STABILITY OF 2D PARTIALLY DISSIPATIVE BOUSSINESQ EQUATIONS AND 3D ROTATING BOUSSINESQ EQUATIONS

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Abstract: Examining the stability of nonlinear partial differential systems near a physically relevant equilibrium with suitable perturbation is a fundamental problem in fluid dynamics. This dissertation solves the stability and large-time behavior of solutions to the nonlinear Boussinesq equations.

We study the two-dimensional Boussinesq equations for buoyancy-driven fluids with degenerate dissipations due to their application in a specific physical scenario. Furthermore, these degenerate dissipations help reveal the inner structure of the system when we perform various interactions between the velocity and temperature. We perturb the solutions of two different two-dimensional Boussinesq systems near the hydrostatic equilibrium in a different domain. We prove that the temperature stabilizes the buoyancy-driven fluids for the first system, which has only vertical dissipation and horizontal thermal diffusion. For the second system containing only horizontal dissipation and vertical thermal diffusion, we establish the stability of the solutions and stratifying patterns of the buoyancy-driven fluids as mathematically rigorous facts.

Along with this, we study the stability of the three-dimensional rotating Boussinesq equations with only horizontal dissipation, which have a special two-dimensional solution that is dynamic and independent of depth. On large scales, this unique solution provides the bulk averaged properties of the fluid motion. To achieve the global existence, uniqueness, and stability result, we perturb the three-dimensional rotating Boussinesq equations near this dynamic solution.

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CHAPTER I

INTRODUCTION

Fluid dynamics studies the movements of the fluids, which tend to deform from their natural position due to external forces. This vast subject includes many active research areas such as hydrodynamics, geophysical fluids, astrophysical fluids, aerodynamics, environmental fluids, and biophysical fluids. This chapter introduces well-known fluid dynamics equations: the Euler Equations, the Navier-Stokes equations, the Boussinesq Equations, and the rotating Boussinesq Equations, along with existing results mainly concerning the stability and large-time behaviors of their solutions. In the subsequent chapters, we present the author's work on the stability and large-time behavior of the solutions of the latter two equations.

1.1 Euler and Navier-Stokes Equations

The Euler and Navier-Stokes equations model the dynamics of inviscid and viscous fluids, respectively. These equations can be derived using the conservation of mass, momentum, and energy ([19], [7], [42], [21]). When the fluid is homogeneous and incompressible, the Navier-Stokes equations without the external forcing term become:

$$\begin{cases} \partial_t v + v \cdot \nabla v = -\nabla P + \nu\, \Delta v \quad x \in \mathbb{R}^N,\, t > 0, \\ \nabla \cdot v = 0, \end{cases} \tag{1.1.1}$$

where $v(x,\, t) = (v_1(x,\, t),\, v_2(x,\, t), \cdots,\, v_N(x,\, t))$ is a vector-valued function which denotes the velocity of the fluid, the scalar-valued function $P(x,\, t)$ is the pressure, ν is the kinematic viscosity of the fluid. In the above system (1.1.1), the first equation represents the conservation of momentum, the statement of Newton's second law of motion, and the second equation represents the conservation of mass also known as the divergence-free condition. When the $\nu = 0$, (1.1.1) becomes the Euler equations.

The nonlinear term $v \cdot \nabla v$ presented in the momentum equation of (1.1.1) complicates analyzing the system. Therefore, whether there exists a global solution to 3D Euler equations or whether they form singularities in a finite time is still unknown. A similar problem is unsolved for the Navier-Stokes equations; the Clay Mathematics Institute announced a one million dollar prize for either the existence and uniqueness or the breakdown of Navier-Stokes solutions on $\mathbb{R}^3$ or $\mathbb{R}^3/\mathbb{Z}^3$ [30]. We can find the solution of the Navier-Stokes equation using a constant function; the problem is that the solution has infinite energy. To be physically relevant, we need a solution with finite energy.

For notational convenience, we shall write ∂_t for $\partial_t = \frac{\partial}{\partial t}$, ∂_j for $\partial_{x_j} = \frac{\partial}{\partial x_j}$ with $j = 1, 2, 3$, and ∂_{jj} or ∂_j^2 for $\partial_{x_j x_j} = \frac{\partial^2}{\partial x_j^2}$ with $j = 1, 2, 3$.

There are insurmountable existence and uniqueness results regarding the Euler and Navier-Stokes equations. We present only some results concerning the stability or the large-time behavior of solutions to these equations. Schonbek and Wiegner ([48], [49]) formulated the Fourier splitting method to study the large-time behavior of the solution of (1.1.1) and proved that the Sobolev norm of the solution decays algebraically in time. However, such a rate does not appear possible for the Euler equations due to the lack of dissipation. The growth of the vorticity gradient has been shown to grow double exponentially in time in various domains for the Euler equations (see, e.g., [25], [35], [64]).

We are mainly interested in the nonlinear systems where the dissipation in one direction is more dominant than in other directions. When the vertical dissipation is significantly smaller compared to the horizontal, the following Navier-Stokes equations is the valuable model for anisotropic geophysical fluids [47]:

$$\begin{cases} \partial_t v + v \cdot \nabla v = -\nabla P + \nu \, \Delta_h v, \quad x \in \mathbb{R}^3, \quad t > 0, \\ \nabla \cdot v = 0. \end{cases} \tag{1.1.2}$$

Here $\Delta_h = \partial_i^2 + \partial_j^2$. The Fourier splitting method mentioned above is unsuitable for partial dissipative systems like (1.1.2). Yang, Wu, and Ji [31] considered the small initial velocity in the Sobolev space $H^4(\mathbb{R}^3) \cap H_h^{-\sigma}(\mathbb{R}^3)$, $\frac{3}{4} \leq \sigma < 1$ with the bootstrapping arguments to study the stability and large-time behavior of (1.1.2). They proved the existence of a unique global solution and extracted the optimal decay rates for the solution as well as its first-order derivatives.

In two-dimension, when only horizontal dissipation is present, Dong, Wu, Xu, and Zhu [28] studied the following anisotropic Navier-Stokes equations:

$$\begin{cases} \partial_t v + v \cdot \nabla v = -\nabla P + \nu \, \partial_{11} v, \quad x \in \Omega = \mathbb{T} \times \mathbb{R}, \, t > 0, \\ \nabla \cdot v = 0. \end{cases} \tag{1.1.3}$$

Due to the lack of vertical dissipation, the stability of the above problem (1.1.3) in the Sobolev space H^2 remains unsolved in the whole space $\mathbb{R}^2$. Therefore, Dong et al. considered the spatial domain $\Omega = \mathbb{T} \times \mathbb{R}$, where $\mathbb{T}$ is a 1D periodic box and decomposed v into horizontal average and the oscillation part to establish the stability and decay results of a solution in the Sobolev spaces.

1.2 Boussinesq Equations

The Boussinesq equations for the buoyancy-driven fluids are suitable models for various lengths and time scales. For instance, it is helpful to model Rayleigh-Bénard convection and study atmospheric and oceanographic flows (see, e.g., [47], [20], [44]). The derivation of the

Boussinesq equations can be found in several books (see, e.g., [19], [47]). One of the most critical assumptions while deriving the Boussinesq equations is that the density variation is considered only on the buoyancy term [53].

This section introduces the 3D rotating Boussinesq equations and the 2D anisotropic Boussinesq equations along with their established results on the stability and large-time behavior of a solution.

1.2.1 The 3D Rotating Boussinesq Equations

The 3D Boussinesq equations combine the 3D rotating Navier-Stokes equations with an additional buoyancy forcing term in the momentum equation and the convection-diffusion equation for the temperature or density depending on the context. Studying 3D rotating Boussinesq equations is essential to better understand fluid flows in the atmosphere and ocean, where the rotation and the stratification are prevalent [44]. The standard 3D rotating Boussinesq equations can be written as ([57], [44])

$$\begin{cases} \partial_t v + v \cdot \nabla v + f\, e_3 \times v = -\nabla p^\star + \nu\, \Delta v - \frac{g}{\rho_b}\, \rho^\star \mathbf{e}_3, \quad x \in \mathbb{R}^3,\, t > 0, \\ \partial_t \rho^\star + v \cdot \nabla \rho^\star = \kappa\, \Delta \rho^\star, \\ \nabla \cdot v = 0, \end{cases} \tag{1.2.1}$$

where $\rho^\star$ is the density, $\kappa^\star$ is the thermal diffusivity, g is the acceleration due to gravity, ρ_b is the reference constant density, and $\mathbf{e}_3$ is the unit vector in the vertical direction. The term $-\frac{g}{\rho_b}\,\rho^\star$ is known as the buoyancy forcing term, and $f = 2\Omega \sin \sigma$, is the rotational frequency, where Ω being the angular frequency of a planetary rotation and σ is the latitude, and the term $f\, e_3 \times v$ is known as the Coriolis force.

The system (1.2.1) have an exact solution when $v \equiv 0$ with p and ρ satisfying so-called hydrostatic balance

$$\partial_3 p^\star(x_3) = -\frac{g\rho^\star(x_3)}{\rho_b}. \tag{1.2.2}$$

When the density depends only on the vertical direction, the pressure can be obtained by integrating (1.2.2). The vertical pressure gradient and buoyancy in the above equation (1.2.2) almost cancel many scales of fluid in the atmosphere and the ocean. Therefore, it is reasonable to consider the perturbation near this hydrostatic balance. Let $(\overline{\rho}, \overline{p})$ be the hydrostatic balance

$$\partial_3 \overline{p}(x_3) = -\frac{g}{\rho_b}\overline{\rho}(x_3). \tag{1.2.3}$$

Indeed, when we consider the perturbation $(v,\, p,\, \rho)$ with

$$p = p^\star - \overline{p} \quad \text{and} \quad \rho = \rho^\star - \overline{\rho},$$

the equation for $(v,\, p,\, \rho)$ becomes

$$\begin{cases} \partial_t v + v\cdot\nabla v + f\, e_3 \times v = -\nabla p + \nu\,\Delta v - \frac{g}{\rho_b}\rho\, e_3, \\ \partial_t \rho + v\cdot\nabla\rho + v_3\partial_3\overline{\rho} = \kappa\,\Delta\rho + \kappa\partial_3^2\overline{\rho}, \\ \nabla\cdot v = 0, \\ v(x,\,0) = v_0(x), \quad \rho(x,\,0) = \rho_0(x). \end{cases} \tag{1.2.4}$$

It is worth remarking that the sign of $\overline{\rho}(x_3)$ in the above equation is important when we consider the stability problem and we assume $\overline{\rho}(x_3) = -x_3$. This assumption can be justified. In fact, in some regions of our atmosphere, sometimes lighter fluid is below heavier fluid, then the sign is positive, and the situation is unstable. This is because the fluid at the bottom is less dense than the fluid above it.

We consider the case when there is only horizontal dissipation, nevertheless the results are also true for the full dissipative case. For simplicity, we assume $g = 1$ and $\rho_b = 1$ in (1.2.4), and write it with only horizontal dissipation as

$$\begin{cases} \partial_t v + v \cdot \nabla v + f\, v_h^{\perp} = -\nabla p + \nu\, \Delta_h v - \rho\, e_3, \\ \partial_t \rho + v \cdot \nabla \rho - v_3 = \kappa\, \Delta_h \rho, \\ \nabla_h \cdot v_h + \partial_3 v_3 = 0, \end{cases} \tag{1.2.5}$$

where we have used

$$e_3 \times v = (-v_2,\, v_1,\, 0) = (v_h^{\perp}, 0), \quad \text{and} \quad v_h = (v_1,\, v_2).$$

The above system (1.2.5) has the following exact solution

$$(v_h^{(0)},\, v_3^{(0)},\, \rho^{(0)})|_{t=0} = (v_{h0}^{(0)}(x_h, t),\, 0,\, 0), \quad p^{(0)} = 0, \tag{1.2.6}$$

with $(v_h^{(0)})$ satisfying

$$\begin{cases} \partial_t v_h^{(0)} + v_h^{(0)} \cdot \nabla_h v_h^{(0)} + f\, v_h^{(0)\perp} = -\nabla_h p + \nu\, \Delta_h v_h^{(0)}, \\ \nabla_h \cdot v_h^{(0)} = 0, \\ v_h^{(0)}(x_h, 0) = v_{h0}^{(0)}(x_h). \end{cases} \tag{1.2.7}$$

This exact solution is two-dimensional, independent of depth, and also known as the equations for barotropic flow. On large scales, this special solution provides the bulk averaged properties of the fluid motion. Moreover, this special solution is an example of dispersive waves where the effects of rotations are essential [44]. Neglecting viscosity in (1.2.7), we can convert it into the vorticity stream form. Furthermore, if we neglect the effect of rotation in the vorticity stream form for barotropic equations, we get the same vorticity stream formulation as in the case of the 2D Navier-Stokes equations.

Due to the nature of above special solution (1.2.7), we separate the horizontal and vertical components in the equation (1.2.5) and write it as the following initial value problem

$$\begin{cases} \partial_t v_h + v_h \cdot \nabla_h v_h + v_3 \partial_3 v_h + f\, v_h^{\perp} = -\nabla_h p + \nu\, \Delta_h v_h, \\ \partial_t v_3 + v_h \cdot \nabla_h v_3 + v_3 \partial_3 v_3 = -\partial_3 p + \nu\, \Delta_h v_3 - \rho, \\ \partial_t \rho + v_h \cdot \nabla_h \rho + v_3 \partial_3 \rho = \kappa\, \Delta_h \rho + v_3, \\ \nabla_h \cdot v_h + \partial_3 v_3 = 0, \\ (v_h(x,0),\, v_3(x,0),\, \rho(x,0))|_{t=0} = (v_{h0}(x),\, v_{30}(x),\, \rho_0(x)). \end{cases} \tag{1.2.8}$$

In the above system, we assume $v = (v_h,\, v_3), \quad \nabla_h = (\partial_1,\, \partial_2)$.

Understanding the stability properties of 3D rotating Boussinesq equations is crucial because of their application in modeling geophysical fluids (see, e.g., [44], [47]). Knowing these issues well may help explain and predict some of the weather phenomena (see, e.g., [11], [20]). We intend to understand the stability properties of general 3D perturbations near the 2D exact solution (1.2.7), and our main result is stated in the following theorem.

Theorem 1.2.1 *Let the initial data $v_{h0}^{(0)} \in C([0,\infty); H^2(\mathbb{R}^2))$ for the special 2D solution in (1.2.7). Consider (1.2.8) with $\nu > 0$ and $\kappa > 0$. Also, assume that $(v_{h0},\, w_0,\, \rho_0) \in H^2(\mathbb{R}^3)$ with $\nabla_h \cdot v_{h0} + \partial_3 v_{30} = 0$. Then, there exists a constant $\varepsilon_0 = \varepsilon_0(\nu, \kappa) > 0$ such that, if*

$$\|v_{h0} - v_{h0}^{(0)}\|_{H^2} + \|v_{30}\|_{H^2} + \|\rho_0\|_{H^2} \leq \varepsilon$$

for sufficiently small $\varepsilon \leq \varepsilon_0$, then (1.2.8) has a unique global solution

$$(v_h,\, v_3,\, \rho) \in C([0,\infty); L^2), \quad (v_h,\, v_3,\, \rho) \in L^\infty([0,\infty); H^2), \quad (\nabla_h v_h,\, \nabla_h v_3,\, \nabla_h \rho) \in L^2([0,\infty); H^2).$$

Moreover, for constants C, $t > 0$,

$$\|v_h - v_h^{(0)}(t)\|_{H^2} + \|v_3(t)\|_{H^2} + \|\rho(t)\|_{H^2} \le C\,\varepsilon,$$

where $v_h^{(0)}$ is the special 2D solution of (1.2.7).

The above theorem states that when the initial data $(v_{h0},\, v_{30},\, \rho_0)$ of 3D rotating Boussinesq equations is close to the initial data $(v_{h0}^{(0)},\, 0,\, 0)$ of the special 2D solution, then (1.2.8) has a unique global solution that is always close to the special 2D solution given by (1.2.7).

To prove Theorem 1.2.1, we need to establish a global a priori bound of the solution in the H^2-norm, and then prove the local existence and uniqueness. First, we use the bootstrapping argument [52] to establish the existence of a global *a priori* bound. We begin with the equation of the difference $(\widetilde{v}_h,\, v_3,\, \rho)$, where

$$\widetilde{v}_h = v_h - v_h^{(0)}.$$

Then, $(\widetilde{v}_h,\, v_3,\, \rho)$ solves

$$\begin{cases} \partial_t \widetilde{v}_h + \widetilde{v}_h \cdot \nabla_h \widetilde{v}_h + \widetilde{v}_h \cdot \nabla_h v_h^{(0)} + v_h^{(0)} \cdot \nabla_h \widetilde{v}_h + v_3 \partial_3 \widetilde{v}_h + f \widetilde{v}_h^{\perp} = -\nabla_h p + \nu\, \Delta_h \widetilde{v}_h, \\ \partial_t v_3 + \widetilde{v}_h \cdot \nabla_h v_3 + v_h^0 \cdot \nabla_h v_3 + v_3 \partial_3 v_3 = -\partial_3 p + \nu\, \Delta_h v_3 - \rho, \\ \partial_t \rho + \widetilde{v}_h \cdot \nabla_h \rho + v_h^0 \cdot \nabla_h \rho + v_3 \partial_3 \rho = \kappa\, \Delta_h \rho + v_3, \\ (\widetilde{v}_h,\, v_3,\, \rho)|_{t=0} = (\widetilde{v}_{h0},\, v_{30},\, \rho_0). \end{cases} \tag{1.2.9}$$

For a solution of (1.2.9), we define the following energy functional

$$\begin{aligned} E(t) = \quad & \sup_{0 \le \tau \le t} (\|\widetilde{v}_h(\tau)\|_{H^2} + \|v_{30}(\tau)\|_{H^2} + \|\rho(\tau)\|_{H^2}) \\ & + \nu \int_0^t (\|\nabla_h \widetilde{v}_h(\tau)\|_{H^2}^2 + \|\nabla_h v_3(\tau)\|_{H^2}^2) d\tau + \kappa \int_0^t \|\nabla_h \rho(\tau)\|_{H^2}^2 d\tau. \end{aligned}$$

Then, we prove that for any $t > 0$, $E(t)$ satisfies

$$E(t) \leq K_0\, E(0) + C\,(\nu^{-4} + \nu^{-1}\kappa^{-3} + \nu^{-2}\kappa^{-2}) K_0\, E(t)^3, \tag{1.2.10}$$

where C is the constant independent of ν and κ, and

$$K_0 := e^{C\,(\nu^{-1}+\kappa^{-1})(\|v_h^{(0)}\|^2_{H^2(\mathbb{R}^2)} + \|\nabla_h v_h^{(0)}\|^2_{H^2(\mathbb{R}^2)})}. \tag{1.2.11}$$

The bootstrapping argument concludes that, if

$$E(0) := \|\widetilde{v}_{h0}\|_{H^2} + \|v_{30}\|_{H^2} + \|\rho_0\|_{H^2} \leq \varepsilon \tag{1.2.12}$$

for sufficiently small $\varepsilon > 0$, then

$$E(t) \leq C\,\varepsilon$$

for a constant $C > 0$ and for all $0 < t < \infty$, which yields the desired global bound on the solution $(\|\widetilde{v}_h(\tau)\|_{H^2} + \|v_{30}(\tau)\|_{H^2} + \|\rho(\tau)\|_{H^2})$.

Second, we use Friedrich's method to prove the local existence and uniqueness of a solution. To do so, we start with finding the regularized systems by performing the Fourier cutoff of the terms in (1.2.9) and then construct a sequence of approximate solutions $\{(\widetilde{v}_h^{(n)},\, v_3^{(n)},\, \rho^{(n)})\}_{n\in\mathbb{N}}$ to these regularized systems. After that with the help of Bernstein's inequality, and the existence and uniqueness theory for ordinary differential equations on Banach Spaces, we prove the global (in time) existence and uniqueness for each fixed $n \in \mathbb{N}$. Next is to establish uniform (in n) local bounds on $(\widetilde{v}_h^{(n)},\, v_3^{(n)},\, \rho^{(n)})$ in the functional setting

$$L^\infty(0, T; H^2) \cap L^2(0, T; H^3)$$

for a uniform time internal $[0, T]$. Then, we show that the sequence $(\widetilde{v}_h^{(n)}, v_3^{(n)}, \rho^{(n)})$ has a convergent subsequence and its limit solves the Boussinesq system (1.2.9). Finally, we prove the uniqueness.

We discuss existing results related to the stability of the 3D rotating Boussinesq equations. Ma, Wu, and Zhang [43] considered the stability of the perturbation of the 3D rotating Boussinesq equation near a special 2D solution without dissipation and heat diffusion. This special solution illustrates the effect of gravity and provides the solution in terms of Brünt-Väisala frequency, which measures the atmospheric stratification.

For the following standard 3D Boussinesq equation

$$\begin{cases} \partial_t v + v \cdot \nabla v = -\nabla p + \nu\, \Delta v + \rho \mathbf{e}_3, \quad x \in \mathbb{R}^3,\, t > 0, \\ \partial_t \rho + v \cdot \nabla \rho = \kappa\, \Delta \rho, \\ \nabla \cdot v = 0, \end{cases} \tag{1.2.13}$$

Brandolese and Schonbek proved that the solution can grow in time [12]. The stability problem for (1.2.13) remains an open problem. When the viscosity and the thermal diffusivity are different in the several directions, we write (1.2.13) in the following anisotropic form

$$\begin{cases} \partial_t v + v \cdot \nabla v = -\nabla p + \nu_1\, \partial_{11} v + \nu_2\, \partial_{22} v + \nu_3\, \partial_{33} + \rho \mathbf{e}_3, \\ \partial_t \rho + v \cdot \nabla \rho = \kappa_1\, \partial_{11} \rho + \kappa_2\, \partial_{22} \rho + \kappa_3\, \partial_{33} \rho, \\ \nabla \cdot v = 0. \end{cases} , \tag{1.2.14}$$

When $\nu_3 = \kappa_1 = \kappa_3 = 0$, Wu and Zhang [59] investigated the stability and large- time behavior problem for the system. The stability problem they considered is difficult due to the lack of vertical velocity dissipation and horizontal thermal diffusion, therefore, they considered the perturbation near the hydrostatic balance with the spatial domain being $\Omega = \mathbb{R}^2 \times \mathbb{T}$ instead of $\mathbb{R}^3$. The idea they employed was to separate the velocity and the temperature into vertical averages and the corresponding oscillation parts. They proved that the oscillation parts decay to zero exponentially, and the system becomes a 2D flow satisfying the 2D Navier-Stokes Equations.

1.2.2 The 2D Boussinesq Equations

The following 2D Boussinesq equations can be derived formally using the 3D rotating Boussinesq equations [57]:

$$\begin{cases} \partial_t v + v \cdot \nabla v = -\nabla P + \nu \, \Delta v + \rho \mathbf{e}_2, \quad x \in \mathbb{R}^2, \, t > 0, \\ \partial_t \rho + v \cdot \nabla \rho = \kappa \, \Delta \rho, \\ \nabla \cdot v = 0, \end{cases} \tag{1.2.15}$$

where $\mathbf{e}$ is the unit vector in the vertical direction. When both ν, $\kappa > 0$, Cannon and DiBenedetto [13] proved the existence of a global solution for any sufficiently smooth data. Whereas, when $\nu = \kappa = 0$, system (2.1.1) is known as the inviscid Boussinesq equations, and the global well-posedness of inviscid Boussinesq equations for general data remains an open problem.

In addition to sharing the above-mentioned ubiquitous applications of Boussinesq equations, the 2D Boussinesq equations also include rich mathematical structures. For example, the 2D Boussinesq equations possess the same vortex stretching mechanism as the 3D incompressible Euler and Navier-Stokes equations. Moreover, these equations are identical to the 3D incompressible Euler equations for axisymmetric swirling flows [45]. Therefore, a good understanding of the 2D Boussinesq equations may assist in solving the outstanding open problems about the global existence or finite-time blow-up of the smooth solution for the 3D Euler and the Navier -Stokes equations.

In the physical scenarios, when the viscosity and/or the thermal diffusivity are negligible in either the horizontal or the vertical direction, the following anisotropic Boussinesq equations are suitable model to consider:

$$\begin{cases} \partial_t v + v \cdot \nabla v = -\nabla P + \nu_1 \, \partial_{11} v + \nu_2 \, \partial_{22} v + \mathbf{e}_2, \\ \partial_t \rho + v \cdot \nabla \rho = \kappa_1 \, \partial_{11} \rho + \kappa_2 \, \partial_{22} \rho, \\ \nabla \cdot v = 0. \end{cases} , \tag{1.2.16}$$

The local and global posed-posedness of (1.2.16) for various values of $\nu_j = 0 = \kappa_j, j = 1, 2$ in several domian can be found in the literatures (see, e.g [2]- [5], [15], [17], [18], [22], [23], [32]-[34], [36], [39], [40], [55], [61]). We provide a summary of existing results related to the stability and large-time behavior of solutions pertaining to (2.1.1) or (1.2.16) in the physically relevant equilibria.

- Doering, Wu, Zhao, and Zheng [26] considered (2.1.1) with $\kappa = 0$ on a bounded domain with stress-free boundary conditions. They considered the perturbation near the hydrostatic equilibrium to study the stability and large-time behavior of solutions. They established the global existence and uniqueness of classical solutions when the initial data lies in the Sobolev spaces. For the general initial data, they proved that the kinetic energy and first derivative of the velocity field go to zero if time goes to infinity. In addition, they provided the criteria for the linear stability and instability of the corresponding linear system.

- In a follow-up work of [26], Tao, Wu, Zhao, and Zheng [54] considered (2.1.1) when $\kappa = 0$ on the periodic domain with hydrostatic equilibrium. They used the spectral method to study the stability problem. For the linear system, they proved that the velocity field converges to zero uniformly, and for the nonlinear system, they established the nonlinear stability results for the L^2 initial data.

- Castro, Córdoba, and Lear [16] studied the stability and large-time behavior on the 2D Boussinesq Equations (2.1.1) with $\kappa = 0$ and also replaced the dissipative term by the velocity damping term in a spatial domain $\mathbb{T} \times [-1, 1]$ with no-slip boundary condition. They considered the hydrodynamic equilibrium in their work and derived asymptotic stability results.

- Tao and Wu [51] studied the linear stability of the 2D Boussinesq equations (1.2.16) when $\nu_1 = 0$ and $\kappa_1 = 0$. They considered the perturbation near the shear flow with the domain $\Omega = \mathbb{T} \times \mathbb{R}$ or $\Omega = \mathbb{T}^2$. Later, Deng, Wu, and Zhang proved the nonlinear stability using the enhanced dissipation [24].

- Lai, Wu, and Zhong [38] established the global existence and the stability of the 2D Boussinesq equation (1.2.16) where $\nu_1 = 0$ and the thermal damping term instead of the thermal diffusive term. They considered the perturbation near the hydrostatic balance in their research and found the large-time behavior of the velocity gradient and temperature via energy method. For a dissipative system, the energy method does not efficiently find the decay rate of a solution or its derivatives. Therefore, in the follow-up work, Lai, Wu, Xu, Zhang, and Zhong [37] established the optimal decay rates using the spectral method.

- Dong, Wu, Xu, and Zhu studied the stability and exponential decay for the 2D anisotropic Boussinesq equations (1.2.16) where $\nu_2 = \kappa_2 = 0$ [27]. Their study used hydrostatic equilibrium with the spatial domain $\Omega = \mathbb{T} \times \mathbb{R}$.

Other related work on various form of Boussinesq equations in a several physically prominent equilibria can also be found (see, e.g., [41], [58], [63], [62], [8], [9], [46], [60]). Our research is oriented within the 2D Boussinesq equations (1.2.16) with $\nu_1 = \kappa_2 = 0$ and when $\nu_2 = \kappa_1 = 0$, namely,

$$\begin{cases} \partial_t v + v \cdot \nabla v = -\nabla P + \nu_2 \, \partial_{22} v + \rho \, \mathbf{e}_2, \quad x \in \mathbb{R}^2, \quad t > 0, \\ \partial_t \rho + v \cdot \nabla \rho = \kappa_1 \, \partial_{11} \rho, \\ \nabla \cdot v = 0, \end{cases} \tag{1.2.17}$$

and

$$\begin{cases} \partial_t v + v\cdot\nabla v = -\nabla P + \nu_1\,\partial_{11} v + \rho\,\mathbf{e}_2, \quad x\in\Omega=\mathbb{T}\times\mathbb{R}, \quad t>0 \\ \partial_t \rho + v\cdot\nabla\rho = \kappa_2\,\partial_{22}\rho, \\ \nabla\cdot v = 0. \end{cases} \tag{1.2.18}$$

The above Boussinesq systems are asymmetrical, and when we swap the vertical and horizontal dissipation, we get different regularity problems. Therefore the mechanism we adopt for the first system (1.2.17) does not work for the second system (1.2.18).

The plan for the rest of this section is as follows. First, we present our assumption and the stability problem. Second, we state the main issues which we have to overcome. Third, we explain the process in which we overcome these problems. Lastly, we present our main results. As we perturb both systems (1.2.17) and (1.2.18) near the hydrodynamic equilibrium, we perform the first step together for both of them. Except for the first step, the remaining steps will be described separately for each problem.

For both systems (1.2.17) and (1.2.18), we intend to understand the stability and large-time behavior of perturbations near so-called hydrostatic equilibrium (v_{he}, ρ_{he}) with

$$v_{he} = 0, \quad \rho_{he} = x_2.$$

When $v_{he} = 0$, the momentum equation in both systems (1.2.17) and (1.2.18) becomes

$$-\nabla P_{he} + \rho_{he}\,\mathbf{e}_2 = 0.$$

Therefore, when the pressure gradient is equal to the buoyancy force, $(v_{he},\, P_{he},\, \rho_{he})$ is a special steady solution to both systems. We consider the perturbation (u, p, θ) with

$$u = v - v_{he}, \quad p = P - P_{he} \quad \text{and} \quad \theta = \rho - \rho_{he}.$$

Then, for the first system (1.2.17), the equation for $(u,\, p,\, \theta)$ satisfies

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nu_2\, \partial_{22} u + \theta \mathbf{e}_2, \\ \partial_t \theta + u \cdot \nabla \theta + u_2 = \kappa_1\, \partial_{11} \theta, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x). \end{cases} \tag{1.2.19}$$

Similarly, for the second system (1.2.18), $(u,\, p,\, \theta)$ solves

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nu_1\, \partial_{11} u + \theta \mathbf{e}_2, \\ \partial_t \theta + u \cdot \nabla \theta + u_2 = \kappa_2\, \partial_{22} \theta, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x). \end{cases} \tag{1.2.20}$$

The above perturbed equations (1.2.19) and (1.2.20) have an extra term u_2 in the temperature equation than their original counterparts. This additional term u_2 comes because of perturbation and helps balance the buoyancy term in the energy estimate.

Our stability problem is to establish that the solution (u, θ) of (1.2.19) or (1.2.20) corresponding to any sufficiently small initial perturbation (u_0, θ_0) measured in the H^2-norm always remains small. This problem is challenging because of the anisotropic dissipation presented in the systems (1.2.19) and (1.2.20).

Next we explain the issue related to (1.2.19) and how we control it. By taking the curl of the momentum equation in (1.2.17), we get the following vorticity $\omega = \nabla \times u$ equation

$$\partial_t \omega + u \cdot \nabla \omega = \nu_2\, \partial_{22} \omega + \partial_1 \theta, \quad x \in \mathbb{R}^2,\ t > 0. \tag{1.2.21}$$

We can attain a uniform bound on the L^2-norm of the vorticity, but it is difficult to retain the L^2-norm of the vorticity gradient due to lack of horizontal dissipation. Assuming $\theta \equiv 0$ in (1.2.21), we obtain the following partially dissipative 2D Navier-Stokes equations

$$\partial_t \omega + u \cdot \nabla \omega = \nu_2 \, \partial_{22} \omega, \quad x \in \mathbb{R}^2, \ t > 0. \tag{1.2.22}$$

If the initial data $\omega_0 \in H^1(\mathbb{R}^2)$, the equation (1.2.22) has a unique global solution. But, the issue of whether the L^2-norm of the vorticity gradient grows or decays as a function of time is still unsolved. This is because when we estimate $\|\nabla \omega(t)\|_{L^2}$, we get the following energy equality

$$\frac{1}{2} \frac{d}{dt} \|\nabla \omega(t)\|_{L^2}^2 + \nu_2 \|\partial_2 \nabla \omega(t)\|_{L^2}^2 = - \int \nabla \omega \cdot \nabla u \cdot \nabla \omega \, dx. \tag{1.2.23}$$

The integral on the right side of (1.2.23) is difficult to estimate, which is the main issue to overcome. With the hope to apply the anisotropic estimates, we decompose it into the following four components

$$\begin{aligned} \int \nabla \omega \cdot \nabla u \cdot \nabla \omega \, dx \;\; &= \;\; \int \partial_1 u_1 \, (\partial_1 \omega)^2 \, dx + \int \partial_1 u_2 \, \partial_1 \omega \, \partial_2 \omega \, dx \\ &\quad + \int \partial_2 u_1 \, \partial_1 \omega \, \partial_2 \omega \, dx + \int \partial_2 u_2 \, (\partial_2 \omega)^2 \, dx. \end{aligned} \tag{1.2.24}$$

Due to the missing horizontal derivatives in the dissipation, it may not be possible to bound the first two terms on the right side of (1.2.24). Therefore, whether the vorticity gradient of (1.2.22) grows in time remains unsolved.

The interesting question is, when we deal with the stability of our problem (1.2.19), we get the same term as in (1.2.24); how would it be possible to deal with the same difficulty for a more complex system like (1.2.19)? We are able to resolve this issue by performing the interaction between the velocity and temperature within the system. In fact, the reason behind this success is due to the effect of temperature on the buoyancy-driven fluids. We now present the summary of the procedure.

The first step of understanding the stability of a nonlinear system is to look at the corresponding linear system. Therefore, we separate the linear terms from the nonlinear ones by eliminating the pressure term in (1.2.19). The tool we use is the Helmholtz-Leray projection $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$. Applying $\mathbb{P}$ to the velocity equation yields

$$\partial_t u = \nu_2 \partial_{22} u + \mathbb{P}(\theta \mathbf{e}_2) - \mathbb{P}(u \cdot \nabla u). \tag{1.2.25}$$

By the definition of $\mathbb{P}$,

$$\mathbb{P}(\theta \mathbf{e}_2) = \theta \mathbf{e}_2 - \nabla\Delta^{-1}\nabla \cdot (\theta \mathbf{e}_2) = \begin{bmatrix} -\partial_1\partial_2\Delta^{-1}\theta \\ \theta - \partial_2^2\Delta^{-1}\theta \end{bmatrix}. \tag{1.2.26}$$

Substituting (1.2.26) in (1.2.25) and writing (1.2.25) in the component equations, we obtain

$$\begin{cases} \partial_t u_1 = \nu_2\, \partial_{22} u_1 - \partial_1\partial_2\Delta^{-1}\theta + N_1, \\ \partial_t u_2 = \nu_2\, \partial_{22} u_2 + \partial_1\partial_1\Delta^{-1}\theta + N_2, \end{cases} \tag{1.2.27}$$

where N_1 and N_2 are the nonlinear terms,

$$N_1 = -(u \cdot \nabla u_1 - \partial_1\Delta^{-1}\nabla \cdot (u \cdot \nabla u)), \quad N_2 = -(u \cdot \nabla u_2 - \partial_2\Delta^{-1}\nabla \cdot (u \cdot \nabla u)).$$

By differentiating the first equation of (1.2.27) in t, using the temperature equation from (1.2.19) and divergence-free condition, we can convert (1.2.19) into the following damped degenerate wave-type system

$$\begin{cases} \partial_{tt} u - (\kappa_1\partial_{11} + \nu_2\partial_{22})\partial_t u + \nu_2\kappa_1\partial_{11}\partial_{22} u + \partial_{11}\Delta^{-1} u = N_3, \\ \partial_{tt}\theta - (\kappa_1\partial_{11} + \nu_2\partial_{22})\partial_t \theta + \nu_2\kappa_1\partial_{11}\partial_{22}\theta + \partial_{11}\Delta^{-1}\theta = N_4, \end{cases} \tag{1.2.28}$$

where N_3 and N_4 contain the nonlinear terms.

This new wave type system (1.2.28) displays all the smoothing and stabilization hidden in the previous system (1.2.19). The momentum equation in (1.2.19) involves only the vertical dissipation, but the system (1.2.28) has the extra horizontal smoothing term, $\partial_{11}\Delta^{-1}u$. This extra horizontal regularization implies that the temperature plays an important role in stabilizing the fluids.

Next, we present our main result corresponding to the nonlinear system (1.2.19). The main results corresponding to the linearized system are presented in Section 2.1.

Theorem 1.2.2 *[10] Consider (1.2.19) with $\nu_2 > 0$ and $\kappa_1 > 0$. Assume that initial data $(u_0, \theta_0) \in H^2(\mathbb{R}^2)$ with $\nabla \cdot u_0 = 0$. Then there exists $\varepsilon = \varepsilon(\nu_2, \kappa_1) > 0$ such that, if (u_0, θ_0) satisfies*

$$\|u_0\|_{H^2} + \|\theta_0\|_{H^2} \leq \varepsilon,$$

then (1.2.19) has a unique global solution (u, θ) satisfying, for any $t > 0$,

$$\begin{aligned}\|u(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2 + \nu_2 \int_0^t \|\partial_2 u\|_{H^2}^2 \, d\tau \\ +\kappa_1 \int_0^t \|\partial_1 \theta\|_{H^2}^2 \, d\tau + C(\nu_2, \kappa_1) \int_0^t \|\partial_1 u_2\|_{L^2}^2 \, d\tau \leq C\, \varepsilon^2,\end{aligned}$$

where $C(\nu_2, \kappa_1) > 0$ and $C > 0$ are constants.

To prove Theorem 1.2.2, we have to use the extra regularization due to the wave structure in (1.2.28). In fact, the control on the time integral of the horizontal derivative of the velocity field, namely

$$\int_0^t \|\partial_1 u(\tau)\|_{L^2}^2 \, d\tau \tag{1.2.29}$$

plays a crucial role in the proof. Clearly, the uniform boundedness of (1.2.29) is not a consequence of the vertical dissipation in the velocity equation but is due to the interaction with the temperature equation. In addition to understanding the time integrability of (1.2.29) from the wave structure derived before, we can also view it as the special coupling in the system (1.2.19), which enables us to transfer the time integrability from one function in the system to another. Our main tool in obtaining the H^2 nonlinear stability is the bootstrapping argument [52].

Now, we present the issue related to (1.2.20) and how we overcome it. The system (1.2.20) can also be converted into the following degenerate wave type system using the similar process described above for (1.2.19):

$$\begin{cases} \partial_{tt}u - (\kappa_2\partial_{22} + \nu_1\partial_{11})\partial_t u + \nu_1\kappa_2\partial_{11}\partial_{22}u + \partial_{11}\Delta^{-1}u = N_5, \\ \partial_{tt}\theta - (\kappa_2\partial_{22} + \nu_1\partial_{11})\partial_t\theta + \nu_1\kappa_2\partial_{11}\partial_{22}\theta + \partial_{11}\Delta^{-1}\theta = N_6, \end{cases} \tag{1.2.30}$$

where N_5 and N_6 contain the nonlinear terms.

The new wave type systems (1.2.30) also display all the smoothing and stabilization hidden in the previous system (1.2.20). We again obtained the additional smoothing term, $\partial_{11}\Delta^{-1}u$, in the horizontal direction. The extra regularization available for the system (1.2.20) is, unfortunately, not helpful as in the previous system (1.2.19). This is because the system (1.2.20) already has the horizontal dissipation in the momentum equation, and we are looking for the vertical regularization. Due to the lack of horizontal dissipation, it does not appear possible to establish the stability of (1.2.20) in the whole space $\mathbb{R}^2$. However, we resolve this problem by considering a different spatial domain $\Omega = \mathbb{T} \times \mathbb{R}$. We explain how to use this domain Ω in our stability problem. Since the horizontal variable lies in a periodic domain, we represent the Fourier transform in the horizontal variable as a sequence of Fourier modes. Then, we separate the zeroth horizontal Fourier mode from the non-zero ones. To explain this separating process, we first introduce several notations.

We consider a function $f = f(x_1, x_2)$ on the domain $\mathbb{T} \times \mathbb{R}$ that can be integrated in x_1 over the 1D periodic box $\mathbb{T} = [0, 1]$ and define its the horizontal average $\overline{f}$ by

$$\overline{f}(x_2) = \int_{\mathbb{T}} f(x_1, x_2)dx_1. \tag{1.2.31}$$

Note that $\overline{f}$ is the zeroth Fourier mode of f. We decompose f into two parts; the horizontal average $\overline{f}$ and the corresponding oscillation part $\widetilde{f}$,

$$f = \overline{f} + \widetilde{f}. \tag{1.2.32}$$

The oscillation part $\widetilde{f}$ contains all non-zero Fourier modes. The decomposition (1.2.32) has the following special properties. This decomposition is orthogonal in the Sobolev space $H^k(\Omega)$ for any non-negative integer k, i.e., we have

$$(\overline{f}, \widetilde{f})_{\dot{H}^k(\Omega)} = 0,$$

where $(g, h)_{\dot{H}^k(\Omega)}$ denotes the inner product in the homogeneous Sobolev space $\dot{H}^k$. Moreover, $\widetilde{f}$ admits strong versions of the Poincaré type inequality

$$\|\widetilde{f}\|_{L^2(\Omega)} \le C\,\|\partial_1 \widetilde{f}\|_{L^2(\Omega)}, \quad \|\widetilde{f}\|_{L^\infty(\Omega)} \le C\,\|\partial_1 \widetilde{f}\|_{H^1(\Omega)}.$$

We prove the following main theorem for the system (1.2.20), which establishes the H^2-stability.

Theorem 1.2.3 *[1] Let $\mathbb{T} = [0,1]$ be a 1D periodic box and let $\Omega = \mathbb{T} \times \mathbb{R}$. Assume $u_0, \theta_0 \in H^2(\Omega)$ and $\nabla \cdot u_0 = 0$. Then there exists $\varepsilon = \varepsilon(\nu_1, \kappa_2) > 0$ such that, if*

$$\|u_0\|_{H^2} + \|\theta_0\|_{H^2} \le \varepsilon,$$

then (1.2.20) has a unique global solution (u, θ) that always remains uniformly bounded, for *any* $t \geq 0$*,*

$$\|u(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2 + 2\nu_1 \int_0^t \|\partial_1 u(\tau)\|_{H^2}^2 d\tau$$
$$+ 2\kappa_2 \int_0^t \|\partial_2 \theta(\tau)\|_{H^2}^2 d\tau + \delta \int_0^t \|\partial_1 \theta(\tau)\|_{L^2}^2 d\tau \leq C\varepsilon^2$$

for some constants $\delta = C(\nu_1, \kappa_2) > 0$ *and* $C > 0$*.*

The above Theorem 1.2.3 implies that the solution of (1.2.20) starting from any small initial perturbation measured in the H^2-norm is always global in time and also remains close to the initial data. The above stability result is possible due to the unique nature of the domain $\Omega = \mathbb{T} \times \mathbb{R}$ we considered. The following decompositions

$$u = \overline{u} + \widetilde{u}, \quad \text{and} \quad \theta = \overline{\theta} + \widetilde{\theta} \tag{1.2.33}$$

help manage the nonlinear terms. However, when the spatial domain is the whole space $\mathbb{R}^2$, we do not have such a decomposition, and the stability problem on (1.2.20) in $\mathbb{R}^2$ remains open.

Theorem 1.2.3 also implies that $\|\partial_1 \theta(\tau)\|_{L^2}^2$ is time integrable. The temperature equation lacks horizontal dissipation in the perturbed system (1.2.20). This extra horizontal regularization shows the effect of temperature due to the coupling and interaction within the system.

The following Theorem 1.2.4 proves the fact that was observed through the numerical simulations of buoyancy-driven stratified fluids by Doering et. al. [26]. They found that the 2D Boussinesq equations without thermal dissipation perturbed near the hydrostatic equilibrium to stratify and finally approach their horizontal averages while the oscillation parts decay to zero. The following Theorem 1.2.4 proves that the oscillation parts $\widetilde{u}$ and $\widetilde{\theta}$ decays to zero at algebraic rates.

Theorem 1.2.4 *[1] Let* $u_0, \theta_0 \in H^2(\Omega)$ *with* $\nabla \cdot u_0 = 0$. *Assume that* (u_0, θ_0) *satisfies*

$$\|u_0\|_{H^2} + \|\theta_0\|_{H^2} \leq \varepsilon\,,$$

for sufficiently small $\varepsilon > 0$. *Let* (u, θ) *be the corresponding solution of (1.2.20). Then the oscillation part* $(\widetilde{u}, \widetilde{\theta})$ *satisfies the following algebraic decay in time,*

$$\|\widetilde{u}\|_{H^1} + \|\widetilde{\theta}\|_{H^1} \leq c(1+t)^{-\frac{1}{2}},$$

for some constant $c > 0$ *and for all* $t \geq 0$. *In addition,* $(\widetilde{u}, \widetilde{\theta})$ *has the asymptotic behavior, as* $t \to \infty$,

$$t\,(\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2) \to 0.$$

Theorem 1.2.4 implies that the solution $(u,\, \theta)$ of (1.2.20) approaches asymptotically to the horizontal average $(\overline{u}, \overline{\theta})$, and the Boussinesq system (1.2.20) ultimately transform to the following 1D system

$$\begin{cases} \partial_t \overline{u} + \overline{u \cdot \nabla \widetilde{u}} + \begin{pmatrix} 0 \\ \partial_2 \overline{p} \end{pmatrix} = \begin{pmatrix} 0 \\ \overline{\theta} \end{pmatrix}, \\ \partial_t \overline{\theta} + \overline{u \cdot \nabla \widetilde{\theta}} = \kappa_2 \partial_2^2 \overline{\theta}. \end{cases}$$

The remaining part of this dissertation is divided into two chapters. In Section 2.1 of Chapter II, we present the results corresponding to the long-time behavior of the linear system (1.2.28) without their proofs and the main lines in the theorem (1.2.2). In Section 2.2 of Chapter II, we prove Theorems 1.2.3 and 1.2.4. Chapter III proves Theorem 1.2.1.

CHAPTER II

2D PARTIALLY DISSIPATIVE BOUSSINESQ EQUATIONS

This chapter summarizes the main results from the author's joint works ([10], [1]). We discuss the stability and large-time behavior of solutions to two different 2D anisotropic Boussinesq equations. Section 2.1 considers the 2D anisotropic Boussinesq equations only with vertical velocity dissipation and horizontal thermal diffusion. Section 2.2 contains the 2D anisotropic Boussinesq equations with horizontal velocity dissipation and vertical thermal diffusion.

2.1 2D Boussinesq Equations with Horizontal Velocity Dissipation and Vertical Thermal Diffusion

In this section, we discuss the stability and large-time behavior of a solution of the 2D anisotropic Boussinesq equations (2.1.1), which are suitable models for specific physical regimes like Prandtl's equation

$$\begin{cases} \partial_t v + v \cdot \nabla v = -\nabla P + \nu_2 \, \Delta v + \rho \mathbf{e}_2, \quad x \in \mathbb{R}^2, \, t > 0, \\ \partial_t \rho + v \cdot \nabla \rho = \kappa_1 \, \Delta \rho, \\ \nabla \cdot v = 0. \end{cases} \tag{2.1.1}$$

We recall from Chapter I that when we perturb (2.1.1) with

$$u = v - v_{he}, \quad p = P - P_{he}, \quad \text{and} \quad \theta = \rho - \rho_{he},$$

$(u,\, p,\, \theta)$ satisfies

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nu_2 \, \partial_{22} u + \theta \mathbf{e}_2, \\ \partial_t \theta + u \cdot \nabla \theta + u_2 = \kappa_1 \, \partial_{11} \theta, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x). \end{cases} \tag{2.1.2}$$

In the following sections, we present the main results concerning the large-time behavior of a solution to the linearized system corresponding to (1.2.28) without proof and the main lines in the proof of Theorem 1.2.2. For the complete proof, we refer to the original article [10].

2.1.1 Large-time Behavior of Linearized System

To understand the large-time behavior of the linearized system corresponding to (2.1.2) or (1.2.28), we first find its explicit solution in terms of Fourier multiplier operators. Then, we calculate the precise upper bound of these Fourier multiplier operators. Furthermore, we exhibit the precise exponential decay rate for the Fourier frequency piece of the solution. We consider the following linearized system of (1.2.28) with the initial conditions, namely

$$\begin{cases} \partial_{tt} u - (\kappa_1 \partial_{11} + \nu_2 \partial_{22}) \partial_t u + \nu_2 \kappa_1 \partial_{11} \partial_{22} u + \partial_{11} \Delta^{-1} u = 0, \\ \partial_{tt} \theta - (\kappa_1 \partial_{11} + \nu_2 \partial_{22}) \partial_t \theta + \nu_2 \kappa_1 \partial_{11} \partial_{22} \theta + \partial_{11} \Delta^{-1} \theta = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x). \end{cases} \tag{2.1.3}$$

The following proposition provides the solution of (2.1.3) explicitly in terms of kernel functions and the initial data.

Proposition 2.1.1 *The solution of (2.1.3) can be explicitly represented as*

$$u_1(t) = K_1(t)\, u_{10} + K_2(t)\, \theta_0, \tag{2.1.4}$$

$$u_2(t) = K_1(t)\, u_{20} + K_3(t)\, \theta_0, \tag{2.1.5}$$

$$\theta(t) = K_4(t)\, u_{20} + K_5(t)\, \theta_0, \tag{2.1.6}$$

where K_1 through K_5 are Fourier multiplier operators with their symbols given by

$$K_1(\xi,t) = G_2(\xi,t) - \nu_2\xi_2^2 G_1(\xi,t), \qquad K_2(\xi,t) = -\frac{\xi_1\xi_2}{|\xi|^2}\, G_1(\xi,t), \tag{2.1.7}$$

$$K_3(\xi,t) = \frac{\xi_1^2}{|\xi|^2}\, G_1(\xi,t), \quad K_4 = -G_1, \quad K_5(\xi,t) = G_2(\xi,t) - \kappa_1\xi_1^2 G_1(\xi,t). \tag{2.1.8}$$

Here G_1 and G_2 are two explicit symbols involving the roots λ_1 and λ_2 of the characteristic equation

$$\lambda^2 + (\kappa_1\xi_1^2 + \nu_2\xi_2^2)\lambda + \nu_2\kappa_1\xi_1^2\xi_2^2 + \frac{\xi_1^2}{|\xi|^2} = 0$$

$$\lambda_1 = -\frac{1}{2}(\kappa_1\xi_1^2 + \nu_2\xi_2^2) - \frac{1}{2}\sqrt{(\kappa_1\xi_1^2 + \nu_2\xi_2^2)^2 - 4\left(\nu_2\kappa_1\xi_1^2\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}\right)},$$

$$\lambda_2 = -\frac{1}{2}(\kappa_1\xi_1^2 + \nu_2\xi_2^2) + \frac{1}{2}\sqrt{(\kappa_1\xi_1^2 + \nu_2\xi_2^2)^2 - 4\left(\nu_2\kappa_1\xi_1^2\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}\right)}.$$

More precisely, when $\lambda_1 \neq \lambda_2$,

$$G_1(\xi,t) = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}, \qquad G_2(\xi,t) = \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2}. \tag{2.1.9}$$

When $\lambda_1 = \lambda_2$,

$$G_1(\xi,t) = te^{\lambda_1 t}, \qquad G_2(\xi,t) = e^{\lambda_1 t} - \lambda_1 t\, e^{\lambda_1 t}. \tag{2.1.10}$$

The proof of the above Proposition 2.1.1 is an immediate consequence of the following Lemma 2.1.1 that solves the degenerate damped wave equation explicitly.

Lemma 2.1.1 *Assume that f satisfies the damped degenerate wave type equation*

$$\begin{cases} \partial_{tt} f - (\nu\partial_{22} + \eta\partial_{11})\partial_t f + \eta\nu\partial_{11}\partial_{22} f + \partial_{11}\Delta^{-1} f = F, \\ f(x,0) = f_0(x), \quad (\partial_t f)(x,0) = f_1(x). \end{cases} \tag{2.1.11}$$

Then f can be explicitly represented as

$$f(t) = G_1(t)\, f_1 + G_2(t)\, f_0 + \int_0^t G_1(t-\tau)\, F(\tau)\, d\tau, \tag{2.1.12}$$

where G_1 and G_2 are two Fourier multiplier operators with their symbols given by

$$G_1(\xi,t) = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}, \qquad G_2(\xi,t) = \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2} \tag{2.1.13}$$

with λ_1 and λ_2 being the roots of the characteristic equation

$$\lambda^2 + (\eta\xi_1^2 + \nu\xi_2^2)\lambda + \nu\eta\xi_1^2\xi_2^2 + \frac{\xi_1^2}{|\xi|^2} = 0 \tag{2.1.14}$$

or

$$\begin{aligned} \lambda_1 &= -\frac{1}{2}(\eta\xi_1^2 + \nu\xi_2^2) - \frac{1}{2}\sqrt{(\eta\xi_1^2 + \nu\xi_2^2)^2 - 4\left(\nu\eta\xi_1^2\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}\right)}, \\ \lambda_2 &= -\frac{1}{2}(\eta\xi_1^2 + \nu\xi_2^2) + \frac{1}{2}\sqrt{(\eta\xi_1^2 + \nu\xi_2^2)^2 - 4\left(\nu\eta\xi_1^2\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}\right)}. \end{aligned} \tag{2.1.15}$$

When $\lambda_1 = \lambda_2$, (2.1.12) remains valid if we replace G_1 and G_2 in (2.1.13) by their corresponding limit form, namely,

$$G_1(\xi,t) = \lim_{\lambda_2 \to \lambda_1} \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} = t e^{\lambda_1 t}$$

and

$$G_2(\xi,t) = \lim_{\lambda_2 \to \lambda_1} \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2} = e^{\lambda_1 t} - \lambda_1 t e^{\lambda_1 t}.$$

To establish the actual large-time behavior of the solutions to (2.1.3), we need to find the upper bounds for the kernel functions K_1 through K_5 in Proposition 2.1.1. These kernels depend on the frequency ξ and are not uniform in every direction. Therefore, to understand the definite behavior of these kernel functions, we divide the whole frequency space $\mathbb{R}^2$ into the following subdomains

$$S_1 = \left\{ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2,\ \nu_2 \kappa_1 \xi_1^2\, \xi_2^2 + \xi_1^2\, |\xi|^{-2} \geq \frac{3}{16} (\nu_2 \xi_2^2 + \kappa_1 \xi_1^2)^2 \right\}, \tag{2.1.16}$$

$$S_2 = \mathbb{R}^2 \setminus S_1. \tag{2.1.17}$$

The following proposition reveals the behavior of the kernel functions in the above subdomains.

Proposition 2.1.1 *Assume the kernel functions K_1 through K_5 are given by (2.1.7) and (2.1.8) with G_1 and G_2 defined in (2.1.9) and (2.1.10). The sets S_1 an S_2 are defined above by (2.1.16) and (2.1.17). Then, the kernel functions K_1 through K_5 can then be bounded as follows.*

(a) Let $\xi \in S_1$. Then

$$Re\lambda_1 \leq -\frac{1}{2}(\nu_2 \xi_2^2 + \kappa_1 \xi_1^2), \qquad Re\lambda_2 \leq -\frac{1}{4}(\nu_2 \xi_2^2 + \kappa_1 \xi_1^2),$$

where Re denotes the real part, and, for constants $c_0 > 0$ and $C > 0$,

$$|K_1(\xi, t)|, |K_5(\xi, t)| \leq C\, e^{-c_0 |\xi|^2 t}, \tag{2.1.18}$$

$$|K_2(\xi, t)|, |K_3(\xi, t)|, |K_4(\xi, t)| \leq C\, t\, e^{-c_0 |\xi|^2 t}. \tag{2.1.19}$$

(b) Let $\xi \in S_2$. Then

$$\lambda_1 \leq -\frac{3}{4}(\nu \xi_2^2 + \eta \xi_1^2), \qquad \lambda_2 \leq -\frac{\nu_2 \kappa_1 \xi_1^2 \xi_2^2 + \xi_1^2 |\xi|^{-2}}{\nu_2 \xi_2^2 + \kappa_1 \xi_1^2},$$

$$|K_1|,\, |K_5| \le C\, e^{-\frac{3}{4}(\nu_2\xi_2^2+\kappa_1\xi_1^2)t} + C\, e^{-\frac{\nu_2\kappa_1\xi_1^2\xi_2^2+|\xi_1|^2|\xi|^{-2}}{\nu_2\xi_2^2+\kappa_1\xi_1^2}t} \tag{2.1.20}$$

and

$$\begin{aligned}
|K_2| &\le \frac{C|\xi_1||\xi_2|}{|\xi|^4}\, e^{-c_0\,|\xi|^2 t} + \frac{C|\xi_1||\xi_2|}{|\xi|^4}\, e^{-c_0\frac{\xi_1^2\xi_2^2}{|\xi|^2}t}\, e^{-c_0\,\frac{\xi_1^2}{|\xi|^4}t}, \\
|K_3| &\le \frac{C|\xi_1|^2}{|\xi|^4}\, e^{-c_0\,|\xi|^2 t} + \frac{C|\xi_1|^2}{|\xi|^4}\, e^{-c_0\frac{\xi_1^2\xi_2^2}{|\xi|^2}t}\, e^{-c_0\,\frac{\xi_1^2}{|\xi|^4}t}, \\
|K_4| &\le \frac{C}{|\xi|^2}\, e^{-c_0\,|\xi|^2 t} + \frac{C}{|\xi|^2}\, e^{-c_0\frac{\xi_1^2\xi_2^2}{|\xi|^2}t}\, e^{-c_0\,\frac{\xi_1^2}{|\xi|^4}t}.
\end{aligned} \tag{2.1.21}$$

To establish the actual large-time behavior of the solutions to (2.1.3), along with the above upper bounds for the kernel functions, we need Lemma 2.1.1, which provides the explicit decay rate for the heat kernel associated with a fractional Laplacian Λ^α ($\alpha \in \mathbb{R}$). The fractional Laplacian operator can be defined using the Fourier transform

$$\widehat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \widehat{f}(\xi). \tag{2.1.22}$$

The proof of the following Lemma can be found in [56].

Lemma 2.1.1 *Let $\alpha \ge 0$, $\beta > 0$ and $1 \le q \le p \le \infty$. Then there exists a constant C such that, for any $t > 0$,*

$$\|\Lambda^\alpha e^{-\Lambda^\beta t} f\|_{L^p(\mathbb{R}^d)} \le\, C\, t^{-\frac{\alpha}{\beta}-\frac{d}{\beta}(\frac{1}{q}-\frac{1}{p})}\, \|f\|_{L^q(\mathbb{R}^d)}.$$

Furthermore, we also make use of the fractional operators Λ_i^σ with $i = 1, 2$ defined by

$$\widehat{\Lambda_i^\sigma f}(\xi) = |\xi_i|^\sigma \widehat{f}(\xi), \quad \xi = (\xi_1, \xi_2).$$

To reflect the anisotropic behavior of the solutions, we need to apply the following anisotropic Sobolev spaces. For $s \ge 0$ and $\sigma \ge 0$, the anisotropic Sobolev space $\mathring{H}_1^{s,-\sigma}(\mathbb{R}^2)$ consists of functions f satisfying

$$\|f\|_{\mathring{H}_1^{s,-\sigma}(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} |\xi|^{2s}\, |\xi_1|^{-2\sigma} |\widehat{f}(\xi)|^2\, d\xi\right)^{\frac{1}{2}} < \infty.$$

Similarly, $\mathring{H}_2^{s,-\sigma}(\mathbb{R}^2)$ consists of functions f satisfying

$$\|f\|_{\mathring{H}_2^{s,-\sigma}(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} |\xi|^{2s}\, |\xi_2|^{-2\sigma} |\widehat{f}(\xi)|^2\, d\xi\right)^{\frac{1}{2}} < \infty.$$

Moreover, we write $\mathring{H}^{s,-\sigma}(\mathbb{R}^2) = \mathring{H}_1^{s,-\sigma}(\mathbb{R}^2) \cap \mathring{H}_2^{s,-\sigma}(\mathbb{R}^2)$ with the norm given by

$$\|f\|_{\mathring{H}^{s,-\sigma}(\mathbb{R}^2)} = \|f\|_{\mathring{H}_1^{s,-\sigma}(\mathbb{R}^2)} + \|f\|_{\mathring{H}_2^{s,-\sigma}(\mathbb{R}^2)}.$$

Theorem 2.1.1 *Consider the linearized system (2.1.3) with the initial data u_0 and θ_0 satisfying $\nabla \cdot u_0 = 0$ and*

$$u_0 \in \mathring{H}^{0,-\sigma} \cap \mathring{H}^{s,-\sigma} \cap \mathring{H}^{s-2,-\sigma}, \quad \theta_0 \in \mathring{H}^{0,-\sigma} \cap \mathring{H}^{s,-\sigma} \cap \mathring{H}^{s-1,-\sigma},$$

where $s \geq 0$ and $\sigma \geq 0$ satisfy $s+\sigma \geq 2$. Then for some constant $C > 0$, the corresponding solution (u,θ) to (2.1.3) satisfies,

$$\begin{aligned}
\|u_1(t)\|_{\mathring{H}^s} &\leq C\, t^{-\frac{1}{2}(s+\sigma)}\, \|u_{10}\|_{\mathring{H}^{0,-\sigma}} + C\, t^{-\frac{\sigma}{2}}\, \|u_{10}\|_{\mathring{H}^{s,-\sigma}} \\
&\quad + C\, t^{-\frac{1}{2}(s+\sigma)+1}\, \|\theta_0\|_{\mathring{H}^{0,-\sigma}} + C\, t^{-\frac{1}{2}-\frac{\sigma}{2}}\, \|\theta_0\|_{\mathring{H}^{s-1,-\sigma}}, \\
\|u_2(t)\|_{\mathring{H}^s} &\leq C\, t^{-\frac{1}{2}(s+\sigma)}\, \|u_{20}\|_{\mathring{H}^{0,-\sigma}} + C\, t^{-\frac{\sigma}{2}}\, \|u_{20}\|_{\mathring{H}^{s,-\sigma}} \\
&\quad + C\, t^{-\frac{1}{2}(s+\sigma)+1}\, \|\theta_0\|_{\mathring{H}^{0,-\sigma}} + C\, t^{-1-\frac{\sigma}{2}}\, \|\theta_0\|_{\mathring{H}^{s,-\sigma}}, \\
\|\theta(t)\|_{\mathring{H}^s} &\leq C\, t^{-\frac{1}{2}(s+\sigma)+1}\, \|u_{20}\|_{\mathring{H}^{0,-\sigma}} + C\, t^{-\frac{\sigma}{2}}\, \|u_{20}\|_{\mathring{H}^{s-2,-\sigma}} \\
&\quad + C\, t^{-\frac{1}{2}(s+\sigma)}\, \|\theta_0\|_{\mathring{H}^{0,-\sigma}} + C\, t^{-\frac{\sigma}{2}}\, \|\theta_0\|_{\mathring{H}^{s,-\sigma}},
\end{aligned}$$

where $\mathring{H}^s$ denotes the standard homogeneous Sobolev space with its norm defined by

$$\|f\|_{\mathring{H}^s} = \||\xi|^s\, |\widehat{f}(\xi)|\|_{L^2(\mathbb{R}^2)}.$$

Next we present the result concerning the effects of stabilizing and regularization of the wave structure via energy method. This can be done by considering a suitable Lyapunov functional and computing their time evolution. We show that the frequencies away from the two axes in the frequency space decay exponentially to zero as $t \to \infty$. First, we define a frequency cutoff function for $a_1 > 0$ and $a_2 > 0$,

$$\widehat{\varphi}(\xi) = \widehat{\varphi}(\xi_1, \xi_2) = \begin{cases} 0, & \text{if} \quad |\xi_1| \le a_1 \ \text{ or } |\xi_2| \le a_2, \\ 1, & \text{otherwise.} \end{cases} \tag{2.1.23}$$

Theorem 2.1.1 *Let $\nu_2 > 0$ and $\kappa_1 > 0$. Consider the linearized system in (2.1.3) or equivalently*

$$\begin{cases} \partial_t u_1 = \nu_2\, \partial_{22} u_1 - \Delta^{-1}\partial_1\partial_2\theta, \\ \partial_t u_2 = \nu_2\, \partial_{22} u_2 + \Delta^{-1}\partial_1\partial_1\theta, \\ \partial_t \theta = \kappa_1\, \partial_{11}\theta - u_2, \\ (u_1, u_2, \theta)(x, 0) = (u_{10}, u_{20}, \theta_0). \end{cases}$$

Let (u, θ) be the corresponding solution. The Fourier frequency piece of (u, θ) away from the two axes of the frequency space decays exponentially in time to zero. More precisely, if $(u_0, \theta_0) \in H^2(\mathbb{R}^2)$ with $\nabla \cdot u_0 = 0$, then there is constant $C_0 = C_0(\nu_2, \kappa_1, a_1, a_2)$ such that, for all $t \ge 0$,

$$\|\partial_t(\varphi * u)(t)\|_{L^2}^2 + \|(\varphi * u)(t)\|_{H^1}^2 \le C\,(\|\varphi * u_0\|_{H^2}^2 + \|\varphi * \theta_0\|_{L^2}^2)\, e^{-C_0 t}, \tag{2.1.24}$$

$$\|\partial_t(\varphi * \theta)(t)\|_{L^2}^2 + \|(\varphi * \theta)(t)\|_{H^1}^2 \le C\,(\|\varphi * \theta_0\|_{H^2}^2 + \|\varphi * u_0\|_{L^2}^2)\, e^{-C_0 t}, \tag{2.1.25}$$

where φ is as defined in (2.1.23) and $C = C(\nu_2, \kappa_1, a_1, a_2) > 0$ is a constant.

2.1.2 Stability of Nonlinear System

This section summarizes the nonlinear stability result corresponding to (2.1.2). As mentioned in Chapter I, the main obstacle is finding a suitable upper bound on the nonlinear term (1.2.24). Due to the same obstacle, the stability problem on the partially dissipative 2D Navier-Stokes equations remains unsolved. Nevertheless, the effect of the temperature on the fluid helps to establish the stability of the coupled nonlinear system (2.1.2). We use the following lemma to take advantage of the anisotropic dissipation, whose proof can be found in [14].

Lemma 2.1.2 *Assume that f, g, $\partial_2 g$, h and $\partial_1 h$ are all in $L^2(\mathbb{R}^2)$. Then, for some constant $C > 0$,*

$$\int_{\mathbb{R}^2} |fgh|\, dx \le C\|f\|_{L^2}\|g\|_{L^2}^{\frac{1}{2}}\|\partial_2 g\|_{L^2}^{\frac{1}{2}}\|h\|_{L^2}^{\frac{1}{2}}\|\partial_1 h\|_{L^2}^{\frac{1}{2}}.$$

Now we outline the main ideas of the proof of Theorem 1.2.2

The Main Ideas of Proof of Theorem 1.2.2. We start with constructing a suitable energy functional

$$\begin{aligned} E(t) &= \max_{0\le\tau\le t}(\|u(\tau)\|_{H^2}^2 + \|\theta(\tau)\|_{H^2}^2) + 2\nu_2\int_0^t \|\partial_2 u\|_{H^2}^2 d\tau \\ &\quad +2\kappa_1\int_0^t \|\partial_1\theta\|_{H^2}^2 d\tau + \delta\int_0^t \|\partial_1 u_2\|_{L^2}^2\, d\tau, \end{aligned} \tag{2.1.26}$$

where $\delta > 0$ is a suitably selected parameter. Then, we show that $E(t)$ satisfies

$$E(t) \le C\,E(0) + C\,E(t)^{\frac{3}{2}}, \tag{2.1.27}$$

and apply the bootstrapping argument to (2.1.27) to obtain the nonlinear stability results. Proving (2.1.27) requires two main steps. The first step is to estimate the H^2-norm of (u, θ)

and the second step is to estimate $\|\partial_1 u\|_{L^2}^2$ with its time integral.

Since u is a divergence-free vector field, we have $\|\nabla u\|_{L^2} = \|\omega\|_{L^2}, \|\Delta u\|_{L^2} = \|\nabla\omega\|_{L^2}$. Therefore, establishing the H^2-norm of u is equivalent to the sum of the L^2-norm of u, ω, and $\nabla\omega$. We begin with estimating the L^2-norm of (u,θ) by taking the inner product of (u,θ) with the first two equations in (2.1.2) which yields

$$\begin{aligned}&\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 + 2\nu_2\int_0^t \|\partial_2 u(\tau)\|_{L^2}^2\,d\tau + 2\kappa_1\int_0^t \|\partial_1\theta(\tau)\|_{L^2}^2 d\tau \\ &= \|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2. \end{aligned} \tag{2.1.28}$$

To estimate the L^2-norm of $(\omega, \nabla\theta)$, we use the following equations for ω and θ,

$$\begin{aligned} &\partial_t\omega + u\cdot\nabla\omega = \nu_2\partial_{22}\omega + \partial_1\theta, \\ &\partial_t\theta + u\cdot\nabla\theta + u_2 = \kappa_1\partial_{11}\theta. \end{aligned} \tag{2.1.29}$$

Applying ∇ to the second equation of (2.1.29) and dotting $(\omega,\nabla\theta)$ with the equations of ω and $\nabla\theta$ yields

$$\frac{1}{2}\frac{d}{dt}(\|\omega\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) + \nu_2\|\partial_2\omega\|_{L^2}^2 + \kappa_1\|\partial_1\nabla\theta\|_{L^2}^2 = -\int \nabla\theta\cdot\nabla u\cdot\nabla\theta\,dx, \tag{2.1.30}$$

where we use the fact that $\int(\partial_1\theta\,\omega - \nabla u_2\cdot\nabla\theta)\,dx = 0$, which comes from writing ω and u in terms of the stream function ψ, namely $\omega = \Delta\psi$ and $u = \nabla^\perp\psi := (-\partial_2\psi, \partial_1\psi)$.

To make a use of anisotropic dissipation, we split right side of (2.1.30) into four terms as

$$-\int \nabla\theta\cdot\nabla u\cdot\nabla\theta\,dx = -\int(\partial_1 u_1(\partial_1\theta)^2 + \partial_1 u_2\partial_1\theta\partial_2\theta + \partial_2 u_1\partial_1\theta\partial_2\theta + \partial_2 u_2(\partial_2\theta)^2)\,dx.$$

The goal is to find upper bounds for each term in the right-hand side of above equation that are time integrable. We can estimate the first three terms in above equation using Lemma (2.1.2). For the last term, we use Lemma (2.1.2) along with the divergence-free condition

$\nabla \cdot u = 0$. Then combining all the estimates, (2.1.30) becomes

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) + 2\nu_2\|\partial_2\nabla u\|_{L^2}^2 + 2\kappa_1\|\partial_1\nabla\theta\|_{L^2}^2 \\
&\qquad \le C\left(\|u\|_{H^1} + \|\nabla\theta\|_{L^2}\right)\left(\|\partial_2 u\|_{H^1}^2 + \|\partial_1\theta\|_{H^1}^2\right). \qquad (2.1.31)
\end{aligned}$$

Integrating (2.1.31) over $[0,t]$ and combining with (2.1.28), we obtain

$$\begin{aligned}
&\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{H^1}^2 + 2\nu_2\int_0^t \|\partial_2 u(s)\|_{H^1}^2 ds + 2\kappa_1\int_0^t \|\partial_1\theta(s)\|_{H^1}^2 ds \\
&\le \|u(0)\|_{L^2}^2 + \|\theta(0)\|_{H^1}^2 + C\int_0^t (\|u\|_{H^1} + \|\nabla\theta\|_{L^2})\left(\|\partial_2 u\|_{H^1}^2 + \|\partial_1\theta\|_{H^1}^2\right) d\tau \\
&\le E(0) + C\,E(t)^{\frac{3}{2}}. \qquad (2.1.32)
\end{aligned}$$

Observe that the upper bound in (2.1.32) depends only on the H^1-norm. As a consequence any initial small H^1 data leads to a global H^1 weak solution. However, uniqueness of these H^1-level solutions are not known. In fact, when we evaluate the difference $(\widetilde{u}, \widetilde{\theta})$ of two solutions $(u^{(1)}, \theta^{(1)})$ and $(u^{(2)}, \theta^{(2)})$, we obtain the following terms

$$-\int \tilde{u}\cdot\nabla u^{(1)}\cdot\tilde{u}\,dx, \quad \text{and} \quad \int \tilde{u}\cdot\nabla\theta^{(1)}\cdot\tilde{\theta}\,dx, \qquad (2.1.33)$$

which are difficult to manage. When we estimated the difference of solutions in L^2- norm, we tried to gain the horizontal dissipation in the velocity to manage the above nonlinear terms. However, this process does not work well because it creates extra bad terms that are impossible to manage. Therefore, we look for the global H^2-solutions.

To control the H^2-norm, it only remains to bound the L^2-norm of $(\nabla\omega, \Delta\theta)$. Applying ∇ to the first equation of (2.1.29) and Δ to the second equation of (2.1.29) and dotting the resultant equations with $(\nabla\omega, \Delta\theta)$ yields

$$\frac{1}{2}\frac{d}{dt}(\|\nabla\omega\|_{L^2}^2 + \|\Delta\theta(t)\|_{L^2}^2) + \nu_2\|\partial_2\nabla\omega\|_{L^2}^2 + \kappa_1\|\partial_1\Delta\theta\|_{L^2}^2 = J_1 + J_2, \qquad (2.1.34)$$

where

$$J_1 = -\int \nabla\omega \cdot \nabla u \cdot \nabla\omega \, dx, \qquad J_2 = -\int \Delta\theta \cdot \Delta(u \cdot \nabla\theta) dx.$$

In the above calculation, we use the fact that $\int (\nabla \partial_1 \cdot \nabla\, \omega - \Delta u_2 \cdot \Delta\theta)\, dx = 0$, which comes from writing ω and u in terms of the stream function ψ as before. The aim is to find an upper bound that is time integrable for each term J_1 and J_2. To apply the anisotropic dissipation, we split J_1 as

$$\begin{aligned} J_1 &= -\int \partial_1 u_1\, (\partial_1\omega)^2\, dx - \int \partial_1 u_2\, \partial_1\omega\, \partial_2\omega\, dx - \int \partial_2 u_1\, \partial_1\omega\, \partial_2\omega dx - \int \partial_2 u_2\, (\partial_2\omega)^2 dx \\ &= \int \partial_2 u_2\, (\partial_1\omega)^2 dx - \int \partial_1 u_2\, \partial_1\omega\, \partial_2\omega\, dx - \int \partial_2 u_1\, \partial_1\omega\, \partial_2\omega\, dx - \int \partial_2 u_2\, (\partial_2\omega)^2\, dx \\ &:= J_{11} + J_{12} + J_{13} + J_{14}. \end{aligned}$$

To bound J_1 suitably, we need the help of the extra regularization term

$$\int_0^t \|\partial_1 u_2\|_{L^2}^2\, d\tau. \tag{2.1.35}$$

Using integration by parts and Lemma 2.1.2, we have

$$\begin{aligned} J_{11} &= -2\int u_2\, \partial_1\omega\, \partial_2\partial_1\omega\, dx \\ &\le C\|\partial_2\partial_1\omega\|_{L^2} \|\partial_1\omega\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}} \|u_2\|_{L^2}^{\frac{1}{2}} \|\partial_1 u_2\|_{L^2}^{\frac{1}{2}} \\ &\le C\left(\|u_2\|_{L^2} + \|\partial_1\omega\|_{L^2}\right) \|\partial_2\partial_1\omega\|_{L^2}^{\frac{3}{2}}\, \|\partial_1 u_2\|_{L^2}^{\frac{1}{2}}. \end{aligned}$$

Similarly, we can estimate other terms in J_1 using Lemma 2.1.2, and combining all the estimates yields,

$$|J_1| \le C\, \|u\|_{H^2} \left(\|\partial_2 \nabla\omega\|_{L^2}^2 + \|\partial_1 u_2\|_{L^2}^2 + \|\partial_2 u_1\|_{L^2}^2\right). \tag{2.1.36}$$

To estimate J_2, using integration by parts, we first write it as follows

$$\begin{aligned} J_2 &= -\int \Delta\theta\, \Delta u_1\, \partial_1\theta\, dx - \int \Delta\theta\, \Delta u_2\, \partial_2\theta\, dx - 2\int \Delta\theta\, \nabla u_1 \cdot \partial_1 \nabla\theta\, dx - 2\int \Delta\theta\, \nabla u_2 \cdot \partial_2\nabla\theta\, dx \\ &:= J_{21} + J_{22} + J_{23} + J_{24}. \end{aligned}$$

By Lemma 2.1.2,

$$\begin{aligned} |J_{21}| &\leq C\, \|\partial_1\theta\|_{L^2}\, \|\Delta\theta\|_{L^2}^{\frac{1}{2}}\, \|\partial_1\Delta\theta\|_{L^2}^{\frac{1}{2}}\, \|\Delta u_1\|_{L^2}^{\frac{1}{2}}\, \|\partial_2\Delta u_1\|_{L^2}^{\frac{1}{2}} \\ &\leq C\, (\|\Delta\theta\|_{L^2} + \|\Delta u_1\|_{L^2})\, \|\partial_1\theta\|_{H^2}^{\frac{3}{2}}\, \|\partial_2\Delta u_1\|_{L^2}^{\frac{1}{2}}. \end{aligned} \tag{2.1.37}$$

Similarly, we can estimate other terms using the divergence-free condition, Lemma 3.1.1, and integration by parts. Then, combining all the estimates, we get

$$|J_2| \leq C\, (\|\theta\|_{H^2} + \|u\|_{H^2})\, \|\partial_1\theta\|_{H^2}^{\frac{3}{2}}\, \|\partial_2 u\|_{H^2}^{\frac{1}{2}}. \tag{2.1.38}$$

Inserting (2.1.38) and (2.1.36) in (2.1.34) yields

$$\begin{aligned} &\frac{d}{dt}(\|\Delta u\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2) + 2\nu_2\|\partial_2\Delta u\|_{L^2}^2 + 2\kappa_1\|\partial_1\Delta\theta\|_{L^2}^2 \\ &\leq C\, (\|\theta\|_{H^2} + \|u\|_{H^2})\, \|\partial_1\theta\|_{H^2}^{\frac{3}{2}}\, \|\partial_2 u\|_{H^2}^{\frac{1}{2}} \\ &\quad + C\, \|u\|_{H^2}\, (\|\partial_2\nabla\omega\|_{L^2}^2 + \|\partial_1 u_2\|_{L^2}^2 + \|\partial_2 u_1\|_{L^2}^2). \end{aligned} \tag{2.1.39}$$

Integrating (2.1.39) over the time interval $[0, t]$ yields

$$\begin{aligned} &\|\Delta u(t)\|_{L^2}^2 + \|\Delta\theta(t)\|_{L^2}^2 + 2\nu_2\int_0^t \|\partial_2\Delta u\|_{L^2}^2 d\tau + 2\kappa_1\int_0^t \|\Delta\partial_1\theta\|_{L^2}^2 d\tau \\ &\leq \|\Delta u_0\|_{L^2}^2 + \|\Delta\theta_0\|_{L^2}^2 + C\int_0^t (\|\theta\|_{H^2} + \|u\|_{H^2})\, \|\partial_1\theta\|_{H^2}^{\frac{3}{2}}\, \|\partial_2 u\|_{H^2}^{\frac{1}{2}}\, d\tau \\ &\quad + C\int_0^t \|u\|_{H^2}\, (\|\partial_2\nabla\omega\|_{L^2}^2 + \|\partial_1 u_2\|_{L^2}^2 + \|\partial_2 u_1\|_{L^2}^2)\, d\tau \\ &\leq E(0) + C\, E(t)^{\frac{3}{2}}. \end{aligned} \tag{2.1.40}$$

Next, we bound the last piece in $E(t)$ defined by (2.1.26)

$$\int_0^t \|\partial_1 u_2\|_{L^2}^2 \, d\tau.$$

as a justification for its use. For this task, we use the equation of θ, which by applying ∂_1 can be written as

$$\partial_1 u_2 = -\partial_t \partial_1 \theta - \partial_1(u \cdot \nabla\theta) + \kappa_1 \partial_{111}\theta. \tag{2.1.41}$$

Multiplying (2.1.41) with $\partial_1 u_2$ and then integrating over $\mathbb{R}^2$ yields

$$\begin{aligned} \|\partial_1 u_2\|_{L^2}^2 &= -\int \partial_t\partial_1\theta \, \partial_1 u_2 \; dx - \int \partial_1 u_2 \, \partial_1(u \cdot \nabla\theta) \; dx + \kappa_1 \int \partial_1 u_2 \, \partial_{111}\theta \; dx \\ &:= K_1 + K_2 + K_3. \end{aligned}$$

To estimate K_1, first, we shift the time derivative and write as

$$K_1 = -\frac{d}{dt}\int \partial_1\theta \, \partial_1 u_2 \, dx + \int \partial_1\theta \, \partial_1\partial_t u_2 \, dx := K_{11} + K_{12}. \tag{2.1.42}$$

Using the equation of the second component of the velocity, we can write K_2 as

$$\begin{aligned} K_{12} &= \int \partial_{11}\theta \, (u \cdot \nabla) u_2 \; dx \; + \int \partial_{11}\theta \; \partial_2 p \; dx \\ &\quad -\nu_2 \int \partial_{11}\theta \; \partial_{22} u_2 \; dx \; - \int \partial_{11}\theta \; \theta \; dx. \end{aligned}$$

Now, we replace the pressure term by applying the divergence operator to the velocity equation, that is

$$p = -\Delta^{-1}\nabla \cdot (u \cdot \nabla u) + \Delta^{-1}\partial_2\theta.$$

Therefore,

$$\begin{aligned} K_{12} &= \int \partial_{11}\theta\,(u\cdot\nabla)u_2\;dx\;+\int \partial_{11}\theta\,(-\partial_2\Delta^{-1}\nabla\cdot(u\cdot\nabla u))\;dx \\ &\quad -\nu_2\int \partial_{11}\theta\;\partial_{22}u_2\;dx\;-\int \partial_{11}\theta\;\partial_{11}\Delta^{-1}\theta\;dx \\ &:= K_{121}+K_{122}+K_{123}+K_{124}. \end{aligned}$$

To bound K_{121}, we further split it,

$$\begin{aligned} K_{121} &= \int \partial_{11}\theta(u_1\partial_1u_2+u_2\partial_2u_2)dx \\ &= \int \partial_{11}\theta\;u_1\;\partial_1u_2\;dx\;+\int \partial_{11}\theta\;u_2\;\partial_2u_2\;dx. \end{aligned}$$

By Lemma 2.1.2,

$$\begin{aligned} |K_{121}| &\le C\,\|\partial_{11}\theta\|_{L^2}\,\|u_1\|_{L^2}^{\frac12}\|\partial_1u_1\|_{L^2}^{\frac12}\|\partial_1u_2\|_{L^2}^{\frac12}\|\partial_2\partial_1u_2\|_{L^2}^{\frac12} \\ &\quad +C\,\|u_2\|_{L^\infty}\,\|\partial_{11}\theta\|_{L^2}\,\|\partial_2u_2\|_{L^2} \\ &\le C\,\|u\|_{H^1}\,\|\partial_2u\|_{H^1}\,\|\partial_{11}\theta\|_{L^2}+\;C\,\|u\|_{H^2}\,\|\partial_2u\|_{L^2}\,\|\partial_{11}\theta\|_{L^2}. \end{aligned}$$

By integration by parts and the boundedness of the double Riesz transform [50], we can estimate K_{122} as

$$\begin{aligned} K_{122} &= -\int \partial_1\theta\;\partial_{12}\Delta^{-1}\nabla\cdot(u\cdot\nabla u)\;dx \\ &\le \|\partial_1\theta\|_{L^2}\,\|\Delta^{-1}\partial_{12}\nabla\cdot(u\cdot\nabla u)\|_{L^2} \\ &\le C\,\|\partial_1\theta\|_{L^2}\,\|\partial_2(u\cdot\nabla u)\|_{L^2} \\ &\le C\,\|\partial_1\theta\|_{L^2}\,\|\partial_2u\cdot\nabla u+u\cdot\nabla\partial_2u\|_{L^2} \\ &\le C\,\|\partial_1\theta\|_{L^2}\,(\,\|\partial_2u\|_{L^4}\,\|\nabla u\|_{L^4}+\|u\|_\infty\|\nabla\partial_2u\|_{L^2}) \\ &\le C\,\|\partial_1\theta\|_{L^2}\,\|\partial_2u\|_{H^1}\,\|\nabla u\|_{H^1}+\;C\,\|\partial_1\theta\|_{L^2}\,\|u\|_{H^2}\|\nabla\partial_2u\|_{L^2}. \end{aligned}$$

By the boundedness of the double Riesz transform, we have

$$K_{124} = \int \partial_1\theta\, \partial_{11}\Delta^{-1}\partial_1\theta\; dx \le C\,\|\partial_1\theta\|_{L^2}^2.$$

K_{123} can be easily bounded,

$$|K_{123}| \le C\,\|\partial_{11}\theta\|_{L^2}\,\|\partial_{22}u_2\|_{L^2}.$$

Using all the above estimates, we have an upper bound for K_{12} as

$$|K_{12}| \le\; C\,\|\partial_1\theta\|_{L^2}^2 + C\,\|\partial_{11}\theta\|_{L^2}\,\|\partial_{22}u_2\|_{L^2} +\; C\,\|u\|_{H^2}\,\|\partial_2 u\|_{H^1}\,\|\partial_1\theta\|_{H^1}. \tag{2.1.43}$$

To bound K_2, we first decompose it into four terms as

$$\begin{aligned} K_2 \;\; &= \;\; -\int \partial_1 u_2\,\partial_1 u_1\,\partial_1\theta\,dx - \int \partial_1 u_2 u_1 \partial_1\partial_1\theta\,dx \\ &\quad\;\; -\int \partial_1 u_2 \partial_1 u_2 \partial_2\theta\,dx - \int \partial_1 u_2 u_2 \partial_1\partial_2\theta\,dx. \end{aligned}$$

By Lemma 2.1.2, all the above terms can be estimated easily to get

$$\begin{aligned} |K_2| \;\; &\le \;\; C\,\|u\|_{H^1}\,(\|\partial_2 u\|_{H^1}^2 + \|\partial_1\theta\|_{H^1}^2) \\ &\quad\;\; +C\,(\|u\|_{H^2} +\; \|\theta\|_{H^2})(\|\partial_1 u_2\|_{L^2}^2 + \|\partial_1\theta\|_{H^1}^2). \end{aligned} \tag{2.1.44}$$

Finally, we estimate K_3 as

$$|K_3| \le\; \kappa_1\|\partial_1 u_2\|_{L^2}\,\|\partial_{111}\theta\|_{L^2} \le \frac{1}{2}\|\partial_1 u_2\|_{L^2}^2 + C\,\|\partial_1\theta\|_{H^2}^2. \tag{2.1.45}$$

Combining (2.1.45), (2.1.42), (2.1.43) and (2.1.44), we have

$$\begin{aligned}\frac{1}{2}\,\|\partial_1 u_2\|_{L^2}^2 &\leq C\,\|\partial_1\theta\|_{H^2}^2 - \frac{d}{dt}\int \partial_1\theta\,\partial_1 u_2\,dx \\ &\quad + C\,\|\partial_{11}\theta\|_{L^2}\,\|\partial_{22}u_2\|_{L^2} + C\,\|u\|_{H^2}\,(\|\partial_2 u\|_{H^1}^2 + \|\partial_1\theta\|_{H^1}^2) \\ &\quad + C\,(\|u\|_{H^2} + \|\theta\|_{H^2})(\|\partial_1 u_2\|_{L^2}^2 + \|\partial_1\theta\|_{H^1}^2).\end{aligned}$$

Integrating over $[0,t]$ yields

$$\begin{aligned}\int_0^t \|\partial_1 u_2\|_{L^2}^2\,d\tau &\leq C\int_0^t \|\partial_1\theta\|_{H^2}^2\,d\tau - 2\int \partial_1\theta\,\partial_1 u_2\,dx + 2\int \partial_1\theta_0\,\partial_1 u_{02}\,dx \\ &\quad + C\int_0^t \|\partial_{11}\theta\|_{L^2}\,\|\partial_{22}u_2\|_{L^2}\,d\tau \\ &\quad + C\int_0^t \|u\|_{H^2}\,(\|\partial_2 u\|_{H^1}^2 + \|\partial_1\theta\|_{H^1}^2)\,d\tau \\ &\quad + C\int_0^t (\|u\|_{H^2} + \|\theta\|_{H^2})(\|\partial_1 u_2\|_{L^2}^2 + \|\partial_1\theta\|_{H^1}^2)\,d\tau \\ &\leq C\int_0^t \|\partial_1\theta\|_{H^2}^2\,d\tau + C\int_0^t \|\partial_2 u\|_{H^2}^2\,d\tau + C\,(\|u\|_{H^1}^2 + \|\theta\|_{H^1}^2) \\ &\quad + C\,(\|u_0\|_{H^1}^2 + \|\theta_0\|_{H^1}^2) + C\,E(t)^{\frac{3}{2}}. \end{aligned} \tag{2.1.46}$$

Next we combine (2.1.32), (2.1.40), and (2.1.46). We have to eliminate the quadratic terms on the right-hand side of (2.1.46) by the corresponding terms on the left-hand side. For this purpose, we multiply both sides of (2.1.46) by a suitable small coefficient δ. Then, (2.1.32) + (2.1.40) + δ (2.1.46) yields

$$\begin{aligned}&\|u(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2 + 2\nu_2\int_0^t \|\partial_2 u\|_{H^2}^2 d\tau + 2\kappa_1\int_0^t \|\partial_1\theta\|_{H^2}^2 d\tau + \delta\int_0^t \|\partial_1 u_2\|_{L^2}^2 \\ &\leq E(0) + C\,E(t)^{\frac{3}{2}} + C\,\delta\,(\|u(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2) + C\,\delta\,(\|u_0\|_{H^2}^2 + \|\theta_0\|_{H^2}^2) \\ &\quad + C\,\delta\int_0^t \|\partial_2 u\|_{H^2}^2 d\tau + C\,\delta\int_0^t \|\partial_1\theta\|_{H^2}^2 d\tau + C\,\delta\,E(t)^{\frac{3}{2}}. \end{aligned} \tag{2.1.47}$$

If $\delta > 0$ is chosen to be sufficiently small, say

$$C\,\delta \le \frac{1}{2}, \quad C\,\delta \le \nu_2, \quad C\,\delta \le \kappa_1,$$

then (2.1.47) is reduced to

$$E(t) \le\, C_1\, E(0) +\, C_2\, E(t)^{\frac{3}{2}}, \tag{2.1.48}$$

where C_1 and C_2 are positive constants. To obtain the desired stability result, we apply the bootstrapping argument to (2.1.48). Indeed, if the initial data (u_0, θ_0) is sufficiently small,

$$\|u(0)\|_{H^2}^2 + \|\theta(0)\|_{H^2}^2 \le \varepsilon := \frac{1}{4\sqrt{C_1}C_2},$$

then using (2.1.48), we can show that

$$\|u(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2 \le \sqrt{2C_1}\,\varepsilon.$$

The bootstrapping argument begins with the ansatz that, for $t < T$

$$E(t) \le \frac{1}{4C_2^2} \tag{2.1.49}$$

and show that

$$E(t) \le \frac{1}{8C_2^2} \quad \text{for all } t \le T. \tag{2.1.50}$$

Then the bootstrapping argument would imply that $T = \infty$ and (2.1.50) actually holds for all t. Using (2.1.48) and (2.1.49), we can easily get (2.1.50). In fact, substituting (2.1.49) in (2.1.48) gives

$$E(t) \le\, C_1\, E(0) +\, C_2\, E(t)^{\frac{3}{2}} \le C_1\,\varepsilon^2 +\, C_2\,\frac{1}{2C_2}\,E(t).$$

That is,

$$\frac{1}{2}E(t) \le C_1\,\varepsilon^2,$$

which is (2.1.50). This establishes global stability. For the proof of uniqueness, we refer to the original article [10]. ■

2.2 2D Boussinesq Equations with Vertical Velocity Dissipation and Horizontal Thermal Diffusion

In this section, we focus on the following 2D anisotropic Boussinesq equations (2.2.1) that model anisotropic buoyancy-driven fluids when the vertical velocity dissipation and the horizontal thermal diffusion are negligible

$$\begin{cases} \partial_t v + v\cdot\nabla v = -\nabla P + \nu_1\,\partial_{11} v + \rho\,\mathbf{e}_2, \quad x\in\Omega=\mathbb{T}\times\mathbb{R}, \quad t>0 \\ \partial_t\rho + v\cdot\nabla\rho = \kappa_2\,\partial_{22}\rho, \\ \nabla\cdot v = 0. \end{cases} \tag{2.2.1}$$

Here, we consider the spatial domain $\Omega = \mathbb{T}\times\mathbb{R}$, with $\mathbb{T}=[0,1]$ being a 1D periodic box and $\mathbb{R}$ being the whole line.

We recall the following perturbed equation from Chapter I

$$\begin{cases} \partial_t u + u\cdot\nabla u = -\nabla p + \nu_1\,\partial_{11} u + \theta\mathbf{e}_2, \\ \partial_t\theta + u\cdot\nabla\theta + u_2 = \kappa_2\,\partial_{22}\theta, \\ \nabla\cdot u = 0, \\ u(x,0)=u_0(x), \quad \theta(x,0)=\theta_0(x). \end{cases} \tag{2.2.2}$$

We start with the basic facts that are useful in the proof of main theorems. As mentioned in Chapter I, we consider a function $f = f(x_1, x_2)$ on the domain $\mathbb{T}\times\mathbb{R}$ that can be integrated in x_1 over the 1D periodic box $\mathbb{T} = [0, 1]$. Then, the horizontal average $\overline{f}$ is defined as

$$\overline{f}(x_2) = \int_{\mathbb{T}} f(x_1, x_2) dx_1. \tag{2.2.3}$$

Here, $\overline{f}$ is the zeroth Fourier mode of f. Then, we decompose f into two parts; horizontal average $\overline{f}$ and the corresponding oscillation part $\widetilde{f}$,

$$f = \overline{f} + \widetilde{f}. \tag{2.2.4}$$

The following Lemma 2.2.1 contains basic properties of the decomposition (2.2.4).

Lemma 2.2.1 *Suppose that $f \in H^2(\Omega)$ defined on $\Omega = \mathbb{T} \times \mathbb{R}$ is sufficiently regular. Consider $\overline{f}$ and $\widetilde{f}$ as defined above in (2.2.3) and (2.2.4). Then*
(a) The average operator $\overline{f}$ and the oscillation operator $\widetilde{f}$ have following properties,

$$\overline{\partial_1 f} = \partial_1 \overline{f} = 0, \quad \overline{\partial_2 f} = \partial_2 \overline{f}, \quad \widetilde{\partial_1 f} = \partial_1 \widetilde{f}, \quad \widetilde{\partial_2 f} = \partial_2 \widetilde{f}, \quad \overline{\widetilde{f}} = 0.$$

(b) For a divergence-free vector field f, $\overline{f}$ and $\widetilde{f}$ are also divergence-free,

$$\nabla \cdot \overline{f} = 0 \quad \textit{and} \quad \nabla \cdot \widetilde{f} = 0.$$

(c) $\overline{f}$ and $\widetilde{f}$ are orthogonal in $\dot{H}^k$ for any integer $k \geq 0$, namely

$$(\overline{f}, \widetilde{f})_{\dot{H}^k(\Omega)} := \int_\Omega \overline{D^k f} \cdot \widetilde{D^k f} dx = 0, \quad \|f\|^2_{\dot{H}^k(\Omega)} = \|\overline{f}\|^2_{\dot{H}^k(\Omega)} + \|\widetilde{f}\|^2_{\dot{H}^k(\Omega)}.$$

In particular,

$$\|\overline{f}\|_{\dot{H}^k(\Omega)} \leq \|f\|_{\dot{H}^k(\Omega)} \quad \textit{and} \quad \|\widetilde{f}\|_{\dot{H}^k(\Omega)} \leq \|f\|_{\dot{H}^k(\Omega)}.$$

In fact, the orthogonality applies for any integrable functions,

$$\int_\Omega \overline{f} \cdot \widetilde{g}\, dx = 0.$$

We can easily verify the properties given in Lemma 2.2.1 using the definition $\overline{f}$ and the decomposition of f defined earlier.

The following Lemma 2.2.2 compares the 1D Sobolev inequalities on the whole line $\mathbb{R}$ and its bounded domain version.

Lemma 2.2.2 *For any 1D function $f \in H^1(\mathbb{R})$,*

$$\|f\|_{L^\infty(\mathbb{R})} \le \sqrt{2}\, \|f\|_{L^2(\mathbb{R})}^{\frac{1}{2}}\, \|f'\|_{L^2(\mathbb{R})}^{\frac{1}{2}}.$$

For any bounded domain such as $\mathbb{T} = [0,1]$ and $f \in H^1(\mathbb{T})$,

$$\|f\|_{L^\infty(\mathbb{T})} \le \sqrt{2}\, \|f\|_{L^2(\mathbb{T})}^{\frac{1}{2}}\, \|f'\|_{L^2(\mathbb{T})}^{\frac{1}{2}} + \|f\|_{L^2(\mathbb{T})},$$

in particular, if the function f has mean zero such as the oscillation part $\tilde{f}$,

$$\|f\|_{L^\infty(\mathbb{T})} \le C\, \|f\|_{L^2(\mathbb{T})}^{\frac{1}{2}}\, \|f'\|_{L^2(\mathbb{T})}^{\frac{1}{2}}.$$

Anisotropic Sobolev inequalities have become an indispensable tool while researching anisotropic equations. For a 2D function, the following lemma offers anisotropic upper bounds for triple products and for the L^∞-norm.

Lemma 2.2.3 *Let $\Omega = \mathbb{T} \times \mathbb{R}$. For any $f, g, h \in L^2(\Omega)$ with $\partial_1 f \in L^2(\Omega)$ and $\partial_2 g \in L^2(\Omega)$, then*

$$\Big|\int_\Omega fgh\, dx\Big| \le C\|f\|_{L^2}^{\frac{1}{2}}(\|f\|_{L^2} + \|\partial_1 f\|_{L^2})^{\frac{1}{2}}\|g\|_{L^2}^{\frac{1}{2}}\|\partial_2 g\|_{L^2}^{\frac{1}{2}}\|h\|_{L^2}\,. \tag{2.2.5}$$

For any $f \in H^2(\Omega)$, we have

$$\begin{aligned}\|f\|_{L^\infty(\Omega)} \leq & C\|f\|_{L^2}^{\frac{1}{4}}(\|f\|_{L^2} + \|\partial_1 f\|_{L^2})^{\frac{1}{4}}\|\partial_2 f\|_{L^2}^{\frac{1}{4}} \\ & \times (\|\partial_2 f\|_{L^2} + \|\partial_1\partial_2 f\|_{L^2})^{\frac{1}{4}}.\end{aligned}$$

When f in Lemma 2.2.3 is replaced by the oscillation part $\tilde{f}$, the lower-order part in (2.2.5) can be removed and we have the following lemma.

Lemma 2.2.4 *Let $\Omega = \mathbb{T} \times \mathbb{R}$. For any $f, g, h \in L^2(\Omega)$ with $\partial_1 f \in L^2(\Omega)$ and $\partial_2 g \in L^2(\Omega)$, then*

$$\Big|\int_\Omega \widetilde{f} g h \, dx\Big| \leq C\|\widetilde{f}\|_{L^2}^{\frac{1}{2}}\|\partial_1 \widetilde{f}\|_{L^2}^{\frac{1}{2}}\|g\|_{L^2}^{\frac{1}{2}}\|\partial_2 g\|_{L^2}^{\frac{1}{2}}\|h\|_{L^2}. \tag{2.2.6}$$

For any $f \in H^2(\Omega)$, we have

$$\|\widetilde{f}\|_{L^\infty(\Omega)} \leq C\|\widetilde{f}\|_{L^2}^{\frac{1}{4}}\|\partial_1 \widetilde{f}\|_{L^2}^{\frac{1}{4}}\|\partial_2 \widetilde{f}\|_{L^2}^{\frac{1}{4}}\|\partial_1\partial_2 \widetilde{f}\|_{L^2}^{\frac{1}{4}}.$$

Lemma 2.2.5 *Let $\overline{f}$ and $\widetilde{f}$ be defined as in (2.2.3) and (2.2.4). If $\|\partial_1 \widetilde{f}\|_{L^2(\Omega)} < \infty$, then*

$$\|\widetilde{f}\|_{L^2(\Omega)} \leq C\|\partial_1 \widetilde{f}\|_{L^2(\Omega)},$$

where C is a pure constant. In addition, if $\|\partial_1 \widetilde{f}\|_{H^1(\Omega)} < \infty$, then

$$\|\widetilde{f}\|_{L^\infty(\Omega)} \leq C\|\partial_1 \widetilde{f}\|_{H^1(\Omega)}.$$

As a direct consequence of Lemma 2.2.5 and the inequality (2.2.6), one has

$$\Big|\int_\Omega \widetilde{f} g h \, dx\Big| \leq C\|\partial_1 \widetilde{f}\|_{L^2}\|g\|_{L^2}^{\frac{1}{2}}\|\partial_2 g\|_{L^2}^{\frac{1}{2}}\|h\|_{L^2}. \tag{2.2.7}$$

The last lemma provides an explicit decay rate in (2.2.9) for functions that are integrable and are decreasing in a general sense, namely (2.2.8).

Lemma 2.2.6 *Let $f = f(t)$ be a nonnegative function satisfying , for two constants $C_0 > 0$ and $C_1 > 0$,*

$$\int_0^\infty f(\tau)d\tau < C_0 \quad \text{and} \quad f(t) \le C_1 f(s) \quad \text{for any} \quad 0 \le s < t. \tag{2.2.8}$$

Then, for $C_2 = \max\{2C_1 f(0), 4C_0C_1\}$ and for any $t > 0$,

$$f(t) \le C_2(1+t)^{-1}. \tag{2.2.9}$$

Furthermore, $f(t)$ has the following large-time asymptotic behavior,

$$\lim_{t\to\infty} t\, f(t) = 0.$$

In the following subsections, we sketch the proof of Theorem 1.2.3 and Theorem 1.2.4.

2.2.1 The H^2 Stability

The ambition of this section is to prove the Theorem 1.2.3.

The Sketch of the Proof of Theorem 1.2.3. Since the local well-posedness of (2.2.2) can be derived using a standard method (see, e.g., [45]), we concentrate on the global H^2-bound of the solution (u, θ). The idea is to construct a suitable energy functional and then use the bootstrapping argument (see, e.g., [52]) to obtain the desired stability results. We construct the following energy functional $E(t)$:

$$\begin{aligned} E(t) : &= \max_{0\le\tau\le t}(\|u(\tau)\|_{H^2}^2 + \|\theta(\tau)\|_{H^2}^2) + 2\nu_1 \int_0^t \|\partial_1 u\|_{H^2}^2 d\tau + 2\kappa_2 \int_0^t \|\partial_2\theta\|_{H^2}^2 d\tau + \int_0^t \|\partial_1\theta\|_{L^2}^2\, d\tau \\ &= E_1(t) + E_2(t). \end{aligned}$$

The first part of the above energy functional $E_1(t)$ contains the H^2-norm of the solution

$(u,\,\theta)$ and the corresponding time integral parts form the partial dissipation, namely

$$E_1(t) := \max_{0\le\tau\le t}(\|u(\tau)\|_{H^2}^2 + \|\theta(\tau)\|_{H^2}^2) + 2\nu_1\int_0^t \|\partial_1 u\|_{H^2}^2 d\tau + 2\kappa_2\int_0^t \|\partial_2\theta\|_{H^2}^2 d\tau.$$

The second part of the above energy functional $E_2(t)$ is because of the extra smoothing reflected in the wave equation (1.2.30),

$$E_2(t) := \int_0^t \|\partial_1\theta\|_{L^2}^2\, d\tau.$$

We apply bootstrapping argument to prove the following inequality, for $t > 0$,

$$E(t) \le c_1 E(0) + c_2 E(t)^{\frac{3}{2}}. \tag{2.2.10}$$

Establishing (2.2.10) requires to prove the following two estimates; for E_1, we prove

$$E_1 \le E_1(0) + c_3\, E_1(t)^{\frac{3}{2}} + c_4\, E_2(t)^{\frac{3}{2}}, \tag{2.2.11}$$

and for E_2, we prove

$$E_2 \le c_5 E_1(0) + c_6\, E_1(t) + c_7\, E_1(t)^{\frac{3}{2}} + c_8\, E_2(t)^{\frac{3}{2}}, \tag{2.2.12}$$

where c_1 through c_8 are all constants.

Dotting $(u,\,\theta)$ with first two equations in (2.2.2), we obtain the following global L^2-bound

$$\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 + 2\nu_1\int_0^t \|\partial_1 u\|_{L^2}^2 d\tau + 2\kappa_2\int_0^t \|\partial_2\theta\|_{L^2}^2 d\tau = \|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2. \tag{2.2.13}$$

To estimate the H^1-norm, we use the vorticity equation and the temperature equation

$$\begin{cases} \partial_t\omega + u\cdot\nabla\omega = \nu_2\partial_{11}\omega + \partial_1\theta, \\ \partial_t\theta + u\cdot\nabla\theta + u_2 = \kappa_2\partial_{22}\theta. \end{cases} \tag{2.2.14}$$

Applying ∇ to the temperature equation and taking the inner product of $(\omega, \nabla\theta)$ with the equations of ω and $\nabla\theta$ yields

$$\frac{1}{2}\frac{d}{dt}(\|\omega\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) + \nu_1\|\partial_1\omega\|_{L^2}^2 + \kappa_2\|\partial_2\nabla\theta\|_{L^2}^2 = -\int \nabla\theta\cdot\nabla u\cdot\nabla\theta\,dx = I_1. \tag{2.2.15}$$

In the above step, we use the fact that $\nabla\cdot u = 0$, there exists a stream function ψ so that $u = \nabla^\perp\psi = (-\partial_2\psi, \partial_1\psi)$ and $\Delta\psi = \omega$.

To apply the anisotropic dissipation, we decompose the right term of (2.2.15) into four components

$$\begin{aligned} I_1 :&= -\int \partial_1 u_1(\partial_1\theta)^2 dx - \int \partial_1 u_2\partial_1\theta\partial_2\theta dx \\ &\quad -\int \partial_2 u_1\partial_1\theta\partial_2\theta dx - \int \partial_2 u_2(\partial_2\theta)^2 dx \\ &:= I_{11} + I_{12} + I_{13} + I_{14}. \end{aligned} \tag{2.2.16}$$

We are looking for the upper bounds for all the terms in the right side of (2.2.16) that are time-integrable. By $\nabla\cdot u = 0$, integration by parts, Lemma 2.2.4 and Young's inequality, we can estimate I_{11} as

$$\begin{aligned} I_{11} :&= -\int \partial_1 u_1(\partial_1\theta)^2 dx = -2\int u_2\partial_1\theta\partial_1\partial_2\theta dx \\ &\le c\,\|\partial_1\partial_1\partial_2\theta\|_{L^2}\|\partial_1\theta\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\theta\|_{L^2}^{\frac{1}{2}}\|u_2\|_{L^2} \\ &\le c\,\|\partial_2\theta\|_{H^2}^{\frac{3}{2}}\|\partial_1\theta\|_{L^2}^{\frac{1}{2}}\|u\|_{H^2} \\ &\le c\,\|u\|_{H^2}\Big(\|\partial_2\theta\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2\Big). \end{aligned} \tag{2.2.17}$$

Similarly, using Lemmas 2.2.1 and 2.2.5, we get

$$
\begin{aligned}
I_{12} :&= -\int \partial_1 u_2 \partial_1\theta \partial_2\theta dx = -\int \partial_1 \widetilde{u_2} \partial_1 \widetilde{\theta} \partial_2 \theta dx \\
&\le c\, \|\theta\|_{H^2} \Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_2 \theta\|_{H^2}^2 \Big).
\end{aligned} \tag{2.2.18}
$$

To estimate I_{13} we need to invoke the decompositions $u = \tilde{u} + \bar{u}$ and $\theta = \tilde{\theta} + \bar{\theta}$ because it contains two terms with "bad" derivatives $\partial_2 u_1$ and $\partial_1\theta$, so

$$
\begin{aligned}
I_{13} :&= -\int \partial_2 u_1 \partial_1\theta \partial_2\theta dx \\
&= -\int \partial_2 \overline{u_1} \partial_1 \widetilde{\theta} \partial_2 \overline{\theta} dx - \int \partial_2 \widetilde{u_1} \partial_1 \widetilde{\theta} \partial_2 \overline{\theta} dx \\
&\quad - \int \partial_2 \overline{u_1} \partial_1 \widetilde{\theta} \partial_2 \widetilde{\theta} dx - \int \partial_2 \widetilde{u_1} \partial_1 \widetilde{\theta} \partial_2 \widetilde{\theta} dx \\
&:= I_{131} + I_{132} + I_{133} + I_{134}.
\end{aligned} \tag{2.2.19}
$$

Since $\partial_2 \overline{u_1}$ and $\partial_2 \overline{\theta}$ depends only on x_2 and $\int_{\mathbb{T}} \partial_1 \widetilde{\theta} = 0$, we have $I_{131} = 0$. Using Lemma 2.2.4 and Young's inequality all the remaining terms in I_{13} can be bounded, and combining all the bounds we get

$$
I_{13} \le\ c\, \|u\|_{H^2} \Big(\|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2 + \|\partial_1 u\|_{H^2}^2 \Big). \tag{2.2.20}
$$

By $\nabla \cdot u = 0$, and Lemmas 2.2.1 and 2.2.4,

$$
\begin{aligned}
I_{14} :&= -\int \partial_2 u_2 (\partial_2\theta)^2 dx = \int \partial_1 \widetilde{u_1} (\partial_2\theta)^2 dx \\
&\le c \|\theta\|_{H^2} \Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_2\theta\|_{H^2}^2 \Big).
\end{aligned} \tag{2.2.21}
$$

Inserting the bounds in (2.2.17), (2.2.18), (2.2.20), (2.2.21) in (2.2.16) leads to

$$
I_1 \le c(\|u\|_{H^2} + \|\theta\|_{H^2}) \Big(\|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2 + \|\partial_1 u\|_{H^2}^2 \Big). \tag{2.2.22}
$$

Then, (2.2.15) can be written as,

$$\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) + \kappa_2\|\partial_2\nabla\theta\|_{L^2}^2 + \nu_1\|\partial_1\omega\|_{L^2}^2$$
$$\leq c\,(\|u\|_{H^2} + \|\theta\|_{H^2})\Big(\|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2 + \|\partial_1 u\|_{H^2}^2\Big). \tag{2.2.23}$$

Integrating (2.2.23) in time over $[0, t]$ yields,

$$\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 + 2\kappa_2\int_0^t \|\partial_2\nabla\theta\|_{L^2}^2 d\tau + 2\nu_1\int_0^t \|\partial_1 w\|_{L^2}^2 d\tau$$
$$\leq \|\nabla u_0\|_{L^2}^2 + \|\nabla\theta_0\|_{L^2}^2 + c\,E(t)^{\frac{3}{2}}. \tag{2.2.24}$$

Next, we estimate the H^2-norm of (u, θ). We apply ∇ to the vorticity equation in (2.1.15) and Δ to the temperature equation in (2.1.15) and taking inner product with $(\nabla\omega,\, \Delta\theta)$ yields

$$\frac{1}{2}\frac{d}{dt}(\|\nabla\omega\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2) + \kappa_2\|\partial_2\nabla\theta\|_{L^2}^2 + \nu_1\|\partial_1\nabla\omega\|_{L^2}^2$$
$$= -\int \nabla\omega\cdot\nabla u\cdot\nabla\omega\,dx - \int \Delta\theta\cdot\Delta(u\cdot\nabla\theta)dx \tag{2.2.25}$$
$$:= J_1 + J_2, \tag{2.2.26}$$

where we have used $\nabla\cdot u = 0$, and the fact that there exists a stream function ψ such that $u = \nabla^\perp\psi = (-\partial_2\psi, \partial_1\psi)$ and $\Delta\psi = \omega$. To find the upper bound of J_1, we first split it as

$$\begin{aligned} J_1 &:= -\int \nabla\omega\cdot\nabla u\cdot\nabla\omega\,dx \\ &= -\int \partial_1 u_1(\partial_1\omega)^2\,dx - \int \partial_1 u_2\partial_1\omega\partial_2\omega\,dx \\ &\quad - \int \partial_2 u_1\partial_1\omega\partial_2\omega\,dx - \int \partial_2 u_2(\partial_2\omega)^2\,dx \\ &:= J_{11} + J_{12} + J_{13} + J_{14}. \end{aligned} \tag{2.2.27}$$

Due to Lemmas 2.2.1 and 2.2.4, divergence free condition, we can bound $J_{1i}, i = 1, 2, 4$ as

$$J_{1i} \leq c\|u\|_{H^2}\|\partial_1 u\|_{H^2}^2 . \tag{2.2.28}$$

To bound J_{13}, we make a use the orthogonal decomposition of u_1 and ω as well as Lemma 2.2.1 to get,

$$\begin{aligned} J_{13} &:= -\int \partial_2 u_1 \partial_1 \omega \partial_2 \omega \, dx = -\int \partial_2 u_1 \partial_1 \widetilde{\omega} \partial_2 \omega \, dx \\ &= -\int \partial_2 \overline{u_1} \partial_1 \widetilde{\omega} \partial_2 \overline{\omega} \, dx - \int \partial_2 \overline{u_1} \partial_1 \widetilde{\omega} \partial_2 \widetilde{\omega} \, dx - \int \partial_2 \widetilde{u}_1 \partial_1 \widetilde{\omega} \partial_2 \omega \, dx \\ &= J_{131} + J_{132} + J_{133}. \end{aligned} \tag{2.2.29}$$

Then, by Lemma 2.2.1,

$$J_{131} := -\int \partial_2 \overline{u_1} \partial_1 \widetilde{w} \partial_2 \overline{w} \, dx = 0. \tag{2.2.30}$$

Use Lemma 2.2.4, $J_{13i}, i = 2, 3$ can be bounded as

$$J_{13i} \leq c\|u\|_{H^2}\|\partial_1 u\|_{H^2}^2. \tag{2.2.31}$$

Combining above estimate, we have

$$J_{13} \leq c\|u\|_{H^2}\|\partial_1 u\|_{H^2}^2. \tag{2.2.32}$$

Notice that all the terms in J_1 can be bounded by the same bound. Therefore, we have

$$J_1 \leq c\|u\|_{H^2}\|\partial_1 u\|_{H^2}^2. \tag{2.2.33}$$

To estimate J_2, we first write as following using integrating by parts,

$$
\begin{aligned}
J_2 :=& -\int \Delta\theta \cdot \Delta(u\cdot\nabla\theta)dx \\
=& -\int \Delta\theta\Delta u_1\partial_1\theta dx - \int \Delta\theta\Delta u_2\partial_2\theta dx \\
& -2\int \Delta\theta\nabla u_1\cdot\partial_1\nabla\theta dx - 2\int \Delta\theta\nabla u_2\cdot\partial_2\nabla\theta dx \\
:=& J_{21}+J_{22}+J_{23}+J_{24}.
\end{aligned}
\tag{2.2.34}
$$

To bound J_{21}, we invoke the decompositions $u=\overline{u}+\widetilde{u}$ and $\theta=\overline{\theta}+\widetilde{\theta}$ and write it as

$$
\begin{aligned}
J_{21} :=& -\int \Delta\theta\Delta u_1\partial_1\theta dx = -\int \Delta\theta\Delta u_1\partial_1\widetilde{\theta}dx \\
=& -\int \Delta\overline{u_1}\partial_1\widetilde{\theta}\Delta\overline{\theta}dx - \int \Delta\overline{u_1}\partial_1\widetilde{\theta}\Delta\widetilde{\theta}dx \\
& -\int \Delta\widetilde{u_1}\partial_1\widetilde{\theta}\Delta\overline{\theta}dx - \int \Delta\widetilde{u_1}\partial_1\widetilde{\theta}\Delta\widetilde{\theta}dx \\
:=& J_{211}+J_{212}+J_{213}+J_{214}.
\end{aligned}
\tag{2.2.35}
$$

By Lemma 2.2.1,

$$
J_{211} := -\int \Delta\overline{u_1}\partial_1\widetilde{\theta}\Delta\overline{\theta}dx = \int_{\mathbb{R}} \Delta\overline{u_1}\Delta\overline{\theta}\int_{\mathbb{T}}\partial_1\widetilde{\theta}\,dx_1dx_2 = \int_{\mathbb{R}}\Delta\overline{u_1}\Delta\overline{\theta}\partial_1\overline{\widetilde{\theta}}dx_2 = 0. \tag{2.2.36}
$$

To bound J_{212}, we first write it as

$$
\begin{aligned}
J_{212} :=& -\int \Delta\overline{u_1}\partial_1\widetilde{\theta}\Delta\widetilde{\theta}dx \\
=& -\int \partial_{11}\overline{u_1}\partial_1\widetilde{\theta}\Delta\widetilde{\theta}dx - \int \partial_{22}\overline{u_1}\partial_1\widetilde{\theta}\partial_{11}\widetilde{\theta}dx - \int \partial_{22}\overline{u_1}\partial_1\widetilde{\theta}\partial_{22}\widetilde{\theta}dx \\
:=& J_{2121}+J_{2122}+J_{2123}.
\end{aligned}
\tag{2.2.37}
$$

Integrating by parts and Lemma 2.2.1,

$$J_{2121} = J_{2122} = 0, \tag{2.2.38}$$

and using Lemma 2.2.4 and Young's inequality, we get

$$J_{2123} \leq c\|u\|_{H^2}\Big(\|\partial_2\theta\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2\Big). \tag{2.2.39}$$

Collecting all the upper bounds, we get

$$J_{212} \leq c\|u\|_{H^2}\Big(\|\partial_2\theta\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2\Big). \tag{2.2.40}$$

By Lemmas 2.2.1 and 2.2.4, $J_{21i}, i = 3, 4$ can be estimated as

$$J_{21i} \leq c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2 + \|\partial_1 u\|_{H^2}^2\Big). \tag{2.2.41}$$

Collecting (2.2.36), (2.2.40), (2.2.41), and inserting them in (2.2.35), we get

$$J_{21} \leq c(\|u\|_{H^2} + \|\theta\|_{H^2})\Big(\|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2 + \|\partial_1 u\|_{H^2}^2\Big). \tag{2.2.42}$$

Using the decompositions of u and θ, we write J_{22} as,

$$\begin{aligned} J_{22} &:= -\int \Delta\theta\Delta u_2\partial_2\theta dx \\ &= -\int \Delta\overline{\theta}\Delta\overline{u_2}\partial_2\theta dx - \int \Delta\overline{\theta}\Delta\widetilde{u_2}\partial_2\theta dx \\ &\quad - \int \Delta\widetilde{\theta}\Delta\overline{u_2}\partial_2\theta dx - \int \Delta\widetilde{\theta}\Delta\widetilde{u_2}\partial_2\theta dx \\ &:= J_{221} + J_{222} + J_{223} + J_{224}. \end{aligned} \tag{2.2.43}$$

By the divergence free condition of u, and Lemmas 2.2.1 and 2.2.4, we get

$$J_{221} = J_{222} = 0, \tag{2.2.44}$$

According to Lemmas 2.2.1 and 2.2.4 and Young's inequality, we can bound $J_{22i}, i = 3, 4$ as

$$J_{222} \le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big). \tag{2.2.45}$$

Inserting (2.2.44), and (2.2.45) in (2.2.43), we get

$$J_{22} \le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big). \tag{2.2.46}$$

To estimate J_{23}, we split it as

$$\begin{aligned}
J_{23} &:= -2\int \Delta\theta\nabla u_1 \cdot \partial_1\nabla\theta dx \\
&== -2\int \Delta\widetilde{\theta}\partial_1 u_1\partial_1\partial_1\theta dx - 2\int \Delta\overline{\theta}\partial_1 u_1\partial_1\partial_1\theta dx \\
&\quad - 2\int \Delta\widetilde{\theta}\partial_2 u_1\partial_1\partial_2\theta dx - 2\int \Delta\overline{\theta}\partial_2 u_1\partial_1\partial_2\theta dx \\
&:= J_{231} + J_{232} + J_{233} + J_{234}.
\end{aligned} \tag{2.2.47}$$

Using Lemma 2.2.1, we can write J_{231} as

$$\begin{aligned}
J_{231} &:= -2\int \Delta\widetilde{\theta}\partial_1 u_1\partial_1\partial_1\theta dx = -2\int \Delta\widetilde{\theta}\partial_1\widetilde{u_1}\partial_1\partial_1\widetilde{\theta} dx \\
&= -2\int \partial_1\partial_1\widetilde{\theta}\partial_1\widetilde{u_1}\partial_1\partial_1\widetilde{\theta} dx - 2\int \partial_2\partial_2\widetilde{\theta}\partial_1\widetilde{u_1}\partial_1\partial_1\widetilde{\theta} dx \\
&:= J_{2311} + J_{2312}.
\end{aligned} \tag{2.2.48}$$

By $\nabla \cdot u = 0$, integration by parts, Lemma 2.2.4, and Young's inequality, both terms in J_{231}

can be bound by the same bound, and we get

$$J_{231} \le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big). \tag{2.2.49}$$

Using Lemmas 2.2.1 and 2.2.4, we bound J_{232} as

$$J_{232} \le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big). \tag{2.2.50}$$

To bound J_{233}, we invoke decompositions $u = \overline{u} + \widetilde{u}$ and $\theta = \overline{\theta} + \widetilde{\theta}$ to get

$$\begin{aligned}
J_{233} &:= -2\int \Delta\widetilde{\theta}\partial_2 u_1\partial_1\partial_2\theta dx \\
&= -2\int \Delta\widetilde{\theta}\partial_2\widetilde{u_1}\partial_1\partial_2\widetilde{\theta}dx - 2\int \partial_1\partial_1\widetilde{\theta}\partial_2\overline{u_1}\partial_1\partial_2\widetilde{\theta}dx - 2\int \partial_2\partial_2\widetilde{\theta}\partial_2\overline{u_1}\partial_1\partial_2\widetilde{\theta}dx \\
&= J_{2331} + J_{2332} + J_{2333}.
\end{aligned} \tag{2.2.51}$$

According to Lemma 2.2.4 and Young's inequality J_{2331} becomes

$$J_{2331} \le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big). \tag{2.2.52}$$

Using integration by parts, Lemma 2.2.1, Hölder's inequality and Lemma 2.2.2, we have

$$\begin{aligned}
J_{2332} &:= -2\int \partial_1\partial_1\widetilde{\theta}\partial_2\overline{u_1}\partial_1\partial_2\widetilde{\theta}dx \\
&= 2\int_{\mathbb{R}} \partial_2\overline{u_1}\Big(\int_{\mathbb{T}} \partial_1\widetilde{\theta}(\partial_1\partial_1\partial_2\widetilde{\theta})dx_1\Big)dx_2 \\
&\le 2\int_{\mathbb{R}} |\partial_2\overline{u_1}|\|\partial_1\widetilde{\theta}\|_{L^2_{x_1}}\|\partial_1\partial_1\partial_2\widetilde{\theta}\|_{L^2_{x_1}}dx_2 \\
&\le 2\|\partial_2\overline{u_1}\|_{L^\infty_{x_2}}\|\partial_1\widetilde{\theta}\|_{L^2_{x_2}L^2_{x_1}}\|\partial_1\partial_1\partial_2\widetilde{\theta}\|_{L^2_{x_2}L^2_{x_1}} \\
&\le c\|\partial_2\overline{u_1}\|_{H^1}\|\partial_1\widetilde{\theta}\|_{L^2}\|\partial_1\partial_1\partial_2\widetilde{\theta}\|_{L^2} \\
&\le c\|u\|_{H^2}\Big(\|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big).
\end{aligned} \tag{2.2.53}$$

By Lemma 2.2.4,

$$J_{2333} \leq c\,\|u\|_{H^2}\|\partial_2\theta\|_{H^2}^2. \tag{2.2.54}$$

Combining (2.2.52), (2.2.53), (2.2.54) and inserting them in (2.2.51), we get

$$J_{233} \leq c\,c(\|u\|_{H^2} + \|\theta\|_{H^2})\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_2\theta\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2\Big). \tag{2.2.55}$$

By Lemmas 2.2.1 and 2.2.4,

$$J_{234} \leq c\|u\|_{H^2}\|\partial_2\theta\|_{H^2}^2. \tag{2.2.56}$$

Inserting (2.2.49), (2.2.50), (2.2.55) and (2.2.56) in (2.2.47), we obtain

$$J_{23} \leq\ c\,(\|u\|_{H^2} + \|\theta\|_{H^2})\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big). \tag{2.2.57}$$

To estimate J_{24}, using the decompositions $u = \overline{u} + \widetilde{u}$ and $\theta = \overline{\theta} + \widetilde{\theta}$ and Lemma 2.2.1, we get

$$\begin{aligned}
J_{24} &:= -2\int \Delta\theta\nabla u_2 \cdot \partial_2\nabla\theta dx \\
&= -2\int(\partial_1 u_2\partial_1\partial_2\theta\Delta\theta + \partial_2 u_2\partial_2\partial_2\theta\Delta\theta)dx \\
&= -2\int \partial_1\widetilde{u_2}\partial_1\partial_2\widetilde{\theta}\Delta\theta - 2\int \partial_2 u_2\partial_2\partial_2\theta\Delta\theta dx \\
&= -2\int \partial_1\widetilde{u_2}\partial_1\partial_2\widetilde{\theta}\Delta\theta dx - 2\int \partial_2\overline{u_2}\partial_2\partial_2\theta\Delta\theta dx - 2\int \partial_2\widetilde{u_2}\partial_2\partial_2\theta\Delta\theta dx \\
&:= J_{241} + J_{242} + J_{243}.
\end{aligned} \tag{2.2.58}$$

By Lemma 2.2.4 and Young's inequality we have

$$J_{241} \leq c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big). \tag{2.2.59}$$

Next, using the divergence free condition of u and Lemma 2.2.1,

$$J_{242} := -2\int \partial_2\overline{u_2}\partial_2\partial_2\theta\Delta\theta dx = 2\int \partial_1\overline{u_1}\partial_2\partial_2\theta\Delta\theta dx = 0. \tag{2.2.60}$$

According to Lemma 2.2.4 and Young's inequality,

$$J_{244} \le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big). \tag{2.2.61}$$

Collecting (2.2.59), (2.2.60), and (2.2.61) and inserting them in (2.2.58), we obtain

$$J_{24} \le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big). \tag{2.2.62}$$

Thus, by (2.2.42), (2.2.46), (2.2.57), (2.2.62), and (2.2.34),

$$J_2 \le c\,(\|u\|_{H^2} + \|\theta\|_{H^2})\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big). \tag{2.2.63}$$

Substituting (2.2.33) and (2.2.63) in (2.2.26), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla\omega\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2) + \kappa_2\|\partial_2\nabla\theta\|_{L^2}^2 + \nu_1\|\partial_1\nabla\omega\|_{L^2}^2 \\
&\qquad \le c\,(\|u\|_{H^2} + \|\theta\|_{H^2})\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big).
\end{aligned} \tag{2.2.64}$$

Integrating (2.2.64) in time over [0,t], we get

$$\begin{aligned}
&\|\nabla\omega\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2 + 2\kappa_2\int_0^t \|\partial_2\nabla\theta\|_{L^2}^2\,d\tau + 2\nu_1\int_0^t \|\partial_1\nabla\omega\|_{L^2}^2\,d\tau \\
&\le c\int_0^t (\|u\|_{H^2} + \|\theta\|_{H^2})\Big(\|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2 + \|\partial_1 u\|_{H^2}^2\Big)\,d\tau + \|\Delta u_0\|_{L^2}^2 + \|\Delta\theta_0\|_{L^2}^2 \\
&\le \|\Delta u_0\|_{L^2}^2 + \|\Delta\theta_0\|_{L^2}^2 + c\,E_1(t)^{\frac{3}{2}} + c\,E_2(t)^{\frac{3}{2}}.
\end{aligned} \tag{2.2.65}$$

The first desired inequality (2.2.11) then follows from (2.2.13), (2.2.24) and (2.2.65).

Now, we prove the second desired inequality (2.2.12), the idea is to estimate the time integral of $\|\partial_1\theta\|_{L^2}^2$. Here, we need to couple the vorticity equation with the temperature equation because of the lack of horizontal dissipation in the temperature equation θ,

$$\begin{cases} \partial_t\omega + u\cdot\nabla\omega = \nu_1\partial_{11}\omega + \partial_1\theta, \\ \partial_t\theta + u\cdot\nabla\theta + u_2 = \kappa_2\partial_{22}\theta. \end{cases} \tag{2.2.66}$$

Dotting the first equation of (2.2.66) by $\partial_1\theta$ and then integrating in space, we get

$$\begin{aligned} \|\partial_1\theta\|_{L^2}^2 &= \int \partial_1\theta(\partial_t\omega - \nu_1\partial_{11}\omega + u\cdot\nabla\omega)dx \\ &= \frac{d}{dt}\int \partial_1\theta\omega dx - \int \omega\partial_1\partial_t\theta dx - \nu_1\int \partial_1\theta\partial_{11}\omega dx + \int \partial_1\theta(u\cdot\nabla\omega)dx \\ &:= A + B + C + D. \end{aligned}$$

Due to Hölder inequality and Cauchy's inequality, we have

$$\begin{aligned} \int_0^t A d\tau &:= \int_0^t \frac{d}{dt}\int \partial_1\theta\omega dx\, d\tau \\ &\le \frac{1}{2}\Big(\|\theta\|_{H^2}^2 + \|u\|_{H^2}^2\Big) + \frac{1}{2}\Big(\|\theta_0\|_{H^2}^2 + \|u_0\|_{H^2}^2\Big). \end{aligned} \tag{2.2.67}$$

We write B into the three terms using integrating by parts and the second equation in (2.2.66),

$$\begin{aligned} B &:= -\int \omega\,\partial_1\partial_t\theta\,dx = \int \partial_1\omega\,\partial_t\theta\,dx \\ &= \int \partial_1\omega\Big(\kappa_2\partial_{22}\theta - u\cdot\nabla\theta - u_2\Big)dx \\ &= \kappa_2\int \partial_1\omega\,\partial_2\partial_2\theta dx - \int \partial_1\omega\,u_2 dx - \int \partial_1\omega\,u\cdot\nabla\theta\,dx \\ &:= B_1 + B_2 + B_3. \end{aligned} \tag{2.2.68}$$

By Hölder's inequality, integrating by parts and Lemmas 2.2.1, 2.2.4, and 2.2.5, we have

$$B \leq 2\|\partial_1 u\|_{H^2}^2 + \frac{\kappa_2^2}{4}\|\partial_2\theta\|_{H^2}^2 + c\|u\|_{H^2}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_2\theta\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2\Big).$$

Hence,

$$\begin{aligned}\int_0^t B d\tau \leq 2\int_0^t \|\partial_1 u\|_{H^2}^2 d\tau + \frac{\kappa_2^2}{4}\int_0^t \|\partial_2\theta\|_{H^2}^2 d\tau \\ + c\int_0^t \|u\|_{H^2}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_2\theta\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2\Big) d\tau. \qquad (2.2.69)\end{aligned}$$

To bound the integral C, we use both Hölder's inequality and Young's inequality

$$C := -\nu_1\int \partial_1\theta\partial_{11}w dx \leq \nu_1\|\partial_1\theta\|_{L^2}\|\partial_{11}w\|_{L^2} \leq \frac{1}{4}\|\partial_1\theta\|_{L^2}^2 + \nu_1^2\|\partial_1 u\|_{H^2}^2.$$

Hence,

$$\int_0^t C d\tau \leq \frac{1}{4}\int_0^t \|\partial_1\theta\|_{L^2}^2 d\tau + \nu_1^2\int_0^t \|\partial_1 u\|_{H^2}^2 d\tau. \qquad (2.2.70)$$

Due to Lemma 2.2.1, D can be written as

$$\begin{aligned}D &:= \int \partial_1\theta(u\cdot\nabla\omega)dx = \int \partial_1\widetilde{\theta}(u\cdot\nabla\omega)dx \\ &= \int \partial_1\widetilde{\theta}u_1\partial_1\partial_1 u_2 dx - \int \partial_1\widetilde{\theta}u_1\partial_1\partial_2 u_1 dx \\ &\quad + \int \partial_1\widetilde{\theta}u_2\partial_2\partial_1 u_2 dx - \int \partial_1\widetilde{\theta}u_2\partial_2\partial_2 u_1 dx \\ &:= D_1 + D_2 + D_3 + D_4. \qquad (2.2.71)\end{aligned}$$

By using Lemmas 2.2.1 and 2.2.4, the fact that $\overline{u_2} = 0$ and the inequality (2.2.7), D can be bounded by

$$D \le c\|u\|_{H^2}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big)$$

Hence,

$$\begin{aligned}\int_0^t D\,d\tau &\le c\int_0^t \|u\|_{H^2}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big)d\tau \\ &\le cE_1(t)^{\frac{3}{2}} + cE_2(t)^{\frac{3}{2}}. \end{aligned} \tag{2.2.72}$$

Therefore, combining the estimates (2.2.67), (2.2.69), (2.2.70) and (2.2.72), we obtain

$$\begin{aligned}\int_0^t \|\partial_1\theta\|_{L^2}^2 d\tau \le\ & \frac{1}{2}\Big(\|\theta\|_{H^2}^2 + \|u\|_{H^2}^2\Big) + \frac{1}{2}\Big(\|\theta_0\|_{H^2}^2 + \|u_0\|_{H^2}^2\Big) \\ & + 2\int_0^t \|\partial_1 u\|_{H^2}^2 d\tau + \frac{\eta^2}{4}\int_0^t \|\partial_2\theta\|_{H^2}^2 d\tau \\ & + \frac{1}{4}\int_0^t \|\partial_1\theta\|_{L^2}^2 d\tau + \delta\nu^2\int_0^t \|\partial_1 u\|_{H^2}^2 d\tau \\ & + c\int_0^t \|u\|_{H^2}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big)d\tau \\ \le\ & \frac{1}{4}\int_0^t \|\partial_1\theta\|_{L^2}^2 d\tau + cE_1(0) + cE_1(t) + cE_1(t)^{\frac{3}{2}} + cE_2(t)^{\frac{3}{2}}, \end{aligned}$$

which is (2.2.12). Adding (2.2.11) with $1/(2c_6)$ of (2.2.12) yields the desired inequality in (2.2.10). The bootstrapping argument applied to (2.2.10), as in the proof of Theorem 1.2.2 yields the desired global H^2-bound on (u, θ). The proof of uniqueness is standard, and we refer to the original paper [1] for the proof. ■

2.2.2 The Decay Rates of Oscillation Part

This subsection proves the Theorem 1.2.4.

The Sketch of the proof of Theorem 1.2.4. First, we write the system governing the horizontal average $(\overline{u}, \overline{\theta})$, namely,

$$\begin{cases} \partial_t \overline{u} + \overline{u \cdot \nabla \widetilde{u}} + \begin{pmatrix} 0 \\ \\ \partial_2 \overline{p} \end{pmatrix} = \begin{pmatrix} 0 \\ \\ \overline{\theta} \end{pmatrix}, \\ \\ \partial_t \overline{\theta} + \overline{u \cdot \nabla \widetilde{\theta}} = \kappa_2 \partial_2^2 \overline{\theta}. \end{cases} \tag{2.2.73}$$

Taking the difference of (1.2.20) and (2.2.73), we get

$$\begin{cases} \partial_t \widetilde{u} + \widetilde{u \cdot \nabla \widetilde{u}} + \widetilde{u}_2 \partial_2 \overline{u} - \nu_1 \partial_1^2 \widetilde{u} + \nabla \widetilde{p} = \widetilde{\theta} e_2 \,, \\ \partial_t \widetilde{\theta} + \widetilde{u \cdot \nabla \widetilde{\theta}} + \widetilde{u}_2 \partial_2 \overline{\theta} - \kappa_2 \partial_2^2 \widetilde{\theta} + \widetilde{u}_2 = 0. \end{cases} \tag{2.2.74}$$

Dotting the system (2.2.74) by $(\widetilde{u}, \widetilde{\theta})$ yields,

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \Big(\|\widetilde{u}\|_{L^2}^2 + \|\widetilde{\theta}\|_{L^2}^2 \Big) + \nu_1 \|\partial_1 \widetilde{u}\|_{L^2}^2 + \kappa_2 \|\partial_2 \widetilde{\theta}\|_{L^2}^2 \\ &= - \int \widetilde{u \cdot \nabla \widetilde{u}} \cdot \widetilde{u} dx - \int \widetilde{u}_2 \partial_2 \overline{u} \cdot \widetilde{u} dx - \int \widetilde{u \cdot \nabla \widetilde{\theta}} \cdot \widetilde{\theta} dx - \int \widetilde{u}_2 \partial_2 \overline{\theta} \cdot \widetilde{\theta} dx \\ &:= A_1 + A_2 + A_3 + A_4. \end{aligned} \tag{2.2.75}$$

By the divergence free condition of u and Lemmas 2.2.1, we have $A_1 = A_3 = 0$. Then, again using the divergence free condition of u, and Lemmas 2.2.1, 2.2.4 and 2.2.5, we can bound A_2 as

$$A_2 \leq c \|u\|_{H^2} \|\partial_1 \widetilde{u}\|_{L^2}^2. \tag{2.2.76}$$

Similarly, by Hölder's inequality, and Lemmas 2.2.2 and 2.2.5, we have

$$A_4 \leq c \|\theta\|_{H^2} \Big(\|\partial_1 \widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2 \Big). \tag{2.2.77}$$

Therefore, we have

$$\frac{1}{2}\frac{d}{dt}\Big(\|\widetilde{u}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2\Big)+\nu_1\|\partial_1\widetilde{u}\|_{L^2}^2+\kappa_2\|\partial_2\widetilde{\theta}\|_{L^2}^2$$
$$\leq\ c\left(\|u\|_{H^2}+\|\theta\|_{H^2}\right)\Big(\|\partial_1\widetilde{u}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2\Big). \qquad (2.2.78)$$

Now, we apply ∇ to (2.2.74) to get

$$\begin{cases}\partial_t\nabla\widetilde{u}+\nabla(\widetilde{u\cdot\nabla\widetilde{u}})+\nabla(\widetilde{u_2}\partial_2\overline{u})-\nu_1\partial_1^2\nabla\widetilde{u}+\nabla\nabla\widetilde{p}=\nabla(\widetilde{\theta}e_2)\,,\\ \partial_t\nabla\widetilde{\theta}+\nabla(\widetilde{u\cdot\nabla\widetilde{\theta}})+\nabla(\widetilde{u_2}\partial_2\overline{\theta})-\kappa_2\partial_2^2\nabla\widetilde{\theta}+\nabla\widetilde{u_2}=0.\end{cases} \qquad (2.2.79)$$

Taking the L^2-inner product of (2.2.79) with $(\nabla\widetilde{u},\nabla\widetilde{\theta})$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|\nabla\widetilde{u}(t)\|_{L^2}^2+\|\nabla\widetilde{\theta}(t)\|_{L^2}^2\Big)+\nu_1\|\partial_1\nabla\widetilde{u}\|_{L^2}^2+\kappa_2\|\partial_2\nabla\widetilde{\theta}\|_{L^2}^2\\ &=-\int\nabla(\widetilde{u\cdot\nabla\widetilde{u}})\cdot\nabla\widetilde{u}dx-\int\nabla(\widetilde{u_2}\partial_2\overline{u})\cdot\nabla\widetilde{u}dx\\ &\quad-\int\nabla(\widetilde{u\cdot\nabla\widetilde{\theta}})\cdot\nabla\widetilde{\theta}dx-\int\nabla(\widetilde{u_2}\partial_2\overline{\theta})\cdot\nabla\widetilde{\theta}dx\\ &:=B_1+B_2+B_3+B_4.\end{aligned} \qquad (2.2.80)$$

According to Lemma 2.2.1, we write B_1 explicitly into the following four integrals,

$$\begin{aligned}B_1&:=-\int\nabla(\widetilde{u\cdot\nabla\widetilde{u}})\cdot\nabla\widetilde{u}dx=-\int\nabla(u\cdot\nabla\widetilde{u})\cdot\nabla\widetilde{u}\,dx\\ &=-\int\partial_1u_1\partial_1\widetilde{u}\cdot\partial_1\widetilde{u}dx-\int\partial_1u_2\partial_2\widetilde{u}\cdot\partial_1\widetilde{u}dx\\ &\quad-\int\partial_2u_1\partial_1\widetilde{u}\cdot\partial_2\widetilde{u}dx-\int\partial_2u_2\partial_2\widetilde{u}\cdot\partial_2\widetilde{u}dx\\ &:=B_{11}+B_{12}+B_{13}+B_{14}.\end{aligned} \qquad (2.2.81)$$

Using Lemma 2.2.1 and the inequality in (2.2.7), we can bound all the above terms with the same bound, therefore we get

$$B_1 \le c\|u\|_{H^2}\|\partial_1\widetilde{u}\|_{H^1}^2.. \tag{2.2.82}$$

We write B_2 explicitly,

$$\begin{aligned} B_2 &:= -\int \nabla(\widetilde{u_2}\partial_2\overline{u})\cdot\nabla\widetilde{u}dx \\ &= -\int \partial_1\widetilde{u_2}\partial_2\overline{u}\cdot\partial_1\widetilde{u}dx - \int \partial_2\widetilde{u_2}\partial_2\overline{u}\cdot\partial_2\widetilde{u}dx \\ &\quad -\int \widetilde{u_2}\partial_1\partial_2\overline{u}\cdot\partial_1\widetilde{u}dx - \int \widetilde{u_2}\partial_2\partial_2\overline{u}\cdot\partial_2\widetilde{u}dx \\ &:= B_{21} + B_{22} + B_{23} + B_{24}. \end{aligned} \tag{2.2.83}$$

By the definition of $\overline{u}$, $B_{23} = 0$ and using Lemma 2.2.1 and the divergence free condition of u all the remaining terms can be bound with the same bound resulting

$$B_2 \le c\|u\|_{H^2}\|\partial_1\widetilde{u}\|_{H^1}^2. \tag{2.2.84}$$

By the definition of $\overline{u}$, we can split B_3 into four integrals,

$$\begin{aligned} B_3 &:= -\int \nabla(\widetilde{u\cdot\nabla\widetilde{\theta}})\cdot\nabla\widetilde{\theta}dx = -\int \nabla(u\cdot\nabla\widetilde{\theta})\cdot\nabla\widetilde{\theta}dx \\ &= -\int \partial_1\widetilde{\theta}\partial_1\widetilde{u_1}\partial_1\widetilde{\theta}dx - \int \partial_2\widetilde{\theta}\partial_1\widetilde{u_2}\partial_1\widetilde{\theta}dx \\ &\quad -\int \partial_1\widetilde{\theta}\partial_2 u_1\partial_2\widetilde{\theta}dx - \int \partial_2\widetilde{\theta}\partial_2\widetilde{u_2}\partial_2\widetilde{\theta}dx \\ &:= B_{31} + B_{32} + B_{33} + B_{34}. \end{aligned} \tag{2.2.85}$$

Integrating by parts and using Lemma 2.2.1 and Young's inequality, we have

$$B_{31} \le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\partial_2\widetilde{\theta}\|_{H^1}^2\Big). \tag{2.2.86}$$

To deal with B_{32}, we use Lemma 2.2.4,

$$B_{32} \leq c\|\theta\|_{H^2}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\partial_2\widetilde{\theta}\|_{H^1}^2\Big). \tag{2.2.87}$$

For B_{33}, we invoke the decomposition $u_1 = \overline{u_1} + \widetilde{u_1}$ to write it into two integrals

$$\begin{aligned} B_{33} &:= -\int \partial_1\widetilde{\theta}\partial_2 u_1 \partial_2\widetilde{\theta}dx \\ &= -\int \partial_1\widetilde{\theta}\partial_2\widetilde{u_1}\partial_2\widetilde{\theta}dx - \int \partial_1\widetilde{\theta}\partial_2\overline{u_1}\partial_2\widetilde{\theta}dx \\ &:= B_{331} + B_{332}. \end{aligned} \tag{2.2.88}$$

By integration by parts, Hölder's inequality, and Lemmas 2.2.1 and 2.2.4,

$$B_{331} \leq c\|\theta\|_{H^2}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2 + \|\partial_2\widetilde{\theta}\|_{H^1}^2\Big). \tag{2.2.89}$$

Due to Lemma 2.2.2 and Hölder's inequality,

$$B_{332} \leq c\|u\|_{H^2}\Big(\|\widetilde{\theta}\|_{L^2}^2 + \|\partial_2\widetilde{\theta}\|_{H^1}^2\Big). \tag{2.2.90}$$

Combining (2.2.89), (2.2.90) and (2.2.88), we obtain,

$$B_{33} \leq c\left(\|u\|_{H^2} + \|\theta\|_{H^2}\right)\Big(\|\widetilde{\theta}\|_{L^2}^2 + \|\partial_2\widetilde{\theta}\|_{H^1}^2 + \|\partial_1\widetilde{u}\|_{H^1}^2\Big). \tag{2.2.91}$$

According to the divergence-free condition of u and Lemma 2.2.4,

$$B_{34} \leq c\|\theta\|_{H^2}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\partial_2\widetilde{\theta}\|_{H^1}^2\Big). \tag{2.2.92}$$

Inserting the estimates (2.2.86), (2.2.87), (2.2.88) and (2.2.92) in (2.2.85) yields

$$B_3 \le c\,(\|u\|_{H^2} + \|\theta\|_{H^2})\Big(\|\partial_1 \widetilde{u}\|_{H^1}^2 + \|\partial_2 \widetilde{\theta}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2\Big). \tag{2.2.93}$$

By integration by parts, we write B_4 into four terms as follows,

$$\begin{aligned}
B_4 &:= -\int \nabla(\widetilde{u_2}\partial_2\overline{\theta}) \cdot \nabla\widetilde{\theta} dx \\
&= -\int \partial_1(\widetilde{u_2}\partial_2\overline{\theta}) \cdot \partial_1\widetilde{\theta} dx - \int \partial_2(\widetilde{u_2}\partial_2\overline{\theta}) \cdot \partial_2\widetilde{\theta} dx \\
&= -\int \partial_1\widetilde{u_2}\partial_2\overline{\theta}\partial_1\widetilde{\theta} dx - \int \widetilde{u_2}\partial_1\partial_2\overline{\theta}\partial_1\widetilde{\theta} dx \\
&\quad -\int \partial_2\widetilde{u_2}\partial_2\overline{\theta}\partial_2\widetilde{\theta} dx - \int \widetilde{u_2}\partial_2\partial_2\overline{\theta}\partial_2\widetilde{\theta} dx \\
&:= B_{41} + B_{42} + B_{43} + B_{44}.
\end{aligned} \tag{2.2.94}$$

Due to the definition of the horizontal average $\overline{\theta}$, $B_{42} = 0$ and using integration by parts, Hölder's inequality, the divergence-free condition of u, Lemmas 2.2.1 and 2.2.4, all remaining terms have same bound.

$$B_4 \le c\Big(\|u\|_{H^2} + \|\theta\|_{H^2}\Big)\Big(\|\partial_1 \widetilde{u}\|_{H^1}^2 + \|\partial_2 \widetilde{\theta}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2\Big). \tag{2.2.95}$$

Combining (2.2.82), (2.2.84), (2.2.93) and (2.2.95) yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\nabla\widetilde{u}(t)\|_{L^2}^2 + \|\nabla\widetilde{\theta}(t)\|_{L^2}^2\Big) + \nu_1\|\partial_1\nabla\widetilde{u}\|_{L^2}^2 + \kappa_2\|\partial_2\nabla\widetilde{\theta}\|_{L^2}^2 \\
&\qquad\qquad \le c\,(\|u\|_{H^2} + \|\theta\|_{H^2})\Big(\|\partial_1 \widetilde{u}(t)\|_{H^1}^2 + \|\partial_2 \widetilde{\theta}(t)\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2\Big).
\end{aligned} \tag{2.2.96}$$

In order to control the norm $\|\widetilde{\theta}\|_{L^2}$ appearing in (2.2.78) and (2.2.96), we need to add the following term,

$$-\frac{d}{dt}\Big(\delta(\widetilde{u_2}, \widetilde{\theta})\Big) = -\delta(\partial_t\widetilde{u_2}, \widetilde{\theta}) - \delta(\widetilde{u_2}, \partial_t\widetilde{\theta}),$$

where $\delta > 0$ is a small constant to be fixed in the end of the proof. The inclusion of this term will produce an extra regularization term to help bound $\|\widetilde{\theta}\|_{L^2}$. This stabilizing term comes due to the interaction between $\widetilde{u}$ and $\widetilde{\theta}$ in (2.2.74). By Hölder's inequality, one easily sees that, for sufficiently small $\delta > 0$,

$$(\|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2) - \delta(\widetilde{u_2}, \widetilde{\theta}) \geq 0.$$

Using the first equation of (2.2.74) and the fact that $\overline{u_2} = 0$, we have

$$\partial_t \widetilde{u_2} + \widetilde{u \cdot \nabla \widetilde{u_2}} - \nu_1 \partial_1^2 \widetilde{u_2} + \partial_2 \widetilde{p} = \widetilde{\theta}. \tag{2.2.97}$$

Applying $\nabla \cdot$ to the first equation of (2.2.74), we get

$$\nabla \cdot (\widetilde{u \cdot \nabla \widetilde{u}}) + \nabla \cdot (\widetilde{u_2} \partial_2 \overline{u}) + \Delta \widetilde{p} = \partial_2 \widetilde{\theta}. \tag{2.2.98}$$

Using (2.2.98), and applying ∂_2, we get

$$\partial_2 \widetilde{p} = -\partial_2 \Delta^{-1} \nabla \cdot (\widetilde{u \cdot \nabla \widetilde{u}}) - \partial_2 \Delta^{-1} \nabla \cdot (\widetilde{u_2} \partial_2 \overline{u}) + \partial_2 \partial_2 \Delta^{-1} \widetilde{\theta}. \tag{2.2.99}$$

Using (2.2.97) and the second equation of (2.2.74), we get

$$\begin{aligned}
-\delta \frac{d}{dt}(\widetilde{u_2}, \widetilde{\theta}) &= -\delta(\partial_t \widetilde{u_2}, \widetilde{\theta}) - \delta(\widetilde{u_2}, \partial_t \widetilde{\theta}) \\
&= -\delta(\widetilde{\theta} - \partial_2 \widetilde{p} + \nu_1 \partial_1^2 \widetilde{u_2} - \widetilde{u \cdot \nabla \widetilde{u_2}}, \widetilde{\theta}) \\
&\quad - \delta(\widetilde{u_2}, -\widetilde{u_2} + \kappa_2 \partial_2^2 \widetilde{\theta} - \widetilde{u_2} \partial_2 \overline{\theta} - \widetilde{u \cdot \nabla \widetilde{\theta}}) \\
&= -\delta \|\widetilde{\theta}\|_{L^2}^2 + \int \partial_2 \widetilde{p} \widetilde{\theta} dx - \delta \nu_1 \int \partial_1^2 \widetilde{u_2} \widetilde{\theta} dx + \delta \int \widetilde{u \cdot \nabla \widetilde{u_2}} \widetilde{\theta} dx \\
&\quad + \delta \|\widetilde{u_2}\|_{L^2}^2 - \delta \kappa_2 \int \partial_2^2 \widetilde{\theta} \widetilde{u_2} dx + \delta \int \widetilde{u_2} \widetilde{u_2} \partial_2 \overline{\theta} dx + \delta \int \widetilde{u \cdot \nabla \widetilde{\theta}} \widetilde{u_2} dx \\
&:= N_1 + \cdots + N_8.
\end{aligned} \tag{2.2.100}$$

We write N_2 as

$$
\begin{aligned}
N_2 &:= \delta \int \partial_2 \widetilde{p} \widetilde{\theta} dx \\
&= -\delta \int \partial_2 \Delta^{-1} \nabla \cdot (\widetilde{u \cdot \nabla \widetilde{u}}) \cdot \widetilde{\theta} dx - \delta \int \partial_2 \Delta^{-1} \nabla \cdot (\widetilde{u_2} \partial_2 \overline{u}) \cdot \widetilde{\theta} dx \\
&\quad + \delta \int \partial_2 \partial_2 \Delta^{-1} \widetilde{\theta} \cdot \widetilde{\theta} dx \\
&:= N_{21} + N_{22} + N_{23}.
\end{aligned} \tag{2.2.101}
$$

By (2.2.99), Hölder's inequality, the boundedness of the Riesz transform and Lemmas 2.2.4 and 2.2.2, N_{21} and N_{22} can be bounded as

$$
N_{21}, N_{22} \leq c\delta \|u\|_{H^2} \Big(\|\partial_1 \widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2 \Big). \tag{2.2.102}
$$

For N_{23}, using integration by parts and Plancherel's theorem yields

$$
\begin{aligned}
N_{23} &:= \delta \int \partial_2 \partial_2 \Delta^{-1} \widetilde{\theta} \cdot \widetilde{\theta} dx \\
&= \delta \int \partial_2 \Delta^{-\frac{1}{2}} \widetilde{\theta} \cdot \partial_2 \Delta^{-\frac{1}{2}} \widetilde{\theta} dx \\
&= \delta \|\partial_2 \Lambda^{-1} \widetilde{\theta}\|_{L^2}^2 \\
&= \delta \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} \int_{\mathbb{R}} \frac{\xi_2^2}{k^2 + \xi_2^2} |\widehat{\widetilde{\theta}}(k, \xi_2)|^2 d\xi_2 \\
&\leq \delta \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} \int_{\mathbb{R}} \xi_2^2 |\widehat{\widetilde{\theta}}(k, \xi_2)|^2 d\xi_2 = \delta \|\partial_2 \widetilde{\theta}\|_{L^2}^2,
\end{aligned} \tag{2.2.103}
$$

where $\Lambda = (-\Delta)^{\frac{1}{2}}$ and we have used the fact that the oscillation part has the horizontal mode equal to 0, or $\widehat{\widetilde{\theta}}(0, \xi_2) = 0$. Combining (2.2.102), (2.2.103) and (2.2.101) yields

$$
N_2 \leq c\delta \Big(\|u\|_{H^2} + \|\theta\|_{H^2} \Big) \Big(\|\partial_1 \widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2 \Big) + \delta \|\partial_2 \widetilde{\theta}\|_{L^2}^2. \tag{2.2.104}
$$

By Hölder's inequality,

$$N_3 := -\delta\nu_1 \int \partial_1^2 \widetilde{u_2}\widetilde{\theta} dx \le \delta\nu_1 \|\partial_1^2 \widetilde{u_2}\|_{L^2} \|\widetilde{\theta}\|_{L^2} \le \delta\nu_1^2 \|\partial_1 \widetilde{u}\|_{H^1}^2 + \frac{\delta}{4}\|\widetilde{\theta}\|_{L^2}^2. \tag{2.2.105}$$

To bound N_4, we use Lemma 2.2.1, Hölder's inequality, and Lemmas 2.2.3 and 2.2.5,

$$\begin{aligned} N_4 &:= \delta \int \widetilde{u \cdot \nabla \widetilde{u_2}} \widetilde{\theta} dx \\ &= \delta \int u \cdot \nabla \widetilde{u_2} \widetilde{\theta} dx \\ &\le c\delta \|u \cdot \nabla \widetilde{u_2}\|_{L^2} \|\widetilde{\theta}\|_{L^2} \\ &\le c\delta \|u\|_{H^2} \Big(\|\partial_1 \widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2 \Big). \end{aligned} \tag{2.2.106}$$

By Lemma 2.2.5,

$$N_5 := \delta \|\widetilde{u_2}\|_{L^2}^2 \le c\delta \|\partial_1 \widetilde{u_2}\|_{L^2}^2 \le c\delta \|\partial_1 \widetilde{u}\|_{H^1}^2. \tag{2.2.107}$$

Due to Hölder's inequality and Lemma 2.2.5,

$$\begin{aligned} N_6 &:= -\delta\kappa_2 \int \partial_2^2 \widetilde{\theta} \widetilde{u_2} dx \\ &\le c\delta \|\partial_2^2 \widetilde{\theta}\|_{L^2} \|\widetilde{u_2}\|_{L^2} \\ &\le c\delta \Big(\|\partial_2 \widetilde{\theta}\|_{H^1}^2 + \|\partial_1 \widetilde{u}\|_{H^1}^2 \Big). \end{aligned} \tag{2.2.108}$$

Using Lemma 2.2.4 and Lemma 2.2.5, we get

$$\begin{aligned} N_7 &:= \delta \int \widetilde{u_2}\widetilde{u_2} \partial_2 \overline{\theta} dx \\ &\le c\delta \|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \overline{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_2 \overline{\theta}\|_{L^2}^{\frac{1}{2}} \|\widetilde{u_2}\|_{L^2} \\ &\le c\delta \|\theta\|_{H^2} \|\partial_1 \widetilde{u}\|_{H^1}^2. \end{aligned} \tag{2.2.109}$$

To deal with N_8, we first split it into three terms using Lemma 2.2.1,

$$
\begin{aligned}
N_8 &:= \delta \int \widetilde{u \cdot \nabla \widetilde{\theta}} \widetilde{u_2} dx \\
&= \delta \int u \cdot \nabla \widetilde{\theta} \widetilde{u_2} dx \\
&= \delta \int \widetilde{u_1} \partial_1 \widetilde{\theta} \widetilde{u_2} dx + \delta \int \overline{u_1} \partial_1 \widetilde{\theta} \widetilde{u_2} dx + \delta \int u_2 \partial_2 \widetilde{\theta} \widetilde{u_2} dx \\
&:= N_{81} + N_{82} + N_{83}. \qquad (2.2.110)
\end{aligned}
$$

By Lemma 2.2.4 and divergence free condition of u, integration by parts, Hölder's inequality and Lemma 2.2.2, (2.2.7), Lemma 2.2.5 and in view of (2.2.110), we get

$$
N_8 \leq c\delta \Big(\|u\|_{H^2} + \|\theta\|_{H^2} \Big) \Big(\|\partial_1 \widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2 \Big). \qquad (2.2.111)
$$

Inserting (2.2.104), (2.2.105), (2.2.106), (2.2.107), (2.2.108), (2.2.109) and (2.2.111) in (2.2.100) leads to

$$
\begin{aligned}
-\delta \frac{d}{dt} (\widetilde{u_2}, \widetilde{\theta}) \leq & -\delta \|\widetilde{\theta}\|_{L^2}^2 + c\delta \Big(\|u\|_{H^2} + \|\theta\|_{H^2} \Big) \Big(\|\partial_1 \widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2 \Big) \\
& + \frac{\delta}{4} \|\widetilde{\theta}\|_{L^2}^2 + c\delta \Big(\|\partial_1 \widetilde{u}\|_{H^1}^2 + \|\partial_2 \widetilde{\theta}\|_{H^1}^2 \Big). \qquad (2.2.112)
\end{aligned}
$$

Combining (2.2.78), (2.2.96) and (2.2.112), we obtain

$$
\begin{aligned}
\frac{d}{dt} \Big(\|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2 - \delta (\widetilde{u_2}, \widetilde{\theta}) \Big) & + 2\nu_1 \|\partial_1 \widetilde{u}\|_{H^1}^2 + 2\kappa_2 \|\partial_2 \widetilde{\theta}\|_{H^1}^2 \\
& \leq c \Big(\|u\|_{H^2} + \|\theta\|_{H^2} \Big) \Big(\|\partial_1 \widetilde{u}\|_{H^1}^2 + \|\partial_2 \widetilde{\theta}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2 \Big) \\
& \quad - \frac{3\delta}{4} \|\widetilde{\theta}\|_{L^2}^2 + c\delta \Big(\|u\|_{H^2} + \|\theta\|_{H^2} \Big) \Big(\|\partial_1 \widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2 \Big) \\
& \quad + c\delta \Big(\|\partial_1 \widetilde{u}\|_{H^1}^2 + \|\partial_2 \widetilde{\theta}\|_{H^1}^2 \Big).
\end{aligned}
$$

By Theorem 1.2.3, we know that if $\varepsilon > 0$ is sufficiently small and $\|u_0\|_{L^2} + \|\theta_0\|_{L^2} \leq \varepsilon$, then

$\Big(\|u\|_{H^2} + \|\theta\|_{H^2}\Big) \leq c\varepsilon$. Hence we get

$$
\begin{aligned}
\frac{d}{dt}\Big(\|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2 - \delta(\widetilde{u_2}, \widetilde{\theta})\Big) &+ 2\nu_1\|\partial_1\widetilde{u}\|_{H^1}^2 + 2\kappa_2\|\partial_2\widetilde{\theta}\|_{H^1}^2 \\
&\leq c\epsilon\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\partial_2\widetilde{\theta}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2\Big) \\
&\quad - \frac{3\delta}{4}\|\widetilde{\theta}\|_{L^2}^2 + c\delta\epsilon\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2\Big) \\
&\quad + c\delta\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\partial_2\widetilde{\theta}\|_{H^1}^2\Big).
\end{aligned}
$$

Choosing $\epsilon > 0$ such that $c\epsilon \leq \min(\frac{1}{4}, \frac{\delta}{4})$, we get

$$
\begin{aligned}
\frac{d}{dt}\Big(\|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2 - \delta(\widetilde{u_2}, \widetilde{\theta})\Big) &+ 2\nu_1\|\partial_1\widetilde{u}\|_{H^1}^2 + 2\kappa_2\|\partial_2\widetilde{\theta}\|_{H^1}^2 \\
&\leq \frac{\delta}{4}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\partial_2\widetilde{\theta}\|_{H^1}^2\Big) + \frac{\delta}{4}\|\widetilde{\theta}\|_{L^2}^2 \\
&\quad - \frac{3\delta}{4}\|\widetilde{\theta}\|_{L^2}^2 + \frac{\delta}{4}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2\Big) \\
&\qquad\qquad + c\delta\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\partial_2\widetilde{\theta}\|_{H^1}^2\Big) \\
&\leq -\frac{\delta}{4}\|\widetilde{\theta}\|_{L^2}^2 + c\delta\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\partial_2\widetilde{\theta}\|_{H^1}^2\Big).
\end{aligned}
$$

Choosing $\delta > 0$ such that $c\delta \leq \min(\nu_1, \kappa_2, \frac{c}{2})$, we obtain

$$
\frac{d}{dt}\Big(\|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2 - \delta(\widetilde{u_2}, \widetilde{\theta})\Big) + \nu_1\|\partial_1\widetilde{u}\|_{H^1}^2 + \kappa_2\|\partial_2\widetilde{\theta}\|_{H^1}^2 + \frac{\delta}{4}\|\widetilde{\theta}\|_{L^2}^2 \leq 0. \tag{2.2.113}
$$

Due to the choice of δ, we have

$$
\frac{1}{2}\Big(\|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2\Big) - \delta(\widetilde{u_2}, \widetilde{\theta}) \geq 0.
$$

or

$$
\frac{1}{2}(\|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2) \leq \|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2 - \delta(\widetilde{u_2}, \widetilde{\theta}) \leq \frac{3}{2}(\|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2).
$$

For any $0 \le s \le t$, integrating (2.2.113) in time yields

$$\frac{1}{2}(\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2) + \int_s^t (\nu_1 \|\partial_1 \widetilde{u}\|_{H^1}^2 + \kappa_2 \|\partial_2 \widetilde{\theta}\|_{H^1}^2 + \frac{\delta}{4}\|\widetilde{\theta}\|_{L^2}^2)\, d\tau$$
$$\le \frac{3}{2}(\|\widetilde{u}(s)\|_{H^1}^2 + \|\widetilde{\theta}(s)\|_{H^1}^2).$$

Especially, for any $0 \le s \le t$,

$$\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2 \le 3(\|\widetilde{u}(s)\|_{H^1}^2 + \|\widetilde{\theta}(s)\|_{H^1}^2) \tag{2.2.114}$$

and

$$\int_0^\infty (\nu_1 \|\partial_1 \widetilde{u}\|_{H^1}^2 + \kappa_2 \|\partial_2 \widetilde{\theta}\|_{H^1}^2 + \frac{\delta}{4}\|\widetilde{\theta}\|_{L^2}^2)\, d\tau \le C < \infty.$$

Combining with the time integral bounds from Theorem 1.2.3,

$$\int_0^\infty \|\partial_1 u\|_{H^2}^2\, dt < \infty, \quad \int_0^\infty \|\partial_1 \theta\|_{L^2}^2\, dt < \infty \quad \text{and} \quad \int_0^\infty \|\partial_2 \theta\|_{H^2}^2\, dt < \infty,$$

we obtain

$$\int_0^\infty (\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2)\, dt < \infty. \tag{2.2.115}$$

Applying Lemma 2.2.6 to (2.2.114) and (2.2.115) yields

$$\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2 \le c(1+t)^{-1},$$

and the asymptotic behavior, as $t \to \infty$,

$$t\,(\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2) \to 0.$$

This completes the main ideas of the proof of Theorem 1.2.4. ■

CHAPTER III

3D ROTATING BOUSSINESQ EQUATIONS

This chapter concerns 3D rotating Boussinesq equations. In particular, we consider the perturbation of the 3D rotating Boussinesq equations near a special 2D solution. The stability result we discuss here is different and maybe more complex than the stability results presented in Chapter II, where we considered the steady special solution. Whereas, this chapter deals with the dynamic special solution. We recall the following anisotropic 3D rotating Boussinesq equations only with the horizontal dissipation

$$\begin{cases} \partial_t v_h + v_h \cdot \nabla_h v_h + v_3\, \partial_3 v_h + f\, v_h^{\perp} = -\nabla_h p + \nu\, \Delta_h v_h, \\ \partial_t v_3 + v_h \cdot \nabla_h v_3 + v_3\, \partial_3 v_3 = -\partial_3 p + \nu\, \Delta_h v_3 - \rho, \\ \partial_t \rho + v_h \cdot \nabla_h \rho + v_3\, \partial_3 \rho = \kappa\, \Delta_h \rho + v_3, \\ \nabla_h \cdot v_h + \partial_3 v_3 = 0, \\ (v_h(x,0),\, v_3(x,0),\, \rho(x,0))|_{t=0} = (v_{h0}(x),\, v_{30}(x),\, \rho_0(x)). \end{cases} \tag{3.0.1}$$

Also, recall that the special solution of (3.0.1) is given by

$$(v_h^{(0)},\, v_3^{(0)},\, \rho^{(0)})|_{t=0} = (v_{h0}^{(0)}(x_h,\, t),\, 0,\, 0), \quad p^{(0)} = 0,$$

with $(v_h^{(0)})$ satisfying

$$\begin{cases} \partial_t v_h^{(0)} + v_h^{(0)} \cdot \nabla_h v_h^{(0)} + f\, v_h^{(0)\perp} = -\nabla_h p + \nu\, \Delta_h v_h^{(0)}, \\ \nabla_h \cdot v_h^{(0)} = 0, \quad v_h^{(0)}(x_h, 0) = v_{h0}^{(0)}(x_h). \end{cases} \tag{3.0.2}$$

The goal of this chapter is to prove Theorem 1.2.1 which states that when the initial data $(v_{h0},\, v_{30},\, \rho_0)$ of (3.0.1) is close to the initial data $(v_{h0}^{(0)},\, 0,\, 0)$ of (3.0.2), then (3.0.1) has a unique global solution that is close to the special 2D solution given by (3.0.2) for all the time. To prove Theorem 1.2.1, we need to establish a global *a priori* bound of the solution in H^2-norm and then prove the local existence and uniqueness of the solution. This task is divided into two sections. In Section 3.1, we use the bootstrapping argument to establish the existence of a global *a priori* bound. In Section 3.2, we use Friedrich's method to prove the local existence and uniqueness.

3.1 Global *a priori* bound

This section aims to prove the following proposition (3.1.1). To prove it, we need the following lemma about the anisotropic upper bound for triple products (see, e.g., [6]).

Lemma 3.1.1 *There exists a constant $C > 0$ such that*

$$\Big| \int_{\mathbb{R}^3} F(x)G(x)H(x)dx \Big| \leq C\, \|F\|_{L^2}^{\frac{1}{2}} \|\partial_1 F\|_{L^2}^{\frac{1}{2}} \|G\|_{L^2}^{\frac{1}{2}} \|\partial_2 G\|_{L^2}^{\frac{1}{2}} \|H\|_{L^2}^{\frac{1}{2}} \|\partial_3 H\|_{L^2}^{\frac{1}{2}}$$

$$\Big| \int_{\mathbb{R}^3} F(x)G(x)H(x)dx \Big| \leq C\, \|F\|_{L^2}^{\frac{1}{4}} \|\partial_3 F\|_{L^2}^{\frac{1}{4}} \|\nabla_h F\|_{L^2}^{\frac{1}{4}} \|\nabla_h \partial_3 F\|_{L^2}^{\frac{1}{4}} \times \|G\|_{L^2} \|H\|_{L^2}^{\frac{1}{2}} \|\partial_3 H\|_{L^2}^{\frac{1}{2}}$$

Proposition 3.1.1 *Assume that $(\widetilde{v}_h,\, v_3,\, \rho)$ solves (1.2.9). Let $E(t)$ be energy functional* *defined as*

$$\begin{aligned} E(t) = \quad & \sup_{0\le\tau\le t} (\|\widetilde{v}_h(\tau)\|_{H^2} + \|v_{30}(\tau)\|_{H^2} + \|\rho(\tau)\|_{H^2}) \\ & +\nu \int_0^t (\|\nabla_h \widetilde{v}_h(\tau)\|_{H^2}^2 + \|\nabla_h v_3(\tau)\|_{H^2}^2) d\tau + \kappa \int_0^t \|\nabla_h \rho(\tau)\|_{H^2}^2 d\tau. \end{aligned}$$

Then $E(t)$ satisfies for any $t > 0$,

$$E(t) \le K_0 E(0) + C\,(\nu^{-4} + \nu^{-1}\kappa^{-3} + \nu^{-2}\kappa^{-2}) K_0 E(t)^3, \tag{3.1.1}$$

where C is the constant independent of ν and κ, and

$$K_0 := e^{C\,(\nu^{-1}+\kappa^{-1})(\|v_h^{(0)}\|_{H^2(\mathbb{R}^2)}^2 + \|\nabla_h v_h^{(0)}\|_{H^2(\mathbb{R}^2)}^2)} \tag{3.1.2}$$

Proof. For the notational simplicity, we replace $\tilde{v}_h$ by v_h and consider the following system satisfied by $(v_h,\, v_3,\, \rho)$,

$$\begin{cases} \partial_t v_h + v_h \cdot \nabla_h v_h + v_h \cdot \nabla_h v_h^{(0)} + v_h^{(0)} \cdot \nabla_h v_h + v_3\, \partial_3 v_h + f v_h^{\perp} = -\nabla_h p + \nu\, \Delta_h v_h, \\ \partial_t v_3 + v_h \cdot \nabla_h v_3 + v_h^0 \cdot \nabla_h v_3 + v_3\, \partial_3 v_3 = -\partial_3 p + \nu\, \Delta_h v_3 - \rho, \\ \partial_t \rho + v_h \cdot \nabla_h \rho + v_h^0 \cdot \nabla_h \rho + v_3\, \partial_3 \rho = \kappa\, \Delta_h \rho + v_3, \\ \nabla_h \cdot v_h + \partial_3 v_3 = 0, \\ (v_h,\, v_3,\, \rho)|_{t=0} = (v_{h0} - v_{h0}^{(0)},\, v_{30},\, \rho_0). \end{cases} \tag{3.1.3}$$

Taking the inner product of (3.0.2) with $v_h^{(0)}$ we get

$$\frac{1}{2}\frac{d}{dt}\|v_h^{(0)}(t)\|_{L^2(\mathbb{R}^2)}^2 + \nu \|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)}^2 = 0, \tag{3.1.4}$$

where we have used the following facts,

$$\int v_h^{(0)} \cdot \nabla_h v_h^{(0)} \cdot v_h^{(0)}\, dx = 0 = \int -\nabla_h p \cdot v_h^{(0)}\, dx, \quad \int \nu\, \Delta_h v_h^{(0)} \cdot v_h^{(0)}\, dx = -\nu\, \|\nabla_h v_h^{(0)}\|_{L^2}^2.$$

The facts mentioned above can be proved easily using the integration by parts and the divergence-free condition of $v_{h0}^{(0)}$. Integrating (3.1.4) in time yields

$$\|v_h^{(0)}(t)\|_{L^2(\mathbb{R}^2)}^2 + 2\nu \int_0^t \|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)}^2 dt = \|v_{h0}^{(0)}\|_{L^2(\mathbb{R}^2)}^2. \tag{3.1.5}$$

Applying Δ_h to (3.0.2) and taking the inner product with $\Delta_h v_h^{(0)}$ we get,

$$\frac{d}{dt}\|\Delta_h v_h^{(0)}(t)\|_{L^2(\mathbb{R}^2)}^2 + 2\nu\|\Delta_h \nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)}^2 = -\int \Delta_h (v_h^{(0)} \cdot \nabla_h v_h^{(0)}) \cdot \Delta_h v_h^{(0)}, \tag{3.1.6}$$

where we have used the following facts

$$\int -\Delta_h \nabla_h p \cdot \Delta_h v_h^{(0)}\, dx = 0, \quad \int \nu\, \Delta_h \Delta_h v_h^{(0)} \cdot \Delta_h v_h^{(0)}\, dx = -\nu\, \|\Delta_h \nabla_h v_h^{(0)}\|_{L^2}^2.$$

To deal with the nonlinear term, we write it as follows

$$\begin{aligned} -\int \Delta_h (v_h^{(0)} \cdot \nabla_h v_h^{(0)}) \cdot \Delta_h v_h^{(0)} \quad &= -\int v_h^{(0)} \cdot \Delta_h \nabla_h v_h^{(0)} \cdot \Delta_h v_h^{(0)} - \int \Delta_h v_h^{(0)} \nabla_h v_h^{(0)} \cdot \Delta_h v_h^{(0)} \\ &\quad -2 \int \nabla_h v_h^{(0)} \cdot \Delta_h v_h^{(0)} \cdot \Delta_h v_h^{(0)}. \end{aligned}$$

The first term in the above equality becomes zero due to divergence free condition $\nabla \cdot v_h^{(0)} = 0$. Therefore,

$$\Big| \int \Delta_h \, (v_h^{(0)} \cdot \nabla_h v_h^{(0)}) \cdot \Delta_h v_h^{(0)} \Big| \leq C \int |\Delta_h v_h^{(0)}| \nabla_h v_h^{(0)}| |\Delta_h v_h^{(0)}|.$$

Using Hölder inequality, Ladyzhenskaya's inequality, and Young's inequality, (3.1.6) becomes

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta_h v_h^{(0)}(t)\|_{L^2(\mathbb{R}^2)}^2 + \nu\|\Delta_h \nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)}^2 &\le C\int |\Delta_h v_h^{(0)}||\nabla_h v_h^{(0)}||\Delta_h v_h^{(0)}| \\ &\le C\,\|\Delta_h v_h^{(0)}\|_{L^4}\|\nabla_h v_h^{(0)}\|_{L^2}\|\Delta_h v_h^{(0)}\|_{L^4} \\ &\le C\,\|\Delta_h v_h^{(0)}\|_{L^2}\|\nabla_h \Delta_h v_h^{(0)}\|_{L^2}\|\nabla_h v_h^{(0)}\|_{L^2} \\ &\le \nu\,\|\Delta_h \nabla_h v_h^{(0)}\|_{L^2}^2 + C\,\nu^{-1}\,\|\nabla_h v_h^{(0)}\|_{L^2}^2\|\Delta_h v_h^{(0)}\|_{L^2}^2.\end{aligned} \tag{3.1.7}$$

The above inequality simplifies to

$$\frac{d}{dt}\|\Delta_h v_h^{(0)}(t)\|_{L^2(\mathbb{R}^2)}^2 + \nu\,\|\Delta_h \nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)}^2 \le C\,\nu^{-1}\|\nabla_h v_h^{(0)}\|_{L^2}^2\|\Delta_h v_h^{(0)}\|_{L^2}^2. \tag{3.1.8}$$

Gronwall's inequality then implies that

$$\|\Delta_h v_h^{(0)}(t)\|_{L^2(\mathbb{R}^2)}^2 \le \|\Delta_h v_h^{(0)}(0)\|_{L^2(\mathbb{R}^2)}^2 e^{\int_0^t C\,\nu^{-1}\|\nabla_h v_h^{(0)}(t)\|_{L^2(\mathbb{R}^2)}^2}. \tag{3.1.9}$$

Since $v_h^{(0)} \in H^2(\mathbb{R}^2)$, we have,

$$\|\Delta_h v_h^{(0)}(t)\|_{L^2(\mathbb{R}^2)}^2 \le C\,\|\Delta_h v_h^{(0)}(0)\|_{L^2(\mathbb{R}^2)}^2. \tag{3.1.10}$$

Taking the inner product of $(v_h,\, v_3,\, \rho)$ in (3.1.3) with $(v_h,\, v_3,\, \rho)$, we get

$$\frac{1}{2}\frac{d}{dt}(\|v_h\|_{L^2}^2 + \|v_3\|_{L^2}^2 + \|\rho\|_{L^2}^2) + \nu\,\|\nabla_h v_h\|_{L^2}^2 + \nu\,\|\nabla_h v_3\|_{L^2}^2 + \kappa\,\|\nabla_h \rho\|_{L^2}^2 = -\int v_h \cdot \nabla_h v_h^{(0)} \cdot v_h, \tag{3.1.11}$$

where we have used the following facts

$$\begin{aligned}
&\int v_h \cdot (v_h \cdot \nabla_h v_h + v_3\, \partial_3 v_h)\, dx = 0, \quad \int v_h^{(0)} \cdot \nabla_h v_h \cdot v_h\, dx = 0,\\
&\int -\nabla_h p \cdot v_h - \partial_3 p \cdot v_3\, dx = 0, \qquad v_h^{\perp} \cdot v_h\, dx = 0,\\
&\int v_3 \cdot (v_h \cdot \nabla_h v_3 + v_3\, \partial_3 v_3)\, dx = 0, \quad \int v_h^0 \cdot \nabla_h v_3 \cdot v_3\, dx = 0,\\
&\int \rho \cdot (v_h \cdot \nabla_h \rho + v_3\, \partial_3 \rho)\, dx = 0, \quad \int v_h^0 \cdot \nabla_h \rho \cdot \rho\, dx = 0,\\
&\int \nu\, \Delta_h v_h \cdot v_h\, dx = -\nu\, \|\nabla_h v_h\|_{L^2}^2, \quad \int \nu\, \Delta_h v_3 \cdot v_3 dx = -\nu \|\nabla_h v_3\|_{L^2}^2,\\
&\int \nu\, \Delta_h \rho \cdot \rho\, dx = -\nu\, \|\nabla_h \rho\|_{L^2}^2, \quad -\int \rho \cdot v_3\, dx + \int v_3 \cdot \rho\, dx = 0.
\end{aligned}$$

Since the rotation does not affect the energy of the system, the Coriolis forcing term does not contribute to the L^2-norm. Using the fact that $v_h^{(0)}$ depends only on x_h as well as Hl̈der's inequality, Ladyzhenskaya's inequality, and Young's inequality, we can estimate the right-hand side of (3.1.11) as

$$\begin{aligned}
\Big| \int v_h \cdot \nabla_h v_h^{(0)} \cdot v_h \Big| \quad &\leq \|v_h\|_{L^2_{x_3} L^4_h} \|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)} \|v_h\|_{L^2_{x_3} L^4_h}\\
&\leq \|v_h\|_{L^2}^{\frac{1}{2}} \|\nabla_h v_h\|_{L^2}^{\frac{1}{2}} \|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)} \|v_h\|_{L^2}^{\frac{1}{2}} \|\nabla_h v_h\|_{L^2}^{\frac{1}{2}}\\
&\leq \|v_h\|_{L^2} \|\nabla_h v_h\|_{L^2} \|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)}\\
&\leq \frac{\nu}{2} \|\nabla_h v_h\|_{L^2}^2 + C\, \frac{1}{\nu} \|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)}^2 \|v_h\|_{L^2}^2.
\end{aligned}$$

Then the energy equation (3.1.11) becomes

$$\begin{aligned}
&\frac{d}{dt}(\|v_h\|_{L^2}^2 + \|v_3\|_{L^2}^2 + \|\rho\|_{L^2}^2) + 2\nu\, \|\nabla_h v_h\|_{L^2}^2 + 2\nu\, \|\nabla_h v_3\|_{L^2}^2 + 2\kappa\, \|\nabla_h \rho\|_{L^2}^2\\
&\qquad \leq\, C\, \nu^{-1}\, \|\nabla_h v_h^{(0)}\|_{L^2}^2 (\|v_h\|_{L^2}^2 + \|v_3\|_{L^2}^2 + \|\rho\|_{L^2}^2). \qquad (3.1.12)
\end{aligned}$$

Applying Δ to the equations of $(v_h,\, v_3,\, \rho)$ in (3.1.3), and dotting with $(\Delta v_h,\, \Delta v_3,\, \Delta\rho)$ yields

$$\frac{1}{2}\frac{d}{dt}\left(\|\Delta v_h\|_{L^2}^2 + \|\Delta v_3\|_{L^2}^2 + \|\Delta\rho\|_{L^2}^2\right) + \nu\,\|\Delta\nabla_h v_h\|_{L^2}^2 + \nu\,\|\Delta\nabla_h v_3\|_{L^2}^2 + \kappa\,\|\Delta\nabla_h\rho\|_{L^2}^2$$
$$= I_1 + \cdots + I_7, \tag{3.1.13}$$

with

$$I_1 = -\int \Delta(v_h\cdot\nabla_h v_h + v_3\,\partial_3 v_h)\cdot\Delta v_h\,dx, \quad I_2 = -\int \Delta(v_h\cdot\nabla_h v_3 + v_3\,\partial_3 w)\cdot\Delta v_3\,dx,$$
$$I_3 = -\int \Delta(v_h\cdot\nabla_h\rho + v_3\,\partial_3\rho)\cdot\Delta\rho\,dx, \quad I_4 = -\int \Delta(v_h^{(0)}\cdot\nabla_h v_h)\cdot\Delta v_h\,dx,$$
$$I_5 = -\int \Delta(v_h\cdot\nabla_h v_h^{(0)})\cdot\Delta v_h\,dx, \quad I_6 = -\int \Delta(v_h^0\cdot\nabla_h v_3)\Delta v_3\,dx, \quad I_7 = -\int \Delta(v_h^0\cdot\nabla_h\rho)\Delta\rho\,dx,$$

where, we have used the following facts

$$\int(\Delta\nabla_h p\cdot\Delta v_h + \Delta\partial_3\, p\Delta v_3)\,dx = \int \Delta p\,\Delta(\nabla_h\cdot v_h + \partial_3 v_3)\,dx = \Delta v_h^{\perp}\cdot\Delta v_h = 0,$$
$$\int \nu\,\Delta\Delta_h v_h\cdot\Delta v_h = -\nu\|\Delta\nabla_h v_h\|_{L^2}^2, \quad \int \nu\,\Delta\Delta_h v_3\cdot\Delta v_3 = -\nu\|\Delta\nabla_h v_3\|_{L^2}^2,$$
$$\int \nu\,\Delta\Delta_h\rho\cdot\Delta\rho = -\nu\|\Delta\nabla_h\rho\|_{L^2}^2.$$

First, we estimate I_1, which can be written as,

$$\begin{aligned} I_1 \quad &= -\int \Delta v_h\cdot\nabla_h v_h\cdot\Delta v_h\,dx - \int v_h\cdot\nabla_h\Delta v_h\cdot\Delta v_h\,dx - 2\int \nabla v_h\cdot\nabla\nabla_h v_h\cdot\Delta v_h\,dx \\ &-\int \Delta v_3\partial_3 v_h\cdot\Delta v_h\,dx - \int v_3\partial_3\Delta v_h\cdot\Delta v_h\,dx - 2\int \nabla v_3\nabla\partial_3 v_h\cdot\Delta v_h\,dx. \end{aligned}$$

Due to divergence-free condition $\nabla_h\cdot v_h + \partial_3 v_3 = 0$, we have

$$\begin{aligned} I_1 \quad &= -\int \Delta v_h\cdot\nabla_h v_h\cdot\Delta v_h\,dx - 2\int \nabla v_h\cdot\nabla\nabla_h v_h\cdot\Delta v_h\,dx \\ &-\int \Delta v_3\partial_3 v_h\cdot\Delta v_h\,dx - 2\int \nabla v_3\nabla\partial_3 v_h\cdot\Delta v_h\,dx \\ &= I_{11} + I_{12} + I_{13} + I_{14}. \end{aligned}$$

Then, using the anisotropic upper bound in Lemma 3.1.1, I_{11} can be estimated as,

$$\begin{aligned}
|I_{11}| \quad &= \Big| \int \Delta v_h \cdot \nabla_h v_h \cdot \Delta v_h \, dx \Big| \\
&\leq C \, \|\Delta v_h\|_{L^2}^{\frac{1}{2}} \|\partial_1 \Delta v_h\|_{L^2}^{\frac{1}{2}} \|\nabla_h v_h\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla_h v_h\|_{L^2}^{\frac{1}{2}} \|\Delta v_h\|_{L^2}^{\frac{1}{2}} \|\partial_2 \Delta v_h\|_{L^2}^{\frac{1}{2}} \\
&\leq C \, \|\nabla_h v_h\|_{H^2}^{\frac{3}{2}} \|\nabla_h v_h\|_{L^2}^{\frac{1}{2}} \|v_h\|_{H^2} \\
&\leq \frac{\nu}{8} \|\nabla_h v_h\|_{H^2}^2 + C \, \nu^{-3} \|\nabla_h v_h\|_{L^2}^2 \|v_h\|_{H^2}^4.
\end{aligned}$$

Similar to I_{11}, we can estimate I_{12} as

$$\begin{aligned}
|I_{12}| \quad &= \Big| \int \nabla v_h \cdot \nabla \nabla_h v_h \cdot \Delta v_h \, dx \Big| \\
&\leq C \, \|\nabla v_h\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla v_h\|_{L^2}^{\frac{1}{2}} \|\nabla \nabla_h v_h\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla \nabla_h v_h\|_{L^2}^{\frac{1}{2}} \|\Delta v_h\|_{L^2}^{\frac{1}{2}} \|\partial_2 \Delta v_h\|_{L^2}^{\frac{1}{2}} \\
&\leq C \, \|\nabla_h v_h\|_{H^2}^{\frac{3}{2}} \|\nabla_h v_h\|_{H^1}^{\frac{1}{2}} \|v_h\|_{H^2} \\
&\leq \frac{\nu}{8} \|\nabla_h v_h\|_{H^2}^2 + C \, \nu^{-3} \|\nabla_h v_h\|_{H^1}^2 \|v_h\|_{H^2}^4.
\end{aligned}$$

To estimate I_{13} we first write it as

$$I_{13} = - \int \Delta_h v_3 \, \partial_3 v_h \cdot \Delta v_h \, dx - \int \partial_{33} v_3 \, \partial_3 v_h \cdot \Delta v_h \, dx = I_{131} + I_{132}.$$

Then using $\nabla_h \cdot v_h + \partial_3 v_3 = 0$ and Lemma 3.1.1, we have

$$\begin{aligned}
|I_{131}| \quad &\leq C \, \|\Delta_h v_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 \Delta_h v_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 v_h\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_3 v_h\|_{L^2}^{\frac{1}{2}} \|\Delta v_h\|_{L^2}^{\frac{1}{2}} \|\partial_2 \Delta v_h\|_{L^2}^{\frac{1}{2}} \\
&\leq C \, \|\nabla_h v_3\|_{H^2}^{\frac{1}{2}} \|\nabla_h v_h\|_{H^2}^{\frac{3}{2}} \|v_h\|_{H^2} \\
&\leq \frac{\nu}{8} \|\nabla_h v_h\|_{H^2}^2 + C \, \nu^{-3} \|\nabla_h v_3\|_{H^2}^2 \|v_h\|_{H^2}^4.
\end{aligned}$$

Similarly, using $\nabla_h \cdot v_h + \partial_3 v_3 = 0$ and Lemma 3.1.1, we have

$$
\begin{aligned}
|I_{132}| \quad &\leq C\,\|\partial_{33} v_3\|_{L^2}^{\frac{1}{2}}\|\partial_{333} v_3\|_{L^2}^{\frac{1}{2}}\|\partial_3 v_h\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_3 v_h\|_{L^2}^{\frac{1}{2}}\|\Delta v_h\|_{L^2}^{\frac{1}{2}}\|\partial_2\Delta v_h\|_{L^2}^{\frac{1}{2}} \\
&\leq C\,\|\nabla_h v_h\|_{H^1}^{\frac{1}{2}}\|\nabla_h v_h\|_{H^2}^{\frac{3}{2}}\|v_h\|_{H^2} \\
&\leq \frac{\nu}{8}\|\nabla_h v_h\|_{H^2}^2 + C\,\nu^{-3}\|\nabla_h v_h\|_{H^1}^2\|v_h\|_{H^2}^4.
\end{aligned}
$$

To estimate I_{14}, we first write it as

$$
I_{14} = -2\int \nabla_h v_3 \nabla\partial_3 v_h \cdot \Delta v_h \, dx - 2\int \partial_3 v_3 \nabla\partial_3 v_h \cdot \Delta v_h \, dx = I_{141} + I_{142}.
$$

Then using divergence-free condition for v, and Lemma 3.1.1, we have

$$
\begin{aligned}
|I_{141}| \quad &\leq C\,\|\nabla_h v_3\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_3 v_3\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_3 v_h\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_3\partial_1 v_h\|_{L^2}^{\frac{1}{2}}\|\Delta v_h\|_{L^2}^{\frac{1}{2}}\|\partial_2\Delta v_h\|_{L^2}^{\frac{1}{2}} \\
&\leq C\,\|\nabla_h v_3\|_{L^2}^{\frac{1}{2}}\|\nabla_h v_h\|_{H^2}^{\frac{3}{2}}\|v_h\|_{H^2} \\
&\leq \frac{\nu}{8}\|\nabla_h v_h\|_{H^2}^2 + C\,\nu^{-3}\|\nabla_h v_3\|_{L^2}^2\|v_h\|_{H^2}^4.
\end{aligned}
$$

Similarly, using divergence-free condition for v, and Lemma 3.1.1, we have

$$
\begin{aligned}
|I_{142}| \quad &\leq C\,\|\partial_3 v_3\|_{L^2}^{\frac{1}{2}}\|\partial_3\partial_3 v_3\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_3 v_h\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_3\partial_1 v_h\|_{L^2}^{\frac{1}{2}}\|\Delta v_h\|_{L^2}^{\frac{1}{2}}\|\partial_2\Delta v_h\|_{L^2}^{\frac{1}{2}} \\
&\leq C\,\|\nabla_h v_h\|_{H^1}^{\frac{1}{2}}\|\nabla_h v_h\|_{H^2}^{\frac{3}{2}}\|v_h\|_{H^2} \\
&\leq \frac{\nu}{8}\|\nabla_h v_h\|_{H^2}^2 + C\,\nu^{-3}\|\nabla_h v_h\|_{H^1}^2\|v_h\|_{H^2}^4.
\end{aligned}
$$

Using divergence-free condition $\nabla_h \cdot v_h + \partial_3 v_3 = 0$, we write I_2 as

$$
\begin{aligned}
I_2 \quad &= -\int \Delta v_h \cdot \nabla_h v_3 \cdot \Delta v_3 \, dx - 2\int \nabla v_h \cdot \nabla\nabla_h v_3 \cdot \Delta v_3 \, dx \\
&- \int \Delta v_3 \, \partial_3 v_3 \cdot \Delta v_3 \, dx - 2\int \nabla v_3 \, \nabla\partial_3 v_3 \cdot \Delta v_3 \, dx. \\
&= I_{21} + I_{22} + I_{23} + I_{24}.
\end{aligned}
$$

Then, using the anisotropic upper bound in Lemma (3.1.1), we have,

$$
\begin{aligned}
|I_{21}| &= \Big| \int \Delta v_h \cdot \nabla_h v_3 \cdot \Delta v_3 \, dx \Big| \\
&\le C \, \|\Delta v_h\|_{L^2}^{\frac{1}{2}} \|\partial_1 \Delta v_h\|_{L^2}^{\frac{1}{2}} \|\nabla_h v_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla_h v_3\|_{L^2}^{\frac{1}{2}} \|\Delta v_3\|_{L^2}^{\frac{1}{2}} \|\partial_2 \Delta v_3\|_{L^2}^{\frac{1}{2}} \\
&\le C \, \|\nabla_h v_h\|_{H^1}^{\frac{1}{2}} \|\nabla_h v_3\|_{H^2}^{\frac{3}{2}} \|v_h\|_{H^2}^{\frac{1}{2}} \|v_3\|_{H^2}^{\frac{1}{2}} \\
&\le \frac{\nu}{8} \|\nabla_h v_3\|_{H^2}^2 + C \, \nu^{-3} \|\nabla_h v_h\|_{H^1}^2 \|v_h\|_{H^2}^2 \|v_3\|_{H^2}^2 .
\end{aligned}
$$

Similarly as I_{21}, we have

$$
\begin{aligned}
|I_{22}| &= \Big| \int \nabla v_h \cdot \nabla \nabla_h v_3 \cdot \Delta v_3 \, dx \Big| \\
&\le C \, \|\nabla v_h\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla v_h\|_{L^2}^{\frac{1}{2}} \|\nabla \nabla_h v_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla \nabla_h v_3\|_{L^2}^{\frac{1}{2}} \|\Delta v_3\|_{L^2}^{\frac{1}{2}} \|\partial_2 \Delta v_3\|_{L^2}^{\frac{1}{2}} \\
&\le C \, \|\nabla_h v_h\|_{H^1}^{\frac{1}{2}} \|\nabla_h v_3\|_{H^2}^{\frac{3}{2}} \|v_h\|_{H^2}^{\frac{1}{2}} \|v_3\|_{H^2}^{\frac{1}{2}} \\
&\le \frac{\nu}{8} \|\nabla_h v_3\|_{H^2}^2 + C \, \nu^{-3} \|\nabla_h v_h\|_{H^1}^2 \|v_h\|_{H^2}^2 \|v_3\|_{H^2}^2 .
\end{aligned}
$$

Using the divergence-free condition of v and Lemma (3.1.1), we estimate I_{23} as

$$
\begin{aligned}
|I_{23}| &= \Big| \int \Delta v_3 \, \partial_3 v_3 \cdot \Delta v_3 \, dx \Big| \\
&\le C \, \|\Delta v_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 \Delta v_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 v_3\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_3 v_3\|_{L^2}^{\frac{1}{2}} \|\Delta v_3\|_{L^2}^{\frac{1}{2}} \|\partial_2 \Delta v_3\|_{L^2}^{\frac{1}{2}} \\
&\le C \, \|\nabla_h v_h\|_{H^2}^{\frac{3}{2}} \|\nabla_h v_3\|_{H^2}^{\frac{1}{2}} \|v_3\|_{H^2} \\
&\le \frac{\nu}{8} \|\nabla_h v_h\|_{H^2}^2 + C \, \nu^{-3} \|\nabla_h v_3\|_{H^2}^2 \|v_3\|_{H^2}^4 .
\end{aligned}
$$

Similarly, as I_{23}, we have

$$
\begin{aligned}
|I_{24}| &= \Big| \int \nabla v_3\, \nabla\partial_3 v_3 \cdot \Delta v_3\, dx \Big| \\
&\le C\, \|\nabla v_3\|_{L^2}^{\frac12} \|\partial_3 \nabla v_3\|_{L^2}^{\frac12} \|\nabla \partial_3 v_3\|_{L^2}^{\frac12} \|\partial_1 \nabla\partial_3 v_3\|_{L^2}^{\frac12} \|\Delta v_3\|_{L^2}^{\frac12} \|\partial_2 \Delta v_3\|_{L^2}^{\frac12} \\
&\le C\, \|\nabla_h v_h\|_{H^2}^{\frac32} \|\nabla_h v_3\|_{H^2}^{\frac12} \|v_3\|_{H^2} \\
&\le \frac{\nu}{8} \|\nabla_h v_h\|_{H^2}^2 + C\,\nu^{-3} \|\nabla_h v_3\|_{H^2}^2 \|v_3\|_{H^2}^4.
\end{aligned}
$$

To estimate I_3, we first write it as follows using the divergence-free condition for v.

$$
\begin{aligned}
I_3 = \quad & -\int \Delta v_h \cdot \nabla_h \rho \cdot \Delta\rho\, dx - 2\int \nabla v_h \cdot \nabla\nabla_h \rho \cdot \Delta \rho\, dx, \\
& -\int \Delta v_3\, \partial_3 \rho \cdot \Delta\rho\, dx - 2\int \nabla v_3\, \nabla\partial_3 \rho \Delta\rho\, dx, \\
& = I_{31} + I_{32} + I_{33} + I_{34}.
\end{aligned}
$$

By using the anisotropic upper bound, we estimate I_{31} as

$$
\begin{aligned}
|I_{31}| &\le C\, \|\Delta v_h\|_{L^2}^{\frac12} \|\partial_1 \Delta v_h\|_{L^2}^{\frac12} \|\nabla_h \rho\|_{L^2}^{\frac12} \|\partial_3 \nabla_h \rho\|_{L^2}^{\frac12} \|\Delta\rho\|_{L^2}^{\frac12} \|\partial_2 \Delta\rho\|_{L^2}^{\frac12} \\
&\le C\, \|\nabla_h v_h\|_{H^1}^{\frac12} \|\nabla_h \rho\|_{H^2} \|\nabla_h \rho\|_{L^2}^{\frac12} \|v_h\|_{H^2}^{\frac12} \|\rho\|_{H^2}^{\frac12} \\
&\le \frac{\nu}{8} \|\nabla_h v_h\|_{H^2}^2 + \frac{\kappa}{8} \|\nabla_h \rho\|_{H^2}^2 + C\,\nu^{-1}\kappa^{-2} \|\nabla_h \rho\|_{L^2}^2 \|v_h\|_{H^2}^2 \|\rho\|_{H^2}^2.
\end{aligned}
$$

Similarly as I_{31}, we estimate I_{32} as

$$
\begin{aligned}
|I_{32}| &\le C\, \|\nabla v_h\|_{L^2}^{\frac12} \|\partial_1 \nabla V_h\|_{L^2}^{\frac12} \|\nabla\nabla_h \rho\|_{L^2}^{\frac12} \|\partial_3 \nabla\nabla_h \rho\|_{L^2}^{\frac12} \|\Delta\rho\|_{L^2}^{\frac12} \|\partial_2 \Delta\rho\|_{L^2}^{\frac12} \\
&\le C\, \|\nabla_h v_h\|_{H^2}^{\frac12} \|\nabla_h \rho\|_{H^2} \|\nabla_h \rho\|_{H^1}^{\frac12} \|v_h\|_{H^2}^{\frac12} \|\rho\|_{H^2}^{\frac12} \\
&\le \frac{\nu}{8} \|\nabla_h v_h\|_{H^2}^2 + \frac{\kappa}{8} \|\nabla_h \rho\|_{H^2}^2 + C\,\nu^{-1}\kappa^{-2} \|\nabla_h \rho\|_{H^1}^2 \|v_h\|_{H^2}^2 \|\rho\|_{H^2}^2.
\end{aligned}
$$

To estimate I_{33}, we first write it as

$$I_{33} = -\int \Delta_h v_3 \partial_3 \rho \cdot \Delta \rho \, dx - \int \partial_{33} v_3 \partial_3 \rho \cdot \Delta \rho \, dx = I_{331} + I_{332}.$$

Then, using the divergence-free condition, $\nabla_h \cdot v_h + \partial_3 v_3 = 0$ and Lemma (3.1.1) yields

$$\begin{aligned} |I_{331}| &\leq C \, \|\Delta_h v_3\|_{L^2}^{\frac{1}{2}} \|\Delta_h \partial_3 v_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 \rho\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_1 \rho\|_{L^2}^{\frac{1}{2}} \|\Delta \rho\|_{L^2}^{\frac{1}{2}} \|\partial_2 \Delta \rho\|_{L^2}^{\frac{1}{2}} \\ &\leq C \, \|\nabla_h v_3\|_{H^2}^{\frac{1}{2}} \|\nabla_h v_h\|_{H^2}^{\frac{1}{2}} \|\nabla_h \rho\|_{H^2} \|\rho\|_{H^2} \\ &\leq \frac{\nu}{8} \|\nabla_h v_3\|_{H^2}^2 + \frac{\kappa}{8} \|\nabla_h \rho\|_{H^2}^2 + C \, \nu^{-1} \kappa^{-2} \|\nabla_h v_h\|_{H^2}^2 \|\rho\|_{H^2}^4. \end{aligned}$$

Similarly as I_{331}, we have

$$\begin{aligned} |I_{332}| &\leq C \, \|\partial_{33} v_3\|_{L^2}^{\frac{1}{2}} \|\partial_{333} v_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 \rho\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_2 \rho\|_{L^2}^{\frac{1}{2}} \|\Delta \rho\|_{L^2}^{\frac{1}{2}} \|\partial_1 \Delta \rho\|_{L^2}^{\frac{1}{2}} \\ &\leq C \, \|\nabla_h \rho\|_{H^2} \|\nabla_h v_h\|_{H^2}^{\frac{1}{2}} \|\nabla_h v_h\|_{H^2}^{\frac{1}{2}} \|\rho\|_{H^2} \\ &\leq \frac{\nu}{8} \|\nabla_h v_h\|_{H^2}^2 + \frac{\kappa}{8} \|\nabla_h \rho\|_{H^2}^2 + C \, \nu^{-1} \kappa^{-2} \|\nabla_h v_h\|_{H^2}^2 \|\rho\|_{H^2}^4. \end{aligned}$$

To estimate I_{34}, we first write it as

$$I_{34} = -2 \int \nabla_h v_3 \, \nabla \partial_3 \rho \cdot \Delta \rho \, dx - 2 \int \partial_3 v_3 \, \nabla \partial_3 \rho \cdot \Delta \rho \, dx = I_{341} + I_{342}.$$

Then, using the divergence-free condition, $\nabla_h \cdot v_h + \partial_3 v_3 = 0$ and the anisotropic upper bound we have

$$\begin{aligned} |I_{341}| &\leq C \, \|\nabla_h v_3\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_3 v_3\|_{L^2}^{\frac{1}{2}} \|\nabla \partial_3 \rho\|_{L^2}^{\frac{1}{2}} \|\nabla \partial_3 \partial_1 \rho\|_{L^2}^{\frac{1}{2}} \|\Delta \rho\|_{L^2}^{\frac{1}{2}} \|\partial_2 \Delta \rho\|_{L^2}^{\frac{1}{2}} \\ &\leq C \, \|\nabla_h v_3\|_{H^2}^{\frac{1}{2}} \|\nabla_h v_h\|_{H^2}^{\frac{1}{2}} \|\nabla_h \rho\|_{H^2} \|\rho\|_{H^2} \\ &\leq \frac{\nu}{8} \|\nabla_h v_3\|_{H^2}^2 + \frac{\kappa}{8} \|\nabla_h \rho\|_{H^2}^2 + C \, \nu^{-1} \kappa^{-2} \|\nabla_h v_h\|_{H^2}^2 \|\rho\|_{H^2}^4. \end{aligned}$$

Similarly as I_{341}, we have

$$\begin{aligned}
|I_{342}| &\leq C\,\|\partial_3 v_3\|_{L^2}^{\frac{1}{2}}\|\partial_3\partial_3 v_3\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_3\rho\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_3\partial_1\rho\|_{L^2}^{\frac{1}{2}}\|\Delta\rho\|_{L^2}^{\frac{1}{2}}\|\partial_2\Delta\rho\|_{L^2}^{\frac{1}{2}}\\
&\leq C\,\|\nabla_h v_h\|_{H^2}^{\frac{1}{2}}\|\nabla_h v_h\|_{H^2}^{\frac{1}{2}}\|\nabla_h\rho\|_{H^2}\|\rho\|_{H^2}\\
&\leq \frac{\nu}{8}\|\nabla_h v_h\|_{H^2}^2+\frac{\kappa}{8}\|\nabla_h\rho\|_{H^2}^2+C\,\nu^{-1}\kappa^{-2}\|\nabla_h v_h\|_{H^2}^2\|\rho\|_{H^2}^4.
\end{aligned}$$

To estimate I_4, we write it as

$$\begin{aligned}
I_4 &= -\int \Delta(v_h^{(0)}\cdot\nabla_h v_h)\cdot\Delta v_h\,dx,\\
&= -\int \Delta v_h^{(0)}\cdot\nabla_h v_h\cdot\Delta v_h\,dx-\int v_h^{(0)}\cdot\Delta\nabla_h v_h\cdot\Delta v_h\,dx-2\int\nabla v_h^{(0)}\cdot\nabla\nabla_h v_h\cdot\Delta v_h\,dx,\\
&= I_{41}+I_{42}+I_{43}.
\end{aligned}$$

Since $v_h^{(0)}$ is divergence-free, we have

$$I_{42} = \int v_h^{(0)}\cdot\Delta\nabla_h v_h\cdot\Delta v_h\,dx = 0,$$

and due to the fact that $v_h^{(0)} = v_h^{(0)}(x_h,t)$ is independent of x_3, and using Hölder inequality and Ladyzhenskaya's inequality, we estimate I_{41} as

$$\begin{aligned}
|I_{41}| &= \Big|\int \Delta v_h^{(0)}\cdot\nabla_h v_h\cdot\Delta v_h\,dx\Big|\\
&\leq C\,\|\nabla_h v_h\|_{L^2_{x_3}L^4_h}\|\Delta_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)}\|\Delta v_h\|_{L^2_{x_3}L^4_h}\\
&\leq C\,\|\nabla_h v_h\|_{L^2}^{\frac{1}{2}}\|\nabla_h\nabla_h v_h\|_{L^2}^{\frac{1}{2}}\|\Delta_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)}\|\Delta v_h\|_{L^2}^{\frac{1}{2}}\|\Delta\nabla_h v_h\|_{L^2}^{\frac{1}{2}}\\
&\leq C\,\|\nabla_h v_h\|_{H^2}\|v_h\|_{H^2}\|\Delta_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)}\\
&\leq \frac{\nu}{8}\|\nabla_h v_h\|_{H^2}^2+C\,\nu^{-1}\|\Delta_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)}^2\|v_h\|_{H^2}^2.
\end{aligned}$$

Similar to I_{41}, we estimate I_{43} as

$$\begin{aligned}
|I_{43}| &= \Big| \int \nabla_h v_h^{(0)} \cdot \nabla\nabla_h v_h \cdot \Delta v_h \, dx \Big| \\
&\le C\, \|\nabla\nabla_h v_h\|_{L^2_{x_3} L^4_h} \|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)} \|\Delta v_h\|_{L^2_{x_3} L^4_h} \\
&\le C\, \|\nabla\nabla_h v_h\|_{L^2}^{\frac{1}{2}} \|\nabla_h \nabla\nabla_h v_h\|_{L^2}^{\frac{1}{2}} \|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)} \|\Delta v_h\|_{L^2}^{\frac{1}{2}} \|\Delta\nabla_h v_h\|_{L^2}^{\frac{1}{2}} \\
&\le C\, \|\nabla_h v_h\|_{H^2} \|v_h\|_{H^2} \|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)} \le \frac{\nu}{8} \|\nabla_h v_h\|_{H^2}^2 + C\, \nu^{-1} \|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)}^2 \|v_h\|_{H^2}^2.
\end{aligned}$$

To estimate I_5, we write it as

$$\begin{aligned}
I_5 &= -\int \Delta(v_h \cdot \nabla_h v_h^{(0)}) \cdot \Delta v_h \, dx, \\
&= -\int \Delta v_h \cdot \nabla_h v_h^{(0)} \cdot \Delta v_h \, dx - \int v_h \cdot \Delta\nabla_h v_h^{(0)} \cdot \Delta v_h \, dx - 2\int \nabla v_h \cdot \nabla\nabla_h v_h^{(0)} \cdot \Delta v_h \, dx, \\
&= I_{51} + I_{52} + I_{53}.
\end{aligned}$$

Due to the fact that $v_h^{(0)} = v_h^{(0)}(x_h, t)$ is independent of x_3, and using Hölder inequality and Ladyzhenskaya's inequality, we estimate I_{51} as

$$\begin{aligned}
I_{51} &= \int \Big| \Delta v_h \cdot \nabla_h v_h^{(0)} \cdot \Delta v_h \, dx \Big| \\
&\le C\, \|\Delta v_h\|_{L^2_{x_3} L^4_h} \|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)} \|\Delta v_h\|_{L^2_{x_3} L^4_h} \\
&\le C\, \|\Delta v_h\|_{L^2} \|\Delta\nabla_h v_h\|_{L^2} \|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)} \\
&\le \frac{\nu}{8} \|\nabla_h v_h\|_{H^2}^2 + C\, \nu^{-1} \|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)}^2 \|v_h\|_{H^2}^2.
\end{aligned}$$

Similar to I_{51}, we have

$$\begin{aligned}
I_{52} &= \Big| \int v_h \cdot \Delta\nabla_h v_h^{(0)} \cdot \Delta v_h \, dx \Big| \\
&\le C\, \|v_h\|_{L^2_{x_3} L^4_h} \|\Delta\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)} \|\Delta v_h\|_{L^2_{x_3} L^4_h} \\
&\le C\, \|v_h\|_{L^2}^{\frac{1}{2}} \|\nabla_h v_h\|_{L^2}^{\frac{1}{2}} \|\Delta\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)} \|\Delta v_h\|_{L^2}^{\frac{1}{2}} \|\nabla_h \Delta v_h\|_{L^2}^{\frac{1}{2}} \\
&\le \frac{\nu}{8} \|\nabla_h v_h\|_{H^2}^2 + C\, \nu^{-1} \|\nabla_h v_h^{(0)}\|_{H^2(\mathbb{R}^2)}^2 \|v_h\|_{H^2}^2.
\end{aligned}$$

Similarly, as in I_{51} and I_{52}, we have

$$
\begin{aligned}
I_{53} \quad &= \Big| \int \nabla v_h \cdot \nabla \nabla_h v_h^{(0)} \cdot \Delta v_h \, dx \Big| \\
&\le C \, \|\nabla v_h\|_{L^2_{x_3} L^4_h} \|\nabla \nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)} \|\Delta v_h\|_{L^2_{x_3} L^4_h} \\
&\le C \, \|\nabla v_h\|_{L^2}^{\frac{1}{2}} \|\nabla_h \nabla v_h\|_{L^2}^{\frac{1}{2}} \|\nabla \nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)} \|\Delta v_h\|_{L^2}^{\frac{1}{2}} \|\nabla_h \Delta v_h\|_{L^2}^{\frac{1}{2}} \\
&\le C \, \|v_h\|_{H^2} \|\nabla_h v_h\|_{H^2} \|\nabla_h v_h^{(0)}\|_{H^1(\mathbb{R}^2)} \\
&\le \frac{\nu}{8} \|\nabla_h v_h\|_{H^2}^2 + C \, \nu^{-1} \|\nabla_h v_h^{(0)}\|_{H^2(\mathbb{R}^2)}^2 \|v_h\|_{H^2}^2.
\end{aligned}
$$

We first write I_6 as

$$
\begin{aligned}
I_6 \quad &= - \int \Delta (v_h^{(0)} \cdot \nabla_h v_3) \cdot \Delta v_3, dx, \\
&= - \int \Delta v_h^{(0)} \cdot \nabla_h v_3 \cdot \Delta v_3 \, dx - \int v_h^{(0)} \cdot \Delta \nabla_h v_3 \cdot \Delta v_3 \, dx - 2 \int \nabla_h v_h^{(0)} \cdot \nabla \nabla_h v_3 \cdot \Delta v_3 \, dx, \\
&= I_{61} + I_{62} + I_{63}.
\end{aligned}
$$

Due to divergence-free condition for $v_h^{(0)}$, we have

$$
I_{62} = - \int v_h^{(0)} \cdot \Delta \nabla_h v_3 \cdot \Delta v_3 \, dx = 0.
$$

Using the fact that $v_h^{(0)} = v_h^{(0)}(x_h, t)$ is independent of x_3, Hölder inequality and Ladyzhenskaya's inequality, we estimate the remaining terms in I_6 as

$$
\begin{aligned}
|I_{61}| \quad &= \Big| \int \Delta_h v_h^{(0)} \cdot \nabla_h v_3 \cdot \Delta v_3 \, dx \Big| \\
&\le C \, \|\nabla_h v_3\|_{L^2_{x_3} L^4_h} \|\Delta_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)} \|\Delta v_3\|_{L^2_{x_3} L^4_h} \\
&\le C \, \|\nabla_h v_3\|_{L^2}^{\frac{1}{2}} \|\nabla_h \nabla_h v_3\|_{L^2}^{\frac{1}{2}} \|\Delta_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)} \|\Delta v_3\|_{L^2}^{\frac{1}{2}} \|\nabla_h \Delta v_3\|_{L^2}^{\frac{1}{2}} \\
&\le C \, \|\nabla_h v_3\|_{H^2} \|\Delta_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)} \|\Delta v_3\|_{L^2} \\
&\le \frac{\nu}{8} \|\nabla_h v_3\|_{H^2}^2 + C \, \nu^{-1} \|\Delta_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)}^2 \|v_3\|_{H^2}^2,
\end{aligned}
$$

and

$$\begin{aligned}
|I_{63}| &= \Big|\int \nabla_h v_h^{(0)} \cdot \nabla\nabla_h v_3 \cdot \Delta v_3 \, dx\Big| \\
&\le C \, \|\nabla\nabla_h v_3\|_{L^2_{x_3}L^4_h} \|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)} \|\Delta v_3\|_{L^2_{x_3}L^4_h} \\
&\le C \, \|\nabla\nabla_h v_3\|_{L^2}^{\frac{1}{2}} \|\nabla_h \nabla\nabla_h v_3\|_{L^2}^{\frac{1}{2}} \|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)} \|\Delta v_3\|_{L^2}^{\frac{1}{2}} \|\Delta\nabla_h v_3\|_{L^2}^{\frac{1}{2}} \\
&\le C \, \|\nabla_h v_3\|_{H^2} \|\Delta v_3\|_{L^2} \|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)} \\
&\le \frac{\nu}{8} \|\nabla_h v_3\|_{H^2}^2 + C \, \nu^{-1} \|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)}^2 \|v_3\|_{H^2}^2.
\end{aligned}$$

To estimate I_7, we first write it as

$$\begin{aligned}
I_7 &= -\int \Delta(v_h^{(0)} \cdot \nabla_h \rho) \cdot \Delta\rho, dx, \\
&= -\int \Delta_h v_h^{(0)} \cdot \nabla_h \rho \cdot \Delta\rho \, dx - \int v_h^{(0)} \cdot \Delta\nabla_h \rho \cdot \Delta\rho \, dx - 2\int \nabla_h v_h^{(0)} \cdot \nabla\nabla_h \rho \cdot \Delta\rho \, dx, \\
&= I_{71} + I_{72} + I_{73}.
\end{aligned}$$

Due to divergence-free condition for $v_h^{(0)}$,

$$I_{72} = \int v_h^{(0)} \cdot \Delta\nabla_h \rho \cdot \Delta\rho \, dx = 0.$$

Using the fact that $v_h^{(0)} = v_h^{(0)}(x_h, t)$ is independent of x_3, Hölder inequality and Ladyzhenskaya's inequality, we estimate the remaining terms in I_7 as

$$\begin{aligned}
|I_{71}| &= \Big|\int \Delta_h v_h^{(0)} \cdot \nabla_h \rho \cdot \Delta\rho \, dx\Big| \\
&\le C \, \|\nabla_h \rho\|_{L^2_{x_3}L^4_h} \|\Delta_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)} \|\Delta\rho\|_{L^2_{x_3}L^4_h} \\
&\le C \, \|\nabla_h \rho\|_{L^2}^{\frac{1}{2}} \|\nabla_h \nabla_h \rho\|_{L^2}^{\frac{1}{2}} \|\Delta_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)} \|\Delta\rho\|_{L^2}^{\frac{1}{2}} \|\nabla_h \Delta\rho\|_{L^2}^{\frac{1}{2}} \\
&\le C \, \|\nabla_h \rho\|_{H^2} \|\Delta_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)} \|\Delta\rho\|_{L^2} \\
&\le \frac{\kappa}{8} \|\nabla_h \rho\|_{H^2}^2 + C \, \kappa^{-1} \|\Delta_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)}^2 \|\rho\|_{H^2}^2,
\end{aligned}$$

and

$$
\begin{aligned}
|I_{73}| &= \Big| \int \nabla_h v_h^{(0)} \cdot \nabla\nabla_h \rho \cdot \Delta\rho \, dx \Big| \\
&\leq C \, \|\nabla\nabla_h \rho\|_{L^2_{x_3} L^4_h} \|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)} \|\Delta\rho\|_{L^2_{x_3} L^4_h} \\
&\leq C \, \|\nabla\nabla_h \rho\|_{L^2}^{\frac{1}{2}} \|\nabla_h \nabla\nabla_h \rho\|_{L^2}^{\frac{1}{2}} \|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)} \|\Delta\rho\|_{L^2}^{\frac{1}{2}} \|\nabla_h \Delta\rho\|_{L^2}^{\frac{1}{2}} \\
&\leq C \, \|\nabla_h \rho\|_{H^2} \|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)} \|\Delta\rho\|_{L^2} \leq \frac{\kappa}{8} \|\nabla_h \rho\|_{H^2}^2 + C \, \kappa^{-1} \|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)}^2 \|\rho\|_{H^2}^2.
\end{aligned}
$$

Inserting all the above estimates in the energy inequality (3.1.13), we get

$$
\begin{aligned}
&\frac{d}{dt}(\|\Delta v_h\|_{L^2}^2 + \|\Delta v_3\|_{L^2}^2 + \|\Delta\rho\|_{L^2}^2) + \nu\|\nabla_h \Delta v_h\|_{L^2}^2 + \nu\|\nabla_h \Delta v_3\|_{L^2}^2 + \kappa\|\nabla_h \Delta\rho\|_{L^2}^2 \\
&\leq C \, ((\nu^{-1} + \kappa^{-1})(\|v_h^{(0)}\|_{H^2(\mathbb{R}^2)}^2 + \|\nabla_h v_h^{(0)}\|_{H^2(\mathbb{R}^2)}^2) \\
&\times (\|v_h\|_{H^2}^2 + \|v_3\|_{H^2}^2 + \|\rho\|_{H^2}^2) \\
&+ C \, \nu^{-3}(\|\nabla_h v_h\|_{H^2}^2 + \|\nabla_h v_3\|_{H^2}^2)(\|v_h\|_{H^2}^2 + \|v_3\|_{H^2}^2)^2 \\
&+ C \, \nu^{-1}\kappa^{-2}(\|\nabla_h v_h\|_{H^2}^2 + \|\nabla_h \rho\|_{H^2}^2)(\|v_h\|_{H^2}^2 + \|\rho\|_{H^2}^2)^2.
\end{aligned} \tag{3.1.14}
$$

Adding (3.1.12) and (3.1.14) we have

$$
\begin{aligned}
&\frac{d}{dt}(\|v_h\|_{H^2}^2 + \|v_3\|_{H^2}^2 + \|\rho\|_{H^2}^2) + \nu\|\nabla_h v_h\|_{H^2}^2 + \nu\|\nabla_h v_3\|_{H^2}^2 + \kappa\|\nabla_h \rho\|_{H^2}^2 \\
&\leq C \, (\nu^{-1} + \kappa^{-1})(\|v_h^{(0)}\|_{H^2(\mathbb{R}^2)}^2 + \|\nabla_h v_h^{(0)}\|_{H^2(\mathbb{R}^2)}^2) \\
&\times (\|v_3\|_{H^2}^2 + \|v_h\|_{H^2}^2 + \|\rho\|_{H^2}^2) \\
&+ C \, \nu^{-3}(\|\nabla_h v_h\|_{H^2}^2 + \|\nabla_h v_3\|_{H^2}^2)(\|v_h\|_{H^2}^2 + \|v_3\|_{H^2}^2)^2 \\
&+ C \, \nu^{-1}\kappa^{-2}(\|\nabla_h v_h\|_{H^2}^2 + \|\nabla_h \rho\|_{H^2}^2)(\|v_h\|_{H^2}^2 + \|\rho\|_{H^2}^2)^2.
\end{aligned} \tag{3.1.15}
$$

The above inequality (3.1.15) is in the form

$$
\frac{d}{dt} f(t) + f_1(t) \leq a(t) f(t) + f_2(t). \tag{3.1.16}
$$

Therefore, Gronwall's inequality implies

$$f(t) + \int_0^t f_1(\tau)d\tau \le e^{\int_0^t a(\tau)d\tau} f(0) + e^{\int_0^t a(\tau)d\tau} \int_0^t f_2(\tau)d\tau, \tag{3.1.17}$$

where

$$a(t) = C\,(\nu^{-1} + \kappa^{-1})(\|v_h^{(0)}\|^2_{H^2(\mathbb{R}^2)} + \|\nabla_h v_h^{(0)}\|^2_{H^2(\mathbb{R}^2)}), \tag{3.1.18}$$

and (3.1.5), (3.1.10) and (3.1.9) implies that

$$e^{\int_0^t a(\tau)d\tau} \le e^{C\,(\nu^{-1}+\kappa^{-1})\left(\|v_h^{(0)}\|^2_{H^2(\mathbb{R}^2)} + \|\nabla_h v_h^{(0)}\|^2_{H^2(\mathbb{R}^2)}\right)} := K_0, \tag{3.1.19}$$

which depends only on the initial data for the special 2D solution, ν, and κ.

We have

$$\begin{aligned} f_2(t) \quad &:= C\,\nu^{-3}(\|\nabla_h v_h\|^2_{H^2} + \|\nabla_h v_3\|^2_{H^2})(\|v_h\|^2_{H^2} + \|v_3\|^2_{H^2})^2 \\ &+ C\,\nu^{-1}\kappa^{-2}(\|\nabla_h v_h\|^2_{H^2} + \|\nabla_h \rho\|^2_{H^2})(\|v_h\|^2_{H^2} + \|\rho\|^2_{H^2})^2. \end{aligned}$$

Then, integrating $f_2(t)$ from 0 to t yields

$$\begin{aligned} \int_0^t f_2(\tau)d\tau \quad &= C\,\nu^{-3}\int_0^t (\|\nabla_h v_h\|^2_{H^2} + \|\nabla_h v_3\|^2_{H^2})(\|v_h\|^2_{H^2} + \|v_3\|^2_{H^2})^2\,d\tau \\ &+ C\,\nu^{-1}\kappa^{-2}\int_0^t (\|\nabla_h v_h\|^2_{H^2} + \|\nabla_h \rho\|^2_{H^2})(\|v_h\|^2_{H^2} + \|\rho\|^2_{H^2})^2\,d\tau \\ &\le C\,\nu^{-3}(\|v_h\|^2_{H^2} + \|v_3\|^2_{H^2})^2 \int_0^t (\|\nabla_h v_h\|^2_{H^2} + \|\nabla_h v_3\|^2_{H^2})\,d\tau \\ &+ C\,\nu^{-1}\kappa^{-2}(\|v_h\|^2_{H^2} + \|\rho\|^2_{H^2})^2 \int_0^t (\|\nabla_h v_h\|^2_{H^2} + \|\nabla_h \rho\|^2_{H^2})\,d\tau \\ &\le C\,\nu^{-3}E^2(t)\nu^{-1}E(t) + C\nu^{-1}\kappa^{-2}E^2(t)(\nu^{-1} + \kappa^{-1})E(t) \\ &= C\,(\nu^{-4} + \nu^{-2}\kappa^{-2} + \nu^{-1}\kappa^{-3})E(t)^3. \end{aligned}$$

Using (3.1.17), we have

$$E(t) \leq K_0\, E(0) + C\,(\nu^{-4} + \nu^{-1}\kappa^{-3} + \nu^{-2}\kappa^{-2}) K_0 E(t)^3. \tag{3.1.20}$$

where C is a constant independent of ν and κ. This establishes the global bound and completes the proof of Proposition (3.1.1). ■

3.2 Local Existence and Uniqueness

In this section, we establish the local existence and uniqueness of solutions to (1.2.9). Our goal is to prove the following proposition:

Proposition 3.2.1 *Consider the initial value problem (1.2.9). Assume the initial data* $(\widetilde{v}_{h0}, v_{30}, \rho_0) \in H^2$ *that satisfies* $\nabla_h \cdot \widetilde{v}_{h0} + \partial_3 v_{30} = 0$. *Then there is a* $T > 0$ *and a unique solution* $(\widetilde{v}_h, v_3, \rho)$ *of (1.2.9) on* $[0, T_0)$ *satisfying*

$$(\widetilde{v}_h, v_3, \rho) \in C([0, T_0); L^2) \quad (\nabla_h \widetilde{v}_h, \nabla_h v_3, \nabla_h \rho) \in L^2([0, T_0); H^2)$$

Proof. For notational convenience, we ignore tilde and write (v_h, v_3, ρ) for $(\widetilde{v}_h, v_3, \rho)$. We use Friedrich's method to show the local existence result. Before starting Friedrich's method, we introduce some notations. First, we define

$$\begin{aligned} L^2_n \quad &= \{f \in L^2(\mathbb{R}^3) |\ \text{supp}\ \hat{f} \subset B(0,n)\}, \\ L^{2,\sigma}_n \quad &= \{u \in L^2(\mathbb{R}^3) | \nabla \cdot u = 0,\ \text{supp}\ \hat{u} \subset B(0,n)\}. \end{aligned}$$

The above spaces L^2_n and $L^{2,\sigma}_n$ are equipped with the L^2-norm. Further, for $n \in \mathbb{N}^+$, we define the following cut-off operator on L^2

$$\mathbb{E}_n f = \mathcal{F}^{-1}(\chi_{B(0,n)} \widehat{f}),$$

where $\widehat{f}$ and $\mathcal{F}^{-1}f$ represent the Fourier transform and the inverse Fourier transform, respectively, and $\chi_{B(0,n)}$ is the characteristics function on the ball $B(0,n)$.

The pressure term can be represented in terms of (v_h, v_3, ρ) by using the divergence-free condition. Taking the divergence of the velocity equation in (1.2.9) yields

$$-\Delta p = \nabla_h \cdot P_1 + \partial_3 P_2,$$

where

$$\begin{aligned} P_1 &= v_h \cdot \nabla_h v_h + v_h \cdot \nabla_h v_h^{(0)} + v_h^{(0)} \cdot \nabla_h v_h + v_3 \partial_3 v_h + f v_h^{\perp}, \\ P_2 &= v_h \cdot \nabla_h v_3 + v_h^{(0)} \cdot \nabla_h v_3 + v_3 \partial_3 v_3 + \rho. \end{aligned}$$

Therefore,

$$p = p(v_h,\, v_3,\, \rho) := (-\Delta)^{-1}(\nabla_h \cdot P_1 + \partial_3 P_2). \tag{3.2.1}$$

In order to construct the solution of (1.2.9), we look a sequence of solutions

$$\{(v_h^{(n)},\, v_3^{(n)},\, \rho^{(n)})\}_{n=1}^{\infty} \quad \text{with} \quad (v_h^{(n)},\, v_3^{(n)}) \in L_n^{2,\sigma}, \quad \rho^{(n)} \in L_n^2 \tag{3.2.2}$$

to the following regularized system:

$$\begin{cases} \partial_t v_h^{(n)} + \mathbb{E}_n\Big(v_h^{(n)} \cdot \nabla_h v_h^{(n)} + v_h^{(n)} \cdot \nabla_h v_h^{(0)} + v_h^{(0)} \cdot \nabla_h v_h^{(n)} + v_3^{(n)} \partial_3 v_h^{(n)} + f (v_h^{\perp})^{(n)}\Big) \\ \qquad\qquad = -\nabla_h p^{(n)} + \nu\, \Delta_h v_h^{(n)}, \\ \partial_t v_3^{(n)} + \mathbb{E}_n\Big(v_h^{(n)} \cdot \nabla_h v_3^{(n)} + v_h^0 \cdot \nabla_h w^{(n)} + w^{(n)} \partial_3 v_3^{(n)}\Big) = -\partial_3 p^{(n)} + \nu\, \Delta_h v_3^{(n)} - \rho^{(n)}, \\ \partial_t \rho^{(n)} + \mathbb{E}_n\Big(v_h^{(n)} \cdot \nabla_h \rho^{(n)} + v_h^0 \cdot \nabla_h \rho^{(n)} + v_3^{(n)} \partial_3 \rho^{(n)}\Big) = \kappa\, \Delta_h \rho^{(n)} + v_3^{(n)}, \\ \nabla_h \cdot v_h^{(n)} + \partial_3 v_3^{(n)} = 0, \\ (v_h^{(n)},\, v_3^{(n)},\, \rho^{(n)})|_{t=0} = \mathbb{E}_n(v_{h0} - v_{h0}^{(0)},\, v_{30},\, \rho_0), \end{cases} \tag{3.2.3}$$

where $p^{(n)} = \mathbb{E}_n\, p(v_h^{(n)},\, v_3^{(n)},\, \rho^{(n)})$, with p defined as in (3.2.1).

We write the system (3.2.3) as an ordinary differential equation as,

$$\frac{d}{dt}(v_h^{(n)},\, w^{(n)},\, \rho^{(n)}) = F_n(v_h^{(n)},\, v_3^{(n)},\, \rho^{(n)}) \tag{3.2.4}$$

on the Banach space

$$(v_h^{(n)},\, v_3^{(n)}) \in L_n^{2,\sigma}, \quad \rho^{(n)} \in L_n^2,$$

where

$$\begin{aligned}
F_n(v_h^{(n)},\, v_3^{(n)},\, \rho^{(n)}) \quad & = -\mathbb{E}_n\big(v_h^{(n)} \cdot \nabla_h v_h^{(n)}\big) - \mathbb{E}_n\big(v_h^{(n)} \cdot \nabla_h v_h^{(0)}\big) - \mathbb{E}_n\big(v_h^{(0)} \cdot \nabla_h v_h^{(n)}\big) - \mathbb{E}_n\big(v_3^{(n)} \partial_3 v_h^{(n)}\big) \\
& -\mathbb{E}_n\big(f(v_h^{\perp})^{(n)}\big) - \nabla_h p^{(n)} + \nu\, \Delta_h v_h^{(n)} - \mathbb{E}_n\big(v_h^{(n)} \cdot \nabla_h v_3^{(n)}\big) - \mathbb{E}_n\big(v_h^{0} \cdot \nabla_h v_3^{(n)}\big) \\
& -\mathbb{E}_n\big(v_3^{(n)} \partial_3 v_3^{(n)}\big) - \partial_3 p^{(n)} + \nu\, \Delta_h v_3^{(n)} - \rho^{(n)} - \mathbb{E}_n\big(v_h^{(n)} \cdot \nabla_h \rho^{(n)}\big) \\
& -\mathbb{E}_n\big(v_h^{0} \cdot \nabla_h \rho^{(n)}\big) - \mathbb{E}_n\big(v_3^{(n)} \partial_3 \rho^{(n)}\big) + \kappa\, \Delta_h \rho^{(n)} + v_3^{(n)} \\
& = F_1 + \cdots + F_{18}.
\end{aligned}$$

Next, we verify that F_n maps $L_n^{2,\sigma} \times L_n^2$ to $L_n^{2,\sigma} \times L_n^2$, and is locally Lipschitz with the help of the following Lemma (3.2.1).

Lemma 3.2.1 *Let B be a ball and $\mathcal{C}$ is a annulus. A constant C exists such that for any nonnegative integer k, any p, $q \in [1, \infty]$ with $q \geq p$, and any function $f \in L^p$, we have*

$$supp\, \hat{f} \subset \lambda B \implies \|D^k f\|_{L^q} := \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^q} \leq C^{k+1} \lambda^{k+d(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p}, \tag{3.2.5}$$

and

$$supp\, \hat{f} \subset \lambda \mathcal{C} \implies C^{-k-1} \lambda^k \|f\|_{L^p} \leq \|D^k f\|_{L^q} \leq C^{k+1} \lambda^k \|f\|_{L^p}. \tag{3.2.6}$$

We show each term in F_n maps $L_n^{2,\sigma} \times L_n^2$ to $L_n^{2,\sigma} \times L_n^2$. By using Hölder's inequality and

Lemma (3.2.1), $F_1 = -\mathbb{E}_n\big(v_h^{(n)} \cdot \nabla_h v_h^{(n)}\big)$, can be estimated as

$$\begin{aligned}
\|\mathbb{E}_n(v_h^{(n)} \cdot \nabla_h v_h^{(n)})\|_{L^2(\mathbb{R}^3)} &\leq \|v_h^{(n)} \cdot \nabla_h v_h^{(n)}\|_{L^2(\mathbb{R}^3)} \\
&\leq \|v_h^{(n)}\|_{L^\infty(\mathbb{R}^3)} \|\nabla_h v_h^{(n)}\|_{L^2(\mathbb{R}^3)} \\
&\leq C\, n^{\frac{3}{2}} \|v_h^{(n)}\|_{L^2(\mathbb{R}^3)} n \|v_h^{(n)}\|_{L^2(\mathbb{R}^3)} = C\, n^{\frac{5}{2}} \|v_h^{(n)}\|^2_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

Similarly, by using Holder's inequality and Lemma (3.2.1), we can show that other terms also maps $L^{2,\sigma}_n \times L^2_n$ to $L^{2,\sigma}_n \times L^2_n$. For $F_2 = -\mathbb{E}_n\big(v_h^{(n)} \cdot \nabla_h v_h^{(0)}\big)$, we have

$$\begin{aligned}
\|\mathbb{E}_n(v_h^{(n)} \cdot \nabla_h v_h^{(0)})\|_{L^2(\mathbb{R}^3)} &= \|\mathbb{E}_n\big(\nabla_h(v_h^{(n)} \cdot v_h^{(0)}) - \nabla_h v_h^{(n)} \cdot v_h^{(0)}\big)\|_{L^2(\mathbb{R}^3)} \\
&\leq \|\nabla_h \mathbb{E}_n(v_h^{(n)} \cdot v_h^{(0)})\|_{L^2(\mathbb{R}^3)} + \|\nabla_h v_h^{(n)} v_h^{(0)}\|_{L^2(\mathbb{R}^3)} \\
&\leq n \|\mathbb{E}_n(v_h^{(n)} \cdot v_h^{(0)})\|_{L^2(\mathbb{R}^3)} + \|\nabla_h v_h^{(n)}\|_{L^\infty(\mathbb{R}^3)} \|v_h^{(0)}\|_{L^2(\mathbb{R}^3)} \\
&\leq n \|v_h^{(n)} \cdot v_h^{(0)}\|_{L^2(\mathbb{R}^3)} + \|\nabla_h v_h^{(n)}\|_{L^\infty(\mathbb{R}^3)} \|v_h^{(0)}\|_{L^2(\mathbb{R}^3)} \\
&\leq n n^{\frac{3}{2}} \|v_h^{(n)}\|_{L^2(\mathbb{R}^3)} \|v_h^{(0)}\|_{L^2(\mathbb{R}^3)} + n^{\frac{5}{2}} \|v_h^{(n)}\|_{L^2(\mathbb{R}^3)} \|v_h^{(0)}\|_{L^2(\mathbb{R}^3)} \\
&\leq C\, n^{\frac{5}{2}} \|v_h^{(n)}\|_{L^2(\mathbb{R}^3)} \|v_h^{(0)}\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

Similarly, for $F_3 = -\mathbb{E}_n\big(v_h^{(0)} \cdot \nabla_h v_h^{(n)}\big)$, we obtain

$$\begin{aligned}
\|\mathbb{E}_n(v_h^{(0)} \cdot \nabla_h v_h^{(n)})\|_{L^2(\mathbb{R}^3)} &\leq \|v_h^{(0)} \cdot \nabla_h v_h^{(n)}\|_{L^2(\mathbb{R}^3)} \\
&\leq \|v_h^{(0)}\|_{L^2(\mathbb{R}^3)} \|\nabla_h v_h^{(n)}\|_{L^\infty(\mathbb{R}^3)} \\
&\leq C\, \|v_h^{(0)}\|_{L^2(\mathbb{R}^3)} n^{\frac{5}{2}} \|v_h^{(n)}\|_{L^2(\mathbb{R}^3)} \leq C\, n^{\frac{5}{2}} \|v_h^{(0)}\|_{L^2(\mathbb{R}^3)} \|v_h^{(n)}\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

For $F_4 = \mathbb{E}_n\big(v_3^{(n)} \partial_3 v_h^{(n)}\big)$ we have

$$\begin{aligned}
\|\mathbb{E}_n(v_3^{(n)} \partial_3 v_h^{(n)})\|_{L^2(\mathbb{R}^3)} &\leq \|v_3^{n}\|_{L^\infty(\mathbb{R}^3)} \|\partial_3 v_h^{(n)}\|_{L^2(\mathbb{R}^3)} \\
&\leq C\, n^{\frac{3}{2}} \|v_3^{(n)}\|_{L^2(\mathbb{R}^3)} n \|v_h^{(n)}\|_{L^2(\mathbb{R}^3)} \leq C\, n^{\frac{5}{2}} \|v_3^{(n)}\|^2_{L^2(\mathbb{R}^3)} \|v_h^{(n)}\|^2_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

We assume that f has a compact support, therefore, for $F_5 = f(v_h^{\perp})^{(n)}$, we have

$$\|f(v_h^{\perp})^{(n)}\|_{L^2(\mathbb{R}^3)} \leq C\|v_h^{(n)}\|_{L^2(\mathbb{R}^3)}.$$

For $F_6 = \nabla_h \mathbb{E}_n\, p(v_h^{(n)},\, v_3^{(n)},\, \rho^{(n)})$, we first write it into the following nine terms

$$\begin{aligned}
\nabla_h \mathbb{E}_n\, p(v_h^{(n)},\, v_3^{(n)},\, \rho^{(n)}) \quad &= \nabla_h \mathbb{E}_n\big((-\Delta)^{-1}\big(\nabla_h \cdot P_1 + \partial_3 P_2\big) \\
&= \mathbb{E}_n\big(\nabla_h(-\Delta)^{-1}\big(\nabla_h \cdot \big(v_h^{(n)} \cdot \nabla_h v_h^{(n)}\big)\big)\big) + \mathbb{E}_n\big(\nabla_h(-\Delta)^{-1}\big(\nabla_h \cdot \big(v_h^{(n)} \cdot \nabla_h v_h^{(0)}\big)\big)\big) \\
&+ \mathbb{E}_n\big(\nabla_h(-\Delta)^{-1}\big(\nabla_h \cdot \big(v_h^{(0)} \cdot \nabla_h v_h^{(n)}\big)\big)\big) + \mathbb{E}_n\big(\nabla_h(-\Delta)^{-1}\big(\big(\nabla_h v_3^{(n)} \partial_3 v_h^{(n)}\big)\big)\big) \\
&+ \mathbb{E}_n\big(\nabla_h(-\Delta)^{-1}\big(\big(\nabla_h \cdot f(v_h^{\perp})^{(n)}\big)\big)\big) + \mathbb{E}_n\big(\nabla_h(-\Delta)^{-1}\partial_3\big(v_h^{(n)} \cdot \nabla_h v_3^{(n)}\big)\big) \\
&+ \mathbb{E}_n\big(\big(\nabla_h(-\Delta)^{-1}\partial_3(v_h^{(0)} \cdot \nabla_h v_3^{(n)}\big)\big) + \mathbb{E}_n\big(\big(\nabla_h(-\Delta)^{-1}\partial_3\big(v_3^{(n)} \partial_3 v_3^{(n)}\big)\big) \\
&+ \mathbb{E}_n\big(\nabla_h(-\Delta)^{-1}\partial_3 \rho^{(n)}\big) = F_{61} + \cdots + F_{69}.
\end{aligned}$$

Them, we take $F_{61} = \mathbb{E}_n\big(\nabla_h(-\Delta)^{-1}\big(\nabla_h \cdot \big(v_h^{(n)} \cdot \nabla_h v_h^{(n)}\big)\big)\big)$ and show that it maps $L_n^{2,\sigma} \times L_n^2$ to $L_n^{2,\sigma} \times L_n^2$. Using the fact that Riesz transforms are boundedness in the L^2 space, Hölder's inequality, and Lemma (3.2.1), we have

$$\begin{aligned}
\|\mathbb{E}_n\big(\nabla_h(-\Delta)^{-1}\big(\nabla_h \cdot \big(v_h^{(n)} \cdot \nabla_h v_h^{(n)}\big)\big)\big)\|_{L^2(\mathbb{R}^3)} \quad &\leq \|v_h^{(n)} \cdot \nabla_h v_h^{(n)}\|_{L^2(\mathbb{R}^3)} \\
&\leq \|v_h^{(n)}\|_{L^\infty(\mathbb{R}^3)} \|\nabla_h v_h^{(n)}\|_{L^2(\mathbb{R}^3)} \\
&\leq C\, n^{\frac{3}{2}} \|v_h^{(n)}\|_{L^2(\mathbb{R}^3)} n \|v_h^{(n)}\|_{L^2(\mathbb{R}^3)} = C\, n^{\frac{5}{2}} \|v_h^{(n)}\|_{L^2(\mathbb{R}^3)}^2.
\end{aligned}$$

All the remaining terms in F_6 can be estimated similarly.

For $F_7 = \nu \Delta_h v_h^{(n)}$, by using Lemma (3.2.1), we have

$$\|\nu \Delta_h v_h^{(n)}\|_{L^2(\mathbb{R}^3)} \leq C n^2 \|v_h^{(n)}\|_{L^2(\mathbb{R}^3)}.$$

For $F_8 = -\mathbb{E}_n(v_h^{(n)} \cdot \nabla_h v_3^{(n)})$, by using Hölder's inequality and Lemma (3.2.1), we have

$$\begin{aligned}
\|\mathbb{E}_n(v_h^{(n)} \cdot \nabla_h v_3^{(n)})\|_{L^2(\mathbb{R}^3)} &\le \|v_h^{(n)} \cdot \nabla_h v_3^{(n)}\|_{L^2(\mathbb{R}^3)} \\
&\le \|v_h^{(n)}\|_{L^\infty(\mathbb{R}^3)} \|\nabla_h v_3^{(n)}\|_{L^2(\mathbb{R}^3)} \\
&\le Cn^{\frac{3}{2}} \|v_h^{(n)}\|_{L^2(\mathbb{R}^3)} n \|v_3^{(n)}\|_{L^2(\mathbb{R}^3)} \le Cn^{\frac{5}{2}} \|v_h^{(n)}\|_{L^2(\mathbb{R}^3)} \|v_3^{(n)}\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

Similarly, for $F_9 = -\mathbb{E}_n(v_h^{(0)} \cdot \nabla_h v_3^{(n)})$, by using the Hölder's inequality and by Lemma (3.2.1),

$$\begin{aligned}
\|\mathbb{E}_n(v_h^{(0)} \cdot \nabla_h v_3^{(n)})\|_{L^2(\mathbb{R}^3)} &\le \|v_h^{(0)} \cdot \nabla_h v_3^{(n)}\|_{L^2(\mathbb{R}^3)} \\
&\le \|v_h^{(0)}\|_{L^2(\mathbb{R}^3)} \|\nabla_h v_3^{(n)}\|_{L^\infty(\mathbb{R}^3)} \le Cn^{\frac{5}{2}} \|v_h^{(0)}\|_{L^2(\mathbb{R}^3)} \|v_3^{(n)}\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

For $F_{10} = -\mathbb{E}_n(v_3^{(n)} \partial_3 v_3^{(n)})$, by using the Hölder's inequality and Lemma (3.2.1), we have

$$\begin{aligned}
\|\mathbb{E}_n(v_3^{(n)} \partial_3 v_3^{(n)})\|_{L^2(\mathbb{R}^3)} &\le \|v_3^{(n)} \partial_3 v_3^{(n)}\|_{L^2(\mathbb{R}^3)} \\
&\le \|v_3^{(n)}\|_{L^\infty(\mathbb{R}^3)} \|\partial_3 v_3^{(n)}\|_{L^2(\mathbb{R}^3)} \\
&\le Cn^{\frac{3}{2}} \|v_3^{(n)}\|_{L^2(\mathbb{R}^3)} n \|v_3^{(3)}\|_{L^2(\mathbb{R}^3)} \le Cn^{\frac{5}{2}} \|v_3^{(n)}\|^2_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

For $F_{11} = \partial_3 \mathbb{E}_n \, p(v_h^{(n)}, \, v_3^{(n)}, \, \rho^{(n)})$, we first write it as

$$\begin{aligned}
\partial_3 \mathbb{E}_n \, p(v_h^{(n)}, \, v_3^{(n)}, \, \rho^{(n)}) &= \partial_3 \mathbb{E}_n \big((-\Delta)^{-1} \big(\nabla_h \cdot P_1 + \partial_3 P_2 \big) \\
&= \mathbb{E}_n \big(\partial_3 (-\Delta)^{-1} \big(\nabla_h \cdot \big(v_h^{(n)} \cdot \nabla_h v_h^{(n)} \big) \big) \big) + \mathbb{E}_n \big(\partial_3 (-\Delta)^{-1} \big(\nabla_h \cdot \big(v_h^{(n)} \cdot \nabla_h v_h^{(0)} \big) \big) \big) \\
&+ \mathbb{E}_n \big(\partial_3 (-\Delta)^{-1} \big(\nabla_h \cdot \big(v_h^{(0)} \cdot \nabla_h v_h^{(n)} \big) \big) \big) + \mathbb{E}_n \big(\partial_3 (-\Delta)^{-1} \big(\big(\nabla_h w^{(n)} \partial_3 v_h^{(n)} \big) \big) \big) \\
&+ \mathbb{E}_n \big(\partial_3 (-\Delta)^{-1} \big(\big(\nabla_h \cdot f(v_h^{\perp})^{(n)} \big) \big) \big) + \mathbb{E}_n \big(\partial_3 (-\Delta)^{-1} \partial_3 \big(v_h^{(n)} \cdot \nabla_h v_3^{(n)} \big) \big) \\
&+ \mathbb{E}_n \big(\partial_3 (-\Delta)^{-1} \partial_3 \big(v_h^0 \cdot \nabla_h v_3^{(0)} \big) \big) + \mathbb{E}_n \big(\big(\partial_3 (-\Delta)^{-1} \partial_3 (v_h^{(0)} \cdot \nabla_h v_3^{(n)} \big) \big) \\
&+ \mathbb{E}_n \big(\big(\partial_3 (-\Delta)^{-1} \partial_3 \big(v_3^{(n)} \partial_3 v_3^{(n)} \big) \big) + \mathbb{E}_n \big(\partial_3 (-\Delta)^{-1} \partial_3 \rho^{(n)} \big).
\end{aligned}$$

We can estimate each term in f_{11} exactly similar to F_6. For $F_{12} = \nu \Delta_h v_3^{(n)}$, similar to F_7, by using Lemma (3.2.1), we have $\|\nu\Delta_h v_h^{(n)}\|_{L^2(\mathbb{R}^3)} \leq Cn^2\|v_h^{(n)}\|_{L^2(\mathbb{R}^3)}$. For $F_{13} = \rho^{(n)}$, we have $\|\rho^{(n)}\|_{L^2(\mathbb{R}^3)} \leq C\,\|\rho^{(n)}\|_{L^2(\mathbb{R}^3)}$. For $F_{14} = -\mathbb{E}_n(v_h^{(n)} \cdot \nabla_h \rho^{(n)})$, by using Hölder's inequality and Lemma (3.2.1), we have

$$\begin{aligned}
\|\mathbb{E}_n(v_h^{(n)} \cdot \nabla_h \rho^{(n)})\|_{L^2(\mathbb{R}^3)} &\leq \|v_h^{(n)} \cdot \nabla_h \rho^{(n)}\|_{L^2(\mathbb{R}^3)} \leq \|v_h^{(n)}\|_{L^\infty(\mathbb{R}^3)}\|\nabla_h \rho^{(n)}\|_{L^2(\mathbb{R}^3)} \\
&\leq Cn^{\frac{3}{2}}\|v_h^{(n)}\|_{L^2(\mathbb{R}^3)} n\|\rho^{(n)}\|_{L^2(\mathbb{R}^3)} \leq Cn^{\frac{5}{2}}\|v_h^{(n)}\|_{L^2(\mathbb{R}^3)}\|\rho^{(n)}\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

For $F_{15} = -\mathbb{E}_n(v_h^{(0)} \cdot \nabla_h \rho^{(n)})$, by using the Hölder's inequality and Lemma (3.2.1) yields

$$\begin{aligned}
\|\mathbb{E}_n(v_h^{(0)} \cdot \nabla_h \rho^{(n)})\|_{L^2(\mathbb{R}^3)} &\leq \|v_h^{(0)} \cdot \nabla_h \rho^{(n)}\|_{L^2(\mathbb{R}^3)} \\
&\leq \|v_h^{(0)}\|_{L^2(\mathbb{R}^3)}\|\nabla_h \rho^{(n)}\|_{L^\infty(\mathbb{R}^3)} \leq Cn^{\frac{5}{2}}\|v_h^{(0)}\|_{L^2(\mathbb{R}^3)}\|\rho^{(n)}\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

For $F_{16} = -\mathbb{E}_n(v_3^{(n)}\partial_3 \rho^{(n)})$, by using the Hölder's inequality and Lemma (3.2.1), we have

$$\begin{aligned}
\|\mathbb{E}_n(v_3^{(n)}\partial_3 \rho^{(n)})\|_{L^2(\mathbb{R}^3)} &\leq \|v_3^{(n)}\partial_3 \rho^{(n)}\|_{L^2(\mathbb{R}^3)} \\
&\leq \|v_3^{(n)}\|_{L^2(\mathbb{R}^3)}\|\partial_3 \rho^{(n)}\|_{L^\infty(\mathbb{R}^3)} \leq Cn^{\frac{3}{2}}\|v_3^{(n)}\|_{L^2(\mathbb{R}^3)} n\|\rho^{(3)}\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

For $F_{17} = \nu\Delta\rho^{(n)} \in L^2(\mathbb{R}^3)$, using Lemma (3.2.1), we have $\|\nu\Delta_h \rho^{(n)}\|_{L^2(\mathbb{R}^3)} \leq C\,n^2\|\rho^{(n)}\|_{L^2(\mathbb{R}^3)}$. For $F_{18} = v_3^{(n)} \in L^2(\mathbb{R}^3)$, we have $\|\nu v_3^{(n)}\|_{L^2(\mathbb{R}^3)} \leq C\,\|v_3^{(n)}\|_{L^2(\mathbb{R}^3)}$.
Now, we show a few terms in F_n are locally Lipschitz, we start with $F_1 = -\mathbb{E}_n\big(v_h^{(n)} \cdot \nabla_h v_h^{(n)}\big)$, by using the Hölder's inequality and by Lemma (3.2.1), we have

$$\begin{aligned}
&\|\mathbb{E}_n(v_h^{(n)} \cdot \nabla_h v_h^{(n)}) - \mathbb{E}_n(u_h^{(n)} \cdot \nabla_h u_h^{(n)})\|_{L^2(\mathbb{R}^3)} \\
&= \|v_h^{(n)} \cdot \nabla_h v_h^{(n)} - u_h^{(n)} \cdot \nabla_h v_h^{(n)} + u_h^{(n)} \cdot \nabla_h v_h^{(n)} - u_h^{(n)} \cdot \nabla_h u_h^{(n)}\|_{L^2(\mathbb{R}^3)} \\
&\leq \|(v_h^{(n)} - u_h^{(n)}) \cdot \nabla_h v_h^{(n)}\|_{L^2(\mathbb{R}^3)} + \|u_h^{(n)} \cdot \nabla_h (v_h^{(n)} - u_h^{(n)})\|_{L^2(\mathbb{R}^3)} \\
&\leq \|v_h^{(n)} - u_h^{(n)}\|_{L^\infty(\mathbb{R}^3)}\|\nabla_h v_h^{(n)}\|_{L^2(\mathbb{R}^3)} + \|u_h^{(n)}\|_{L^\infty(\mathbb{R}^3)}\|\nabla_h (v_h^{(n)} - u_h^{(n)})\|_{L^2(\mathbb{R}^3)} \\
&= C\,n^{\frac{5}{2}}\big(\|v_h^{(n)}\|_{L^2(\mathbb{R}^3)} + \|u_h^{(n)}\|_{L^2(\mathbb{R}^3)}\big)\|v_h^{(n)} - u_h^{(n)}\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

For $F_2 = -\mathbb{E}_n\big(v_h^{(n)} \cdot \nabla_h v_h^{(0)}\big)$, by using the Hölder's inequality and Lemma (3.2.1) yields

$$
\begin{aligned}
&\|\mathbb{E}_n(v_h^{(n)} \cdot \nabla_h v_h^{(0)}) - \mathbb{E}_n(u_h^{(n)} \cdot \nabla_h v_h^{(0)})\|_{L^2(\mathbb{R}^3)} \\
&= \|\mathbb{E}_n\big(\nabla_h(v_h^{(n)} \cdot v_h^{(0)}) - \nabla_h u_h^{(n)} \cdot v_h^{(0)}\big) - \mathbb{E}_n\big(\nabla_h(u_h^{(n)} \cdot v_h^{(0)}) - \nabla_h v_h^{(n)} \cdot v_h^{(0)}\big)\|_{L^2(\mathbb{R}^3)} \\
&\leq \|\mathbb{E}_n\big(\nabla_h(v_h^{(n)} - u_h^{(n)}) \cdot v_h^{(0)}\big)\|_{L^2(\mathbb{R}^3)} + \|\nabla_h(v_h^{(n)} - u_h^{(n)}) \cdot v_h^{(0)}\|_{L^2(\mathbb{R}^3)} \\
&\leq n\|(v_h^{(n)} - u_h^{(n)}) \cdot v_h^{(0)}\|_{L^2(\mathbb{R}^3)} + \|\nabla_h(v_h^{(n)} - u_h^{(n)}) \cdot v_h^{(0)}\|_{L^2(\mathbb{R}^3)} \\
&\leq C\, n^{\frac{5}{2}}\|v_h^{(n)} - u_h^{(n)}\|_{L^2(\mathbb{R}^3)}\|v_h^{(0)}\|_{L^2(\mathbb{R}^3)} + C\, n^{\frac{5}{2}}\|v_h^{(0)}\|_{L^2(\mathbb{R}^3)}\|v_h^{(n)} - u_h^{(n)}\|_{L^2(\mathbb{R}^3)} \\
&\leq C\, n^{\frac{5}{2}}\|v_h^{(0)}\|_{L^2(\mathbb{R}^3)}\|v_h^{(n)} - u_h^{(n)}\|_{L^2(\mathbb{R}^3)}.
\end{aligned}
$$

For $F_3 = -\mathbb{E}_n\big(v_h^{(0)} \cdot \nabla_h v_h^{(n)}\big)$, by using the Hölder's inequality and by Lemma (3.2.1) gives

$$
\begin{aligned}
\|\mathbb{E}_n(v_h^{(0)} \cdot \nabla_h v_h^{(n)}) - \mathbb{E}_n(v_h^{(0)} \cdot \nabla_h u_h^{(n)})\|_{L^2(\mathbb{R}^3)} \quad &\leq \|v_h^{(0)} \cdot \nabla_h v_h^{(n)} - v_h^{(0)} \cdot \nabla_h u_h^{(n)}\|_{L^2(\mathbb{R}^3)} \\
&\leq \|v_h^{(0)}\|_{L^2(\mathbb{R}^3)}\|\nabla_h(v_h^{(n)} - u_h^{(n)})\|_{L^\infty(\mathbb{R}^3)} \\
&\leq C\, n^{\frac{5}{2}}\|v_h^{(0)}\|_{L^2(\mathbb{R}^3)}\|v_h^{(n)} - u_h^{(n)}\|_{L^2(\mathbb{R}^3)}.
\end{aligned}
$$

For $F_4 = \mathbb{E}_n\big(v_3^{(n)}\partial_3 v_h^{(n)}\big)$, by using the Hölder's inequality and by Lemma (3.2.1), we have

$$
\begin{aligned}
&\|\mathbb{E}_n(v_3^{(n)}\partial_3 v_h^{(n)}) - \mathbb{E}_n(u_3^{(n)}\partial_3 u_h^{(n)})\|_{L^2(\mathbb{R}^3)} \\
&= \|v_3^{(n)}\partial_3 v_h^{(n)} - u_3^{(n)}\partial_3 v_h^{(n)} + u^{(n)}\partial_3 v_h^{(n)} - u^{(n)}\partial_3 u_h^{(n)}\|_{L^2(\mathbb{R}^3)} \\
&\leq \|(v_3^{(n)} - u_3^{(n)})\partial_3 v_h^{(n)} + u_3^{(n)}\partial_3(v_h^{(n)} - u_h^{(n)})\|_{L^2(\mathbb{R}^3)} \\
&\leq \|v_3^{(n)} - u_3^{(n)}\|_{L^2(\mathbb{R}^3)}\|\partial_3 v_h^{(n)}\|_{L^\infty(\mathbb{R}^3)} + \|q^{(n)}\|_{L^\infty(\mathbb{R}^3)}\|\partial_3 v_h^{(n)} - u_h^{(n)}\|_{L^2(\mathbb{R}^3)} \\
&\leq n^{\frac{5}{2}}\|\partial_3 v_h^{(n)}\|_{L^2(\mathbb{R}^3)}\|v_3^{(n)} - u_3^{(n)}\|_{L^2(\mathbb{R}^3)} + n^{\frac{3}{2}}\|q^{(n)}\|_{L^2(\mathbb{R}^3)} n\|v_h^{(n)} - u_h^{(n)}\|_{L^2(\mathbb{R}^3)} \\
&\leq n^{\frac{5}{2}}\|v_h^{(n)}\|_{L^2(\mathbb{R}^3)}\|v_3^{(n)} - u_3^{(n)}\|_{L^2(\mathbb{R}^3)} + n^{\frac{5}{2}}\|u_3^{(n)}\|_{L^2(\mathbb{R}^3)}\|v_h^{(n)} - u_h^{(n)}\|_{L^2(\mathbb{R}^3)}.
\end{aligned}
$$

Similarly, all remaining terms can be shown to be locally Lipschitz. Then, the the following theorem (3.2.1) implies that, for any $n \in \mathbb{N}$, there is $T_n > 0$ and a unique solution

$(v_h^{(n)}, w^{(n)}, \rho^{(n)})$ of the regularized system (3.2.3) satisfying

$$(v_h^{(n)}, w^{(n)}, \rho^{(n)}) \in C^\infty([0, T_n); L_n^{2,\sigma}) \times C^\infty([0, T_n); L_n^2).$$

Theorem 3.2.1 *Let E be a Banach space, U be an open subset of E, I an open interval of $\mathbb{R}$, and $(t_0, x_0) \in I \times U$. Let $F \in L^1_{loc}(I, C_\mu(U, E))$, where μ is an Osgood module of continuity and $C_\mu(U, E)$ is the set of bounded, continuous map from U to E such that*

$$\|F(t,x)\|_{C_\mu} := \sup_{x\in U}\|F(t,\, x)\|_E + \sup_{0\le\|x-y\|_E\le 1} \frac{\|F(t,\, x) - F(t,\, y)\|_E}{\mu(\|x-y\|_E)} < \infty.$$

Then there exist an interval $J \subset I$ such that the ODE

$$x(x) = x_0 + \int_{t_0}^{t} F(\tau, x(\tau))d\tau$$

has a unique continuous solution.

To demonstrate the solution $(v_h^{(n)}, v_3^{(n)}, \rho^{(n)})$ is global in time, we perform the L^2 estimate as before in (3.1.11). In fact, dotting the equations in (3.2.3) with $(v_h^{(n)},\, v_3^{(n)},\, \rho^{(n)})$ and using the divergence-free condition, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|v_h^{(n)}\|_{L^2}^2 + \|v_3^{(n)}\|_{L^2}^2 + \|\rho^{(n)}\|_{L^2}^2) + \nu\|\nabla_h v_h^{(n)}\|_{L^2}^2 + \nu\|\nabla_h v_3^{(n)}\|_{L^2}^2 + \kappa\|\nabla_h \rho^{(n)}\|_{L^2}^2 \qquad (3.2.7)\\
&= -\int v_h^{(n)} \cdot \mathbb{E}_n\Big(v_h^{(n)} \cdot \nabla_h v_h^{(n)} + v_h^{(n)} \cdot \nabla_h v_h^{(0)} + v_h^{(0)} \cdot \nabla_h v_h^{(n)} + v_3^{(n)}\partial_3 v_h^{(n)} + f(v_h^\perp)^{(n)}\Big)dx\\
&- \int w^{(n)} \cdot \mathbb{E}_n\Big(v_h^{(n)} \cdot \nabla_h v_3^{(n)} + v_h^{(0)} \cdot \nabla_h v_3^{(n)} + v_3^{(n)}\partial_3 v_3^{(n)} - \partial_3\rho^{(n)}\Big)dx\\
&- \int \rho^{(n)} \cdot \mathbb{E}_n\Big(v_h^{(n)} \cdot \nabla_h \rho^{(n)} + v_h^{0} \cdot \nabla_h \rho^{(n)} + v_3^{(n)}\partial_3 \rho^{(n)} + \partial_3 v_3^{(n)}\Big)dx\\
&= -\int v_h^{(n)} \cdot \nabla_h v_h^{(0)} \cdot v_h^{(n)} dx.
\end{aligned}$$

Using the fact that $v_h^{(0)}$ is independent of x_3, and Hölder, Ladyzhenskaya's, and Young's inequalities, the right-hand side of (3.2.7) can be estimated as

$$\begin{aligned}\Big|\int v_h^{(n)}\cdot\nabla_h v_h^{(0)}\cdot v_h^{(n)}\Big| &\le \|v_h^{(n)}\|_{L^2_{x_3}L^4_h}\|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)}\|v_h^{(n)}\|_{L^2_{x_3}L^4_h}\\ &\le \|v_h^{(n)}\|_{L^2}^{\frac12}\|\nabla_h v_h^{(n)}\|_{L^2}^{\frac12}\|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)}\|v_h^{(n)}\|_{L^2}^{\frac12}\|\nabla_h v_h^{(n)}\|_{L^2}^{\frac12}\\ &\le \frac{\nu}{2}\|\nabla_h v_h^{(n)}\|_{L^2}^2 + C\,\nu^{-1}\|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)}^2\|v_h^{(n)}\|_{L^2}^2.\end{aligned}$$

Then (3.2.7) becomes

$$\begin{aligned}&\frac{d}{dt}(\|v_h^{(n)}\|_{L^2}^2+\|v_3^{(n)}\|_{L^2}^2+\|\rho^{(n)}\|_{L^2}^2)+\nu\|\nabla_h v_h^{(n)}\|_{L^2}^2+\nu\|\nabla_h v_3^{(n)}\|_{L^2}^2+\kappa\|\nabla_h\rho^{(n)}\|_{L^2}^2\\ &\qquad\le C\,\nu^{-1}\|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)}^2(\|v_h^{(n)}\|_{L^2}^2+\|v_h^{(n)}\|_{L^2}^2+\|v_h^{(n)}\|_{L^2}^2).\end{aligned} \tag{3.2.8}$$

Gronwall's inequality then implies the following global upper bound

$$\begin{aligned}\|v_h^{(n)}\|_{L^2}^2+\|v_3^{(n)}\|_{L^2}^2+\|\rho^{(n)}\|_{L^2}^2 &\le A_0\,(\|v_h^{(n)}(0)\|_{L^2}^2+\|v_3^{(n)}(0)\|_{L^2}^2+\|\rho^{(n)}(0)\|_{L^2}^2)\\ &\le A_0\,(\|v_{h0}-v_{h0}^{(0)}\|_{L^2}+\|v_{30}\|_{L^2}+\|\rho_0\|_{L^2}),\end{aligned}$$

where

$$A_0 := e^{\int_0^t C\,\nu^{-1}\|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)}^2} = e^{C\,\nu^{-2}\|v_{h0}^{(0)}\|_{L^2(\mathbb{R}^2)}^2}. \tag{3.2.9}$$

Then, Theorem 3.2.2 implies that the solution $(v_h^{(n)}, v_3^{(n)}, \rho^{(n)})$ is global in time and

$$(v_h^{(n)}, v_3^{(n)}, \rho^{(n)}) \in C^\infty([0,\infty); L_n^{2,\sigma}) \times C^\infty([0,\infty); L_n^2).$$

Theorem 3.2.2 *Let (T_*, T^*) be the maximal interval of existence. If F satisfies*

$$\|F(t,x)\|_E \le M\|x\|_E^2$$

for some constant M, then for any $t_0 \in (T_*, T^*)$, we have

$$\int_{T_*}^{t_0} \|x\|_E \, dt - T_* = T^* + \int_{t_0}^{T_*} \|x\|_E \, dt = \infty.$$

The next step is to show there is a time $T > 0$ independent of n such that

$$(v_h^{(n)}, v_3^{(n)}, \rho^{(n)}) \in L^\infty(0, T; H^2) \times L^2(0, T; H^3)$$

with its norm bounded uniformly in the above space.

Applying Δ to the equations of $(v_h^{(n)},\, v_3^{(n)},\, \rho^{(n)})$ in (3.2.3) with $(\Delta v_h^{(n)},\, \Delta v_3^{(n)},\, \Delta \rho^{(n)})$ we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\Delta v_h^{(n)}\|_{L^2}^2 + \|\Delta v_3^{(n)}\|_{L^2}^2 + \|\Delta \rho^{(n)}\|_{L^2}^2) + \nu \|\Delta \nabla_h v_h^{(n)}\|_{L^2}^2 \\ &+\nu \|\Delta \nabla_h v_3^{(n)}\|_{L^2}^2 + \kappa \|\Delta \nabla_h \rho^{(n)}\|_{L^2}^2 = A_1 + \cdots + A_7 \end{aligned} \tag{3.2.10}$$

with

$$\begin{aligned} A_1 &= -\int \Delta \mathbb{E}_n (v_h^{(n)} \cdot \nabla_h v_h^{(n)} + v_3^{(n)} \, \partial_3 v_h^{(n)}) \cdot \Delta v_h^{(n)} \, dx, \quad & A_5 &= -\int \Delta \mathbb{E}_n (v_h^{(n)} \cdot \nabla_h v_h^{(0)}) \cdot \Delta v_h^{(n)} \, dx, \\ A_2 &= -\int \Delta \mathbb{E}_n (v_h^{(n)} \cdot \nabla_h v_3^{(n)} + v_3^{(n)} \, \partial_3 w^{(n)}) \cdot \Delta v_3^{(n)} \, dx, \quad & A_6 &= -\int \Delta \mathbb{E}_n (v_h^{0} \cdot \nabla_h v_3^{(n)}) \Delta v_3^{(n)} \, dx, \\ A_3 &= -\int \Delta \mathbb{E}_n (v_h^{(n)} \cdot \nabla_h \rho^{(n)} + v_3^{(n)} \, \partial_3 \rho^{(n)}) \cdot \Delta \rho^{(n)} \, dx, \quad & A_7 &= -\int \Delta \mathbb{E}_n (v_h^{0} \cdot \nabla_h \rho^{(n)}) \Delta \rho^{(n)} \, dx, \\ A_4 &= -\int \Delta \mathbb{E}_n (v_h^{(0)} \cdot \nabla_h v_h^{(n)}) \cdot \Delta v_h^{(n)} \, dx, \end{aligned}$$

where, we have used the following facts

$$\begin{aligned} &\int (\Delta \nabla_h p^{(n)} \cdot \Delta v_h^{(n)} + \Delta \partial_3 p^{(n)} \Delta v_3^{(n)}) \, dx = \int \Delta p^{(n)} \Delta (\nabla_h \cdot v_h^{(n)} + \partial_3 v_3^{(n)}) \, dx = \Delta v_h^{(n)\perp} \cdot \Delta v_h^{(n)} = 0, \\ &\int \nu \, \Delta \Delta_h v_h^{(n)} \cdot \Delta v_h^{(n)} = -\nu \|\Delta \nabla_h v_h^{(n)}\|_{L^2}^2, \quad \int \nu \, \Delta \Delta_h v_3 \cdot \Delta v_3^{(n)} = -\nu \|\Delta \nabla_h v_3^{(n)}\|_{L^2}^2, \\ &\int \nu \, \Delta \Delta_h \rho^{(n)} \cdot \Delta \rho^{(n)} = -\nu \|\Delta \nabla_h \rho^{(n)}\|_{L^2}^2. \end{aligned}$$

Then, as in (3.1.15), (3.2.10) becomes

$$
\begin{aligned}
&\frac{d}{dt}(\|v_h^{(n)}\|_{H^2}^2 + \|v_3^{(n)}\|_{H^2}^2 + \|\rho^{(n)}\|_{H^2}^2) + \nu\|\nabla_h v_h^{(n)}\|_{H^2}^2 + \nu\|\nabla_h v_3^{(n)}\|_{H^2}^2 + \kappa\|\nabla_h \rho^{(n)}\|_{H^2}^2 \\
&\leq\; C\left(\nu^{-1} + \kappa^{-1})(\|v_h^{(0)}\|_{H^2(\mathbb{R}^2)}^2 + \|\nabla_h v_h^{(0)}\|_{H^2(\mathbb{R}^2)}^2\right) \\
&\times (\|v_h^{(n)}\|_{H^2}^2 + \|v_3^{(n)}\|_{H^2}^2 + \|\rho^{(n)}\|_{H^2}^2) \\
&+\; C\,\nu^{-3}(\|\nabla_h v_h^{(n)}\|_{H^2}^2 + \|\nabla_h v_3^{(n)}\|_{L^2}^2)(\|v_h^{(n)}\|_{H^2}^2 + \|v_3^{(n)}\|_{H^2}^2)^2 \\
&+\; C\,\nu^{-1}\kappa^{-2}(\|\nabla_h v_h^{(n)}\|_{H^2}^2 + \|\nabla_h \rho^{(n)}\|_{L^2}^2)(\|v_h^{(n)}\|_{H^2}^2 + \|\rho^{(n)}\|_{H^2}^2)^2. \qquad (3.2.11)
\end{aligned}
$$

The above inequality (3.2.11) can be written as

$$
\begin{aligned}
&\frac{d}{dt}(\|v_h^{(n)}\|_{H^2}^2 + \|v_3^{(n)}\|_{H^2}^2 + \|\rho^{(n)}\|_{H^2}^2) + \nu\|\nabla_h v_h^{(n)}\|_{H^2}^2 + \nu\|\nabla_h v_3^{(n)}\|_{H^2}^2 + \kappa\|\nabla_h \rho^{(n)}\|_{H^2}^2 \\
&\leq\; C\,(\nu^{-1} + \kappa^{-1})(\|v_h^{(0)}\|_{H^2(\mathbb{R}^2)}^2 + \|\nabla_h v_h^{(0)}\|_{H^2(\mathbb{R}^2)}^2) \\
&\times (\|v_h^{(n)}\|_{H^2}^2 + \|v_3^{(n)}\|_{H^2}^2 + \|\rho^{(n)}\|_{H^2}^2) \\
&+\; C\left(\nu^{-3}(\|\nabla_h v_h^{(n)}\|_{H^2}^2 + \|\nabla_h v_3^{(n)}\|_{L^2}^2) + \nu^{-1}\kappa^{-2}(\|\nabla_h v_h^{(n)}\|_{L^2}^2 + \|\nabla_h \rho^{(n)}\|_{L^2}^2\right) \\
&\times (\|v_h^{(n)}\|_{H^2} + \|v_3^{(n)}\|_{H^2} + \|\rho^{(n)}\|_{H^2})^4. \qquad (3.2.12)
\end{aligned}
$$

Assuming

$$
\begin{aligned}
a(t) :=&\quad C\,(\nu^{-1} + \kappa^{-1})(\|v_h^{(0)}\|_{H^2(\mathbb{R}^2)}^2 + \|\nabla_h v_h^{(0)}\|_{H^2(\mathbb{R}^2)}^2), \\
A(t) :=&\quad e^{-\int_0^t a(\tau)d\tau}(\|v_h^{(n)}\|_{L^2}^2 + \|v_3^{(n)}\|_{H^2}^2 + \|\rho^{(n)}\|_{H^2}^2), \qquad (3.2.13)
\end{aligned}
$$

we can convert (3.2.11) into the following inequality

$$
\begin{aligned}
\frac{d}{dt}A(t) \quad&\leq\; C\left(\nu^{-3}(\|\nabla_h v_h^{(n)}\|_{H^2}^2 + \|\nabla_h v_3^{(n)}\|_{L^2}^2) + \nu^{-1}\kappa^{-2}(\|\nabla_h v_h^{(n)}\|_{L^2}^2 + \|\nabla_h \rho^{(n)}\|_{L^2}^2\right)e^{\int_0^t a(\tau)d\tau}A^2(t) \\
&\leq\; C\,K_0 A^2(t)\,C\left(\nu^{-3}(\|\nabla_h v_h^{(n)}\|_{H^2}^2 + \|\nabla_h v_3^{(n)}\|_{L^2}^2) + \nu^{-1}\kappa^{-2}(\|\nabla_h v_h^{(n)}\|_{L^2}^2 + \|\nabla_h \rho^{(n)}\|_{L^2}^2\right),
\end{aligned}
$$

where K_0 is defined as before. Integrating in time yields

$$-\frac{1}{A(t)} + \frac{1}{A(0)} = L(t), \tag{3.2.14}$$

where,

$$L(t) := C\,K_0 \int_0^t \Big(\nu^{-3}(\|\nabla_h v_h^{(n)}\|_{H^2}^2 + \|\nabla_h v_3^{(n)}\|_{L^2}^2) + \nu^{-1}\kappa^{-2}(\|\nabla_h v_h^{(n)}\|_{L^2}^2 + \|\nabla_h \rho^{(n)}\|_{L^2}^2\Big) d\tau.$$

To estimate $L(t)$, we choose the sufficiently large integer n_0 independent of n and define

$$v_F = e^{\nu\Delta_h t}\mathbb{E}_{n_0}(v_{h0} - v_{h0}^{(0)}), \quad v_{3F} = e^{\nu\Delta_h t}\mathbb{E}_{n_0}(v_{30}), \quad \rho_F = e^{\nu\Delta_h t}\mathbb{E}_{n_0}(\rho_0).$$

It is not difficult to prove that the v_F and v_{3F} satisfies the divergence-free condition

$$\nabla_h \cdot v_F + \partial_3 v_{3F} = 0. \tag{3.2.15}$$

Using the divergence-free conditions $\nabla_h \cdot v_{h0} + \partial_3 v_{30} = 0$ and on $\nabla_h \cdot v_{h0}^{(0)}$, we have

$$\begin{aligned}
\nabla_h \cdot v_F + \partial_3 v_{3F} &= \nabla_h \cdot e^{\nu\Delta_h t}\mathbb{E}_{n_0}(v_{h0} - v_{h0}^{(0)}) + \partial_3 e^{\nu\Delta_h t}\mathbb{E}_{n_0}(\partial_3 v_{30}) \\
&= e^{\nu\Delta_h t}\mathbb{E}_{n_0}(\nabla_h \cdot (v_{h0} - v_{h0}^{(0)})) + e^{\nu\Delta_h t}\mathbb{E}_{n_0}(\partial_3 v_{30}) \\
&= e^{\nu\Delta_h t}\mathbb{E}_{n_0}(\nabla_h \cdot v_{h0} + \partial_3 v_{30}) = 0. \qquad (3.2.16)
\end{aligned}$$

Now we split $v_h^{(n)}, v_3^{(n)}$ and $\rho^{(n)}$ into the following two parts

$$v_h^{(n)} = u_h^{(n)} + v_F, \quad v_3^{(n)} = u_3^{(n)} + v_{3F}, \quad \rho^{(n)} = \theta^{(n)} + \rho_F.$$

Then, for $n \geq n_0$, $(u_h^{(n)},\, u_3^{(n)},\, \theta^{(n)})$ solves

$$\begin{cases} \partial_t u_h^{(n)} + \mathbb{E}_n\Big(v_h^{(n)} \cdot \nabla_h v_h^{(n)} + v_h^{(n)} \cdot \nabla_h v_h^{(0)} + v_h^{(0)} \cdot \nabla_h v_h^{(n)} + v_3^{(n)} \partial_3 v_h^{(n)} + f(v_h^{\perp})^{(n)}\Big) \\ = -\nabla_h p^{(n)} + \nu\, \Delta_h u_h^{(n)}, \\ \partial_t u_3^{(n)} + \mathbb{E}_n\Big(v_h^{(n)} \cdot \nabla_h v_3^{(n)} + v_h^{0} \cdot \nabla_h v_3^{(n)} + v_3^{(n)} \partial_3 v_3^{(n)}\Big) = -\partial_3 p^{(n)} + \nu\, \Delta_h u_3^{(n)} - \rho^{(n)}, \\ \partial_t \theta^{(n)} + \mathbb{E}_n\Big(v_h^{(n)} \cdot \nabla_h \rho^{(n)} + v_h^{0} \cdot \nabla_h \rho^{(n)} + v_3^{(n)} \partial_3 \rho^{(n)}\Big) = \kappa\, \Delta_h \theta^{(n)} + v_3^{(n)}, \\ \nabla_h \cdot v_h^{(n)} + \partial_3 v_3^{(n)} = 0, \quad (u_h^{(n)},\, u_3^{(n)},\, \theta^{(n)})|_{t=0} = (\mathbb{E}_n - \mathbb{E}_{n0})(v_{h0} - v_{h0}^{(0)},\, v_{30},\, \rho_0). \end{cases} \tag{3.2.17}$$

Taking the inner product of (3.2.17) with $(u_h^{(n)},\, u_3^{(n)},\, \theta^{(n)})$ we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|u_h^{(n)}\|_{L^2}^2 + \|u_3^{(n)}\|_{L^2}^2 + \|\theta^{(n)}\|_{L^2}^2) + \nu\|\nabla_h u_h^{(n)}\|_{L^2}^2 + \nu\|\nabla_h u_3^{(n)}\|_{L^2}^2 + \kappa\|\nabla_h \theta^{(n)}\|_{L^2}^2 \\ &= -\int u_h^{(n)} \cdot \mathbb{E}_n\Big(v_h^{(n)} \cdot \nabla_h v_h^{(n)} + v_h^{(n)} \cdot \nabla_h v_h^{(0)} + v_h^{(0)} \cdot \nabla_h v_h^{(n)} + v_3^{(n)} \partial_3 v_h^{(n)} + f(v_h^{\perp})^{(n)}\Big) dx \\ &- \int u_3^{(n)} \cdot \mathbb{E}_n\Big(v_h^{(n)} \cdot \nabla_h v_3^{(n)} + v_h^{(0)} \cdot \nabla_h v_3^{(n)} + v_3^{(n)} \partial_3 v_3^{(n)}\Big) dx \\ &- \int \theta^{(n)} \cdot \mathbb{E}_n\Big(v_h^{(n)} \cdot \nabla_h \rho^{(n)} + v_h^{0} \cdot \nabla_h \rho^{(n)} + v_3^{(n)} \partial_3 \rho^{(n)}\Big) dx \\ &- \int (q^{(n)} \rho^{(n)} - v_3^{(n)} \theta^{(n)}) dx, \qquad (3.2.18) \\ &= L_1 + L_2 + L_3 + L_4, \qquad (3.2.19) \end{aligned}$$

where we have used the divergence free condition $\nabla_h \cdot u_h + \partial_3 u_3 = 0$ to eliminate the pressure term. To estimate the terms on the right-hand side of (3.2.19), we first notice that

$$\begin{aligned} \|u_h^{(n)}\|_{L^2} &\leq \|v_h^{(n)}\|_{L^2} + \|v_F\|_{L^2} \\ &\leq A_0(\|u_{h0} - v_{h0}^{(0)}\|_{L^2} + \|v_{30}\|_{L^2} + \|\rho_0\|_{L^2}) + \|u_{h0} - v_{h0}^{(0)}\|_{L^2} \leq M_0, \end{aligned}$$

where

$$M_0 := (A_0 + 1)(\|u_{h0} - v_{h0}^{(0)}\|_{L^2} + \|v_{30}\|_{L^2} + \|\rho_0\|_{L^2}).$$

Similarly,

$$\|u_3^{(n)}\|_{L^2} \leq M_0, \quad \|\theta^{(n)}\|_{L^2} \leq M_0.$$

We assume that $n \geq n_0$. Clearly, we have

$$\mathbb{E}_n u_h^{(n)} = u_h^{(n)}, \quad \mathbb{E}_n q^{(n)} = q^{(n)}, \quad \mathbb{E}_n \theta^{(n)} = \theta^{(n)}. \tag{3.2.20}$$

To estimate L_1, using the properties of $\mathbb{E}_n$, we first it as

$$\begin{aligned}
L_1 =& -\int u_h^{(n)} \cdot \mathbb{E}_n\Big(v_h^{(n)} \cdot \nabla_h v_h^{(n)} + v_h^{(n)} \cdot \nabla_h v_h^{(0)} + v_h^{(0)} \cdot \nabla_h v_h^{(n)} + v_3^{(n)} \partial_3 v_h^{(n)} + f(v_h^{\perp})^{(n)}\Big) dx \\
=& -\int u_h^{(n)} \cdot \Big(v_h^{(n)} \cdot \nabla_h v_h^{(n)} + v_h^{(n)} \cdot \nabla_h v_h^{(0)} + v_h^{(0)} \cdot \nabla_h v_h^{(n)} + v_3^{(n)} \partial_3 v_h^{(n)} + f(v_h^{\perp})^{(n)}\Big) dx \\
=& -\int u_h^{(n)} \cdot \Big(v_h^{(n)} \cdot \nabla_h u_h^{(n)} + v_3^{(n)} \partial_3 u_h^{(n)}\Big) + u_h^{(n)} \cdot \Big(v_h^{(n)} \cdot \nabla_h v_F + v_3^{(n)} \partial_3 v_F\Big) dx \\
& -\int u_h^{(n)} \cdot \Big(u_h^{(n)} \cdot \nabla_h v_h^{(0)} + v_F \cdot \nabla_h v_h^{(0)}\Big) dx \\
& -\int u_h^{(n)} \cdot \Big(v_h^{(0)} \cdot \nabla_h u_h^{(n)} + v_h^{(0)} \cdot \nabla_h v_F + f(u_h^{(n)})^{\perp} + f(v_F)^{\perp})\Big) dx \\
=& -\int u_h^{(n)} \cdot \Big(v_h^{(n)} \cdot \nabla_h v_F + v_3^{(n)} \partial_3 v_F\Big) dx - \int u_h^{(n)} \cdot \Big(u_h^{(n)} \cdot \nabla_h u_h^{(0)} + v_F \cdot \nabla_h u_h^{(0)}\Big) dx \\
& -\int u_h^{(n)} \cdot \Big(v_h^{(0)} \cdot \nabla_h v_F + f(v_F)^{\perp})\Big) dx \\
=& \, L_{11} + L_{12} + L_{13}.
\end{aligned} \tag{3.2.21}$$

In the above calculations we have used the following facts,

$$-\int u_h^{(n)} \cdot \Big(v_h^{(n)} \cdot \nabla_h u_h^{(n)} + v_3^{(n)} \partial_3 u_h^{(n)}\Big) = -\int u_h^{(n)} \cdot \Big(v_h^{(0)} \cdot \nabla_h u_h^{(n)}\Big) = 0, \quad u_h^{(n)} \cdot (u_h^{(n)})^{\perp} = 0,$$
$$\|\nabla_h v_F\|_{L^\infty} \leq C\, n_0^{\frac{5}{2}} \|u_{h0}\|_{L^2}, \quad \|\partial_3 v_F\|_{L^2_{x_3} L^\infty_h} \leq C\, n_0^2 \|u_{h0}\|_{L^2}, \quad \|v_F\|_{L^2_{x_3} L^\infty_h} \leq C\, n_0 \|u_{h0}\|_{L^2}.$$

We can estimate L_{11} as

$$
\begin{aligned}
|L_{11}| &= \Big| \int u_h^{(n)} \cdot \Big(v_h^{(n)} \cdot \nabla_h v_F + v_3^{(n)} \partial_3 v_F \Big) dx \Big| \\
&\le \|u_h^{(n)}\|_{L^2} \|v_h^{(n)}\|_{L^2} \|\nabla_h v_F\|_{L^\infty} + \|u_h^{(n)}\|_{L^2} \|v_3^{(n)}\|_{L^2} \|\partial_3 v_F\|_{L^\infty} \\
&\le C\, n_0^{\frac{5}{2}} M_0^3,
\end{aligned}
\tag{3.2.22}
$$

where we have used the following bound using Lemma (3.2.1)

$$
\|\nabla_h v_F\|_{L^\infty} \le C\, n_0^{\frac{5}{2}} \|u_{h0}\|_{L^2}.
$$

For L_{12}, we need slightly different treatment,

$$
\begin{aligned}
L_{12} &= -\int u_h^{(n)} \cdot \Big(u_h^{(n)} \cdot \nabla_h v_h^{(0)} + v_F \cdot \nabla_h v_h^{(0)} \Big) dx \\
&= -\int \Big(u_h^{(n)} \cdot (u_h^{(n)} \cdot \nabla_h v_h^{(0)}) - v_F \cdot \nabla_h u_n^{(n)} v_h^{(0)} - u_n^{(n)} \cdot \nabla_h \cdot v_F v_h^{(0)} \Big) dx \\
&\le \|u_h^{(n)}\|_{L^2_{x_3} L^4_h}^2 \|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)} + \|\nabla_h u_h^{(n)}\|_{L^2} \|v_h^{(0)}\|_{L^2(\mathbb{R}^2)} \|v_F\|_{L^2_{x_3} L^\infty_h} + \|u_h^{(n)}\|_{L^2} \|v_h^{(0)}\|_{L^2(\mathbb{R}^2)} \|\nabla_h \cdot v_F\|_{L^2_{x_3} L^\infty_h} \\
&\le \|u_h^{(n)}\|_{L^2} \|\nabla_h u_h^{(n)}\|_{L^2} \|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)} + C\, n_0 M_0 \|\nabla_h u_h^{(n)}\|_{L^2} \|v_h^{(0)}\|_{L^2(\mathbb{R}^2)} + C\, n_0^2 M_0^2 \|v_h^{(0)}\|_{L^2(\mathbb{R}^2)} \\
&\le \frac{\nu}{4} \|\nabla_h u_h^{(n)}\|_{L^2} + C\, \nu^{-1} \|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)} \|u_h^{(n)}\|_{L^2} + C\, n_0^2 M_0^2 (\|v_h^{(0)}\|_{L^2(\mathbb{R}^2)} + \nu^{-1} \|v_h^{(0)}\|_{L^2(\mathbb{R}^2)}^2),
\end{aligned}
\tag{3.2.23}
$$

where we have used the following bounds using Lemma (3.2.1)

$$
\|\nabla_h v_F\|_{L^\infty} \le C\, n_0^{\frac{5}{2}} \|u_{h0}\|_{L^2}, \quad \|\partial_3 v_F\|_{L^2_{x_3} L^\infty_h} \le C\, n_0^2 \|u_{h0}\|_{L^2}, \quad \|v_F\|_{L^2_{x_3} L^\infty_h} \le C\, n_0 \|u_{h0}\|_{L^2}.
$$

As in L_{11}, we can estimate L_{13} as

$$
\begin{aligned}
|L_{13}| &= \Big| \int u_h^{(n)} \cdot \Big(v_h^{(0)} \cdot \nabla_h v_F + f (v_F)^\perp) \Big) dx \Big| \\
&\le C\, n_0^2 M_0^3 + C M_0^2,
\end{aligned}
\tag{3.2.24}
$$

where we have used the following bound using Lemma (3.2.1)

$$\|\nabla_h v_F\|_{L^\infty} \le C n_0^{\frac{5}{2}} \|u_{h0}\|_{L^2}.$$

Substituting (3.2.22), (3.2.23) and (3.2.24) in (3.2.21), we have

$$\begin{aligned} |L_1| \le\ & C\, n_0^{\frac{5}{2}} M_0^3 + \frac{\nu}{4}\|\nabla_h u_h^{(n)}\|_{L^2} + C\,\nu^{-1}\|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)}\|u_h^{(n)}\|_{L^2} \\ & + C\, n_0^2 M_0^2(\|v_h^{(0)}\|_{L^2(\mathbb{R}^2)} + C\,\nu^{-1}\|v_h^{(0)}\|^2_{L^2(\mathbb{R}^2)}) + C\, n_0^2 M_0^3 + C\, M_0^2. \end{aligned} \tag{3.2.25}$$

Now we estimate L_2,

$$\begin{aligned} L_2 = & -\int u_3^{(n)} \cdot \mathbb{E}_n\Big(v_h^{(n)} \cdot \nabla_h v_3^{(n)} + v_h^{(0)} \cdot \nabla_h v_3^{(n)} + v_3^{(n)} \partial_3 v_3^{(n)}\Big) dx \\ & = -\int u_3^{(n)} \cdot \Big(v_h^{(n)} \cdot \nabla_h v_3^{(n)} + v_h^{(0)} \cdot \nabla_h v_3^{(n)} + v_3^{(n)} \partial_3 v_3^{(n)}\Big) dx \\ & = -\int u_3^{(n)} \cdot \Big(v_h^{(n)} \cdot \nabla_h u_3^{(n)} + v_3^{(n)} \partial_3 u_3^{(n)}\Big) + u_3^{(n)} \cdot \Big(v_h^{(n)} \cdot \nabla_h v_{3F} + v_3^{(n)} \partial_3 v_{3F}\Big) dx \\ & - \int \Big(u_3^{(n)} \cdot v_h^{(0)} \cdot \nabla_h u_3^{(n)} + u_3^{(n)} \cdot v_h^{(0)} \cdot \nabla_h v_{3F}\Big) dx \\ & = -\int u_3^{(n)} \cdot \Big(v_h^{(n)} \cdot \nabla_h v_F + v_3^{(n)} \partial_3 v_{3F}\Big) dx - \int \Big(u_3^{(n)} \cdot v_h^{(0)} \cdot \nabla_h u_3^{(n)}\Big) dx - \int \Big(u_3^{(n)} \cdot v_h^{(0)} \cdot \nabla_h v_{3F}\Big) dx \\ & \le \|u_3^{(n)}\|_{L^2}\|v_h^{(n)}\|_{L^2}\|\nabla_h v_{3F}\|_{L^\infty} + \|u_3^{(n)}\|_{L^2}\|v_3{}^{(n)}\|_{L^2}\|\partial_3 v_{3F}\|_{L^\infty} \\ & + \|u_3^{(n)}\|_{L^2}\|v_h^{(0)}\|_{L^2}\|\nabla_h u_3^{(n)}\|_{L^2_{x_3}L^\infty_h} + \|u_3^{(n)}\|_{L^2}\|v_h^{(0)}\|_{L^2}\|\nabla_h v_{3F}\|_{L^2_{x_3}L^\infty_h} \\ & \le C\, n_0^{\frac{5}{2}} M_0^3 + C\, n_0^2 M_0^2 \|v_h^{(0)}\|_{L^2(\mathbb{R}^2)}. \end{aligned} \tag{3.2.26}$$

In the above calculations, we have used the following facts,

$$-\int u_3^{(n)} \cdot \Big(v_h^{(n)} \cdot \nabla_h u_3^{(n)} + v_3^{(n)} \partial_3 u_3^{(n)}\Big) = 0,$$

$$\|\nabla_h v_F\|_{L^\infty} \le C\, n_0^{\frac{5}{2}}\|u_{h0}\|_{L^2} \quad \|\partial_3 v_F\|_{L^2_{x_3}L^\infty_h} \le C\, n_0^2\|u_{h0}\|_{L^2}, \quad \|v_F\|_{L^2_{x_3}L^\infty_h} \le C\, n_0\|u_{h0}\|_{L^2}.$$

Similarly, we estimate L_3,

$$
\begin{aligned}
L_3 &= -\int \theta^{(n)} \cdot \mathbb{E}_n\Big(v_h^{(n)} \cdot \nabla_h \rho^{(n)} + v_h^0 \cdot \nabla_h \rho^{(n)} + v_3^{(n)} \partial_3 \rho^{(n)}\Big) dx \\
&= -\int \theta^{(n)} \cdot \Big(v_h^{(n)} \cdot \nabla_h \theta^{(n)} + v_3^{(n)} \partial_3 \theta^{(n)}\Big) + \theta^{(n)} \cdot \Big(v_h^{(n)} \cdot \nabla_h \rho_F + v_3^{(n)} \partial_3 \rho_F\Big) dx \\
&- \int \Big(\theta^{(n)} \cdot v_h^{(0)} \cdot \nabla_h \theta^{(n)} + \theta^{(n)} \cdot v_h^{(0)} \cdot \nabla_h \rho_F\Big) dx \\
&= -\int \theta^{(n)} \cdot \Big(v_h^{(n)} \cdot \nabla_h \rho_F + v_3^{(n)} \partial_3 \rho_F\Big) dx - \Big(\theta^{(n)} \cdot v_h^{(0)} \cdot \nabla_h \rho_F\Big) dx \\
&\leq \|\theta^{(n)}\|_{L^2} \|v_h^{(n)}\|_{L^2} \|\nabla_h \rho_F\|_{L^\infty} + \|\theta^{(n)}\|_{L^2} \|v_3^{(n)}\|_{L^2} \|\partial_3 \rho_F\|_{L^\infty} \\
&+ \|\theta^{(n)}\|_{L^2} \|v_h^{(0)}\|_{L^2} \|\nabla_h \rho_F\|_{L^2_{x_3} L^\infty_h} \\
&\leq C\, n_0^{\frac{5}{2}} M_0^3 + C\, n_0^2 M_0^2 \|v_h^{(0)}\|_{L^2(\mathbb{R}^2)}. \qquad (3.2.27)
\end{aligned}
$$

In the above calculations, we have used the following facts,

$$
\begin{aligned}
&-\int u_3^{(n)} \cdot \Big(v_h^{(n)} \cdot \nabla_h \theta^{(n)} + v_3^{(n)} \partial_3 \theta^{(n)}\Big) = 0, \\
&\|\nabla_h v_F\|_{L^\infty} \leq C\, n_0^{\frac{5}{2}} \|u_{h0}\|_{L^2} \quad \|\partial_3 v_F\|_{L^2_{x_3} L^\infty_h} \leq C\, n_0^2 \|u_{h0}\|_{L^2}, \quad \|v_F\|_{L^2_{x_3} L^\infty_h} \leq C\, n_0 \|u_{h0}\|_{L^2}.
\end{aligned}
$$

The above inequalities are due to Lemma (3.2.1) on the Bernstein inequality.

Combining (3.2.21), (3.2.21) and (3.2.25), we can write (3.2.19) as

$$
\begin{aligned}
&\frac{d}{dt}(\|u_h^{(n)}\|_{L^2}^2 + \|u_3^{(n)}\|_{L^2}^2 + \|\theta^{(n)}\|_{L^2}^2) + \nu \|\nabla_h u_h^{(n)}\|_{L^2}^2 \qquad (3.2.28) \\
&+\nu \|\nabla_h u_3^{(n)}\|_{L^2}^2 + \kappa \|\nabla_h \theta^{(n)}\|_{L^2}^2 \leq C\, \nu^{-1} \|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)} \|u_h^{(n)}\|_{L^2} \\
&+ C\, n_0^2 M_0^2 (\|v_h^{(0)}\|_{L^2(\mathbb{R}^2)} + C\nu^{-1} \|v_h^{(0)}\|_{L^2(\mathbb{R}^2)}^2) + C\, n_0^{\frac{5}{2}} M_0^3 + C\, n_0^2 M_0^2 \|v_h^{(0)}\|_{L^2(\mathbb{R}^2)} + C\, M_0^2.
\end{aligned}
$$

The upper bound we obtained above depends on n_0 and the initial data, which is independent of n. This fact is vital to obtaining a time interval independent of n. We write (3.2.28) as

$$\frac{d}{dt}(\|u_h^{(n)}\|_{L^2}^2 + \|q^{(n)}\|_{L^2}^2 + \|\theta^{(n)}\|_{L^2}^2) + \nu\|\nabla_h u_h^{(n)}\|_{L^2}^2 + \nu\|\nabla_h u_3^{(n)}\|_{L^2}^2 + \kappa\|\nabla_h \theta^{(n)}\|_{L^2}^2$$
$$\leq C\nu^{-1}\|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)}(\|u_h^{(n)}\|_{L^2} + \|u_3^{(n)}\|_{L^2} + \|\theta^{(n)}\|_{L^2}) + Q_0,$$

where

$$Q_0 : \quad = C\, n_0^2 M_0^2(\|u_h^{(0)}\|_{L^2(\mathbb{R}^2)} + C\,\nu^{-1}\|u_h^{(0)}\|_{L^2(\mathbb{R}^2)}^2) + C\, n_0^{\frac{5}{2}} M_0^3 + C n_0^2 M_0^2 \|v_h^{(0)}\|_{L^2(\mathbb{R}^2)} + C\, M_0^2.$$

Then using Gronwall's inequality we have

$$\|u_h^{(n)}(t)\|_{L^2}^2 + \|u_3^{(n)}(t)\|_{L^2}^2 + \|\theta^{(n)}(t)\|_{L^2}^2 + \nu\int_0^t \|\nabla_h u_h^{(n)}\|_{L^2}^2 d\tau + \nu\int_0^t \|\nabla_h u_3^{(n)}\|_{L^2}^2 d\tau + \kappa\int_0^t \|\nabla_h \theta^{(n)}\|_{L^2}^2 d\tau$$
$$\leq e^{C\,\nu^{-1}\int_0^t \|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)} d\tau}(\|u_h^{(n)}(0)\|_{L^2} + \|u_3^{(n)}(0)\|_{L^2} + \|\theta^{(n)}(0)\|_{L^2}) + Q_0 t),$$
$$\leq C\, A_0 (I - \mathbb{E}_{n_0})\|(\|v_{h0} - v_{h0}^{(0)}\|_{L^2} + \|v_3^{(0)}\|_{L^2} + \|\rho^{(0)}\|_{L^2}) + C\, A_0 Q_0 t, \tag{3.2.29}$$

where we have used,

$$e^{C\,\nu^{-1}\int_0^t \|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)} d\tau} \leq C\, A_0$$

and

$$(\|u_h^{(n)}(0)\|_{L^2} + \|u_3^{(n)}(0)\|_{L^2} + \|\theta^{(n)}(0)\|_{L^2})$$
$$= (\|(v_h^{(n)}(0) - v_F(0)\|_{L^2} + \|v_3^{(n)}(0) - v_{3F}(0)\|_{L^2} + \|\rho^{(n)}(0) - \rho_F(0)\|_{L^2})$$
$$\leq (I - \mathbb{E}_{n_0})\|(\|v_{h0} - v_{h0}^{(0)}\|_{L^2} + \|v_3^{(0)}\|_{L^2} + \|\rho^{(0)}\|_{L^2})$$

Now we estimate $L(t)$, namely

$$L(t) := C\, K_0 \int_0^t \Big(\nu^{-3}(\|\nabla_h v_h^{(n)}\|_{H^2}^2 + \|\nabla_h v_3^{(n)}\|_{L^2}^2) + \nu^{-1}\kappa^{-2}(\|\nabla_h v_h^{(n)}\|_{L^2}^2 + \|\nabla_h \rho^{(n)}\|_{L^2}^2)\Big) d\tau$$

Clearly, by the definition of $v_F = e^{\nu\Delta_h t}\mathbb{E}_{n0}(u_{h0} - u_{h0}^{(0)})$,

$$\int_0^t \|\nabla_h v_F\|_{L^2}^2 \le n_0^2 \int_0^t \|\mathbb{E}_{n0}(u_{h0} - u_{h0}^{(0)})\|_{L^2}^2 d\tau \le n_0^2 \|u_{h0} - u_{h0}^{(0)}\|_{L^2}^2 t.$$

Similarly,

$$\int_0^t \|\nabla_h v_{3F}\|_{L^2}^2 \le n_0^2 \|v_{30}\|_{L^2}^2 t, \quad \int_0^t \|\nabla_h \rho_F\|_{H^2}^2 \le n_0^2 \|\rho_0\|_{L^2}^2 t.$$

Therefore, we can bound $L(t)$ by

$$\begin{aligned}
L(t): \quad &= C\,K_0 \int_0^t \Big(\nu^{-3}(\|\nabla_h v_h^{(n)}\|_{H^2}^2 + \|\nabla_h v_3^{(n)}\|_{L^2}^2) + \nu^{-1}\kappa^{-2}(\|\nabla_h v_h^{(n)}\|_{L^2}^2 + \|\nabla_h \rho^{(n)}\|_{L^2}^2)\Big) d\tau \\
&\le C\,K_0(\nu^{-3} + \nu^{-1}\kappa^{-2})\Big(\int_0^t (\|\nabla_h u_h^{(n)}\|_{H^2}^2 + \|\nabla_h u_3^{(n)}\|_{H^2}^2 + \|\nabla_h \theta_h^{(n)}\|_{H^2}^2) \\
&+ \int_0^t (\|\nabla_h v_F\|_{H^2}^2 + \|\nabla_h v_{3F}\|_{H^2}^2 + \|\nabla_h v_F\|_{H^2}^2) d\tau\Big) \\
&\le C\,K_0^2(\nu^{-4} + \nu^{-1}\kappa^{-3})\|(I - \mathbb{E}_{n_0})(\|v_{h0} - v_{h0}^{(0)}\|_{L^2}^2 + \|v_3^{(0)}\|_{L^2}^2 + \|\rho^{(0)}\|_{L^2}^2) \\
&+ C\,K_0(\nu^{-3} + \nu^{-1}\kappa^{-2}) n_0^2 \Big(\|v_{h0} - v_{h0}^{(0)}\|_{L^2}^2 + \|v_3^{(0)}\|_{L^2}^2 + \|\rho^{(0)}\|_{L^2}^2 + K_0 Q_0\Big) t.
\end{aligned}$$

In order to obtain an upper bound for $L(t)$, we recall (3.2.13) to get

$$A(0) = \|\mathbb{E}_n(v_{h0} - v_{h0}^{(0)})\|_{H^2}^2 + \|\mathbb{E}_n(v_3^{(0)})\|_{H^2}^2 + \|\mathbb{E}_n(\rho^{(0)})\|_{H^2}^2$$

If we choose n_0 sufficiently large and $t \le T$ for sufficiently small $T > 0$, then the upper bound for $L(t)$ in (3.2.30) can be made sufficiently small so that

$$1 - L(t)A(0) \ge \frac{1}{2}, \quad 0 < t \le T.$$

Then using (3.2.14), for any $0 < t \le T$,

$$\frac{1}{A(t)} = \frac{1 - L(t)A(0)}{A(0)} \ge \frac{1}{2A(0)} \tag{3.2.30}$$

or

$$A(t) \leq 2A(0) \leq 2(\|v_{h0} - v_{h0}^{(0)}\|_{H^2}^2 + \|v_3^{(0)}\|_{H^2}^2 + \|\rho^{(0)}\|_{H^2}^2)$$

Then by definition of $L(t)$ in (3.2.13)

$$\|v_h^{(n)}\|_{H^2}^2 + \|v_3^{(n)}\|_{H^2}^2 + \|\rho^{(n)}\|_{H^2}^2 = e^{\int_0^t a(\tau)d\tau} A(t) \leq 2K_0(\|v_{h0} - v_{h0}^{(0)}\|_{H^2}^2 + \|v_3^{(0)}\|_{H^2}^2 + \|\rho^{(0)}\|_{H^2}^2).$$

Integrating (3.2.11) in time yields the upper bound, for any $t \in [0, T]$,

$$\begin{aligned} &\int_0^t \nu\|\nabla_h v_h^{(n)}\|_{H^2}^2 + \int_0^t \nu\|\nabla_h v_3^{(n)}\|_{H^2}^2 + \int_0^t \kappa\|\nabla_h \rho^{(n)}\|_{H^2}^2 \\ &\leq C\,(\|v_{h0} - v_{h0}^{(0)}\|_{H^2}^2 + \|v_3^{(0)}\|_{H^2}^2 + \|\rho^{(0)}\|_{H^2}^2). \end{aligned} \tag{3.2.31}$$

Thus, we have shown that there is $T > 0$ independent of n such that

$$(v_h^{(n)},\, v_3^{(n)},\, \rho^{(n)}) \in L^\infty(0, T; H^2) \cap L^2(0, T; H^3)$$

with its norm in the above space bounded uniformly in terms of n.

We now show that $(v_h^{(n)},\, v_3^{(n)},\, \rho^{(n)})$ has a convergent subsequence whose limit solves (3.0.1). Due to Banach-Alaoglu theorem, there exists a subsequence, which is still denoted by $(v_h^{(n)},\, v_3^{(n)},\, \rho^{(n)})$, and $(v_h,\, v_3,\, \rho) \in L^\infty(0, T; H^2) \cap L^2(0, T; H^3)$ satisfying

$$\begin{aligned} &(v_h^{(n)},\, v_3^{(n)},\, \rho^{(n)}) \rightharpoonup (v_h,\, v_3,\, \rho) \quad \text{in} \quad H^2 \quad \text{for almost every} \quad t, \\ &(v_h^{(n)},\, v_3^{(n)},\, \rho^{(n)}) \rightharpoonup (v_h,\, v_3,\, \rho) \quad \text{in} \quad L^2(0, T; H^3). \end{aligned}$$

The above weak convergence are not sufficient to show that $(v_h,\, v_3,\, \rho)$ solves the Boussinesq system (3.0.1). We need to prove the strong convergence. To do so, we will show that

$$(v_h^{(n)},\, v_3^{(n)},\, \rho^{(n)}) \to (v_h,\, v_3,\, \rho), \quad \text{in} \quad L^2(0, T; L^2)$$

using the Aubin-Lions-Simon lemma (3.2.2).

Lemma 3.2.2 *(Aubin-Lions-Simon). Let X_0, X and X_1 be three Banach spaces with $X_0 \hookrightarrow X \hookrightarrow X_1$. Suppose X_0 is compactly embedding in X and X is continuously embedded in X_1. Let $1 \leq p, q \leq \infty$. Set*

$$W = \{f \in L^p(0,T;X_0) | \partial_t f \in L^q(0,T;X_1)\}.$$

1. *If $p < \infty$, then the embedding of W into $L^p(0,T;X)$ is compact.*

2. *If $p = +\infty$ and $q > 1$, then the embedding of W into $C([0,T];X)$ is compact.*

To use the Lemma (3.2.2, we need to prove

$$(\partial_t v_h^{(n)},\, \partial_t v_3^{(n)},\, \partial_t \rho^{(n)}) \in L^2(0,T;H^{-1}(\mathbb{R}^3)).$$

In the previous step, we have shown that

$$(v_h^{(n)},\, v_3^{(n)},\, \rho^{(n)}) \in L^\infty(0,T;H^2(\mathbb{R}^3)), \qquad (\nabla_h v_h^{(n)},\, \nabla_h v_3^{(n)},\, \nabla_h \rho^{(n)}) \in L^2(0,T;H^2(\mathbb{R}^3))$$

with uniform bound. Now we show that

$$\partial_t v_h^{(n)} \in L^2(0,T;H^{-1}(\mathbb{R}^3)).$$

We take $g \in H^1$ with $\|g\|_{H^1} = 1$, then using the first equation of regularized system (3.2.3), we have

$$\begin{aligned}
\int \partial_t v_h^{(n)} \cdot g\,dx \qquad &= \int -\Big(\mathbb{E}_n(v_h^{(n)} \cdot \nabla_h v_h^{(n)} + v_h^{(n)} \cdot \nabla_h v_h^{(0)} + v_h^{(0)} \cdot \nabla_h v_h^{(n)} \\
&+ v_3^{(n)} \partial_3 v_h^{(n)} + f(v_h^{\perp})^{(n)}) - \nabla_h p^{(n)} + \nu\,\Delta_h v_h^{(n)}\Big) \cdot g\,dx \\
&= \int -\mathbb{E}_n\Big(v_h^{(n)} \cdot \nabla_h v_h^{(n)}\Big) \cdot g dx - \int \mathbb{E}_n\Big(v_h^{(n)} \cdot \nabla_h v_h^{(0)}\Big) \cdot g\,dx \\
&- \int \mathbb{E}_n\Big(v_h^{(0)} \cdot \nabla_h v_h^{(n)}\Big) \cdot g dx - \int \mathbb{E}_n\Big(v_3^{(n)} \partial_3 v_h^{(n)}\Big) \cdot g\,dx \\
&- \int \mathbb{E}_n\Big(f(v_h^{\perp})^{(n)}\Big) \cdot g dx - \int \nabla_h p^{(n)} \cdot g\,dx - \int \nu\,\Delta_h v_h^{(n)} \cdot g\,dx \\
&= K_1 + \cdots + K_7.
\end{aligned}$$

Note that

$$K_1 = \int -\mathbb{E}_n(v_h^{(n)} \cdot \nabla_h v_h^{(n)}) \cdot g\,dx = -\int (v_h^{(n)} \cdot \nabla_h v_h^{(n)}) \cdot g\,dx.$$

Holder's and Sobolev inequality yield

$$\begin{aligned}
|K_1| \qquad &\leq \|v_h^{(n)}\|_{L^3} \|\nabla_h v_h^{(n)}\|_{L^3} \|g\|_{L^3} \\
&\leq C\,\|v_h^{(n)}\|_{L^2}^{\frac{1}{2}} \|\nabla v_h^{(n)}\|_{L^2}^{\frac{1}{2}} \|\nabla_h v_h^{(n)}\|_{L^2}^{\frac{1}{2}} \|\nabla\nabla_h v_h^{(n)}\|_{L^2}^{\frac{1}{2}} \|g\|_{L^2}^{\frac{1}{2}} \|\nabla g\|_{L^2}^{\frac{1}{2}} \\
&\leq C\,\|v_h^{(n)}\|_{H^1} \|\nabla_h v_h^{(n)}\|_{H^1} \|g\|_{H^1} \leq C \|v_h^{(n)}\|_{H^2} \|\nabla_h v_h^{(n)}\|_{H^1} \|g\|_{H^1}.
\end{aligned}$$

First, we write K_2 as

$$K_2 = \int -\mathbb{E}_n(v_h^{(n)} \cdot \nabla_h v_h^{(0)}) \cdot g\,dx = -\int (v_h^{(n)} \cdot \nabla_h v_h^{(0)}) \cdot \mathbb{E}_n g\,dx.$$

Applying Hölder's inequality and Ladyzhenskaya's inequality yields,

$$\begin{aligned}|K_2| &\leq C\,\|v_h^{(n)}\|_{L^2}^{\frac{1}{2}}\|\nabla_h v_h^{(n)}\|_{L^2}^{\frac{1}{2}}\|\nabla_h v_h^{(0)}\|_{L^2}\|g\|_{L^2}^{\frac{1}{2}}\|\nabla_h g\|_{L^2}^{\frac{1}{2}} \\ &\leq C\,\|v_h^{(n)}\|_{H^1}\|\nabla_h v_h^{(0)}\|_{L^2}\|g\|_{H^1} \leq C\,\|v_h^{(n)}\|_{H^2}\|\nabla_h v_h^{(0)}\|_{L^2}\|g\|_{H^1}.\end{aligned}$$

For K_3, we write it as

$$K_3 = \int -\mathbb{E}_n(v_h^{(0)} \cdot \nabla_h v_h^{(n)}) \cdot g\,dx = -\int v_h^{(0)} \cdot \nabla_h v_h^{(n)} \cdot \mathbb{E}_n g\,dx.$$

Then, by Hölder's inequality and Ladyzhenskaya's inequality, we have

$$\begin{aligned}|K_3| &\leq C\,\|\nabla_h v_h^{(n)}\|_{L^2}^{\frac{1}{2}}\|\nabla_h \nabla_h v_h^{(n)}\|_{L^2}^{\frac{1}{2}}\|v_h^{(0)}\|_{L^2}\|g\|_{L^2}^{\frac{1}{2}}\|\nabla_h g\|_{L^2}^{\frac{1}{2}} \\ &\leq C\,\|\nabla_h v_h^{(n)}\|_{H^1}\|v_h^{(0)}\|_{L^2}\|g\|_{H^1} \leq C\,\|\nabla_h v_h^{(n)}\|_{H^1}\|v_h^{(0)}\|_{L^2}\|g\|_{H^1}.\end{aligned}$$

Now, for K_4, we write it as

$$K_4 = \int -\mathbb{E}_n(v_3^{(n)} \cdot \partial_3 v_h^{(n)}) \cdot g\,dx = \int -v_3^{(n)} \cdot \partial_3 v_h^{(n)} \cdot g\,dx.$$

Then, by Hölder's inequality and Sobolev inequality, we have

$$\begin{aligned}|K_4| &\leq \|v_3^{(n)}\|_{L^3}\|\partial_3 v_h^{(n)}\|_{L^3}\|g\|_{L^3} \\ &\leq C\,\|v_3^{(n)}\|_{L^2}^{\frac{1}{2}}\|\nabla v_3^{(n)}\|_{L^2}^{\frac{1}{2}}\|\partial_3 v_h^{(n)}\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_3 v_h^{(n)}\|_{L^2}^{\frac{1}{2}}\|g\|_{L^2}^{\frac{1}{2}}\|\nabla g\|_{L^2}^{\frac{1}{2}} \\ &\leq C\,\|v_3^{(n)}\|_{H^1}\|v_h^{(n)}\|_{H^2}\|g\|_{H^1} \leq C\,\|v_3^{(n)}\|_{H^2}\|v_h^{(n)}\|_{H^2}\|g\|_{H^1}.\end{aligned}$$

Now, for K_5,

$$K_5 = \int \mathbb{E}_n\Big(f(v_h^{\perp})^{(n)}\Big) \cdot g\,dx = \int f(v_h^{\perp})^{(n)} \cdot \mathbb{E}_n g\,dx.$$

Since f is a smooth function with compact support, we have

$$|K_5| \leq \|f(v_h^{\perp})^{(n)}\|_{L^2}\|g\|_{L^2} \leq C\,\|v_h^{(n)}\|_{L^2}\|g\|_{H^1}.$$

For K_6, by using the integration by parts, we have

$$K_6 = -\int \nabla_h p^{(n)} \cdot g\,dx = \int p^{(n)} \cdot \nabla_h g\,dx.$$

Then, by Hölder's inequality

$$|K_6| \leq C\,\|p^{(n)}\|_{L^2}\|g\|_{H^1}.$$

For K_7, by using the integration by parts, we have

$$K_7 = \int -\nu\,\Delta_h v_h^{(n)} \cdot gdx = \int \nu\,\nabla_h v_h^{(n)} \cdot \nabla_h g dx.$$

Then, by Hölder's inequality

$$|K_7| \leq \nu\|\nabla_h v_h^{(n)}\|_{L^2}\|g\|_{H^1} \leq C\,\|\nabla_h v_h^{(n)}\|_{H^1}\|g\|_{H^1}.$$

Therefore,

$$|\int \partial_t v_h^{(n)} \cdot gdx| \leq C\,\|g\|_{H^1}\Big(\|v_h^{(n)}\|_{H^2}(\|\nabla_h v_h^{(n)}\|_{H^2} + \|\nabla_h v_h^{(0)}\|_{L^2} + \|v_h^{(0)}\|_{L^2} + \|v_3^{(n)}\|_{H^2} + 2) + \|p^{(n)}\|_{L^2}\Big)$$

Which is,

$$\|\partial_t v_h^{(n)}\|_{H^{-1}} \leq C\,\Big(\|v_h^{(n)}\|_{H^2}(\|\nabla_h v_h^{(n)}\|_{H^1} + \|\nabla_h v_h^{(0)}\|_{L^2} + \|v_h^{(0)}\|_{L^2} + \|v_3^{(n)}\|_{H^2} + 2) + \|p^{(n)}\|_{L^2}\Big).$$

Then,

$$
\begin{aligned}
&\|\partial_t v_h^{(n)}\|_{L^2(0,T;H^{-1})}^2 = \int_0^T \|\partial_t v_h^{(n)}\|_{H^{-1}}^2 dt \\
&\leq C \int_0^T \left(\|v_h^{(n)}\|_{H^2}(\|\nabla_h v_h^{(n)}\|_{H^1} + \|\nabla_h v_h^{(0)}\|_{L^2} + \|v_h^{(0)}\|_{L^2} + \|v_3^{(n)}\|_{H^2} + 2) + \|p^{(n)}\|_{L^2}\right)^2 dt \\
&\leq C \int_0^T \|v_h^{(n)}\|_{H^2}^2(\|\nabla_h v_h^{(n)}\|_{H^1} + \|\nabla_h v_h^{(0)}\|_{L^2} + \|v_h^{(0)}\|_{L^2} + \|v_3^{(n)}\|_{H^2} + 2)^2 + C \int_0^T \|p^{(n)}\|_{L^2}^2 d\tau \\
&\leq \sup_{0\leq t\leq T} \|v_h^{(n)}\|_{H^2}^2 \int_0^T (\|\nabla_h v_h^{(n)}\|_{H^1} + \|\nabla_h v_h^{(0)}\|_{L^2} + \|v_h^{(0)}\|_{L^2} + \|v_3^{(n)}\|_{H^2} + 2)^2 d\tau \\
&+ C \int_0^T \|p^{(n)}\|_{L^2}^2 dt \\
&\leq \infty.
\end{aligned}
$$

In the above calculation, we have used fact that $p^{(n)} \in L^2(0,T;L^2(\mathbb{R}^3))$ which can be seen using (3.2.1). Thus we have shown that

$$\partial_t v_h^{(n)} \in L^2(0,T;H^{-1}(\mathbb{R}^3)).$$

Similarly, we can show that

$$\partial_t w^{(n)}, \partial_t \rho^{(n)} \in L^2(0,T;H^{-1}(\mathbb{R}^3)).$$

The Aubin-Lions-Simon Lemma can not be applied to the spaces

$$H^1(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3) \hookrightarrow H^{-1}(\mathbb{R}^3) \tag{3.2.32}$$

since the above embeddings are not compact due to unbounded domain $\mathbb{R}^3$. Therefore, we take the following spaces:

$$H^1(B(0,m)) \hookrightarrow L^2(B(0,m)) \hookrightarrow H^{-1}(B(0,m)), \tag{3.2.33}$$

where $B(0,m)$ is the ball with radius $m > 0$ being an integer. In (3.2.33), the first embedding is compact, and the second is continuous. Then the Aubin-Lions-Simon Lemma says that for each positive integer m (fixed), there is a subsequence

$$(v_h^{(n_{m,k})},\, v_3^{(n_{m,k})},\, \rho^{(n_{m,k})}) \to (v_h\,, v_3\,, \rho) \in L^2(0,T;L^2(B(0,m))) \quad \text{as} \quad k \to \infty. \tag{3.2.34}$$

Then, we obtain the following desired subsequence using the Cantor diagonal process:

$$(v_h^{(n_{m,m})},\, v_3^{(n_{m,m})},\, \rho^{(n_{m,m})}) \to (v_h\,, v_3\,, \rho) \in L^2(0,T;L^2(\mathbb{R}^3)) \quad \text{as} \quad m \to \infty. \tag{3.2.35}$$

We can now show that $(v_h,\, v_3,\, \rho)$ solves the Boussinesq equation in (1.2.9) by taking the limit in (3.2.3). We will not present this part here since the process is standard. Now we combine the facts that, for any $T > 0$,

$$u \in L^2(0,T;H^2) \hookrightarrow L^2(0,T;H^1) \tag{3.2.36}$$

and $\partial_t u \in L^2(0,T;H^{-1})$ to obtain $(v_h, w, \rho) \in C([0,\infty);L^2)$ using the following Lemma (3.2.3) which can be found in [29].

Lemma 3.2.3 *Let $T > 0$. Suppose $u \in L^2(0,T;H^1(\mathbb{R}^d))$ with $\partial_t u \in L^2(0,T;H^{-1}(\mathbb{R}^d))$. Then $u \in C([0,T];L^2(\mathbb{R}^d))$.*

Finally, we prove the uniqueness. Assume that

$$(v_h,\, v_3,\, \rho), (U_h,\, V_3,\, \Theta) \in L^\infty(0,T;H^2) \cap L^2(0,T;H^3)$$

solve (1.2.9). Consider the difference $(\delta v_h,\, \delta w,\, \delta\rho)$ with

$$\delta v_h = v_h - U_h, \quad \delta v_3 = v_3 - V_3, \quad \delta\rho = \rho - \Theta.$$

Then $(\delta v_h,\, \delta w,\, \delta\rho)$ satisfies,

$$\begin{cases} \partial_t \delta v_h + \delta v_h \cdot \nabla_h v_h + U_h \cdot \nabla_h \delta v_h + \delta v_h \cdot \nabla_h v_h^{(0)} + v_h^{(0)} \cdot \nabla_h \delta v_h + \delta v_3 \partial_3 v_h \\ +V_3 \partial_3 \delta v_h + f(\delta v_h)^{\perp} = -\nabla_h \delta p + \nu\, \Delta_h \delta v_h, \\ \partial_t \delta v_3 + \delta v_h \cdot \nabla_h v_3 + U_h \cdot \nabla_h \delta v_3 + v_h^0 \cdot \nabla_h \delta v_3 + \delta v_3 \partial_3 v_3 + V_3 \partial_3 \delta v_3 = -\partial_3 \delta p + \nu\, \Delta_h \delta v_3 - \delta\rho, \\ \partial_t \delta\rho + \delta v_h \cdot \nabla_h \rho + U_h \cdot \nabla_h \delta\rho + v_h^0 \cdot \nabla_h \delta\rho + \delta v_3 \partial_3 \rho + V_3 \partial_3 \delta\rho = \kappa\, \Delta_h \delta\rho + \delta v_3, \\ \nabla_h \cdot \delta v_h + \partial_3 \delta v_3 = 0, \\ (\delta v_h,\, \delta v_3,\, \delta\rho)|_{t=0} = 0, \end{cases} \tag{3.2.37}$$

where δp denotes the corresponding pressure difference. Dotting (3.2.37) with $(\delta v_h,\, \delta v_3,\, \delta\rho)$ yields,

$$\frac{1}{2}\frac{d}{dt}(\|\delta v_h,\, \delta v_3,\, \delta\rho\|_{L^2}^2) + \nu\|\nabla_h \delta v_h\|_{L^2}^2) + \nu\|\nabla_h \delta v_3\|_{L^2}^2 + \kappa\|\nabla_h \delta\rho\|_{L^2}^2 = R_1 + \cdots + R_8, \tag{3.2.38}$$

where

$$\begin{aligned} R_1 &= -\int \delta v_h \cdot \nabla_h v_h \cdot \delta v_h \, dx, \quad R_2 = -\int \delta v_h \cdot \nabla_h v_h^{(0)} \cdot \delta v_h \, dx, \\ R_3 &= -\int \delta v_3 \partial_3 v_h \cdot \delta v_h \, dx, \quad R_4 = -\int \delta v_h \cdot \nabla_h v_3 \delta v_3 \, dx, \\ R_5 &= -\int \delta v_3 \partial_3 v_3 \delta v_3 \, dx, \quad R_6 = -\int \delta v_h \cdot \nabla_h \rho \delta\rho \, dx, \\ R_7 &= -\int \delta v_3 \partial_3 \rho \delta\rho \, dx. \end{aligned}$$

Here we have used the facts,

$$\begin{aligned} &-\int (U_h \cdot \nabla_h \delta v_h + V_3 \partial_3 \delta v_h) \cdot \delta v_h \, dx = 0, \quad -\int (U_h \cdot \nabla_h \delta v_3 + V_3 \partial_3 \delta v_3) \cdot \delta w \, dx = 0, \\ &-\int (U_h \cdot \nabla_h \delta\rho + V_3 \partial_3 \delta\rho) \cdot \delta\rho \, dx = 0, \end{aligned}$$

due to the divergence-free condition $\nabla_h \cdot U_h + \partial_3 V_3 = 0$. Then by Lemma (3.1.1), we have

$$
\begin{aligned}
|R_1| &\leq C \|\delta v_h\|_{L^2}^{\frac{1}{2}} \|\partial_1 \delta v_h\|_{L^2}^{\frac{1}{2}} \|\nabla_h v_h\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla_h v_h\|_{L^2}^{\frac{1}{2}} \|\delta v_h\|_{L^2}^{\frac{1}{2}} \|\partial_2 \delta v_h\|_{L^2}^{\frac{1}{2}} \\
&\leq C \|\nabla_h \delta v_h\|_{L^2} \|\nabla_h v_h\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla_h v_h\|_{L^2}^{\frac{1}{2}} \|\delta v_h\|_{L^2} \\
&\leq \frac{\nu}{8} \|\nabla_h \delta v_h\|_{L^2}^2 + C \|\nabla_h v_h\|_{L^2} \|\partial_3 \nabla_h v_h\|_{L^2} \|\delta v_h\|_{L^2}^2.
\end{aligned}
$$

Since $\nabla_h v_h^{(0)}$ is independent of x_3, we can estimate R_2 as,

$$
\begin{aligned}
|R_2| &\leq \|\delta v_h\|_{L^2_{x_3} L^4_h} \|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)} \|\delta v_h\|_{L^2_{x_3} L^4_h} \\
&\leq C \|\delta v_h\|_{L^2}^{\frac{1}{2}} \|\nabla_h \delta v_h\|_{L^2}^{\frac{1}{2}} \|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)} \|\delta v_h\|_{L^2}^{\frac{1}{2}} \|\nabla_h \delta v_h\|_{L^2}^{\frac{1}{2}} \\
&\leq C \|\delta v_h\|_{L^2} \|\nabla_h \delta v_h\|_{L^2} \|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)} \\
&\leq \frac{\nu}{8} \|\nabla_h \delta v_h\|_{L^2}^2 + C \|\nabla_h v_h^{(0)}\|_{L^2(\mathbb{R}^2)}^2 \|\delta v_h\|_{L^2}^2.
\end{aligned}
$$

For R_3, using divergence-free condition $\partial_3 \delta v_3 = -\nabla_h \cdot \delta v_h$, and Lemma (3.1.1),

$$
\begin{aligned}
|R_3| &\leq C \|\delta v_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 \delta v_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 v_h\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_3 v_h\|_{L^2}^{\frac{1}{2}} \|\delta v_h\|_{L^2}^{\frac{1}{2}} \|\partial_2 \delta v_h\|_{L^2}^{\frac{1}{2}} \\
&\leq C \|\delta v_3\|_{L^2}^{\frac{1}{2}} \|\nabla_h \delta v_h\|_{L^2} \|\partial_3 v_h\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_3 v_h\|_{L^2}^{\frac{1}{2}} \|\delta v_h\|_{L^2}^{\frac{1}{2}} \\
&\leq \frac{\nu}{8} \|\nabla_h \delta v_h\|_{L^2}^2 + C \|\partial_3 v_h\|_{L^2} \|\partial_3 \nabla_h v_h\|_{L^2} (\|\delta v_h\|_{L^2} + \|\delta v_3\|_{L^2})^2.
\end{aligned}
$$

For R_4, using divergence-free condition $\partial_3 \delta v_3 = -\nabla_h \cdot \delta v_h$, and Lemma (3.1.1),

$$
\begin{aligned}
|R_4| &\leq C \|\delta v_h\|_{L^2}^{\frac{1}{2}} \|\partial_2 \delta v_h\|_{L^2}^{\frac{1}{2}} \|\nabla_h v_3\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla_h v_3\|_{L^2}^{\frac{1}{2}} \|\delta v_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 \delta v_3\|_{L^2}^{\frac{1}{2}} \\
&\leq C \|\delta v_h\|_{L^2}^{\frac{1}{2}} \|\nabla_h \delta v_h\|_{L^2} \|\nabla_h v_3\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla_h v_3\|_{L^2}^{\frac{1}{2}} \|\delta v_3\|_{L^2}^{\frac{1}{2}} \\
&\leq \frac{\nu}{8} \|\nabla_h \delta v_h\|_{L^2}^2 + C \|\nabla_h v_3\|_{L^2} \|\partial_1 \nabla_h v_3\|_{L^2} (\|\delta v_h\|_{L^2} + \|\delta v_3\|_{L^2})^2..
\end{aligned}
$$

For R_5, using divergence-free condition $\partial_3\delta v_3 = -\nabla_h\cdot\delta v_h$, and Lemma (3.1.1),

$$
\begin{aligned}
|R_5| \quad &\le C\,\|\delta v_3\|_{L^2}^{\frac12}\|\partial_3\delta v_3\|_{L^2}^{\frac12}\|\partial_3 v_3\|_{L^2}^{\frac12}\|\partial_1\partial_3 v_3\|_{L^2}^{\frac12}\|\delta v_3\|_{L^2}^{\frac12}\|\partial_2\delta v_3\|_{L^2}^{\frac12}\\
&\le C\,\|\nabla_h\delta v_h\|_{L^2}^{\frac12}\|\partial_3 v_3\|_{L^2}^{\frac12}\|\partial_3\nabla_h v_3\|_{L^2}^{\frac12}\|\delta v_3\|_{L^2}\|\nabla_h\delta v_3\|_{L^2}^{\frac12}\\
&\le \frac{\nu}{8}\|\nabla_h\delta v_h\|_{L^2}^2+\frac{\nu}{8}\|\nabla_h\delta v_3\|_{L^2}^2+C\,\|\partial_3 v_3\|_{L^2}\|\partial_3\nabla_h v_3\|_{L^2}\|\delta v_3\|_{L^2}^2.
\end{aligned}
$$

For R_6, using Lemma (3.1.1),

$$
\begin{aligned}
|R_6| \quad &\le C\,\|\delta v_h\|_{L^2}^{\frac12}\|\partial_1\delta v_h\|_{L^2}^{\frac12}\|\nabla_h\rho\|_{L^2}^{\frac12}\|\partial_3\nabla_h\rho\|_{L^2}^{\frac12}\|\delta\rho\|_{L^2}^{\frac12}\|\partial_2\delta\rho\|_{L^2}^{\frac12}\\
&\le C\,\|\delta v_h\|_{L^2}^{\frac12}\|\nabla_h\delta v_h\|_{L^2}^{\frac12}\|\nabla_h\rho\|_{L^2}^{\frac12}\|\partial_3\nabla_h\rho\|_{L^2}^{\frac12}\|\delta\rho\|_{L^2}^{\frac12}\|\nabla_h\delta\rho\|_{L^2}^{\frac12}\\
&\le \frac{\nu}{8}\|\nabla_h\delta v_h\|_{L^2}^2+\frac{\nu}{8}\|\nabla_h\delta\rho\|_{L^2}^2+C\,\|\nabla_h\rho\|_{L^2}\|\partial_3\nabla_h\rho\|_{L^2}(\|\delta v_h\|_{L^2}+\|\delta\rho\|_{L^2})^2.
\end{aligned}
$$

For R_7, using divergence-free condition $\partial_3\delta v_3 = -\nabla_h\cdot\delta v_h$, and Lemma (3.1.1),

$$
\begin{aligned}
|R_7| \quad &= -\int \delta v_3\partial_3\rho\delta\rho\,dx\\
&\le C\,\|\delta v_3\|_{L^2}^{\frac12}\|\partial_3\delta v_3\|_{L^2}^{\frac12}\|\partial_3\rho\|_{L^2}^{\frac12}\|\partial_1\partial_3\rho\|_{L^2}^{\frac12}\|\delta\rho\|_{L^2}^{\frac12}\|\partial_2\delta\rho\|_{L^2}^{\frac12}\\
&\le C\,\|\delta v_3\|_{L^2}^{\frac12}\|\nabla_h\delta v_h\|_{L^2}^{\frac12}\|\partial_3\rho\|_{L^2}^{\frac12}\|\partial_3\nabla_h\rho\|_{L^2}^{\frac12}\|\delta\rho\|_{L^2}^{\frac12}\|\nabla_h\delta\rho\|_{L^2}^{\frac12}\\
&\le \frac{\nu}{8}\|\nabla_h\delta v_h\|_{L^2}^2+\frac{\nu}{8}\|\nabla_h\delta\rho\|_{L^2}^2+C\,\|\partial_3\rho\|_{L^2}\|\partial_3\nabla_h\rho\|_{L^2}(\|\delta v_3\|_{L^2}+\|\delta\rho)\|_{L^2})^2.
\end{aligned}
$$

Substituting all the above estimates in (3.2.38) yields,

$$
\frac{d}{dt}(\|\delta v_h\|_{L^2}^2+\|\delta w\|_{L^2}^2+\|\delta\rho\|_{L^2})^2 \le B(t)(\|\delta v_h\|_{L^2}^2+\|\delta w\|_{L^2}^2+\|\delta\rho\|_{L^2})^2,
$$

where

$$
\begin{aligned}
B(t):\quad &= C\,\|\nabla_h v_h\|_{L^2}\|\partial_3\nabla_h v_h\|_{L^2} + C\|\nabla_h v_h^{(0)}\|^2_{L^2(\mathbb{R}^2)} + C\|\partial_3 v_h\|_{L^2}\|\partial_3\nabla_h v_h\|_{L^2} \\
&+ C\,\|\nabla_h v_3\|_{L^2}\|\partial_1\nabla_h v_3\|_{L^2} + C\|\partial_3 v_3\|_{L^2}\|\partial_3\nabla_h v_3\|_{L^2} \\
&\ C\,\|\nabla_h\rho\|_{L^2}\|\partial_3\nabla_h\rho\|_{L^2} + C\|\partial_3\rho\|_{L^2}\|\partial_3\nabla_h\rho\|_{L^2}.
\end{aligned}
$$

Then Gronwall's inequality implies uniqueness and completes the proof of Proposition 3.2.1.

■

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APPENDICES

Basic Inequalities and Sobolev Space

This section contains basic inequalities and the definition of Sobolev space that we used in the proof of various theorems.

0.0.1 Basic Inequalities

- **Hölder inequality**.

  For a, $b \in \mathbb{R}$,

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}.$$

- **Hölder inequality with ϵ**.

  For a, $b \in \mathbb{R}$, and $\epsilon > 0$

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}.$$

- **Young's inequality**.

  For a, $b > 0$, let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

- **Young's inequality with ϵ**.

For $a,\, b > 0$, $\epsilon > 0$, let $1 < p,\, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab \leq \epsilon a^p + C(\epsilon)\, b^q,$$

where $C(\epsilon) = (\epsilon p)^{-\frac{q}{p}} q^{-1}$.

- **Hölder inequality**. Suppose $1 \leq p,\, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then if $u \in L^p$ and $v \in L^q$, we have

$$\int |uv|\, dx \leq \|u\|_{L^p} \|v\|_{L^q}$$

where $\|\cdot\|_{L^p}$ represents the standard L^p norm, which is defined as

$$\|u\|_{L^p} = \begin{cases} \left(\int_{\mathbb{R}^d} |u(x)|^p dx \right)^{\frac{1}{p}}, & \text{if} \quad 1 \leq p < \infty \\ \operatorname{ess\,sup} |u|, & \text{if} \quad p = \infty. \end{cases} \tag{A.1}$$

- **Gronwall inequality**

Let $a(t) \geq 0$ and $\int_0^T a(t)\, dt < \infty$. Assume $g \geq 0$ and is locally integrable on $(0, T)$. If $f \geq 0$ satisfies

$$\frac{d}{dt} f \leq af + g,$$

then

$$f(t) \leq e^{\int_0^t d\tau} f(0) + \int_0^t e^{\int_s^t a\tau\, d\tau}\, g(s)\, ds$$

for any $t \in [0, T)$.

0.0.2 Sobolev Space

Definition 0.0.1 *Let $p \in [1, \infty]$ and m is a non-negative integer. Let U be an open subset of $\mathbb{R}^d$. Assume f is locally integrable function such that for each multi-index α with $|\alpha| \leq m$, $D^\alpha f \in L^p(U)$ exists in a weak sense. Then, the Sobolev space $W^{m,p}(U)$ contains all f with $\|f\|_{W^{m,p}(U)} < \infty$ where*

$$\|f\|_{W^{m,p}(U)} = \begin{cases} \Big(\sum_{|\alpha| \leq m} \int_U |D^\alpha f|^p dx \Big)^{\frac{1}{p}}, & \text{if} \quad 1 \leq p < \infty, \\ \sum_{|\alpha| \leq m} \operatorname{ess\,sup}_U |D^\alpha f|, & \text{if} \quad p = \infty. \end{cases} \tag{A.2}$$

For $p = 2$, we write $W^{m,2} = H^m$, for $m = 0, 1, \cdots$.

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