COMPRESSION OF ORIENTABLE 3-MANIFOLD TRIANGULATIONS AND PACHNER PATHS

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COMPRESSION OF ORIENTABLE 3-MANIFOLD
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Abstract: Three dimensional triangulations can be described by giving a set of gluing maps between faces of tetrahedra (subject to some mild constraints). While this is a natural way to describe triangulations, it becomes computationally expensive to recognize when two triangulations are isomorphic. Here isomorphic triangulations are equivalent up to relabelling. To solve this problem, Burton created an isomorphism signature, which associates a string canonically to a triangulation that is shared by all triangulations isomorphic to it. However, this representative labelling never corresponds to an oriented triangulation. In computational topology, it is often important to deal with oriented triangulations if possible so we present a similar encoding for orientable 3-manifolds known as an oriented isomorphism signature that will always encode an oriented triangulation. We also present an encoding scheme for describing a path in the Pachner graph, an object for relating all triangulations of a fixed 3-manifold, as a string of printable characters that can be appended to the end of an oriented isomorphism signature. This allows us to easily store and describe how one isomorphism class of triangulations can be transformed into another via a series of local operations without losing any topological data.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

INTRODUCTION

1.1 Outline

For any given 3-manifold $\mathcal{M}$, there are infinitely many triangulations that describe the manifold. These triangulations can be partitioned into isomorphism classes where two triangulations are isomorphic if one can be achieved by relabelling the tetrahedra and vertices of the other. Often times when working in disciplines such as computational knot theory and low-dimensional topology it is useful to work with isomorphism classes of triangulations rather than with the triangulations themselves. In particular, this approach is helpful in the *recognition problem* where we are asked if a given triangulation $\mathcal{T}$ triangulates $\mathcal{M}$.

To aid in answering this question, Burton introduces an isomorphism signature. This is a printable string of ASCII characters that encodes the complete gluing data for a specific triangulation that is used as a representative for the entire isomorphism class. The construction of this isomorphism signature is outlined in Section 2. It is easy to compute and because it is a string it allows for fast insertion and look-up in sorting algorithms [1]. Burton's isomorphism signature is only defined for connected triangulations so in Section 2.1 we introduce a natural extension of this encoding scheme for disconnected triangulations.

Burton's isomorphism signature always encodes the gluing data for a non-oriented triangulation even if the manifold it describes is orientable. When dealing with an orientable 3-manifold $\mathcal{M}$ it is often useful to work with an oriented triangulation that describes that manifold. To this end, we introduce an oriented isomorphism signature which encodes the gluing data for an oriented triangulation that is used as a representative for all the triangulations in its isomorphism class. A detailed explanation of what it means for a triangulation to be oriented, what it means for a 3-manifold to be orientable, as well as the construction of this oriented isomorphism signature are outlined in Section 3.

In Section 4 we describe the relationship between isomorphism classes of a 3-manifold $\mathcal{M}$. Starting with a triangulation in one isomorphism class we can perform a sequence of local operations to obtain a triangulation of a different isomorphism class in a way that preserves topological information. These operations are known as Pachner moves and the relationship between isomorphism classes of triangulations of $\mathcal{M}$ via these operations can be represented in the Pachner graph of $\mathcal{M}$ where each node of the graph represents an isomorphism class and each arc in the graph represents a Pachner move. In this section, we focus on the Pachner graphs of orientable 3-manifolds and use the oriented isomorphism signature to describe each isomorphism class in the graph. We then construct an encoding scheme that describes a directed path through the Pachner graph as a string of printable ASCII characters. This string is appended to the end of the oriented isomorphism signature that represents the base node of the path.

The key contributions of this paper are the introduction of the oriented isomorphism signature and the directed Pachner path encoding. Together these compression schemes provide a new way for computational topologists to study and interact with triangulations of orientable manifolds in a way that brings their orientability to the forefront of the representation.

CHAPTER II

THE ISOMORPHISM SIGNATURE

2.1 Background

We begin by giving a faithful summary of [1] making only slight departures for the sake of clarity. Throughout our conversation we will assume that a given triangulation $\mathcal{T}$ is connected. We will briefly address disconnected triangulations in Section 2.1.

A triangulation of a 3-manifold is a set of tetrahedra usually glued to one another via a set of face pairings. These face pairings can glue the faces of distinct tetrahedra together or can glue one face of a tetrahedron onto another face of the same tetrahedron. If the latter occurs anywhere in our triangulation we call it a *pseudo-triangulation*. Given a triangulation, a set of tetrahedron vertices that are identified with one another via our face pairings is referred to as a *vertex of the triangulation*. A *face of the triangulation* and *edge of the triangulation* are similarly defined.

Given a triangulation with n tetrahedra, there are $n!24^n$ possible labellings. This factorial growth in the number of labellings makes it difficult to compare any two given triangulations to see if they are triangulations of the same manifold. In order to reduce the computational complexity of this problem we will restrict ourselves to looking at a certain kind of labelling called a canonical labelling.

Definition 2.1.1 (Burton's canonical labellings) *Given a labelling of a connected triangulation of size n, let $A_{t,f}$ denote the tetrahedron glued to face f of tetrahedron t (so that $A_{t,f} \in \{0, \cdots, n-1, \partial\}$ for all $t = 0, \cdots, n-1$ and $f = 0, ..., 3$). If face f of tetrahedron t is a boundary component then we say $A_{t,f} = \partial$. The labelling is canonical if, when we write out the sequence $A_{0,0}, A_{0,1}, A_{0,2}, A_{0,3}, A_{1,0}, \cdots, A_{n-1,3}$, the following properties hold:*

1. *For each $1 \leq i < j$, tetrahedron i first appears before tetrahedron j first appears.*
2. *For each $i \geq 1$, suppose tetrahedron i first appears as the entry $A_{t,f} = i$. Then the corresponding gluing uses the identity map: face f of tetrahedron t is glued to face f of tetrahedron i so that vertex v of tetrahedron t maps to vertex v of tetrahedron i for each $v \neq f$.*

The canonical labelling scheme above depends only on our choice of tetrahedron 0 and its vertices. Once these choices have been made, property 1 and the identity map of property 2 induce a relabelling on all remaining tetrahedra and vertices as long as the manifold being triangulated is a connected manifold. This canonical labelling scheme will never produce a an oriented triangulation of the manifold, if one is possible, but it does reduce the number of

Labelling of Elements in S_4							
0123	0	1023	6	2013	12	3012	18
0132	1	1032	7	2031	13	3021	19
0213	2	1203	8	2103	14	3102	20
0231	3	1230	9	2130	15	3120	21
0312	4	1302	10	2301	16	3201	22
0321	5	1320	11	2310	17	3210	23

Table 1: Labelling of elements in S_4

ways to describe a triangulation of size n from $n!24^n$ different ways to $24n$ possible canonical labellings.

While this greatly reduces the number of ways to describe a triangulation, it doesn't necessarily allow for fast insertion or fast lookup of the triangulation in a data set. To overcome this, Burton defines an *isomorphism signature* for a given triangulation $\mathcal{T}$. This isomorphism signature is unique, easy to compute, and describes the isomorphism class of our original triangulation [1]. For each of our $24n$ canonical labellings, we can find a $proto - signature$ as detailed in Definition 2.5. Burton defines but does not name this proto-signature. We have given it a name for the clarity of the reader. The isomorphism signature is then simply defined to be the smallest lexicographically $(1 < Z < a)$ of the $24n$ proto-signatures.

Our proto-signature needs to concisely encode the gluing information of a canonical labelling. Burton uses the following specific sequences and method to come up with proto-signatures:

Assume that face f of tetrahedron t is glued to face f' of tetrahedron t'. We can compare these two faces lexicographically to determine which is larger and which is smaller. That is if $t < t'$ then $(t, f) < (t', f')$ and if $t = t'$ and $f < f'$ then $(t, f) < (t', f')$ and we say that (t, f) is lexicographically smaller than (t', f').

Now for each pair of faces that are glued together in a canonical labelling we only need to encode this information once in order to keep our proto-signature as concise as possible. In order to encode the gluing we begin by defining several sequences.

Definition 2.1.2 (Destination sequence) *We can create a new sequence by removing from $A_{t,f}$ the term corresponding to the lexicographically larger of each pair of faces that are glued together in the canonical labelling. This resulting sequence is called the destination sequence.*

This destination sequence tells us what tetrahedron a certain face is glued to but not how it is glued to that tetrahedron. Because each tetrahedron has 4 vertices, anytime we glue two faces together we can encode that information as an element of S_4.

Definition 2.1.3 (Permutation sequence) *The permutation sequence of a canonical labelling assigns to each element in the destination sequence an element of S_4 that describes the permutation needed to send the vertices of face f to the vertices of face f'. In Table 1, each element of S_4 is described by where it sends 0123 and is labelled* $0, \cdots, 23$.

Finally, we need one last sequence to be able to precisely encode a canonical labelling of a triangulation. This final sequence ensures that a canonical labelling can be recreated from any given proto-signature which is useful as we want our proto-signatures and our isomorphism signature to completely and uniquely encode the gluings of a triangulation.

Definition 2.1.4 (Type sequence) *The type sequence assigns to each term in the destination sequence a typing as follows:*

1. *A term in the destination sequence is of type 0 if the corresponding face is a boundary component, i.e. $A_{t,f} = \partial$ for that particular term.*
2. *A term in the destination sequence is of type 1 if it is the first time a tetrahedron (aside from tetrahedron 0) appears in the destination sequence.*
3. *Otherwise, a term in the destination sequence is of type 2.*

Notice that each term in the destination, permutation, and type sequence when considered together encode the information of a pair of faces that are glued together under our canonical labelling. In particular if a face pairing is of type 1, then we know that the same term in the permutation sequence must be 0 (corresponding to the identity permutation 0123) as according to property 2 for canonical labellings if a face is glued to a new tetrahedron then it must use the identity map.

The number of tetrahedra in the triangulation combined with the information in the destination, type, and permutation sequences is enough to completely describe any triangulation with a canonical labelling [1]. In order to make this information easier to wield and to encode in a computer system we can use a function π that takes input of a natural number and matches it with a printable character as seen in Table 2:

Natural number i	$0 \cdots 25$	$26 \cdots 51$	$52 \cdots 61$	62	63
Printable character $\pi(i)$	a $\cdots$ z	A $\cdots$ Z	0 $\cdots$ 9	+	-

Table 2: The π function

For $i \geq 64$ we can still use the function ϵ to encode i using $d = \lfloor \log_{64}(i) \rfloor + 1$ printable characters. we calculate $\epsilon(i)$ in the following manner. We write i as a $d-$digit number in base 64. We then encode each of the d base 64 digits using π, beginning with the least significant digit. For example, if $i = 3863$, then we can encode i in $d = \lfloor \log_{64}(3863) \rfloor + 1 = 2$ digits. $3863 = 23 + 64 \cdot 60$ so $\epsilon(3863) = \pi(23)\pi(60) = x8$. Notice that if $0 < i < 64$ then $\epsilon(i) = \pi(i)$.

Definition 2.1.5 (Proto-signature) *Given a canonical labelling of n tetrahedra, its proto-signature can be found in the following way.*

- *First, we encode n and d. Specifically if $n \geq 63$ then we begin with the marker $\pi(63)$ followed by $\pi(d)$ and then $\epsilon(n)$. If $n < 63$ then we simply begin with $\pi(n)$ and it is understood that $d = 1$.*

- *Next, taking three terms in the type sequence, $b_0, b_1, \cdots, b_k$, at a time and using a tail of one or two zeros in order to have the number of terms in our type sequence be divisible by three we encode the type sequence as $\pi(b_0 + b_1 * 4 + b_2 * 16)\pi(b_3 + b_4 * 4 + b_5 * 16)\cdots$*

- *Then we encode the terms in our destination sequence, $d_0, d_1, d_2, \cdots d_k$, as $\epsilon(d_i)\epsilon(d_j)\cdots$ where d_i and d_j refer to terms in the destination sequence of type 2 with $0 \le i < j \le k$.*

- *Finally, we encode the terms in our permutation sequence $p_0, p_1, p_2, \cdots p_k$, as $\pi(p_i)\pi(p_j)\cdots$ where p_i and p_j refer to terms in the permutation sequence of type 2 with $0 \le i < j \le k$.*

Although alluded to before, we give a formal definition of the isomorphism signature.

Definition 2.1.6 (Isomorphism Signature) *The lexicographically smallest of the of the 24n proto-signatures is the isomorphism signature.*

In this definition when we say lexicographically smallest we are following the ASCII convention of $1 < Z < a$ as this is simple to implement in computer systems.

Remark 2.1.1 *The ASCII convention of $1 < Z < a$ means that the sequence $0, 1, 2, \cdots, 9, A, B, \cdots, Z, a, b, \cdots, z$ is in increasing order. This ordering is used because programming languages such as C and Python, as well as others, use this ordering of the ASCII tables in their built in string comparison functions. Note that this ordering is distinct from the ordering that the π function introduces when mapping integers to printable characters.*

As an explicit example consider the following triangulation:

Example Triangulation				
Tetrahedron	Face 012	Face 013	Face 023	Face 123
0	1(032)	1(213)	0(312)	0(230)
1	2(203)	2(132)	0(021)	0(103)
2	3(312)	3(023)	1(102)	1(031)
3	3(130)	3(201)	2(013)	2(120)

Table 3: Original labelling of example triangulation

This labelling is not canonical because face 013 of tetrahedron 0 is glued to 1(213). This is the first time tetrahedron 1 appears in our sequence $A_{t,f}$ so according to property 2 of canonical labellings we should use the identity gluing. Thus, face 013 of tetrahedron 0 should be glued to 1(013).

To establish a canonical labelling of the above triangulation we simply need to choose which tetrahedron will be tetrahedron 0 as well as how to label its vertices. Let's apply the relabelling that begins by sending 3(0123) to 0(0123). This is an arbitrary choice to demonstrate how relabelling and canonical labelling affects a triangulation. This induces the canonical labelling presented in detail in Table 4.

Now that we have a canonical labelling we can determine the proto-signature of this labelling following Burton's encoding. $A_{t,f}$ for the gluing above is the sequence 1, 1, 0, 0, 0,

Relabelling Sending 3(0123) to 0(0123)				
Tetrahedron	Face 012	Face 013	Face 023	Face 123
0	0(130)	0(201)	1(310)	1(123)
1	2(321)	0(320)	2(023)	0(123)
2	3(230)	3(013)	1(023)	1(210)
3	3(123)	2(013)	2(201)	3(012)

Table 4: Relabelling sending 3(0123) to 0(0123)

2, 0, 2, 1, 1, 3, 3, 3, 2, 2, 3. The first thing to notice is that Table 4 has repeated information as each face of a tetrahedron is glued to a different face of a tetrahedron. After dropping the terms in $A_{t,f}$ that have redundant information we see that our destination sequence is $1, 1, 0, 2, 2, 3, 3, 3$ which can easily be seen in Table 5 where the redundant information has been crossed out.

Relabelling Sending 3(0123) to 0(0123)				
Tetrahedron	Face 012	Face 013	Face 023	Face 123
0	~~0(130)~~	0(201)	1(310)	1(123)
1	2(321)	~~0(320)~~	2(023)	~~0(123)~~
2	3(230)	3(013)	~~1(023)~~	~~1(210)~~
3	~~3(123)~~	~~2(013)~~	~~2(201)~~	3(012)

Table 5: Essential information of the relabelling

Now that we have our destination sequence we can easily find our type sequence which is $1, 2, 2, 1, 2, 1, 2, 2$. Lastly using Table 1 above and Table 5 we see that the permutation sequence for this labelling is $0, 23, 13, 0, 23, 0, 16, 18$.

This triangulation consists of $n = 4$ tetrahedra so the first term in our proto-signature is $\pi(4) = e$. Next, we encode the type sequence as $\pi(1 + 2 \cdot 4 + 2 \cdot 16)\pi(1 + 2 \cdot 4 + 1 \cdot 16)\pi(2 + 2 \cdot 4 + 0 \cdot 16) = \pi(41)\pi(25)\pi(10) = Pzk$. Then we encode the face pairings of type 2 in the destination sequence as $\epsilon(1)\epsilon(0)\epsilon(2)\epsilon(3)\epsilon(3) = bacdd$. Finally, we encode the face pairings of type 2 in the permutation sequence as $\pi(23)\pi(13)\pi(23)\pi(16)\pi(18) = xnxqs$. Concatenating all this together in order we find that the proto-signature for our canonical labelling is ePzkbacddxnxqs.

This proto-signature, however, is not the isomorphism signature of this triangulation. We obtain the labelling that gives us the isomorphism signature when we relabel 2(0123) in the original labelling as 0(3102) and then construct the induced canonical labelling. The resulting proto-signature (and isomorphism signature of this triangulation) is eLAkbcbddhhwqj.

Another relabelling that would give us a proto-signature that is not the isomorphism signature but is lexicographically smaller than our labelling worked out in detail above would be obtained by relabelling 0(3012) in the original labelling to 0(0123) and then following the rules for canonical labellings. The resulting labelling has proto-signature ePzkabcddjqxxn. We can see that this is greater than our isomorphism signature because eL<eP, but this proto-signature is less than the proto-signature worked out in detail because ePzka<ePzkb.

Proposition 2.1.1 *Let $\mathcal{T}$ be a connected triangulation consisting of at least two tetrahedra. The smallest possible first distinguishing term in a proto-signature of $\mathcal{T}$ is H if our triangulation has at least one boundary component. For closed and ideal triangulations, the smallest possible first distinguishing term is L.*

Proof. We stress that the first place that our proto-signatures can differ is in their encoding of the first 3 terms of the type sequence. As there are three possible types for each tetrahedron in the destination sequence then there are at most 3^3 possible different ways to encode this information. An exhaustive list of them is found below where $a + 4 \cdot b + 16 \cdot c$ is the natural number associated with the first three terms a, b, c in the type sequence.

Encodings of the first three terms in the type sequence					
i	$\pi(i)$	i	$\pi(i)$	i	$\pi(i)$
0+4·0+16·0	a	1+4·0+16·0	b	2+4·0+16·0	c
0+4·0+16·1	q	1+4·0+16·1	r	2+4·0+16·1	s
0+4·0+16·2	G	1+4·0+16·2	H	2+4·0+16·2	I
0+4·1+16·0	e	1+4·1+16·0	f	2+4·1+16·0	g
0+4·1+16·1	u	1+4·1+16·1	v	2+4·1+16·1	w
0+4·1+16·2	K	1+4·1+16·2	L	2+4·1+16·2	M
0+4·2+16·0	i	1+4·2+16·0	j	2+4·2+16·0	k
0+4·2+16·1	y	1+4·2+16·1	z	2+4·2+16·1	A
0+4·2+16·2	O	1+4·2+16·2	P	2+4·2+16·2	Q

Table 6: Encodings of the first three terms in the type sequence

Now according to our lexicographical ordering $1 < Z < a$ and Table 6, the smallest such first distinguishing term in a proto-signature would be A. We achieve this with a type sequence that begins with 2,2,1. However, this is only possible if every face on tetrahedron 0 is glued to another face on tetrahedron 0 meaning we have a disconnected component in $\mathcal{T}$. Thus, there are fewer than 27 possible combinations of the first three terms in a type sequence that give rise to a canonical labelling of $\mathcal{T}$. Table 7 shows the same information as Table 6 but with cells corresponding to type sequences of non-canonical labellings and cells corresponding to triangulations having a disconnected component crossed out.

Looking at the remaining possible encodings of the first three terms of our type sequence we see that the the lexicographically smallest possible first distinguishing term in a proto-signature is H. This occurs when the first three terms of the type sequence are 1, 0, 2. Recall that faces of type 0 are boundary components so this "H" is only possible if our triangulation is of a manifold with boundary. If our manifold is closed, meaning that 0 doesn't appear anywhere in the type sequence, then the smallest possible first distinguishing term is L corresponding to a type sequence beginning with 1, 1, 2.

■

In the case that we have some sort of geometric information or other data associated with our original (possibly non-canonical) labelling of a triangulation then it would be useful to be able to associate this information with our canonical labellings. This can be achieved because each relabelling is an isomorphism.

Encodings of the first three terms in the type sequence					
i	$\pi(i)$	i	$\pi(i)$	i	$\pi(i)$
0+4·0+16·0	a	1+4·0+16·0	b	2+4·0+16·0	c
0+4·0+16·1	q	1+4·0+16·1	r	2+4·0+16·1	s
~~0+4·0+16·2~~	~~G~~	1+4·0+16·2	H	~~2+4·0+16·2~~	~~I~~
0+4·1+16·0	e	1+4·1+16·0	f	2+4·1+16·0	g
0+4·1+16·1	u	1+4·1+16·1	v	2+4·1+16·1	w
0+4·1+16·2	K	1+4·1+16·2	L	2+4·1+16·2	M
0+4·2+16·0	i	1+4·2+16·0	j	~~2+4·2+16·0~~	~~k~~
0+4·2+16·1	y	1+4·2+16·1	z	~~2+4·2+16·1~~	~~A~~
~~0+4·2+16·2~~	~~O~~	1+4·2+16·2	P	~~2+4·2+16·2~~	~~Q~~

Table 7: Possible and impossible encodings of the first three terms in the type sequence

Once we decide which tetrahedron t in $\mathcal{T}$ to relabel as tetrahedron zero in our canonical labelling $\mathcal{C}$ as well as how to relabel the vertices of this of tetrahedron t then we have fixed the labellings of all other tetrahedra and vertices in our canonical labelling via the canonical labelling rules. The relabelling to the vertices of tetrahedron 0 in $\mathcal{C}$ can be described as an element of S_4. This element has an inverse so our mapping can be reversed as long as we know what tetrahedron in $\mathcal{T}$ was mapped to tetrahedron 0 in $\mathcal{C}$. Thus, each canonical relabelling preserves the structure of our triangulation and has an inverse map so each canonical relabelling of $\mathcal{T}$ is an isomorphism.

2.2 Disconnected Triangulations

Burton restricted his labelling scheme to connected triangulations. However, a natural place where disconnected triangulations arise is when cutting a triangulation along a normal surface.

In the connected case, we simply had to choose which tetrahedron to label as tetrahedron zero and choose an ordering on the vertices. Then following Burton's canonical labelling scheme this would fix a labelling on all other tetrahedra in our only connected component. Thus, choosing a tetrahedron zero and an ordering on its vertices only fixes the labelling for all other tetrahedra in the same connected component as tetrahedron 0.

If our triangulation T has i connected components then we can label each component $j \in \{1, \cdots, i\}$. Now let n_j represent the number of tetrahedra in component j. Each component has $24n_j$ possible canonical labellings when considered by itself. When we consider all the components together we can't have a tetrahedron zero in each component so our choice of tetrahedron from which to build out our canonical labelling in each component will be the lowest indexed tetrahedron in that component. In particular the lowest indexed tetrahedron in component j will be $\sum_{k=1}^{j-1} n_k$ unless $j = 1$ in which case the lowest indexed tetrahedron is tetrahedron 0. There are $i!$ different ways to order our disconnected components so we have that the total number of canonical labellings for a disconnected triangulation is $24^i i! \prod_{k=1}^{i} n_k$.

A disconnected triangulation consisting of n tetrahedra can have at worst n distinct connected components. Thus, the worst case scenario is that a given triangulation has $n!24^n$ possible canonical labellings. Obviously this doesn't reduce in any significant way the number of labellings we have to consider. So this is really only an effective method of storing the information in a triangulation if there are relatively few connected components.

When it comes to creating an isomorphism signature for disconnected triangulations we only need to change one thing from Burton's definitions. For disconnected triangulations we redefine the type sequence to the following.

Definition 2.2.1 (Type sequence) *The type sequence assigns to each term in the destination sequence a typing as follows:*

1. *A term in the destination sequence is of type 0 if the corresponding face is a boundary component, i.e.* $A_{t,f} = \partial$ *for that particular term.*
2. *A term in the destination sequence is of type 1 if it is the first time a tetrahedron appears in the destination sequence unless it is the lowest indexed tetrahedron in a connected component of the triangulation.*
3. *Otherwise, a term in the destination sequence is of type 2.*

While there is only one thing we need to change in order to create a canonical labelling scheme that functions for disconnected triangulations allowing these disconnected components does change the result to Proposition 2.1.1 above.

Remark 2.2.1 *Proposition 2.1.1 is restricted to only connected triangulations. If we allow for triangulations with disconnected components then Table 8 details a triangulation that has* A *as its first distinguishing term in its proto-signature:*

Disconnected Triangulation				
Tetrahedron	Face 012	Face 013	Face 023	Face 123
0	0(013)	0(012)	0(123)	0(023)
1	1(013)	1(012)	2(013)	2(123)
2	2(023)	1(023)	2(012)	1(123)

Table 8: Disconnected triangulation example

This canonical labelling has destination sequence $0, 0, 2, 2, 1, 2$ *and type sequence* $2, 2, 1, 2, 2, 2$. *We can see that* $\pi(2+4\cdot 2+16\cdot 1) = \pi(26) = A$. *Thus, the smallest possible first distinguishing term in a proto-signature of a triangulation with disconnected components is* A.

CHAPTER III

THE ORIENTED ISOMORPHISM SIGNATURE

3.1 Orientation of Tetrahedra and 3-Manifolds

We begin with a discussion of orientability and orientation of tetrahedra in order to later define an oriented canonical labelling before moving to the orientation of a smooth manifold as a whole. Please note that we have moved back into the assumption that $\mathcal{T}$ is a connected triangulation.

A solid tetrahedron t is a 3-manifold with boundary meaning that for x an interior point of t there is a local neighborhood that resembles $\mathbb{R}^3$. This means that in order to orient the tangent space at x we need an ordered basis consisting of three vectors. A natural choice to describe this basis are the three vectors from a vertex of t to the three other vertices.

Proposition 3.1.1 *A tetrahedron has two possible orientations.*

Proof. Let t be a tetrahedron embedded in $\mathbb{R}^3$ and a, b, and c vectors from v, a vertex of tetrahedron t, in the direction of each of the other vertices of t. Now let e_a, e_b, and e_c be vectors based a point x in the interior of t in the direction of a, b, and c respectively. This ensures that e_a, e_b, and e_c are not all co-planar and thus form a basis for the tangent space at x. Because a tetrahedron is a 3-manifold then the plane containing e_a and e_b splits the tangent space into two spaces. The vector e_c can lie on either side of this plane so there are two possible orientations. ■

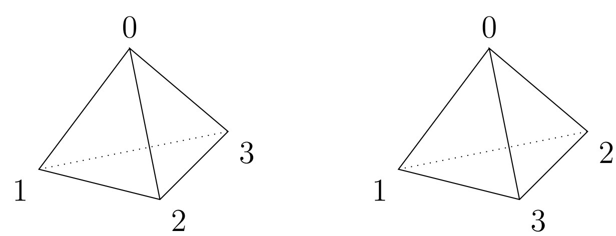

Figure 1: A negatively oriented tetrahedron (left) and a positively oriented tetrahedron (right)

Definition 3.1.1 (Orientation of a Tetrahedron) *Let t be a tetrahedron and a, b, and c an ordered basis. We say that t is positively oriented if c and $a \times b$, obtained by using the right-hand rule, lie on the same side of the plane containing a and b. Otherwise, we say that t is negatively oriented.*

From this definition we can see the usefulness of using vectors that are coincident with certain edges in a tetrahedron as a basis. Thus, a natural choice for the basis in order to determine the orientation of a tetrahedron is the ordered basis $\{01, 02, 03\}$ where 01 is the vector in the direction from vertex 0 to vertex 1 and 02 and 03 are similarly defined.

Each of the faces of a tetrahedron t is a boundary component and thus it inherits an orientation from t. However, each face is a $2-$manifold so an ordered basis for its orientation only consists of two vectors. This means there is a choice in how each face inherits it's orientation as it will either have an inward pointing normal vector or an outward pointing normal vector.

Definition 3.1.2 (Induced Orientation) *Each face inherits an orientation and a basis from its parent tetrahedron. This is called the induced orientation. If t is positively oriented then it induces a basis on each face that has an outward pointing normal vector and if t is negatively oriented then it induces a basis on each face that has an inward pointing normal vector.*

This means we can assign a canonical ordered basis to each face. For face 0 this ordered basis is $\{12, 13\}$ where 12 represents the vector from vertex 1 to vertex 2 and 13 is similarly defined. For face 1 this basis is $\{03, 02\}$. For face 2 this basis is $\{01, 03\}$. For face 3 this basis is $\{02, 01\}$. Note that these canonical ordered bases allow us to determine the orientation of an entire tetrahedron based off whether the canonical ordered basis for any given face has an inward or outward pointing normal vector.

Lemma 3.1.1 *Applying a 2-cycle to the labelling of a tetrahedron swaps the orientation of the tetrahedron.*

Proof. Let t be a tetrahedron that is positively oriented. This means that the vector 03 in the ordered basis $\{01, 02, 03\}$ is on the same side of the plane containing 01 and 02 as the vector 01×02. Applying a single 2-cycle to the labelling of vertices in t has the effect of swapping two vertices in t. Let P be the plane containing 01 and 02.

First if this 2-cycle is (13) or (23), then 01×02 remains on the same side of P as it was in the original but 03 swaps sides making the resulting tetrahedron negatively oriented. Next if this 2-cycle is (12), then 01×02 is now on the other side of P but 03 remains on its original side giving a negatively oriented tetrahedron. Then if the 2-cycle is (01) or (02) then it reverses the direction of one of the basis vectors so 01×02 also reverses direction and now points to the opposite side of P as 03. Lastly, if the 2-cycle is (03), then the vector 03 swaps directions and 01×02 is unaffected so the resulting tetrahedron is negatively oriented.

By a similar argument if t is negatively oriented than any of the 2-cycles listed above will have the same effect but will result in 03 and 01×02 to lie on the same side of P. This is an exhaustive list of the two-cycles of four elements so applying any single 2-cycle to the vertices of a tetrahedron swaps the orientation of the tetrahedron. ■

Thus far we have discussed the orientation of tetrahedra but more generally we can define the orientation of any manifold. First lets review some aspects of a general $n-$dimensional manifold $\mathcal{M}$. Any manifold can be described by an atlas.

Definition 3.1.3 *An atlas $\mathcal{A}$ for an $n-$dimensional manifold $\mathcal{M}$ consists of an indexed set of coordinate charts $\{(U_i, \phi_i) : i \in I\}$ where the set $\{U_i : i \in I\}$ is a cover for $\mathcal{M}$ and $\phi_i : U_i \to \mathbb{E}^n$ where $\mathbb{E}^n$ is the $n-$dimensional Euclidean space is a homeomorphism.*

In other words $\mathcal{M}$ is locally identifiable with some subset of $\mathbb{E}^n$ which has exactly two possible orientations. A transition map allows us to compare coordinate charts of a manifold.

Definition 3.1.4 *Given two coordinate charts (U_i, ϕ_i) and (U_j, ϕ_j) with $U_i \cap U_j \neq \emptyset$ from an atlas $\mathcal{A}$ for $\mathcal{M}$, the transition map $\tau_{i,j} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is the map $\tau_{i,j} = \phi_j \circ \phi_i^{-1}$.*

Now that we have a more concrete way of describing manifolds we can define what it means for any manifold to be orientable. This definition comes from Lee [2].

Definition 3.1.5 (Orientable) *A smooth manifold $\mathcal{M}$ is orientable if there exists an atlas $\mathcal{A}$ such that the transition maps between all overlapping coordinate charts in $\mathcal{A}$ have positive Jacobian determinant and we call $\mathcal{A}$ an oriented atlas.*

Now that we understand what it means for a smooth 3-manifold $\mathcal{M}$ to be orientable we can begin to see how this relates to the orientation of tetrahedra within a triangulation of $\mathcal{M}$.

Definition 3.1.6 (Propagated Orientations) *Once we assign an orientation to one tetrahedron in our triangulation then we can use it to determine the orientation of its neighbors based off gluing data and normal vectors to each face of our tetrahedra. We say that the adjacent tetrahedra have been assigned the propagated orientation from the original tetrahedron.*

Please note that the propagated orientation need not be the same as the orientation of the original tetrahedron. The way in which the orientation is propagated out is detailed in the following proposition.

Proposition 3.1.2 *Given two distinct tetrahedra t_1 and t_2 in a triangulation of an oriented 3-manifold with at least one face glued together then the propagated orientation t_2 receives from t_1 is the same orientation as t_1 if and only if the face pairing corresponds to an odd element of S_4.*

Proof. Without loss of generality assume that t_1 is positively oriented. Let face f be the face of t_1 that is glued to face f' in t_2. Recall that changing the orientation of a tetrahedron also changes the direction of the induced orientation on its faces and thus changes the direction of the normal vector for these faces from outward pointing to inward pointing or vice versa. We also know that because t_1 and t_2 are tetrahedra in a triangulation of an orientable 3-manifold then each tetrahedron can be assigned a consistent orientation of either positive or negative. This means once we establish the direction of the normal vector for any face of our tetrahedron then we have established the only possible orientation for the tetrahedron as a whole within the given triangulation.

We begin by showing that an odd face pairing implies that t_1 and t_2 have the same orientation. From Lemma 3.1.1 we know that applying a single 2-cycle to the ordering of vertices swaps the orientation of the tetrahedron and thus swaps the direction of the normal vector on all faces of the tetrahedron from outward to inward or vice versa. Thus applying an odd element of S_4 to the labeling of t_1 will swap the direction of the normal vector of all faces from outward to inward pointing. Face f' in the relabelling is now coincident with what was originally face f before the relabelling. This means that f' must be oriented such that it has a normal vector that points into the interior of t_1 but f' is exactly the face of t_2 that is glued to f in t_1 so the normal vector for face f' points outward from t_2 meaning that t_2 is also positively oriented. This means that we have assigned t_2 the propagated orientation from t_1 and these orientations are the same if the face pairing is an odd element of S_4.

Now to show that if t_1 and t_2 have the same orientation and share a face then their gluing must correspond with an odd element of S_4. We will use a proof by contrapositive to show that if the gluing corresponds to an even element of S_4 then t_1 and t_2 have opposite orientations. Once again using Lemma 3.1.1 we see that applying an even element of S_4 to the labelling of t_1 means that will result in a tetrahedron that has an outward pointing normal vector. After applying this relabelling what was originally face f in t_1 will now be labelled as face f' and will have a normal vector that points outward from t_1, but this means that the normal vector to face f' on t_2 must point into t_2 so t_2 is negatively oriented while t_1 is positively oriented. This means that we have assigned t_2 the propagated orientation from t_1 mentioned in Definition 3.1.6 but in this instance the orientations are different because the face pairing came from an even element of S_4.

Thus if t_1 and t_2 are tetrahedra with at least one face glued together then the face pairing corresponds to an odd element of S_4 if and only if t_1 and t_2 have the same orientation. ■

The way in which the orientation of our original oriented tetrahedron propagates out to its neighbors is the same in a non-orientable manifold; however in this case, there will be some contradiction in the way this orientation is propagated out in that at least one tetrahedron will be assigned a positive orientation and a negative orientation based off its face pairings to adjacent tetrahedra. Thus when we say a 3-manifold is non-orientable, we mean that in a triangulation of the manifold there must be some tetrahedron that will be assigned both a positive and negative orientation when using our propagated orientation.

Although using an odd element of S_4 when gluing two tetrahedra together means that the two tetrahedra will have the same orientation this gluing swaps the orientation of the normal vectors of the face pairing. Thus we call such elements *orientation reversing face pairings* and these are precisely the elements 1, 2, 5, 6, 9, 10, 13, 14, 17, 18, 21, and 22 in Table 1 in Section 2. Now we stress the importance of orientation reversing face pairings in triangulations of orientable manifolds.

Proposition 3.1.3 *A 3-manifold $\mathcal{M}$ is orientable if it permits a triangulation $\mathcal{T}$ such that the gluing map for each face pairing is an orientation reversing face pairing.*

Proof. Let $\mathcal{T}$ be a triangulation with all orientation reversing face pairings. Then there is an atlas $\mathcal{A}$ that describes $\mathcal{M}$ such that $\mathcal{A} = \{(t, \phi_t) : t \in \mathcal{T}\}$ with $\phi_t : t \to \mathbb{R}^3$ such that the canonical basis $\{01, 02, 03\}$ for t is embedded such that it is coincident with the standard ordered basis for $\mathbb{R}^3$.

Now let i and j be two tetrahedra that are adjacent to one another. Because they are glued together along a face f by an orientation reversing face pairings then if the normal vector determined by the canonical basis and labelling of f in tetrahedron i points inward to i then so must the normal vector for the face f in j point inward to j. Similarly, if the normal vector determined by the canonical basis and labelling of f in i points outward from i then so must the normal vector for the face f in j point outward from j. Thus, when we embed these tetrahedra in $\mathbb{R}^3$ using our maps ϕ_i and ϕ_j these maps will have the same sign for the determinant of their Jacobian matrices.

Recall the transition map from tetrahedron i to tetrahedron j is defined by the following composition of functions $\tau_{i,j} = \phi_j \circ \phi_i^{-1}$. This transition map will have the Jacobian determinant $\det(Jac(\phi_j \circ \phi_i^{-1})) = \det(Jac(\phi_j(\phi_i^{-1}))Jac(\phi_i^{-1})) = \det(Jac(\phi_j(\phi_i^{-1})))\det(Jac(\phi_i^{-1}))$. Because $\det(Jac(\phi_j(\phi_i^{-1})))$ and $\det(Jac(\phi_i^{-1}))$ will have the same sign then $\det(Jac(\tau_{i,j})) > 0$.

As all our face pairings in $\mathcal{T}$ are orientation reversing then the transition maps between any two adjacent tetrahedra will have positive Jacobian determinant. Thus $\mathcal{A} = \{(t, \phi_t) : t \in \mathcal{T}\}$ is an oriented atlas describing $\mathcal{M}$ so $\mathcal{M}$ is an oriented 3-manifold. ■

Although Proposition 3.1.2 shows that the orientability of a manifold and the orientation of tetrahedra in a triangulation of a manifold have some relationship to one another, it is important to note that orientability is a feature of the manifold independent of any choice of triangulation. In fact for an orientable manifold, most given triangulations will consist of both positively and negatively oriented tetrahedra. If every tetrahedra in a triangulation has only orientation reversing face pairings then we say this is an *oriented triangulation* as it describes an oriented manifold according to Proposition 3.1.3.

3.2 Encoding of the Oriented Isomorphism Signature

Burton's canonical labelling is very powerful in that for a triangulation $\mathcal{T}$ consisting of n tetrahedra it greatly reduces the number of labellings that we need to consider from $n!24^n$ to $24n$. However because this labelling scheme implements the identity map, which is not an orientation reversing face pairing, when gluing a new tetrahedra to an existing one we obtain a non-oriented triangulation even if the manifold is orientable. Often times it is useful to deal with an oriented triangulation if one is possible so we diverge from Burton and introduce the following definition.

Definition 3.2.1 (Oriented canonical labellings) *Given a labelling of a triangulation of size n, let $A_{t,f}$ denote the tetrahedron glued to face f of tetrahedron t (so that $A_{t,f} \in \{0, \cdots, n-1, \partial\}$ for all $t = 0, \cdots, n-1$ and $f = 0, ..., 3$). If face f of tetrahedron t is a boundary component then we say $A_{t,f} = \partial$. The labelling is an oriented canonical labelling if, when we write out the sequence $A_{0,0}, A_{0,1}, A_{0,2}, A_{0,3}, A_{1,0}, \cdots, A_{n-1,3}$, the following properties hold:*

1. *For each $1 \leq i < j$, tetrahedron i first appears before tetrahedron j first appears.*
2. *For each $i \geq 1$, suppose tetrahedron i first appears as the entry $A_{t,f} = i$. Then the corresponding gluing uses the map swapping the lexicographically largest vertices of face f of tetrahedron t.*
3. *The gluing between vertices of face f or tetrahedron t to $A_{t,f}$ is an orientation reversing face pairing.*

These gluings in condition 2 correspond to the following elements $s \in S_4$ labelled in Table 1 above: if $f = 0$ then $s = 1$, if $f = 1$ then $s = 1$, if $f = 2$ then $s = 5$, and if $f = 3$ then $s = 2$. Notice that these are all orientation reversing face pairings so they satisfy condition 3.

Theorem 3.2.1 *Every orientable manifold $\mathcal{M}$ permits an oriented canonical labelling and every oriented canonical labelling describes an oriented manifold.*

Proof. Assume $\mathcal{M}$ is an oriented manifold with triangulation $\mathcal{T}$. We can construct a relabelling of $\mathcal{T}$ using the first two conditions in the definition of oriented canoncial labellings. From condition 2 we see that the in the relabelling each new tetrahedron is glued on via an orientation reversing face pairing. Now because $\mathcal{M}$ is orientable then Proposition 3.1.2 applies and each tetrahedron in our relabelling must have the same orientation as it has an orientation reversing face pairing. Because each tetrahedron in the relabelling has the same orientation then Proposition 3.1.2 applies again and every gluing must be an orientation reversing face pairing. Thus, condition 3 in the definition is met so $\mathcal{M}$ permits an oriented canonical labelling.

Now assume that we have a triangulation $\mathcal{T}$ that is described by an oriented canonical labelling. Then from condition 3 and Proposition 3.1.3 we can easily see that the manifold described by an oriented canonical labelling must be an orientable manifold. ■

With our understanding of orientability of manifolds and triangulations we can now define an oriented proto-signature and oriented isomorphism signature similar but distinct from definitions 2.1.5 and 2.1.6 above.

Definition 3.2.2 (Oriented Proto-signature) *Given an oriented canonical labelling of n tetrahedra, its oriented proto-signature can be found in the following way.*

- *First, we begin with the character "\$" to denote that this is an oriented proto-signature as opposed to Burton's proto-signature. If we want to specify the orientation of our triangulation we instead begin with the character "!" to denote that the tetrahedra are positively oriented or we begin with the character "?" to denote that the tetrahedra are negatively oriented.*

- *Then, we encode n and d. Specifically if $n \geq 63$ then we begin with the marker $\pi(63)$ followed by $\pi(d)$ and then $\epsilon(n)$. If $n < 63$ then we simply begin with $\pi(n)$ and it is understood that $d = 1$.*

- *Next, taking three terms in the type sequence, $b_0, b_1, \cdots, b_k$, at a time and using a tail of one or two zeros in order to have the number of terms in our type sequence be divisible by three we encode the type sequence as $\pi(b_0 + b_1 * 4 + b_2 * 16)\pi(b_3 + b_4 * 4 + b_5 * 16)\cdots$*

- *Then we encode the terms in our destination sequence, $d_0, d_1, d_2, \cdots d_k$, as $\epsilon(d_i)\epsilon(d_j)\cdots$ where d_i and d_j refer to terms in the destination sequence of type 2 with $0 \leq i < j \leq k$.*

- *Finally, we encode the terms in our permutation sequence $p_0, p_1, p_2, \cdots p_k$, as $\pi(p_i)\pi(p_j)\cdots$ where p_i and p_j refer to terms in the permutation sequence of type 2 with $0 \leq i < j \leq k$.*

Definition 3.2.3 (Oriented Isomorphism Signature) *The lexicographically smallest of the of the 24n oriented proto-signatures is known as the oriented isomorphism signature.*

To see how this differs from Burton's isomorphism signature let us consider the same triangulation as in section 2.1.

Original Labelling				
Tetrahedron	Face 012	Face 013	Face 023	Face 123
0	1(032)	1(213)	0(312)	0(230)
1	2(203)	2(132)	0(021)	0(103)
2	3(312)	3(023)	1(102)	1(031)
3	3(130)	3(201)	2(013)	2(120)

Table 9: Original labelling of example triangulation

As we have already stated this triangulation has isomorphism signature eLAkbcbddhhwqj. This labelling is not an oriented canonical labelling because face 013 of tetrahedron 0 is glued to 1(213). This is the first time tetrahedron 1 appears in our sequence $A_{t,f}$ so according to property 2 of oriented canonical labellings we should use the gluing corresponding to element 5 in the S_4 above. In other words 0(013) should be glued to 1(031).

To establish an oriented canonical labelling of the above triangulation we need to choose which tetrahedron will be tetrahedron 0 as well as how to label its vertices. Let's apply the relabelling that begins by sending 2(0123) to 0(3102). This induces the following oriented canonical labelling detailed in Table 10.

Relabelling sending 2(0123) to 0(3102)				
Tetrahedron	Face 012	Face 013	Face 023	Face 123
0	2 (102)	1 (103)	2 (032)	1 (132)
1	1 (320)	0 (103)	1 (210)	0 (132)
2	0 (102)	3 (230)	0 (032)	3 (132)
3	3 (301)	3 (120)	2 (301)	2 (132)

Table 10: Relabelling sending 2(0123) to 0(3102)

Now that we have an oriented canonical labelling we can determine its oriented protosignature. For this example we choose not to determine the orientation of the tetrahedra in this triangulation and instead use $ to denote that this is an oriented isomorphism signature.

$A_{t,f}$ for the gluing above is the sequence $1, 2, 1, 2, 0, 1, 0, 1, 3, 0, 3, 0, 2, 2, 3, 3$. Notice once again that Table 10 has repeated information as each face of a tetrahedron is glued to a different face of a tetrahedron. After dropping the terms in $A_{t,f}$ that have redundant information we see that our destination sequence is $1, 2, 1, 2, 1, 3, 3, 3$ which can easily be seen in Table 11 where the redundant information has been crossed out.

Relabelling sending 2(0123) to 0(3102)				
Tetrahedron	Face 012	Face 013	Face 023	Face 123
0	2 (102)	1 (103)	2 (032)	1 (132)
1	~~1 (320)~~	~~0 (103)~~	1 (210)	~~0 (132)~~
2	~~0 (102)~~	3 (230)	~~0 (032)~~	3 (132)
3	~~3 (301)~~	3 (120)	~~2 (301)~~	~~2 (132)~~

Table 11: Essential information of relabelling

Now that we have our destination sequence we can easily find our type sequence which is $1, 1, 2, 2, 2, 2, 1, 2, 2$. Lastly using our S_4 table above and our crossed out table above we see that the permutation sequence for this labelling is $1, 1, 6, 6, 17, 1, 17, 9$.

Recall that our first character for our oriented isomorphism signature will be $ as we decided not to provide a basis to orient our tetrahedra. This triangulation consists of $n = 4 < 63$ tetrahedra so the second term in our proto-signature is $\pi(4) = e$. Next, we encode the type sequence as $\pi(1 + 1 \cdot 4 + 2 \cdot 16)\pi(2 + 2 \cdot 4 + 1 \cdot 16)\pi(2 + 2 \cdot 4 + 0 \cdot 16) = \pi(37)\pi(26)\pi(10) = LAk$. Then we encode the face pairings of type 2 in the destination sequence as $\epsilon(1)\epsilon(2)\epsilon(1)\epsilon(3)\epsilon(3) = bcbdd$. Finally, we encode the face pairings of type 2 in the permutation sequence as $\pi(6)\pi(6)\pi(17)\pi(17)\pi(9) = ggrrj$. Concatenating all this together in order we find that the proto-signature for our canonical labelling is $eLAkbcbddggrrj. This is in fact the oriented isomorphism signature as it is the lexicographically smallest of the 24n oriented proto-signatures for this triangulation following the ASCII convention that $1 < Z < a$.

Clearly this encoding, $eLAkbcbddggrrj, of our original triangulation differs from Burton's isomorphism signature which is eLAkbcbddhhwqj. While both the oriented isomorphism signature and Burton's isomorphism signature are representations of triangulations that are isomorphic to the same original triangulation, our oriented isomorphism signature has the advantage that the triangulation that it encodes is oriented.

CHAPTER IV

COMPRESSION OF PACHNER PATHS

4.1 Pachner Moves and Pachner Graphs

A 3-manifold $\mathcal{M}$ can be described by many triangulations. The number of tetrahedra in these triangulations are not restricted meaning that it is possible to have a triangulation consisting of p tetrahedra and another triangulation consisting of q tetrahedra both describing $\mathcal{M}$ with $p \neq q$. If there is a relabelling that sends one triangulation of $\mathcal{M}$ to another triangulation we say that these triangulations are isomorphic as the relabelling constitutes an isomorphism between the two. Notice that two triangulations having differing numbers of tetrahedra are never isomorphic despite the fact that they might triangulate the same 3-manifold. Changing the number of tetrahedra in a triangulation but preserving what manifold the triangulation represents can be achieved by local changes to the triangulation called *Pachner moves*.

Definition 4.1.1 (Pachner Moves) *The four simplest Pachner moves are performed as follows:*

- *The 2-3 move replaces two distinct tetrahedra that are glued together with three distinct tetrahedra that are glued along a common edge.*
- *The 4-1 move replaces 4 distinct tetrahedra containing a common vertex of degree 4 with a single tetrahedron.*
- *The 3-2 move and 1-4 move are inverses of the 2-3 move and 4-1 move respectively.*

In Figure 2 notice that the 2-3 move increases the number of tetrahedra in our triangulation by one and thus its inverse the 3-2 move will decrease the number of tetrahedra by one. Similarly the 4-1 move decreases the number of tetrahedra by three so its inverse the 1-4 move increases the number of tetrahedra by 3. We can think of the 2-3 move as occurring at a face pairing in the triangulation, the 3-2 move at an edge of the triangulation, the 1-4 move at a tetrahedron of the triangulation, and the 4-1 move at a vertex of the triangulation. While these moves are quite powerful in allowing us to increase or decrease the number of tetrahedra in a triangulation, they do have restrictions. A Pachner move can only be applied at a sub-simplex of dimension k for $k \in \{0, 1, 2\}$ of the triangulation if the k-simplex has exactly $4 - k$ distinct adjacent tetrahedra in the triangulation. In the case of the 1-4 move occurring at a tetrahedron in the triangulation this move is always possible.

While two triangulations with differing numbers of tetrahedra may describe the same manifold $\mathcal{M}$, determining what Pachner moves need to be done to convert one triangulation to the other is no small feat. Any two triangulations of the same closed 3-manifold can be

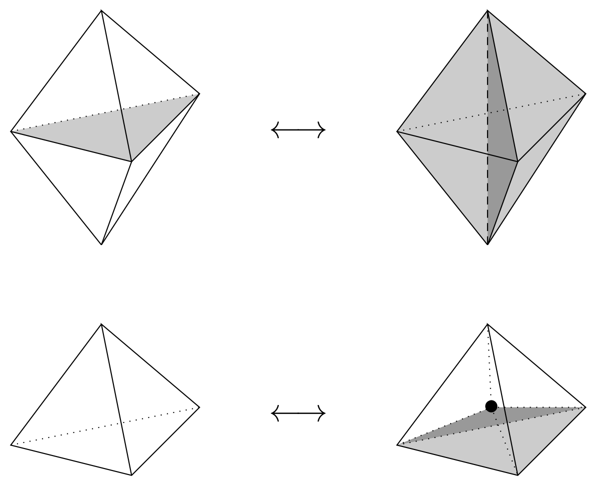

Figure 2: The 2-3 and 3-2 moves (top) and the 1-4 and 4-1 moves (bottom)

made isomorphic through a sequence of Pachner moves [3] and this sequence can be seen in the Pachner graph for a 3-manifold. The following definition comes from [1].

Definition 4.1.2 (The Pachner Graph) *Let $\mathcal{M}$ be any 3-manifold. The Pachner graph of $\mathcal{M}$, denoted $\mathcal{P}(\mathcal{M})$, is an infinite graph constructed as follows. The nodes of $\mathcal{P}(\mathcal{M})$ correspond to isomorphism classes of triangulations of $\mathcal{M}$. Two nodes of $\mathcal{P}(\mathcal{M})$ are joined by an arc if and only if there is a Pachner move that converts one class of triangulations into the other.*

The nodes of $\mathcal{P}(\mathcal{M})$ are partitioned into finite levels 1, 2, 3, $\cdots$, where each level n contains the nodes corresponding to triangulations consisting of n tetrahedra.

The oriented isomorphism signature of a triangulation can be used as a representation for the isomorphism class of the triangulation also known as the triangulation class. Thus, each node in the Pachner graph can be described by an oriented isomorphism signature which encodes a specific triangulation, namely the oriented canonical labelling that gives rise to the proto-signature that is the oriented isomorphism signature. Because each arc in $\mathcal{P}(\mathcal{M})$ describes a Pachner move then if two nodes are connected via a path, there is a sequence of Pachner moves that can be applied to convert one triangulation class to the other. Because each Pachner move increases or decreases the number of tetrahedra in a triangulation by a fixed amount we can determine what type of move is being applied by looking at the level of our nodes within the graph before and after traversing the arc. For example if we start at a level 7 node in the Pachner graph and there is an arc to a level 4 node, then these two nodes are related by a 4-1 move as this is the only Pachner move that decreases the number of tetrahedra, and thus the level of our node, by three.

While we can determine what type of Pachner move needs to be performed to get from one node to the next in a path by observing the difference in levels of consecutive nodes, the Pachner graph alone doesn't tell us where to apply this move. Applying a move at the wrong location even if the move applied gets us to a node of the same level as what arc we intended to traverse can send us down a completely different path. The labelling of the triangulation we apply the Pachner move to also changes the location of where we need to apply a specific move to traverse a certain arc in our graph. Because of this we need to create a way to accurately and consistently describe a path regardless of what labelling you want to apply the move to.

4.2 Encoding Pachner Paths

We choose to let the triangulation encoded by the oriented isomorphism signature always be the labelling to which we apply a Pachner move. Let $\mathcal{T}$ denote this triangulation. With a consistent choice of how to label the triangulation representing each node we can now make a consistent choice in how to describe the locations within a node where we can apply a Pachner move.

We will order the vertices, faces, and edges of $\mathcal{T}$ in lexicographical order.

Definition 4.2.1 (Order of Sub-simplices of a Triangulation) *Let $v_{t,k}$ for all $t \in \mathcal{T}$ and $k \in \{0,1,2,3\}$ represent vertex k of tetrahedron t. Let $e_{t,k}$ for all $t \in \mathcal{T}$ and $k \in \{0,1,2,3,4,5\}$ represent the edges of tetrahedron t with $k = 0$ corresponding with edge 01, $k = 1$ with edge 02, $k = 2$ with edge 03, $k = 3$ with edge 12, $k = 4$ with edge 13, and $k = 5$ with edge 23. Let $f_{t,k}$ for all $t \in \mathcal{T}$ and $k \in \{0,1,2,3\}$ represent face k of tetrahedron t. Now let V represent the sequence $v_{0,0}, v_{0,1}, v_{0,2}, v_{0,3}, v_{1,0}, \cdots$, let E represent the sequence $e_{0,0}, e_{0,1}, e_{0,2}, \cdots, e_{0,6}, e_{1,0}, \cdots$, and let F represent the sequence $f_{0,0}, f_{0,1}, f_{0,2}, f_{0,3}, f_{1,0}, \cdots$.*

The sub-simplices of $\mathcal{T}$ are in lexicographical order (or equivalently we can say that $\mathcal{T}$ is ordered) if for each $0 \leq i < j$, vertex i of $\mathcal{T}$ first appears before vertex j of $\mathcal{T}$ in V. Similarly, edge i of $\mathcal{T}$ first appears before edge j of $\mathcal{T}$ in E, and face i of $\mathcal{T}$ first appears before face j of $\mathcal{T}$ in F.

Notice that any given labelling of $\mathcal{T}$ will have exactly one such lexicographical ordering of the vertices, faces, and edges of the triangulation. Thus, we can use this ordering to describe precisely where to perform a specific Pachner move within $\mathcal{T}$ as we know it always references the triangulation encoded by the oriented isomorphism signature. Also see that according to the definition above if $\mathcal{T}$ is ordered then vertex 0 of $\mathcal{T}$ will always be the vertex of $\mathcal{T}$ corresponding to $v_{0,0}$. The same is true for face 0 and edge 0 of $\mathcal{T}$ corresponding to $f_{0,0}$ and $e_{0,0}$ respectively. With these ordering conventions in place we can describe how to traverse an arc in the Pachner graph via the following definition.

Definition 4.2.2 (Pachner Arc Encoding) *Given an oriented isomorphism signature corresponding to triangulation $\mathcal{T}$ and representing a node in the Pachner graph of a 3-manifold $\mathcal{M}$ we can encode an arc from this node to another corresponding to a Pachner move performed at a specific k-simplex of $\mathcal{T}$ in the following manner.*

- *First if i, the total number of $k-$simplices of $\mathcal{T}$, is greater than 63 we begin with the marker $\pi(63)$ followed by $\pi(d)$ with $d = \lfloor \log_{64}(i) \rfloor + 1$. If $i < 63$ we don't encode anything for this step and it is understood that $d = 1$.*

- *Next, we encode the location L corresponding to a specific vertex, edge, face, or tetrahedron within our triangulation where this Pachner move occurs using d printable characters as $\epsilon(L)$*

- *Finally, we encode what type of Pachner move we are doing at this location L as $\pi(k)$ so if $k = 0$ we are encoding a 4-1 move, if $k = 1$ we are encoding a 3-2 move, if $k = 2$ we are encoding a 2-3 move, and if $k = 3$ we are encoding a 1-4 move as each move is only possible at a sub-simplices of a specific dimension.*

Theorem 4.2.1 *Pachner arc encodings represent only one arc in the Pachner graph and are well defined.*

Proof. Let $\mathcal{T}$ be the base node for our arc in the Pachner graph. The arc describes a Pachner move of type k at a specific $k-$simplex, P, of $\mathcal{T}$. Each Pachner arc encoding encodes the dimension k of the sub-simplex where the Pachner move occurs and thus encodes what type of move should be performed. We also encode the location P of this Pachner move as L the integer representing P in the lexicographical ordering of $k-$simplices of $\mathcal{T}$. The triangulation $\mathcal{T}$ representing the base node of our arc is a fixed triangulation, meaning there is no ambiguity in what its labelling should be. Combining all this information means that a Pachner arc encoding at a specific node tells us what dimension of sub-simplex to apply a Pachner move at, what kind of move to apply, and the location of the move within the ordering of sub-simplices of $\mathcal{T}$ so it is a well defined encoding. ■

Note that although a Pachner arc encoding is well defined, each arc in the Pachner graph will have two such encodings as the encoding is dependent on what node you start at or in other words its encoding is dependent on which direction you are traversing the arc. This is because each arc represents a Pachner move and it's inverse.

If we know our current node than there is only one possible encoding of any arc off of it as we can only traverse it in one possible direction. This works to our advantage as it allows us to concatenate Pachner arc encodings to describe a directional path through the Pachner graph.

Definition 4.2.3 (Directed Pachner Path Encoding) *A directed Pachner path encoding is constructed from a base node and sequence of Pachner arc encodings. The encoding is the string beginning with the oriented isomorphism signature concatenated with the character '#' to denote the end of the oriented isomorphism signature and the beginning of the Pachner arc encodings, and finally this is concatenated with each of the Pachner arc encodings in order.*

As an explicit example we construct a directed Pachner path encoding with the beginning node \$eLAkbcbddggrrj. This is the oriented isomorphism signature that we constructed as an example in Section 3. Let's apply a 1-4 move at tetrahedron one of \$eLAkbcbddggrrj to get a new triangulation $\mathcal{T}'$. The location for this Pachner move is 1 and the move is of type 3 because it occurs at a 3-simplex of \$eLAkbcbddggrrj and this triangulation has less than 63 3-simplices so $d = 1$ and this Pachner arc can be encoded as $\epsilon(1)\pi(3) = bd$. Next let's encode the Pachner arc corresponding to applying a 3-2 move at edge 01 in tetrahedron 3 of $\mathcal{T}$. This edge corresponds with edge 8 in the ordering of edges in $\mathcal{T}$. This triangulation has less than 63 edges so $d = 1$ and this Pachner arc can be encoded as $\epsilon(8)\pi(1) = hb$. Thus, the directed Pachner path encoding for this path is \$eLAkbcbddggrrj#bdhb.

We can easily describe traversing more edges in the Pachner graph by adding more Pachner arc encodings onto the end of any given directed Pachner path encoding. In this way we can consider a Pachner arc encoding as a suffix that can be appended to the end of an oriented isomorphism signature or an already existing directed Pachner path encoding.

Remark 4.2.1 *In this section we have made the choice to focus on analyzing the Pachner graph of orientable manifolds, but this encoding scheme for directed Pachner paths also works for triangulations of non-orientable 3-manifolds. Instead of representing each node as the oriented isomorphism signature we instead use the isomorphism signature and we replace the oriented isomorphism signature in the encoding with the triangulation's isomorphism signature. Please note that this will likely change the actual encoding of each Pachner arc as the indexing of subsimplices in the isomorphism signature and oriented isomorphism signature can be different.*

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