.

Finally, a *morphism of (co)lax strong monoidal functors* is a morphism of the underlying (co)lax monoidal functors.

Proposition 2.5.2.

(1) (Benabou [7]) *If F is a (co)lax monoidal functor from $\mathcal{C}$ to $\mathcal{D}$, and $\mathbf{h}$ is a (co)monoid in $\mathcal{C}$ then $F(\mathbf{h}) \in \mathcal{D}$ is a (co)monoid. If $f : \mathbf{h} \to \mathbf{h}'$ is a morphism of (co)monoids in $\mathcal{C}$, then $F(f)$ is a morphisms of (co)monoids in $\mathcal{D}$. Finally, a morphism of (co)lax monoidal functors $F \to G$ yields a morphism of (co)monoids $F(\mathbf{h}) \to G(\mathbf{h})$ if $\mathbf{h} \in \mathcal{C}$ is a (co)monoid.*

(2) (Aguiar and Mahajan [3]) *If F is a bilax monoidal functor from $\mathcal{C}$ to $\mathcal{D}$ and $\mathbf{h} \in \mathcal{C}$ is a bimonoid, then $F(\mathbf{h}) \in \mathcal{D}$ is a bimonoid. If $f : \mathbf{h} \to \mathbf{h}'$ is a morphism of bimonoids in $\mathcal{C}$ then $F(f)$ is a morphism of bimonoids in $\mathcal{D}$. Finally, a morphism of bilax monoidal functors, $F \to G$, yields a morphism of bimonoids $F(\mathbf{h}) \to G(\mathbf{h})$ if $\mathbf{h} \in \mathcal{C}$ is a bimonoid.*

Proposition 2.5.3. (Aguiar and Mahajan [3]) *If F is a bistrong monoidal functor from $\mathcal{C}$ to $\mathcal{D}$ and $\mathbf{h} \in \mathcal{C}$ is a Hopf monoid with antipode $s : \mathbf{h} \to \mathbf{h}$, then $F(\mathbf{h}) \in \mathcal{D}$ is a Hopf monoid with antipode $F(s)$. If $f : \mathbf{h} \to \mathbf{h}'$ is a morphism of Hopf monoids in $\mathcal{C}$, then $F(f)$ is a morphism of Hopf monoids in $\mathcal{D}$. Finally, a morphism of bistrong monoidal functors, $F \to G$, yields a morphism of Hopf monoids $F(\mathbf{h}) \to G(\mathbf{h})$ if $\mathbf{h} \in \mathcal{C}$ is a Hopf monoid.*

CHAPTER 3

Graded Vector Spaces and Hopf Algebras

In this chapter, we give a brief overview of several different possible monoidal structures on the category of graded vector spaces, $\mathbf{gVec}_{\mathbb{K}}$ and what familiar objects correspond to (bi/co)monoids in $\mathbf{gVec}_{\mathbb{K}}$. We discuss the notion of invariance and coinvariance in representations of algebras and how groups fit into the picture. Please reference [3] Chapter 2 for an exposition on $\mathbb{K}G$-modules. We end by discussing two examples of Hopf algebras–the ring of symmetric functions and the ring of symmetric functions in noncommutative colored variables.

3.1. $\mathbf{gVec}_{\mathbb{K}}$

An $\mathbb{N}$-*graded vector space* is a vector space V with a decomposition as a direct sum in the form

$$V = \bigoplus_{n\geq 0} V_n$$

where V_n is the homogeneous component of degree n in V. We say a linear map, $f : V \to W$, is a *morphism of graded vector spaces* if $f(V_n) \subseteq W_n$ for all $n \in \mathbb{N}$, we write $f = \bigoplus_{n\geq 0} f_n$ where $f_n : V_n \to W_n$. Together, these make up the objects and morphisms of the category of graded vector spaces, $\mathbf{gVec}_{\mathbb{K}}$.

We can consider two different operations that make $\mathbf{gVec}_{\mathbb{K}}$ into a monoidal category:

Definition 3.1.1. Given graded vector spaces V and W, we can define the *Cauchy Product*, $V \cdot W$, and the *Hadamard Product*, $V \times W$, by:

$$(V \cdot W)_n = \bigoplus_{i=0}^{n} V_i \otimes W_{n-i} \tag{22}$$

$$(V \times W)_n = V_n \otimes W_n \tag{23}$$

where $\otimes$ denotes the usual tensor product of $\mathbb{K}$-vector spaces.

The Cauchy Product, $\cdot$, turns $\mathbf{gVec}_{\mathbb{K}}$ into a monoidal category with unit being

$$\mathbf{1}_{\mathbb{K}} := \bigoplus_{n\geq 0} \mathbf{1}_n$$

where

$$\mathbf{1}_n = \begin{cases} \mathbb{K} & \text{if n=0} \\ 0 & \text{otherwise} \end{cases}.$$

The Hadamard Product, $\times$ turns $\mathbf{gVec}_{\mathbb{K}}$ into a monoidal category with unit being

$$E := \bigoplus_{n\geq 0} E_n,$$

where $E_n := \mathbb{K}$ for all n.

Both $(\mathbf{gVec}_{\mathbb{K}}, \cdot)$ and $(\mathbf{gVec}_{\mathbb{K}}, \times)$ are symmetric categories with the braiding $\beta : V \cdot W \to W \cdot V$ and $\beta : V \times W \to W \times V$ given on pure tensors by

$$v \otimes w \mapsto w \otimes v,$$

which swaps the tensor factors.

3.1.1. Hopf Algebras

A monoid in $\mathbf{gVec}_{\mathbb{K}}$ is a *graded algebra*, i.e., a graded vector space $A = \bigoplus_{n \geq 0} A_n$ with morphisms

$$\mu : A \cdot A \to A \quad \text{and} \quad \iota : \mathbb{K} \to A,$$

called the *coproduct* and *counit* respectively, where both μ and ι preserve the grading, i.e., $\mu(A_n \otimes A_m) \subseteq A_{n+m}$ for all $n, m \in \mathbb{N}$, and $\iota(\mathbb{K}) \subseteq A_0$. A comonoid in $\mathbf{gVec}_{\mathbb{K}}$ is a *graded coalgebra*, i.e., a graded vector space $C = \bigoplus_{n \geq 0} C_n$ with morphisms

$$\Delta : C \to C \cdot C \quad \text{and} \quad \varepsilon : C \to \mathbb{K},$$

where both Δ and ε preserve grading, i.e., $\Delta(C_n) \subseteq \bigoplus_{s+t=n} C_s \otimes C_t$ for all $s + t = n \in \mathbb{N}$ and $\varepsilon : C_0 \to \mathbb{K}$. A bimonoid in $\mathbf{gVec}_{\mathbb{K}}$ is a *graded bialgebra*, i.e., an algebra (H, μ, ι) and coalgebra (H, Δ, ε) such that Δ and ε are algebra morphisms. A Hopf monoid in $\mathbf{gVec}_{\mathbb{K}}$ is a *graded Hopf algebra*, i.e., a bialgebra H with a unique antipode $s : H \to H$ that preserves grading. For more details regarding Hopf algebras, please reference the following: [15], [32], and [13].

Remark 3.1.2. When working with the coproduct, Δ, it is useful to use Sweedler notation,

$$\Delta(c) = \sum_{(c)} c_1 \otimes c_2 = \sum c_1 \otimes c_2$$

to abbreviate formulas involving Δ. Using this notation, the following formulas can be expressed in terms of the diagram axioms (or deduced from them):

$$\sum_{(c)} c_1 \varepsilon(c_2) = c = \sum_{(c)} \varepsilon(c_1) c_2 \tag{24}$$

$$\sum_{(c)} \sum_{(c_1)} (c_1)_1 \otimes (c_1)_2 \otimes c_2 = \sum_{(c)} \sum_{(c_2)} c_1 \otimes (c_2)_1 \otimes (c_2)_2 = \sum_{(c)} c_1 \otimes c_2 \otimes c_3 \tag{25}$$

$$\sum_{(c)} s(c_1) c_2 = \varepsilon(c) = \sum_{(c)} c_1 s(c_2) \tag{26}$$

$$\varepsilon(s(c)) = \varepsilon(c) \tag{27}$$

We encourage the reader to reference [15] for more Hopf algebra identities expressed using Sweedler notation.

Viewing $\mathbf{gVec}_{\mathbb{K}}$ as a monoidal category under the Hadamard product, $\times$, all of the above remarks still hold true.

3.2. Representations of Algebras

DEFINITION 3.2.1. For an algebra A, a *representation* of A is a vector space V together with an algebra homomorphism $\rho : A \to \mathrm{End}(V)$.

We often call V an A-*module*, more precisely a left A-module and write $a.v = \rho(a)v$ for $a \in A, v \in V$. We are also interested in maps that respect this action:

DEFINITION 3.2.2. Let A be an algebra, $\rho_V : A \to \mathrm{End}(V)$ and $\rho_W : A \to \mathrm{End}(W)$ be two representations of A. We say a linear map $f : V \to W$ is an A-*module map* if for all $a \in A$ we have

$$(\rho_W \circ f)(a) = (f \circ \rho_V)(a).$$

In module notation, the above condition is

$$a.f(v) = f(a.v) \text{ for all } a \in A, v \in V.$$

REMARK 3.2.3. Notions of subrepresentations (submodules), irreducible representations (simple modules), quotient representations (quotient modules), and direct sums of representations are defined in the same way as for finite groups. However, in order to make sense of the tensor product of representations and the trivial representation, more structure is needed. For this, the coproduct and counit are needed.

3.2.1. Trivial Representation

For a finite group G, the trivial representation is as follows:

$$\rho : G \to GL(\mathbb{K}) \cong \mathbb{K}^\times$$

$$g \mapsto \rho(g) = \mathrm{id}_{\mathbb{K}}\,.$$

For a general algebra, we need extra structure in order to make $\mathbb{K}$ a trivial representation. The counit, $\varepsilon : A \to \mathbb{K}$, is that structure that is needed. The *trivial representation of an algebra* is the following:

$$a.z = \varepsilon(a)z \text{ for all } a \in A, z \in \mathbb{K}.$$

Note that using the counit from the group algebra structure, i.e., $\varepsilon(g) = 1$ for all $g \in G$ gives us the notion of the trivial representation of a group (as above).

3.2.2. Tensor Product of Representations

For a finite group, G, and two representations V and W, we can turn the tensor product $V \otimes W$ into a representation in the following way:

$$g.(v \otimes w) = g.v \otimes g.w \text{ for all } g \in G, v \in V, w \in W.$$

In order to make sense of a tensor product of two representations, V and W, of an algebra A, we need the coproduct, $\Delta : A \to A \otimes A$. We can turn $V \otimes W$ into an A-module in the following way:

$$a.(v \otimes w) = \sum_{(a)} a_1.v \otimes a_2.w \text{ for all } a \in A, v \in V, w \in W.$$

Note, that when using the coproduct of the group algebra we recover $V \otimes W$ as a G-module as seen above.

3.3. (Co)invariance

Let $\mathbb{K}$ be a field, A be a Hopf algebra, and V an A-module. Let

$$V^A := \{x \in V \mid a.x = \varepsilon(a)x \ \ \forall a \in A\}$$

denote the space of *A-invariants of V*, i.e., the A-submodule of V in which A acts via the counit.

Consider the subspace $\mathcal{I} := \langle a.x - \varepsilon(a)x \mid x \in V,\ a \in A\rangle$. This is an A-submodule: for all $b \in A$, we have that

$$\begin{aligned} b.(a.x - \varepsilon(a)x) &= ba.x - \varepsilon(a)bx \\ &= ba.x - \varepsilon(ba)x + \varepsilon(ba)x - \varepsilon(a)bx \\ &= ba.x - \varepsilon(ba)x + \varepsilon(b)\varepsilon(a)x - \varepsilon(a)bx \\ &\in \mathcal{I}. \end{aligned}$$

Let $V_A := V/\mathcal{I}$. This A-module is the space of *A-coinvariants of V*, in other words the largest quotient that A acts by the counit.

Now consider the vector space $A \otimes V$. We can turn this into an A-module by letting A act via the coproduct, i.e.,

$$a.(x \otimes v) = \sum_{(a)} a_1.x \otimes a_2.v \qquad \forall\, a \in A,\ \forall\, x \otimes v \in A \otimes V.$$

Denote this A-module by $A \otimes_\Delta V$, and let $\mathcal{I}_\Delta := \langle a.(x \otimes v) - \varepsilon(a)x \otimes v \mid x \otimes v \in A \otimes V,\ a \in A\rangle$.

Let $A \otimes_m V$ denote the A-module where A acts via left multiplication on A, i.e.,

$$a.(x \otimes v) = a.x \otimes v \qquad \forall\, a \in A,\ \forall\, x \otimes v \in A \otimes_\Delta V,$$

and let $\mathcal{I}_m := \langle a.(x \otimes v) - \varepsilon(a)x \otimes v \mid x \otimes v \in A \otimes V,\ a \in A\rangle$.

We wish to show that $(A \otimes_\Delta V)_A \cong V$, but in order to do so we need the following lemmas.

Lemma 3.3.1. *$A \otimes_\Delta V \cong A \otimes_m V$ as A-modules.*

Proof. Define

$$\varphi : A \otimes_\Delta V \to A \otimes_m V$$

$$x \otimes v \mapsto \sum_{(x)} x_1 \otimes x_2.v$$

and

$$\rho : A \otimes_m V \to A \otimes_\Delta V$$

$$x \otimes v \mapsto \sum_{(x)} x_1 \otimes s(x_2).v,$$

where $x \otimes v$ is a pure tensor in $A \otimes V$, and then extend linearly.
We must show that φ and ρ are inverses to each other.

$$\begin{aligned}
\varphi(\rho(x \otimes v)) &= \varphi\left(\sum_{(x)} x_1 \otimes s(x_2).v\right) \\
&= \sum_{(x)} \sum_{(x_1)} (x_1)_1 \otimes (x_1)_2 s(x_2).v \\
&= \sum_{(x)} \sum_{(x_2)} x_1 \otimes (x_2)_1 s((x_2)_2).v \\
&= \sum_{(x)} x_1 \otimes \varepsilon(x_2).v \\
&= \sum_{(x)} x_1 \varepsilon(x_2) \otimes v \\
&= x \otimes v,
\end{aligned}$$

where the third equality comes from Equation 25, the fourth equality is from Equation 26, the fifth equality is obtained since the tensor product is over $\mathbb{K}$, and finally by Equation 27 we get the final equality. Using the same argument, gives:

$$\begin{aligned}
\rho(\varphi(x \otimes v)) &= \rho\left(\sum_{(x)} x_1 \otimes x_2.v\right) \\
&= \sum_{(x)} \sum_{(x_1)} (x_1)_1 \otimes s((x_1)_1) x_2.v \\
&= \sum_{(x)} \sum_{(x_2)} x_1 \otimes s((x_2)_1)(x_2)_2.v \\
&= \sum_{(x)} x_1 \otimes \varepsilon(x_2).v \\
&= \sum_{(x)} x_1 \varepsilon(x_2) \otimes v \\
&= x \otimes v.
\end{aligned}$$

Thus $A \otimes_\Delta V \cong A \otimes_m V$ as vector spaces.
Finally, we must show that φ is an A-module morphism. Let $a \in A$, $x \otimes v \in A \otimes_m V$, then

$$\varphi(a.x \otimes v) = \varphi(ax \otimes v) = \sum_{(ax)} (ax)_1 \otimes (ax)_2.v = \sum_{(a)} \sum_{(x)} a_1 x_1 \otimes a_2 x_2.v$$

and

$$a.\varphi(x \otimes v) = a. \sum_{(x)} x_1 \otimes x_2.v = \sum_{(a)} \sum_{(x)} a_1 x_1 \otimes a_2 x_2.v.$$

Therefore, $A \otimes_\Delta V \cong A \otimes_m V$ as A-modules.

$\square$

Corollary 3.3.2. *There is an induced isomorphism*

$$\overline{\varphi} : (A \otimes_\Delta V)_A \to (A \otimes_m V)_A.$$

Proof. Because φ is an isomorphism we get the induced isomorphism $\overline{\varphi}$ given by the following diagram:

$$\begin{tikzcd} A \otimes_\Delta V \arrow[rr, "\varphi"] \arrow[d, two heads] & & A \otimes_m V \arrow[d, two heads] \\ (A \otimes_\Delta V)/\mathcal{I}_\Delta \arrow[rr, dashed, "\overline{\varphi}"'] & & (A \otimes_m V)/\varphi(\mathcal{I}_\Delta). \end{tikzcd}$$

All that remains to show is $\varphi(\mathcal{I}_\Delta) = \mathcal{I}_m$, which is immediate since φ is an A-module isomorphism. Clearly, $\varphi(\mathcal{I}_\Delta) \subseteq \mathcal{I}_m$. For the reverse containment, since φ is an isomorphism, there exists an $x \otimes v \in A \otimes V$ such that $\varphi(x \otimes v) = y \otimes w$ for all $y \otimes w \in A \otimes_m V$. Using this element gives $\mathcal{I}_m \subseteq \varphi(\mathcal{I}_\Delta)$.

□

Lemma 3.3.3. *Let V be an A-module, then*

$$(A \otimes_m V)_A \cong V$$

as vector spaces. In other words, the space of A-coinvariants of $A \otimes_m V$ is isomorphic as vector spaces to V.

Proof. Define maps

$$\tau : V \to (A \otimes_m V)/\mathcal{I}_m$$
$$v \mapsto \overline{1 \otimes v},$$

where the overline denotes the projection to coinvariants, and

$$\pi : A \otimes V \to V$$

$$x \otimes v \mapsto \varepsilon(x)v.$$

First, observe that $\mathcal{I}_m \subseteq ker(\rho)$:

$$\varphi(a.(x \otimes v) - \varepsilon(a)x \otimes v) = \varepsilon(ax)v - \varepsilon(a)\varepsilon(x)v = \varepsilon(ax)v - \varepsilon(ax)v = 0.$$

So there is an induced map $\overline{\pi} : (A \otimes_m V)/\mathcal{I}_m \to V$.

Finally to show that τ and $\overline{\varphi}$ are isomorphisms:

$$\overline{\varphi}(\tau(v)) = \overline{\varphi}(\overline{1 \otimes v}) = \varepsilon(1)v = v$$

and

$$\tau(\overline{\varphi}(x \otimes v)) = \tau(\varepsilon(x)v) = \overline{1 \otimes \varepsilon(x)v} = \varepsilon(x)\overline{1 \otimes v} = \overline{x \otimes v}.$$

Therefore, $(A \otimes_m V)_A \cong V$ as desired.

□

Proposition 3.3.4. *Let V be an A-module and A act on $A \otimes_\Delta V$ via the coproduct. We have the following isomorphism of vector spaces:*

$$(A \otimes_\Delta V)_A \cong V.$$

Proof. Combine Lemmas 3.3.3, 3.3.1 and Corollary 3.3.2.

□

3.4. The Ring of Symmetric Functions

We follow the exposition in [20]. Let $\mathbb{K}$ be a field. Given an infinite variable set $X = (x_1, x_2, ..)$, consider the $\mathbb{K}$-algebra $\mathbb{K}[[X]] := \mathbb{K}[[x_1, x_2, ..]]$ of all formal power series in the indeterminates $x_1, x_2, .x_3, ..$ over $\mathbb{K}$. An element here has form $f(X) = \sum_\alpha c_\alpha x^\alpha$ where $c_\alpha \in \mathbb{K}$ and $x^\alpha := x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_n}^{\alpha_n}$ is the monomial indexed by the weak composition $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$. Let $\deg(x^\alpha) := \sum_{i=1}^{n} \alpha_i$ be the degree, and we say f has *bounded degree* if there exists a $d \in \mathbb{N}$ for which $\deg(x^\alpha) > d$ implies $c_\alpha = 0$. We consider a $\mathbb{K}$-subalgebra, $R(X) \subset \mathbb{K}[[X]]$, consisting of all formal power series of bounded degree. For every n, the symmetric group S_n acts via

$$\sigma.f(x_1, x_2, ..) = f(x_{\sigma(1)}, x_{\sigma(2)}, ...) \text{ for all } f \in R(X), \sigma \in S_n,$$

where $x_{\sigma(k)} = x_k$ for all $k > n$. In other words, S_n, acts on the first n variables and fixes the remaining. Let $S_{(\infty)} := \cup_{n \geq 0} S_n$, this also acts on $R(X)$.

Definition 3.4.1. The *ring of symmetric functions in X with coefficients in* $\mathbb{K}$, denoted Λ, is the $S_{(\infty)}$-invariant subalgebra of $R(X)$:

$$\Lambda \;\; := \;\; \{f \in R(X) \mid \sigma.f = f \;\; \forall\, \sigma \in S_{(\infty)}\}.$$

We can also define a coproduct structure on Λ as follows. We have the ring homomorphism

$$R(X) \otimes R(X) \to R(X, Y)$$
$$f(X) \otimes g(X) \mapsto f(X)g(Y)$$

where $(X, Y) = (x_1, x_2, ..., y_1, y_2, ...)$. This restricts to the isomorphism

$$\Lambda \otimes \Lambda \mapsto R(X, Y)^{S_{(\infty)} \times S_{(\infty)}}$$

. Since $S_{(\infty)} \times S_{(\infty)}$ is a subgroup of $S_{(\infty,\infty)}$[1], we get the following inclusion

$$\Lambda(X, Y) \hookrightarrow \Lambda \otimes \Lambda.$$

This gives a coproduct

$$\Lambda(X) \xrightarrow{\Delta} \Lambda(X, Y) \hookrightarrow \Lambda \otimes \Lambda$$
$$f(X) \mapsto f(X, Y).$$

We have that Λ is a Hopf Algebra by Proposition 1.4.14 in [20].

3.4.1. Monomial Symmetric Functions

The ring of symmetric functions has a number of distinguished bases, each with their own advantage. These bases are all labelled by (integer) partitions as defined in Section 1.3. The simplest basis to consider is the basis given by the *monomial symmetric functions*, $\{m_\lambda\}$. Given a partition λ,

$$m_\lambda := \sum_{\alpha \in S_{(\infty)}\alpha} x^\lambda,$$

where the action of $S_{(\infty)}$ on a partition λ is given by permuting the entries of the partition.

[1] $S_{(\infty,\infty)}$ is the group of all permutations of $\{x_1, x_2, ..., y_1, y_2, ...\}$ leaving all but finitely many variables invariant.

EXAMPLE 3.4.2.

$$m_{(3)} = x_1^3 + x_2^3 + \cdots ,$$
$$m_{(2,1)} = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + \cdots ,$$
$$m_{(1,1,1)} = x_1x_2x_3 + x_1x_2x_4 + x_2x_3x_4 + \cdots .$$

The Hopf algebra structure using the monomial basis is as follows:

- The product is given by: let $\lambda \vdash n$ and $\mu \vdash k$, then
$$m_\lambda \otimes m_\nu \mapsto \sum_{\nu \vdash (n+k)} r^{\nu}_{\lambda,\mu} m_\nu,$$
where $r^{\nu}_{\lambda,\mu}$ is the number of pairs of sequences (α, β) with $\alpha_i, \beta_i \geq 0$ where $\alpha \in S_{(\infty)}\lambda$ and $\beta \in S_{(\infty)}\mu$.

- The coproduct is given by: let $\lambda \vdash n$, then
$$\Delta(m_\lambda) = \sum_{\mu \sqcup \nu = \lambda} m_\mu \otimes m_\nu,$$
where $\mu \sqcup \nu$ is the partition obtained from the multiset union of μ and ν.

3.5. The Hopf Algebra $\tilde{\Pi}^{(r)}$

Here, we review the ring of symmetric functions in noncommutative C_r-colored variables, please reference [1] and [2]; we denote this by $\tilde{\Pi}^{(r)}$. Recall, $C_r = \langle a \mid a^r = 1 \rangle$ is the cyclic group of order r.

3.5.0.1. *Definition of* $\tilde{\Pi}^{(r)}$. As before, we let $\mathbb{K}$ be a field. Given an infinite noncommutative variable set $X = \{x_1, x_2, ...\}$, let

$$X^{(r)} := X \times C_r.$$

Here we view the elements of C_r as coloring the variable set X. Given a coloring $\xi = (\xi_1, ..., \xi_n) \in C_r^n$, a set partition $\pi \vdash [n]$, and variables $x_{i_1}, ..., x_{i_n}$ with $x_{i_j} = x_{i_k}$ if and only if i_j and i_k are in the same block of π, we can form a word

$$\omega_{(\pi,\xi)}(x_{i_1}, .., x_{i_n}) := (x_{i_1}, \xi_1)(x_{i_2}, \xi_2) \cdots (x_{i_n}, \xi_n).$$

Let $\deg(\omega_{(\pi,\xi)}(x_{i_1}, .., x_{i_n})) := n$ be the degree. We consider $\mathbb{K}\langle\langle X^{(r)} \rangle\rangle$, the associative algebra of formal power series in the noncommuting variables $X^{(r)}$. An element in $\mathbb{K}\langle\langle X^{(r)} \rangle\rangle$ is of the form:

$$f(X^{(r)}) = \sum_{(\pi,\xi)} c_{(\pi,\xi)} \omega_{(\pi,\xi)}(x_{i_1}, .., x_{i_n}),$$

where $c_{(\pi,\xi)} \in \mathbb{K}$ and the sum ranges over all colored set partitions and all allowable choices of variables. We say f has *bounded degree* if there exists a $d \in \mathbb{N}$ for which $\deg(\omega_{(\pi,\xi)}) > d$ implies $c_{(\pi,\xi)} = 0$.

For each positive integer n, there is an action of S_n on $\mathbb{K}\langle\langle X^{(r)} \rangle\rangle$, coming from the permutation of variables action above, via

$$\sigma.f((x_1, \xi_1), (x_2, \xi_2), ...) = f((x_{\sigma(1)}, \xi_1), (x_{\sigma(2)}, \xi_2), ...)$$

where $\sigma(i) = i$ for $i > n$, i.e., S_n fixes x_i whenever $i > n$. Denote $S_{(\infty)} := \cup_{n \geq 0} S_n$. Because S_n acts on $\mathbb{K}\langle\langle X^{(r)} \rangle\rangle$ for all n, this implies that $S_{(\infty)}$ acts on $\mathbb{K}\langle\langle X^{(r)} \rangle\rangle$.

Definition 3.5.1. The *ring of symmetric functions in the noncommuting variables* $X^{(r)}$ *with coefficients in* $\mathbb{K}$, denoted $\tilde{\Pi}^{(r)}$, is the $S_{(\infty)}$-invariant subalgebra of $\mathbb{K}\langle\langle X^{(r)}\rangle\rangle$ consisting of elements of bounded degree, i.e.,

$$\tilde{\Pi}^{(r)} := \left\{ f \in \mathbb{K}\langle\langle X^{(r)}\rangle\rangle \mid \sigma.f = f \text{ for all } \sigma \in S_{(\infty)}\ ,\ \deg(f) < \infty \right\}.$$

$\tilde{\Pi}^{(r)}$ is graded based on the C_r-colored set partition $(\pi, (\xi_1, ..., \xi_n))$ where $\pi \vdash [n]$ and $(\xi_1, ..., \xi_n) \in C_r^n$, i.e.,

$$\tilde{\Pi}^{(r)} = \bigoplus_{n \geq 0} \tilde{\Pi}_n^{(r)},$$

where $\tilde{\Pi}_n^{(r)} := \{f \in \tilde{\Pi}^{(r)} \mid f(X^{(r)}) = \sum_{(\pi,\xi)} c_{(\pi,\xi)}\omega_{(\pi,\xi)}\ s.t\ \deg(\omega_{(\pi,\xi)}) = n\}$. A basis of $\tilde{\Pi}^{(r)}$ is given by monomials indexed by colored set partitions:

$$m_{\pi,\xi} := \sum w,$$

where $\pi \vdash [n], \xi \in C_r^n$, and the sum is over the set of words $w = (x_{i_1}, \xi_1) \cdots (x_{i_n}, \xi_n)$ where $x_i = x_j$ if and only if i and j are in the same block of $\pi \vdash [n]$ and the colors are arbitrary.

For a colored variable, we will interchangeably use the notation (x_i, ξ_i) and x_{i,ξ_i}.

Remark 3.5.2.

(1) When $r = 1$, all partitions are trivially colored and will drop the coloring from the notation, $m_{\pi,(1,...,1)} = m_\pi$.

(2) When $r = 2$, we view C_2 as the multiplicative group of order two consisting of the following elements $\{1, -1\}$ where we often write $\overline{1} = -1$. When it's clear by context, C_2-colored variables will interchangeably be denoted as

$$(x_i, 1) = x_{i,1} = x_i$$

$$(x_i, -1) = x_{i,-1} = x_{\bar{i}}.$$

The following are some examples of C_2-colored monomials.

Example 3.5.3.

- $m_{13|24,(1,\bar{1},1,1)} = x_1 x_{\bar{2}} x_1 x_2 + x_2 x_{\bar{1}} x_1 x_1 + x_1 x_{\bar{3}} x_1 x_3 \cdots$
- $m_{12|3,(\bar{1},\bar{1},1)} = x_{\bar{1}} x_{\bar{1}} x_2 + x_{\bar{2}} x_{\bar{2}} x_1 + x_{\bar{1}} x_{\bar{1}} x_3 \cdots$
- $m_{12|3,(1,\bar{1},1)} = x_1 x_{\bar{1}} x_2 + x_2 x_{\bar{2}} x_1 + x_1 x_{\bar{1}} x_3 + \cdots$

3.5.1. (Co)Product:

Now we describe the Hopf structure of $\tilde{\Pi}^{(r)}$. In order to do so, we must first understand how tuples of C_r colorings multiply. The product of colors is given by concatenation, i.e., $(\xi_1, .., \xi_n) \cdot (\delta_1, ..., \delta_m) = (\xi_1, ..., \xi_n, \delta_1, ..., \delta_m) \in C_r^{n+m}$. We refer the reader to Chapter 1 for a reminder on notations and operations involving set partitions.

Let $\pi \vdash [k]$ and $\mu \vdash [m]$ where $k+m=n$, then the product μ is given by

$$m_{\pi,\xi} \otimes m_{\mu,\delta} \mapsto \sum_{\substack{\nu \vdash [n] \\ \nu \wedge ([k]|[m]) = \pi|\mu}} m_{\nu,\xi\cdot\delta}.$$

Let $\pi \vdash [n]$ and coloring ξ, then the coproduct Δ is given by

$$m_{\pi,\xi} \mapsto \sum_{\mu \sqcup \nu = \pi} m_{st(\mu),\xi|_\mu} \otimes m_{st(\nu),\xi|_\nu}.$$

Here $\xi|_\mu$ denotes the coloring on $st(\mu)$, i.e., the subsequence $(\xi_{i_1}, ..\xi_{i_\ell})$ with $i_1 < i_2 < \cdots < i_\ell$ and i_j is in a block of μ.

Example 3.5.4. Consider the set partition $13|2 \vdash [3]$ with coloring $(1,\bar{1},1)$ and the set partition $12 \vdash [2]$ with coloring $(\bar{1},1)$, then the product is as follows:

$$m_{13|2,(1,\bar{1},1)} \otimes m_{12,(\bar{1},1)} \mapsto m_{13|2|45,(1,\bar{1},1,\bar{1},1)} + m_{1345|2,(1,\bar{1},1,\bar{1},1)} + m_{13|245,(1,\bar{1},1,\bar{1},1)}$$

Example 3.5.5. Consider the set partition $13|245 \vdash [5]$ with the coloring $(1,\bar{1},1,\bar{1},1)$, the coproduct is as follows:

$$m_{13|245,(1,\bar{1},1,\bar{1},1)} \mapsto \mathbf{1} \otimes m_{13|245,(1,\bar{1},1,\bar{1},1)} + m_{12,(1,1)} \otimes m_{123,(\bar{1},\bar{1},1)} + m_{13|245,(1,\bar{1},1,\bar{1},1)} \otimes \mathbf{1}$$

Remark 3.5.6.

- When $r=1$, we recover the Hopf Algebra of Symmetric functions in noncommutative variables as defined in [11] and [35]. In the literature, this is denoted as Π.

- We could have easily colored $\tilde{\Pi}$ with any finite group G instead of restricting ourselves to C_r, we denote this by $\tilde{\Pi}^{(G)}$. In fact, if G is infinite there isn't any problem putting colorings on set partitions since each set partition consists of finitely many entries. We would just have infinitely many ways to color a given set partition.

- Given an algebra A, with fixed basis B, we can color set partitions by the basis of A. We denote this by $\tilde{\Pi}^{(B)}$. When $A = \mathbb{K}G$ and we take our basis to be G, we are back to $\tilde{\Pi}^{(G)}$.

CHAPTER 4

Species

In this chapter, we give a brief introduction to the theory of species. In the first portion, we follow the notation of Aguiar and Mahajan ([3]); please see the following sources for more treatments of the theory of species:[25], [9] [6], and [4].

4.1. Species

Fix a field $\mathbb{K}$, and let I be a finite set.

DEFINITION 4.1.1. A *(vector) species* $\mathbf{p}$ is a functor

$$\mathbf{p} : \mathbf{Set}^{\times} \to \mathbf{Vec}_{\mathbb{K}}$$

where $\mathbf{Set}^{\times}$ and $\mathbf{Vec}_{\mathbb{K}}$ are as defined in Example 2.1.3.

Recall, we write $[n]$ to denote the set $\{1, 2, ..., n\}$. Correspondingly, we write $\mathbf{p}[n]$ for $\mathbf{p}[\{1, 2, ..., n\}]$. Let S_n denote the symmetric group on n letters. Each element $\sigma \in S_n$ is a bijection $\sigma : [n] \to [n]$ thus induces a linear map $\mathbf{p}[\sigma] : \mathbf{p}[n] \to \mathbf{p}[n]$. Hence, $\mathbf{p}[n]$ is an S_n module via $\sigma.v = \mathbf{p}[\sigma]v$ for all $v \in \mathbf{p}[n]$ and $\sigma \in S_n$.

DEFINITION 4.1.2. A *morphism of species*, $\mathbf{p}$ and $\mathbf{q}$ is a natural transformation $\alpha : \mathbf{p} \to \mathbf{q}$, i.e., for all finite sets I, we have a linear map $\alpha_I : \mathbf{p}[I] \to \mathbf{q}[I]$, such that for each bijection $\sigma : I \to J$ the following diagram commutes:

$$\begin{tikzcd} \mathbf{p}[I] \arrow[r, "\alpha_I"] \arrow[d, "\mathbf{p}[\sigma]"'] & \mathbf{q}[I] \arrow[d, "\mathbf{q}[\sigma]"] \\ \mathbf{p}[J] \arrow[r, "\alpha_J"'] & \mathbf{q}[J]. \end{tikzcd}$$

In other words, a species consists of a family of vector spaces $\mathbf{p}[I]$ one for each finite set $I \in \mathbf{Set}^{\times}$, together with linear maps $\mathbf{p}[f] : \mathbf{p}[I] \to \mathbf{p}[J]$ for all bijections $f : I \to J$. We also have that $\mathbf{p}[\mathrm{id}_I] = \mathrm{id}_{\mathbf{p}[I]}$ and $\mathbf{p}[\tau \circ \sigma] = \mathbf{p}[\tau] \circ \mathbf{p}[\sigma]$ whenever τ and σ are composable bijections.

EXAMPLE 4.1.3. Here, we briefly introduce some examples of species that will be used consistently throughout this thesis, and one can reference Section 5 for more details regarding these examples.

- **Exponential Species**
  On objects I and for all morphisms $f : I \to J$:
  $$\mathbf{E}[I] := \mathbb{K}$$
  $$\mathbf{E}[f] := \mathrm{id}_{\mathbb{K}}\,.$$

- **Linear Order Species**
  On objects I:

$$\begin{aligned}\mathbf{L}[I] &:= \mathbb{K}\text{-vector space with basis indexed by the linear orders on } I\\ &= \langle H_\ell \mid \ell \text{ a linear order order on } I\rangle,\end{aligned}$$

where H_ℓ denotes the basis element labelled by the linear order ℓ. When we need to specify the linear order, we write $\ell = \ell_1 \cdots \ell_{|I|}$ for the linear order $\ell_1 < \cdots < \ell_{|I|}$ and $\ell_k \in I\ \forall k$.

On morphisms $f : I \to J$:

$$\begin{aligned}\mathbf{L}[f] \quad : \quad & \mathbf{L}[I] \to \mathbf{L}[J]\\ & H_\ell \mapsto H_{f(\ell)},\end{aligned}$$

where $f(\ell)$ denotes the linear order on J obtained by applying f to each letter of ℓ. That is, if $\ell = \ell_1 \cdots \ell_{|I|}$, then $f(\ell) = f(\ell_1) \cdots f(\ell_{|I|})$.

For example, $\mathbf{L}[\{a, b, c\}]$ has basis elements labeled by

$$\langle abc,\ acb,\ bac,\ bca,\ cab,\ cba\rangle,$$

and $\mathbf{L}[3]$ has basis elements labeled by

$$\langle 123,\ 132,\ 213,\ 231,\ 312,\ 321\rangle.$$

Consider the set bijection $f : \{a, b, c\} \to \{1, 2, 3\}$ given by $f(a) = 2$, $f(b) = 1$, and $f(c) = 3$; this gives rise to a linear map $\mathbf{L}[f] : \mathbf{L}[\{a, b, c\}] \to \mathbf{L}[3]$ given by

$$\begin{array}{ll} H_{abc} \mapsto H_{213} & H_{acb} \mapsto H_{231}\\ H_{bac} \mapsto H_{123} & H_{bca} \mapsto H_{132}\\ H_{cab} \mapsto H_{321} & H_{cba} \mapsto H_{312}\end{array}.$$

- **Set Partition Species**
  On objects I,

$$\begin{aligned}\mathbf{\Pi}[I] &:= \mathbb{K}\text{-vector space with basis indexed by the set partitions of } I\\ &= \langle H_\pi \mid \pi \vdash I\rangle,\end{aligned}$$

where H_π denotes the basis element labelled by the set partition $\pi \vdash I$, as defined in Section 1.1.

On morphisms $f : I \to J$, this gives rise to the linear map:

$$\begin{aligned}\mathbf{\Pi}[f] \quad : \quad & \mathbf{\Pi}[I] \to \mathbf{\Pi}[J]\\ & H_\pi \mapsto H_{f(\pi)},\end{aligned}$$

where $f(\pi) \vdash J$ obtained by applying f to each element of the blocks of π.

For example, $\mathbf{\Pi}[\{a, b, c\}]$ has basis elements labeled by

$$\langle abc,\ ab|c,\ ac|b,\ a|bc,\ a|b|c\rangle.$$

$\mathbf{\Pi}[3]$ has basis elements labeled by

$$\langle 123,\ 12|3,\ 13|2,\ 1|23,\ 1|2|3\rangle.$$

Consider the set bijection $f : \{a, b, c\} \to \{1, 2, 3\}$ given by $f(a) = 2$, $f(b) = 1$, and $f(c) = 3$; this gives rise to a linear map $\mathbf{L}[f] : \mathbf{L}[\{a, b, c\}] \to \mathbf{L}[3]$ given by

$$\begin{array}{rcl} H_{abc} \mapsto H_{123} & & H_{ab|c} \mapsto H_{12|3} \\ H_{ac|b} \mapsto H_{1|23} & & H_{a|bc} \mapsto H_{13|2} \\ H_{a|b|c} \mapsto H_{1|2|3} & & \end{array}.$$

Remark 4.1.4. When there is no confusion, we often only denote the basis element by the combinatorial object that labels it; for example, H_π will be denoted as π.

4.2. Monoidal Structures

Let **Sp** denote the category of species, whose objects are species and morphisms as above. The following are two operations that turn **Sp** into a monoidal category.

(1) **Sp** is a monoidal category under the *Cauchy Product*, $\cdot$, defined by

$$(\mathbf{p} \cdot \mathbf{q})[I] := \bigoplus_{S \sqcup T = I} \mathbf{p}[S] \otimes \mathbf{q}[T].$$

The unit $\mathbf{1}$ is defined by

$$\mathbf{1}[I] = \begin{cases} \mathbb{K} & \text{if } I = \emptyset \\ 0 & \text{otherwise} \end{cases}.$$

It is braided and symmetric with the braiding given on pure tensors by

$$\beta_{\mathbf{p},\mathbf{q}} : \mathbf{p}[S] \otimes \mathbf{q}[T] \to \mathbf{q}[T] \otimes \mathbf{p}[S]$$
$$x \otimes y \mapsto y \otimes x.$$

(2) **Sp** is a monoidal category under the *Hadamard Product*, $\times$, defined by

$$(\mathbf{p} \times \mathbf{q})[I] := \mathbf{p}[I] \otimes \mathbf{q}[I].$$

The unit is given by $\mathbf{E}$, defined in Example 4.1.3. $(\mathbf{Sp}, \times)$ is also braided and symmetric with the braiding given on pure tensors by

$$\beta_{\mathbf{p},\mathbf{q}} : \mathbf{p}[S] \otimes \mathbf{q}[S] \to \mathbf{q}[S] \otimes \mathbf{p}[S]$$
$$x \otimes y \mapsto y \otimes x.$$

Here, we summarize the definitions of (co/bi/Hopf) monoid, given in Section 2.3 in terms of the Cauchy product; replacing the Cauchy product with the Hadamard product would be defined similarly.

A *monoid* in $(\mathbf{Sp}, \cdot)$ is a species $\mathbf{p}$ together with morphisms $\mu : \mathbf{p} \cdot \mathbf{p} \to \mathbf{p}$ and $\iota : \mathbf{1} \to \mathbf{p}$ such that the diagrams in (5) commute. For each decomposition $S \sqcup T = I$, there is a linear map

$$\mu_I : \mathbf{p}[S] \otimes \mathbf{p}[T] \to \mathbf{p}[I].$$

There is also a linear map $\iota_\emptyset : \mathbb{K} \to \mathbf{p}[\emptyset]$. A *comonoid* in $(\mathbf{Sp}, \times)$ is a species $\mathbf{p}$ together with morphisms $\Delta : \mathbf{p} \to \mathbf{p} \cdot \mathbf{p}$ and $\varepsilon : \mathbf{p} \to \mathbf{1}$ such that the diagrams with arrows reversed in (5) commute. For each decomposition $S \sqcup T = i$, there is a linear map

$$\Delta_{S,T} : \mathbf{p}[I] \to \mathbf{p}[S] \otimes \mathbf{q}[T].$$

There is also a linear map $\varepsilon_\emptyset : \mathbf{p}[\emptyset] \to \mathbb{K}$. A *bimonoid* is a species $\mathbf{p}$ such that $\mathbf{p}$ is both a monoid and comonoid and the Diagrams (8), (9), and (10) commute. A *Hopf monoid* is a species $\mathbf{p}$ that is a bimonoid with an antipode such that the diagrams in (12) commute.

PROPOSITION 4.2.1. (Stover [37]) *A bimonoid $\mathbf{h}$ for which $\mathbf{h}[\emptyset]$ is a Hopf algebra is a Hopf monoid. In particular, if $\mathbf{h}[\emptyset] = \mathbb{K}$, that is $\mathbf{h}$ is connected, then $\mathbf{h}$ is a Hopf monoid.*

4.3. Fock Functors

In [3], Aguiar and Mahajan defined four bilax monoidal functors from the category of species to the category of graded vector spaces, called Fock functors, two of which correspond to the notion of invariance and the other two corresponding to coinvariance:

$$K, \overline{K} : \mathbf{Sp} \to \mathbf{gVec}_{\mathbb{K}} \qquad \overline{K}^\vee, K^\vee : \mathbf{Sp} \to \mathbf{gVec}_{\mathbb{K}}$$

$$K(\mathbf{p}) := \bigoplus_{n \geq 0} \mathbf{p}[n] \qquad K^\vee(\mathbf{p}) := \bigoplus_{n \geq 0} \mathbf{p}[n]$$

$$\overline{K}(\mathbf{p}) := \bigoplus_{n \geq 0} \mathbf{p}[n]_{S_n} \qquad \overline{K}^\vee(\mathbf{p}) := \bigoplus_{n \geq 0} \mathbf{p}[n]^{S_n}.$$

Note that even though $K = K^\vee$, it useful to keep a distinction between the two when considering (co)invariance, as you will see when we consider the bilax structure. Here, $\mathbf{p}[n]_{S_n}$ is the space of S_n-coinvariants and $\mathbf{p}[n]^{S_n}$ is the space of S_n-invariants, as defined in Section 3.3. The importance of these categorical definitions is demonstrated in the following theorems. Recall that bilax monoidal structure was defined in Section 2.4.

THEOREM 4.3.1 (Aguiar and Mahajan [3]). *The functors $(K, \varphi, \varphi_0, \psi, \psi_0)$, $(\overline{K}, \overline{\varphi}, \overline{\varphi}_0, \overline{\psi}, \overline{\psi}_0)$, $(K^\vee, \varphi^\vee, \varphi_0^\vee, \psi^\vee, \psi_0^\vee)$ and $(\overline{K}^\vee, \overline{\varphi}^\vee, \overline{\varphi}_0^\vee, \overline{\psi}^\vee, \overline{\psi}_0^\vee)$ are bilax monoidal functors.*

The natural transformations for the bilax structure that correspond to K and $\overline{K}$ are as follows: We have $\varphi_0 = \mathrm{id}$ and $\psi_0 = \mathrm{id}$. For all $\mathbf{p}, \mathbf{q}$ and $s + t = n$, we have

$$\varphi_{\mathbf{p},\mathbf{q}} : \mathbf{p}[s] \otimes \mathbf{q}[t] \xrightarrow{\mathbf{p}[\mathrm{id}] \otimes \mathbf{q}[\mathrm{cano}_s]} \mathbf{p}[s] \otimes \mathbf{q}[[s+t]]$$

where $\mathrm{id} : [s] \to [s]$ and cano_s is the order preserving bijection $\mathrm{cano}_s : [s] \to [s+t] = \{1+s, ..., t+s\}$. For decompositions $S \sqcup T = [n]$ such that $|S| = s$ and $|T| = t$,

$$\psi_{\mathbf{p},\mathbf{q}} : \mathbf{p}[S] \otimes \mathbf{q}[T] \xrightarrow{\mathbf{p}[st] \otimes \mathbf{q}[st]} \mathbf{p}[s] \otimes \mathbf{q}[t]$$

where st denotes the order preserving bijections $st : S \to [s]$ and $st : T \to [t]$. The maps $\overline{\varphi}, \overline{\varphi}_0, \overline{\psi}$, and $\overline{\psi}_0$ are induced by φ and ψ on coinvariants.

The natural transformations for the bilax structure that correspond to $K^\vee$ and $\overline{K}^\vee$ are as follows: We have $\varphi_0^\vee = \mathrm{id}$ and $\psi_0^\vee = \mathrm{id}$. For all $\mathbf{p}, \mathbf{q}$ and $s + t = n$, we have

$$\varphi^\vee_{\mathbf{p},\mathbf{q}} : \mathbf{p}[s] \otimes \mathbf{q}[t] \xrightarrow{\oplus \mathbf{p}[\mathrm{cano}] \otimes \mathbf{q}[\mathrm{cano}]} \bigoplus_{\substack{S \sqcup T = [n] \\ |S| = s \\ |T| = t}} \mathbf{p}[S] \otimes \mathbf{q}[T].$$

For the sections of $\psi^\vee$, we have

$$\psi^\vee_{\mathbf{p},\mathbf{q}} : \mathbf{p}[s] \otimes \mathbf{q}[s+1, s+t] \xrightarrow{\mathbf{p}[\mathrm{id}] \otimes \mathbf{q}[st]} \mathbf{p}[s] \otimes \mathbf{q}[t]$$

and the zero map on all other decompositions of I. These structure maps restrict to the invariants, defining $\overline{\varphi}^\vee, \overline{\varphi}^\vee_0, \overline{\psi}^\vee$, and $\overline{\psi}^\vee_0$.

Theorem 4.3.2 (Aguiar and Mahajan [3]). *If* $\mathbf{h}$ *is a Hopf monoid in* $\boldsymbol{Sp}$, *then* $K(\mathbf{h})$, $\overline{K}(\mathbf{h})$, $K^\vee(\mathbf{h})$, *and* $\overline{K}^\vee(\mathbf{h})$ *are graded Hopf algebras.*

The above theorem gives us a way of constructing Hopf algebras from a given Hopf monoid; in particular, a given Hopf monoid can have multiple Hopf algebras associated to it depending on which functor is applied to it. Many interesting combinatorial Hopf algebras can be constructed in this fashion. For example, the ring of symmetric functions and the ring of symmetric functions in noncommutative variables. The goal of the remainder of this thesis is to generalize these constructions.

CHAPTER 5

Examples of Hopf Monoids

In this section, we describe the Hopf monoid structure in detail of a few examples that will be used throughout the remaining sections. We are particularly interested in species that are Hopf monoids; as every Hopf monoid corresponds to a Hopf algebra by applying a Fock functor. One can refer to [3], [9], [4] and [6] to see more properties of these examples.

5.1. The Hopf Monoid of Linear Orders

The species of linear orders, $\mathbf{L}$, is a surprisingly simple example of a Hopf monoid with a very rich structure. As we will explain later, the components of $\mathbf{L}$ correspond to the regular representation of the symmetric groups. Some other things to note, $\mathbf{L}$ is self dual, it is used in constructing the free monoid of a species (analogous to the tensor algebra of a vector space), and can be thought of geometrically as chambers. Finally, $\mathbf{L}$ also plays well with the exponential species defined in Example 4.1.3–these species can be combined in various ways to produce new Hopf Monoids. Please reference the above for a thorough exposition on this topic.

Recall, the species of linear orders is defined on objects by:

$$\mathbf{L} : \mathbf{Set}^{\times} \to \mathbf{Vec}_{\mathbb{K}}$$

$$I \mapsto \mathbf{L}[I] := \langle H_\ell \mid \ell \text{ a linear order on I} \rangle$$

Recall, that we often write ℓ to denote H_ℓ, the basis element indexed by the linear order ℓ of I. For a decomposition, $S \sqcup T = I$, the product and coproduct maps are given by:

$$\mu_{S,T}(\ell_S, \ell_T) = \ell_S \cdot \ell_T \qquad\qquad \Delta_{S,T}(\ell) = \ell|_S \otimes \ell|_T$$

where ℓ_S is a linear order on S and ℓ_T is a linear order on T. $\ell_S \cdot \ell_T$ denotes the concatenation of the two linear orders to form a linear order on I. Also, $\ell|_S$ denotes the linear order on S formed by the restriction of ℓ to the subset S. The antipode is given by

$$s(\ell) = (-1)^{|I|}\overline{\ell}$$

where $|I|$ is the cardinality of I (or equivalently the length of ℓ), and $\overline{\ell}$ is the linear order formed by reversing the entries of ℓ.

Example 5.1.1. Consider $[3] = \{1, 2, 3\}$. Then

$$\mathbf{L}[3] = \langle H_{123}, H_{132}, H_{213}, H_{231}, H_{312}, H_{321} \rangle$$

Let $S = \{1, 3\}$ and $T = \{2\}$, the following are examples of the product, coproduct, and antipode:

$$\mu_{S,T}(31 \otimes 2) = 312$$

$$\Delta_{S,T}(213) = 13 \otimes 2$$

$$s(213) = -(312)$$

5.2. The Hopf Monoid of Set Partitions

The Hopf monoid of set partitions is another example of a Hopf monoid that is rich with structure. It has both a combinatorial flavor and geometric flavor. Geometrically, for each finite set I, this corresponds to the permutahedron, see [6]. Combinatorially, this Hopf monoid has four distinguished bases labelled by set partitions. We will be focusing on the basis that generalizes the complete homogeneous functions ([11], [35], [20], and [29]). To see more detail regarding the other bases and properties of such, please see [31].

Let

$$\mathbf{\Pi} : I \to \mathbf{\Pi}[I] := \langle H_\pi \mid \pi \vdash I \rangle.$$

Again, when there is no confusion, we will use π to denote H_π. Given a decomposition $S \sqcup T = I$, the product and coproduct are as follows:

$$\mu_{S,T} : \mathbf{\Pi}[S] \otimes \mathbf{\Pi}[T] \to \mathbf{\Pi}[I]$$
$$\pi \otimes \sigma \mapsto \pi \sqcup \sigma$$
$$\Delta_{S,T} : \mathbf{\Pi}[I] \mapsto \mathbf{\Pi}[S] \otimes \mathbf{\Pi}[T]$$
$$\pi \mapsto \pi|_S \otimes \pi|_T$$

The unit $\iota_\emptyset : \mathbb{K} \to \mathbf{\Pi}[\emptyset]$ given by $\iota_\emptyset(1) = \mathbf{1}$ where $\mathbf{1}$ is the empty set partition. The counit $\varepsilon_\emptyset : \mathbf{\Pi}[\emptyset] \to \mathbb{K}$ is given by $\varepsilon_\emptyset(\mathbf{1}) = 1$ and zero for all other H_π.

Theorem 5.2.1. (Aguiar, Mahajan, [3]) *The antipode for $\mathbf{\Pi}$, $s : \mathbf{\Pi} \to \mathbf{\Pi}$, has components given by*

$$s_I : \mathbf{\Pi}[I] \to \mathbf{\Pi}[I]$$
$$\pi \mapsto \sum_{\sigma : \pi \leq_{op} \sigma} (-1)^{\ell(\sigma)} (\pi : \sigma)! \sigma$$

where the sum is over all partitions that refine π and $(\pi : \sigma)! = \prod_{X_S \in \pi}(n_S!)$ where n_S is the number of blocks of σ that partition the blocks X_S of π.

Remark 5.2.2. Here we are using the opposite of the usual refinement ordering as given in Subsection 1.1.1. We say $x \leq_{op} y$ if y refines x, i.e., every block of x is a union of blocks of y.

Example 5.2.3. Consider $[4] = \{1, 2, 3, 4\}$, then

$$\mathbf{\Pi}[4] = \langle 1234, 123|4, 124|3, \cdots, 1|2|3|4 \rangle$$

Let $S = \{1, 2, 4\}$ and $T = \{3\}$. The following are examples of the product, coproduct, and antipode:

$$\mu_{S,T}(14|2 \otimes 3) = 14|2|3$$
$$\Delta_{S,T} 123|4 = 12|4 \otimes 3$$
$$s_{[4]}(14|23) = 14|23 - 2(1|23|4) - 2(14|2|3) + 4(1|2|3|4)$$

Theorem 5.2.4 ([31]). *There are Hopf algebra isomorphisms f and $\overline{f}$ such that the following diagram commutes*

$$\begin{tikzcd} K(\mathbf{\Pi}) \arrow[r, "f"] \arrow[d, two heads] & \Pi \arrow[d, two heads] \\ \overline{K}(\mathbf{\Pi}) \arrow[r, "\overline{f}"'] & \Lambda, \end{tikzcd}$$

where Π *is as in Section 3.5 when* $r = 1$ *and* Λ *is the ring of symmetric functions as in Section 3.4. The isomorphisms are given by* $f(H_\pi) = h_\pi$ *and* $\overline{f}(H_\pi) = \lambda! h_\pi$.

5.3. The Hopf Monoid of Super Class Functions on Unitriangular Groups

In [1], Aguiar and friends construct a Hopf algebra from the supercharacters of $UT_n(\mathbb{F}_q)$ and show that this is isomorphic to the Hopf algebra of the ring of symmetric functions in noncommutative variables. Aguiar-Bergeron-Thiem later show that this Hopf algebra is the result of applying a certain Fock functor to a Hopf monoid of set partitions, see [2]. Later, we will show that viewing this Hopf monoid as a Hopf monoid in the category of A-species via the functors described in Chapter 9, gives rise to a Hopf algebra that is isomorphic to the Hopf algebra of B-colored symmetric functions in noncommutative variables. In this section, we discuss the background needed for the Hopf monoid of Super Class functions on Unitriangular Groups.

5.3.1. Super Character Theory of $U(I, \ell)$

There are different ways to construct a supercharacter theories for groups, but we will restrict ourselves to the technique used for algebraic groups as done in [2], and will recall only the minimum amount of information needed. Provided below is the formal definition; however, please reference [16] for a thorough exposition on this topic.

Definition 5.3.1. Let G be a finite group, $\mathcal{K}$ be a partition of G into unions of conjugacy classes, and $\mathcal{X}$ be a set of characters of G. We say the pair $(\mathcal{K}, \mathcal{X})$ is a *supercharacter theory* of G if

- $|\mathcal{X}| = |\mathcal{K}|$
- the characters $\chi \in \mathcal{X}$ are constant on the members of $\mathcal{K}$, and
- $\{1\} \in \mathcal{K}$, where 1 is the identity element of G, and $\mathbf{1} \in \mathcal{X}$ where $\mathbf{1}$ is the trivial character of G.

The characters $\chi \in \mathcal{X}$ are called *supercharacters* and the blocks $K \in \mathcal{K}$ are called *superclasses*.

Let $\mathfrak{n}$ be a nilpotent $\mathbb{F}_q$-Lie algebra. The *algebra group* associated to $\mathfrak{n}$ is the set

$$G(\mathfrak{n}) = \{1 + x \mid x \in \mathfrak{n}\}.$$

Define an equivalence relation on $G(\mathfrak{n})$ given by

$$x \sim y \text{ if there exists } g \text{ and } h \in G(\mathfrak{n}) \text{ s.t } y - 1 = g(x-1)h.$$

Diaconis and Isaacs refer to the equivalence classes as the *superclasses* and the functions $G(\mathfrak{n}) \to \mathbb{K}$ that are constant on these classes as *superclass functions*. They denote the set of these as $\mathbf{scf}(G(\mathfrak{n}))$, where

$$\mathbf{scf} : \{\text{algebra groups}\} \to \mathbf{Vec}_{\mathbb{K}}$$

via

$$\mathbf{scf}(G) = \mathbb{K} - \text{vector space of superclass functions on } G$$

is a contravariant functor. Under the equivalence relation, each superclass is a union of conjugacy classes since

$$gxg^{-1} - 1 = g(x-1)g^{-1}$$

hence $x \sim gxg^{-1}$ for any x and $g \in G(\mathfrak{n})$. From this, we see that every superclass function is a class function.

5.3.2. Unitriangular Groups, $U(I, \ell)$

Let $\mathbb{K}$ be a field. Given a finite set I, let $M(I)$ denote the algebra of matrices indexed by I:

$$M(I) := \{A = (a_{i,j})_{i,j \in I}, \ \ a_{i,j} \in \mathbb{K} \ \forall \ i, j \in I\}.$$

Given a linear order ℓ on I, then $U(I, \ell)$ is the subgroup of upper ℓ-triangular matrices given by

$$U(I,\ell) = \left\{ (u_{i,j})_{i,j\in I} \ \middle| \ u_{i,j} \in \mathbb{K} \ \ s.t \ \ u_{i,j} = \left\{ \begin{array}{ll} 1 & \text{if } i = j \\ 0 & \text{whenever } i >_\ell j \end{array} \right. \right\}.$$

We denote $\mathfrak{n}(I, \ell)$ to be the subalgebra of $M(I)$ consisting strictly upper triangular matrices with respect to the order ℓ. These are clearly nilpotent. Define $U(I, \ell)$ as $G(\mathfrak{n}(I, \ell))$; hence, $U(I, \ell)$ has a supercharacter theory.

We can define the *species of superclass functions on unitriangular groups*, $\mathbf{scf}(U)$, as follows: for each finite set I and bijection $f : I \to J$,

$$\mathbf{scf}(U)[I] := \bigoplus_{\ell \in L[I]} \mathbf{scf}(U(I, \ell))$$

$$\mathbf{scf}(U)[f] : \mathbf{scf}(U)[I] \to \mathbf{scf}(U)[J].$$

For a given linear order $\ell \in L[I]$, we have

$$\mathbf{scf}(U)[f] : \mathbf{scf}(U(I, \ell)) \to \mathbf{scf}(U(J, f(\ell)))$$

where $f(\ell)$ is as defined in Example 4.1.3.

Throughout, we will only consider matrices of such form with entries in a finite field.

5.3.2.1. *Hopf Monoid* ***scf(U)***. Following [2] and [1], each superclass has a unique matrix, $U_{(X,\alpha)}$ of the following form:

- Upper triangular with 1's on diagonal.
- At most one nonzero entry in each row and column, excluding diagonal entry.

It is shown that the set of such matrices are in a one-to-one correspondence with arc diagrams, (X, α)–hence the notation used for the unique matrix. The set of *arcs* is

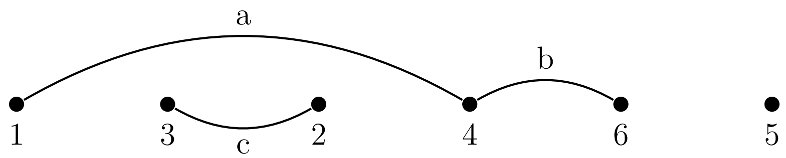
$$A(X, \ell) := \{(i, j) \mid i <_\ell j, \ i, j \in \text{same block, } S, \text{ of } X \vdash I, \text{ and } \nexists \ k \in S \ s.t \ i <_\ell k <_\ell j, \}$$

We also have function

$$\alpha : A(X, \ell) \to \mathbb{F}_q^\times$$

The pair (X, α) is an *arc diagram*.
As the next example shows, it is convenient to represent an arc diagram as a labeled graph.

Example 5.3.2. Let $a, b, c \in \mathbb{F}_q^\times$.



The following data corresponds to the arc diagram above :

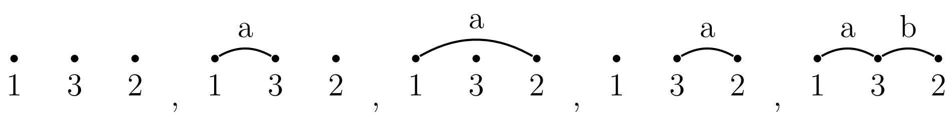
$$\ell = 1 < 3 < 2 < 4 < 6 < 5, \text{ and } X = 146|32|5, \ \alpha(1,4) = a \ \ \alpha(4,6) = b, \ \alpha(3,2) = c$$

The matrix that corresponds to this is:

$$U_{(X,\alpha)} = \begin{pmatrix} 1 & 0 & 0 & a & 0 & 0 \\ 0 & 1 & c & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & b \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Let $\kappa_{(X,\alpha)}$ denote the characteristic function for the superclass containing $U_{(X,\alpha)}$. As we range over all arc diagrams, (X, α), on (I, ℓ), these form a basis for $\mathbf{scf}(U(I, \ell))$.

EXAMPLE 5.3.3. Let $I = [3], \ell = 132$, and $a, b \in \mathbb{F}_q^\times$. The basis elements of $\mathbf{scf}(U([3], 132))$ are labelled by the following arc diagrams.



When we restrict ourselves to the field $\mathbb{F}_2$, then the arcs are labeled by elements of $\mathbb{F}_2^\times = \{1\}$. Thus, they are in bijection with set partitions.

Let $S \subseteq I$ be an arbitrary set. Given a partition of I, say X, let

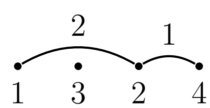
$$A(X, \ell)|_S := \{(i, j) \in A(X, \ell) \mid i \text{ and } j \text{ belong to } S\}$$

and $\alpha|_S$ to denote the restriction of α to $A(X, \ell)|_S$.

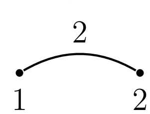
EXAMPLE 5.3.4. Let $\mathbb{F}_3$ be the field with 3 elements, $I = [4]$, and $\ell = 1324$. We have that

$$A(124|3, \ell) = \{(1, 2), (2, 4)\}.$$

When $\alpha(1, 2) = 2$ and $\alpha(2, 4) = 1$, this data corresponds to the arc diagram



Let $S = \{1, 2\} \subset [4]$. Then $A(124|3, 1324)|_S = \{(1, 2)\}$ and $\alpha|_S$ is given by $\alpha|_S(1, 2) = 2$. This data corresponds to the digram given by



If $S = \{1, 3, 4\}$ then $A(124|3, 1324)|_S = \emptyset$.

5.3.3. (Co)Product

We will use the combinatorial descriptions of the basis $\kappa_{(X,\alpha)}$, as seen above, to describe the (co)product.

Let $I = S_1 \sqcup S_2$ and $\ell_i \in L[S_i]$, $i = 1, 2$, the product is given by

$$\mu_{S_1,S_2} : \mathbf{scf}(U(S_1,\ell_1)) \otimes \mathbf{scf}(U(S_2,\ell_2)) \to \mathbf{scf}(U(I, \ell_1 \cdot \ell_2))$$

$$\kappa_{X_1,\alpha_1} \otimes \kappa_{X_2,\alpha_2} \mapsto \sum_{\substack{X|_{S_i} = X_i \\ \alpha|_{S_i} = \alpha_i}} \kappa_{X,\alpha}.$$

The coproduct is given by

$$\Delta_{S_1,S_2} : \mathbf{scf}(U(S,\ell)) \otimes \to \mathbf{scf}(U(S_1,\ell|_{S_1})) \otimes \mathbf{scf}(U(S_2,\ell|_{S_2}))$$

$$\kappa_{X,\alpha} \mapsto \begin{cases} \kappa_{X|_{S_1},\alpha|_{S_1}} \otimes \kappa_{X|_{S_2},\alpha|_{S_2}} & \text{if } S_1 \text{ is the union of some blocks of } X \\ 0 & \text{otherwise} \end{cases}$$

EXAMPLE 5.3.5. Let $\mathbb{F}_q$ be our field. Let $= [3]$ and let $S_1 = \{1, 2\}$ with $\ell_1 = 12$ and $S_2 = \{3\}$ with $\ell_2 = 3$. For the product, we have:

$$\kappa_{1\overset{a}{\frown}2} \otimes \kappa_{3} \mapsto \kappa_{1\overset{a}{\frown}2\;\;3} + \sum_{b\in\mathbb{F}_q^\times} \kappa_{1\overset{a}{\frown}2\overset{b}{\frown}3}$$

For the coproduct, we have:

$$\Delta(\kappa_{1\overset{a}{\frown}2\;\;3}) = \mathbf{1} \otimes \kappa_{1\overset{a}{\frown}3;\;2} + \kappa_{1} \otimes \kappa_{1\overset{a}{\frown}2} + \kappa_{1\overset{a}{\frown}2} \otimes \kappa_{1} + \kappa_{1} \otimes \kappa_{1\overset{a}{\frown}2} + \kappa_{1\overset{a}{\frown}3;\;2} \otimes \mathbf{1}$$

The next result shows that when we consider the Hadamard product (see Section 4.2) of the species of linear orders, $\mathbf{L}$, with the species of set partitions, $\mathbf{\Pi}$, there is a relationship between the basis elements of $\mathbf{L} \times \mathbf{\Pi}$ with the basis elements of $\mathbf{scf}(U)$. Please see [2] for the proof.

PROPOSITION 5.3.6 (Aguiar [2]). *Let $\mathbb{F}$ be an arbitrary finite field. The map*

$$\varphi : \boldsymbol{L} \times \boldsymbol{\Pi} \to \boldsymbol{scf}(U),$$

whose sections are given by

$$\varphi_I(\ell \otimes m_X) = \sum_{\alpha : A(X,\ell) \to \mathbb{F}^\times} \kappa_{X,\alpha},$$

is an injective morphism of Hopf monoids.

The morphism φ is adding labels from $\mathbb{F}_q^\times$ to the underlying arcs in all possible ways. In particular, when $q = 2$, there is only one such way. This observation leads to the following result.

COROLLARY 5.3.7 (Aguiar, Mahajan). *There is an isomorphism of Hopf monoids*

$$\boldsymbol{scf}(U) \cong \boldsymbol{L} \times \boldsymbol{\Pi},$$

i.e., an isomorphism between the Hopf monoid of superclass functions on unitriangular matrices with entries in $\mathbb{F}_2$ *and the Hadamard product of the Hopf monoid of linear orders and set partitions.*

5.3.4. Relationship to the Hopf Algebra Π

For any Hopf monoid $\mathbf{p}$, we have [3], Thm 15.13,

$$\overline{K}(\mathbf{L} \times \mathbf{p}) \cong K(\mathbf{p}).$$

This, together with Corollary 5.3.7, we have

$$\overline{K}(\mathbf{scf}(U)) \cong \overline{K}(\mathbf{L} \times \mathbf{\Pi}) \cong K(\mathbf{\Pi}).$$

in other words, our Fock functors given by the S_n-coinvariants applied to the Hopf monoid of superclass functions on unitriangular matrices with entries in $\mathbb{F}_2$ is isomorphic to the Hopf algebra of symmetric functions in noncommuting variables. This is the main result of [1].

CHAPTER 6

Generalizations of Species

Here we give a brief overview of two generalizations of species: $\mathcal{H}$-species and $\mathbf{B_r}$-species; the latter of which is a a generalization of $\mathcal{H}$-species. Please reference [10] for $\mathcal{H}$-species, and [22] and [21] for $\mathbf{B_r}$-species. Later when we define a notion of an A-species, we will see how both of these generalizations fit inside of our definition.

6.1. $\mathcal{H}$-Species

Here, we follow the notation and work of Choquette and Bergeron [10].

DEFINITION 6.1.1. An *$\mathcal{H}$-set*, (I, σ), is a finite set I together with an involution σ on I, where σ is without fixed points. An *$\mathcal{H}$-bijection*, is a bijection between $\mathcal{H}$-sets (I, σ) and (J, τ) such that the following diagram commutes:

$$\begin{array}{ccc} I & \xrightarrow{f} & J \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\tau} \\ I & \xrightarrow[f]{} & J, \end{array}$$

where f is a bijection between the finite sets I and J.

Denote $\mathbf{B}^{\mathcal{H}}$ to be the category whose objects are $\mathcal{H}$-sets and whose morphisms are $\mathcal{H}$-bijections. Often times, we consider the skeleton of $\mathbf{B}^{\mathcal{H}}$, denote this $\mathbf{B}^{\mathcal{H}}_s$, whose objects are of the form $([\overline{n}, n], \sigma_0)$ where $[\overline{n}, n] = \{\overline{n}, ..., \overline{1}, 1, .., n\}$ where $\overline{i} = -i$, and σ_0 is the natural involution on non-zero integers, i.e., $\sigma_0(i) = \overline{i}$ for all $i \in \mathbb{Z} \setminus \{0\}$. The $\mathcal{H}$-bijections of objects from $\mathbf{B}^{\mathcal{H}}_s$ are isomorphic to $\mathcal{B}_n$, the hyperoctrahedral group or the group of signed permutations, see [10] for more on the hyperoctrahedral group.

REMARK 6.1.2. When working with $\mathcal{H}$-sets of the form $([\overline{n}, n], \sigma_0)$, the involution is almost always suppressed and is denoted $[\overline{n}, n]$.

DEFINITION 6.1.3. Let I be a finite set and σ an involution on I. A *section* is a map

$$s : I/\sigma \to I$$

which is a right inverse for the projection $I \to I/\sigma$. In particular, $s([i]) \in \{i, \sigma(i)\}$, i.e., taking the coset made from $i \in I$ and mapping it to either itself or $\sigma(i)$.

DEFINITION 6.1.4. An $\mathcal{H}$-subset (S, σ_S) of an $\mathcal{H}$-set (I, σ) is a set S such that $S \subseteq I$ and $\sigma_S(S) = S$, where $\sigma_S : S \to S$ is the restriction of σ to the set S.

The condition, $\sigma_S(S) = S$ ensures that an $\mathcal{H}$-subset is again an $\mathcal{H}$-set.

DEFINITION 6.1.5. An *$\mathcal{H}$-decomposition* F of (I, σ) is an ordered sequence $F = (F_1, ..., F_\ell)$ of disjoint $\mathcal{H}$-subsets of (I, σ) such that $\bigsqcup\limits_{i=1}^{\ell} F_i = A$.

DEFINITION 6.1.6. A *hyperoctrahedral species*, is a functor

$$\mathbf{p} : \mathbf{B}^{\mathcal{H}} \to \mathbf{Vec}_{\mathbb{K}}.$$

A *morphism* of $\mathcal{H}$-species is a natural transformation between $\mathcal{H}$-species. Let $\mathbf{Sp}^{\mathcal{H}}$ denote the category of $\mathcal{H}$-species.

The following is an analogue of the fact that species can be defined as sequence of S_n-modules.

PROPOSITION 6.1.7 (Choquette, Bergeron [10]). *An $\mathcal{H}$-species can be defined as a sequence of modules of the hyperoctrahedral groups.*

PROPOSITION 6.1.8 (Choquette, Bergeron [10]). *$(\boldsymbol{Sp}^{\mathcal{H}}, \cdot, \circ, \beta)$ is a symmetric monoidal category under the* Cauchy Product

$$\mathbf{p} \cdot \mathbf{q}[I, \sigma] = \bigoplus_{S \sqcup T = I} \mathbf{p}[S, \sigma_S] \otimes \mathbf{q}[T, \sigma_T],$$

where the direct sum is over $\mathcal{H}$-decompositions. The unit for this product is given by

$$\circ[I, \sigma] = \begin{cases} \mathbb{K} & \textit{if } I = \emptyset \\ 0 & \textit{otherwise.} \end{cases}$$

The braiding $\beta_{\mathbf{p},\mathbf{q}} : \mathbf{p} \cdot \mathbf{q} \to \mathbf{q} \cdot \mathbf{p}$ is given by

$$\mathbf{p}[S, \sigma_S] \otimes \mathbf{q}[T, \sigma_T] \to \mathbf{q}[T, \sigma_T] \otimes \mathbf{p}[S, \sigma_S]$$
$$x \otimes y \mapsto y \otimes x.$$

Under the Cauchy product, the notions of monoid, comonoid, bimonoid, and Hopf monoid can be defined using Definitions 2.3.1, 2.3.2, 2.3.3, and 2.3.5.

DEFINITION 6.1.9. The functor

$$\mathcal{S} : \mathbf{Sp} \to \mathbf{Sp}^{\mathcal{H}}$$

is defined for a species $\mathbf{p}$, an $\mathcal{H}$-set (I, σ), and $\mathcal{H}$-bijection f, by

$$\mathcal{S}\mathbf{p}[I, \sigma] := \bigoplus_{s : I/\sigma \to I} \mathbf{p}[s(I/\sigma)]$$

$$\mathcal{S}[f] := \bigoplus_{s : I/\sigma \to I} \mathbf{p}[f|_{s(I/\sigma)}],$$

where the direct sums are over all section maps as in Definition 6.1.3.

PROPOSITION 6.1.10 (Choquette, Bergeron). *$\mathcal{S}$ is a bistrong monoidal functor with bilax structures given by $\varphi_0 = \mathrm{id}, \psi_0 = \mathrm{id}$, and $\varphi = \mathrm{id} \otimes \mathrm{id}$ and $\psi = \mathrm{id} \otimes \mathrm{id}$.*

6.2. B_r-Species

We now introduce the category $\mathbf{B_r}$, as defined by Henderson in [22] and [21], whose objects are finite sets with a free action of the cyclic group of order r, C_r, and the morphisms are the bijections that respect this action. Every object is isomorphic to an object $[n]_r := C_r \times [n]$ for some $n \in \mathbb{N}$ and $\mathrm{End}([n]_r) \cong C_r \wr S_n$. With this data, this forms a skeleton of $\mathbf{B_r}$. Henderson defined $\mathbf{B_r}$-species to be the functor

$$\mathbf{B_r} \to \mathbf{Vec}_{\mathbb{K}}.$$

A $\mathbf{B_r}$-species can be viewed as modules for $C_r \wr S_n$.

(1) When $r = 1$, $\mathbf{B_1}$ is the category of finite sets and bijections–the usual notion of species as in Aguiar, Mahajan.

(2) When $r = 2$, by identifying $C_2 \times [n]$ with $[\overline{n}, n]$, we recover $\mathcal{H}$-species as defined by Choquette and Bergeron.

Remark 6.2.1. Henderson makes a comment that all of his results would hold true if we replaced C_r with any finite group G, making $G \wr S_n$-modules instead of $C_r \wr S_n$-modules.

Above, Choquette and Bergeron constructed a functor $\mathcal{S}$ that went from $\mathbf{Sp}$ to $\mathbf{Sp}^{\mathcal{H}}$; this functor can be easily generalized to a functor from $\mathbf{Sp}$ to $\mathbf{Sp}^{G}$ via:

$$\begin{aligned} \mathbf{p} \;\mapsto\; & \mathcal{S}\mathbf{p}[G \times [n]] := \bigoplus_{s:[n]\to G\times[n]} \mathbf{p}[s([n])] \\ & \mathcal{S}\mathbf{p}[(\vec{g}, \sigma)] := \bigoplus_{s:[n]\to G\times[n]} \mathbf{p}[(\vec{g}, \sigma)|_{s([n])}] \end{aligned}$$

where $(\vec{g}, \sigma) \in G \wr S_n$ with $\vec{g} := (g_1, ..., g_n) \in G^n$ and the direct sum ranges over all section maps.

CHAPTER 7

A-Species

We now introduce the main object of study in this thesis. Let A be a unital associative Hopf algebra over a field $\mathbb{K}$. In this chapter, we define the notion of an A-species. In particular, we are replacing the S_n-module structure with the wreath product algebra, $A \wr S_n$, i.e., an A-species will give a family of modules for $A \wr S_n$ for $n \geq 0$. We show that both the Cauchy Product and Hadamard Product make the category of A-species a monoidal category. We end by showing the category of $\mathbb{K}G$-species is equivalent to the category of G-species (see Section 6.2). For different choices of G, we are able to recover the classical version of species (Section 4.1), $\mathcal{H}$-species (Section 6.1), and B_r-species (Section 6.2.)

7.1. A-Species

Before defining A-species, we need to set up notation.

7.1.1. Notation

Let $\mathbb{K}$ be an arbitrary field, and A be a Hopf algebra[1] over $\mathbb{K}$ with a fixed basis given by $B = \{b_t \mid t \in T\}$, where T is not necessarily finite. Define $c_{i,j}^k$ to be the structure constant given by the following product

$$b_i b_j = \sum_{k \in T} c_{i,j}^k b_k.$$

7.1.2. Wreath Product of Algebras

We consider the *wreath product*, denoted $A \wr S_n$. As a vector space, $A \wr S_n = A^{\otimes n} \otimes \mathbb{K}S_n$ and the multiplication is given by

$$(a_1 \otimes \cdots \otimes a_n \otimes \sigma)(c_1 \otimes \cdots \otimes c_n \otimes \pi) = a_1 c_{\sigma^{-1}(1)} \otimes \cdots \otimes a_n c_{\sigma^{-1}(n)} \otimes \sigma \circ \pi$$

for all $a_i, c_i \in A$ and $\sigma, \pi \in S_n$.

When A is a Hopf algebra, $A \wr S_n$ can be endowed with Hopf structure built from the Hopf structure of A and $\mathbb{K}S_n$. The coproduct for $\mathbb{K}S_n$ and A are as follows:

$$\Delta_{S_n} : S_n \to S_n \otimes S_n$$
$$\Delta_{S_n}(\sigma) = \sigma \otimes \sigma \ \ \forall\, \sigma \in S_n$$

and

$$\Delta_A : A \to A \otimes A$$
$$\Delta_A(a) = \sum_{(a)} a_1 \otimes a_2 \ \ \forall\, a \in A.$$

[1]Many of our definitions and constructions only require A to be an algebra, coalgebra, or bialgebra. For clarity's sake, we choose that our algebra A, is a Hopf algebra. The first time we rely on the fact that A needs to be a Hopf algebra happens in Section 3.3. We need the Hopf structure, specifically relationships between the antipode and counit, to prove Lemma 3.3.1.

The coproduct for $A \wr S_n$ is

$$\Delta : A \wr S_n \to A \wr S_n \otimes A \wr S_n$$
$$a_1 \otimes \cdots \otimes a_n \otimes \sigma \mapsto \sum_{(a_i)} [(a_1)_1 \otimes \cdots \otimes (a_n)_1 \otimes \sigma] \otimes [(a_1)_2 \otimes \cdots \otimes (a_n)_2 \cdot \sigma].$$

The counit of $A \wr S_n$ is give by

$$\varepsilon : A \wr S_n \to \mathbb{K}$$
$$\varepsilon(a_1 \otimes \cdots \otimes a_n \otimes \sigma) = \varepsilon_A(a_1) \cdots \varepsilon_A(a_n) \varepsilon_{S_n}(\sigma) = \varepsilon_A(a_1) \cdots \varepsilon_A(a_n),$$

where $\varepsilon_A : A \to \mathbb{K}$ and $\varepsilon_{S_n} : \mathbb{K}S_n \to \mathbb{K}$ denote the counits for A and $\mathbb{K}S_n$ respectively.

To condense notation, from now on write $a_1 \cdots a_n \otimes \sigma := a_1 \otimes \cdots \otimes a_n \otimes \sigma \in A \wr S_n$.

We also use the following for brevity: for our fixed basis $B = \{b_t \mid t \in T\}$ for A and for $\underline{t} = (t_1, ..., t_n) \in T^n$, let $b_{\underline{t}} = b_{t_1} \otimes \cdots \otimes b_{t_n}$. A basis for $A \wr S_n$ is given by

$$\{b_{\underline{t}} \otimes \sigma \mid \forall\, \sigma \in S_n, \underline{t} \in T^n\}.$$

For example, $b_{(1,2,1)} \otimes (132)$ would correspond to the basis element $b_1 \otimes b_2 \otimes b_1 \otimes (132)$ in $A^{\otimes 3} \otimes S_3$.
Let $c_{\underline{i},\underline{j}}^{\underline{k}}$ be the structure constant given by the following product of generators:

$$(b_{\underline{i}} \otimes \mathrm{id})(b_{\underline{j}} \otimes \mathrm{id}) = \sum c_{\underline{i},\underline{j}}^{\underline{k}} (b_{\underline{k}} \otimes \mathrm{id}). \tag{28}$$

On the other hand, we have:

$$\begin{aligned}
(b_{\underline{i}} \otimes \mathrm{id})(b_{\underline{j}} \otimes \mathrm{id}) &= (b_{i_1} \otimes \cdots b_{i_n} \otimes \mathrm{id})(b_{j_1} \otimes \cdots b_{j_n} \otimes \mathrm{id}) \\
&= b_{i_1} b_{j_1} \otimes \cdots \otimes b_{i_n} b_{j_n} \otimes \mathrm{id} \\
&= (\sum_{k \in T} c_{i_1,j_1}^{k} b_k) \otimes \cdots \otimes (\sum_{k \in T} c_{i_n,j_n}^{k} b_k) \otimes \mathrm{id} \\
&= \sum_{t=1}^{n} \sum_{k_t \in T} c_{i_1,j_1}^{k_1} c_{i_2,j_2}^{k_2} \cdots c_{i_n,j_n}^{k_n} (b_{k_1} \otimes b_{k_2} \otimes \cdots \otimes b_{k_n} \otimes \mathrm{id}).
\end{aligned}$$

Thus we have that the structure constants are given by $c_{\underline{i},\underline{j}}^{\underline{k}} = c_{i_1,j_1}^{k_1} c_{i_2,j_2}^{k_2} \cdots c_{i_n,j_n}^{k_n}$.

Remark 7.1.1. Observe that when $A = \mathbb{K}G$ and we choose $B = G$, we have that the product of two basis elements is yet again a basis element, i.e., $b_i b_j = b_k$ for some k which implies $c_{i,j}^{k} = 1$ and $c_{i,j}^{\ell} = 0$ for all $\ell \neq k$. Hence in the product $(b_{\underline{i}} \otimes \mathrm{id})(b_{\underline{j}} \otimes \mathrm{id})$, we have that $c_{\underline{i},\underline{j}}^{\underline{k}} = 0$ for all $\underline{k} \in T^n$ except for one $\underline{k}' \in T^n$ which corresponds to $c_{\underline{i},\underline{j}}^{\underline{k}'} = 1$. Thus $(b_{\underline{i}} \otimes \mathrm{id})(b_{\underline{j}} \otimes \mathrm{id}) = b_{\underline{k}'} \otimes \mathrm{id}$.

When performing calculations, we often work with the following generating set for $A \wr S_n$:

$$\{b_{t_1} \otimes \cdots \otimes b_{t_n} \otimes \mathrm{id}, 1_A \otimes \cdots \otimes 1_A \otimes \sigma \mid \forall \sigma \in S_n, t_r \in T\}.$$

In our shorthand notations, this looks like:

$$\{b_{t_1} \cdots b_{t_n} \otimes \mathrm{id}, 1_A \cdots 1_A \otimes \sigma \mid \forall \sigma \in S_n, t_r \in T\}$$

and

$$\{b_{\underline{t}} \otimes \mathrm{id}, 1_A \otimes \cdots \otimes 1_A \otimes \sigma \mid \forall \sigma \in S_n, t_r \in T\}.$$

From here on out, we will drop the A in 1_A when it's clear by context.

7.1.3. The Category $\mathbf{Set}^A$

We start by defining the category of A-sets, this will be the category that our notion of A-species is built from. We set the notation that given a set X, $\mathbb{K}[X]$ denotes the $\mathbb{K}$-vector space with basis X.

DEFINITION 7.1.2. Let $\mathbb{K}$ be an arbitrary field and A be a $\mathbb{K}$-algebra. $\mathbf{Set}^A$ is the category that consists of the following data:

- Objects: $I_A := A^{\otimes I} \otimes \mathbb{K}[I] \ \ \forall\, I \in \mathbf{Set}^{\times}$ where $\mathbb{K}[I]$ denotes the linearization of I.
- Morphisms: Let $\Gamma := \{f : I \to J \mid f \text{ is a bijection}\}$. For all $I_A, J_A \in \mathbf{Set}^A$,

$$\mathrm{Hom}(I_A, J_A) = \begin{cases} \left(\bigotimes\limits_{j\in J} A_j\right) \otimes \mathbb{K}[\Gamma] & \text{if } |I| = |J| \\ 0 & \text{otherwise} \end{cases},$$

where $A_j = A$ for all j and composition as defined in (29). Given a basis element $\bigotimes\limits_{j\in J} b_{t_j} \otimes f \in \mathrm{Hom}(I_A, J_A)$, we define it to be the following linear map:

$$\bigotimes_{j\in J} b_{t_j} \otimes f : I_A \to J_A$$

$$b_{r_{i_1}} \otimes \cdots b_{r_{i_n}} \otimes v \mapsto c_{j_1} \otimes \cdots \otimes c_{j_n} \otimes f(v)$$

where $c_{j_k} := b_{t_{j_k}} b_{r_{f^{-1}(i_k)}}$ for all $k \in [n]$.

REMARK 7.1.3. We let $A^{\otimes I}$ denote the $|I|$-fold tensor product of A with tensors indexed by I. When necessary for calculations, we choose an ordering on I, and use this ordering to denote the positions of A, i.e.,

$$A^{\otimes I} = A_{i_1} \otimes A_{i_2} \otimes \cdots \otimes A_{i_{|I|}}$$

Now, we show that the above data defines a category:

PROOF. The objects are defined as above.
For any $I_A, J_A, K_A \in \mathbf{Set}^A$ and $\hat{f} := \bigotimes\limits_{k\in K} b_k \otimes f \in \mathrm{Hom}(J_A, K_A)$ and $\hat{g} := \bigotimes\limits_{j\in J} b_j \otimes g \in \mathrm{Hom}(I_A, J_A)$, we have a function:

$$\circ : \mathrm{Hom}(J_A, K_A) \times \mathrm{Hom}(I_A, J_A) \to \mathrm{Hom}(I_A, K_A)$$

$$(\hat{f}, \hat{g}) \mapsto \hat{f} \circ \hat{g}$$

With composition defined as:

$$\hat{f} \circ \hat{g} := \Big(\bigotimes_{k\in K} b_k \otimes f\Big) \circ \Big(\bigotimes_{j\in J} b_j \otimes g\Big) = \bigotimes_{\substack{j\in J \\ f(j)=k}} b_k b_j \otimes f \circ g = \bigotimes_{k\in K} b_k b_{f^{-1}(k)} \otimes f \circ g \tag{29}$$

and extend by linearity to general morphisms.

We must show that $\circ$ is associative and that $\hat{f} \circ \mathrm{id}_{I_A} = \hat{f} = \mathrm{id}_{I_A} \circ \hat{f}$.

1. $\circ$ is associative.
   Let $\hat{f} \in \mathrm{Hom}(K_A, T_A)$, $\hat{g} \in \mathrm{Hom}(J_A, K_A)$, and $\hat{h} \in \mathrm{Hom}(I_A, J_A)$, then:

$$\begin{aligned}
\hat{f} \circ (\hat{g} \circ \hat{h}) &= \left(\bigotimes_{t \in T} c_t \otimes f\right) \circ \left(\bigotimes_{k \in K} b_k a_{g^{-1}(k)} \otimes g \circ h\right) \\
&= \bigotimes_{t \in T} c_t b_{f^{-1}(t)} a_{g^{-1}(f^{-1}(t))} \otimes f \circ (g \circ h) \\
&= \bigotimes_{t \in T} c_t b_{f^{-1}(t)} a_{(f \circ g)^{-1}(t)} \otimes (f \circ g) \circ h \\
&= \left(\bigotimes_{t \in T} c_t b_{f^{-1}(t)} \otimes f \circ g\right) \circ \left(\bigotimes_{k \in K} b_k \otimes h\right) \\
&= (\hat{f} \circ \hat{g}) \circ \hat{h}
\end{aligned}$$

2. $f \circ \mathrm{id}_{I_A} = f$
   Let $\hat{f} \in \mathrm{End}(I_A)$, i.e., $\hat{f} = \bigotimes_{i \in I} a_i \otimes f$. Note $\mathrm{id}_{I_A} := \bigotimes_{i \in I} 1_{A_i} \otimes \mathrm{id}$. Then,

$$\begin{aligned}
f \circ \mathrm{id}_{I_A} &= \left(\bigotimes_{i \in I} a_i \otimes f\right)\left(\bigotimes_{i \in I} 1_{A_i} \otimes \mathrm{id}\right) \\
&= \bigotimes_{i \in I} a_i 1_{A_{f^{-1}(i)}} \otimes f \circ \mathrm{id} \\
&= \bigotimes_{i \in I} a_i \otimes f \\
&= \hat{f}.
\end{aligned}$$

   Similarly for $\mathrm{id}_{I_A} \circ \hat{f}$.

Therefore, $\mathbf{Set}^A$ defines a category.

□

When we restrict to objects of the form $[n]_A = A^{\otimes n} \otimes \mathbb{K}[n]$, these form a skeleton for $\mathbf{Set}^A$. We define $\widetilde{\mathbf{Set}}^A$ to be as follows:

Definition 7.1.4. $\widetilde{\mathbf{Set}}^A$ is the category that consists of the following data:

- Objects: $[n]_A = A^{\otimes n} \otimes \mathbb{K}[n] \;\; \forall n$
- Morphisms: $\mathrm{End}([n]_A) = A \wr S_n$

Proposition 7.1.5. *$\widetilde{\mathbf{Set}}^A$ is a skeleton of $\mathbf{Set}^A$.*

Proof. We must show that $\widetilde{\mathbf{Set}}^A$ is a full subcategory of $\mathbf{Set}^A$ that is skeletal. Clearly, $\widetilde{\mathbf{Set}}^A$ is a subcategory of $\mathbf{Set}^A$: every object in $\widetilde{\mathbf{Set}}^A$ is an object in $\mathbf{Set}^A$, and for every pair of objects $[n]_A, [r]_A \in \widetilde{\mathbf{Set}}^A$ we have:

$$\mathrm{Hom}_{\widetilde{\mathbf{Set}}^A}([n]_A, [r]_A) \subseteq \mathrm{Hom}_{\mathbf{Set}^A}([n]_A, [r]_A).$$

In fact, the reverse inclusion holds by definition of $\mathrm{Hom}_{\widetilde{\mathbf{Set}}^A}([n]_A, [r]_A)$, thus the inclusion functor is full. Now to show that the inclusion functor is essentially surjective. Consider the object $I_A \in \mathbf{Set}^A$ where $|I| = n$ for some n, then there is an isomorphism from $I \to [n]$. We can use this isomorphism to show

$$I_A \cong [n]_A = \iota([n]_A).$$

Finally, to show that no two objects of $\widetilde{\mathbf{Set}}^A$ are isomorphic. It's easy to see that each object in $\widetilde{Set}^A$ is distinct, for if $[n]_A$ and $[r]_A$ were isomorphic that would mean $n = r$ contradicting their distinctness.

$\square$

We will occasionally work with $\widetilde{\mathbf{Set}}^A$ instead of $\mathbf{Set}^A$ since they are equivalent categories [23].

Let $I_A, J_A \in \mathbf{Set}^A$ such that $|I| = |J$. We choose an order on the underlying sets I and J. We wish to consider two order preserving bijections between them, the *standardization map*, *st*, and the *canonical map*, cano. These are as follows:

$$\mathrm{st}_I : I_A \to [|I|]_A$$

and

$$\mathrm{cano}_J : I_A \to J_A.$$

On the tensor product of our algebra, both of these maps essentially act as the identity while only renaming the indices with elements in $[|I|]$ and J respectively.

Remark 7.1.6.

1. We will let cano_t denote the map that shifts the underlying set by t:
$$\mathrm{cano}_t : [n]_A \to [1+t, n+t]_A$$
2. If $I = [n]$, then $\mathrm{st}_I = \mathrm{id}_{[n]_A}$.

Before defining our notion of an A-species and operations that go along with its structure, we need to define what a decomposition of objects in $\mathbf{Set}^A$ looks like:

Definition 7.1.7. We say S_A is an A-*subset* of I_A if $S \subseteq I$ and the order of S is inherited from I.

Definition 7.1.8. An A-*decomposition* of $I_A \in \mathbf{Set}^A$ is an ordered sequence of disjoint A-subsets $(F_i)_A := (A^{\otimes F_i} \otimes \mathbb{K}[F_i])_{i=1}^{\ell}$ such that the underlying sets are a decomposition of I. We often write $\bigsqcup_{i=1}^{\ell} F_i = I$ to denote the A-decomposition $\bigsqcup_{i=1}^{\ell} (F_i)_A := \bigsqcup_{i=1}^{\ell} A^{\otimes |F_i|} \otimes \mathbb{K}[F_i] = I_A$ when it's clear by context.

7.1.4. A-Species

Here, we define the notion of an A-species and show that these correspond to a family of modules for $A \wr S_n$ for $n \geq 0$.

Definition 7.1.9. An A-*species* is a functor

$$\mathbf{p} : \mathbf{Set}^A \to \mathbf{Vec}_{\mathbb{K}}.$$

Specifically, an A-species consists of a family of vector spaces $\mathbf{p}[I_A]$, one for each $I_A \in \mathbf{Set}^A$, together with linear maps $\mathbf{p}[\hat{f}] : \mathbf{p}[I_A] \to \mathbf{p}[J_A]$, one for each morphism $\hat{f} : I_A \to J_A$, satisfying

$$\mathbf{p}[\mathrm{id}_{I_A}] = \mathrm{id}_{\mathbf{p}[I_A]} \quad \textit{and} \quad \mathbf{p}[\hat{f} \circ \hat{g}] = \mathbf{p}[\hat{f}] \circ \mathbf{p}[\hat{g}]$$

whenever $\hat{f}$ and $\hat{g}$ are composable bijections.

Definition 7.1.10. We say an A-species is *finite-dimensional* if each vector space $\mathbf{p}[I_A]$ is of finite dimension.

Definition 7.1.11. We say an A-species is *connected* if $\mathbf{p}[\emptyset] = \mathbb{K}$.

Definition 7.1.12. Let $\mathbf{p}$ and $\mathbf{q}$ be two A-species. An *A-species morphism*, $\alpha : \mathbf{p} \to \mathbf{q}$, is a natural transformation, i.e., a family of maps $\alpha_{I_A} : \mathbf{p}[I_A] \to \mathbf{q}[I_A]$, one for each $I_A \in \mathbf{Set}^A$ such that for each bijection $\hat{f} : I_A \to J_A$, the following diagram commutes:

$$\begin{tikzcd} \mathbf{p}[I_A] \arrow[r, "\alpha_{I_A}"] \arrow[d, "\mathbf{p}[\hat{f}]"'] & \mathbf{q}[I_A] \arrow[d, "\mathbf{q}[\hat{f}]"] \\ \mathbf{p}[J_A] \arrow[r, "\alpha_{J_A}"'] & \mathbf{q}[J_A]. \end{tikzcd}$$

Proposition 7.1.13. *An A-species $\mathbf{p}$ defines a sequence of $A \wr S_n$ modules.*

Proof. It suffices to use objects from the skeleton of the category $\mathbf{Set}^A$. First, let $\mathbf{p}$ be an A-species. Define

$$. : A \wr S_n \times \mathbf{p}[n_A] \to \mathbf{p}[n_A]$$
$$(a_1 \cdots a_n \otimes \sigma).v = \mathbf{p}[(a_1 \cdots a_n \otimes \sigma)](v)$$

for any pure tensor $(a_1 \cdots a_n \otimes \sigma) \in A \wr S_n$ and any $v \in p[n_A]$. We wish to show this defines a left action. Recall, for shorthand we use $a_1 \cdots a_n \otimes \sigma$. Since $\mathbf{p}$ is a functor, we have for all $(a_1 \cdots a_n \otimes \sigma), (c_1 \cdots c_n \otimes \tau) \in A \wr S_n$ and $v, w \in p[n_A]$,

1.
$$\begin{aligned} (a_{\underline{t}} \otimes \sigma).(v + w) &= \mathbf{p}[(a_1 \cdots a_n \otimes \sigma)](v + w) \\ &= \mathbf{p}[(a_1 \cdots a_n \otimes \sigma)](v) + \mathbf{p}[(a_1 \cdots a_n \otimes \sigma)](w) \\ &= (a_1 \cdots a_n \otimes \sigma).v + (a_1 \cdots a_n \otimes \sigma).w \end{aligned}$$

2.
$$\begin{aligned} [(a_1 \cdots a_n \otimes \sigma) + (c_1 \cdots c_n \otimes \tau)].v &= \mathbf{p}[(a_1 \cdots a_n \otimes \sigma) + (c_1 \cdots c_n \otimes \tau)](v) \\ &= \mathbf{p}[(a_1 \cdots a_n \otimes \sigma)](v) + \mathbf{p}[(c_1 \cdots c_n \otimes \tau)](v) \\ &= (a_1 \cdots a_n \otimes \sigma).v + (c_1 \cdots c_n \otimes \tau).v \end{aligned}$$

3.
$$\begin{aligned} (a_1 \cdots a_n \otimes \sigma).((c_1 \cdots c_n \otimes \tau).v) &= (a_1 \cdots a_n \otimes \sigma).(\mathbf{p}[(c_1 \cdots c_n \otimes \tau)](v)) \\ &= \mathbf{p}[(a_1 \cdots a_n \otimes \sigma)](\mathbf{p}[(c_1 \cdots c_n \otimes \tau)](v)) \\ &= \mathbf{p}[(a_1 \cdots a_n \otimes \sigma)(c_1 \cdots c_n \otimes \tau)](v) \\ &= ((a_1 \cdots a_n \otimes \sigma)(c_1 \cdots c_n \otimes \tau)).v \end{aligned}$$

4.
$$(1_A \cdots 1_A \otimes \mathrm{id}).v = \mathbf{p}[(1_A \cdots 1_A \otimes \mathrm{id})](v) = v.$$

Therefore, $\mathbf{p}[n_A]$ is a left $A \wr S_n$-module.

Now, given a sequence of $A \wr S_n$-modules, say

$$V_0,\ V_1,\ V_2, V_3 ...$$

Define a functor $\mathbf{p}$ via

$$\mathbf{p}[n_A] := V_n$$

$$\mathbf{p}[(a_1 \cdots a_n \otimes \sigma)](v) = (a_1 \cdots a_n \otimes \sigma).v$$

This defines a functor since for $(a_1 \cdots a_n \otimes \sigma), (c_1 \cdots c_n \otimes \tau) \in A \wr S_n$ and $v \in V_n$ we have:

$$\begin{aligned} \mathbf{p}[(a_1 \cdots a_n \otimes \sigma)(c_1 \cdots c_n \otimes \tau)](v) &= ((a_1 \cdots a_n \otimes \sigma)(c_1 \cdots c_n \otimes \tau)).v \\ &= (a_1 \cdots a_n \otimes \sigma).(c_1 \cdots c_n \otimes \tau).v \\ &= \mathbf{p}[(a_1 \cdots a_n \otimes \sigma)](\mathbf{p}[(c_1 \cdots c_n \otimes \tau)](v)) \end{aligned}$$

and

$$\mathbf{p}[(1_A \cdots 1_A \otimes \mathrm{id})](v) = (1_A \cdots 1_A \otimes \mathrm{id}).v = v = \mathrm{id}_{V_n}(v)$$

Therefore an A-species is equivalent to having a sequence of $A \wr S_n$-modules.

□

REMARK 7.1.14. Morphisms of A-species are morphisms of $A \wr S_n$-modules. Let $\alpha : \mathbf{p} \to \mathbf{q}$ be a morphism of A-species, $(c_1 \cdots c_n \otimes \sigma) \in A \wr S_n$, and $x \in \mathbf{p}[n_A]$. We have that $A \wr S_n$ acts on $\mathbf{p}[n_A]$ via the functor $\mathbf{p}$ and since α is a natural transformation, we further have

$$\alpha_{[n_A]}((c_1 \cdots c_n \otimes \sigma).x) =$$

$$\alpha_{[n_A]}(\mathbf{p}[(c_1 \cdots c_n \otimes \sigma)](x)) = \mathbf{q}[(c_1 \cdots c_n \otimes \sigma)](\alpha_{[n_A]}(x)) = (c_1 \cdots c_n \otimes \sigma).\alpha_{[n_A]}(x).$$

The following lemma shows that any action coming from I_A can be transformed into an action of J_A.

LEMMA 7.1.15. $\forall I_A, J_A \in \mathbf{Set}^A$ *s.t* $|I| = |J|$, $\mathrm{End}(I_A) \cong \mathrm{End}(J_A)$ *as algebras.*

PROOF. Let $\hat{f} \in \mathrm{Hom}(I_A, J_A)$. It suffices to assume that $\hat{f}$ has form $\hat{f} = \bigotimes_{j \in J} a_j \otimes f$ where $f : I \to J$ is a bijection. Define mutually inverse maps:

$$\varphi : \mathrm{End}(I_A) \to \mathrm{End}(J_A)$$

$$\bigotimes_{i \in I} a_i \otimes \sigma \mapsto \bigotimes_{j \in J} a_{f^{-1}(j)} \otimes f \circ \sigma \circ f^{-1}$$

and

$$\rho : \mathrm{End}(J_A) \to \mathrm{End}(I_A)$$

$$\bigotimes_{j \in J} a_j \otimes \sigma \mapsto \bigotimes_{i \in I} a_{f(i)} \otimes f^{-1} \circ \sigma \circ f$$

Will show that φ is in fact an algebra morphism (to show ρ is an algebra morphism is similar), and that they are inverses to each other.

First to show φ is an algebra morphism.

$$\begin{aligned}
\varphi((\bigotimes_{i\in I} a_i \otimes \sigma)(\bigotimes_{i\in I} c_i \otimes \tau)) &= \varphi(\bigotimes_{i\in I} a_i c_{\sigma^{-1}(i)} \otimes \sigma \circ \tau) \\
&= \bigotimes_{j\in J} a_{f^{-1}(j)} c_{\sigma^{-1}(f^{-1}(j))} \otimes f \circ \sigma \circ \tau \circ f^{-1} \\
&= \bigotimes_{j\in J} a_{f^{-1}(j)} c_{(f^{-1}\circ f)\circ(\sigma^{-1}(f^{-1}(j)))} \otimes f \circ \sigma \circ (f^{-1} \circ f) \circ \tau \circ f^{-1} \\
&= \bigotimes_{j\in J} a_{f^{-1}(j)} c_{f^{-1}\circ(f\circ\sigma^{-1}\circ f^{-1})^{-1}(j)} \otimes f \circ \sigma \circ f^{-1} \circ f \circ \tau \circ f^{-1} \\
&= (\bigotimes_{j\in J} a_{f^{-1}(j)} \otimes f \circ \sigma \circ f^{-1})(\bigotimes_{j\in J} c_{f^{-1}(j)} \otimes f \circ \tau \circ f^{-1}) \\
&= \varphi(\bigotimes_{i\in I} a_i \otimes \sigma)\varphi(\bigotimes_{i\in I} c_i \otimes \tau)
\end{aligned}$$

$$\varphi(\bigotimes_{i\in I} 1_i \otimes \mathrm{id}_I) = \bigotimes_{j\in J} 1_{f^{-1}(j)} \otimes f \circ \mathrm{id}_I \circ f^{-1} = \bigotimes_{j\in J} 1_{f^{-1}(j)} \otimes \mathrm{id}_J$$

Showing that ρ is an algebra morphism is a similar calculation as above.

Finally, to show that φ and ρ are mutually inverse to each other.

$$\rho(\varphi(\bigotimes_{i\in I} a_i \otimes \sigma)) = \rho(\bigotimes_{j\in J} a_{f^{-1}(j)} \otimes f \circ \sigma \circ f^{-1}) = \bigotimes_{i\in I} a_{f^{-1}(f(i))} \otimes f^{-1} \circ f \circ \sigma \circ f^{-1} \circ f = \bigotimes_{i\in I} a_i \otimes \sigma$$

$$\varphi(\rho(\bigotimes_{j\in J} a_j \otimes \tau)) = \varphi(\bigotimes_{i\in I} a_{f(i)} \otimes f^{-1} \circ \tau \circ f) = \bigotimes_{j\in J} a_{f(f^{-1}(j))} \otimes f \circ f^{-1} \circ \tau \circ f \circ f^{-1} = \bigotimes_{j\in J} a_j \otimes \tau$$

Therefore, $\mathrm{End}(I_A) \cong \mathrm{End}(J_A)$ as algebras.

□

Corollary 7.1.16. *$\mathrm{End}(I_A) \cong A \wr S_n$ where $|I| = n$.*

Proof. Take $J_A := [n]_A$.

□

Proposition 7.1.17. *For all $I_A,\ J_A \in \mathbf{Set}^A$ such that $|I| = |J|$,*

$$\mathbf{p}[I_A] \cong \mathbf{p}[J_A]$$

as $A \wr S_n$-modules.

Proof. From Proposition 7.1.13, we have that $\mathbf{p}[I_A]$ is a left $A \wr S_n$-module for all I_A. Consider the morphism $\hat{f} = \bigotimes_{j\in J} 1_J \otimes f \in \mathrm{Hom}(I_A, J_A)$, it's easy to see this is a bijection between I_A and J_A so there is a $\hat{f}^{-1} \in \mathrm{Hom}(J_A, I_A)$. We have that $\hat{f}$ induces a linear map

$$\mathbf{p}[I_A] \xrightarrow{\mathbf{p}[\hat{f}]} \mathbf{p}[J_A].$$

It's clear that $\mathbf{p}[\hat{f}]$ is an isomorphism of vector spaces with $\mathbf{p}[\hat{f}]^{-1} = \mathbf{p}[\hat{f}^{-1}]$ (this follows from functoriality of $\mathbf{p}$, $\mathbf{p}[g] \circ \mathbf{p}[f] = \mathbf{p}[\mathrm{id}] \iff \mathbf{p}[g \circ f] = \mathbf{p}[\mathrm{id}] \iff g \circ f = \mathrm{id}$). Thus $\mathbf{p}[I_A] \cong \mathbf{p}[J_A]$ as vector spaces.
Now to show isomorphic as $A \wr S_n$-modules. In order to do so, we must show the following diagram commutes:

$$
\begin{tikzcd}
\mathbf{p}[I_A] \arrow[r, "\bigotimes\limits_{j\in J} 1_J \otimes f"] \arrow[d, "\mathbf{p}[\bigotimes\limits_{i\in I} a_i \otimes \sigma]"'] & \mathbf{p}[J_A] \arrow[d, "\mathbf{p}[\bigotimes\limits_{j\in J} a_{f^{-1}(j)} \otimes f\circ\sigma\circ f^{-1}]"] \\
\mathbf{p}[I_A] \arrow[r, "\bigotimes\limits_{j\in J} 1_J \otimes f"'] & \mathbf{p}[J_A].
\end{tikzcd}
$$

From functoriality of $\mathbf{p}$, we only need to check what happens at the set level. Following the top right corner yields:

$$(\bigotimes_{j\in J} a_{f^{-1}(j)} \otimes f\circ\sigma\circ f^{-1})(\bigotimes_{j\in J} 1_J \otimes f) = \bigotimes_{j\in J} a_{f^{-1}(j)} 1_{f^{-1}\circ(f\circ\sigma\circ f^{-1})^{-1}(j)} \otimes f\circ\sigma = \bigotimes_{j\in J} a_{f^{-1}(j)} \otimes f\circ\sigma.$$

Following the bottom left corner yields:

$$(\bigotimes_{j\in J} 1_j \otimes f)(\bigotimes_{i\in I} a_i \otimes \sigma) = \bigotimes_{j\in J} 1_j a_{f^{-1}(j)} \otimes f\circ\sigma = \bigotimes_{j\in J} a_{f^{-1}(j)} \otimes f\circ\sigma.$$

Thus the diagram commutes. Therefore $\mathbf{p}[I_A] \cong \mathbf{p}[J_A]$ as $A \wr S_n$-modules.

□

We can also consider the category of all A-species, denoted by $\mathbf{Sp}^A$.

Here we give some examples of A-species that will be of particular interest to us in the following sections.

Example 7.1.18. **Exponential A-Species**

On objects $I_A \in \mathbf{Set}^A$,

$$\mathbf{E}_A[I_A] := \mathbb{K}.$$

On morphisms,

$$\mathbf{E}_A[f] := \varepsilon(f),$$

i.e., scalar multiplication by $\varepsilon(f)$ where ε is the counit of the wreath product algebra, $A \wr S_I$. This will correspond to the trivial representation of $A \wr S_n$. When $A = \mathbb{K}$, we have that $\mathbf{E}_A[f] := \mathrm{id}_{\mathbb{K}}$.

Example 7.1.19. **Linear Order A-Species**

On objects:

$$\mathbf{L}_A[I_A] := \bigoplus_{s:I\to B\times I} \mathbf{L}[s(I)],$$

where the direct sum is over all section maps, $s : I \to B \times I$ such that $s(i) \in B \times [i]$.
On morphisms:

$$\mathbf{L}_A[(b_{i_1}\cdots b_{i_n} \otimes \sigma)] := \bigoplus_{s:[n]\to B\times[n]} \mathbf{L}[(b_{i_1}\cdots b_{i_n} \otimes \sigma)|_{s([n])}].$$

Please see Example 11.1.1, for a detailed example of this functor applied to morphisms.

EXAMPLE 7.1.20. **Set Partition A-Species**

On objects:

$$\mathbf{\Pi}_A[I_A] := \bigoplus_{s:I\to B\times I} \mathbf{\Pi}[s(I)],$$

where the direct sum is over all section maps, $s : I \to B \times I$ such that $s(i) \in B \times [i]$.
On morphisms:

$$\mathbf{\Pi}_A[(b_{i_1} \cdots b_{i_n} \otimes \sigma)] := \bigoplus_{s:[n]\to B\times[n]} \mathbf{\Pi}[(b_{i_1} \cdots b_{i_n} \otimes \sigma)|_{s([n])}].$$

Please see Subsection 11.2 for a detailed description of $\mathbf{\Pi}_A$.

7.2. The Cauchy Product

DEFINITION 7.2.1. Let $\mathbf{p}$ and $\mathbf{q}$ be A-species. The *Cauchy product*, $\cdot : \mathbf{Sp}^A \times \mathbf{Sp}^A \to \mathbf{Sp}^A$, is the species defined by

$$(\mathbf{p} \cdot \mathbf{q})[I_A] := \bigoplus_{S\sqcup T=I} \mathbf{p}[S_A] \otimes \mathbf{q}[T_A]$$

The direct sum is over all decompositions of the underlying set I.
On a bijection $\hat{f} : I_A \to J_A$, the map $(\mathbf{p} \cdot \mathbf{q})$ is defined to be the direct sum of the maps

$$\mathbf{p}[S_A] \otimes \mathbf{q}[T_A] \xrightarrow{\mathbf{p}[\hat{f}|_S]\otimes\mathbf{q}[\hat{f}|_T]} \mathbf{p}[\hat{f}(S_A)] \otimes \mathbf{q}[\hat{f}(T_A)]$$

over all decompositions of the set I.

PROPOSITION 7.2.2. *$(\boldsymbol{Sp}^A, \cdot, \mathbf{1}_{\mathbb{K}}, \alpha, \lambda, \rho, \beta)$ is a symmetric monoidal category with the braiding $\beta_{\mathbf{p},\mathbf{q}} : \mathbf{p} \cdot \mathbf{q} \to \mathbf{q} \cdot \mathbf{p}$ given by*

$$\mathbf{p}[S_A] \otimes \mathbf{q}[T_A] \to \mathbf{q}[T_A] \otimes \mathbf{p}[S_A]$$

$$x \otimes y \mapsto y \otimes x.$$

PROOF. We first need to define a natural isomorphism $\alpha : (\_ \cdot \_) \cdot \_ \to \_ \cdot (\_ \cdot \_)$ where we view $(\_ \cdot \_) \cdot \_$ and $\_ \cdot (\_ \cdot \_)$ as functors from $\mathbf{Sp}^A \times \mathbf{Sp}^A \times \mathbf{Sp}^A \to \mathbf{Sp}^A$. The section maps of α will be defined as follows. Let $\mathbf{p}, \mathbf{q}, \mathbf{h}$ be A-species, then α must have section maps $\alpha_{\mathbf{p},\mathbf{q},\mathbf{h}}$ that are natural transformations. We must show that $\alpha_{\mathbf{p},\mathbf{q},\mathbf{h}} : (\mathbf{p} \cdot \mathbf{q}) \cdot \mathbf{h} \to \mathbf{p} \cdot (\mathbf{q} \cdot \mathbf{h})$ is a natural transformation. Observe that for all $I_A \in \mathbf{Set}^A$, we have that
$(\mathbf{p} \cdot \mathbf{q}) \cdot \mathbf{h}[I_A]$

$$\begin{aligned}
&= \bigoplus_{S\sqcup T=I} (\mathbf{p} \cdot \mathbf{q})[S_A] \otimes \mathbf{h}[T_A] \\
&= \bigoplus_{\substack{S \sqcup T = I \\ R \sqcup K = S}} (\mathbf{p}[R_A] \otimes \mathbf{q}[K_A]) \otimes \mathbf{h}[T_A] \\
&\cong \bigoplus_{R\sqcup K\sqcup T=I} \mathbf{p}[R_A] \otimes (\mathbf{q}[K_A] \otimes \mathbf{h}[T_A]) \\
&= \bigoplus_{R\sqcup U=I} \mathbf{p}[R_A] \otimes (\mathbf{q} \cdot \mathbf{h})[U_A] \\
&= \mathbf{p} \cdot (\mathbf{q} \cdot \mathbf{h})[I_A]
\end{aligned}$$

The isomorphism holds from the fact that the tensor product is associative on vector spaces. The above gives the isomorphism of the section maps of $\alpha_{\mathbf{p},\mathbf{q},\mathbf{h}}$:

$$\alpha_{\mathbf{p},\mathbf{q},\mathbf{h}}[I_A] : (\mathbf{p}\cdot\mathbf{q})\cdot\mathbf{h}[I_A] \to \mathbf{p}\cdot(\mathbf{q}\cdot\mathbf{h})[I_A]$$

which is the associator from the category of **Vec**, i.e., $(x\otimes y)\otimes z \mapsto x\otimes(y\otimes z)$. Now to show that for all $f : I_A \to J_A$, the following diagram commutes:

$$\begin{tikzcd}
(\mathbf{p}\cdot\mathbf{q})\cdot\mathbf{h}[I_A] \arrow[r, "(\mathbf{p}\cdot\mathbf{q})\cdot\mathbf{h}(f)"] \arrow[d, "\alpha_{\mathbf{p},\mathbf{q},\mathbf{h}}"'] & (\mathbf{p}\cdot\mathbf{q})\cdot\mathbf{h}[J_A] \arrow[d, "\alpha_{\mathbf{p},\mathbf{q},\mathbf{h}}"] \\
\mathbf{p}\cdot(\mathbf{q}\cdot\mathbf{h})[I_A] \arrow[r, "\mathbf{p}\cdot(\mathbf{q}\cdot\mathbf{h})(f)"'] & \mathbf{p}\cdot(\mathbf{q}\cdot\mathbf{h})[J_A]
\end{tikzcd}$$

Fix $R\sqcup K\sqcup T = I$, and let $(x\otimes y)\otimes z \in (\mathbf{p}[R_A]\otimes\mathbf{q}[K_A])\otimes\mathbf{h}[T_A]$. Then

$$\begin{aligned}
\mathbf{p}\cdot(\mathbf{q}\cdot\mathbf{h})(f)(\alpha_{\mathbf{p},\mathbf{q},\mathbf{h}}((x\otimes y)\otimes z)) &= \mathbf{p}\cdot(\mathbf{q}\cdot\mathbf{h})(f)(x\otimes(y\otimes z)) \\
&= \mathbf{p}[f|_R]x\otimes(\mathbf{q}[f|_K]y\otimes\mathbf{h}[f|_T]z) \\
&= \alpha_{\mathbf{p},\mathbf{q},\mathbf{h}}\left((\mathbf{p}[f|_R]x\otimes\mathbf{q}[f|_K]y)\otimes\mathbf{h}[f|_T]z\right) \\
&= \alpha_{\mathbf{p},\mathbf{q},\mathbf{h}}\circ(\mathbf{p}\cdot\mathbf{q})\cdot\mathbf{h}(x\otimes y\otimes z)
\end{aligned}$$

Thus each section map of α, $\alpha_{\mathbf{p},\mathbf{q},\mathbf{h}}$, is a natural transformation. Now that we have the section maps $\alpha_{\mathbf{p},\mathbf{q},\mathbf{h}}$ defined and are the appropriate maps, we need to show that α is a natural transformation.
Let $\beta_{\mathbf{p}} : \mathbf{p}\to\mathbf{p}'$,$\beta_{\mathbf{q}} : \mathbf{q}\to\mathbf{q}'$, and $\beta_{\mathbf{h}} : \mathbf{h}\to\mathbf{h}'$ be morphisms of A-species, i.e., they are natural transformations. We must show that the following diagram commutes:

$$\begin{tikzcd}
(\mathbf{p}\cdot\mathbf{q})\cdot\mathbf{h} \arrow[r, "(\beta_{\mathbf{p}}\cdot\beta_{\mathbf{q}})\cdot\beta_h"] \arrow[d, "\alpha_{\mathbf{p},\mathbf{q},\mathbf{h}}"'] & (\mathbf{p}'\cdot\mathbf{q}')\cdot\mathbf{h}' \arrow[d, "\alpha_{\mathbf{p}',\mathbf{q}',\mathbf{h}'}"] \\
\mathbf{p}\cdot(\mathbf{q}\cdot\mathbf{h}) \arrow[r, "\beta_{\mathbf{p}}\cdot(\beta_{\mathbf{q}}\cdot\beta_{\mathbf{h}})"'] & \mathbf{p}'\cdot(\mathbf{q}'\cdot\mathbf{h}').
\end{tikzcd}$$

On each object I_A, this diagram commutes since $\alpha_{\mathbf{p},\mathbf{q},\mathbf{h}}$ and β's are natural transformations.Therefore α is a natural transformation.

Now to define the left and right unitators. I will show all the details for the left unitator and the details for the right unitator are done in the same fashion.
We must define a natural isomorphism $\lambda : \mathbf{1}_{\mathbb{K}}\cdot\_\_ \to \mathrm{id}\_$ where we view $\mathbf{1}_{\mathbb{K}}\cdot\_\_$ and $\mathrm{id}\_$ as functors from $\mathbf{Sp}^A \to \mathbf{Sp}^A$. The sections of λ again need to be natural transformations, i.e., $\lambda_{\mathbf{p}} : \mathbf{1}_{\mathbb{K}}\cdot\mathbf{p}\to\mathrm{id}_{\mathbf{p}} = \mathbf{p}$ is a natural transformation. For all $I_A \in \mathbf{Set}^A$, we have that

$$(\mathbf{1}_{\mathbb{K}}\cdot\mathbf{p})[I_A] = \bigoplus_{S\sqcup T=I}\mathbf{1}_{\mathbb{K}}[S_A]\otimes\mathbf{p}[T_A] = \mathbb{K}\otimes\mathbf{p}[I_A]\cong\mathbf{p}[I_A]$$

thus we define the sections of $\lambda_{\mathbf{p}}$ by the vector space isomorphisms

$$\lambda_{\mathbf{p}[I_A]} : \mathbb{K}\otimes\mathbf{p}[I_A] \xrightarrow{\cong} \mathbf{p}[I_A]$$

$$c\otimes v\mapsto cv.$$

Now, let $f : I_A \to J_A$, we must show the following diagram commutes:

$$\begin{tikzcd}
(\mathbf{1}_{\mathbb{K}} \cdot \mathbf{p})[I_A] \arrow[r, "(\mathbf{1}_{\mathbb{K}} \cdot \mathbf{p})(f)"] \arrow[d, "\lambda_{\mathbf{p}[I_A]}"'] & (\mathbf{1}_{\mathbb{K}} \cdot \mathbf{p})[J_A] \arrow[d, "\lambda_{\mathbf{p}[J_A]}"] \\
\mathbf{p}[I_A] \arrow[r, "\mathrm{id}_{\mathbf{p}[f]}"'] & \mathbf{p}[J_A],
\end{tikzcd}$$

which reduces to the following diagram

$$\begin{tikzcd}
\mathbb{K} \otimes \mathbf{p}[I_A] \arrow[r, "\mathrm{id}_{\mathbb{K}} \otimes \mathbf{p}[f]"] \arrow[d, "\lambda_{\mathbf{p}[I_A]}"'] & \mathbb{K}| \otimes \mathbf{p}[J_A] \arrow[d, "\lambda_{\mathbf{p}[J_A]}"] \\
\mathbf{p}[I_A] \arrow[r, "\mathbf{p}[f]"'] & \mathbf{p}[J_A].
\end{tikzcd}$$

Hence

$$\lambda_{\mathbf{p}} \circ (\mathrm{id}_{\mathbb{K}} \otimes \mathbf{p}[f])(c \otimes v) = \lambda_{\mathbf{p}}(c \otimes \mathbf{p}[f]v) = c\mathbf{p}[f]v = \mathbf{p}[f](cv) = \mathbf{p}[f](\lambda_{\mathbf{p}}(c \otimes v)).$$

Thus $\lambda_{\mathbf{p}}$ is a natural transformation. Finally, to show that λ is a natural transformation. For all $\beta : \mathbf{p} \to \mathbf{p}'$, we must show the following diagram commutes:

$$\begin{tikzcd}
\mathbf{1}_{\mathbb{K}} \cdot \mathbf{p} \arrow[r, "\mathbf{1}_{\mathbb{K}} \cdot \beta"] \arrow[d, "\lambda_{\mathbf{p}}"'] & \mathbf{1}_{\mathbb{K}} \cdot \mathbf{p}' \arrow[d, "\lambda_{\mathbf{p}'}"] \\
\mathbf{p} \arrow[r, "\beta"'] & \mathbf{p}'.
\end{tikzcd}$$

Since for each object, I_A, $\lambda_{\mathbf{p}}$ and $\lambda_{\mathbf{p}'}$ are isomorphisms and since $\beta \cong \mathbf{1}_{\mathbb{K}} \otimes \beta$ is a natural transformation, we get the above diagram to commute. Therefore λ is a natural isomorphism.
Similarly for the right unitator, ρ whose sections are defined by the vector space isomorphisms:

$$\mathbf{p}[I_A] \otimes \mathbb{K} \cong \mathbf{p}[I_A].$$

Showing that all the coherence conditions hold, takes places in the category of vector spaces which we know are satisfied there. Thus the diagrams in 4 commutes.

We have that it is braided. We have that for all A-species, $\mathbf{p}$ and $\mathbf{q}$, $\beta_{\mathbf{p},\mathbf{q}}$ is an isomorphism since each section map is an isomorphism of the following vector spaces

$$\mathbf{p}[S_A] \otimes \mathbf{q}[T_A] \cong \mathbf{q}[T_A] \otimes \mathbf{p}[S_A].$$

Finally, it is symmetric since $\beta_{\mathbf{p},\mathbf{q}} \circ \beta_{\mathbf{q},\mathbf{p}} = \mathrm{id}$.

□

A *monoid*, $\mathbf{p}$ in $\mathbf{Sp}^A$ consists of morphisms of A-species

$$\mu : \mathbf{p} \cdot \mathbf{p} \to \mathbf{p} \quad \text{and} \quad \iota : \mathbf{1}_{\mathbb{K}} \to \mathbf{p},$$

where μ is the product and ι is the unit. The morphism μ consists of a family of linear maps

$$\mu_{S,T} : \mathbf{p}[S_A] \otimes \mathbf{p}[T_A] \to \mathbf{p}[I_A],$$

one for each decomposition of the underlying set I.The unit ι reduces to only one nontrivial linear map when our underlying set is empty:

$$\iota_{\emptyset} : \mathbb{K} \to \mathbf{p}[\emptyset].$$

For each bijection $\hat{f} : I_A \to J_A$ and each decomposition $S \sqcup T = I$, if $\hat{f}_{\hat{S}}(S_A) = S'_A$ and $\hat{f}_{T_A}(T_A) = T'_A$ where $S' \sqcup T' = J$, then for μ to be a natural transformation, we must have the following diagram commute:

$$\begin{tikzcd}
\mathbf{p}[S_A] \otimes \mathbf{p}[T_A] \arrow[r, "\mu_{S,T}"] \arrow[d, "\mathbf{p}[\hat{f}_{\hat{S}}] \otimes \mathbf{p}[\hat{f}_{T_A}]"'] & \mathbf{p}[I_A] \arrow[d, "\mathbf{p}[\hat{f}]"] \\
\mathbf{p}[S'_A] \otimes \mathbf{p}[T'_A] \arrow[r, "\mu_{S',T'}"'] & \mathbf{p}[J_A].
\end{tikzcd}$$

We must have that μ satisfies the associativity axiom. For each decomposition $S \sqcup T \sqcup R = I$, the following diagram must commute:

$$\begin{tikzcd}
\mathbf{p}[S_A] \otimes \mathbf{p}[T_A] \otimes \mathbf{p}[R_A] \arrow[r, "\mathrm{id} \otimes \mu_{T,R}"] \arrow[d, "\mu_{S,T} \otimes \mathrm{id}"'] & \mathbf{p}[S_A] \otimes \mathbf{p}[(T \sqcup R)_A] \arrow[d, "\mu_{S,T \sqcup R}"] \\
\mathbf{p}[(S \sqcup T)_A] \otimes \mathbf{p}[R_A] \arrow[r, "\mu_{S \sqcup T,R}"'] & \mathbf{p}[I_A].
\end{tikzcd}$$

Finally, ι must be a left and right unit for μ, i.e., for each I_A, the following diagrams must commute:

$$\begin{tikzcd}
\mathbf{p}[I_A] & \mathbf{p}[\emptyset] \otimes \mathbf{p}[I_A] \arrow[l, "\mu_{\emptyset,I}"'] & & \mathbf{p}[I_A] \otimes \mathbf{p}[\emptyset] \arrow[r, "\mu_{I,\emptyset}"] & \mathbf{p}[I_A] \\
& \mathbb{K} \otimes \mathbf{p}[I_A] \arrow[ul, equal] \arrow[u, "\iota_\emptyset \otimes \mathrm{id}"'] & & \mathbf{p}[I_A] \otimes \mathbb{K} \arrow[u, "\mathrm{id} \otimes \iota_\emptyset"] \arrow[ur, equal] &
\end{tikzcd} .$$

Definition 7.2.3. A *comonoid*, $\mathbf{p}$, in $\mathbf{Sp}^A$ consists of morphisms of A-species

$$\Delta : \mathbf{p} \to \mathbf{p} \cdot \mathbf{p} \quad \text{and} \quad \varepsilon : \mathbf{p} \to \mathbf{1}_{\mathbb{K}}.$$

The morphism Δ consists of a family of linear maps

$$\Delta_{S,T} : \mathbf{p}[I_A] \to \mathbf{p}[S_A] \otimes \mathbf{p}[T_A],$$

one for each decomposition of the underlying set I. Only one map is nontrivial for the counit ϵ, when our underlying set is the empty set:

$$\varepsilon_\emptyset : \mathbf{p}[\emptyset] \to \mathbb{K}.$$

Δ and ε both satisfy the usual coassociativity and counital diagrams as stated in the monoid definition with arrows reversed and replacing $\mu_{S,T}$ with $\Delta_{S,T}$ and ι with ε.

The following lemma from [3] is originally stated with finite sets, I. However, we can replace their sets I with $I_A \in \mathbf{Set}^A$ with no change since decompositions of A-sets are in bijection with set decompositions of I. This lemma is needed for the compatibility of the product and coproduct in the below definition of a bimonoid.

Lemma 7.2.4 (Aguir and Majahan [3]). *Let $S \sqcup T = I = S' \sqcup T'$ be two decompositions of the underlying set I, from I_A. Then there are unique subsets B, C, D and E of I such that*

$$S = B \sqcup C,\ T = D \sqcup E,\ S' = B \sqcup D,\ T' = C \sqcup E.$$

PROOF. The only choice is $B = S \cap S'$, $C = S \cap T'$, $D = T \cap S'$ and $E = T \cap T'$.

$\square$

A *bimonoid*, $\mathbf{p}$ in $\mathbf{Sp}^A$ is an A-species such that $\mathbf{p}$ has both a monoid and comonoid structure, with the additional condition that Δ and ϵ are morphisms of monoids, or equivalently, μ and ι are morphisms of comonoids.

The compatibility conditions are given by requiring the following diagrams to commute. Given I_A and any A-decompositions, $S \sqcup T = I = S' \sqcup T'$, the following diagram must commute:

$$\begin{tikzcd}
\mathbf{p}[\hat{B}] \otimes \mathbf{p}[\hat{C}] \otimes \mathbf{p}[\hat{D}] \otimes \mathbf{p}[\hat{E}] \arrow[rr, "\mathrm{id} \otimes \beta \otimes \mathrm{id}"] & & \mathbf{p}[\hat{B}] \otimes \mathbf{p}[\hat{D}] \otimes \mathbf{p}[\hat{C}] \otimes \mathbf{p}[\hat{E}] \arrow[d, "\mu_{B,D} \otimes \mu_{C,E}"] \\
\mathbf{p}[\hat{S}] \otimes \mathbf{p}[T_A] \arrow[u, "\Delta_{B,C} \otimes \Delta_{D,E}"] \arrow[r, "\mu_{S,T}"'] & \mathbf{p}[I_A] \arrow[r, "\Delta_{S',T'}"'] & \mathbf{p}[\hat{S'}] \otimes \mathbf{p}[\hat{T'}],
\end{tikzcd}$$

where B, C, D and E are as in Lemma 7.2.4. Recall, all sets decorated with a hat are to be understood as objects in $\mathbf{Set}^A$, i.e., $\hat{B} := B_A$. We also require the following diagrams to commute:

$$\begin{tikzcd}
\mathbf{p}[\emptyset] \otimes [\emptyset] \arrow[r, "\epsilon_\emptyset \otimes \epsilon_\emptyset"] \arrow[d, "\mu_{\emptyset,\emptyset}"] & \mathbb{K} \otimes \mathbb{K} \arrow[d, equal] \\
\mathbf{p}[\emptyset] \arrow[r, "\epsilon_\emptyset"'] & \mathbb{K},
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{K} \arrow[r, "\iota_\emptyset"] \arrow[d, equal] & \mathbf{p}[\emptyset] \arrow[d, "\Delta_{\emptyset,\emptyset}"] \\
\mathbb{K} \otimes \mathbb{K} \arrow[r, "\iota_\emptyset \otimes \iota_\emptyset"'] & \mathbf{p}[\emptyset] \otimes \mathbf{p}[\emptyset],
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathbf{p}[\emptyset] \arrow[dr, "\epsilon_\emptyset"] & \\
\mathbb{K} \arrow[ur, "\iota_\emptyset"] \arrow[rr, equal] & & \mathbb{K}.
\end{tikzcd}$$

A *Hopf monoid* $\mathbf{p}$ in $\mathbf{Sp}^A$ is a bimonoid with the morphism (the *antipode*), $s : \mathbf{p} \to \mathbf{p}$. For each I_A, $s_I : \mathbf{p}[I_A] \to \mathbf{p}[I]$ must be such that

$$\mathbf{p}[I_A] \xrightarrow{\oplus \Delta_{S,T}} \bigoplus_{S \sqcup T = I} \mathbf{p}[\hat{S}] \otimes \mathbf{p}[T_A] \xrightarrow{\mathrm{id} \otimes s_T} \bigoplus_{S \sqcup T = I} \mathbf{p}[\hat{S}] \otimes \mathbf{p}[T_A] \xrightarrow{\oplus \mu_{S,T}} \mathbf{p}[I_A],$$

$$\mathbf{p}[I_A] \xrightarrow{\oplus \Delta_{S,T}} \bigoplus_{S \sqcup T = I} \mathbf{p}[\hat{S}] \otimes \mathbf{p}[T_A] \xrightarrow{s_S \otimes \mathrm{id}} \bigoplus_{S \sqcup T = I} \mathbf{p}[\hat{S}] \otimes \mathbf{p}[T_A] \xrightarrow{\oplus \mu_{S,T}} \mathbf{p}[I_A]$$

are zero, and the following diagrams must commute:

$$\begin{tikzcd}
\mathbf{p}[\emptyset] \otimes \mathbf{p}[\emptyset] \arrow[rr, "\mathrm{id} \otimes s_\emptyset"] & & \mathbf{p}[\emptyset] \otimes \mathbf{p}[\emptyset] \arrow[d, "\mu_{\emptyset,\emptyset}"] \\
\mathbf{p}[\emptyset] \arrow[u, "\Delta_{\emptyset,\emptyset}"] \arrow[r, "\epsilon_\emptyset"'] & \mathbb{K} \arrow[r, "\iota_\emptyset"'] & \mathbf{p}[\emptyset]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{p}[\emptyset] \otimes \mathbf{p}[\emptyset] \arrow[rr, "s_\emptyset \otimes \mathrm{id}"] & & \mathbf{p}[\emptyset] \otimes \mathbf{p}[\emptyset] \arrow[d, "\mu_{\emptyset,\emptyset}"] \\
\mathbf{p}[\emptyset] \arrow[u, "\Delta_{\emptyset,\emptyset}"] \arrow[r, "\epsilon_\emptyset"'] & \mathbb{K} \arrow[r, "\iota_\emptyset"'] & \mathbf{p}[\emptyset].
\end{tikzcd}$$

7.3. The Hadamard Product

In this section, we introduce the Hadamard product. This operation also turns the category of A-species, into a monoidal category with unit $\mathbf{E}$, defined in Example 7.1.18. Whenever we consider the Hadamard product, we also assume our algebra is a bialgebra.

DEFINITION 7.3.1. Let $\mathbf{p}$ and $\mathbf{q}$ be A-species. The *Hadamard product*, $\times : \mathbf{Sp}^A \times \mathbf{Sp}^A \to \mathbf{Sp}^A$, is the species defined as follows:
For all I_A,

$$(\mathbf{p} \times \mathbf{q})[I_A] := \mathbf{p}[I_A] \otimes \mathbf{q}[I_A].$$

On morphism generators $(1 \cdots 1 \otimes f)$ and $(b_{i_1} \cdots b_{i_n} \otimes \mathrm{id})$ we have:

$$(\mathbf{p} \times \mathbf{q})[(1 \cdots 1 \otimes f)] := \mathbf{p}[(1 \cdots 1 \otimes f)] \otimes \mathbf{q}[(1 \cdots 1 \otimes f)]$$

$$(\mathbf{p} \times \mathbf{q})[(b_{i_1} \cdots b_{i_n} \otimes \mathrm{id})] := \sum_{(b_{i_k})\ \forall k} \mathbf{p}[((b_{i_1})_1 \cdots (b_{i_n})_1 \otimes \mathrm{id})] \otimes \mathbf{q}[((b_{i_1})_2 \cdots (b_{i_n})_2 \otimes \mathrm{id})].$$

In other words, $A \wr S_n$ acts on the Hadamard product via the coproduct; when we restrict ourselves to S_n the action is diagonal.

PROPOSITION 7.3.2. *$(\boldsymbol{Sp}^A, \times, \boldsymbol{E}_A, \alpha, \lambda, \rho, \beta)$ is a symmetric monoidal category with the braiding $\beta_{\mathbf{p},\mathbf{q}} : \mathbf{p} \times \mathbf{q} \to \mathbf{q} \times \mathbf{p}$ given by*

$$\mathbf{p}[I_A] \otimes \mathbf{q}[I_A] \to \mathbf{q}[I_A] \otimes \mathbf{p}[I_A]$$

$$x \otimes y \mapsto y \otimes x.$$

PROOF. We first need to define a natural isomorphism $\alpha : (\_\_ \times \_\_) \times \_\_ \to \_\_ \times (\_\_ \times \_\_)$ where we view $(\_\_ \times \_\_) \times \_\_$ and $\_\_ \times (\_\_ \times \_\_)$ as functors from $\mathbf{Sp}^A \times \mathbf{Sp}^A \times \mathbf{Sp}^A \to \mathbf{Sp}^A$. The section maps of α will be defined as follows, which themselves must be natural transformations. Let $\mathbf{p}, \mathbf{q}, \mathbf{h} \in \mathbf{Sp}^A$. Observe that

$$\begin{aligned}
(\mathbf{p} \times \mathbf{q}) \times \mathbf{h}[I_A] &= (\mathbf{p} \times \mathbf{q})[I_A] \times \mathbf{h}[I_A] \\
&= (\mathbf{p}[I_A] \times \mathbf{q}[I_A]) \times \mathbf{h}[I_A] \\
&\cong \mathbf{p}[I_A] \times (\mathbf{q}[I_A] \times \mathbf{h}[I_A]) \\
&= \mathbf{p}[I_A] \times (\mathbf{q} \times \mathbf{h})[I_A] \\
&= \mathbf{p} \times (\mathbf{q} \times \mathbf{h})[I_A].
\end{aligned}$$

The isomorphism follows from the fact that the tensor product is associative on vector spaces. The above defines the isomorphism of the section maps $\alpha_{\mathbf{p},\mathbf{q},\mathbf{h}}$. Now to show that for all $f : I_A \to J_A$, the following diagram commutes:

$$\begin{tikzcd}
(\mathbf{p} \times \mathbf{q}) \times \mathbf{h}[I_A] \arrow[r, "(\mathbf{p}\times\mathbf{q})\times\mathbf{h}[f]"] \arrow[d, "\alpha_{\mathbf{p},\mathbf{q},\mathbf{h}}"'] & (\mathbf{p} \times \mathbf{q}) \times \mathbf{h}[J_A] \arrow[d, "\alpha_{\mathbf{p},\mathbf{q},\mathbf{h}}"] \\
\mathbf{p} \times (\mathbf{q} \times \mathbf{h})[I_A] \arrow[r, "\mathbf{p}\times(\mathbf{q}\times\mathbf{h})[f]"'] & \mathbf{p} \times (\mathbf{q} \times \mathbf{h})[J_A].
\end{tikzcd}$$

It suffices to show on the morphism generators, $(1 \cdots 1 \otimes f)$ and $(b_{i_1} \cdots b_{i_n} \otimes \mathrm{id})$.
Let $x \in \mathbf{p}[I_A]$, $y \in \mathbf{q}[I_A]$, and $z \in \mathbf{h}[I_A]$. First

$\mathbf{p} \times (\mathbf{q} \times \mathbf{h})[(1 \cdots 1 \otimes f)](\alpha_{\mathbf{p,q,h}}((x \otimes y) \otimes z))$:

$$\begin{aligned}
&= \mathbf{p} \times (\mathbf{q} \times \mathbf{h})[(1 \cdots 1 \otimes f)](x \otimes (y \otimes z)) \\
&= \mathbf{p}[(1 \cdots 1 \otimes f)]x \otimes (\mathbf{q} \times \mathbf{h})[(1 \cdots 1 \otimes f)](y \otimes) \\
&= \mathbf{p}[(1 \cdots 1 \otimes f)]x \otimes (\mathbf{q}[(1 \cdots 1 \otimes f)]y \otimes \mathbf{h}[(1 \cdots 1 \otimes f)]z) \\
&= \alpha_{\mathbf{p,q,h}}((\mathbf{p} \times \mathbf{q})[(1 \cdots 1 \otimes f)](x \otimes y) \otimes \mathbf{h}[(1 \cdots 1 \otimes f)]z) \\
&= \alpha_{\mathbf{p,q,h}}((\mathbf{p} \times \mathbf{q}) \times \mathbf{h}[(1 \cdots 1 \otimes f)]((x \otimes y) \otimes z).
\end{aligned}$$

Now,
$\mathbf{p} \times (\mathbf{q} \times \mathbf{h})[(b_{i_1} \cdots b_{i_n} \otimes \mathrm{id})])(\alpha_{\mathbf{p,q,h}}((x \otimes y) \otimes z))$:

$$\begin{aligned}
&= \mathbf{p} \times (\mathbf{q} \times \mathbf{h})[(b_{i_1} \cdots b_{i_n} \otimes \mathrm{id})](x \otimes (y \otimes z)) \\
&= \sum \mathbf{p}[((b_{i_1})_1 \cdots (b_{i_n})_1 \otimes \mathrm{id})]x \otimes (\mathbf{q} \times \mathbf{h})[((b_{i_1})_2 \cdots (b_{i_n})_2 \otimes \mathrm{id})](y \otimes) \\
&= \sum \mathbf{p}[((b_{i_1})_1 \cdots (b_{i_n})_1 \otimes \mathrm{id})]x \otimes (\mathbf{q}[((b_{i_1})_2 \cdots (b_{i_n})_2 \otimes \mathrm{id})]y \otimes \mathbf{h}[((b_{i_1})_3, ..., (b_{i_n})_3, \mathrm{id})]z) \\
&= \alpha_{\mathbf{p,q,h}}\left(\sum (\mathbf{p} \times \mathbf{q})[((b_{i_1})_1 \cdots (b_{i_n})_1 \otimes \mathrm{id})](x \otimes y) \otimes \mathbf{h}[((b_{i_1})_2 \cdots (b_{i_n})_2 \otimes \mathrm{id})]z\right) \\
&= \alpha_{\mathbf{p,q,h}}((\mathbf{p} \times \mathbf{q}) \times \mathbf{h}[((b_{i_1} \cdots b_{i_n} \otimes \mathrm{id}))]((x \otimes y) \otimes z).
\end{aligned}$$

Thus the diagram commutes, and each section map $\alpha_{\mathbf{p,q,h}}$ is a natural transformation. The naturality diagram for α reduces to the naturality diagram of the section maps. Thus α is a natural transformation, moreover a natural isomorphism.

Now to define the left and right unitators. I will show the details for left unitator and the right unitator is done in a similar fashion. We must define a natural isomorphism $\lambda : \mathbf{E}_A \times \_\_ \to \mathrm{id}$ where we view $\mathbf{E}_A \times \_\_$ and id as functors from $\mathbf{Sp}^A \to \mathbf{Sp}^A$. The section maps of λ must again be natural transformations, i.e., for all $\mathbf{p} \in \mathbf{Sp}^A$, $\lambda_{\mathbf{p}} : \mathbf{E}_A \times \mathbf{p} \to \mathrm{id}(\mathbf{p})$ is a natural transformation. For all I_A, we have that

$$(\mathbf{E}_A \times \mathbf{p})[I_A] = \mathbf{E}_A[I_A] \otimes \mathbf{p}[I_A] = \mathbb{K} \otimes \mathbf{p}[I_A] \cong \mathbf{p}[I_A],$$

where the isomorphism is given by scalar multiplication. Thus we define the section maps of $\lambda_{\mathbf{p}}$ from the above isomorphism. We must show that the following diagram commutes for all $f : I_A \to J_A$,

$$\begin{tikzcd}
(\mathbf{E}_A \times \mathbf{p})[I_A] \arrow[r, "(\mathbf{E}_A \times \mathbf{p})[f]"] \arrow[d, "\lambda_{\mathbf{p}[I_A]}"'] & (\mathbf{E}_A \times \mathbf{p})[J_A] \arrow[d, "\lambda_{\mathbf{p}[J_A]}"] \\
\mathbf{p}[I_A] \arrow[r, "\mathbf{p}[f]"'] & \mathbf{p}[J_A].
\end{tikzcd}$$

It suffices to show on the morphism generators, $(1 \cdots 1 \otimes f)$ and $(b_{i_1} \cdots b_{i_n} \otimes \mathrm{id})$. Let $x \in \mathbf{E}_A[I_A]$ and $y \in \mathbf{p}[I_A]$.
First,

$$\begin{aligned}
\lambda_{\mathbf{p}[I_A]} \circ (\mathbf{E}_A \times \mathbf{p})[(1 \cdots 1 \otimes f)](x \otimes y) &= \lambda_{\mathbf{p}[I_A]} \circ (\mathbf{E}_A[(1 \cdots 1 \otimes f)]x \otimes \mathbf{p}[(1 \cdots 1 \otimes f)]y) \\
&= \lambda_{\mathbf{p}[\hat{J}]} \circ (\varepsilon(f)x \otimes \mathbf{p}[(1 \cdots 1 \otimes f)]y) \\
&= \lambda_{\mathbf{p}[J_A]} \circ (x \otimes \mathbf{p}[(1 \cdots 1 \otimes f)]y) \\
&= x\mathbf{p}[(1 \cdots 1 \otimes f)](y) \\
&= \mathbf{p}[(1 \cdots 1 \otimes f)](\lambda_{\mathbf{p}[I_A]}(x \otimes y))
\end{aligned}$$

as desired. Now, $\lambda_{\mathbf{p}[I_A]} \circ (\mathbf{E}_A \times \mathbf{p})[(b_{i_1} \cdots b_{i_n} \otimes \mathrm{id})](x \otimes y)$:

$$\begin{aligned}
&= \lambda_{\mathbf{p}[I_A]} \left(\sum \mathbf{E}_A[(b_{i_1} \cdots b_{i_n} \otimes \mathrm{id})_1]x \otimes \mathbf{p}[(b_{i_1} \cdots b_{i_n} \otimes \mathrm{id})_2]y \right) \\
&= \lambda_{\mathbf{p}[I_A]} \left(\sum \varepsilon((b_{i_1} \cdots b_{i_n} \otimes \mathrm{id})_1)x \otimes (b_{i_1} \cdots b_{i_n} \otimes \mathrm{id})_2.y \right) \\
&= \sum \varepsilon((b_{i_1} \cdots b_{i_n} \otimes \mathrm{id})_1)x(b_{i_1} \cdots b_{i_n} \otimes \mathrm{id})_2.y \\
&= x \sum \varepsilon((b_{i_1} \cdots b_{i_n} \otimes \mathrm{id})_1)(b_{i_1} \cdots b_{i_n} \otimes \mathrm{id})_2).y \\
&= x(b_{i_1} \cdots b_{i_n} \otimes \mathrm{id}).y \\
&= x\mathbf{p}[(b_{i_1} \cdots b_{i_n} \otimes \mathrm{id})]y \\
&= \mathbf{p}[(b_{i_1} \cdots b_{i_n} \otimes \mathrm{id})](xy) \\
&= \mathbf{p}[(b_{i_1} \cdots b_{i_n} \otimes \mathrm{id})] \circ \lambda_{\mathbf{p}[I_A]}(x \otimes y),
\end{aligned}$$

where we use the Sweedler Identity (25) for the fifth equality. Thus $\lambda_{\mathbf{p}}$ is a natural transformation. Showing the naturality of λ, reduces to the naturality diagram of $\lambda_{\mathbf{p}}$. Thus λ is a natural transformation, moreover a natural isomorphism. To show that the right unitator is natural isomorphism is similar, with the section maps defined by the following vector space isomorphisms:

$$\mathbf{p}[I_A] \otimes \mathbb{K} \cong \mathbf{p}[I_A].$$

Showing that all the coherence conditions hold takes place in the category of vector spaces; thus, the diagrams in 4 commute.

We have that it is braided. For all $\mathbf{p}, \mathbf{q} \in \mathbf{Sp}^A$, β is a natural isomorphism since each section map is an isomorphism of the following vector spaces:

$$\mathbf{p}[I_A] \times \mathbf{q}[I_A] \cong \mathbf{q}[I_A] \times \mathbf{p}[I_A].$$

Clearly $\beta_{\mathbf{p},\mathbf{q}} \circ \beta_{\mathbf{q},\mathbf{p}} = \mathrm{id}$, thus symmetric.

□

Remark 7.3.3. For every $I_A \in \mathbf{Set}^A$ and $\mathbf{p}, \mathbf{q} \in \mathbf{Sp}^A$, $A \wr S_n$ acts on the Hadamard product, $(\mathbf{p} \times \mathbf{q})[I_A] = \mathbf{p}[I_A] \otimes \mathbf{q}[I_A]$, since $\mathrm{End}(I_A) \cong A \wr S_n$ for some $n = |I|$. See Section 3.2.

7.4. Relationship to Generalized Species

To end this chapter, we let $A = \mathbb{K}G$ and take G to be our basis, and we will show that the category of $\mathbb{K}G$-species is equivalent to the category of G-species, as defined by Henderson (see Section 6.2, [22], and [21]).

7.4.1. G-Species to $\mathbb{K}G$-Species

First, we define a functor that constructs a $\mathbb{K}G$-species from a G-species.

Lemma 7.4.1. *For each G-species, $\mathbf{p}$, we can define a functor $F : \boldsymbol{Sp}^G \to \boldsymbol{Sp}^{\mathbb{K}G}$ via*

$$\begin{aligned} \mathbf{p} &\mapsto F\mathbf{p}[\mathbb{K}G^{\otimes n} \otimes \mathbb{K}[n]] := \mathbf{p}[G \times [n]] \\ & F\mathbf{p}[\sum a_{(\vec{g},\sigma)} v_{(\vec{g},\sigma)}] := \sum a_{(\vec{g},\sigma)} \mathbf{p}[(\vec{g},\sigma)] \\ \alpha &\mapsto F(\alpha)_{[n_A]} := \alpha_{[n]} \end{aligned}$$

where the direct sum is over all $(\vec{g},\sigma) \in G \wr S_n$ where $\vec{g} := (g_1, ..., g_n) \in G^n$.

PROOF.

- For all $\mathbf{p} \in \mathbf{Sp}^G$, $F\mathbf{p} \in \mathbf{Sp}^{\mathbb{K}G}$ since
  (1) $F\mathbf{p}[\mathbb{K}G^{\otimes n} \otimes \mathbb{K}[n]] := \mathbf{p}[G \times [n]]$ which is a vector space by definition of $\mathbf{p}$. Furthermore, this is a $\mathbb{K}G \wr S_n$ module by extending linearly since $\mathbf{p}[G \times [n]]$ is a $G \wr S_n$ module.
  (2) $F\mathbf{p}[\sum a_{(\vec{g},\sigma)} v_{(\vec{g},\sigma)}] := \sum a_{(\vec{g},\sigma)} \mathbf{p}[(\vec{g},\sigma)]$ is a linear map since it is a sum of linear maps.
- Let $\alpha : \mathbf{p} \to \mathbf{q}$ be a morphism of G-species. We want to show that $F(\alpha) : F\mathbf{p} \to F\mathbf{q}$ is a natural transformation. For all $\mathbb{K}G^{\otimes n} \otimes \mathbb{K}[n] \in \mathbf{Set}^{\mathbb{K}G}$, define the section maps as follows:
$$F(\alpha)_{[n_A]} := \alpha_{[n]}$$
  We must show the following diagram commutes:
$$\begin{tikzcd} F\mathbf{p}[\mathbb{K}G^{\otimes n} \otimes \mathbb{K}[n]] \arrow[r, "F\mathbf{p}[\sum a_{(\vec{g},\sigma)} v_{(\vec{g},\sigma)}]"] \arrow[d, "F(\alpha)_{[n_A]}"'] & F\mathbf{p}[\mathbb{K}G^{\otimes n} \otimes \mathbb{K}[n]] \arrow[d, "F(\alpha)_{[n_A]}"] \\ F\mathbf{q}[\mathbb{K}G^{\otimes n} \otimes \mathbb{K}[n]] \arrow[r, "F\mathbf{q}[\sum a_{(\vec{g},\sigma)} v_{(\vec{g},\sigma)}]"'] & F\mathbf{q}[\mathbb{K}G^{\otimes n} \otimes \mathbb{K}[n]] \end{tikzcd}$$
  which reduces to
$$\begin{tikzcd} \mathbf{p}[G \times [n]] \arrow[r, "\sum a_{(\vec{g},\sigma)} \mathbf{p}[(\vec{g},\sigma)]"] \arrow[d, "\alpha_{[n]}"'] & \mathbf{p}[G \times [n]] \arrow[d, "\alpha_{[G \times [n]]}"] \\ \mathbf{q}[G \times [n]] \arrow[r, "\sum a_{(\vec{g},\sigma)} \mathbf{q}[(\vec{g},\sigma)]"'] & \mathbf{q}[G \times [n]] \end{tikzcd}$$
  Since α is a natural transformation, we have that this diagram commutes. Thus $F(\alpha) : F\mathbf{p} \to F\mathbf{q}$ is a natural transformation.
- Now to show that $F(\mathrm{id}_{\mathbf{p}}) = \mathrm{id}_{F\mathbf{p}}$. Note that for all $\mathbf{p} \in \mathbf{Sp}^G$, $\mathrm{id}_{\mathbf{p}} : \mathbf{p} \to \mathbf{p}$ is a natural transformation whose section maps, $\mathrm{id}_{\mathbf{p}[G \times [n]]}$, are given by the usual identity map. By definition, $F(\mathrm{id}_{\mathbf{p}})_{[n_A]} = \mathrm{id}_{\mathbf{p}[n]}$. Now note that, $\mathrm{id}_{F\mathbf{p}} : F\mathbf{p} \to F\mathbf{p}$ is a natural transformation, whose section maps are as follows $\mathrm{id}_{F\mathbf{p}[n_A]} = \mathrm{id}_{\mathbf{p}[n]}$. Thus $F(\mathrm{id}_{\mathbf{p}}) = \mathrm{id}_{F\mathbf{p}}$.
- Finally, to show $F(\alpha \circ \beta) = F(\alpha) \circ F(\beta)$ for all $\alpha : \mathbf{p} \to \mathbf{q}$ and $\beta : \mathbf{q} \to \mathbf{h}$. Since α and β are natural transformations, we have that
$$F(\alpha \circ \beta)_{[n_A]} := (\alpha \circ \beta)_{[n]} = \alpha_{[n]} \circ \beta_{[G \times [n]]} = F(\alpha)_{[n_A]} \circ F(\beta)_{[n_A]}$$

Thus F defines a functor from $\mathbf{Sp}^G$ to $\mathbf{Sp}^{\mathbb{K}G}$.

□

PROPOSITION 7.4.2. *F is a bilax monoidal functor with natural transformations φ^F and ψ^F whose sections are given by*

$$F(\mathbf{p})\cdot F(\mathbf{q}) \underset{\psi^F_{\mathbf{p},\mathbf{q}}}{\overset{\varphi^F_{\mathbf{p},\mathbf{q}}}{\rightleftarrows}} F(\mathbf{p}\cdot\mathbf{q})$$

where both $\varphi^F_{\mathbf{p},\mathbf{q}}$ and $\psi^F_{\mathbf{p},\mathbf{q}}$ are given by the identity.

PROOF. Observe that for an object $\mathbb{K}G^{\otimes n}\otimes\mathbb{K}[n]$, we have:

$$\begin{aligned}
F(\mathbf{p})\cdot F(\mathbf{q})[\mathbb{K}G^{\otimes n}\otimes\mathbb{K}[n]] &= \bigoplus_{S\sqcup T=[n]} F\mathbf{p}[\mathbb{K}G^{\otimes|S|}\otimes\mathbb{K}[S]]\otimes F\mathbf{q}[\mathbb{K}G^{\otimes|T|}\otimes\mathbb{K}[T]]\\
&= \bigoplus_{S\sqcup T=[n]} \mathbf{p}[G\times S]\otimes\mathbf{q}[G\times T]\\
&= (\mathbf{p}\cdot\mathbf{q})[G\times[n]]\\
&= F(\mathbf{p}\cdot\mathbf{q})[\mathbb{K}G^{\otimes n}\otimes\mathbb{K}[n]]
\end{aligned}$$

By the equalities above, on a degree n piece, we can define the section maps of φ^f and ψ^f via $\varphi^F_{\mathbf{p},\mathbf{q}} := \mathbf{p}[\mathrm{id}]\otimes\mathbf{q}[\mathrm{id}]$ and $\psi^F_{\mathbf{p},\mathbf{q}} := \mathbf{p}[\mathrm{id}]\otimes\mathbf{q}[\mathrm{id}]$.
Now to define φ^F_0 and ψ^F_0. Observe that

$$F(\mathbf{1}_{\mathbb{K}})[\mathbb{K}G^{\otimes n}\otimes\mathbb{K}[n]] = \mathbf{1}_{\mathbb{K}}[G\times[n]] = \begin{cases}\mathbb{K} & \text{if } n=0\\ 0 & \text{otherwise}\end{cases} = \mathbf{1}_{\mathbb{K}}\in\mathbf{Sp}^A$$

Thus φ^F_0 and ψ^F_0 are the identity maps.

Showing that $\varphi^F_{\mathbf{p},\mathbf{q}}$ and $\psi^F_{\mathbf{p},\mathbf{q}}$ are natural transformations and satisfy the bilax conditions is straightforward to check since in both cases $\mathbf{p}$ and $\mathbf{q}$ are being applied to the identity functor; the proof is done in a similar fashion as in Proposition 10.1.3.

□

COROLLARY 7.4.3. *F is a bistrong functor.*

PROOF. It's clear that $\psi^F = (\varphi^F)^{-1}$ and $\psi^F_0 = (\varphi^F_0)^{-1}$, thus by Proposition 3.46 in [3] we have that F is bistrong.

□

7.4.2. $\mathbb{K}G$-species to G-species

Now we construct a functor $\hat{H}$ from $\mathbb{K}G$-species to G-species.

DEFINITION 7.4.4. For each G-set and morphism of G-sets, we can define a functor $H:\mathbf{Set}^G\to\mathbf{Set}^{\mathbb{K}G}$ via

$$\begin{aligned}
G\times[n] &\mapsto \mathbb{K}G^{\otimes n}\otimes\mathbb{K}[n]\\
(\vec{g},\sigma) &\mapsto v_{(\vec{g},\sigma)}
\end{aligned}$$

PROOF.

(1) By definition, $H(G\times[n])$ is an object in $\mathbf{Set}^{\mathbb{K}G}$ for all $G\times[n]\in\mathbf{Set}^G$.

(2) Let $(\vec{g},\sigma) : G\times[n] \to G\times[n]$. Note that it suffices to use endomorphisms since if $m \neq n$ then the only map would be the zero map. We have that $H(\vec{g},\sigma) : H(G\times[n]) \to H(G\times[n])$ is a morphism in $\mathbf{Set}^{\mathbb{K}G}$ since $H(\vec{g},\sigma) = v_{(\vec{g},\sigma)} \in \mathrm{End}(\mathbb{K}G^{\otimes n} \otimes \mathbb{K}G[n])$. Now to show that:

- $H(\mathrm{id}_{G\times[n]}) = \mathrm{id}_{H(G\times[n])}$.
  We have:
  $$H(\mathrm{id}_{G\times[n]}) = H((\vec{\mathbf{1}}_G,\mathrm{id})) = v_{(\vec{\mathbf{1}}_G,\mathrm{id})} = \mathrm{id}_{\mathbb{K}G^{\otimes n}\otimes\mathbb{K}[n]} = \mathrm{id}_{H(G\times[n])}$$
- $H((\vec{g},\sigma)\circ(\vec{r},\tau)) = H((\vec{g},\sigma))\circ H((\vec{r},\tau))$
  $$\begin{aligned} H((\vec{g},\sigma)\circ(\vec{r},\tau)) &= H((g_1r_{\sigma^{-1}(1)},..,g_nr_{\sigma^{-1}(n)},\sigma\circ\tau)) \\ &= v_{(g_1r_{\sigma^{-1}(1)},..,g_nr_{\sigma^{-1}(n)},\sigma\circ\tau)} \\ &= v_{(\vec{g},\sigma)}v_{(\vec{r},\tau)} \\ &= H((\vec{g},\sigma))\circ H((\vec{r},\tau)) \end{aligned}$$

Thus, H is a functor. □

Definition 7.4.5. For each $\mathbb{K}G$-species, $\mathbf{p}$, we can define a functor $\hat{H} : \mathbf{Sp}^{\mathbb{K}G} \to \mathbf{Sp}^G$ via

$$\begin{aligned} \mathbf{p} \;\mapsto\; & \hat{H}\mathbf{p}[G\times[n]] := \mathbf{p}\circ H(G\times[n]) \\ & \hat{H}\mathbf{p}[(\vec{g},\sigma)] := \mathbf{p}\circ H((\vec{g},\sigma)) \end{aligned}$$

Proof. We immediately have that $\hat{H}$ is a functor since both $\mathbf{p}$ and H are functors and the compositions of two functors is again a functor.

□

7.4.2.1. *Equivalence of Categories.*

Proposition 7.4.6. *$\mathbf{Sp}^G$ and $\mathbf{Sp}^{\mathbb{K}G}$ are equivalent categories.*

Proof. In order to show that $\mathbf{Sp}^G$ and $\mathbf{Sp}^{\mathbb{K}G}$ are equivalent categories we must show that there exists two natural isomorphisms

$$\eta : \mathrm{id}_{\mathbf{Sp}^G} \to \hat{H}\circ F$$

and

$$\epsilon : F\circ\hat{H} \to \mathrm{id}_{\mathbf{Sp}^{\mathbb{K}G}}.$$

First to show η is a natural transformation. Let $\mathbf{p}\in\mathbf{Sp}^G$, then the section maps

$$\eta_{\mathbf{p}} : \mathrm{id}_{\mathbf{Sp}^G}(\mathbf{p}) \to \hat{H}\circ F(\mathbf{p})$$

must again be a natural transformation. Let $G\times[n] \in \mathbf{Set}^G$ and define $\eta_{\mathbf{p}[G\times[n]]}$ by the following equalities:

$$\begin{aligned} \hat{H}(F\mathbf{p})[G\times[n]] &= F\mathbf{p}\circ H(G\times[n]) \\ &= F\mathbf{p}[\mathbb{K}G^{\otimes n}\otimes\mathbb{K}[n]] \\ &= \mathbf{p}[G\times[n]]. \end{aligned}$$

Now for all $(\vec{g},\sigma) : G\times[n] \to G\times[n]$ the following diagram must commute

$$\begin{tikzcd}
\mathrm{id}_{\mathbf{Sp}^G}(\mathbf{p})[G\times[n]] \arrow[r, "\eta_{\mathbf{p}[G\times[n]]}"] \arrow[d, "\mathrm{id}_{\mathbf{Sp}^G}(\mathbf{p})_{(\vec{g},\sigma)}"'] & \hat{H}\circ F(\mathbf{p})[G\times[n]] \arrow[d, "\hat{H}\circ F(\mathbf{p})_{(\vec{g},\sigma)}"] \\
\mathrm{id}_{\mathbf{Sp}^G}(\mathbf{p})[G\times[n]] \arrow[r, "\eta_{\mathbf{p}[G\times[n]]}"'] & \hat{H}\circ F(\mathbf{p})[G\times[n]].
\end{tikzcd}$$

This reduces to

$$\begin{tikzcd}
\mathbf{p}[G\times[n]] \arrow[r, "\eta_{\mathbf{p}[G\times[n]]}"] \arrow[d, "\mathbf{p}_{(\vec{g},\sigma)}"'] & \mathbf{p}[G\times[n]] \arrow[d, "\mathbf{p}_{(\vec{g},\sigma)}"] \\
\mathbf{p}[G\times[n]] \arrow[r, "\eta_{\mathbf{p}[G\times[n]]}"'] & \mathbf{p}[G\times[n]],
\end{tikzcd}$$

which clearly commutes. Thus $\eta_{\mathbf{p}}$ is a natural transformation. The naturality diagram of η reduces to the naturality diagram of $\eta_{\mathbf{p}}$, thus η is a natural transformation; moreover, it's a natural isomorphism by the equalities used to define $\eta_{\mathbf{p}}$.

To show ϵ is also a natural isomorphism is done in a similar fashion as above. The section maps, $\epsilon_{\mathbf{p}}$ are again natural transformations given by the following equalities:

$$\begin{aligned}
F(\hat{H}\mathbf{p})[\mathbb{K}G^{\otimes n}\otimes\mathbb{K}[n]] &= \hat{H}\mathbf{p}[G\times[n]] \\
&= \mathbf{p}\circ H[G\times[n]] \\
&= \mathbf{p}[\mathbb{K}G^{\otimes n}\otimes\mathbb{K}[n]].
\end{aligned}$$

Therefore $\mathbf{Sp}^G$ and $\mathbf{Sp}^{\mathbb{K}G}$ are equivalent categories.

□

REMARK 7.4.7. When we let $G = C_2$, we get a linearized version of $\mathcal{H}$-species. When we take G to be the trivial group, we get the classical notion of vector species.

CHAPTER 8

Decorated A-species Examples

In this chapter, we give different notions of decorated A-species. We define a bilax bistrong monoidal functor that constructs the most naive example of an A-species from a classical species. We end by defining a decorated version of an A-species that is a generalization of the decorated species as defined in Chapter 16 of [3].

8.1. A naive example of an A-Species

In this section, we give a naive way to construct an A species from any species $\mathbf{p} \in \mathbf{Sp}$. Given $I_A \in \mathbf{Set}^A$ and $\bigotimes_{j\in J} c_j \otimes f \in \mathrm{Hom}(I_A, J_A)$, we define a functor $H : \mathbf{Sp} \to \mathbf{Sp}^A$ on objects by

$$\begin{aligned} H\mathbf{p}[I_A] &:= A^{\otimes I} \otimes \mathbf{p}[I] \\ H\mathbf{p}[\bigotimes_{j\in J} c_j \otimes f] &:= \bigotimes_{j\in J} c_j \otimes \mathbf{p}[f]. \end{aligned}$$

On a pure tensor $\bigotimes_{i\in I} a_i \otimes v \in A^I \otimes \mathbf{p}[I]$, we have $(\bigotimes_{j\in J} c_j \otimes \mathbf{p}[f])(\bigotimes_{i\in I} a_i \otimes v) = \bigotimes_{j\in J} c_j a_{f^{-1}(j)} \otimes \mathbf{p}[f]v$ and extend by linearity.
Given a morphism of species $\alpha : \mathbf{p} \to \mathbf{q}$, define:

$$H\alpha_{[I_A]} := \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \alpha_I,$$

where on a pure tensor $\bigotimes_{i\in I} a_i \otimes v \in A^I \otimes \mathbf{p}[I]$, we have

$$(\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \alpha_I)(\bigotimes_{i\in I} a_i \otimes v) = \bigotimes_{i\in I} a_i \otimes \alpha_I(v).$$

Proof. We must show the above is indeed a functor.
First to show that $H\mathbf{p} \in \mathbf{Sp}^A$.

- It's clear that $H\mathbf{p}[I_A] := A^I \otimes \mathbf{p}[I] \in \mathbf{Vec}_{\mathbb{K}}$.
- We have that $H\mathbf{p}[\bigotimes_{j\in J} c_j \otimes f]$ is a linear map by construction.

Now let $\mathbf{p}$ and $\mathbf{q}$ be species and consider $\alpha : \mathbf{p} \to \mathbf{q}$ a morphism of species. We want to show that $H\alpha : H\mathbf{p} \to H\mathbf{q}$ is a morphism of A-species, i.e., a natural transformation. For all $[n]_A \in \mathbf{Set}^A$, define the sections of $H\alpha$ as follows:

$$H\alpha_{I_A} := \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \alpha_I$$

Let $\bigotimes_{j\in J} c_j \otimes f : I_A \to J_A$. We must show the following diagram commutes:

$$\begin{tikzcd}
H\mathbf{p}[I_A] \arrow[r, "H\alpha[I_A]"] \arrow[d, "H\mathbf{p}[\bigotimes_{j\in J} c_j \otimes f]"'] & H\mathbf{q}[I_A] \arrow[d, "H\mathbf{q}[\bigotimes_{j\in J} c_j \otimes f]"] \\
H\mathbf{p}[J_A] \arrow[r, "H\alpha[J_A]"'] & H\mathbf{q}[J_A].
\end{tikzcd}$$

The diagram above reduces to the following diagram commuting:

$$\begin{tikzcd}
A^I \otimes \mathbf{p}[I] \arrow[r, "\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \alpha_{[I]}"] \arrow[d, "\bigotimes_{j\in J} c_j \otimes \mathbf{p}[f]"'] & A^I \otimes \mathbf{q}[I] \arrow[d, "\bigotimes_{j\in J} c_j \otimes \mathbf{q}[f]"] \\
A^J \otimes \mathbf{p}[J] \arrow[r, "\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \alpha_{[J]}"'] & A^J \otimes \mathbf{q}[J].
\end{tikzcd}$$

Let $\bigotimes_{i\in I} a_i \otimes v \in A^I \otimes \mathbf{p}[I]$ be a pure tensor. Because of the naturality of α we get the following equalities:

$$\begin{aligned}
(\bigotimes_{j\in J} c_j \otimes \mathbf{q}[f])(\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \alpha_I)(\bigotimes_{i\in I} a_i \otimes v) &= (\bigotimes_{j\in J} c_j \otimes \mathbf{q}[f])(\bigotimes_{i\in I} a_i \otimes \alpha_I(v)) \\
&= \bigotimes_{j\in J} c_j a_{f^{-1}(j)} \otimes \mathbf{q}[f] \circ \alpha_I(v) \\
&= \bigotimes_{j\in J} c_j a_{f^{-1}(j)} \otimes \alpha_J \circ \mathbf{p}[f](v) \\
&= (\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \alpha_J)(\bigotimes_{j\in J} c_j a_{f^{-1}(j)} \otimes \mathbf{p}[f])(v) \\
&= (\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \alpha_J)(\bigotimes_{j\in J} c_j \otimes \mathbf{p}[f])(\bigotimes_{i\in I} a_i \otimes v)
\end{aligned}$$

Thus the diagram commutes, and we have that $H\alpha_A$ is a morphism in $\mathbf{Sp}^A$.

Now, let $\mathrm{id}_{\mathbf{p}} : \mathbf{p} \to \mathbf{p}$ be the identity morphism in $\mathbf{Sp}$. We must show that $H\,\mathrm{id}_{\mathbf{p}} = \mathrm{id}_{H\mathbf{p}}$. Note that $H\,\mathrm{id}_{\mathbf{p}} : H\mathbf{p} \to H\mathbf{p}$ has sections given by

$$H\,\mathrm{id}_{\mathbf{p}}[I_A] := \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \mathrm{id}_{\mathbf{p}}$$

which is exactly the sections of $\mathrm{id}_{H\mathbf{p}}$.

Finally to show that for morphisms $\alpha : \mathbf{q} \to \mathbf{h}$ and $\beta : \mathbf{p} \to \mathbf{q}$, we have $H\alpha \circ H\beta = H(\alpha \circ \beta)$.

$$\begin{aligned}
(H\alpha \circ H\beta)[I_A] &= (\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \alpha_I) \circ (\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \beta_I) \\
&= (\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \alpha_I \circ \beta_I) \\
&= (\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes (\alpha \circ \beta)_I) \\
&= H(\alpha \circ \beta)[I_A]
\end{aligned}$$

Thus $H : \mathbf{Sp} \to \mathbf{Sp}^A$ is in fact a functor.

$\square$

Now, we show that this functor is a bistrong bilax monoidal functor, which implies it preserves Hopf monoids. For the remainder of this section, it suffices to work with the skeleton, $\tilde{\mathbf{Set}}^{A}$.

Proposition 8.1.1. *H is a bilax monoidal functor.*

Proof. In order to show that H is a bilax monoidal functor, we need to define natural transformations

$$\mathcal{M} \circ (H \times H) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} H \circ \mathcal{M}$$

where $\mathcal{M}$ denotes the tensor product of functors and $\mathcal{M} \circ (H \times H)$ and $H \circ \mathcal{M}$ are both functors from $\mathbf{Sp} \times \mathbf{Sp} \to \mathbf{Sp}^{A}$.
Let $\mathbf{p}, \mathbf{q} \in \mathbf{Sp}$, then

$$H\mathbf{p} \cdot H\mathbf{q} \underset{\psi_{\mathbf{p},\mathbf{q}}}{\overset{\varphi_{\mathbf{p},\mathbf{q}}}{\rightleftarrows}} H(\mathbf{p} \cdot \mathbf{q})$$

Note that $\varphi_{\mathbf{p},\mathbf{q}}$ and $\psi_{\mathbf{p},\mathbf{q}}$ themselves must be natural transformations. Observe, on an object

$$\begin{aligned} H\mathbf{p} \cdot H\mathbf{q}[n_A] &= \bigoplus_{R \sqcup T = [n]} H\mathbf{p}[R_A] \otimes H\mathbf{q}[T_A] \\ &= \bigoplus_{R \sqcup T = [n]} A^{\otimes R} \otimes \mathbf{p}[R] \otimes A^{\otimes T} H\mathbf{q}[T] \end{aligned}$$

and

$$\begin{aligned} H(\mathbf{p} \cdot \mathbf{q})[n_A] &= A^{\otimes N} \otimes (\mathbf{p} \cdot \mathbf{q})[n] \\ &= A^{\otimes N} \otimes \left(\bigoplus_{R \sqcup T = [n]} \mathbf{p}[R] \otimes \mathbf{q}[T] \right) \\ &= \bigoplus_{R \sqcup T = [n]} A^{\otimes N} \otimes \bigoplus_{R \sqcup T = [n]} \mathbf{p}[R] \otimes \mathbf{q}[T] \end{aligned}$$

Define the sections of $\varphi_{\mathbf{p},\mathbf{q}}$ and $\psi_{\mathbf{p},\mathbf{q}}$ on a fixed decomposition as follows:

$$\varphi_{\mathbf{p},\mathbf{q}} : A^{\otimes R} \otimes \mathbf{p}[R] \otimes A^{\otimes T} \otimes \mathbf{q}[T] \mapsto \beta(A^{\otimes R} \otimes A^{\otimes T}) \otimes \mathbf{p}[R] \otimes \mathbf{q}[T]$$

where β is the permutation that shuffles the R and T positions back into the natural order of $[n]$.

$$\psi_{\mathbf{p},\mathbf{q}} : A^{\otimes n} \otimes \mathbf{p}[R] \otimes \mathbf{q}[T] \mapsto A^{\otimes R} \otimes \mathbf{p}[R] \otimes A^{\otimes T} \otimes \mathbf{q}[T]$$

Now, observe that

$$\begin{aligned} H\mathbf{1}_{\mathbb{K}}[n_A] &:= A^{\otimes n} \otimes \mathbf{1}_{\mathbb{K}}[n] \\ &= \begin{cases} A^{\otimes \emptyset} \otimes \mathbb{K} & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \mathbb{K} & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases} \\ &= \mathbf{1}_{\mathbb{K}} \in \mathbf{Sp}^{A}. \end{aligned}$$

Thus $\varphi_0 = \mathrm{id}$ and $\psi_0 = \mathrm{id}$.

First, we will show the lax monoidal structure of H. In order to do so, we must show that φ is a natural transformation, is associative, and is left/right unital.

- Claim: φ is a natural transformation.
  Let $\mathbf{p}, \mathbf{q} \in \mathbf{Sp}$, define the sections of $\varphi_{\mathbf{p},\mathbf{q}} : H\mathbf{p} \cdot H\mathbf{q} \to H(\mathbf{p}\cdot\mathbf{q})$ as above which must also be natural transformations. For $1^{\otimes n} \otimes \sigma : [n]_A \to [n]_A$, we need the following diagram to commute:

$$\begin{tikzcd} H\mathbf{p}\cdot H\mathbf{q}[n_A] \arrow[r, "\varphi_{\mathbf{p},\mathbf{q}}[n]"] \arrow[d, "(H\mathbf{p}\cdot H\mathbf{q})[1^{\otimes n}\otimes\sigma]"'] & H(\mathbf{p}\cdot\mathbf{q})[n_A] \arrow[d, "H(\mathbf{p}\cdot\mathbf{q})[1^{\otimes n}\otimes\sigma]"] \\ H\mathbf{p}\cdot H\mathbf{q}[n_A] \arrow[r, "\varphi_{\mathbf{p},\mathbf{q}}[n]"'] & H(\mathbf{p}\cdot\mathbf{q})[n_A]. \end{tikzcd}$$

  For a fixed decomposition $S \sqcup T = [n]$, this diagram reduces to:

$$\begin{tikzcd} A^{\otimes S}\otimes\mathbf{p}[S]\otimes A^{\otimes T}\otimes\mathbf{q}[T] \arrow[r, "\varphi_{\mathbf{p},\mathbf{q}}[n]"] \arrow[d, "(H\mathbf{p}\cdot H\mathbf{q})[(1\cdots 1\otimes\sigma)]"'] & A^{\otimes n}\otimes\mathbf{p}[S]\otimes\mathbf{q}[T] \arrow[d, "H(\mathbf{p}\cdot\mathbf{q})[(1\cdots 1\otimes\sigma)]"] \\ A^{\otimes S}\otimes\mathbf{p}[S]\otimes A^{\otimes T}\otimes\mathbf{q}[T] \arrow[r, "\varphi_{\mathbf{p},\mathbf{q}}[n]"'] & A^{\otimes n}\otimes\mathbf{p}[S]\otimes\mathbf{q}[T]. \end{tikzcd}$$

  Following the top right composition yields:

$$A^{\otimes S}\otimes\mathbf{p}[S]\otimes A^{\otimes T}\otimes\mathbf{q}[T] \mapsto A^{\otimes n}\otimes\mathbf{p}[\sigma(S)]\otimes\mathbf{q}[\sigma(T)]$$

  Following the bottom left composition yields:

$$\begin{aligned} A^{\otimes S}\otimes\mathbf{p}[S]\otimes A^{\otimes T}\otimes\mathbf{q}[T] &\mapsto A^{\otimes\sigma(S)}\otimes\mathbf{p}[\sigma(S)]\otimes A^{\otimes\sigma(T)}\otimes\mathbf{q}[\sigma(T)] \\ &\mapsto A^{\otimes\sigma(S)\sqcup\sigma(T)}\otimes\mathbf{p}[\sigma(S)]\otimes\mathbf{q}[\sigma(T)] \end{aligned}$$

  Clearly, $\sigma(S)\sqcup\sigma(T) = [n]$ since $\sigma \in S_n$ and $S \sqcup T = [n]$. Thus the diagram commutes. Now to show the diagram commutes for $(b_{i_1}\cdots b_{i_n}\otimes\mathrm{id}) : [n]_A \to [n]_A$.

$$\begin{tikzcd} A^{\otimes S}\otimes\mathbf{p}[S]\otimes A^{\otimes T}\otimes\mathbf{q}[T] \arrow[r, "\varphi_{\mathbf{p},\mathbf{q}}[n]"] \arrow[d, "(H\mathbf{p}\cdot H\mathbf{q})[(b_{i_1}\cdots b_{i_n}\otimes\mathrm{id})]"'] & A^{\otimes n}\otimes\mathbf{p}[S]\otimes\mathbf{q}[T] \arrow[d, "H(\mathbf{p}\cdot\mathbf{q})[(b_{i_1}\cdots b_{i_n}\otimes\mathrm{id})]"] \\ A^{\otimes S}\otimes\mathbf{p}[S]\otimes A^{\otimes T}\otimes\mathbf{q}[T] \arrow[r, "\varphi_{\mathbf{p},\mathbf{q}}[n]"'] & A^{\otimes n}\otimes\mathbf{p}[S]\otimes\mathbf{q}[T] \end{tikzcd}$$

  Consider element $(b_{j_{k_1}}\cdots b_{j_{k_s}}\otimes v)\otimes(b_{j_{r_1}}\cdots b_{j_{r_t}}\otimes w) \in A^{\otimes S}\otimes\mathbf{p}[S]\otimes A^{\otimes T}\otimes\mathbf{q}[T]$, where $\{k_1 < \cdots < k_s\} = S$ and $\{r_1 < \cdots < r_t\} = T$. Then following the top right hand side of the composition yields:

$$\begin{aligned} &= (b_{i_1}\cdots b_{i_n}\otimes\mathbf{p}[\mathrm{id}\,|_S]\otimes\mathbf{q}[\mathrm{id}\,|_T])(b_{j_1}\cdots b_{j_n}\otimes v\otimes w) \\ &= b_{i_1}b_{j_1}\cdots b_{i_n}b_{j_n}\otimes v\otimes w \end{aligned}$$

  Following the bottom left hand side of the composition yields:

$$\begin{aligned} &= \varphi_{\mathbf{p},\mathbf{q}}[n]\circ(b_{i_{k_1}}b_{j_{k_1}}\cdots b_{i_{k_s}}b_{j_{k_s}}\otimes v\otimes b_{i_{r_1}}b_{j_{r_1}}\cdots b_{i_{r_t}}b_{j_{r_t}}\otimes w) \\ &= b_{i_1}b_{j_1}\cdots b_{i_n}b_{j_n}\otimes v\otimes w. \end{aligned}$$

Thus the diagram commutes and $\varphi_{\mathbf{p},\mathbf{q}}$ is a natural transformation.

Finally, let $\alpha : \mathbf{p} \to \mathbf{p}'$ and $\beta : \mathbf{q} \to \mathbf{q}'$ be two morphisms of species. To show that φ is a natural transformation we need the following diagram to commute:

$$\begin{tikzcd} H\mathbf{p}\cdot H\mathbf{q} \arrow[r, "\varphi_{\mathbf{p},\mathbf{q}}"] \arrow[d, "H\alpha\cdot H\beta"'] & H(\mathbf{p}\cdot\mathbf{q}) \arrow[d, "H(\alpha\cdot\beta)"] \\ H\mathbf{p}'\cdot H\mathbf{q}' \arrow[r, "\varphi_{\mathbf{p}',\mathbf{q}'}"'] & H(\mathbf{p}'\cdot\mathbf{q}'). \end{tikzcd}$$

On an object $[n]_A$ and decomposition $S \sqcup T = [n]$ this reduces to:

$$\begin{tikzcd} A^{\otimes S}\otimes\mathbf{p}[S]\otimes A^{\otimes T}\otimes\mathbf{q}[T] \arrow[r, "\varphi_{\mathbf{p},\mathbf{q}}[n_A]"] \arrow[d, "1^{\otimes S}\otimes\alpha_S\otimes 1^{\otimes T}\otimes\beta_T"'] & A^{\otimes n}\otimes\mathbf{p}[S]\otimes\mathbf{q}[T] \arrow[d, "1^{\otimes n}\otimes\alpha_S\otimes\beta_T"] \\ A^{\otimes S}\otimes\mathbf{p}'[S]\otimes A^{\otimes T}\otimes\mathbf{q}'[T] \arrow[r, "\varphi_{\mathbf{p}',\mathbf{q}'}[n_A]"'] & A^{\otimes n}\otimes\mathbf{p}'[S]\otimes\mathbf{q}'[T]. \end{tikzcd}$$

Consider an element $b_{j_{k_1}}\otimes\cdots\otimes b_{j_{k_s}}\otimes v\otimes b_{j_{r_1}}\cdots b_{j_{r_t}}\otimes w \in A^{\otimes S}\otimes\mathbf{p}[S]\otimes A^{\otimes T}\otimes\mathbf{q}[T]$ Following both the right hand and left hand corners of the diagram, yields the same desired result:

$$b_{j_{k_1}}\otimes\cdots\otimes b_{j_{k_s}}\otimes v\otimes b_{j_{r_1}}\cdots b_{j_{r_t}}\otimes w \mapsto b_{j_1}\otimes\cdots\otimes b_{j_n}\otimes\alpha_S(v)\otimes\beta_T(w)$$

Thus the above diagram commutes for each decomposition, and hence the sum. Therefore φ is a natural transformation.

- Claim: φ associative.

  Let $\mathbf{p},\mathbf{q}$, and $\mathbf{h}\in\mathbf{Sp}$. We must show that the following diagram commutes:

$$\begin{tikzcd} H\mathbf{p}\cdot H\mathbf{q}\cdot H\mathbf{h} \arrow[r, "\mathrm{id}\otimes\varphi_{\mathbf{q},\mathbf{h}}"] \arrow[d, "\varphi_{\mathbf{p},\mathbf{q}}\otimes\mathrm{id}"'] & H\mathbf{p}\cdot H(\mathbf{q}\cdot\mathbf{h}) \arrow[d, "\varphi_{\mathbf{p},\mathbf{q}\cdot\mathbf{h}}"] \\ H(\mathbf{p}\cdot\mathbf{q})\cdot H\mathbf{h} \arrow[r, "\varphi_{\mathbf{p}\cdot\mathbf{q},\mathbf{h}}"'] & H(\mathbf{p}\cdot\mathbf{q}\cdot\mathbf{h}). \end{tikzcd}$$

  For an object $[n]_A$, we will show that each component corresponding to a fixed decomposition commutes, say $R\sqcup S\sqcup T = [n]$. Before doing so, we need to understand the natural transformations $\varphi_{\mathbf{p},\mathbf{q}\cdot\mathbf{h}}$ and $\varphi_{\mathbf{p}\cdot\mathbf{q},\mathbf{h}}$.

  For $\varphi_{\mathbf{p},\mathbf{q}\cdot\mathbf{h}}$ the sections are given by:

$$A^{\otimes R}\otimes\mathbf{p}[R]\otimes A^{\otimes M}\otimes(\mathbf{q}\cdot\mathbf{h})[M] \mapsto \beta_{R,M}(A^{\otimes R}\otimes A^{\otimes M})\otimes\mathbf{p}[R]\otimes(\mathbf{q}\cdot\mathbf{h})[M]$$

  On a decomposition $S\sqcup T = M$, we have:

$$A^{\otimes R}\otimes\mathbf{p}[R]\otimes A^{\otimes M}\mathbf{q}[S]\cdot\mathbf{h}[T] \mapsto \beta_{R,M}(A^{\otimes R}\otimes A^{\otimes M})\otimes\mathbf{p}[R]\otimes\mathbf{q}[S]\cdot\mathbf{h}[T].$$

  Similarly, given decomposition $R\sqcup S\sqcup T = [n]$, the sections of $\varphi_{\mathbf{p}\cdot\mathbf{q},\mathbf{h}}$ are:

$$A^{\otimes R}\otimes\mathbf{p}[R]\otimes A^{\otimes S}\otimes\mathbf{q}[S]\otimes A^{\otimes T}\otimes\mathbf{h}[T] \mapsto \beta_{R\sqcup S,T}(A^{\otimes R\sqcup S}\otimes A^{\otimes T})\otimes\mathbf{p}[R]\otimes\mathbf{q}[S]\otimes\mathbf{h}[T].$$

  Finally, to show that

$$\varphi_{\mathbf{p},\mathbf{q}\cdot\mathbf{h}}\circ(\mathrm{id}\otimes\varphi_{\mathbf{q},\mathbf{h}}) = \varphi_{\mathbf{p}\cdot\mathbf{q},\mathbf{h}}\circ(\varphi_{\mathbf{p},\mathbf{q}}\otimes\mathrm{id})$$

For the lefthand side, $\varphi_{\mathbf{p},\mathbf{q}\cdot\mathbf{h}} \circ (\mathrm{id} \otimes \varphi_{\mathbf{q},\mathbf{h}})(A^{\otimes R} \otimes \mathbf{p}[R] \otimes A^{\otimes S} \otimes \mathbf{q}[S] \otimes A^{\otimes T} \otimes \mathbf{h}[T])$:

$$\begin{aligned}
&= \varphi_{\mathbf{p},\mathbf{q}\cdot\mathbf{h}}(A^{\otimes R} \otimes \mathbf{p}[R] \otimes \beta_{S,T}(A^{\otimes S} \otimes A^{\otimes T}) \otimes \mathbf{q}[S] \otimes \mathbf{h}[T]) \\
&= \varphi_{\mathbf{p},\mathbf{q}\cdot\mathbf{h}}(A^{\otimes R} \otimes \mathbf{p}[R] \otimes A^{\otimes S\sqcup T} \otimes \mathbf{q}[S] \otimes \mathbf{h}[T]) \\
&= \beta_{R,S\sqcup T}(A^{\otimes R} \otimes A^{\otimes S\sqcup T})\mathbf{p}[R] \otimes \mathbf{q}[S] \otimes \mathbf{h}[T] \\
&= A^{\otimes n} \otimes \mathbf{p}[R] \otimes \mathbf{q}[S] \otimes \mathbf{h}[T].
\end{aligned}$$

For the righthand side, $\varphi_{\mathbf{p}\cdot\mathbf{q},\mathbf{h}} \circ (\varphi_{\mathbf{p},\mathbf{q}} \otimes \mathrm{id})(A^{\otimes R} \otimes \mathbf{p}[R] \otimes A^{\otimes S} \otimes \mathbf{q}[S] \otimes A^{\otimes T} \otimes \mathbf{h}[T])$:

$$\begin{aligned}
&= \varphi_{\mathbf{p}\cdot\mathbf{q},\mathbf{h}}(\beta_{R,S}(A^{\otimes R} \otimes A^{\otimes S}) \otimes \mathbf{p}[R] \otimes \mathbf{q}[S] \otimes A^{\otimes T} \otimes \mathbf{h}[T]) \\
&= \varphi_{\mathbf{p}\cdot\mathbf{q},\mathbf{h}}(A^{\otimes R\sqcup S} \otimes \mathbf{p}[R] \otimes \mathbf{q}[S] \otimes A^{\otimes T} \otimes \mathbf{h}[T]) \\
&= \beta_{R\sqcup S,T}(A^{\otimes R\sqcup S} \otimes A^{\otimes T}) \otimes \mathbf{p}[R] \otimes \mathbf{q}[S] \otimes \mathbf{h}[T] \\
&= A^{\otimes n} \otimes \mathbf{p}[R] \otimes \mathbf{q}[S] \otimes \mathbf{h}[T].
\end{aligned}$$

Thus, φ is associative.

- Claim: φ is left/right unital. In order to show that φ is left unital, we must show that the following diagram commutes:

$$\begin{tikzcd}
\mathbf{1}_{\mathbb{K}} \cdot H\mathbf{p} \arrow[d, "\varphi_0 \otimes \mathrm{id}"'] & H\mathbf{p} \arrow[l, "\lambda_{H\mathbf{p}}"'] \arrow[d, "H(\lambda_{\mathbf{p}})"] \\
H\mathbf{1}_{\mathbb{K}} \cdot H\mathbf{p} \arrow[r, "\varphi_{\mathbf{1}_{\mathbb{K}},\mathbf{p}}"'] & H(\mathbf{1}_{\mathbb{K}} \cdot \mathbf{p})
\end{tikzcd}$$

On an object $[n]_A$ this reduces to:

$$\begin{tikzcd}
\bigoplus_{S\sqcup T=[n]} \mathbf{1}_{\mathbb{K}}[[s]_A] \otimes A^{\otimes T} \otimes \mathbf{p}[T] \arrow[d, "\varphi_0 \otimes \mathrm{id}"'] & A^{\otimes n} \otimes \mathbf{p}[n] \arrow[l, "\lambda_{H\mathbf{p}}"'] \arrow[d, "H(\lambda_{\mathbf{p}})"] \\
\bigoplus_{S\sqcup T=[n]} A^{\otimes S} \otimes \mathbf{1}_{\mathbb{K}}[S] \otimes A^{\otimes T} \otimes \mathbf{p}[T] \arrow[r, "\varphi_{\mathbf{1}_{\mathbb{K}},\mathbf{p}}"'] & \bigoplus_{S\sqcup T=[n]} A^{\otimes n} \otimes \mathbf{1}_{\mathbb{K}}[S] \otimes \mathbf{p}[T]
\end{tikzcd}$$

By definition of $\mathbf{1}_{\mathbb{K}}$, we only need to consider $S = \emptyset$. $\lambda_{H\mathbf{p}}$ is the vector space isomorphism for the left unitator map in the monoidal structure. We also have that $H(\lambda_{\mathbf{p}}) = 1_A^{\otimes n} \otimes \lambda_{\mathbf{p}}$, $\varphi_0 = \mathrm{id}$ yields the following diagram:

$$\begin{tikzcd}
\mathbb{K} \otimes A^{\otimes n} \otimes \mathbf{p}[n] \arrow[d, "\mathrm{id} \otimes \mathrm{id}"'] & A^{\otimes n} \otimes \mathbf{p}[n] \arrow[l, "\lambda_{H\mathbf{p}}"'] \arrow[d, "H(\lambda_{\mathbf{p}})"] \\
\mathbb{K} \otimes A^{\otimes n} \otimes \mathbf{p}[n] \arrow[r, "\varphi_{\mathbf{1}_{\mathbb{K}},\mathbf{p}}"'] & A^{\otimes n} \otimes \mathbb{K} \otimes \mathbf{p}[n].
\end{tikzcd}$$

Now to show that H is a colax monoidal functor.

- Claim: ψ is a natural transformation.
Let $\mathbf{p}, \mathbf{q} \in \mathbf{Sp}$, define the sections $\psi_{\mathbf{p},\mathbf{q}} : H(\mathbf{p} \cdot \mathbf{q}) \to H\mathbf{p} \cdot H\mathbf{q}$ as above; we must show that these sections are natural transformations. For $(1 \cdots 1 \otimes \sigma) : [n]_A \to [n]_A$, we need the following diagram to commute:

$$\begin{tikzcd}
H(\mathbf{p}\cdot\mathbf{q})[n_A] \arrow[r, "\psi_{\mathbf{p},\mathbf{q}}[n_A]"] \arrow[d, "H(\alpha\cdot\beta)[(1\cdots 1\otimes\sigma)]"'] & H\mathbf{p}\cdot H\mathbf{q}[n_A] \arrow[d, "H\alpha\cdot H\beta[(1\cdots 1\otimes\sigma)]"] \\
H(\mathbf{p}\cdot\mathbf{q})[n_A] \arrow[r, "\psi_{\mathbf{p},\mathbf{q}}[n_A]"'] & H\mathbf{p}\cdot H\mathbf{q}[n_A].
\end{tikzcd}$$

For a fixed decomposition $S \sqcup T = [n]$, this diagram reduces to:

$$\begin{tikzcd}
A^{\otimes n}\otimes\mathbf{p}[S]\otimes\mathbf{q}[T] \arrow[r, "\psi_{\mathbf{p},\mathbf{q}}[n_A]"] \arrow[d, "H(\alpha\cdot\beta)[(1\cdots 1\otimes\sigma)]"'] & A^{\otimes S}\otimes\mathbf{p}[S]\otimes A^{\otimes T}\otimes\mathbf{q}[T] \arrow[d, "H\mathbf{p}\cdot H\mathbf{q}[(1\cdots 1\otimes\sigma)]"] \\
A^{\otimes n}\otimes\mathbf{p}[S]\otimes\mathbf{q}[T] \arrow[r, "\psi_{\mathbf{p},\mathbf{q}}[n_A]"'] & A^{\otimes S}\otimes\mathbf{p}[S]\otimes A^{\otimes T}\otimes\mathbf{q}[T].
\end{tikzcd}$$

Following the top right and bottom left compositions unambiguously yields:

$$A^{\otimes n}\otimes\mathbf{p}[S]\otimes\mathbf{q}[T] \mapsto A^{\otimes\sigma(S)}\otimes\mathbf{p}[\sigma(S)]\otimes A^{\sigma(T)}\otimes\mathbf{q}[\sigma(T)].$$

For $(b_{i_1}\cdots b_{i_n}\otimes\mathrm{id}) : [n]_A \to [n]_A$ and fixed decomposition $S \sqcup T = [n]$,we need the following to commute:

$$\begin{tikzcd}
A^{\otimes n}\otimes\mathbf{p}[S]\otimes\mathbf{q}[T] \arrow[r, "\psi_{\mathbf{p},\mathbf{q}}[n_A]"] \arrow[d, "H(\alpha\cdot\beta)[(b_{i_1}\cdots b_{i_n}\otimes\mathrm{id})]"'] & A^{\otimes S}\otimes\mathbf{p}[S]\otimes A^{\otimes T}\otimes\mathbf{q}[T] \arrow[d, "H\mathbf{p}\cdot H\mathbf{q}[(b_{i_1}\cdots b_{i_n}\otimes\mathrm{id})]"] \\
A^{\otimes n}\otimes\mathbf{p}[S]\otimes\mathbf{q}[T] \arrow[r, "\psi_{\mathbf{p},\mathbf{q}}[n_A]"'] & A^{\otimes S}\otimes\mathbf{p}[S]\otimes A^{\otimes T}\otimes\mathbf{q}[T].
\end{tikzcd}$$

Consider the element $b_{j_1}\cdots b_{j_n}\otimes v\otimes w \in A^{\otimes n}\otimes\mathbf{p}[S]\otimes\mathbf{q}[T]$. Following the top right corner of the diagram yields:

$$\begin{aligned}
b_{j_1}\cdots b_{j_n}\otimes v\otimes w &\mapsto (b_{j_{k_1}}\cdots b_{j_{k_s}}\otimes v)\otimes(b_{j_{r_1}}\cdots b_{j_{r_t}}\otimes w)\\
&\mapsto (b_{i_{k_1}}b_{j_{k_1}}\cdots b_{i_{k_s}}b_{j_{k_s}}\otimes v)\otimes(b_{i_{r_1}}b_{j_{r_1}}\cdots b_{i_{r_t}}b_{j_{r_t}}\otimes w)
\end{aligned}$$

Following the bottom left corner yields:

$$\begin{aligned}
b_{j_1}\cdots b_{j_n}\otimes v\otimes w &\mapsto b_{i_1}b_{j_1}\cdots b_{i_n}b_{j_n}\otimes v\otimes w\\
&\mapsto (b_{i_{k_1}}b_{j_{k_1}}\cdots b_{i_{k_s}}b_{j_{k_s}}\otimes v)\otimes(b_{i_{r_1}}b_{j_{r_1}}\cdots b_{i_{r_t}}b_{j_{r_t}}\otimes w)
\end{aligned}$$

Since the diagram commutes for each decomposition, we have that the sections $\psi_{\mathbf{p},\mathbf{q}}$ are natural transformations.

Finally, let $\alpha : \mathbf{p} \to \mathbf{p}'$ and $\beta : \mathbf{q} \to \mathbf{q}'$ be two morphisms of species. We want to show that the following diagram commutes:

$$\begin{tikzcd}
H(\mathbf{p}\cdot\mathbf{q}) \arrow[r, "\psi_{\mathbf{p},\mathbf{q}}"] \arrow[d, "H(\alpha\cdot\beta)"'] & H\mathbf{p}\cdot H\mathbf{q} \arrow[d, "H\alpha\cdot H\beta"] \\
H(\mathbf{p}'\cdot\mathbf{q}') \arrow[r, "\psi_{\mathbf{p}',\mathbf{q}'}"'] & H\mathbf{p}'\cdot H\mathbf{q}'.
\end{tikzcd}$$

On an object $[n]_A$, a decomposition $S \sqcup T = [n]$, this reduces to the following diagram:

$$\begin{tikzcd}
A^{\otimes n} \otimes \mathbf{p}[S] \otimes \mathbf{q}[T] \arrow[r, "\psi_{\mathbf{p},\mathbf{q}}[n_A]"] \arrow[d, "1^{\otimes n} \otimes \alpha_S \otimes \beta_T"'] & A^{\otimes S} \otimes \mathbf{p}[S] \otimes A^{\otimes T} \otimes \mathbf{q}[T] \arrow[d, "1^{\otimes S} \otimes \alpha_S \otimes 1^{\otimes T} \otimes \beta_T"] \\
A^{\otimes n} \otimes \mathbf{p}'[S] \otimes \mathbf{q}'[T] \arrow[r, "\psi_{\mathbf{p}',\mathbf{q}'}[n_A]"'] & A^{\otimes S} \otimes \mathbf{p}'[S] \otimes A^{\otimes T} \otimes \mathbf{q}'[T].
\end{tikzcd}$$

Let $b_{j_1} \cdots b_{j_n} \otimes v \otimes w \in A^{\otimes n} \otimes \mathbf{p}[S] \otimes \mathbf{q}[T]$, then following the right hand and left hand corners of the diagram both yield:

$$b_{j_1} \cdots b_{j_n} \otimes v \otimes w \mapsto (b_{j_{k_1}} \cdots b_{j_{k_s}} \otimes \alpha_S(v)) \otimes (b_{j_{r_1}} \cdots b_{j_{r_t}} \otimes \beta_T(w)).$$

The above diagram commutes for each decomposition, thus ψ is a natural transformation.

- Claim: φ coassociative.
  Let $\mathbf{p}, \mathbf{q}$ and $\mathbf{h} \in \mathbf{Sp}$. We must show that the following diagram commutes:

$$\begin{tikzcd}
H(\mathbf{p} \cdot \mathbf{q} \cdot \mathbf{h}) \arrow[r, "\psi_{\mathbf{p}\cdot\mathbf{q},\mathbf{h}}"] \arrow[d, "\psi_{\mathbf{p},\mathbf{q}\cdot\mathbf{h}}"'] & H(\mathbf{p} \cdot \mathbf{q}) \cdot H\mathbf{h} \arrow[d, "\psi_{\mathbf{p},\mathbf{q}} \otimes \mathrm{id}"] \\
H\mathbf{p} \cdot H(\mathbf{q} \cdot \mathbf{h}) \arrow[r, "\mathrm{id} \otimes \psi_{\mathbf{q},\mathbf{h}}"'] & H\mathbf{p} \cdot H\mathbf{q} \cdot H\mathbf{h}.
\end{tikzcd}$$

  For an object $[n]_A$, we will show that each component corresponding to a fixed decomposition, $R \sqcup S \sqcup T = [n]$ commutes. Before doing so, we must understand the natural transformations $\psi_{\mathbf{p}\cdot\mathbf{q},\mathbf{h}}$ and $\psi_{\mathbf{p},\mathbf{q}\cdot\mathbf{h}}$.

  For $\psi_{\mathbf{p}\cdot\mathbf{q},\mathbf{h}}$, the sections are given by:

$$A^{\otimes n} \otimes (\mathbf{p} \cdot \mathbf{q})[M] \otimes \mathbf{h}[T] \mapsto A^{\otimes M} \otimes (\mathbf{p} \cdot \mathbf{q})[M] \otimes A^{\otimes T} \otimes \mathbf{h}[T].$$

  On a decomposition $R \sqcup S = M$, we have:

$$A^{\otimes n} \otimes \mathbf{p}[R] \otimes \mathbf{q}[S] \otimes \mathbf{h}[T] \mapsto A^{\otimes R \sqcup S} \otimes \mathbf{p}[R] \otimes \mathbf{q}[S] \otimes A^{\otimes T} \otimes \mathbf{h}[T].$$

  Similarly, for $\psi_{\mathbf{p},\mathbf{q}\cdot\mathbf{h}}$ and decomposition $S \sqcup T = [M]$ we have:

$$A^{\otimes n} \otimes \mathbf{p}[R] \otimes \mathbf{q}[S] \otimes \mathbf{h}[T] \mapsto A^{\otimes R} \otimes \mathbf{p}[R] \otimes A^{\otimes S \sqcup T} \otimes \mathbf{q}[S] \otimes \mathbf{h}[T].$$

  Finally, to show that

$$(\psi_{\mathbf{p},\mathbf{q}} \otimes \mathrm{id}) \circ \psi_{\mathbf{p}\cdot\mathbf{q},\mathbf{h}} = (\mathrm{id} \otimes \psi_{\mathbf{q},\mathbf{h}}) \circ \psi_{\mathbf{p},\mathbf{q}\cdot\mathbf{h}}.$$

  Both compositions, for a fixed decomposition, yields the same map:

$$A^{\otimes n} \otimes \mathbf{p}[R] \otimes \mathbf{q}[S] \otimes \mathbf{h}[T] \mapsto A^{\otimes R} \otimes \mathbf{p}[R] \otimes A^{\otimes S} \otimes \mathbf{q}[S] \otimes A^{\otimes T} \otimes \mathbf{h}[T].$$

  Therefore, ψ is coassociative.

- Claim: φ is left/right counital.

Finally, to show that H is a bilax monoidal functor, we must show that the braiding and unitality conditions are satisfied.

- In order for the braiding condition to hold, we must show the following:

  $$\psi_{\mathbf{p}\cdot\mathbf{r},\mathbf{q}\cdot\mathbf{h}} \circ H(\mathrm{id}\cdot\beta\cdot\mathrm{id}) \circ \varphi_{\mathbf{p}\cdot\mathbf{q},\mathbf{r}\cdot\mathbf{h}} = (\varphi_{\mathbf{p},\mathbf{r}}\cdot\varphi_{\mathbf{q},\mathbf{h}}) \circ (\mathrm{id}\cdot\beta\cdot\mathrm{id}) \circ (\psi_{\mathbf{p},\mathbf{q}}\cdot\psi_{\mathbf{r},\mathbf{h}})$$

  The above are natural transformations from

  $$H(\mathbf{p}\cdot\mathbf{q})\cdot H(\mathbf{r}\cdot\mathbf{h}) \to H(\mathbf{p}\cdot\mathbf{r})\cdot H(\mathbf{q}\cdot\mathbf{h})$$

  We must first understand the natural transformations $\varphi_{\mathbf{p}\cdot\mathbf{q},\mathbf{r}\cdot\mathbf{h}}$ and $\psi_{\mathbf{p}\cdot\mathbf{r},\mathbf{q}\cdot\mathbf{h}}$.

  The sections of $\varphi_{\mathbf{p}\cdot\mathbf{q},\mathbf{r}\cdot\mathbf{h}}$ on a decomposition $S\sqcup T=[n]$ are given by:

  $$A^{\otimes S}\otimes(\mathbf{p}\cdot\mathbf{q})[S]\otimes A^{\otimes T}\otimes(\mathbf{r}\cdot\mathbf{h})[T] \mapsto A^{\otimes n}\otimes(\mathbf{p}\cdot\mathbf{q})[S]\otimes(\mathbf{r}\cdot\mathbf{h})[T]$$

  On decompositions $B\sqcup C=S$ and $U\sqcup V=T$, we have

  $$A^{\otimes S}\otimes\mathbf{p}[B]\otimes\mathbf{q}[C]\otimes A^{\otimes T}\otimes\mathbf{r}[U]\otimes\mathbf{h}[V] \mapsto A^{\otimes n}\otimes\mathbf{p}[B]\otimes\mathbf{q}[C]\otimes\mathbf{r}[U]\otimes\mathbf{h}[V]$$

  The sections of $\psi_{\mathbf{p}\cdot\mathbf{r},\mathbf{q}\cdot\mathbf{h}}$ on a decomposition $S\sqcup T=[n]$ are given by:

  $$A^{\otimes n}\otimes(\mathbf{p}\cdot\mathbf{r})[S]\otimes(\mathbf{q}\cdot\mathbf{h})[T] \mapsto A^{\otimes S}\otimes(\mathbf{p}\cdot\mathbf{r})[S]\otimes A^{\otimes T}\otimes(\mathbf{q}\cdot\mathbf{h})[T]$$

  On decompositions $B\sqcup C=S$ and $U\sqcup V=T$, we have

  $$A^{\otimes n}\otimes\mathbf{p}[B]\otimes\mathbf{r}[C]\otimes\mathbf{q}[U]\otimes\mathbf{h}[V] \mapsto A^{\otimes S}\otimes\mathbf{p}[B]\otimes\mathbf{r}[C]\otimes A^{\otimes T}\otimes\mathbf{q}[U]\otimes\mathbf{h}[V]$$

  Recall that $H(\mathrm{id}\cdot\beta\cdot\mathrm{id}) := 1_A^{\otimes n}\otimes\mathrm{id}\otimes\beta\otimes\mathrm{id}$ where $\beta:\mathbf{p}[S]\otimes\mathbf{q}[T]\mapsto\mathbf{q}[T]\otimes\mathbf{p}[S]$. Following the left and right hand sides of the desired equality, for a fixed decomposition $B\sqcup C\sqcup U\sqcup V=[n]$, leads to the same map:

  $$A^{\otimes S}\mathbf{p}[B]\otimes\mathbf{q}[C]\otimes A^{\otimes T}\otimes\mathbf{r}[U]\otimes\mathbf{h}[V] \mapsto A^{\otimes B\sqcup U}\otimes\mathbf{p}[B]\otimes\mathbf{r}[U]\otimes A^{\otimes C\sqcup V}\otimes\mathbf{q}[C]\otimes\mathbf{h}[V]$$

- Finally, we must show the unitalty conditions.
  Since both $\varphi_0=\mathrm{id}$ and $\psi_0=\mathrm{id}$, and $\lambda_{\mathbf{1}_{\mathbb{K}}}$ and $\rho_{\mathbf{1}_{\mathbb{K}}}$ are isomorphisms, the unitality conditions in Diagram (16) are satisfied.

Therefore H is a bliax monoidal functor.

□

PROPOSITION 8.1.2. *The functor H is a bistrong functor.*

PROOF. To show that H is a bistrong functor, it suffices to show that $\varphi_0\circ\psi_0=\mathrm{id}$ and that $\psi\circ\varphi=\mathrm{id}$. It's clear that $\varphi_0\circ\psi_0=\mathrm{id}$ since $\varphi_0=\mathrm{id}$ and $\psi_0=\mathrm{id}$. Now, consider the decomposition $S\sqcup T=[n]$, then we have

$$A^{\otimes S}\otimes\mathbf{p}[S]\otimes A^{\otimes T}\otimes\mathbf{q}[T] \xrightarrow{\varphi_{\mathbf{p},\mathbf{q}}} A^{\otimes n}\otimes\mathbf{p}[S]\otimes\mathbf{q}[T] \xrightarrow{\psi_{\mathbf{p},\mathbf{q}}} A^{\otimes S}\otimes\mathbf{p}[S]\otimes A^{\otimes T}\otimes\mathbf{q}[T]$$

Therefore H is a bistrong functor. □

COROLLARY 8.1.3. *If $\mathbf{p}\in\boldsymbol{Sp}$ is a Hopf monoid, then $H\mathbf{p}\in\boldsymbol{Sp}^A$ is a Hopf monoid.*

PROOF. Since H is a bistrong bilax functor, we have that $H\mathbf{p}\in\mathbf{Sp}^A$ is a Hopf monoid since $\mathbf{p}\in\mathbf{Sp}$ was a Hopf monoid. □

Remark 8.1.4. The above results hold true if instead of decorating a species with A, but instead with any A-module V. Then

$$H_V : \mathbf{Sp} \to \mathbf{Sp}^A$$

$$\begin{aligned} H_V\mathbf{p}[I_A] &:= V^{\otimes n} \otimes \mathbf{p}[I] \\ H_V\mathbf{p}[\bigotimes_{j\in J} c_j \otimes f] &:= \bigotimes_{j\in J} c_j \otimes \mathbf{p}[f], \end{aligned}$$

where on a pure tensor $\bigotimes_{i\in I} v_i \otimes w \in V^I \otimes \mathbf{p}[I]$, we have

$$(\bigotimes_{j\in J} c_j \otimes \mathbf{p}[f])(\bigotimes_{i\in I} v_i \otimes w) = \bigotimes_{j\in J} c_j.v_{f^{-1}(j)} \otimes \mathbf{p}[f](w)$$

and extend by linearity.
Given $\alpha : \mathbf{p} \to \mathbf{q}$, a morphism of species, we have:

$$H_V\alpha_{I_A} \;\; := \;\; \mathrm{id} \otimes \cdots \mathrm{id} \otimes \alpha_I.$$

8.1.1. Decorating an A-species

The above section gave two ways to decorate a classical species to turn it into an A-species; this relied on decorating with an A-module. In this section, we start with an A-species and decorate it with an $A \wr S_n$ module. Let V be an A-module. There is an action of $A \wr S_n$ on $V^{\otimes n}$ given by

$$a_1 \cdots a_n \otimes \sigma.(v_1 \cdots \otimes v_n) = a_1.v_{\sigma^{-1}(1)} \cdots a_n.v_{\sigma^{-1}(n)}$$

for all $a_1 \cdots a_n \otimes \sigma \in A \wr S_n$ and $v_1 \cdots v_n \in V^{\otimes n}$. Let $\mathbf{p} \in \mathbf{Sp}^A$, then $A \wr S_n$ acts on $V^{\otimes n} \otimes \mathbf{p}[n_A]$ via its coproduct:

$$(a_1 \cdots a_n \otimes \sigma).(v_1 \cdots v_n \otimes x) = \sum_{(a_1 \cdots a_n \otimes \sigma)} (a_1 \cdots a_n \otimes \sigma)_1.(v_1 \cdots v_n) \otimes (a_1 \cdots a_n \otimes \sigma)_2.x$$

for all $(a_1 \cdots a_n \otimes \sigma) \in A \wr S_n$, $v_1 \cdots v_n \in V^{\otimes n}$, and $x \in \mathbf{p}[n_A]$.

Remark 8.1.5. Here, we only define what this decorated A-species is. One could do a similar analysis to that of [3] to see what results may come about. A final thing to observe, is that when $A = \mathbb{K}$, we recover Aguiar and Mahajan's notion of decorated species.

CHAPTER 9

A Functor from Species to A-Species

In this section, we will construct a functor from the category of species to the category of A-species. It is more subtle and interesting than the functor defined in Chapter 8.1. This functor will be bilax and bistrong monoidal and have the additional property that it sends the regular representation of S_n to the regular representation of $A \wr S_n$.

9.1. The Functor $\mathcal{S}^A$

We generalize the notion of a section map as defined in [10] .

Definition 9.1.1. Let B be a fixed basis for A and I be a finite set. A *section* for B is a map $s : I \to B \times I$ such that $s(i) \in B \times \{i\} \;\; \forall i \in I$.

Remark 9.1.2. This definition of a section encompasses the definition given in [10]; take $A = \mathbb{K}C_2$ and $B = C_2$. See Subsection 9.2 and Definition 6.1.3 in Section 6.1 for a reminder of section maps.

Using the section maps as in Definition 9.1.1, we can define a bistrong monoidal functor $\mathcal{S}^A : \mathbf{Sp} \to \mathbf{Sp}^A$.

Definition 9.1.3. The functor

$$\mathcal{S}^A : \mathbf{Sp} \to \mathbf{Sp}^A$$

is defined for a species $\mathbf{p} \in \mathbf{Sp}$, $I_A \in \mathbf{Set}^A$, endomorphisms $1 \cdots 1 \otimes g$ and $b_{i_1} \cdots b_{i_n} \otimes \mathrm{id} \in A \wr S_{|I|}$, and a morphism of species $\alpha : \mathbf{p} \to \mathbf{q}$ by:

$$\mathcal{S}^A\mathbf{p}[I_A] := \bigoplus_{s:I\to B\times I} \mathbf{p}[s(I)]$$

$$\mathcal{S}^A\mathbf{p}[1\cdots 1 \otimes g] := \bigoplus_{s:I\to B\times I} \mathbf{p}[(1\cdots 1 \otimes g)|_{s(I)}]$$

$$\mathcal{S}^A\mathbf{p}[(b_{i_1}\cdots b_{i_n} \otimes \mathrm{id})] := \bigoplus_{s:I\to B\times I} \mathbf{p}[(b_{i_1}\cdots b_{i_n} \otimes \mathrm{id})|_{s(I)}]$$

$$\mathcal{S}^A\alpha[I_A] := \bigoplus_{s:I\to B\times I} \alpha_{[s(I)]},$$

where the sums are taken over all sections $s : I \to B \times I$.

For a fixed section, $s(I) = \{(b_{j_1}, j_1), ..., (b_{j_n} j_n)\}$, the linear maps are as follows:

$$\mathbf{p}[1\cdots 1 \otimes g] : \mathbf{p}[s(I)] \to \mathbf{p}[\{(b_{j_1}, j_{g(1)}), ..., (b_{j_n}, j_{g(n)})\}]$$

$$\mathbf{p}[(b_{i_1}\cdots b_{i_n} \otimes \mathrm{id})] := \sum_{\underline{k}\in[m]^n} c^{\underline{k}}_{\underline{i},\underline{j}} \mathbf{p}[f^{\underline{k}}_s] : \mathbf{p}[s(I)] \to \mathbf{p}[\{(b_{k_1}, j_1), ..., (b_{k_n}, j_n)\}]$$

where

$$f_s^{\underline{k}} : s(I) \to \{(b_{k_1}, i_1), ..., (b_{k_n}, i_n)\}$$

$$(b_{j_t}, t) \mapsto (b_{k_t}, i_t) \tag{30}$$

and the $c^{\underline{k}}_{\underline{i},\underline{j}}$ are as defined in Equation (28) .

PROOF.
To show that $\mathcal{S}^A$ is a functor, it suffices to use the skeleton $\mathbf{Set}^A$:
Let $\mathbf{p} \in \mathbf{Sp}$, we must show that $\mathcal{S}^A\mathbf{p} \in \mathbf{Sp}^A$.

- It's clear that $\mathcal{S}^A\mathbf{p}[n_A] := \bigoplus_{s:[n]\to B\times[n]} \mathbf{p}[s([n])] \in \mathbf{Vec}_{\mathbb{K}}$.
- $\mathcal{S}^A\mathbf{p}[1\cdots 1\otimes\sigma] := \bigoplus_{s:[n]\to B\times[n]} \mathbf{p}[(1\cdots 1\otimes\sigma)|_{s([n])}]$ is a linear map: For a fixed section, $s([n]) = \{(b_{i_1}, 1), .., (b_{i_n}, n)\}$,$\mathbf{p}[(1\cdots 1\otimes\sigma)|_{s([n])}]$ is induced from the set bijection

  $$1\cdots 1\otimes\sigma : \{(b_{i_1}, 1), .., (b_{i_n}, n)\} \to \{(b_{i_1}, \sigma(1)), .., (b_{i_n}, \sigma(n))\}$$

  thus is a linear map since $\mathbf{p} \in \mathbf{Sp}$. Ranging over all sections, gives a sum of linear maps.

- $\mathcal{S}^A\mathbf{p}[(b_{i_1}\cdots b_{i_n}\otimes \mathrm{id})] := \bigoplus_{s:[n]\to B\times[n]} \mathbf{p}[(b_{i_1}\cdots b_{i_n}\otimes \mathrm{id})|_{s([n])}]$ is a linear map: For each fixed section, $s([n]) = \{(b_{i_1}, 1), .., (b_{i_n}, n)\}$, we have that

  $$\mathbf{p}[(b_{i_1}\cdots b_{i_n}\otimes \mathrm{id})|_{s([n])}] := \sum_{\underline{k}\in[m]^n} c^{\underline{k}}_{\underline{i},\underline{j}} \mathbf{p}[f_s^{\underline{k}}]$$

  is a direct sum of linear maps $\mathbf{p}[f_s^{\underline{k}}]$. Note that $\mathbf{p}[f_s^{\underline{k}}]$ is a linear map since $f_s^{\underline{k}} : s([n]) \to \{(b_{k_1}, 1), ..., (b_{k_n}, n)\}$ is a set bijection and $\mathbf{p} \in \mathbf{Sp}$.
  Therefore, $\mathcal{S}^A\mathbf{p}[(b_{i_1}\cdots b_{i_n}\otimes \mathrm{id})]$ is a linear map because it is the direct sum of linear maps.

Now, let $\mathbf{p}$ and $\mathbf{q} \in \mathbf{Sp}$ and let $\alpha : \mathbf{p} \to \mathbf{q}$ be a morphism of species. We want to show that $\mathcal{S}^A\alpha : \mathcal{S}^A\mathbf{p} \to \mathcal{S}^A\mathbf{q}$ is a natural transformation. For all $[n]_A \in \mathbf{Set}^A$, define the section maps as follows:

$$\mathcal{S}^A\alpha_{[n_A]} := \bigoplus_{s:[n]\to B\times[n]} \alpha_{[s([n])]}$$

First let $1\cdots 1\otimes\sigma : [n]_A \to [n]_A$. We must show the following diagram commutes:

$$\begin{tikzcd}
\mathcal{S}^A\mathbf{p}[n_A] \arrow[r, "\mathcal{S}^A\alpha_{[n_A]}"] \arrow[d, "\mathcal{S}^A\mathbf{p}[(1\cdots 1\otimes\sigma)]"'] & \mathcal{S}^A\mathbf{q}[n_A] \arrow[d, "\mathcal{S}^A\mathbf{q}[(1\cdots 1\otimes\sigma)]"] \\
\mathcal{S}^A\mathbf{p}[n_A] \arrow[r, "\mathcal{S}^A\alpha_{[n_A]}"'] & \mathcal{S}^A\mathbf{q}[n_A]
\end{tikzcd}$$

This diagram reduces to:

$$\begin{tikzcd}
\bigoplus_{s:[n]\to B\times[n]} \mathbf{p}[s([n])] \arrow[r, "\bigoplus_{s:[n]\to B\times[n]} \alpha_{s([n])}"] \arrow[d, "\bigoplus_{s:[n]\to B\times[n]} \mathbf{p}[(1\cdots 1\otimes\sigma)|_s]"'] & \bigoplus_{s:[n]\to B\times[n]} \mathbf{q}[s([n])] \arrow[d, "\bigoplus_{s:[n]\to B\times[n]} \mathbf{q}[(1\cdots 1\otimes\sigma)|_s]"] \\
\bigoplus_{s:[n]\to B\times[n]} \mathbf{p}[s([n])] \arrow[r, "\bigoplus_{s:[n]\to B\times[n]} \alpha_{s([n])}"'] & \bigoplus_{s:[n]\to B\times[n]} \mathbf{q}[s([n])].
\end{tikzcd}$$

For a given section, the corresponding component of the diagram commutes since α is a natural transformation and $(1\cdots 1\otimes\sigma) : s([n]) \to s'([n])$ is a set bijection. Thus the overall diagram commutes.

Now let $(b_{i_1}\cdots b_{i_n}\otimes \mathrm{id}) : [n]_A \to [n]_A$. We must show that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{S}^A\mathbf{p}[n_A] \arrow[r, "\mathcal{S}^A\alpha_{[n_A]}"] \arrow[d, "\mathcal{S}^A\mathbf{p}[(b_{i_1}\cdots b_{i_n}\otimes\mathrm{id})]"'] & \mathcal{S}^A\mathbf{q}[n_A] \arrow[d, "\mathcal{S}^A\mathbf{q}[(b_{i_1}\cdots b_{i_n}\otimes\mathrm{id})]"] \\
\mathcal{S}^A\mathbf{p}[n_A] \arrow[r, "\mathcal{S}^A\alpha_{[n_A]}"'] & \mathcal{S}^A\mathbf{q}[n_A].
\end{tikzcd}$$

This diagram reduces to:

$$\begin{tikzcd}
\bigoplus_{s:[n]\to B\times[n]} \mathbf{p}[s([n])] \arrow[r, "\bigoplus_{s:[n]\to B\times[n]} \alpha_{s([n])}"] \arrow[d, "\bigoplus_{s:[n]\to B\times[n]} \mathbf{p}[(b_{i_1}\cdots b_{i_n}\otimes\mathrm{id})|_s]"'] & \bigoplus_{s:[n]\to B\times[n]} \mathbf{q}[s([n])] \arrow[d, "\bigoplus_{s:[n]\to B\times[n]} \mathbf{q}[(b_{i_1}\cdots b_{i_n}\otimes\mathrm{id})|_s]"] \\
\bigoplus_{s:[n]\to B\times[n]} \mathbf{p}[s([n])] \arrow[r, "\bigoplus_{s:[n]\to B\times[n]} \alpha_{s([n])}"'] & \bigoplus_{s:[n]\to B\times[n]} \mathbf{q}[s([n])].
\end{tikzcd}$$

Recall, $\mathbf{p}[(b_{i_1}\cdots b_{i_n}\otimes \mathrm{id})|_s] = \sum_{\underline{k}\in T^n} c^{\underline{k}}_{\underline{i},\underline{j}}\mathbf{p}[f^{\underline{k}}_{s}]$. Now, fix a section $s([n]) = \{(b_{j_1},1),...,(b_{j_n},n)\}$ and $\underline{k}\in T^n$ where $\underline{k} = (k_1,..,k_n)$. Then the corresponding component in the above diagram is

$$\begin{tikzcd}
\mathbf{p}[s([n])] \arrow[r, "\alpha_{s([n])}"] \arrow[d, "c^{\underline{k}}_{\underline{i},\underline{j}}\mathbf{p}[f^{\underline{k}}_{s}]"'] & \mathbf{q}[s([n])] \arrow[d, "c^{\underline{k}}_{\underline{i},\underline{j}}\mathbf{q}[f^{\underline{k}}_{s}]"] \\
\mathbf{p}[s'([n])] \arrow[r, "\alpha_{s'([n])}"'] & \mathbf{q}[s'([n])]
\end{tikzcd}$$

where $s'([n]) = \{(b_{k_1},1),...,(b_{k_n},n)\}$. This diagram commutes since α is a natural transformation and both $\mathbf{p}[f^{\underline{k}}_{s}]$ and $\mathbf{q}[f^{\underline{k}}_{s}]$ are the linear maps induced from the set bijection $f^{\underline{k}}_{s} : \{(b_{j_1},1),...,(b_{j_n},n)\} \to \{(b_{k_1},1),...,(b_{k_n},n)\}$. Since the diagram commutes as we range over all $\underline{k}\in T^n$ and all section maps, we have that the desired original diagram commutes. Thus $\mathcal{S}^A\alpha$ is a natural transformation.

Now to show that $\mathcal{S}^A \operatorname{id}_{\mathbf{p}} = \operatorname{id}_{\mathcal{S}^A\mathbf{p}}$. First, note that $\operatorname{id}_{\mathbf{p}} : \mathbf{p} \to \mathbf{p}$ is a natural transformation whose section maps, $\operatorname{id}_{\mathbf{p}[n]} : \mathbf{p}[n] \to \mathbf{p}[n]$ are given by the usual identity map. Now note that, $\operatorname{id}_{\mathcal{S}^A\mathbf{p}} : \mathcal{S}^A\mathbf{p} \to \mathcal{S}^A\mathbf{p}$ is a natural transformation whose sections are given by

$$\operatorname{id}_{\mathcal{S}^A\mathbf{p}[n_A]} = \bigoplus_{s:[n]\to B\times[n]} \operatorname{id}_{\mathbf{p}[s([n])]} .$$

By definition, we have that for all $[n]_A \in \mathbf{Set}^A$,

$$\mathcal{S}^A \operatorname{id}_{\mathbf{p}[n_A]} := \bigoplus_{s:[n]\to B\times[n]} \operatorname{id}_{\mathbf{p}[s([n])]}$$

Thus $\mathcal{S}^A \operatorname{id}_{\mathbf{p}} = \operatorname{id}_{\mathcal{S}^A\mathbf{p}}$.

Finally, we must show that $\mathcal{S}(\alpha \circ \beta) = \mathcal{S}\alpha \circ \mathcal{S}\beta$ for all morphisms of species $\beta : \mathbf{p} \to \mathbf{q}$ and $\alpha : \mathbf{q} \to \mathbf{h}$.

$$\begin{aligned} \mathcal{S}^A(\alpha \circ \beta)_{[n_A]} &= \bigoplus_{s:[n]\to B\times[n]} (\alpha \circ \beta)_{s([n])} \\ &= \bigoplus_{s:[n]\to B\times[n]} \alpha_{s([n])} \circ \bigoplus_{s:[n]\to B\times[n]} \beta_{s([n])} \\ &= \mathcal{S}^A\alpha \circ \mathcal{S}^A\beta \end{aligned}$$

where the second equality holds since both α and β are natural transformations.

Therefore, $\mathcal{S}^A : \mathbf{Sp} \to \mathbf{Sp}^A$ is indeed a functor.

$\square$

The following proposition shows that when A is a group algebra, for each $\underline{i}$ and $\underline{j}$ there is only one nonzero $f_s^{\underline{k}}$ as in Equation (9.1.3).

PROPOSITION 9.1.4. *Let $A = \mathbb{K}G$ for some group $G = \{b_1, b_2, ...\}$. For a fixed section, $s([n]) = \{(b_{j_1}, 1), .., (b_{j_n}, n)\}$, and $(b_{i_1} \cdots b_{i_n} \otimes \operatorname{id}) \in \mathbb{K}G \wr S_n$, there is only on such $\underline{k} \in T^{|G|}$ such that $c_{\underline{i},\underline{j}}^{\underline{k}}$ is nonzero. Specifically, $c_{\underline{i},\underline{j}}^{\underline{k}} = 1$.*

PROOF. Assume there are two $\underline{k} \in T^{|G|}$ such that the corresponding coefficient is nonzero, say $\underline{k}$ and $\hat{\underline{k}}$. First note that $\underline{i}$ and $\underline{j}$ are fixed tuples that correspond to our chosen $(b_{i_1} \cdots b_{i_n} \otimes \operatorname{id}) \in \mathbb{K}G \wr S_n$ and fixed section $s([n]) = \{(b_{j_1}, 1), .., (b_{j_n}, n)\}$ respectively. Also note that each $c_{i_t,j_t}^{k_t} = 1$ or 0 since the product of two basis elements yields another basis element. So it must be that

$$\begin{matrix} 1 = c_{\underline{i},\underline{\jmath}}^{\underline{k}} = \prod_t c_{i_t,j_t}^{k_t} \\ 1 = c_{\underline{i},\underline{j}}^{\hat{\underline{k}}} = \prod_t c_{i_t,j_t}^{\hat{k}_t} \end{matrix} \iff \begin{matrix} b_{i_t}b_{j_t} = c_{i_t,j_t}^{k_t} b_{k_t} = b_{k_t} \\ b_{i_t}b_{j_t} = c_{i_t,j_t}^{\hat{k}_t} b_{\hat{k}_t} = b_{\hat{k}_t} \end{matrix} \forall t \iff b_{k_t} = b_{\hat{k}_t} \; \forall t \iff k_t = \hat{k}_t \; \forall t$$

Therefore it must be that $\underline{k} = \hat{\underline{k}}$.

$\square$

Now, the following lemma shows that ranging over section maps defined on the parts of a given A-decomposition is equivalent to fixing a section map and then ranging over its respective decompositions. This will be needed in the following proposition when showing $\mathcal{S}^A$ is bilax monoidal.

LEMMA 9.1.5. *Let $R_A \sqcup T_A = [n]_A$ be an A-decomposition as in Definition 7.1.8, then the following sets are in bijection with one another:*

$$\left\{ s'(R) \sqcup s''(T) \mid \substack{R \sqcup T = [n] \\ s' : R \to B \times R \\ s'' : T \to B \times T} \right\} \leftrightarrow \{U \sqcup V = s([n]) \mid s : [n] \to B \times [n]\}.$$

PROOF. Given an A-decomposition, $R \sqcup T = [n]$ and section maps

$$s' : R \to B \times R \ \text{ and } \ s'' : T \to B \times T,$$

we can define a section map $s : [n] \to B \times [n]$ via

$$s([n]) = \begin{cases} s'(R) & \text{if } i \in R \\ s''(T) & \text{if } i \in T \end{cases}.$$

Let $U := Im(s|_r) = Im(s')$ and $V := Im(s|_T) = Im(s'')$, then clearly $U \sqcup V = Im(s)$ as an A-decomposition.

Now, consider a section map $s : [n] \to B \times [n]$ given by $s([n]) = \{(b_{i_1}, 1), ..., (b_{i_n}, n)\}$ and a decomposition $U \sqcup V = s([n])$.
Note that U has the following form: $U = \{(b_{i_{\alpha_1}}, \alpha_1), .., (b_{i_{\alpha_m}}, \alpha_m)\}$ where $\alpha_j \in [n]$ for all $j \in [1, m]$ and distinct, but $b_{i_{\alpha_j}}$ not necessarily distinct; in other words, $U \subset \bigsqcup_{j=1}^{m} B \times \{\alpha_j\}$.
Similarly, $V = \{(b_{i_{\beta_1}}, \beta_1), .., (b_{i_{\beta_s}}, \beta_s)\}$ where $\beta_k \in [n]$ for all $k \in [n]$ and distinct, but $b_{i_{\beta_k}}$ not necessarily distinct.
Define

$$R := A_{\alpha_1} \otimes \cdots \otimes A_{\alpha_m} \otimes \mathbb{K}[\{\alpha_1, ..., \alpha_m\}]$$

and

$$T := A_{\beta_1} \otimes \cdots \otimes A_{\beta_s} \otimes \mathbb{K}[\{\beta_1, ..., \beta_s\}].$$

Then $R \sqcup T = [n]_A$ as A-decomposition, and we can define section maps $s'(\{\alpha_1, ..., \alpha_m\}) = U$ and $s''(\{\beta_1, ..., \beta_s\}) = V$.

$\square$

Now to show that $\mathcal{S}^A$ is bilax monoidal.

PROPOSITION 9.1.6. *$\mathcal{S}^A$ is a bilax monoidal functor with natural transformations φ^A and ψ^A whose sections are given by:*

$$\mathcal{S}^A\mathbf{p} \cdot \mathcal{S}^A\mathbf{q} \underset{\varphi^A_{\mathbf{p},\mathbf{q}}}{\overset{\psi^A_{\mathbf{p},\mathbf{q}}}{\leftrightarrows}} \mathcal{S}^A(\mathbf{p} \cdot \mathbf{q})$$

where both $\varphi^A_{\mathbf{p},\mathbf{q}}$ and $\psi^A_{\mathbf{p},\mathbf{q}}$ are given by the identity natural transformation.

PROOF. Observe that for an object $[n]_A \in \mathbf{Set}^A$, we have:

$$\begin{aligned}
(\mathcal{S}^A\mathbf{p}\cdot\mathcal{S}^A\mathbf{q})[n_A] &= \bigoplus_{R\sqcup T=[n]} \mathcal{S}^A\mathbf{p}[R_A]\otimes\mathcal{S}^A\mathbf{q}[T_A] \\
&= \bigoplus_{R\sqcup T=[n]} \left(\bigoplus_{s'}\mathbf{p}[s'(R)]\right)\otimes\left(\bigoplus_{s''}\mathbf{q}[s''(T)]\right) \\
&= \bigoplus_{R\sqcup T=[n]} \bigoplus_{\substack{s' : R\to B\times R \\ s'' : T\to B\times T}} \mathbf{p}[s'(R)]\otimes\mathbf{q}[s''(T)] \\
&= \bigoplus_{s:[n]\to B\times[n]} \bigoplus_{U\sqcup V=s([n])} \mathbf{p}[U]\otimes\mathbf{q}[V] \\
&= \bigoplus_{s}(\mathbf{p}\cdot\mathbf{q})[s([n])] \\
&= \mathcal{S}^A(\mathbf{p}\cdot\mathbf{q})[[n]_A]
\end{aligned}$$

where the fourth equality holds from Lemma 9.1.5.

By the equalities above, on the degree n component, we can define the section maps of φ^A and ψ^A as follows: $\varphi^A_{\mathbf{p},\mathbf{q}} := \mathbf{p}[\mathrm{id}]\otimes\mathbf{q}[\mathrm{id}]$ and $\psi^A_{\mathbf{p},\mathbf{q}} := \mathbf{p}[\mathrm{id}]\otimes\mathbf{q}[\mathrm{id}]$.
Now, we have

$$\mathcal{S}^A\mathbf{1}_{\mathbb{K}}[n_A] = \bigoplus_{s:[n]\to B\times[n]} \mathbf{1}_{\mathbb{K}}[s([n])] = \begin{cases} \mathbb{K} & \text{if } n=0 \\ 0 & \text{otherwise} \end{cases}.$$

Hence, we can define

$$\varphi^A_0 = \mathrm{id} \quad \text{and} \quad \psi^A_0 = \mathrm{id}\,.$$

Showing that $\varphi^A_{\mathbf{p},\mathbf{q}}$ and $\psi^A_{\mathbf{p},\mathbf{q}}$ are natural transformations and satisfy the bilax monoidal conditions is straightforward to check and can be done in a similar fashion as in Proposition 10.1.3.
Therefore, $\mathcal{S}^A$ is bilax monoidal.

$\square$

COROLLARY 9.1.7. *$\mathcal{S}^A$ is a bistrong functor.*

PROOF. It's easy to see that $\varphi^A = (\psi^A)^{-1}$ and $\varphi^A_0 = (\psi^A_0)^{-1}$, thus by Proposition 3.46 in [3] we have that $\mathcal{S}^A$ is bistrong.

$\square$

THEOREM 9.1.8. *Given a Hopf monoid $\mathbf{p}\in\boldsymbol{Sp}$, then $\mathcal{S}^A(\mathbf{p})\in\boldsymbol{Sp}^A$ is a Hopf monoid.*

PROOF. Let $\mathbf{p}$ be a Hopf monoid in $\mathbf{Sp}$. From Proposition 9.1.6 and 9.1.7, $\mathcal{S}^A$ is a bistrong bilax monoidal functor. Thus, by Proposition [3], $\mathcal{S}^A(\mathbf{p})$ is a Hopf monoid in $\mathbf{Sp}^A$.

$\square$

REMARK 9.1.9. As mentioned earlier, it's usually very difficult to find a closed form for the antipode. However, since $\mathcal{S}^A$ is a bistrong functor, we know that the antipode of $\mathbf{p}\in\mathbf{Sp}$ is preserved. In other words, the antipode of $\mathcal{S}^A(\mathbf{p})$ is $\mathcal{S}^A(s)$, where s is the antipode of $\mathbf{p}$.

9.2. Motivation Behind Section Maps

Here we will explain the motivation for the use of sections maps by Choquette and Bergeron in [10] when constructing the functor $\mathcal{S} : \mathbf{Sp} \to \mathbf{Sp}^{\mathcal{H}}$. Before explaining the role of the section maps, we will first analyze the species of linear orders, the regular representation of S_n and the regular representation of $C_2 \wr S_n$ to determine what would be the appropriate $\mathcal{H}$-species that gives the regular representation of $C_2 \wr S_n$ when evaluated on $[-n, n]$.

9.2.1. The Regular Representation of S_n and the Species of Linear Orders

First, recall that the *regular representation* of a finite group G is the vector space $\mathbb{K}G = \langle v_g \mid g \in G \rangle$ where the linear representation is given by G acting on itself via $g.h = gh$, i.e., a group homomorphism

$$\rho : G \to GL(\mathbb{K}G)$$

$$g \mapsto \rho_g(v_h) = v_{gh}.$$

This vector space has dimension $|G|$. We will denote this representation by R_G.

Example 9.2.1. Let $n = 3$. The regular representation of S_3 is

$$R_{S_3} = \langle v_{()}, v_{(12)}, v_{(13)}, v_{(23)}, v_{(123)}, v_{(132)} \rangle,$$

where the action of S_3 is given by $\pi.v_\sigma = v_{\pi\sigma}$. This representation has dimension $3! = 6$.

Now, recall the species of linear orders as defined in Example 4.1.3;

$$\mathbf{L}[I] := \langle H_\ell \mid \ell \text{ a linear order order on } I \rangle.$$

Example 9.2.2.

$$\mathbf{L}[3] = \langle H_{123}, H_{132}, H_{213}, H_{231}, H_{312}, H_{321} \rangle$$

In the above two examples, the isomorphism of S_3 modules is given by:

$$v_{()} \mapsto H_{123}$$

$$v_{(12)} \mapsto H_{213}$$

$$v_{(13)} \mapsto H_{321}$$

$$v_{(23)} \mapsto H_{132}$$

$$v_{(123)} \mapsto H_{231}$$

$$v_{(132)} \mapsto H_{312}$$

It's easy to check that this yields an isomorphism of S_3 modules. In general, we have an isomorphism of S_n modules between R_{S_n}, the regular representation of S_n, and $\mathbf{L}[n]$, the components of the species of linear orders. This isomporphism is given by

$$v_\sigma \mapsto H_{\sigma(1)\sigma(2)\cdots\sigma(n)}$$

and is shown by combining Lemma 9.2.8, Propositions 9.2.6 and 9.2.11 when we restrict ourselves to the elements of S_n.

9.2.2. The Regular Representation of $C_2 \wr S_n$

Let G be a group and H a subgroup. From representation theory, we have that the induced representation of the regular representation of any subgroup of a group, is the regular representation of the group. S_n is a subgroup of $C_2 \wr S_n$, thus

$$R_{C_2 \wr S_n} = \mathrm{Ind}_{S_n}^{C_2 \wr S_n} R_{S_n} = \mathbb{K}(C_2 \wr S_n) \otimes_{\mathbb{K}S_n} R_{S_n}$$

An easy calculation shows that the basis of $R_{C_2 \wr S_n}$ is labelled by $(\delta_1 \otimes \cdots \otimes \delta_n \otimes \mathrm{id}) \otimes \sigma$ where $\delta_i \in C_2$ and $\sigma \in S_n$. Hence,

$$\dim(R_{C_2 \wr S_n}) = |C_2|^n n! = 2^n n!$$

EXAMPLE 9.2.3. Let $n = 3$. Note that a basis element in $\mathbb{K}[C_2 \wr S_3] \otimes_{\mathbb{K}S_3} R_{S_3}$ looks like

$$(\delta_1 \cdot \delta_2 \cdot \delta_3 \otimes \pi) \otimes \sigma = (\delta_1 \cdot \delta_2 \cdot \delta_3 \otimes \mathrm{id})(1 \cdot 1 \cdot 1 \otimes \pi) \otimes \sigma = (\delta_1 \cdot \delta_2 \cdot \delta_3 \otimes \mathrm{id}) \otimes \pi.\sigma,$$

where $\pi.\sigma = \pi \circ \sigma \in S_3$. Thus every basis element can be written in the form

$$(\delta_1 \cdot \delta_2 \cdot \delta_3 \otimes \mathrm{id}) \otimes \sigma.$$

The dimension of $R_{C_2 \wr S_3}$ is $2^3 3! = 48$. Note, we can further say

$$K(C_2 \wr S_3) \otimes_{\mathbb{K}S_3} R_{S_3} \cong K[C_2 \wr S_3] \otimes_{\mathbb{K}S_3} \mathbf{L}[3],$$

In general, we have

$$R_{C_2 \wr S_n} \cong \mathbb{K}[C_2 \wr S_n] \otimes \mathbf{L}[n].$$

Thus, when trying to construct the appropriate H-species to correspond to $R_{C_2 \wr S_n}$, it's only natural to somehow build off the species of linear orders in some way. This is where section maps come into the picture.

9.2.3. Section Maps

First, we recall the definition of a section map given in [10].

Let I be a set and σ an involution on I. A section is a map

$$s : I/\sigma \to I$$

which is a right inverse for the projection $I \to I/\sigma$. In particular, $s([i]) \in \{i, \sigma(i)\}$ $\forall i \in I$, and $[i]$ denotes the coset made from $i \in I$.

EXAMPLE 9.2.4. When $A = [-n, n]$, we have a natural involution $\sigma_0(i) = -i$. Then $[-n, n]/\sigma_0$ can be identified with $[n]$. Thus $s(i) = i$ or $-i$, here i is identified with the coset $\{i, -i\}$. It's easy to see that there are 2^n many possible section maps for any given n.

We can also think about the above example in the following way:

EXAMPLE 9.2.5. Consider $X = C_2 \times [n] = \bigsqcup_{i=1}^{n} C_2 \times \{i\}$ with a free action of C_2, i.e., C_2 acts on the first coordinate by left multiplication. Then

$$s : (C_2 \times [n])/C_2 \to C_2 \times [n]$$

$$s(\{C_2 \times \{i\}\}) = (1, i) \text{ or } (-1, i).$$

Again, there are 2^n many possible section maps.

In the above examples, it's easy to see that the $|\operatorname{Im}(s)| = n$, thus $\mathbf{L}[\operatorname{Im}(s)]$ corresponds to the regular representation of S_n. Using the section maps, gave us a way to turn an object used for $\mathcal{H}$-species into an object used for species. If we range over all 2^n many possible section maps, we get a vector space with dimension $2^n n!$.

At its core, a section map is a way of assigning an index i to a basis element of $\mathbb{K}[C_2]$ indexed by i. Using this line of thinking gave us our generalized definition of a section map:

Let B be a fixed basis for A. A section is a map $s : [n] \to B \times [n]$ s.t $s(i) \in B \times \{i\}$.

9.2.4. $\mathcal{H}$-Species of Linear Orders

We first consider the special case of the $\mathcal{H}$-species of linear orders. Fix a section map, $s : [n] \to C_2 \times [n]$ whose image is given by $s([n]) = \{(\delta_1, 1), ..., (\delta_n, n)\}$, where the δ_i are not necessarily distinct. Applying $\mathbf{L}$ to the image of the fixed section map yields:

$$\mathbf{L}[\{(\delta_1, 1), ..., (\delta_n, n)\}],$$

which has basis labelled by all possible linear orders of the tuples, i.e., all linear orders on the set $[n]$ where each $i \in [n]$ is colored by $\delta_i \in C_2$. Notice that the basis elements here feel similar to the basis elements of the regular representation of $C_2 \wr S_n$ from above.
Thus,

PROPOSITION 9.2.6.

$$\varphi : R_{C_2 \wr S_n} \to \bigoplus_{s:[n]\to C_2\times[n]} \boldsymbol{L}[s([n])]$$

via

$$(\delta_1 \cdots \delta_n \otimes \sigma) \mapsto (\delta_{\sigma(1)}, \sigma(1)) \cdots (\delta_{\sigma(n)}, \sigma(n))$$

is an isomorphism of vector spaces.

PROOF. This is easy to check that this is an isomorphism of vector spaces. $\square$

EXAMPLE 9.2.7. Let $n = 3$ and $(-1 \cdot 1 \cdot -1 \otimes (132)) \in C_2 \wr S_3$, then

$$\varphi((-1 \cdot 1 \cdot -1 \otimes (132))) = (-1, 2)(-1, 1)(1, 3) \in \mathbf{L}[\{(-1, 1), (-1, 2), (1, 3)\}].$$

You can also think of this as the linear order $\overline{2}\ \overline{1}\ 3$, i.e., 2 colored by -1, 1 colored by -1, and 3 colored by 1.

LEMMA 9.2.8. $\displaystyle\bigoplus_{s:[n]\to C_2\times[n]} \boldsymbol{L}[s([n])]$ *is a $C_2 \wr S_n$-module.*

PROOF. Let $(\delta_1 \cdots \delta_n \otimes \sigma) \in C_2 \wr S_n$. Note that $(\delta_1 \cdots \delta_n \otimes \sigma) = (\delta_1 \cdots \delta_n \otimes \mathrm{id})(1 \cdots 1 \otimes \sigma)$; hence, it suffices to show how $(\delta_1 \cdots \delta_n \otimes \mathrm{id})$ and $(1 \cdots 1 \otimes \sigma)$ act individually since they generate $C_2 \wr S_n$. Fix a section map, s, say $s([n]) = \{(\epsilon_1, 1), .., (\epsilon_n, n)\}$. At the object level, for all $(\epsilon_i, i) \in s([n])$, we have that

$$(1 \cdots 1 \otimes \sigma).(\epsilon_i, i) = (\epsilon_i, \sigma(i))$$

and

$$(\delta_1 \cdots \delta_n \otimes \mathrm{id}).(\epsilon_i, i) = (\delta_i \epsilon_i, i),$$

which can be viewed as elements of some section s' and s'' respectively, where $s'([n]) = \{(\epsilon_1, \sigma(1)), .., (\epsilon_n, \sigma(n))\}$ and $s''([n]) = \{(\delta_1\epsilon_1, 1), ..., (\delta_n\epsilon_n, n)\}$. So each can be viewed as bijections of $\mathcal{H}$-sets that induce linear maps of vector spaces of linear orders:

$$(1 \cdots 1 \otimes \sigma) : s([n]) \to s'([n]) \rightsquigarrow \mathbf{L}[(1 \cdots 1 \otimes \sigma)] : \mathbf{L}[s([n])] \to \mathbf{L}[s'([n])],$$

$$(\delta_1 \cdots \delta_n \otimes \mathrm{id}) : s([n]) \to s''([n]) \rightsquigarrow \mathbf{L}[(\delta_1 \cdots \delta_n \otimes \mathrm{id})] : \mathbf{L}[s([n])] \to \mathbf{L}[s''([n])].$$

By the functoriality of $\mathbf{L}$, we have an action given by: for all $v \in \mathbf{L}[s([n])]$,

$$(1 \cdots 1 \otimes \sigma).v = \mathbf{L}[(1 \cdots 1 \otimes \sigma)](v)$$

and

$$(\delta_1 \cdots \delta_n \otimes \mathrm{id}).v = \mathbf{L}[(\delta_1 \cdots \delta_n \otimes \mathrm{id})](v)$$

Since each element $(\delta_1 \cdots \delta_n \otimes \sigma) \in C_2 \wr S_n$ can be written as

$$(\delta_1 \cdots \delta_n \otimes \mathrm{id})(1 \cdots 1 \otimes \sigma) = (\delta_1 \cdots \delta_n \otimes \sigma) = (1 \cdots 1 \otimes \sigma)(\delta_{\sigma^{-1}(1)} \cdots \delta_{\sigma^{-1}(n)} \otimes \mathrm{id}),$$

we must check the following:

$$(\delta_1 \cdots \delta_n \otimes \mathrm{id}).(1 \cdots 1 \otimes \sigma).v = (1 \cdots 1 \otimes \sigma).(\delta_{\sigma^{-1}(1)} \cdots \delta_{\sigma^{-1}(n)} \otimes \mathrm{id}).v.$$

We have:

$$\begin{aligned}
(\delta_1 \cdots \delta_n \otimes \mathrm{id}).(1 \cdots 1 \otimes \sigma).v &= \mathbf{L}[(\delta_1 \cdots \delta_n \otimes \mathrm{id})] \circ \mathbf{L}[(1 \cdots 1 \otimes \sigma)](v) \\
&= \mathbf{L}[(\delta_1 \cdots \delta_n \otimes \mathrm{id})(1 \cdots 1 \otimes \sigma)](v) \\
&= \mathbf{L}[(\delta_1 \cdots \delta_n \otimes \sigma)](v) \\
&= \mathbf{L}[(1 \cdots 1 \otimes \sigma)(\delta_{\sigma^{-1}(1)} \cdots \delta_{\sigma^{-1}(n)} \otimes \mathrm{id})](v) \\
&= \mathbf{L}[(1 \cdots 1 \otimes \sigma)] \circ \mathbf{L}[(\delta_{\sigma^{-1}(1)} \cdots \delta_{\sigma^{-1}(n)} \otimes \mathrm{id})](v) \\
&= (1 \cdots 1 \otimes \sigma).(\delta_{\sigma^{-1}(1)} \cdots \delta_{\sigma^{-1}(n)} \otimes \mathrm{id}).v
\end{aligned}$$

as desired.

$\square$

REMARK 9.2.9. In general, let $(\epsilon_1, \ell_1) \cdots (\epsilon_n, \ell_n) \in \mathbf{L}[s([n])]$ for the appropriate section s, then

$$\mathbf{L}[(1 \cdots 1 \otimes \sigma)]((\epsilon_1, \ell_1) \cdots (\epsilon_n, \ell_n)) = (\epsilon_1, \sigma(\ell_1)) \cdots (\epsilon_n, \sigma(\ell_n))$$

and

$$\mathbf{L}[(\delta_1 \cdots \delta_n \otimes \mathrm{id})]((\epsilon_1, \ell_1) \cdots (\epsilon_n, \ell_n)) = (\epsilon_1 \cdot \delta_{\ell_1}, \ell_1) \cdots (\epsilon_n \cdot \delta_{\ell_n}, \ell_n).$$

EXAMPLE 9.2.10. Let $n = 3$ and $(1,1)(-1,3)(1,2) \in \mathbf{L}[\{(1,1),(1,2),(-1,3)\}]$ (*equivalently one could think of this as* $1\overline{3}2 \in \mathbf{L}[\{1, 2, \overline{3}\}]$). The following will show how particular elements of $C_2 \wr S_n$ act on this colored linear order.

$$\begin{aligned}
\bullet(1 \cdot 1 \cdot 1 \otimes (123)).(1,1)(-1,3)(1,2) &= \mathbf{L}[(1 \cdot 1 \cdot 1 \otimes (123))](1,1)(-1,3)(1,2) \\
&= (1, (123)(1))(-1, (123)(3))(1, (123)(2)) \\
&= (1,2)(-1,1)(1,3)
\end{aligned}$$

$$\begin{aligned}
\bullet(-1 \cdot 1 \cdot -1 \otimes \mathrm{id}).(1,1)(-1,3)(1,2) &= \mathbf{L}[(-1 \cdot 1 \cdot -1 \otimes \mathrm{id})](1,1)(-1,3)(1,2) \\
&= (1 \times -1, 1)(-1 \times -1, 3)(1 \times 1, 2) \\
&= (-1,1)(1,3)(1,2)
\end{aligned}$$

Equivalently can think of as

$$(1 \cdot 1 \cdot 1 \otimes (123)).1\overline{3}2 = \boldsymbol{L}[(1 \cdot 1 \cdot 1 \otimes (123))](1\overline{3}2) = 2\overline{1}3$$
$$(-1 \cdot 1 \cdot -1 \otimes \mathrm{id}).1\overline{3}2 = \boldsymbol{L}[(-1, 1, -1, \mathrm{id})](1\overline{3}2) = \overline{1}32$$

PROPOSITION 9.2.11. *The regular representation of $C_2 \wr S_n$ and $\bigoplus_{s:[n]\to C_2\times[n]} \boldsymbol{L}[s([n])]$ are isomorphic as $C_2 \wr S_n$ modules.*

PROOF. Let $R_{C_2\wr S_n}$ denote the regular representation of $C_2 \wr S_n$. Recall, $C_2 \wr S_n$ acts on $R_{C_2\wr S_n}$ by left multiplication. We must show that for all $(\epsilon_1 \cdots \epsilon_n \otimes \pi) \in R_{C_2\wr S_n}$ we have

$$(1 \cdots 1 \otimes \sigma).\varphi((\epsilon_1 \cdots \epsilon_n \otimes \pi)) = \varphi((1 \cdots 1 \otimes \sigma)(\epsilon_1 \cdots \epsilon_n \otimes \pi))$$

and

$$(\delta_1 \cdots \delta_n \otimes \mathrm{id}).\varphi((\epsilon_1 \cdots \epsilon_n \otimes \pi)) = \varphi((\delta_1 \cdots \delta_n \otimes \mathrm{id}).(\epsilon_1 \cdots \epsilon_n \otimes \pi)).$$

To show that the first equation holds, we have:

$$\begin{aligned}
\varphi((1 \cdots 1 \otimes \sigma).(\epsilon_1 \cdots \epsilon_n \otimes \pi)) &= \varphi((\epsilon_{\sigma^{-1}(1)} \cdots \epsilon_{\sigma^{-1}(n)} \otimes \sigma \circ \pi)) \\
&= (\epsilon_{\sigma^{-1}(\sigma(\pi(1)))}, \sigma \circ \pi(1)) \cdots (\epsilon_{\sigma^{-1}(\sigma(\pi(n)))}, \sigma \circ \pi(n)) \\
&= (\epsilon_{\pi(1)}, \sigma \circ \pi(1)) \cdots (\epsilon_{\pi(n)}, \sigma \circ \pi(n)) \\
&= \mathbf{L}[(1 \cdots 1 \otimes \sigma)]((\epsilon_{\pi(1)}, \pi(1)) \cdots (\epsilon_{\pi(n)}, \pi(n))) \\
&= (1 \cdots 1 \otimes \sigma).(\epsilon_{\pi(1)}, \pi(1)) \cdots (\epsilon_{\pi(n)}, \pi(n)) \\
&= (1 \cdots 1 \otimes \sigma).\varphi((\epsilon_1 \cdots \epsilon_n \otimes \pi)).
\end{aligned}$$

For the second equation,

$$\begin{aligned}
\varphi((\delta_1 \cdots \delta_n \otimes \mathrm{id}).(\epsilon_1 \cdots \epsilon_n \otimes \pi)) &= \varphi((\delta_1\epsilon_1 \cdots \delta_n\epsilon_n \otimes \pi)) \\
&= (\delta_{\pi(1)}\epsilon_{\pi(1)}, \pi(1)) \cdots (\delta_{\pi(n)}\epsilon_{\pi(n)}, \pi(n)) \\
&= \mathbf{L}[(\delta_1 \cdots \delta_n \otimes \mathrm{id})]((\epsilon_{\pi(1)}, \pi(1)) \cdots (\epsilon_{\pi(n)}, \pi(n))) \\
&= (\delta_1 \cdots \delta_n \otimes \mathrm{id}).(\epsilon_{\pi(1)}, \pi(1)) \cdots (\epsilon_{\pi(n)}, \pi(n)) \\
&= (\delta_1 \cdots \delta_n \otimes \mathrm{id}).\varphi((\epsilon_1 \cdots \epsilon_n \otimes \pi)).
\end{aligned}$$

Therefore

$$R_{C_2\wr S_n} \cong \bigoplus_{s:[n]\to C_2\times[n]} \mathbf{L}[s([n])]$$

as $C_2 \wr S_n$ modules.

□

Thus the appropriate linear order species in the category of $\mathcal{H}$-species should be defined as follows:

$$\mathbf{L}_{\mathcal{H}}[C_2 \times n] := \bigoplus_{s:[n]\to[-n,n]} \mathbf{L}[s([n])].$$

DEFINITION 9.2.12 (Choquette, Bergeron [10]). The $\mathcal{H}$-species of linear orders, is defined to be the functor

$$\mathbf{L}_{\mathcal{H}}[C_2 \times [n]] := \bigoplus_{s:[n]\to[-n,n]} \mathbf{L}[s([n])]$$

$$\mathbf{L}_{\mathcal{H}}[(1 \cdots 1 \otimes \sigma)] := \bigoplus_{s:[n]\to[-n,n]} \mathbf{L}[(1 \cdots 1 \otimes \sigma)|_{s([n])}]$$

$$\mathbf{L}_{\mathcal{H}}[(\delta_1 \cdots \delta_n \otimes \mathrm{id})] := \bigoplus_{s:[n]\to[-n,n]} \mathbf{L}[(\delta_1 \cdots \delta_n \otimes \mathrm{id})|_{s([n])}].$$

Remark 9.2.13. Observe that nothing special about C_2 was used here. C_2 could be replaced with any finite group and we would get the appropriate result. Thus for any G-species, we have that the G-species of linear orders is

$$\mathbf{L}_G[G \times [n]] := \bigoplus_{s:[n]\to G\times[n]} \mathbf{L}[s([n])]$$

$$\mathbf{L}_G[(1 \cdots 1 \otimes \sigma)] := \bigoplus_{s:[n]\to G\times[n]} \mathbf{L}[(1 \cdots 1 \otimes \sigma)|_{s([n])}]$$

$$\mathbf{L}_G[(g_1 \cdots g_n \otimes \mathrm{id})] := \bigoplus_{s:[n]\to G\times[n]} \mathbf{L}[(g_1 \cdots g_n \otimes \mathrm{id})|_{s([n])}]$$

and corresponds to the regular representation of $G \wr S_n$.

9.2.5. A-Species of Linear Orders and Regular Representation of $A \wr S_n$

The above gave us motivation for our construction of $\mathcal{S}^A$. When we evaluate our functor $\mathcal{S}^A$ on the species of linear orders, as in Subsection 5.1, we would like to get an A-species whose components are isomorphic to the regular representation of $A \wr S_n$. Hence our construction of $\mathcal{S}^A$ seems to be the reasonable thing.We define the A-species of linear orders in the following way; please refer to Section 11.1 for a thorough description of this example.

Definition 9.2.14. **A-Species of Linear Orders, $\mathbf{L}_A$**
Let $\mathbf{L}_A := \mathcal{S}^A(\mathbf{L})$.

$$\mathbf{L}_A[n_A] := \bigoplus_{s:[n]\to B\times[n]} \mathbf{L}[s([n])]$$

i.e., the $\mathbb{K}$-span of linear orders on $s([n])$ for all sections $s : [n] \to B \times [n]$.

On endomorphisms of $[n]_A$, it suffices to see what happens on generators:

$$\mathbf{L}_A[(1 \cdots 1 \otimes \sigma)] := \bigoplus_{s:[n]\to B\times[n]} \mathbf{L}[(1 \cdots 1 \otimes \sigma)|_{s([n])}],$$

$$\mathbf{L}_A[(b_{i_1} \cdots b_{i_n} \otimes \mathrm{id})] := \bigoplus_{s:[n]\to B\times[n]} \mathbf{L}[(b_{i_1} \cdots b_{i_n} \otimes \mathrm{id})|_{s([n])}].$$

Now, we begin to show that the components of the A-species of linear orders corresponds to the regular representation of $A \wr S_n$ for all n.

Proposition 9.2.15.

$$\varphi : A \wr S_n \to \bigoplus_{s:[n]\to B\times[n]} \boldsymbol{L}[s([n])]$$

via

$$(b_{i_1} \cdots b_{i_n} \otimes \sigma) \mapsto (b_{i_{\sigma(1)}}, \sigma(1)) \cdots (b_{i_{\sigma(n)}}, \sigma(n))$$

is an isomorphism of vector spaces.

PROOF. First to show that φ is injective. Assume $\varphi((b_{i_1}\cdots b_{i_n}\otimes\sigma)) = \varphi((b_{j_1}\cdots b_{j_n}\otimes\tau))$. Then

$$(b_{i_{\sigma(1)}},\sigma(1))\cdots(b_{i_{\sigma(n)}},\sigma(n)) = (b_{j_{\tau(1)}},\tau(1))\cdots(b_{j_{\tau(n)}},\tau(n))$$

which implies that $(b_{i_{\sigma(k)}},\sigma(k)) = (b_{j_{\tau(k)}},\tau(k))\ \ \forall k$. This happens if and only if $b_{i_{\sigma(k)}} = b_{j_{\tau(k)}}$ and $\sigma(k)=\tau(k)$ for all k which implies $\sigma=\tau$. Thus $(b_{i_1}\cdots b_{i_n}\otimes\sigma) = (b_{j_1}\cdots b_{j_n}\otimes\tau)$ which shows φ is injective.

Finally to show surjective. Let $(b_{i_1},\ell_1)\cdots(b_{i_n},\ell_n)\in\mathbf{L}[s([n])]$ for some section s. There exists a $\sigma\in S_n$ such that $\sigma(k)=\ell_k$ for all k. Consider $(b_{i_{\sigma^{-1}(1)}}\cdots b_{i_{\sigma^{-1}(k)}}\otimes\sigma)$, then

$$\begin{aligned}\varphi((b_{i_{\sigma^{-1}(1)}}\cdots b_{i_{\sigma^{-1}(k)}}\otimes\sigma)) &= (b_{i_{\sigma^{-1}(\sigma(1))}},\sigma(1))\cdots(b_{i_{\sigma^{-1}(\sigma(n))}},\sigma(n))\\ &= (b_{i_1},\ell_1)\cdots(b_{i_n},\ell_n)\end{aligned}$$

Thus φ is surjective.

Therefore, φ is a vector space isomorphism.

$\square$

PROPOSITION 9.2.16. *The regular representation of $A\wr S_n$ and $\boldsymbol{L}_A[n_A]$ are isomorphic as $A\wr S_n$ modules.*

PROOF. We must show that φ is morphism of $A\wr S_n$ modules. It suffices to show on the generators of $A\wr S_n$, $(1\cdots1\otimes\tau)$ and $(b_{i_1}\cdots b_{i_n}\otimes\mathrm{id})$.
Given basis element $(b_{j_1}\cdots b_{j_n}\otimes\sigma)\in A\wr S_n$, then

$$\varphi((b_{j_1}\cdots b_{j_n}\otimes\sigma)) = (b_{j_{\sigma(1)}},\sigma(1))\cdots(b_{j_{\sigma(n)}},\sigma(n))\in\mathbf{L}[s([n])]$$

where $s([n]) = \{(b_{j_{\sigma(1)}},\sigma(1)),...,(b_{j_{\sigma(n)}},\sigma(n))\}$.

- First to show that $(1\cdots1\otimes\tau).\varphi((b_{j_1}\cdots b_{j_n}\otimes\sigma)) = \varphi((1\cdots1\otimes\tau).(b_{j_1}\cdots b_{j_n}\otimes\sigma))$

$$\begin{aligned}\varphi((1\cdots1\otimes\tau).(b_{j_1}\cdots b_{j_n}\otimes\sigma)) &= \varphi((b_{j_{\tau^{-1}(1)}}\cdots b_{j_{\tau^{-1}(n)}}\otimes\tau\circ\sigma))\\ &= (b_{j_{\tau^{-1}(\tau(\sigma(1)))}},\tau(\sigma(1)))\cdots(b_{j_{\tau^{-1}(\tau(\sigma(n)))}},\tau(\sigma(n)))\\ &= (b_{j_{\sigma(1)}},\tau(\sigma(1)))\cdots(b_{j_{\sigma(n)}},\tau(\sigma(n)))\\ &= \mathbf{L}[(1\cdots1\otimes\tau)|_s]((b_{j_{\sigma(1)}},\tau(\sigma(1)))\cdots(b_{j_{\sigma(n)}},\tau(\sigma(n))))\\ &= (1\cdots1\otimes\tau).(b_{j_{\sigma(1)}},\tau(\sigma(1)))\cdots(b_{j_{\sigma(n)}},\tau(\sigma(n)))\\ &= (1\cdots1\otimes\tau).\varphi((b_{j_1}\cdots b_{j_n}\otimes\sigma))\end{aligned}$$

- Finally, to show that

$$(b_{i_1}\cdots b_{i_n}\otimes\mathrm{id}).\varphi((b_{j_1}\cdots b_{j_n}\otimes\sigma)) = \varphi((b_{i_1}\cdots b_{i_n}\otimes\mathrm{id}).(b_{j_1}\cdots b_{j_n}\otimes\sigma)).$$

$$\begin{aligned}
\varphi((b_{i_1}\cdots b_{i_n}\otimes \mathrm{id}).(b_{j_1}\cdots b_{j_n}\otimes\sigma)) &= \varphi((b_{i_1}b_{j_1}\cdots b_{i_n}b_{j_n}\otimes\sigma))\\
&= \sum_{\underline{k}\in T^n} c^{\underline{k}}_{\underline{i},\underline{j}}\varphi((b_{k_1}\cdots b_{k_n}\otimes\sigma))\\
&= \sum_{\underline{k}\in T^n} c^{\underline{k}}_{\underline{i},\underline{j}}(b_{k_{\sigma(1)}},\sigma(1))\cdots(b_{k_{\sigma(n)}},\sigma(n))\\
&= \sum_{\underline{k}\in T^n} c^{\underline{k}}_{\underline{i},\underline{j}}\mathbf{L}[f^{\underline{k}}_{s}](b_{j_{\sigma(1)}},\sigma(1))\cdots(b_{j_{\sigma(n)}},\sigma(n))\\
&= \mathbf{L}[f(b_{i_1}\cdots b_{i_n}\otimes\mathrm{id})|_s](b_{j_{\sigma(1)}},\sigma(1))\cdots(b_{j_{\sigma(n)}},\sigma(n))\\
&= (b_{i_1}\cdots b_{i_n}\otimes\mathrm{id}).(b_{j_{\sigma(1)}},\sigma(1))\cdots(b_{j_{\sigma(n)}},\sigma(n))\\
&= (b_{i_1}\cdots b_{i_n}\otimes\mathrm{id}).\varphi((b_{j_1}\cdots b_{j_n}\otimes\sigma))
\end{aligned}$$

Therefore, φ is a morphism of $A\wr S_n$-modules as desired.

$\square$

CHAPTER 10

A-Fock Functors

In this chapter, we define six monoidal functors from A-species to graded vector spaces. These will be A-versions of the full and bosonic fock functors, $K^\vee, \overline{K}^\vee, K$, and $\overline{K}$, defined in [3], and as described in Section 4.1. For this reason, we will follow a similar notation. First we will define three bilax monoidal functors $K_A^\vee, \widetilde{K_A^\vee}$, and $\overline{K}_A^\vee$ that correspond to $A \wr S_n$-invariance. Then we will define three bilax functors $K_A, \widetilde{K}_A$, and $\overline{K}_A$ that correspond to $A \wr S_n$-coinvariance. We end this section by showing a natural transformation between these two constructions.

10.1. A-Invariance

Let the counit of A be $\varepsilon : A \to \mathbb{K}$. We start by defining the analogue of the invariant Fock functor defined in [3].

Definition 10.1.1. For each $\mathbf{p} \in \mathbf{Sp}^A$ and morphism $f : \mathbf{p} \to \mathbf{q}$ of A-species, we can define the functor $K_A^\vee : \mathbf{Sp}^A \to \mathbf{gVec}$ via

$$K_A^\vee(\mathbf{p}) := \bigoplus_{n\geq 0} \mathbf{p}[n_A]$$

$$K_A^\vee(f) := \bigoplus_{n\geq 0} f_{[n_A]}$$

Remark 10.1.2. Clearly, by definition of $\mathbf{p}$, $K_A^\vee(\mathbf{p}) \in \mathbf{gVec}$. Since $K_A^\vee$ applied to a morphism of A-species yields a family of linear maps so on each graded piece composition makes sense, thus making sense overall. $K_A^\vee(\mathrm{id}_\mathbf{p})_{[[n]_A]} := \mathrm{id}_{\mathbf{p}[[n]_A]}$ for all $[n]_A$, thus $K_A^\vee(\mathrm{id}_\mathbf{p}) = \mathrm{id}_{K_A^\vee(\mathbf{p})}$.

Proposition 10.1.3. *The functor $K_A^\vee$ is a bilax monoidal functor.*

Proof.
Recall from Section 2.4 that in order to show that $K_A^\vee$ is a bilax monoidal functor, we need to define natural transformations

$$\mathcal{M} \circ (K_A^\vee \times K_A^\vee) \underset{\psi^\vee}{\overset{\varphi^\vee}{\rightleftarrows}} K_A^\vee \circ \mathcal{M}$$

where $\mathcal{M}$ denotes the tensor product functor defined in Section 2.4 and $\mathcal{M} \circ (K_A^\vee \times K_A^\vee)$ and $K_A^\vee \circ \mathcal{M}$ are both functors from $\mathbf{Sp}^A \times \mathbf{Sp}^A \to \mathbf{gVec}$.

Let $\mathbf{p}, \mathbf{q} \in \mathbf{Sp}^A$, then

$$K_A^\vee(\mathbf{p})\cdot K_A^\vee(\mathbf{q}) \underset{\psi^\vee_{\mathbf{p},\mathbf{q}}}{\overset{\varphi^\vee_{\mathbf{p},\mathbf{q}}}{\rightleftarrows}} K_A^\vee(\mathbf{p}\cdot\mathbf{q}).$$

Note that

$$K_A^\vee(\mathbf{p})\cdot K_A^\vee(\mathbf{q}) = \bigoplus_{n\geq 0}\bigoplus_{r+t=n}\mathbf{p}[r_A]\otimes\mathbf{q}[t_A]$$
$$K_A^\vee(\mathbf{p}\cdot\mathbf{q}) = \bigoplus_{n\geq 0}\bigoplus_{R\sqcup T=[n]}\mathbf{p}[R_A]\otimes\mathbf{q}[T_A].$$

On the degree n piece, we define the sections of $\varphi^\vee$ and $\psi^\vee$ as follows:

$$\varphi^\vee_{\mathbf{p},\mathbf{q}} : \mathbf{p}[r_A]\otimes\mathbf{q}[t_A] \to \bigoplus_{\substack{R\sqcup T=[n]\\ |R|=r,|T|=t}} \mathbf{p}[R_A]\otimes\mathbf{q}[T_A],$$

$$\psi^\vee_{\mathbf{p},\mathbf{q}} : \mathbf{p}[r_A]\otimes\mathbf{q}[[1+r,t+r]_A] \xrightarrow{\mathbf{p}[\mathrm{id}]\otimes\mathbf{q}[\mathrm{cano}_r]} \mathbf{p}[r_A]\otimes\mathbf{q}[t_A],$$

where $|R|=r$ and $|T|=t$. When $R\neq[r]$ and $T\neq[1+r,t+r]$, $\psi^\vee_{\mathbf{p},\mathbf{q}}=0$. Note that cano_r is the order preserving bijection that shifts all the values in the set down by r.

Now, observe that $K_A^\vee(\mathbf{1}_\mathbb{K}) = \bigoplus_{n\geq 0}\mathbf{1}_\mathbb{K}[n_A] = \mathbb{K}\oplus 0\oplus\cdots\cong\mathbb{K}$ which is the unit of **gVec**. Thus we can define $\varphi^\vee_0=\mathrm{id}$ and $\psi^\vee_0=\mathrm{id}$.

First to show the lax monoidal structure of $K_A^\vee$. In order to do so, we must show that φ is a natural transformation, is associative, and is left and right unital.

1. Claim: $\varphi^\vee$ is a natural transformation.
   Let $\mathbf{p},\mathbf{q}\in\mathbf{Sp}^A$, define the sections $\varphi^\vee_{\mathbf{p},\mathbf{q}}$ as above. Now let $\alpha:\mathbf{p}\to\mathbf{p}'$ and $\beta:\mathbf{q}\to\mathbf{q}'$ be two A-species morphisms. We must show the following diagram commutes:

$$\begin{tikzcd} K_A^\vee(\mathbf{p})\cdot K_A^\vee(\mathbf{q}) \arrow[r, "\varphi^\vee_{\mathbf{p},\mathbf{q}}"] \arrow[d, "K_A^\vee(\alpha)\cdot K_A^\vee(\beta)"'] & K_A^\vee(\mathbf{p}\cdot\mathbf{q}) \arrow[d, "K_A^\vee(\alpha\cdot\beta)"] \\ K_A^\vee(\mathbf{p}')\cdot K_A^\vee(\mathbf{q}') \arrow[r, "\varphi^\vee_{\mathbf{p}',\mathbf{q}'}"'] & K_A^\vee(\mathbf{p}'\cdot\mathbf{q}'). \end{tikzcd}$$

   For each fixed $r+t=n$, we have:

$$\begin{tikzcd} \mathbf{p}[r_A]\otimes\mathbf{q}[t_A] \arrow[r, "\oplus\mathbf{p}[\mathrm{cano}]\otimes\mathbf{q}[\mathrm{cano}]"] \arrow[d, "\alpha_{[r_A]}\otimes\beta_{[t_A]}"'] & \bigoplus_{\substack{R\sqcup T=[n]\\ |R|=r,|T|=t}} \mathbf{p}[R_A]\otimes\mathbf{q}[T_A] \arrow[d, "\oplus\alpha_{R_A}\otimes\beta_{T_A}"] \\ \mathbf{p}'[r_A]\otimes\mathbf{q}'[t_A] \arrow[r, "\oplus\mathbf{p}'[\mathrm{cano}]\otimes\mathbf{q}'[\mathrm{cano}]"'] & \bigoplus_{\substack{R\sqcup T=[n]\\ |R|=r,|T|=t}} \mathbf{p}'[R_A]\otimes\mathbf{q}'[T_A]. \end{tikzcd} \tag{31}$$

Note, that $\alpha : \mathbf{p} \to \mathbf{p}'$ is a natural transformation, i.e., for all $f : I_A \to J_A$, the following diagram commutes:

$$\begin{tikzcd}
\mathbf{p}[I_A] \arrow[r, "\alpha_{I_A}"] \arrow[d, "\mathbf{p}[f]"'] & \mathbf{p}'[I_A] \arrow[d, "\mathbf{p}'[f]"] \\
\mathbf{p}[J_A] \arrow[r, "\alpha_{J_A}"'] & \mathbf{p}'[J_A].
\end{tikzcd}$$

Similarly for β. Thus for each decomposition, $R \sqcup T = [n]$, we have that Diagram 31 commutes–this is because α and β are natural transformations and $\mathbf{p}[\text{cano}] \otimes \mathbf{q}[\text{cano}]$ are bijections. Thus the entire square commutes; hence, $\varphi^\vee$ is a natural transformation.

2. Claim: $\varphi^\vee$ is associative.

We must show Diagram (13) commutes. We will show that it commutes on each component of degree n of $K_A^\vee(\mathbf{p}) \cdot K_A^\vee(\mathbf{q}) \cdot K_A^\vee(\mathbf{h})$. In order to show that the associativity axioms holds, we must first understand $\varphi^\vee_{\mathbf{p}\cdot\mathbf{q},\mathbf{h}}$ and $\varphi^\vee_{\mathbf{p},\mathbf{q}\cdot\mathbf{h}}$.

$$\varphi^\vee_{\mathbf{p}\cdot\mathbf{q},\mathbf{h}} : (\mathbf{p}\cdot\mathbf{q})[m_A] \otimes \mathbf{h}[r_A] \xmapsto{\oplus(\mathbf{p}\cdot\mathbf{q})[\text{cano}_M]\otimes\mathbf{h}[\text{cano}_R]} \bigoplus_{\substack{M \sqcup R = [m+r] \\ |M| = m,\ |R| = r}} (\mathbf{p}\cdot\mathbf{q})[M_A] \otimes \mathbf{h}[R_A].$$

On the order preserving bijection $\text{cano}_M : A^{\otimes m} \otimes \mathbb{K}[m] \to A^{\otimes |M|} \otimes \mathbb{K}[M]$, $(\mathbf{p}\cdot\mathbf{q})$ is defined to be the direct sum of maps

$$\mathbf{p}[S_A] \otimes \mathbf{q}[T_A] \mapsto \mathbf{p}[(\text{cano}_M(S))_A] \otimes \mathbf{q}[(\text{cano}_M(T))_A]$$

ranging over all decompositions $S \sqcup T = M$.Thus, on a decomposition $S \sqcup T = [m]$, we have $\varphi^\vee_{\mathbf{p}\cdot\mathbf{q},\mathbf{h}}$ defined as follows:

$$\mathbf{p}[S_A] \otimes \mathbf{q}[T_A] \otimes \mathbf{h}[[r]_A] \mapsto \bigoplus_{\substack{M \sqcup R = [m+r] \\ |M| = m,\ |R| = r}} \mathbf{p}[\text{cano}_M(S_A)] \otimes \mathbf{q}[\text{cano}_M(T_A)] \otimes \mathbf{h}[R_A],$$

where $\text{cano}_M(S) \sqcup \text{cano}_M(T) = M$ is one of the decompositions of the Cauchy product on the right hand side. As we range over all such decompositions of $[m]$, we get every possible decomposition of M.

Similarly, $\varphi^\vee_{\mathbf{p},\mathbf{q}\cdot\mathbf{h}}$ is defined as follows:

$$\varphi^\vee_{\mathbf{p},\mathbf{q}\cdot\mathbf{h}} : \mathbf{p}[\hat{s}] \otimes (\mathbf{q}\cdot\mathbf{h})[\hat{u}] \xmapsto{\oplus\mathbf{p}[\text{cano}_S]\otimes(\mathbf{q}\cdot\mathbf{h})[\text{cano}_U]} \bigoplus_{\substack{S \sqcup U = [s+u] \\ |S| = s,\ |U| = u}} \mathbf{p}[\hat{S}] \otimes (\mathbf{q}\cdot\mathbf{h})[\hat{U}]$$

$$\mathbf{p}[\hat{[s]}] \otimes \mathbf{q}[T_A] \otimes \mathbf{h}[R_A] \longmapsto \bigoplus_{\substack{S \sqcup U = [s+u] \\ |S| = s,\ |U| = u}} \mathbf{p}[\hat{S}] \otimes \mathbf{q}[\text{cano}_U(T_A)] \otimes \mathbf{h}[\text{cano}_U(R_A)],$$

where $T \sqcup R = [u]$ and $\text{cano}_U(T) \sqcup \text{cano}_U(R) = U$.

Finally, to show

$$\varphi^\vee_{\mathbf{p},\mathbf{q}\cdot\mathbf{h}} \circ (\text{id} \otimes \varphi^\vee_{\mathbf{q},\mathbf{h}}) = \varphi^\vee_{\mathbf{p}\cdot\mathbf{q},\mathbf{h}} \circ (\varphi^\vee_{\mathbf{p},\mathbf{q}} \otimes \text{id}).$$

Fix a decomposition $s + t + r = n$.

For the lefthand side $\varphi^{\vee}_{\mathbf{p},\mathbf{q}\cdot\mathbf{h}} \circ (\mathrm{id} \otimes \varphi^{\vee}_{\mathbf{q},\mathbf{h}})(\mathbf{p}[s_A] \otimes \mathbf{q}[r_A] \otimes \mathbf{h}[t_A])$:

$$= \varphi^{\vee}_{\mathbf{p},\mathbf{q}\cdot\mathbf{h}} \left(\mathbf{p}[s_A] \otimes \bigoplus_{\substack{R \sqcup T = [r+t] \\ |R| = r,\ |T| = t}} \mathbf{q}[R_A] \otimes \mathbf{h}[T_A] \right)$$

$$= \bigoplus_{\substack{S \sqcup M = [n] \\ |S| = s \\ |M| = r+t}} \mathbf{p}[S_A] \otimes \left(\bigoplus_{\substack{c_M(R) \sqcup c_M(T) = M \\ |c_M(R)| = r,\ |c_M(T)| = t}} \mathbf{q}[\mathrm{cano}_M(R_A)] \otimes \mathbf{h}[\mathrm{cano}_M(T_A)] \right)$$

Let $M_1 := \mathrm{cano}_M(R_A)$ and $M_2 := \mathrm{cano}_M(T_A)$

$$= \bigoplus_{\substack{S \sqcup M = [n] \\ |S| = s \\ |M| = r+t}} \mathbf{p}[S_A] \otimes \left(\bigoplus_{\substack{M_1 \sqcup M_2 = M \\ |M_1| = r,\ |M_2| = t}} \mathbf{q}[M_1] \otimes \mathbf{h}[M_2] \right)$$

$$= \bigoplus_{\substack{S \sqcup M_1 \sqcup M_2 = [n] \\ |S| = s \\ |M_1| = r \\ |M_2| = t}} \mathbf{p}[S_A] \otimes \mathbf{q}[M_1] \otimes \mathbf{h}[M_2].$$

For the righthand side $\varphi^{\vee}_{\mathbf{p}\cdot\mathbf{q},\mathbf{h}} \circ (\varphi^{\vee}_{\mathbf{p},\mathbf{q}} \otimes \mathrm{id})(\mathbf{p}[s_A] \otimes \mathbf{q}[r_A] \otimes \mathbf{h}[t_A])$:

$$= \varphi^{\vee}_{\mathbf{p}\cdot\mathbf{q},\mathbf{h}} \left(\left(\bigoplus_{\substack{S \sqcup R = [s+r] \\ |S| = s \\ |R| = r}} \mathbf{p}[S_A] \otimes \mathbf{q}[R_A] \right) \otimes \mathbf{h}[t_A] \right)$$

$$= \bigoplus_{\substack{U \sqcup T = [n] \\ |U| = s+r \\ |T| = t}} \left(\bigoplus_{\substack{c_U(S) \sqcup c_U(R) = U \\ |c_U(S)| = s \\ |c_U(R)| = r}} \mathbf{p}[\mathrm{cano}_U(S_A)] \otimes \mathbf{q}[\mathrm{cano}_U(R_A)] \right) \otimes \mathbf{h}[T_A]$$

Let $U_1 := \text{cano}_U(S_A)$ and $U_2 := \text{cano}_U(R_A)$,

$$\begin{aligned}
&= \bigoplus_{\substack{U \sqcup T = [n] \\ |U| = s+r \\ |T| = t}} \left(\bigoplus_{\substack{U_1 \sqcup U_2 = U \\ |U_1| = s \\ |U_2| = r}} \mathbf{p}[U_1] \otimes \mathbf{q}[U_2] \right) \otimes \mathbf{h}[T_A] \\
&= \bigoplus_{\substack{U_1 \sqcup U_2 \sqcup T = [n] \\ |U_1| = s \\ |U_2| = r \\ |T| = t}} \mathbf{p}[U_1] \otimes \mathbf{q}[U_2] \otimes \mathbf{h}[T_A].
\end{aligned}$$

Since we are ranging over all possible decompositions of n, the lefthand side and the righthand side are the same.

Thus the diagram needed commutes, and we get that φ is associative.

3. Claim: $\varphi^\vee$ is left and right unital.

   To show that $\varphi^\vee$ is left and right unital, we must show that Diagram (14). For a fixed $r + t = n$, this is equivalent to showing
   $$\varphi^\vee_{\mathbf{p},\mathbf{1}_{\mathbb{K}}} \circ (\text{id}_{\mathbf{p}} \cdot \varphi^\vee_0) = \text{id}_{\mathbf{p}} = \varphi^\vee_{\mathbf{1}_{\mathbb{K}},\mathbf{p}} \circ (\varphi^\vee_0 \cdot \text{id}_{\mathbf{p}})$$
   On the right hand side:
   $$\begin{aligned}
   \varphi^\vee_{\mathbf{1}_{\mathbb{K}},\mathbf{p}} \circ (\varphi^\vee_0 \cdot \text{id}_{\mathbf{p}}) &= \left(\bigoplus_{\substack{R \sqcup T = [n] \\ |R| = r, |T| = t}} \mathbf{1}_{\mathbb{K}}[\text{cano}_R] \otimes \mathbf{p}[\text{cano}_T] \right) \circ (\text{id} \otimes \text{id}_{\mathbf{p}}) \\
   &= \left(\mathbf{1}_{\mathbb{K}}[\text{cano}_\emptyset] \otimes \mathbf{p}[\text{cano}_{[n]}] \oplus \big(\bigoplus_{R \neq \emptyset} 0 \otimes p[\text{cano}_T] \big) \right) \circ (\text{id} \otimes \text{id}) \\
   &= \text{id}_{\mathbb{K}} \otimes \mathbf{p}[\text{cano}_{[n]}] \\
   &= \text{id}_{\mathbb{K}} \otimes \text{id}_{\mathbf{p}} \\
   &\cong \text{id}_{\mathbf{p}}
   \end{aligned}$$
   Similarly for the left hand side. Thus showing that $K^\vee_A$ is a lax monoidal functor.

Now to show that $K^\vee_A$ is a colax monoidal functor.

1. Claim: $\psi^\vee$ natural transformation.

   Let $\alpha : \mathbf{p} \to \mathbf{p}'$ and $\beta : \mathbf{q} \to \mathbf{q}'$ be two A-species morphisms. We want to show the following diagram commutes:

$$\begin{tikzcd} K_A^\vee(\mathbf{p}\cdot\mathbf{q}) \arrow[r, "\psi_{\mathbf{p},\mathbf{q}}"] \arrow[d, "K_A^\vee(\alpha\cdot\beta)"'] & K_A^\vee(\mathbf{p})\cdot K_A^\vee(\mathbf{q}) \arrow[d, "K_A^\vee(\alpha)\cdot K_A^\vee(\beta)"] \\ K_A^\vee(\mathbf{p}'\cdot\mathbf{q}') \arrow[r, "\psi_{\mathbf{p}',\mathbf{q}'}"'] & K_A^\vee(\mathbf{p}')\cdot K_A^\vee(\mathbf{q}'). \end{tikzcd}$$

For each decomposition $R \sqcup T = [n]$, we have:

$$\begin{tikzcd} \mathbf{p}[R_A]\otimes\mathbf{q}[T_A] \arrow[r, "\mathbf{p}[\mathrm{id}]\otimes\mathbf{q}[\mathrm{cano}]"] \arrow[d, "\alpha_{[R]}\otimes\beta_{[T]}"'] & \mathbf{p}[r_A]\otimes\mathbf{q}[t_A] \arrow[d, "\alpha_{[r_A]}\otimes\beta_{[t_A]}"] \\ \mathbf{p}'[R_A]\otimes\mathbf{q}'[T'_A] \arrow[r, "\mathbf{p}'[\mathrm{id}]\otimes\mathbf{q}'[\mathrm{cano}]"'] & \mathbf{p}'[r_A]\otimes\mathbf{q}'[t_A]. \end{tikzcd}$$

This diagram commutes since both α and β are natural transformations and cano is a bijection of the underlying objects. Thus $\psi^\vee_{\mathbf{p},\mathbf{q}}$ is a natural transformation, hence $\psi^\vee$ is a natural transformation since the naturality diagram boils down to the naturality of $\psi^\vee_{\mathbf{p},\mathbf{q}}$.

2. Claim: $\psi^\vee$ is coassociative.

We must show that the diagram formed by reversing the arrows in Diagram (13) and replacing φ with ψ commutes. We will show that it commutes on each piece of degree n of $K_A^\vee(\mathbf{p}\cdot\mathbf{q}\cdot\mathbf{h})$. First, in order to show that $\psi^\vee$ is a coassociative, we must understand the maps $\psi^\vee_{\mathbf{p}\cdot\mathbf{q},\mathbf{h}}$ and $\psi^\vee_{\mathbf{p},\mathbf{q}\cdot\mathbf{h}}$.
First,

$$\psi^\vee_{\mathbf{p}\cdot\mathbf{q},\mathbf{h}} : (\mathbf{p}\cdot\mathbf{q})[s_A]\otimes\mathbf{h}[[1+s,t+s]_A] \xmapsto{(\mathbf{p}\cdot\mathbf{q})[\mathrm{id}]\otimes\mathbf{h}[\mathrm{cano}_s]} (\mathbf{p}\cdot\mathbf{q})[s_A]\otimes\mathbf{h}[t_A]$$

$$\mathbf{p}[(S_1)_A]\otimes\mathbf{q}[(S_2)_A]\otimes\mathbf{h}[[1+s,t+s]_A] \longmapsto \mathbf{p}[(S_1)_A]\otimes\mathbf{q}[(S_2)_A]\otimes\mathbf{h}[t_A],$$

where $S_1 \sqcup S_2 = [s]$.
Now,

$$\psi^\vee_{\mathbf{p},\mathbf{q}\cdot\mathbf{h}} : \mathbf{p}[s_A]\otimes(\mathbf{q}\cdot\mathbf{h})[[1+s,t+s]_A] \xmapsto{\mathbf{p}[\mathrm{id}]\otimes(\mathbf{q}\cdot\mathbf{h})[\mathrm{cano}_s]} \mathbf{p}[s_A]\otimes(\mathbf{q}\cdot\mathbf{h})[t_A]$$

$$\mathbf{p}[s_A]\otimes\mathbf{q}[(T_1)_A]\otimes\mathbf{h}[(T_2)_A] \longmapsto \mathbf{p}[s_A]\otimes\mathbf{q}[\mathrm{cano}_s(T_1)_A]\otimes\mathbf{h}[\mathrm{cano}_s(T_2)_A],$$

where $T_1 \sqcup T_2 = [1+s,t+s]$ and $\mathrm{cano}_s(T_1)_A \sqcup \mathrm{cano}_s(T_2)_A = [t]$.

Finally to show

$$(\mathrm{id}\otimes\psi^\vee_{\mathbf{q},\mathbf{h}})\circ\psi^\vee_{\mathbf{p},\mathbf{q}\cdot\mathbf{h}} = (\psi^\vee_{\mathbf{p},\mathbf{q}}\otimes\mathrm{id})\circ\psi^\vee_{\mathbf{p}\cdot\mathbf{q},\mathbf{h}}.$$

Fix a decomposition $S \sqcup R \sqcup T = [n]$.

Note that the lefthand side is only nonzero when $S = [s]$ and $R \sqcup T = [1+s,n]$. We have $(\mathrm{id}\otimes\psi^\vee_{\mathbf{q},\mathbf{h}})\circ\psi^\vee_{\mathbf{p},\mathbf{q}\cdot\mathbf{h}}(\mathbf{p}[s_A]\otimes\mathbf{q}[R_A]\otimes\mathbf{h}[T_A])$

$$\begin{aligned} &= (\mathrm{id}\otimes\psi^\vee_{\mathbf{q},\mathbf{h}})(\mathbf{p}[s_A]\otimes\mathbf{q}[\mathrm{cano}_s(R_A)]\otimes\mathbf{h}[\mathrm{cano}_s(T_A)]) \\ &= \mathbf{p}[s_A]\otimes\psi^\vee_{\mathbf{q},\mathbf{h}}(\mathbf{q}[\mathrm{cano}_s(R_A)]\otimes\mathbf{h}[\mathrm{cano}_s(T_A)]) \end{aligned}$$

where $\mathrm{cano}_s(R) \sqcup \mathrm{cano}_s(T) = [n-s]$. Note $\psi^\vee_{\mathbf{q},\mathbf{h}}$ is only nonzero when the underlying sets $\mathrm{cano}_s(R) = [r]$ and $\mathrm{cano}_s(T) = [1+r, n-s]$, which implies the original sets had to have been as follows $R = [1+s, r+s]$ and $T = [1+r+s, n]$. Then applying $\psi^\vee_{\mathbf{q},\mathbf{h}}$ yields:

$$\begin{aligned} &= \mathbf{p}[s_A] \otimes \mathbf{q}[r_A] \otimes \mathbf{h}[\mathrm{cano}_r([1+r, n-s]_A)] \\ &= \mathbf{p}[s_A] \otimes \mathbf{q}[r_A] \otimes \mathbf{h}[[n-s-r]_A] \end{aligned}$$

For the righthand side: $(\psi^\vee_{\mathbf{p},\mathbf{q}} \otimes \mathrm{id}) \circ \psi^\vee_{\mathbf{p}\cdot\mathbf{q},\mathbf{h}}(\mathbf{p}[s_A] \otimes \mathbf{q}[[1+s, r+s]_A] \otimes \mathbf{h}[[1+r+s, n]_A])$

$$\begin{aligned} &= (\psi^\vee_{\mathbf{p},\mathbf{q}} \otimes \mathrm{id})(\mathbf{p}[s_A] \otimes \mathbf{q}[[1+s, r+s]_A] \otimes \mathbf{h}[[1, n-s-r]_A]) \\ &= \mathbf{p}[s_A] \otimes \mathbf{q}[r_A] \otimes \mathbf{h}[[n-s-r]_A] \end{aligned}$$

Thus $\psi^\vee$ is coassociative as desired.

3. <u>Claim:</u> $\psi^\vee$ is left and right counital.
In order to show that $\psi^\vee$ is left and right counital, we need Diagrams (14) to commute when the arrows are reversed with $\psi^\vee$ inserted in the appropriate places. For a fixed n, that amounts to showing:

$$(\mathrm{id}_{\mathbf{p}} \cdot \psi^\vee_0) \circ \psi^\vee_{\mathbf{p},\mathbf{1}_{\mathbb{K}}} = \mathrm{id}_{\mathbf{p}} = (\psi^\vee_0 \cdot \mathrm{id}_{\mathbf{p}}) \circ \psi^\vee_{\mathbf{1}_{\mathbb{K}},\mathbf{p}}$$

First notet by definition of $\mathbf{1}_{\mathbb{K}}$, we have that $\psi^\vee_{\mathbf{1}_{\mathbb{K}},\mathbf{p}}$:

$$\bigoplus_{S \sqcup T = [n]} \mathbf{1}_{\mathbb{K}}[S_A] \otimes \mathbf{p}[T_A] \xrightarrow{\oplus \mathbf{1}_{\mathbb{K}}[\mathrm{id}] \otimes \mathbf{p}[cano_s]} \bigoplus_{s+t=n} \mathbf{1}_{\mathbb{K}}[s_A] \otimes \mathbf{p}[t_A]$$

reduces to

$$\mathbb{K} \otimes \mathbf{p}[n_A] \xrightarrow{\oplus \mathbf{1}_{\mathbb{K}}[\mathrm{id}] \otimes \mathbf{p}[\mathrm{cano}_s]} \mathbb{K} \otimes \mathbf{p}[n_A]$$

So $\psi^\vee_{\mathbf{1}_{\mathbb{K}},\mathbf{p}} = \mathrm{id}_{\mathbb{K}} \otimes \mathrm{id}_{\mathbf{p}}$.

On the right hand side:

$$\begin{aligned} (\psi^\vee_0 \cdot \mathrm{id}_{\mathbf{p}}) \circ \psi^\vee_{\mathbf{1}_{\mathbb{K}},\mathbf{p}} &= (\mathrm{id}_{\mathbb{K}} \otimes \mathrm{id}_{\mathbf{p}}) \circ (\mathrm{id}_{\mathbb{K}} \otimes \mathrm{id}_{\mathbf{p}}) \\ &= \mathrm{id}_{\mathbb{K}} \otimes \mathrm{id}_{\mathbf{p}} \\ &\cong \mathrm{id}_{\mathbf{p}} \end{aligned}$$

Similarly for the left hand side. Thus showing $K^\vee_A$ is a colax monoidal functor.

Finally to show that $K^\vee_A$ is a bilax monoidal functor we must show that the braiding and unitality conditions are satisifed.

1. In order for the braiding condition to hold, we must show the following:

$$\psi_{\mathbf{p}\cdot\mathbf{r},\mathbf{q}\cdot\mathbf{h}} \circ K^\vee_A(\mathrm{id} \cdot \beta \cdot \mathrm{id}) \circ \varphi_{\mathbf{p}\cdot\mathbf{q},\mathbf{r}\cdot\mathbf{h}} = (\varphi_{\mathbf{p},\mathbf{r}} \cdot \varphi_{\mathbf{q},\mathbf{h}}) \circ (\mathrm{id} \cdot \beta \cdot \mathrm{id}) \circ (\psi_{\mathbf{p},\mathbf{q}} \cdot \psi_{\mathbf{r},\mathbf{h}}).$$

The above are natural transformations from

$$K^\vee_A(\mathbf{p} \cdot \mathbf{q}) \cdot K^\vee_A(\mathbf{r} \cdot \mathbf{h}) \to K^\vee_A(\mathbf{p} \cdot \mathbf{r}) \cdot K^\vee_A(\mathbf{q} \cdot \mathbf{h}).$$

Before showing the above equality holds, we need to understand both $\varphi_{\mathbf{p}\cdot\mathbf{q},\mathbf{r}\cdot\mathbf{h}}$ and $\psi_{\mathbf{p}\cdot\mathbf{r},\mathbf{q}\cdot\mathbf{h}}$.

First

$$\varphi_{\mathbf{p}\cdot\mathbf{q},\mathbf{r}\cdot\mathbf{h}} : K_A^{\vee}(\mathbf{p}\cdot\mathbf{q}) \cdot K_A^{\vee}(\mathbf{r}\cdot\mathbf{h}) \to K_A^{\vee}(\mathbf{p}\cdot\mathbf{q}\cdot\mathbf{r}\cdot\mathbf{h}).$$

Fix $n+m$, then

$$(\mathbf{p}\cdot\mathbf{q})[n_A] \otimes (\mathbf{r}\cdot\mathbf{h})[m_A] \xrightarrow{\oplus \mathbf{p}[\text{cano}]\otimes\mathbf{q}[\text{cano}]} \bigoplus_{N\sqcup M=[n+m]} (\mathbf{p}\cdot\mathbf{q})[N_A] \otimes (\mathbf{r}\cdot\mathbf{h})[M_A].$$

After doing the Cauchy products on the lefthand side and considering a specific decomposition $B \sqcup C = [n]$ and $U \sqcup V = [m]$, we get that $\varphi^{\vee}_{\mathbf{p}\cdot\mathbf{q},\mathbf{r}\cdot\mathbf{h}}$ is as follows:

$$\mathbf{p}[B_A]\otimes\mathbf{q}[C_A]\otimes\mathbf{r}[U_A]\otimes\mathbf{h}[V_A] \mapsto \mathbf{p}[\text{cano}_N(B_A)]\otimes\mathbf{q}[\text{cano}_N(C_A)]\otimes\mathbf{r}[\text{cano}_M(U_A)]\otimes\mathbf{h}[\text{cano}_M(V_A)]$$

Where $\text{cano}_N(B_A)$ is one of the decompositions of N_A after performing the Cauchy product on the right hand side; similarly for the rest of the cano maps applied to the remaining sets.

Now,

$$\psi_{\mathbf{p}\cdot\mathbf{r},\mathbf{q}\cdot\mathbf{h}} : K_A^{\vee}(\mathbf{p}\cdot\mathbf{r}\cdot\mathbf{q}\cdot\mathbf{h}) \to K_A^{\vee}(\mathbf{p}\cdot\mathbf{r}) \cdot K_A^{\vee}(\mathbf{q}\cdot\mathbf{h}).$$

On the n^{th} degree ψ is zero everywhere except on:

$$(\mathbf{p}\cdot\mathbf{r})[s_A] \otimes (\mathbf{q}\cdot\mathbf{h})[[1+s,t+s]_A] \xrightarrow{(\mathbf{p}\cdot\mathbf{r})[\text{id}]\otimes(\mathbf{q}\cdot\mathbf{h})[\text{cano}_s]} (\mathbf{p}\cdot\mathbf{r})[s_A] \otimes (\mathbf{q}\cdot\mathbf{h})[t_A].$$

After doing the Cauchy products on the lefthand side and for a specific decomposition of $B \sqcup C = [s]$ and $U \sqcup V = [1+s, t+s]$, we have:

$$\mathbf{p}[B_A] \otimes \mathbf{r}[C_A] \otimes \mathbf{q}[U_A] \otimes \mathbf{h}[V_A] \mapsto \mathbf{p}[B_A] \otimes \mathbf{r}[C_A] \otimes \mathbf{q}[\text{cano}_s(U_A)] \otimes \mathbf{h}[\text{cano}_s(V_A)],$$

where B and C are one of the pairs of decompositions of $[s]$ after performing the Cauchy product $\mathbf{p}\cdot\mathbf{r}$, and $\text{cano}_s(U)$ and $\text{cano}_s(V)$ is a pair of decompositions of $[t]$ after performing the Cauchy product $\mathbf{q}\cdot\mathbf{h}$.

For the right hand side:

$$(\varphi_{\mathbf{p},\mathbf{r}} \cdot \varphi_{\mathbf{q},\mathbf{h}}) \circ (\text{id} \circ \beta \circ \text{id}) \circ (\psi_{\mathbf{p},\mathbf{q}} \cdot \psi_{\mathbf{r},\mathbf{h}})(\mathbf{p}[s_A] \otimes \mathbf{q}[[1+s,t+s]_A] \otimes \mathbf{r}[u_A] \otimes \mathbf{h}[[1+u,v+u]_A])$$

$$\begin{aligned}
&= (\varphi_{\mathbf{p},\mathbf{r}} \cdot \varphi_{\mathbf{q},\mathbf{h}}) \circ (\text{id} \cdot \beta \cdot \text{id})(\mathbf{p}[s_A] \otimes \mathbf{q}[t_A] \otimes \mathbf{r}[u_A] \otimes \mathbf{h}[v_A]) \\
&= (\varphi_{\mathbf{p},\mathbf{r}} \cdot \varphi_{\mathbf{q},\mathbf{h}})(\mathbf{p}[s_A] \otimes \mathbf{r}[u_A] \otimes \mathbf{q}[t_A] \otimes \mathbf{h}[v_A]) \\
&= \bigoplus_{\substack{S \sqcup U = [s+u] \\ T \sqcup V = [t+v] \\ |S| = s,\ |U| = u \\ |T| = t,\ |V| = v}} \mathbf{p}[S_A] \otimes \mathbf{r}[U_A] \otimes \mathbf{q}[T_A] \otimes \mathbf{h}[V_A].
\end{aligned}$$

For the left hand side:

$$(\psi_{\mathbf{p}\cdot\mathbf{r},\mathbf{q}\cdot\mathbf{h}}) \circ K_A^{\vee}(\text{id} \cdot \beta \cdot \text{id}) \circ (\varphi_{\mathbf{p}\cdot\mathbf{q},\mathbf{r}\cdot\mathbf{h}})(\mathbf{p}[s_A] \otimes \mathbf{q}[[1+s,t+s]_A] \otimes \mathbf{r}[u_A] \otimes \mathbf{h}[[1+u,v+u]_A])$$

$$= (\psi_{\mathbf{p}\cdot\mathbf{r},\mathbf{q}\cdot\mathbf{h}})\left(\bigoplus_{\substack{N\sqcup M=[s+t+u+v]\\|N|=s+t\\|M|=u+v}}\mathbf{p}[c_N(S_A)]\otimes\mathbf{r}[c_M(U_A)]\otimes\mathbf{q}[c_N(T_A)]\otimes\mathbf{h}[c_M(V_A)]\right).$$

For a fixed $N\sqcup M=[s+t+u+v]$,

$$\mathbf{p}[c_N(S_A)]\otimes\mathbf{r}[c_M(U_A)]\otimes\mathbf{q}[c_N(T_A)]\otimes\mathbf{h}[c_M(V_A)]\subseteq\bigoplus_{N'\sqcup M'=[s+t+u+v]}(\mathbf{p}\cdot\mathbf{r})[N'_A]\otimes(\mathbf{q}\cdot\mathbf{h})[M'_A]$$

$\psi_{\mathbf{p}\cdot\mathbf{r},\mathbf{q}\cdot\mathbf{h}}$ is zero everywhere except when $N'=[s+u]$ and $M'=[1+s+u,t+v+s+u]$. In our equation, in order to get a nonzero vector space we need that $N':=\mathrm{cano}_N(S_A)\sqcup\mathrm{cano}_M(U_A)=[s+u]$ and $M':=\mathrm{cano}_N(T_A)\sqcup\mathrm{cano}_M(V_A)=[1+s+u,t+v+s+u]$ Thus

$$\begin{aligned}&= (\psi_{\mathbf{p}\cdot\mathbf{r},\mathbf{q}\cdot\mathbf{h}})\left(\bigoplus_{\substack{N\sqcup M=[s+t+u+v]\\|N|=s+t\\|M|=u+v}}\mathbf{p}[c_N(S_A)]\otimes\mathbf{r}[c_M(U_A)]\otimes\mathbf{q}[c_N(T_A)]\otimes\mathbf{h}[c_M(V_A)]\right)\\&= \bigoplus\mathbf{p}[c_N(S_A)]\otimes\mathbf{r}[c_M(U_A)]\otimes\mathbf{q}[st|_{M'}(c_N(T_A))]\otimes\mathbf{h}[st|_{M'}(c_M(V_A))],\end{aligned}$$

where the direct sum is over $c_N(S_A)\sqcup c_M(U_A)=[s+u]$, $st|_{M'}(c_N(T_A))\sqcup st|_{M'}(c_M(V_A))=[t+v]$ such that $|c_N(S)|=s$, $|c_M(U_)|=u$, $|st|_{M'}(c_N(T_A))|=t$, and $|st|_{M'}(c_M(V_A))|=v$. Reindexing the sum, yields the same output as the right hand side. Thus, the braiding condition is satisfied.

2. Finally, we must show that the unitality condition.
Since $\varphi_0^\vee=\mathrm{id}$, $\psi_0^\vee=\mathrm{id}$ and $\lambda_{\mathbf{1}_\mathbb{K}}$ and $\rho_{\mathbf{1}_\mathbb{K}}$ are isomorphisms, the unitality conditions in (16) are satisfied.
Therefore, $K_A^\vee$ is bilax.

□

Now, since for each $\mathbf{p}\in\mathbf{Sp}^A$, $\mathbf{p}[I_A]$ is a $A\wr S_n$ module, we can consider the space of $A\wr S_n$ invariants, i.e.,

$$\mathbf{p}[I_A]^{A\wr S_n}=\langle v\in\mathbf{p}[I_A]\mid(a_1,..,a_n,\sigma).v=\varepsilon_A(a_1)\cdots\varepsilon_A(a_n)v\rangle$$

.

Definition 10.1.4. For each $p\in\mathbf{Sp}^A$ and morphism $f:p\to q$ of A-species, we can define the functor $\widetilde{K_A^\vee}:\mathbf{Sp}^A\to\mathbf{gVec}$ via

$$\widetilde{K_A^\vee}(p):=\bigoplus_{n\geq 0}\mathbf{p}[n_A]^{A\wr S_n}$$

$$\widetilde{K_A^\vee}(f):=\bigoplus_{n\geq 0}f_{[n_A]}.$$

Proof. We immediately have $\widetilde{K_A^\vee}(p) \in \mathbf{gVec}$. Since $\widetilde{K_A^\vee}$ applied to a morphism of A-species is just the restriction to the $A \wr S_n$ invariant subspace of $K_A^\vee$ applied to a morphism of A-species, it follows that composition makes sense and identity is sent to identity. $\square$

Proposition 10.1.5. *The functor $\widetilde{K_A^\vee} : \boldsymbol{Sp}^A \to \mathbf{gVec}$ is bilax monoidal.*

Proof. For the bilax structure of $\widetilde{K_A^\vee}$, we define the maps $\tilde{\varphi}^\vee$ and $\tilde{\psi}^\vee$ by the diagram below:

$$\begin{tikzcd}
K_A^\vee(\mathbf{p}) \cdot K_A^\vee(\mathbf{q}) \arrow[r, shift left, "\varphi^\vee_{\mathbf{p},\mathbf{q}}"] & K_A^\vee(\mathbf{p} \cdot \mathbf{q}) \arrow[l, shift left, "\psi^\vee_{\mathbf{p},\mathbf{q}}"] \\
\widetilde{K_A^\vee}(\mathbf{p}) \cdot \widetilde{K_A^\vee}(\mathbf{q}) \arrow[u, hook] \arrow[r, dashed, shift left, "\tilde{\varphi}^\vee_{\mathbf{p},\mathbf{q}}"] & \widetilde{K_A^\vee}(\mathbf{p} \cdot \mathbf{q}) \arrow[u, hook] \arrow[l, dashed, shift left, "\tilde{\psi}^\vee_{\mathbf{p},\mathbf{q}}"]
\end{tikzcd}$$

We must first show that $\tilde{\varphi}^\vee$ and $\tilde{\psi}^\vee$ sends invariant elements to invariant elements. I will show this for $\tilde{\varphi}^\vee$ and the proof for $\tilde{\psi}^\vee$ is simpler, since if an element is invariant under $A \wr S_n$ it is clearly invariant under the subgroup $A \wr (S_r \times S_t)$.
Fix $r + t = n$, then the diagram becomes:

$$\begin{tikzcd}
\mathbf{p}[r_A] \otimes \mathbf{q}[t_A] \arrow[r, "\oplus \mathbf{p}[\text{cano}] \otimes \mathbf{q}[\text{cano}]"] & \displaystyle\bigoplus_{\substack{R \sqcup T = [n] \\ |R| = r \\ |T| = t}} \mathbf{p}[R_A] \otimes \mathbf{q}[T_A] \\
\mathbf{p}[r_A]^{A \wr S_r} \otimes \mathbf{q}[t_A]^{A \wr S_t} \arrow[u, hook] \arrow[r, dashed] & \left(\displaystyle\bigoplus_{\substack{R \sqcup T = [n] \\ |R| = r \\ |T| = t}} \mathbf{p}[R_A] \otimes \mathbf{q}[T_A] \right)^{A \wr S_n} \arrow[u, hook]
\end{tikzcd}.$$

Consider an element $\sum v_{[r]_A} \otimes w_{[t]_A} \in \mathbf{p}[r_A]^{A \wr S_r} \otimes \mathbf{q}[t_A]^{A \wr S_t}$. Notice that $v_{[r]_A}$ is invariant under $A \wr S_r$, i.e., for all $(a_1 \cdots a_r \otimes \sigma) \in A \wr S_r$ we have that $(a_1 \cdots a_r \otimes \sigma).v_{[r]_A} = \prod_{i=1}^{r} \varepsilon_A(a_i) v_{[r]_A}$. Similarly, $(c_1 \cdots c_t \otimes \tau).w_{[t]_A} = \prod_{j=1}^{t} \varepsilon_A(c_j) w_{[t]_A}$ for all $(c_1 \cdots c_t \otimes \tau) \in A \wr S_t$. Applying $\varphi^\vee$, we get

$$\sum v_{[r]_A} \otimes w_{[t]_A} \mapsto \sum_{\substack{R \sqcup T = [r+t] \\ |R| = r \\ |T| = t}} \sum v_{R_A} \otimes w_{T_A}.$$

We want to show that the image is invariant under $A \wr S_n$, i.e., for all $(a_1 \cdots a_n \otimes \sigma) \in A \wr S_n$, we have

$$(a_1 \cdots a_n \otimes \sigma). \sum_{\substack{R \sqcup T = [r+t] \\ |R| = r \\ |T| = t}} \sum v_{R_A} \otimes w_{T_A} = \prod_{i=1}^{n} \varepsilon_A(a_i) \sum v_{R_A} \otimes w_{T_A} \tag{32}$$

It suffices to show invariant under the following elements that generate $A \wr S_n$:

$$\{(a_1 \cdot 1 \cdots 1 \otimes \mathrm{id}) \mid a_1 \in A_1\} \sqcup \{(1 \cdots 1 \otimes (ij)) \mid (ij) \text{ a simple transposition}\}$$

By the functoriality of $\mathbf{p}$ and $\mathbf{q}$, we can work at the level of our objects of $\mathbf{Set}^A$.

Fix decomposition of $[n]$, say $R = \{\delta_1, ..., \delta_r\}$ and $T = \{\xi_1, ..., \xi_t\}$.

First, consider $(1 \cdots 1 \otimes (ij)) \in A \wr S_n$. We must show that this element acts by identity since $\varepsilon(1 \cdots 1 \otimes (ij)) = 1$. We first need to understand how $(1 \cdots 1 \otimes (ij))$ acts on $\bigotimes_{z=1}^{r} A_{\delta_z} \otimes \mathbb{K}[R]$ and $\bigotimes_{k=1}^{t} A_{\xi_k} \otimes \mathbb{K}[T]$. There are two cases we must check:

<u>Case 1:</u> WLOG, let $i, j \in R$, then for some $s \in [r]$, we have $\delta_s = i$ and $\delta_{s+1} = j$. First we look at the restriction of our simple transposition (ij) on our underlying sets R and T. Since $i, j \in R$, $(ij).R = R$, it's the permutation from $R \to R$ that swaps i and j, and (ij) acts as the identity map on T since $i, j \notin T$.

- $1^{\otimes n} \otimes (ij)$ acts on $A_{\delta_1} \otimes \cdots \otimes A_{\delta_r} \otimes \mathbb{K}[R]$ by the map $\bigotimes_{z=1}^{r} 1_{\delta_z} \otimes (ij)|_R$, i.e.,

$$\bigotimes_{z=1}^{r} 1_{\delta_k} \otimes (ij)|_R\Big(\bigotimes_{k=1}^{r} A_{\delta_z} \otimes \mathbb{K}[R]\Big) = \bigotimes_{z=1}^{r} 1_{\delta_z} A_{\delta_{(ij)^{-1}(z)}} \otimes \mathbb{K}[(ij)|_R.R] = \bigotimes_{z=1}^{r} 1_{\delta_z} A_{\delta_{(ij)^{-1}(z)}} \otimes \mathbb{K}[R].$$

- $1^{\otimes n} \otimes (ij)$ acts on $A_{\xi_1} \otimes \cdots \otimes A_{\xi_t} \otimes \mathbb{K}[T]$ by the map $\bigotimes_{k=1}^{t} 1_{\xi_k} \otimes (ij)|_T$, i.e.,

$$\bigotimes_{k=1}^{t} 1_{\xi_k} \otimes (ij)|_T\Big(\bigotimes_{k=1}^{t} A_{\xi_k} \otimes \mathbb{K}[T]\Big) = \bigotimes_{k=1}^{t} 1_{\xi_k} A_{\xi_{(ij)^{-1}(k)}} \otimes \mathbb{K}[(ij)|_T.T] = \bigotimes_{k=1}^{t} 1_{\xi_k} A_{\xi_{(ij)^{-1}(k)}} \otimes \mathbb{K}[T].$$

Note that $(ij)^{-1} = (ij)$, thus $(ij)^{-1}(z) = \begin{cases} \delta_s & \text{if } z = \delta_{s+1} \\ \delta_{s+1} & \text{if } z = \delta_s \\ z & \text{otherwise} \end{cases}$.

Hence, for all $k \in [t]$, $(ij)^{-1}(k) = k$. Thus, when our underlying set is R, we have that $(1 \cdots 1 \otimes (ij))$ acts by identity on all A's except in the positions $\delta_s = i$ and $\delta_{s+1} = j$, in which case (ij) acts by place permutation. Because this is a bijection and cano_R is a bijection, we can define the map $f_r \in A \wr S_r$ that makes the following diagram commute.

$$\begin{tikzcd}
A_1 \otimes \cdots \otimes A_r \otimes \mathbb{K}[r] \arrow[r, "\mathrm{cano}_R"] \arrow[d, "f_r"] & A_{\delta_1} \otimes \cdot\cdot A_{\delta_s} \otimes A_{\delta_{s+1}} \otimes \cdot\cdot \otimes A_{\delta_r} \otimes \mathbb{K}[R] \arrow[d, "1^{\otimes n} \otimes (ij)"] \\
A_1 \otimes \cdots \otimes A_r \otimes \mathbb{K}[r] \arrow[r, "\mathrm{cano}_R"] & A_{\delta_1} \otimes \cdot\cdot \otimes A_{\delta_{s+1}} \otimes A_{\delta_s} \otimes \cdot \otimes A_{\delta_r} \otimes \mathbb{K}[R].
\end{tikzcd}$$

Moreover, we have that $f_r = (1 \cdots 1 \otimes (ij)) \in A \wr S_r$.

Furthermore, we have that $(1 \cdots 1 \otimes (ij))$ acts as the identity on $\bigotimes_{k=1}^{t} A_{\xi_k} \otimes \mathbb{K}[T]$. Again, since this is a bijection and cano_T is a bijection, we can define the map $f_t \in A \wr S_t$ to be the map that makes the following diagram commute.

$$\begin{tikzcd}
A_1 \otimes \cdots \otimes A_t \otimes \mathbb{K}[t] \arrow[r, "\text{cano}_T"] \arrow[d, "f_t"] & A_{\xi_1} \otimes \cdots \otimes A_{\xi_t} \otimes \mathbb{K}[T] \arrow[d, "1^{\otimes n} \otimes (ij)"] \\
A_1 \otimes \cdots \otimes A_t \otimes \mathbb{K}[t] \arrow[r, "\text{cano}_T"] & A_{\xi_1} \otimes \cdots \otimes A_{\xi_t} \otimes \mathbb{K}[T].
\end{tikzcd}$$

Moreover, we have that $f_t = (1, ..., 1, \text{id}) \in A \wr S_t$.

From the above diagrams commuting and by the functoriality of $\mathbf{p}$ and $\mathbf{q}$, we have the following commuting diagram

$$\begin{tikzcd}
\sum v_{[r]_A} \otimes w_{[t]_A} \arrow[r, "\mathbf{p}[\text{cano}_R] \otimes \mathbf{q}[\text{cano}_T]"] \arrow[d, "f_r \times f_t"] & \sum v_{R_A} \otimes w_{T_A} \arrow[d, "1^{\otimes n} \otimes (ij)"] \\
(f_r \times f_t). \sum v_{[r]_A} \otimes w_{[t]_A} \arrow[r, "\mathbf{p}[\text{cano}_R] \otimes \mathbf{q}[\text{cano}_T]"] & (1 \cdots 1 \otimes (ij)). \sum v_{R_A} \otimes w_{T_A}.
\end{tikzcd}$$

Observe that

$$f_r \times f_t. \sum v_{[r]_A} \otimes w_{[t]_A} = \varepsilon(f_r)\varepsilon(f_t) \sum v_{[r]_A} \otimes w_{[t]_A} = \sum v_{[r]_A} \otimes w_{[t]_A}$$

since $\sum v_{[r]_A} \otimes w_{[t]_A}$ is invariant under $A \wr S_r \otimes A \wr S_t$.
Thus, we have that

$$\mathbf{p}[\text{cano}_R] \otimes \mathbf{q}[\text{cano}_T](\sum v_{[r]_A} \otimes w_{[t]_A}) = (1 \cdots 1 \otimes (ij)). \sum v_{R_A} \otimes w_{T_A}$$

Which implies

$$(1 \cdots 1 \otimes (ij)). \sum v_{R_A} \otimes w_{T_A} = \sum v_{T_A} \otimes w_{T_A}.$$

Thus $(1 \cdots 1 \otimes (ij))$ fixes elements labeled by R_A and T_A when both i, j are contained in one of the underlying sets.

Case 2: WLOG, say $i \in R$ and $j \in J$, then for some $s \in [r]$ we have $\delta_s = i$ and for some $h \in [t]$ we have that $\xi_h = j$. Then (ij) is an order preserving bijection that replaces δ_s with ξ_h in R and replaces ξ_h with δ_s in T. On the tuple of A's, we are just renaming the i^{th} position with the j^{th} position while preserving the original order. Since this is a bijection and cano is a bijection, we can define f_r by the following commuting diagram:

$$\begin{tikzcd}
A_1 \otimes \cdots \otimes A_r \otimes \mathbb{K}[r] \arrow[r, "\text{cano}_R"] \arrow[d, "f_r"] & A_{\delta_1} \otimes \cdot \cdot \otimes A_{\delta_s} \otimes \cdot \cdot \otimes A_{\delta_r} \otimes \mathbb{K}[R] \arrow[d, "1^{\otimes n} \otimes (ij)"] \\
A_1 \otimes \cdots \otimes A_r \otimes \mathbb{K}[r] \arrow[r, "\text{cano}''_R"] & A_{\delta_1} \otimes \cdot \cdot \otimes A_{\xi_h} \otimes \cdot \cdot \otimes A_{\delta_r} \otimes \mathbb{K}[R']
\end{tikzcd}$$

Moreover, $f_r = (1, ..., 1, \mathrm{id}) \in A \wr S_r$ since both maps in the right hand corner of the diagram are order preserving.
We can also define f_t by the following commuting diagram:

$$\begin{tikzcd}
A_1 \otimes \cdots \otimes A_r \otimes \mathbb{K}[t] \arrow[r, "\mathrm{cano}_T"] \arrow[d, "f_t"] & A_{\xi_1} \otimes \cdot \cdot \otimes A_{\xi_h} \otimes \cdot \cdot \otimes A_{\xi_t} \otimes \mathbb{K}[T] \arrow[d, "1^{\otimes n} \otimes (ij)"] \\
A_1 \otimes \cdots \otimes A_r \otimes \mathbb{K}[t] \arrow[r, "\mathrm{cano}''_T"] & A_{\xi_1} \otimes \cdot \cdot \otimes A_{\delta_s} \otimes \cdot \cdot \otimes A_{\xi_t} \otimes \mathbb{K}[T']
\end{tikzcd}$$

where $f_t = (1 \cdots 1 \otimes \mathrm{id}) \in A \wr S_t$ since both maps in the right hand corner are order preserving.

From the above diagrams commute and the functoriality of $\mathbf{p}$ and $\mathbf{q}$, we have the following commuting diagram

$$\begin{tikzcd}
\sum v_{[r]_A} \otimes w_{[t]_A} \arrow[r, "\mathbf{p}[\mathrm{cano}_R] \otimes \mathbf{q}[\mathrm{cano}_T]"] \arrow[d, "f_r \times f_t"] & \sum v_{R_A} \otimes w_{T_A} \arrow[d, "1^{\otimes n} \otimes (ij)"] \\
(f_r \times f_t). \sum v_{[r]_A} \otimes w_{[t]_A} \arrow[r, "\mathbf{p}[\mathrm{cano}_R] \otimes \mathbf{q}[\mathrm{cano}_T]"] & (1 \cdots 1 \otimes (ij)). \sum v_{R_A} \otimes w_{T_A}.
\end{tikzcd}$$

Observe that

$$f_r \times f_t. \sum v_{[r]_A} \otimes w_{[t]_A} = \varepsilon(f_r)\varepsilon(f_t) \sum v_{[r]_A} \otimes w_{[t]_A} = \sum v_{[r]_A} \otimes w_{[t]_A},$$

since $\sum v_{[r]_A} \otimes w_{[t]_A}$ is invariant under $A \wr S_r \otimes A \wr S_t$.
Thus, we have that

$$\mathbf{p}[\mathrm{cano}'_R] \otimes \mathbf{q}[\mathrm{cano}'_T] \left(\sum v_{[r]_A} \otimes w_{[t]_A} \right) = (1 \cdots 1 \otimes (ij)). \sum v_{R_A} \otimes w_{T_A}.$$

This implies

$$(1 \cdots 1 \otimes (ij)). \sum v_{R_A} \otimes w_{T_A} = \sum v_{\hat{R}'} \otimes w_{\hat{T}'},$$

which is a term in the sum in Equation 32.

This shows that an individual term when acted on by $(1 \cdots 1 \otimes (ij))$ either is fixed or is again another term in the sum. Now we need to show that we get every possible term in the sum, i.e., we get a term corresponding to each decomposition $R \sqcup T = [n]$ such that $|R| = r$ and $|T| = t$. First, note that we won't get any extra terms since the underlying action consists of bijections of those sets. Furthermore, for two distinct decompositions $R_1 \sqcup T_1$ and $R_2 \sqcup T_2$, we get distinct pairs $R'_1 \sqcup T'_1$ and $R'_2 \sqcup T'_2$ after acting by (ij), for if we didn't then $R_1 = R_2$ and $T_1 = T_2$. Since (ij) is a bijection, given any $R' \sqcup T'$, we can find the original decomposition by applying $(ij)^{-1}$. Thus we get every possible term in the sum.

Now consider $(b \cdot 1 \cdots 1 \otimes \mathrm{id}) \in A \wr S_n$. We must show that this element acts by $\varepsilon(b \cdot 1 \cdots 1 \otimes \mathrm{id}) = \varepsilon(b)$. First, we need to understand how $(b \cdot 1 \cdots 1 \otimes \mathrm{id})$ acts on $\bigotimes_{z=1}^{r} A_{\delta_z} \otimes \mathbb{K}[R]$ and $\bigotimes_{k=1}^{t} A_{\xi_k} \otimes \mathbb{K}[T]$.
WLOG, say $1 \in R$, where 1 denotes the position of b in the tuple. Then $b \cdot 1 \cdots 1 \otimes \mathrm{id}$ acts on $A_{\delta_1} \otimes \cdots A_{\delta_r} \otimes \mathbb{K}[R]$ by

$$\left(b \otimes \left(\bigotimes_{z=2}^{r} 1_{\delta_z} \right) \otimes \mathrm{id}_R \right) \left(\bigotimes_{z=1}^{r} A_{\delta_z} \otimes \mathbb{K}[R] \right) = bA_{\delta_1} \otimes A_{\delta_2} \otimes \cdots \otimes A_{\delta_r} \otimes \mathbb{K}[R]$$

i.e., by multiplication by b on the left. Because $b \otimes \left(\bigotimes_{z=2}^{r} 1_{\delta_z}\right) \otimes \mathrm{id}_R$ and cano_R are order preserving bijections, we can define f_r by the following commuting diagram

$$\begin{tikzcd}
A_1 \otimes \cdots \otimes A_r \otimes \mathbb{K}[r] \arrow[r, "\mathrm{cano}_R"] \arrow[d, "f_r"] & A_{\delta_1} \otimes \cdots \otimes A_{\delta_r} \otimes \mathbb{K}[R] \arrow[d, "b\otimes\left(\bigotimes_{i=2}^{n} 1_i\right)\otimes\mathrm{id}"] \\
A_1 \otimes \cdots \otimes A_r \otimes \mathbb{K}[r] \arrow[r, "\mathrm{cano}_R"] & bA_{\delta_1} \otimes \cdots \otimes A_{\delta_r} \otimes \mathbb{K}[R]
\end{tikzcd}$$

where $f_r = b \otimes \left(\bigotimes_{z=2}^{r} 1_{\delta_z}\right) \otimes \mathrm{id}_R \in A \wr S_r$

Since $1 \notin T$, we have that $b \otimes 1 \otimes \cdots \otimes 1 \otimes \mathrm{id}$ acts as the identity on $A_{\xi_1} \otimes \cdots \otimes A_{\xi_t} \otimes \mathbb{K}[T]$, since $(b \cdot 1 \cdots 1 \otimes \mathrm{id})|_T$ is the map $\bigotimes_{k=1}^{t} 1_{\xi_k} \otimes \mathrm{id}$. Thus, we can define the bijection $f_t = \bigotimes_{k=1}^{t} 1_k \otimes \mathrm{id}_T \in A \wr S_t$, coming from the following commuting diagram:

$$\begin{tikzcd}
A_1 \otimes \cdots \otimes A_t \otimes \mathbb{K}[t] \arrow[r, "\mathrm{cano}_T"] \arrow[d, "f_t"] & A_{\xi_1} \otimes \cdots \otimes A_{\xi_t} \otimes \mathbb{K}[T] \arrow[d, "b\otimes\left(\bigotimes_{i=2}^{n} 1_i\right)\otimes\mathrm{id}"] \\
A_1 \otimes \cdots \otimes A_t \otimes \mathbb{K}[t] \arrow[r, "\mathrm{cano}_T"] & A_{\xi_1} \otimes \cdots \otimes A_{\xi_t} \otimes \mathbb{K}[T]
\end{tikzcd}$$

By functoriality of $\mathbf{p}$ and $\mathbf{q}$, we get the following commuting square:

$$\begin{tikzcd}
\sum v_{[r]_A} \otimes w_{[t]_A} \arrow[r, "\mathbf{p}[\mathrm{cano}_R]\otimes\mathbf{q}[\mathrm{cano}_T]"] \arrow[d, "f_r \times f_t"] & \sum v_{R_A} \otimes w_{T_A} \arrow[d, "b\otimes\left(\bigotimes_{i=2}^{n} 1_i\right)\otimes\mathrm{id}"] \\
(f_r \times f_t).\sum v_{[r]_A} \otimes w_{[t]_A} \arrow[r, "\mathbf{p}[\mathrm{cano}_R]\otimes\mathbf{q}[\mathrm{cano}_T]"] & b \otimes \left(\bigotimes_{i=2}^{n} 1_i\right) \otimes \mathrm{id}\,.\sum v_{R_A} \otimes w_{T_A}
\end{tikzcd}$$

Observe that since $\sum v_{[r]_A} \otimes w_{[t]_A}$ invariant under $A \wr S_r \otimes A \wr S_t$, we have

$$(f_r \times f_t).\sum v_{[r]_A} \otimes w_{[t]_A} = \varepsilon(f_r)\varepsilon(f_t) \sum v_{[r]_A} \otimes w_{[t]_A} = \varepsilon(b) \sum v_{[r]_A} \otimes w_{[t]_A}.$$

Thus we have that

$$\mathbf{p}[\mathrm{cano}_R] \otimes \mathbf{q}[\mathrm{cano}_T] \left(\varepsilon(b) \sum v_{[r]_A} \otimes w_{[t]_A}\right) = b \otimes \left(\bigotimes_{i=2}^{n} 1_i\right) \otimes \mathrm{id}\,.\sum v_{R_A} \otimes w_{T_A}$$

which yields

$$b \otimes \left(\bigotimes_{i=2}^{n} 1_i\right) \otimes \mathrm{id}\,.\sum v_{R_A} \otimes w_{T_A} = \varepsilon(b) \sum v_{R_A} \otimes w_{T_A}$$

Ranging over all decompositions yields the desired result. Thus $K_A^\vee$ restricts to invariants.

Finally,to show that $\widetilde{K_A^\vee}$ is bilax monoidal. First note that $\widetilde{K_A^\vee}(\mathbf{1}_\mathbb{K}) = K_A^\vee(\mathbf{1}_\mathbb{K})$, thus $\tilde{\varphi}_0^\vee = \varphi_0^\vee$ and similarly $\tilde{\psi}_0^\vee = \psi_0^\vee$. Since $\tilde{\varphi}_{\mathbf{p},\mathbf{q}}^\vee$ and $\tilde{\psi}_{\mathbf{p},\mathbf{q}}^\vee$ are defined to be the restriction of $\varphi^\vee$ and $\psi^\vee$, they satisfy all the axioms to make $\widetilde{K_A^\vee}$ a bilax monoidal functor.

$\square$

Proposition 10.1.6. *The functor $\widetilde{K_A^\vee}$ is a bistrong functor.*

Proof. To show that $\widetilde{K_A^\vee}$ is a bistrong functor, it suffices to show that $\tilde{\varphi}_0^\vee \circ \tilde{\psi}_0^\vee = \mathrm{id}$ and $\tilde{\psi}^\vee \circ \tilde{\varphi}^\vee = \mathrm{id}$. First, since $\tilde{\varphi}_0^\vee = \mathrm{id}$ and $\tilde{\psi}_0^\vee = \mathrm{id}$ we have that $\tilde{\varphi}_0^\vee \circ \tilde{\psi}_0^\vee = \mathrm{id}$. Now, let $S \sqcup T = [n]$, then

$$\mathbf{p}[s_A] \xrightarrow{\mathbf{p}[\mathrm{cano}_S]} \bigoplus_{\substack{S \subseteq [n] \\ |S| = s}} S_A \xrightarrow{\mathbf{p}[\mathrm{id}_{[s]}]} \mathbf{p}[s_A]$$

$$\mathbf{q}[t_A] \xrightarrow{\mathbf{q}[\mathrm{cano}_T]} \bigoplus_{\substack{T \subseteq [n] \\ |T| = t}} \mathbf{q}[T_A] \xrightarrow{\mathbf{q}[\mathrm{cano}_t]} \mathbf{q}[t_A]$$

Tensoring the diagrams together yields the desired result. Thus $\widetilde{K_A^\vee}$ is a bistrong monoidal functor. See Prop 3.46 in [3].

□

10.2. A-Coinvariance

For each $\mathbf{p} \in \mathbf{Sp}^A$ and morphism $f : \mathbf{p} \to \mathbf{q}$ of A-species, we consider functor $K_A : \mathbf{Sp}^A \to \mathbf{gVec}$, as defined in 10.1.1 via

$$K_A(\mathbf{p}) := \bigoplus_{n \geq 0} \mathbf{p}[n_A]$$

$$K_A(f) := \bigoplus_{n \geq 0} f_{[n_A]}.$$

The functor K_A admits another bilax structure, different from the one used in Section 10.1. This new bilax structure allows for the coinvariance functor, described in the following section, to be a bistrong bilax monoidal functor. We describe this bilax structure in the following proposition.

Proposition 10.2.1. *The functor K_A is a bilax monoidal functor.*

Proof. In order to show that K_A is a bilax monoidal functor, we need to define natural transformations

$$\mathcal{M} \circ (K_A \times K_A) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} K_A \circ \mathcal{M}$$

Again, $\mathcal{M}$ denotes the tensor product of functors and both $\mathcal{M} \circ (K_A \times K_A)$ and $K_A \circ \mathcal{M}$ are functors from $\mathbf{Sp}^A \times \mathbf{Sp}^A \to \mathbf{gVec}$.
Let $\mathbf{p}, \mathbf{q} \in \mathbf{Sp}^A$, then

$$K_A(\mathbf{p}) \cdot K_A(\mathbf{q}) \underset{\psi_{\mathbf{p},\mathbf{q}}}{\overset{\varphi_{\mathbf{p},\mathbf{q}}}{\rightleftarrows}} K_A(\mathbf{p} \cdot \mathbf{q})$$

Note that

$$K_A(\mathbf{p}) \cdot K_A(\mathbf{q}) = \bigoplus_{n \geq 0} \bigoplus_{r+t=n} \mathbf{p}[r_A] \otimes \mathbf{q}[t_A]$$

$$K_A(\mathbf{p}\cdot\mathbf{q}) = \bigoplus_{n\geq 0}\bigoplus_{R\sqcup T=[n]} \mathbf{p}[R_A]\otimes\mathbf{q}[T_A]$$

On the degree n piece, we define the sections of φ and ψ as follows:

$$\varphi_{\mathbf{p},\mathbf{q}} : \mathbf{p}[r_A]\otimes\mathbf{q}[t_A] \xrightarrow{\mathbf{p}[\mathrm{id}]\otimes\mathbf{q}[\mathrm{cano}]} \mathbf{p}[r_A]\otimes\mathbf{q}[[r+1,r+t]_A]$$

$$\psi_{\mathbf{p},\mathbf{q}} : \mathbf{p}[R_A]\otimes\mathbf{q}[T_A] \xrightarrow{\mathbf{p}[\mathrm{st}]\otimes\mathbf{q}[\mathrm{st}]} \mathbf{p}[[|R|]_A]\otimes\mathbf{q}[[|T|]_A].$$

Now, observe that $K_A(\mathbf{1}_{\mathbb{K}}) = \bigoplus_{n\geq 0}\mathbf{1}_{\mathbb{K}}[n_A] = \mathbb{K}\oplus 0\oplus\cdots = \mathbb{K}$ which is the unit of **gVec**. Thus we can define $\varphi_0 = \mathrm{id}$ and $\psi_0 = \mathrm{id}$.

Showing the lax/colax structure of φ/ψ is an analogous argument as in Proposition 10.1.3 and can reference [3] for a proof of this in the classical version of species.

$\square$

Definition 10.2.2. For each $\mathbf{p}\in\mathbf{Sp}^A$ and morphism $f:\mathbf{p}\to\mathbf{q}$ of A-species, we can define the functor $\widetilde{K}_A^{\vee} : \mathbf{Sp}^A \to \mathbf{gVec}$ via

$$\widetilde{K_A}(\mathbf{p}) := \bigoplus_{n\geq 0}\mathbf{p}[n_A]_{A\wr S_n}$$

$$\widetilde{K_A}(f) := \bigoplus_{n\geq 0}\overline{f}_{[n_A]}$$

Proof. $\square$

For the bilax structure, we define the maps $\tilde\varphi$ and $\tilde\psi$ by the commutativity of the following diagram:

$$\begin{tikzcd}
K_A(\mathbf{p})\cdot K_A(\mathbf{q}) \arrow[r, shift left, "\varphi_{\mathbf{p},\mathbf{q}}"] \arrow[d, two heads, "\pi"] & K_A(\mathbf{p}\cdot\mathbf{q}) \arrow[l, shift left, "\psi_{\mathbf{p},\mathbf{q}}"] \arrow[d, two heads, "\pi"] \\
\widetilde{K_A}(\mathbf{p})\cdot\widetilde{K_A}(\mathbf{q}) \arrow[r, shift left, dashed, "\tilde\varphi_{\mathbf{p},\mathbf{q}}"] & \widetilde{K_A}(\mathbf{p}\cdot\mathbf{q}) \arrow[l, shift left, dashed, "\tilde\psi_{\mathbf{p},\mathbf{q}}"]
\end{tikzcd}$$

where π is the obvious quotient map.

Proposition 10.2.3. *The maps $\tilde\varphi$ and $\tilde\psi$ are well-defined and inverses of each other.*

Proof. First consider the natural transformation ψ. On the degree n component, the above diagram reduces to

$$\begin{tikzcd}
\bigoplus_{S\sqcup T=[n]} \mathbf{p}[S_A]\otimes \mathbf{q}[T_A] \arrow[r, "\psi_{\mathbf{p},\mathbf{q}}"] \arrow[d, two heads, "\pi"] & \bigoplus_{s+t=n} \mathbf{p}[s_A]\otimes \mathbf{q}[t_A] \arrow[d, two heads, "\oplus\pi_s\otimes\pi_t"] \\
\left(\bigoplus_{S\sqcup T=[n]} \mathbf{p}[S_A]\otimes \mathbf{q}[T_A]\right)_{A\wr S_n} \arrow[r, dashed, "\tilde{\psi}_{\mathbf{p},\mathbf{q}}"] & \bigoplus_{s+t=n} \mathbf{p}[s_A]_{A\wr S_s}\otimes \mathbf{q}[t_A]_{A\wr S_t}
\end{tikzcd}$$

First, we will describe the kernels of the projection maps. Observe that the kernel of $\oplus\pi_s \otimes \pi_t$ is spanned by elements of the form

$$\langle (v_s - \sigma.v_s)\otimes w_t\rangle \oplus \langle v_s \otimes (w_t - \tau.w_t)\rangle \oplus \langle (v_s-\sigma.v_s)\otimes (w_t - \tau.w_t)\rangle$$

where $v_s \in \mathbf{p}[[s]_A]$, $\sigma \in A\wr S_s$, $w_t \in \mathbf{q}[[t]_A]$, and $\tau \in A \wr S_t$. A general element in the kernel of π has form

$$\sum_{S\sqcup T=[n]}\sum_i v_S^i \otimes w_T^i - \sigma.\left(\sum_{S\sqcup T=[n]}\sum_i v_S^i\otimes w_T^i\right)$$

Since $\sigma \in A\wr S_n$ acts linearly, for a fixed $S\sqcup T=[n]$, we get that a pure tensor has the following form $v_S\otimes w_T - \sigma.v_S\otimes w_T$.

Consider a decomposition $S\sqcup T=[n]$ where $|S|=s$ and $|T|=t$, and let $(\vec{a}\otimes\sigma)\in A\wr S_n$ and suppose that $(\vec{a}\otimes\sigma)(S_A)=R_A$ and $(\vec{a}\otimes\sigma)(T_A)=U_A$. This defines bijections $(\vec{a}\otimes\sigma)_S : S_A \to R_A$ and $(\vec{a}\otimes\sigma)_T : T_A\to U_A$. Because these are bijections and st is an order preserving bijection we can define bijections $(\vec{a}\otimes\sigma)'_S\in A\wr S_s$ and $(\vec{a}\otimes\sigma)'_T \in A\wr S_t$ by the following commutative diagrams:

$$\begin{tikzcd}
S_A \arrow[r, "\mathrm{st}"] \arrow[d, "(\vec{a}\otimes\sigma)_S"'] & {[s]_A} \arrow[d, "(\vec{a}\otimes\sigma)'_S"] & T_A \arrow[r, "\mathrm{st}"] \arrow[d, "(\vec{a}\otimes\sigma)_T"'] & {[t]_A} \arrow[d, "(\vec{a}\otimes\sigma)'_T"] \\
R_A \arrow[r, "\mathrm{st}"] & {[s]_A} & U_A \arrow[r, "\mathrm{st}"] & {[t]_A}
\end{tikzcd} \tag{33}$$

By the above squares commuting, and by functoriality of $\mathbf{p}$ and $\mathbf{q}$ we have:

$$(\mathbf{p}[(\vec{a}\otimes\sigma)'_S]\otimes\mathbf{q}[(\vec{a}\otimes\sigma)'_T])\circ(\mathbf{p}[\mathrm{st}]\otimes\mathbf{q}[\mathrm{st}]) = (\mathbf{p}[\mathrm{st}]\otimes\mathbf{q}[\mathrm{st}])\circ(\mathbf{p}[(\vec{a}\otimes\sigma)_S]\otimes\mathbf{q}[(\vec{a}\otimes\sigma)_T]).$$

Now, to show that $\tilde{\psi}_{\mathbf{p},\mathbf{q}}$ is well-defined we must show that an element

$$v_S\otimes w_T - (\vec{a}\otimes\sigma).v_S\otimes w_T \in \ker(\pi)\subseteq \ker((\oplus\pi_s\otimes\pi_t)\circ(\mathbf{p}[\mathrm{st}]\otimes\mathbf{q}[\mathrm{st}]))$$

then $\mathbf{p}[\mathrm{st}]\otimes\mathbf{q}[\mathrm{st}](v_S\otimes w_T - (\vec{a}\otimes\sigma).v_S\otimes w_T)$

$$\begin{aligned}
&= \mathbf{p}[\mathrm{st}]v_S\otimes\mathbf{q}[\mathrm{st}]w_T - \mathbf{p}[\mathrm{st}](\vec{a}\otimes\sigma)_S v_S\otimes \mathbf{q}[\mathrm{st}](\vec{a}\otimes\sigma)_T w_T\\
&\overset{\text{eqn (33)}}{=} v_s\otimes w_t - (\vec{a}\otimes\sigma)'_S\mathbf{p}[\mathrm{st}]v_S\otimes(\vec{a}\otimes\sigma)'_T\mathbf{q}[\mathrm{st}]w_T\\
&= v_s\otimes w_t - (\vec{a}\otimes\sigma)'_S v_s\otimes(\vec{a}\otimes\sigma)'_T w_t
\end{aligned}$$

Adding in zero (denoted in colored text) gives:

$$\begin{aligned} &= v_s \otimes w_t + v_s \otimes (\vec{a} \otimes \sigma)'_T w_t - v_s \otimes (\vec{a} \otimes \sigma)'_T w_t - (\vec{a} \otimes \sigma)'_S v_s \otimes (\vec{a} \otimes \sigma)'_T w_t \\ &= v_s \otimes (w_t - (\vec{a} \otimes \sigma)'_T w_t) + (v_s - (\vec{a} \otimes \sigma)'_S v_s) \otimes (\vec{a} \otimes \sigma)'_T w_t \end{aligned}$$

which is a sum of elements of $\ker(\oplus \pi_s \otimes \pi_t)$ as desired. Thus $\tilde{\psi}$ is well-defined.

Finally, to show that $\tilde{\varphi}$ is well-defined. On the degree n component, the diagram reduces to

$$\begin{tikzcd} \bigoplus_{s+t=n} \mathbf{p}[s_A] \otimes \mathbf{q}[t_A] \arrow[r, "\mathbf{p}[\mathrm{id}]\otimes\mathbf{q}[\mathrm{cano}]"] \arrow[d, "\oplus\pi_s\otimes\pi_t"] & \bigoplus \mathbf{p}[s_A] \otimes \mathbf{q}[[1+s, t+s]_A] \arrow[d, "\pi"] \\ \bigoplus_{s+t=n} \mathbf{p}[s_A]_{A\wr S_S} \otimes \mathbf{q}[t_A]_{A\wr S_t} \arrow[r, dashed, "\tilde{\varphi}_{\mathbf{p},\mathbf{q}}"] & (\bigoplus \mathbf{p}[s_A] \otimes \mathbf{q}[[1+s, t+s]_A])_{A\wr S_n} \end{tikzcd}$$

Fix $s + t = n$, and let $\vec{a} \otimes \sigma_s \in A \wr S_s$ and $\vec{b} \otimes \sigma_t \in A \wr S_t$. Since cano, $\vec{a} \otimes \sigma_s$ and $\vec{b} \otimes \sigma_t$ are bijections we can define $\vec{a} \otimes \sigma'_s$ and $\vec{b} \otimes \sigma'_t$ by the following commutative diagrams:

$$\begin{tikzcd} {[s]_A} \arrow[r, "\mathrm{id}"] \arrow[d, "\vec{a}\otimes\sigma_s"'] & {[s]_A} \arrow[d, "\vec{a}\otimes\sigma'_s"] \\ {[s]_A} \arrow[r, "\mathrm{id}"'] & {[s]_A} \end{tikzcd} \qquad \begin{tikzcd} {[t]_A} \arrow[r, "\mathrm{cano}_s"] \arrow[d, "\vec{b}\otimes\sigma_t"'] & {[1+s, t+s]_A} \arrow[d, "\vec{b}\otimes\sigma'_t"] \\ {[t]_A} \arrow[r, "\mathrm{cano}_s"] & {[1+s, t+s]_A} \end{tikzcd}$$

By the above two diagrams commuting and by functoriality of $\mathbf{p}$ and $\mathbf{q}$, we have:

$$(\mathbf{p}[\mathrm{id}] \otimes \mathbf{q}[\mathrm{cano}]) \circ (\mathbf{p}[\vec{a} \otimes \sigma'_s] \otimes \mathbf{q}[\vec{b} \otimes \sigma'_t]) = (\mathbf{p}[\vec{a} \otimes \sigma_s] \otimes \mathbf{q}[\vec{b} \otimes \sigma_t]) \circ (\mathbf{p}[\mathrm{id}] \otimes \mathbf{q}[\mathrm{cano}]).$$

Now, to show that $\tilde{\varphi}$ is well-defined we must show that an element in the kernel lives in the kernel of $\pi \circ \mathbf{p}[\mathrm{id}] \otimes \mathbf{q}[\mathrm{cano}]$. I will show on an element of the form $v_s - (\vec{a} \otimes \sigma_s.v_s) \otimes w_t$ for some $v_s \in \mathbf{p}[[s]_A], w_t \in \mathbf{q}[[t]_A]$ and $(\vec{a} \otimes \sigma_s.v_s) \in A \wr S_s$. It will be a symmetric argument for the other description of elements in the kernel.
$\mathbf{p}[\mathrm{id}] \otimes \mathbf{q}[\mathrm{cano}](v_s - (\vec{a} \otimes \sigma_s.v_s) \otimes w_t)$

$$\begin{aligned} &= v_s \otimes w_{s+t} - \mathbf{p}[\mathrm{id}](\vec{a} \otimes \sigma_s.v_s).v_s \otimes \mathbf{q}[\mathrm{cano}](\vec{1}, \mathrm{id}_t) w_t \\ &= v_s \otimes w_{s+t} - (\vec{a} \otimes \sigma_s.v_s)' \mathbf{p}[\mathrm{id}].v_s \otimes (\vec{1}, \mathrm{id}_t)' \mathbf{q}[\mathrm{cano}] w_t \\ &= v_s \otimes w_{s+t} - ((\vec{a} \otimes \sigma_s.v_s)' \times (\vec{1}, \mathrm{id}_t)').v_s \otimes w_{s+t} \end{aligned}$$

which is an element in the kernel. Thus $\tilde{\varphi}$ is well-defined.

From $\psi_{\mathbf{p},\mathbf{q}} \circ \varphi_{\mathbf{p},\mathbf{q}} = \mathrm{id}$, we can deduce that $\tilde{\psi}_{\mathbf{p},\mathbf{q}} \circ \tilde{\varphi}_{\mathbf{p},\mathbf{q}} = \mathrm{id}$. Finally, to show that $\tilde{\varphi} \circ \tilde{\psi} = \mathrm{id}$. Let $S \sqcup T = [n]$, $x \in \mathbf{p}[A^{\otimes S} \otimes \mathbb{K}[S]]$, and $y \in \mathbf{q}[T_A]$, we must show that

$$\overline{x \otimes y} = \overline{\mathbf{p}[\mathrm{id} \circ \mathrm{st}] \otimes \mathbf{q}[\mathrm{cano}_s \circ \mathrm{st}](x \otimes y)}.$$

Showing that $\mathbf{p}[\mathrm{id} \circ \mathrm{st}] \otimes \mathbf{q}[\mathrm{cano} \circ \mathrm{st}]$ is given by a permutation would guarantee that this composite is the identity on coinvariants–this is because for any vector space, V_{S_n} surjects to $V_{A\wr S_n}$. If we look at the set level, it's not hard to see that there exists an element of $A \wr S_n$ $(1 \cdots 1 \otimes \sigma) : [n]_A \to [n]_A$ such that the restrictions

$$(1 \cdots 1 \otimes \sigma)|_S : A^{\otimes S} \otimes \mathbb{K}[S] \to [s]_A$$

which is just the standardization map st, and

$$(1 \cdots 1 \otimes \sigma)|_T : T_A \to A^{\otimes t} \otimes \mathbb{K}[s + [t]]$$

which can be thought of as the standardization map st shifted by s. Applying the functors $\mathbf{p}$ and $\mathbf{q}$, give $\mathbf{p}[\mathrm{id} \circ \mathrm{st}] \otimes \mathbf{q}[\mathrm{cano}_s \circ \mathrm{st}] = \mathbf{p}[(1 \cdots 1 \otimes \sigma)|_S] \otimes \mathbf{q}[(1 \cdots 1 \otimes \sigma)|_T]$.
Thus,

$$\begin{aligned}
\overline{\mathbf{p}[\mathrm{id} \circ \mathrm{st}] \otimes \mathbf{q}[\mathrm{cano}_s \circ \mathrm{st}](x \otimes y)} &= \overline{\mathbf{p}[(1 \cdots 1 \otimes \sigma)|_S] \otimes \mathbf{q}[(1 \cdots 1 \otimes \sigma)|_T](x \otimes y)} \\
&= \overline{\varepsilon(1 \cdots 1 \otimes \sigma)x \otimes y} \\
&= \overline{x \otimes y}
\end{aligned}$$

Therefore the maps $\tilde{\varphi}$ and $\tilde{\psi}$ are inverses of each other.

$\square$

REMARK 10.2.4. The proof of Proposition 10.2.3 follows the proof of Proposition 15.2 in [3] with more details given.

PROPOSITION 10.2.5. *The functor $\widetilde{K_A}$ is a bilax monoidal functor.*

PROOF. For the bilax structure, we define the maps $\tilde{\varphi}$ and $\tilde{\psi}$ by the commutativity of the following diagrams above. Since the maps are defined from $\varphi_{\mathbf{p},\mathbf{q}}$ and $\psi_{\mathbf{p},\mathbf{q}}$, we have that the bilax conditions are satisfied.

$\square$

PROPOSITION 10.2.6. *The functor $\widetilde{K_A}$ is a bistrong monoidal functor.*

PROOF. We have that $\tilde{\varphi}_0 \circ \tilde{\psi}_0 = \mathrm{id}$ since $\tilde{\varphi}_0 = \mathrm{id}$ and $\tilde{\psi}_0 = \mathrm{id}$. We showed that $\tilde{\varphi}\tilde{\psi} = \mathrm{id}$ within Proposition 10.2.3.

$\square$

S_n is naturally viewed as a subgroup of $A \wr S_n$, hence we can consider the space of S_n coinvariants.

DEFINITION 10.2.7. For each $\mathbf{p} \in \mathbf{Sp}^A$ and morphism $f : \mathbf{p} \to \mathbf{q}$ of A-species, we can define the functor $\overline{K}_A : \mathbf{Sp}^A \to \mathbf{gVec}_{\mathbb{K}}$ via:

$$\overline{K}_A(\mathbf{p}) := \bigoplus_{n \geq 0} \mathbf{p}[n_A]_{S_n}$$

$$\overline{K}_A(f) := \bigoplus_{n \geq 0} \overline{f}_{[n_A]}$$

where $\overline{f}_{[n_A]}(v) = [f_{[n_A]}(v)]$, i.e., the coset formed by $f_{[n_A]}(v)$.

PROPOSITION 10.2.8. *The functor $\overline{K}_A$ is a bilax monoidal functor.*

PROOF. In the proof of Proposition 10.2.3, we showed that the natural transformation φ factored through the space of $A \wr S_t \times A \wr S_r$- coinvariants. Thus φ factors through the space of $S_t \times S_r$-coinvariants. We also showed that ψ factored through the space of $A \wr S_n$-coinvariants, thus factors through the space of S_n-coinvariants. $\square$

Corollary 10.2.9. *The functor $\overline{K}_A$ is a bistrong monoidal functor.*

Proof. This follows directly from Proposition 10.2.3 when showing $\tilde{\varphi}$ and $\tilde{\psi}$ are inverses to each other. $\square$

10.3. Morphisms Between These Functors

Proposition 10.3.1. *The maps*

$$K_A \twoheadrightarrow \widetilde{K}_A \quad \text{and} \quad \widetilde{K_A^\vee} \hookrightarrow K_A^\vee$$

are natural transformations of bilax functors.

Proof. First, we must show that $\iota : \widetilde{K_A^\vee} \to K_A^\vee$ is a natural transformation.

- Let $\mathbf{p} \in \mathbf{Sp}^A$ and define the sections to be given by

$$\iota_{\mathbf{p}} : \widetilde{K_A^\vee}(\mathbf{p}) \to K_A^\vee(\mathbf{p})$$
$$\bigoplus_{n \geq 0} (\mathbf{p}[[n]_A])^{A \wr S_n} \hookrightarrow \mathbf{p}[[n]_A]$$

  where $\iota_{\mathbf{p}}$ is the inclusion map.

- Now, ι must be such that for all $\alpha : \mathbf{p} \to \mathbf{q} \in \mathbf{Sp}^A$, the following diagram commutes:

$$\begin{tikzcd}
\widetilde{K_A^\vee}(\mathbf{p}) \arrow[r, hook, "\iota_{\mathbf{p}}"] \arrow[d, "\widetilde{K_A^\vee}(\alpha)"'] & K_A^\vee(\mathbf{p}) \arrow[d, "K_A^\vee(\alpha)"] \\
\widetilde{K_A^\vee}(\mathbf{q}) \arrow[r, hook, "\iota_{\mathbf{q}}"'] & K_A^\vee(\mathbf{q})
\end{tikzcd}$$

  On a component of degree n, this diagram becomes:

$$\begin{tikzcd}
\mathbf{p}[n_A]^{A \wr S_n} \arrow[r, hook, "\iota_{\mathbf{p}[n_A]}"] \arrow[d, "\alpha_{[n_A]}"'] & \mathbf{p}[n_A] \arrow[d, "\alpha_{[n_A]}"] \\
\mathbf{q}[n_A]^{A \wr S_n} \arrow[r, hook, "\iota_{\mathbf{q}[n_A]}"'] & \mathbf{q}[n_A]
\end{tikzcd}$$

Since α is a natural transformation and we have inclusion maps, we have that the above diagram commutes. Therefore ι is a natural transformation of $\widetilde{K_A^\vee} \hookrightarrow K_A^\vee$.

Now we must show that ι is a morphism of bilax monoidal functors. We will show that diagrams in Definition 2.5.1 commute.
To show a morphism of lax functors, let $\mathbf{p}, \mathbf{q} \in \mathbf{Sp}^A$ and $s + t = n$. The diagram on the right of Diagram (20) commutes trivially since $\tilde{\varphi^\vee}_0 = \varphi_0^\vee = \mathrm{id}$. Now to show the diagram on the left commutes, let $x \otimes y \in \mathbf{p}[s_A]^{A \wr S_s} \otimes \mathbf{q}[t_A]^{A \wr S_t}$ then following the right top corner yields:

$$x \otimes y \xrightarrow{\tilde{\varphi^\vee}_{\mathbf{p},\mathbf{q}}} \bigoplus_{S \sqcup T = [n]} \mathbf{p}[\mathrm{cano}_S]x \otimes \mathbf{q}[\mathrm{cano}_T]y \xrightarrow{\iota_{\mathbf{p},\mathbf{q}}} \bigoplus_{S \sqcup T = [n]} \mathbf{p}[\mathrm{cano}_S]x \otimes \mathbf{q}[\mathrm{cano}_T]y$$

The bottom left corner yields:

$$x \otimes y \xrightarrow{\iota_{\mathbf{p}} \otimes \iota_{\mathbf{q}}} x \otimes y \xrightarrow{\varphi^\vee_{\mathbf{p},\mathbf{q}}} \bigoplus_{S \sqcup T = [n]} \mathbf{p}[\mathrm{cano}_S]x \otimes \mathbf{q}[\mathrm{cano}_T]y$$

Thus ι is a morphism of lax monoidal functors.
To show a morphism of colax functors, let $\mathbf{p}, \mathbf{q} \in \mathbf{Sp}^A$. Again the diagram on the right of Diagram (21) commutes trivially since ${\tilde{\psi}^{\vee}}_0 = \psi^{\vee}_0 = \mathrm{id}$. For the diagram on the left, let $\sum x_S \otimes y_T \in (\bigoplus_{S \sqcup T = [n]} \mathbf{p}[A^{\otimes S} \otimes \mathbb{K}[S]] \otimes \mathbf{q}[T_A])^{A \wr S_n}$. Following the right top corner yields:

$$\sum x_S \otimes y_T \xrightarrow{{\tilde{\psi}^{\vee}}_{\mathbf{p},\mathbf{q}}} x_{[s]} \otimes \mathbf{q}[\mathrm{cano}_s] y_{[s+1,s+t]} \xrightarrow{\iota_{\mathbf{p}} \otimes \iota_{\mathbf{q}}} x_{[s]} \otimes \mathbf{q}[\mathrm{cano}_s] y_{[s+1,s+t]}$$

Following the bottom left corner yields:

$$\sum x_S \otimes y_T \xrightarrow{\iota_{\mathbf{p} \cdot \mathbf{q}}} \sum x_S \otimes y_T \xrightarrow{\psi^{\vee}_{\mathbf{p},\mathbf{q}}} x_{[s]} \otimes \mathbf{q}[\mathrm{cano}_s] y_{[s+1,s+t]}$$

Thus ι is a morphism of colax monoidal functors.
Therefore ι is a morphism of bilax monoidal functors.

Finally, to show that $\pi : K_A \twoheadrightarrow \widetilde{K}_A$ is a natural transformation.

- Let $\mathbf{p} \in \mathbf{Sp}^A$ and define the sections to be given by
$$\pi_{\mathbf{p}} : K_A(\mathbf{p}) \to \widetilde{K}_A(\mathbf{p})$$
where $\pi_{\mathbf{p}}$ is the projection map.
- Now, for all $\alpha : \mathbf{p} \to \mathbf{q} \in \mathbf{Sp}^A$ the following diagram must commute:
$$\begin{array}{ccc} K_A(\mathbf{p}) & \xrightarrow{\pi_{\mathbf{p}}} & \widetilde{K}_A(\mathbf{p}) \\ {\scriptstyle K_A(\alpha)}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{K}_A(\alpha)} \\ K_A(\mathbf{q}) & \xrightarrow{\pi_{\mathbf{q}}} & \widetilde{K}_A(\mathbf{q}) \end{array}$$
On a degree n component, the diagram reduces to:
$$\begin{array}{ccc} \mathbf{p}[n_A] & \xrightarrow{\pi_{\mathbf{p}[n]}} & \mathbf{p}[n_A]_{A \wr S_n} \\ {\scriptstyle \alpha_{[n]}}\big\downarrow & & \big\downarrow{\scriptstyle \overline{\alpha}_{[n]}} \\ \mathbf{q}[n_A] & \xrightarrow{\pi_{\mathbf{q}[n]}} & \mathbf{q}[n_A]_{A \wr S_n} \end{array}$$
where $\overline{\alpha}_{[n]}(\overline{x}) = \overline{\alpha_{[n]}(x)}$. Now
$$\begin{aligned} \overline{\alpha}_{[n]} \circ \pi_{\mathbf{p}[n_A]}(x) &= \overline{\alpha}_{[n]}(\overline{x}) \\ &= \overline{\alpha_{[n_A]}(x)} \\ &= \pi_{\mathbf{q}[n_A]} \circ (\alpha_{[n]}(x)) \end{aligned}$$
Therefore π is a natural transformation.

To show that π is a morphism of bilax functors, we check the same diagrams as showing ι was a morphism of bilax functors.

□

In the classical setting, for any species $\mathbf{p} \in \mathbf{Sp}$, the functor K could be written in terms of $\overline{K}$ via $K(\mathbf{p}) \cong \overline{K}(\mathbf{L} \times \mathbf{p})$. We show that our functor $\mathcal{S}^A$ lets us extend this result to A-species. As in the classical case, this is a useful to for describing A-species.

THEOREM 10.3.2. *There exists an isomorphism of the following bilax monoidal functors from* $\boldsymbol{Sp} \to \boldsymbol{Vec}_{\mathbb{K}}$,

$$\overline{K}_A(\mathcal{S}^A(\boldsymbol{L} \times \_\_)) \cong K_A(\mathcal{S}^A\_\_).$$

PROOF. First, we have that $\overline{K}_A(\mathcal{S}^A(\mathbf{L} \times \_\_))$ and $K_A(\mathcal{S}^A\_\_)$ are both functors from $\mathbf{Sp} \to \mathbf{Vec}_{\mathbb{K}}$.

We wish to define a natural isomorphism $\alpha : K_A(\mathcal{S}^A\_\_) \to \overline{K}_A(\mathcal{S}^A(\mathbf{L} \times \_\_))$.
Now, let $\mathbf{p} \in \mathbf{Sp}$. The section maps are given by

$$\alpha_{\mathbf{p}} : K_A(\mathcal{S}^A\mathbf{p}) \to \overline{K}_A(\mathcal{S}^A(\mathbf{L} \times \mathbf{p}))$$

Observe that

$$\begin{aligned}
\overline{K}_A(\mathcal{S}^A(\mathbf{L} \times \mathbf{p})) &= \bigoplus_{n\geq 0}(\mathcal{S}^A(\mathbf{L} \times \mathbf{p})[n_A])_{S_n} \\
&= \bigoplus_{n\geq 0}\left(\bigoplus_{s:[n]\to B\times[n]} (\mathbf{L} \times \mathbf{p})[s([n])]\right)_{S_n} \\
&= \bigoplus_{n\geq 0}\left(\bigoplus_{s:[n]\to B\times[n]} \mathbf{L}[s([n])] \otimes \mathbf{p}[s([n])]\right)_{S_n}
\end{aligned}$$

and

$$\begin{aligned}
K_A(\mathcal{S}^A\mathbf{p}) &= \bigoplus_{n\geq 0}\mathcal{S}^A\mathbf{p}[[n]_A] \\
&= \bigoplus_{n\geq 0}\bigoplus_{s:[n]\to B\times[n]} \mathbf{p}[s([n])]
\end{aligned}$$

On a degree n piece and for a given section $s : [n] \to B \times [n]$, we define the components of α by:

$$\begin{aligned}
\alpha_{\mathbf{p}}^{s,n} : \mathbf{p}[s([n])] &\to \left(\bigoplus_{s:[n]\to B\times[n]} \mathbf{L}[s([n])] \otimes \mathbf{p}[s([n])]\right)_{S_n} \\
v &\mapsto \overline{C^s_{(n)} \otimes v}
\end{aligned}$$

where $C^s_{(n)}$ is the canonical linear order on $[n]$ whose coloring is determined by the section map $s : [n] \to B \times [n]$, i.e., $s(1)s(2)\cdots s(n)$ and the overline denotes the projection to the coinvariants.

We have that $\alpha_{\mathbf{p}}$ is an isomorphism of graded vector spaces with the inverse map defined on the basis of $\left(\bigoplus_{s:[n]\to B\times[n]} \mathbf{L}[s([n])] \otimes \mathbf{p}[s([n])]\right)_{S_n}$, given by $\varphi_{\mathbf{p}}(\overline{C^s_{(n)}) \otimes v^s_k} = v^s_k$. These maps are clearly mutual inverses to each other.

Now, for all $\beta : \mathbf{p} \to \mathbf{q} \in \mathbf{Sp}$ the following diagram must commute:

$$\begin{array}{ccc}
K_A(\mathcal{S}^A\mathbf{p}) & \xrightarrow{\alpha_{\mathbf{p}}} & \overline{K}_A(\mathcal{S}^A(\mathbf{L} \times \mathbf{q})) \\
{\scriptstyle K_A(\mathcal{S}^A\beta)}\downarrow & & \downarrow{\scriptstyle \overline{K}_A(\mathcal{S}^A(\mathrm{id} \times \beta))} \\
K_A(\mathcal{S}^A\mathbf{q}) & \xrightarrow[\alpha_{\mathbf{q}}]{} & \overline{K}_A(\mathcal{S}^A(\mathbf{L} \times \mathbf{q}))
\end{array}$$

On a degree n component and for a given section, this reduces to:

$$\begin{tikzcd} \mathbf{p}[s([n])] \arrow[r, "\alpha_{\mathbf{p}}^{s,n}"] \arrow[d, "\beta_{s([n])}"'] & \left(\bigoplus_{s:[n]\to B\times[n]} \mathbf{L}[s([n])]\otimes\mathbf{p}[s([n])]\right)_{S_n} \arrow[d, "\overline{\mathrm{id}\otimes\beta_{s([n])}}"] \\ \mathbf{q}[s([n])] \arrow[r, "\alpha_{\mathbf{q}}^{s,n}"'] & \left(\bigoplus_{s:[n]\to B\times[n]} \mathbf{L}[s([n])]\otimes\mathbf{q}[s([n])]\right)_{S_n} \end{tikzcd}$$

Because $\alpha_{\mathbf{p}}^{s,n}$ is an isomorphism and β is a natural transformation, we have that this diagram commutes. Thus, α is a natural isomorphism.

All that remains to show is that α is a natural isomorphism of bilax monoidal functors. Note that from Proposition 8.66 in [3], $\mathbf{L}\times\_\_$ is a bilax monoidal functor from $\mathbf{Sp}$ to $\mathbf{Sp}$. We have shown that $\overline{K}_A$, K_A, and $\mathcal{S}^A$ are all bilax monoidal functors, thus by Theorem 3.22 in [3]. Thus their compositions are bilax monoidal. Now, because the bilax structure of $\mathcal{S}^A$ is given by the identity map for both colax and lax structure, the proof technique of Proposition 15.9 in [3] remains the same with the small change of looking at a decomposition the image of a section map instead of just $[n]$. Will show briefly that the colax structures are preserved and the lax structure is check similarly.
The colax structure of $\overline{K}_A(\mathcal{S}^A(\mathbf{L}\times\_\_))$ is given by the following: Let $\mathbf{p},\mathbf{q}\in\mathbf{Sp}$

$$\begin{tikzcd} \overline{K}_A(\mathcal{S}^A(\mathbf{L}\times(\mathbf{p}\cdot\mathbf{q}))) \arrow[d, "\overline{K}_A(\mathcal{S}^A(\Delta_{\mathbf{L}}\times\mathrm{id}_{\mathbf{p}\cdot\mathbf{q}}))"] & \\ \overline{K}_A(\mathcal{S}^A((\mathbf{L}\cdot\mathbf{L})\times(\mathbf{p}\cdot\mathbf{q}))) \arrow[r] & \overline{K}_A(\mathcal{S}^A((\mathbf{L}\times\mathbf{p})\cdot(\mathbf{L}\cdot\mathbf{q}))) \arrow[d, "\overline{\psi}\circ\overline{K}_A(\psi^A)"] \\ & \overline{K}_A(\mathcal{S}^A(\mathbf{L}\times\mathbf{p}))\cdot\overline{K}_A(\mathcal{S}^A(\mathbf{L}\times\mathbf{q})) \end{tikzcd}$$

where $\Delta_{\mathbf{L}}$ is the coproduct of $\mathbf{L}$ as in Section 5.1. The horizontal map in the center is $\overline{K}_A\circ\mathcal{S}^A$ applied to the colax structure of the Hadamard functor as described in Chapter 8 of [3], specifically Equation 8.73. Finally, $\overline{\psi}\circ\overline{K}_A(\psi^A)$ gives the colax structure of $\overline{K}_A\circ\mathcal{S}^A$ where $\overline{\psi}$ is the projection of ψ as described in Proposition 10.2.1 and ψ^A as in Proposition 9.1.6.

Finally, we must show that the above composite matches the colax structure of $K_A\circ\mathcal{S}^A$. Let $S\sqcup T=s([n])$, i.e., S is a subset of $[n]$ whose coloring is determined by the section map $s:[n]\to B\times[n]$ (similarly for T). Let $x\in\mathbf{p}[S]$ and $y\in\mathbf{q}[T]$. Applying the composite above to the element $\overline{C^s_{(n)} \otimes x \otimes y}$ yields:

$$\begin{aligned}
\overline{C^s_{(n)} \otimes x \otimes y} &\mapsto \sum_{U \sqcup V = s([n])} \overline{C^s_{(n)}|_U \otimes C^s_{(n)}|_V \otimes x \otimes y} \\
&\mapsto \overline{C^s_{(n)}|_S \otimes x \otimes C^s_{(n)}|_T \otimes y} \\
&\mapsto \overline{C^s_{|S|} \otimes \mathrm{st}_S(x) \otimes C^s_{|T|} \otimes \mathrm{st}_T(y)}
\end{aligned}$$

where the second mapping is zero unless U matches S and V matches T, in which case it is the identity. which matches the colax structure of $K_A \circ \mathcal{S}^A$.

Therefore α is a natural isomorphism of bilax monoidal functors. □

We end this section by showing a result similar to Proposition 15.9 in [3], this states $K \cong \overline{K}(\mathbf{L} \times (\_\_))$ as bilax monoidal functors; in other words the S_n-coinvariants of the Hadamard product of the linear order species with any species $\mathbf{p} \in \mathbf{Sp}$ is isomorphic to the Fock functor K applied to $\mathbf{p}$. This isomorphism relies on the fact that the $\mathbf{L}[n]$ corresponds to the regular representation for S_n. Here, we show a similar result using $\mathbf{L}_A$, and that $\mathbf{L}_A[n]$ is the regular representation of $A \wr S_n$ for every $n \geq 0$.

Lemma 10.3.3. *$\boldsymbol{L}_A \times \_\_$ is a bilax monoidal functor.*

Proof. Recall, $\mathbf{L}_A := \mathcal{S}^A(\mathbf{L})$, where $\mathbf{L} \in \mathbf{Sp}$ as in Section 5.1. We can view $\mathbf{L}_A \times \_\_$ as a functor from $\mathbf{Sp}^A$ to $\mathbf{Sp}^A$. By Proposition 9.1.8, $\mathbf{L}_A$ is a bimonoid; hence, according to Proposition 8.66 in [3], it can be viewed as a bilax monoidal functor.

□

Theorem 10.3.4. *There is an isomorphism of bilax monoidal functors from $\boldsymbol{Sp}^A \to$ $\boldsymbol{Vec}_{\mathbb{K}}$,*

$$K_A \cong \widetilde{K}_A(\boldsymbol{L}_A \times \_\_).$$

Proof. First, we have that $\widetilde{K}_A(\mathbf{L}_A \times \_\_)$ and K_A are both functors from $\mathbf{Sp}^A$ to $\mathbf{Vec}_{\mathbb{K}}$. Now, we wish to define a natural isomorphism $\alpha : K_A \to \widetilde{K}_A(\mathbf{L}_A \times \_\_)$.

Let $\mathbf{p} \in \mathbf{Sp}^A$, the section maps $\alpha_{\mathbf{p}}$ are given by

$$K_A(\mathbf{p}) \to \widetilde{K}_A(\mathbf{L}_A \times \mathbf{p}).$$

On the components of degree n, we have:

$$\mathbf{p}[n_A] \to (\mathbf{L}_A[n_A] \otimes \mathbf{p}[n_A])_{A \wr S_n}.$$

Because $\mathbf{L}_A[[n]_A]$ corresponds to the regular representation of $A \wr S_n$, we can let $\alpha_{\mathbf{p}}$ be the isomorphism from Proposition 3.3.4 to define the natural isomorphism of the functors.
Now for all $\beta : \mathbf{p} \to \mathbf{q} \in \mathbf{Sp}^A$, we need the the following diagram to commute:

$$\begin{tikzcd}
K_A(\mathbf{p}) \arrow[r, "\alpha_{\mathbf{p}}"] \arrow[d, "K_A(\beta)"'] & \widetilde{K}_A(\mathbf{L}_A \times \mathbf{p}) \arrow[d, "\widetilde{K}_A(\mathrm{id} \times \beta)"] \\
K_A(\mathbf{q}) \arrow[r, "\alpha_{\mathbf{q}}"'] & \widetilde{K}_A(\mathbf{L}_A \times \mathbf{q}).
\end{tikzcd}$$

On a degree n component, this diagram is equivalent to:

$$\begin{tikzcd}
\mathbf{p}[n_A] \arrow[r, "\alpha_{\mathbf{p}}"] \arrow[d, "\beta_{[n]}"'] & (\mathbf{L}_A[n_A] \otimes \mathbf{p}[n_A])_{A\wr S_n} \arrow[d, "\overline{\mathrm{id} \otimes \beta_{[n]}}"] \\
\mathbf{q}[n_A] \arrow[r, "\alpha_{\mathbf{q}}"] & (\mathbf{L}_A[n_A] \otimes \mathbf{q}[n_A])_{A\wr S_n} .
\end{tikzcd}$$

This diagrams commutes since β is a natural transformation and $\alpha_{\mathbf{p}}$, $\alpha_{\mathbf{q}}$ are isomorphisms. Thus $\alpha : K_A \to \widetilde{K}_A(\mathbf{L}_A \times \_\_)$ is a natural isomorphism.

Finally, we need to show that this is a morphism of bilax monoidal functors. Observe that both K_A and $\widetilde{K}_A$ are bilax monoidal functors by Propositions 10.2.1 and 10.2.5. By Lemma 10.3.3 above, we have $\mathbf{L}_A \times \_\_$ is bilax, and hence the composition $\widetilde{K}_A \circ \mathbf{L}_A \times \_\_$ is a bilax monoidal functor.
We now check that the colax structures are preserved. To check that the lax structures are preserved can be done in a similar way.

Let $\mathbf{p}, \mathbf{q} \in \mathbf{Sp}^A$, as seen in Theorem 10.3.2, the colax structure is given by:

$$\begin{tikzcd}
\widetilde{K}_A(\mathbf{L}_A \times (\mathbf{p} \cdot \mathbf{q})) \arrow[d, "\widetilde{K}_A(\Delta \times \mathrm{id}_{\mathbf{p}\cdot\mathbf{q}})"] & \\
\widetilde{K}_A((\mathbf{L}_A \cdot \mathbf{L}_A) \times (\mathbf{p} \cdot \mathbf{q})) \arrow[r] & \widetilde{K}_A((\mathbf{L}_A \times \mathbf{p}) \cdot (\mathbf{L}_A \times \mathbf{q})) \arrow[d, "\tilde{\psi}"] \\
& \widetilde{K}_A(\mathbf{L}_A \times \mathbf{p}) \cdot \widetilde{K}_A(\mathbf{L}_A \times \mathbf{q})
\end{tikzcd}$$

where Δ is the coproduct of $\mathbf{L}_A$ as described in Section 5.1. The horizontal map in the middle is $\overline{K}_A$ applied to the colax structure of the Hadamard functor as described in Chapter 8 of [3], specifically Equation 8.73. Finally, $\tilde{\psi}$ gives the colax structure of $\widetilde{K}_A$, as described in Proposition 10.2.1.

Finally, we must show that the above composite matches the colax structure of K_A. Let $S \sqcup T = [n]$ be an A-decomposition of $[n]_A$, and $x \in \mathbf{p}[S_A]$ and $y \in \mathbf{q}[T_A]$. Applying the composite above to the element $\overline{C_{(n)} \otimes x \otimes y}$ yields:

$$\begin{aligned}
\overline{C_{(n)} \otimes x \otimes y} &\mapsto \sum_{U \sqcup V = s([n])} \overline{C_{(n)}|_U \otimes C_{(n)}|_V \otimes x \otimes y} \\
&\mapsto \overline{C_{(n)}|_S \otimes x \otimes C_{(n)}|_T \otimes y} \\
&\mapsto \overline{C_{|S|} \otimes \mathrm{st}_S(x) \otimes C_{|T|} \otimes \mathrm{st}_T(y)}
\end{aligned}$$

where the second mapping is zero unless U matches S and V matches T, in which case it is the identity. which matches the colax structure of $K_A \circ \mathcal{S}^A$.

□

10.4. Hopf Algebras from A-Hopf Monoids

In this section, we write out the explicit Hopf algebra structure after applying the bilax monoidal functors defined above.

THEOREM 10.4.1. *Given a Hopf monoid* $\mathbf{h} \in \boldsymbol{Sp}^A$*, then* $K_A(\mathbf{h})$*,* $\widetilde{K}_A(\mathbf{h})$*,* $K_A^\vee(\mathbf{h})$*, and* $\widetilde{K_A^\vee}(\mathbf{h})$ *are graded Hopf Algebras.*

PROOF. In Propositions 10.1.3, 10.1.5, 10.2.1, and 10.2.5 we showed that $K_A, \tilde{K}_A, K_A^\vee$, and $\widetilde{K_A^\vee}$ were all bilax monoidal functors. Thus $K_A(\mathbf{h})$, $\tilde{K}_A(\mathbf{h})$, $K_A^\vee(\mathbf{h})$, and $\widetilde{K_A^\vee}(\mathbf{h})$ are graded bialgebras by Proposition (2.5.2). From Propositions 10.1.6 and 10.2.6, $\tilde{K_A^\vee}(\mathbf{h})$ and $\tilde{K}_A(\mathbf{h})$ are bistrong monoidal functors, hence by Proposition (2.5.3) these are graded Hopf Algebras.
Now, since $K_A(\mathbf{h})$ and $K_A^\vee(\mathbf{h})$ are graded bialgebras, we only need to show that $K_A^\vee(\mathbf{h})_0$ and $K_A(\mathbf{h})_0$ are Hopf algebras, i.e., the degree zero components of the respective graded bialgebras. Observe that $K_A^\vee(\mathbf{h})_0 = \mathbf{h}[\emptyset]$ and $K_A(\mathbf{h})_0 = \mathbf{h}[\emptyset]$ which are by definition Hopf Algebras. Thus $K_A^\vee(\mathbf{h})$ and $K_A(\mathbf{h})$ are graded Hopf algebras by Proposition 8.10 in [3]. $\square$

10.4.1. Hopf Algebra Structure

Let μ, ι, δ, and ε be the structure maps for a Hopf monoid $\mathbf{h} \in \mathbf{Sp}^A$. We will make explicit the structure maps for $K_A(\mathbf{h})$ and the others will follow similarly. The structure maps are as follows:

$$K_A(\mathbf{h}) \cdot K_A(\mathbf{h}) \xrightarrow{\varphi_{\mathbf{h},\mathbf{h}}} K_A(\mathbf{h} \cdot \mathbf{h}) \xrightarrow{K_A(\mu)} K_A(\mathbf{h})$$

$$\mathbb{K} \xrightarrow{\varphi_0} K_A(\mathbf{1}_\mathbb{K}) \xrightarrow{K_A(\iota)} K_A(\mathbf{h})$$

$$K_A(\mathbf{h}) \xrightarrow{K_A(\Delta)} K_A(\mathbf{h} \cdot \mathbf{h}) \xrightarrow{\psi_{\mathbf{h},\mathbf{h}}} K_A(\mathbf{h}) \cdot K_A(\mathbf{h})$$

$$K_A(\mathbf{h}) \xrightarrow{K_A(\varepsilon)} K_A(\mathbf{1}_\mathbb{K}) \xrightarrow{\psi_0} \mathbb{K}$$

In particular, the components of the product and coproduct of $K_A(\mathbf{h})$ are the following compositions

$$\mathbf{h}[n_A] \otimes \mathbf{h}[m_A] \to \mathbf{h}[[n+m]_A]$$

$$x \otimes y \mapsto \mu(x \otimes \mathbf{h}[\mathrm{cano}]y)$$

and

$$\mathbf{h}[n_A] \to \bigoplus_{s+t=n} \mathbf{h}[s_A] \otimes \mathbf{h}[t_A]$$

$$x \mapsto \sum \mathbf{h}[\mathrm{st}]x_{(1)} \otimes \mathbf{h}[\mathrm{st}]x_{(2)}$$

where Sweedler notation is used when computing the coproduct, i.e., $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$ (see Subsection 3.1.1).

Remark 10.4.2. On Antipodes:
Above, we did not state the structure of the antipodes. In general, antipodes are not preserved. Let s denote the antipode of our Hopf monoid $\mathbf{h}$. By Proposition (2.5.3), $\tilde{K}_A^\vee(s)$ and $\tilde{K}_A(s)$ are the antipodes of $\tilde{K}_A^\vee(\mathbf{h})$ and $\tilde{K}_A(\mathbf{h})$ respectively. However, since $K_A^\vee$ and K_A are not bistrong monoidal functors, these need not preserve the antipode structure. To have an explicit description is often very difficult. See [8] to read further on work done in various settings.

CHAPTER 11

A-Hopf Monoids: Examples

In this chapter, we give three examples of Hopf monoids in the category of A-species. We recall the functor from Section 9.1, $\mathcal{S}^A$ which constructs an A-species from a species:

$$\mathcal{S}^A : \mathbf{Sp} \to \mathbf{Sp}^A$$

$$\mathbf{p} \mapsto \mathcal{S}^A(\mathbf{p})[I_A] := \bigoplus_s \mathbf{p}[s(I)]$$

where $s : I \to B \times I$ such that $s(i_k) \in B \times \{i_k\}$.

We then apply this to the Hopf monoids described in Chapter 5 to get Hopf monoids in $\mathbf{Sp}^A$. In turn, we will show how these three A-Hopf monoids relate to the Hopf algebra of symmetric functions in B-colored noncommutative variables, $\tilde{\Pi}^{(B)}$.

11.1. A-Hopf Monoid of Linear Orders

In this section, we describe the Hopf structure of the A-Hopf monoid of Linear Orders in detail, which will be denoted by $\mathbf{L}_A$. At the end of the section, we will see how $\mathbf{L}_A$ interacts with the other examples given.

Recall, $\mathbf{L}_A := \mathcal{S}^A(\mathbf{L})$. Applying $\mathcal{S}^A$ to $\mathbf{L}$ yields:

$$\mathbf{L}_A[n_A] := \bigoplus_{s:[n]\to B\times[n]} \mathbf{L}[s([n])]$$

that is, the $\mathbb{K}$-span of linear orders on $s([n])$ for all sections $s : [n] \to B \times [n]$.
On generating morphisms,

$$\mathbf{L}_A[(1 \cdots 1 \otimes \sigma)] := \bigoplus_{s:[n]\to B\times[n]} \mathbf{L}[(1 \cdots 1 \otimes \sigma)|_{s([n])}]$$

$$\mathbf{L}_A[(b_{i_1} \cdots b_{i_n} \otimes \mathrm{id})] := \bigoplus_{s:[n]\to B\times[n]} \mathbf{L}[(b_{i_1} \cdots b_{i_n} \otimes \mathrm{id})|_{s([n])}],$$

where $\mathbf{L}[(1 \cdots 1 \otimes \sigma)|_{s([n])}]$ and $\mathbf{L}[(1 \cdots 1 \otimes \sigma)|_{s([n])}]$ are as defined in Definition 9.1.3.

In the following example, we make explicit what $\mathbf{L}_A$ does on objects and morphisms.

Example 11.1.1. **$\mathbb{K}C_2$-Species of Linear Orders, $\mathbf{L}_{\mathbb{K}C_2}$**
Let $n = 2$ and consider $A = \mathbb{K}C_2$ with basis $B = \{b_1 = 1, b_2 = -1\}$, hence $T = \{1, 2\}$. The sections, $s : [2] \to B \times [2]$, are given by

$$s_1 : \begin{matrix} 1 \mapsto 1 \\ 2 \mapsto 2 \end{matrix} \quad , \quad s_2 : \begin{matrix} 1 \mapsto \overline{1} \\ 2 \mapsto 2 \end{matrix} \quad , \quad s_3 : \begin{matrix} 1 \mapsto 1 \\ 2 \mapsto \overline{2} \end{matrix} \quad , \quad s_4 : \begin{matrix} 1 \mapsto \overline{1} \\ 2 \mapsto \overline{2} \end{matrix}$$

where $\bar{i}$ denotes the image $s(i) = (-1, i)$ and i denotes the image $s(i) = (1, i)$ for all $i \in [2]$. The endomorphism ring of $\mathbb{K}C_2^{\otimes 2} \otimes \mathbb{K}[2]$ is generated by the elements $(1 \cdot 1 \otimes (12))$, $(-1 \cdot 1 \otimes \mathrm{id}) \in \mathbb{K}C_2 \wr S_2$.

$$\mathbf{L}_{\mathbb{K}C_2}[\mathbb{K}C_2^{\otimes 2} \otimes \mathbb{K}[2]] = \mathbf{L}[\{1,2\}] \oplus \mathbf{L}[\{\bar{1},2\}] \oplus \mathbf{L}[\{1,\bar{2}\}] \oplus \mathbf{L}[\{\bar{1},\bar{2}\}]$$

$$\mathbf{L}_{\mathbb{K}C_2}[(1 \cdot 1 \otimes (12))] := \bigoplus_{s:[2]\to B\times[2]} \mathbf{L}[(1 \cdot 1 \otimes (12))|_{s([2])}]$$

$$\mathbf{L}_{\mathbb{K}C_2}[(-1 \cdot 1 \otimes \mathrm{id})] := \bigoplus_{s:[2]\to B\times[2]} \mathbf{L}[(-1 \cdot 1 \otimes \mathrm{id})|_{s([2])}]$$

We want to understand how the linear maps are defined. Fix a section, say s_2 as above, then we only need to look at the component that corresponds to the restriction to s_2:

For $\mathbf{L}[(1 \cdot 1 \otimes (12))|_{s_2([2])}]$, we have:

$$(1 \cdot 1 \otimes (12)) : s_2([2]) \to s_3([2]) \quad \rightsquigarrow \quad \mathbf{L}[(1 \cdot 1 \otimes (12))] : \mathbf{L}[s_2([2])] \to \mathbf{L}[s_3([2])]$$

$$\begin{array}{c} \bar{1} \mapsto \bar{2} \\ 2 \mapsto 1 \end{array} \quad \rightsquigarrow \quad \begin{array}{c} \bar{1}2 \mapsto \bar{2}1 \\ 2\bar{1} \mapsto 1\bar{2} \end{array}$$

For $\mathbf{L}[(-1 \cdot 1 \otimes \mathrm{id})|_{s_2}] = \sum_{\underline{k}\in T^2} c^{\underline{k}}_{\underline{i},\underline{j}} \mathbf{L}[f^{\underline{k}}_{s_2}]$, we must first determine the values of the $c^{\underline{k}}_{\underline{i},\underline{j}}$ and understand the corresponding $\mathbf{L}[f^{\underline{k}}_{s_2}]$. Recall $f^{\underline{k}} : \{(b_{j_1}, 1), .., (b_{j_n}, n)\} \to \{(b_{k_1}, 1), .., (b_{k_n}, n)\}$ and throughout this $\underline{i} = (2, 1)$ coming from $(-1 \cdot 1 \otimes \mathrm{id})$ and $\underline{j} = (2, 1)$ coming from our section s_2.

- $\underline{k} = (1, 1)$,

  $$f^{\underline{k}}_{s_2} : \{\bar{1}, 2\} \to \{1, 2\} \rightsquigarrow \mathbf{L}[f^{\underline{k}}_{s_2}] : \mathbf{L}[\{\bar{1}, 2\}] \to \mathbf{L}[\{1, 2\}]$$

  $$\begin{array}{c} \bar{1} \mapsto 1 \\ 2 \mapsto 2 \end{array} \quad \rightsquigarrow \quad \begin{array}{c} \bar{1}2 \mapsto 12 \\ 2\bar{1} \mapsto 21 \end{array}$$

  Now $c^{\underline{k}}_{\underline{i},\underline{j}} = c^1_{2,2} c^1_{1,1} = 1 \cdot 1 = 1$ since $c^1_{2,2}$ is the coefficient in front of b_1 in the product $b_2 \cdot b_2 = b_1$ and $c^1_{1,1}$ is the coefficient in front of b_1 in $b_1 \cdot b_1 = b_1$.

- $k = (1, 2)$,

  $$f^{\underline{k}}_{s_2} : \{\bar{1}, 2\} \to \{1, \bar{2}\} \rightsquigarrow \mathbf{L}[f^{\underline{k}}_{s_2}] : \mathbf{L}[\{\bar{1}, 2\}] \to \mathbf{L}[\{1, \bar{2}\}]$$

  $$\begin{array}{c} \bar{1} \mapsto 1 \\ 2 \mapsto \bar{2} \end{array} \quad \rightsquigarrow \quad \begin{array}{c} \bar{1}2 \mapsto 1\bar{2} \\ 2\bar{1} \mapsto \bar{2}1 \end{array}$$

  Now, $c^{\underline{k}}_{\underline{i},\underline{j}} = c^1_{2,2} c^2_{1,1} = 1 \cdot 0 = 0$

- $k = (2, 1)$

  $$f^{\underline{k}}_{s_2} : \{\bar{1}, 2\} \to \{\bar{1}, 2\} \rightsquigarrow \mathbf{L}[f^{\underline{k}}_{s_2}] : \mathbf{L}[\{\bar{1}, 2\}] \to \mathbf{L}[\{\bar{1}, 2\}]$$

  $$\begin{array}{c} \bar{1} \mapsto \bar{1} \\ 2 \mapsto 2 \end{array} \quad \rightsquigarrow \quad \begin{array}{c} \bar{1}2 \mapsto \bar{1}2 \\ 2\bar{1} \mapsto 2\bar{1} \end{array}$$

Now, $c^{\underline{k}}_{\underline{i},\underline{j}} = c^2_{2,2}c^1_{1,1} = 0 \cdot 1 = 0$

- $k = (2,2)$

$$f^{\underline{k}}_{s_2} : \{\overline{1}, 2\} \to \{\overline{1}, \overline{2}\} \rightsquigarrow \mathbf{L}[f^{\underline{k}}_{s_2}] : \mathbf{L}[\{\overline{1}, 2\}] \to \mathbf{L}[\{\overline{1}, \overline{2}\}]$$

$$\begin{array}{ccc} \overline{1} \mapsto \overline{1} & & \overline{1}2 \mapsto \overline{12} \\ 2 \mapsto \overline{2} & \rightsquigarrow & 2\overline{1} \mapsto \overline{21} \end{array}$$

Now, $c^{\underline{k}}_{\underline{i},\underline{j}} = c^2_{2,2}c^2_{1,1} = 0 \cdot 0 = 0$

Thus $\mathbf{L}[(-1 \cdot 1 \otimes \mathrm{id})|_{s_2}] = \sum_{\underline{k} \in T^2} c^{\underline{k}}_{\underline{i},\underline{j}} \mathbf{L}[f^{\underline{k}}_{s_2}] = \mathbf{L}[f^{(1,1)}_{s_2}]$ since only one $c^{\underline{k}}_{\underline{i},\underline{j}}$ accounts towards the sum.

Remark 11.1.2. In Example 11.1.1 above, our algebra was special in the sense that it was a group algebra. Products of basis elements in group algebras yields a single basis element, hence why all the $c^{\underline{k}}_{\underline{i},\underline{j}}$ were zero except for one. In general, when we are working with an algebra that is not a group algebra, products of basis elements yields a linear combinations of basis elements. Meaning that more than a single $\mathbf{L}[f^{\underline{k}}_{s}]$ will count towards the sum.

We immediately have that $\mathbf{L}_A$ is a Hopf monoid since it is the image of the Hopf monoid, $\mathbf{L}$, under the bilax bistrong monoidal functor $\mathcal{S}^A$, see Propositions 2.5.3 and 2.5.2. The following describes the Hopf monoid structure of $\mathbf{L}_A$.

11.1.1. Algebra Structure

To determine the product structure on $\mathbf{L}_A$, $\hat{\mu} : \mathbf{L}_A \cdot \mathbf{L}_A \to \mathbf{L}_A$ we need the following diagram to commute:

$$\begin{tikzcd} \mathbf{L}_A \cdot \mathbf{L}_A \arrow[rr, dashed, "\hat{\mu}"] \arrow[dr, "\varphi_{\mathbf{L},\mathbf{L}}"'] & & \mathbf{L}_A \\ & \mathcal{S}^A(\mathbf{L} \cdot \mathbf{L}) \arrow[ur, "\mathcal{S}^A(\mu)"'] & \end{tikzcd}$$

Note that the map in blue is the map in question. We have the maps in black, $\varphi_{\mathbf{L},\mathbf{L}}$ and $\mathcal{S}^A(\mu)$ (see Section 9.1, Proposition 9.1.6). We have that

$$\mathbf{L}_A \cdot \mathbf{L}_A[I_A] \to \mathbf{L}_A[I_A]$$

reduces to:

$$\bigoplus_{S \sqcup T = I} \bigoplus_{\substack{s' : S \to B \times S \\ s'' : T \to B \times T}} \mathbf{L}[s'(S)] \otimes \mathbf{L}[s''(T)] \to \bigoplus_{s} \mathbf{L}[s(I)]$$

Thus, given a decomposition $S \sqcup T = I$, the product is as follows:

$$\hat{\mu}_{S,T} : \mathbf{L}[s'(S)] \otimes \mathbf{L}[s''(T)] \to \mathbf{L}[s(I)]$$
$$\ell_1 \otimes \ell_2 \mapsto \ell_1 \cdot \ell_2$$

where

- ℓ_1 is a linear order on $s'(S)$ for some section s'
- ℓ_2 is a linear order on $s''(T)$ for some section s''

- s is the section determined by s' and s'' where $s(S) = s'(S)$and $s(T) = s''(T)$, and
- $\ell_1 \cdot \ell_2$ is the linear order on $[n]$ formed by concatenation, as defined in Section 5.1.

The unit $\iota_\emptyset^A : \mathbb{K} \to \mathbf{L}_A[\emptyset]$ is given by $\iota_\emptyset^A(1) = e$, where e is the distinguished basis element of $\mathbf{L}_A[\emptyset]$, i.e., the empty linear order.

11.1.2. Coalgebra Structure

To determine the coproduct on $\mathbf{L}_A$, $\hat{\Delta} : \mathbf{L}_A \to \mathbf{L}_A \cdot \mathbf{L}_A$ we need the following diagram to commute

$$\begin{tikzcd}
\mathbf{L}_A \arrow[rr, dashed, "\hat{\Delta}"] \arrow[dr, "\mathcal{S}^A(\Delta)"'] & & \mathbf{L}_A \cdot \mathbf{L}_A \\
& \mathcal{S}^A(\mathbf{L} \cdot \mathbf{L}) \arrow[ur, "\psi_{\mathbf{L},\mathbf{L}}"'] &
\end{tikzcd}$$

This is the diagram for the product where the arrows are reversed and replacing the appropriate maps with $\mathcal{S}^A(\Delta)$ and $\psi_{\mathbf{L},\mathbf{L}}$. Thus, given a section map $s : I \to B \times I$ and decomposition $S \sqcup T = I$, the coproduct structure is as follows:

$$\hat{\Delta}_{S,T}^s : \mathbf{L}[s(I)] \to \mathbf{L}[s(S)] \otimes \mathbf{L}[s(T)]$$

$$\ell \mapsto \ell|_{s(S)} \otimes \ell|_{s(T)}$$

where $\ell|_{s(S)}$ is the subset of ℓ consisting of elements of $s(S)$.

The counit $\varepsilon_\emptyset^A : \mathbf{L}_A[\emptyset] \to \mathbb{K}$ is given by $\varepsilon_\emptyset^A(e) = 1$.

11.1.3. Antipode

Since $\mathcal{S}^A$ is a bistrong bilax monoidal functor, we have that the antipode of $\mathbf{L}$ is preserved. Hence:

$$s_{I_A} : \bigoplus_{s:I\to B\times I} \mathbf{L}[s(I)] \to \bigoplus_{s:I\to B\times I} \mathbf{L}[s(I)]$$

$$\ell \mapsto (-1)^{|I|}\overline{\ell}$$

where $\overline{\ell}$ is obtained by reversing the order (as defined in Section 5.1).

Example 11.1.3. Let $A = \mathbb{K}C_3$ where $C_3 = \langle r \mid r^3 = 1\rangle$. Let $S = \{1, 3\}$ and $T = \{2, 4, 5\}$ be a decomposition of $[5]$, and fix a section $s : [5] \mapsto \{(1, 1),(r, 2),(1, 3),(r^2, 4),(r, 5)\}$. Then

$$\hat{\mu}_{S,T} : (1, 1)(1, 3) \otimes (r, 2)(r, 5)(r^2, 4) \mapsto (1, 1)(1, 3)(r, 2)(r, 5)(r^2, 4)$$

$$\hat{\Delta}_{S,T} : (1, 3)(r, 5)(1, 1)(r^2, 4)(r, 2) \mapsto (1, 3)(1, 1) \otimes (r, 5)(r^2, 4)(r, 2)$$

$$s((1, 3)(1, 1)(r, 5)(r^2, 4)(r, 2)) = -(r, 2)(r^2, 4)(r, 5)(1, 1)(1, 3)$$

Remark 11.1.4. When $A = \mathbb{K}C_2$, we recover the notion of $\mathcal{H}$-species of linear orders as defined in Definition (9.2.12) and [10].

11.2. A-Hopf Monoid of Colored Set Partitions

In this section, we will describe the structure of the A-Hopf monoid of Set Partitions in detail; we will denote this by $\mathbf{\Pi}_A$.

Recall the vector species of set partitions, $\mathbf{\Pi}$, as described in Section 5.2.

We define $\mathbf{\Pi}_A : \mathcal{S}^A(\mathbf{\Pi})$. Applying $\mathcal{S}^A$ to $\mathbf{\Pi}$ yields:

$$\mathcal{S}^A(\mathbf{\Pi})[n_A] := \bigoplus_{s:[n]\to B\times[n]} \mathbf{\Pi}[s([n])]$$

$$\mathcal{S}^A(\mathbf{\Pi})[(1\cdots 1\otimes\sigma)] := \bigoplus_{s:[n]\to B\times[n]} \mathbf{\Pi}[(1\cdots 1\otimes\sigma)|_{s([n])}]$$

$$\mathcal{S}^A(\mathbf{\Pi})[(b_{t_1}\cdots b_{t_n}\otimes \mathrm{id})] := \bigoplus_{s:[n]\to B\times[n]} \mathbf{\Pi}[(b_{t_1}\cdots b_{t_n}\otimes \mathrm{id})|_{s([n])}]$$

where $\mathbf{\Pi}[(1\cdots 1\otimes\sigma)|_{s([n])}]$ and $\mathbf{\Pi}[(1\cdots 1\otimes\sigma)|_{s([n])}]$ are as defined in Definition 9.1.3.

In the following example, we make explicit what $\mathbf{\Pi}_A$ does on objects and morphisms.

Example 11.2.1. **$\mathbb{K}C_2$-Species of Set Partitions, $\mathbf{\Pi}_{\mathbb{K}C_2}$**
Let $n = 2$ and consider $A = \mathbb{K}C_2$ with basis $B = \{b_1 = 1, b_2 = -1\}$, hence $T = \{1, 2\}$. The sections, $s : [2] \to B \times [2]$, are given by

$$s_1 : \begin{array}{c} 1\mapsto 1 \\ 2\mapsto 2\end{array} \;,\; s_2 : \begin{array}{c} 1\mapsto \overline{1} \\ 2\mapsto 2\end{array} \;,\; s_3 : \begin{array}{c} 1\mapsto 1 \\ 2\mapsto \overline{2}\end{array} \;,\; s_4 : \begin{array}{c} 1\mapsto \overline{1} \\ 2\mapsto \overline{2}\end{array}$$

where $\overline{i}$ denotes the image $s(i) = (-1, i)$ and i denotes the image $s(i) = (1, i)$ for all $i \in [2]$. The endomorphism ring of $\mathbb{K}C_2^{\otimes 2}\otimes\mathbb{K}[2]$ is generated by the elements $(1\cdots 1\otimes(12))$, $(-1\cdot 1\otimes \mathrm{id}) \in \mathbb{K}C_2 \wr S_2$.

$$\mathbf{\Pi}_{\mathbb{K}C_2}[\mathbb{K}C_2^{\otimes 2}\otimes\mathbb{K}[2]] = \mathbf{\Pi}[\{1,2\}] \oplus \mathbf{\Pi}[\{\overline{1},2\}] \oplus \mathbf{\Pi}[\{1,\overline{2}\}] \oplus \mathbf{\Pi}[\{\overline{1},\overline{2}\}]$$

$$\mathbf{\Pi}_{\mathbb{K}C_2}[(1\cdot 1\otimes(12))] := \bigoplus_{s:[2]\to B\times[2]} \mathbf{\Pi}[(1\cdots 1\otimes(12))|_{s([2])}]$$

$$\mathbf{\Pi}_{\mathbb{K}C_2}[(-1\cdot 1\otimes \mathrm{id})] := \bigoplus_{s:[2]\to B\times[2]} \mathbf{\Pi}[(-1\cdot 1\otimes \mathrm{id})|_{s([2])}]$$

We want to understand how the linear maps are defined. Fix a section, say s_2 as above, then we only need to look at the component that corresponds to the restriction to s_2:

For $\mathbf{\Pi}[(1\cdots 1\otimes(12))|_{s_2([2])}]$, we have:

$$(1\cdots 1\otimes(12)) : s_2([2]) \to s_3([2]) \quad\rightsquigarrow\quad \mathbf{\Pi}[(1\cdots 1\otimes(12))] : \mathbf{\Pi}[s_2([2])] \to \mathbf{\Pi}[s_3([2])]$$

$$\begin{array}{c} \overline{1}\mapsto\overline{2} \\ 2\mapsto 1\end{array} \quad\rightsquigarrow\quad \begin{array}{c} \overline{1}2\mapsto 1\overline{2} \\ \overline{1}|2\mapsto 1|\overline{2}\end{array}$$

For $\mathbf{\Pi}[(-1\cdot 1\otimes \mathrm{id})|_{s_2}] = \sum_{\underline{k}\in T^2} c^{\underline{k}}_{\underline{i},\underline{j}}\mathbf{\Pi}[f^{\underline{k}}_{s_2}]$, we must first determine the values of the $c^{\underline{k}}_{\underline{i},\underline{j}}$ and understand the corresponding $\mathbf{\Pi}[f^{\underline{k}}_{s_2}]$. Recall $f^{\underline{k}} : \{(b_{j_1}, 1), .., (b_{j_n}, n)\} \to \{(b_{k_1}, 1), .., (b_{k_n}, n)\}$ and throughout this $\underline{i} = (2, 1)$ coming from $(-1\cdot 1\otimes \mathrm{id})$ and $\underline{j} = (2, 1)$ coming from our section s_2.

- $\underline{k} = (1,1)$,

$$f_{s_2}^{\underline{k}} : \{\overline{1}, 2\} \to \{1, 2\} \rightsquigarrow \mathbf{\Pi}[f_{s_2}^{\underline{k}}] : \mathbf{\Pi}[\{\overline{1}, 2\}] \to \mathbf{\Pi}[\{1, 2\}]$$

$$\begin{array}{c} \overline{1} \mapsto 1 \\ 2 \mapsto 2 \end{array} \qquad \rightsquigarrow \qquad \begin{array}{c} \overline{1}2 \mapsto 12 \\ \overline{1}|2 \mapsto 1|2 \end{array}$$

  Now $c_{\underline{i},\underline{j}}^{\underline{k}} = c_{2,2}^{1} c_{1,1}^{1} = 1 \cdot 1 = 1$ since $c_{2,2}^{1}$ is the coefficient in front of b_1 in the product $b_2 \cdot b_2 = b_1$ and $c_{1,1}^{1}$ is the coefficient in front of b_1 in $b_1 \cdot b_1 = b_1$.

- $k = (1,2)$,

$$f_{s_2}^{\underline{k}} : \{\overline{1}, 2\} \to \{1, \overline{2}\} \rightsquigarrow \mathbf{\Pi}[f_{s_2}^{\underline{k}}] : \mathbf{\Pi}[\{\overline{1}, 2\}] \to \mathbf{\Pi}[\{1, \overline{2}\}]$$

$$\begin{array}{c} \overline{1} \mapsto 1 \\ 2 \mapsto \overline{2} \end{array} \qquad \rightsquigarrow \qquad \begin{array}{c} \overline{1}2 \mapsto 1\overline{2} \\ \overline{1}|2 \mapsto 1|\overline{2} \end{array}$$

  Now, $c_{\underline{i},\underline{j}}^{\underline{k}} = c_{2,2}^{1} c_{1,1}^{2} = 1 \cdot 0 = 0$

- $k = (2,1)$

$$f_{s_2}^{\underline{k}} : \{\overline{1}, 2\} \to \{\overline{1}, 2\} \rightsquigarrow \mathbf{\Pi}[f_{s_2}^{\underline{k}}] : \mathbf{\Pi}[\{\overline{1}, 2\}] \to \mathbf{\Pi}[\{\overline{1}, 2\}]$$

$$\begin{array}{c} \overline{1} \mapsto \overline{1} \\ 2 \mapsto 2 \end{array} \qquad \rightsquigarrow \qquad \begin{array}{c} \overline{1}2 \mapsto \overline{1}2 \\ \overline{1}|2 \mapsto \overline{1}|2 \end{array}$$

  Now, $c_{\underline{i},\underline{j}}^{\underline{k}} = c_{2,2}^{2} c_{1,1}^{1} = 0 \cdot 1 = 0$

- $k = (2,2)$

$$f_{s_2}^{\underline{k}} : \{\overline{1}, 2\} \to \{\overline{1}, \overline{2}\} \rightsquigarrow \mathbf{\Pi}[f_{s_2}^{\underline{k}}] : \mathbf{\Pi}[\{\overline{1}, 2\}] \to \mathbf{\Pi}[\{\overline{1}, \overline{2}\}]$$

$$\begin{array}{c} \overline{1} \mapsto \overline{1} \\ 2 \mapsto \overline{2} \end{array} \qquad \rightsquigarrow \qquad \begin{array}{c} \overline{1}2 \mapsto \overline{12} \\ \overline{1}|2 \mapsto \overline{1}|\overline{2} \end{array}$$

  Now, $c_{\underline{i},\underline{j}}^{\underline{k}} = c_{2,2}^{2} c_{1,1}^{2} = 0 \cdot 0 = 0$

Thus $\mathbf{\Pi}[(-1 \cdot 1 \otimes \mathrm{id})|_{s_2}] = \sum_{\underline{k} \in T^2} c_{\underline{i},\underline{j}}^{\underline{k}} \mathbf{\Pi}[f_{s_2}^{\underline{k}}] = \mathbf{\Pi}[f_{s_2}^{(1,1)}]$ since only one $c_{\underline{i},\underline{j}}^{\underline{k}}$ accounts towards the sum.

We immediately have that $\mathbf{\Pi}_A$ is a Hopf monoid since it is the image of the Hopf monoid, $\mathbf{\Pi}$, under the bilax bistrong monoidal functor $\mathcal{S}^A$, see Propositions 2.5.3 and 2.5.2.The following describes the Hopf monoid structure of $\mathbf{\Pi}_A$.

11.2.1. Algebra Structure

To determine the product structure on $\mathbf{\Pi}_A$, $\hat{\mu} : \mathbf{\Pi}_A \cdot \mathbf{\Pi}_A \to \mathbf{\Pi}_A$ we need the following diagram to commute:

$$\begin{tikzcd}
\mathbf{\Pi}_A \cdot \mathbf{\Pi}_A \arrow[rr, dashed, "\hat{\mu}"] \arrow[dr, "\varphi_{\mathbf{\Pi},\mathbf{\Pi}}"'] & & \mathbf{\Pi}_A \\
& \mathcal{S}^A(\mathbf{\Pi} \cdot \mathbf{\Pi}) \arrow[ur, "\mathcal{S}^A(\mu)"'] &
\end{tikzcd}$$

Note that the map in blue is the map in question. We have the maps in black, $\varphi_{\mathbf{\Pi},\mathbf{\Pi}}$ and $\mathcal{S}^A(\mu)$ (see Section 9.1 Proposition 9.1.6).

We have that

$$\mathbf{\Pi}_A \cdot \mathbf{\Pi}_A[I_A] \to \mathbf{\Pi}_A[I_A]$$

reduces to:

$$\bigoplus_{S \sqcup T = I} \bigoplus_{\substack{s' : S \to B \times S \\ s'' : T \to B \times T}} \mathbf{\Pi}[s'(S)] \otimes \mathbf{\Pi}[s''(T)] \to \bigoplus_{s} \mathbf{\Pi}[s(I)]$$

Thus, given a decomposition $S \sqcup T = I$, the product is as follows:

$$\hat{\mu}_{S,T} : \mathbf{\Pi}[s'(S)] \otimes \mathbf{\Pi}[s''(T)] \to \mathbf{\Pi}[s(I)]$$
$$\pi \otimes \sigma \mapsto \pi \sqcup \sigma$$

where

- π is a set partition on $s'(S)$ for some section s'
- σ a set partiiton on $s''(T)$ for some section s''
- s is the section determined by s' and s'' where $s(S) = s'(S)$and $s(T) = s''(T)$

The unit $\iota_{\emptyset}^A : \mathbb{K} \to \mathbf{\Pi}_A[\emptyset]$ is given by $\iota_{\emptyset}^A(1) = e$, where e is the distinguished basis element of $\mathbf{\Pi}_A[\emptyset]$, i.e., the empty set partition.

11.2.2. Coalgebra Structure

To determine the coproduct on $\mathbf{\Pi}_A$, $\hat{\Delta} : \mathbf{\Pi}_A \to \mathbf{\Pi}_A \cdot \mathbf{\Pi}_A$ we need the following diagram to commute

$$\begin{tikzcd} \mathbf{\Pi}_A \arrow[rr, dashed, "\hat{\Delta}"] \arrow[dr, "\mathcal{S}^A(\Delta)"'] & & \mathbf{\Pi}_A \cdot \mathbf{\Pi}_A \\ & \mathcal{S}^A(\mathbf{\Pi} \cdot \mathbf{\Pi}) \arrow[ur, "\psi_{\mathbf{\Pi},\mathbf{\Pi}}"'] & \end{tikzcd}$$

This is the diagram for the product where the arrows are reversed and replacing the appropriate maps with $\mathcal{S}^A(\Delta)$ and $\psi_{\mathbf{\Pi},\mathbf{\Pi}}$. Thus, given a section map $s : I \to B \times I$ and decomposition $S \sqcup T = I$, the coproduct structure is as follows:

$$\hat{\Delta}_{S,T}^s : \pi \mapsto \pi|_{s(S)} \otimes \pi|_{s(T)}$$

The counit $\varepsilon_{\emptyset}^A : \mathbf{\Pi}_A[\emptyset] \to \mathbb{K}$ is given by $\varepsilon_{\emptyset}^A(e) = 1$.

11.2.3. Antipode

Since $\mathcal{S}^A$ is a bistrong bilax monoidal functor, we have that the antipode of $\mathbf{\Pi}$ is preserved. Hence:

$$s_{I_A} : \bigoplus_{s : I \to B \times I} \mathbf{\Pi}[s(I)] \to \bigoplus_{s : I \to B \times I} \mathbf{\Pi}[s(I)]$$
$$H_\pi \mapsto \sum_{\substack{\sigma \vdash s(I) \\ \sigma \leq \pi}} (-1)^{\ell(\sigma)} (\pi : \sigma)! H_\sigma$$

as defined in Section 5.2.

REMARK 11.2.2. Notice, that there are not any extra rules involving the color of a set partition. When computing the antipode recursively, the multiplication and coproduct definitions determine the colorings of the output of the antipode.

EXAMPLE 11.2.3.

1. Let $A = \mathbb{K}$ then we get the linearization of the vector species of set partitions, as defined in Section 5.2.

2. Let $A = \mathbb{K}C_2$, then we get the linearization of the $\mathcal{H}$-species defined in [10]. e.g, when $n = 5$,

$$\begin{aligned}\mathbf{\Pi}_{\mathbb{K}C_2}[\mathbb{K}C_2^{\otimes 5} \otimes \mathbb{K}[5]] &= \bigoplus_{s:[5]\to C_2\times[5]} \mathbf{\Pi}[s([5])] \\ &= \mathbf{\Pi}[\{1,2,3,4,5\}] \oplus \mathbf{\Pi}[\{\overline{1},2,3,4,5\}] \oplus \cdots \oplus \mathbf{\Pi}[\{\overline{1},\overline{2},\overline{3},\overline{4},\overline{5}\}]\end{aligned}$$

Let $U = \{1,3,5\}$ and $V = \{2,4\}$ be a decomposition of [5].

$$\hat{\mu}_{U,V}(1\overline{5}|3 \otimes \overline{2}4) = 1\overline{5}|3|\overline{2}4$$

$$\hat{\Delta}_{U,V}(1\overline{5}|3|\overline{2}4) = 1\overline{5}|3 \otimes \overline{2}4$$

$$s_{[\hat{5}]}(1\overline{2}3|4\overline{5}) = -1\overline{2}3|4\overline{5} + 1\overline{2}3|4\overline{5} + 1\overline{2}|3\overline{4}5$$

$\mathbf{\Pi}_{\mathbb{K}C_2}$ is the A-species that gives the vector space formed from all C_2-colored set partitions of a set I.

11.3. A-Hopf Monoid of Super Class Functions on Unitriangular Groups

In this section, we will describe the structure of the A-Hopf monoid of Superclass functions on unitriangular groups in detail–we will denote this by $\mathbf{scf}_A(U)$.

Recall, the species of superclass functions on unitriangular groups, $\mathbf{scf}(U)$, as defined in Section 5.3.

We define $\mathbf{scf}_A(U) := \mathcal{S}^A(\mathbf{scf}(U))$. Applying $\mathcal{S}^A$ to $\mathbf{scf}_A(U)$ yields:

$$\mathbf{scf}_A(U)[n_A] := \bigoplus_{s:[n]\to B\times[n]} \mathbf{scf}(U)[s([n])]$$

i.e., the $\mathbb{K}$-span of linear orders on $s([n])$ for all sections $s : [n] \to B \times [n]$.

$$\mathbf{scf}_A(U)[(1\cdots 1 \otimes \sigma)] := \bigoplus_{s:[n]\to B\times[n]} \mathbf{scf}(U)[(1\cdots 1 \otimes \sigma)|_{s([n])}]$$

$$\mathbf{scf}_A(U)[(b_{i_1}\cdots b_{i_n} \otimes \mathrm{id})] := \bigoplus_{s:[n]\to B\times[n]} \mathbf{scf}(U)[(b_{i_1}\cdots b_{i_n} \otimes \mathrm{id})|_{s([n])}]$$

where $\mathbf{scf}(U)[(1\cdots 1 \otimes \sigma)|_{s([n])}]$ and $\mathbf{scf}(U)[(1\cdots 1 \otimes \sigma)|_{s([n])}]$ are as defined in Definition 9.1.3.

In the following example, we make explicit what $\mathbf{scf}_A(U)$ does on objects and morphisms.

Example 11.3.1. **$\mathbb{K}C_2$-Species of Superclass functions, $\mathbf{scf}_{\mathbb{K}C_2}(U)$**
Let $n = 2$ and consider $A = \mathbb{K}C_2$ with basis $B = \{b_1 = 1, b_2 = \overline{1}\}$, hence $T = \{1, 2\}$. The sections, $s : [2] \to B \times [2]$, are given by

$$s_1 : \begin{array}{c} 1 \mapsto 1 \\ 2 \mapsto 2 \end{array} \quad , \quad s_2 : \begin{array}{c} 1 \mapsto \overline{1} \\ 2 \mapsto 2 \end{array} \quad , \quad s_3 : \begin{array}{c} 1 \mapsto 1 \\ 2 \mapsto \overline{2} \end{array} \quad , \quad s_4 : \begin{array}{c} 1 \mapsto \overline{1} \\ 2 \mapsto \overline{2} \end{array}$$

where $\overline{i}$ denotes the image $s(i) = (-1, i)$ and i denotes the image $s(i) = (1, i)$ for all $i \in [2]$. The endomorphism ring of $\mathbb{K}C_2^{\otimes 2} \otimes \mathbb{K}[2]$ is generated by the elements $(1 \cdot 1 \otimes (12))$, $(-1 \cdot 1 \otimes \mathrm{id}) \in \mathbb{K}C_2 \wr S_2$.

$$\mathbf{scf}_{\mathbb{K}C_2}(U)[[2]_{\mathbb{K}C_2}] = \mathbf{scf}(U)[\{1,2\}] \oplus \mathbf{scf}(U)[\{\overline{1},2\}] \oplus \mathbf{scf}(U)[\{1,\overline{2}\}] \oplus \mathbf{scf}(U)[\{\overline{1},\overline{2}\}]$$

$$\mathbf{scf}_{\mathbb{K}C_2}(U)[(1 \cdot 1 \otimes (12))] := \bigoplus_{s:[2]\to B\times[2]} \mathbf{scf}(U)[(1 \cdot 1 \otimes (12))|_{s([2])}]$$

$$\mathbf{scf}_{\mathbb{K}C_2}(U)[(-1 \cdot 1 \otimes \mathrm{id})] := \bigoplus_{s:[2]\to B\times[2]} \mathbf{scf}(U)[(-1 \cdot 1 \otimes \mathrm{id})|_{s([2])}]$$

We want to understand how the linear maps are defined. Fix a section, say s_2 as above, then we only need to look at the component that corresponds to the restriction to s_2. Observe that:

$$\mathbf{scf}(U)[\{\overline{1},2\}] = \mathbf{scf}(U(\{\overline{1},2\},\overline{1}2)) \bigoplus \mathbf{scf}(U(\{\overline{1},2\},2\overline{1})).$$

In this example, we will restrict ourselves to the component labelled by the linear order $\overline{1}2$. For $\mathbf{scf}(U)[(1 \cdot 1 \otimes (12))|_{s_2([2])}]$, we have:

$$(1 \cdot 1 \otimes (12)) : s_2([2]) \to s_3([2])$$

$$\begin{array}{c} \overline{1} \mapsto \overline{2} \\ 2 \mapsto 1 \end{array}$$

induces the following linear map

$$\mathbf{scf}(U)[(1 \cdot 1 \otimes (12))] : \mathbf{scf}(U(\{\overline{1},2\},\overline{1}2)) \to \mathbf{scf}(U(\{\overline{1},2\},2\overline{1}))$$

$$\kappa \underset{\overline{1}}{\bullet} \; \underset{2}{\bullet} \quad \mapsto \quad \kappa \underset{\overline{2}}{\bullet} \; \underset{1}{\bullet}$$

$$\kappa \underset{\overline{1}}{\bullet}\!\frown\!\underset{2}{\bullet} \quad \mapsto \quad \kappa \underset{\overline{2}}{\bullet}\!\frown\!\underset{1}{\bullet}$$

For $\mathbf{scf}(U)[(-1 \cdot 1 \otimes \mathrm{id})|_{s_2}] = \sum_{\underline{k}\in T^2} c^{\underline{k}}_{\underline{i},\underline{j}} \mathbf{scf}(U)[f^{\underline{k}}_{s_2}]$, we must first determine the values of the $c^{\underline{k}}_{\underline{i},\underline{j}}$ and understand the corresponding $\mathbf{scf}(U)[f^{\underline{k}}_{s_2}]$. Recall $f^{\underline{k}} : \{(b_{j_1}, 1), .., (b_{j_n}, n)\} \to \{(b_{k_1}, 1), .., (b_{k_n}, n)\}$ and throughout this $\underline{i} = (2, 1)$ coming from $(-1 \cdot 1 \otimes \mathrm{id})$ and $\underline{j} = (2, 1)$ coming from our section s_2.

- $\underline{k} = (1, 1)$,

$$f^{\underline{k}}_{s_2} : \{\overline{1}, 2\} \to \{1, 2\}$$
$$\begin{array}{c} \overline{1} \mapsto 1 \\ 2 \mapsto 2 \end{array}$$

  leads to the linear map

$$\mathbf{scf}(U)[f^{\underline{k}}_{s_2}] : \mathbf{scf}(U)[\{\overline{1}, 2\}] \to \mathbf{scf}(U)[\{1, 2\}]$$

$$\begin{array}{ccc} \kappa\ \underset{\overline{1}}{\bullet}\ \underset{2}{\bullet} & \mapsto & \kappa\ \underset{1}{\bullet}\ \underset{2}{\bullet} \\ \kappa\ \underset{\overline{1}}{\bullet}\!\frown\!\underset{2}{\bullet} & \mapsto & \kappa\ \underset{1}{\bullet}\!\frown\!\underset{2}{\bullet} \end{array}$$

Now $c^{\underline{k}}_{\underline{i},\underline{j}} = c^1_{2,2}c^1_{1,1} = 1 \cdot 1 = 1$ since $c^1_{2,2}$ is the coefficient in front of b_1 in the product $b_2 \cdot b_2 = b_1$ and $c^1_{1,1}$ is the coefficient in front of b_1 in $b_1 \cdot b_1 = b_1$.

- $k = (1, 2)$,

$$\begin{aligned} f^{\underline{k}}_{s_2} : \{\overline{1}, 2\} &\to \{1, \overline{2}\} \\ \overline{1} &\mapsto 1 \\ 2 &\mapsto \overline{2} \end{aligned}$$

leads to the linear map

$$\mathbf{scf}(U)[f^{\underline{k}}_{s_2}] : \mathbf{scf}(U)[\{\overline{1}, 2\}] \to \mathbf{scf}(U)[\{1, \overline{2}\}]$$

$$\begin{array}{ccc} \kappa\ \underset{\overline{1}}{\bullet}\ \underset{2}{\bullet} & \mapsto & \kappa\ \underset{1}{\bullet}\ \underset{\overline{2}}{\bullet} \\ \kappa\ \underset{\overline{1}}{\bullet}\!\frown\!\underset{2}{\bullet} & \mapsto & \kappa\ \underset{1}{\bullet}\!\frown\!\underset{\overline{2}}{\bullet} \end{array}$$

Now, $c^{\underline{k}}_{\underline{i},\underline{j}} = c^1_{2,2}c^2_{1,1} = 1 \cdot 0 = 0$.

- $k = (2, 1)$

$$\begin{aligned} f^{\underline{k}}_{s_2} : \{\overline{1}, 2\} &\to \{\overline{1}, 2\} \\ \overline{1} &\mapsto \overline{1} \\ 2 &\mapsto 2 \end{aligned}$$

leads to the linear map

$$\mathbf{scf}(U)[f^{\underline{k}}_{s_2}] : \mathbf{scf}(U)[\{\overline{1}, 2\}] \to \mathbf{scf}(U)[\{\overline{1}, 2\}]$$

$$\begin{array}{ccc} \kappa\ \underset{\overline{1}}{\bullet}\ \underset{2}{\bullet} & \mapsto & \kappa\ \underset{\overline{1}}{\bullet}\ \underset{2}{\bullet} \\ \kappa\ \underset{\overline{1}}{\bullet}\!\frown\!\underset{2}{\bullet} & \mapsto & \kappa\ \underset{\overline{1}}{\bullet}\!\frown\!\underset{2}{\bullet} \end{array}$$

Now, $c^{\underline{k}}_{\underline{i},\underline{j}} = c^2_{2,2}c^1_{1,1} = 0 \cdot 1 = 0$.

- $k = (2, 2)$

$$\begin{aligned} f^{\underline{k}}_{s_2} : \{\overline{1}, 2\} &\to \{\overline{1}, \overline{2}\} \\ \overline{1} &\mapsto \overline{1} \\ 2 &\mapsto \overline{2} \end{aligned}$$

leads to the linear map

$$\mathbf{scf}(U)[f^{\underline{k}}_{s_2}] : \mathbf{scf}(U)[\{\overline{1}, 2\}] \to \mathbf{scf}(U)[\{\overline{1}, \overline{2}\}]$$

$$\begin{array}{cccc} \kappa & \underset{\bar{1}}{\bullet}\ \underset{2}{\bullet} & \mapsto & \kappa\ \underset{\bar{1}}{\bullet}\ \underset{\bar{2}}{\bullet} \\ \kappa & \underset{\bar{1}\ \ 2}{\overset{\frown}{\bullet\ \bullet}} & \mapsto & \kappa\ \underset{\bar{1}\ \ \bar{2}}{\overset{\frown}{\bullet\ \bullet}} \end{array}$$

Now, $c^{\underline{k}}_{\underline{i},\underline{j}} = c^2_{2,2} c^2_{1,1} = 0 \cdot 0 = 0$.

Thus $\mathbf{scf}(U)[(-1 \cdot 1 \otimes \mathrm{id})|_{s_2}] = \sum_{\underline{k} \in T^2} c^{\underline{k}}_{\underline{i},\underline{j}} \mathbf{scf}(U)[f^{\underline{k}}_{s_2}] = \mathbf{scf}(U)[f^{(1,1)}_{s_2}]$ since only one $c^{\underline{k}}_{\underline{i},\underline{j}}$ accounts towards the sum.

We immediately have that $\mathbf{scf}_A(U)$ is a Hopf monoid since it is the image of the Hopf monoid, $\mathbf{scf}(U)$, under the bilax bistrong monoidal functor $\mathcal{S}^A$, see Propositions 2.5.3 and 2.5.2. The following describes the Hopf monoid structure of $\mathbf{scf}_A(U)$.

11.3.1. Algebra Structure

To determine the product structure on $\mathbf{scf}_A(U)$, $\hat{\mu} : \mathbf{scf}_A(U) \cdot \mathbf{scf}_A(U) \to \mathbf{scf}_A(U)$ we need the following diagram to commute:

$$\begin{tikzcd} \mathbf{scf}_A(U) \cdot \mathbf{scf}_A(U) \arrow[rr, dashed, "\hat{\mu}"] \arrow[dr, "\varphi_{\mathbf{scf}(U),\mathbf{scf}(U)}"'] & & \mathbf{scf}_A(U) \\ & \mathcal{S}^A(\mathbf{scf}(U) \cdot \mathbf{scf}(U)) \arrow[ur, "\mathcal{S}^A(\mu)"'] & \end{tikzcd}$$

Note that the map in blue is the map in question. We have the maps in black, $\varphi_{\mathbf{scf}(U),\mathbf{scf}(U)}$ and $\mathcal{S}^A(\mu)$ (see Section 9.1, Proposition 9.1.6).
For a decomposition $S \sqcup T = I$, and section maps $s' : S \to B \times S$ and $s'' : T \to B \times T$, we have that

$$\mathbf{scf}_A(U) \cdot \mathbf{scf}_A(U)[I_A] \to \mathbf{scf}_A(U)[I_A]$$

reduces to:

$$\bigoplus_{\substack{\ell_S \in L[s'(S)] \\ \ell_T \in L[s''(T)]}} \mathbf{scf}[U(s'(S), \ell_S)] \otimes \mathbf{scf}[U(s''(T), \ell_T)] \to \bigoplus_{\ell} \mathbf{scf}[U(s(I), \ell_S \cdot \ell_T)]$$

The product is as follows:

$$\hat{\mu}_{S,T} : \mathbf{scf}[U(s'(S), \ell_S)] \otimes \mathbf{scf}[U(s''(T), \ell_T)] \to \mathbf{scf}[U(s(I), \ell_S \cdot \ell_T)]$$

$$\kappa_{X_1,\alpha_1} \otimes \kappa_{X_2,\alpha_2} \mapsto\!\!\mapsto \sum_{\substack{X|_A = X_A \\ \alpha|_A = \alpha_A}} \kappa_{X,\alpha}.$$

where

- κ_{X_S,α_S} is the basis element corresponding the the arc diagram (X_S, α_S) on $s'(S)$ for some section s' and $\ell_S \in L[s'(S)]$.
- κ_{X_T,α_T} is the basis element corresponding the the arc diagram (X_T, α_T) on $s''(T)$ for some section s'' and $\ell_T \in L[s''(T)]$.
- s is the section determined by s' and s'' where $s(S) = s'(S)$and $s(T) = s''(T)$

- $X|_S$ is the set partition formed restricting the set partition $X \vdash s(I)$ to values in $s'(S)$.

Note: Please refer back to Section 5.3 for a reminder of the combinatorics of arc diagrams.

11.3.2. Coalgebra Structure

To determine the coproduct on $\mathbf{scf}_A(U)$, $\hat{\Delta} : \mathbf{scf}_A(U) \to \mathbf{scf}_A(U) \cdot \mathbf{scf}_A(U)$ we need the following diagram to commute:

$$\begin{tikzcd}
\mathbf{scf}_A(U) \arrow[rr, dashed, "\hat{\Delta}"] \arrow[dr, "\mathcal{S}^A(\Delta)"'] & & \mathbf{scf}_A(U) \cdot \mathbf{scf}_A(U) \\
& \mathcal{S}^A(\mathbf{scf}(U) \cdot \mathbf{scf}(U)) \arrow[ur, "\psi_{\mathbf{scf}(U),\mathbf{scf}(U)}"'] &
\end{tikzcd}$$

This is the diagram for the product where the arrows are reversed and replacing the appropriate maps with $\mathcal{S}^A(\Delta)$ and $\psi_{\mathbf{scf}A(U),\mathbf{scf}(U)}$. Thus, given a section map $s : I \to B \times I$ and decomposition $S \sqcup T = I$, the coproduct structure is as follows:

$$\hat{\Delta}^s_{S,T} : \kappa_{X,\alpha} \mapsto \begin{cases} \kappa_{X|_{S_1},\alpha|_{S_1}} \otimes \kappa_{X|_{S_2},\alpha|_{S_2}} & \text{if } S_1 \text{ is the union of some blocks of } X \\ 0 & \text{otherwise.} \end{cases}$$

Example 11.3.2. Let $A = \mathbb{K}C_2$ and our field be $\mathbb{F}_2$. We have

$$\mathbf{scf}_A(U)[\mathbb{K}C_2^{\otimes 2} \otimes \mathbb{K}[2]] = \bigoplus_{s:[2]\to C_2\times[2]} \bigoplus_{\ell\in L[s([2])]} \mathbf{scf}[U(s([2]),\ell)].$$

Notice that this is a 16-dimensional vector space. There are 2! many linear orders that correspond to each section, and there are 2^2 many section maps. For a fixed section and linear order, each component has dimension corresponding to the number of arc diagrams, which in this case is the number of set partitions of [2] because our field in $\mathbb{F}_2$.

Remark 11.3.3. In general, if working over the field $\mathbb{F}_2$, we have that the

$$\dim(\mathbf{scf}_A(U)[n_A]) = (n!)|B|^r dim(\Pi[n])$$

as long as the basis B is a finite set.

CHAPTER 12

Relationships to the Hopf Algebra $\tilde{\Pi}^{(B)}$

In this section, we show a string of relationships by applying the Fock functors K_A, $\widetilde{K}_A$, and $\overline{K}_A$ to $\mathbf{scf}_A(U)$, $(\mathbf{L} \times \mathbf{\Pi})_A$, $\mathbf{\Pi}_A$, and $\mathbf{L}_A \times \mathbf{\Pi}_A$; all of which end up being isomorphic as Hopf Algebras to $\tilde{\Pi}^{(B)}$.

12.1. K_A applied to $\mathbf{\Pi}_A$

We show that we get the associated Hopf Algebra, $\tilde{\Pi}^{(B)}$ (Section 3.5) by applying the Fock functor K_A, defined in Section 10.2, to $\mathbf{\Pi}_A$.
To determine the Hopf Algebra structure (see Section 10.4), we will be utilizing the following maps:

$$K_A(\mathbf{\Pi}_A) \cdot K_A(\mathbf{\Pi}_A) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} K_A(\mathbf{\Pi}_A \cdot \mathbf{\Pi}_A)$$

Which reduces to:

$$\bigoplus_{s+t=n} \mathbf{\Pi}_A[s_A] \otimes \mathbf{\Pi}_A[t_A] \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \bigoplus_{S \sqcup T=[n]} \mathbf{\Pi}_A[S_A] \otimes \mathbf{\Pi}_A[T_A]$$

where φ and ψ are defined as follows:

$$\varphi : \mathbf{\Pi}_A[s_A] \otimes \mathbf{\Pi}_A[t_A] \xmapsto{\mathrm{id} \otimes can} \mathbf{\Pi}_A[s_A] \otimes \mathbf{\Pi}_A[[1+t, s+t]_A]$$

$$\psi : \mathbf{\Pi}_A[S_A] \otimes \mathbf{\Pi}_A[T_A] \xmapsto{\quad \mathrm{st} \otimes \mathrm{st} \quad} \mathbf{\Pi}_A[s_A] \otimes \mathbf{\Pi}_A[t_A]$$

12.1.1. Algebra Structure:

The product, μ, is given by the following composition:

$$K_A(\mathbf{\Pi}_A) \cdot K_A(\mathbf{\Pi}_A) \xrightarrow{\mathrm{id} \otimes \mathrm{cano}} K_A(\mathbf{\Pi}_A \cdot \mathbf{\Pi}_A) \xrightarrow{K_A(\hat{\mu})} K_A(\mathbf{\Pi}_A)$$

$$\pi \otimes \sigma \mapsto \pi \otimes \mathrm{cano}(\sigma) \mapsto \pi \sqcup \mathrm{cano}(\sigma)$$

where

- π is a set partition on $s'([s])$ for some section s'
- σ a set partition on $s''([t])$ for some section s''
- $\mathrm{cano}(\sigma)$ is a set partition on $\mathrm{cano}(s''([t]))$

The unit, ι, is given by the following composition:

$$\begin{array}{ccccc} \mathbb{K} & \longrightarrow & K_A(\mathbf{1}) & \longrightarrow & K_A(\mathbf{\Pi}_A) \\ \mathbb{K} & \longrightarrow & \mathbb{K} & \longrightarrow & \bigoplus_{n\geq 0} \mathbf{\Pi}_A[n_A] \\ 1 & \longmapsto & 1 & \longmapsto & \mathbf{1}_{K_A(\mathbf{\Pi}_A)} \end{array}$$

where $\mathbf{1}_{K_A(\mathbf{\Pi}_A)}$ is the empty set partition.

12.1.2. Coalgebra Structure:

The coproduct, Δ, is given by the following composition:

$$\begin{array}{ccccc} K_A(\mathbf{\Pi}_A) & \xrightarrow{K_A(\hat{\Delta})} & K_A(\mathbf{\Pi}_A \cdot \mathbf{\Pi}_A) & \xrightarrow{\mathrm{st}_S \otimes \mathrm{st}_T} & K_A(\mathbf{\Pi}_A) \cdot K_A(\mathbf{\Pi}_A) \\ \pi & \longmapsto & \pi|_S \otimes \pi|_T & \longmapsto & \mathrm{st}(\pi|_S) \otimes \mathrm{st}(\pi|_T) \end{array}$$

with $\mathrm{st}(\pi|_S)$ being as in Section 5.2.

The counit, ε, is given by the following composition:

$$\begin{array}{ccccc} K_A(\mathbf{\Pi}_A) & \longrightarrow & K_A(\mathbf{1}) & \longrightarrow & \mathbb{K} \\ \mathbf{1}_{K_A(\mathbf{\Pi}_A)} & \longmapsto & 1 & \longmapsto & 1 \end{array}$$

where $\mathbf{1}_{K_A(\mathbf{\Pi}_A)}$ is the empty set partition.

12.1.3. Antipode:

Recall, that in general K_A does not preserve the antipode; the antipode is only preserved by bistrong bilax monoidal functors.

Example 12.1.1.
Let $A = \mathbb{K}C_2$ and $n = 3$. Let $s+t = 3$ such that $s = 2$ and $t = 1$. Fix section maps $s'([2]_A) = \{1, 2\}$ and $s''([1]_A) = \{\overline{1}\}$; together, these determine the section map $s([3]_A) = \{1, 2, \overline{3}\}$. The following are examples of the product and coproduct on elements from the corresponding components.

$$\mu : \mathbf{\Pi}[\{1, 2\}] \otimes \mathbf{\Pi}[\{\overline{1}\}] \to \mathbf{\Pi}[\{1, \overline{2}, 3\}]$$
$$1|2 \otimes \overline{1} \mapsto 1|2|\overline{3}$$
$$12 \otimes \overline{1} \mapsto 12|\overline{3}$$
$$\Delta : \mathbf{\Pi}[\{1, \overline{2}, 3\}] \to \mathbf{\Pi}[\{1, 2\}] \otimes \mathbf{\Pi}[\{\overline{1}\}]$$
$$1\overline{2}|3 \mapsto 1|2 \otimes \overline{1}$$
$$1\overline{2}3 \mapsto 12 \otimes \overline{1}$$
$$13|\overline{2} \mapsto 12 \otimes \overline{1}$$

The following proposition relates the Hopf algebra of B-colored set partitions to the Hopf Algebra associated to the A-Hopf monoid of set partitions. But first, we need some notation. We have functions D and c given by

$$D : \mathbf{\Pi}[s([n])] \to \Pi[n]$$
$$\pi \mapsto D(\pi)$$

where $D(\pi) :=$ underlying set partition of $s([n])$ whose values are in $[n]$.

$$c : \mathbf{\Pi}[s([n])] \to B^n$$
$$\pi \mapsto c(\pi)$$

where $c(\pi) := \xi = (\xi_1, .., \xi_n) \in B^n$ where the elements of B color $D(\pi)$, in other words ξ_i is the color on i.

Example 12.1.2. Let $A = \mathbb{K}C_2$ and consider the section map given by $s([3]) = \{1, \overline{2}, 3\}$. Let $\pi = 13/\overline{2} \vdash s([3])$. Then $D(\pi) = 13/2$ and $c(\pi) = (1, \overline{1}, 1)$.

The *product* of two colors is given by concatenation, $\xi \cdot \mu = (\xi_1, ..., \xi_n, \mu_1, ..., \mu_n)$.

Proposition 12.1.3. *We have that*

$$K_A(\mathbf{\Pi}_A) \cong \tilde{\Pi}^{(B)}$$

as Hopf algebras. On a degree n component and for a fixed section map $s([n]_A)$, the isomorphism is given by

$$H_\pi \mapsto \sum_{\sigma \vdash [n]} (\sigma \wedge D(\pi))! m_{\sigma,\xi},$$

where $(\sigma \wedge D(\pi))!$ is as defined in Section.

Proof. When we consider Π_A as being trivially colored, this is just the Hopf monoid of Set Partitions, $\mathbf{\Pi}$. It's well known that $K(\mathbf{\Pi}) := \bigoplus_{n \geq 0} \Pi[n]$ is isomorphic to the ring of symmetric functions in noncommuting variables, Π, via the Hopf algebra isomorphism (on a degree n component) [31].

$$\varphi : H_\pi \mapsto \sum_{\sigma \vdash [n]} (\sigma \wedge \pi)! m_\sigma$$

Note that $\sum_\sigma (\sigma \wedge \pi)! m_\sigma := h_\pi$, the complete homogenous basis element for Π of degree n. Now to show that $\tilde{\varphi} : K_A(\mathbf{\Pi}_A) \to \tilde{\Pi}^{(B)}$ is a Hopf algebra isomorphism, where on a degree n piece and a fixed section map $s([n]_A)$, it is defined as

$$H_\pi \mapsto \sum_{\sigma \vdash [n]} (\sigma \wedge D(\pi))! m_{\sigma,\xi}$$

where $H_\pi \in \Pi[s([n])]$ for some section s. First to show that $\tilde{\varphi}$ is injective. Asumme $\tilde{\varphi}(H_\pi) = \tilde{\varphi}(H_\nu)$ then

$$\sum_{\sigma \vdash [n]} (\sigma \wedge D(\pi))! m_{\sigma,\xi} = \sum_{\tau \vdash [n]} (\tau \wedge D(\nu))! m_{\tau,\xi}.$$

Given a σ, then the corresponding term on the right hand side is when $\tau = \sigma$. If $c(\pi) \neq c(\nu)$ then by definition of a colored monomial, $m_{\sigma,c(\pi)} \neq m_{\sigma,c(\nu)}$ contradicting $\tilde{\varphi}(H_\pi) = \tilde{\varphi}(H_\nu)$. Thus it must also be that $c(\pi) = c(\nu)$. Now to show that $(\sigma \wedge D(\pi))! = (\sigma \wedge D(\nu))!$ implies $D(\pi) = D(\nu)$. If $\sigma = 12 \cdots n$, then $\sigma \wedge \eta = \eta$ for all $\eta \vdash [n]$. Thus $(\sigma \wedge D(\pi))! = (\sigma \wedge D(\nu))!$

implies $D(\pi)! = D(\nu)!$. This happens if and only if $D(\pi)$ and $D(\nu)$ have the same integer partition type. Now, if $\sigma = D(\pi)$, and $D(\pi) \neq D(\nu)$ but have same integer partition type we have:

$$(\sigma \wedge D(\pi))! = (D(\pi) \wedge D(\pi))! = D(\pi)!$$

and

$$(\sigma \wedge D(\nu))! = (D(\pi) \wedge D(\nu))! = 1|2|\cdots|n! = 1.$$

This implies that $D(\pi) = 1|2|\cdots|n$ or $D(\pi) = D(\nu)$. Since π was arbitrary, it must be that $D(\pi) = D(\nu)$. Hence $H_\pi = H_\nu$, yielding injectivity. We have $\tilde{\varphi}$ is surjective since $\sum(\sigma \wedge D(\pi))! m_{\sigma,\xi}$ form a basis for $\tilde{\Pi}^{(B)}$. Therefore $\tilde{\varphi}$ is an isomorphism of vector spaces.

Now to show that $\tilde{\varphi}$ is a Hopf morphism, it suffices to show that $\tilde{\varphi}$ is a bialgebra morphism Lemma 4.04 [32].
We must show that

$$\tilde{\varphi} \circ \mu = \mu \circ (\tilde{\varphi} \otimes \tilde{\varphi}).$$

Let $H_\pi \in \Pi[s([m])]$ and $H_\sigma \in \Pi[s([n])]$, then for the left hand side we have:

$$\tilde{\varphi} \circ \mu(H_\pi \otimes H_\nu) = \tilde{\varphi}\left(H_{\pi \sqcup \mathrm{cano}[\nu]}\right) = \sum_{\sigma} (\sigma \wedge D(\pi \sqcup \mathrm{cano}(\nu)))! m_{\sigma, c(\pi \sqcup \mathrm{cano}(\nu))}$$

For the right hand side of the equation:

$$\begin{aligned}
\mu \circ (\tilde{\varphi} \otimes \tilde{\varphi})(H_\pi \otimes H_\nu) &= \mu\left(\sum_{\sigma_1} (\sigma_1 \wedge D(\pi))! m_{\sigma_1, c(\pi)} \otimes \sum_{\sigma_2} (\sigma_2 \wedge D(\nu))! m_{\sigma_2, c(\nu)}\right) \\
&= \sum_{\substack{\tau \vdash [n+m] \\ \tau \wedge [n]|[m] = \sigma_1|\sigma_2}} (\sigma_1 \wedge D(\pi))! (\sigma_2 \wedge D(\nu))! m_{\tau, c(\pi)\cdot c(\nu)}
\end{aligned}$$

Because of the Hopf isomorphism φ, we only need to show that $c(\pi) \cdot c(\nu) = c(\pi \sqcup \mathrm{cano}(\nu))$. By definition $c(\pi) = \vec{b}_\pi = (b_{i_1}, \ldots b_{i_n})$ for some $\vec{b}_\pi \in B^n$ and $c(\nu) = \vec{b}_\nu = (b_{k_1}, \ldots, b_{k_m})$ fro some $\vec{b}_\nu \in B^m$. Then

$$c(\pi) \cdot c(\nu) = (b_{i_1}, \ldots b_{i_n}, b_{k_1}, \ldots, b_{k_m}) =: c(\pi \sqcup \mathrm{cano}(\nu))$$

Thus $\tilde{\varphi}$ is an algebra morphism. Next, to show that $\tilde{\varphi}$ is a coalgebra morphism, i.e.,

$$\Delta \circ \tilde{\varphi} = (\tilde{\varphi} \otimes \tilde{\varphi}) \circ \Delta$$

Again, let $H_\pi \in \Pi[s([n])]$. For the left hand side, we have:

$$\begin{aligned}
\Delta(\tilde{\varphi}(H_\pi)) &= \sum_{\sigma} (\sigma \wedge D(\pi))! \Delta(m_{\sigma, c(\pi)}) \\
&= \sum_{\sigma} (\sigma \wedge D(\pi))! \sum_{\mu \sqcup \nu = \sigma} m_{\mathrm{st}(\mu), c(\pi)|_\mu} \otimes m_{\mathrm{st}(\nu), c(\pi)|_\nu}
\end{aligned}$$

For the right hand side, we have $(\tilde{\varphi} \otimes \tilde{\varphi})(\Delta(H_\pi))$ is:

$$\begin{aligned}
&= (\tilde{\varphi} \otimes \tilde{\varphi})\left(\sum_{\mu \sqcup \nu} H_{\mathrm{st}(\mu)} \otimes H_{\mathrm{st}(\nu)}\right) \\
&= \sum_{\mu \sqcup \nu} \left(\sum_{\sigma_1} (\sigma_1 \wedge D(\mathrm{st}(\mu)))! m_{\sigma_1, c(\mathrm{st}(\mu))}\right) \otimes \left(\sum_{\sigma_2} (\sigma_2 \wedge D(\mathrm{st}(\nu))! m_{\sigma_2, c(\mathrm{st}(\nu))})\right)
\end{aligned}$$

Again, because of φ being an isomorphism it amounts to showing that $c(\mathrm{st}(\mu)) = c(\pi)|_\mu$ and $c(\mathrm{st}(\nu)) = c(\pi)|_\nu$ We have

$$c(\mathrm{st}(\mu)) = c(\mu) = c(\pi)|_\mu$$

Similarly for ν.

Therefore, $\tilde{\varphi}$ is an isomorphism of Hopf algebras.

□

12.2. $\overline{K}_A$ applied to $\mathbf{scf}_A(U)$

Here, we show the relationship between $\mathbf{scf}_A(U)$ and $\tilde{\Pi}^{(B)}$.

Proposition 12.2.1. *There is an isomorphism of Hopf algebras*

$$\overline{K}_A(\boldsymbol{scf}_A(U)) \cong \tilde{\Pi}^{(B)},$$

i.e., the S_n-coinvariants of the A-Hopf monoid of superclass functions on unitriangular matrices with entries in $\mathbb{F}_2$ is isomorphic to the Hopf algebra of symmetric functions in colored noncommuting variables, $\tilde{\Pi}^{(B)}$.

Proof. First, recall that Corollary 5.3.7 state $\mathbf{scf}(U) \cong \mathbf{L} \times \mathbf{\Pi}$ when our matrix entries are from $\mathbf{F}_2$. By Proposition 9.1.6 and Corollary 9.1.7, we have that $\mathcal{S}^A$ is a bilax bistrong monoidal functor and thus preserves Hopf monoids. Hence,

$$\mathbf{scf}_A(U) := \mathcal{S}^A(\mathbf{scf}(U)) \cong \mathcal{S}^A(\mathbf{L} \times \mathbf{\Pi}) := (\mathbf{L} \times \mathbf{\Pi})_A.$$

By Propositions 10.2.8 and Corollary 10.2.9, we have that $\overline{K}_A$ is a bilax bistrong functor; thus $\overline{K}_A(\mathbf{scf}_A(U)) \cong \overline{K}_A((\mathbf{L} \times \mathbf{\Pi})_A)$.By Theorem 10.3.2, we have that $\overline{K}_A((\mathbf{L} \times \mathbf{\Pi})_A) \cong K_A(\mathbf{\Pi}_A)$. By Proposition 12.1.3, $K_A(\mathbf{\Pi}_A) \cong \tilde{\Pi}^{(B)}$. Putting it all together, yields:

$$\overline{K}_A(\mathbf{scf}_A(U)) \cong \overline{K}_A((\mathbf{L} \times \mathbf{\Pi})_A) \cong K_A(\mathbf{\Pi}_A) \cong \tilde{\Pi}^{(B)}.$$

□

12.3. $\widetilde{K}_A$ applied to $\mathbf{L}_A \times \mathbf{\Pi}_A$

Consider $\mathbf{\Pi}_A$ and using Theorem 10.3.4, we get that

$$\widetilde{K}_A(\mathbf{L}_A \times \mathbf{\Pi}_A) \cong K_A(\mathbf{\Pi}_A) \cong \tilde{\Pi}^{(r)}.$$

12.4. A String of Relationships

To summarize: as seen above, for the Hopf algebra $\tilde{\Pi}^{(B)}$ we have shown that there at at least four A-Hopf monoids that can be associated to it; in general for a given Hopf algebra there could be many more A-Hopf monoids that can be associated to it. We focused on three Hopf monoids in the category of species to construct the A-Hopf monoids needed to give the following string of isomorphisms:

$$\overline{K}_A(\mathbf{scf}_A(U)) \cong \overline{K}_A((\mathbf{L} \times \mathbf{\Pi})_A) \cong K_A(\mathbf{\Pi}_A) \cong \widetilde{K}_A(\mathbf{L}_A \times \mathbf{\Pi}_A) \cong \tilde{\Pi}^{(B)}.$$

CHAPTER 13

B_r-Invariant Polynomials and C_r-Colored Set Partitions

The ring of symmetric functions, Λ, can be lifted to the ring of symmetric functions in noncommutative variables, Π, by essentially forgetting the commutativity property. In Π, there is an analogous theory to that of the ordinary symmetric functions. In [35], Rosas showed that there are analogues to the bases given monomial, elementary, primitive, complete homogeneous and schur functions which are now labelled by set partitions rather than integer partitions (also see [11]). She relates these two sets of bases via the canonical projection map $\rho : \mathbb{C}\langle\langle X\rangle\rangle \to \mathbb{C}[[X]]$ which lets the variables commute. Rosas also defines a right inverse to ρ called the lifting map.

In this chapter, we end by doing an analogous construction. We define a projection map from a quotient of the Hopf algebra of C_r-colored set partitions to the Hopf algebra of B_r-invariant functions, where $B_r := C_r \wr S_n$. As seen in Section 11.2, $\tilde{\Pi}^{(r)}$ is the Hopf algebra of C_r-colored set partitions associated to the $\mathbb{K}C_r$-Hopf monoid, $\mathbf{\Pi}_{\mathbb{K}C_r}$.

13.1. A Quotient of $\tilde{\Pi}^{(r)}$

First, recall $\tilde{\Pi}^{(r)}$ from Section 3.5. Let the variable set be denoted by $X = \{x_1, x_2, ...\}$. A basis is given by monomials indexed by colored set partitions $\{m_{\pi,\xi} \mid \pi \vdash [n], \xi \in C_r^n\}$, where

$$m_{\pi,\xi} := \sum w,$$

where w is the set of words $w = (x_{i_1}, \xi_1)\cdots(x_{i_n}, \xi_n)$ where $x_i = x_j$ if and only if i and j are in the same block of $\pi \vdash [n]$. For a colored variable, we will interchangeably use the notation (x_i, ξ_i) and x_{i,ξ_i}.

Remark 13.1.1.

- When $r = 1$, all partitions are trivially colored and we will drop the coloring from the notation, $m_{\pi,(1,\ldots,1)} = m_\pi$.

- When $r = 2$, colored variables will interchangeably be denoted as

$$(x_i, 1) = x_{i,1} = x_i$$
$$(x_i, -1) = x_{i,-1} = x_{\bar{i}}$$

Example 13.1.2. Recall examples of C_2-colored monomials:

- $m_{13/24,(1,\bar{1},1,1)} = x_1x_{\bar{2}}x_1x_2 + x_2x_{\bar{1}}x_1x_1 + x_1x_{\bar{3}}x_1x_3\cdots$

- $m_{12/3,(\bar{1},\bar{1},1)} = x_{\bar{1}}x_{\bar{1}}x_2 + x_{\bar{2}}x_{\bar{2}}x_1 + x_{\bar{1}}x_{\bar{1}}x_3\cdots$

- $m_{12/3,(1,\bar{1},1)} = x_1 x_{\bar{1}} x_2 + x_2 x_{\bar{2}} x_1 + \cdots$

We first look to see how B_r acts on $\tilde{\Pi}^{(r)}$.

PROPOSITION 13.1.3. *The action of B_r on $\tilde{\Pi}^{(r)}$ is given by:*

$$\begin{aligned}(\delta_1, ..., \delta_n, \sigma).x_{i,\xi_1} &= (\delta_1 \cdots \delta_n \otimes \mathrm{id})(1 \cdots 1 \otimes \sigma).x_{i,\xi_i} \\ &= (\delta_1 \cdots \delta_n \otimes \mathrm{id}).x_{\sigma(i),\xi_i} \\ &= x_{\sigma(i),\xi_i \cdot \delta_{\sigma(i)}}.\end{aligned} \tag{34}$$

PROOF. First to show that the identity element, $(1, ..., 1, \mathrm{id}) \in B_r$ acts as the identity

$$(1, ..., 1, \mathrm{id}).(x_i, \xi_i) = (x_{\mathrm{id}(i)}, \xi_i \cdot 1_{\mathrm{id}(i)}) = (x_i, \xi_i).$$

Now consider $(\delta_1, ..., \delta_n, \sigma), (\varepsilon_1, ..., \varepsilon_n, \tau) \in B_r$, then

$$\begin{aligned}(\delta_1, ..., \delta_n, \sigma).(\varepsilon_1, ..., \varepsilon_n, \tau).(x_i, \xi_i) &= (\delta_1, ..., \delta_n, \sigma).(x_{\tau(i)}, \xi_i \cdot \varepsilon_{\tau(i)}) \\ &= (x_{\sigma(\tau(i))}, \xi_i \cdot \varepsilon_{\tau(i)} \cdot \delta_{\sigma(\tau(i))}) \\ &= (x_{\sigma(\tau(i))}, \xi_i \cdot \varepsilon_{\sigma^{-1}(\sigma(\tau(i)))} \cdot \delta_{\sigma(\tau(i))}) \\ &= (\delta_1 \varepsilon_{\sigma^{-1}}, ..., \delta_n \varepsilon_{\sigma^{-1}(n)}, \sigma \circ \tau)(x_i, \xi_i) \\ &= ((\delta_1, ..., \delta_n, \sigma)(\varepsilon_1, ..., \varepsilon_n, \tau)).(x_i.\xi_i),\end{aligned}$$

thus B_r acts on $\tilde{\Pi}^{(r)}$.

□

For the remainder of this section, we only want to consider certain colorings on set partitions, which we call good colorings.

DEFINITION 13.1.4. We say that a *good coloring* on $\pi = \pi_1|\pi_2|\cdots|\pi_m \vdash [n]$ is a $\xi = (\xi_1, \xi_2, .., \xi_n) \in C_r^n$ such that $\xi_j = \xi_k$ if $j, k \in \pi_s$. We say that ξ is a *bad coloring* on π if it is not good.

EXAMPLE 13.1.5.
Consider the set partition $\pi = 13|2|45$ colored by C_2. An example of a good coloring on this partition is

$$(1, \bar{1}, 1, \bar{1}, \bar{1}).$$

Here, since 1,3 in the same block and we have that $\xi_1 = \xi_3$. Similarly, 4 and 5 are in the same block and $\xi_4 = \xi_5$.

Consider the set partition $\pi = 13|2|45$ colored by C_2. An example of a bad coloring on this partition is

$$(1, 1, \bar{1}, 1, 1).$$

This coloring is not good because 1 and 3 are in the same block, but $\xi_1 \neq \xi_3$.

EXAMPLE 13.1.6.
Consider the set partition $\pi = 12|3|45$ colored by C_2. Some good colorings on this partition include, but not limited to,

$$(11|1|11),\ (\bar{1}\bar{1}|1|11),\ (\bar{1}\bar{1}|1|\bar{1}\bar{1}),\ (11|\bar{1}|11).$$

There are 2^3 many ways to put a good coloring on this π.

Consider the set partition $\pi = 12|3|45$ colored by C_2. Some bad colorings on this partition include, but not limited to,

$$(\bar{1}1|1|11),\ (\bar{1}\bar{1}|1|\bar{1}1), (\bar{1}1|1|1\bar{1}).$$

REMARK 13.1.7. In general, if coloring by C_r there are $r^{\ell(\pi)}$ many ways to put a good coloring on π.

For $\pi \vdash [n]$, consider $a_\pi = \sum_{\xi \in C_r^n} m_{\pi,\xi}$ and the space spanned by such elements. Since we are ranging over all colorings on a given $\pi \vdash [n]$, it's easy to see that this polynomial is invariant under the action of B_r, as defined in Equation (34). Observe that we can break up each a_π into a sum ranging over good and bad colorings:

$$a_\pi = \sum_{\substack{\xi \\ \text{good color}}} m_{\pi,\xi} + \sum_{\substack{\xi \\ \text{bad color}}} m_{\pi,\xi}.$$

If we consider the portion of a_π labelled by good colorings, these are also invariant under the action of B_r (as we will see in Proposition 13.2.1). The space generated by such elements does not form a Hopf subalgebra. However, by quotienting out by the bad colorings on π we get a B_r invariant Hopf algebra.

PROPOSITION 13.1.8. *$I := \langle m_{\pi,\xi} \mid \xi$ bad color on $\pi\rangle$ ranging over all $\pi \vdash [n]$ is a bi-ideal; that is a a two-sided ideal and coideal, of $\tilde{\Pi}^{(r)}$.*

PROOF. Note that $I = \bigoplus_{n\geq 0} \langle m_{\pi,\xi} \mid \xi \text{ bad color on } \pi\rangle$ is graded. It suffices to show that each graded piece is a bi-ideal of the respective graded piece of $\tilde{\Pi}^{(r)} = \bigoplus_{n\geq 0}\langle m_{\pi,\delta} \mid \pi \vdash [n], \delta \in C_r^n\rangle$. To show this is a bi-ideal, we must show that it's a two sided ideal and coideal.
Fix an n and bad colors $\delta, \xi \in C_r^n$. Obviously, $\langle m_{\pi,\xi} \mid \xi \text{ bad color on } \pi\rangle$ is a subspace.
Now we must show that $\langle m_{\pi,\xi} \mid \xi \text{ bad color on } \pi\rangle$ is a two sided ideal. Let $\pi \vdash [n]$ and $\sigma \vdash [m]$, then

$$\mu(m_{\pi,\xi} \otimes m_{\sigma,\delta}) = \sum_{\substack{\nu \vdash [n+m] \\ \nu \wedge [n]|[m] = \pi|\sigma}} m_{\nu,\xi\cdot\delta}$$

where $\xi \cdot \delta = (\xi_1, ...\xi_n, \delta_1, ..., \delta_m)$. The join (greatest lower bound) is an intersection condition between all the blocks of ν and $[n]|[m]$. So a block of π must be contained entirely in a block of ν for all blocks of π. Since ξ was a bad coloring on π, i.e., $\xi_j \neq \xi_k$ for at least one block of π this block must be contained in an entire block of ν. Thus $\xi \cdot \delta$ is a bad coloring on ν. Note that this argument is regardless of the type of coloring of δ. Similarly, this argument works if δ had been the bad coloring instead of ξ. Thus $\langle m_{\pi,\xi} \mid \xi \text{ bad color on } \pi\rangle$ is a two sided ideal.

Finally, to show that I is a coideal. Let $\xi \in C_r^n$ be a bad color. Observe that the first instance of a bad coloring happens when $\pi \vdash [2]$, in particular $m_{12,(1,a^k)}, m_{12,(a^k,1)}$. Since

$\tilde{\Pi}^{(r)}$ is connected, i.e., $(\tilde{\Pi}^{(r)})_0 \cong \mathbb{K}$, we know that $\varepsilon(m_{\pi,\xi}) = 0$ for all $m_{\pi,\xi} \in \bigoplus_{n\geq 1} \tilde{\Pi}^{(r)}$ and $\langle m_{\pi,\xi} \mid \xi \text{ bad color on } \pi\rangle \subseteq \bigoplus_{n\geq 1} \tilde{\Pi}^{(r)}$ thus $\varepsilon(\langle m_{\pi,\xi} \mid \xi \text{ bad color on } \pi\rangle) = 0$.
Now,

$$\Delta(m_{\pi,\xi}) = \sum_{\mu \sqcup \nu = \pi} m_{\mathrm{st}(\mu)\xi|_\mu} \otimes m_{\mathrm{st}(\nu)\xi|_\nu}.$$

Since ξ is a bad coloring on π, then for at least one block, say π_s we have $\xi_j \neq \xi_k$ when $j,k \in \pi_s$. Both μ and ν consists of entire blocks of π whose disjoint union is π. Thus when we consider the restriction of the bad coloring ξ to μ and ν, one of them will be a bad coloring since $\pi_s \subseteq \mu$ or ν. When $\pi_s \subseteq \mu$, $m_{\mathrm{st}(\mu),\xi|_\mu} \in \langle m_{\pi,\xi} \mid \xi \text{ bad color on } \pi\rangle$ and $m_{\mathrm{st}(\nu),\xi|_\nu} \in \tilde{\Pi}^{(r)}$. Similarily, if $\pi_s \subseteq \nu$. Ranging over all decompositions $\mu \sqcup \nu = \pi$, yields

$$\Delta(m_{\pi,\xi}) \subseteq \tilde{\Pi}^{(r)} \otimes \langle m_{\pi,\xi} \mid \xi \text{ bad color on } \pi\rangle + \langle m_{\pi,\xi} \mid \xi \text{ bad color on } \pi\rangle \otimes \tilde{\Pi}^{(r)}.$$

Thus a coideal.
Therefore $\langle m_{\pi,\xi} \mid \xi \text{ bad color on } \pi\rangle$ is a bi-ideal.

□

COROLLARY 13.1.9. *$\tilde{\Pi}^{(r)}/I$ is a Hopf Algebra.*

PROOF. We have that $\tilde{\Pi}^{(r)}/I$ is a bialgebra since I is a bi-ideal. We also have that $\tilde{\Pi}^{(r)}/B$ is connected since quotienting out by degree two pieces and higher. Note that $\tilde{\Pi}^{(r)}/I$ is graded, $\tilde{\Pi}^{(r)}/I = \bigoplus_{n\geq 0}\langle m_{\pi,\xi} \mid \xi \text{ good color on } \pi \vdash [n]\rangle$. Therefore $\tilde{\Pi}^{(r)}/I$ is a Hopf Algebra with basis given by $\{m_{\pi,\xi} \mid \xi \text{ good color}\}$.

□

13.2. A B_r-Invariant Hopf subalgebra of $\tilde{\Pi}^{(r)}/I$

Let $\pi \vdash [n]$ and b_π denote the image of a_π under this quotient,

$$b_\pi := \sum_{\substack{\xi \\ \text{good color}}} m_{\pi,\xi} + I.$$

We will show that the subspace formed by the b_π's are invariant under the action of B_r, form a Hopf subalgebra, and under a push down map go to the basis elements of $\mathbb{C}\langle\langle \mathbf{x}\rangle\rangle^{\mathbf{B_r}}$ with action defined in subsection 13.3.

PROPOSITION 13.2.1. *b_π is invariant under the action of $B_r = C_r \wr S_n$.*

PROOF. Given $b_\pi = \sum_{\substack{\xi \\ \text{good color}}} m_{\pi,\xi}$. Let $(\delta_1, .., \delta_n, \sigma) \in B_r$. To show that b_π is invariant under the action defined in 34, we will first show that $(\delta_1, .., \delta_n, \sigma).m_{\pi,\xi} = m_{\pi,\chi}$ for some good coloring χ.

$$
\begin{aligned}
(\delta_1,..,\delta_n,\sigma).m_{\pi,\xi} &= \sum_{\substack{\xi \\ \text{good color}}} (\delta_1,..,\delta_n,\sigma).x_{i_1,\xi_1}\cdots x_{i_n,\xi_n} \\
&= \sum_{\substack{\xi \\ \text{good color}}} x_{\sigma(i_1),\xi_1\cdot\delta_{\sigma(i_1)}}\cdots x_{\sigma(i_n),\xi_n\cdot\delta_{\sigma(i_n)}} \\
&= \sum_{\substack{\chi \\ \text{good color}}} x_{\sigma(i_1),\chi_1}\cdots x_{\sigma(i_n),\chi_n} \\
&= m_{\pi,\chi} \\
&\in b_\pi
\end{aligned}
$$

First, observe that if we forget the coloring, $x_{\sigma(i_1)}\cdots x_{\sigma(i_n)}$ is a term of $m_\pi \in \Pi$, the ring of symmetric functions in noncommuting variables. Since by definition of $m_\pi \in \Pi$, $i_j = i_k \iff j,k$ in same block of π. This implies that $\sigma(i_j) = \sigma(i_k) \iff j,k$ in same block. Yielding:

$$\sum_{\substack{\xi \\ \text{good color}}} x_{\sigma(i_1),\xi_1\cdot\delta_{\sigma(i_1)}}\cdots x_{\sigma(i_n),\xi_n\cdot\delta_{\sigma(i_n)}} = m_{\pi,\xi\cdot\delta_\sigma}$$

Now to show that $(\xi_1\cdot\delta_{\sigma(i_1)},...,\xi_n\delta_{\sigma(i_n)})$ is a good coloring on π. From above, we showed that $\sigma(i_j) = \sigma(i_k) \iff j,k$ in same block of π. Thus further implies that $\delta_{\sigma(i_j)} = \delta_{\sigma(i_k)}$ whenever j,k in the same block, hence $\xi_j\cdot\delta_{\sigma(i_j)} = \xi_k\cdot\delta_{\sigma(i_k)}$ whenever j,k in same block of π. Thus $(\xi_1\cdot\delta_{\sigma(i_1)},...,\xi_n\delta_{\sigma(i_n)})$ is a good coloring on π, denote this tuple by χ, yielding the third equality.
As we range over all good colorings ξ on π, each new coloring after acting by $(\delta_1,...,\delta_n,\sigma)$ will be distinct. If not distinct, then for two distinct colors ξ and ξ' we would have

$$(\xi_1\cdot\delta_{\sigma(i_1)},...,\xi_n\delta_{\sigma(i_n)}) = (\xi'_1\cdot\delta_{\sigma(i_1)},...,\xi'_n\delta_{\sigma(i_n)}) \iff \xi_i = \xi'_i \ \forall i$$

after applying the action, but this contradicts $\xi \neq \xi'$. Thus we get every possible good coloring on π, therefore the b_π are invariant under the action of $C_r \wr S_n$.

$\square$

We wish to show that the subspace spanned by these b_π's will be a sub Hopf algebra. Before doing so, we need the following lemma and corollary which will be used to show that B is closed under the (co)product inherited from $\tilde{\Pi}^{(r)}/I$.

Lemma 13.2.2. *Given $\pi \vdash [n]$ and decomposition $\mu \sqcup \nu = \pi$. Let $\xi = \{\xi^1,\xi^2,...,\xi^m\}$ be the set of good colorings on π. Consider the restriction of ξ to μ, i.e., $\xi|_\mu := \{\xi^1|_\mu,...,\xi^m|_\mu\}$. The coloring $\xi^i|_\mu \in \xi|_\mu$ appears $r^{\ell(\pi)}/r^{\ell(\mu)}$ many times. That is, $r^{\ell(\pi)}/r^{\ell(\mu)}$ is the number of ways to extend a good coloring on μ to π.*

Proof. Let $\pi = \pi_1|\cdots|\pi_j$ then $\mu = \pi_{\alpha_1}|\cdots|\pi_{\alpha_r}$ for some $r \geq 0$ such that $r \in [1,j]$ and each π_{α_i} is a full block of π. Color the blocks of π indexed by μ with the coloring $\xi^i|_\mu$. Observe that there are $r^{\ell(\pi)}/r^{\ell(\mu)}$ many ways to color the remaining blocks of π. Hence, there are $r^{\ell(\pi)}/r^{\ell(\mu)}$ many good colorings on π that restrict to the good coloring $\xi^i|_\mu$ of μ. Thus, we have that $\xi^i|_\mu$ appears $r^{\ell(\pi)}/r^{\ell(\mu)}$ many times in the set $\xi|_\mu$.

$\square$

COROLLARY 13.2.3. *$\xi|_\mu$ is a multiset of all good colorings on μ, each appearing $r^{\ell(\pi)}/r^{\ell(\mu)}$ many times.*

PROOF. Since μ is a subset of full blocks of π, every good coloring on μ is a restriction of some good coloring on π. There are $r^{\ell(\mu)}$ many ways to put a good color on the blocks of π indexed from μ. Therefore, each good coloring on μ appears $r^{\ell(\pi)}/r^{\ell(\mu)}$ many times in the set $\xi|_\mu$.

□

PROPOSITION 13.2.4. *$B := \bigoplus_{n\geq 0} \langle b_\pi \mid \pi \vdash [n]\rangle$ is a B_r-invariant Hopf subalgebra of $\tilde{\Pi}^{(r)}/I$. For $b_\pi \in B_n$, $b_\sigma \in B_m$ the product is given by:*

$$\overline{\mu}(b_\pi \otimes b_\sigma) = \sum_{\substack{\nu \vdash [n+m] \\ \nu \wedge [n]|[m] = \pi|\sigma}} b_\nu.$$

For $b_\pi \in B_n$, the coproduct is given by:

$$\overline{\Delta}(b_\pi) = \sum_{\mu \sqcup \nu = \pi} b_{\mathrm{st}(\mu)} \otimes b_{\mathrm{st}(\nu)}.$$

PROOF. Invariance follows from Proposition 13.2.1. First, notice that it's easy to see that this is a graded subspace with the degree zero component being the field.Now to show that B is closed under the product and coproduct. In doing so, we show that the prodcut and coproduct are as defined above.
We have that

$$\begin{aligned}
\overline{\mu}(b_\pi \otimes b_\sigma) &= \overline{\mu}\Big(\sum_{\substack{\xi \\ \text{good color}}} m_{\pi,\xi} \otimes \sum_{\substack{\delta \\ \text{good color}}} m_{\sigma,\delta}\Big) \\
&= \sum_{\substack{\xi \\ \text{good color on } \pi}} \sum_{\substack{\delta \\ \text{good color on } \sigma}} \overline{\mu(m_{\pi,\xi} \otimes m_{\sigma,\delta})} \\
&= \sum_{\substack{\xi \\ \text{good color on } \pi}} \sum_{\substack{\delta \\ \text{good color on } \sigma}} \sum_{\substack{\nu \vdash [n+m] \\ \nu \wedge [n]|[m] = \pi|\sigma}} \overline{m_{\nu,\xi\cdot\delta}} \\
&= \sum_{\substack{\xi\cdot\delta \\ \text{good color on } \nu}} \sum_{\substack{\nu \vdash [n+m] \\ \nu \wedge [n]|[m] = \pi|\sigma}} m_{\nu,\xi\cdot\delta} \\
&= \sum b_\nu \in B_{n+m},
\end{aligned}$$

where ν is such that $\nu \vdash [n+m]$ and $\nu \wedge [n]|[m] = \pi|\sigma$. For the fourth equality: Note that in general, $\xi \cdot \delta$ is not necessarily a good coloring on ν, but since we are quotienting by monomials labeled by bad colorings, we have that $\xi \cdot \delta$ is a good coloring on ν. Moreover, as we range over all possible good colorings ξ and δ on π and σ respectively, we have that $\xi\cdot\delta$ is a complete list of good colorings on ν after quotienting. For if we had a good color $\chi = (\chi_1, .., \chi_{n+m})$ on ν; the intersection conditions on ν, i.e., $\nu \cap [n] = \pi$ and $\nu \cap \mathrm{st}([m]) = \mathrm{st}(\sigma)$ imply that the restrictions $(\chi_1, .., \chi_n)$ and $(\chi_{n+1}, ..., \chi_{n+m})$ are good colorings on π and σ respectively. Thus

$\chi = \xi \cdot \delta$ for some good colorings ξ and δ.

Finally, to show that B is closed with respect to the coproduct.

$$\begin{aligned}
\overline{\Delta}(b_\pi) &= \overline{\Delta}\Big(\sum_{\substack{\xi \\ \text{good color on } \pi}} m_{\pi,\xi}\Big) \\
&= \sum_{\substack{\xi \\ \text{good color on } \pi}} \overline{\Delta(m_{\pi,\xi})} \\
&= \sum_{\substack{\xi \\ \text{good color on } \pi}} \overline{\sum_{\mu \sqcup \nu = \pi} m_{\mathrm{st}(\mu),\xi|_\mu} \otimes m_{\mathrm{st}(\nu),\xi|_\nu}} \\
&= \sum_{\substack{\xi \\ \text{good color on } \pi}} \sum_{\mu \sqcup \nu = \pi} m_{\mathrm{st}(\mu),\xi|_\mu} \otimes m_{\mathrm{st}(\nu),\xi|_\nu} \\
&= \sum_{\mu \sqcup \nu = \pi} b_{\mathrm{st}(\mu)} \otimes b_{\mathrm{st}(\nu)} \\
&\in \bigoplus_{|\mu|+|\nu|=n} B_{|\mu|} \otimes B_{|\nu|}.
\end{aligned}$$

For the fourth equality: As we are range over all good colorings ξ on π, there will be no bad colorings that get killed off in the quotient. Let $\pi = \pi_1|\cdots|\pi_j$ then $\mu = \pi_{\alpha_1}|\cdots|\pi_{\alpha_r}$ for some $r \geq 0$ such that $r \in [1, j]$, and each π_{α_i} is a full block of π. Thus restricting to μ yields a good coloring on μ. Similarly, for $\xi|_\nu$. Hence there are no restrictions on ξ after taking the quotient.

For the fifth equality: Fix a decomposition $\mu \sqcup \nu = \pi$. Let $\xi = \{\xi^1, ..., \xi^m\}$ be the set of good colorings on $\pi = \pi_1|\cdots|\pi_j$. Note that the number of good colorings on π is

$$|\xi| = m = r^{\ell(\pi)}.$$

Restrict the set of good colorings on π to both μ and ν:

$$\xi|_\mu := \{\xi^1|_\mu, ..., \xi^m|_\mu\} \qquad \text{and} \qquad \xi|_\nu := \{\xi^1|_\nu, ..., \xi^m|_\nu\}.$$

From the fourth equality, we have that each element in $\xi|_\mu$ and $\xi|_\nu$ are good colorings on μ and ν respectively.

From Lemma 13.2.2 and Corollary 13.2.3, we have that $\xi|_\mu$ is a multiset of good colorings on μ, with each distinct coloring appearing $r^{\ell(\pi)}/r^{\ell(\mu)}$ many times. Similarly, $\xi|_\nu$ is a multiset of good colorings on ν, with each distinct coloring appearing $r^{\ell(\pi)}/r^{\ell(\nu)}$ many times.

Now, for this fixed $\mu \sqcup \nu = \pi$ and ranging over all good colorings on π (and suppressing obvious notation on right hand side), we get the following:

$$\sum_\xi m_{\mathrm{st}(\mu),\xi|_\mu} \otimes m_{\mathrm{st}(\nu),\xi|_\nu} = \xi^1|_\mu \otimes \xi^1|_\nu + \xi^2|_\mu \otimes \xi^2|_\nu + \cdots + \xi^m|_\mu \otimes \xi^m|_\nu.$$

From above, the good coloring $\xi^1|_\mu$ appears $r^{\ell(\pi)}/r^{\ell(\mu)}$ many times, thus we can group together appropriate terms:

$$\xi^1|_\mu \otimes (\xi^{i_1}|_\nu + \xi^{i_2}|_\nu + \cdots \xi^{i_s}|_\nu).$$

There are $r^{\ell(\pi)}/r^{\ell(\mu)}$ many terms in the sum to the right of the tensor product. Notice that

$$r^{\ell(\pi)}/r^{\ell(\mu)} = r^{\ell(\pi)-\ell(\mu)} = r^{\ell(\nu)} = \text{no. of good colorings on } \nu$$

Further, we have that each element in the sum is distinct. Let $\xi^i \neq \xi^j \in \xi$. If $\xi^i|_\nu = \xi^j|_\nu$, then we would have that $\xi^i = \xi^j \in \xi$ since $\xi^i|_\mu = \xi^j|_\mu = \xi^1|_\mu$. But this contradicts $\xi^i \neq \xi^j \in \xi$. Thus each coloring in the sum is distinct. Thus we have that:

$$\xi^1|_\mu \otimes (\xi^{i_1}|_\nu + \xi^{i_2}|_\nu + \cdots \xi^{i_s}|_\nu) = \xi^1|_\mu \otimes b_{\mathrm{st}(\nu)}$$

by definition of $b_{\mathrm{st}(\nu)}$.
Repeating this for each distinct coloring in $\xi|_\mu$, will result in

$$\xi^1|_\mu \otimes b_{\mathrm{st}(\nu)} + \cdots + \xi^{r^{\ell(\mu)}}|_\mu \otimes b_{\mathrm{st}(\nu)} = (\xi^1|_\mu + \cdots \xi^{r^{\ell(\mu)}}) \otimes b_{\mathrm{st}(\nu)} = b_{\mathrm{st}(\mu)} \otimes b_{\mathrm{st}(\nu)}.$$

Therefore as we range over all decompositions of π, we get the desired result that

$$\sum_\xi \sum_{\mu \sqcup \nu = \pi} m_{\mathrm{st}(\mu),\xi|_\mu} \otimes m_{\mathrm{st}(\nu),\xi|_\nu} = \sum_{\mu \sqcup \nu = \pi} b_{\mathrm{st}(\mu)} \otimes b_{\mathrm{st}(\nu)}.$$

$\square$

13.3. The ring of B_r-invariant functions in the noncommutative variables

Given an infinite noncommutative variable set $X = \{x_1, x_2, ..\}$, we can consider $\mathbb{C}\langle\langle x_1, x_2, x_3, ...\rangle\rangle = \mathbb{C}\langle\langle \mathbf{X}\rangle\rangle$ the associative algebra of formal power series in the noncommuting variables $\mathbf{x}$. In [12], an action of $B_2 = C_2 \wr S_n$ was defined on $\mathbb{C}\langle\langle \mathbf{x}\rangle\rangle$. A signed permutation, $\delta_1 \otimes \cdots \otimes \delta_n \otimes \sigma \in B_2$, sends a variable x_i to $\pm x_{\sigma(i)}$ where the sign in front of the variable is determined by the element $\delta_i \in C_2$.

We can extend this action to an action of $B_r := C_r \wr S_n$ in the following proposition. Recall, $C_r = \langle a \mid a^r = 1\rangle$ is the cyclic group of order r, and we can identify a with $\omega = e^{\frac{2\pi i}{r}}$, the primitive r^{th} root of unity.

Proposition 13.3.1. *For every $n \geq 0$, the action of B_r is given by*

$$(\delta_1, ..., \delta_n, \sigma).x_i = w^k x_{\sigma(i)}$$

where w^k is the r^{th} root of unity that corresponds to the element $\delta_{\sigma(i)} \in C_r$ and $\delta_{\sigma(i)} = a^k$ for some $k \in [1, r]$. When $i > n$, then x_i is fixed.

Proof. First to show that the identity element, $(1, ..., 1, \mathrm{id}) \in B_r$ acts as the identity. Note that $1 = a^r \in C_r$ which corresponds to the root of unity ω^r, thus for every variable x_i we have

$$(1, ..., 1, \mathrm{id}).x_i = \omega^r x_{\mathrm{id}(i)} = x_i.$$

Now let $(\delta_1, ..., \delta_n, \sigma), (\varepsilon_1, ..., \varepsilon_n, \tau) \in B_r$. Note that for all i, $\delta_i = a^{k_i} \in C_r$ for some $k_i \in [1, r]$. This corresponds to a r^{th} root of unity, ω^{k_i}. For ease of computation, I will denote both by the element a^{k_i} and ω^{k_i} by δ_i and will specify which if not clear by context.

$$\begin{aligned}((\delta_1,...,\delta_n,\sigma)(\varepsilon_1,...,\varepsilon_n,\tau)).x_i &= (\delta_1\varepsilon_{\sigma^{-1}(1)},...,\delta_n\varepsilon_{\sigma^{-1}(n)},\sigma\circ\tau) \\ &= \delta_{\sigma(\tau(i))}\varepsilon_{\sigma^{-1}(\sigma(\tau(i)))}x_{\sigma(\tau(i))} \\ &= \delta_{\sigma(\tau(i))}\varepsilon_{\tau(i)}x_{\sigma(\tau(i))} \\ &= (\delta_1,...,\delta_n,\sigma).\varepsilon_{\tau(i)}x_{\tau(i)} \\ &= (\delta_1,...,\delta_n,\sigma).((\varepsilon_1,...,\varepsilon_n,\tau).x_i).\end{aligned}$$

Thus this defines an action of B_r on $\mathbb{C}\langle\langle X\rangle\rangle$.

□

Let $B_r^{(\infty)} := \cup_{n\geq 0} C_r \wr S_n$. Because B_r acts on $\mathbb{C}\langle\langle X\rangle\rangle$ for all n, we have that $B_r^{(\infty)}$ acts on $\mathbb{C}\langle\langle X\rangle\rangle$.

Definition 13.3.2. (Brlek [12])The *ring of B_r-invariant functions in the noncommutative variables X with coefficients in $\mathbb{C}$*, denoted $\mathbb{C}\langle\langle X\rangle\rangle^{B_r}$, is the $B_r^{(\infty)}$-invariant subalgebra of $\mathbb{C}\langle\langle X\rangle\rangle$ consisting of elements of bounded degree, i.e.,
$\mathbb{C}\langle\langle X\rangle\rangle^{B_r} :=$

$$\{f \in \mathbb{C}\langle\langle X\rangle\rangle \mid (\delta_1,...,\delta_n,\sigma).f = f \text{ for all } \delta_1,...,\delta_n,\sigma \in C_r\wr S_n,\ \ \deg(f)<\infty\}.$$

Now, consider $\{m_\pi \mid r\big||\pi_i|\ \forall \text{ blocks of } \pi\}$ where m_π is defined as usual, i.e., $m_\pi = \sum x_{i_1}x_{i_2}\cdots x_{i_n}$ where $i_j = i_k$ iff j,k in same block of π. In the following proposition, we show that these form a basis for the $\mathbb{C}\langle\langle X\rangle\rangle_r^B$.

Proposition 13.3.3. *$\{m_\pi \mid r\big||\pi_i|\ \forall$ blocks of $\pi\}$ is a basis for $\mathbb{C}\langle\langle X\rangle\rangle^{B_r}$.*

Proof. First to show that $\{m_\pi \mid r\big||\pi_i|\ \forall \text{ blocks of } \pi\}$ is a spanning set for $\mathbb{C}\langle\langle X\rangle\rangle^{B_r}$. Consider $f = \sum\limits_{x_{i_k}\in\mathbf{x}} c_\pi x_{i_1}\cdots x_{i_n}$, where $c_\pi\in\mathbb{C}$ and π denotes the set partition that corresponds to its respective monomial. We wish to show that f can be written as a sum of the m_π's such that $r\big||\pi_i|$. Since f was chosen to be invariant under B_r, it suffices to look at the action on f of the following elements that generate B_r:

$$\{(1,..,\delta_t,..1,\mathrm{id}) \mid 1\text{'s everywhere but in position } t,\ \delta_t\in C_r\} \sqcup \{(1,...,1,\sigma)\mid \sigma\in S_n\}$$

First, from [35] the polynomials invariant under the action of S_n can be written as a sum of the monomial basis m_π (with no restrictions on π).

Now to see how $(1,..,\delta_t,..1,\mathrm{id})$ acts on a term, $c_\pi x_{i_1}\cdots x_{i_n}$ in f.
We have:

- If $i_j\neq t$ for all j, we have that the monomial is already invariant under $(1,..,\delta_t,..1,\mathrm{id})$ since we are not picking up any power of a root of unity:
$$(1,..,\delta_t,..1,\mathrm{id}).c_\pi x_{i_1}\cdots x_{i_n} = c_\pi x_{i_1}\cdots x_{i_n}.$$
- If $i_j = t$ for at least one j, then some block of π, say π_s corresponds to t, specifically the block with values of these j's such that $i_j=t$. Thus yielding:
$$(1,..,\delta_t,..1,\mathrm{id}).c_\pi x_{i_1}\cdots x_{i_n} = c_\pi\omega^{|\pi_s|}x_{i_1}\cdots x_{i_n}.$$
Solving yields $c_\pi\omega^{|\pi_s|} = c_\pi \implies \omega^{|\pi_s|}=1$ which happens if and only if $r\big||\pi_s|$.

As we range over all positions t and values δ_t, we get that the monomials invariant under $(1, .., \delta_t, ..1, \mathrm{id})$ are the monomials indexed by π such that $r \Big| |\pi_i|$. Therefore $f = \sum c_\pi m_\pi$.

We have that for each π such that $r \Big| |\pi_i|$, the m_π are invariant under the action of B_r. Cleary the m_π invariant under all elements of the form $(1, ..., 1, \sigma)$, since these m_π are a subset of the monomial basis from Π. Now to show invariant under elements of the form $(1, .., \delta_t, ..1, \mathrm{id})$.

$$\begin{aligned}
(1, .., \delta_t, ..1, \mathrm{id}). \sum x_{i_1} \cdots x_{i_n} &= \sum \omega_t^{|\pi_s|} x_{i_1} \cdots x_{i_n} \\
&= \sum \omega_t^{ra} x_{i_1} \cdots x_{i_n} \\
&= \sum x_{i_1} \cdots x_{i_n} \\
&= m_\pi.
\end{aligned}$$

Finally, to show that $\{m_\pi \mid r \Big| |\pi_i| \ \forall \text{ blocks of } \pi\}$ is linearly independent.
Assume $\sum c_\pi m_\pi = 0$. Each monomial is indexed by set partitions, and it can only appear in exactly one m_π. In order for this term to vanish it must be that $c_\pi = 0$. Thus $\{m_\pi \mid r \Big| |\pi_i| \ \forall i\}$ linearly independent.

□

REMARK 13.3.4. When $r = 2$, we recover the polynomials invariant under the hyperoctrahedral group as in [**12**].

13.3.1. Push Down Map

Now consider the algebra morphism

$$\rho : \tilde{\Pi}^{(r)}/I \to \mathbb{C}\langle\langle X \rangle\rangle$$

given by

$$x_{i,a^k} \mapsto \omega^k x_i,$$

where $\omega^k = e^{\frac{2\pi i k}{r}}$, i.e., the r^{th} root of unity corresponding to a^k, and a is the generator of $C_r = \langle a \mid a^r = 1\rangle$.

The goal is to show that under this push down map, the basis of $\tilde{\Pi}^{(r)}/I$ gets sent to the basis of $\mathbb{C}\langle\langle X \rangle\rangle^{B_r}$, up to some constant. In order to do so, we need the following lemmas. First, we will see what happens to the $m_{\pi,\xi}$ under this push down map and then extend to the b_π.

LEMMA 13.3.5. *Let $\pi = 12 \cdots n \vdash [n]$, where $n = ra + b$. The image of $m_{\pi,\xi}$ where $\xi = (\xi_1, .., \xi_n)$ is a good coloring on π under ρ is*

$$\rho(m_{\pi,\xi}) = \begin{cases} m_\pi & \textit{if } r|n \\ (w^k)^b m_\pi & \textit{if } n = ra + b \end{cases}$$

PROOF. First note that since $\xi = (\xi_1, .., \xi_n)$ is a good coloring on $\pi = 12\cdots n$, we have that $\xi_i = \xi_j$ for all i, j. Let $\xi = (a^k, a^k, .., a^k)$ for some $k \in [1, r]$ denote said good coloring.

$$\begin{aligned}
\rho(m_{\pi,\xi}) &= \sum \rho(x_{i_1,a^k})\cdots\rho(x_{i_n,a^k}) \\
&= (w^k)^n \sum x_{i_1}\cdots x_{i_n} \\
&= (w^k)^{ra+b} m_\pi \\
&= w^{kb} m_\pi.
\end{aligned}$$

If $r|n$, then $b = 0$, then $\rho(m_{\pi,\xi}) = m_\pi$ as desired.
Ir $r \nmid n$, then $b \neq 0$, then $\rho(m_{\pi,\xi}) = w^{kb} m_\pi$ as desired.

□

We will need the following lemma to help determine what the image of a b_π will be under ρ.

LEMMA 13.3.6. *Let $n = rs + b$ for some $s \in \mathbb{Z}$ and $b \in [0, 1 - r]$, we have the homomorphism*

$$\begin{aligned}
f_b : C_r &\to C_r \\
a &\mapsto a^b
\end{aligned}$$

For different values of b, we have

$$\operatorname{Im}(f_b) \cong \begin{cases} \{1\} & \text{if } b = 0 \\ C_r & \text{if } gcd(b, r) = 1 \\ C_{\frac{r}{gcd(b,r)}} & \text{if } \gcd(b, r) \neq 1 \end{cases}.$$

PROOF. By the First Isomorphism Theorem, we have

$$C_r / \ker(f_b) \cong \operatorname{Im}(f_b).$$

Furthermore, $Im(f_b)$ is a cyclic subgroup of C_r since every subgroup of a cyclic subgroup is cyclic. We want to determine what $Im(f_b)$ is for different values of b. In order to do so, we must first determine what the kernel is.

$$\begin{aligned}
\ker(f_b) &= \{a^k \in C_r \mid f_b(a^k) = 1\} \\
&= \{a^k \in C_r \mid a^{bk} = 1\} \\
&= \{a^k \in C_r \mid bk \mod r \equiv 0\} \\
&= \{a^k \in C_r \mid bk = rs \text{ for some } s \in \mathbb{Z}\}.
\end{aligned}$$

Consider the prime factorization of r, i.e., $r = \prod_{i=1}^m p_i^{\varepsilon_i}$ where for all i, p_i is prime with multiplicity ε_i and looking at different values of b yields:

(1) $b = 0$:
If $b = 0$, then for all $k \in [1, r]$ we have that $bk \mod r \equiv 0$. Thus $\ker(f_0) = C_r$.

(2) $b \neq 0$:
Say $\gcd(b, r) = \prod\limits_{i=1}^{m} p_i^{\delta_i}$ where δ_i may possibly be zero and at least one $\delta_i \neq \varepsilon_i$. So there exists $x, y \in \mathbb{Z}$ such that

$$b = \prod_{i=1}^{m} p_i^{\delta_i} x,$$

and

$$r = \prod_{i=1}^{m} p_i^{\delta_i} y.$$

Since $b, r \geq 0$, we have that $x, y \geq 0$. We also have that $y = \prod_{i=1}^{m} p_i^{\varepsilon_i - \delta_i}$. Solving for k in $bk = rs$ yields

$$k = \frac{rs}{b} = \frac{\prod_{i=1}^{m} p_i^{\delta_i} \prod_{i=1}^{m} p_i^{\varepsilon_i - \delta_i} s}{\prod_{i=1}^{m} p_i^{\delta_i} x} = \frac{\prod_{i=1}^{m} p_i^{\varepsilon_i - \delta_i} s}{x}.$$

Observe that x and $\prod_{i=1}^{m} p_i^{\varepsilon_i - \delta_i}$ are relatively prime. For if they were not, then the prime factorization of x would contain at least one of the p_i's in $\prod_{i=1}^{m} p_i^{\varepsilon_i - \delta_i}$, call this prime q. This would imply that $\prod_{i=1}^{m} p_i^{\delta_i}$ was not the gcd, but instead $\prod_{i=1}^{m} p_i^{\delta_i} \times q$. Since $k \in [1, r]$, we have that x must divide s, i.e., for some $s' \in \mathbb{Z}_{\geq 0}$ $s = xs'$. So

$$k = \frac{\prod_{i=1}^{m} p_i^{\varepsilon_i - \delta_i} s}{x} = \frac{\prod_{i=1}^{m} p_i^{\varepsilon_i - \delta_i} xs'}{x} = \prod_{i=1}^{m} p_i^{\varepsilon_i - \delta_i} s'.$$

Note that once $s' = \gcd(b, r)$, we have that $k = r$ and $a^r = 1$ which is already in the kernel. If $s' = \gcd(b, r) + j$ for some $j < \gcd(b, r)$, we have that

$$a^k = a^{\prod_{i=1}^{m} p_i^{\varepsilon_i - \delta_i} s'} = a^{\prod_{i=1}^{m} p_i^{\varepsilon_i - \delta_i} (\gcd(b,r) + j)} = a^{\prod_{i=1}^{m} p_i^{\varepsilon_i - \delta_i} (\gcd(b,r))} a^j = a^r a^j = a^j,$$

thus $s' \in [1, \gcd(b, r)]$.Therefore,

$$\ker(f_b) \cong \{a^k \in C_r \mid k = \prod_{i=1}^{m} p_i^{\varepsilon_i - \delta_i} s' \text{ for } s' = 1, 2, ..., \gcd(b, r)\}.$$

- When $\gcd(b, r) = 1$, then $\ker(f_b) \cong \{1\}$.
- When $\gcd(b, r) \neq 1$, then $\ker(f_b) \cong C_{\gcd(b,r)}$.

Therefore giving us our desired result:

$$\operatorname{Im}(f_b) \cong \begin{cases} \{1\} & \text{if } b = 0 \\ C_r & \text{if } gcd(b, r) = 1 \\ C_{\frac{r}{gcd(b,r)}} & \text{if } \gcd(b, r) \neq 1 \end{cases}.$$

□

LEMMA 13.3.7. *Let $\pi = 12 \cdots n$, the set partition consisting of a single block of n, then*

$$\rho(b_\pi) = \begin{cases} rm_\pi & \text{if } r \big| n \\ 0 & \text{otherwise} \end{cases}.$$

Proof. $\pi = 12\cdots n$ colored by C_r, so any good coloring will have form $(\underbrace{a^k, ..., a^k}_{n \text{ many times}})$ for some $k \in [1, r]$. Recall, $r = \prod_{i=1}^{m} p_i^{\varepsilon_i}$ where p_i a prime. We have that $n = rs + b$ for some $s \in \mathbb{Z}$ and $b \in [0, r-1]$. Observe that

$$\begin{aligned} \rho(b_\pi) &= \sum_{i=1}^{r} \rho(m_{\pi,(a^k)^n}) \\ &= \sum_{i=1}^{r} \omega^{k(rs+b)} m_\pi \\ &= \sum_{i=1}^{r} \omega^{kb} m_\pi \\ &= (\omega^b + \omega^{2b} + \cdots + \omega^{rb}) m_\pi. \end{aligned} \tag{35}$$

Using the isomorphism between C_r and the r^{th} roots of unity, $\langle \omega \mid \omega^r = 1 \rangle$ and Lemma 13.3.6, we get that $(\omega^b + \omega^{2b} + \cdots + \omega^{rb})$ is the following:

- If b=0, then $\omega^{bk} = 1$ for all k since $Im(f_0) \cong \{1\}$. Thus $\sum_{k=1}^{r} \omega^{kb} = r$.
- If $\gcd(b, r) = 1$, then $Im(f_b) \cong C_r \cong \langle \omega \mid \omega^r = 1 \rangle$, thus $\sum_{k=1}^{r} \omega^{kb} = 0$.
- If $\gcd(b, r) \neq 1$, then $Im(f_b) \cong C_{\frac{r}{gcd(b,r)}} \cong \langle \omega \mid \omega^{\frac{r}{gcd(b,r)}} = 1 \rangle$, thus

$$\sum_{k=1}^{r} \omega^{kb} = \gcd(b, r)(\omega^b + \omega^{2b} + \cdots + \omega^{\frac{r}{gcd(b,r)} b}) = 0$$

Therefore

$$\rho(b_\pi) = \begin{cases} rm_\pi & \text{if } r | n \\ 0 & \text{otherwise} \end{cases}.$$

$\square$

Now we can prove the desired Proposition.

Proposition 13.3.8. *For all $\pi \vdash [n]$, we have:*

$$\rho(b_\pi) = \begin{cases} r^{\ell(\pi)} m_\pi & \textit{if } r | n_i \ \forall i \textit{ where } n_i = |\pi_i| \\ 0 & \textit{otherwise} \end{cases}.$$

PROOF. Observe that:

$$\begin{aligned}
\rho(b_{\pi_1|\cdots|\pi_t}) &= \sum_{\substack{\xi \\ \text{good color on } \pi}} \rho(m_{\pi_1|\cdots|\pi_t,\xi}) \\
&= \sum_{\substack{\xi \\ \text{good color on } \pi}} \prod_{i=1}^{t} \omega^{(k_i b_i)} m_{\pi_1|\cdots|\pi_t} \\
&= \sum_{\substack{\xi \\ \text{good color on } \pi}} \omega^{\sum_{i=1}^{t}(k_i b_i)} m_{\pi_1|\cdots|\pi_t}
\end{aligned}$$

Note that for a given good coloring ξ on π, we write $\xi = (\xi_1, \xi_2, ..., \xi_t)$ where $\xi_i = (a^{k_i}, ..., a^{k_i}) \in C_r^{n_i}$ is a good coloring on π_i. We range over each coloring, ξ_i, individually, while holding the other colors constant. The coloring in the sum that we are ranging over will be in blue, while the ones in black are the ones being held constant. For example, as we range over all the possible good colorings ξ_1 on π_1, the k_1's range through $[1, r]$ because we are getting a different k_1 for each ξ_1, yielding the sum $\beta_1 = \omega^{b_1} + \omega^{2b_1} + \cdots + \omega^{rb_1}$. From Lemma **??** and 13.3.6, we get that

$$\begin{aligned}
\sum_{\substack{(\xi_1,..,\xi_t) \\ \text{good color on } \pi}} \omega^{\sum_{i=1}^{t}(k_i b_i)} m_{\pi_1|\cdots|\pi_t} &= \sum_{\substack{({\color{blue}\xi_1},..,\xi_t) \\ \text{good color on } \pi_1|\cdots|\pi_t}} \omega^{{\color{blue}k_1} b_1} \omega^{\sum_{i=2}^{t} k_i b_i} m_{\pi_1|\cdots|\pi_t} \\
&= \sum_{\substack{(\xi_2,..,\xi_t) \\ \text{good color on } \pi_2|\cdots|\pi_t}} (\omega^{b_1} + \cdots + \omega^{rb_1}) \omega^{\sum_{i=2}^{t} k_i b_i} m_{\pi_1|\cdots|\pi_t} \\
&= \sum_{\substack{({\color{blue}\xi_2},..,\xi_t) \\ \text{good color on } \pi_2|\cdots|\pi_t}} \beta_1 \cdot \omega^{{\color{blue}k_2} b_2} \omega^{\sum_{i=3}^{t} k_i b_i} m_{\pi_1|\cdots|\pi_t} \\
&= \sum_{\substack{({\color{blue}\xi_3},..,\xi_t) \\ \text{good color on } \pi_3|\cdots|\pi_t}} (\beta_1 \cdot \beta_2) \omega^{{\color{blue}k_3} b_3} \omega^{\sum_{i=4}^{t} k_i b_i} m_{\pi_1|\cdots|\pi_t} \\
&\ \ \vdots \\
&= (\beta_1 \cdot \beta_2 \cdots \beta_t) m_{\pi_1|\cdots|\pi_t}.
\end{aligned}$$

Finally, by Lemma 13.3.7 we have that $\beta_i = r$ if $r \Big| n$, otherwise zero, thus we get:

$$\rho(b_\pi) = \begin{cases} r^{\ell(\pi)} m_\pi & \text{if } r \Big| |\pi_i| \ \forall i \\ 0 & \text{otherwise} \end{cases},$$

as desired.

$\square$

PROPOSITION 13.3.9. *$\overline{\rho}$ is a B_r-module homomorphism.*

PROOF. We must show that for all $(\delta_1 \otimes \cdots \otimes \delta_n \otimes \sigma) \in B_r := C_r \wr S_n$ and $b_\pi \in B$, we have that $\rho((\delta_1, ..., \delta_n, \sigma).b_\pi) = (\delta_1, ..., \delta_n, \sigma).\rho(b_\pi)$. From Proposition 13.2.1 and the definition of $m_\pi \in \mathbb{C}\langle\langle X\rangle\rangle^{B_r}$, we have:

$$\begin{aligned}
\rho((\delta_1, ..., \delta_n, \sigma).b_\pi) &= \rho(b_\pi) \\
&= r^{\ell(\pi)} m_\pi \\
&= r^{\ell(\pi)} (\delta_1, ..., \delta_n, \sigma).m_\pi \\
&= (\delta_1, ..., \delta_n, \sigma).\rho(b_\pi)
\end{aligned}$$

as desired.

$\square$

We can also define a one sided right inverse, $\tilde{\rho}$, to ρ. Define a *lifting map*

$$\tilde{\rho} : \mathbb{C}\langle\langle X\rangle\rangle^{B_r} \to B := \bigoplus_{n \geq 0} \langle b_\pi \rangle$$

by linearly extending

$$\tilde{\rho}(m_\pi) = \frac{1}{r^{\ell(\pi)}} b_\pi.$$

PROPOSITION 13.3.10. *The map $\rho\tilde{\rho}$ is the identity map on $\mathbb{C}\langle\langle \mathbf{x}\rangle\rangle^{B_r}$.*

PROOF.

$$\rho(\tilde{\rho}(m_\pi)) = \frac{1}{r^{\ell(\pi)}} \rho(b_\pi) = \left(\frac{1}{r^{\ell(\pi)}}\right) r^{\ell(\pi)} m_\pi = m_\pi.$$

$\square$

REMARK 13.3.11. Observe that $\tilde{\rho}$ is not a left inverse to ρ. This is because in $\mathbb{C}\langle\langle X\rangle\rangle^{B_r}$, the set partitions that label the basis elements are restricted to having form $r\Big||\pi_i|$ for all blocks π_i and there are no restrictions on the type of set partitions allowed in B.

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