WEYL'S LAW FOR CUSP FORMS
OF ARBITRARY ARCHIMEDEAN TYPE

By

AYAN MAITI

Bachelor of Science in Mathematics and Computer Science
Chennai Mathematical Institution
Chennai, Tamil Nadu, India
2013

Master of Science in Mathematics
Central European University
Budapest, Hungary
2016

Submitted to the Faculty of the
Graduate College of the
Oklahoma State University
in partial fulfillment of
the requirements for
the Degree of
DOCTOR OF PHILOSOPHY
July, 2022

WEYL'S LAW FOR CUSP FORMS

OF ARBITRARY ARCHIMEDEAN TYPE

Dissertation Approved:

Dr. Mahdi Asgari

Dissertation Advisor

Dr. Baski Balasundaram

Dr. David Wright

Dr. Roger Zierau

ACKNOWLEDGMENTS

I would like to dedicate my dissertation to my parents . Without their unequivocal and unwavering support during the time of my Ph.d, the completion of this journey would not have been possible. I would like to thank my supervisor, Dr. Mahdi Asgari, for his dedicated support and guidance. Dr. Asgari continuously provided encouragement and was always willing and enthusiastic to assist in any way he could throughout the research project. I would also like to thank Dr. Roger Zierau for providing advice regarding representation theory of real groups. Finally I would like to thank Dr. David Wright for helping me understand some of the core concepts of algebraic number theory pertinent to this research.

Acknowledgments reflect the views of the author and are not endorsed by committee members or Oklahoma State University.

Name: AYAN MAITI

Date of Degree: JULY, 2022

Title of Study: WEYL'S LAW FOR CUSP FORMS OF ARBITRARY ARCHIMEDEAN TYPE

Major Field: MATHEMATICS

Abstract: We generalize the work of E. Lindenstrauss and A. Venkatesh establishing Weyl's Law for cusp forms from the spherical spectrum to arbitrary Archimedean type. Weyl's law for the spherical spectrum gives an asymptotic formula for the number of cusp forms that are bi-K_∞ invariant in terms of eigenvalue of the Laplacian. We prove an analogous asymptotic holds for cusp forms with Archimedean type τ, where the main term is multiplied by $\dim \tau$. While in the spherical case the surjectivity of the Satake Map was used, in the more general case that is not available and we use Arthur's Paley-Wiener theorem and multipliers.

TABLE OF CONTENTS

Chapter Page

CHAPTER I

INTRODUCTION

In the introduction we explain the Weyl's Law in the Eucildean case. Then we provide a brief history concerning other generalized domains. For more details on the history and the statements of the conjectures refer to [LV], [Mü2] and [Mü1].

1.1 A brief history

Let M be a compact Riemannian manifold. Let ω_M be the unit ball in M. Weyl proved that the number of eigenfunctions of the Laplacian with eigenvalues less than T, is asymptotic to $C(M)T^{\dim(M)/2}$, where

$$C(M) = \frac{\mathrm{Vol}(\omega_M)}{(2\pi)^{\dim(M)}}\mathrm{Vol}(M),$$

[LV]. Let Γ be an arithmetic subgroup of $SL_2(\mathbb{Z})$ and $\mathbb{H}$ be the upper-half plane. Selberg [Se] using his celebrated trace formula proved Weyl's asymptotic for the discrete spectrum of Laplacian when the space is $M = \Gamma\backslash\mathbb{H}$.

Let G be a semisimple linear algebraic group of adjoint and split type over $\mathbb{Q}$. Let $G(\mathbb{R})$ be the set of $\mathbb{R}$-points of G. For simplicity of this exposition let us assume $\Gamma \subset G(\mathbb{R})$ to be a torsion free arithmetic subgroup. Let K_∞ be a maximal compact subgroup. We denote by $L^2(\Gamma\backslash G(\mathbb{R}))$ the space of Γ invariant, square integrable functions on $G(\mathbb{R})$. We will denote the cuspidal subspace of the above space by $L^2_{\text{cusp}}(\Gamma\backslash G(\mathbb{R}))$. Let $M = \Gamma\backslash G(\mathbb{R})/K_\infty$ be a locally symmetric space. Suppose $d = \dim(\Gamma\backslash G/K_\infty)$. Then it was proved by Lindenstrauss and Venkatesh [LV], that number of spherical, i.e. bi-K_∞ invariant cuspidal Laplacian eigenfunctions, whose eigenvalues are less than T is asymptotic to $C(M)T^{\dim(M)/2}$, where $C(M)$

is the same constant as above.

In my thesis I prove an asymptotic estimate, in terms of the Laplacian eigenvalue, for the number of cusp forms of arbitrary K_∞-type on a semisimple, split, adjoint linear algebraic group over $\mathbb{Q}$, generalizing the work of E. Lindenstrauss and A. Venkatesh in [LV] for the trivial K_∞-type (i.e. spherical cusp forms). Asymptotic formulas with remainder term (i.e. Weyl's law with a remainder term) for arbitrary K_∞-type will be proved by J. Matz and W. Müller [Mü3] in their upcoming work.

Now we describe the Weyl's law using the notation introduced by [Mü2]. Let $\mathbb{H}$ be the upper-half plane as above and let Γ be a congruence subgroup of $SL(2,\mathbb{Z})$. Let Δ be the hyperbolic Laplacian on $\mathbb{H}$. Let $N^\Gamma_{\text{cusp}}(T)$ be the number of cuspidal Δ-eigenfunctions [Bmp], whose eigenvalue is less than T. Selberg, using his celebrated trace formula [Se] for the group $SL(2,\mathbb{R})$, proved the following analogue of the classical Weyl's law:

$$N^\Gamma_{\text{cusp}}(T) \sim \frac{\mathrm{Vol}(\Gamma\backslash\mathbb{H})}{4\pi}T, \quad \text{as} \quad T \to \infty.$$

Let G be a semi-simple linear algebraic group over $\mathbb{Q}$. Let K_∞ be a maximal compact subgroup of $G(\mathbb{R})$. Let Γ be an arithmetic subgroup of $G(\mathbb{Q})$. Let $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ be the center of universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$, the complexified Lie algebra of $G(\mathbb{R})$. A cusp form for Γ [La] is defined via following properties: 1. It is a smooth and K_∞-finite complex-valued function, 2. It is a simultaneous eigenfunction of $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ and 3. it satisfies

$$\int_{\Gamma\bigcap N_p(\mathbb{R})\backslash N_P(\mathbb{R})} f(nx)dn = 0,$$

for all unipotent radicals N_P of proper rational parabolic subgroups P of G. It can be shown that cusp forms are square-integrable. Let $L^2_{\text{cusp}}(\Gamma\backslash G(\mathbb{R}))$ be the closure of the linear span of all cusp forms. If $r = \mathrm{rank}(G(\mathbb{R})/K_\infty)$ and $d = \dim(G(\mathbb{R})/K_\infty)$, then it has been conjectured by Sarnak [Sa] that for $r > 1$ and for an irreducible lattice Γ

$$\frac{N^\Gamma_{\text{cusp}}(T)}{T^{d/2}} \sim \frac{\mathrm{vol}(\Gamma\backslash G)}{(4\pi)^{d/2}\mathbf{\Gamma}(d/2+1)}, \quad \text{as} \quad T \to \infty,$$

where $\mathbf{\Gamma}(\cdot)$ denotes the Gamma function. Let R be the right regular representation of $G(\mathbb{R})$ on the space of square integrable automorphic forms denoted by $L^2(\Gamma\backslash G(\mathbb{R}))$. Suppose

(τ, V_τ) denotes an irreducible finite-dimensional representation of K_∞. Let $d_\tau = \dim(V_\tau)$. Let

$$(L^2(\Gamma\backslash G(\mathbb{R})) \otimes V_\tau)^{K_\infty}$$

be the space of the homogeneous vector bundle on the Riemannian symmetric space $G(\mathbb{R})/K_\infty$. Functions in this space satisfy the following condition:

$$f(gk) = \tau(k^{-1})f(g).$$

Let $\Omega_{G(\mathbb{R})}$ be the Casimir operator in $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$, the center of the universal enveloping algebra. Then $-\Omega_{G(\mathbb{R})} \otimes Id$ induces a self adjoint operator Δ_τ whose restriction to the cuspidal subspace

$$L^2_{\text{cusp}}(\Gamma\backslash G(\mathbb{R}), \tau) := (L^2_{\text{cusp}}(\Gamma\backslash G(\mathbb{R})) \otimes V_\tau)^{K_\infty}$$

has pure point spectrum

$$0 \leq \lambda_1(\tau) < \lambda_2(\tau) < \cdots \to \infty$$

with finite multiplicities. Suppose $\mathcal{E}(\lambda_i(\tau))$ denotes the respective eigenspace corresponding to the eigenvalue $\lambda_i(\tau)$. We define the counting function for the cuspidal spectrum of Δ_τ as

$$N^\Gamma_{\text{cusp}}(T, \tau) = \sum_{\lambda_i(\tau) \leq T} \dim(\mathcal{E}(\lambda_i(\tau))).$$

Let $N^\Gamma_{\text{disc}}(T, \tau)$ be the counting function of the discrete spectrum of Δ_τ. Müller [Mü1] made the following generalization of Sarnak's conjecture. For any arithmetic subgroup Γ and any irreducible K_∞-type τ, we have:

$$\frac{N^\Gamma_{\text{disc}}(T, \tau)}{T^{d/2}} \sim \frac{\text{vol}(\Gamma\backslash G)\dim(\tau)}{(4\pi)^{d/2}\Gamma(d/2+1)}, \quad \text{as} \quad T \to \infty.$$

Donnelly [Do] proved the upper bound of $N^\Gamma_{\text{cusp}}(T, \tau)$ with the same constant terms in more general settings. Therefore, to establish the above formula for the cuspidal spectrum, one has to prove the lower bound with the same constant and the correct asymptotic terms. The conjectures of Sarnak and Müller have been proved for the following cases: for congruence subgroups of $SO(n, 1)$ by Reznikov [Rez]; for congruence subgroups of $\text{Res}_{F/\mathbb{Q}}SL_2$, where

F is a totally real field by Efrat [Ef]; for $\Gamma = SL_3(\mathbb{Z})$ by Steve Miller [Mi]; for torsion free arithmetic subgroups of $SL_n(\mathbb{R})$ by Müller [Mü2]; and for the torsion free arithmetic subgroups of semisimple linear algebraic groups of split and adjoint type in the case of spherical cusp form by Lindenstrauss and Venkatesh [LV].

Our theorem, which is a generalization of the result of Lindenstrauss and Venkatesh in the case of cusp forms of arbitrary K_∞-type is the following:

Theorem 1.1.1 *Let G be a semisimple, split, adjoint type linear algebraic group over $\mathbb{Q}$. Let $G_\infty = G(\mathbb{R})$ be the real points of G, and let Γ be a torsion free arithmetic subgroup of $\mathbf{G}$. Suppose $d = dim(G_\infty/K_\infty)$. Let (τ, V_τ) be an irreducible finite-dimensional representation of K_∞. Let $N^\Gamma_{cusp}(T,\tau)$ be the number of cuspidal eigenfunctions of Δ_τ with eigenvalue $\leq T$, counted with multiplicities. Then we have the following asymptotic formula:*

$$\frac{N^\Gamma_{cusp}(T,\tau)}{T^{d/2}} \sim \frac{dim(\tau)\, vol(\Gamma\backslash G_\infty)}{(4\pi)^{d/2}\Gamma(d/2+1)}, \quad as \quad T \to \infty. \tag{1.1.1}$$

The usual methodology is to apply Arthur's trace formula (or some variant thereof) to a suitable family of test functions. However, there are several new features in the non-trivial K_∞-type case that I address in my thesis work, as I explain below. Although all the conjectures are made with the assumption that Γ is an arithmetic subgroup, for simplicity we will work with Γ being a congruence subgroup in this thesis.

CHAPTER II

NOTATIONS AND PRELIMINARIES

In this section we recall some basic facts of Harmonic Analysis on real and p-adic groups (see [LV]). We also define the notations and review the background necessary for this thesis.

2.1 Parabolic subgroups

For more details on parabolic subgroups refer to [Kn]. Let G be a semisimple linear algebraic group of split and adjoint type over $\mathbb{Q}$. Let S be a finite set of places of $\mathbb{Q}$ containing ∞. We fix a minimal parabolic subgroup, i.e. a Borel subgroup, $P_0 \supset A_0$, where A_0 is a maximal $\mathbb{Q}$-split torus. Suppose $N_0 = R_u(P_0)$ is the unipotent radical of P_0. We have a Levi decomposition $P_0 = M_0 N_0$ with $M_0 \supset A_0$. Let P be a parabolic subgroup containing P_0 with a Levi decomposition $P = M_P N_P$. Such a parabolic subgroup is called a standard parabolic with respect to P_0. Moreover we let $A_{M_P} =$ Split part of $Z(M_P)$, where Z denotes the center.

Let $F = \mathbb{Q}_p$ or $\mathbb{R}$. For simplicity we will use the above notation for the choices of subgroups defined over F. Let us denote the Weyl group of $G(F)$ with respect to A_0 by $W = W(G, A_0)$. Let $\Phi = \Phi(G, A_0)$ be the set of roots. The set of simple roots $\Pi \subset \Phi$ and the set of positive roots $\Phi^+ \subset \Phi$ can be determined by the fixed choice of P_0. If $\alpha \in \Phi^+$, without loss of generality we write $\alpha > 0$.

Let $P = MN \subset G(F)$ be a standard parabolic subgroup of $G(F)$. Such a parabolic has one to one correspondence with subsets of set of simple roots [Kn]. The corresponding set of

simple roots is denoted by $\Pi_M \subset \Phi$. We denote by A_M the split component of the center of M and let $X(M)_F$ be the group of F-rational characters of M. From the discussion above if $\Pi_M = \Theta$, we can use A_Θ to denote A_M. In particular we have, $A_\emptyset = A$ and $A_\Pi = A_G$.
Due to the injectivity of the restriction homomorphism $X(M)_F \mapsto X(A_M)_F$ and existence of a finite cokernel (as the above homomorphism has finite index), we have a canonical linear isomorphism

$$\mathfrak{a}_M^* = X(M)_F \otimes_{\mathbb{Z}} \mathbb{R} \cong X(A_M)_F \otimes_{\mathbb{Z}} \mathbb{R}.$$

If L is the Levi component of a standard parabolic subgroup such that $L \subset M$, then

$$A_M \subset A_L \subset L \subset M.$$

The restriction $X(M)_F \mapsto X(L)_F$ induces an injective map and its restriction induces a linear injection $i_M^L : \mathfrak{a}_M^* \mapsto \mathfrak{a}_L^*$. We have a linear surjection $r_M^L : \mathfrak{a}_L^* \mapsto \mathfrak{a}_M^*$ induced by the restriction map $X(A_L)_F \mapsto X(A_M)_F$. Let $(\mathfrak{a}_M^L)^*$ be the kernel of the restriction r_M^L. Then

$$\mathfrak{a}_L^* = i_M^L(\mathfrak{a}_M^*) \oplus (\mathfrak{a}_M^L)^*.$$

There is a homomorphism $H_M : M \mapsto \mathfrak{a}_M = Hom(X(M), \mathbb{R})$ such that:

$$|\nu(m)|_F = \begin{cases} q^{\langle \nu, H_M(m) \rangle}, & \text{if } F = \mathbb{Q}_p \\ e^{\langle \nu, H_M(m) \rangle}, & \text{if } F = \mathbb{R}, \end{cases}$$

for all $m \in M$ and $\nu \in X(M)_F$.
We set $G_S = G(\mathbb{Q}_S)$, $A_{0,S} = A_0(\mathbb{Q}_S)$, $M_{0,S} = M_0(\mathbb{Q}_S)$ and $N_{0,S} = N_0(\mathbb{Q}_S)$. We denote a parabolic subgroup over $\mathbb{Q}_S$ as $P_S = M(\mathbb{Q}_S)N(\mathbb{Q}_S)$ with its corresponding Levi decomposition. We can think of this parabolic as a direct product of parabolic subgroups of a product of groups. Let $G_\infty = G(\mathbb{R})$. We have an Iwasawa decomposition $G_\infty = N_\infty A_\infty^o K_\infty$, where K_∞ is a maximal compact subgroup of $G(\mathbb{R})$.
Let $K_S = K_\infty \prod_{p \in S \setminus \infty} G(\mathbb{Z}_p)$, where $G(\mathbb{Z}_p)$ is a maximal compact subgroup of $G(\mathbb{Q}_p)$ for all prime p. We will choose S such that it has following the property: for each finite $p \in S$ and

for each parabolic $P(\mathbb{Q}_p) \supset A_0(\mathbb{Q}_p)$, $K_p \bigcap M_P(\mathbb{Q}_p)$ is the stabilizer in $M_P(\mathbb{Q}_p)$ of a special vertex in the building of $M_P(\mathbb{Q}_p)$ and this vertex belongs to the apartment associated to the maximal torus $A_0(\mathbb{Q}_p)$ [LV]. For almost all finite p, This condition is automatically satisfied. Moreover, $K_\infty \bigcap M_P(\mathbb{R})$ is a maximal compact subgroup of $M_P(\mathbb{R})$, and $K_S \bigcap M(\mathbb{Q}_S)$ is a maximal compact subgroup of $M(\mathbb{Q}_S)$.

From the Iwasawa decomposition we have the map $N(\mathbb{Q}_S)\times M(\mathbb{Q}_S)\times K_S \mapsto G_S$ is surjective. We equip each $G(\mathbb{Q}_p)$, for p finite, with the Haar measure such that the volume of $G(\mathbb{Z}_p)$ is 1. We equip K_∞ with the Haar measure of volume 1, and then choose the Haar measure on G_∞ which is compatible with the Riemannian metric defined on the Riemannian symmetric space G_∞/K_∞. Let Φ^+ be the system of positive roots of $A_{0,S}$ with respect to $N_{0,S}$ and let $\Delta \subset \Phi^+$ be the set of simple roots. Let δ_S be the square root of the modulus character of $A_{0,S}$.

The following lemma which is due to Harish-Chandra [HC3] going to describe the correspondence between parabolic subgroups of $G(\mathbb{Q}_p)$ contained in some parabolic subgroup $Q(\mathbb{Q}_p)$ and the parabolic subgroups of $M_Q(\mathbb{Q}_p)$ for all $p \in S$.

Lemma 2.1.1 *There is a one to one correspondence between parabolic subgroups $P(\mathbb{Q}_p)$ of $G(\mathbb{Q}_p)$ which are contained in $Q(\mathbb{Q}_p)$, and parabolic subgroups ${}^*P(\mathbb{Q}_p)$ of $M_Q(\mathbb{Q}_p)$. The correspondence is as follows. If $Q(\mathbb{Q}_p) = M_Q(\mathbb{Q}_p)N_Q(\mathbb{Q}_p)$ and $P(\mathbb{Q}_p) = M_P(\mathbb{Q}_p)N_P(\mathbb{Q}_p)$ is the corresponding Levi decomposition, then the Levi decomposition of ${}^*P_Q = P(\mathbb{Q}_p)\cap M_Q(\mathbb{Q}_p)$ is ${}^*P_Q = M_P(\mathbb{Q}_p)N_Q^P(\mathbb{Q}_p)$, where $A_P(\mathbb{Q}_p) = A_Q^P(\mathbb{Q}_p)A_Q(\mathbb{Q}_p)$ and $N_P(\mathbb{Q}_p) = N_Q^P(\mathbb{Q}_p)N_Q(\mathbb{Q}_p)$.*

2.2 Congruence subgroup

We choose a congruence subgroup $\Gamma \subset G(\mathbb{Z}[S^{-1}])$, which is torsion free. The number of Γ- orbits of proper $\mathbb{Q}$-parabolic subgroups is finite. Let us denote their representatives as $\{P_1, P_2, ..., P_r\}$. We conjugate them by appropriate elements of $G(\mathbb{Q})$ so that the $P_i(\mathbb{Q}_S)$ contain the minimal parabolic subgroup $M_{0,S}A_{0,S}N_{0,S}$. We denote them as $Q_{i,S} = M_{i,S}A_{i,S}N_{i,S}$, and their corresponding conjugating elements as $\delta_i \in G(\mathbb{Q})$ (i.e.

$\delta_i P_i \delta_i^{-1} = Q_i$). Let $M_{i,S} = M_{Q_i,S}$, $N_{i,S} = N_{Q_i,S}$ and $A_{i,S} = A_{Q_i,S}$. Moreover, we put $\Gamma_i = \delta_i \Gamma \delta_i^{-1}$, $\Gamma_{N_{i,S}} = \Gamma_i \cap N_{i,S}$ and $\Gamma_{A_{i,S}} = \Gamma_i \cap A_{i,S}$.

Let $X^*(M(\mathbb{Q}_S))_{\mathbb{Q}_S}$ be the set of $\mathbb{Q}_S$ characters of $M(\mathbb{Q}_S)$, the Levi subgroup of P_S. The dual of this space, which can be identified with the Lie algebra of the maximal split part of the center of $M(\mathbb{Q}_S)$ is

$$\mathfrak{a}_{M(\mathbb{Q}_S)} = \mathrm{Hom}(X^*(M(\mathbb{Q}_S))_{\mathbb{Q}_S}, \mathbb{R})$$

For $\nu_S \in X^*(M(\mathbb{Q}_S))_{\mathbb{Q}_S}$, we have the Harish-Chandra homomorphism

$$H_{M(\mathbb{Q}_S)} : M(\mathbb{Q}_S) \longmapsto \mathfrak{a}_{M(\mathbb{Q}_S)}$$

given by [PR]

$$e^{\langle H_{M(\mathbb{Q}_S)}(m), \nu_S \rangle} = \prod_{p \in S} |\nu_p(m_p)|_p.$$

Let ω_S be an irreducible unitary square-integrable admissible representation of $M(\mathbb{Q}_S)$ which is trivial on $A(\mathbb{Q}_S)$. We define the set of equivalence classes of ω_S as $\mathcal{E}_2(M(\mathbb{Q}_S))$. For $\nu_S \in X^*(M(\mathbb{Q}_S))_{\mathbb{Q}_S} \otimes \mathbb{C} = \mathfrak{a}^*_{M(\mathbb{Q}_S),\mathbb{C}}$, we can define the following induced representation on G_S with parameters (ω_S, ν_S):

$$\mathrm{Ind}(\omega_S, \nu_S) = \prod_{p \in S} \mathrm{Ind}(\omega_p, \nu_p).$$

2.3 Test functions

Let (τ, V_τ) be an irreducible K_∞-type, i.e. an irreducible finite dimensional representation of K_∞. Suppose d_τ and χ_τ denote the dimension and the character of the above representation, respectively. Let $C_c^\infty(G(\mathbb{R}), \tau, \tau)$ be the following space of functions [Cmp2]

$$\left\{ \phi_\infty : G(\mathbb{R}) \to \mathrm{End}(V_\tau), \phi_\infty(k_1 g k_2) = \tau(k_2^{-1}) \phi_\infty(g) \tau(k_1^{-1}) \right\}.$$

A function $\Phi_\infty \in C_c^\infty(G(\mathbb{R}))$ is called bi-K_∞-finite if the following condition is satisfied:

$$\Phi_\infty(x) = \int_{K_\infty} \int_{K_\infty} d_\tau \chi_\tau(k) \Phi_\infty(k^{-1} x k') d_\tau \chi_\tau({k'}^{-1}) dk' dk.$$

A function Φ_∞ is called K_∞-central if $\Phi_\infty(kxk^{-1}) = \Phi_\infty(x)$ for all $k \in K_\infty$ and for all $x \in G(\mathbb{R})$. We denote the convolution algebra of bi-K_∞-finite and K_∞-central functions as $C_c^\infty(G(\mathbb{R}))_{K_\infty}^{K_\infty}$. This convolution algebra is isomorphic to the End(V_τ)-valued algebra defined above via the following isomorphism [Cmp1] :

$$\begin{aligned} C_c^\infty(G(\mathbb{R}), \tau, \tau) &\cong C_c^\infty(G(\mathbb{R}))_{K_\infty}^{K_\infty} \\ \phi_\infty &\mapsto \Phi_\infty = d_\tau \mathrm{Tr}\phi_\infty \\ \int\limits_{K_\infty} \Phi_\infty(gk)\tau(k)dk = \phi_\infty(g) &\leftarrow\!\shortmid \Phi_\infty(g) \end{aligned}$$

At the non-Archimedean places of $S' = S\backslash\infty$ we define the Hecke algebra as the space of compactly supported, locally constant functions. We denote this space as $C_c^\infty(G(\mathbb{Q}_{S'}))$. We define the co-center of this Hecke algebra as the following quotient:

$$\bar{\mathcal{H}}(G(\mathbb{Q}_{S'})) := \frac{C_c^\infty(G(\mathbb{Q}_{S'}))}{[C_c^\infty(G(\mathbb{Q}_{S'})), C_c^\infty(G(\mathbb{Q}_{S'}))]}.$$

Moreover we choose the functions from this space which are bi-${K'}_{S'}$ invariant, where ${K'}_{S'}$ is an arbitrary compact subgroup of the maximal compact subgroup $K_{S'}$. We denote this subspace as $\bar{\mathcal{H}}({K'}_{S'}\backslash G_{S'}/{K'}_{S'})$. Let $\Phi_{S'} \in \bar{\mathcal{H}}({K'}_{S'}\backslash G_{S'}/{K'}_{S'})$. Hence we can combine the End(V_τ)-valued function ϕ_∞ at the Archimedean place with $\Phi_{S'}$ to obtain an endomorphism valued test function on G_S and denote the set containing these functions as:

$$C_c^\infty(G(\mathbb{R}), \tau, \tau) \otimes \bar{\mathcal{H}}({K'}_{S'}\backslash G_{S'}/{K'}_{S'}).$$

We denote the scalar valued counterpart of the above space as

$$C_c^\infty(G(\mathbb{R}))_{K_\infty}^{K_\infty} \otimes \bar{\mathcal{H}}({K'}_{S'}\backslash G_{S'}/{K'}_{S'}).$$

Let $L^2(\Gamma\backslash G_S, V_\tau)$ be the following set:

$$\left\{ f : \Gamma\backslash G_S \mapsto V_\tau : f(gk_\infty) = \tau(k_\infty)^{-1}f(g), (f_1, f_2) = \int_{\Gamma\backslash G_S} \langle f_1(x), f_2(x)\rangle_{V_\tau} dx \right\}.$$

Here the inner product makes sense as $\mathrm{Vol}(\Gamma\backslash G_S) < \infty$. Also as Γ is chosen to be a torsion free congruence subgroup, $\Gamma\backslash G(\mathbb{R})$ is a manifold. Elements of $C_c^\infty(G(\mathbb{R}), \tau, \tau) \otimes \mathcal{C}({K'}_{S'}\backslash G_{S'}/{K'}_{S'})$ act on this space via convolution.

Let R be the right regular representation of $G(\mathbb{R})$ on $L^2(\Gamma\backslash G(\mathbb{R}))$. Let $\Omega_{G(\mathbb{R})}$ be the Casimir operator in $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$, the center of the universal enveloping algebra. Then $-\Omega_{G(\mathbb{R})}\otimes Id$ induces a self adjoint operator Δ_τ whose restriction to

$$L^2_{\text{cusp}}(\Gamma\backslash G(\mathbb{R}),\tau) := (L^2_{\text{cusp}}(\Gamma\backslash G(\mathbb{R}))\otimes V_\tau)^{K_\infty}$$

has pure point spectrum with finite multiplicities. Let us denote them as

$$0 \leq \lambda_1(\tau) < \lambda_2(\tau) < \cdots \to \infty$$

with finite multiplicities. Suppose $\mathcal{E}(\lambda_i(\tau))$ denotes the eigenspace corresponding to the eigenvalue $\lambda_i(\tau)$. For $T\geq 0$, we have the counting function as

$$N^{\Gamma}_{\text{cusp}}(T,\tau) = \sum_{\lambda_i(\tau)\leq\sqrt{T}} \dim(\mathcal{E}(\lambda_i(\tau))).$$

We can redefine the above counting function by representation theoretic means in the following way. Let $\Pi_{\text{cusp}}(G(\mathbb{Q}_S))$ be the set of unitary irreducible cuspidal subrepresentations of the regular representation of $G(\mathbb{Q}_S)$ on $L^2_{\text{cusp}}(\Gamma\backslash G(\mathbb{Q}_S)/K^{'}_{S'})$. Let $\Pi_{\text{cusp}}(G(\mathbb{R}))$ be the set of subrepresentations of the regular representations of $G(\mathbb{R})$ acting on $L^2_{\text{cusp}}(\Gamma\backslash G(\mathbb{Q}_S)/K^{'}_{S'})$. Any element $\pi\in\Pi_{\text{cusp}}(G(\mathbb{Q}_S))$ can be written as $\pi=\pi_\infty\otimes\pi_{S\backslash\infty}$, where $\pi_\infty\in\Pi_{\text{cusp}}(G(\mathbb{R}))$. Let $H_{\pi_\infty}(\tau)$ be the τ-isotypical subspace of $(\pi_\infty, H_{\pi_\infty})$. Let $H^{K'}_{\pi_{S\backslash\infty}}$ be the subspace of $K^{'}_{S'}$-fixed vectors in $(\pi_{S\backslash\infty}, H_{\pi_{S\backslash\infty}})$. Let $m(\pi_\infty)$, resp $m(\pi)$, be the multiplicity with which π_∞, resp π, occurs as a subrepresentation of $G(\mathbb{R})$, resp $G(\mathbb{Q}_S)$, in the cuspidal subspace $L^2_{\text{cusp}}(\Gamma\backslash G(\mathbb{Q}_S)/K^{'}_{S'})$. Then we have

$$m(\pi_\infty) = \sum_{\pi^{'}\in\Pi_{\text{cusp}}(G(\mathbb{Q}_S))} m(\pi^{'})\dim H^{K'}_{\pi_{S\backslash\infty}}$$

for all $\pi^{'}$ such that $\pi^{'}_\infty=\pi_\infty$. Suppose ν_π denotes the Casimir eigenvalue of π_∞. Then we take the subcollection $\Pi_{\text{cusp}}(G(\mathbb{Q}_S))_T$ whose elements satisfies $|\nu_\pi|^2\leq T$. Similarly, we define $\Pi_{\text{cusp}}(G(\mathbb{R}))_T$. Then we have

$$\sum_{\pi_\infty\in\Pi_{\text{cusp}}(G(\mathbb{R}))_T} m(\pi_\infty)\dim\text{Hom}_{K_\infty}(H_{\pi_\infty}(\tau),V_\tau) = N^{\Gamma}_{\text{cusp}}(T,\tau).$$

2.4 Fourier transform

We now define the scalar-valued Fourier transform of functions on $C_c^\infty(G(\mathbb{R}))_{K_\infty}^{K_\infty}$. Let $P_\infty = M_\infty^1 A_\infty N_\infty$ be the Langlands decomposition of a standard cuspidal parabolic subgroup of $G(\mathbb{R})$. Choose $\omega_\infty \in \mathcal{E}_2(M_\infty^1)$. Suppose θ_{ω_∞} denotes its character. Let d_{ω_∞} be the formal degree of ω_∞. Let τ be the double representation of K_∞ on $L^2(K_\infty \times K_\infty)$ obtained from (τ, V_τ). Let τ_{M_∞} be the restriction of τ to $K_\infty \bigcap M_\infty$. We let $L^2_\omega(M_\infty, \tau_{M_\infty})$ be the set of τ_{M_∞}-spherical functions on $L^2(M_\infty) \otimes L^2(K_\infty \times K_\infty)$. The norm in this space is defined as:

$$\|\psi\|^2 = \int_{M_\infty} \int_{K_\infty \times K_\infty} \|\psi(k_1 : m : k_2)\|^2 dk_1 dk_2 dm.$$

It can be made into a Hilbert algebra with the multiplication via

$$(\psi_1 \psi_2)(k_1 : m : k_2) = \int_{M_\infty} \int_{K_\infty} \psi_1(k_1 : \tilde{m} : k^{-1}) \psi_2(k : \tilde{m}^{-1} m : k_2) dk d\tilde{m}.$$

The Fourier transform of a function $\Phi_\infty \in C_c^\infty(G(\mathbb{R}))$ is defined as in [Ar4]:

$$\Phi_\infty \mapsto \widehat{\Phi_\infty}(\omega_\infty, \nu_\infty) \in L^2_\omega(M_\infty, \tau_{M_\infty}),$$

where $\widehat{\Phi_\infty}(\omega_\infty, \nu_\infty)$ is given by

$$\widehat{\Phi_\infty}(\omega_\infty, \nu_\infty)(k_1 : m : k_2) = d_{\omega_\infty} \int\limits_{M_\infty} \int\limits_{A_\infty} \int\limits_{N_\infty} \Phi_\infty(k_1 n a m \widetilde{m} k_2) \theta_{\omega_\infty}(\widetilde{m}^{-1}) e^{(-\nu_\infty + \rho_\infty)\ln(a)} dn da d\widetilde{m}.$$

Next we define the operator valued Fourier transform of $\Phi_\infty \in C_c^\infty(G(\mathbb{R}))_{K_\infty}^{K_\infty}$. Let $\pi_\infty \in \widehat{G(\mathbb{R})}(\tau)$, the unitary irreducible representation of $G(\mathbb{R})$ that contains τ upon restriction to K_∞. Then $\Phi_\infty \mapsto \pi_\infty(\Phi_\infty)$ defines the operator valued Fourier transform on the space of endomorphisms of finite dimensional vector space. From Harish-Chandra's sub-representation theorem we know that π_∞ is isomorphic to an irreducible subrepresentation of an induced representation from a cuspidal parabolic P_∞ with parameters $(\omega_\infty, \nu_\infty)$. Let us denote the induced representation as $\mathrm{Ind}(\omega_\infty, \nu_\infty)$. Then we have the following relation [Ar4]:

$$d_{\omega_\infty} \mathrm{Tr}(\mathrm{Ind}(\omega_\infty, \nu_\infty)\Phi_\infty) = \int\limits_{K_\infty} \widehat{\Phi_\infty}(\omega_\infty, \nu_\infty)(k^{-1} : 1 : k) dk.$$

Let $m_{\pi_\infty}(\tau)$ be the multiplicity with which τ appears in the decomposition of π_∞ restricted to K_∞. Suppose $\pi_{\infty,\tau}(\Phi_\infty)$ is the restriction of π_∞ to $\mathcal{H}_{\pi_\infty}(\tau)$. Then we can define the spherical Fourier transform $\mathcal{F}(\Phi_\infty)(\pi_\infty) \in \mathrm{End}(\mathbb{C}^{m_{\pi_\infty}(\tau)})$ as follows [Cmp1]:

$$\pi_{\infty,\tau}(\Phi_\infty) = \mathbb{1}_\tau \otimes \mathcal{F}(\Phi_\infty)(\pi_\infty).$$

Let $\widetilde{\Phi_\infty}(x) = \overline{\Phi_\infty(x^{-1})}$. Then $\pi_\infty(\widetilde{\Phi_\infty}) = \pi_\infty(\Phi_\infty)^*$, the conjugate transpose of $\pi_\infty(\Phi_\infty)$. Moreover, $\mathrm{Tr}\pi_\infty(\Phi_\infty \star \widetilde{\Phi_\infty}) = ||\pi_\infty(\Phi_\infty)||^2_{\mathrm{HS}}$. Let $\mu_\infty(\omega_\infty, \nu_\infty)$ be the Harish-Chandra μ function [HC3] corresponding to the induced parameters $(\omega_\infty, \nu_\infty)$. Let $\mathcal{P}$ be the set of associated classes of parabolic subgroups. The Plancherel inversion of Φ_∞ has the following formula:

$$\Phi_\infty \star \widetilde{\Phi_\infty}(e) = \sum_{\mathcal{P}} n(\mathcal{P})^{-1} \sum_{P\in\mathcal{P}} \sum_{\mathcal{E}_2(M_\infty)} d_\omega (\frac{1}{2\pi i})^q \int\limits_{i\mathfrak{a}^*_\infty} ||\mathrm{Ind}(\omega_\infty, \nu)(\Phi_\infty)||^2_{\mathrm{HS}}\, \mu_\infty(\omega_\infty, \nu) d\nu.$$

Similarly, at a non-Archimedean place p if we assume the induced parameters are $(\omega_p \otimes \nu_p)$, then the the Plancherel measure $\mu_p(\omega_p)$ is defined over a connected compact manifold $\mathcal{O}_2(M(\mathbb{Q}_p))$ for each $\omega_p \in \mathcal{E}_2(M(\mathbb{Q}_p))$. We denote by d_{ω_p} the Euclidean measure of the connected compact manifold. Then we have the following Plancherel inversion formula for $\Phi_p \in \mathcal{C}(K'_p \backslash G(\mathbb{Q}_p)/K'_p)$:

$$\Phi_p \star \widetilde{\Phi_p}(e) = \sum_{\mathcal{P}} n(\mathcal{P})^{-1} \sum_{P\in\mathcal{P}} \sum_{\mathcal{E}_2(M_p)} d_{\omega_p} \int\limits_{\mathcal{O}_2(M(\mathbb{Q}_p))} ||\mathrm{Ind}(\omega_p)(\Phi_p)||^2_{\mathrm{HS}}\, \mu_p(\omega_p) d\omega_p.$$

CHAPTER III

EXAMPLES AND METHODOLOGY OF VARIOUS CASES

3.1 Euclidean case

Suppose $D \subset \mathbb{R}^n$ is a bounded domain. Then the Laplacian has discrete eigenvalues accumulating at $+\infty$, the only accumulating point. If we denote $N(\lambda) = \text{card}\{\lambda_i \leq \lambda\}$ to be the eigenvalue counting function of the Laplacian, then Weyl's Law states that

$$N(\lambda) \sim \frac{\omega_n}{(2\pi)^n} \text{vol}(D) \lambda^{\frac{n}{2}},$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. Here are some examples:

- When $D = [0, m]$, $m \in \mathbb{R}$ and $m > 0$ then $\sin \frac{i\pi x}{m}$, $i \in \mathbb{N}$ gives us the family of the solution to the Dirichlet eigenvalue problem, where the euclidean Laplacian is $\frac{d^2}{dx^2}$ with the eigenvalues $\frac{i^2\pi^2}{m^2}$. Then $N(\lambda) = \text{card}\{i | i \leq \frac{m\sqrt{\lambda}}{\pi}\} \sim \frac{m\sqrt{\lambda}}{\pi}$

- When $D = [0, m] \times [0, n]$, $m, n \in \mathbb{R}$, $m, n > 0$ then similarly as above we have $\sin \frac{i\pi x}{m} \sin \frac{j\pi y}{n}$ as the solution to the Dirichlet eigenvalue problem with the Laplacian $\frac{d^2}{dx^2} + \frac{d^2}{dy^2}$. Then, $N(\lambda) = \text{card}\{(i, j) : i, j \geq 0, \left(\frac{i\pi}{m\sqrt{\lambda}}\right)^2 + \left(\frac{j\pi}{n\sqrt{\lambda}}\right)^2 \leq 1\}$. Hence,

$$N(\lambda) \sim \frac{\pi}{4} \frac{m\sqrt{\lambda}}{\pi} \frac{n\sqrt{\lambda}}{\pi} = \frac{\text{area}(D)}{4\pi} \lambda,$$

  where the right hand side is the area of the ellipse.

3.2 When $G = SL(2)$

Let $\mathbb{H}$ be the upper half plane. It can be realized as the quotient space $SL(2,\mathbb{R})/SO(2,\mathbb{R})$. Let Γ be a congruence subgroup of $SL(2,\mathbb{Z})$. In this section we discuss the methodology to prove the Weyl's law for the special case of $\Gamma\backslash\mathbb{H}$. For more details we refer to [LM1].

Let Δ be the hyperbolic Laplacian corresponding to the surface $\Gamma\backslash\mathbb{H}$. It acts on $L^2(\Gamma\backslash\mathbb{H})$. Let $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \cdots$ be the eigenvalues of Δ. We put $\lambda_j = 1/4 + r_j^2$ for $r_j \in \mathbb{R}_{\geq 0} \cup [0, 1/2]i$. When $\lambda > 0$, we put

$$N^{\Gamma}(\lambda) = \mathrm{card}\{j : \lambda_j \leq \lambda\},$$

as the eigenvalue counting function. Our job is to find an asymptotic formula for this function as $\lambda \to \infty$. Here, each eigenvalue is counted with their respective multiplicity. The constant function is the only noncuspidal eigenfunction as Γ is a congruence subgroup.

Now we recall Selberg's trace fomrula [Se]. Let $E_k(z,s)$ be the Eisenstein series attached to the k-th cusp (suppose a_k). It is defined by:

$$E_k(z,s) = \sum_{\gamma \in \Gamma_{a_k}\backslash\Gamma} \mathrm{Im}(\sigma_k \gamma z)^s.$$

Here, $\sigma_k \in SL(2,\mathbb{R})$ is such that $\sigma_k(a_k) = \infty$. The Eisenstein series is an eigenfunction of the hyperbolic Laplacian with $s(1-s)$ as eigenvalue. The series converges locally uniformly for all $z, s \in \mathbb{H}$ and for $\mathrm{Re}(s) \geq 1$. The zeroth coefficient of $E_k(z,s)$ in the cusp a_l is:

$$\int_0^1 E_k(\sigma_l(x+iy), s)dx = \delta_{k,l} y^s + \phi_{k,l}(s) y^{1-s}.$$

here, $\phi_{k,l}(s)$ is a meromorphic function of s. The matrix $\Phi(s) = (\phi_{k,l})_{k,l}$ is called the scattering matrix. Let $\phi(s) = \det(\Phi(s))$. Let $h \in C_c^\infty(\mathbb{R})$. The Fourier transform of h is defined as

$$\hat{h}(z) = \int_{\mathbb{R}} h(t) e^{itz} dt.$$

From the Paley-Wiener theorem in the Euclidean case we know that $\hat{h}$ is entire and rapidly decreasing on horizontal strips. For $t \in \mathbb{R}$ we put

$$\hat{h}_t(z) = \hat{h}(t+z) + \hat{h}(t-z).$$

Let us also assume for simplicity that Γ is torsion free. Now we apply the trace formula to the family of function $\hat{h}_t$. It gives the following identity:

$$\begin{aligned}\sum_{j=-\infty}^{\infty} \hat{h}(t-r_j) &= \frac{\mathrm{Vol}(\Gamma\backslash\mathbb{H})}{2\pi} \int\limits_{\mathbb{R}} \hat{h}(t-r) r \tanh \pi r dr + \sum_{\{\gamma\}_\Gamma} \frac{2l(\gamma_0)}{\sinh l(\gamma)/2} h(l(\gamma)) \cos tl(\gamma) \\ &+ \frac{1}{2\pi} \int\limits_{\mathbb{R}} \hat{h}(t-r) \frac{\phi'}{\phi}(1/2+ir) dr + \frac{1}{2}\hat{h}(t) \mathrm{Tr}(I - \Phi(1/2)) - 2mh(0)\ln 2 \\ &- \frac{m}{\pi} \int\limits_{\mathbb{R}} \hat{h}(t-r) \frac{\Gamma'}{\Gamma}(1+ir) dr \end{aligned} \tag{3.2.1}$$

Here, m denotes the number of cusps. $\{\gamma\}_\Gamma$ denotes the hyperbolic conjugacy classes. $l(\gamma)$ is the length of closed geodesic $\eta(\gamma)$ on $\Gamma\backslash\mathbb{H}$, determined by each of these conjugacy classes. Let γ_0 denote the primitive geodesic element. Then it can be shown that each γ is some power of γ_0. As h is compactly supported and smooth function on $\mathbb{R}$, each of the series and the integrals are absolutely and uniformly convergent.

We now let $t \to \infty$ and investigate the asymptotic behaviours of the both sides of this above formula. As $|\tanh x| \le 1$, for $x \in \mathbb{R}$, we have that

$$\int\limits_{\mathbb{R}} \hat{h}(t-r) r \tanh \pi r dr = \mathcal{O}(|t|), \quad \text{as} \quad |t| \to \infty.$$

The second sum is bounded by a finite number as cosine function is bounded and the sum is over finitely many conjugacy classes as h is compactly supported.

Next for the term corresponding to the scattering matrix we use Huxley's computation [Hux] to determine the formula for ϕ. It is a finite product of quotients of Dirichlet L-functions and $\mathbf{\Gamma}$-functions. To estimate the logarithmic derivative of the gamma function, we use Stirling's approximation formula. Then we estimate the logarithmic derivative of

Dirichlet L-functions on the line $\mathrm{Re}(s) = 1$ using standard estimation. Therefore we get

$$\int_{\mathbb{R}} \hat{h}(t-r)\frac{\phi'}{\phi}(1/2+r)dr = \mathcal{O}(\ln|t|), \quad \text{as} \quad |t| \to \infty.$$

Similarly, we get

$$\int_{\mathbb{R}} \hat{h}(t-r)\frac{\mathbf{\Gamma}'}{\mathbf{\Gamma}}(1+ir)dr = \mathcal{O}(\ln|t|), \quad \text{as} \quad |t| \to \infty.$$

The remaining terms are bounded as $|t| \to \infty$. Therefore we obtain

$$\sum_{j=-\infty}^{\infty} \hat{h}(t-r_j) = \mathcal{O}(|t|), \quad \text{as} \quad |t| \to \infty.$$

Now we choose h such that $\hat{h} \geq 0$ on $\mathbb{R}$ and strictly positive on $[-1,1]$. It will be enough for us to consider $r_j \in \mathbb{R}$, as there are finitely many of imaginary set of eigenvalues of Δ on $L^2(\Gamma\backslash\mathbb{H})$. For a given $\lambda \in \mathbb{R}$ we estimate the number of eigenvalues in the neighbourhood of λ as follows:

$$\mathrm{card}\{j : |\lambda - r_j| \ \leq 1\} \cdot \min\{\hat{h}(z) : z \in [-1,1]\} \leq \sum_{r_j \in \mathbb{R}} \hat{h}(t-r_j).$$

Therefore, we obtain the following:

$$\mathrm{card}\{j : |\lambda - r_j| \ \leq 1\} = \mathcal{O}(1+|\lambda|), \quad \forall \lambda \in \mathbb{R}.$$

Using this result we obtain the following auxiliary result:

$$\sum_{r_j \leq \lambda} \left| \int_{\mathbb{R}-[-\lambda,\lambda]} \hat{h}(t-r_j)dt \right| + \sum_{r_j \geq \lambda} \left| \int_{-\lambda}^{\lambda} \hat{h}(t-r_j)dt \right| = \mathcal{O}(1+|\lambda|), \quad \forall \lambda > 1.$$

Let $p(r)$ be an even continuous function on $\mathbb{R}$ such that $p(r) = \mathcal{O}(1+|r|)$. We integrate both sides of (3.2.1) on the interval $[-\lambda,\lambda]$ and then let $\lambda \to \infty$ to study the asymptotic behaviour of both sides. Assume that h is chosen such that $h(0) = 1$. Then we obtain

$$\int_{[-\lambda,\lambda]} \int_{\mathbb{R}} \hat{h}(t-r)p(r)drdt = \int_{[-\lambda,\lambda]} p(r)dr + \mathcal{O}(\lambda), \quad \text{as} \quad \lambda \to \infty$$

Now we can apply the above equation by replacing $p(r)$ with either of $r\tanh\pi r$, $\frac{\phi'}{\phi}(1/2+ir)$ or $\frac{\mathbf{\Gamma}'}{\mathbf{\Gamma}}(1+ir)$. Therefore we arrive at the following equations respectively

$$\int\limits_{[-\lambda,\lambda]}\int\limits_{\mathbb{R}}\hat{h}(t-r)r\tanh\pi r drdt = \lambda^2+\mathcal{O}(\lambda), \quad \text{as} \quad \lambda\to\infty,$$

$$\int\limits_{[-\lambda,\lambda]}\int\limits_{\mathbb{R}}\hat{h}(t-r)\frac{\phi'}{\phi}(1/2+ir)drdt = \mathcal{O}(\lambda\ln\lambda), \quad \text{as} \quad \lambda\to\infty,$$

$$\int\limits_{[-\lambda,\lambda]}\int\limits_{\mathbb{R}}\hat{h}(t-r)\frac{\mathbf{\Gamma}'}{\mathbf{\Gamma}}(1+ir)drdt = \mathcal{O}(\lambda\ln\lambda), \quad \text{as} \quad \lambda\to\infty.$$

As the other terms are finite, the integral over the interval $[-\lambda,\lambda]$ is of order $\mathcal{O}(\lambda)$. Therefore the left hand side of (3.2.1) becomes

$$\int\limits_{[-\lambda,\lambda]}\sum_{j=-\infty}^{\infty}\hat{h}(t-r_j) = \frac{\mathrm{Vol}(\Gamma\backslash\mathbb{H})}{2\pi}\lambda^2+\mathcal{O}(\lambda\ln\lambda).$$

By rearranging the sum and using the auxiliary result above we arrive at the final formula:

$$N^{\Gamma}(\lambda) = \frac{\mathrm{Vol}(\Gamma\backslash\mathbb{H})}{4\pi}\lambda^2+\mathcal{O}(\lambda\ln\lambda).$$

3.3 When $G = SL(n)$

We rephrase the Weyl's law in this case in adelic language. For more details see [Mü2]. Let $G = GL(n)$, as an algebraic group defined over $\mathbb{Q}$. Suppose $\mathbb{A}$ is the ring of adeles of $\mathbb{Q}$. Let A_G be the split part of the center of G and $A_G(\mathbb{R})_0$ be its identity component of its real points. Let $\Pi(G(\mathbb{A}))$ be the set of equivalence classes of irreducible unitary representations of $G(\mathbb{A})$ whose central character is trivial on the identity component defined above. We denote the subspace of cusp forms in $L^2(G(\mathbb{Q})A_G(\mathbb{R})_0\backslash G(\mathbb{A}))$ by $L^2_{\text{cusp}}(G(\mathbb{Q})A_G(\mathbb{R})_0\backslash G(\mathbb{A}))$. we define the set of subrepresentations of the regular representation in $L^2_{\text{cusp}}(G(\mathbb{Q})A_G(\mathbb{R})_0\backslash G(\mathbb{A}))$ by $\Pi_{\text{cusp}}(G(\mathbb{A}))$ which is subset of $\Pi(G(\mathbb{A}))$. Let $\mathbb{A}_f$ be the finite adele ring. Any unitary irreducible $\pi\in\Pi(G(\mathbb{A}))$ can be written as a product $\pi = \pi_\infty\otimes\pi_f$, where π_∞ and π_f are the unitary irreducible representation of $G(\mathbb{R})$ and $G(\mathbb{A}_f)$, respectively. Let $\mathcal{H}_{\pi_\infty}$ and $\mathcal{H}_{\pi_f}$

be the Hilbert space representations corresponding to π_∞ and π_f, respectively. Let K_f be an arbitrary open compact subgroup of $G(\mathbb{A}_f)$. Let the subspace of K_f-fixed elements in $\mathcal{H}_{\pi_f}$ be denoted by $\mathcal{H}_{\pi_f}^{K_f}$. Let $G(\mathbb{R})^1$ be the group of elements in $G(\mathbb{R})$ whose determinant is 1. Let λ_{π_∞} be the Casimir eigenvalue of π_∞ restricted to $G(\mathbb{R})^1$ for a fix $\pi \in \Pi(G(\mathbb{A}))$. Let $\Pi_{\text{cusp}}(G(\mathbb{A}))_T$ be the set of π such that $|\lambda_{\pi_\infty}| \leq T$, for $T > 0$. Let ϵ_{K_f} be a constant such that $\epsilon_{K_f} = 1$ if $-1 \in K_f$ and 0 otherwise. Then adelic Weyl's law states that [Mü1]:

Theorem 3.3.1 *Let $G = GL(n)$, and $d_n = dim(SL(n)/SO(n))$. Let K_f be an arbitrary open compact subgroup of $G(\mathbb{A}_f)$. Let (τ, V_τ) be an irreducible $SO(n,\mathbb{R})$-type. Then*

$$\sum_{\pi \in \Pi_{cusp}(G(\mathbb{A}))_T} dim(\mathcal{H}_{\pi_f}^{K_f}) dim(\mathcal{H}_{\pi_\infty} \otimes V_\tau)^{SO(n)}$$
$$\sim dim(\tau) \frac{Vol(G(\mathbb{Q})A_G(\mathbb{R})_0 \backslash G(\mathbb{A})/K_f)}{(4\pi)^{d_n/2}\mathbf{\Gamma}(1+d_n/2)}(1+\epsilon_{K_f})T^{d_n/2}.$$

The heat equation method is based on the study of the asymptotic behaviour of the trace of the heat operator. This method combined with Arthur's trace formula proves the above theorem. Here we provide some basic ideas of this method. For more details we refer to [Mü1].

Let $G(\mathbb{A})^1$ be the group of elements $x \in G(\mathbb{A})$ such that $|\det(x)|_{\mathbb{A}} = 1$. The non-invariant trace formula provides an identity between distribution on $G(\mathbb{A})^1$. The identity is of the form [Ar1]

$$\sum_{\chi \in \mathfrak{X}} J_\chi(f) = \sum_{o \in \mathfrak{O}} J_o(f), \quad f \in C_c^\infty(G(\mathbb{A})^1).$$

The left and the right hand sides are called the spectral side ($J_{\text{spec}}(f)$) and the geometric side ($J_{\text{geom}}(f)$), respectively. The left hand side is parameterized by the cuspidal data $\mathfrak{X}$ and the right hand hand side is parameterized by the semisimple conjugacy classes in $G(\mathbb{Q})$.

Here we only consider the case when τ is trivial. For $t > 0$, We choose a family of test functions $\tilde{\phi}_t^1 \in C_c^\infty(G(\mathbb{A})^1)$ to apply on the above trace formula. At the infinite place we choose this family to be the product of heat kernel $h_t \in C^\infty(G(\mathbb{R})^1)$ of the Laplacian and a smooth, compactly supported family of functions ϕ_t, and at the finite places it is given by $\mathbb{1}_{K_f}$, where K_f is an open compact subgroup of $G(\mathbb{A}_f)$. We apply the trace formula to $\tilde{\phi}_t^1$. Then the idea is to study the asymptotic behaviour of the both sides when $t \to 0$. We denote the set of irreducible unitary representations which occur discretely in the regular representation of $G(\mathbb{A})$ on $L^2(G(\mathbb{Q})A_G(\mathbb{R})_0 \backslash G(\mathbb{A}))$ by $\Pi_{\text{disc}}(G(\mathbb{A}))$. Let $m(\pi)$ be the multiplicity of π. Let $\mathcal{H}_{\pi_\infty}^{K_\infty}$ be the space of K_∞-invariant vectors in $\mathcal{H}_{\pi_\infty}$.

To study the asymptotic behaviour of the geometric side we appeal to the fine $\mathfrak{o}$-expansion:

$$J_{\text{geom}}(\tilde{\phi}_t^1) = \sum_{M \in \mathcal{L}} \sum_{\gamma \in (M(\mathbb{Q}_S))_{M,S}} a^M(S,\gamma) J_M(\tilde{\phi}_t^1, \gamma).$$

Here $J_M(\tilde{\phi}_t^1, \gamma)$ denotes the weighted orbital integrals, M runs over the set of Levi subgroups $\mathcal{L}$ containing the Levi component M_0 of the standard minimal parabolic subgroup P_0, S is a finite set of places of $\mathbb{Q}$, and $(M(\mathbb{Q}_S))_{M,S}$ is a certain set of equivalence classes in $M(\mathbb{Q}_S)$. Müller [Mü2] proves the following limit: for $M \neq G$ and $\gamma \neq 1$

$$\lim_{t \to 0} t^{d_n/2} J_M(\tilde{\phi}_t^1, \gamma) = 0.$$

Therefore it is enough to study the orbital integral when $M = G$ and $\gamma = 1$. Using the behaviour of the heat kernel h_t evaluated at 1 and letting $t \to 0$, we arrive at the following asymptotic formula:

$$J_{\text{geom}}(\tilde{\phi}_t^1) \sim \frac{\text{Vol}(G(\mathbb{Q})A_G(\mathbb{R})_0 \backslash G(\mathbb{A})/K_f)}{(4\pi)^{d_n/2}} (1 + \epsilon_{K_f}) T^{d_n/2}.$$

Now one moves to estimate the spectral side. The spectral side can be written as a finite linear combination of distributions $J_{M,P}^L(\tilde{\phi}_t^1, s)$:

$$J_{\text{spec}}(\tilde{\phi}_t^1) = \sum_{M \in \mathcal{L}} \sum_{L \in \mathcal{L}(M)} \sum_{P \in \mathcal{P}(M)} \sum_{s \in W^L(\mathfrak{a}_M)_{\text{reg}}} a_{M,s} J_{M,P}^L(\tilde{\phi}_t^1, s).$$

Here $\mathcal{L}(M)$ denotes the set of Levi subgroups which contains M, $\mathcal{P}(M)$ denotes the set of parabolic subgroups whose Levi subgroup coincides with M and $W^L(\mathfrak{a}_M)_{\text{reg}}$ is a subgroup of the full Weyl group. For each $M \in \mathcal{L}$, the important terms associated with $J_{M,P}^L(\tilde{\phi}_t^1, s)$ are the logarithmic derivatives of the intertwining operators

$$M_{Q|P}(\lambda) : \mathcal{A}^2(P) \longmapsto \mathcal{A}^2(Q), \quad P, Q, \in \mathcal{P}(M), \quad \lambda \in \mathfrak{a}_{M,\mathbb{C}},$$

where $\mathcal{A}^2(P)$ and $\mathcal{A}^2(Q)$ are the square-integrable automorphic forms attached to parabolic subgroups P, Q. When $M = L = G$ and $s = 1$ we have

$$J^G_{G,G}(\tilde{\phi}^1_t, 1) = \sum_{\pi \in \Pi_{\text{disc}}(G(\mathbb{A}))} m(\pi) e^{t\lambda_{\pi_\infty}} \dim(\mathcal{H}^{K_\infty}_{\pi_\infty}) \dim(\mathcal{H}^{K_f}_{\pi_f}).$$

For a fixed $\pi \in \Pi_{\text{disc}}(M(\mathbb{A}))$, let $M_{Q|P}(\pi, \lambda)$ be the restriction of the $M_{Q|P}(\lambda)$ to the subspace of automorphic forms $\mathcal{A}^2_\pi(P)$ of $\pi-$type. Let $r_{Q|P}(\pi, \lambda)$ be meromorphic functions satisfying

$$M_{Q|P}(\pi, \lambda) = r_{Q|P}(\pi, \lambda)^{-1} N_{Q|P}(\pi, \lambda),$$

where $N_{Q|P}(\pi, \lambda)$ are the normalized intertwining operators. Now to estimate the logarithmic derivatives of these intertwining operators on $i\mathfrak{a}_M$, one uses the available knowledge of Rankin-Selberg L- functions and corresponding ϵ-factors on $GL(n)$. Using these estimates one gets the following limit:

$$\lim_{t \to 0} J^L_{M,P}(\tilde{\phi}^1_t, s) = \mathcal{O}(t^{-\frac{d_n - 1}{2}}),$$

for all proper Levi subgroups M, all $L \in \mathcal{L}(M)$, $P \in \mathcal{P}(M)$ and $s \in W^L(\mathfrak{a}_M)_{\text{reg}}$. Therefore equating estimates on the both the geometric and the spectral sides we get:

$$\sum_{\pi \in \Pi_{\text{disc}}(G(\mathbb{A}))} m(\pi) e^{t\lambda_{\pi_\infty}} \dim(\mathcal{H}^{K_\infty}_{\pi_\infty}) \dim(\mathcal{H}^{K_f}_{\pi_f}) \sim \frac{\text{Vol}(G(\mathbb{Q}) A_G(\mathbb{R})_0 \backslash G(\mathbb{A})/K_f)}{(4\pi)^{d_n/2}} (1 + \epsilon_{K_f}) t^{-d_n/2}.$$

Now to prove Weyl's law for the discrete spectrum with trivial $K_\infty-$ type we use Karamata's theorem on the left hand side.

3.4 The method of Lindenstrauss and Venkatesh

In this section, we summarize the method and the proof of Lindenstrauss and Venkatesh [LV]. Let S be a finite set of places including the Archimedean places. Let $G_S = G(\mathbb{Q}_S)$ be the $\mathbb{Q}_S$-points of $\mathbf{G}$. Let $\Gamma \subset G(\mathbb{Z}[S]^{-1})$ be a torsion free congruence subgroup. Let $Q_0 = M_0 N_0$ be the Levi decomposition of a minimal parabolic subgroup of G defined over $\mathbb{Q}$, where

$M_0 \supset A_0$. Let K_S be a maximal compact subgroup of $G(\mathbb{Q}_S)$. Let $\{Q_1, Q_2, \cdots Q_i\}$ be the set of inequivalent representatives of Γ-orbits on the set of $\mathbb{Q}$-parabolic subgroups of G. We consider $\mathbb{Q}_S$-points of these representatives and conjugate them by suitable elements from Γ to make them standard parabolics. We rename these representatives as $P_i(\mathbb{Q}_S) = P_{i,S}$. We have the following Iwasawa and Langlands decomposition of $G(\mathbb{Q}_S)$ and $P_i(\mathbb{Q}_S)$ respectively :

$$G(\mathbb{Q}_S) = P_i(\mathbb{Q}_S)K_S, \quad P_i(\mathbb{Q}_S) = M_i^1(\mathbb{Q}_S)A_i(\mathbb{Q}_S)N_i(\mathbb{Q}_S).$$

Let $\Gamma_{A_{i,S}} = \Gamma \bigcap A_i(\mathbb{Q}_S)$. Let $C_c^\infty(K_S\backslash G(\mathbb{Q}_S)/K_S)$ be a convolution ring of bi-K_S invariant test functions acting on the space $L^2(\Gamma\backslash G(\mathbb{Q}_S))$, via convolution. Let $\nu \in \mathfrak{a}_S^*$, where $\mathfrak{a}_S = \mathrm{Lie}(A_0(\mathbb{Q}_S))$. Suppose $\mathfrak{a}_{S,\mathrm{temp}}^*$ is the subset where the Plancherel measure for spherical functions is supported. Let $\dim(A_0) = r$. Let E_ν be the corresponding spherical eigenfunction of the Laplacian. For $\Phi \in C_c^\infty(K_S\backslash G_S/K_S)$, it can be shown that E_ν is also an eigenfunction of the corresponding convolution operator "$\star\Phi$". We denote by $\hat{\Phi}(\nu)$ the spherical transform of Φ, defined as the eigenvalue of the convolution operator:

$$E_\nu \star \Phi = \hat{\Phi}(\nu)E_\nu.$$

We recall the following properties of the Plancherel measure and Plancherel inversions for spherical functions Φ [LV]. Let $\mu(\nu)d\nu$ be the Plancherel measure.

a. We have the inversion formula:

$$\Phi(e) = \int\limits_{\mathfrak{a}_{S,\mathrm{temp}}^*} \hat{\Phi}(\nu)\mu(\nu)d\nu. \tag{3.4.1}$$

b. There is a constant $C_1 > 0$ such that for any positive function $\psi(\nu)$ on $\mathfrak{a}_{S,\mathrm{temp}}^*$,

$$\int\limits_{\mathfrak{a}_{S,\mathrm{temp}}^*} \psi(\nu)\mu(\nu)d\nu \leq C_1 \int\limits_{\mathfrak{a}_{S,\mathrm{temp}}^*} \psi(\nu)(1 + \|\nu_\infty\|)^{d-r}d\nu. \tag{3.4.2}$$

c. There exists a constant $\alpha(G)$ such that,

$$\int\limits_{\nu\in\mathfrak{a}_{S,\mathrm{temp}}^*, \|\nu\|^2\leq T} \mu(\nu) \sim \alpha(G)T^{\frac{d}{2}}, \quad \text{as} \quad T \to \infty. \tag{3.4.3}$$

For $\Phi \in C_c^\infty(K_S\backslash G_S/K_S)$ we define the Abel-Satake transformation

$$\mathcal{S}(\Phi)(a) = \delta(a)^{-1} \int\limits_{n \in N_S} \Phi(na)dn,$$

for all $a \in A_S$. We can prove that $\mathcal{S}(\Phi)$ is a $A_S \cap K_S$ invariant function. Therefore, we can regard this as a function on $A_S/A_S \cap K_S$, which are W_S- invariant. Consequently we have the following commutative diagram:

$$\begin{array}{ccc} C_c^\infty(K_S\backslash G_S/KS) & \xrightarrow{\hat{\Phi}} & \mathcal{H}(\mathfrak{a}_S^*) \\ \Big\downarrow{\scriptstyle \mathcal{S}} & & \Big\downarrow{\scriptstyle Id} \\ C_c^\infty(\mathfrak{a}_S)^{W_S} & \xrightarrow{FT} & \mathcal{H}(\mathfrak{a}_S^*) \end{array}$$

Here, $\mathcal{H}(\mathfrak{a}_S^*)$ denotes the space of holomorphic functions on $\mathfrak{a}_S^*$ and FT denotes the Euclidean Fourier transform on $\mathfrak{a}_S$.

3.4.1 Cuspidality condition

Lindenstrauss and Venkatesh [LV, (3.3)−(3.4)] prove a necessary condition on a test function so that the convolution operator generated by it will have a purely cuspidal image on the space of the spherical automorphic forms. One of the key differences between their method and the method of using test functions to apply simple trace formula [FK] is that these convolution operators do not factor into the composition of independent convolution operators at each place. Hence the possibility of having everywhere unramified cusp forms in their kernel is avoided. The condition is described as follows:

Lemma 3.4.1 *(Lindenstrauss - Venkatesh) [LV]: Suppose $\Phi \in C_c^\infty(K_S\backslash G_S/K_S)$ satisfies the following condition*

$$\hat{\Phi}(\nu) = 0, \forall \nu|_{\Gamma_{A_{i,S}}} = 1, \forall i, \tag{3.4.4}$$

or equivalently

$$\sum_{\gamma \in \Gamma_{A_{i,S}}} \mathcal{S}\Phi(\gamma a) = 0,$$

then Φ *maps* $L^1_{loc}(\Gamma\backslash G_S)$ *to* $L^2_{cusp}(\Gamma\backslash G_S/K_S)$*.*

Here $L^1_{\text{loc}}(\Gamma\backslash G_S)$ and $L^2_{\text{cusp}}(\Gamma\backslash G_S/K_S)$ denote the space of locally integrable and square-integrable cuspidal functions respectively.

The existence of such non-zero Φ can be proved using the surjectivity of the Abel-Satake transform for bi-K_S invariant functions. It turns out that Φ is a linear combination of W_S invariant point masses.

3.4.2 The pre-trace formula

Following [Mi] Lindenstrauss and Venkatesh used the pre-trace formula instead of using the full Arthur's trace formula as only the lower bound of the counting function to be achieved. Suppose $H \in C_c^\infty(K_S\backslash G_S/K_S)$ satisfies the condition of Lemma 3.4.1 and acts as a convolution operator on $L^1_{\text{loc}}(\Gamma\backslash G_S)$. Let Ω be a compact set whose measure is arbitrarily close to $\text{Vol}(\Gamma\backslash G_S)$. Then we have the following inequality:

$$H(1)\text{Vol}(\Omega) + \sum_{\gamma\in Z}\int_\Omega H(x^{-1}\gamma x)dx \le \sum_\nu \hat{H}(\nu). \tag{3.4.5}$$

Here, $Z = \Big(\Gamma\backslash\{e\}\Big)\bigcup\Big(xgx^{-1} : x \in \tilde{\Omega}, x \in \operatorname{supp} H\Big)$.

3.4.3 Test functions

Let us fix $\epsilon > 0$ and $0 < t \le 1$. Using Arthur's [Ar5] Paley-Wiener theorem at the Archimedean place, we can choose an entire Schwarz function $h_\epsilon(t\nu)$ on $\mathfrak{a}_S^*$ such that its Plancherel inversion H_t will have the following estimate for sufficiently small t, and a constant $C_1 > 0$:

$$\left|t^d H_t(e) - \alpha(G)\right| \le C_1\epsilon. \tag{3.4.6}$$

We multiply H_t with $\mathbb{1}_{K_{S\backslash\infty}}$ to make it a function on $G(\mathbb{Q}_S)$. Moreover, we can choose a sequence Φ_n, such that Φ_n satisfies the condition of Lemma 3.4.1 for each n. We now form a family of test functions $\Phi_n \star H_t$ and then plug them in (3.4.5). Letting $n \to \infty$ and $t \to 0$, using (3.4.6) on the geometric side, and using the bounds of spherical functions [DKV1] we have the correct lower bound of $N^{\Gamma}_{\mathrm{cusp}}(T,\tau)$ as $T \to \infty$. Notice that in the case of spherical cusp forms we have $d_\tau = 1$.

3.5 Method for arbitrary K_∞-type

To extend the above method of Lindenstrauss and Venkatesh for cusp forms of arbitrary K_∞-type, we choose our test functions from the convolution ring of $\mathrm{End}(V_\tau)$-valued, compactly supported, smooth functions on $G(\mathbb{R})$. Such a function h satisfies the following radial condition:

$$h(k_1 g k_2) = \tau(k_2^{-1}) h(g) \tau(k_1^{-1}).$$

We denote this ring as $C_c^\infty(G(\mathbb{R}),\tau,\tau)$. Let $C_c^\infty(G(\mathbb{R}))^{K_\infty}_{K_\infty}$ denote the scalar-valued convolution ring of bi-K_∞-finite, K_∞-central functions. Let $L^2_{\mathrm{cusp}}(\Gamma\backslash G_S)_{K_\infty}$ be the space of scalar valued cusp forms of K_∞-type τ. We recall the following properties from R. Camporesi's work [Cmp2]:
1. $h \mapsto H^\sharp := d_\tau \mathrm{Tr}(h)$ is a ring bijection from $C_c^\infty(G(\mathbb{R}),\tau,\tau)$ to $C_c^\infty(G(\mathbb{R}))^{K_\infty}_{K_\infty}$. 2. We have the following convolution identity:

$$d_\tau \mathrm{Tr}(h_1 \star h_2) = d_\tau \mathrm{Tr}(h_1) \star d_\tau \mathrm{Tr}(h_2). \tag{3.5.1}$$

Let K' be an arbitrary open compact subgroup of $G(\mathbb{Q}_{S\backslash\infty})$. We multiply h and $H^\sharp$ by a function from the ring $C_c^\infty(K'\backslash G(\mathbb{Q}_{S\backslash\infty})/K')$ to define them as functions on $G(\mathbb{Q}_S)$. We have the following analogue of Plancherel inversion [Ar3] for arbitrary compactly supported smooth functions on $G(\mathbb{R})$:

$$H^\sharp_\infty \star \tilde{H}^\sharp_\infty(1) = \sum_{\mathcal{P}} n(\mathcal{P})^{-1} \sum_{P\in\mathcal{P}} \sum_{\omega_\infty \in \mathcal{E}_2(M^1_\infty)} (\frac{1}{2\pi i})^q \int_{i\mathfrak{a}^*_\infty} d_{\omega_\infty} \left|\left|\pi_\infty(\omega_\infty,\nu_\infty)(H^\sharp_\infty)\right|\right|^2_{\mathrm{HS}}$$
$$\mu(\omega_\infty,\nu_\infty) d\nu_\infty.$$

Here, $\mathcal{P}$ denotes the associated classes of parabolic subgroups, $\mathcal{E}_2(M^1_{P,\infty})$ denotes the equivalence classes of square integrable representations of $M^1_{P,\infty}$, $\mu(\omega_\infty, \nu_\infty)d\nu$ is the Plancherel measure, and $\pi_\infty(\omega_\infty, \nu_\infty)$ is the induced representation with parameters $\omega_\infty, \nu_\infty$.

The operator-valued spherical Fourier transform $\mathcal{F}$ could be defined for any $H^\sharp_\infty$ in the following way. For any $\pi_\infty \in \widehat{G(\mathbb{R})}(\tau)$, the set of unitary irreducible representations containing τ, let $\mathcal{H}_{\pi_\infty}(\tau)$ be its τ-isotypic subspace. Then

$$\pi_\infty(H^\sharp_\infty)|_{\mathcal{H}_{\pi_\infty}(\tau)} = \mathbb{1}_\tau \otimes \mathcal{F}(H^\sharp_\infty)(\pi_\infty).$$

At a non-Archimedean place p, we have a similar Plancherel inversion formula [HC3] for compactly supported smooth function on $G(\mathbb{Q}_p)$, which is bi-K'_p invariant:

$$H_p \star \tilde{H}_p(1) = \sum_{A_p \in \mathcal{C}} c(G(\mathbb{Q}_p)/A_p)^{-2}\gamma(G(\mathbb{Q}_p)/A_p)^{-1}[\mathcal{M}(G(\mathbb{Q}_p)/A_p)]^{-1} \int\limits_{\omega_p \in \mathcal{E}_2(M_p)^{K'_p}} ||\pi_p(\omega_p)(H_p)||^2_{\mathrm{HS}}\, \mu(\omega_p) d\omega_p.$$

Here, $\mathcal{C}$ is a complete set of standard tori in $G(\mathbb{Q}_p)$ such that no two of which are conjugate in $G(\mathbb{Q}_p)$. $M_p \supset A_p$ $\mathcal{E}_2(M_p)^{K'_p}$ denotes the set of unitary, irreducible, admissible, square-integrable representations ω_p which contain the trivial representation of $K'_p \bigcap M_p$. The measure n this space is defined in [HC3]. This set is a union of finitely many compact manifolds [BDK, Section 2.3]. Therefore the Plancherel measure $\mu(\omega_p)d\omega_p$ is defined over a compact manifold. Also $\pi_p(\omega_p)$ denotes the induced representation with the parameter ω_p.

We have the following estimates of the Plancherel measure defined on the parameters corresponding to the minimal parabolic at the Archimedean place and arbitrary standard parabolics at a non-Archimedean place p:

a. There exists a positive constant C' such that:

$$\mu(\omega_\infty, \nu_\infty)\mu_p(\omega_p) \le C'(1 + |\nu_\infty\|)^{d-r}. \tag{3.5.2}$$

b. There exists a positive constant $\alpha(G)$ such that:

$$\int\limits_{\nu_\infty \in i\mathfrak{a}^*_{0,\infty}, \|\nu_\infty\|^2 \le T} \int\limits_{\omega_p \in \mathcal{E}_2(M_{0,p})^{K'_p}} \mu(\omega_\infty, \nu_\infty)\mu_p(\omega_p) \sim \alpha(G) T^{d/2}, \tag{3.5.3}$$

as $T \to \infty$.

3.5.1 Cuspidality condition

We have a generalization of Lemma 1 for bi-K_∞-finite test functions. Assume that $\Phi \in C_c^\infty(G_S)_{K_\infty}^{K_\infty}$ is such that

$$\mathrm{Ind}(\omega_S, \nu)(\Phi) = 0 \quad \text{whenever} \quad \nu|_{\Gamma_{A_{i,S}}} = 1 \quad \forall i, \tag{3.5.4}$$

for all parabolics $P_S \subset P_{i,S}$, for all ω_S, equivalence classes of unitary irreducible representation of $M_{P,S}^1$ for $\nu \in \mathfrak{a}_{P_{0,S},\mathbb{C}}^*$ in an appropriate countable set of elements in $\mathfrak{a}_{P_{0,S},\mathbb{C}}^*$. ω_S is such that it is discrete series at the Archimedean place and supercuspidal at the non-Archimedean places with $\tau|_{K_\infty \cap M_{P,\infty}^1} \subset \omega_\infty$. Then Φ maps elements in $L^1_{\mathrm{loc}}(\Gamma \backslash G_S)$ to $L^2_{\mathrm{cusp}}(\Gamma \backslash G_S)_{K_\infty}$ as convolution operator.

3.5.2 The pre-trace formula

Suppose $\phi \in C_c^\infty(G(\mathbb{Q}_S), \tau, \tau)$ satisfies the conditions of Section 2.2.1. Let Ω be a compact set whose measure is arbitrarily close to $\mathrm{Vol}(\Gamma \backslash G_S)$. Let e_λ be the eigenfunction of Δ_τ with eigenvalue λ in $L^1_{\mathrm{loc}}(\Gamma \backslash G(\mathbb{Q}_S), \tau)$. Then we have the inequality

$$\mathrm{Tr}\phi(1)\mathrm{Vol}(\Omega) + \sum_Z \int_\Omega \mathrm{Tr}\phi(x^{-1}\gamma x)dx \leq \sum_\lambda (e_\lambda \star \phi, e_\lambda). \tag{3.5.5}$$

Here $Z = \Big(\Gamma \backslash \{e\}\Big) \bigcup \Big\{xgx^{-1} : x \in \Omega, \text{x lies in support of } \mathrm{Tr}(\phi)\Big\}$. We can rewrite the right hand side of (3.5.5) as [BM, Cor. 2.2]:

$$\sum_\lambda (e_\lambda \star \phi, e_\lambda) = \sum_\lambda \sum_{-\nu_\pi = \lambda} m(\pi_S)\mathrm{Tr}(\pi_S(\mathrm{Tr}(\phi))).$$

3.5.3 Test functions

Let us fix $\epsilon > 0$ and $0 < t \leq 1$. Using Arthur's Paley-Wiener theorem [Ar5] at the Archimedean place for bi-K_∞-finite functions, we can choose an entire Schwarz function

$\pi_\infty(\omega_\infty,\nu_\infty)(H^\sharp_\infty)$ on $\mathcal{E}_2(M^1_{0,\infty})\times\mathfrak{a}^*_{0,\mathbb{C}}$ such that following inequality holds:

$$\left| t^d \sum_{\mathcal{P}\in\mathrm{Cl}(G_\infty)} n(\mathcal{P})^{-1}\sum_{P\in\mathcal{P}}\sum_{\omega_\infty\in\mathcal{E}_2(M^1_\infty)}\left(\frac{1}{2\pi i}\right)^q \int_{i\mathfrak{a}^*_\infty} d_{\omega_\infty}\left|\left|\pi_\infty(\omega_\infty,t\nu_\infty)(H^\sharp_\infty)\right|\right|^2_{\mathrm{HS}} \mu(\omega_\infty,\nu_\infty)d\nu_\infty-\alpha(\mathbf{G})\right|\le C_1\epsilon. \tag{3.5.6}$$

We denote by $H^\sharp_{\infty,t}$, the family of K_∞-central test functions whose Fourier transform is $\pi_\infty(\omega_\infty,t\nu_\infty)(H^\sharp_\infty)$, defined on $\mathcal{E}_2(M_{0,\infty})\times\mathfrak{a}^*_{0,\mathbb{C}}$. We multiply $H^\sharp_{\infty,t}$ by $\mathbb{1}_{K'}$ to get a function $H^\sharp_t$ defined on $G(\mathbb{Q}_S)$. We choose a sequence of functions $\Phi^\sharp_{n,S}$ satisfying (3.5.4), such that $\pi_S(\Phi_{n,S})$ converges to $\mathbb{1}_{\mathcal{H}_{\pi_S}(\tau)}$. Let $\widetilde{\phi}=\overline{\phi(x^{-1})^T}$. We now plug $(\phi_n\star h_t\star\widetilde{h}_t)$ in (3.5.4). Using (3.5.3), (3.5.6) and taking $T=1/t$ the term corresponding to trivial conjugacy class on the geometric side will be asymptotic to

$$d_\tau\alpha(G)T^d, \tag{3.5.7}$$

as $T\to\infty$. This is the lower bound of $N^\Gamma_{\mathrm{cusp}}(T,\tau)$ as $T\to\infty$. Also the terms corresponding to the non-trivial conjugacy classes will tend to 0 as $T\to\infty$, using the same bounds on spherical functions [DKV1]. I use the same proof provided by Lindenstrauss and Venkatesh [LV, Section 6.3] to obtain the correct constant term $\alpha(G)$.

3.6 The method of Labesse and Müller

Labesse and Müller [LM] proved a weak version of Weyl's law for semisimple algebraic groups which are almost simple and simply connected. Their idea of proving the asymptotic formula for the counting functions is to apply the Arthur-Selberg trace formula for a family of test functions, whose Archimedean part arise from the integral kernel function of the integral operator $e^{-t\Delta_\tau}$ for $0<t\le 1$ and the non-Archimedean parts are idempotents $\mathbb{1}_{K_f}$, where K_f is an arbitrary compact subgroup at the non-Archimedean place. In the spectral side the terms contributing to the inner-product of Eisenstein series will contribute trivially when $t\to 0$ as shown by W. Müller [Mü2] in the case of $SL_n(\mathbb{R})$. But the calculation is delicate for

arbitrary groups. In this regard it will be useful to find test functions such that convolution operators with respect to them have purely cuspidal image. Let S be the finite set of non-Archimedean places. Then following the simple trace formula introduced by Flicker and Kazhdan [FK], Labesse and Müller [LM] considered the test functions decomposed as $f = f_\infty \otimes f_S \otimes \mathbb{1}_{K_f^S}$, where f_S are the pseudo-coefficients of Steinberg representation of $G(\mathbb{Q}_S)$ acting on $L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f)$. Hence the image of the right regular representation with respect to the above test function projects into subspace $L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, S)$, generated by the vectors of automorphic representations which are Steinberg at places in S. Define the eigenvalue counting function $N^\Gamma_{\text{cusp}}(T,\tau,S)$ with respect to S in $L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, S)$. Using this idea they were able to show that:

$$\limsup_{T\to\infty} \frac{N^\Gamma_{\text{cusp}}(T,\tau,S)}{T^{d/2}} = \frac{C_S(\Gamma)\text{vol}(\Gamma\backslash G)\dim(\tau)}{(4\pi)^{d/2}\Gamma(d/2+1)}.$$

But the non-triviality of the constant $C_S(\Gamma)$ would depend on the choice of the compact set K_f, as $\Gamma = G(\mathbb{Q}) \cap K_f$, where K_p for $p \in S$ lies inside the minimal parahoric compact subgroup. To achieve the Weyl's law one would need to know that the constant above is the one provided by Donnelly, which unfortunately is not always possible because of the appearance of $C_S(\Gamma)$.

3.7 Isolation of cuspidal spectrum

Recently Beuzart-Plessis, Liu, Zhang, and Zhu in their paper [BLZZ] proved the existence of integral operators with a purely cuspidal image. Their choice of test functions are from a Schwartz space of functions whose partial derivatives are bounded, but not compactly supported. Therefore, it is a subspace of Harish-Chandra Schwartz space [HC1]. It would be interesting to apply Arthur's trace formula to the composition of this test function and the kernel of the heat operator $e^{-t\Delta_\tau}$ [LM], to count the number of cusp forms.

3.8 Future work

There are several directions in which this work could be extended and strengthened as possible future projects.

The first direction involves attempting to extend my work to non-linear cover groups of algebraic groups. It would be interesting to see whether the method described in section 3.5 could be generalized for arithmetic quotients of cover groups e.g. double cover of $SL(2, \mathbb{Q})$ and more generally for any arithmetic subgroups of more general cover groups of a semi-simple algebraic groups over any number fields.

Lapid and Müller [LM1] proved Weyl's law for spherical cusp forms with a remainder term in the case of arithmetic quotient of $SL(n, \mathbb{R})$, generalizing the result of [DKV2] for the compact arithmetic quotients. They considered the distribution of eigenvalues of the center of the universal enveloping algebra. Later Finis and Lapid [FL] improved the remainder term in Weyl's law for the congruence subgroups of any Chevalley group. It will be interesting to see whether those same error terms could be achieved for the cusp forms of arbitrary K_∞-type for more general groups.

On the other hand, J. Matz [Ma] and later J. Matz and N. Templier [MT] proved the asymptotic bounds of distribution of Hecke eigenvalues in the case of arithmetic subgroups of $GL(n, \mathbb{F})$ for various cases of $\mathbb{F}$ (i.e. for when $\mathbb{F}$ is an imaginary quadratic field extension of $\mathbb{Q}$ or when $\mathbb{F} = \mathbb{Q}$). They used a more generalized bound for spherical functions than that provided by [DKV1]. Also they worked with test functions which are non-compactly supported. It would also be interesting to extend their result for general groups and arbitrary K_∞-types.

CHAPTER IV

PROOF OF THE MAIN THEOREM

In this chapter we prove our main theorem 1.2.1.

4.1 Plancherel measure and estimates

In this section we review the explicit formula of Plancherel measure in the case of reductive Lie groups (both Real and $p-$adic) and their various estimates which provide an essential part of the proof of the main term of the Weyl's law. The references for this section are [HC1] and [HC2].

4.1.1 The real case

Product formula

For this section let us fix a parabolic subgroup (P, A) of $G(\mathbb{R})$, with the corresponding Langlands decomposition $P = M^1AN$. Let $\omega_\infty \in \mathcal{E}_2(M^1)$ be a square integrable unitary irreducible class of representation of M^1. Let $\mu(\omega_\infty, \nu_\infty)$ for $\omega_\infty \in \mathcal{E}_2(M^1)$ and $\nu_\infty \in \mathfrak{a}^*_{\mathbb{C}}$ be the Harish-Chandra μ function of the pair $(G(\mathbb{R}), P)$. Denote by Σ the set of all roots of (P, A). A root $\alpha \in \Sigma$ is called reduced if $k\alpha \notin \Sigma$ for $0 < k \leq 1/2$. Let Φ be the set of all reduced roots. For any $\alpha \in \Phi$ put

$$\mathfrak{n}_\alpha = \bigoplus_{k\alpha : k \geq 1} \mathfrak{n}(\alpha), \tag{4.1.1}$$

where $\mathfrak{n}(\alpha) = \{x \in \mathfrak{g} : [H, X] = \alpha(H)X, \forall H \in \mathfrak{a}\}$. Let N_α be the analytic subgroup corresponding to $\mathfrak{n}_\alpha$. Let σ_α be the hyper-plane given by $\alpha = 0$ in $\mathfrak{a}$. Let Z_α be the centralizer of σ_α. We put $M_\alpha = {}^0Z_\alpha$, $A_\alpha = M_\alpha \cap A$ and $\theta(N_\alpha) = \bar{N}_\alpha$. Then we can define the following parabolic subgroups with their corresponding Langlands decompositions

$$^*P_\alpha = M^1 A_\alpha N_\alpha, \quad ^*\bar{P}_\alpha = M^1 A_\alpha \bar{N}_\alpha, \quad P = M^1 AN.$$

We can now define the product formula for the Harish-Chandra μ function.

Suppose $\mu(\omega_\infty, \nu_\infty^\alpha)$ denotes the corresponding $\mu(\omega_\infty, \nu_\infty)$ for the group $(M_\alpha, {}^*P_\alpha)$. Here $\nu_\infty \in \mathfrak{a}_{\mathbb{C}}^*$ and ν_∞^α denotes the restriction of ν_∞ to A_α, and $\omega \in \mathcal{E}_2(M^1)$. Then for a suitable constant C_G, depending on G, we have the following product formula [HC2, Theorem 12, p. 145]

$$\mu(\omega_\infty, \nu_\infty) = C_G \prod_{\alpha \in \Phi} \mu(\omega_\infty, \nu_\infty^\alpha). \tag{4.1.2}$$

Here we have that prk $M_\alpha = 0$, prk$^*P_\alpha = 1$.

Explicit formula

When prk $G = 0$, and prk $P = 1$ we have the following two possibilities

- $\mathcal{E}_2(G) \neq \emptyset$
- $\mathcal{E}_2(G) = \emptyset$

Here the first condition is equivalent to Rank G =Rank K. We write down the formula for the μ function in each case.

- [HC1, Theorem 1, Sec. 24] We consider the second possibility first. Let us introduce some notation. Let Q be the set of positive roots of $(\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{h}$ is a θ-stable Cartan subalgebra of $\mathfrak{g}$. Now Q is the union of three disjoint parts Q_I, Q_R, Q_C, set of imaginary, real and complex roots respectively. Let H_α be the unique element in $\mathfrak{h}$ such that $(H_\alpha, H) = \alpha(H)$, for all $H \in \mathfrak{h}$, where $(,)$ denotes the Killing form. With respect to the Cartan involution we have the decomposition $\mathfrak{g} = \mathfrak{k} \bigoplus \mathfrak{p}$. Moreover we have $\mathfrak{h} = \mathfrak{h}_I \oplus \mathfrak{h}_R$, and a fixed parabolic subgroup with Langlands decomposition $P = M^1 e^{\mathfrak{h}_R} N$. We put

$$\widetilde{w_I} = \prod_{\alpha \in Q_I} H_\alpha, \quad \widetilde{w_R} = \prod_{\alpha \in Q_R} H_\alpha, \quad \widetilde{w_+} = \prod_{\alpha \in Q_C} H_\alpha.$$

Suppose $\mathfrak{h}_I$ is a cartan subalgebra of $\mathfrak{k}$. Then the second condition is satisfied. From the theorem of Harish-Chandra [HC1, Sec. 23,Theorem 1] we know that $\omega_\infty \in \mathcal{E}_2(M^1)$ corresponds to an element in orbit of $H_I^{*'}$ under the action of $W(M^1/H_I)$, where $H_I^{*'}$ is a subspace of H_I^*, which is the Cartan subgroup of M^1 in $K \cap M^1$(we call this element λ which lies in the lie algebra of $H_I^{*'}$). Let a^* be the element in $H_I^{*'}$ that corresponds to ω_∞. Put $\lambda = \lambda(a^*) \in i\mathfrak{h}_I^{*'}$. Then we have the following formula of the μ function:

$$\mu(\omega_\infty, \nu_\infty) = C\widetilde{w_+}(\lambda + \nu_\infty) = C \prod_{\alpha \in Q_C} |(\lambda + \nu_\infty, \alpha)| \,. \tag{4.1.3}$$

Here the constant C depends on $G(\mathbb{R})$ and M^1. In this case one can see that if Q_R is empty [Wal, p. 58, (2.3.5)], then dim $N = |Q_C|$. So μ is a polynomial in ν_∞ of degree dim N. Hence as $t \to \infty$ we have

$$\mu(\omega_\infty, t\nu_\infty) \sim C_{\nu_\infty} t^{\dim(N)},$$

for a non-zero positive constant C_{ν_∞}.

Therefore when $G = M_\alpha$, $N = N_\alpha$, $e^{\mathfrak{h}_\mathbf{R}} = A_\alpha$ we have,

$$\mu(\omega_\infty, t\nu_\infty^\alpha) \sim C_{\nu_\infty^\alpha} t^{\dim(N_\alpha)} \quad \text{as} \quad t \to \infty.$$

- [HC1, Sec. 36, p. 190] Next, we consider the case of rank G= rank K. Let $\mathfrak{h}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$ with $\mathfrak{h} = \mathfrak{h}_I \oplus \mathfrak{h}_R$, dim $(e^{\mathfrak{h}_\mathbf{R}}) = 1$, and a fixed parabolic subgroup with Langlands decomposition $P = M^1 e^{\mathfrak{h}_R} N$. So dim $N = 1 + |Q_C|$.

  Let H_I be the analytic subgroup corresponding to $\mathfrak{h}_I$. Let Q be the set of positive roots of $(\mathfrak{g}, \mathfrak{h})$ which is the disjoint union of imaginary (Q_I), real (Q_R) and complex roots (Q_C). Let us denote by α the unique root in Q_R. For $a^* \in H_I^*$, let us define

$$\mu_0(a^*, \nu_\infty) := d(a^*)^{-1} \mathrm{Tr}\left(\frac{\pi i \nu_\infty^\alpha \sinh \pi i \nu_\infty^\alpha}{\cosh \pi i \nu_\infty^\alpha - \frac{(-1)^{\rho_\alpha}}{2}(\sigma_{a^*}(\gamma) + \sigma_{a^*}(\gamma^{-1}))}\right).$$

Here $d(a^*)$ is the degree, $\nu_\infty^\alpha = 2\frac{(\nu_\infty,\alpha)}{(\alpha,\alpha)}$, $\rho_\alpha = 2\frac{(\rho,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$, where ρ is the half of the sum of the positive roots of $(\mathfrak{g},\mathfrak{h}))$, σ_α is the irreducible representation of H_I whose character is a^* and γ is a fixed element in H_I. So the above expression has the form

$$u(z) = \frac{z\sinh \pi z}{\cosh \pi z + k},$$

for a fixed real number $k \geq -1$.

We know by Harish-Chandra [HC3] $\omega_\infty \in \mathcal{E}_2(M^1)$ corresponds to an element $a^* \in H_I^{*'}$, which we denote by $\lambda = \lambda(a^*) \in \mathfrak{h}_I^{*'}$ as the corresponding element in the Lie algebra of H_I. As before we define:

$$\widetilde{w_+}(\omega_\infty : \nu_\infty) = \prod_{\alpha \in Q_C} |(\lambda + \nu_\infty, \alpha)|\,.$$

Moreover, we define

$$\mu_0(\omega_\infty, \nu_\infty) = \frac{1}{|W(M^1/H_I)|} \sum_{W(M^1/H_I)} \mu_0(sa^*, \nu_\infty).$$

With these notations we can write the Harish-Chandra μ-function as follows

$$\mu(\omega_\infty, \nu_\infty) = C|\alpha|\mu_0(\omega_\infty, \nu_\infty)\widetilde{w_+}(\omega_\infty : \nu_\infty). \tag{4.1.4}$$

We can show that $u(tz) \sim C_z t$, as $t \to \infty$, when $z \in \mathbb{R}$ and $C_z > 0$. On the other hand $\widetilde{w_+}(\omega_\infty : \nu_\infty)$ is a polynomial in ν_∞ with degree $\dim(N) - 1$. Therefore we obtain the same asymptotic expression

$$\mu(\omega_\infty, t\nu_\infty) \sim C_\nu t^{\dim N} \quad \text{as} \quad t \to \infty.$$

With the previous notation (i.e. when $G = M_\alpha$, $N = N_\alpha$, $e^{\mathfrak{h}_\mathbf{R}} = A_\alpha$) we have, $\mu(\omega_\infty, t\nu_\infty^\alpha) \sim C_{\nu^\alpha} t^{\dim(N_\alpha)}$ as $t \to \infty$.

Asymptotic estimate

Because we have the polar decomposition $G = KP$, where $P = MAN$ is the Langlands decomposition, we have

$$\dim(G/K) = \dim(\frac{M^1}{K \cap M^1}) + \dim(A) + \dim(N).$$

Therefore in the case where $P = P_0$ is the minimal parabolic we have

$$\dim(G/K) = \dim(A_0) + \dim(N_0) = \dim(A_0) + \sum_{\alpha \in \Phi} \dim(N_{0,\alpha}),$$

and we have the following estimate of the density of the Plancherel measure:

$$\int_{\nu_\infty \in i\mathfrak{a}^*_{0,\mathbb{R}}:(\nu_\infty,\nu_\infty) \leq t^2} \mu(\omega_\infty, \nu_\infty) d\nu \sim C_G C_{\nu_\infty} t^{\dim(G/K)}, \quad \text{as} \quad t \to \infty. \tag{4.1.5}$$

Now for the pair (M_α, A_α) we have a polynomial bound. Here we invoke ([HC1, theorem 1, Sec. 25]). The theorem states that μ can be extended to the whole complex plane meromorphically. Moreover there exists $C, r \geq 0$ such that:

$$|\mu(\omega_\infty, \nu_\infty)| \leq C(1 + \|\mathrm{Im}(\nu_\infty)\|)^r.$$

Our job is to find an explicit value of r in the inequality mentioned in Harish-Chandra's paper.

In the case where $\mathcal{E}_2(M_\alpha) = \emptyset$, we have that $\mu(\omega_\infty, \nu^\alpha_\infty)$ is a polynomial in ν_∞ of degree $\dim(N_\alpha)$. So in this scenario we can take $r = \dim(N_\alpha)$. And similarly for the other case we can arrive at the same estimate, as $\sup_{z\in\mathbb{C}}|u(z)| \leq (1+|z|)$, whenever z is real and the other part, namely $\widetilde{w_+}(\omega_\infty : \nu_\infty)$, is a polynomial in ν_∞ of degree $\dim(N_\alpha) - 1$.

Hence combining with the product formula mentioned above we conclude that for some $C' > 0$ we have

$$\mu(\omega_\infty, \nu_\infty) \leq C'(1 + ||\nu_\infty||)^{\dim(N)}. \tag{4.1.6}$$

Remark: An important point to note here is that the constant C' does not really matter in terms of finding the main term of Weyl's law. The only constant that could matter is C_G [LV, Sec. 6.3]).

4.1.2 The $p-$adic case

The following observation is due to [HC3, p. 355]. The Plancherel measure in this case is defined on $\mathcal{E}_2(M_p)$, as is evident from the formulas above. For this case $\mathcal{E}_2(M_p)$ is compact, (and it can be written as $\sqcup_{\omega_p \in \mathcal{E}_2(M_p)} \mathcal{O}_{\omega_p}$) hence the asymptotic estimate will not change if we are to consider the plancherel inversion formula for the group G_S.

Hence combining the above two subsections we arrive at the estimate that

$$\int_{\nu \in i\mathfrak{a}_{0,\infty}^* \times \mathcal{E}_2(M_p) : (\nu_\infty, \nu_\infty) \leq t^2} \mu(\omega, \nu) d\nu d\omega_p \sim \alpha(G) t^{\dim(G_\infty / K_\infty)}, \quad \text{as} \quad t \to \infty. \tag{4.1.7}$$

4.2 Condition for purely cuspidal image

In this section we provide sufficient condition on the space of scalar-valued test functions so that the image of convolution operator on scalar-valued K_∞-finite automorphic forms only consists of cuspidal K_∞-finite automorphic forms . We closely follow [LV, prop. 3, second proof].

We need some preparation. We recall a couple of lemmas due to Harish-Chandra regarding vanishing conditions of Schwartz functions.

Let us recall some of the notations mentioned already in the preliminaries. Let $Q = M_Q N_Q$ be a standard parabolic subgroup of $G(\mathbb{R})$ and $G(\mathbb{Q}_p)$. Let $\mathcal{C}(G(\mathbb{R}), \tau)$ be the Harish-Chandra Schwartz space of vector-valued function which are τ-spherical. These are functions from $G(\mathbb{R})$ to $V_\tau \subset L^2(K_\infty \times K_\infty)$, where V_τ is viewed as a double representation of τ. The action of τ is given by

$$\tau(k)\phi(k_1 : g : k_2)\tau(k') = \phi(k_1 k : g : k' k_2).$$

Let $\mathcal{C}_{\text{cusp}}(M_Q, \tau_{M_Q})$ be the space of functions which are cuspidal, $\tau_{M_Q \cap K_\infty}$-spherical, $\mathfrak{Z}(M_Q)$-finite and A_Q-invariant. The L^2-completion of this space is generated by the square integrable matrix coefficients of finitely many classes of isomorphic unitary irreducible discrete series representations of M_Q which are A_Q-invariant. For more information about this space see

[Ar5, Chap. I, Sec. 2]. Define

$$\phi_\infty^Q(la) = \int_{N_Q} \phi_\infty(nla)dn, \quad \phi_\infty \in \mathcal{C}(G(\mathbb{R}), \tau),$$

for all $la \in M_Q^1 A_Q$. Moreover, we write $\phi_\infty^Q \sim 0$ if the following holds:

$$\int_{M_Q/A_Q} \Big(f(l), \phi_\infty^Q(la)\Big) dl = 0, \quad \forall f \in \mathcal{C}_{\text{cusp}}(M_Q, \tau_{M_Q^1})$$

for all $a \in A_Q$, where $(,)$ denotes the inner product in the vector space V_τ (which is viewed as a double representation space of the action of τ on $L^2(K_\infty \times K_\infty)$). Then by [HC3, Vol IV, p. 149] we have the following.

Lemma 4.2.1 (Archimedean case) *Let ϕ_∞ be an element in $\mathcal{C}(G(\mathbb{R}), \tau)$ such that $\phi_\infty^Q \sim 0$ for all parabolic subgroups $Q(\mathbb{R}) \subset G(\mathbb{R})$. Then $\phi_\infty = 0$.*

We now modify the above conditions for scalar valued bi-K_∞-finite functions, by using the ideas mentioned in [HC3, Vol IV, p. 175]. We denote the corresponding scalar valued function as Φ_∞. Again we write $\Phi_\infty^Q \sim 0$ if

$$\int_{M_Q^1} f(l)\Phi_\infty^Q(la)dl = 0, \quad \forall a \in A_Q.$$

The above integral is a function defined on A_Q, the split part of the center of M_Q. Hence if Φ_∞ is a compactly supported smooth function on $G(\mathbb{R})$, then the integral above is also a compactly supported function defined on A_Q. We recall some characterization of parabolic subgroups of standard Levi subgroups due to Harish-Chandra

Lemma 4.2.2 *Let $p \in S$. There is an one to one correspondence between parabolic subgroup $P(\mathbb{Q}_p)$ of $G(\mathbb{Q}_p)$ which are contained in $Q(\mathbb{Q}_p)$, and parabolic subgroups ${}^*P(\mathbb{Q}_p)$ of $M_Q(\mathbb{Q}_p)$. The correspondence are as follows: If $Q(\mathbb{Q}_p) = M_Q(\mathbb{Q}_p)N_Q(\mathbb{Q}_p)$ and $P(\mathbb{Q}_p) = M_P(\mathbb{Q}_p)N_P(\mathbb{Q}_p)$ are the corresponding Levi decompositions, then ${}^*P_Q = P(\mathbb{Q}_p) \cap M_Q(\mathbb{Q}_p) = M_P(\mathbb{Q}_p)N_Q^P(\mathbb{Q}_p)$ is the corresponding Levi decomposition, where $A_P(\mathbb{Q}_p) = A_Q^P(\mathbb{Q}_p)A_Q(\mathbb{Q}_p)$, $N_P(\mathbb{Q}_p) = N_Q^P(\mathbb{Q}_p)N_Q(\mathbb{Q}_p)$.*

Lemma 4.2.3 *Let $\Phi \in C_c^\infty(G(\mathbb{R}))_{K_\infty}^{K_\infty} \otimes \bar{\mathcal{H}}(G(K'_{S'}\backslash \mathbb{Q}_{S'}/K'_{S'}))$. Assume that:*

$$Ind(\omega_S, \nu_S)(\Phi) = 0, whenever \quad \nu|_{\Gamma_{A_{i,S}}} = 1 \quad \forall i \tag{4.2.1}$$

for all parabolic $P_S \subset Q_{i,S}$ whose Archimedean part is the chosen minimal parabolic subgroup, for all ω_S, equivalence classes of unitary irreducible representation of $M(\mathbb{Q}_S)$ whose Archimedean component is a discrete series and the non-Archimedean components are supercuspidal, such that $\omega_\infty \subset \tau|_{K_\infty \cap M_\infty}$. Then Φ satisfies the following equation for all i, for all $k_1, k_2 \in K_S$ and for all $m \in M_{i,S}$:

$$\sum_{\Gamma_{A_{i,S}}} \int_{N_{i,S}} \Phi(k_1 n \gamma m k_2) dn = 0. \tag{4.2.2}$$

Proof. Equation (4.2.1) implies that

$$\int\limits_{M(\mathbb{Q}_S)} \int\limits_{N(\mathbb{Q}_S)} \int\limits_{K_S} \Phi(k_1 mnk_2)\omega_S(m^{-1})e^{(-\nu_S+\rho_S)H_{P_S}(m^{-1})} dk_2 dndm = 0,$$

for all $k_1 \in K_S$, for all $P(\mathbb{Q}_S) = P_S \subset Q_{i,S} = Q_i(\mathbb{Q}_S)$, and for all $\nu_S \in \mathfrak{a}^*_{P_{0,S},\mathbb{C}}$ such that $\nu_S|_{\Gamma_{A_{i,S}}} = 1$. (we will suppress the iteration i in our discussion that follows) Hence we can drop the integration on K_S to get:

$$\int\limits_{M(\mathbb{Q}_S)} \int\limits_{N(\mathbb{Q}_S)} \Phi(k_1 mnk_2)\omega_S(m^{-1})e^{(-\nu_S+\rho_S)H_{P_S}(m^{-1})} dndm = 0.$$

Breaking the group $N(\mathbb{Q}_S)$ as product of $N_Q^P(\mathbb{Q}_S)$ and $N_Q(\mathbb{Q}_S)$ we have the following:

$$\int\limits_{M(\mathbb{Q}_S)} \int\limits_{N_Q^P(\mathbb{Q}_S)} \int\limits_{N_Q(\mathbb{Q}_S)} \Phi(k_1 mn'n''k_2)\omega_S(m^{-1})e^{(-\nu_S+\rho_S)H_{P_S}(m)} dn''dn'dm = 0. \tag{4.2.3}$$

As ${}^*P_Q(\mathbb{Q}_S) = M(\mathbb{Q}_S)N_Q^P(\mathbb{Q}_S) \subset M_{Q,S}$, is a parabolic subgroup of $M_{Q,S}$, we have $\Gamma_{A,S} \subset M(\mathbb{Q}_S)$ and $\Gamma_{A,S}$ centralizes $M(\mathbb{Q}_S)$. Hence (4.2.3) implies that

$$\begin{aligned} \int\limits_{\frac{M(\mathbb{Q}_S)}{\Gamma_{A,S}}} \int\limits_{N_Q^P(\mathbb{Q}_S)} \int\limits_{N_Q(\mathbb{Q}_S)} \sum_{\Gamma_{A,S}} \Phi(k_1 m_1 \gamma n'n''k_2)\omega_S(m_1^{-1}) \\ e^{(-\nu_S+\rho_S)H_{P_S}(m_1)} dn''dn'dm_1 = 0. \end{aligned} \tag{4.2.4}$$

We will apply Fubini's theorem to change the order of the integral on $N_Q(\mathbb{Q}_S)$ and sum on $\Gamma_{A,S}$ (as Φ is compactly supported the integral above is convergent). Moreover we can break down the set $\frac{M(\mathbb{Q}_S)}{\Gamma_{A,S}}$ into product of $\frac{M(\mathbb{Q}_S)}{A(\mathbb{Q}_S)}$ and $\frac{A(\mathbb{Q}_S)}{\Gamma_{A,S}}$. Hence we can think of the above integral as the Fourier transform of the integral

$$\int\limits_{\frac{M(\mathbb{Q}_S)}{A(\mathbb{Q}_S)}} \int\limits_{N_Q^P(\mathbb{Q}_S)} \int\limits_{N_Q(\mathbb{Q}_S)} \sum_{\Gamma_{A,S}} \Phi(k_1 m_1 \gamma n' n'' k_2) f_s(m_1) dn'' dn' dm_1,$$

where f_s is the product of the coefficient of discrete series (at the Archimedean place) and cuspidal representations (at the non-Archimedean places) of $M(\mathbb{Q}_S)$. Therefore, using the injectivity of Fourier transform on functions defined on $M_{Q,S}$ which are compactly supported modulo the central direction of $M_{Q,S}$ (which holds for non-Archimedean case by combining [B, Th. 25], [BDK] and [CH, Section 5.7]), (4.2.4) implies that

$$\sum_{\Gamma_{A_{i,S}}} \int_{N_{i,S}} \Phi(k_1 n \gamma m k_2) dn = 0. \tag{4.2.5}$$

■

Lemma 4.2.4 *If $\Phi \in C_c^\infty(G(\mathbb{R}))_{K_\infty}^{K_\infty} \otimes \bar{\mathcal{H}}(G(K'_{S'} \backslash \mathbb{Q}_{S'} / K'_{S'}))$ satisfies the eq.* (4.2.2) *then Φ maps the elements in $L^1_{loc}(\Gamma \backslash G_S)$ to $L^2_{cusp}(\Gamma \backslash G_S)$ as a convolution operator.*

Proof. We fix $1 \le i \le r$ and let $\Psi \in L^2(\Gamma \backslash G_S)_{K_\infty}$ and $\Psi_i(g) = \Psi(\delta_i g)$. Then it is easy to see that $\Psi_i \in L^2(\Gamma_i \backslash G_S)_{K_\infty}$. To get the purely cuspidal image we need the following:
For $\Phi = \Phi_\infty \Phi_{S \backslash \infty}$, where $\Phi_\infty \in C_c^\infty(G(\mathbb{R}))_{K_\infty}^{K_\infty}$, $\Phi_{S\backslash\infty} \in \bar{\mathcal{H}}(G(K'_{S'} \backslash \mathbb{Q}_{S'} / K'_{S'}))$ and $\Psi_i \in L^2(\Gamma_i \backslash G_S)_{K_\infty}$,

$$\int_{\Gamma_{N_{i,S}} \backslash N_{i,S}} \Psi_i \star \Phi(nx) dn = 0.$$

This is equivalent to:

$$\int_{\Gamma_{N_{i,S}} \backslash N_{i,S}} \int_{G_S} \Phi(y^{-1} n x) \Psi_i(y) dy dn = 0.$$

Now we break down the integral on G_S as an integral on $\Gamma_i \backslash G_S$ and a discrete sum on Γ_i to get the equivalent relation:

$$\int\limits_{\Gamma_{N_{i,S}} \backslash N_{i,S}} \sum_{\Gamma_i} \int_{\Gamma_i \backslash G_S} \Phi(y^{-1} \gamma^{-1} n x) \Psi(y) dy dn = 0,$$

for all x, y. This is equivalent to:

$$\int\limits_{\Gamma_{N_{i,S}}\backslash N_{i,S}} \sum_{\Gamma_i} \Phi(y^{-1}\gamma^{-1}nx)dn = 0.$$

After swapping the integral and the sum (as the sum over Γ_i is locally finite, and Φ is compactly supported, we can use the Monotone Convergence Theorem or Dominated Convergence Theorem) and rearranging the domains, we get a sum over $(\Gamma_i \bigcap N_{i,S})\backslash\Gamma_i$ and an integral on $N_{i,S}$ Therefore, we arrive at the equivalent condition:

$$\sum_{\Gamma_{N_{i,S}}\backslash\Gamma_i} \int_{N_{i,S}} \Phi(y^{-1}\gamma nx)dn = 0. \tag{4.2.6}$$

We replace x and y with their respective Iwasawa decompositions, i.e. $x = m_1n_1k_1$ and $y = m_2n_2k_2$. Moreover, we can write the sum over $\frac{\Gamma}{\Gamma\bigcap N_{i,S}}$ as sum over $\Gamma\bigcap A_{i,S} = \Gamma_{A_{i,S}}$ cosets. Therefore as $M_{i,S}$ centralizes $A_{i,S}$, we can write $k_2^{-1}n_2^{-1}m_2^{-1}\gamma nm_1n_1k_1 = k_2^{-1}n_2^{-1}\gamma m_2^{-1}nm_1n_1k_1$. As $M_{i,S}$ normalizes $N_{i,S}$, with a modular factor we have:

$$k_2^{-1}n_2^{-1}\gamma m_2^{-1}nm_1n_1k_1 = k_2^{-1}n_2^{-1}n'\gamma m_2^{-1}m_1n_1k_1.$$

But as the modular factor, a scalar, only depends on m_2, and γ belongs to a discrete subgroup, we can ignore that above. Therefore we can finally write the argument of Φ as $k_2^{-1}n_2^{-1}n'n''\gamma m_2^{-1}m_1k_1 = k_2n\gamma mk_2$. Hence, we can rewrite the condition in (4.2.6) as follows: For all $k_1, k_2 \in K_S$ and for all $m \in M_{i,S}$ [LV, eq. 4.6]

$$\sum_{\Gamma_{A_{i,S}}} \int_{N_{i,S}} \Phi(k_1n\gamma mk_2)dn = 0. \tag{4.2.7}$$

Consequently (4.2.2) is the sufficient condition. ■

In the next step we need to find a non-zero combined test functions on G_S, which is bi-K_∞-finite,K_∞-central and compactly supported at the Archimedean place and a function from the Hecke algebra at the non-Archimedean places that satisfies the above conditions. For parabolic subgroups P_S, whose Archimedean component is the chosen standard minimal parabolic subgroup, (4.2.1) would become as follows: For all $\nu_S \in \mathfrak{a}^*_{P_S} \supset \mathfrak{a}^*_{Q_{i,S}}$ such that whenever for all i, $\nu_S|_{\Gamma_{A_{i,S}}} = 1$, we have

$$\mathrm{Ind}_{P_S}^{G_S}(\omega_S, \nu_S)(\Phi) = 0, \tag{4.2.8}$$

for all ω_S discrete series representation at the Archimedean place and supercuspidal at the non-Archimedean places of $M_{P,S}$ such that $\tau|_{M_{0,\infty}} \supset \omega_\infty$. By the description of arithmetic tori [PR, Theorem 5.12], ν_S should have the property that $\nu_p = \nu_q$, for all $p, q \in S$. Let $\mathfrak{Z}(G(\mathbb{Q}_p))$ be the ring of regular functions defined on union of Benrstein components $\Omega(G(\mathbb{Q}_p))$ ([MT, 2.3.1]). Combining for all $p \in S\backslash\infty$ we define $\mathfrak{Z}(\mathbb{Q}_{S\backslash\infty})$ to be the set of Fourier transform of Bernstein center for $G(\mathbb{Q}_{S\backslash\infty})$. Let z_i be the elements in the Bernstein center for each $Q_i(\mathbb{Q}_{S\backslash\infty})$ [MT, 2.2.1]. Let $\hat{z}_i \in \mathfrak{Z}(G(\mathbb{Q}_p))$. We can form the test function

$$\Phi = \Phi_\infty \prod_i (z_i \star \mathbb{1}_{K'}).$$

We have to find a regular function R defined on $\Omega(G(\mathbb{Q}_{S\backslash\infty}))$ such that $R(\nu_{S\backslash\infty}) = 0$, whenever $\nu_p = \nu_q$, for all $p, q \in S$ and $\nu_p \in \mathfrak{a}^*_{P_S}$. We can find a polynomial that satisfies this property for every $P_S \subset Q_{i,S}$. Hence, we could apply the Arthur's Paley-Wiener Theorem at the Archimedean place and matrix Paley-Wiener Theorem for Hecke alegebra by Bernstein [B, Theorem 25] at the non-archimedean place to construct a non-zero test function Φ.

4.3 The pre-trace formula

In this section we write the pre-trace formula. We choose a test function whose Archimedean component is a $\tau-$spherical function belonging to the convolution algebra $C_c^\infty(G(\mathbb{R}), \tau, \tau)$, satisfying the identity

$$\phi(k_1 g k_2) = \tau(k_2)^{-1}\phi(g)\tau(k_1)^{-1}$$

and a scalar valued function at the non-Archimedean places from the Hecke algebra $\bar{\mathcal{H}}(G(K'_{S'}\backslash\mathbb{Q}_{S'}/K'_{S'}))$. Suppose it also satisfies the condition of cuspidality described in the previous section. It acts on $\Gamma-$invariant L^2 eigensection (K_∞-finite,$K'_{S'}$-fixed) $e_\lambda(x)$ (orthonormal with respect to the inner product mentioned in the introduction) of the Casimir operator(defined for sections of vector bundles), with the eigenvalue parameter defined as λ. We define the convolution action as follows:

$$\begin{aligned} e_\lambda \star \phi(x) &= \int_{G_S} \phi(y^{-1}x) e_\lambda(y) dy \\ &= \int_{\Gamma \backslash G_S} \sum_{\gamma^{-1}\in\Gamma} \phi(y^{-1}\gamma^{-1}x) e_\lambda(y) dy \\ &= \int_{\Gamma \backslash G_S} K(x,y) e_\lambda(y) dy \\ &= \int_{\Gamma \backslash G_S / K_\infty} K(x,y) e_\lambda(y) dy \end{aligned}$$

In the last equation we have used the fact that $K(x,y)e_\lambda(y)$ is $K_\infty-$invariant on y. Then the spectral expansion of $K(x,y)$ can be written as follows:

$$K(x,y) = \sum_{\lambda,\mu} (e_\lambda \star \phi, e_\mu) e_\mu(x) \otimes e_\lambda(y)^*,$$

where $e_\mu(y)^*$ denotes the dual vector which acts on $f(y)$ through the pairing $\langle f(y), e_\mu(y)\rangle_{V_\tau}$ on the fiber $(E_\tau)_y$ [Dui, eq. 7.3]. If we let $x = y$, then the spectral side will have the following form:

$$K(x,x) = \sum_{\lambda,\mu} (e_\lambda \star \phi, e_\mu) e_\mu(x) \otimes e_\lambda(x)^*.$$

Therefore, taking the trace on both sides we get

$$\mathrm{Tr} K(x,x) = \sum_{\lambda} (e_\lambda \star \phi, e_\lambda) e_\lambda(x) \otimes e_\lambda(x)^*.$$

Consider a compact subset $\Omega \subset \Gamma\backslash G_S$, whose measure is arbitrarily close to $\mathrm{Vol}(\Gamma\backslash G_S)$. We take the pre-image of Ω in G_S and call it $\tilde{\Omega}$. Unwinding the sum on the left hand side we get

$$TrK(x,x) = Tr\phi(e) + \sum_{\gamma\in Z} \mathrm{Tr}\phi(x^{-1}\gamma x),$$

The set Z will have the following form :

$$Z = \Big(\Gamma\backslash\{e\}\Big) \bigcup \Big(xgx^{-1} : x \in \tilde{\Omega}, \text{x lies in support of } \mathrm{Tr}(\phi)\Big).$$

The cardinality of Z would be finite, and would depend only on $\tilde{\Omega}$ and the support of $\mathrm{Tr}(\phi)$. Integrating both sides over $\tilde{\Omega}$, we obtain the following:

$$\int_{\tilde{\Omega}} TrK(x,x)dx \leq \int_{\Gamma\backslash G_S/K_\infty} TrK(x,x)dx = \sum_\lambda (e_\lambda \star \phi, e_\lambda)$$

To make sure we have a self adjoint convolution operator we need $\phi(x) = \overline{\phi(x^{-1})^T}$. To achieve the self-adjointness we replace ϕ with $\phi \star \widetilde{\phi}$, where $\widetilde{\phi}(x) = \overline{\phi(x^{-1})^T}$. Hence, the right hand side of the above inequality becomes $\sum_\lambda (e_\lambda \star \phi, e_\lambda \star \phi)$.

Now by a theorem of Gelfand, Graev and Piatetski-Shapiro [Bmp, prop. 3.2.3] which states that the convolution operator on the scalar-valued automorphic forms is a compact operator. Therefore we obtain: $\sum_\lambda (e_\lambda \star \phi, e_\lambda \star \phi) < \infty$. Hence, we have

$$\mathrm{Tr}(\phi \star \widetilde{\phi}(e))\mathrm{Vol}(\Omega) + \sum_{\gamma \in Z} \int_\Omega \mathrm{Tr}(\phi \star \tilde{\phi})(x^{-1}\gamma x) \leq \sum_\lambda (e_\lambda \star \phi, e_\lambda \star \phi). \tag{4.3.1}$$

We now give a representation theoretic interpretation of $\sum_\lambda (e_\lambda \star \phi, e_\lambda \star \phi)$. Let $\pi_\infty \in \Pi_{\mathrm{cusp}}(G(\mathbb{R}), \tau)$ be the Archimedean part of irreducible unitary representation π_S which appears as a subrepresentation of right regular representation of $G(\mathbb{Q}_S)$ on $L^2_{\mathrm{cusp}}(\Gamma\backslash G(\mathbb{Q}_S), \tau)$ with multiplicities $m(\pi_\infty)$, and let H_{π_∞} be the corresponding Hilbert space. Let $H_{\pi_\infty}(\tau)$ be the $\tau-$isotypic subspace. Then using [BM, Thm. 3.3] we have:

$$\sum_\lambda (e_\lambda \star \phi, e_\lambda \star \phi) = \sum_{\Pi_{\mathrm{cusp}}(G(\mathbb{R}),\tau)} m(\pi_\infty)\left(\sum_{i=1}^{m} (e_i \star \phi, e_i \star \phi)\right), \tag{4.3.2}$$

where $m = \dim(\mathrm{Hom}_{K_\infty}(\mathcal{H}_{\pi_\infty}(\tau), V_\tau))$.

4.4 An approximation lemma

In this section we find a family of test functions

$$H_{S,t} = H_{\infty,t} \cdot \mathbb{1}_{K'},$$

for $0 \leq t < 1$ that satisfy certain approximations. For the rest of the section and beyond we will write $S\backslash\infty = S'$ and $K' = K'_{S\backslash\infty}$. We prove a slight generalization of [LV, Lemma 2] below.

Let $\mathfrak{h} = i\mathfrak{h}_{K_\infty} + \mathfrak{a}_{0,\infty}$ be the Cartan subalgebra of $\mathfrak{U}(\mathfrak{g}_\infty)$. Let $\mathfrak{h}_\mathbb{C} = \mathfrak{h} \otimes \mathbb{C}$ be the complexification of the Cartan subalgebra. Let $\mathfrak{h}_\mathbb{C}^*$ be the dual of $\mathfrak{h}_\mathbb{C}$. We fix $0 < \epsilon < 1$. By [LV, (5.12)-(5.15)] we know there exist a nonempty open set of Schwarz functions ψ defined on cylinders

$$\{\lambda_\infty \in \mathfrak{h}_\mathbb{C}^* : |\mathrm{Re}(\lambda_\infty)| \leq a\}$$

that satisfy the following conditions:

- $0 \leq \psi(\lambda_\infty) < 1, \quad \text{when} \quad \|\lambda_\infty\| \leq 1, \lambda_\infty \in \mathfrak{h}_{K_\infty}^* + i\mathfrak{a}_{0,\infty}^*.$
- $\int\limits_{i\mathfrak{a}_{0,\infty}^*} |\psi(\nu_{K_\infty} + \nu_\infty) - \chi(\nu_{K_\infty} + \nu_\infty)| \, (1 + \|\nu_\infty\|)^{\dim(N_0)} d\nu_\infty \leq \epsilon$, for fixed $\nu_{K_\infty} \in \mathfrak{h}_{K_\infty,\mathbb{C}}^*$ such that $\mathrm{Re}(\nu_{K_\infty})$ is bounded.
- $\sup\limits_{\|\lambda_\infty\|>1} (1 + \|\lambda_\infty\|)^{d+1} \, |\psi(\lambda_\infty)| \leq \epsilon.$

Here $\chi(\nu_{K_\infty} + \nu_\infty)$ denotes the characteristic function of the sphere $\|\nu_{K_\infty} + \nu_\infty\| \leq 1$. Without loss of generality we may assume that ψ can be extended to a holomorphic function on $\mathfrak{h}_\mathbb{C}^*$, as the Fourier transform of ψ has compact support. Let $\mathbb{O}_\mathbb{C}$ be the orthogonal group of $\mathfrak{h}_\mathbb{C}^*$ with respect to the inner product $\langle , \rangle$. This inner product is induced from the Killing form on $\mathfrak{h}_\mathbb{C}$. By averaging we can make ψ as $\mathbb{O}_\mathbb{C}$-invariant function. Therefore $\psi(\lambda_\infty)$ depends only on $\langle \lambda_\infty, \lambda_\infty \rangle$. Let d_{ω_∞} denote the degree of equivalent classes of square integrable irreducible representations ω_∞ of $M_{0,\infty}^1$. Using Frobenius Reciprocity we see that for the case of minimal parabolic $P_{0,\infty}$, we have

$$[\mathrm{Ind}(\omega_\infty, \nu_\infty)|_{K_\infty} : \tau] = [\tau|_{M_{0,\infty}^1} : \omega_\infty].$$

Therefore,

$$\sum_{\omega_\infty \in \mathcal{E}_2(M_{0,\infty}^1)} d_{\omega_\infty} [\mathrm{Ind}(\omega_\infty, \nu_\infty)|_{K_\infty} : \tau] = \sum_{\omega_\infty \in \mathcal{E}_2(M_{0,\infty}^1)} d_{\omega_\infty} [\tau|_{M_{0,\infty}^1} : \omega_\infty] = d_\tau.$$

Put $m_{\omega_\infty} = [\tau|_{M_{0,\infty}^1} : \omega_\infty]$. Let $C_c^\infty(\mathfrak{h})$ be the set of compactly supported smooth functions (i.e. set of multipliers for the convolution algebra $C_c^\infty(G(\mathbb{R}))_{K_\infty}^{K_\infty}$). Then by the Euclidean

Paley-Wiener theorem there exists $\zeta \in C_c^\infty(\mathfrak{h})$ such that its Laplace-Fourier transform $\hat{\zeta}(\lambda_\infty)$ satisfies:

$$\hat{\zeta}(\lambda_\infty) = \psi(\lambda_\infty), \quad \forall \lambda_\infty \in \mathfrak{h}_{\mathbb{C}}^*.$$

There exists a function $H_\infty^\sharp \in C_c^\infty(G(\mathbb{R}))_{K_\infty}^{K_\infty}$ such that $\mathrm{Ind}(\omega_\infty, \nu_\infty)(H_\infty^\sharp) = \mathrm{Ind}(\omega_\infty, \nu_\infty)(d_\tau \chi_\tau)$ [GV, Lemma 1.3.2]. Now using the Arthur's theorem on multipliers [Ar5] we can choose a family of functions $H_{\infty,t,\zeta}^\sharp \in C_c^\infty(G(\mathbb{R}))_{K_\infty}^{K_\infty}$, such that their operator valued Fourier transforms are

$$\mathrm{Ind}(\omega_\infty, \nu_\infty)(H_{\infty,t,\zeta}) = \hat{\zeta}(\nu_{\omega_\infty} + t\nu_\infty)\mathrm{Ind}(\omega_\infty, \nu_\infty)(d_\tau \chi_\tau),$$

for $0 < t \leq 1$. Let $H_{S,t,\zeta}^\sharp = H_{\infty,t,\zeta}^\sharp \cdot \mathbb{1}_{K'}$, for K' an arbitrarily chosen open compact subgroup of $G_{S'}$. Then

$$\left|\left|\widehat{H_{\infty,t,\zeta}^\sharp}(\omega_\infty, \nu_\infty)\right|\right|^2 = \int\limits_{K_\infty} \left|\widehat{H_{\infty,t,\zeta}^\sharp}(\omega, \nu)(1:1:k)\right|^2 dk = d_\omega \left|\left|\mathrm{Ind}(\omega_\infty, \nu_\infty)(H_{\infty,t,\zeta}^\sharp)\right|\right|_{\mathrm{HS}}^2.$$

Therefore, from the above choice of Schwartz function we have

$$\left|\left|\widehat{H_{\infty,t,\zeta}^\sharp}(\omega_\infty, \nu_\infty)\right|\right|^2 = d_\tau d_\omega m_{\omega_\infty} \|\psi(\nu_{\omega_\infty} + t\nu_\infty)\|^2.$$

The following estimate will be instrumental in proving the main estimate in Weyl's law. Let $0 < \epsilon < 1$ and choose $H_{\infty,t,\zeta}^\sharp$ depending on ϵ.

Lemma 4.4.1 *There exists $C_1 > 0$ such that for sufficiently small $0 < t \leq 1$ and for the minimal Parabolic $P_{0,\infty} = M_{0,\infty}^1 A_{0,\infty} N_{0,\infty}$ we have*

$$\left| t^d \sum_{\omega \in \mathcal{E}_2(M_{0,\infty}^1)} \int_{i\mathfrak{a}_{0,\infty}^*} \left|\left|\widehat{H_{\infty,t,\zeta}^\sharp}(\omega_\infty, \nu_\infty)\right|\right|^2 \mu(\omega_\infty, \nu_\infty) d\nu - d_\tau^2 \alpha(G_\infty) \right| \leq C_1 \epsilon.$$

Proof. Recall the Plancherel inversion formula at the real place

$$f \star \tilde{f}(1) = \sum_{\mathcal{P}} n(\mathcal{P})^{-1} \sum_{P \in \mathcal{P}} \sum_{\omega \in \mathcal{E}_2(M_\infty^1)} (\frac{1}{2\pi i})^q \int_{i\mathfrak{a}_{M_\infty}^*} \left|\left|\widehat{f}(\omega, \nu)\right|\right|^2 \mu(\omega, \nu) d\nu. \tag{4.4.1}$$

Here, $\mathcal{P}$ denotes the associated classes pf parabolic subgroups. The integer q denotes the dimension of respective $i\mathfrak{a}_{M_\infty}^*$. We are interested on the summand that corresponds to the minimal parabolic. For the sum and integral involving the minimal parabolic subgroup $P_{0,\infty} = M_{0,\infty}A_{0,\infty}N_{0,\infty}$, and $\dim(i\mathfrak{a}^*_{0,\infty}) = r$ we have

$$\limsup_{t\to 0}\left| t^d \sum_{\omega_\infty \in \mathcal{E}_2(M^1_{0,\infty})} (\frac{1}{2\pi i})^r \int_{i\mathfrak{a}^*_{0,\infty}} \|\widehat{H^\sharp_{\infty,t,\zeta}}(\omega_\infty,\nu_\infty)\|^2 \mu(\omega_\infty,\nu_\infty)d\nu_\infty - d^2_\tau \alpha(G_\infty)\right|$$

$$= \limsup_{t\to 0}\left| t^d \sum_{\omega_\infty \in \mathcal{E}_2(M^1_{0,\infty})} (\frac{1}{2\pi i})^r \int_{i\mathfrak{a}^*_{0,\infty}} \|\widehat{H^\sharp_{\infty,t,\zeta}}(\omega_\infty,\nu_\infty)\|^2 \mu(\omega_\infty,\nu_\infty)d\nu_\infty - d^2_\tau \alpha(G_\infty)\right|$$

$$\leq \left| \limsup_{t\to 0} t^d \sum_{\omega_\infty \in \mathcal{E}_2(M^1_{0,\infty})} (\frac{1}{2\pi i})^r \int_{i\mathfrak{a}^*_{0,\infty}} d_\tau d_\omega m_{\omega_\infty} \left| \|\psi(\nu_{\omega_\infty} + t\nu_\infty)\|^2 - \chi(\nu_{\omega_\infty} + t\nu_\infty)\right| \mu(\omega_\infty,\nu_\infty)d\nu_\infty\right|$$

$$\leq \left| \limsup_{t\to 0} t^d \sum_{\omega_\infty \in \mathcal{E}_2(M^1_{0,\infty})} (\frac{1}{2\pi i})^r \int_{i\mathfrak{a}^*_{0,\infty}} d_\tau d_\omega m_{\omega_\infty} \left| \|\psi(\nu_{\omega_\infty} + \nu_\infty)\|^2 - \chi(\nu_{\omega_\infty} + \nu_\infty)\right| \mu(\omega_\infty,t^{-1}\nu_\infty)d(t^{-1}\nu_\infty)\right|$$

$$\leq \left| \sum_{\omega_\infty \in \mathcal{E}_2(M^1_{0,\infty})} (\frac{1}{2\pi i})^r \int_{i\mathfrak{a}^*_{0,\infty}} d_\tau d_\omega m_{\omega_\infty} \left| \|\psi(\nu_{\omega_\infty} + \nu_\infty)\|^2 - \chi(\nu_{\omega_\infty} + \nu_\infty)\right| \Big(1 + \|\nu_\infty\|\Big)^{\dim(N_0)} d(\nu_\infty)\right|$$

$$\leq C_1\epsilon$$

In the last step we use the second condition of ψ given at the beginning of this section. ■

4.5 Plancherel inversion and test functions

In this section we describe the choice of test functions. We start by recalling a result of Camporesi, which identifies the endomorphism valued convolution algebra with scaler valued functions.

Proposition 4.5.1 *[Cmp1, Prop 2.1] Let τ be the irreducible K_∞-type as before of dimension d_τ. then the endomorphism valued convolution algebra isomorphic to scalar valued bi-K_∞-finite, K_∞-central function space. The anti-isomorphism is given by the following map*

$$f \mapsto F = d_\tau Tr(f). \tag{4.5.1}$$

Moreover it satisfies the following relations:

$$d_\tau\, Tr(f_1 \star f_2) = d_\tau\, Tr(f_2) \star d_\tau\, Tr(f_1),$$

$$d_\tau \chi_\tau \star F = F = F \star d_\tau \chi_\tau.$$

4.5.1 Test functions

The following steps will describe our test functions. We closely follow [LV, Lemma 2,3].

● We choose a function $\Phi^\sharp_S = \Phi^\sharp_\infty \Phi_{S'}$, where $\Phi^\sharp_\infty \in C_c^\infty(G(\mathbb{R}))^{K_\infty}_{K_\infty}$ and $\Phi_{S'} \in \bar{\mathcal{H}}(K^{'}_{S'} \backslash G(\mathbb{Q}_{S'})/K^{'}_{S'})$, so that the cuspidality condition holds true, i.e. $\Phi^\sharp_S$ satisfies (4.2.1). By the isomorphism in (4.5.1) we have a function $\phi_\infty \in C_c^\infty(G(\mathbb{R}), \tau, \tau)$ so that $d_\tau \mathrm{Tr}\phi_\infty = \Phi^\sharp_\infty$. Let ϕ_S be the product of ϕ_∞ and $\Phi_{S'} \in \bar{\mathcal{H}}(K^{'}_{S'} \backslash G(\mathbb{Q}_{S'})/K^{'}_{S'})$.

● Next we choose a family of functions

$$h_{\infty,t,\zeta} \in C_c^\infty(G(\mathbb{R}), \tau, \tau) \quad \text{for} \quad 0 < t \leq 1.$$

From the properties mentioned in the previous section, we can choose an entire Schwartz function $\widehat{H_{\infty,\zeta}}(\omega, \nu)$ that satisfies Lemma 4.4.1. We form a family $h_{\infty,t,\zeta}$ for $0 < t \leq 1$, so that $d_\tau \mathrm{Tr} h_{\infty,t,\zeta} = H^\sharp_{\infty,t,\zeta}$. We multiply $h_{\infty,t,\zeta}$ with $\mathbb{1}_{K'}$, and call this function $h_{S,t,\zeta}$.

● Finally, choose a sequence $\Phi^\sharp_{n,S}$ that satisfies the condition of cuspidality. Let $Z_\infty \in C_c^\infty(\mathfrak{h})$ be an element in the set of Archimedean multipliers. Then by the Euclidean Paley-Wiener theorem, $\hat{Z}_\infty$ is bounded on the set $\{\lambda_\infty \in \mathfrak{h}^*_{\mathbb{C}} : \mathrm{Im}(\lambda_\infty) = 0\}$. If we choose Φ_S so that it satisfies the condition of cuspidality, then the fourier transform of elements of the Bernstein center at the non-Archimedean places is bounded on the set of unitary unramified characters. Suppose the bound is $B > 0$. Following [LV, pp. 245-246] we construct such a sequence. Let

$$P_n(Z_S) = 1 - \Big(1 - \frac{(Z_S)^2}{B^2}\Big)^n.$$

Let $\Phi^\sharp_S = Z_S \star f_S$, where $\|\pi_\infty(f_\infty)\|^2_{\mathrm{HS}} = 1$ and $f_{S\backslash\infty} = \mathbb{1}_{K'}$. As the multipliers on the space of bi-$K_\infty-$finite compactly supported smooth functions on $G(\mathbb{R})$ and multipliers on space of locally constant functions on $G(\mathbb{Q}_{S'})$ are equipped with convolution, (thought of as a multiplication) we can define a sequence $\Phi^\sharp_{n,S} = P_n(Z_S) \star f_S \star \widetilde{f_S}$. P_n will satisfy the following properties

- $P_n(0) = 0$

- $\mathrm{Ind}(\omega_S, \nu_S)(\Phi^\sharp_{n,S}) = P_n(\hat{Z}_S)(\mathrm{Ind}(\omega_S, \nu_S)(f_S)\mathrm{Ind}(\omega_S, \nu_S)(f_S)^*)$.

  Notice that here multiplication of Fourier transforms are defined as [Ar3, part II,pp. 1.1]. Therefore $\Phi^\sharp_{n,S}$ will satisfy (4.2.1).

Let $\phi_{n,S}$ be the endomorphism valued test functions corresponding to $\Phi^\sharp_{n,S}$ as $\phi_{n,S}$. We apply the partial trace formula on the family of test functions $\phi_{n,S}\star h_{S,t,\zeta}$. Write $\phi_{n,S,t,\zeta} = \phi_{n,S}\star h_{S,t,\zeta}$. We obtain

$$\begin{aligned} d_\tau \mathrm{Tr}((\phi_{n,S,t,\zeta} \star \widetilde{\phi_{n,S,t,\zeta}})(e))\mathrm{Vol}(\Omega) + d_\tau \textstyle\sum_{\gamma\in Z} \int_\Omega \mathrm{Tr}(\phi_{n,S,t,\zeta} \star \widetilde{\phi_{n,t,S}})(x^{-1}\gamma x) \\ \leq d_\tau \sum_\lambda (e_\lambda \star \phi_{n,S,t,\zeta} \star \widetilde{\phi_{n,S,t,\zeta}}, e_\nu) \\ = d_\tau \sum_\lambda \sum_{-\nu_{\pi_\infty}=\lambda} m(\pi)\mathrm{Tr}(\pi(\mathrm{Tr}(\phi_{n,,S,t,\zeta} \star \widetilde{\phi_{n,S,t,\zeta}}))). \end{aligned} \tag{4.5.2}$$

4.5.2 Plancherel inversion

We now recall the Plancherel theorem at Archimedean place as in [Ar3, part II, (2.1)]

$$\begin{aligned} \widetilde{\Phi^\sharp_{n,\infty,t,\zeta}} \star \Phi^\sharp_{n,\infty,t,\zeta}(e)\widetilde{\Phi_{n,S'}} \star \Phi_{n,S'}(e) = \textstyle\sum_{\mathcal{P}\in \mathrm{Cl}(G_\infty)} n(\mathcal{P})^{-1} \sum_{P\in\mathcal{P}} \sum_{\omega_\infty\in\mathcal{E}_2(M_\infty)} (\frac{1}{2\pi i})^q \\ \int_{i\mathfrak{a}^*_\infty} \left|\left|\mathrm{Ind}(\omega_\infty, \nu_\infty)(\Phi^\sharp_{n,\infty,t,\zeta})\right|\right|^2_{\mathrm{HS}} \mu(\omega_\infty, \nu_\infty) d\nu_\infty \widetilde{\Phi_{n,S'}} \star \Phi_{n,S'}(e). \end{aligned}$$

We have the following convergences as $n \to \infty$

$$\mathrm{Ind}(\omega_{S'})(\Phi_{n,S'}) \to \mathrm{Ind}(\omega_{S'})(\mathbb{1}_{K'}), \quad \text{and} \quad \|\mathrm{Ind}(\omega_\infty, \nu_\infty)(\Phi^\sharp_{n,\infty})\| \to 1.$$

Now if we take the limit inside the norm (due to continuity) and inside the Fourier transformation (due to isometry)[Ar4, p. 4719] the above integrand converges to

$$\left|\left|\widehat{H^\sharp_{\infty,t,\zeta}}(\omega_\infty, \nu_\infty)\right|\right|^2 \mu(\omega_\infty, \nu_\infty).$$

Therefore, we see that integrand corresponding to the minimal parabolic summand in the Plancherel Formula can be divided into two sets $X = \{\nu \in i\mathfrak{a}^*_{0,\infty} : P_n(\hat{Z}_S) \leq \epsilon\}$ and its complement X^c. As we take $\lim \epsilon \to 0$, the set X will have measure 0, and on X^c the integrand will become $\|\widehat{H^\sharp_{\infty,t,\zeta}}(\omega_\infty, \nu_\infty)\|^2 \mu(\omega_\infty, \nu_\infty)$. From the discussion above, it is clear that the following estimate will be enough for us to arrive at the main term as $\limsup t \to 0$.

Lemma 4.5.1 *For all n, there exists $C_1 > 0$ such that*

$$\left| t^d \sum_{\mathcal{P} \in Cl(G_\infty)} n(\mathcal{P})^{-1} \sum_{P \in \mathcal{P}} \sum_{\omega_\infty \in \mathcal{E}_2(M_\infty)} (\frac{1}{2\pi i})^q \int_{i\mathfrak{a}^*_\infty} \left\| \widehat{H^\sharp_{\infty,t,\zeta}}(\omega_\infty, \nu_\infty) \right\|^2 \mu(\omega_\infty, \nu_\infty) d\nu_\infty \right.$$
$$\left. \widetilde{\Phi_{n,S'}} \star \Phi_{n,S'}(e) - d_\tau^2 \alpha(G_\infty) \right| \leq C_1 \epsilon.$$

Proof. We can ignore the terms related to the $p-$adic Plancherel formula as the tempered parameters in this case are finite disjoint unions of compact orbifolds, hence those terms will be automatically bounded. Therefore, we only concentrate on the Archimedean part. We have

$$\left| t^d \sum_{\mathcal{P} \in \mathrm{Cl}(G_\infty)} n(\mathcal{P})^{-1} \sum_{P \in \mathcal{P}} \sum_{\omega_\infty \in \mathcal{E}_2(M_\infty)} (\frac{1}{2\pi i})^q \int_{i\mathfrak{a}^*_\infty} \left\| \widehat{H^\sharp_{\infty,t,\zeta}}(\omega_\infty, \nu_\infty) \right\|^2 \mu(\omega_\infty, \nu_\infty) d\nu_\infty - d_\tau^2 \alpha(G_\infty) \right|$$

$$= \left| t^d \times (\text{non-minimal terms}) + t^d \sum_{\omega_\infty \in \mathcal{E}_2(M_{0,\infty})} (\frac{1}{2\pi i})^r \int_{i\mathfrak{a}^*_{0,\infty}} \left\| \widehat{H^\sharp_{\infty,t,\zeta}}(\omega_\infty, \nu_\infty) \right\|^2 \mu(\omega_\infty, \nu_\infty) d\nu_\infty - d_\tau^2 \alpha(G_\infty) \right|$$

The Plancherel density corresponding to the non-minimal parabolic subgroups will have the following asymptotic estimate. For some integer $l < d$, we have

$$\int_{i\mathfrak{a}^*_{P,\infty}} \mu(\omega_\infty, t^{-1}\nu_\infty) d(t^{-1}\nu_\infty) \sim t^{-l} \quad \text{as} \quad t \to 0.$$

Therefore, the non-minimal terms will tend to 0 as $t \to 0$. And the approximation for the other term was dealt with in Lemma 4.4.1. ■

Therefore, We see from (4.5.2) the main term corresponding to the trivial conjugacy class is asymptotic to

$$d_\tau^2 \alpha(G) \mathrm{Vol}(\Omega) t^{-d} \quad \text{as} \quad t \to 0.$$

4.6 Bounds for the non-trivial classes

To get the estimates for non-trivial conjugacy classes on the geometric side we write the Fourier inversion formula of Harish-Chandra with respect to Eisenstein integrals. To this end we use the formula (1.1) in [Ar5, Chap. III Sec. 1]. It gives

$$H_{\infty,t}(x)|_{(1,1)}$$
$$= \sum_{\mathcal{P}} |\mathcal{P}|^{-1} \sum_{P\in\mathcal{P}} |W(\mathfrak{a}_P)|^{-1} \int\limits_{i\mathfrak{a}^*_\infty} E_P(x_\infty, \mu_P(\nu_\infty)\widehat{H_{\infty,P}}(t\nu_\infty), \nu_\infty)_{(1:1)} d\nu_\infty,$$

where $\mathcal{P}$ denotes an associated class of parabolic subgroups and the function $H_{\infty,t}$ lies in $C_c^\infty(G(\mathbb{R}),\tau)$. Note that $C_c^\infty(G(\mathbb{R}),\tau)$ is isomorphic to $C_c^\infty(G(\mathbb{R}))_{K_\infty}$ via the relation:

$$H_{\infty,t}(x)|_{k_1,k_2} = H_{\infty,t}(k_1 x k_2).$$

We concentrate on the part of the above series and integral corresponding to the minimal parabolic. We obtain the following inequality for the summand corresponding to minimal parabolic $P_{0,\infty}$:

$$\int\limits_{i\mathfrak{a}^*_{0,\infty}} |E_{P_0}(x_\infty, \mu_{P_0}(\nu_\infty)\widehat{H_{\infty,P_0}}(t\nu_\infty), \nu_\infty)|_{(1:1)}| d\nu_\infty \tag{4.6.1}$$
$$\leq \int\limits_{i\mathfrak{a}^*_{0,\infty}} \int\limits_{K_\infty} |\widehat{H_{\infty,P_0}}(t\nu)(1:m(kx):1)||\mu_{P_0}(\nu)|e^{(\nu+\rho)H(kx_\infty)}|dkd\nu$$

The right hand side of the above inequality is bounded by

$$\int\limits_{i\mathfrak{a}^*_{0,\infty}} \left(\|\widehat{H_{\infty,P_0}}(t\nu)\|\mu_{P_0}(\nu) \int\limits_{K_\infty} |e^{(\nu+\rho)H(kx)}|dk \right) d\nu.$$

Moreover, we have $\int_{K_\infty} |e^{(\nu+\rho)H(kx)}|dkd\nu \leq C_1(1+\|\nu\|)^{-1/2}$ when $x \notin K_\infty$ and lies in a fixed compact set [LV, Lemma 3]. Now making a change of variable $\nu_\infty \to \frac{\nu_\infty}{t}$, using the Paley-Wiener bound of $\|\widehat{H_{\infty,P}}(\nu_\infty)\|$, and using the bound of the Plancherel Measure from (4.1.6), we obtain the following inequality:

$$\int\limits_{i\mathfrak{a}^*_{0,\infty}} |E_P(x_\infty, \mu_P(\nu)\widehat{H_{\infty,P}}(t\nu)(1:1), \nu)| d\nu \leq C_2 t^{-d+1/2}. \tag{4.6.2}$$

Hence, we have the bound

$$\left|H_{\infty,t}(x)|_{(1,1)}\right| \leq C_2 t^{-d+1/2}. \tag{4.6.3}$$

Now we apply (4.6.3) for the Archimedean part of the integrand corresponding to the non-trivial conjugacy class of $\gamma \in \Gamma$ in (4.5.2) on the geometric side. Notice that similar to [LV, (6.3)] we can assume that the support of $d_\tau \mathrm{Tr}(\phi_{n,S,t,\zeta} \star \widetilde{\phi_{n,S,t,\zeta}})$ that lies inside K_∞ will have measure zero when projected onto $G(\mathbb{R})$. Moreover,

$$d_\tau \mathrm{Tr}(\phi_{n,\infty,t,\zeta} \star \widetilde{\phi_{n,\infty,t,\zeta}}) = \Phi^\sharp_{n,\infty,t,\zeta} \star \widetilde{\Phi^\sharp_{n,\infty,t,\zeta}}.$$

This is a bi-K_∞-finite function. Hence we can apply the above discussion. Arguing as in [LV, pg. 243] for the lower bound of Weyl's law, we can see that the terms corresponding to a non-trivial conjugacy class in (4.5.2) is bounded by $c|Z|t^{-d+1/2}$. Notice that as Γ injects into G_S diagonally, the cardinality of Z is finite. Moreover the L^1 norm of $\Phi_{n,S'}\mathbb{1}_{K'}$ are bounded by a constant for all n. Therefore it follows from (4.5.2) that

$$\left| d_\tau \mathrm{Tr}((\widetilde{\phi_{n,S,t,\zeta}} \star \phi_{n,S,t,\zeta})(e))\mathrm{Vol}(\Omega) + d_\tau \sum_{\gamma \in Z} \int_\Omega \mathrm{Tr}(\widetilde{\phi_{n,S,t,\zeta}} \star \phi_{n,S,t,\zeta})(x^{-1}\gamma x) \right| < \sum_\lambda C_{\phi_{n,S,t,\zeta}}$$

or,

$$\left| d_\tau \mathrm{Tr}((\widetilde{\phi_{n,S,t,\zeta}} \star \phi_{n,S,t,\zeta})(e))\mathrm{Vol}(\Omega) \right| - c|Z|t^{-d+1/2} < \sum_\lambda C_{\phi_{n,S,t,\zeta}}$$

or,

$$\left| t^d \widetilde{\Phi^\sharp_{n,S,t,\zeta}} \star \Phi^\sharp_{n,S,t,\zeta}(e)\mathrm{Vol}(\Omega) \right| - c|Z|t^{1/2} < t^d \sum_\lambda C_{\phi_{n,S,t,\zeta}}.$$

Here, $C_{\phi_{n,S,t,\zeta}} = (e_\lambda \star \phi_{n,S,t,\zeta}, e_\lambda \star \phi_{n,S,t,\zeta})$. From (21) and [BM, Lemma 3.3] we have that

$$\sum_\lambda (e_\lambda \star \phi_{n,S,t,\zeta}, e_\lambda \star \phi_{n,S,t,\zeta}) = \sum_{\Pi_{\mathrm{cusp}}(G(\mathbb{Q}_S),\tau)} m(\pi) \left|\left|\pi(\Phi^\sharp_{n,S,t,\zeta})\right|\right|^2_{\mathrm{HS}}. \tag{4.6.4}$$

The multiplicities of π_∞ can be written as:

$$m(\pi_\infty) = \sum'_{\Pi_{\mathrm{cusp}}(G(\mathbb{Q}_S),\tau)} m(\pi') \dim \mathcal{H}^{K'_{S'}}_{\pi'}, \tag{4.6.5}$$

where the sum is over $\pi^{'}$ whose Archimedean component is π_∞. If we let $n \to \infty$, we have that $\pi(\Phi^\sharp_{n,t,S,\zeta}) \to \dim\mathcal{H}^{K^{'}_{S'}}_{\pi^{'}}\pi_\infty(H^\sharp_{\infty,t,\zeta})$. Hence rewriting the (25) we get

$$\sum_\lambda (e_\lambda \star \phi_{n,S,t,\zeta}, e_\lambda \star \phi_{n,S,t,\zeta}) = \sum_{\Pi_{\text{cusp}}(G(\mathbb{R}),\tau)} m(\pi_\infty) \left|\left|\pi_\infty\left(H^\sharp_{\infty,t,\zeta}\right)\right|\right|^2_{\text{HS}}. \tag{4.6.6}$$

The sum on the right hand side of (4.6.6), could be divided into two parts, $\|\lambda_{\pi_\infty}\|^2 \leq t^{-2}$ and $\|\lambda_{\pi_\infty}\|^2 \geq t^{-2}$. Here λ_{π_∞} denotes the infinitesimal character of π_∞. The representations π_∞ is a subrepresentations of a non-unitary principle series representation with parameters $\omega_\infty \otimes \nu_\infty \otimes 1$. The infinitesimal character of λ_{π_∞} can be written as $\nu_{\omega_\infty} + \nu_\infty$, where ν_{ω_∞} is the infinitesimal character of ω_∞ [Kn, Prop. 8.22]. Let d_{ω_∞} be the formal degree of ω_∞. We have the following inequality of Hilbert-Schmidt norm in terms of Fourier transform $\widehat{H^\sharp_{\infty,t,\zeta}}(\omega_\infty, \nu_\infty)$:

$$\left|\left|\widehat{H^\sharp_{\infty,t,\zeta}}(\omega_\infty, \nu_\infty)\right|\right|^2 \geq d_{\omega_\infty} \left|\left|\pi_\infty\left(H^\sharp_{\infty,t,\zeta}\right)\right|\right|^2_{\text{HS}}.$$

Using the choice of the Schwartz function the right hand side of (4.6.6) is bounded by

$$\sum_{\|\nu_{\omega_\infty}+\nu_\infty\| \leq t^{-2}} d_\tau m(\pi_\infty) \quad \dim(\text{Hom}(\mathcal{H}_{\pi_\infty}(\tau), V_\tau))\, |\psi(t\nu_\infty)|^2 \tag{4.6.7}$$

Hence, using our earlier notations, we have

$$\sum_{\|\nu_{\omega_\infty}+\nu_\infty\| \leq t^{-2}} d_\tau m(\pi_\infty) \quad \dim(\text{Hom}_{K_\infty}(\mathcal{H}_{\pi_\infty}(\tau), V_\tau))\, |\psi(t\nu_\infty)|^2 \ \leq d_\tau N^\Gamma_{\text{cusp}}(t^{-2}, \tau)$$

4.7 Main theorem

In this last section we put all our earlier results together to prove our main asymptotic formula. Suppose Δ_τ is the self-adjoint Casimir operator acting on $L^2_{\text{cusp}}(\Gamma\backslash G_S, \tau)$ with pure point spectrum $0 < \nu_0(\tau) \leq \nu_1(\tau) \leq \nu_2(\tau) \leq \ldots \to \infty$. Let $\mathcal{E}(\nu_i(\tau))$ denote the space of eigenvectors with eigenvalue $\nu_i(\tau)$. Define

$$N^\Gamma_{\text{cusp}}(T^2, \tau) = \sum_{\nu_i(\tau) \leq T^2} \dim\mathcal{E}(\nu_i(\tau)).$$

Let M be a Riemannian manifold. Suppose $C(M)$ denotes the product of volume of M, the volume of the Euclidean unit ball in $\mathbb{R}^{\dim(M)}$ and $(2\pi)^{-\dim(M)}$. Collecting all the results in the previous section, we prove the following:

Theorem 4.7.1 *Let G be a semisimple, connected, algebraic group over $\mathbb{Q}_S$. Assume that G is also split over $\mathbb{Q}$ and of adjoint type. Let $\Gamma \subset G(\mathbb{Z}[S^{-1}])$ be a congruence subgroup with no torsion element. Let $X_\infty = G_\infty/K_\infty$ and $d = dim_{\mathbb{R}} X_\infty$. Let τ be an irreducible representation of K_∞ of dimension d_τ. Then there exists a constant $C(\Gamma\backslash X_\infty) > 0$, such that*

$$N^{\Gamma}_{cusp}(T^2, \tau) \sim d_\tau C(\Gamma\backslash X_\infty) T^d \quad as \quad T \to \infty.$$

Proof. We make a change of variable $t = \frac{1}{T}$, and prove the asymptotic as $\limsup_{t\to 0}$. Let us apply the partial trace formula in (4.3.1) with ϕ being the test function $\phi_{n,S,t,\zeta} \star \widetilde{\phi_{n,S,t,\zeta}}$ in Section 4.5. Taking the limit as $n \to \infty$ and using (4.5.2) the inequality becomes

$$\mathrm{Tr}((h_{S,t,\zeta} \star \widetilde{h_{S,t,\zeta}})(e))\mathrm{Vol}(\Omega) + \sum_{\gamma \in Z} \int_{\Omega} \mathrm{Tr}(h_{S,t,\zeta} \star \widetilde{h_{S,t,\zeta}})(x^{-1}\gamma x) \leq N^{\Gamma}_{\mathrm{cusp}}(t^{-2}, \tau). \quad (4.7.1)$$

Now from Section 4.4 and Lemma 4.4.1 we can conclude that the term corresponding to the identity class will be asymptotic to $d_\tau \alpha(G) t^{-d} \mathrm{Vol}(\Gamma\backslash G_S)$ as $\limsup_{t\to 0}$ and as $\lim_{n\to\infty}$. And from (4.6.3) we can show that as we take $\limsup_{t\to 0}$ the terms corresponding to non-identity classes will converge to 0. This is done exactly as in the proof of the lower bound in Weyl's law [LV, page 243].

There exist $\Gamma_{\infty,i}$, for finitely many i, such that

$$\Gamma\backslash G_S/K_S = \bigcup_i \Gamma_{\infty,i}\backslash G(\mathbb{R})/K_\infty.$$

For each i, let $N^{\Gamma}_i(T, \tau)$ be the eigenvalue counting function for the space $\Gamma_{\infty,i}\backslash G(\mathbb{R})/K_\infty$. That this same asymptotic term along with the constant $C(\Gamma_{\infty,i}\backslash X_\infty)$ is an upper bound for the right hand side has been proved in greater generality by Donnelly [Do]. To prove

$$\alpha(G)\mathrm{Vol}(\Gamma\backslash G_S) = \sum_i C(\Gamma_{\infty,i}\backslash X_\infty),$$

we argue as in [LV, Sec. 6.3]. Therefore it establishes the asymptotic formula in the statement of the theorem.

■

REFERENCES

[Ar1] J. Arthur. A trace formula for reductive groups I: Terms associated to classes in $G(\mathbb{Q})$, Duke Math. J . 45 (1978), 911–952

[Ar2] James Arthur. A Paley-Wiener theorem for real reductive groups, Acta Mathematica volume 150, pages1–89(1983).

[Ar3] J. Arthur. Harmonic Analysis of the Schwartz space on a Reductive Lie group I and II, Preprint.

[Ar4] J. Arthur. A theorem on the Schwartz space of a reductive Lie group, Proc. Nat. Acad. Sci. USA 72, No. 12 (1975), 4718-4719.

[Ar5] J. Arthur. A Paley-Wiener theorem for real reductive groups, Acta Mathematica 150 (1983), 1-89.

[Ar6] J. Arthur. The Trace Paley Wiener Theorem for Schwartz Functions, Contemporary Mathematics 177 (1994), 171-180.

[B] J. Bernstein. Draft of : Representations of p-adic groups, Fall 1992. Lectures by Joseph Bernstein, Written by Karl E. Rumelhart.

[BDK] J. Bernstein, P. Deligne, D. Kazhdan. Trace Paley-Wiener theorem for reductive p-adic groups, J. d'Analyse Math. 47 (1986), 180–192.

[BLZZ] R. Beuzart-Plessis, Y. Liu, W. Zhang, and X. Zhu. Isolation of cuspidal spectrum, with application to the Gan–Gross–Prasad conjecture, Annals of Mathematics, Issue 2, 194 (2021), 519-584.

[BM] D. Barbasch, H. Moscovici. L2-Index and the Selberg Trace Formula" by in Journal Of Functional Analysis 53 (1983), 151-201.

[Bmp] D. Bump. Automorphic forms and Representations, Cambridge studies in Advanced Mathematics 55.

[Cmp1] R. Camporesi. The spherical transform for homogeneous vector bundles over Riemannian symmetric spaces, Journal of Lie Theory 7 (1997) 29–60.

[Cmp2] R. Camporesi. The Helgason Fourier Transform For Homogeneous Vector Bundles Over Riemannian Symmetric Spaces, Pacific journal of mathematics 179, No. 2, 1997.

[CH] D. Ciubotaru, X. He. Cocenters of p-adic Groups, III: Elliptic and Rigid Cocenters, Peking Mathematical Journal 4(2) 2017.

[DKV1] J. Duistermaat, J. Kolk, V. Varadarajan. Functions, flows and oscillatory integrals on flag manifolds and conjugacy classes in real semisimple lie groups, Compositio Mathematica, tome 49, no.3 (1983) 309-398.

[DKV2] J. Duistermaat, J. Kolk, V. Varadarajan. Spectra of compact locally symmetric manifolds of negative curvature. Invent. Math., 52(1) (1979) 27–93.

[Dui] J. Duistermaat. The heat Kernel Lefschetz fixed point formula for the spin-C dirac operator, Modern Birkhäuser Classics.

[Do] H. Donnelly. On the cuspidal spectrum for finite volume symmetric spaces, J. Differential Geom. 17 (1982), 239–253.

[Ef] I. Efrat. The Selberg Trace Formula for $PSL_2(\mathbb{R})$, Mem. Amer. Math. Soc. 65, no. 359 (1987), A. M. S., Providence, RI.

[FK] Y. Flicker, D. Kazhdan. A simple trace formula, J. Analyse Math. 50 (1988), 189–200.

[FL] T. Finis, E. Lapid. On the remainder term of the Weyl law for congruence subgroups of Chevalley groups, Duke Math. J. 170 (4), 653-695.

[GV] R. Gangolli, V.S. Varadarajan. Harmonic Analysis of Spherical Functions on Real Reductive Groups, Springer-Verlag Berlin Heidelberg New York London Paris Tokyo.

[HC1] Harish-Chandra. Harmonic Analysis on Real Reductive Groups III. The Maass-Selberg Relations and the Plancherel Formula, Annals of Mathematics, Second Series, Vol. 104, No. 1 (Jul., 1976), 117-201.

[HC2] Harish-Chandra. ”On the theory of the Eisenstein integral” in Conference on Harmonic Analysis (Univ. Maryland, College Park), Lecture Notes in Mathematics 266 (1971), Berlin, New York: Springer-Verlag, 123–149.

[HC3] Harish-Chandra. Collected Papers, Springer, Berlin 1984.

[Hux] M. HUXLEY. “Scattering matrices for congruence subgroups” in Modular Forms (Durham, England, 1983), Ellis Horwood Ser. Math. Appl.: Statist. Oper. Res., Horwood, Chichester, England, 1984, 141 – 156.

[Kn] A. Knapp. Representation theory of Semisimple groups, an Overview based on examples,Princeton University Press, Princeton, New jersey, 1986.

[La] R. Langlands. On the Functional Equations Satisfied by Eisenstein Series, Lecture Notes in Math. 544, Springer-Verlag, New York, 1976.

[LM] J.-P. Labesse, W. Müller. Weak Weyl’s law for congruence subgroups, math.RT/0404037.

[LM1] E. Lapid, W. Müller. Spectral asymptotics for arithmetic quotients of $SL(n,\mathbb{R})/SO(n,\mathbb{R})$, Duke Mathematical Journal 149 (2009).

[LV] E. Lindenstrauss, A. Venkatesh. Existence and Weyl's law for spherical cusp forms,GAFA, Geom. func. anal. 17 (2007) 220-251.

[Rez] A. Reznikov. Eisenstein matrix and existence of cusp forms in rank one symmetric spaces, Geom. Funct. Anal. 3 (1993), 79–105.

[Ma] J. Matz. Weyl's law for Hecke operators on $GL(n)$ over imaginary quadratic number fields, Amer. J. Math., 139 (2017), no. 1, 57-145.

[Mi] S. Miller. On the existence and temperedness of cusp forms for $SL(3,\mathbb{Z})$. J. Reine Angew. Math. 533 (2001), 127–169.

[Mü1] W. Müller. Eigenvalue estimates for locally symmetric spaces of finite volume, in Partial Differential Equations (Holzhau, 1988) 179–196, Teubner-Texte Math. 112, Leipzig, 1989.

[Mü2] W. Müller. Weyl's law for the cuspidal spectrum of $SL(n)$, Annals of Mathematics, 165 (2007), 275–333.

[Mü3] W. Müller. Asymptotics of automorphic spectra and the trace formula, Families of automorphic forms and trace formula, Simon Symposia (2016), 477-531.

[MT] A. Moy, M. Tadic. The Bernstein Center In Terms Of Invariant Locally Integrable Functions, Representation Theory An Electronic Journal of the American Mathematical Society 6, (November 19, 2002), 313–329.

[PR] V. Platonov, A. Rapinchuk, Rachel Rowen. Algebraic Groups and Number Theory, Academic Press, Dec 7, 1993.

[Sa] P. Sarnak. On cusp forms, in The Selberg Trace Formula and Related Topics (Brunswick, Maine, 1984) (D. Hejhal et al., eds.), Contemp. Math. 53, 393–407.

[Se] A. Selberg. Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series, J. Indian Math. Soc. (N.S.) 20 (1956), 47–87.

[Wal] N. Wallach. Real Reductive Groups I, Volume 132,1st Edition, Academy Press, 1988.

[W] G. Warner. Harmonic Analysis on Semi-Simple Lie Groups II, Springer-Verlag Berlin Heidelberg New York 1972.

VITA

Ayan Maiti

Candidate for the Degree of

Doctor of Philosophy

Thesis: WEYL'S LAW FOR CUSP FORMS OF ARBITRARY ARCHIMEDEAN TYPE

Major Field: Mathematics

Biographical:

Education:

Completed the requirements for the Doctor of Philosophy in Mathematics at Oklahoma State University, Stillwater, Oklahoma in July, 2022.

Completed the requirements for the Master of Science in Mathematics and its application at Central European University, Budapest, Hungary in 2016.

Completed the requirements for the Bachelor of Science in Mathematics and computer science at Chennai Mathematical Institute, Chennai, India in 2013.