AN INVESTIGATION OF THE IDENTITY TRAJECTORIES OF MATHEMATICS INSTRUCTORS PARTICIPATING IN AN INQUIRY-ORIENTED PROFESSIONAL DEVELOPMENT INITIATIVE

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I conclude with the single verse that guided how I conducted my work this final year:

Colossians 3:17: And whatever you do, in word or deed, do everything in the name of the Lord Jesus, giving thanks to God the Father through him.

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Abstract: The enactment of a large-scale, long-term professional development initiative designed to enhance participants' knowledge base by influencing their identity trajectories as mathematics instructors is complex. Effectively influencing participants' trajectories could be enhanced by understanding the nature of participants' goal structures, priorities, and commitments. The focus of my research is to reveal features of participants' distal goals as mathematics instructors by eliciting their contributions and takeaways from their participation in MIP activities and by uncovering their interpretations of three elements of mathematical inquiry.

I collected data primarily from conducting semi-structured interviews with MIP participants. I interviewed eight participants who had participated on one of the Collaborative Research and Development Teams (CoRDs). I asked them to discuss their contributions and takeaways from their involvement in MIP activities and to describe their vision for future collaborations among mathematics faculty. In addition to these interviews, I also conducted two exploratory case studies. The focus of these case studies was to reveal participants' identity trajectories by uncovering their interpretations of three elements of mathematical inquiry and the nature and purpose of conducting a conceptual analysis. I analyzed these interviews in different ways but my approach centrally relied on generating open codes from the transcripts and organizing them into categories.

One finding from my analysis of the eight interviews was that participants valued opportunities to collaborate to share experiences and struggles with other colleagues. This result reveals the importance of providing opportunities to continue cultivating a community of practice among MIP participants. Both case studies revealed instructors' commitments to support students' engagement in productive mathematical practices by engaging in critical thinking and problem-solving activities. Considering the focus of the MIP is to support faculty to critically evaluate the nature of the meanings they intend students to construct, there is need to perturb and extend MIP participants' knowledge base. Participants need to recognize (1) the differences between supporting students' engagement in productive mathematical practices and supporting students' construction of productive meanings and (2) the affordances of the latter.

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INTRODUCTION AND STATEMENT OF THE PROBLEM

A Need for Reform

In a recent government report, the Committee on STEM Education of the National Science & Technology Council outlined a five-year strategy for STEM education reform based on the vision that "all Americans will have lifelong access to high-quality STEM education and the United States will be the global leader in STEM literacy, innovation, and employment" (2018, p. v). Even more recently, the STEM Opportunities Act of 2019 focused on supporting the STEM education of women and minority groups.[1]

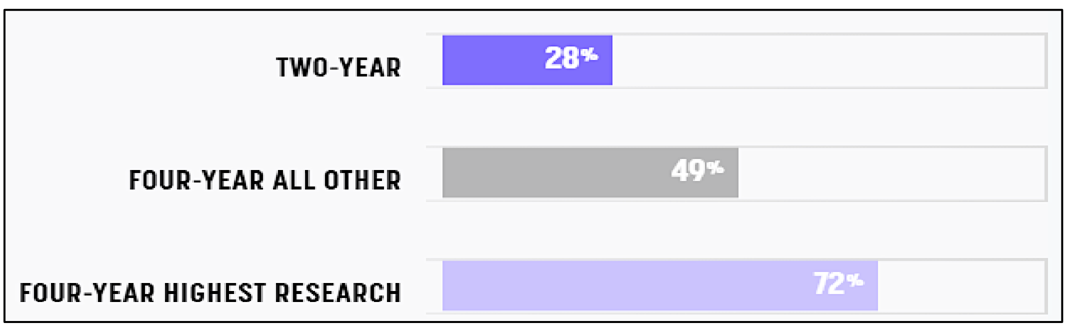
In response to calls for reform, there have been a range of recent efforts across the United States focusing on improving students' access to and success in STEM education. The need for nationwide reform is also reflected in Oklahoma. According to the Data Dashboard from Complete College America (2008 cohort),[2] the national average percentile for first-time full-time students receiving a bachelor's degree (not corresponding to a higher research category) in four years is 20% as compared to 12% in Oklahoma. Certain gateway courses impose a strong barrier

[1] The STEM Opportunities Act was under consideration by the U.S. Senate at the time of writing.
[2] https://completecollege.org/data-dashboard/

impacting college students' graduation rates. As indicated in Figure 1, while more than 70% of students working on a four-year degree at a research institution in Oklahoma have completed both their mathematics and English gateway courses within two years, the completion rate is less than 50% for those seeking to complete their bachelor's degree at the state's other four-year institutions (Complete College America, 2012 cohort). Completion of the mathematics gateway course is particularly problematic. While 28% of students attending two-year institutions have completed both their mathematics and English gateway courses within two years, only 3% have completed their mathematics gateway course (and not English) as opposed to 29% who have completed their English gateway course (and not mathematics) (Complete College America, 2012 cohort).

Figure 1

Completion of English and Mathematics Gateway Courses in Oklahoma Within Two Years by Institution



Note. From *Data Dashboard, completion*, by Complete College America, 2012 (https://completecollege.org/data-dashboard/).

The Oklahoma State Regents for Higher Education (OSRHE) recognized the imminence of these problems, identifying "low success rates in remedial and gateway math courses as a significant barrier to student success" and committing to "improve the effectiveness and efficiency of remediation and freshmen gateway courses" (Oklahoma State Regents for Higher

Education, 2017 February, p. 1). OSHRE formed the Math Pathways Task Force and after several meetings in 2016, the Task Force produced five goals and presented five essential challenges for achieving them. Additionally, the Task Force offered the following five recommendations that were refined through their interactions with Oklahoma's post-secondary institutions (Oklahoma State Regents for Higher Education, 2017, p. 2):

1. Establish statewide college meta-majors and corresponding math pathways, ensuring transferability across institutions;
2. Improve student preparation, including efforts in K-12 education and remediation reform;
3. Increase student engagement and the teaching of applications in gateway math classes;
4. Increase support for important academic success skills in gateway math classes; and
5. Provide faculty and advisor professional development and resources.

While the first two recommendations have been addressed through the state's math pathways and corequisite reform initiatives, the Mathematical Inquiry Project (MIP)—funded by the National Science Foundation—was designed to support OSHRE in accomplishing the latter three objectives by cultivating a community of mathematics faculty at institutions of higher education across Oklahoma to develop curricular resources for entry-level mathematics courses centered around three elements of mathematical inquiry—*active learning*, *meaningful applications*, and *academic success skills*. Specifically, the focus on the first two elements of inquiry, active learning and meaningful applications, addresses recommendation (3), while including academic success skills addresses recommendation (4). Finally, the MIP is a large-scale professional development initiative for faculty (recommendation (5)) centered around gateway mathematics courses (recommendation (3) and (4)). Faculty participating in the MIP receive opportunities for professional development by attending workshops, engaging on research teams, and developing peer-mentoring relationships.

Structural Design of the MIP

Faculty were first afforded an opportunity to participate in the MIP by attending an Initiation Workshop in the summer of 2019. Throughout these five-day events (four full days), instructors worked collaboratively to identify key conceptual threads related to the content focus of the workshop. Some of these topics will later be developed into curricular modules and instructional resources by small teams of faculty working on Collaborative Research and Development Teams (CoRDs). The formation of CoRDs to design these resources on topics highlighted during the Initiation Workshops signifies the next phase of the MIP. Once these modules have been developed (and possibly piloted), emerging leaders in the MIP community will help facilitate one-day Regional Workshops, the third phase, to help support interested instructors to implement these modules into their instruction. The final phase of the MIP consists of cultivating peer mentoring relationships between MIP personnel and faculty seeking to incorporate the curricular artifacts developed by CoRDs into their teaching. In the following sections, I discuss each of these four phases in more detail beginning with the Initiation Workshops.

Initiation Workshops

The first feature of the MIP design involves a series of Initiation Workshops occurring over multiple summers. Four of the five workshops focus on gateway mathematics courses (Functions and Modeling, Quantitative Reasoning, College Algebra and Precalculus, and Calculus I) and the fifth workshop addresses the topic of students’ academic success skills. The

MIP held three workshops during summer 2019. Due to the COVID-19 pandemic, the other two workshops were postponed to summer 2021, instead of summer 2020.

The first workshop conducted in May 2019 focused on academic success skills, affording participating faculty the opportunity to leverage insights about various categories of students' affect (e.g., growth mindset) prior to attending the other, subject-matter focused, workshops. Accordingly, the MIP team encouraged faculty who were interested in attending one of the content-based workshops (the other four) to also attend the academic success skills workshop for this purpose. Some faculty who attended this first workshop gave presentations during subsequent Initiation Workshops that addressed a particular topic related to students' affect (e.g., grit).

The five-day Initiation Workshops leveraged expertise from three primary sources—outside experts, members of the MIP Team, and the participants themselves. Those attending the workshops experienced opportunities to gain insight from outside sources by reading journal articles and listening to virtual presentations. The MIP Team intended the pre-workshop readings to expose participants to particular ideas in the mathematics education literature to equip them to engage in discussions and collaborative activities during the workshop.

Besides leveraging knowledge from experts through the literature, the MIP also recruited outside experts to offer insights during the Academic Success Skills workshop. While the MIP Team is highly qualified, none specialize in the areas of affect or academic success. Tammi Marshall (2019, May), the mathematics department chair at Cuyamaca College in California, shared her knowledge and materials for supporting students' affective engagement in mathematics, and Dr. Katherine Good (2019, May), Associate Professor of Psychology at Baruch

College, provided further expertise during her virtual presentation, *Making Mindsets Matter: Classroom Cultures that Increase Student Engagement, Learning, and Achievement*.

The second source of expertise leveraged during the workshops was held by the MIP Team. These five researchers (Drs. Cook, Dorko, Jaco, Oehrtman, and Tallman), who collectively have extensive experience teaching gateway courses as well as conducting research in mathematics education, frequently gave presentations throughout these workshops. One presentation on conceptual analysis and one discussion on the three elements of inquiry were consistently given at each of the Initiation Workshops to begin the process of equipping participating instructors with the knowledge base required for CoRDs to effectively design their curricular modules. For example, a member of the MIP Team facilitated a discussion of the three elements of inquiry to unpack the components of the definitions, which were grounded in constructivist epistemology.

Finally, the Initiation Workshops, modeled after the American Institute of Mathematics (AIM) Squares, were also designed to leverage the expertise of participants by providing opportunities for participating faculty to collaborate in small groups and brainstorm about topics and activities. More prevalent in the latter half of the workshop, faculty joined small breakout sessions to discuss topics developed and refined by the entire group of participants. From these group discussions, which were facilitated by a member of the MIP Team, participants identified a list of conceptual threads to guide the subsequent activity of CoRDs (see Table 1).

Table 1

Conceptual Threads Identified During the Summer 2019 and Summer 2021 Initiation Workshops

Academic Success Skills Workshop	Functions and Modeling Workshop	College Algebra and Precalculus Workshop

• Mathematics Anxiety • Problem Solving and Critical Thinking • Developing Classroom Communities • Mindset • Productive Struggle, Persistence, and Perseverance • Motivation and Interest • Beliefs about Mathematics	• Function • Modeling and Quantitative Reasoning • Rate of Change • Function Classes	• Rate of Change and Covariation • Functions and Their Fundamental Characteristics • Multiple Problem-Solving Strategies and Representational Equivalence • Quantitative Reasoning and Modeling

Calculus I	**Quantitative Reasoning**
• Functions • Limits • Local Linearity, Differentials, Infinity, and Infinitesimals • Rate of Change • Continuity • Accumulation, Integrals, and the Fundamental Theorem of Calculus • Modeling	• Information Presentation and Consumption • Ratios, Proportions, and Proportional Reasoning • Quantification • Critical Thinking • Spreadsheets • Modeling • Problem Solving

Many professional development initiatives offer some type of workshop training for participants. After analyzing their notes from discussions with National Science Foundation program directors in the Transforming Undergraduate Stem Education initiative, Khatri et al (2013) concluded that for "spreading an innovation beyond the developers," program directors "consider workshops to be the most *effective* [emphasis added] method of propagation" (p. 218). Moreover, the "most successful workshops are in-depth, multi-day, immersive experiences with follow-up interaction with the PI as participants implement the new strategy in their own institutional circumstances" (ibid., pp. 218-219). Supporting these conclusions, Garet et al. (2001) argued that while workshops may be a productive first step, they need to be supplemented by other opportunities to allow for productive growth. The design of the MIP affords faculty opportunities for further participation as some of these finalized topics would be developed into curricular modules by CoRDs during the next phase of the project.

Collaborative Research and Development Teams (CoRDs)

A CoRD is a team of three to six faculty working together to construct a curricular module related to a conceptual thread identified during one of the Initiation Workshops. Structurally, there are two types of CoRDs—content-based CoRDs and academic success skill CoRDs. Content-based CoRDs address a conceptual thread associated with one of the four content focused workshops—Functions and Modeling, Quantitative Reasoning, College Algebra and Precalculus, and Calculus I—while academic success skills CoRDs focus on topics within the affective domain (e.g., grit, motivation, identity, perseverance, mindset, beliefs).

After the summer workshops, the MIP Team sent out recruitment letters in fall 2019 to encourage MIP participants to join CoRDs (see the appendix for part of a sample request for proposal for a Functions and Modeling CoRD). Interested instructors contacted the MIP Team either as an individual wanting to join a CoRD or perhaps with another MIP participant(s) or colleague(s) they identified who might want to form their own CoRD. The MIP team helped foster the formation of CoRDs for instructors who wanted to participate in the development of these modules but did not know who else might be interested. At the beginning of 2022, seven CoRDs had been formed.

Once a CoRD was formed, they began the process of writing a proposal related to the chosen topic among the conceptual threads identified during the Initiation Workshops. They sent the completed proposal to the MIP Team, and it was distributed to an MIP Correspondent, a member of the Project Team who was assigned to support that CoRD. After reviewing their proposal, this correspondent provided feedback, which contained a series of probing questions and prompts for clarification, and requested that the CoRD re-submit their proposal after having considered the initial review. The CoRD was officially formed upon resubmission of a satisfactory proposal. While the Project Team provided deadlines for MIP participants to submit proposals, they were flexible in adjusting the initial deadlines. Moreover, the COVID-19 pandemic, which began impacting universities in March 2020, further softened the Project Team's expectations regarding the timeline and overall progress of CoRDs.

After a CoRDs' proposal had been accepted, CoRD members continued the process of reading the research literature and meeting together, strategizing how to design their module. The MIP Correspondent offered support and guidance by periodically interacting with them through email and virtual meetings as well as providing feedback on draft materials. The correspondent's guidance was intentionally limited in these interactions. While the MIP Correspondent sought to empower the CoRD members to develop the expertise in a particular topic, they also offered suggestions, provided feedback, or supplied resources in service of supporting CoRD members in attending to the three components of mathematical inquiry on which the project is based. Maintaining instructors' individual agency and respecting their expertise is an essential aspect of the professional learning experience the MIP seeks to engender.

Regional Workshops

Many mathematics faculty in Oklahoma will not be working on CoRDs to develop these curricular resources. Instructors who were not able to attend an Initiation Workshop or participate on a CoRD but are interested in learning more about the curricular resources CoRDs develop can attend a regional workshop. These workshops are one-day events in which faculty learn how to incorporate the modules developed by CoRDs into their classrooms. Prior to these one-day meetings, the MIP Team may hold workshops to prepare and equip these emerging leaders to effectively coordinate these events.

Peer Mentoring

While the regional workshops present opportunities to expand the MIP Community of Practice (CoP), there are limitations to these one-day events. Despite efforts from emerging MIP leaders to offer guidance to support faculty in incorporating the developed curriculum modules into their teaching, fidelity of implementation remains uncertain. Yet, the development of semester-long peer-mentoring relationships between these MIP leaders and faculty interested in implementing the curricular modules developed by the CoRDs provides further opportunities for continued discussions and support.

There are several potential affordances offered by cultivating these mentoring relationships. First, through the development of these interactions, the MIP CoP will continue to expand in regions throughout Oklahoma. Moreover, instructors who are engaging in these mentoring relationships with MIP leaders may mentor other faculty. The emergence of these communities in regional pockets might help sustain the cultural practices of the MIP and increase broad support for learning mathematics through inquiry. Second, through these relationships, emerging MIP leaders will be able to provide expertise and guidance (with the support of the MIP Team) regarding learning through inquiry and the implementation of conceptually-focused curricular resources.

Overview of the Five Chapters

In this opening chapter, I compare the MIP with other professional development initiatives in STEM education and discuss the central focus of the MIP design. In chapter two, I present my analysis of MIP participants' responses to seven prompts prior to attending an Initiation Workshop. Each of chapters three and four focus on a case study with a single MIP participant. In these two chapters, I examine the participants' professional identities as mathematics instructors, illuminating the features they associate with competent mathematics instruction. I do this by investigating their conceptions of three elements of mathematical inquiry—*active learning*, *meaningful applications*, and *academic success skills*—and Thompson's (2008) notion of conceptual analysis. I expect the participants' expressed interpretation of these three elements of inquiry, and their relation to conceptual analysis, to elucidate central aspects of their identity as mathematics instructors by revealing the goals, priorities, commitments, and orientations that guide their practice. Additionally, I compare their images of these constructs with how they are conceptualized according to the MIP. In the final chapter, I review and conclude the previous four chapters.

In the following sections, I compare the MIP with other research-generating STEM professional development initiatives according to the following two criteria:

1. What is the nature of the problem being addressed?
2. What are the theoretical implications of the research?

Criterion 1: The Nature of the Problem Being Addressed

Many professional development programs designed to improve STEM education seek to address a similar problem—inadequacy of students' learning or low success rates. For example, Du et al. (2018) investigated the extent to which a three-year professional development effort influenced middle school teachers' instructional practices. They opened their article by referencing a U.S. report highlighting poor academic success in science and mathematics on standardized tests. Eddy et al. (2019) examined how geoscience faculty at two-year colleges experienced change as learners while undergoing an instructional improvement effort. Referencing the Center for Community College Student Engagement (2015) regarding student learning, Eddy et al. (2019) stated that student learning is at the "center of national discussions, especially as it relates to the completion agenda" (p. 540). Pelletreau et al. (2019) studied 16 science instructors from five institutions who designed instructional units to support student learning. In their opening sentence, they identified a common understanding among faculty regarding students' misconceptions about ideas in science. As these studies illustrate, the fundamental problem that the professional development initiatives (and many others like them) were designed to address is student learning and achievement. The diverse professional development programs examined in these studies, demonstrate a need to explore the nature of the common problem by asking the following question: Why are students underperforming in STEM education?

Addressing this question is nontrivial but providing support for faculty to engage in more student-centered approaches is one feature of some researchers' efforts. These three researchers (Du et al. 2018, Eddy et al., 2019, & Pelletreau et al., 2019) each included active learning as a feature of their design. Du et al. (2019) used Desimone's (2009) framework for effective professional development that included active learning as one of the five features, and Eddy et al. (2019) stated that the initial year for their program "provided development of active learning strategies and metacognition practices to improve student learning" (p. 541). Additionally, Pelletreau et al. (2019) designed their work "based on professional development models that have been reported to increase faculty use of active learning" (p. 2). The implementation of a professional development initiative could provide opportunities for faculty to learn how to engage students in active learning, and faculty participation may lead to a transformation of their identities as instructors. Similarly, the principal affordance of the MIP is not the final production of curricular products, but rather the transformation of the participants' professional identities and the resulting cultural shift in post-secondary mathematics teaching required for sustained improvements in students' learning and success.[3] Noticing distinctions between the central issues addressed by other professional development programs and the MIP requires consideration of the following question: Toward what underlying goals, priorities, and commitments are the designers of professional development initiatives influencing their participants' identities as mathematics instructors? To help address this question more concretely, I illustrate findings from several studies.

Du et al. (2019) investigated the extent to which a three-year professional development effort improved middle school teachers' pedagogical practices and STEM content knowledge. After presenting despairing statistics regarding low success rates for STEM students in the United States compared to other nations, they offered a solution—supporting teachers' development of an integrated STEM approach. A key component of their design was to incorporate active learning, and Du et al. (2019) observed a shift in teachers' perceptions towards student-centered instructional practices. The following quote illustrated one of their participants' refined conceptions:

> So many ways I was saying. Different students learn in different ways that's why I always use a combination, reading and lectures. Hands-on though is a good way for most students although not all students are really good with the hands-on kind of movement types of things. (Du et al., 2019, p. 110)

During the post-interview, another participant in their program offered the following remarks: "Small class, small group setting where it is a participatory environment where the kids are learning from their questions that they're asking as opposed to dictating here's how you do the examples" (Du et al., 2019, p. 110). These comments made by participants after their

[3] I define *identity* more thoroughly in a later chapter.

involvement in the professional development program reflect a change towards embracing a particular kind of learning environment by focusing on a "more learner-centered classroom culture in which students were engaged in meaninful [sic] group work, PBL, discussion, interacting, and all students' ideas, questions, and contributions could be respected" (Du et al., 2019, p. 110).

Eddy et al. (2019) investigated the impact of an intervention informed by situated learning theory on 22 geoscience faculty from 17 colleges throughout the United States. In discussing their project background, they highlighted the main priority of the intervention:

> The intention of the project is that these faculty members will implement high-impact, evidence-based instruction and broader teaching and learning practices at their own institutions, which will lead to improved STEM learning for students, broadened participation in the geosciences, and help build a more robust STEM workforce. (p. 541)

Again, notice that one of their primary objectives is to support faculty towards implementing "high-impact, evidence-based" pedagogical strategies learned through professional development. The researchers concluded that the design of their intervention supported participants in advancing "their repertoire of effective teaching strategies as well as their understandings of what they could do within their roles as geoscience faculty" (Eddy et al., 2019, p. 551). Some of these strategies included "think-pair-share, gallery walks, exam wrappers, metacognition, working with diverse students, and career pathways" (ibid., p. 548). In contrast to these other initiatives, a central objective of the MIP is to transform participants' professional identities as instructors with respect to the features they associate with competent teaching practices (e.g., their constructed meanings of the process entailed in engaging students in active learning). The aim of the MIP is to foster participants' transformations of their identities as instructors by supporting their capacity to design and implement curricular artifacts informed by a conceptual analysis and in alignment with the MIP vision.

As a point of clarification, this focus differs slightly from an investigation of the self-efficacy of participating instructors, although that is a related component. Rather than examining participants' confidence in their ability to support students' learning, the purpose of this study is to explore participants' *identities* as instructors by investigating their construction of three elements of mathematical inquiry and conceptual analysis.[4]

In this context, the MIP seeks to transform participants' identities (as instructors) towards recognizing the importance of critically reflecting on, and consequentially recognizing the importance for engendering, the *conceptual activity* necessary for students to construct productive mathematical meanings. In this way, the MIP has a particular meaning for 'conceptual activity' that differs from others' perspectives. Many instructors seek to design activities or facilitate classroom discussions that engage students in "mathematical thinking." In responding to an MIP online survey prompt asking participants to describe their expertise related to the College Algebra and Precalculus Pathway, one instructor remarked, "The most important area in this course (and every course) is to make the student a 'math thinker'." Additionally, the previously-discussed professional development initiatives encouraged instructors to engage students in group work or reflective activities, providing opportunities to enhance students' critical thinking skills. In these ways, some of the differences between the MIP and other initiatives may appear inconsequential. Distinguishing between the objectives of many of these

[4] I discuss literature on identity research in Chapter 2.

initiatives and the goals of the MIP requires careful examination of the *nature* and *purpose* of students' thinking.

To clarify the point, consider the various ways that instructors could support students' engagement in mathematical thinking. An instructor can present a problematic situation where students are challenged to "think" in ways that foster less productive conceptual activity. For instance, an instructor might engage their students in a task requiring them to apply certain algebraic skills. Students engaging in this activity will be challenged to "think," but their mental activity is directed towards associating the appropriate skill or technique as opposed to thinking more conceptually. As another example, consider an instructor seeking to support students in solving standard integration problems when encountering different situations (e.g., if a student is in situation *A*, they may try using a trig substitution). While supporting students' *problem-solving* ability is certainly important, these instructors may not be advancing students' *conceptual* activity. The MIP Team seeks to support instructors in intentionally designing tasks and facilitating classroom discussions in ways that enhance students' *thinking*, conceptualized as their abstraction and generalization of mathematical relationships or productive understandings. In other words, there is a difference between merely "getting students to think" by engaging them in group work activities and supporting students' thinking by first identifying and clarifying the nature of the mental actions required to construct a productive meaning.

In contrast to the previous two studies discussed, research conducted by Pelletreau et al. (2018) documented the outcomes of a professional development program, which focused more intentionally on supporting students' construction of productive meanings. They designed their intervention using a data-driven approach in which participants made iterative changes to instruction based on student work. During some of their meetings, faculty would "talk about student responses to these questions and brainstorm ideas for in-class activities that could help students with common conceptual difficulties identified using the questions" (Pelletreau et al., 2018, p. 3). Hence, there appeared to be a central focus on supporting faculty to think critically about the nature of students' conceptions and strategically incorporate tasks to foster students' conceptual development.

In their discussion of the Inquiry Oriented Differential Equations project, Rasmussen and Kwon (2007) made the following remarks:

> Moreover, as our understanding of student thinking evolves, so does our understanding of the kinds of teacher knowledge that would be important for promoting student learning. Beyond content knowledge, such teacher knowledge includes an awareness of students' informal and intuitive ways of reasoning. (p. 192)

In this respect, the MIP Team seeks to influence how faculty conceptualize features of competent mathematics instructional design by engaging them in professional development experiences intended to enhance their knowledge base regarding student thinking. I discuss an interpretation of knowledge base in the context of *pedagogical content knowledge* in the following section.

Pedagogical Content Knowledge

I begin with a brief discussion of two interpretations of pedagogical content knowledge (PCK) articulated by Tallman (2021). I then discuss a third interpretation in more detail, leveraging constructs from constructivist epistemology. This third interpretation is consistent with the goals of the MIP to transform participants' identities as instructors.

The first interpretation Tallman (2021) offers is considering PCK as a *transformation of content knowledge into pedagogical representations*. Tallman (2021) characterized research in the area of mathematical knowledge for teaching (MKT) as predominantly emphasizing *behavioral* features of teachers' instruction: "The attention in MKT research on teachers' behavior has contributed to the development of a variety of knowledge constructs introduced to label and categorize behavior, not explain it" (p. 12). Rather than addressing the mental actions required for instructors to support students' construction of productive mathematical meanings, proponents of PCK in alignment with this interpretation conceptualize knowledge based on these behaviors: "knowledge of *X* is the knowledge required to do *X*" (ibid., p. 11). Tallman (2021) described another interpretation of PCK as an *integration of pedagogical and content knowledge*. Supporting instructors' enactment of PCK that is compatible with this perspective entails developing teachers' knowledge of effective pedagogical practices and specific content separately. Tallman (2021) identified the limitations of this interpretation: "Leveraging mathematical contexts to instruct pre-service teachers in pedagogy . . . might result only in teachers' capacity to superficially and inflexibly enact specific behaviors in their uncritical efforts to imitate 'evidence-based pedagogical practices.' "(p. 13).

In contrast to these two interpretations, Tallman (2021) articulated a description of PCK that supports instructors' capacity to provide experiences for students that engage them in the cognitive activity required to construct productive meanings. To support students' conceptual mathematics learning, it is imperative that instructors are cognizant of their own mathematical schemes.

Unpacking this conceptualization requires articulating what is meant by a scheme. Piaget articulated reflective abstraction according to three distinct meanings, two of which are relevant to this discussion. First, reflective abstraction can be described as *reflecting* abstraction, referring to a projection and subsequent coordination of the individual's activities which may or may not include their awareness (von Glasersfeld, 1995). Engaging in reflecting abstractions produces cognitive schemes as "organizations of internalized actions and operations" (Tallman, 2021, p. 15). Additionally, reflecting *on* these mental operations results in the individual's awareness of these actions and thus represents a higher form of reflective abstraction called *reflected* abstraction. Tallman (2021) described the process of engaging in reflected abstraction as "necessary to establish the connection between the content of the mathematical subject matter and its origin in the cognitive experience of the learner" (p. 21).

An affordance of becoming cognizant of one's own mathematical schemes is that it equips an instructor with the capacity to utilize their content knowledge purposefully (e.g., developing a hypothetical learning trajectory) (ibid). In other words, this process of reflecting on one's mental operations serves as a pivotal link enabling a teacher to construct *pedagogical content knowledge* (i.e., content knowledge with pedagogical affordances). In this context, a central goal of the MIP is to transform the identities of participating faculty by enhancing their knowledge base of the features entailed in competent instructional practices. In discussing the MIP intervention in the following section, I present different ways participants' experiences in the MIP might support their construction of PCK.

The MIP Intervention

In the previous section, I described the central issue being addressed by the MIP—the lack of instructors' requisite knowledge base required to support students' construction of meaning—and I contrasted this focus with other professional development initiatives that emphasize the accumulation or incorporation of pedagogical strategies. Additionally, I clarified my notion of "knowledge base," relying on the conceptualization articulated by Tallman (2021), and I reiterated the objective of the MIP to cultivate a shift in the professional identities of MIP instructors by enhancing their knowledge base. In the following sections, I describe this knowledge base specific to the MIP design by discussing three elements of inquiry and conceptual analysis. I conclude this discussion by identifying ways that participants' experiences in the MIP might support their construction of PCK.

Three Elements of Mathematical Inquiry

The MIP is centrally focused on cultivating a community of mathematics instructors to support students in learning mathematics through inquiry. Specifically, the MIP is centered around the enactment of three elements of mathematical inquiry—*active learning*, *meaningful applications*, and *academic success skills*. I discuss each of these three components and their relationship to features of radical constructivism and Piaget's genetic epistemology, and I specifically contrast the MIP definition of active learning with other interpretations in research.

Before this discussion, I offer a clarification. My research question is centered around investigating the experiences of participating faculty in the MIP with respect to their *identities as instructors* (i.e., related to their conception of the features entailed in acting as a competent mathematics instructor). In my description of these three elements of inquiry, there will be some discussion of *students' identities as learners* (e.g., the definition of academic success skills). While I will briefly discuss students' identities as learners, I clarify that my research questions are centered around participants' identities as mathematics instructors.

Active learning. Chickering and Gamson (1987) offered seven principles for improving undergraduate education, one of which is active learning. In addition to providing examples, they articulated their conception of active learning:

> Learning is not a spectator sport. Students do not learn much just by sitting in classes listening to teachers, memorizing pre-packaged assignments, and spitting out answers. They must talk about what they are learning, write about it, relate it to past experiences,

apply it to their daily lives. They must make what they learn part of themselves. (Chickering & Gamson, 1987, p. 5)

According to Chickering and Gamson (1987), active learning involves students' active engagement with disciplinary content in a variety of different ways to provide more meaningful connections. More recently, Lugosi and Uribe (2020) investigated the effectiveness of incorporating six different active learning strategies in College Algebra and Business Calculus courses: interactive presentation style, group-work with discussion and feedback, volunteer presentations of solutions by groups, raise students' learning interest towards specific topics, involve students in mathematical explorations, experiments, and projects, and continuous motivation and engagement of students.[5] Many of these principles are pedagogical strategies and offer general prescriptions for how to engage students in active learning by doing group work, giving presentations in an interactive style, or having students give presentations. In contrast, the MIP definition of active learning highlights the nature of students' conceptual activity required to understand a specific concept:

> *Students engage in active learning when they work to resolve a problematic situation whose resolution requires them to select, perform, and evaluate actions whose structures are equivalent to the structures of the concepts to be learned.*

These three components of the MIP definition of active learning, *select*, *perform*, and *evaluate*, are compatible with von Glasersfeld's (1995) description of action schemes. As a simple illustration, suppose a student is working to resolve a situation they conceive as problematic in some way. Engaging a student in a problematic situation may result in their perturbation. A student who is perturbed is afforded an opportunity to modify their current cognitive structures to

[5] These six active learning strategies become a central focal point of discussion in Chapter 4.

reestablish conceptual equilibrium (von Glasersfeld, 1995, p. 68). In their chapter on reflected abstraction, Tallman and O'Bryan (2022) described this process:

> When a knowing subject does not possess an appropriate assimilatory scheme to inform their actions when confronted with a novel stimulus or experience, a state of cognitive perturbation, or disequilibrium results. This state can induce a need for the individual to accommodate the stimulus or experience by modifying an existing scheme or creating a new scheme so that their conceptual structures remain viable with their experiential reality. (pp. 24-25)

Tallman and O'Bryan (2022) highlighted an affordance of providing problems which engage students in problematic situations as it may result in the construction of new schemes or the refinement of existing ones. While the situation may be problematic, students engaging *reflecting abstraction* (see Chapter 3 for a discussion of abstractions) are positioned to engage productively in solving the problem by "constructing actions at the reflected level of thought and organizing these actions into cognitive structures at this higher cognitive level" (Tallman & O'Brian, 2022, p. 14).

Assuming the student successfully assimilates sufficient information, the student *selects* or *identifies* (an action) an approach to solve the problem (a goal-oriented perspective), resulting in some outcome or effect. The result could be successful or unsuccessful, and the student's interpretation of this outcome may or may not represent a productive reflection of their actions. If the result produces a negative experience (e.g., an unexpected outcome), then the student may be perturbed, positioning them to *evaluate* their actions. This may lead to the adaption of an existing scheme or the creation of a new one. The evaluation process is dependent on the context to the extent that it enables students to think critically about the answers they construct. Students engaging in an exercise with no "real-life context" in which they are manipulating symbols are less equipped to evaluate the plausibility of their approach or solution.

Appropriately interpreting these first three components of the MIP definition of active learning requires careful attention to the last phrase in the definition. Engaging students in active learning is not only dependent on students' mental and physical activity but also on the specific nature of that activity in supporting students' construction of productive meanings. Students are actively learning the targeted idea if the structures of their actions are *equivalent to the structures of the concepts to be learned*.

It is counterintuitive that the structures of students' actions could be equivalent to the structures of *concepts*. During a discussion of active learning at the Functions and Modeling workshop, a member of the MIP Team accentuated this feature of the definition:

> So, if one reads, uh, this definition of active learning, one interprets action as, as, simply meaning what students do and concepts, um, as being, more or less the topics in sections of the textbook. There's a category error in the criterion for active learning in, in the sense that how can one establish an equivalence between the structure of action and the structure of concepts? Um, there, there needs to be something about how we're conceptualizing activity in mathematical concepts that make, uh, that allow us to compare the structure of these two entities.

Unpacking these ideas depends on interpreting the phrase "conceptualizing activity in mathematical concepts." According to Piaget's constructivist epistemology, a concept is actively constructed, not an external source of information waiting to be conveyed or transferred into the mind of a student who passively receives it. In this sense, the structures of students' actions or their construction of the concept relates to the structures of the concept to be learned. More precisely, the structures of students' actions might become equivalent to the structures of the concept through the refinement and extension of students' mathematical schemes. This could be achieved if students' engagement in the activity fosters their abstraction of these mathematical relationships and subsequent modification of their schemes to be equivalent to the structures of the concept to be learned. In other words, operationalizing this component of the definition requires an instructor to develop a knowledge base that equips them to support students' conceptual understanding (e.g., by developing a sequence of tasks) of the mathematical idea.

Before discussing the MIP definition of meaningful applications, I offer two clarifying remarks. First, it is possible operationalize some components of the MIP definition of active learning but not all of them. For instance, an instructor could give their students an exercise that is problematic (i.e., requires certain algebraic skills to solve) but does not support students' construction of a scheme that reflects the "structures of the concepts to be learned." While students engaging in this activity might be challenged to "think," their thinking is not in the service of abstracting mathematical relationships resulting in the refinement or extension of their cognitive schemes.

Second, there are situations in which students may appear to be actively learning by other definitions but would not be according to the MIP definition. For instance, students who are working together in groups or answering formative assessment questions during class might not be actively learning. Engaging students in active learning requires the instructor to support students' conceptual activity, and this becomes more achievable if the problem incorporates a meaningful application.

Meaningful applications. The second element of mathematical inquiry is meaningful applications. The MIP definition of meaningful applications is as follows:

> *Applications are meaningfully incorporated in a mathematics class to the extent that they support students in identifying mathematical relationships, making and justifying claims, and generalizing across contexts to extract common mathematical structure.*

Meaningful applications support students in acting on their schemes to identify mathematical relationships and reflecting on their constructed solution after making claims. Reflection is an essential aspect of evaluating one's actions and a "mysterious" process involving remembering and re-presentation (von Glasersfeld, 1995, p. 90). Students' capacity to justify their claims and think critically about the answers they construct is enhanced by having a meaningful context. Students engaging in an activity without a story-like context in which they are "pushing symbols" are less equipped to evaluate the plausibility of their approach or solution.

Finally, meaningful applications support students' capacity to *generalize findings across contexts to extract common mathematical structures* by providing opportunities for them to engage in reflective abstraction (see Chapter 3 for a discussion of abstractions). As a student implements similarly structured schemes for solving problems with different contexts, they may begin to engage in empirical abstraction. Through repetition and the instructor's guidance, the student may eventually become conscious of their efforts as they abstract their actions into schemes, positioning them to extract the common structure of the problem and engage in reflected abstraction. von Glasersfeld (1995) framed it this way:

> This simple form of the principle of induction, namely 'to retain what has functioned successfully in the past', can be abstracted and turned upon itself: because the inductive procedure has been a successful one, it may be advantageous to generate situations in which it could be employed. (p. 70)

To illustrate the MIP definition of meaningful applications more concretely, I discuss two related rates problems from Reed et al. (2021). In relation to the MIP definition, the first example represents a more meaningful application than the second example:

> Example 1: A man starts walking north at 4ft/sec from a point P. Five minutes later a woman starts walking south at 5 ft/sec from a point 500 feet due east of P. At what rate are the people moving apart 15 minutes after the woman starts walking?
>
> Example 2: If the radius of a spherical balloon is increasing at a rate of 2 inches per minute, at what rate is the volume increasing when the radius is 3 inches?

Both examples represent related rates problems, but Example 1 supports students to engage in quantitative reasoning while constructing the context and identifying mathematical relationships based on their construction. Reed et al. (2021) discussed students' required reasoning to solve this problem:

> The solution requires students to first conceptualize the constant rate at which one side length of a right triangle changes with respect to elapsed time. Specifically, the sum of the constant speeds at which the man and woman are walking is the rate of change of the side length of the right triangle (with respect to elapsed time) with acute vertices determined by the man and woman's positions. Moreover, substituting the correct values into the related rate formula that results from implicitly differentiating the Pythagorean theorem requires students to conceptualize the triangle's other side length as invariant as both the man and woman are in motion. A *dynamic conception* of the context is therefore necessary to solve this problem. (p. 15).

In contrast, solving Example 2 does not require students to contextualize the situation but rather use a formula (volume of a sphere) and procedurally solve using implicit differentiation. Hence, the latter example represents a less meaningful application according to the MIP definition.

The MIP definition of meaningful applications does not reflect a typical interpretation that the *context* or *setting* is meaningful, because it relates to some "real-world" example or is interesting or relatable to students. In other words, having a "real-world" context or an intriguing problem is not a sufficient condition for an application to be meaningful according to the MIP definition. Yet, engaging students in intriguing problems is an important feature for supporting students' motivation, which relates to the third element of inquiry: academic success skills.

Academic success skills. The MIP definition of academic success skills is as follows:

> *Academic success skills foster students' construction of their identity as learners in ways that enable productive engagement in education and the associated academic community.*

First, academic success skills foster students' *construction* of their academic identity. Conceptualizing students' academic success skills through the lens of radical constructivism may seem irrelevant with regards to offering substantive contributions towards advancing instructional practices, but consider remarks from Tallman and Uscanga (2020). In their theoretical analysis of students' mathematical anxiety (MA), they stated that the "constructivist conceptualization of identity has implications for students' experience of MA and for instructional and curricular innovations that seek to minimize its negative influence" (p. 15). Later in their article, they offered more details:

> Specifically, an instructor can support students in enhancing their appraisal of their psychological resources by providing repeated opportunities for them to reflect on how the cognitive components of their desired identities as mathematics learners contributed

> to their achievement of particular cognitive states, or learning goals. Doing this enables students to recognize that they possess the intellectual capabilities necessary to understand mathematical ideas in a meaningful way, and contributes to their development of a disposition to appraise the challenges they encounter as manageable. (Tallman & Uscanga, 2020, p. 24)

These comments highlight the importance of providing students with opportunities to reflect on the meanings they *construct* from their mathematical activity.

Additionally, students' academic success skills enable them to construct their *identity as learners.* Cribbs et. al. (2021) analyzed relationships between mathematics mindset, mathematics identity, mathematics anxiety, and mathematics self-efficacy impacting choices for STEM careers. They concluded that mathematics mindset positively influenced mathematics identity whereas mathematics anxiety negatively influenced mathematics identity. Instructors intending to foster students' construction of their identity as learners seek to empower them with ownership, instilling confidence in their ability to learn and perceiving them as intellectually capable. One striking illustration is presented in McGee and Martin (2011) as an example of how a black student " 'fronted' to maintain the appearance of conformity":

> Walking into the first day of a higher level mathematics or engineering class with the book outside of the book bag so (hopefully) no one would ask ''Are you in the right class?'': ''I walked into the class [Calculus III] and they [classmates] just looked shocked. Then a girl slivered up to me and asked if she could see my [Calculus III] book. . . . Now, I always walk in [on the first day of class] with my book in my hand and I slam it down on my desk!'' (p. 1370)

Additionally, Tallman and Uscanga (2020) articulated how students' conceptions of their own mathematical competence in relation to their desired identities may engender mathematical anxiety. Students experience mathematical anxiety, "because they recognize the possibility that their genuine mathematical activity will reveal characteristics of their mathematical competence in particular, and intellectual potential in general, that are antithetical to their desired identities as students and/or future professionals" (Tallman & Uscanga, 2020, p. 15).

Finally, students construct their identities as learners *in ways that enable productive engagement in education and the associated academic community.* This latter part of the definition is important, since students may perceive themselves as learners and simultaneously be lazy or unproductively engage with other classmates (i.e., talking in class when the instructor is speaking). Cultivating some of the academic success skills identified during the first Initiation Workshop (i.e., productive struggle, growth mindset, problem solving) and supporting students to engage in reflective abstraction, instructors can equip them with confidence derived from engaging in higher-level thinking (i.e., constructing and abstracting productive mathematical meanings.)

To make this discussion more concrete, I now discuss an example from Simon and Tzur (2004) to illustrate a sequence of tasks which incorporate aspects of all three components of inquiry. Simon and Tzur (2004) developed these activities to support elementary school students to abstract the mathematical relationship between equivalent fractions. The first two tasks are written as follows:

> *Task 1*: Draw a rectangle with 1/2 shaded. Draw lines on the rectangle so that it is divided into sixths. Determine how many sixths are in 1/2.

Task 2: Draw a rectangle with 2/3 shaded. Draw lines on the rectangle so that it is divided into twelfths. Determine 2/3 = ?/12.

Now consider how these tasks incorporate aspects of the three elements of inquiry. They are necessarily *problematic* for an elementary school student who has not abstracted the relationship between equivalent fractions and thus require the student to *select* how to divide the rectangle and *perform* the operation envisioned to find the solution. Consider a later task in the sequence:

Task 4: Drawing diagrams to solve equivalent fractions problems is not much fun when the numbers get large. For the following do not draw a diagram. Rather describe what would happen at each step if you were to draw a diagram. Use that thinking to answer the following:

a. 5/9 = ?/90

b. 7/9 = ?/72

After completing the activity, the student is required to *evaluate* the effects of their actions to produce an equivalent fraction by reflecting on their activities. Through this abstraction, the student might begin *identifying the mathematical relationship* of a common factor between the two fractions. Engaging in this process supports the structure of the student's actions to become equivalent to the structures of the concept. Simon and Tzur (2004) articulated their envisioned learning process from students' engagement:

1. Using their current knowledge of fractions, the students will draw a diagram of the original fraction and subdivide the parts in the original fraction to create the number of parts in the new denominator.

2. Using their knowledge of whole number multiplication, the students will multiply the number of shaded parts (old numerator) by the number of subdivisions in each part to determine the new numerator.

3. Through reflection on their activity and its effects, the students will determine that the activity of multiplying the denominator (subdividing each part) causes the numerator (the number of shaded parts) to be increased by the same factor.

A student engaging in Task 4 will also be required to make *claims*. Moreover, while this activity might be enhanced if the student also needed to *justify* their answer, a component in a later task, it would be difficult to provide alternative contexts (e.g., different shapes) considering the audience of students. Finally, the successful completion of this activity (perhaps requiring more iterations or more tasks) might support this student's *construction of their identity as learners*, because the tasks are purposefully designed to foster a student's reflection on their actions in service of identifying the mathematical relationship between equivalent fractions.

Mutual Influence

In this section, I briefly describe different relationships between the MIP definitions of active learning, meaningful applications, and academic success skills. The connections between these three inquiry elements can be contextualized within the framework of the previous discussion in which I identified the influence of constructivist epistemology and radical constructivism on the construction of these definitions.

Active learning. How does designing tasks that incorporate meaningful applications and support students' academic success skills influence students' capacity to engage in active learning? Recall the three definitions (see Table 2).

Table 2

Operational Definitions of the Three Elements of Inquiry

Three Elements of Inquiry	Operationalized Definitions
Active Learning	*Students engage in active learning when they work to resolve a problematic situation whose resolution requires them to select, perform, and evaluate actions whose structures are equivalent to the structures of the concepts to be learned.*
Meaningful Applications	*Applications are meaningfully incorporated in a mathematics class to the extent that they support students in identifying mathematical relationships, making and justifying claims, and generalizing across contexts to extract common mathematical structure.*
Academic Success Skills	*Academic success skills foster students' construction of their identity as learners in ways that enable productive engagement in education and the associated academic community.*

By providing a context that supports students to make and justify claims, students are positioned to *evaluate their actions* to identify the extent to which they produce plausible solutions. Additionally, incorporating meaningful applications supports students to engage in the conceptual activity required to abstract mathematical relationships (i.e., students are the ones *selecting and performing actions*). Related to academic success skills, supporting students' construction of their identities as learners requires the instructor to clarify the nature of the actions that are needed to accomplish this outcome, and to engage learners in experiences by which they come to recognize that they possess the capacity to engage in these actions. In particular, this may influence their capacity to identify the *structures of the concepts to be learned*.

Meaningful applications. How does designing tasks that engage students in active learning and support students' academic success skills necessitate the incorporation of meaningful applications? First, consider the influence of engaging students in active learning on the incorporation of meaningful applications. Engaging students to select, perform, and evaluate their actions, an instructor is positioned to designs tasks that support students in *identifying mathematical relationships*. Additionally, as students engage in the process of selecting and performing actions, they are *making claims* from their mathematical activity. Finally, instructors who design tasks that support students to evaluate their actions are fostering their capacity to *justify claims* previously made. On the other hand, promoting students' development of academic success skills also influences an instructor's capacity to incorporate meaningful applications. Instructors who design tasks that support students' construction of their identities as learners may consider ways that they can support students in *identifying mathematical relationships.*

Academic success skills. How does designing tasks that engage students in active learning and incorporate meaningful applications influence the development of students' academic success skills? Designing tasks that support students to engage in mathematical activity (i.e., they select, perform, and evaluate their actions) that fosters their construction of productive mathematical meanings (i.e., the structures of their actions are equivalent with the structures of the concepts to be learned) may simultaneously encourage students' *construction of their identities as learners*. Additionally, the process of engaging in this mathematical activity may support students' productive engagement in the surrounding community by enhancing their critical thinking and problem-solving skills. Similarly, designing tasks that incorporate meaningful applications, an instructor is positioned to support students' development of academic success skills. Providing contexts and examples that encourage students to justify claims, an instructor may support students' development of critical thinking and communication skills, enhancing their *productive engagement in the community*.

This commentary naturally prompts the consideration of the following question: What is the nature of the activity required to operationalize *one* of these three inquiry elements? Addressing this question motivates a discussion of *conceptual analysis*.

Conceptual Analysis

Operationalizing the three elements of inquiry requires specific preparation. Consider the last phrase in the definition of active learning:

> Students engage in active learning when they work to resolve a problematic situation whose resolution requires them to select, perform, and *evaluate actions whose structures are equivalent to the structures of the concepts to be learned.*

An instructor seeking to design tasks which engage students in active learning needs their students to perform mental *actions* in alignment (i.e., *equivalent*) with the instructor's image of the *structures of the concepts to be learned*. In other words, an instructor intending to promote active learning as defined by the MIP would necessarily need to conduct a conceptual analysis to first unpack their own conceptions (i.e., "structures of the concepts to be learned").

Thompson (2008) articulated *conceptual analysis* as a description of the reflective process of specifying the cognitive activity involved in understanding an idea in a particular way. MIP participants are exposed to conceptual analysis at each of the Initiation Workshops, and faculty who choose to participate on a CoRD are expected to conduct one on their chosen topic. By engaging in the reflective thinking required to perform a conceptual analysis, faculty become cognizant of their own mathematical schemes and are better equipped to support students' conceptual learning through purposeful enactment of mathematically informed pedagogical practices (Tallman, 2021).

Experiences Supporting Participants' Construction of PCK

I now describe the experiences in which MIP participants have (or will) engage to support their construction of PCK. Specifically, I discuss faculty's participation at the Initiation Workshops and their involvement on CoRDs. While this discussion is not intended to be comprehensive, it will illuminate the nature of participants' engagement in the MIP CoP that may support their awareness (and hopefully enactment) of the importance of reflecting on and clarifying the nature of their own conceptions through the process of conducting a conceptual analysis.

Four of the five Initiation Workshops, which were held over summer 2019 and summer 2021, focused on identifying and clarifying key conceptual threads in entry-level mathematics courses—Functions and Modeling, Quantitative Reasoning, College Algebra and Precalculus, and Calculus I—and the fifth workshop addressed the topic of students' academic success skills. Faculty attending these workshops experienced opportunities to gain insight from outside sources by reading journal articles and listening to virtual presentations.

The MIP Team intended the pre-workshop readings to expose participants to ideas in mathematics education literature (e.g., conceptual analysis, learning trajectories, supporting students' quantitative and covariational reasoning) so that they might be better prepared to participate in discussions during the workshop.

During the Initiation Workshops, the MIP Team introduced participants to the three elements of inquiry and conceptual analysis. Additionally, the MIP team modeled these workshops after the American Institute of Mathematics 'Squares' to be highly interactive, providing opportunities for faculty to engage in reflected abstraction during break-out sessions by identifying and clarifying important features of understanding a specific topic.

These specified topics might then be the focus of a future CoRD: a team of three to six MIP faculty working together to construct a curricular module related to a conceptual thread identified during one of the Initiation Workshops. Structurally, there are two types of CoRDs—content-based CoRDs and academic success skill CoRDs. Content-based CoRDs construct a module associated with one of the four content focused workshops—Functions and Modeling, Quantitative Reasoning, College Algebra and Precalculus, and Calculus I—while the academic success skill CoRDs develop resources related to affective topics (e.g., grit, motivation).

Once a CoRD has been formed, they can begin the process of writing a proposal for their specified topic. During encounters between the CoRD and the MIP Correspondent, the Correspondent might support participants' construction of PCK by introducing artifacts (e.g., research articles) or guiding discussions. After reviewing participants' proposals and eventual submission of the module, the Correspondent provides feedback (often through the form of probing questions) in the service of directing the CoRDs' attention to engage in reflected abstraction by encouraging them to articulate and clarify the nature of the mental activities required to support students' construction of productive mathematical meanings. Finally, the distribution of the *artificial* CoRD module designed by the MIP Team provides another opportunity for MIP faculty to reflect on *how* these tasks were designed and for what *purpose* were they created.

Criterion 2: Implications of the Research

The primary goal of this section is to highlight distinctions between the nature of the research being conducted in association with the MIP compared with other professional development initiatives. I begin this section with a discussion of design research in mathematics education and highlight two important features of this research. I then provide concrete illustrations of research, associated with STEM professional development initiatives for instructors, measured according to these two features. I conclude with a discussion of the research being conducted in the context of the MIP.

Categorizing Recent Research

Traditionally, many researchers did not use theoretically-guided qualitative methods to investigate questions related to the teaching and learning of mathematics. Kilpatrick (1992), in his article *A History of Research in Mathematics Education*, provided an excellent synthesis of the historical context of research in mathematics education by detailing its emergence, considering significant contributions from other disciplines, and providing key insights during pivotal moments in its tumultuous development. As the field continued to develop and evolve in universities during the 1900s, it depended heavily on the influence of two other disciplines—mathematics and psychology. Kilpatrick (1992) observed that mathematics education naturally attracted the involvement of some mathematicians, and in 1908 the International Commission on the Teaching of Mathematics was formed during the Fourth International Congress of Mathematicians in Rome. The significance of this commission should not be overlooked. Kilpatrick (1992) stated that their reports "marked the beginning of efforts by mathematicians and mathematics educators not only to reform school mathematics but also to gather information that could be used in that reform" (p. 7).

While a productive first step, early research products produced by the commission proved markedly inadequate: "The data-gathering activities of the international commission were monumental, politically motivated, methodologically unsophisticated, and conceptually weak. The reports that resulted were more compilations of data than analyses or interpretations" (Kilpatrick, 1992, p. 6). Over time, research in the field has evolved to include more comprehensive methodologies and coherent theoretical frameworks. With an ever-increasing number of mathematics education studies focusing on a wide range of phenomena and involving combinations of quantitative and qualitative approaches, some attention has been given to reflecting on the state and purpose of research in the field.

In the opening chapter of the *Compendium for Research in Mathematics Education*, Confrey (2017) identified the difficult task of responding to the question, "What is research?". In addressing this question, she categorized research according to three "buckets" based on its purpose: to inform, to deform, or to reform. I omit a discussion of the latter two buckets and focus my attention on Confrey's (2017) description of the first one.

Confrey (2017) characterized the first bucket, described as research to *inform*, based on three questions relating to scientific research. Research relating to the first question, "What is happening?" encompasses large-scale, quantitative analyses of trends and is not pertinent to this discussion. The other two questions, "Is there a systematic effect?" and "How or why is it happening?" highlight two fundamentally different approaches to conducting research that are relevant to consider.

On one end of the spectrum, mathematics education researchers who are inclined toward more "scientific studies" favor research categorized under the question, "Is there a systematic effect?". Those conducting studies according to this classification seek to corroborate the effectiveness of an intervention with a causal relationship. Design research is positioned on the other end of the spectrum and in the service of addressing the question, "How or why is it happening?". Cobb et al. (2017) described five common features of design research studies: (1) they address problems identified by practitioners, (2) they have a methodology that is "highly interventionist," (3) they have theoretical and practical grounding, (4) they are designed iteratively, and (5) they seek to generalize (p. 209).

Considering this first bucket, I illustrate different types of research being conducted by the MIP and other large-scale professional development programs along a continuum where classical experimental designs and design research are at opposing ends. Specifically, I analyze

STEM professional development research studies according to two criteria: the extent to which they (1) incorporate theory to help explain the underlying mechanisms that guide teacher change and (2) identify modifications being made to the design of their intervention. As an important note, while the first study is most closely associated with an experimental design and the last study is an example of design research, the other studies in between the first and the last are not necessarily increasingly more design based.

This first criterion relates to the theory of teacher change (Wayne et al., 2008). Wayne et al. (2008) outlined five design issues facing experimental and quasi-experimental studies. In their discussion of the first design issue, they elaborated on two theories accompanying any professional development effort, the latter being the theory of teacher change:

> The theory of teacher change is the intervention's theory about the features of PD [professional development] that will promote change in teacher knowledge and/or teacher practice, including its theory about the assumed mechanisms through which features of the PD are expected to support teacher learning. (Wayne et al., 2008, p. 472)

While "any given PD intervention requires these two theories" (Wayne et al., 2008, p. 472), the extent to which the mechanisms described in the theory of teacher change are explicit and adequately informed by epistemological perspectives greatly vary among research studies. Hence, I use this metric to assess where large-scale, PD interventions might be positioned on the continuum.

Equipped with an understanding of the mechanisms that contribute to the effectiveness of their intervention, researchers are better positioned to make modifications (related to the second criterion) to their design based on theoretically informed rationale. Cobb et al. (2017) described the use of ongoing analysis for occasioning the refinement of a design:

> This process of testing and revising conjectures and thus of improving the associated design for supporting learning involves iterative cycles of design and analysis. At any point in a design study, the evolving design reflects then-current conjectures about the process of the participants' individual and collective development and the means of supporting it. Ongoing analyses of both students' activity and the enacted supports for their learning provide opportunities to test, refine, and revise the underlying conjectures, and these revisions in turn inform the modification of the design. (p. 209)

To this end, I illustrate studies documenting STEM professional development interventions targeted towards improving instructional practices according to these two criteria. I do not intend my discussion to be comprehensive since these two criteria are a subset of the five features of design research that Cobb et al. (2017) identified. Yet, the relationship between the two criteria is evident: "Importantly, design-based research goes beyond merely designing and testing particular interventions. Interventions embody specific theoretical claims about teaching and learning, and reflect a commitment to understanding the relationships among theory, designed artifacts, and practice" (The Design-Based Research Collective, 2003). Additionally, I do not intend my description of these studies to convey a linear trajectory from experimental studies towards more design-based research. Hence, while the first study is most closely associated with an experimental design and the last study is an example of design research, the ones in between the first and the last are not increasingly more design based.

A Continuum of Studies

Du et al. (2018) investigated the extent to which a three-year professional development effort on integrated STEM influenced the instructional practices of middle school teachers involved in the program. Specifically, they examined both the quality of participants' teaching practices as well as the evolution of their conceptions regarding effecting instruction (p. 107). As their title suggests, their investigation aligns more with experimental design studies than design research. Using a mixed methods approach involving classroom videos and teacher interviews, they measured the effectiveness of their intervention, concluding that teachers "significantly ($p < 0.005$) improved their STEM teaching practice" and "teachers reported more frequent use and improved attitudes toward implementing STEM practice" (Du et al., 2018, p. 112).

Regarding criterion one, Du et al. (2018) were not positioned to make strong theoretical contributions from their research design. First, they evaluated their interviews and observations in a pre/post format. Second, their conceptual framework for conducting the intervention, articulated by Desimone (2009), was based on five effective features that have been shown to be impactful for professional development. Hence, Du et al. (2018) did not incorporate a theory to guide them in understanding the underlying mechanisms associated with teacher change. Since their purpose was to evaluate the effectiveness of the intervention and report their findings, their study illustrates research more closely associated with experimental design. Importantly, these comments are not intended to diminish or devalue this research but to clarify the purpose and potential implications of the study based on its design.

Moreover, Du et al. (2018) do not indicate modifications being made to the design of the intervention, perhaps a reflection of their methodology. While they incorporated interviews and classroom observations into their research design, these were conducted according to a pre/post assessment. Hence, they were not positioned to modify their design from this research. They did require these teachers to submit five videos throughout the program, but it was unclear the extent to which these were examined, much less analyzed, for the purpose of refining their intervention.

As another illustration, Borda et al. (2020) investigated instructional changes (and constraints on those changes) from faculty participating in a professional development program as well as students' perceptions of these new approaches. The objectives of the project, *Change at the Core*, which involved a university and two community colleges, were to foster productive engagement from underrepresented STEM majors, improve instructional practices and offer a viable model focused on student learning.

Regarding the first criterion, Borda et al. (2020) were guided by constructivist theory, incorporating formative assessment as a "framework for selecting and sequencing content and pedagogy to align with constructivist theory" (Borda et al., 2020, p. 3). Presumptively, they used this theory and their underlying framework primarily for the purpose of supporting faculty participating in the intervention rather than as a lens to investigate the mechanisms of transformation. There are several reasons which give support to this claim.

First, they did not explicate the mechanisms of constructivist epistemology as described by Piaget and others. Second, their framework was "extremely relevant to our faculty participants when it came to connecting high-impact pedagogies with important curricular goals" (ibid., p. 3). Third, in their section on implications, they did not discuss how incorporating constructivist theory enabled them to better understand how instructors were learning. More generally, they recommended that "professional development should be grounded in a theory of learning" so that "faculty have a vision in mind and can experiment with strategies to move toward that vision," and it should "offer ongoing opportunities for learning and practice, spaced out over a long enough period of time to give instructors a chance to evolve" (Borda et al., 2020, p. 15). Now consider the second criterion. Lacking a coherent theory restricted their capacity to make theoretically informed modifications to their design, although some of their coding was used to inform future analysis. Moreover, it was unclear if they intended to leverage their research to modify future design aspects of the project.

Next, I illustrate studies that provided theoretical implications after analyzing their findings. McCourt et al. (2017) conducted a study of nineteen biology instructors through the first half of a five-year professional development program. Their theoretical framework was expectancy value theory (EVT), "a framework for considering what motivates humans to engage and persist in certain behaviors" (ibid., p. 3). According to EVT, expectancy (beliefs about success and the ease of the task) and value (attainment, intrinsic, utility, and cost) determine one's motivation (ibid., p. 3). After analyzing their data, McCourt et al. (2017) offered refinements to this theory. Their findings not only indicated that expectancy should be considered in one category (as opposed to two), but they also highlighted the difficulty of dividing value into intrinsic, utility, and attainment since instructors often described all three together:

> As an example, in one instance of dialogue, an instructor would talk about enjoying the AACR group (intrinsic value) because it gave her a chance to talk about and improve teaching (utility), as well as improving teaching to improve job performance (attainment). However, instructors almost always discussed costs separately. Our findings support the idea that EVT can be applied to help explain instructor motivation, while considering cost as a separate subcategory of value (Figure 4). (McCourt et al., 2017, p. 12)

Hence, McCourt et al. (2017) leveraged their findings to offer insights regarding the use of EVT in research. Informed by these theoretical implications, future researchers seeking to incorporate EVT are equipped to potentially modify the theory based on the shortcomings outlined by McCourt et al. (2017). Additionally, they could use the adapted model of the EVT equation:

motivation is the product of expectancy with the difference of value minus cost (ibid., p. 12). Hence, these implications can guide future researchers seeking to use this framework.

McCourt et al. (2017) also leveraged their data to inform subsequent aspects of their research. Conducting yearly interviews, they "generated questions for year 2 interviews by examining both EVT and the anticipated benefits identified in the first interviews" (ibid., p. 4). While they made these refinements, these adjustments concerned their *data collection methods* and not the design of the intervention.

An investigation by Cobb et al. (2009) offers an excellent example of design research centered around statistic teachers' professional development. Cobb et al. (2009) explicated three conceptual challenges they confronted during a five-year investigation focusing on enhancing instructional practices for middle school mathematics instructors teaching statistical data analysis (Cobb et al., 2009, p, 168). Overall, their research aim was to "improve the design for supporting learning that we had formulated at the outset" (Cobb et al., 2009, p. 170).

Theoretically, their research was heavily influenced by a Communities of Practice approach: "Wenger's (1998) discussion of organizations as lived oriented us to identify the groups within the district that were pursuing agendas for how mathematics should be taught and learned" (ibid., p. 175). Moreover, their focus on the interactions between groups accentuated some of the interconnections (boundary encounters, brokers, and boundary objects) Wenger (1998) discussed. Cobb et al. (2009) noted that these connections between groups "came to life" after listening to interviews (p. 177).

Sensitivity to the constructs outlined by Wenger (1998) enabled Cobb et al. (2009) to interpret the mechanisms of change through a theoretical lens. To illustrate this point, consider their analysis of the construct of brokering: "connections provided by people who can introduce elements of one practice into another" (Wenger, 1998, p. 105). After a process of testing and making modifications, Cobb et al. (2009) acknowledge that they had

> initially underestimated the critical role of brokers. It was only as we analyzed the developments that occurred once the two mathematics leaders began to participate in the activities of the professional teaching community that we began to appreciate more fully the important contributions that brokers can make in bridging between perspectives. (p. 179)

Consequently, they modified their approach to provide more support for brokers.

The purpose of this discussion has been to distinguish between types of research being conducted along an experimental design and design research continuum. I analyzed studies associated with STEM professional development initiatives according to two criteria: the extent to which (1) incorporate theory to help explain the underlying mechanisms that guide teacher change and (2) identify modifications being made to the design of their intervention.

Concerning the latter criterion, it was not always evident if (or how) researchers were modifying the design of their intervention. Moreover, among those studies that documented certain adjustments to the intervention, not all refinements were theoretically informed or targeted towards improving the design of the initiative. In some studies, the researchers either did not leverage a coherent theoretical framework, or their theory was not applied to interpret the mechanisms of change of the intervention (related to criterion one). In other studies, research was informed by a coherent framework, positioning the researchers to offer theoretical implications. By examining such a paucity of studies, I am not suggesting that these examples are representative of STEM professional development research along the aforementioned continuum. Yet, I refer to remarks by Cobb et al. (2017) who stated that “there are relatively few published accounts of PD [professional development] design studies” (p. 228).

I consider three implications from this brief analysis. First, this discussion highlights a *discrepancy* regarding how qualitative research is being conducted. Some researchers are more focused on assessing the impact or evaluating the effectiveness of their intervention while other researchers prioritize understanding the mechanisms contributing to the observed change. (And, of course, many studies prioritize both types of research in their mixed methods approach). Both types of research are important but operate with fundamentally different objectives.

Second, my review highlights the *difficulty* in conducting design research to provide theoretical contributions and theoretically informed modifications to the design of the intervention. Doing design research requires the researcher to use a theory that is “sophisticated enough” to enable the researcher to offer meaningful contributions when analyzing the mechanisms of change. Some researchers incorporate theory without that sophistication, hindering their capacity to articulate their findings. Moreover, researchers using more sophisticated theory may not be positioned to offer theoretical contributions.

Finally, my overview of different studies highlights the *need* for more researchers to conduct design research. Certainly, assessing the effectiveness or impact of an intervention is a necessary type of research. Yet, research that prioritizes evaluating the success of an initiative without attending to theoretical implications may be limited in its capacity to advance research in the field.

I briefly comment on how the MIP is using design-research to investigate the evolution of a community of mathematics faculty participating in the MIP CoP by leveraging findings from two exploratory case studies. I conducted these case studies to investigate faculty’s conceptions of the three elements of inquiry and conceptual analysis and to assess the extent to which these conceptions align with the descriptions articulated by the MIP Team. The incorporation of sophisticated learning theory on multiple scales, social learning theory more wholistically and radical constructivism on an individual level, positions the MIP Team to better articulate the mechanisms related to the theory of teacher change and offer potential modifications to the design of the intervention from my analysis of these case studies. These modifications might include the strategic introduction of a reified artifact, design adaptations at future workshops, changes to how the brokers interact with CoRDs, etc.

CHAPTER 2

MATHEMATICS FACULTY MEMBERS' PERCEIVED CONTRIBUTIONS, TAKEAWAYS, AND VISION FOR FUTURE COLLABORATIONS AFTER THEIR INVOLVEMENT IN THE MIP

Introduction

Identity research has roots stemming back to the 1930s (Graven & Heyd-Metzuyanim, 2019) and has continued to evolve over time. In recent years, identity has become a focus of research in mathematics education. As Darragh (2016) pointed out, the overarching theme of identity encompassed more papers at the Mathematics Education in Society conference in Capetown, South Africa in 2013 than any other topic. In the 2017 *Compendium for Research in Mathematics Education* published by NCTM, Langer-Osuna and Esmonde (2017) wrote a chapter specifically discussing identity research in mathematics education. More recently, a special issue in the *International Journal on Mathematics Education* (*ZDM*) focused exclusively on identity. This issue featured sixteen articles about a wide range of topics on identity including race and gender, a networked approach to portray identity, reflectivity and teachers' identities. Identity research can be broadly categorized and has a wide range of implications. Generally, some researchers focus on students' identities while others (including myself) analyze the identities of teachers. Studies focusing on teacher identity can be beneficial to educational policymakers, school administrators, instructional coaches, course coordinators, researchers, etc. Increased knowledge in this area might influence how training programs are structured for newer faculty or how a coordinator supports their colleagues. Findings about teacher identity might also impact researchers' design of professional development initiatives by providing insights about factors that impede or promote teachers' learning and implementation of effective instructional practices. Most importantly, instructor involvement in these programs might ultimately impact student success and improve learning outcomes (Capraro, 2016; Pelletreau et al., 2018).

In their extensive review of 47 recent studies focusing on identity research in mathematics education, Graven and Heyd-Metzuyanim (2019) identified four central objectives of this research area (pp. 369-370):

1. Making a socio-political claim.
2. Providing a relatively holistic lens to examine learners' experiences in relation to their social context.
3. In studies focusing on teachers' identities, identity is used to examine teachers' experiences in various stages of their career.
4. To make pedagogical claims.

Graven and Heyd-Metzuyanim (2019) found that 54% of the articles focused on learners and only 32% focused on teachers. Among the latter category, 25% of the research focused on one teacher, and no study examined more than eight of them (although there may have been more teachers involved in a larger study). This current study adds to this body of research on teacher identity by offering insights complied from eight interviews with collegiate mathematics faculty primarily in Oklahoma.[6]

Literature Review

In this review, I omit a description of the MIP, since it was discussed in the opening chapter. Initially, I describe different ways that identity can be classified: according to its theoretical orientation, how it is defined, or whether it is conceptualized as an *acquisition* or an *action* (Darragh, 2016). After this discussion, I examine specific studies focusing on teachers' identity, describing their purposes and contributions to the field. I conclude this synthesis by examining how my research relates to this previous work and highlight potential insights it offers, both to mathematics education and the MIP. I present my conception of identity and relate it to the focus of my research in the Theoretical Framing section that follows my review of the literature on identity research in mathematics education.

Classifying Identity Research in Mathematics Education

Fundamentally, identity research in mathematics education has been categorized according to its focus on teacher identity or learner identity (Graven & Heyd-Metzuyanim, 2019). In the following sections, I offer different characterizations of identity by considering theory, definitions, and other forms of classification.

Classifying identity by theory. In their chapter published in NCTM's *Compendium for Research in Mathematics Education*, Langer-Osuna and Esmonde (2017) provided an analysis of identity research in mathematics education. The purpose of their chapter was to "tease apart the different definitions of identity currently at play" (p. 637) in the field, and they examined identity

[6] One interviewee is currently working at an institution in another state.

according to four theoretical approaches (see Table 3): poststructural, positioning, narrative, and psychoanalytic. Not intending to provide a comprehensive analysis, they argued that identity research is incoherent in the field and focused their attention on it being contextually bound (pp. 637-638). In their review of 47 articles focusing on identity research in mathematics education, Graven and Heyd-Mezuyanim (2019) concluded that "almost all papers relied on some form of socio-cultural frame" (p. 367).

Table 3

The Nature of Identity According to Four Theories

	Poststructural theory	**Positioning theory**	**Narrative theory**	**Psychoanalytic theory**
Nature of identity	Broad discourses make certain subject positions available and constrain the ways in which people exercise agency. People construct identifications in relation to these discourses.	Individuals construct subject positions in relationship with others during interactions, in particular through talk, that make claims about who people are within particular social contexts. Relative to post-structural theories, positioning theories emphasize agency.	Individuals and the people around them create stories about themselves as they make sense of their experiences across social settings.	Individuals are governed by strong unconscious forces of fear and desire and develop identities as they cope with these inner forces.

Note. Adapted from *Compendium for Research in Mathematics Education* (p. 639), by J. M. Langer-Osuna and I. Esmonde, 2017, National Council of Teachers of Mathematics. Copyright 2017 by the National Council of Teachers of Mathematics, Inc.

First, consider poststructural (or discursive) theory. This perspective seeks to account for power and structural effects influencing an individual's identity formation. Often relying on narratives (e.g., interviews), this theory "foregrounds how structures shape the arenas in which people exercise agency" (Langer-Osuna & Esmonde, 2017, p. 638). The second perspective is positioning theory. Langer-Osuna and Esmonde (2017) described it this way:

> As people interact with one another, they act to position themselves as displaying certain qualities (e.g., friendly, smart, authoritative) or as having a particular role (e.g., teacher,

student, group leader). People also act to position others and accept or reject acts of positioning about themselves or others. (p. 639)

Wenger's (1998) interpretation of identity within his framework of social learning theory is an example of positioning theory of identity.

While poststructural and positioning theory might use stories to *approximate* one's identity, narrative theory *equates* the two. Portraying identity in this way has some benefits: "Rather than being internal and unknowable, identities as stories are human made; shaped by society; changeable across time, context, and narrator; and accessible to researchers through empirical methods such as interviews, observations, or collecting written biographies" (p. 641). Finally, the fourth theory Langer-Osuna and Esmonde (2017) described is psychoanalytic. More prevalent among researchers in the United Kingdom, this theory articulates identity by considering its emergence from affective and subconscious components (p. 642).

Classifying identity by definitions and other categorizations. Overlapping with the approach of Graven and Heyd-Metzuyanim (2019) to classify identity research based on theoretical orientations, Darragh (2016) presented five possible definitions for identity: *participative*, *narrative*, *discursive*, *psychoanalytic*, or *performative*. In the first definition, identity is "constructed through participation and engagement in a social group" (Darragh, 2016, p. 24). Identity can also be viewed as narrative, highlighted in the influential article by Sfard and Prusak (2005) in which they offered an operationalized definition of identity as stories (Graven & Heyd-Matzuyanim, 2019, p. 363). Discursive identity, Darragh (2016) observed, can be viewed in different ways. According to Gee (1999), discourses represent "different ways in which we humans integrate language with non-language 'stuff' " (p. 13). In other words, Gee's

(1999) description encompasses how our feelings, actions, and use of symbols are used to meaningfully connect with the world around us (p. 13). Additionally, discourse can also be viewed according to a poststructural perspective in which discourse represents "wider societal meta-narratives" (p. 25). Identity can also be defined as psychoanalytic, focusing on the subconscious features of the self, or performative in reference to the "stylised repetition of acts over time" (Darragh, 2016, p. 26).

Another way to classify identity is by whether it is something that is inherent in one's nature (an *acquisition* as Erikson described) or more dynamic (an *action* as Mead described) (Darragh, 2016, pp. 26-27). To George Herbert Mead, identity is "multiple, contradictory and socially constituted," whereas Erikson's notion of identity was "something that one has and that becomes coherent and consistent" (Darragh, 2016, p. 27). Darragh (2016) noted that Mead's interpretation has been prominent among mathematics education researchers:

> Wenger (1998) sees identity as "not an object, but a constant becoming" (pp. 153–4). Holland et al. (1998) define identity as "self-understandings" but go on to describe "identity-making processes" (p. 3), which treat identity as a verb. Gee (2000) claims that identity is making a bid to be recognised as a certain type of person. These all describe identity as an action. (p. 27)

In sum, there are a variety of ways to classify identity research in mathematics education. One can categorize it by examining how it fits into an overarching theory (i.e., poststructural, positioning, narrative, or psychoanalytic), by how it is defined (i.e., participative, narrative, discursive, psychoanalytic, or performative), by whether it is considered an acquisition or an action, or in other ways. Before discussing my theoretical framing for conceptualizing identity (according to positioning theory, the participative definition, and viewing it as an action), I present common critiques surrounding how identity is defined or operationalized.

Critiques of identity research in mathematics education. In the following (non-comprehensive) discussion, I offer insights on some critiques of various conceptions of identity expressed in the mathematics education research literature that Darragh (2016) and Graven and Heyd-Metzuyanim (2019) highlight. The commentary presented by these authors is relevant and meaningful, since both articles are recently published, and both offer a detailed overview of identity research.

The first issue concerns how identity is defined. Sfard and Prusak (2005) criticized identity research as lacking operationalized definitions, arguing that identity in the literature "is rarely preceded by any explanations" (p. 15). Darragh (2016) observed that some researchers conceived identity similar to affect and cautioned against such an approach:

> Mathematics education has a strong research tradition in the affective domain. Research which looks at beliefs, goals, motivation, attitude and mathematics anxiety, for example, is undoubtedly an important area. But is it about identity? Beliefs, goals and motivation are all things that people have and as such may fit with a psychological view of identity. Alternatively, beliefs, knowledge and attitude can be seen to influence people's identity enactments—without being considered identity in itself (see also Goos, 2013). The area of affect makes a valuable contribution to mathematics education, but the domain does not need to be re-branded as identity. Doing so muddies waters already filled with a variety of definitions. (p. 28)

A more pressing concern, Darragh (2016) noted, arises when researchers describe identity in one way (e.g. from a sociological perspective) and then operationalize it differently (e.g. according to a psychological approach), leading them to "talk about identity in theoretically inconsistent ways" (p. 28). Moreover, even though researchers frequently constructed a definition of identity from previous literature, Darragh (2016) noticed that they did not always justify their choice of a particular framework: "Authors who followed the definitions of Wenger (1998), Sfard and Prusak (2005) or Martin (2000) in general made very clear the theoretical frame they were drawing from but not always their choice of that particular theoretical frame" (p. 24). While acknowledging the unified consistency with how identity was conceptualized by some researchers, "this level of theoretical coherence was not found across the range of articles" (Darragh, 2016, p. 24).

In their article characterizing the current state and future of identity research in mathematics education, Graven and Heyd-Metzuyanim (2019) referenced these issues identified by Darragh (2016) but offered an alternative perspective. They analyzed 20 prominent mathematics education research journals from 2014 through 2018, ultimately selecting 47 papers to review. Graven and Heyd-Metzuyanim concluded that while researchers often *defined* identity, they did not always articulate the mechanisms by which it exerts an effect on behavior: "The weakness, instead, often lies in the operationalization of identity, namely, in stating precisely what identity is (for the researchers) and how it can be empirically studied" (2019, p. 368). Echoing similar thoughts, Gee (2000) stated that he did not "think it is important what terms we use" but focused his attention on identity "as an analytic tool for studying important issues of theory and practice in education" (p. 100).

Research on Teacher Identity in Mathematics Education

Graven and Heyd-Metzuyanim (2019) presented four themes that arose from their analysis of 47 articles focusing on identity research with the third theme related specifically to teachers' identities. I previously discussed different ways that identity research can be classified—according to the underlying theory, the definition, or whether it is viewed as an action or an acquisition. Another way to categorize this research is by its focus on the learner's identity or the teacher's identity. While Graven and Heyd-Metzuyanim (2019) examined studies related to identity research in mathematics education, Lutovac and Kaasila (2018) reviewed studies which specifically focused on *teacher identity* from mathematics education research. In sum, they analyzed 40 articles in peer-reviewed scholarly journals which were published from 2000 to 2015.

From their analysis, Lutovac and Kaasila (2018) identified six themes related to theoretical models, contextual influences, opportunities for identity growth, affect and teacher identity, issues of power and equity, and the connection between instructional practices and identity. The aim of my study is to examine mathematics instructors' interview responses three years after participating in a professional development workshop to better understand their distal goals as a mathematics instructor (e.g., what do they expect to gain from their participation?) and to infer what these goals reveal about participants' identities as mathematics instructors. My research focus shares similarities to the theme of *theoretical models and what constitutes identity* (Lutovac & Kaasila, 2018) described above, specifically relating to the development of the professional identity of teachers.

For example, van Putten et al. (2015) examined the professional mathematics teacher identity (PMTI) of mathematics education students using case study methodology. The PMTI is described as "involving an individual who has studied the subject for the specific purpose of teaching it" (van Putten et al., 2014, p. 371). Among many possible interpretations, van Putten et al. (2014) defined professional teacher identity (PTI) more generally as the "crossroads between the personal and the social self, the 'who I am at this moment'" (p. 370). Adapting work from Beijaard et al. (2000), van Putten et al. (2015) assessed the extent to which prospective mathematics teachers' perception of their identity as teachers was actualized in their instructional practices. Six teachers completed a questionnaire (other data was collected in this study) to elicit biographical data and to investigate how they characterized their subject matter expertise, didactical expertise, or pedagogical expertise (as an adaption of the question from Beijaard et al., 2000).

Independent of how PTI is conceptualized, different variables that might influence how a teacher perceives their identity include teaching context, teaching experience, and the biography of the teacher (Beijaard et al., 2000, pp. 752-754). Others have examined contextual factors associated with preservice teachers' professional identities which include "learning experiences before they enter into the teacher preparation program, school placement and the teacher education program itself" (Akkoc & Yesildere-Imre, 2017, p. 56).

Ultimately, these factors may influence teachers' perception of their PTI regarding the features they associate with competent teaching. Relatedly, Stols et al. (2015) investigated mathematics educators' perceptions of effective mathematics instruction by showing 46 teachers eight short vignettes of instructors teaching fractions and then asking them (among other things) to choose the lesson that might have been most effective for the pupils and the one that might have been the least effective for them. From their analysis, two major categories emerged: professional competence and affective characteristics. The latter category was less prominent with themes of *discourse* and *teacher attributes* (i.e., confidence, communication, teacher preparation, classroom environment), totaling slightly more than 20% of the teachers' remarks.

Under the first category, however, there were two themes that encompassed more than 50% of the comments. These themes, *use of material* (i.e., skittles, paper circles, pictures of pizza) and *modes of instruction* (i.e., teachers' capability to explain, their inclusion of real-life applications, their support for students with hands-on activities) reflect the professional identities of these teachers by exemplifying the features they associate with effective instruction. In other words, these themes reveal their distal goals and priorities for what they value as instructors. While Stols et al. (2015) concluded that "general pedagogical skills are perceived to be most important when it comes to effective teaching and learning of mathematics" (p. 232), prescribing specific strategies to be incorporated into practice is not necessarily straightforward:

> The results suggest a diversity of opinion on which vignette represented the most effective mode of teaching and learning. A policy document might attempt to define 'effectiveness', but what is actually implemented will be modified by what is feasible and consistent with cultural norms. . . . Given the diversity of schools, there is no one effective type lesson. (Stols et al., 2015, p. 233)

While Stols et al. (2015) focused on mathematics teachers' identities with respect to their instructional practices, others have investigated mathematics teachers' or educators' evolving identities in relation to social influences.

For example, Arslan et al. (2021) conducted two case studies of middle school mathematics teachers' identities who had participated in a teacher education program designed to "develop teachers with reform-oriented mathematics teacher identities" (p. 6). Participating teachers were provided support to design curricular resources supporting students' conceptual learning. Arslan et al. (2021) analyzed two participants' identities as mathematics teachers and the working community's influence. During the first two interviews, Arslan et al. (2021) asked participants to identify and describe influential courses in the program that influenced their role as a mathematics teacher and to discuss expectations from school administrators based on their mathematics teaching. These questions reveal features of teachers' evolving identities and goal structures from their participation in this program.

As another example, Goos and Bennison (2018) used zone theory analysis to study identity formation for "mathematics teacher educators who cross disciplinary boundaries" (pp. 409-412). They conducted interviews with a mathematician and a mathematics educator (Leonard and Joanne, respectively) participating in the Inspiring Mathematics and Science in Teacher Education (IMSITE) project. The IMSITE project focused on cultivating continued collaboration among mathematicians, scientists and mathematics and science educators supporting future teachers and "identifying and institutionalising new ways of integrating the content expertise of mathematicians and scientists with the pedagogical expertise of mathematics and science educators" (p. 412). Among other questions, Goos and Bennison (2018) asked Leonard and Joanne to describe their previous collaboration experiences and their perception of collaboration (including influencing factors) between mathematicians and mathematics educators at their university. They analyzed their findings according to different developmental zones and temporal dimensions of past, present, and future that captured their evolving trajectories.

In this study, I contribute to the literature on professional teacher identity by analyzing a survey administered to a unique population: collegiate mathematics faculty throughout Oklahoma participating in a large-scale professional development initiative. In the discussion of these interviews, I offer insights about the professional identities of these teachers by revealing features of their distal goals as instructors. Specifically, respondents identify takeaways that they expect to gain from their involvement and their how they envision future collaborations among mathematics faculty in Oklahoma.

In the first chapter, I discussed the nature of the research being conducted by other STEM professional development initiatives in comparison to the MIP. I discussed my focus on conducting design-based research to better understand theoretical mechanisms and to provide implications that may be leveraged to modify or refine future design aspects of the MIP. The focus of this chapter is to analyze participants' responses to interview questions to reveal features their distal goals as mathematics instructors, and hence, possible identity trajectories.

Theoretical Framing

While most qualitative research writings will contain a theoretical component, the purposes surrounding the use of theory is not as consistent. Collins and Stockton (2018) examined the primary role of theory in qualitative research and summarized its various purposes: theory can be used to (1) provide clarity regarding epistemological approaches, (2) serve as a guiding framework for the research, (3) offer logical motivation behind methodological choices, or (4) help build theory from research findings (Collins & Stockton, 2018, p. 1). In this section, I discuss theory relative to the first two of these items listed. First, I describe how I conceptualize identity according to a particular epistemological approach. Following this presentation, I discuss how my formation of an initial model of the participants' responses offers insights regarding their underlying goal structures.

Conceptualization of the Self

In this study, I define identity in alignment with Blumer's (1986) conception. Blumer, in his articulation of *symbolic interactionism*, described the human as an acting organism, consistent with Mead's notion of self, stating that a human being "is an object to himself" (1986, p. 12). Through the process of role-taking articulated by Mead, humans interact with themselves from (their image of) the perspectives of others. Blumer (1986) elaborated on this self-interaction:

> This process is in play continuously during one's waking life, as one notes and considers one or another matter, or observes this or that happening. Indeed, for the human being to be conscious or aware of anything is equivalent to his indicating the thing to himself—he is identifying it as a given kind of object and considering its relevance or importance to his line of action. (Blumer, 1986, p. 13)

Blumer (1986) continued, stating that humans are constantly self-assessing and use these indications to guide their action; he described the nature of these actions:

> Fundamentally, action on the part of a human being consists of taking account of various things that he notes and forging a line of conduct on the basis of how he interprets them. The things taken into account cover such matters as his wishes and wants, his objectives,

> the available means for their achievement, the actions and anticipated actions of others, his image of himself, and the likely result of a given line of action. (ibid, p. 15)

According to Blumer's description, human beings act in accordance with their goal structures and belief systems since humans' actions are influenced by their "wishes and wants" and their "objectives." (ibid., p. 15). In this way, Blumer's interpretation of a human's actions is consistent with that of von Glasersfeld (1995), who argued that "action schemes are explicitly goal-directed" (von Glasersfeld, 1995, p. 73).

From Blumer's (1986) perspective, an individual's actions are guided by goal oriented and affective components (e.g., their "wishes and wants" and "objectives" (p. 15)). The relationship between goal structures and identity is clarified in Middleton et al. (2015):

> The goals we set for ourselves define a state of being we desire to enter—an end state representing the culmination of a complex chain of behaviors over time. In essence, they are a projection of a possible future we might shoot for. The end state of a goal can be as simple as obtaining the intercept of a line in a homework problem. It could also entail an affective state like a sense of achievement, or a state as long-term and complex as adoption of a new identity. (p. 7)

Middleton et. al. (2015) described an individual's end goals in relation to their desire of who they want to become. Characterized as *distal goals*, they "often involve desired states centered on an ideal identity the individual wants to attain" (ibid., p. 8). In this way, I characterize respondents' distal goals and beliefs in accordance with their future, desired identity as a mathematics instructor. Using this characterization, I discuss interview responses from mathematics faculty in Oklahoma to illuminate their distal goals by generating inferences from their remarks. Their responses to these prompts reveal features of their distal goals as mathematics instructors evolving from their current commitments and desired identities.

At this point, I address one of the critiques surrounding the use of identity discussed previously. By clarifying how identity is being operationalized, I intend to reduce possible confusion or incoherence. In the literature review, I discussed common critiques of identity research, noting Darragh's (2016) concern when researchers describe identity in one way (e.g., from a sociological perspective) and then operationalize it differently (e.g., according to a psychological approach):

> However, a bigger problem exists when researchers fail to consider the difference between action and acquisition within the same article. These writers discuss identity as if it were an acquisition despite having defined identity using a theoretical frame that views identity as an action. In doing so, they talk about identity in theoretically inconsistent ways. The problem is that many writers seem to draw from the broad theories aligning with the Meadian, sociological, approach to identity, and yet discuss their data and participants as if identity resided within the individual's core and then attempt to measure it. . . . Taking a psychological perspective on research about individuals' relationships with mathematics is not necessarily problematic: the problem lies in the act of defining identity in a contradictory manner to the methods used and conclusions drawn. This happens, for example, when identity is defined using a sociological frame and then a psychological understanding is used to analyse the individual. (p. 28)

Darragh's (2016) critique is warranted since defining identity in multiple ways could result in an incoherent framework. In this paper, I present possible identity trajectories inferred from my interviews with eight participants by discussing features of participants' distal goals as mathematics instructors (e.g., what are their takeaways from participating in an Initiation Workshop?) which influence and are influenced by their identities as mathematics instructors. In particular, their comments reveal features of their distal goals as mathematics instructors at a moment in time, influenced by previous experiences and evolving from their future interactions towards the attainment of their desired identity. In defining identity in this way, I conceptualize it as an *action*.

Given this theoretical framing, my research questions are as follows:

Research Question 1: What do participants' interview responses reveal about their professional identities with respect to their mathematics instruction?

Research Question 2: How can these inferences be leveraged to modify future design aspects of the MIP, in consonance with social learning theory?

The first research question should be contextualized according to my theoretical framing of Blumer (1986) and Middleton et al. (2015) for how I conceptualize identity and characterize it in relation to one's distal goals. Research Question 2 is related to design-based research since I offer theoretical implications for mechanisms articulated in Wenger's (1998) *communities of practice* framework.

The remainder of this chapter is divided into two parts. In Part I, I briefly highlight some results from my analysis of participants' survey responses to the first of six prompts prior attending an Initiation Workshop in summer 2019. In Part II, I present my results from eight interviews conducted spring 2022. I prompted eight faculty who had participated or were currently participating on a CORD to discuss their contribution role in MIP activities, their takeaways from these experiences, and their vision for future collaborations. These questions were similar to the questions in Prompt 1 of the survey. Asking these questions three years later provides insight into participants' identity trajectories by revealing features of their distal goals as mathematics instructors.

PART I: The Survey

Methodology

Faculty interested in attending summer Initiation Workshop were required to complete a Workshop Application Form (WAF) consisting of three basic sections—a general information section (e.g., contact information), a research section (the focus of this chapter), and a workshop section. Participants responding to questions from the research section answered prompts related to their prior teaching experience and expertise, their reason for attending the workshop, their advice for the MIP team in supporting collaboration, etc. I present the results from my analysis of the first of six prompts in the research section Prompt 1 is important since I present results in Part II to interviewees responses to similar questions:

> **Prompt 1:** The MIP is designed to support faculty collaboration across institutions. Please describe how you envision being part of such collaborations, what you hope to contribute, and what you expect to gain from them.

Analysis

I coded the five prompts differently according to two divisions. First, I analyzed a combined set of responses from participants across *all* three workshops for some of the prompts, and I examined other prompts according to their association with a *particular* workshop. Second, some prompts were designed to elicit participants' responses to several sources of information delineated within the prompt, whereas other prompts included only a single question. These two distinctions require further elaboration.

I combined the responses from participants across all three workshops in analyzing Prompt 1. To be meticulous and avoid overcounting multiple responses from the same person, I chose to combine responses from faculty members who filled out the WAF multiple times. For instance, if someone attended two Initiation Workshops and completed the WAF twice, the results would be fourteen responses to these seven prompts. Responses from the two forms were then combined if the participant's response were not identical in both cases. Additionally, this procedure of combining responses was also applied if faculty inadvertently filled out the WAF twice at the same workshop.

Additionally, I coded Prompt 1 according to multiple components:

> **Prompt 1**: The MIP is designed to support faculty collaboration across institutions. Please describe how you envision being part of such collaborations, what you hope to contribute, and what you expect to gain from them.

This prompt was intended to elicit responses related to the envisioning, contributing, and expecting to gain categories. Accordingly, I analyzed participants' remarks based primarily on these three blocks.

I first analyzed these categories according to an exploratory approach in which I generated categories from codes and then refined these categories to provide further clarity. Some respondents' answers were not included in the analysis for various reasons (e.g., less relevant, not addressing the prompt specifically). Among the more meaningful responses, some encompassed only one code while others had multiple codes. After the responses were coded, I began to analyze them and form categories. After all (or most) of the categories had been generated based on a color-coded scheme, I organized the categories according to their respective color. This process enabled me to better visualize the data and either refine these categories further or examine the codes that did not initially align with a group. I present my discussion of the study's results based on this iterative refinement of themes. Depending on the modifications that were made, these themes were not necessarily different from the categories identified in the previous phase of analysis.

Results

In the following sections, I briefly discuss themes from Prompt 1. I focus my discussion and present the results from the more prominent themes that emerged from my analysis (see 4).

Table 4

Prominent Themes from Prompt 1 Per Division

Hope to Contribute	Expect to Gain
Teaching experience/strategies	Curriculum Development/Alignment
Curriculum/Course	General Awareness/Ideas
General Knowledge	Teaching Help/Strategies

Since my coding of the first component of this prompt (how participants envision being a part of collaborations facilitated by the MIP) required more subjective inference and the analysis was not particularly insightful, I do not discuss it.

In total, there were 49 individuals who completed the WAF and attended these workshops, and 36 of the 49 responses were coded according to this component of the prompt, *hope to contribute*. Responses suggest that participants desired to share their ideas and knowledge related to three primary themes: *Curriculum/Course*, *General Knowledge*, and

Teaching Experience/Strategies. Related to the first theme, respondents indicated experience related to the design of activities (e.g., "I would love to share the activities and assignments I have already created") or courses (e.g., "I hope to contribute ideas about the structure of the course and resources for creating a successful Functions & Modeling course," "I have been working collaboratively on many academic workshops including curriculum designing and designing a pacing guide"). The second theme encompassed vague responses related to general knowledge (e.g., "I hope to share the experiences I have had—both good and bad"). There were also some remarks referencing previous experience attending workshops or participating in prior state-wide initiatives (e.g., the Pathways task force, a previous MIP workshop). The theme *Teaching Experience/Strategies* indicates that faculty intended to offer their knowledge from their experience teaching (e.g., "I have been teaching math for 20+ years, ranging from elementary level to collegiate math courses. I will be glad to exchange ideas with other math instructors during the workshop") and strategies they have accumulated throughout their professional experience (e.g., I hope to contribute "instructional strategies I have used to promote student interaction during class lessons"). In addition to these more explicit remarks, there were other comments from faculty describing their experience teaching or coordinating a course that were not coded according to this division, because their responses did not offer clear connections between their experience and their expectations to contribute.

With respect to the final component of this prompt, *expect to gain*, I coded 38 out of 49 responses. Three primary themes emerged: *Course Development/Alignment*, *General Awareness/Ideas,* and *Teaching Help/Strategies.* Related to the first theme, some responses were tailored toward gaining knowledge related to designing activities or activities more generally (e.g., “I hope to gain additional strategies and activities”), math pathways or corequisite courses (e.g., “I expect to learn new ideas and gain insight into what works best in a functions and modeling class”), or pertaining to having more unified course content (e.g., “I hope to see us (Faculty) come together and agree on a uniform standard”). Remarks related to the *General Awareness/Ideas* theme were particularly vague (e.g., “I feel like I am in a good position to take ‘home’ new knowledge and support for my instructors”). Finally, some indicated a desire to improve their teaching (e.g., “I hope that we can all gain practical skills that we can use in the classroom and focus on what is best for students”), gain additional strategies, or enhance student engagement (e.g., “I would like to gain insights on the best approach to teaching this course and ways to make this course ‘fun’ ”).

Participants’ responses indicate their desire to share their knowledge and experience related to curriculum design or development, share their experiences more generally, and share their strategies they have implemented. These comments reveal how they anticipated to contribute based on their expectations and their priorities as mathematics instructors. For example, participants’ desire to gain additional strategies may suggest that their interpretation of effective mathematics instruction consists of accumulating different techniques to implement in the classroom.

While these results from the survey data highlighted general themes and insights regarding participants’ identities prior to their involvement in MIP activities, participants’ responses were often brief and ambiguous. Additionally, they do not reveal features of an *individual’s* identity trajectory. Hence, three years later, I conducted eight interviews with participants who had participated on CoRDs and asked them similar questions to those in Prompt

1. From my analysis of their responses, I identify possible identity trajectories by eliciting features of their distal goals as mathematics instructors.

PART II: The Interviews

Methodology

I emailed all 22 faculty who either had been on or were currently on a CoRD to determine their willingness to participate in a short interview. Among these 22, eight were willing to participate. I conducted eight, virtual, semi-structured interviews, each lasting for around 25 minutes. These eight participants were from five different MIP CoRDs. I asked participants several questions during these interviews, and I focus my results on participants' responses to two questions (see Table 5).

Table 5

Two Interview Questions and Their Usefulness

Questions	Usefulness
1. Can you talk about your participation and involvement in the MIP? a. How did you see your role in contributing to these efforts? b. Why did you participate? c. What did you get out of it?	These questions provide insight into participants' distal goals relative to their participation in MIP activities.
2. Is it important to have opportunities for math faculty to collaborate across institutions in Oklahoma? a. If so, what should these collaborations look like? b. If not, why not?	These questions provide an opportunity for participants' to identify their distal goals relative to how then envision opportunities for collaboration.

Analysis

I recorded these interviews and downloaded their timestamp transcription provided through Zoom.[7] After rewatching and editing these transcriptions, I organized their responses in an Excel spreadsheet by adding labels that indicated the question being asked. For instance, the

[7] I conducted two follow-up interview with Jack and Finn. After editing the transcriptions, I created broad codes to capture features of their responses to a few clarification questions.

first label was *Previous Participation*, the second was *Role in Contributing*, etc. Within these labels I began by performing open coding on their responses, developing codes that encompassed broad excerpts of participants' responses. Once all eight interviews had been conducted, I combined their codes together for each label. That is, the label of *Role in Contributing* would contain all codes from each participant related to this question. I then began by refining these codes into categories. During this process, sometimes one code might be sorted into multiple categories. Once the categories were formed for each label, I organized them according to different properties based on the codes.

Leveraging the codes that I created from 1(c) and 2(a),[8] I created a table (see Table 6) in which I presented three features for each of the eight interviewees: features of their takeaways, envisioned collaboration, and possible identity trajectory. Participants' takeaways may inform their future identity trajectory by revealing some of their values and commitments as instructors from their participation in Initiation Workshops three years prior, and their expectations of future collaboration also indicates features of their priorities as mathematics instructors. I initially developed five broad classifications of possible identity trajectories from my analysis. After presenting rough drafts for each of the eight interviewees based on their responses to 1(a), 1(c), and 2(a), I began to organize components of each rough draft according to a particular classification and then refined these classifications further. Each of these classifications represents a particular domain that indicates features of a participant's possible identity trajectory by revealing meaningful comments related to at least one interviewee.

[8] All interviewees responded affirmatively to question (2).

Wenger (1998) discussed an individual's identity trajectory as being "not a path that can be foreseen or charted but a continuous motion—one that has a momentum of its own in addition to a field of influences." (p. 154). Given these remarks, how can one discuss an individual's identity trajectory with confidence given that their future course of action represents an uncharted evolution negotiated from their participation and interaction with reified artifacts? Moreover, is an individual's trajectory influenced by their past experiences or future ways of engaging in pursuit of the community's joint enterprise? I discuss my response to these questions briefly in the following section.

When discussing an individual's possible identity trajectory, it represents my model characterizing their *possible* future course of action. Participants' identities are influenced by their experiences in the community of practice interacting with other participants and reified artifacts, and from these experiences, they have opportunities to negotiate their interpretation of the normative standard of competence being established in the community. I conceptualize an individual's interpretation of these perceived norms as a property of the individual's constructed interpretation. As Wenger (1998) discussed, an identity trajectory "has a coherence through time that connects the past, present, and future." (p. 154). Hence, an individual's identity trajectory is influenced by both past experiences and future expectations and is conceptualized as a *possible* trajectory to the extent that it represents my model of future actions they might take based on their stated remarks.

Participants' comments related to their takeaways and vision for future collaborations from their MIP experiences reveal features of participants' priorities, valuations, commitments, and expectations. These comments are not prescriptive indications of participants' future participation. For example, Pippa recognized a potential need to incorporate more projects into her teaching. Her comments suggest that a commitment to facilitate a collaborative learning environment represents a *possible* course of future action. On the other hand, Pippa's awareness of ways to improve her teaching is not prescriptive of her future development of instructional resources. While my characterizations of participants' identity trajectories are not intended to be prescriptive since these trajectories are fundamentally unknowable, interviewees' remarks give indications of their priorities and goal structures as mathematics instructors from their perceived takeaways and envisioned opportunities for future collaboration. Their comments do not *prescribe* their future course of action but clarify and suggest a *possible* identity trajectory based on their values and commitments inferred from their verbal remarks.

While an MIP participant's identity trajectory as a mathematic instructor is dynamic and evolving and guided by their future goals and expectations as they negotiate meaning from their interactions with other colleagues, members of the MIP Team, and from their interpretation of reified artifacts, their identity trajectory is also influenced by their previous experiences. Prior to their involvement in the MIP CoP, participants have negotiated meanings for how to consider designing instructional resources and develop curricular artifacts according to their standard of instructional design competence. Their interpretations of these interactions with others and reified artifacts may have evolved from their participation in other communities of practice (e.g., in their mathematics department), based on their own priorities and identities as students, and from their perception of their best (and worst) mathematics instructors. Hence, these past experiences helped forge their current priorities and influence their evolving identities.

Table 6

Takeaways, Envisioned Collaborations, and Possible Identity Trajectories

Name	Takeaways, Envisioned Collaborations, and Possible Identity Trajectories		
	Takeaways	**Envisioned Collaborations**	**Possible Identity Trajectory**
Katie	• Frustrated experience with involvement in professional development activities (e.g., communication, research impacting participants' work) • Frustrated with the MIP definition of active learning • Recognizes positive mission of the MIP	• Leaders less focused on research • Leaders having better communication • Inter-institutional collaboration to share experiences and learn from others	Engage in inter-institutional collaboration to share experiences and learn from colleagues while remaining skeptical of these experiences from her involvement in MIP activities
Sarah	• Identified the importance of collaboration opportunities with colleagues to identify productive ways to teach mathematics	• Developing curriculum on specific topics and ideas • Working on a project to help students	Productively work with faculty to engage in practical discussions around improving the teaching of mathematics and perhaps work on projects to support student success
Robert	• Learned about academic success skills (e.g., inequality, productive struggle) • Provided meaningful contributions to his CoRD	• Productive accountability • Collaborate on an interesting project	Possibly reignited to supporting students' engagement in productive academic success skills (e.g., grit) and perhaps engage in opportunities to collaborate on interesting projects
Jack	• Supported to engage in opportunities to do research and become a better researcher • Learned new ideas to implement in class	• In-person workshops with digital check ins periodically • Sharing experiences and struggles	Continue improving as a researcher and gaining new ideas to help improve his teaching while also sharing experiences and struggles with other colleagues
Ellison	• More sensitive and supportive of students' affective responses	• Learning from a master teacher implementing active learning	Act in ways that are attentive and supporting to students' affective responses and perhaps seek opportunities to learn from experts implementing active learning strategies

Adam	**Takeaways** • Frustrated with experience communicating and administrative experience while participating on a CoRD • Productive collaboration with colleagues on a CoRD • Recognition of differences in how active learning is defined • Unity among colleagues despite differences of opinion	**Envisioned Collaborations** • Longitudinal collaborations to revisit old ideas • In person meetings • MIP focus was productive (big topics to little topics)
	Possible Identity Trajectory: Be skeptical of having productive communication from his involvement on a CoRD, recognize instructors' commitments and values, and perhaps seek opportunities to revisit MIP discussions	
Pippa	**Takeaways** • Need to incorporate projects into her instruction • Aware of poor communication skills • Enhanced confidence of her knowledge of mathematics • Acceptance that others can polish her unfinished products	**Envisioned Collaborations** • Good focus on projects but also discussing experiences • Collaboration with just mathematics faculty (specifically math, not other courses)
	Possible Identity Trajectory: Incorporate more projects into her instruction, participate in future collaboration opportunities to share experiences, and engage with increased confidence of her knowledge of mathematics	
Finn	**Takeaways** • Need to set specific goals for CoRDs • Value in pre-readings and open-ended exercises • Importance of modeling approach discussed at Initiation Workshop	**Envisioned Collaborations** • In person meetings (the purpose could be to receive insight from a colleague to mentor students) • Collaborate on creating supplemental materials • Sharing experiences and struggles
	Possible Identity Trajectory: Consider opportunities to emphasize the modeling approach in mathematics, include open-ended activities in his courses, and perhaps engage in collaboration opportunities to develop supplemental materials while sharing experiences and struggles	

Importantly, while participants' takeaways and envisioned collaborations reveal aspects of their distal goals as instructors, their values expressed do not guarantee that they will be motivated to pursue these commitments in future experiences.

Results

I present some of these interviewee's responses to Prompt 1 (pre-workshop) and some of their comments from their post-workshop remarks.[9] I supplement this data with their responses to questions 1(a), 1(c), and 2(a).[10] Participants' survey responses reflect their expectations prior to attending an Initiation Workshop, whereas their responses from the interview reflect their contributions and takeaways from their involvement in MIP activities (i.e., including their participation on a CoRD). I present this data according four broad topics that are not intended to describe their identity trajectories but reveal features of their possible identity trajectories based on their comments related to that specific topic: *Academic Success Skills and Active Learning*, *Frustrated Experience*, *Content Development*, and *Collaboration*. More generally, their remarks under each of these topics reveal features of their priorities and commitments as mathematics instructors. Participants' characterization according to a particular classification does not suggest that they were the only ones who share features related to that classification or that they did not have experiences related to other ones, since participants' identity trajectories are complex and nuanced.

Academic Success Skills and Active Learning

Academic success skills (Ellison, Robert, and Finn). Ellison attended the Academic Success Skills Workshop and her response to Prompt 1 was brief and general: "I'm interesting [sic] in working with other professionals to improve the quality of mathematics education in Oklahoma and at my institution in particular." Ellison stated that she envisioned her role as a "participant" to finish her work on a CoRD. After prompting her to elaborate on her role in

[9] For the Academic Success Skills Workshop, there were no post-workshop remarks. Additionally, some of the participants were not recorded providing post-workshop comments at the other two workshops.
[10] Notice that participants' response to question 1(b) would not be expected to change after their involvement in MIP activities. Additionally, notice that 1(a) and 1(c) are similar questions to those in Prompt 1.

participating on the CoRD, she described her role as a collaborator with her other CoRD member.

When I prompted her to discuss her takeaways from her involvement at workshops, on a CoRD, and possibly giving a presentation at a workshop,[11] Ellison stated that she had given a presentation on academic success skills (e.g., discussing stereotype threat) and discussed the impact of having this opportunity. She stated that

> since I've gave— given that talk on the emotional component of learning, I feel like I've been more in tune with my students, um, affective responses to things and try and be more supportive of their, um, trying to uh, be, uh, uh, um attuned to what their emotional state is and helping them to get through things. Trying to give them little pep talks here and there, and, you know, set up their expectation, so they know that, you know, struggle is part of learning, and they shouldn't be surprised. Trying to be proactive and, um, addressing some of the common emotional roadblocks students have when they, when they don't particularly like math. . . . I feel like my participating in that talk in the first workshop was helpful in— to just bring to the forefront, uh, the importance of helping students when they're feeling depress— uh, down or discouraged in a mathematical setting about the, the work they're doing. Trying to bring support, um, and encouragement to them in a, in a more deliberate way.

These comments highlight Ellison's identity trajectory reflected through her distal goals to be more attentive to features of students' affective engagement by being "in tune" with her students and their emotions and proactively supporting them. Presenting on this affective domain, Ellison

[11] I intentionally reminded her of possible leadership opportunities.

discussed, fostered her heightened sense of awareness to these struggles. The focus of her dissertation related to stereotype threat for women in mathematics indicates that she was positioned to be influenced by participation in the Academic Success Skills Workshop.

Robert attended both the Academic Success Skills and College Algebra and Precalculus Workshop and his response to Prompt 1 for both workshops was brief. Regarding the former, he wrote, “[His institution] is looking to add a Math Modeling course. This workshop would benefit us in the startup.” For the College Algebra and Precalculus workshop, he responded to Prompt 1 with “I enjoy sharing ideas and improving my teaching craft.” In his post workshop remarks at the conclusion of the College Algebra and Precalculus Workshop, he stated that this workshop “has helped with, maybe, uh, with some new ideas to maybe reignite some of that engagement.”

Robert’s expectations and takeaways during our interview revealed more insight from his experiences that may have led to this reignition. While he anticipated that the Academic Success Skills Workshop would be “really boring” and centered around note taking skills, “it turned out, I mean there’s a lot of things about, uh, uh, gender inequality, or just kind of e— even, even racial inequality with, within the math classroom, uh, that that highlighted that.” He then discussed his appreciation for supporting students’ productive struggle (e.g., giving students’ problems that are too hard to solve) and learning to use the word “grit” instead of courage to support students’ determination to keep trying even their idea is unsuccessful. He stated that the “academic success skills really shocked me the most and really helped me the— more than any of the other ones.” Robert’s comments suggest further affirmation of his identity trajectory to value grit and determination reflected in his practices as a student.[12] During my case study with

[12] I discuss this more during my presentation of my case study with Robert in the following chapter.

Robert, he indicated that he was introduced to *productive struggle* problems (i.e., problems that are too difficult for students to solve) at the Academic Success Skills Workshop. Hence, Robert's participation in the Academic Success Skills Workshop contributed to his commitment to fostering students' tenacity and determination.

Finn attended both the Academic Success Skills and College Algebra and Precalculus Workshops. Consider his response to Prompt 1, prior to attending the Academic Success Skills Workshop:

> I envision that I will be part of a group that comes up with various templates to augment academic success, and I will be responsible for trying to implement them in a few of my classes. I also realize that I will need to share collected data with members of MIP. I expect to gain new knowledge and skills especially in academic success that I can transfer to any of my co-requisite courses be it Stats, College Algebra, or Survey of Mathematics.

In addition to describing his role as a research participant, Finn highlighted his expectations to create templates to support students' success and gain knowledge that could be useful for his co-requisite courses. In his post-workshop remarks after the College Algebra and Precalculus Workshop, he discussed potential affordances resulting from his participation in MIP activities. Academic success skills gave him ideas to "reduce fear which increases their prioritization maybe a little bit more. And then, uh, having active engagement with your students, now, gives me sort of an idea of how I can increase that sort of encouragement." Finn's comments indicate that he is aware of students' affective struggles in the context of mathematics. His remarks also suggest that his participation in MIP activities has enhanced his knowledge for how to attend to students' struggles.

Active learning (Ellison, Adam, and Jack). In discussing her vision for what collaboration for mathematics faculty should look like, Ellison originally echoed the MIP plan, indicating what these collaboration experiences had looked like: coming together, guided by a facilitator, with specific goal to share ideas. She has appreciated these experiences, leaving them more motivated and with another tool to possibly implement in the classroom. Recognizing her response echoed her participation, I invited her to discuss other suggestions for the focus of these collaborations. She described the importance of having an opportunity to witness (or even participate in) a master instructor teaching a class with active learning, stating that we have great conversations, "but they never quite get to the 'How do you do this? What does this actually look like in the classroom?' There's sort of always a gap in that." She stated that she is familiar with lecture style classrooms and wants to see

> what a good interactive, uh, exploring type session looks like. And, I can only imagine, because I haven't really seen it in— I haven't really seen anyone modeling it for me. So, um, um, that's the pieces I feel is missing.

Later, I followed up by asking her to discuss the value she derives from seeing this enactment, and she stated that she has this "aspirational goal" to support students' engagement in active learning. Having the experience "from the inside" of participating as a student for this expert teacher facilitating this class, she indicated, would help her teach her students in similar ways. These comments reveal features of her distal goals to continue improving ways to support students' engagement in active learning and to seek opportunities to learn from an expert instructor who facilitates an interactive or exploratory classroom session.

Adam participated on a CORD but did not attend any of the first three Initiation Workshops, although he did attend the Calculus I workshop in the following summer. He described his role in contributing to MIP activities directly: his role in general was to "possibly connect with other faculty around the state," his role on a CoRD was to create resources as part of a collective, and his role at the workshops was to brainstorm topics for CoRDs to consider developing.[13] Adam had experience developing curriculum in the past (e.g., modifying his course notes to provide more active learning questions), and his comments about his role on the CoRD seem to implicitly reveal an opportunity to critical evaluate students' ways of reasoning about an idea, since he thought about "possible activities and, uh, different trajectories that things might go."[14]

In our interviews, Adam stated that active learning "does not have a clear obvious definition, uh, as I thought," and he indicated that it represented a different definition than he had seen in the literature. Effectively operationalizing the MIP definition of active learning first requires that participants critically evaluate the MIP definition to begin noticing that the MIP definition is constructed differently. Aware of these distinctions, participants may be positioned to wonder what values and commitments relate to the MIP definition. Adam's comments later in the interview reveal this curiosity.

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[13] He paused before discussing his role at the workshops, and said "I think," indicating a tentative response about his role at the workshop.

[14] Adam had experience developing curriculum in the past in modifying his course notes (e.g., provide more active learning questions).

In discussing what collaboration among mathematics faculty should look like across institutions in Oklahoma, he stated that the implementation could be different, highlighting the disadvantages of virtual collaboration.[15] In addition to indicating that the MIP model is good (bring people together and begin discussing big ideas leading and eventually smaller ideas), he highlighted his interest in having opportunities to revisit these old ideas, including the MIP definition of active learning:

> I think it would be interesting to revisit that same— those same sort of ideas later after like— especially with, with working with the MIP definition of active learning, because it is, from my perspective, so different, um, having time to ruminate about what it is and how it looks, especially after I've gone and— after I've gone through some teaching since then. Um, I think it would be interesting to revisit (*pause*), revisit the workshop. Come back together, do we have any new ideas? These are the old ideas. Can we improve on them? Are they still good? Things like that. Um, so, I guess long story short is, it would be cool, I think, to have, um, some sort of backbone for a longitudinal collaboration, rather than collaboration at a specific moment in time, because I think that's going to allow— I think that might allow for more robust, um, and developed ideas and, uh, more developed relationships between people.

His comments indicate a curiosity to continue ruminating on the MIP definition of active learning, revealing (1) his identity as a mathematics instructor to critically evaluate his work, and (2) the difficulty in understanding the MIP definition of active learning. The second revelation highlights the importance of supporting MIP participants to understand the MIP definitions. (In

[15] The first three workshops were in person, but the MIP Team, influenced by the Covid-19 pandemic, decided to have the latter two workshops, one of which was the only one that Adam attended, to be completely virtual.

Chapter 4, I present a way this might be productively achieved by asking participants to analyze contrasting reified artifacts.) These comments also affirm, albeit implicitly, the importance of the MIP design for supporting ongoing collaboration at different stages of the project.

Jack attended the Functions and Modeling Workshop. He responded to Prompt 1 as follows:

> Networking is vital for improvement in any field, particularly mathematics education. I would not be the teacher I am today without having shared ideas and strategies with colleagues regarding pedagogy and/or content. If I want to make my product better, I need to try different ingredients. This is true with teaching. If I want to find better ways to reach my students, I need to self-evaluate constantly and try new things to prevent stagnation.

I asked if there are any principles that motivate his selection of new things and what he values, and Jack discussed the driving force behind his decisions:

> What drives my decision is mostly engagement. Um, if I think that something's going to translate into my class and engage my students to the point to where they're doing this on their own, without me having to be there. That's a great activity for me to do. If— the less I have to prod them on something, the more likely I am going to use it in my class. So, it's a self gen— if an activity has a self-generating kind of purpose, then I'm more than likely going to use that. So that's going to inform my decision on whether or not to incorporate it into the curriculum.

Jack's comments reveal his commitments to cultivate an environment that supports student engagement with minimal instructor support. The features of an activity that inform Jack's decision for whether or not to implement it are its capacity to be "self-generating." Jack's discussion of his course goals provide more insight into why he possesses these commitments:

> Um, if it's something that I have to oversee constantly, I'm probably not going to use it, because I want them to become independent math thinkers. That's my goal in a course. I want them to be able to come up with the questions on their own without having to depend on me. I also want them to be confident in their responses, without having to ask me if it's right. And so, any kind of activity that's going to generate that kind of r— of engagement is going to be what I would choose.

Jack's remarks reveal his desire to support students to become "independent math thinkers" to solve problems and engage confidently. His comments suggest that he has distal goals centered around fostering students development of mathematical orientations that enable them to engage in productive mathematical practices. The MIP definition of academic success skills reflects Jack's goals as a mathematics instructor (i.e., fostering "students' construction of their identity as learners in ways that enable productive engagement in education and the associated academic community.) Achieving this goal motivates him to select new ideas that align with his vision.

Frustrated Experience (Katie and Adam)

Katie attended both the Academic Success Skills and College Algebra and Precalculus Workshop and her response to Prompt 1 for both workshops was the same:

> In many of the first-year math courses at [institution], emphasis is placed on active learning, collaboration, and growth mindset. On a meta level, I think we should be able to practice what we preach, so to speak. We tell students that they have the potential to learn more when they work with others, asking for different perspectives and

> explanations, and I think we can and should develop our materials with the same approach! I work with first-year college students every semester and most of the recent developments in curriculum I have put into place have been aimed at helping them transition to college life, expectations, and skills. I would love to share the activities and assignments I have already created and I would love to see what other people are doing, possibly with better result!

Her response highlights her experience in developing materials and her excitement to learn from colleagues and listen to different perspectives to support student success.

Since Katie did not have post-workshop remarks,[16] I now discuss her comments in our interview. In discussing her leadership role in contributing to MIP activities, Katie highlighted her initial excitement to collaborate with other institutions, but then highlighted negative takeaways from her experiences related to organization and guidance. As some examples, she indicated frustrations related to understanding the MIP definition of active learning, expectations to train other people not coming to fruition, difficulties in receiving payments, and lack of organization on the end goal. She felt that the MIP approach was presented in a way that it is not operating: she was aware of the outer research layer of MIP leaders watching MIP participants make products, "but it appears that there was actually no inner layer."

I asked her to follow up on these comments from her CoRD experience making products. She stated that the focus of the MIP at these workshops was framed as desiring to see what participants' produce, and unsure of the products that would look like, or priorities participants

[16] There were no post-workshop comments after the Academic Success Skills workshop; she did not provide comments after the College Algebra and Precalculus Workshop (she may have left the workshop before its final conclusion).

will have but an approach of "crowdsourcing." At the workshops, she indicated, there appeared to be disorganization regarding the end goal. Additionally, she stated that the MIP definition of active learning is "so starkly different than the commonly used definition of active learning" and it became an "obstacle." She later stated that the communication was unhelpful and highlighted their difficulty in receiving payments.

She also highlighted frustrations with the MIP influence on her CoRD's work. Katie stated that

> when we make a move, a lot of times we were met with, 'Well, that's not where we wanted to go.' Well, well, then just tell us where you want to go (*frustrated laugh*). That was the frustration, I think a lot of times was, I think the— outer layer sometimes seeped in and was affecting the inner layer to— in a way that we were, having a hard time navigating.

In sum, Katie's takeaways were primarily negative and reveal opportunities to improve communication with CoRDs. Her remarks also indicate that she may not have expected her participation on the CoRD to be a professional development experience since she was frustrated that the "outer layer sometimes seeped in and was affecting the inner layer." Despite these criticisms, Katie expressed potential for positives: "You know just talking to the people that I've interacted with, we really see a lot of positive things that could come out of it. We just haven't really experienced any of it if that makes sense." These remarks reveal features of her identity trajectory engage in inter-institutional collaboration to share experiences and learn from colleagues while remaining skeptical of these experiences from her involvement in MIP activities.

In discussing his takeaways, Adam identified the different values and commitments among colleagues at different institutions, and yet, "there is still a community of instructors who are, uh, concerned about making sure that students understand the material independent of their utility as potential researchers or potential mathematicians. Um, so that I think was good to see." Adam, who was on Katie's CoRD, also indicated frustration with the communication and administrative process while working on a CoRD.

Content Related

Develop or incorporate content—specific (Robert, Finn, and Pippa). Participating on a CoRD provided Robert an opportunity to contribute by sharing ideas (see his pre-workshop comments related to the College Algebra and Precalculus Workshop) with his CoRD about a pedagogical approach by mathematics educator, Dan Meyer: presenting an interesting problem with a basic question and intentionally omitting information so that students' must gather information to solve the problem. These contributions reveal features of his distal goals as an instructor: sharing interesting and perhaps realistic problems with colleagues that may require students' participation. These problems may be more realistic since they are not contrived problems designed to test students' procedural fluency but are more open-ended. Additionally, these problems might support students' participation since they may need to request information if it is omitted. Robert's remarks suggest that he prioritizes problems which support students' curiosity, interest, and problem-solving skills. The extent to which these problems support students' construction of mathematical meanings necessary to support their understanding of an idea seems to be less specified, and hence, less important.

Finn had takeaways related to the organizational setup of the CoRDs (i.e., desiring more opportunities for meetings to discuss the nature of the product the CoRDs are creating) and he liked one of the invited speakers from an Initiation Workshop who gave a presentation on the modeling approach to concepts in mathematics in his textbook (which you cannot confuse for a College Algebra textbook). In addition to these, Finn discussed takeaways related to *Content or Curriculum* and *Collaboration and Connections More Generally*. Related to the former domain, he stated that he was "very bad" about reading the pre-readings prior to attending the workshops but "most of those papers have been something— has been something that I routinely go back and read it" since they "have some fairly interesting viewpoints." Finn also enjoyed the activities discussed during the workshops.

> I've enjoyed going through and, and— we've had some various open-ended exercises. So, whenever I was talking about the precalculus, uh, precalculus workshop, the open-ended discussions are things that I, that I've enjoyed. I've come back and have shared them with my faculty here. So, Dr. [instructor's name] has you know— We talk about that, and we've actually started going, well actually some of these work— some of these exercises actually exist in Stewart's calculus. Uh, maybe we need to start putting them in rotation more.

In a follow-up interview, I asked Finn about these readings, and he discussed two papers. Finn indicated that one of the papers was interesting, because the authors encouraged the use of open problems, a practice Finn identified as uncommon to do in a mathematics course. When I prompted him to elaborate later in the interview, Finn presented the dilemma: there is too much content to cover and open-ended problems take a lot of class time. Related to his final comments in the previous excerpt, Finn indicated that there are problems in Stewart's Calculus that are similar to open-ended problems discussed during one of the Initiation Workshops: filling up a jar with water and producing a graph representing the rate of change between two quantities specified. Finn highlighted students' dislike for these problems, since they are accustomed to solving more traditional problems (e.g., take a definite integral and get an answer). While Finn admitted that "it'd be really nice to have students be able to explain that and be able to explain the concept and be comfortable with the concept," he indicated that Calculus I assessments typically focus on more procedural skills (i.e., knowing how to take a derivative or knowing it gives the slope of a tangent line).

His comments suggest that discussion from these workshops around the nature of the activities has filtered into his department and may potentially influence the types of problems they considered implementing. His goals to support students' creativity and their capacity to understand a concept more productively is constrained by students' probable dislike of open-ended problems, his job to "cover" the requisite content with limited time, and his desire to focus on problems that are related to the assessments. Finn's apprehensions are warranted and reveal challenges that mathematics instructors encounter from the demands of their institution.

Pippa attended the College Algebra and Precalculus Workshop and her response to Prompt 1 is given as follows:

> I want to help the state of OK and my school personally prepare students who need college algebra for their degree to be on the right path. I also hope to gain a clearer perspective of what direction we are headed with math education.

Pippa's response indicates a general desire to help the state and her institution to better support students' efforts gain insight on the future direction of mathematics education. Her comments after attending the College Algebra and Precalculus Workshop align with her initial goals:

> Well, I know for myself, uh, seeing us work together, but also seeing some of the concepts and stuff and the, the future. I can kind of see the future where we're going with this, and I really like it, because I think our— all of us have the same purpose, and we want to really reach our students. We have a passion to reach our students. And, I think this kind of envelopes our passion, and it allows us to share that with the entire state, and going— I think this is definitely a positive direction for us to go. And, that's what I got out of it.

Pippa's response highlights the unity of participants' efforts in the MIP movement and common goals to support student success. Pippa described her contributing role in MIP activities as a "partner": "You know, I came up with some ideas. Other people came up with some ideas. We all refined different parts." In the interviews, she discussed her motivation for developing her linear equations tasks for her CoRD. She noticed that linear equations often are presented in the first quadrant (i.e., have positive values), and so she created a problem that used both the second and fourth quadrants, enabling students to see that *x* values and *y* values can be negative. These comments reveal that she possessed commitments to present linear equations tasks highlighting alternative symbolic and graphical attributes. Pippa's focus on these attributes reveals her values to support students understanding of particular meanings, and her comments also suggest that there remain opportunities to enhance her commitments to design tasks intended to engage students in actions necessary to enhance their covariational reasoning.

Develop or Incorporate Content—general (Finn, Pippa, Sarah, and Jack). Related to collaboration, Finn discussed the importance of developing supplemental materials, since some publishers just provide a Canvas Shell in their appendix with resources to support students who need to be "caught up" with the material. Pippa acknowledged her impoverished communication skills when developing the instructions for curricular resources and indicated the value in allowing colleagues to critically evaluate her work:

> I discovered my communication skills are horrible compared to (*little exhale laugh*)— I can teach math. I can't put things on paper very well where other people understand what my thoughts are. Um, I kind of think a little backwards, sometimes, and so having someone go through, and say, oh, let's polish this up. Let's have this say this, because that makes more sense, and I'd be like, oh yeah that does make more sense for other people (*little laugh*). So, uh, that was really good. And it's good to, you know, for me to take note, okay when I'm trying to write out instructions, I may not explain things in ways people will understand, and I might want to have someone look over my written instructions, make sure it makes sense for other people.

She also discussed how her involvement in MIP activities has allowed her to "fine tune" her courses and identified an opportunity to enhance her instruction by incorporating projects-based assignments. Pippa's interview comments suggest distal goals, supported by increased confidence, to possibly be more attentive to her communication skills and to incorporate more projects in her teaching. Moreover, her remarks illustrate how one MIP faculty negotiated meaning from her interactions with other colleagues around the development of a curricular resource.

While Pippa's remarks reveal value she derived from her experiences developing content, Sarah, in discussing takeaways from her experiences collaborating, identified the features of developing content that she values:

> Um, (*long pause*), I don't know I guess it, it. is more of, um, just how nice it is to work with other math educators, and you know, rather than anything specific for my course I felt like a lot can be done if we work together, and I think there's just not enough of that, in general, in, in math departments across the state or, at least where I've been. I've been at two different universities, and we talked to each other, but I don't think we really work together, and, and I feel like that would be really beneficial if more math educators could, could work together and, and not just talk sort of in theory about things that should happen but actually work and, and find out good ways to, to teach math. (*long pause*) And bou— and bounce ideas off of each other, I think would be great if we could do more of that.

Sarah's comments reveal that she values opportunities to collaborate with colleagues to discuss ways to enhance the teaching of mathematics on a more practical level. Her final remark highlights features of her distal goals for mathematics faculty to engage in intentional and productive collaboration with other colleagues throughout the state. Earlier in the interview, she stated that her CoRD had a "really good experience" designing their curricular module and also indicated her dissatisfaction with the interactions she has had with colleagues from her department: we discuss topics, identify frustrations, and share advice, "but we don't really sit down and work together on, on a project at all, um, that's aimed toward helping students, and I thought that that was a really nice thing that we did in the MIP project."

I also asked Sarah to discuss how she envisions collaborations among mathematics faculty across institutions in Oklahoma might look like, Her experience working at two universities in Oklahoma has revealed to her different priorities among departments regarding the breadth of content required to be covered in a mathematics course. She also stated that it would be great to talk about why instructors might teach or decide not to teach a topic, and then maybe discuss more specifics about

> what exactly do we want students to learn about particular topics and not just say, you know, they need to learn what a function is, but, um, talk about all the different ways that we can represent functions and why those are important.

Her comments reveal an attentiveness to not only discussing the inclusion and exclusion of overarching topics, but to also collaborate with colleagues to discuss meaningful ways of supporting students' understanding of ideas.

In a follow-up interview with Jack, I asked him to elaborate on his remarks to "self-evaluate" from his written response to Prompt 1 and discuss the purpose for doing so. Jack's responses to these questions provided insight into his priorities and commitments as a mathematics instructor. Jack stated that he leverages students' specific comments to inform his instruction, and he gave examples of positive comments from students in his geometry class who say (paraphrased by Jack)

> well, now, the way that you're presenting this to me, I'm actually having to think about it, and I'm having to learn it for real instead of just being told what it is. And I said, okay. So, that's some positive reinforcement for me when I'm having them work on these

> constructive struggle inducing problems that that's the way they want to learn. That's the way their brain wants to learn. And so that's positive for me. So, I'll keep using that.

Jack's comments reveal that students' remarks influence his decision to continue using a particular activity. Other times, he may use a video that he thinks is "cool," but students indicate that it made them fall asleep, leading him to not use it the following semester and revealing his evolving approach:

> So, I never have something set in stone. I, I try to keep it fluid, but once something works, man, I'm going to hold on to it as long as I can. So, I have these certain pillars that I will keep in my class.

These comments reveal important features of Jack's identity as a mathematics instructor. Students' willingness or disinterest to productively engage with a resource he incorporated into his class informs Jack's decisions to either continue implementing it in future semesters or perhaps discontinue using that particular activity or video. Jack's remarks reveal his commitments to prioritize students' reactions and comments as an important (and perhaps sometimes determining) feature in assessing which resources to implement.

I asked him to identify the pivotal factor influencing his decision to keep something or disregard it, and he identified students' engagement in mathematical thinking as the "driving force":

> Um, well, it's going to be a rehash of what I had said before. It's this philosophy that I have, um, where I tell them at the beginning of the semester when you're done with this course, I want you to be able to think mathematically about anything. So, whether I keep it in or not is am, am I able to see that they are thinking mathematically and by doing—

> by that phrase, I mean are they thinking critically? Are they able to, to, um, see a solution to a problem or multiple solutions to a problem, and can they see the value in what they're learning? So, all of this idea of being a mathematical thinker means I'm just not going to take this at face value. . . . So that math thinking mentality that I want them to leave with is what drives— that's the driving force behind everything I do.

These comments align with Jack's comments earlier when he discussed his goal to support students to become mathematical thinkers. Jack's remarks reveal his desire to incorporate ideas that help him foster a classroom of students who develop mathematical orientations that enable them to engage in productive mathematical practices (e.g., critical thinking, problem-solving).

Collaboration

Sharing experiences and struggles (Katie, Finn, Jack, Pippa, and Sarah). I asked Katie to discuss what she envisions collaborations across institutions in Oklahoma should look, and she highlighted her approval of what she called the "original" plan of the MIP to travel to or host faculty from other institutions throughout the state and collaborate with colleagues. While there are differences between large and small institutions in Oklahoma, Katie discussed, there are also some shared similarities. She stated that it would be productive to share experiences as well as listen and learn from others' experiences to enhance our growth. Amy also expressed her frustration with witnessing the focus of MIP research on individual's unproductive interpretations of active learning and with not communicating with MIP leaders.

Amy's frustrations also illuminate features of her desired identity trajectory to participate in a leadership position if future opportunities arise. After discussing her frustrations with MIP experiences, Katie acknowledged the exciting opportunity presented by the MIP mission:

> Um, you know, like I said the mission of trying to meet and collaborate with other institutions and other people I think would be really great. Um, you know, especially because most of the other opportunities that I personally have experienced have had to be at sort of Regents level, um, you know, Regents level sort of projects. And I would love them to be a little less formal, uh, a little more collaborative, a little less, uh, policymaking and a little more, you know, like curriculum building and design (*little laugh*). So, I think that could be, I think that could be really cool given the opportunity.

Katie's comments highlight the potential opportunity for productive MIP experiences through collaborating with other institutions to develop curriculum. Her remarks reflect her distal goals, given the opportunity and better experiences, to collaborate with other colleagues to develop curriculum and share experiences to continue her professional growth and learning.

Finn also discussed the importance of collaborating with other colleagues to share experiences through informal conversations. For Finn, the workshop provided an opportunity to meet with other people that "know Oklahoma." Finn stated that

> we weren't working on the breakout session work, um, it became something very, very much along the lines of, um, what are your schools doing, like, uh this year? How, how, did your classes work in this situation? Um, kind of the unfortunate part about, um, about the pandemic was that was actually kind of the more important parts that I actually looked at, because I was like, okay, how did you really do your class this year? Oh, okay, by the way you started functions and modeling last year. How is that working out, because half the schools in the state have college algebra textbook that— which is not

> college— which is not a functions and modeling textbook. So, (*little exhale laugh*), um, you know, turning— being able to turn to that person and ask those questions.

He later discussed the impact of his connections with one member on his CoRD:

> But one of the things I've learned about with, with her was again, it was a connection thing. So, um, she started teaching a functions and modeling, and she hates Cengage as much as I do. So, when it came to setting up that course, um, it was a connection where she called me and she's like, how do we set this up? I know it's not part of the CoRD, but it again became one of those things like, oh, I have a contact. They're not in my school anymore, but I have a contact.

While some of Finn's takeaways were more content focused (e.g., the readings), his remarks reveal the value he derived from opportunities to interact with other colleagues and highlight the affordances of developing these connections. When I asked him if it is important to have opportunities for math faculty to collaborate across institutions in Oklahoma, he said, "Oh, yeah. Yeah. No, as I said, that's the, that's the entire reason I do this." Collectively, these comments reveal that Finn's willingness to collaborate with other colleagues are central to his identity as a mathematics instructor.

Towards the end of the interview, I asked Jack to discuss what he envisioned collaborations across Oklahoma should look like, and he highlighted the value of having in-person meetings once a year with periodic opportunities for virtual check-in meetings. Jack's suggestions attended to features of the *medium* for these collaborations, and I pressed him to discuss his priorities for the *focus* of these sessions. He stated that the working sessions are helpful but has benefited from having opportunities to share experiences with other faculty:

I've gotten a lot out of personal experience sharing, uh, from colleagues and other educators. Like, sometimes I can go to a presentation, where you know they may not be talking about, um, specific ways to teach mathematics, but they might be sharing experiences and what they have gone through. That can give me a lot. Um, just a lot of, uh, mental fuel. In other words, sometimes I'm not in the mood to hear another way to talk about factoring, for instance, right, I mean. Okay, yeah, okay that's great.

Sometimes I want to hear stories about how the teachers have struggled [audio cut out]

I asked Jack to discuss his meaning for "mental fuel," and he discussed the personal value he experiences in having support from colleagues in *vent sessions*: venting with "other people that are in my same possession [sic] that are dealing with— or same position, dealing with the same frustrations" (e.g., students with a fixed mindset). Jack's comments reveal an affordance of collaborating with colleagues to share experiences and admit struggles for enhancing unity among mathematics faculty in similar positions and possessing similar frustrations.

When I asked Pippa to discuss her vision for collaborations between mathematics faculty across institutions in Oklahoma, she stated that collaborations centered around projects are "good" and acknowledged the value in sharing experiences with colleagues. She discussed the importance of mathematics faculty having opportunities to collaborate together to

> discuss what's working what's not, hear what people have done in their situation, and, you know, what other pe— you know, because you can pick stuff up from other people and learn from that and say, Oh, I want to add that to my— what I'm doing, or I want to change and do what they're doing.

Having opportunities to interact with similar-minded colleagues who are excited to discuss mathematics, Pippa discussed, can be a relaxing experience. Pippa's comments reveal her distal goals to share experiences to learn new ideas that she might implement in her class.

Sarah attended the College Algebra and Precalculus Workshop and her response to Prompt 1 was as follows:

> I expect the collaborations to help each of us gain from the experiences and expertise of others to produce relevant inquiry-based materials that can be used in classrooms across the state. It would be beneficial to learn from other instructors some of the challenges they've had and how they address those challenges in their classrooms. My own contributions include experience of teaching entry-level mathematics for 26 years. In addition, my experience in coordinating a large multi-section course can be shared to give insight into some of the challenges in maintaining course consistency with a large and diverse set of instructors.

Sarah's response reveals her initial goals to learn from challenges her colleagues have encountered and overcome as well as contribute from her experiences maintaining alignment as a course coordinator. These comments suggest that she prioritized opportunities to collaborate with colleagues and glean insight from their general experiences. At the conclusion of the College Algebra and Precalculus Workshop, she provided these closing remarks:

> Um, I like what everybody has said, and won't repeat any of that. I've enjoyed meeting everybody, and, um, really impressed with how we all came together, and the, the sort of chaotic way we threw ideas at, at the board, and wrote lists of things. And, I thought, oh my gosh, this is (*little laugh*), this is never going to come together, and then it did. And

> that just, really, I'm impressed by the way that we all worked together so well and made some good documents.

Sarah's comments reveal the value she derived from her experiences engaging in productive collaborations with other MIP colleagues.

Leadership role, mediator, and increased confidence (Sarah, Jack, Finn and Pippa). I prompted Sarah to discuss her role in contributing on a CoRD and in MIP activities and her takeaways from her experience on a CoRD. Regarding the former, she indicated that she was the organizer for her CoRD. Robert and Pippa, who were on the same CoRD with Sarah, also characterized Sarah's role as the CoRD leader or manager. Sarah described her leadership in this way:

> Um, I'm the one that got everyone together to start with, and I sort of organized all the meetings, and, and sent out the invitations for the meetings and, uh, set up the—I think we were doing Google hangouts for a while and then we switched to zoom I think toward the end, and— Anyway, I, I always sort of put together everything and, um, and sort of, I guess was sort of the, the person who initiated all of all of our discussion and, and everything.

Sarah's acknowledgement of her leadership role in managing her CoRD led me to ask her at the end of the interview if her participation in this leadership role, which I highlighted may be different from her participation on other initiatives, has impacted her mindset or might impact her future participation in other initiatives. Sarah admitted that she did not know and discussed how her role as a course coordinator motivated her to engage in a leadership position on a CoRD:

I'm not sure I would have done that, prior to becoming a coordinator here. Um, but I think that served me well in, in helping organize our CoRD. . . . So I don't, I don't know that, that I learned it from, from being the sort of the, the organizer of the CoRD so much, but, that's not a great answer (*little laugh*).

Sarah's remarks suggest that her previous experience in the position of mathematics course coordinator at her institution supplied her with confidence that enabled her to lead the CoRD.

During the interviews,[17] Jack tentatively described himself as a CoRD leader and discussed his contributions as someone who is bringing "tools" into this "strategy session" to work as a team to develop resources that might support students. Jack revealed his image of his involvement in the MIP: develop curricular resources, test them in the future, and make them available to other instructors as supplemental materials designed to help students to develop critical thinking and problem-solving skills in group dynamics.

After Jack admitted that he did not initially think that he would be creating anything as an MIP participant, I asked him if his participation in MIP activities to create resources has impacted how he views his roles in other initiatives, or if his involvement in a leadership role may impact his future participation or involvement in other activities. His experiences, Jack stated, refined his perspective more than it changed it.. He described his role in his department at his institution as someone who is a "reluctant spearhead" who takes the initiative to find solutions to problems that would go unresolved otherwise. Hence, his overall identity trajectory seemed to remain stable but was refined through his participation in MIP activities. Jack acknowledged that the MIP provided opportunities for him to engage in original research (and

[17] While many participants gave post workshop responses after the Functions and Modeling Workshop, Jack did not, and so he must not have been present at the conclusion of the workshop.

become a better researcher in the process) and valued the accountability in developing these resources.

During our interviews, Finn's description of his role was encompassed under the theme of *Collaborating or Supporting Colleagues*. While Finn acknowledged that he might not have the "best idea in the room," his willingness to say something "incredibly stupid (*little exhale laugh* by Josiah and Finn)" could encourage other colleagues to engage in the discussion. He later described his role on a CoRD as relating to communication. Finn's comments suggested that he acted as a mediator in his group to help direct members of his CoRD to focus on the type of product they were expected to produce among differing opinions.[18] Finn's comments suggest that he valued providing directional guidance to his CoRD.

When I prompted Pippa to discuss her takeaways from her involvement in attending workshops and participating on a CoRD, she described her participation as a "mind stretching" experience to develop a project and indicated her increased confidence from her involvement:

> It was surprising that I was able to keep up with the math because most— I think everyone else had a PhD. So, I always consider myself— I only have a Masters, you know. I'm not that smart. And to be able to keep up on the same level with them with no, no problems, I was like okay well, maybe I'm a little smarter than I thought (*little laugh*). Um, and when we were doing those uh, some of those, uh, problems that we had at that one meeting, um, in Oklahoma city, that's when I was like, maybe I'm not as dumb as I thought, because you and I got the problems and then come to find out that you and I were the only ones that got them (*little laugh*). I was like the two without PhDs got the

[18] Finn decided to quit his CoRD work after the first round of revisions because "there was no more going forward" and he was busy.

problems done. . . . Um, so, personally, it was very, you know, self-building for me. Uh, self— you know building my confidence as a mathematician.

Pippa's comments suggest that she gained increased confidence from her participation at the Initiation Workshops to solve assigned calculus problems.

Discussion and Implications

Participants' responses to the interviews suggested some faculty are on (or at least were on) trajectories that value and seek to support students' affective engagement . Ellison described how her presentation helped "bring to the forefront" the importance of supporting discouraged students. She stated that she had "been more in tune" with her students' affective responses. Moreover, Robert was pleasantly surprised at the focus of the Academic Success Skills Workshop and valued supporting students' grit and determination. There were also other trajectories that encompassed participants' willingness to collaborate with colleagues and create connections. Finn discussed the value in having discussions with colleagues about their experiences (e.g., starting a Functions and Modeling course) and described how a connection with a colleague on a CoRD generated discussions outside of only developing curriculum.

Moreover, Jack discussed the value in sharing experiences and struggles with other colleagues for reigniting his "mental fuel." Katie was on a trajectory to be skeptical from her s previous involvement in MIP activities but remained positive about opportunities to engage in future collaborations. Adam also had criticisms but was interested in revisiting the ideas that they had developed from workshops and considering the MIP definition of active learning. Finally, Sarah discussed the value in having purposeful collaborations on a more practical level, and

Pippa had increased confidence from her experience solving problems and heightened awareness of her poor communication skills while engaging on a CoRD.

As specified in my research questions, the focus of this study is to (1) identify features of MIP participants' professional identities as mathematics instructors and (2) identify ways these findings might be leveraged to modify future design features of the MIP, in consonance with social learning theory. To address the first question, I interviewed eight faculty who had or were currently participating on a CoRD to uncover their perceived contributions and takeaways from their previous MIP experiences and their vision for what future collaborations among faculty across institutions in Oklahoma might look like (see Table 7).

I now discuss four implications from these findings. The first two are practical implications, and the latter two relate to social learning theory. In discussing the fourth implication, I present a hypothetical trajectory highlighting the MIP Team's roles as brokers to facilitate meaningful discussions and strategically introduce reified artifacts designed to support participants to operationalize the three elements of inquiry by conducting a conceptual analysis.

First, Katie's negative experiences reveal opportunities to enhance our communication and organization for future MIP participants by streamlining payments and engaging in more consistent communication about the status of their work and upcoming events. Additionally, there remain opportunities to support CoRDs in learning how to operationalize the MIP definition of active learning. Katie stated that it was an "obstacle" and Adam discussed how it was different than traditional definitions of active learning. This latter point is a central feature in my discussion of the next implication, in which I discuss ways in which some participants' distal goals, reflected in their responses in these interviews, might be valued and leveraged to motivate participants to recognize the affordances of operationalizing these definitions and supporting students' construction of productive meanings.

Table 7

Possible Identity Trajectories

Name	Possible Identity Trajectories
Katie	**Possible Identity Trajectory**: Engage in inter-institutional collaboration to share experiences and learn from colleagues while remaining skeptical of these experiences from her involvement in MIP activities
Sarah	**Possible Identity Trajectory**: Productively work with faculty as a leader to engage in practical discussions around improving the teaching of mathematics and perhaps work on projects to support student success
Robert	**Possible Identity Trajectory**: Possibly reignited to supporting students' engagement in productive academic success skills (e.g., grit), engage students in interesting problems requiring critical thinking, and perhaps participate in opportunities to collaborate on interesting projects
Jack	**Possible Identity Trajectory**: Act as a "reluctant spearhead" and improving researcher who seeks to share experiences and struggles with colleagues and gain new ideas that support his goal to help students become mathematical thinkers and to help improve his teaching.
Ellison	**Possible Identity Trajectory**: Act in ways that are attentive and supporting to students' affective responses and perhaps seek opportunities to learn from experts implementing active learning strategies
Adam	**Possible Identity Trajectory**: Be skeptical of having productive communication from his involvement on a CoRD, recognize instructors' commitments and values, and perhaps seek opportunities to revisit MIP discussions
Pippa	**Possible Identity Trajectory**: Incorporate more projects into her instruction, participate in future collaboration opportunities to share experiences, and engage with increased confidence of her knowledge of mathematics
Finn	**Possible Identity Trajectory**: Consider opportunities to emphasize the modeling approach in mathematics or include open-ended activities in his courses, and perhaps engage in collaboration opportunities to develop supplemental materials while sharing experiences and struggles

First, Katie's negative experiences reveal opportunities to enhance our communication and organization for future MIP participants by streamlining payments and engaging in more consistent communication about the status of their work and upcoming events. Additionally, there remain opportunities to support CoRDs in learning how to operationalize the MIP definition of active learning. Katie stated that it was an “obstacle” and Adam discussed how it was different than traditional definitions of active learning. This latter point is a central feature in my discussion of the next implication, in which I discuss ways in which some participants’ distal goals, reflected in their responses in these interviews, might be valued and leveraged to motivate participants to recognize the affordances of operationalizing these definitions and supporting students’ construction of productive meanings.

Second, Ellison’s comments reveal the affordances of providing opportunities for MIP participants’ to give presentations:

> I feel like my participating in that talk in the first workshop was helpful in— to just bring to the forefront, uh, the importance of helping students when they’re feeling depress— uh, down or discouraged in a mathematical setting about the, the work they’re doing. Trying to bring support, um, and encouragement to them in a, in a more deliberate way.

The MIP Team provide future opportunities for MIP leaders to give presentations at Regional Workshops and discuss the products their CoRDs have developed.

Third, participants’ willingness to share experiences and struggles illuminates the importance of the MIP mission to foster a community of practice of mathematics instructors throughout Oklahoma. Their comments reflecting these desires reveal the value in providing more opportunities for MIP participants to continue building connections between other colleagues. The MIP Team will provide opportunities for further collaboration at later stages in the project (i.e., Regional Workshops, the development of peer mentoring relationships) I now discuss the fourth implication.

Several interviewees' comments suggest that they prioritized having opportunities to share experiences with other colleagues and others discussed the value in sharing struggles. Related to the latter, one participant discussed the value in giving a presentation at the academic success skills workshop, making her more attentive and support to important features of students' affective engagement. Leveraging these findings, the MIP Team could purposefully create opportunities in future MIP events to allow participants opportunities to share their experiences with other colleagues and then specifically discuss their frustrations and challenges that they face at their institution in supporting student success and *learning*. These discussions will be meaningful considering participants' willingness to share their experiences and struggles and their expressed attentiveness to or support for students' academic success skills. At the end of this discussion, they would address the following question: Given these frustrations, how can we better support students' learning?

Some participants (if not most) are likely to discuss affective features of student engagement (i.e., interest, motivation, perseverance, critical thinking skills, problem-solving). My case studies in the following two chapters revealed that these were priorities of two mathematicians. Additionally, Jack's comments to support students to become "independent math thinkers" and Ellison's remarks indicating a willingness to learn from a master instructor who facilitates active learning, suggest that some of the interviewees also value these features. In response to these suggestions, MIP leaders could affirm and value participants' priorities and commitments to support students' affective engagement.

In sum, the MIP Team could provide opportunities for participants to share experiences and struggles related to their institution and supporting student learning. Upon discussing how they could support student learning, some participants may discuss more affective features of student engagement given the struggles at their institution. The purpose of facilitating this sequence of discussions is to (1) affirm participants' priorities and commitments and (2) foster a discussion supporting participants to recognize (a) features of contrast and (b) potential affordances of operationalizing the three elements of mathematical inquiry, informed by conducting a conceptual analysis.

Regarding (a), Robert and Adam provide a meaningful contrast. Robert's comments during my case study with him discussed in the following chapter, revealed that was less perturbed by difference between the MIP definition of active learning and his own conceptions than Adam's comments expressed during these interviews. If faculty remain unaware of these differences, then they may be less equipped to be influenced by MIP activities since their distal goals could influence their takeaways from their participation.

After MIP leaders affirm participants' commitments to enhance affective features of student engagement to support student learning (e.g., foster critical thinking skills, motivation) and discuss the importance of the MIP definition of academic success skills, they could extend participants' commitments by presenting sequences of two mathematical tasks related to the same topic. The first task might support students' development of procedural skills or critical thinking skills, whereas the second task would support students' construction of particular meanings. Contrasting these sequences of tasks would help illuminate features of the MIP definitions of active learning and meaningful applications.

During this discussion, the MIP Team could highlight some implications for supporting students' construction of particular meanings. These conversations would (1) provide support to participants like Katie and Adam who had difficulty in understanding the MIP definition of active learning and (2) perhaps stimulate a curiosity for learning how to design tasks which support students' construction of particular meanings (i.e., conducting a conceptual analysis) to enhance participants knowledge base discussed in Chapter 1.

In this first chapter, I examined research conducted on STEM professional development initiatives along a continuum where experimental design and design research were at opposing ends of the spectrum. One way I am conducting design research is by leveraging findings from eight interviews to provide practical and theoretical implications. In addition to discussing opportunities to enhance communication between the MIP leaders and the CoRDs and the potential value of providing ownership opportunities for MIP participants, I presented a hypothetical trajectory for how the MIP Team could facilitate meaningful discussions and strategically introduce reified artifacts designed to enhance participants' knowledge base. The focus of this chapter was on eight participants' perceived contributions and takeaways from their involvement in MIP activities, and their vision for future collaboration opportunities with faculty across institutions in Oklahoma.

In the following chapter, I present my results from a case study that I conducted with one of the interviewees: Robert. While the focus of the eight interviews was to uncover features of participants' identity trajectories by revealing their distal goals as mathematics instructors from their reflections after participating in MIP activities, the focus of the case study is more narrowly focused on Robert's interpretation of the three elements of mathematical inquiry and conceptual analysis.

CHAPTER 3

AN INVESTIGATION OF ROBERT'S IDENTITY TRAJECTORY IN THE MIP: EXPLORATORY CASE STUDY #1

Introduction

As discussed in Chapter 1, there is a central focus on improving STEM education in the United States. Enhancing STEM education is merited considering the high demand for positions in these fields. STEM careers in the United States are expected to increase 13% (compared to 9% for other occupations) between 2017 and 2027, and the median STEM job earnings in the United States are double the median income of all other jobs ($38.85 per hour as compared to $19.30 per hour) (Economic Modeling Specialists International, 2017, as cited in Education Commission of the States, 2021b). Finally, STEM unemployment has been lower than non-STEM unemployment: 2.2% compared to 5.5% from 2014 to 2017 (Education Commission of the States, 2021b).

Government incentives and high demands for STEM improvement have resulted in a recent cascade of reform efforts, some of which include opportunities for professional development for STEM teachers. Supovitz and Turner (2000) summarized the logic underlying teacher professional development (PD) initiatives for improving student success: "high quality professional development will produce superior teaching in classrooms, which will, in turn, translate into higher levels of student achievement" (p. 965). Teacher PD programs target a variety of student populations from elementary school (Brown et al., 2019), middle school, high school (Du et al., 2019; Al Salami et al., 2017; Singer et al., 2011), or colleges and universities (Andrews & Lemons, 2015; Borda et al., 2020; Czajka & McConnell, 2019; Pelch & McConnell, 2016). Within these different academic levels, initiatives might target instructors from a single department, school, or institution (Auerbach & Schussler, 2017), instructors from multiple of these entities (Andrews & Lemons, 2015; Borda et al., 2020; Czajka & McConnell, 2019; Pelch & McConnell, 2016), instructors (or post docs) from across regions (Ebert-May, et al., 2015; Eddy et al., 2019) or an entire state (Schnittka, 2014).

Magliaro and Ernst (2018) highlighted many statewide initiatives in their comprehensive report of 302 STEM education networks. These scholars noted that "STEM networks with widely varying partnership configurations appear to be operational in all states" (Magliaro & Ernst, 2018, p. 8). Among these networks, they discovered 90 at the statewide and territory-wide level (i.e., "Organizations that provide services across an entire state (e.g., state departments of education, governors' advisory boards)" (p. 8)). Their findings indicated that substantial "effort and resources have been and continue to be invested in the advancement of high quality STEM-related experiences *primarily* [emphasis added] for the P-12 sector" (p. 11).

Fewer statewide STEM reform efforts (and even fewer addressing mathematics specifically) have targeted faculty at colleges and universities. The Mathematical Inquiry Project (MIP) seeks to address this need. The MIP is a six-year (an additional year was added to the project timeline due to COVID-19), three million-dollar NSF-funded professional development initiative designed to support faculty teaching entry-level mathematics courses at colleges and universities throughout Oklahoma.

A Necessary Step in Oklahoma

The Education Commission of the States (2021a) identified STEM initiatives at the high school level in Oklahoma. The Oklahoma State Regents for Higher Education (OSHRE) designed the Future Teachers Scholarship Program to incentivize teacher preparation in critical areas (OK College Start). Moreover, OSHRE also created the Teacher Shortage Employment Incentive Program (TSEIP) to "recruit and retain mathematics and science teachers in Oklahoma" (Oklahoma State Regents for Higher Education, A Brief Summary section).

Despite these efforts at the high school level, the Education Commission of the States (2021a) did not identify any adoption of rigorous graduation criteria for mathematics in Oklahoma. Moreover, there is no indication that Oklahoma has any programs focusing on supporting underprivileged and minority groups regarding STEM achievement. Student assessments also highlight need for improvement. For instance, in 2018, the average ACT math score for the graduating class in Oklahoma was 18.8, the same score as in 2017 and 1.1 points lower than in 2014 (ACT, 2018).

These disappointing statistics, although highlighted in Oklahoma, reflect common struggles for students in STEM education across the United States. In addition to demonstrating a need for reform in the P-12 sector, these data also communicates an urgent warning to post-secondary educational institutions. The overarching goal for STEM reform should encompass a dual focus on both preparing students *to* attend a college or university (if they so choose) while also supporting them to successfully graduate *from* that college or university in a timely manner.

According to the Data Dashboard from Complete College America (2008 cohort), the national average percentile for first time full-time students receiving a bachelor's degree (not corresponding to a higher research category) in four years is 20% compared to 12% in Oklahoma. More specifically, certain mathematics gateway courses impose a barrier that impacts college students' graduation rates. While more than 70% of students working on a four-year degree at a higher research institution have completed both their mathematics and English gateway courses within two years, the completion rate is less than 50% for those seeking to complete their bachelor's degree at the other four-year institutions (Complete College America, 2012 cohort). Of these two types of gateway courses, completion of the mathematics course is particularly problematic. While 28% of students attending two-year institutions have completed both their mathematics and English gateway course within two years, only 3% have completed their mathematics gateway course (and not English), as opposed to 29% who have completed their English gateway course (and not math) (Complete College America, 2012 cohort).

Other statistics indicate that gaps remain across races. In Oklahoma, Asian and white students (first time full-time) have completed their bachelor's degree (associated with a higher research degree) 34% and 36% respectively within four years as opposed to African American rates of only 17% (Complete College America, 2008 cohort). In addition, African American students in Oklahoma (first time full-time) have completed their mathematics requirements (associated with higher research degree) at much lower rates than Asians and whites: 11% compared to 22% and 20%, respectively, within two years (Complete College America, 2012 cohort).

Solutions to the Mathematics Barrier

The MIP is one of many STEM reform efforts attempting to address the "mathematics" barrier impeding student success at many colleges and universities by offering students multiple math pathways. The Dana Center Mathematics Pathways project identified four issues impacting student success (McIver et al., 2017, p. 2):

1. Students' performance in developmental math courses.
2. Student's ability to complete required math courses.
3. Students delay their enrollment in math courses.
4. Students' enrollment in math courses that are not useful for their degree of study.

The Dana Center intended to address these four issues through their pathways model to "incorporate rigorous and relevant mathematics aligned to programs of study and informed by guidance from major professional organizations" (McIver et al., 2017, p. 2). Another model targeting similar issues, the American Association of Community Colleges (AACC) Pathways Project, was developed to support 30 colleges in following their guided pathways approach to map out student degree plans, support students' program selection, monitor student progress, and enhance student learning (Jenkins et al., 2017, pp. 1-2). Their guided pathways approach has been implemented in at least 15 states and by at least 200 community colleges as estimated in the AACC report (ibid., p. 3).

Within Oklahoma, there have also been reform initiatives targeting these issues. The Oklahoma State Regents for Higher Education (OSRHE) recognized the imminence of these problems, identifying "low success rates in remedial and gateway math courses as a significant barrier to student success" and committing to "improve the effectiveness and efficiency of remediation and freshmen gateway courses" (Oklahoma State Regents for Higher Education, 2017, p. 1). OSHRE formed the Math Pathways Task Force and after several meetings in 2016, the Task Force produced five goals and presented five essential challenges for achieving them. Additionally, the Task Force offered the following five recommendations that were refined from their interactions with representatives from Oklahoma's post-secondary institutions (Oklahoma State Regents for Higher Education, 2017):

1. Establish statewide college meta-majors and corresponding math pathways, ensuring transferability across institutions;
2. Improve student preparation, including efforts in K-12 education and remediation reform;
3. Increase student engagement and the teaching of applications in gateway math classes;
4. Increase support for important academic success skills in gateway math classes; and
5. Provide faculty and advisor professional development and resources.

While the first two recommendations have been addressed through the math pathways and corequisite reform initiatives, the Mathematical Inquiry Project (MIP)—funded by the National Science Foundation—was designed to support OSHRE in accomplishing the latter three objectives by cultivating a community of mathematics faculty across the state to develop curricular resources for entry-level mathematics courses centered around three elements of inquiry—*active learning*, *meaningful applications*, and *academic success skills*. The MIP Team's operational definitions of these three components are listed in Table 8. For a thorough description of these three elements of inquiry, refer to the discussion in Chapter 1.

Table 8

Operational Definitions of the Three Elements of Inquiry

Active Learning	*Students engage in active learning when they work to resolve a problematic situation whose resolution requires them to select, perform, and evaluate actions whose structures are equivalent to the structures of the concepts to be learned.*

Meaningful Applications	*Applications are meaningfully incorporated in a mathematics class to the extent that they support students in identifying mathematical relationships, making and justifying claims, and generalizing across contexts to extract common mathematical structure.*
Academic Success Skills	*Academic success skills foster students' construction of their identity as learners in ways that enable productive engagement in education and the associated academic community.*

In sum, inadequate student success rates in mathematics in Oklahoma and nationwide suggest the need for large-scale reform initiatives. A plethora of efforts have focused on improving these disappointing statistics to enhance student learning and support student enrollment in appropriate courses by offering multiple math pathways. Within Oklahoma, the Math Pathways initiative has identified a need for *structural* changes by implementing alternative gateway mathematics courses and corequisite support. Further reform priorities identified by OSHRE incentivized the need for *cultural* changes through professional development. In response to this charge, faculty at Oklahoma State University designed the Mathematical Inquiry Project (MIP) to address these issues by cultivating a community of mathematics faculty across the state to participate in professional development workshops centered around mathematics learning through inquiry.

Research Question

In this study, I address the following research question:

Research Question: What is the trajectory of one MIP participant from his engagement in the MIP Community of Practice? What are his interpretations of three elements of mathematical inquiry and conceptual analysis and how do these conceptions reveal features of his identity as an instructor?

Theoretical Perspective

While most qualitative research reports will contain a theoretical component, the purpose for which theory is used is not consistent across qualitative research. Collins and Stockton (2018) examined the primary role of theory in qualitative studies, remarking that theory can serve to (1) provide clarity regarding epistemological approaches, (2) serve as a guiding framework for the research, (3) offer logical motivation behind methodological choices, or (4) help build theory from research findings (Collins & Stockton, 2018, p. 1). In this section on theory, I address the first two of these items listed. I articulate my conception of social learning theory and constructivist epistemology and discuss their relevance and application to the present study. Addressing my research questions first requires an understanding of how professional learning might be occasioned through one's involvement in a *community of practice* (Wenger 1998).

Social Learning Theory and the MIP

The MIP Team leverages a Communities of Practice (Wenger, 1998) approach for engineering the cultivation and evolution of a community of mathematics instructors to support their professional learning. Central to this theory is an individual's interpretation of the practice in which they are engaged: "Practice is, first and foremost, a process by which we can experience the world and our engagement with it as meaningful" (Wenger, 1998, p. 51). More specifically, as an individual participates in a community of practice, they begin to negotiate meaning through their engagement with other members and through their interaction with the community's established set of reified artifacts. Hence, *negotiation*, described as "a flavor of continuous interaction, of gradual achievement, and of give-and-take," occurs in two primary ways: through interactions with others (participation) or interactions with artifacts (reification) (Wenger, 1998, p. 53).

Participation is the mechanism by which an individual negotiates meaning with others in the practice, but it is not intended to be synonymous with collaboration or have a particular positive or negative connotation. Rather, participation is used more generally to characterize "the social experience of living in the world in terms of membership in social communities and active involvement in social enterprises" (Wenger, 1998, p. 55). Participation is sustained by interactions with reified artifacts, where reification is defined as "the process of giving form to our experience by producing objects that congeal this experience into 'thingness' " (p. 58).

The interplay of participation and reification relate to an individual's *practice* but not necessarily to their *community of practice*. Wenger (1998) listed three dimensions by which the operations of practice connect back to the concept of community—*mutual engagement, joint enterprise*, and *shared repertoire*. Each of these dimensions serves a pivotal role in defining a community of practice. *Mutual engagement* refers to the process by which members of the practice engage with one another in their practice, and they may do so while engaged in a *joint enterprise*. Hence, individuals engaging with one another also share common goals as a community. The final dimension is the *shared repertoire* or reified artifacts that exist within the community such as "routines, words, tools, ways of doing things, stories, gestures, symbols, genres, actions, or concepts" (ibid., p. 83).

Theory related to the research question. Addressing my research question requires a discussion of Wenger's (1998) characterization of learning as an *identity trajectory*. According to Wenger (1998), an identity in practice "arises out of an interplay of participation and reification," "is always going on", and something humans "constantly renegotiate during the course of our lives" (pp. 153-154). This social interpretation of identity portrays it as fundamentally dynamic and continually evolving. As individuals negotiate their identity while engaging in the practices of a community, a notion of competence emerges and knowing becomes defined:

> Again, it is by its very practice—not by other criteria—that a community establishes what it is to be a competent participant, an outsider, or somewhere in between. In this regard, *a community of practice acts as a locally negotiated regime of competence*. Within such a regime, knowing is no longer undefined. It can be defined as what would be recognized as competent participation in the practice. (Wenger, 1998, p. 137)

Through the process of enculturation into the norms established by the community, individuals' identities evolve into trajectories (Wenger, 1998, p. 154). Wenger (1998) clarified his interpretation of this construct:

> In using the term 'trajectory' I do not want to imply a fixed course or a fixed destination. To me, the term trajectory suggests not a path that can be foreseen or charted but a continuous motion—one that has a momentum of its own in addition to a field of influences. It has a coherence through time that connects the past, the present, and the future. (p. 154).

Wenger (1998) briefly described five different types of trajectories: *peripheral*, *inbound*, *insider*, *boundary*, and *outbound*. For the purpose of this case study, I examined an individual who appeared to be engaged on an inbound trajectory: "Newcomers are joining the community with the prospect of becoming full participants in its practice. Their identities are invested in their future participation, even though their present participation may be peripheral" (Wenger, 1998, p. 154). Hence, an identity trajectory characterizes the ongoing process by which an individual negotiates their identity, relative to the normative definition of competence established by the community, in pursuit of its joint enterprise.

While it is useful to apply social learning theory as a broad framework to investigate the evolution of the entire MIP community of practice, my interest in the psychological experiences of participants suggests a need incorporate multiple theoretical lenses to address my research question. In the following section, I justify this approach.

Coping with Multiple Theories

In the opening chapter of the *Second Handbook of Research on Mathematics Teaching and Learning*, Cobb (2007) addressed the complex issue of choosing a particular theoretical perspective given the plethora of choices available. Cobb (2007) presented two criteria for comparing and contrasting four fundamentally different theoretical orientations. The first criterion relates to how theoretical perspectives "orient and constrain the types of questions that are asked about the learning and teaching of mathematics, and thus, the nature of the phenomena that are investigated and the forms of knowledge produced" (Cobb, 2007, p. 7). In other words, this criterion concerns how the individual is conceptualized within a particular theoretical perspective (p. 15). Cobb's second criterion focused on the usefulness of a particular theory. Crucially, Cobb's (2007) notion of usefulness is situated within the context of viewing mathematics education as a design science, "the collective mission of which involves developing, testing, and revising conjectured designs for supporting envisioned learning processes" (p. 7). In discussing different theoretical perspectives, Cobb (2007) identified limitations of each approach, acknowledging that it is "unreasonable to expect that any one of these perspectives is ready-made for the collective enterprise of developing, testing, and revising designs" (p. 29).

In this research study, I incorporate multiple theoretical lenses by supplementing social learning theory with radical constructivist epistemology. In the following sections, I justify my approach by examining (1) the focus of this research study (related to Cobb's (2007) first criterion) and (2) the MIP Team's interpretation of highly effective mathematics instruction (related to Cobb's (2007) second criterion).

The individual component. The first criterion Cobb (2007) described corresponds to how the individual is characterized. Critically, the purpose of this research study is to explore how the participant *experienced* their involvement in the MIP community of practice.

While not attending to the cognitive or affective mechanisms influencing an individual's identity formation and transformation, Wenger (1998) recognized the individuality of a participant in the community: "More generally, each participant in a community of practice finds a unique place and gains a unique identity, which is both further integrated and further defined in the course of engagement in practice" (pp. 75-76). Moreover, Wenger (1998) also discussed an ownership prescribed to the members of a community participating in a joint enterprise. The enterprise is "defined by the participants in the very process of pursuing it" (p. 77). He continued by saying that it is their "negotiated response to their situation and thus belongs to them in a profound sense, in spite of all the forces and influences that are beyond their control" (p. 77).

Other scholars have advocated for the inclusion of the individual component when conducting social research. Clarifying remarks from Bogdanov, von Glasersfeld makes the following comments:

> In other words, no analysis of social phenomena can be successful if it does not fully take into account that the mind that constructs viable concepts and schemes is under all circumstances an individual mind. Consequently, also 'others' and 'society' are concepts constructed by individuals on the basis of their own subjective experience. (von Glasersfeld, 1995, p. 121)

These remarks highlight the importance of investigating how an individual constructs meaning, even when exploring social phenomena. This is because an individual's interpretation of these experiences is a property and a consequence of the meanings they construct.

Additionally, consider the remarks from Lutovac and Kaasila (2018), who highlighted the importance of acknowledging the psychological component in addition to the socio-cultural perspective. Based on their analysis of 40 articles focusing on teacher identity in mathematics education, they observed that few studies sought to incorporate the psychological perspective, noting that most were socio-cultural. Identity research, they argued, could be enhanced by considering multiple lenses:

> Although an individual's identity is greatly shaped by the social contexts in which he or she evolves, we believe that by neglecting the individual, i.e. [sic] how one thinks and feels and who one is, is at odds with the core concept of identity itself.

My perspective aligns with Lutovac and Kaasila (2018) and von Glasersfeld (1995). Wenger's (1998) particular version of social learning theory offered a perspective on the mechanisms of individual learning from and through their engagement in communal practices more holistically. An examination of how an individual experiences their involvement in the community, however, requires careful attention to how *they* construct meanings from *their* engagement, which is influenced by their goal structures and belief systems. In using multiple frameworks in this way, these theories work cohesively. That is, by exploring an individual's constructed conceptions and the extent to which they are compatible with the goals of the MIP, I am positioned to offer insights and clarity regarding the mechanisms of the MIP (related to social learning theory) designed to engineer these transformations. The second reason for applying constructivist theory to address my research focus relates to how the MIP Team conceptualizes *competent* instruction.

Competence defined by the MIP. As individuals negotiate their identity while engaging in the practices of the community, a notion of competence emerges and knowing becomes defined. Through an individual's interactions with members (participation) and artifacts (reification) of the community, they become cognizant of the community's culture and subsequently negotiate their identity relative to (their construction of) the knowledge and skillset that characterizes productive participation within the community of practice. It is important to note a difference between the community of mathematics faculty being cultivated by the MIP and the communities of practice described by Wenger (1998). Critically, Wenger described a community of practice as an interplay between agency and structure (O'Donoghue, 2011). In this interplay, while participating individuals have agency to negotiate competence, there is an intersubjective notion of competence *already established* in the existing structure of the community of practice:

> The space of knowledgeability has already been colonized and territory has already been claimed, and so if I want to enter community where there is a strong sense of competence that is already established, I may contest that sense of competence. But even to have the

legitimacy to contest it, I'm going to have to acknowledge the history that is embodied in that competence. (O'Donoghue, 2011)

In contrast, instructors participating in the MIP *are* establishing the community of practice through their involvement in the Initiation Workshops and engagement on the CoRDs. While there is no "history" of an established community of practice that confronts new MIP participants attending workshops or working on CoRDs (at least initially), the MIP Team offers guiding expertise targeted towards a focus on conducting a conceptual analysis in the service of operationalizing the three elements of inquiry to design instructional resources for entry-level mathematics courses. At each of the five Initiation Workshops, the MIP Team facilitated the discussions about the definitions of active learning, meaningful applications, and academic success skills by highlighting key phrases for each construct. Additionally, the design of these workshops enabled participants to discuss these features among partners and as an entire group. Similarly, one member of the project team gave a presentation on conceptual analysis during each workshop. Within this context, one way that the MIP Team might conceptualize competent participation is by an individual's capacity to operationalize these three elements of inquiry in their development of a curricular resource.

While seemingly simplistic, operationalizing active learning, meaningful applications, and academic success skills is nontrivial. For instance, students who are working together in groups may not be actively learning; a teacher designing a task with a real-world application may not be incorporating meaningful applications; an instructor helping students to develop effective study habits may not be supporting students' academic success skills[19].

[19] See Chapter 1 for a thorough discussion of these three components of mathematical inquiry, their mutual influence, and an example of how they might be operationalized in curriculum design.

Competent participation in the MIP Community of Practice is defined in terms of participants' cognitive development, and my research question reflects this interpretation:

> *Research Question:* What is the trajectory of one MIP participant from his engagement in the MIP Community of Practice? What are his interpretations of three elements of mathematical inquiry and conceptual analysis and how do these conceptions reveal features of his identity as an instructor?

The purpose of exploring Robert's conceptions of these three elements of mathematical inquiry and conceptual analysis is to reveal features of his *distal goals*, which reflect his desired identity as an instructor (Middleton et al., 2015).

In this recent discussion, I justified the appropriateness of exploring these sub-research questions, which focus on an *individual's conceptions*, within the context of a professional development initiative based on *social learning theory* designed to foster transformation in participants' identities as mathematics instructors. First, I argued that a central purpose of this study is to explore how an individual *experiences* their participation in the MIP community of practice and cited recent research advocating for the incorporation of the psychological domain when conducting identity research. Moreover, I noted that while Wenger (1998) focused on individual's membership in a *social* community of practice, he acknowledged the uniqueness of an individual's identity negotiated through their engagement. Second, I explained how I conceptualize competent participation according to participants' capacity to operationalize the three elements of inquiry and conceptual analysis, both of which are grounded in a cognitivist tradition. Hence, this discussion motivates my approach to supplement social learning theory with constructivist epistemology.

Before continuing, I make an important remark. From Wenger's (1998) perspective, competence is described as "normative" and viewed as being a shared characteristic of the community. From a constructivist perspective, however, competence is not described as a property of the community but as a construction made by each individual as they negotiate their membership in the community. This latter conception will be my interpretation of "normative" when this term is used henceforth.

This distinction is relevant to von Glasersfeld's (1995) discussion of intersubjective viability. From our own experience, we are able to form models for how another person might react or respond in a particular situation and, if found to be reliable, construct knowledge of a second-order. This higher form of knowledge enables us to construct a more viable experiential reality:

> It helps to create that intersubjective level on which one is led to believe that concepts, schemes of action, goals, and ultimately feelings and emotions are shared by others and, therefore, more real than anything experienced only by oneself. It is the level on which one feels justified in speaking of 'confirmed facts', of 'society', 'social interaction', and 'common knowledge'. (von Glasersfeld, 1995, p. 120)

Within this context, competence is considered "normative" to the extent that it is interpreted intersubjectively across participants.

In the following sections, I first briefly introduce central ideas within radical constructivism. Then, I leverage constructivist epistemology to discuss the process by which an MIP participant's identity as an instructor might evolve through their participation in the MIP community of practice.

Constructivist Epistemology

I address the first two research questions above through the lens of radical constructivism, which was developed from Piaget's genetic epistemology. Radical constructivism is a theory of knowing in which knowledge is constructed in the mind of the individual as opposed to being passively received from an external environment. While acknowledging the difficulty in interpreting Piaget's works, von Glasersfeld (1995), who spent six or seven years focusing on these writings, also expressed discontent with how academics interpreted Piaget's work. A productive interpretation of Piaget's writings, von Glasersfeld (1995) contended, is dependent on Piaget's claim that the nature of reality is in "continual construction instead of consisting of an accumulation of ready-made structures" (Piaget, 1970b, pp. 57-58, as cited in von Glasersfeld, 1995, p. 57). von Glasersfeld (1995) later articulated rationale for such a "radical" position: "The space and time in which we move, measure and, above all, in which we map our movements and operations, are our own construction, and no explanation that relies on them can transcend our experiential world" (p. 74). Echoing thoughts of early thinkers, von Glasersfeld (1995) phrased it another way:

> These thinkers saw with admirable clarity that, in order to judge the goodness of a representation that is supposed to depict something else, one would have to compare it to what it is supposed to represent. In the case of 'knowledge' this would be impossible, because we have no access to the 'real' world except through experience and yet another act of knowing and this, by definition, would simply yield another representation. (p. 93)

From this perspective, Piaget's intentional use of the term re-presentation (with a hyphen) instead of representation can be properly interpreted. According to Piaget, our ability to produce images does not reflect an ontological reality, and hence, should not be considered a *representation* of a real world. Instead, re-presentation "is intended as a mental act that brings a prior experience to an individual's consciousness" (von Glasersfeld, p. 95)

Constructivists contend that knowledge is constructed in the mind of the individual, and von Glasersfeld (1995) succinctly summarized Piaget's description of how this process might be occasioned:

> cognitive change and learning in a specific direction take place when a scheme, instead of producing the expected result, leads to perturbation, and perturbation, in turn, to an accommodation that maintains or re-establishes equilibrium. Learning and the knowledge it creates, thus, are explicitly instrumental. . . . On the level of reflective abstraction . . . operative schemes are instrumental in helping organisms achieve a relatively coherent conceptual network of structures. (p. 68)

To unpack von Glasersfeld's statement, I examine some of the constructs he references in more detail, beginning with a discussion of *assimilation* and *accommodation*.

Assimilation and accommodation. Consider the common phrases, "You don't know, what you don't know" or "You can't see what you can't see." A phrase to describe assimilation might convey the following message: "You can't perceive what you can't [because of your existing cognitive structures] perceive" (or in the opposite sense, "You can only perceive what you *can* perceive"). von Glasersfeld (1995) defined assimilation as "treating new material *as an instance of something known*" and relates this process to a mechanical card sorting machine (pp. 62-63). The machine sorts cards based on some specified criteria from the model card, and if a card "meets" these criteria (independent of whether it has other differences), then it is sorted or otherwise disregarded. Phrased another way, cognitive assimilation represents the process in which “a subject incorporates experiences into existing cognitive structures, and thus consists of the meanings the subject holds” (Tallman & O’Bryan, 2022) and occurs “when a cognizing organism fits an experience into a conceptual structure it already has” (von Glasersfeld, 1995, p. 62).

It is important to clarify that Piaget’s conceptualization of the process of assimilation differs from a biological interpretation of the term. In biology, assimilation refers to the process in which our body’s chemical structure adapts to the *external sources* brought in from the environment. Contrastingly, in cognitive assimilation, our perceptions adapt to our *existing cognitive structures*. von Glasersfeld (1995) described this distinction in this way: “perception modifies what is perceived in order to fit it into the organism's conceptual structures, whereas in the general biological sense, natural selection modifies the structure of organisms so that they fit within the constraints inherent in their environment” (p. 63). Another difference between these two types of assimilation relates to the point of focus. In cognitive assimilation, the emphasis is on the perceptions and interpretations of the organism as opposed to a transferal of information from an external source (von Glasersfeld, 1995).

Properly understanding accommodation first requires a discussion of action schemes. Action schemes consist of three phases: a recognition, a response (through an activity), and a result (beneficial or expected). During the recognition phase, criteria are “met” (assimilated into existing structures) and a response is initiated. Piaget’s articulation of action schemes differs from a stimulus-response perspective articulated by behaviorists (von Glasersfeld, 1995, p. 73). In contrast to random or reflexive reactions, action schemes are “explicitly goal-directed” and are likened to “feedback loops” (p. 73). For instance, suppose an individual’s response produces an “unexpected” outcome resulting in some perturbation. If the individual is able to retrieve information regarding this situation, then accommodation, characterized as the “modification of an individual’s cognitive schemes to enable their assimilation of novel experiences” (Tallman & O’Bryan, 2022) might occur through the creation of a new scheme (e.g., if the result was desirable) or new constraints might be added to the individual’s cognitive structures affecting future responses (e.g., if the outcome was undesirable or unexpected) (ibid.). In other words, perturbations are removed, and equilibration occurs. The nature of an individual’s accommodation is affected by many factors, including an individual’s goal structures.

Before discussing different types of abstraction (the mechanisms of accommodation, and thus equilibration), I provide some clarification. Up to this point, my presentation of Piaget’s description of assimilation and accommodation has been locally focused on an individual’s cognition. I would be remiss to ignore Piaget’s remarks regarding social interactions and equilibration (the reduction of perturbations). Cognitive development is a process of eliminating perturbations through a process called expanding equilibration (von Glasersfeld, p. 67). Elaborating on the concept of equilibration, von Glasersfeld (1995) explained:

> There is a further aspect of equilibration which, although not explicitly stated, is implicit in Piaget's repeated observation that the most frequent occasions for accommodation are provided by interactions with others. Insofar as these accommodations eliminate perturbations, they generate equilibrium not only among the conceptual structures of the individual, but also in the domain of *social interaction*. (p. 67, my emphasis)

Hence, despite his emphasis on knowledge being constructed in an individual's mind, Piaget did not naively ignore the influence of social interactions in facilitating accommodations.

Abstraction. Piaget articulated abstraction using several different French words conveying different meanings. I briefly discuss two—*empirical abstraction* and *pseudo-empirical abstraction*—and then center my discussion around the latter two forms of abstraction—*reflecting* and *reflected*. Empirical abstraction, which can be conceptualized as when a student "constructs knowledge that originates in their perception of exogenic objects (objects other than the self), which need not be physical," relies on students' abstraction characteristics "not from the object, but from the subject's *perception* of the object" (p. 6). In pseudo-empirical abstraction, conceptualized as "a property of the object that is produced by and, crucially, *represents* the action of the subject," the subject's abstraction results from their engagement with a particular object (ibid., p. 9).

Reflective abstraction can be described as *reflecting* abstraction, referring to a projection and subsequent coordination of the individual's activities which may or may not include their awareness (von Glasersfeld, 1995). Reflecting on these reflecting abstractions is characterized as *reflected* abstraction. I clarify the nature of these abstractions by discussing interpretations from three researchers. Tallman (2021) discussed it this way:

> Reflected abstraction is a higher form of abstraction that involves operating on the internalized actions that result from prior reflecting abstractions, which results in a coherence of actions and operations accompanied by conscious awareness. It is the act of deliberately operating on the coordinated actions that result from prior reflecting abstractions that engenders such cognizance. (pp. 15-16)

An affordance of becoming cognizant of one's own mathematical schemes is that it positions an instructor to enact their content knowledge purposefully (e.g., developing a hypothetical learning trajectory) (Tallman, 2021).

Second, Cohen (1986) stated the importance of reflective abstraction in Piaget's theory, describing it as the "linking mechanism in equilibration that moves the individual from one level to the next" and the "mechanism that constructs novelty" (p. 2). He offered an illustration for how one might interpret the reflecting and reflected abstraction:

> The first is a physical sense of reflecting or projecting (like a mirror) to a higher level what is known on a lower level. From toddling, for example, to thinking about toddling is a transposition to a higher level of construction, providing a wider, greater view of one's actions. The second aspect, reflection, is a reorganization or reconstruction (more a conscious aspect) at that higher level of what has been projected, enriched with new elements. The toddling baby has enriched his action of walking to representing to himself the room in which he toddles and where he is at each point in the room. . . . Through reflective abstraction, his actions are both transposed and enlarged. (Cohen, 1986, p. 2)

Cohen's (1986) description presented different levels of sophistication and contrasted the two levels of abstraction by highlighting the different mechanisms characterizing reflecting abstraction (i.e., projection) and reflected abstraction (i.e., reflection).

Thirdly, Simon et al. (2004) offered a physical depiction of these mechanisms. As we engage in an activity, there is an effect which is interpreted as either a positive or a negative outcome. A single activity and its result can be conceptualized as being stored in a jar, which represents our mental records, and the jar is mentally labeled as positive or negative. Simon et al. (2004) used this illustration to articulate Piaget's first two phases of reflective abstraction:

> In the first phase of Piaget' s reflective abstraction, the projection phase, jars are sorted according to their labels (i.e., learners mentally—though not necessarily consciously—compare/sort records based on the results). In the second phase, the reflection phase, the contents of the jars that have been grouped together are compared and patterns observed. Thus, within each subset of the records of experience (positive versus negative results), the learners' mental comparison of the records allows for recognition of patterns, that is, abstraction of the relationship between activity and effect . . . An abstracted activity-effect relationship is the first stage in the development of a new conception. (Simon et al., 2004, p. 319)

In this third interpretation, Simon et. al. (2004) offered a physical description of these two types of abstraction. In particular, their perspective highlighted the activity-effect relationship of action schemes discussed previously.

von Glasersfeld (1995) discussed an important consequence of engaging in reflective abstraction "that has been eminently fertile in the conceptual organization of our experiential world" (p. 69). He explained it this way:

> This simple form of the principle of induction, namely 'to retain what has functioned successfully in the past', can be abstracted and turned upon itself: because the inductive procedure has been a successful one, it may be advantageous to generate situations in which it could be employed. Consequently, a thinking subject that has reached this point

by reflective abstraction … can imaginatively create material and generate reflective abstractions from it that may become useful in some future situation. (p. 70)

His comments suggest that the "successful" retainment of patterns developed from empirical abstraction can become a requisite foundation for an individual to engage in reflective abstraction in productive ways.

An instructor's capacity to support students' productive understanding of an idea depends on the extent to which they have *reflected* on their own thinking and the mathematical knowledge they possess. Instructors who reflect on their mathematical schemes and enhance their mathematical knowledge, enabling them to effectively conduct a conceptual analysis to guide their instruction and curriculum design, are better positioned to effectively operationalize the three elements of inquiry; they are equipped to engage their students in active learning while incorporating meaningful applications by supporting students to make the abstractions necessary to identify and construct mathematical relationships.

The necessity of engaging in this reflective process, a central component of conducting a conceptual analysis, emerges from the following question: How can an instructor support students' productive conceptions about a particular mathematical topic if they have not first reflected on the nature of their own conceptions? According to Thompson (2008), conceptual analysis refers to a description of the mental actions involved in conceptualizing about an idea in a particular way. Reflected abstraction is required when an instructor conducts a conceptual analysis. Without this reflection, an instructor is ill-equipped to purposefully support students' conceptual activity.

Even after reflecting on their own mathematical schemes, an instructor may be constrained by the mathematical knowledge they possess. For example, an instructor whose conception of constant rate of change is limited to "rise over run" will be constrained in their capacity to conduct a conceptual analysis, unless they enhance or expand their mathematical knowledge to encompass more robust ways of conceptualizing this idea.

In sum, I have discussed the process by which an MIP participant's identity as an instructor might evolve from their involvement in MIP activities. First, it is critical that they recognize (i.e., through perturbation) differences between their conception of supporting students' learning through inquiry and the MIP definitions of active learning, meaningful applications, and academic success skills. Promoting this awareness may position participants to be *curious* about the priorities and commitments informing the MIP definition and why they might differ from their own. Hence, it will be important for the MIP Team to consider the mechanisms required to achieve this goal by strategically introducing reified artifacts and generating discussions around these resources to subtly perturb participants' conceptions in service of engendering this awareness to promote curiosity.

Once perturbed, accommodations can occur resulting in the creation of a new scheme or adaption of existing schemes. Once these accommodations occur, participants may begin to conceive of the mechanisms required (i.e., conducting a conceptual analysis) to operationalize the three elements of inquiry. Having this awareness, they will need guidance and support from the MIP Correspondent as they work on their CoRD and engage in conducting a conceptual analysis.

Providing an opportunity to engage in these practices may not be sufficient to influence participants' priorities or values. Participants' accommodations are naturally influenced by their goal structures and their incentives to reconceptualize their practice (or not), and they might not recognize the affordances of conducting a conceptual analysis for their teaching, especially considering the difficulty of the task. Hence, it will also be important for the MIP Team to discuss implications for supporting students' specific understandings of ideas in the same course or future ones. Assuming their goal structures align with the goals of the MIP Team, the extent to which they successfully operationalize these three elements of inquiry and conduct a conceptual analysis depends on, among other factors, (1) their capacity to engage in reflected abstraction and (2) their mathematical knowledge.

Methodology

Introduction

In this study, I am seeking to better understand the current and developing pedagogical practices of a mathematics instructor participating in the MIP. Specifically, the goals of this exploratory case study are to investigate how Robert, a pseudonym, conceptualized the three elements of inquiry, interpreted conceptual analysis, and operationalized these features in the design of his lesson plans and instructional practices. To this end, I address the following research question:

> *Research Question:* What is the trajectory of one MIP participant from his engagement in the MIP Community of Practice? What are his interpretations of three elements of mathematical inquiry and conceptual analysis and how do these conceptions reveal features of his identity as an instructor?

The advantages of conducting quantitative and qualitative research have been well documented, but the latter method has faced more scrutiny and often been deemed as less "scientific." Despite this shortcoming, researchers conducting qualitative research are positioned to uncover findings obscured by analysis of quantitative data alone:

> If you want to know how much people weigh, use a scale. . . . If you want to know what their weight *means* to them, how it affects them, how they think about it, and what they do about it, you need to ask them questions, find out about their experiences, and hear their stories. (Patton, 2002, p. 13).

Within qualitative research, the nature of my research questions (associated with contemporary phenomena, requiring no control of events, and addressing complex issues) suggest that a case study is an appropriate method (Yin, 2009). After discussing different uses of case studies in a variety of fields, Yin (2009) highlighted an underlying theme: "In all these situations, the distinctive need for case studies arises out of the desire to understand complex social phenomena" (p. 4).

Selecting the Case

The rationale for selecting cases in qualitative inquiries is fundamentally different from quantitative approaches. While qualitative studies involve the purposeful selection of cases, recruiting a sample of research participants representative of some population for a quantitative study relies on randomness. Patton (2002) phrased it this way: "What would be 'bias' in statistical sampling, and therefore a weakness, becomes intended focus in qualitative sampling and therefore a strength. The logic and power of purposeful sampling lie in selecting *information-rich cases* for study in depth" (p. 230). Now, I detail the selection process.

There were fourteen potential candidates (the individuals who had attended at least one initiation workshop and were currently participating on at least one CoRD) for the case study. From this list of fourteen, eight were eliminated. These eight instructors were from Oklahoma State University (Stillwater campus), had attended only one summer workshop (not the academic success skills workshop), or they (only one participant) attended all three summer workshops. Among the six faculty (five females and one male) remaining, four had attended multiple initiation workshops (one of whom did not attend the academic success skills workshop) and two had attended only the academic success skills workshop.

To the qualifications of the remaining six candidates, I evaluated them according to three questions: (1) Is this instructor interested in learning about an opportunity to be a participant in a research study during the spring 2021 semester? (2) Is this instructor teaching at least one section of an entry-level mathematics course next semester? (3) To what extent does this instructor have control choosing or designing (i) the activities they use in the classroom from their lesson plans and (ii) the assessment materials for the course (quizzes, homework, and exams)?

To address these three questions, I created a survey and administered it to these six instructors in November 2020. While all who answered the survey expressed interest in learning more about the opportunity to participate in the study, I eliminated two instructors. One did not complete the form, and the other will not be teaching one of the five entry-level mathematics courses (targeted by the MIP) next semester. I organized information about these four remaining candidates in a Microsoft Excel file and began comparing and contrasting them from their responses to the survey and other background information. In addition, I considered the following criteria: (1) to what extent are they motivated to participate (and relatedly, receptive to change), (2) what is the regularity of their meetings and status of their CoRD's progress, (3) how are their classes being taught (virtually, in-person, or a mix between the two), and (4) what is their time availability to be a case study participant?

First, it was important that the case study participant had engaged in the MIP and demonstrate an interest in participating in the project (I discuss this later) as well as display a receptiveness to thinking critically about their instructional practices. I sent the four remaining candidates a brief survey to address the next two criteria related to their CoRD and medium of instruction. Then, I sent participants a recruitment script and an informed consent form (attached in the appendix) to determine their willingness to participate in the study if selected. Finally, I selected Robert.

Before making the final decision, I talked with a member of the MIP Team who had been a previous colleague with Robert. From our discussion, the MIP member did not indicate serious problems with proceeding and offered other reasons why Robert might be a good candidate: he is honest, and he might be a useful "broker" since he does not do pure mathematics research.

Trustworthiness

Conducting a case study is a complex task and requires measures to be taken to enhance the trustworthiness of the research. I discuss three criteria—construct validity, internal validity, and external validity—and specific ways I enhance these measures in my study. I enhanced the construct validity of this study by collecting it from multiple sources: interviews and artifact assignments. Studies using a single data-collection methods "are more vulnerable to errors linked to that particular method. . . than studies that use multiple methods in which different types of data provide cross-data validity checks" (Patton, 2002, p. 248).

I enhanced the internal validity by carefully analyzing data and presenting the results to fairly depict Robert's conceptions of these three elements of inquiry and conceptual analysis. While certain themes were emphasized more in the writing, my presentation of the results encompasses a variety of different characterizations for Robert's image of these components.

To analyze the data, I rewatched each of the interviews and generated data bits as shortened or abbreviated excerpts from Roberts' remarks that captured the essence of his point at a particular moment. From these initial data bits, I carefully abstracted these ideas into larger codes and categories. Sometimes a single data bit encaptured a longer excerpt depending on the flow of the conversation, and hence, was divided into multiple codes or shortened data bits. I combined and related these codes and categories into larger clusters, often through several iterations, until I settled on my final themes. I performed this iterative abstraction in Excel, enabling me to visualize the associations of different clusters within this abstraction process.

I generated these data bits, codes, and subcategories in two different ways. Sometimes, I generated them via open coding disassociated from the context or question that I asked, while other times I associated his remarks with my particular question. Prior to each interview, I prepared intentional questions to ask Robert to test previous hypotheses or generate new insights. Associating his responses with a question allowed me to monitor the trajectory of the sequence of interviews connected with a particular concept which might be otherwise lost in open coding by being divorced of context. Finally, after writing an initial draft of the results, I re-examined the codes and categories from the eleven interviews to assess the extent to which my results provided a fair characterization of Robert's conceptions from the data.

An important distinction between my approach described above and grounded theory is that grounded theorists generate categories by examining different properties and dimensions such as causal conditions, context, etc. (Corbin & Strauss, 1990). An affordance of my approach is the clarity it offered for addressing my research questions. For instance, one of my codes from interview 10 was *Modified definition of AL* (active learning). This code was subsumed under the subcategory, *AL definition,* and eventually under a larger theme encompassing definitions. These simplified abstractions enabled me to address my research question with clarity.

The third criterion, external validity, tests the extent to which the study is generalizable to other contexts, an aspect of case studies that has been problematic:

> The external validity problem has been a major barrier in doing case studies. Critics typically state that single cases offer a poor basis for generalizing. However, such critics are implicitly contrasting the situation to survey research, in which a sample is intended to generalize to a larger universe. *This analogy to samples and universes is incorrect when dealing with case studies*. (Yin, 2009, p. 43)

Yin (2009) highlighted a fundamental distinction between two types of generalization: "Survey research relies on *statistical* generalization, whereas case studies (as with experiments) rely on *analytic* generalization. In analytical generalization, the investigator is striving to generalize a particular set of results to some broad theory…" (p. 43). I enhanced the external validity of this study through *analytic* generalization by leveraging a coherent theoretical framework. In particular, I utilized aspects of social learning theory and constructivist epistemology to address my research questions, affording opportunities for my results to generalize to theoretical propositions, not populations.

Ethical Considerations

I sent my case study participant, Robert, a recruitment script, and he signed an informed consent form in February 2021. The form included a description of the purpose of the study, an explanation of the requirements associated with participating in the case study, an option to quit participating at any time during the study, and other pertinent information (see the appendix).

For his participation, Robert received financial compensation at $50/hour. Patton (2002) articulated the importance of showing reciprocity to research subjects: "Participants in research provide us with something of great value, their stories and their perspectives on their world. We show that we value what they give us by offering something in exchange" (Patton, 2002, p. 415).

Data Collection

In the following sections, I discuss different forms of data collection.

Assignments. During this case study, Robert completed four assignments including one task analysis, one video lecture assignment, one reading assignment, and one written reflection. Robert analyzed tasks from his finance lectures, evaluated two short Youtube videos on the first fundamental theorem of calculus (Calcvids, 2019, December 11) and definite integrals (Calcvids, 2020, March 9), examined excerpts from Thompson (2008) and Tallman and Uscanga (2020), and wrote one reflection after reading an excerpt from Tallman & Uscanga (2020). The purposes of assigning Robert these assignments were to provide an alternative source of data, reduce the data load from interviews (in the case of the written reflection), and allow him time to contemplate how to express a particular thought prior to an interview.

Interviews. I conducted 11 interviews with Robert (typically lasting around one hour) and followed a semi-structured interview guide approach combining the interview guide and the standardized open-ended interview methods (Patton, 2002). While maintaining the structure of the latter approach by having a list of predetermined questions prior to these interviews, I also remained flexible in these discussions, incorporating aspects of the former method:

> The interview guide provides topics or subject areas within which the interviewer is free to explore, probe, and ask questions that will elucidate and illuminate that particular subject. Thus, the interviewer remains free to build a conversation within a particular subject area, to word questions spontaneously, and to establish a conversational style but with the focus on a particular subject that has been predetermined. (Patton, 2002, p. 343)

Table 9 illustrates how I addressed my research question from these two data collection methods.

Table 9

Data Collection Methods Associated with the Research Questions

	Research Question A			Research Question B
	Active Learning	Meaningful Applications	Academic Success Skills	Conceptual Analysis
Interviews	1, 2, 6, 10	1, 3, 7, 9	1, 4, 8	5, 11
Assignments	Task analysis	Task analysis	Reading excerpt Written reflection	Video evaluation

Results

In the following four sections, I discuss Robert's image of active learning, meaningful applications, academic success skills, and conceptual analysis. I begin each section by presenting his definition of each of these constructs followed by our discussions from various interviews[20].

Active Learning

[20] The excerpts I provide from our interviews reflect Robert's speech. I use a dash to illustrate Robert's repetition of the same word, difficult to understand utterances, changes in thought, or the starting syllables of a new word.

I conducted four interviews with Robert during which I asked him questions specifically related to active learning (Interviews 1, 2, 6, and 10). At the beginning of the second interview, Robert constructed the following definition of active learning:

> *Students are engaging in active learning when they are asked to engage with a problem themselves (as opposed to passively observing an instructor solve the problem).*

Robert's initial definition of active learning prioritizes students' activity and engagement as a contrast to students' passive observation of another's work. Moreover, Robert does not specify in his definition the nature of students' engagement necessary for students to be engaging in active learning—engagement itself (which I later discover involves students' critical thinking and struggling) seems to be sufficient.

Student engagement and many "forms." According to Robert, active learning is fundamentally about student *engagement* and *participation*. In contrast, students are passively learning when they "sit back and observe, uh, what other people doing" and they are "not really, uh, actively trying to solve the problem." In discussing a finance problem from one of his courses, Robert indicated that the "students who are answering the question, uh, are engaged in active learning." Later in the interview, he described how he could facilitate a classroom that would be "full on active" by breaking students into groups or using Pear Deck[21]. His comments seemed to suggest that these extent to which an activity supports students' engagement in active learning depends, at least to some extent, on how it is facilitated by the instructor. I asked this question pointedly later in the interview:

[21] Pear Deck is an interactive resource that instructors can use to monitor students' responses to questions.

Josiah: I wanted you to evaluate the extent that these examples support students in actively learning, and I think you talked a lot about a lot of it is how the instructor approaches it.

Robert: Yes.

Josiah: Um, so, in a sense, any of these problems could support students in actively learning depending on how you teach it. Is that correct?

Robert: Yes.

Consistent with Robert's initial definition of active learning, his remarks suggest that the extent to which these examples offer an opportunity for students to engage in active learning relates principally to the pedagogical practices of the instructor. These comments indicate that his interpretation of students' engagement in active learning values students' activity more broadly. During Interview 9, I provided opportunities for Robert to examine the extent to which two instructional videos in calculus (designed to support students' construction of productive meanings about the fundamental theorem of calculus and the definite integrals) engaged students in active learning according to the MIP definition. Robert indicated that it depends on the presentation.

I think my answer is pretty much the same. It really depends on how they teach it. Right, so, if they were to teach it where there's lot, you know— where they pause and ask questions from the student, make, making the students answers ques— answer, uh, give some answers and move onto the next step, then I think that will be a, a certain level of active learning, whereas if it's just presented just, you know, purely lecture form then,

> like the video, like the vi— like the video offers, the, the video is in purely lecture form, uh, then it's not active learning.

Robert's comments suggest that he characterizes the extent to which students are engaging in active learning according to the instructional medium in which the content is presented. Students watching the video in purely lecture form, he indicated, are not actively learning. On the other hand, an instructor could be engaging students in active learning by providing opportunities for students to answer questions. Hence, the environmental conditions provided by the instructor (i.e., teaching in lecture format or asking questions) help characterize the extent to which students are engaging in active learning. While this video would support students in actively learning according to the MIP definition (i.e., it supports construction of productive meanings), Robert's focus on students' activity reveals his commitment to value the environmental conditions required to engage students in critical reflection.

Robert's approach to engage students in active learning. While Robert prefers the traditional lecture style and recognized disadvantages for engaging students in active learning (e.g., time constraints, in-class pressure), he strives to facilitate student engagement from his interactions with them in the classroom:

> So, the way I perform active learning is I, I pose questions, I pose interesting questions, intriguing questions, and, and they gi— they give me ans— students give me ans— their answers, and I, I go with their answers. So, they're actively participating, giving their input, and I'm not just brushing their eq— their answers aside, uh, if its wrong, but I mean, I, I pursue it.

Later in the interview, he discussed a specific example from a first day of class that could support students to engage in active learning. Imagine the earth is perfectly spherical, and tightly tie a rope around the equator of the earth. Next, cut the rope with scissors and add one yard to the rope to make an evenly distributed halo around the earth. He then asked students to determine the largest animal that could fit underneath the rope. Robert considered this activity to be an effective example of active learning, because he began the first day of class with an intriguing problem, solicited student input, gave time for students to work individually, allowed students to talk to their neighbor, and then discussed the problem collectively as an entire class. Robert indicated his satisfaction with this approach: “I mean for me, in my mind, that’s as, that’s as good as it gets (*little laugh*)”.

These comments suggest that Robert conceptualizes active learning according to students’ engagement and general activity based on the environmental conditions that he facilitates. After introducing an interesting problem to solicit student motivation, he provided environmental conditions to support student engagement according to different sizes (i.e., individual work, talking with their neighbor, and whole class discussions). From this example, Robert’s approach to engage students in active learning suggests that he values (1) students’ interactions with each other and (2) his interactions with the students.

Students’ mental activities. While Robert’s initial descriptions of active learning included references to students verbalizing their responses and engaging with the instructor, he indicated that students are not required to be talking to be actively learning; hence, students’ activity could either be physical or *mental*. During Interview 2, Robert described a situation in which an instructor solves a problem and then assigns students a similar problem with the same structure but different numbers. Robert's illustration represented an example of students engaging in a task and *not* actively learning: "they're just kind of copying down the process, right, just, just, just kind of, uh, just not really *thinking* [emphasis added] why they're doing what they're doing." Later in the same interview, Robert indicated that talking is not a requisite condition for students to be actively learning: "No, I don't, I don't require them to say something to actively learn. I think they— as long as they're, they're trying to solve the problem in their he— either in their head or on their, on their, on their own, if they're engaging with the problem, then they are actively learning." Several minutes later, I proposed a hypothetical scenario of Robert observing another classroom in which students appear to be engaged in active learning, but in fact are not. I asked him to describe the criteria that are required for students to be actively learning if their observable behavior alone is insufficient for making such a determination. After a few remarks, Robert stopped himself: "Oh, but (*pause*), it has to be observa— observable then?. . . I can't read their minds, right, so, I mean, so, I'm, you're, you're saying are their certain benchmarks, observable benchmarks to determine if a student is actively learning?"

These excerpts highlight that Robert's conception of active learning is not restricted to the observation of students' physical actions of talking or writing on pencil and paper but also includes characteristics of students' mental activity. In an earlier interview, he indicated that fostering students' active engagement helps establish confidence that students are indeed *learning*. Next, I discuss Robert's conception of the nature of the conceptual activity required for students to be engaged in active learning.

Nature of the mental activity required for active learning. During the second interview, I asked Robert to specify the conditions necessary for a student to go from "not actively learning” to “actively learning,” Robert stated that students “have to be asked to think critically about the problem; they have to make some decisions; they have to make some determinations. So, decision making would be, for me would be the— line.” Robert stated that these decisions, which represent students’ engagement in particular mental actions, need to be “strong” decisions involved with a struggle (e.g., the decision to use *u*-substitution instead of integration by parts). Robert’s comments align with features of the MIP definition of active learning involving students’ *selection* of mental actions. He later made the following conclusion: “As long as they’re actively, they’re, they’re trying hard, they’re stru— they’re struggling with it, if theirs— maybe, maybe the defining line should be there has to be a, a level of struggle with the problem.”

While Robert’s use of the word “struggling” does not specify the nature of the cognitive activity required for students to be actively learning, he offered clarity regarding the *productivity* of the struggle. Initially, Robert categorized students who were determined to solve a problem in a “brute force” approach and could not be deterred as *not* actively learning. Later, he revised his remarks:

> I’ll take that back. The brute force method is fine. Alright because, once they try their method, any method they have, they— try, even, uh,— you know they— give it a go, they struggle with it, and then, of course, after the fa— after they struggle with it, the instructor would provide them a more, more clever method, right, a— shortcut. And then they— then they’ll realize, hey, there’s something there.

He comments reveal (after a revision) that students’ struggle need not encompass a productive attempt to solve a problem, and students can learn from their struggles. This sentiment was echoed in a later interview: "I think struggling with these problems, uh, really adds to your, to a student's intellectual growth, right. Whether, whether they actually resolve it or not, they can, they can, they can learn a lot from their struggle." Hence, Robert's comments seem to suggest that his conception of active learning does not entail a critical evaluation of the mathematical meanings students are positioned to construct through their activity. I asked this question pointedly during Interview 10:

> Josiah: So, let's say now we have this scenario. Two students are working on an active learning task, and they're both trying ideas. So, they're not stuck. So, they're not only working, but they're not stuck on this active learning task. Is it possible for one to be engaged in active learning and the other not?
>
> Robert: No ("okay" by Josiah). I bel— I believe both students if they're— trying things, uh, you know, maybe one is going towards, going towards the right direction, and get, will be, get— towards the answer, get close to the answer, um, and the other one is just trying random things that has, maybe has no, no relation to what they're doing, but they're, they're actually trying things, uh, that they've tried in the past and maybe it gets nowhere. I, I believe they're both equiva— equivalently active learning.

His comments suggest that a sufficient condition for students to engage in active learning is their attempt to solve a problem by applying previous ideas. Importantly, attempting to solve the problem using techniques previously learned, even if these approaches are neither successful nor productive, will position the student to be more attentive to, and thus better retain, the method that led to the correct solution[22]. These comments reveal Robert's valuation of affective features

[22] For a specific example, see Robert's discussion of the quadratic formula towards the end of the following section.

of students' engagement (e.g., perseverance) that make them receptive to learning what he intends to teach them. By how the MIP defines active learning, students who try ideas but engage in random actions may not be engaging in the cognitive activity required for them to construct meanings that are equivalent to the structures of the concepts to be learned.

Robert's comments reveal a nuance into his interpretation of active learning and more specifically, his image of learning. Students can be learning by understanding the idea, or they can be learning by their efforts to apply previous techniques, even if their attempts are unsuccessful. As a consequence of this latter conception of learning, by engaging in this struggling and strong decision making (i.e., perseverance, tenacity, grit), students are better *positioned* to value and understand the idea that he demonstrates or conveys through his instruction since "struggling with these problems" adds "to a student's intellectual growth." I consider the first interpretation as *learning through success* and the latter interpretation as *learning through struggle and previous application*. I discuss the former interpretation in the implications section.

Discussion of the MIP definition. Recall the MIP definition of active learning:

> *Students engage in active learning when they work to resolve a problematic situation whose resolution requires them to select, perform, and evaluate actions whose structures are equivalent to the structures of the concepts to be learned.*

In the opening interview, I showed Robert the MIP definition of active learning, and he said that he "preferably agree[s]. I mean, that sou— that sounds about correct. I mean that, that sounds a more articulate way of saying what I said." In my final interview with Robert, I probed him to address this definition more specifically. After showing him the MIP definition of active learning, I asked Robert to compare and contrast the two definitions. The primary difference Robert noticed was the specificity of the MIP definition compared to his, which was broader. As I anticipated, Robert stated that this difference was minor. When asked if these differences were (1) not significant, (2) a little bit significant, (3) moderately significant, (4) significant, or (5) very significant, Robert ranked them according to (1) or (2):

> Our themes are very similar, and— I think that, they're— trying to say the same thing. I mean, the, I mean, the impor—, the important thing is distinguishing it, distinguishing, uh, active learning from passive learning, and I think they both do that.

These sentiments validated his previous remarks about his interpretation of active learning being similar to the MIP definition. In the same interview, I proposed a hypothetical scenario of two students are working on a task and asked Robert if it was possible that one student is actively learning and the other is not actively learning *according to the MIP definition*. Initially, he imagined that both students would be actively learning, but after examining specific components of the MIP definition, he became perturbed. After indicating that this scenario satisfied the criteria of being (1) a problematic situation in which students were (2) selecting, performing, and evaluating their (3) actions, he discussed the final part of the definition:

> The structures of their actions are equivalent to the structures of the concept to be learned. Uh, well, I mean, maybe, but I, I think that's, uh, that's kind of hard to say, because they don't know what the concepts to be learned is. Right, so, maybe they're trying different things and then, um, [*pause*]. Yeah, maybe, maybe, I don't like this aspect of the def— of the MIP definition, because, uh, even if they're— what they're

> trying to do, right, uh, is— does not amount to anything that works, right, uh, and maybe it's not equivalent to the concepts to be learned, uh— doesn't mean it's not valuable.

Robert then gave an example of students attempting to solve a quadratic equation before being introduced to the quadratic formula. While he indicated that a student applying similar techniques when solving a linear equation (isolating the variable) to solving a quadratic equation will be unsuccessful, the experience will be valuable to see why this technique does not work.

Taken without context, these remarks seem to suggest that he valued students' engagement in productive mathematical practices independent of the initial meanings that they were constructing from their engagement. Yet, his comments should be contextualized as a *reaction* to his dislike of the word "equivalent." His remarks and his example do reveal, however, that he views the concept to be learned similar to learning a new technique or method to solve a problem (i.e., the quadratic formula).

Meaningful Applications

I conducted four interviews with Robert during which I asked him questions specifically related to meaningful applications (Interviews 1, 3, 7, and 9). At the beginning of Interview 3, Robert constructed his definition of meaningful applications:

> *Applications are meaningfully incorporated in a mathematics class when problems are presented that piques student interest and highlights a key concept (or some key concepts) of the lesson.*

There are two fundamental components of Robert's definition—piquing student interest and highlighting a key concept of the lesson.

Piquing student interest. When Robert was shown the MIP definition for meaningful applications in Interview 1, he recognized that it was useful for instructors, but "from a student view, meaningful application would be something that, just a problem that they're— that engages them, in— intrigues them, has to be some kind of, spark some kind of interest." A meaningful application need not be real-world, could relate to them personally, be paradoxical, intuitively simple to understand, interesting to Robert, or an applied, real-world problem. In Interview 7, he described how the quadratic function is abstract, but it becomes more apparently useful to someone when applied to a velocity or acceleration problem.

During our initial interview, Robert provided an example of a meaningful application about a Harvard graduation ceremony. An individual is attending a graduation on behalf of his friend. He is tired and wants to take a nap, but he does not want to be asleep when his friend's name is called. Robert showed a video of the first fifteen seconds of the graduation video, provided a program, and then splits students in groups. Their goal is to predict the maximum time allowed for his nap so that he still wake up to see his friend. They strategize in groups, and then he facilitated a classroom discussion analogous to the *Price is Right* game. The winning team received bonus points on the first exam and is asked to present their strategy.

Robert's emphasis on piquing students' interest reveals features of his distal goals as an instructor to prioritize students' affective engagement. Similarly, recall from his interpretation of active learning that he valued students' willingness to struggle and persevere. Ultimately, students who are intrigued (e.g., the problem is paradoxical, applicable) are better positioned to engage with the content individually or interact with others more productively. His comments suggest that a value of engaging students in problems that represent a meaningful application is that they initiate students' mathematical activity by stimulating their interest. His initial remarks do not reveal that his image of meaningful applications consists only of features of students' affective (and not cognitive) engagement since our initial discussion was focused on the stimulating interest phrase in his constructed definition. Uncovering his image of more cognitive features associated with his interpretation of meaningful applications requires examining his meanings for the other part of his definition: highlight a key concept.

Highlight (more specifically) or demonstrate. A meaningful application should not only pique students' interest, but it should *also highlight a key concept (or some key concepts) of the lesson*. Robert stated that another interpretation for the word "highlights" is *demonstrates*. He said that demonstrates "shows the usefulness of these concepts. It demonstrates why— we're doing what we're doing." Consider the following example he offered illustrating how an instructor might highlight or demonstrate the usefulness of a concept:

> If I can come with an, a, with an application, a problem that forces you, really encourages you to use one over the other, that's a meaningful application, right. . . . If I try to teach you a shell method, right, usually a shell after the washer method, right, but if I give you a problem where I can, where students can just, as just as easily solve it using the washer method, then what's the point, right?

Later in the same interview, he discussed how an instructor who provides students with repetitive examples might bore them. After a generic example, Robert might provide another example that highlights *subtleties* of mathematical content. Examples of subtleties, which are related to limitations, include understanding why division by zero is undefined, or the hypothesis that the Fundamental Theorem of Calculus only applies to continuous functions. These comments reflect an attentiveness to foster students' critical reflection and the former example illustrates his efforts to support students' construction of productive meanings. Rather than ignoring why division by zero is undefined, he addressed and discussed this question directly with his students.

The nature of learning. In responding to a question about the knowledge base required for an instructor to incorporate a meaningful application, Robert indicated the importance of having a "bank" of problems. After prompting from me, Robert identified his criteria for selecting a problem from this bank: it highlights the key concept and, after that criterion is satisfied, it piques students' interest most profoundly. I pressed Robert to identify other criteria for selecting a problem other than demonstrating the advantages of using a particular technique. After gathering his thoughts for more than thirty seconds, Robert expressed the following:

> I don't, I don't know. I mean, I can't think of anything. I mean, so, we— covered two things: either highlights the advantageous, the usefulness of a certain method, cer— certain technique, or if there isn't a problem, just kind of, something that high— that highlights the usefulness of, uh, well, one is highlights the usefulness of that technique over any other technique, or just highlights the usefulness of that tech— that technique in general. Um, I can't think of any other reason I would choose one method over the other.

He then added that another criterion for selecting a problem might be its simplicity. These comments reveal that he prioritized problem-solving activities to determine when to apply a certain technique to solve a particular type of problem.

Now, consider a specific example from his Math in the Modern World course. When asked about how a simple and compound interest problem supports students' conceptual understanding, he stated that it does "quite a bit" since compound interest becomes "greatly superior." While attending to the nature of graphs, he did not discuss constant or non-constant rates of change more specifically. These omissions cannot confirm that he does not emphasize these ideas but perhaps suggest in this context that he may be primarily interested in supporting students' understanding of more broad, overarching takeaways of the central idea being discussed.

I recognized, however, that his teaching style could depend on the audience and aptitude of his students. When asked if his approach to support students' conceptual understanding depended on the content in Interview 9, he indicated that his general approach to motivate students and develop the idea remains consistent. The extent to which he provides more rigor in developing formulas, however, depends on his audience. I asked if there was anything besides consideration to deriving formulas that might differ across courses Robert teaches.

> Josiah: In calculus class you derive a formula. This class you don't. Anything else in the calculus class that might look different in terms of developing the concept of simple or compound interest for inst— for example?
>
> Robert: Umm (*pause*). Maybe not, I mean, I— can't think of anything that would be much more different.

While the examples would be similar, he indicated that he could talk about more advanced problems (e.g., related to compound interest, annuities). Robert's responses suggest that he prioritizes cultivating students' interest, highlighting the usefulness of a concept, and supporting their understanding of derivations. Importantly, Robert's comments are contextualized within his interpretation of meaningful applications and not learning more generally. Within this framework, however, they do reveal insights into his commitments regarding the features of learning he seeks to support by giving a meaningful application: supporting students' learning through giving them a meaningful application consists of conveying to them the usefulness of a technique, the advantageousness of applying different techniques in different contexts, and the understanding of an idea through derivations. These priorities reflect the practices of a curious mathematician who values productive ways to solve problems and understandings why an idea is true (i.e., derivations).

In Interview 7, I provided Robert with a concrete example of a context (Riemann Sums) affording him the opportunity to identify the importance of supporting students' development of productive meanings they construct from their engagement in specific mathematical content facilitated by their reasoning about an applied context. Without a meaningful application, he conjectured, students may not be able to grasp the significance of a concept. He indicated that a meaningful application with a concept allows students to "grasp" or "appreciate" the process or technique or concept being studied. When I pressed him to discuss his meaning for significance, he clarified that it related to the "use— Answering the question why are we doing what we're doing? What, what, what's the point of all this?" Elaborating on this idea, he discussed the example of the shell versus washer techniques to highlight the utility of a computational method. After providing him another opportunity to discuss advantages to facilitate students' learning from their engagement in a meaningful application, he talked about the value of students analyzing their process while engaging in productive struggle problems. Students' engagement in generic productive struggle tasks is independent of the meanings that they are constructing, but their activity represents productive orientations of an ideal student and perhaps reveals features of Robert's priorities and commitments as a mathematics instructor.

A little later in the interview, I directed the conversation towards the *instructor's* role and provided Robert an opportunity to respond by offering a scenario of two students engaging in a task. Asking these questions provided Robert an opportunity to explain the importance of the context for supporting students' conceptual activity:

> Josiah: So, you have Student A who engages in this activity which incorporates a meaningful application, and Student B who engages in this activity related to Riemann sums which does not incorporate a meaningful application. What else does Student A have an opportunity to learn, um, different maybe from Student B? Or, from your perspective as an instructor, how does incorporating a meaningful application facilitate the learning process for Student A better than Student B?
>
> Robert: Okay, uhh, I mean, I, I think it's pretty much what we've discussed already.

Robert discussed the importance of highlighting the utility or subtlety of Riemann sums, offered an example of a productive struggle task (finding the area of a circle given that you know the area of polygons and you cannot use area formula), and discussed the importance of appreciating the process (i.e., not given the formula or process to solve a problem). His example of finding the area of the circle knowing the areas of any size polygon illustrated a focus on supporting students' conceptual understanding of the summation process for Riemann sums

Discussion of the MIP definition. Recall the MIP definition of meaningful applications:

> *Applications are meaningfully incorporated in a mathematics class to the extent that they support students in identifying mathematical relationships, making and justifying claims, and generalizing across contexts to extract common mathematical structure.*

In the final interview, I provided Robert an opportunity to compare his definition of meaningful applications with the MIP definition and identify similarities and differences. Robert identified two primary similarities: (1) both definitions are student-centric, and (2) both definitions convey to students a mathematical concept. Regarding the former, he stated that these definitions center around students "accepting" these meaningful applications (e.g., students could be engaged, happy to "receive this new, new knowledge," or be uninterested). When I asked him to describe what would be required for a student to accept a meaningful application, he discussed piquing students' interest and providing rigor when explaining how to solve the problem. As for differences, he indicated that the MIP definition "doesn't really mention hooking students" and is more specific. Relatedly, when I first showed Robert the MIP definition of meaningful applications during the end of the initial interview, he noticed the absence of student interest in its description.

While it is insightful to hear Robert's comments concerning the compatibility between the MIP definition and his own, these discussions are limited, because they are abstract (i.e., separated from his instructional practices). In addition to these questions, I asked Robert about the process entailed in operationalizing the MIP definition of meaningful applications and asked him to analyze this process to evaluate its compatibility with his conception of incorporating meaningful applications. His initial response was "No" followed shortly by "Yes and no." He discussed how being "on top his game" is difficult to be consistent but sufficient:

> Robert: Sometimes, if you're on top of your game, right, uh, then you— put a lot more thought into it, then yes. I mean, uh, I really, uh, I really think about, uh, these concepts. I think about the motivation. I walk, I say, this how I real— would really think about it. I

say, this, I start here, I start here, this what, this the, this is what I know. This is what I know the students should know, right. So, eve— all the students should know this much. I, I give them the hook, the motivation. I tell them, okay, this is why we're interested in doing what we're about to do. How do we get from, from where we are to what we want? Alright, and I, and I explain the motivation is why we want that. Alright, so they— I make that connection with them. They understand that connection. And based on where we're at, alright, how do we get there? If I— am on top of my game, I deliberately choose problems that walk students through this process well, okay. So, yes, uh, my teaching uh, my teaching lectures, uh, incorporates all of these.

Josiah: So, if you were to design tasks, um, to incorporate a meaningful application as defined by the MIP, you're saying that if I'm on top of my game, then the way I incorporate my meaningful applications aligns with the MIP definition. Is that correct?

Robert: Yes, that's correct.

These comments suggest that Robert believed to operationalize the MIP definition of meaningful applications when he is at the "top of his game." His definition of meaningful applications, however, suggests that he conceptualized meaningful applications primarily according to affective characteristics and highlighting the usefulness of an idea. These distal goals to support students' engagement and provide high-level interpretations of ideas is productive and some of his comments suggest an attention to support students' construction of meanings (e.g., explaining, rather than ignoring, why division by zero is undefined). Overall, however, his remarks suggest that he values supporting students' affective engagement.

Academic Success Skills

I conducted three interviews with Robert during which I asked him questions specifically related to academic success skills (Interviews 1, 4, and 8). At the beginning of Interview 4, Robert constructed his definition:

> *Academic Success Skills are behaviors/actions that help people/students succeed academically (i.e., in their studies/research). Examples include: detailed note taking, a sense of curiosity, the grit/determination to tackle/solve a problem—from several approaches if necessary, and to think critically.*

Robert's descriptions of behaviors and actions encompassed acting with confidence, grit, curiosity, and taking notes. While his image includes note taking skills, he did not take detailed notes as a student, because it hindered his capacity to pay attention during lectures. His definition prioritizes students engaging in critical thinking and exercising grit and tenacity to satisfy curiosity. According to Robert, thinking critically entails making "deliberate decisions" and operating intentionally when solving problems, as opposed to using a particular technique because peers used it, the book encouraged it, or it proved useful in another context. It encompasses an individual's capacity to act as a problem-solver to decide which technique to use, and critical thinking is a key facet of the academic success skills he values most:

> For me, the most important academic success skill is being engaged, and being, uh, curious, and being, and just, really getting, getting down and exploring the concept, right, having the grit, uh, and tenacity to, to, work on the problem. I— that to me, that, for me, hands down, that's the most important, that's the most important academ—, uh, academic success, success skill is being, having the tenacity to, to try, to try a problem even if you

> don't conquer it. Right, so, uh, so, pro— I mean, just offering students problems, uh, that encourages, uh, a certain tenacity is— good.

Robert seeks to encourage students' development of grit and tenacity by verbally encouraging them and illustrating practices of grit. He reminds students to continue trying even if their first attempt is unsuccessful, since little time has been spent, and he illustrates grit by pursuing students' suggestions in class (which may be unsuccessful) or maybe by illustrating his attempt to solve a problem in class without the aid of his notes.

Designing tasks to support the development of students' academic success skills. Robert credited the academic success skills workshop for introducing him to *productive struggle* problems or problems intentionally given that are beyond students' ability to solve. By engaging students in these tasks and encouraging them to reflect on their attempts to solve the problem, an instructor can support their students to learn from the difficulties they encounter. According to Robert, these types of problems are not accessible in textbooks which are comprised of simple, contrived exercises that can often be solved quickly. Additionally, developing problems that engage students in productive struggle often requires the instructor to have knowledge of and experience with applied contexts of mathematical ideas. The difficulty in creating these problems perhaps influenced his preference for a CoRD to create a "bank" of productive struggle problems.

While students might be able to enhance their determination from engaging in productive struggle tasks more generally, I asked him if an instructor's approach to teaching a specific mathematical topic can promote students' curiosity, tenacity, or grit? His response highlighted the ways an instructors' actions can hinder students' development of these affective qualities. He stated that making math “mechanical” also makes it become artificial and tedious: “Students don’t really know why they’re doing what they’re doing; they’re just doing, just because someone else tells them to do it.” He then offered an example from his class in which he probed students to explain why division by zero is undefined. This example illustrates students’ blind acceptance of mathematical ideas without knowing why it is true. Robert’s comments suggest that he values providing students with mathematical knowledge to reduce the tediousness of accepting and memorizing mathematical ideas by supporting students to understand the reasons for why mathematical ideas are true.

During an earlier interview, I asked Robert more specifically about his process for developing tasks that support students’ academic success skills. After selecting a context that he expected might interest students (e.g., money), Robert might reflect on how he could create a task that is obviously complex to students. Once students recognize this complexity, they may begin determining the information necessary to solve the problem, since Robert might intentionally omit some data and only provide it upon students’ requests. These types of problems could support students’ affective engagement because they are (1) applicable and (2) intriguing. After choosing an applicable context (i.e., money), an instructor could present an unorthodox problem with certain information omitted. Consequently, these problems may stimulate students’ curiosity to uncover the missing data required to solve the problem.

Readings. During our final interview on academic success skills, Robert and I discussed two groups of excerpts from two readings: Thompson’s (2008) article on conceptual analysis and an article by Tallman and Uscanga (2020) on students’ mathematics anxiety. I sent these excerpts prior to our interview and prompted Robert to identify important features of these sections to discuss in our interview. The purpose of providing these readings was to allow opportunities for him to (1) examine an example from Thompson (2008) highlighting productive conceptions of constant rate of change and (2) read excerpts from a mathematics anxiety paper that make explicit the importance of developing students' productive conceptions for supporting their academic success skills. I focus this discussion around Robert's interpretation of Thompson's (2008) paper.

Linear functions. Robert described Thompson's interpretation of average speed as representing a constant speed someone must travel to cover the same distance in the same amount of time. While Robert genuinely liked Thompson's (2008) definition, he disapproved of Thompson's critique of conceptualizing average speed as distance divided by time. He considered these two to be equivalent definitions (albeit different approaches) to understand the same idea, and he identified an affordance for describing average speed as distance over time: it is more intuitive for students to understand.

Failing to validate or encourage students' intuitive reasoning, Robert argued, could negatively impact students' motivation. A little later in the interview, I asked Robert if he would recommend one approach to teaching average speed the other. He discussed how the ways of understanding average speed Thompson (2008) contrasted have implications for other ideas, but he would probably start by describing it as distance divided by time. From Robert's perspective, discussing average speed as distance divided by time has implications for discussing derivatives and discussing it in Thompson's (2008) way has implications for discussing the average value of a function and ideas of instantaneous and average velocity in Calculus II.

Robert later discussed Thompson's (2008) section on understanding division as a proportional relationship involving a ratio of quantities. He identified Thompson's (2008) discussion as distinguishing between viewing this division as a unary operation instead of a binary one. Again, however, he questioned if there was "harm" in students questioning these as distinct and expressed concern in teaching this way:

> My only concern is, if I— really, really want them to really understand the difference, this— very strong subtlety here, it, it will be, it will take some effort, and, I, I'm afraid that that amount of effort might end up confusing and pushing away more students than it really enlightens (*pause*) is my biggest concern.

Aware of his concern, I asked him to describe the extent to which he conceptualized these understandings as having implications for future classes. He responded by saying that for advanced courses, this exercise would be beneficial practice for supporting students' capacity to dissect and understand the subtleties of definitions. Finally, when asked if he identified any conceptual advantages of viewing average speed as distance divided by time or how Thompson (2008) described it, he paused for about ten seconds and then remarked: "Um (*pause*), (softly whispers to himself, "Is there an advantage?") I don't know, I don't— I'm not sure if there's an advantage. I don't, I don't know."

Robert's comments in discussing Thompson (2008) reveal features of his goal structures related to students' academic success skills as an instructor. His remarks from this discussion suggest that he values students' motivation and prioritizes supporting students' intuitive reasoning over their construction of more complex meanings, although he did offer implications for teaching Thompson's (2008) approach in Calculus II. He indicated concern in presenting

Thompson's (2008) characterization of average rates of change since "that amount of effort might end up confusing and pushing away more students than it really enlightens (*pause*) is my biggest concern." His comments reflect a prioritization of *affective* considerations and are consistent with his description of *piquing student interest* in his constructed definition of meaningful applications. These remarks indicate that the cost of presenting average rates of change according to Thompson outweigh the advantages Thompson (2008) described:

> When students understand the ideas of average rate of change and constant rate of change with the meanings described here they see immediately the relationships among average rate of change, constant rate of change, slope, secant to a graph, tangent to a graph, and the derivative of a function. They are related by virtue of their common reliance on meanings of average rate of change and constant rate of change. (p. 38)

Importantly, this analysis is not intended to criticize or devalue Robert's interpretations of these ideas but to infer features of his distal goals that emerge from his comments. His comments regarding Thompson's comparison of understanding average rates of change in different ways reveal his understandable hesitation that promoting more complex (and potentially more productive) meanings might negatively influence students' motivation, interest, and engagement by "pushing" them away.

Discussion of the MIP definition. Recall the MIP definition of academic success skills:

> *Academic success skills foster students' construction of their identity as learners in ways that enable productive engagement in their education and the associated academic community.*

When presented with the MIP definition of academic success skills during Interview 1, Robert identified two issues. More generally, the MIP definition did not represent his expectation of success *skills* (e.g., note taking skills). Additionally, he indicated that the phrase “construction of their identities as learners” seemed more abstract and psychological. While Robert highlighted issues with the MIP definition, he liked the phrase “enabling productive engagement,” connecting it to his curious and tenacious identity.

One way to enhance students’ construction of their identity as learners, which he interpreted as students’ affective characteristics (e.g., confidence in their ability, affection for mathematics), is to engage students in mental activity:

> In my mind, any— almost, almost any work is, is good work. If you— if you can engage students into— if you can convince students to do any kind of activity, any kind of brain work, uh, that’s better than doing nothing, and that, that only adds to, uh, their academic, their academic success, and then it adds to their construction of their identity as learners (*he may be reading this phrase from the MIP definition*). Uh, if they’re doing something, and they’re achieving things, uh, that can, that can only help.

This excerpt suggests that Robert emphasizes students’ affective engagement for enhancing their construction of their identity as learners independent of the nature of students’ activity. While these comments cannot reveal that he designs tasks which do not support students’ construction of meanings, they do suggest his priorities as an instructor which beg the following question: If any activity is “good” activity, then is there a need to critically reflect on the meanings that one intends to support or to design curricula and instruction informed by a conduct a conceptual analysis of mathematical ideas?

Conceptual Analysis

I conducted two interviews with Robert during which I asked him questions specifically related to conceptual analysis (Interviews 5 and 11). At the beginning of Interview 5, I asked Robert to construct his definition of what it means to conduct a conceptual analysis, and he was initially confused: “Does it mean like, uh, the concept, whatever concept I’m trying to teach, how I, how I approach teaching it?” Unwilling to provide a technical definition of conceptual analysis, I prompted him to describe the process entailed in teaching a particular concept. Robert then constructed his definition of conceptual analysis:

> *In planning to teach a mathematical concept, I start off with motivation, then move onto development, then conclusion, and finally examples.*
>
> *Motivation: Why would people be interested in “this” concept. Historical background is always nice, but ultimately, an understanding of what this concept can do for “people” (not necessarily the students, but “people” in general) or the “people” at the time of development.*
>
> *Development: The development of the concept. Such as the proof of a theorem.*
>
> *Conclusion: Giving a refined statement of the concept, such as the final statement of a theorem*
>
> *Examples: Show students this concept in action. How and when is it used. Show when it is not used (i.e, limitations of the concept)*

Towards the end of Interview 5, Robert illustrated how each of these four features might be operationalized in teaching a lesson on the quadratic equation:

> So, the motivation is I want to solve this quadratic equation. The development is getting to the, uh, completing the square. The conclusion would be finally I would get the quadratic formula. If I wou—, if I replaced all the coefficients with a, b, and c, this is what it would look like. And that, that— I would get the conclusion. Then we'd go, go onto some examples.

He also indicated that one who has not conducted a conceptual analysis may use the "plug and chug" method for teaching the quadratic formula without discussing how the formula was derived. I briefly discuss some of his comments on the *development* phase as it is most pertinent to this discussion around conceptual analysis.

Development. To Robert, the development of a concept is the vehicle that leads students from the motivation to the final form. I prompted him to discuss the process that he engaged in prior to developing a concept (i.e., related to conceptual analysis). After a long pause, I asked if the question made sense, and he clarified his thinking. He indicated that he did not know and that he usually tries to "jump into it— into actually developing it" knowing what he wants to do. Robert then discussed a detailed example articulating the development of the derivative using secant lines, and later in the interview, he offered an example of developing the quadratic formula.

During Interview 11, I tasked Robert to examine two calculus videos based on the fundamental theorem of calculus and definite integrals (Calcvids, 2019, December 11; Calcvids, 2020, March 9). This task allowed me to collect an additional type of data and provided a concrete context for Robert to discuss the meanings he intended to support. Robert's comments revealed features of his distal goals as an instructor after critically evaluating two artifacts designed to support students' productive construction of specific understandings . Both videos were carefully developed to support students' by applying productive understandings of rates of change (i.e., meanings grounded in quantitative and covariational reasoning). Prior to Interview 11, I asked Robert to analyze these two videos and be prepared to discuss his thoughts.

First example: the definite integral. Robert stated that the fundamental theorem of calculus video presented a "common textbook problem" and described it as a standard approach to teaching the accumulation function. While Robert indicated that the Mars Rover context would appeal to students, he was critical of the video, because it was an artificial problem presenting a complex, irrelevant function whose formula was not derived.

> It felt like it was a little bit artificial, I mean, it's just, in that, or here, here's a, here's a Mars Rover, here's a formula $r(p)$ and that's pretty much it. That's all— you get for it, get, uh, about the Mars Rover, right. I mean you're, you're— so students are hooked. Okay, I want to know more about this Mars Rover, but then you— we, provide them the, the function already, alright. So, they don't get to, they don't, they don't really get to see what's going on with the Mars Rover.

Additionally, Robert stated that discussing the derivation of the formula would generate more student investment in this function. Thirteen minutes later he offered similar remarks:

> But then, they take a big jump, big step by— when they, when they give you that formula, and right when you, right— right when they give you that formula, I could see students just kind of losing interest, saying well, well, that's, that's a big step. I (*little laugh*), I went from— being intrigued and know— and really knowing nothing and then you give me this formula where I have no idea where that came from, right.

Robert's comments suggest an attentiveness to the affective features of students' motivation and interest. Students are "hooked" by the original context, but the introduction of the formula may discourage students' motivation. His recommendation to present the derivation of the rate function r reveals features of his own curiosities as a mathematician and his attentiveness to fostering and maintaining students' affective engagement and the limitations of the video. His frustrations were centered around the random introduction of a complex function since it may reduce students' motivation and interest. The designers of the video introduced the rate function to construct an accumulation of some quantity in service of supporting students' construction of productive meanings to understand the definite integral, but he considered the video to be a standard approach. Upon my prompting, he did affirm that the video highlighted the key concept of the lesson, but his critical evaluation of the task reveals central features of his priorities and commitments to support students' affective engagement.

Second example: the fundamental theorem of calculus. Robert liked the approach presented in the video and stated that he may alter his lecture notes to incorporate some of its material. In particular, Robert liked the (1) geometric presentation of the video and the (2) detailed guidance leading students to develop it. On the other hand, Robert disliked the video's inattention to motivating student interest by providing more explanations. In the video, the presenter describes a context with an air scrubber and indicates that the goal is to find the amount of carbon dioxide removed from an environment over a 12-month interval, represented by $A(12) - A(0)$, given a formula for the first derivative of A. Robert was displeased with how the derivative function was magically introduced considering it would be more difficult to derive. He indicated that from a student's perspective, he would find this confusing

> because I don't— why is *A* prime so easy, a lot easier to find than *A* ("mm-hmm" by Josiah), right? And, and then they never really go into this formulation of finding *A* prime. Show me how you, how you find out, find *A* prime, and, you know, show me how you would find *A* prime. And then, then convince me, that *A*— you need to convince students that *A* prime is easier to com— to compute than *A* is, right. That's a, that— that for me, that would be the first hurdle, um. Alright, so that's one, that would be one of the things I would tackle early on.

Similar to the first example, Robert's critiques reveal features of his distal goals and commitments as a mathematics instructor. His remarks suggesting the need to "convince students" highlights his attentiveness to supporting student motivation and interest. Robert's initial comments focused on the negative consequences of introducing this rate formula for students' curiosity and interest. And yet, the initial purpose of introducing the rate formula is to support students to construct productive ways of understanding the FTC.

Robert's initial reactions do not suggest that he did not value the video's approach to support students' understanding of the FTC. Not only did he acknowledge that the video illustrated a nice geometric presentation, and he liked the detailed guidance leading students to develop it, but at least three times he indicated his interest in altering his lecture notes and "stealing" (i.e., using) this video. Yet, while he may have inferred the epistemological justification for introducing a rate formula initially to calculate linear approximations, his critiques also reveal his attentiveness to critically evaluate the extent to which an instructional resource maintains and fosters students' affective engagement.

In addition to these concerns, Robert also stated that since the video does not graph $y = A'(t)$, students are prevented from recognizing the connection to the area under the curve. Emphasizing this connection is an important component of his instruction:

> Students in my class, they would say, they— you know they'd say— they're, they're made to understand quickly that there is this really wonderful relationship between the area under the curve and the, and the, um, antiderivative, right? Not knowing how that's related, but knowing that they, they're e— in a way equivalent. That's what the Fundamental Theorem of Calculus is. And, we go, we go about doing some examples and say, oh yeah this is easy. I just take the antiderivative and I, I got, I get the answers. That's really neat. Uh, they don't really— at first, they wouldn't really grasp how these are connected, but they just know how to go get, get the answer. But then after I, after I will go through those— some examples, then I would go through the, uh, the derivation of it, why that's true, and I, I prove it.

Robert indicated that he would prove the FTC, and he also recognized that his approach, while providing students with confidence to get answers quickly, does not support students in understanding the formulation of the FTC. Robert's approach reflects his values as a mathematician: he supports students' understanding of an idea by providing a proof. He stated that students, *at first*, would not understand these connections, *but* later he would prove the theorem and help them understand why it is true. Later in the interview, I provided Robert with an opportunity to discuss the learning goals for this video and to hypothesize why the designers might have created it as they did, which differs from a typical textbook approach. He stated that it was a clear explanation for why the definite integral is equal to $F(b) - F(a)$, and it presented a derivation for the formula.

During Interview 11, I asked Robert to describe the process entailed in designing the lesson plan from the video for the fundamental theorem of calculus. These questions allowed Robert to comment on his image of the work required to conduct a conceptual analysis:

> Josiah: What do you think this instructor had to go, um, what'd they have to, how did they design this lesson plan, the process? The textbook says it one way, but how would you even think about developing it? What would that look like?
>
> Robert: Yeah, so I would say, hey— so the textbook wants to teach this, fundamental theorem of calculus, and the way that it's exposition is just kind of maybe unclear or not really, uh, do— doesn't really jive with the, with your style of teaching. It doesn't really catch students.

In addition to his vague description of the textbook's "unclear" presentation, notice Robert's emphasis on pedagogical and affective considerations that might have informed the development of the video. He stated that the presentation in the textbook may not "jive" with one's teaching style or characterize the content in a way that "doesn't really catch students." This occurs in his teaching, and there are many lessons in which he does not like the way it is presented in the book:

> I want to show, I want to show a completely different way, uh, that I think that makes a lot more sense to me. I think it makes more sense to students, uh, and a, and, a lot of the reasons why I do that is a, hey, I can relate, I can bring back things, uh, things that you've done before in completely different field, or a compl— something completely different.

> And you remember how to do that, you like that method, right. Rather than teach the same— this new method, I'm going to expand that method, and— move it into this context. And that, I think that's valuable.

Robert emphasized supporting students' abstraction of methods to new contexts using approaches that they have used before. His remarks reveal his distal goals as an instructor to be attentive to students' desires. In addition to making sense to him, teaching a different approach also makes sense to students and represents a method that they like. Robert commented about continuing to apply "things you have done before," because it presents an easier approach also appeared later in the interview:

> So, what goes in my mind when I change a lesson plan from the textbook, is hey, is there an easier way of making students understand this concept, a more basic method, a more basic way of doing it? Um, can I exploit something that they already know, make them, make them expand that knowledge of what they knew before, make it richer. Say hey, what I do before, this is, now, this is, I— this is more power, I can really, really expand this, uh fr— uh, from what I've been using. Now I can use for, to o— for other things. For me, I think that's more powerful.

These remarks echo Robert's distal goals to make the content more accessible for students to understand and support their capacity to abstract and generalize previous approaches to novel contexts. His rationale for adapting a lesson presented in the textbook, he stated, is it presents an idea in an "easier way of making students understand this concept." Robert's comments suggest that he may prioritize his role as an instructor to provide easier, and perhaps more simplistic, concepts or problem-solving techniques instead of supporting students' construction of more complex or nuanced mathematical meanings that may be more difficult for students to understand. Additionally, these remarks reveal features of his identity as a mathematician to *abstract* an approach from one context and apply it in another situation.

Accepting new knowledge. In the previous excerpt, Robert used "*expand that knowledge*", but during Interview 5, he talked about "*accepting*" a concept and a students' willingness to "*receive*" new knowledge. In another interview when we were discussing meaningful applications, he talked about "*accepting new knowledge*." I asked him to elaborate:

> Josiah: So, how do you think students accept these meaningful applications? Like, can you just talk a little bit about what that means for students to accept meaningful applications?
>
> Robert: Yeah, so, an, an instructor puts a lot of time and energy in developing their notes in developing the ex— their examples, coming up, coming up with which examples they want to use in the class. Right, they make— deliberate decis—these deliberate decisions. And, how that plays with students is different. I mean, do— are the— students engaged? Do these, do these examples, do these ideas, are, do they engage the students? Are they, are they happy to receive this new, new knowledge or do they just not, uhh, whatever, I don't care. Right, so, how do they, how, how well do they accept these, uh, these ideas that you're trying to convey. So, if you have, the more me— you have, if you have, the I, I guess— I feel like the mean— the whole point of meaningful applications is that hey, this is meaningful to you. So, please accept it.

Robert's comments again highlight his emphasis on affective features of students' engagement which is consistent with his constructed definition of meaningful applications. He described students' capacity to accept these meaningful applications as being influenced by their engagement and happiness to accept this "new knowledge," and he does not attend to nuances between students' willingness to receive new knowledge and students' learning the idea. Robert's remarks indicate that by "accept," he is referring to students' willingness to be interested to understand the idea that he is teaching, and the purpose of incorporating a meaningful application is to elicit student interest so that they can "accept" knowledge. Hence, while he does not attend to meaningful applications providing contexts to stimulate and support students' capacity to identify mathematical relationships or construct productive meanings, Robert's comments alone do not reveal that students' interest is a sufficient criterion for their learning. I pressed Robert to elaborate further.

> Josiah: What's required for a student— you talked about accepting this new knowledge, what's required for a student to accept the knowledge that an instructor gives?
>
> Robert: Well, uh, that's one thing I— can kind of pi— I, I mean, I say pique so— present that pique students' interest. So I, that, that's, that's kind of my catch all, right. Because I'm trying to say that, uh, talking about how students receive these meaningful applications. So, if they, for students to want to receive it, right, they have to be interested. You have to pique their interest.

Robert emphasizes that students' engagement is necessary for them to learn new ideas, since uninterested students will not be positioned to engage, and hence, learn. I pressed him to clarify the nature of students' learning once these affective components are satisfied, affording him an opportunity to characterize his own epistemology for students' learning given the condition that students are sufficiently interested to engage.

Josiah: So, let's go a level deeper. So, suppose they're interested. Is that enough for them to receive this, this knowledge or information or does it go deeper?

Robert: It's a start, alright (*little laugh*) ("okay" by Josiah). I mean, that, that's, I mean, when I say pique, I mean, I guess I use that more as a hook. Right, hey— and that's what, that's where motiva— for me, motivation is key, like, can, can my, I, I want to motivate this idea. Can my— attempt in motivating you, does that pique your interest? Alright, uh, is that enough to pique your interest, uh, right? Is— this interesting to you?

Despite my efforts to elicit Robert's image of the process entailed in students' learning a new idea, he reiterated the importance of promoting students' affect through incorporating meaningful applications into his instruction. He clarified that students' interest represents a "start" and discussed the importance of students' motivation. I continued to ask Robert to clarify *how* students' learn once they are motivated. Robert's answer to this question provides important insight into his personal epistemology for students' learning, reflecting his distal goals as an instructor.

Josiah: Okay, okay, so, so that's helpful. So, the, the pique talks, that's the hook that gets them interested. So, now we have these students, student A and student B, and, they are both— maybe I just have one student, I don't know. But they're, they're now interested. And so I— so you talked about accepting this new knowledge, and so I'm trying to say, what does it take for a student to get to the level of accepting this new knowledge? So, first, they're interested, ("hooked" by Robert). There's a hook. So, once there's a hook, you said that's a start. Um, I'm just trying to get an idea of what it would take them to accept this new knowledge as you've talked about. They're interested now. Then what?

> Robert: Yeah, so that— part is hard, because, um, now that they, now that you've got them— if you can get them hooked, now, usually, usually what's next is the rigor, right. I mean, first, mathematics— let's say mathematics, the i— the ideas that we talk about is pretty rigorous, right. Let's— say for instance that we're talking about the quadratic formula. That's a pretty rigorous formula.

Robert's comments reflect his priorities as an instructor: supporting students' motivation and engagement and their capacity to understand the derivation of mathematical topics (e.g., the quadratic formula). This primary focus suggests an attentiveness to students' affective engagement and their mathematical practices (understanding the derivation of a formula).

Now consider Robert's image of how he supports students to engage in active learning and incorporates a meaningful application. Seemingly, his purpose for incorporating a meaningful application is to pique students' interest, so that they are positioned to learn the technique or derivation that he intends to convey. Robert's conception of students' engagement in active learning consists of students making strong decisions and engaging in critical thinking. By engaging in these practices, students, if they do not learn the concept successfully, may have a better understanding of why their approach was unsuccessful. Not intended to be comprehensive, his remarks indicated that his image of supporting students' learning entails illustrating problem-solving approaches and understanding derivations, and students' capacity to learn is influenced by their willingness to engage and their attentiveness to recognize the value of a new method or the rigor of an idea.

Summary and Mutual Influence

It became evident from Robert's responses that he considered his conceptions of the three elements of inquiry in alignment (in general) with the MIP definitions. Since he had exposure to inquiry learning from his previous interactions with an MIP colleague, he seemed to interpret his experiences in the MIP as confirming his current instructional practices. Even when he was perturbed when discussing the MIP definitions, his identity as an instructor remained mostly stable.

Some of this stability should be attributed to his firmly-established goals as an instructor. Influenced from a discussion with another MIP colleague, Robert's perspective on teaching is firmly rooted in his identity as an instructor:

> One of the things he, we talk about is, the most important thing you can do in, in changing your teaching, or adjusting your teaching is, uh, stay genuine to who you are. And I— really believe in that. Um, so, I mean, don't, don't just change to another style just because someone else— someone says it's— been successful with the other people. Right, uh, you have to stay true to yourself and what and— really continue to do what you enjoy doing and maybe tweak, um, tweak your, your teaching a little, uh, to incorporate some of these ideas.

These comments are revealing. They suggest that Robert's identity as a mathematics instructor, influenced by his distal goals, are firmly grounded in his prior experiences and his hesitancy to modify his priorities and commitments based on others' successes. His desire to "stay genuine to who you are" suggests that his identity trajectory will remain mostly consistent throughout his participation in MIP activities with opportunities for a "tweak."

The purpose of this discussion is not to contest or devalue Robert's conception of active learning or to categorize it as wrong. Robert is a thoughtful instructor who is attentive to the importance of engaging students and supporting how students' are thinking about an idea:

> As an educator, that's usually what I go with. Right, my, my, uh, when students ask me how to solve something, rather than giving, giving them my way of solving it, I— generally say the way that you see first is usually the best. Right, it may not be the easiest, but, the way that you see first— cause that make, that, that's— it makes a lot of sense to you. So, I usually try to a— approach problems from how people view it, and— or how students view it.

While it would be inaccurate to claim that Robert does not support students' understanding of ideas, his comments throughout these interviews suggest that he prioritizes supporting students' affective engagement so that they are positioned to be receptive to the skills and understandings he conveys in his teaching.

In the following sections, I discuss his identity as an instructor reflected in his goal structures and belief systems regarding how he conceptualizes the three elements of inquiry and conceptual analysis. I conclude this chapter by discussing implications from these findings.

Active Learning.

As specified in Robert's constructed definition, active learning is fundamentally about student engagement and participation. In alignment with the MIP definition, he identified students' activity involving their *mental* actions. His comments, however, suggest that engaging students in active learning requires the instructor to provide conditions that may engender student activity (e.g., using Pear deck, asking questions, conducting group work) and is independent of the nature of the task design:

> Robert: If they don't finish, well, this process of, of moving these— your brain cells around. That's, that's good, too. I mean that's, that's just brain exercise that is, that, that's good. I mean, it's just like any kind of muscular exercise. Any kind of exercise, any kind of activity is good, is good activity.
>
> Josiah: Do you consider that learning?
>
> Robert: Yes, I do.
>
> Josiah: So that activity of them wrestling or trying, even if they are not conceptually learning what's intended or approaching it in a productive way, um, that could still be learning if they are attempting and their brain cells are as you described moving. Is that correct?
>
> Robert: Yes, yes. I— what I'm saying is I believe there's value in that.
>
> Josiah: So, there, there's value in that. Do you consider that learning though?
>
> Robert: Uh, (*pause*), yes, I do.

Moreover, Robert did not differentiate between the "learning" aspect when distinguishing between active learning and passive learning: "Since they both have learning in it, so I, I didn't, I didn't even, I didn't even worry about distinguishing those two. Learning, learning is learning, I mean, can— are you absorbing new material or not?" I would not expect Robert to discuss students' construction of productive meanings or use common language reflective of constructivist epistemology. However, these comments reveal that he characterizes learning more broadly as a consequence of students' engagement and willingness to learn. Moreover, Robert’s remarks suggest that he may not prioritize (or perhaps differentiate for some topics) the affordances of supporting students’ understanding of particular meanings. His response begs the question: What can students learn from active learning?

In the final interview discussing active learning, I posed this question to Robert in the Zoom chat: When students are actively learning (again, consistent with the MIP definition), how do their actions support their learning of a particular mathematical concept? He indicated that active learning “leads to (*pause*) a— more solid understanding of the mathematical— mathematical concept, a more— leads to a, a more solid or, or greater appreciation for the mathematical concept.” After probing Robert about what he intended by “solid understanding” in the context of the MIP definition, he described this understanding as “may— maybe, uh, maybe grasping the— subtleties of the concept more.” Robert offered an example, stating that the Fundamental Theorem of Calculus *only* works for continuous functions.

Meaningful Applications

Robert’s remarks throughout the four interviews discussing meaningful applications reflected important aspects of his identity as an instructor. Fundamentally, Robert conceptualized the incorporation of meaningful applications as requiring an attentiveness to students’ *affective* characteristics, specifically motivation and interest. Robert attempts to garner student interest in task design both in the *context* as well as the *content*. A context that is relatable, paradoxical, or applied, such as the finance task from his Math in the Modern World course, may encourage students to be interested in the problem. As in the Harvard graduation example, he also provides opportunities for students to *make and justify claims*, an important component of the MIP definition of meaningful applications. Additionally, he supports students’ understanding of a theorem or formula. More generally, motivating students is a critical component of his success. He stated that

> for most, for the most part, in most of my teaching, I like to, uh, talk— begin with why we're interested in what we're doing, right. Can I set up a story, can I start up— set up a scenario where, where people are interested in what we're about to do.

Robert's attentiveness to supporting students' motivation and interest reflects his experience as a student. He indicated that seeing the theorem was insufficient to satisfy his curiosity; he needed to see the proof to understand *why* the theorem is true.

Academic Success Skills

Robert's interpretation of academic success skills reflects his identity as a mathematics student realized through his higher-level goals. Robert does not value note taking skills, and he did not take detailed notes as a student. Robert values curiosity, and some of his best skills include "being curious, being able to ask questions, and being— having the grit to, to mess around with it." Later in the same interview he remarked, "As I mentioned, the mo— hands down, the most, the most important succ— uh, success skill I have personally as a student is just being curious and having the tenacity to, uh, to work on the problem." Finally, Robert also emphasized the importance of thinking critically and offered a personal anecdote. He described a time he attended a Calculus IV class without knowing that there would be an exam that day. One of the problems on the exam was to find the volume of a cone. Although no method for solving the problem was specified, the instructor expected the students to use a triple integral. Robert, however, simply calculated the volume using the basic formula: multiplying the base times the height times one third. In these ways, Robert's perspective on taking notes, exemplifying grit and tenacity, and thinking critically all reveal aspects of his personal identity as a mathematics student and also reflect important characteristics of the MIP definition of academic success skills.

Conceptual Analysis

His remarks suggest that he focused on the mechanics and derivation of formulas and his notion of derivation seemed to include an instructors' capacity to support students' construction of productive mathematical meanings. For instance, I asked Robert about the learning goals for students and to hypothesize why an instructor might teach the fundamental theorem of calculus in this way? He discussed how the video offered a clear explanation of why the definite integral is equal to $F(b) - F(a)$ by graphing $F(x)$ as opposed to $f(x)$. I then asked a similar question about the learning goals for students engaging in this lesson. He indicated that he thinks it would just represent an explanation of where the formula comes from and then continued: "Why is this definite integral equal to $F(b) - F(a)$? Where is that, where that— where is that coming from? Uh, it basically is deri— is derivation, and that, that's very important."

Towards the end of Interview 5, Robert described the process for developing a concept as "fascinating" and discussed how his own conceptions influence his instructional practices:

> I want to know where this come from, came from. As a student I was curio— I, I needed that as a student. So, I've, I've always gone through this process personally, individually, and I want students to experience this process with me. So— when I develop a lesson plan, I think about how I, how did I go, how I went about stu— learning, learning and accepting this concept. And I want to guide the— my students along that, that same path that I took.

Earlier in the same interview, I asked a question about the nature of the process entailed in developing the concept, and Robert echoed a similar sentiment:

> Yeah, I mean, so I usually think about my own trial, my— own path to there. I mean, because, being the kind of student that I was, I was very curious, I, I kept delving deeper. I, I was very tenacious with my study. I, I wanted to fill in, I wanted to fill in all the gaps. Right, so, I start off— so as a student, I start off from scratch, and then they, they try and get me over to the conclusion. Then, I got to make sure each, each proc— each step I take I have to believe it. Alright, I don't want to take anything for granted, so I— walk through this process, and I ask—and I, and, I figure out, I ask myself certain questions and make sure I'm able to answer these questions. So, I, a lot of times I anticipate a lot of questions that students should ask or at least I anticipate the kind of questions I would ask if I were a student. Uh, most of the time, students don't ask the questions that I anticipate.

These comments suggest an attentiveness to supporting students' meanings by anticipating potential questions. Other comments, such as "any kind of activity is good, is good activity," and his desire for students to "accept" new knowledge, however, indicate that there are opportunities to enhance Robert's knowledge base.

In sum, Robert's interpretations of the three elements of inquiry reveal different commitments that he supports. When considered in unison, however, they reflect similar goals he possesses as a mathematics instructor. Robert's interpretation of active learning consists of students' struggling or making strong decisions while engaging in a problem. His comments suggest that he had two interpretations of learning, which I described as *learning from success* and *learning from previous application.* As a consequence of the latter interpretation, students engaging in tenacity and grit are positioned to be receptive to the idea that Robert conveys through his instruction. Robert interpreted a meaningful application as a problem that (1) piques students' interest and (2) highlights the key concept to be learned. The latter idea entails providing students with problems that demonstrate the usefulness of a technique or idea, or the usefulness of using one technique over another (e.g., the shell method over the washer method). Finally, Robert's conception of academic success skills consisted of students' engagement in curiosity, tenacity, and grit to solve a problem, even if you are unsuccessful. These are also the academic success skills that Robert values the most.

I describe the coherence between these three elements of inquiry in this way. Suppose Robert presents a meaningful application to students, and he designs the problem to be interesting and paradoxical. Students with productive academic success skills will be willing to engage with tenacity in grit in solving the problem, and even students with typically less productive academic success skills might be more enticed to participate since the problem may *pique* their interest. Students who successfully learn the idea are engaging in active learning. The students who are unsuccessful, as a consequence of their application of previous ideas stemming from their productive academic success skills, are then positioned to learn the technique or understanding that he intends to support. Finally, the nature of the problem (i.e., highlighting the concept to be learned since it is a meaningful application), suggests that the revelation will be more meaningful to students.

My image of Robert's evaluation of the three elements of inquiry and their mutual influence characterize the case that Robert represents. Robert's interpretations of the three elements of inquiry reveal the importance of maintaining and fostering students' affective engagement, so that they can understand the idea he is teaching or learn a technique or perhaps even the usefulness of a technique. As a consequence of their engagement in productive academic success skills (i.e., tenacity, grit, determination), regardless of whether they are successful in solving the problem, they are positioned to learn the technique or idea that Robert intends to teach. In sum, Robert represents a case of an instructor who values ways to support or enhance students' motivation, prioritizes academic success skills that reflect determination and grit, provides applications of tasks that highlight the usefulness of a particular technique, and seeks to elicit students' thinking to guide them to the correct solution or development of a particular understanding.

Discussion and Implications

Critically, the purpose of highlighting these distinctions is neither to diminish or devalue Robert's interpretations of these three elements of inquiry or conceptual analysis. Robert's conceptions of active learning, meaningful applications and academic success skills convey features of his identity as a mathematics instructor by revealing his distal goals. This discussion is intended to highlight ways in which his instruction might be *extended.*

There is a difference between an MIP participant who does *not know* if their curricular design operationalizes an element of inquiry as defined by the MIP, and it does not, and one who *knows* their curricular design operationalizes an element of inquiry defined by the MIP, and it does not. Robert's comments suggest that he shares similarities with those in the latter group. For example, while Robert's conception of active learning appears to diverge from the MIP definition in meaningful ways, he considered these differences to be insignificant. Robert's goal structures as an instructor to stay “true” to himself limits the MIP Team’s capacity to positively influence his identity trajectory. Influencing his identity trajectory will require an attentiveness to Robert’s commitments and priorities as a mathematics instructor and perturbing and guiding these values in productive ways.

I described Robert’s conception of active learning in two ways: *learning from success* and *learning from previous application.* I focus my attention on the former. During Interview 10, I asked Robert to determine whether students are actively learning after presenting the following scenario: two students are working on an active learning task, and both students are trying ideas (i.e., not stuck). I prompted Robert to determine if it is possible for one student to be engaged in active learning and the other not, and he stated that both are actively learning. I then asked him to consider that same scenario and evaluate if the students are actively learning according to the MIP definition:

> I think so. I think, I think they are, they are. Let’s, let’s see. The problematic situation is still there. Uh, students are selecting, performing, and evaluating. Alright so even, so let, the— one student is doing a good job. The get— they’re going to get the answer. That’s, uh, that’s given that they’re active. How— let’s talk about the student who is just trying random things, who may be— who may amount to nowhere close to the ri— to the right solution. Uh, I think that’s the student that’s in question. Are they performing active learning or not?

Robert’s comments seem to suggest, although he did not specifically say “active learning,” that for the first student who gets the correct answer, it is a “given” that they are engaged in active learning. During our interviews, Robert also did not distinguish between active learning and passive learning by the "learning" component: "Since they both have learning in it, so I, I didn't, I didn't even, I didn't even worry about distinguishing those two. Learning, learning is learning, I mean, can— are you absorbing new material or not?" These comments reveal features of his personal epistemology, and hence, highlight opportunities to perturb and guide Robert's image of effective instruction and his conception of learning by engaging him in discussions and strategically introducing reified artifacts.

First, his remarks suggest that shifting Robert's image of learning may require engaging him in discussions centered around identifying and clarifying different nuances associated with how an instructor can support students' learning (e.g., demonstrating skills, conveying particular meanings) and the constraints and affordances for doing so. Robert is an intelligent mathematician and an experienced instructor, and so he would participate productively in these conversations. His comments reveal broad features of his image of learning, but he would be able to identify different ways that an instructor can support students' learning.

Robert implicitly supports students' learning of skills and procedures, useful techniques, and particular meanings, but his general comments suggest a potential inattentiveness to the nature of students' learning he intends to support; his focus, as was demonstrated by his interpretation of the three elements of inquiry, is on affective features of students' engagement. Consequently, this inattention could limit Robert's capacity to consistently and purposefully support students' construction of more productive ways of learning an idea.

Regarding the introduction of reified artifacts, the MIP Team could provide an example which illustrates the incorporation of a meaningful application compatible with the MIP. Participants may not be perturbed by this task however, if they perceive it as being in alignment with their approach to support students' construction of particular meanings. A more productive alternative may be to strategically introduce reified artifacts designed to elicit this perturbation by presenting examples that *seemingly* incorporate meaningful applications, but are less mathematically meaningful, and facilitate a conversation highlighting these distinctions. Leveraging the findings from these interviews, these examples should include interesting, paradoxical, or applied, real-world contexts while also highlighting the usefulness of a particular approach or certain subtleties. Additionally, facilitating discussions centered around identifying common misconceptions between the MIP definition of active learning and how active learning is conceptualized by most instructors may provide cognitive dissonance in service of promoting more awareness of these distinctions.

Additionally, it may be beneficial to surround Robert with MIP participants with strong personalities who conceptualize instruction in ways that are more compatible with the MIP vision. The subsequent interactions may support Robert's capacity to negotiate meaning from the normative definition of competence being established by other members in the community.

Perturbing participants by providing such examples or guiding them through conversations of nuances for how an instructor might conceptualize learning, however, may not motivate MIP participants; they may not be interested to construct new schemes in alignment with the MIP vision to accommodate this dissonance. Their identity as instructors has been cultivated from their years as a student, their experience teaching, their conversations with other colleagues, their attendance of professional development workshops emphasizing the enactment of pedagogical strategies, etc. Consequently, their goal structures and belief systems are less malleable to be impacted than perhaps a first-year instructor. Even if MIP faculty recognize the knowledge base that would be required to conduct a conceptual analysis, they may not recognize the *need* for conducting one. Hence, it may be productive if the MIP Team facilitates discussions highlighting implications for student learning by exemplifying the affordances of being positioned to purposefully guide these interactions by being attentive to the nature of students' conceptual activity informed by having a clear image of the meanings an instructor intends to support.

In addition to perturbing their conceptions and offering implications for operationalizing the three elements of inquiry, additional support will be required to support them in *developing* tasks which effectively incorporate meaningful applications. For example, after providing a task which *seemingly* incorporates a meaningful application, it may be productive to discuss how to adapt that task to incorporate a meaningful application by discussing specific components of the definition. Supporting participants in this guided process provides a necessary scaffolding by illustrating the features of the task which make it *mathematically meaningful* while also engendering the need for developing the knowledge base required to operationalize this definition by conducting a conceptual analysis or reading mathematics education literature.

Naively, an instructor could design a task that implements *most* of the components of the MIP definition of active learning, and yet not recognize the importance of operationalizing the last component (i.e., students' actions are "equivalent to the structures of the concept to be learned"). I define this process as *pseudo-operationalize*. For instance, an instructor could design a task which is *problematic* and assign students to work on it individually so that they are *selecting* and *performing*. To satisfy the *evaluate* criterion, the instructor could add the question to the problem: “Does your response make sense. Why or why not?.” Students are working on this problem individually, and so it involves their *actions*.

These actions, however, are directly dependent on the last (and perhaps most important) component of the definition. Epistemologically, the MIP definition of active learning is grounded in constructivist epistemology and operationalizing this definition requires the instructor to conduct a conceptual analysis in order to support students’ actions to be “equivalent to the structures of the concepts to be learned.” Robert strives to provide opportunities for students which allow them to engage in a problematic situation and supports students’ capacity to select, perform, and evaluate their actions. Comments such as “any kind of activity is good, is good activity,” however, reveal different priorities for instructional competence.

CHAPTER 4

AN INVESTIGATION OF AMY'S IDENTITY TRAJECTORY IN THE MIP: EXPLORATORY CASE STUDY #2

Introduction and Review

In the introduction to the previous chapter, I discussed the importance of and increasing need for equipping future generations to be successful in STEM. Consequently, a cascade of STEM professional development initiatives have been created to address these needs. I then highlighted comments from Magliaro and Ernst (2018) that STEM networks primarily focused on the P-12 sector. While addressing students' needs at the P-12 sector is essential, as these early education years are foundational for students' construction of important ideas in STEM education, I argued that universities also need to be equipped to support underprepared STEM students' in being successful and graduating in a timely manner. I then discussed the motivation leading to the development of the Mathematical Inquiry Project (MIP), and the goals that the project hopes to accomplish: to foster the development of a community of mathematics faculty throughout Oklahoma to engage in discussions around high quality mathematics instruction and to operationalize the three elements of inquiry in their development of curricular resources for entry-level undergraduate mathematics courses.

The demand for improvements in STEM education is increasing the need to provide professional development opportunities designed to equip instructors to better support student learning and success. Designing these interventions is a complex process which, for theoretically informed initiatives, will require careful consideration of the theoretical mechanisms intended to engineer these changes. One prominent theoretical framework associated with social learning theory is Wenger's (1998) articulation of a *community of practice*.

Cultivating and fostering the development of a community of practice is an essential feature of the design of the MIP, a large-scale, professional development initiative focused on supporting instructors to design curricular resources, centered around inquiry learning, for entry-level mathematics courses. As discussed previously, faculty across the 27 public institutions of higher education in Oklahoma can engage in MIP activities through their participation on a Collaborative Research and Development Team (CoRD). In this study, I examine the identity trajectory of one faculty participating on a Calculus I CoRD. Specifically, I investigate (1) the nature of her conceptions about the three elements of inquiry and conceptual analysis and (2) the evolution of her conceptions from her participation on a CoRD and the associated theoretical design features which might have influenced these changes. To make this presentation more clearly defined, after describing the study's theoretical perspective, I divide the chapter into two parts. In Part I and Part II, I discuss the methodology, results, and implications pertaining to Research Question 1 and Research Question 2, respectively:

Research Question 1: What is the trajectory of one MIP participant from her engagement in the MIP Community of Practice? What are her interpretations of three elements of mathematical inquiry and conceptual analysis and how do these conceptions reveal features of her identity as an instructor?

Research Question 2: How does one MIP participants' involvement on a CoRD influence how she conceptualizes active learning? In what ways are the mechanisms enacted to promote this transformation successful or unsuccessful.

By investigating the case study participant's interpretation of three elements of mathematical inquiry and conceptual analysis, I uncover features of their priorities, values, goals, and commitments as a mathematics instructor.

Theoretical Perspective

In Chapter 3, I discussed Wenger's (1998) *communities of practice* framework by briefly highlighting the three central components—*mutual engagement*, *joint enterprise*, and *shared repertoire*—and then discussing Wenger's characterization of learning within a community of practice as an *identity trajectory*. I then motivated my presentation of multiple theories (social learning theory and constructivist epistemology) by discussing Cobb's (2007) two criteria for comparing different theoretical orientations. I leveraged these criteria to justify my inclusion of radical constructivism since (1) the focus of this study is on the experience of the individual and (2) competence is defined according to one's capacity to operationalize the MIP three elements of inquiry, which are grounded in constructivist epistemology. I then provided a thorough discussion of key constructs within radical constructivism that serve as theoretical mechanisms that characterize the process by which an individual learns.

I direct my focus in this chapter on social learning theory. I first discuss Wenger's (1998) interpretation of a community of practice and participation and reification. Then, I offer a description of the theoretical mechanisms that are intended to initiate a shift in the professional identities of MIP participants by enhancing their knowledge base.

Participation, Reification, and Communities of Practice.

There are two principal mechanisms, *participation* and *reification*, that are critical to understanding identity formation according to Wenger's (1998) *communities of practice* framework. Participation—the means by which an individual negotiates meaning with others—should not be viewed as a binary operator that is turned "on" when an individual is engaging in a specific practice and then turned "off" after they leave that practice. Rather, it is a "constituent of our identities" (ibid., p. 57). Participation is sustained by interactions with reified artifacts, where reification is defined as "the process of giving form to our experience by producing objects that congeal this experience into 'thingness'" (p. 58). Each reified artifact is located across a continuum of time with both a history and a future; someone created the product, and it may be a source around which an individual might later negotiate meaning.

The interplay of participation and reification corresponds to an individual's practice but not necessarily to their *community* of practice. Wenger (1998) listed three dimensions by which the operations of practice connect to the community: through *mutual engagement* in a *joint enterprise* with a *shared repertoire* of resources. These three dimensions characterize a community of practice, and one's identity is a form of *membership* in this practice. As a new participant engages in the activities of a community of practice, they begin to identify a "normative" competence negotiated in the community's pursuit of its joint enterprise.[23] This ongoing negotiation of meaning positions the participant to evaluate their own competence in

[23] I intend normative to be interpreted as an individual's intersubjective appraisal they construct from their interpretation of competent participation in that community.

relation to other members in the practice. In the following sections, I discuss features of the MIP design that may enhance participants' knowledge base as mathematics instructors.

Design Mechanisms That Might Enhance Instructors' Knowledge Base

I first describe competence before discussing the design mechanisms that may to engender transformations in participants' identities as mathematics instructors. As individuals negotiate their identity while engaging in the practices of the community, a notion of competence emerges and knowing becomes defined:

> Again, it is by its very practice—not by other criteria—that a community establishes what it is to be a competent participant, an outsider, or somewhere in between. In this regard, *a community of practice acts as a locally negotiated regime of competence*. Within such a regime, knowing is no longer undefined. It can be defined as what would be recognized as competent participation in the practice. (Wenger, 1998, p. 137)

Through an individual's interactions with members of a community (participation) and artifacts of the community (reification), they negotiate their identity in relation to their evolving image of competent participation. While MIP participants are not joining a pre-existing community of mathematics faculty with an established set of norms and defined notion of competence, the MIP Team offers guiding expertise targeted towards supporting participating instructors to reconceptualize their priorities and commitments regarding the features they associate with acting as a competent instructor. More generally, the MIP Team conceptualizes competent participation by an individual's capacity to operationalize the three elements of inquiry in instruction and curriculum design. Operationalizing these three elements of inquiry may require a deeper transformation in their professional identities as teachers, and hence, will not happen quickly. My description of competent participation is from the perspective of the ambitious goals and overarching vision of the project. Individuals participating in the MIP CoP will negotiate what they interpret as competent participation, and these perspectives may differ dramatically.

In the following sections, I discuss initial and future design mechanisms of the MIP, in terms of reification and participation, intended to occasion shifts in participants' identities. The initial mechanisms are contextualized in the design of the Initiation Workshops and participants' initial involvement on CoRDs, while the future mechanisms are contextualized through brokering relationships and the transmission of objects during boundary encounters.

Initial reificative and participative mechanisms. The MIP Team introduced reified artifacts to support participants in conceptualizing features of instruction in ways that are compatible with their vision. One presentation on conceptual analysis and one discussion centering around the three elements of inquiry were consistently given at each of the Initiation Workshops to begin the process of equipping these instructors with the knowledge base required for CoRDs to effectively design their modules. For example, at each workshop, a member of the MIP Team facilitated a lengthy discussion on the three elements of inquiry—active learning, meaningful applications, and academic success skills. Since these three definitions are technically written and grounded in constructivist epistemology, it is important to unpack their individual components by highlighting key phrases for each element of inquiry. In terms of participative mechanisms, faculty voiced their opinions and ideas among the entire group (and sometimes among partners) regarding these highlighted features of the definitions. Moreover, faculty participating on a CoRD are afforded opportunities to operationalize these elements of inquiry in the design of their module, which entails conducting a conceptual analysis of their selected curricular topic.

Future reificative and participative mechanisms. MIP faculty participating on a CoRD experience opportunities for their identities as teachers to evolve through their continued participation and periodic interactions with their MIP Correspondent. The interactions between members of a CoRD and the MIP Team represent a variation of an *immersion boundary encounter* (Wenger, 1998). According to Wenger (1998), this one-way interaction involves individuals visiting a practice to better understand the communities' engagement. Wenger (1998) stated that "visitors must 'background' their home membership in order to advance the boundary relation and maximize exposure to or influence on the practice of the visited community" (p. 112). In this context, the MIP Correspondent represents the "visitor" who seeks to better understand the practices of the CoRD through their interactions (occurring primarily through email correspondence and by attending virtual meetings).

Through these boundary encounters, participants from both practices can negotiate meaning through both reification (via boundary objects) and participation (via brokering). In the former way, reified products have the potential to become *boundary objects*: "artifacts, documents, terms, concepts, and other forms of reification around which communities of practice can organize their interconnections" (Wenger, 1998, p. 105). Boundary objects are a centerpiece around which members of different practices, those who designed the artifact and others who are interpreting it, negotiate meaning (ibid., p. 108).

There are several boundary objects that could be effective in engineering the intended identity shifts among MIP participants. First, once a CoRD completes their proposal and eventually their draft module, they send it to the MIP Correspondent to be reviewed. In this process, the MIP Correspondent offers suggestions that are informed by constructivist principles (i.e., in alignment with the principles of conducting a conceptual analysis and operationalizing the three elements of inquiry). In reviewing a CoRDs' module, these Correspondents offer probing questions in service of supporting the CoRD members to clarify how their proposed activities might foster students' abstraction or generalization of the structures of the targeted concept(s), as opposed to only supporting particular behavioral proficiencies. By reflecting on these questions, participants are positioned to modify their module in alignment with the instructional principles outlined by the MIP Team. Other boundary objects could include select mathematics education research literature, the introduction of other definitions of active learning, different resources created by others that effectively (or ineffectively) support students' engagement in active learning, etc.

While boundary objects represent reificative forms of communication between practices, the CoRDs and the MIP Correspondent can also bridge their communities through *participative* mechanisms. In this situation, the MIP Correspondent acts as an agent that connects these two practices through the process of *brokering*: "connections provided by people who can introduce elements of one practice into another" (p. 105). Brokers must delicately balance their multimembership by straddling boundaries to provide both distance and legitimacy.

The MIP Correspondents, who act as brokers, contribute to occasioning shifts in the identities of MIP faculty participating on CoRDs. By offering resources from mathematics education literature or providing probing questions or suggestions in alignment with constructivist principles, the MIP brokers seek to support participants to design tasks informed by conducting a conceptual analysis of a mathematical topic (Thompson, 2008).

Three Important Features for Change

While this discussion of theory presents design mechanisms that may engender participants' transformation in how they operationalize these three elements of inquiry informed by conducting a conceptual analysis, these constructs are presented in generalities. To clarify the mechanisms by which Amy's identity trajectory may evolve from her participation on a CoRD, I discuss three central features—*curiosity*, *practice*, and *reflection*—that I identified as critical for supporting Amy's identity trajectory towards a conceptualization of competent instruction in alignment with the MIP vision. During my and Amy's eight meetings, the central focus was on the first feature, but I still discuss the other two briefly and contextualize their importance.

Curiosity. The MIP Team conceptualized the three elements of mathematical inquiry according to constructivist epistemology and carefully crafted three definitions which reflect this epistemological characterization. Nevertheless, the abundance of teachers' meanings associated with these three components that contrast with the purposeful construction of these definitions by the MIP Team has presented obstacles for fostering participants' productive interpretations of the definitions. In other words, some participants (as in the case of Robert) have assimilated their images of these three components to their existing schemes, and hence, have little awareness of differences between their vision of effective mathematics instruction and the vision reflected in the MIP definition of learning through inquiry.

Consequentially, the CoRD meetings need to first stimulate Amy's *curiosity* to notice discrepancies between her conception of active learning and the MIP definition. Becoming aware of these distinctions might position Amy to hypothesize reasons for the different priorities, values, and commitments associated with the MIP definitions.

The MIP Correspondent and I presented different opportunities for Amy to develop this curiosity by (1) supporting her critical evaluation the MIP definition, and (2) illuminating misconceptions regarding its primary focus. To accomplish the latter, we compared and contrasted different definitions of active learning from the mathematics education literature with the MIP definition. I elaborate on these two approaches in Meeting 3, 4, and 5 of Part II of the results.

Practice. Once an MIP participant becomes cognizant of the differences between their conceptualization of active learning and how it is defined by the MIP, they might become curious about the foundational beliefs, values, and commitments on which they are based. Consequently, this curiosity might stimulate participants' interest to engage in the process of attaining a deeper understanding of these three components, and eventually to appreciate the need to critically evaluate the nature of the meanings required for students to construct robust understandings of curricular content. Noticing these differences and becoming curious, however, is insufficient for engendering the necessary changes required for one to achieve this goal. Hence, additional attention is needed to support participants' engagement in this instructional design process. Due to the time constraints of my research, I did not observe meetings in which Amy began crafting a curricular module with her CoRD.

Amy's CoRD meetings should also create opportunities for her to *practice* operationalizing the three elements of inquiry in the instructional design process. The MIP Correspondent provided opportunities for Amy and Kyle to construct a curricular module during some of the meetings and helped facilitate discussions by providing guidance and prodding them to critically reflect on the meanings they intend to support. This support is particularly important since it is easy to *pseudo-operationalize* these three definitions by superficially conducting a conceptual analysis of the targeted mathematical topic.

Reflection. Engaging in the process of conducting a conceptual analysis is a nontrivial task that requires a critical evaluation of the nature of meanings required to construct productive understandings of an idea. As a result, even participants whose curiosity was stimulated might not recognize the affordances of conducting a conceptual analysis for their capacity to support students' conceptual learning. Hence, participants also need to be supported to *reflect* on the implications of engaging in this process. As an illustration, fostering this reflection could be facilitated by the MIP Correspondent offering his perspective on the benefits of conducting a conceptual analysis.

In summary, the MIP Team is incorporating social learning theory to help cultivate the evolution of a community of practice. I first discussed Wenger's (1998) interpretation of a community of practice, the essential processes of *participation* and *reification*. After presenting initial and future reificative and participatory design mechanisms more generally, I then identified three specific features (*curiosity*, *practice*, and *reflection*) that might support Amy's capacity to effectively operationalize these three elements of inquiry.

As with the study reported in Chapter 3, I investigated the nature of an MIP participant's conceptions of the three elements of inquiry and conceptual analysis by conducting an exploratory case study (Part I). In Part II, I explore the extent to which Amy's constructed image of these three elements of inquiry and conceptual analysis evolve from her participation on a CoRD.

Part I: The Case Study

As I discussed in the Introduction, the remainder of this chapter will be divided into Part I and Part II. In Part I, I discuss the methodology, results, and implications associated with investigating Research Question 1. Similarly, I address Research Question 2 in Part II by discussing the methodology, results, and implications. I begin discussing my methodology for Part I.

Methodology

Selecting the Case

A few days prior to the start of the virtual Calculus I Initiation Workshop in early August 2021, I sent registered participants a survey to assess their interest in participating in an exploratory case study as an extension of the MIP research. Additionally, respondents also indicated the courses that they would be teaching during fall 2021, and the extent to which they exercise control in designing curriculum for the class (e.g., homework assignments and exams). In total, 15 faculty completed the survey, and I eliminated five of them based on their responses: three had participated on a CoRD, one indicated he was not willing to participate, and one was not teaching one of the entry-level courses identified by the MIP.

I followed up by sending a second survey to the remaining ten participants to evaluate (1) which (if any) faculty intended to join a CoRD in fall 2021, and (2) the nature of how their class was being conducted (virtual or in-person) and if their classes were being recorded. Five responded to the survey, and I eliminated two candidates who indicated that they did not intend to join a CoRD in fall 2021. Among the remaining three, I sent two of the candidates (the third candidate was less suitable) an informed consent form with a link where candidates could indicate their willingness to participate in the study. Both respondents indicated their willingness to participate. Upon discussion with the MIP Team, I decided to select Amy. Some advantages of this selection were that she would be teaching Calculus I in fall 2021 which could inform her involvement in her CoRD, that she appeared to critically evaluate her own practices, and that she exerted control in designing her curricular materials. Additionally, Amy's gender provided a contrast to the previous case study candidate, Robert.

Ethical Considerations

After reading the informed consent form, Amy agreed to participate in the study in August 2021. The form included a description of the purpose of the study, an explanation of the requirements associated with participating in the case study, an option to quit participating at any time during the study, and other pertinent information. For her participation, Amy received financial compensation at $50/hour.

Data Collection

I collected data by giving assignments and conducting interviews (see Table 10).

Table 10

Data Collection Methods Associated with the Research Questions

<table>
<tr><th></th><th>Active Learning</th><th>Meaningful Applications</th><th>Academic Success Skills</th><th>Conceptual Analysis</th></tr>
<tr><td>Interviews</td><td>2, 6, 10</td><td>3, 7, 11</td><td>4, 8, 12</td><td>5, 9, 13</td></tr>
<tr><td rowspan="2">Assignments</td><td colspan="3">Meaningful application comparison assignment</td><td>None</td></tr>
<tr><td colspan="4">MIP definition comparison assignment</td></tr>
</table>

Assignments. I gave Amy two assignments to complete between interviews. In the first assignment, I sent Amy a document which articulated different interpretations of her image of meaningful applications. I presented her constructed characterizations of meaningful applications, her comments about associating "meaningful" with "beneficial," an example of her description of a meaningful and a non-meaningful application, and ways she described how a mathematical task might be modified to become more meaningful. I then provided her the following instructions:

> *Please take some time and answer the questions provided below. The purpose of this task was to allow her to articulate her conception of meaningful applications based on your responses from a prior interview and any additional thoughts you might not have previously had the opportunity to express. The information above is organized based on our past conversation, but feel free to modify/arrange/organize it as you work.*

Finally, I provided four questions (some questions also had sub-questions) which prompted her to conceptualize her different characterizations of meaningful applications and integrate her responses between these different characterizations of meaningful applications to provide more clarity.[24] For instance, Question 2 stated that she viewed meaningful applications as being beneficial, and it prompted her to reflect on how the affordances of incorporating meaningful applications related to her constructed definitions addressed in Question 1.

For the second assignment, I sent Amy a document with her eight descriptions of active learning, four descriptions of meaningful applications, seven descriptions of academic success, and the three MIP definitions. I then prompted her to compare her descriptions of these three

[24] I provided the assignment and Amy's responses to each question in Appendix A.

components with the MIP definition by identifying their similarities, differences, significance of these differences, and an elaboration on the significance level that she described. A central purpose of this assignment was to provide an opportunity for her to critically evaluate the differences and significance of these differences between her own image of these three components in comparison to the MIP definitions.

Interviews. I conducted 13 interviews with Amy during the fall of 2021, three associated with each element of inquiry and conceptual analysis and one initial interview to discuss her teaching philosophy more generally. These interviews typically lasted one hour and followed a semi-structured interview guide approach (Patton 2002). Table 10 illustrates how I addressed my research questions from these two data collection methods.

During the first interview for each of the four topics (the three elements of inquiry and conceptual analysis), I asked Amy to construct a definition of that particular component. I provided a prompt (e.g., *students engage in active learning when …*) for some of them, and for others I did not (e.g., conceptual analysis). She constructed a precise definition of conceptual analysis but offered descriptions (at least four) characterizing her image of the three elements of inquiry. After she typed her characterization of these topics, I prompted her to discuss them (and sometimes identify the most prominent features) during that initial interview. During later interviews I asked her more questions related to specific tasks or a specific context to afford her opportunities to discuss her image of these topics in a more concrete context.

Trustworthiness and Data Analysis

My discussion of trustworthiness in the subsection in the Methodology section of Chapter 3 applies to the present case study as well, although my analysis of this study differed in some ways. I began by rewatching the video of the interview and methodically extracting Amy's comments to generate data bits. This nature of this process is primarily descriptive as opposed to analytical since I segmented Amy's comments into smaller parts. As an illustration, the first interview lasted 56 minutes, and I generated more than 200 data bits. These data bits were recorded in Microsoft Excel and encapsulated between colored cells indicating that I was asking a question. These sections, perhaps around a minute in length, provided natural breaks to code the data. In describing their strategy for analyzing data, Corbin and Strauss (2015) stated that coding sections too large could be burdensome. Sometimes I would combine sections and code them together (e.g., if they were small).

I coded these segments between questions for concepts that encompassed at least one data bit. Sometimes I described the whole segment with one concept and other times I used multiple concepts. Then, for each concept, I wrote a dated memo describing features of the concept described from the data bits, and I also generated further thoughts and questions. Many times, after writing a memo, I would rewatch the segment of the video pertaining to that concept, affording me the opportunity to be sensitive to how Amy expressed these ideas and potentially add to my memos. Considering the breadth of information generated from the data bits, I naturally interpreted some of the data bits between questions as less relevant, and hence, did not provide conceptual labels, generate memos, or rewatch excerpts for this data to potentially add more to my memo. Other times, I would do some combination of these three but not all of them. These strategic decisions prevented unnecessary attention and energy being offered towards less meaningful remarks.

Corbin and Strauss (2015) stated that memos (and diagrams) are initially "rudimentary representations of thought and grow in complexity, density, clarity, and accuracy as the research progresses" (p. 117). While the articulation of memos varies, they are an essential part of the analysis process (ibid.). Corbin and Strauss (2015) described some of the functions of memos:

> Among the most important of these [functions] is that they force analysts to work with concepts rather than raw data. Also, they enable analysts to use creativity and imagination, often stimulating new insights into data. Another function of memos is that they are reflections of analytic thought. A lack of logic and coherence quickly manifests itself when analysts are forced to put ideas down on paper. (p. 122)

As an illustration, sometimes I would write more than twenty-five memos for a single interview. After some interviews, I would write one final memo highlighting one or two important theme(s) identified from the previous memos.

After describing the data bits with concepts and elaborating on the concepts by writing memos, I then grouped the concepts together based on similar ideas to create categories. As an illustration, there were four categories formed from the first interview. Sometimes these categories were defined according to the focus of a particular topic (e.g., based on one characterization of academic success skills or her discussion of one task that supported students' engagement in active learning).

After creating these categories, I examined each one to identify different properties, described by Corbin and Strauss (2015) as "characteristics or qualities of concepts that define, give specificity, and differentiate one concept from another" (p. 57). It was less useful to describe the dimensions of these properties considering the nature of the data being analyzed.

For instance, one of the properties of the category Feedback was *Avenues for generating student feedback*. These avenues were characterized by their descriptiveness as opposed to their measurability on a particular spectrum. On other occasions, however, the nature of the properties enabled me to articulate specific dimensions (e.g., the dimensions past and future for the property, *Reconstruction design*). Finally, sometimes after describing the categories by articulating various properties and dimensions, I created a diagram to illustrate the connections between categories.

Results

Active Learning

I conducted three interviews with Amy virtually, each lasting approximately 55 minutes, during which I asked specific questions regarding her conception of active learning (Interviews 2, 6, and 10). Active learning was the entire focus of these three interviews, and a small segment of Interview 1 encompassed questions specific to that topic. While my presentation of the results is generally chronological, my synthesis of each interview influenced the overall organization of this discussion.

During Interview 2, I asked Amy to construct a definition of active learning based on the following prompt: *Students engage in active learning when they…* She offered eight different descriptions (see Table 11).

Table 11

Amy's Eight Descriptions of Active Learning

(1) They are physically and mentally engaged in an activity that promotes learning/understanding of material.
(2) They are communicating their ideas/thoughts with their peers and instructor.
(3) Working in small groups or presenting material to the class.
(4) They take part in problem solving that leads them to a deeper understanding of where theorems came from.
(5) They are asking questions to themselves, their peers, or their instructor

(6) They are thinking more deeply about why they use a particular procedure or method to approach a problem. They may wonder if that procedure can be expended [sic] to other problems or modified to suit another situation.
(7) They are encouraged to think about a problem in multiple ways. Then they can analyze which technique is most appropriate in a particular setting. (I see this especially being true with integration techniques and some with limits.)
(8) They collaborate with their peers/they bounce ideas off of their peers.

Characterization of active learning: the practices of a mathematician. Amy identified the three most important characterizations of students' engagement in active learning as (5), (6), and (7) in Table 11 These descriptions represent students' activities more broadly encompassed under (1). I begin with a discussion of her more general description of active learning before discussing these three characterizations.

Regarding students' physical engagement, Amy indicated that students are not required to be talking to be actively learning. Alternatively, students who are mentally active, she discussed, are not engaged in a mindless or routine activity. Computing a derivative, she discussed, could become a mindless activity for students, but processing the sequence of procedures (e.g., difference rule, power rule, constant multiple rule) required to solve this problem characterizes her interpretation of students' *mental* engagement. Notice that this illustration of her interpretation of students' mental activity while engaging in active learning centered around procedural mastery. A more general interpretation of the mental activity she associates with students who are actively learning is evident from items (5), (6), and (7) in Table 11. Properties of these three descriptions entail (a) students' engagement with notes and generating questions and (b) students' problem-solving activities using multiple tools and abstracting mathematics.

I asked her to provide an example of the types of questions she would like students to be asking related to (5). She discussed how there are different approaches to solve limit questions (i.e., using the dominating term approach or dividing by x to the degree of the denominator), and students have asked about her reason for proffering the dominating term method. After affirming that these questions were centered around problem-solving, she indicated that these questions could also be about notation (e.g., the use of limits with functions), and hence, conceptual. In discussing (6), she stated that students often view mathematics procedurally. While "going through that procedure is an [sic] form of active learning," it is "more important that they can bring that procedure and take it other places with them. Right, it's one tool that we're providing them." She described (7) as students taking ownership of solving problems according to their use of an appropriate technique. She practiced problems outside of class, helping her to recognize patterns. She stated that students should have "multiple tricks up their sleeve."

Given the broad characterization of active learning according to (5), (6), and (7), Amy synthesized these three descriptions into one statement: *Students engage in active learning when they become practitioners/"researchers"/critical thinkers in their field of mathematics*. Consistent with her synthesis of these three items, I characterize her image of students' mental engagement as a reflection of the practices of a mathematician. This involves students asking questions, considering possible approaches, seeking to generalize the use of these techniques in other situations, and thinking critically about a problem in multiple ways.

Identity as a student and an instructor. During our discussion of (7) in the second interview, she described her identity as a mathematics student as "super good at spotting patterns," because she worked on problems outside of class. After affirming her intentions to support students to develop an arsenal of tricks to solve a problem, Amy identified her characterization as reflecting the practices of a mathematician explicitly:

> I think the only thing that I would add is that I do try to tell students I think— this is, right how mathematicians think in a way. Or, at least this is how I think as a mathematician; I'm very much a pattern spotter. Um, so, I— want to convey to students that I'm not good at math. It's the fact that I sit down and work through things and then I start to analyze them to spot patterns is what makes me good at math.

These comments highlight her priorities and commitments as a mathematics student to act in ways that align with the practices of a mathematician. Amy seeks to support students to develop a growth mindset by acknowledging that she is "not good at math," and her comments reveal important features of her learning theory emerging from her identity as a mathematics student: she is successful in solving mathematics problems from her critical evaluation of ideas and persistent engagement in productive mathematical practices (e.g., spotting patterns).

Student's engagement or an instructor's practices? During Interview 2, Amy described students' engagement in active learning as reflecting the practices of a mathematician and highlighted the activities of the student:

> Okay, um, so I see this as that students are kind of synthesizing all their material. Um, they're realizing, okay, we have these different ways of approaching things, and I'm going to attempt to use as many techniques as I can, um, and then eventually, you know, put together, well, this is good for this setting. This is not good for this setting. I would definitely want to avoid it. Um, so I think it's putting some of the, I mean, it's, it's taking away ownership from me, the instructor, and putting it on them.

Her comments highlight students engaging in activities to abstract and generalize problem-solving techniques. In addition to the activities of the student, engaging students in active learning as defined in the MIP also requires the instructor to engage in *thoughtful reflection to design activities* that support students' construction of meanings that are "equivalent to the structures of the concepts to be learned." During Interview 6, I provided Amy a prompt to identify these connections more explicitly. After she affirmed that a student could be engaging in a task and actively learning, I asked her to describe the extent to which the student's engagement in active learning is based on the design of the task or the activity of the student? She indicated that the primary onus is on the activity of the student but highlighted the importance of thoughtful task design:

> Um, so I guess in a way it probably depends (*pause*)— I mean, I could probably put very little thought into an activity and students still could be engaged in active learning. Um, right, I mean even if I just throw a problem at them, that's, in a sense they're do— they're actively learning. They're going through the process. They're figuring things out on their own … . However, I think a lot more can be done, and they can truly engage in active learning, and, and— to help them make those connections, and that sort of stuff, it does require a lot of thought process.

Amy's comments suggest that her conception of active learning prioritizes students' problem-solving activity, but she also recognizes the instructor's role in enhancing students' learning through thoughtful task design. Similarly, in the MIP definition, students' engagement and thoughtful task design are both necessary for students to be engaged in active learning. Her loose conception of learning does not imply that she is inattentive when designing tasks to supporting students' ways of thinking, as our subsequent discussions demonstrate. Rather, Amy's comments suggests that her meanings for learning are loosely conceptualized since students still could be engaging in active learning while engaged with a task designed with minimal thought or purpose.

Higher standard for active learning: task design and open discussions. During our first interview, Amy indicated that she had a "higher," loftier standard of active learning in which students have an "aha" moment, and a lesser standard for active learning consisting of student engagement and discussion. Cognizant of these distinctions from my analysis of Interview 2, I asked Amy if her description of designing tasks to engage students in active learning would change if she considered her higher standard. The purpose of asking this question was to address potential criticism that her more critical and refined description of active learning might reflect a deeper awareness of and attention to the nature and purpose of designing tasks to support students' engagement in active learning. She stated that more "nuance" would be required to understand students' thinking, and students would be more comfortable talking than writing: "Yeah, so, there is a nuance I think to get to that gold standard. Um, but in my mind, I don't necessarily see those nuances happening within the exact activity. I think it's the discussion that follows the activity."

I followed up by asking her to discuss the features of this interaction that are important and to describe how such features might be different when incorporated into mathematical tasks. She discussed the importance of initially eliciting students' thoughts and then supporting them to make connections between ideas. Her comments highlight her thoughtful approach of eliciting students' thinking in service of supporting their capacity to make connections and abstract relationships from these conversations. While her remarks suggest more attentiveness to students' construction of meanings than a general pedagogical approach to simply ask questions, these reflections do not reveal the *specific nature of the connections* she intended to support. In other words, what specific meanings or abstractions does she support for a particular mathematical topic? In the final interview, I asked her to identify specific connections from tasks she designed that support students' engagement in active learning.

Specific tasks. During Interview 10, I asked Amy to discuss different problems that she proposed have the potential to engage students in active learning. Among the three examples that she discussed, I identified general themes regarding her description of the benefits and limitations of these problems. Related to benefits, Amy indicated that her examples provided opportunities for students to break down problems and rely on previous knowledge. She also discussed limitations that might hinder students' capacity to make these connections, the difficulty and newness of components of the problem, and a concern about "pigeonholing" students. From her perspective, these three examples (1) relied on previous knowledge and (2) leveraged students' understanding of functions.

After describing her second example of a problem that engages students in active learning, she indicated that she did not like it as much as the first example. For her third example, she described students computing the derivative of power functions using the limit definition, spotting patterns, and then articulating the power rule. Notice that her third example is consistent with her description of active learning to engage in problem-solving activities and her identity as a pattern-spotting mathematician.

I now discuss her first example in more detail, beginning with Amy's description of the problem: "Provide an integral to students and ask them to find a function and an interval such that when you evaluate it you get a particular number" (E.g., Define a, b, and $f(x)$ such that $\int_a^b f(x)\,dx = k$ for some constant k).

Amy stated that she likes the problem, because it requires students to engage in reverse engineering, "relying on the fact that there's this connection to derivatives but they can also be thinking about it in terms of area" and supports them in connecting derivatives and antiderivatives or area and its relation to definite integrals. When I asked about how this problem supports students' engagement in active learning, she stated that it was "probably not designed well. I pretty much just lay it out for them, and I say find a function, find an interval, such that this is satisfied." She later clarified her meaning, stating that the students may not recognize the relevance of this question to other ideas. According to her new grading approach, however, she seeks to enhance students' active learning by asking more questions:

> And so I see on that, if it's it in that arena, my asking questions is where they're more engaging in the active learning. Um, it's that follow up discussion. Um, and it, it's not necessarily happening in class, but it's, you know, those questions that are on— as feedback, um, for the student."

Her example and her comments suggest one way she attempted to support students' engagement in active learning: provide a nuanced, open-ended problem and provide feedback to support students to make connections. This example represents an unguided problem that supports students' capacity to be creative, which is important to foster students' problem-solving capacity.

Another important feature of her comments about supporting students' active learning through task design was her concern about *pigeonholing* students. In discussing her first example, she acknowledged the difficulty in qualifying students' engagement in active learning since she does not know their thought processes. I asked her if she could design a sequence of tasks so that students' engagement in it would give her more confidence that they were actively learning. While acknowledging the possibility, she expressed concern that she might "pigeonhole" students. The way the problem is presented, students could conceptualize a solution in multiple ways (i.e., by calculating the area under a curve or by applying their knowledge of derivatives). Providing more specific questions, she admitted. might pigeonhole students to solve the problem by thinking about the solution in one particular way: "So, I think it is possible. I just worry about funneling them all in the same category, um, or the same way of thinking which isn't necessarily— I don't think that's necessarily beneficial." These comments suggest that she might be interpreting an instructor's capacity to support students' construction of productive understandings as limiting students' creativity or reducing the opportunity for students to make strategic decisions in solving a problem

Amy's concern suggests that she might have conflated two different domains: students' construction of productive *meanings* and students' creativity while engaging in *problem-solving* activities. The former domain is carefully guided by an instructor's own understanding of a particular idea, while the latter domain reflects the problem-solving activities of the student. Amy's comments, at least in the context of this example, suggest that she prioritizes tasks that allow students to creatively think about problems in multiple ways, consistent with one of her initial descriptions of active learning. Moreover, her remarks in the context of this example seem to be more about her commitments to not stifle students' creativity and less about her inattentiveness to the mechanisms of learning. This notion of pigeonholing becomes an important feature of our discussion in one of the latter meetings in Part II.

Discussion of the MIP definition. Recall the MIP definition of active learning:

Students engage in active learning when they work to resolve a problematic situation whose resolution requires them to select, perform, and evaluate actions whose structures are equivalent to the structures of the concepts to be learned.

During our initial interview, Amy identified the importance of being able to *select* information from related rates problems and of talking to students line by line about the usefulness of each bit of information provided in a task statement, breaking it down, and analyzing each part to determine its usefulness. Later, when she talked about *u*-substitution, she discussed the trial and error associated with students' selection of a function to be expressed as a variable *u* to express an integrand as a non-composite function, and she stated that one may have to do a "few *u*-subs," attending to students' *evaluation* of these transformations.

Her use of the word "daunting" in describing these tasks suggest that she may consider these types of exercises to present a *problematic situation*. In Interview 10, she identified the "biggest drawback" of her first example she presented that engaged students in active learning is that it was a non-traditional problem. These types of problems, Amy discussed, can cause discomfort to students who are accustomed to doing textbook problems and getting an answer in the back of the book. In these non-standard problems, however, students cannot look at online resources to help them solve it and may get different answers from their classmates. After I asked Amy for other comments about how her three examples support students' engagement in active learning, she highlighted the problematic nature of these activities: "Well, the only thing I, I think, and I pointed it this out with the other ones is I think these problems do make students uncomfortable. Um, and, and I think it's good for students to be uncomfortable." She later stated that “if a student is willing to persist, I guarantee they will have no problem with the problem.” Once some students get “uncomfortable,” however, they quit.

These comments suggest that she implicitly incorporates different features of the definition of active learning in her instructional practices and reveal her image of the relationship between active learning and academic success skills. By engaging in a problematic situation that might make them “uncomfortable,” students are afforded opportunities to persist and overcome mathematical anxiety that they may be feeling. Finally, notice her “guarantee” that students’ persistence equates with their success. While she would likely acknowledge that these comments are hyperbolic, they reveal features of her image of students’ learning emerging from persistent efforts. Her comments focus on the engagement of the student and not the meanings their persistence might enable them to construct.

Between clinical interviews, I assigned Amy the task of identifying similarities and differences between her descriptions of active learning and the MIP definition. In her written responses, she stated that they both focus on the student, but the MIP definition has an “end goal”:

> Students are trying to solve a particular problem (overcome a problematic situation). With my definition, analysis and questioning are highlighted. The end goal isn’t as important. Questioning one’s self is a big part of my definition. I’m sure it is a part of the MIP definition, but it doesn’t stand out as much, in my opinion. The MIP definition almost reads like a formula, which may oversimplify the process. Whereas my definition is more convoluted.

Consistent with her previous descriptions of active learning, Amy's comments reveal her image of active learning consists of the enactment of productive mathematical practices (e.g., analyzing and questioning) with the "end goal" not being "as important." In highlighting her description, she does not discuss the nature of students' engagement being in service of supporting students' construction of particular meanings. Hence, her remarks suggest that students' engagement in active learning is primarily centered around *students'* productive mathematical practices (e.g., analyzing, problem-solving) and less dependent on the *instructors'* explicit goals (informed by a conceptual analysis in the MIP definition) to support students' construction of particular meanings.

In this assignment, I also provided Amy four choices to characterize the importance of these differences, and she chose the option, "A little bit significant." She stated that students' analyses relate to resolving the problematic situation identified in the MIP definition but should be emphasized more.

First, notice that Amy's critique of the MIP definition is consistent with her own image of active learning. Moreover, her comments suggest that she identified minor differences between the MIP definition and her own descriptions of active learning, and she did not seem to recognize the importance of supporting students to engage in actions whose "structures are equivalent to the structures of the concepts to be learned." Rather, she remained focused on enhancing students' problem-solving capabilities. During multiple meetings in Part II, Amy participated in discussions focused on the MIP definition of active learning. Hence, this topic will be revisited later in the chapter.

Meaningful Applications

I conducted three interviews with Amy virtually, each lasting approximately 55 minutes, in which I asked specific questions regarding her conception of meaningful applications (Interviews 3, 7, and 11). Meaningful applications was the entire focus of the latter three interviews, and a small segment of Interview 1 encompassed questions specific to that topic. While my presentation of the results is generally chronological, my synthesis of each interview influenced the overall organization of this discussion.

During Interview 2, I asked Amy to construct a definition of meaningful applications based on the following prompt: *Applications are meaningfully incorporated in a mathematics class when …* Amy offered four different descriptions (see Table 12).

Table 12

Amy's Four Descriptions of Meaningful Applications

(1)	Students are actively engaged in the material. That can be done through worksheets, discussion questions, guided examples/problems.
(2)	Students come up with their own questions based on the material that they've covered thus far.
(3)	The prompts encourage the students to connect various areas/ideas/concepts within the class.
(4)	The prompts foreshadow/motivate upcoming material.

The first two descriptions of meaningful applications in Table 12 reflect the practices of the student, while the latter two illustrate the instructor's role in designing prompts that incorporate meaningful applications.

Affective interpretation. Amy's image of meaningful applications during Interview 3 could be broadly categorized according to *affective* features and *cognitive* features, although there is overlap between these two domains. Regarding the former, she described prompts that motivate ideas, are relatable, or realistic. Amy indicated that a meaningful application need not be a real-world application but something that supports students in making connections (from material previously or recently learned) to illustrate their understanding. I asked Amy to identify an example of a task that is not a meaningful application, and she discussed related rates problems as not being meaningful since they are "contrived" problems that provide students opportunities to practice procedures:

> They can be made more meaningful, I like— I mean, I— right, I think they're more meaningful when they have to come up with the formula that represents the cost. Um, but right, I mean, a ladder sliding down is not meaningful; it doesn't resonate. And I shouldn't say it's not— it's partially meaningful, but I wouldn't throw it into this category. That, that sort of task seems like, I don't know, I'm just giving you some sort of thing to practice the procedure on. Um, right, because, for us, that sort of task is totally mindless, I think.

She indicated that she could make her related rates problems better by providing ones that challenge students to select appropriate information or to have students collect data (e.g., bringing her bike into the classroom). Notice here, that Amy is emphasizing the contrived nature of the problem and her preference that students derive a formula that expresses relationships between quantities in an applied context. Amy's comments reveal her attention to how different contexts for related rates problem could support students' engagement more productively (if prompted to derive the formula) or less productively (if given a contrived example). In addition to being relatable, she also talked about the importance of designing tasks with *realistic* numbers to enable students to infer the reasonableness of their solutions:

> And I, and I want them to think about do these numbers make sense, and what do I expect my answer to look like? Do I expect a negative number? Um, do I have a range for what I think it should be? I, I think that stuff is more important than actually, oh let's take the

> derivative using implicit differentiation and then plug and chug, um, because a lot of the times they get unrealistic answers. Um, if I'm speeding away from a police officer, I—the distance between us should be increasing … . So, it's not necessarily the calculus that I think I want them to learn (*little exhale laugh*).

In this excerpt, Amy highlighted the importance of designing tasks with realistic numbers and contexts that enable students to evaluate the appropriateness of their solution. Her final comment—"it's not necessarily the calculus that I think I want them to learn"—suggests that she values the sense-making and critical thinking requirements of related rates problems. This idea relates to the *justifying claims* component of the MIP definition of meaningful applications.[25]

In the following interview discussing meaningful applications, I asked Amy about the features of tasks that might help encourage students to have an expectation of their final solution. In her response, Amy discussed an approach to understanding a problem she saw posted on a website:

> So, I mean even just, without talking about derivatives, if a radius is this, what does that tell us about the volume. If the radius is a third of the height, what does that tell us about the volume? Or— stuff like that, um, to help— so it's like these little baby steps to help us get a better feel for what's going on, and then so we can finally dive into the real question: how is the volume changing with respect to time or something. Um, so, I like the idea of granular questions.

The creators of the tasks, Amy discussed, were intentional about scaffolding the development of students' productive meanings by engaging students with opportunities to examine the

[25] While Amy did not make this association to the MIP definition directly, her comments reveal features of the MIP definition of meaningful applications.

relationship between the volume and radius sequentially. Amy values the "baby steps" she discussed (i.e., supporting students to engage in "sense-making" activities that enable them to understand the problem before addressing the procedural question) because students' engagement in these activities might give them "a better feel for the material or the, the concepts, or the shape that they're dealing with." This last excerpt segues into a discussion of the *cognitive* features associated with incorporating a meaningful application.

Cognitive interpretation. In discussing how she could improve her design of related rates problems, she liked the notion of supporting students to critically think about the information given in the problem, and she later identified the importance of appropriately interpreting the problem. In Interview 3, she provided an anecdote of a conversation with a former calculus student who appreciated being supported to engage in critical thinking:

> Um, I ran into a Calc 1 student actually today, a former Calc 1 student today and he goes I appreciated the fact that you made us think about things more deeply. Like, it's really paying off in my other classes. Um, and so, right, meaningful applications don't necessarily have to be practical, but the knowledge that students gain on top of— I mean, e— the right, there's probably some internal growth happening to or more maturity or something that benefits them in their other classes, um, or the other aspects of their life.

I asked her to characterize this description of deeper thinking and knowledge that students' gain, and she had a difficult time succinctly conveying her thoughts. She connected parts of this description of deeper thinking and knowledge that students' gain with having a growth mindset, internalizing, or being inquisitive. In her written response to one assignment, she included in her definition of meaningful applications that they "allow students to internalize a procedure."

Aware that students who are internalizing, hypothesizing, and self-evaluating reflect attributes of the students, I asked her to comment on the instructor's role in designing tasks to support students to internalize:

> Yeah, so, um, I like the idea of asking students to approach problems in multiple ways, um, and I think that could go even further to talk about— and, and asking them not telling them, which method do you think is most beneficial and why? Um, I think that really gets at a lot of things.

Notice that her comments centered around supporting students' capacity to solve problems and reflect one of her descriptions to support students' engagement in active learning by encouraging them to think in multiple ways.

In addition to thinking more deeply, Amy explained that a meaningful application also supports students in *making connections between ideas and concepts.* I asked Amy to comment on item (3) from Table 12 (i.e., her description of a meaningful application as prompts that "encourage the students to connect various areas/ideas/concepts within the class"), and she stated that questions that arise in her head may not elicit productive responses. She discussed an example in which a task did not elicit the intended response from students, revealing her image of how an instructor might support students to make connections between ideas and concepts:

> Well, first I asked them to graph a function with specific endpoints that had to cross the *x*-axis anywhere in that interval, um, and the function had to be defined for every *x*-value. And then I asked the same question, but I said now the function can't cross *x*-axis, um, and keep in mind my endpoints made sure that one was negative, and one was positive. Um, and then my third question was what characteristics do you notice, or what are the

differences between the two graphs? Why does one have *x*-intercept and one doesn't? Um, right, so, the idea was that I was hoping they would recognize this is like the Intermediate Value Theorem. Um, the main difference is their function is not continuous. If the function is not continuous, we're not guaranteed to cross the *x*-axis. Um, but, because of the way I framed the third question, um, that didn't necessarily come across.

While her questions did not elicit the intended responses, her example reflects a thoughtfully designed task designed to support students understanding of the hypotheses of the Intermediate Value Theorem, namely that the function must be continuous.

Context. In the MIP definition of meaningful applications, the context of a problem provides a background to allow students to make and justify claims as a basis for their abstraction of mathematical relationships. By engaging in similarly structured problems with different contexts, students are afforded opportunities to generalize the structures of the concepts by engaging in repeated reasoning. Providing Amy an opportunity to identify these features, I prompted her to discuss the importance of engaging students in tasks for a particular topic or idea that describe different applied contexts. In addition to identifying how such tasks might elicit students' motivation, she also discussed how students' engagement in these activities can lead to a discussion about arriving at the same conclusion despite different backgrounds:

Um, so, I could imagine posing a problem or putting a problem within various contexts. And so, yeah, say one is about cycling, and one is about walking from one classroom to the other class— something. Um, and so, right, I would probably pick the cycling one, and then, I don't know, someone who's constantly having to go from cross campus to different classrooms might pick the other one. Um, but then it's, it'd be really neat to come together and be like, well, what was your process? Um, what did you do to get there to your final answer? Here's what I did to get to mine. Oh, my goodness they're the same, or they're pretty gosh-darn close. Um, we're relying on the same ideas.

She stated that it "creates an avenue for discussion" and echoed similar sentiments when I asked her about the implications of providing different contexts in terms of the *content* being taught:

> Um, I mean, the benefit too I think is— right, it's the redundancy. Um, they don't appear to be redundant, but they are very redundant; I'm doing the exact same thing in every single problem. Um, and so, familiarity, right. You're getting familiar with process. So, that is beneficial. Um, once you extract the co— the background information from it. Um, so I mean yeah, the redundancy, seeing process over and over and over again regardless of the context is beneficial.

These comments from the previous two excerpts reflect attributes of *the common mathematical structures component* of the MIP definition of meaningful applications. Amy discussed one's familiarity with the process, their capacity to extract background information from it, and seeing the "process over and over and over again regardless of the context." Amy's remarks reveal her attentiveness to the affordances of students' engagement in similarly structure tasks with different contexts: they might enable students to dissociate process from context and abstract particular meanings from their repeated engagement.

Specific content. During Interview 3, I prompted Amy to discuss item (4) in Table 12 (i.e., her description of meaningful applications as consisting of prompts that "foreshadow/motivate upcoming material"). In our discussion, Amy mentioned that her students struggle understanding average velocity and rates of change. She stated that part of the problem is that she covers topics quickly, because “the student should know how to co— compute the slope of a line, the student, I mean, and instantaneous— or average velocity is, I mean, it’s the same as slope, rate of change. Um, it shouldn’t be that difficult.” In response to these challenges, she indicated that she needs to be more thoughtful.

Aware of the complexities associated with supporting students to construct productive meanings about rates of change, I asked her in a future interview to elaborate on her meaning of being more thoughtful. She initially identified the lapse between semesters (either Christmas or summer) as a factor contributing to students’ difficulties. One minute into her response she stated her new approach to teaching these ideas:

> I’m going to actually just start with slope. Period. Um, yes, the slope is the, the same equation as, you know, average rate of change, um, but I’m going to really start on something they know, um, as opposed to starting with average velocity, um, and so, I think the thing is, is I want to build their confidence early on. Let’s compute a bunch of slopes. Okay, let’s embed them within a function so that these slopes correspond to secant lines.

Supporting students to recognize the mathematical relationships between slopes and secant lines, Amy discussed, may alleviate a potential disconnect regarding students’ understanding of these ideas or students’ understanding of slope and average velocity. Amy’s comments reveal her commitments to enhance students’ confidence by supporting their (1) procedural fluency in calculating slopes of lines and their (2) awareness of symbolic and geometric representations of slope. While Amy’s comments indicate some attentiveness to supporting students’ meanings for understanding rates of change, they do so more broadly without revealing the specific nature of the meanings (or their associated implications for other ideas in calculus) she intends to support. I later asked her specifically about how she intends to support students' thinking about rates of change:

> So, I think they just think about them as two independent variables. Right, and with rates of change, we're saying— I mean, derivatives we're talking about change in *y* with respect to *x*, um, and so, and, it made me think going back, do they even realize like when we're computing the slope of a secant line, *y* is dependent upon *x*. Like, *x* is our independent variable, *y* is the dependent variable; it depends on *x*. And so, it, it, made me really think yesterday, that there's some sort of disconnect. They're thinking of these purely as formulas, um, and not understanding the context around it, possibly.

Her comments suggest that she is attending more productively to the nature of the relationships between the two variables, but her understanding might be extended by conceptualizing average rates of change in terms of constant rates of change (Thompson, 2008). I provided Amy another opportunity to identify productive ways of thinking about rates of change from an instructor's perspective, and she offered additional clarity:

> Um, I mean I think it goes with this visual aspect, and it's probably something I overlook, but like the steepness of a slope, right. We should, we should probably be able to look at a line or even just a curve and say, oh that slope is pretty steep at that point. Um, and what do I mean by steep? I mean, the slope would be a— yeah, um, that's relative but— or, you know, even just being i— be able to identify it's increasing, it's decreasing; um, it's constant; it has a horizontal tangent line. That sort of stuff.

Amy's comments perhaps suggest that there may be opportunities to enhance her knowledge regarding how she might support students' conceptions of rates of change. For example, Thompson (2008) described constant rates of change as representations of a proportional relationship between the change in two quantities and highlighted the affordances of supporting students construction of meanings in these ways:

> When students understand the ideas of average rate of change and constant rate of change with the meanings described here they see immediately the relationships among average rate of change, constant rate of change, slope, secant to a graph, tangent to a graph, and the derivative of a function. They are related by virtue of their common reliance on meanings of average rate of change and constant rate of change. (p. 38)

During Interview 11, I asked Amy to discuss problems or tasks that incorporate a meaningful application. The first problem that she discussed was a related rates problems divided into a sequence of questions:

(1) What is the volume formula for a sphere?

(2) How fast does the volume of a spherical balloon change with respect to its radius? (How could this question be rephrased?)

(3) How fast does the volume of a balloon change with respect to time? (How could this question be rephrased?)

(4) If the radius of the balloon is increasing at a constant rate of 0.1 inches per minute, how fast is the volume of the balloon changing at the time when its radius is 4 inches?

Students become overwhelmed while attempting to solve these problems, and so Amy tried to "break down those problems" to make them more accessible to students. She stated that the second question was motivated by her desire to support students in making little connections (i.e., avoid telling students to calculate *dv/dr* directly). Her example seems to reflect features of her fourth description of meaningful applications presented in Table 12: "(3) The prompts encourage the students to connect various areas/ideas/concepts within the class." The two primary properties that emerged from my analysis of her discussion of this example were *Breaking down the problem* and *Making connections*.

Related to the former, Amy's comments reveal her attentiveness to students' cognitive activity by asking them to rephrase a particular question (presumably in the form of mathematical notation (e.g., express *dv*/*dt*)). I followed up by asking Amy to describe the meanings she expected students to construct from their engagement in these problems. After acknowledging students' aptitude to take derivatives at this point, she identified the "biggest takeaway": "And so, admittedly, I think the biggest takeaway that I want in this section is that they're fully capable of doing these problems. Um, they're— right, I separated it for them, but that's the takeaway I want for them." Notice that the "biggest takeaway" she hopes students' learn from their engagement in this activity centers around enhancing students' development of a growth mindset. She demonstrates problem-solving strategies by separating this activity into sub questions and prompting students to reflect on the meanings of the sub questions in service of illustrating productive problem-solving practices that might reflect her own as a mathematician.

The latter property that emerged was *Making connections*. I asked Amy to discuss how the task is mathematically meaningful and she identified the affordances of the context:

> Um, so I think it's mathematically meaningful is because— I mean, especially (2) and (3), um, paired together or, or juxtaposed, even, um. Right, when we take derivatives,

> they're typically in terms of 'Well here's just *f(x)*. Take the derivative with respect to *x*,' or you know, um. And so, I think these are more mathematically meaningful, because they're realizing there's context behind them, um. And, and right, we don't tend to do things in isolation, um. And so— I don't know if that answered your question super well, but I think— I mean I don't think of math as just procedures, um, and so I think they're having to have a better understanding mathematically of what's going on in order to do the procedures.

Her comments suggest that an affordance of a meaningful application is that it provides students an opportunity to make connections between the procedures they are doing (i.e., implicit differentiation) and their relationship to a broader context (i.e., determining how fast the volume of the balloon is changing). The two properties that emerged from my analysis together with Amy's remarks suggest that her image of a meaningful application consists of an activity that provides a context for students to practice procedures (make connections) and enhance their confidence while learning problem-solving strategies (break down the problem).

Amy's second example was a Mean Value Theorem (MVT) Worksheet she assigned to students. I asked her how this project is meaningful, and she stated that it supported students in making connections and taking "baby steps to get the bigger picture" to support students' confidence and their capacity to make connections. I now discuss a few problems from her worksheet.

Students are prompted to graph a continuous function that is presented, identify characteristics of it, draw and calculate the slope of the secant line between two specific points provided, and determine if there is another point, *c*, in the interval such that the secant line has the same slope as $f'(c)$ (i.e., determine if the MVT is applicable to this particular function). In the next question students are provided another example of a function that is not continuous and prompted to answer a similar sequence of questions. After students are asked to identify ways in which the two functions are similar and different, they are presented with the MVT and prompted to identify what $f'(c)$, $\frac{f(b)-f(a)}{b-a}$, and $f'(c) = \frac{f(b)-f(a)}{b-a}$ represent graphically.

I asked Amy to discuss the meanings she expects students to develop from their engagement in her MVT worksheet, and she stated her hope that students will learn to understand the meanings of a hypothesis in relation to if–then statements:

> Um, I mean, admittedly, a big part of it is, I hope that they understand what a hypothesis is, um. I mean, right, so many theorems are stated as if-then. Um, and so, the big takeaway that I want is the 'if' part is what has to be satisfied. If this is satisfied, then we're guaranteed the then part. Um, and so, admittedly, I— yeah, I think that's the biggest takeaway that I want: if we have this part, we automatically get this part.

Her comment reveals that a feature of her identity as an instructor is a commitment to support students' engagement in mathematical practices that reflect those of a research mathematician. As a research mathematician, she recognizes and values the importance of understanding the hypotheses in a theorem that must be satisfied for the theorem to be true. I asked her to discuss how students' engagement supports them to make connections from a cognitive standpoint, and towards the end of her response, she highlighted the importance of redundancy for supporting students to solidify connections in their head. Upon my prompting, she elaborated on how the redundancy is beneficial;

> Yeah. Yeah. I mean, right, so (*sigh)*— I mean, it goes back to just my philosophy of math kind of to begin with. Right, I mean, how do— with redundancy is how we spot patterns. How we spot patterns is how we create theorems, um— it— leads us to theorems. Um, so, (*sigh*). Yeah, so that— I mean, I think, that's really in the back of my mind is, yes, I'm teaching a particular concept but I'm also teaching them in a way to be mathematicians. I'm teaching them, essentially, how to think and how to, um— Yes, once again, I'm teaching them procedures as well, but it, it's the thought process, um, that I think is probably the most important aspect, because if they can think about it the right way, they're going to be able to break it down. They're going to be able to pick things apart and figure things out. Um, and so, it's with that redundancy that they can realize 'Oh, this is just this. This is just this, and I'm going to repeat the process, because it does break down the same way.

Her comments reveal her priorities as a commitments as an mathematics instructor to support students' to engage in problem-solving practices reflecting her identity as a mathematician. While she recognizes her role as an instructor to support students' understanding of concepts and procedures, cultivating students to develop a mathematical mindset centered around critically evaluating how to solve problems and spot patterns are features of her instruction she prioritizes the most.

While her comments reveal her commitments to support students to productively engage in practices representative of a research mathematician, her sequence of problems highlight her attentiveness to provide opportunities for students to construct symbolic and graphical meanings of the MVT. Students are prompted to construct an example and non-example illustrating the

MVT and then later asked to represent the symbolic meanings of $f'(c)$, $\frac{f(b)-f(a)}{b-a}$, and $f'(c) = \frac{f(b)-f(a)}{b-a}$ graphically.

In sum, her related rates problem and her MVT worksheet reveal features of her identity as a mathematician from her commitments to support students to understand hypotheses, to learn productive strategies to simply problems, and to develop identities (with confidence) as critical thinkers and problem-solvers. As her first example illustrated, a meaningful application might afford students an opportunity to make connections between the procedures they perform and their association to a broader context. Finally, her MVT worksheet revealed her attentiveness to engage students in prompts designed to support them to graphically represent the symbolic notation of the mathematics presented in a theorem.

Academic Success Skills

I conducted three interviews with Amy virtually, each lasting approximately 55 minutes, during which I asked specific questions regarding her conception of academic success skills (Interviews 4, 8, and 12). Academic success skills was the entire focus of the these three interviews, and a small segment of Interview 1 encompassed questions specific to this topic. While my presentation of the results is generally chronological, my synthesis of each interview influenced the overall organization of this discussion.

During Interview 4, I asked Amy to construct a definition of academic success skills. She offered eight different descriptions (see Table 13)

Table 13

Amy's Seven Descriptions of Academic Success Skills

(1) Knowing how and when to take notes
(2) Knowing when to ask a question and when to study/review the material before asking a question.

(3) Knowing how to properly study.
(4) Knowing that quality of study time is more important than quantity.
(5) Knowing what resources are available and how to utilize them.
(6) Knowing your ideal study time/location (and working/studying then).
(7) Being able to come up with your own questions to help you study/better understand the material.

After prompting Amy to identify three of these seven that she considered most important, she highlighted (2), (5), and (6).

Amy's image of academic success skills. Amy's three descriptions reveal her belief that students who possess productive academic success skills are self-aware of their thoughts, resources, and identity as a student. By learning academic success skills, she later remarked, students not only become self-aware of their own strengths and weaknesses, but they may also learn how to ask questions and think critically, which she identified as "probably more important than just the material that I'm teaching." These remarks suggest that possessing the academic success skills Amy identified as productive positions students to become better problem-solvers through their awareness of their cognitive habits and characteristics.

Amy identified different influences from her experience as a student that impacted her habits related to item (6) in Table 13

> So, I mean, growing up, I wa— we were super structured, I mean, between— so, my high school wasn't a prep school, but if you asked anyone, they would have said it was a prep school. So, right, there's that pressure of performing well in school, and then there was the pressure of well, I mean— my soccer team traveled all over country, and we were never home. So, I had to be very structured and on top of things, and ("student athlete, yes" by Josiah). Yeah, as soon as there's free time, you sit and you do your homework and you focus on that and then, um ("sure" by Josiah). And so, I had that structure growing up.

Moreover, Amy stated that her parents would not allow her to play outside until her homework was completed. She also identified her husband as influencing her prioritization of these three academic success skills. She discussed how he adapted his study habits after failing a math course to work on math immediately after class. These comments reveal that Amy's structured background supported her development of productive study habits that she now values and is committed to fostering in her students.

In the following sections, I organize the discussion centered around two broad questions:

1. What is the nature of students' learning goals and how does this impact productive and unproductive behaviors related to academic success skills?
2. How does an instructor's actions impact students' responsiveness to feedback?

I highlight some of Amy's remarks from our three interviews related to these questions. Her comments reveal her image of students' goals and her role as an instructor in facilitating students' productive responses to instructor feedback.

Students' learning goals and consequential behaviors. In our final interview, I asked Amy to classify different goals students tend to possess for learning mathematics, and she highlighted two major extremes: attempting to pass the classes and viewing them as required prerequisites or seeking to understand meanings for why things might be true. She later classified different goals that students develop and how students' goals influence their actions in the classroom:

> Yeah, so grade driven, they're going to do the extra work, um, but they tend to ask surface level questions. Um, as a pre-req course, and the students that just view it as a pre-req course, I think do more of the bare minimum, they're— yeah, do more of the bare

> minimum. You know, not as engaged, not as willing to ask questions. That sort of thing. Waiting till last minute to do things. Um, the more inquisitive group who care more about learning as opposed to the grade. They're the ones that are posing questions in class, out of class, whenever. Um, they're the ones who ha— come to office hours and don't just ask how to do a procedure but more of— they want the information behind things.

She indicated that students may not be receptive to feedback if it is not a priority for them to receive it. Amy described a student-athlete in her class who wanted to meet with her, because his grades were low, but he did not have time to meet with her, because he used his time practicing. She inferred that he prioritized his sport over his class. Amy's comments reveal her awareness that students could possess learning goals or performance goals and was attuned to the potential impacts of these goals on students' behaviors related to academic success skills (Middleton et al., 2015).

During our second interview discussing academic success skills, Amy discussed students' expectations and learning goals for engaging in mathematical activities. Students have expectations for both the speed and nature of getting solutions (i.e., procedurally). In contrast to less productive learning goals, she also discussed more productive student orientations. She stated that it is really helpful "for students that realize that math is about patterns, um, and recognizing what patterns are occurring and being able to ask questions about those patterns. Is it consistent? Does this only happen in certain cases?" Students' inquisitiveness is important for enhancing their motivation, their critical thinking skills, and their learning. I asked several questions prompting Amy to discuss a response from an earlier interview in which she stated that a student's inquisitiveness is probably more important than the material she is teaching. In her remarks, she offered a personal anecdote:

> Um, right, as an undergrad, I took linear algebra. I just thought of it as one of— another course. I'm just taking it, whatever. Um, but right it was in grad school that it's like, holy mackerel. This stuff is so important. Um, and it's when I started asking those questions and thinking about, you know, individually how this could impact various things that I began to realize, gosh this is probably the best class. This is the class every student should take regardless. Um, it's because of asking those questions, I started to see the big picture, and it didn't just become a process. Um, and so I think, the big part of it is its— it gives us motivation to understand the material and dive deeper into it.

After prompting Amy to identify if students' inquisitiveness is more important for future classes (e.g., linear algebra) or something else, she responded generally, stating that it's "important for everything": "Right, I think we should always be questioning everything. … So yes, it does directly impact our future classes and I think it makes our future classes probably that much easier." These comments suggest her valuation of students' developing productive orientations in alignment with normative practices of a mathematician. Her personal anecdote illuminated the importance she attributed to students engaging in these activities to question, analyze, and hypothesize for supporting their understanding of mathematics.

I also prompted Amy to discuss features of tasks that might engender students' development of productive academic success skills. She responded by discussing the importance of students' engagement in self-reflection and questioning while working on these tasks—a response that addressed students' behaviors as opposed to features of mathematical tasks that might encourage or promote these behaviors. I later asked the question more pointedly, essentially asking the following: *Is there something about the instructor's role in designing the task independent of this ideal student who possesses these productive success skills?*

An instructor's feedback and its impact on students' work. She was initially unsure of how she might support students' development of academic success skills through task design, and then proposed that she might ask more directed questions and foster a classroom environment that encourages student responses.

> So, I think there is a way that you could ask questions like that sort of stuff, so that they're starting to make connections, but they probably need (*pause*)— something has to be done in class, too, to make them feel comfortable to start doing that sort of thing. Right, otherwise I could just imagine, like, students just skipping it or saying I'm not going to think about this or whatever. I don't know. There is something that has to be said for the environment that you create to, um, that helps elicit that sort of stuff in tasks.

Amy's comments demonstrate the significance she attributes to cultivating a learning environment that supports students to willingly respond to questions and participate in class discussions. During Interview 8, Amy discussed how students' negative responses could be a consequence of receiving unproductive or unnecessarily critical feedback:

> Um, but right, also part of it could be an instructor issue, too. So, I'm thinking about that opposite end of the spectrum an— as to why students aren't getting feedback or receptive to feedback. I mean maybe the feedback is not good. Maybe the feedback has a negative—causes them to have a negative response. Um, so, so why would I continue to

review something if it's not even helpful or if it's causing me a headache some way. Um, so part of it could be on the instructor's end.

While Amy acknowledged that students are responsible for how they prioritize their time and efforts in the classroom, she indicated that the nature of an instructor's feedback (both *what* is being said and *how* it is being said) could be detrimental by being disrespectful, having negative connotations, or being unproductive. As one illustration, she recalled an example of when a former student who had someone write something similar to "this is terrible" on her work. In discussing the influence of Dr. Katherine Good (a speaker during the Initiation Workshops) on her conception of academic success skills, Amy identified the importance of having an awareness of negative connotations of our words and their potential impact on students. She gave an illustration of changing a related rates problem involving *cops*, suggesting this was related to the potential cultural implications of using that context. These remarks reveal Amy's attentiveness to supporting students' affective engagement by her awareness of how we give feedback and the context surrounding the words we use.

Not only did Amy acknowledge that instructors' words can have negative connotations, but she also indicated that their approach to grading could also be unproductive in terms of fostering students' development of academic success skills. For example, she indicated that it might not be beneficial for an instructor to mark "x" through wrong answers on a multiple-choice assignment. Relatedly, in the last interview, Amy highlighted how instructors sometimes grade based on *performance* rather than *process* and the impact of this assessment practice for students' engagement and mindset:

> Right, we reward them for right answers but, but not for the work that is appropriate. Um, and so I think it's this background stuff that gets in their head, like, 'Oh, I don't know how to do math. I got the wrong answer.' Well, it's not the wrong answer. You, you had the right process. I'd rather have you have the right process than— I mean, I want the correct answer to, but right, it's what's been emphasized to them previously. So, I think it's a lot of— I think the baggage, it's, it's what they bring with them that can affect them, um, currently. It's, um— I, I think it's what's been stressed. We value the final answer. No, not necessarily. Um, we value that you do things quickly. No, that's not necessarily true, either. Um, we— I, I don't think students are necessarily encouraged to ask questions. Um, and I, I think that's something that needs to change. Um, and I, I think that affects them. Right, if they're not comfortable asking questions, how are they going to grow and learn? Um, so I think a lot of it has to do with baggage.

In addition to valuing the students' correct solutions, Amy explained that she also values the ideas students construct and express. Amy's comments suggest that students' actions are influenced from their experiences receiving performance-focused feedback from instructors. She explained that this practice results in students are entering mathematics classes carrying negative "baggage" that influences their goals and their conception of what mathematical proficiency entails: solving problems quickly and getting correct solutions.

On the other hand, Amy also made comments related to motivating students, eliciting productive engagement, and offering productive feedback and its potential affordances for students' learning. Regarding the former, she identified the importance of providing a personal story of overcoming challenges or inviting students to discuss how they have used resources productively in their learning. She expressed that an instructor can elicit productive engagement by valuing students' input and responses by encouraging them to interject and by responding with initial positive feedback (i.e., conveying appreciation for students' participation.) Before students' ever respond, an instructor can value their input by patiently waiting: "even it's those pauses and breaks. Um— or and when we're prodding them to answer a question, um, even in that awkward silence. Right, I think that's showing that we value their input and their thought process." Amy also discussed cultivating a classroom environment wherein students can interrupt her if they have question or comments. Moreover, I prompted Amy to describe how the nature of the feedback she offers relates to her image of academic success skills, and Amy stated that the feedback helps students learn and understand the material better. Collectively, Amy's comments reveal her commitment to help foster an environment that enhances students' willingness to ask questions by valuing their responses and empowering them to be vocal.

While she stated that the best type of feedback is verbal, she also discussed students' development of a productive identity from their engagement in mathematical tasks. In our final interview, Amy identified two characteristics of a productive mathematics identity: (1) feeling uncomfortable and not always knowing the answer, and (2) feeling comfortable to ask questions. She highlighted the implications for the former, after I asked her how students' engagement in the specific mathematical content that she has designed from a task or in teaching might have implications for their identities, goals, and beliefs:

> Yeah, so I think, um (*pause*), I think that when, um, students are more engaged, right— Well, part of it to is also to get the buy in to get them to be engaged, right. When we assign just procedural things, they tend to be quick. Right, I'm in. I'm out. I know the

answer. I can check it on Wolfram Alpha or whatever. Um, but when they're engaged, I, I think that has huge potential for that growth mindset. Um, and also, right, we're building persistence, um, because they are having to sit down and, um wrestle with these ideas and these concepts. Um, and so I will admit, I think initially they're probably not super happy about all of that. Um, and I don't— and, and while it is beneficial to them at that point in time, I don't think they necessarily that they see the benefits until later on in their career, um, their academic career or their actual career. Um, and so, right, I think it's creating those uncomfortable situations where they have to think, they have to analyze, they have to go deeper, um, is really beneficial for that growth mindset and that persistence. There is a bit where you have to balance it though, right, because if it has such a high overhead, or the, the question appears to be— have such a high overhead there— right, you could just have students who don't attempt at all. So, there has to be some sort of middle ground so that you're not turning students away, um, but you're actually engaging them. But, I think it has huge repercussion, positive repercussions.

Amy's comments suggest that an instructor's assignment of a task might influence the nature and productivity of students' engagement. She expressed that engaging students in these "uncomfortable situations" has implications for their persistence and growth mindset, which quick procedural problems are unable to effectively foster. These remarks highlight Amy's conception of relationship between the MIP definitions of active learning and academic success skills since problematic situations provide students with opportunities to enhance their growth mindset with persistent efforts. Her comments seem to be compatible with features of the MIP definition of academic success skills and its focus on supporting students' *construction of their identity as learners.*

Conceptual Analysis

I conducted two interviews with Amy virtually, during which I asked specific questions regarding her conception of conceptual analysis (Interviews 5 and 13). Conceptual analysis was the entire focus of these two interviews, and a small segment of Interview 1 encompassed questions specific to that topic. While my presentation of the results is generally chronological, my synthesis of each interview influenced the overall organization of this discussion. During Interview 5, Amy constructed a definition of conceptual analysis:

> Conceptual analysis entails a process of understanding where a concept comes from, how it came about, and how it is applicable. (Applicable does not necessarily mean how it is used in the real world.)

Her description naturally led to a discussion of three components: (1) where a concept comes from, (2) how it came about, and (3) how it is applicable. Exemplifying the first component of her definition, she discussed the importance of supporting students' understanding of the limit definition of the derivative by illustrating the relationship between a succession of secant lines (i.e., lines intersecting a graph over an interval of decreasing width such as $(a, a + \Delta x)$ where Δx approaches zero) and a tangent line (i.e., a line tangent to the graph of a function f at a particular point $(a, f(a))$). She discussed how the difference quotient represents the slope of these secant lines and highlighted the process of decreasing the distance of the secant lines as they approach the "target point" (i.e., the limit of the difference quotient as Δx approaches zero represents the slope of the tangent line.) After my prompting, she offered two more examples of specifying "where a concept comes from":

> It, it's very true of Riemann sums as well. Right, I mean, um, especially if you break the definition down. Right, this is just the area of a rectangle. Um, and— yeah, I mean, in particular, yeah, this repre— represents the width. This represents the height. Um, and, we can see visually that as we increase the number of rectangles, we're doing a better job approximating. Um— or we generally do a better job approximating. Um, and so, this is the idea of we're decreasing the width on the rectangle, um, and so we're getting infinitely many rectangles. Um, so I see it coming into play really big there. I mean, but even with shortcuts for derivatives, right, the power rule. The students can see how the power rule comes about. Um, I mean, even just okay compute the derivative of *x* using the limit of the difference quotient. Compute x^2, x^3, x^4. You start to recognize patterns.

In these three examples, all of which illustrate common interpretations, Amy discussed constructing meaning for definitions and abstracting a computational rule. The first two examples illustrate more of a conceptual focus while the latter is more procedurally oriented. If

If Amy supports students to understand Riemann sums as representations of bounded areas on the Cartesian plane, then there may be opportunities to enhance her knowledge base of ways to support students to construct a more coherent set of meanings across different topics in Calculus I.[26] While it is productive for students to possess a geometric understanding of these ideas, having geometric meanings alone limits students' capacity to develop coherent meanings

[26] Amy, herself, acknowledged in our opening interview that she needed to designate more time to discussing rates of change.

for understanding the relationships between instantaneous rates of change and Riemann sums. In other words, students who only possess a geometric understanding of instantaneous rates of change and Riemann sums as bounded areas under a curve are not positioned to recognize connections between these ideas by possessing a coherent set of meanings for rates of change. On the other hand, students who construct meanings of Riemann sums as representations of an accumulation of changes leveraging ideas of linear approximation are better equipped to develop a coherent set of meanings across these two topics. Hence, there may be opportunities for Amy's interpretation of conceptual analysis to be enhanced by not only attending to the meanings students' might construct related to a particular topic in Calculus I but also how these understandings might be leveraged to enable students to construct productive ways of understanding future topics.

Later in the interview, Amy described her image of conceptual understanding as providing a foundation for derivations:

> Yeah, so, and I think, a conceptual understanding, that's a tricky definition, um (*exhale laugh*), is— I think a big part of it is understanding ho— why we can do certain things. Um— right, so when I think about this in terms of like trig contexts. Right, I don't have all trig identities memorized, but I have this understanding of how the triangle works, and because of that, I can derive that other stuff. Um, so, I don't have a good solid definition for conceptual understanding. but it's this idea that ultimately it, it eliminates the need for a bunch of memorization, because I have the tools, or I have that conceptual understanding so that I can then derive the things I need.

This example illustrates Amy's belief that possessing a conceptual understanding of a mathematical idea provides a foundation of knowledge that can be leveraged to derive formulas or make inferences, eliminating the need for rote memorization. Amy's comments reveal an attentiveness to support students' understanding of the origin or purpose of a definition or formula (i.e., why it exists). During Interview 13, I asked Amy to briefly describe her image of conceptual analysis, and in her response, she discussed students constructing a justification or rationale for the underlying theory of mathematics: "To me, it means that students have a grasp of essentially the theory or the ideas behind why something holds, or yeah, why something works. Um, or why we can do certain things." Amy's remarks suggest that a consequence of having conducted a conceptual analysis is that it positions students to develop a conceptual understanding for why the theorem "holds" or why a mathematical idea is true. Instead of only being procedurally fluent, students may also possess an understanding of the underlying mathematical theory.

Knowledge base. Conducting a conceptual requires an instructor to reflect on their own meanings for understanding an idea and consider how students' engagement in mathematical experiences might support their construction of similar understandings. Amy responded to my prompt for her to discuss the knowledge base required for an instructor to effectively conduct a conceptual analysis by stating that instructors need to be (1) comfortable with the material and be (2) aware of students' trajectories with respect to prerequisite courses they have taken and future courses that they will take. These two points are fundamental features entailed in the process of conducting a conceptual analysis. In the following sections, I organize my presentation of our interviews centered around these two components, focusing on the latter item first, and then discussing Amy's analysis of a sequence of tasks and a video that were designed to support students' construction of productive meanings.

Predicting and supporting students' thinking. During Interview 5, I asked Amy to clarify her conception of the differences between lesson planning and conducting a conceptual analysis, or if one of these activities subsumes the other. She stated that the "process can be different" and her response suggested her attentiveness to supporting students' mathematical reasoning:

> Um, yeah, so I, I think that (*pause*) there is an interplay between the two, um, but— and I, I do think lesson planning plays a huge role, um, but I, I do think it is more than just lesson planning. But, I guess, I mean, anyone could say, right, part of good lesson planning is predicting what students might think about and, yes it is, um, and part of good lesson planning is opening these avenues or these channels for them to do these homework assignments that guide them along the way. Um, so, I, I mean, in all reality, I guess I would say that yes it is all lesson planning then, because good lesson planning can create these avenues and channels.

Her comments suggest that conducting a conceptual analysis involves anticipating students' conceptions about a particular idea. Later in the interview, I prompted Amy to describe her first thoughts while she is engaging in the process of conducting a conceptual analysis. In her response, she highlighted the importance of being attentive to students' thinking:

> So, I, I think that's really— there's some motivating incentive for me to think about what are the— by, by thinking about what the students are thinking about, I think that emphasizes the fact that it, it's more than just a formula. Um, or— you know, it's more

than just this equation, um, but I— it doesn't seem that they're making these connections. Um, and so I think having that insight creates the motivation for how should we approach this. Um, so, I guess that's the first thing that comes to mind.

Her comments reveal a willingness to consider students' thinking and seem to be motivated by her desire to support students' understanding. Amy appeared to be cognizant that her students might not make the connections or construct the meanings between various mathematical ideas and their representations that she intends to support and indicated a willingness to anticipate students' thinking and interpretations.

Amy's motivation to hypothesize about students' thinking, and her expectation for how the resulting insights might inform her teaching, is significant considering the difficulty and importance of the task. During a discussion of meaningful applications, she highlighted the difficulty of supporting students to engage in exploration, develop hypotheses, and ask questions since these mental activities are not easy for an instructor to infer through observation. Amy also explained that her content knowledge and the time removed from her experience learning the material makes it difficult to identify struggles that students might experience:

Amy: Um, and, and, I learned this material long time ago, um, and so, um, it's hard for me to go back say, 'Oh yeah, this is where I really struggled', and, I, I don't know, and, right but, maybe it wouldn't be so hard if I did more self-evaluation too; that's probably something is— I'm maybe not super good at either, which is why I think it's so beneficial for students, um, but.

Josiah: Self-evaluation in what sense?

> Amy: Um, right, so, admittedly, like I know what I'm good at. I, I joke all the time that I'm very prideful. Um (*little exhale laugh*), but it— I mean, I do know what I'm good at, but I struggle to identify, oh yeah, I struggle with that sort of thing. Um, and then, you know, that breakthrough. Just like students, they struggle with things and it's hard to get that breakthrough.

In these comments, Amy identified her mastery of the material as a barrier hindering her capacity to become aware of obstacles that students' encounter and need to overcome to have a "breakthrough" in their understanding. As a result, she recognized a possible need to engage in more self-reflection. Reflecting on one's mathematical schemes is an important step when conducting a conceptual analysis.

An instructor's knowledge. In describing the knowledge base necessary for an instructor to effectively conduct a conceptual analysis, Amy stated that an instructor needs to be comfortable with the material. She later commented that a conceptual analysis

> forces you to analyze your thought process. Um, so having a conceptual understanding, I mean— and then translating it to some sort of conce— conceptual activity, um, yeah, I think is the big component, um, because you can then think about how you think about it and how you might address that with students.

Amy indicated that her experience (or preferences for) learning mathematics might not be compatible with students' priorities and commitments. In one of our interviews focused on active learning, Amy stated that "just because it's the way I learned something doesn't mean it will translate well to students." She explained that her approach is based on "thinking about how I learned or how I would have liked to learn." Amy's comments reveal her commitment to reflect on her own meanings as a basis for supporting students' learning, while also recognizing potential limitations of doing so.[27]

Designing tasks. In the following two sections, I present Amy's analysis of and comments related to specific topics: inverse trigonometric functions and the Fundamental Theorem of Calculus.

Inverse trigonometric functions. I showed Amy the first four problems of a sequence of tasks designed by a member of the MIP Team to support students construction of productive meanings around inverse trigonometric functions (see Figure 2, Figure 3, and Figure 4).[28] I present Question 2, part (3a), and Question 4 below. Question 3 had three parts with the sine value being 0.7 and –0.3 in parts (b) and (c), respectively. In this discussion, I asked Amy to infer the goals of these tasks and the purpose for which they were designed, identify ways they might be improved, and suggest questions she might ask the designer. Amy's responses to these questions reveal aspects of her interpretation of conceptual analysis, as well as important features of her identity as a mathematics instructor.

Figure 2

Question 2 from an Inverse Trigonometric Function Assignment

[27] While her comments are meaningful, analyzing how one thinks about an idea might appear different from identifying and clarifying the nature the mental actions necessary to construct a productive way of understanding an idea.

[28] Amy stated that she does not prefer to teach trigonometry courses, and she does not often think about trigonometric functions, since it is not a priority for her.

2. The following diagram displays an angle with a circle centered at the vertex of the angle. On this image, represent the input and output quantities that the sine function relates.

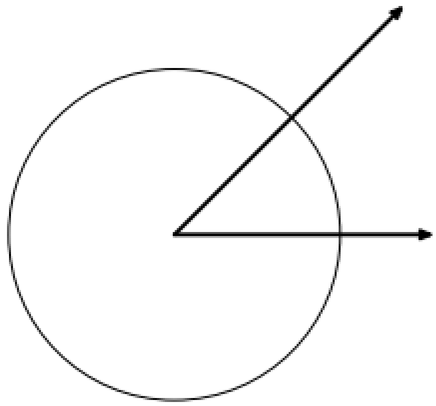


Figure 3

Question 3a from an Inverse Trigonometric Function Assignment

3. a. On the figure below, sketch the terminal ray of the angle that has a sine value of 0.5.

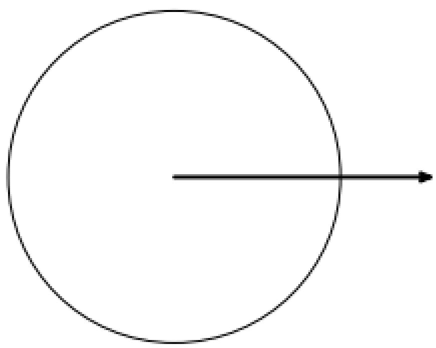
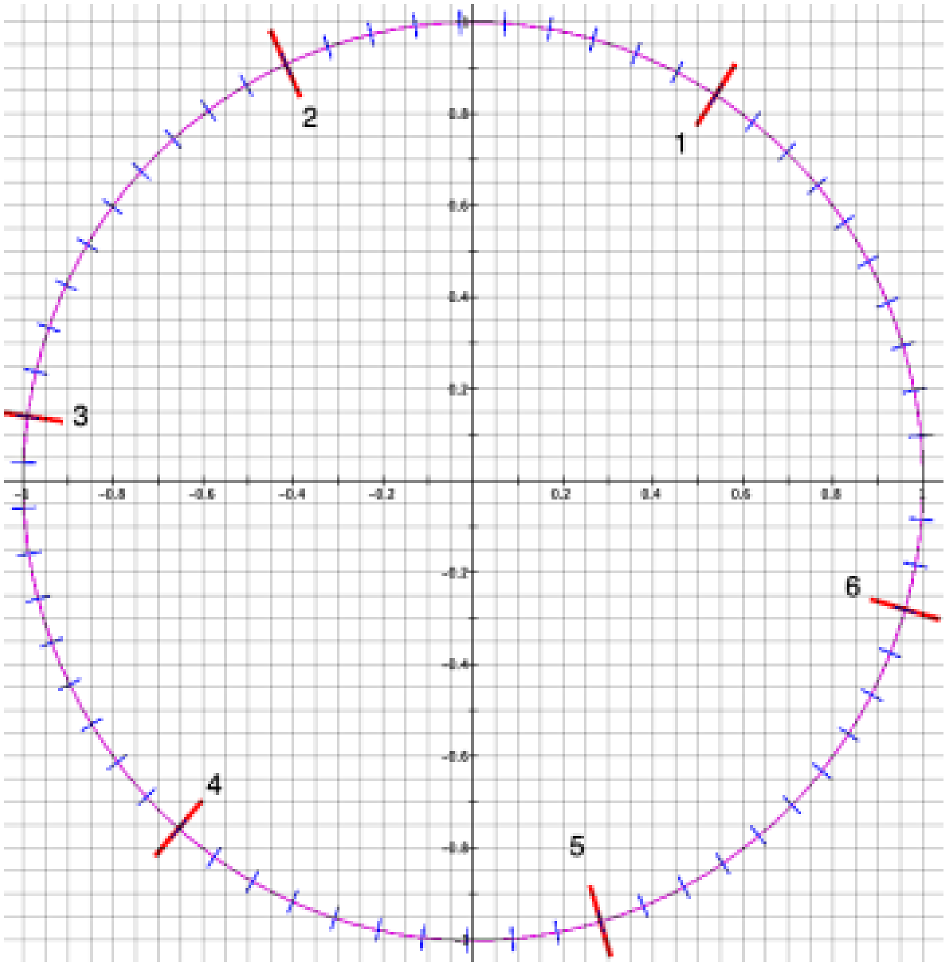


Figure 4

Question 4 from an Inverse Trigonometric Function Assignment

4. Use the diagram below to estimate arc lengths (measured in units of radius lengths) that produce the following sine values.



a. An arc length of ________ will produce 0.5 as a value of sine of that arc length.

b. An arc length of ________ will produce 0.7 as a value of sine of that arc length.

c. An arc length of ________ will produce –0.3 as a value of sine of that arc length.

d. An arc length of ________ will produce –0.8 as a value of sine of that arc length.

e. Suppose that $\cos(a) = -0.8$ and $\pi \le a \le 3\pi/2$. What, approximately, is $\sin(2a)$?

In her initial interpretation of these tasks, Amy discussed affective and cognitive features of student engagement. Regarding the former, she highlighted students' potential anxiety in working on tasks with trigonometric functions and discussed the lack of relatability of the context, which she later identified as an area for improvement. Moreover, Amy indicated that Problems 2 through 4 support students to recognize the relationship between inputs and outputs of the sine function while promoting covariational reasoning, in contrast to memorizing the unit circle. These comments suggest Amy's awareness of the affordances of these problems for supporting students' construction of productive meanings.

I also asked Amy to hypothesize about the nature of the goals informing this instructor's design of this sequence of problems. In response, Amy stated that the instructor could have goals to support students in recognizing the purposefulness of the output (i.e., it is not arbitrary). After I prompted her to talk about this relationship more specifically, she discussed the proportionality of the circle:

> Right, so, 1 more so relies on here we have this right triangle. And, I mean, that happens in 2, 3, and 4 as well that we can create this right triangle, um, but it's putting it in context of a relationship and the arc length. Um, and I think that component is particularly missed. Um, and so it doesn't matter what the unit— what the radius of our circle is. Right, and I think that's lost when students tend to do, like a unit circle. Right, it's a unit circle. It has length one, um, or the radius has length one. Um, and so, I think it's making that connection, um, that we do have an arbitrary circle, and, um, you can still create that right triangle, but— and it's not influenced by the length of our arcs, or excuse me, the length of our radius.

Her comments suggest an awareness of designing these tasks to support students' abstraction of the measurements independent of the size of the circle's radius length. Later, I asked her about the process of designing a task like this one, and she discussed trigonometric identities:

> Um, so, this is partially influenced by how I think, um, but I'm in particular looking at problem (4e), um, what approximately is *2sin(a)* or, sin— excuse me, *sin(2A)*. Um, I, I think— and, my students mess this up all the time. Right, they, they just want to pull out a two and then be content with substituting an *a*. Um, but, I think this idea that we have all these trig identities. In particular, right, there's a trig identity, the double angle formula. Um, it really does rely on the fact that it's input, output— all these other questions are input, output-based. And, if we can see what that relation— or if we can see what's happening as we vary our inputs, what's happening to our outputs, it gives us a better understanding for those identities. So, I'm going to answer that question in terms of how I think. I think that— the instructor is thinking about the trig identities and how to make them, um, resonate better or stick with the students better. Um, and in order to do that it involves input and output of a trig function.

The designer of the activity identified five major ideas of the lesson, and the first two are presented as follows:

1. The sine function takes angle measures as input and produces proportions of the circle's radius as output.
2. The inverse of sine is simply the same set of input-output pairs, but with input and output reversed.

Amy's comments reveal that she attended to features of the second major idea by discussing how "we can see what's happening as we vary our inputs, what's happening to our outputs." She recognized, at least to some extent, that the instructor possessed commitments to support students to construct meanings around the relationship between the input and output quantities of the sine function. Amy's remarks also reveal that she anticipated the designer possessed goals to support students' understanding of trigonometric identities, whereas the designer's intentions centered around their commitments to engage students in a sequence of tasks designed to support students' construction of particular meanings associated with understanding the relationship between the input quantity and the output quantity, as a proportional relationship of the circle's radius, of the sine function.

Her anticipation of the instructor's goals, she admitted, are influenced by how she thinks and perhaps her students' difficulty in understanding the meaning of *sin(2a)*. Amy hypothesized that the instructor intends to make trigonometric identities "resonate better or stick" with students. While her comments do not use words like "construction of meanings" or "abstraction," Amy's remarks may indicate an attentiveness to these features. More generally, her comments reveal features of her goal structures to support students to develop meanings for understanding trigonometric identifies.

When I asked her how she might consider creating Problems 2 and 3, given the first and fourth questions, she stated that she may have incorporated a *t*-chart to support students' understanding of inputs and outputs. Since a *t*-chart only illustrates the input-output relationship, her comments suggest that she might not have recognized the implications or potential value of supporting students' covariational reasoning (i.e., reasoning about the relationship between the simultaneous varying of the input and output quantities of the sine function). Amy's attentiveness to supporting students' understanding and ways of reasoning more generally, however, became evident in her remarks about questions she might ask the designer of this task:

> Um, I think I would like to know if this is the final iteration. Um, I'd also like to know what, what led to this iteration or if there were previous iterations. Um, and what sort of things that— or what sort of questions students asked, um, as, as they were working through it. Um (*pause*), yeah, and once again, that's a bit influenced by the fact that I don't teach trig stuff, and I tend to not deal with a lot of trig stuff.

By asking these questions, one becomes sensitive to the nature of the understandings the designer intends to support through the creation of this sequence of tasks. Additionally, she highlighted the value of this task: "Um, and, part of— it's a learning, right. I mean, I want to create something that looks good like this. So, how am I going to create something that looks like this?"

Fundamental theorem of calculus. Amy discussed an assignment she developed to support students' understanding of the Fundamental Theorem of Calculus (FTC).[29] In this activity, students' primary objective is to find the area under the curve using familiar shapes over the interval [–1, *x*], using –1 (instead of zero) and *x* (instead of a definitive endpoint) to cause some discomfort for students. Students are prompted to take the derivative of the area function they calculated to discover that their derivative function is the original one. She discussed her

[29] Before discussing her assignment related to the Fundamental Theorem of Calculus (FTC), I asked her if she conducted a conceptual analysis in designing this assignment. She stated that it was not at that "stage" but is a "building block."

motivation for creating this task: to support students in recognizing the relationship between area and derivatives.

After her discussion, I showed her a video from the Calculus Videos Project (calcvids.org) designed to support students' conceptual understanding of the FTC by leveraging productive meanings for linearization and local linearity (Calcvids, 2019, December 11). After this presentation, I invited Amy to identify similarities and differences between the two curricular resources. While both approaches incorporate the derivative of a function, how this derivative is used, she stated, represents a "big difference":

> Um, and so, in a way, right, it's using the derivative in a productive sense to begin with, whereas mine is more of an aha moment. Um, and, and so, the intent in which we use the derivative is different. Um, we have the same end goal. But, theirs is a practical, and mine is more of an 'aha' type thing.

These comments reveal her critical evaluation of her own practices and an awareness of the affordances for leveraging productive meanings for the derivative (i.e., constructing a relationship between rate of change and accumulation.) Amy's use of the derivative enabled students to have "more of an aha moment," perhaps supporting students' pseudo-empirical abstraction of the relationship between bounded areas in the Cartesian plane and a function's antiderivative, and the video used the derivative to support students' construction of particular meanings.

Later in the interview, I asked Amy to hypothesize about the nature of the goals and purposes for creating this video, and to identify potential implications for using these approaches. Prompting her to discuss these goals may reveal her priorities and commitments as an instructor by revealing certain features of the task that she identifies as being most important. Her comments were particularly vague: the designer's goal is to understand the theorem, their approach is to simplify a complex idea, and their focus is on area. Her remarks about area reveal features of her priorities as a mathematics instructor to support students in recognizing the relationship between the FTC and the area under the curve as an "aha" moment. Her comments suggest that she may be supporting students to engage in *pseudo-empirical abstraction*, since students are positioned to construct abstractions from their activity (i.e., taking the derivative of the area function and getting the original function) on a particular object (i.e., the area function) with perhaps less meaning for why these abstractions are true.

Knowing these goals and implications, I prodded her to identify the reasoning required for her to think about the FTC from the perspective presented in the video. She identified her background in discrete mathematics as having a prominent influence on her teaching practices:

> Yes, I am teaching about functions. Yes, I— incorporate functions, um, but I, I think my mindset is I do like discrete things, and I, I like to look for patterns and that sort of stuff. Um, and so I think that influences how I approach things, um, and so, and I, I that could also influence this person as well.

Her comments reflect the importance of her identity as a mathematician in influencing her priorities to discuss (or not discuss) functions. They also reveal her identity as a pattern-spotter.

Summary and Mutual Influence

After I provide a brief overview of the results from our discussion of the three elements of inquiry and conceptual analysis, I discuss Amy's image of the relationship between and the mutual influence among each of the three elements of inquiry.

Active Learning

Amy's characterization of active learning has been influenced by her identity as a pure mathematics researcher and reflects the practices of a mathematician (i.e., asking questions, thinking critically about how to approach a problem, generalizing these procedures, thinking about a problem in multiple ways, recognizing patterns). She condensed her image of active learning into a single statement reflecting these practices: *Students engage in active learning when they become practitioners/"researchers"/critical thinkers in their field of mathematics*. I later asked Amy to discuss the nature of the *instructor's* role in designing tasks to support students' engagement in active learning, since these characteristics depended primarily on the onus of the student. While she acknowledged that the design of a task could enhance students' capacity to engage in active learning, she expressed that an instructor could "probably put very little thought into an activity and students still could be engaged in active learning." Aware that she identified higher and lower standards for active learning, I provided opportunities for Amy to describe her higher standard for active learning, and she indicated that more "nuance" would be needed to understand students' thinking in discussions following students' engagement in an activity.

Amy also presented tasks that she considered examples of supporting students' engagement in active learning. In this discussion, Amy cautioned against the potential for over-prescriptive task design to "pigeonhole" students to think in a particular and consistent ways. Her comments reveal that she might have considered an instructor's capacity to support students' construction of specific meanings as perhaps limiting students' creativity and ability to critically think about a problem in multiple ways. On the other hand, one of the activities she discussed that she considered might support students' engagement in active learning—"Provide an integral to students and ask them to find a function and an interval such that when you evaluate it you get a particular number)"—represented an open-ended, exploratory task that allowed students to creatively think about solving the problem in multiple ways. Her throughout these interviews suggest that she valued certain parts of the MIP definition of active learning (e.g., the problematic situation) more than others (e.g., the structures of students' actions become equivalent to the structures of the concepts to be learned.).

Amy's remarks reveal that she prioritized students acting in ways that reflect the practices of a mathematician. Students engage in active learning by asking questions, thinking deeply about different approaches, and seeking to generalize and extend their problem-solving strategies. Moreover, Amy's concern about pigeonholing students indicates her commitment to supporting students' creativity to think in multiple ways. Her comments also reveal that she possessed an unrefined conception of learning since students who engage in activity and "in a sense" are "actively learning," but the instructor can enhance this engagement with thoughtful task design. Amy's comments suggest that she conceptualized active learning as consisting of students' engagement in productive mathematical practices (e.g., analyzing, questioning, critically thinking) with less emphasis on the specific meanings that students are positioned to construct from that engagement. These remarks are not surprising and should not be viewed as a criticism of Amy's instruction, but they reveal the lens through which she interpreted and critiqued the MIP definition of active learning.

Meaningful Applications

Amy provided four descriptions of meaningful applications with the latter two focused on the instructor's role in engaging students with meaningful applications. During our third interview, she discussed meaningful applications according to affective and cognitive characteristics. In the former domain, she described prompts that motivate ideas and are relatable, or realistic. Amy indicated that she wanted students to interpret and evaluate the appropriateness of their answers based on their expectations. In the latter domain, she recalled an example of a conversation with a former calculus student who thanked her for supporting his critical thinking skills.

In our discussion of specific mathematical content, she described how she supports students' understanding of rates of change as an example of a meaningful application. She discussed the relationship between the independent and the dependent variables, the "steep" nature of a function's graph, and having an awareness of increasing, decreasing or constant slope on a graph. Additionally, she presented a related rates example and discussed her Mean Value Theorem project as illustrations of engaging students in meaningful applications. As a later addition to her description of meaningful applications, she wrote that meaningful applications "allow students to internalize a procedure." Related to her conception of active learning, Amy stated that she liked "asking students to approach problems in *multiple ways*" as a way of supporting students' to engage in this process of internalizing.

Amy's comments reveal that she interpreted meaningful applications as being mathematically meaningful for students. Amy's remarks suggest that she valued students' engagement in critical thinking since she recognized limitations in problems that are "some sort of thing to practice the procedure on" and considered students' critical evaluation of their solutions to be “more important than actually, oh let’s take the derivative using implicit differentiation and then plug and chug, um, because a lot of the times they get unrealistic answers.” Consistent with her interpretation of active learning, she valued students’ engagement in problem-solving activities (e.g., appropriately interpretation data in a problem) and recognized the affective implications from their engagement (e.g., build their confidence). Amy’s Mean Value Theorem Worksheet provided an excellent illustration of her efforts to support students’ construction of meanings from their engagement in a sequence of tasks. Some of her other comments discussing rates of change, however, suggest that her knowledge base could be enhanced by being more attentive to supporting students’ engagement in quantitative and covariational reasoning and critically evaluating the specific understandings that she might support students to construct through their engagement with tasks that incorporate a meaningful application.

Academic Success Skills

Amy’s descriptions of academic success skills centered around supporting students to have proper study habits by recognizing the appropriate time to ask questions, knowing the availability and use of different resources, and being self-aware of their own ideal study time and location. Amy’s comments indicate that her structured background as a student supported her to develop productive study habits that she aspired to promote in her teaching.

Amy characterized students’ learning goals according to two major extremes: performance based (i.e., passing the class and viewing it as a prerequisite) versus learning focused (i.e., striving to understand meanings for why an idea might be true). She lamented that some students possess negative learning goals of speed and performance, impacting their orientations. On the other hand, Amy observed that students with more productive orientations are willing to ask questions and be inquisitive.

When I asked Amy to discuss the instructor's role in designing tasks, she highlighted the importance of cultivating an environment that supports students to ask questions. She identified negative feedback that students could receive from their instructor (i.e., disrespectful, having negative connotations, or unproductive), and she discussed how instructors' emphases on performance over process exerts a negative influence students' mindsets. On the other hand, she expressed that an instructor could value students' responses by providing opportunities for them to answer and offering initial positive feedback.

Amy's comments reveal her desire to counteract students' unproductive goals and orientations. She acknowledged that students often believe that mathematics is about solving problems quickly and getting correct solutions, and she indicated that she values students' ideas, even if they result in incorrect solutions, and wants to encourage students to ask questions. Students may not want to ask questions, but she seeks to elicit student engagement by prioritizing their comments. Amy's willingness to support students to ask questions seems to be positively influenced by her experience taking a Linear Algebra course, where she experienced the value of connecting ideas. Finally, attending the academic success skill workshop increased Amy's sensitivity to words that may have negative connotations for students' affective engagement.

Conceptual Analysis

Amy defined conceptual analysis as a *process of understanding where a concept comes from, how it came about, and how it is applicable.* (Applicable does not necessarily mean how it is used in the real world.) Her illustrations of explaining "where a concept comes from" centered around constructing meaning for a definition (e.g., limit definition of the derivative), or formula (e.g., power rule).

When I asked Amy to describe the knowledge base required for an instructor to effectively conduct a conceptual analysis, she identified an awareness of students' trajectories and the importance of an instructor's knowledge. Regarding the former, she discussed the importance of supporting students' thinking and acknowledged that this task is difficult for her since she has a robust understanding of the material and is disconnected from her initial experiences learning the content. In discussing an instructor's knowledge, she stated that a conceptual analysis "forces you to analyze your thought process." I also prompted Amy to discuss how she would support students to internalize, question, hypothesize, or self-evaluate, and she identified the strategy of asking students to think about approaching a problem in multiple ways.

I provided a more concrete context in our final interview by asking Amy to examine a sequence of tasks and an instructional video developed to support students' construction of productive meanings about inverse trigonometric functions and the FTC, respectively. Additionally, she contrasted her approach to support students' understanding of the FTC with the resources I asked her to evaluate. Related to the former topic, I prompted her to infer the instructor's goals in creating these resources, identify areas for improvement, and contemplate questions she might ask the designer. Finally, she hypothesized the instructor's goals and purpose for creating the FTC video, and I prompted her to identify implications of presenting the content in the ways represented in the instructional videos.

Amy's evaluation of these curricular resources indicated her awareness of the importance and difficulty of anticipating students' thinking as a means of motivating her approach to teaching a particular idea. Our discussion of the inverse trigonometric function assignment revealed that she recognized the implications of tasks to support students' abstraction of measurements and engagement in covariational reasoning but also seemed to suggest some limitations in her own approach (i.e., using a *t*-chart to support students in making connections between discrete values of the input and output quantities). The questions Amy proposed asking the designer of the instructional resources she evaluated indicated her attention to students' thinking and demonstrated that she valued these resources, claiming that she aspired to "create something that looks good like this." Finally, our discussion of her FTC assignment and the FTC video indicated her awareness of supporting students' meanings in more or less productive ways.

Mutual Relationship

At the beginning of our final interview discussing conceptual analysis, I asked Amy to describe her image of the relationship between three elements of inquiry and their mutual influence. Specifically, I asked her to discuss six bidirectional relationships between each of the three components. After discussing these six relationships, I prompted her to identify the strongest influence or connection among these six possible directions:

> Um, I'll admit, so, I think the thing that stands out to me the most is that I— i— in my mind, in order to create a good meaningful application, it's going to rely heavily on active learning. But, in order to have productive active learning, um, I, I think there does need to be some of that— the correct mindset; they need to have some, um, success skills that are productive. Um, because otherwise, some of these things are just going to be

surface level type stuff. So, if I'm willing to struggle, um, if I'm willing to push myself, right, it's much easier to be challenged. It's— right, it's easier for me to throw problems at them that make them uncomfortable. … And so, I would say that if I were creating a chart, it pretty much does start with success skills to active learning to meaningful applications. That would be the thing that stands out to me the most.

Amy's comments indicating the strongest implications between these three elements of inquiry suggests that students' academic success skills influence their capacity to productively engage in active learning. These remarks highlight important implications between the MIP definition of active learning contextualized in a "problematic situation" and the MIP definition of academic success skills, which encompasses students' growth mindset and strong self-efficacy. Throughout our interviews, Amy's comments highlighted connections between the MIP definitions for the three components of learning mathematics through inquiry. For instance, during the initial interview, I asked Amy to discuss if a real-world application might not be a *meaningful* application, and in response she attended to features of the other two MIP definitions:

> Amy: Um, I think it— well (*pause*). I think it could possibly not be a meaningful application, right, if a student gets overwhelmed with everything that they're given. Because it, if it's a real-world application, I would imagine that there's a lot of information that's not useful. Students have hard time parsing out what they need, what they don't need, um, and— they could run into brick wall pretty quickly and become overwhelmed. Um, I think you could take a real-world application and break it down and talk with them through why you're breaking it down the way you are to help guide

them, and I, I think that would be meaningful, um, going through that entire process. But, I, I don't think just throwing— here's a real world application. Have a go at it. I, I think that could be daunting and actually detrimental. Um, but, but it, once again, it also depends on the types of students you have. Um, so, yeah.

Josiah: Depends on the type of students you have.

Amy: Well, I mean, in terms of, are they willing to persevere, um, are they willing to push through, ask questions, analyze things, think about, well this isn't necessary, but this is. Um, I, I, think— I mean, I think about my kid, right, he's young, he— I don't want work on something I'm bad at. Um, and, um, I'm afraid that some of our students also have that mentality. … So, there's no room for growth. Um, I mean in terms of— well, because they're not even willing to work on it. So, we— if we work on the mindset changing, then it could be meaningful.

In this excerpt, Amy identified students' struggle to select relevant pieces of information from a story problem, which relates to the *selecting* component of the MIP definition of active learning. Moreover, a "daunting" problem reflects characteristics of a *problematic* situation. Related to academic success skills, her remarks highlight the importance of cultivating a *growth mindset* among her students.

Identity Trajectory and Implications

In this section, I discuss Amy's identity trajectory and implications that may inform the MIP Team in designing professional development experiences for other participants. First, I discuss Amy's adaptability as an instructor. Second, I discuss her image of active learning to support students' engagement in mathematical activity in alignment with the practices of a mathematician. I then provide implications for how the MIP Team might *extend* her image of active learning by affirming her commitments and clarifying the nature and purpose of the MIP definition.

Amy represents a case of an instructor who values students' engagement mathematical practices reflecting the practices of a mathematician, since students can engage in active learning by asking questions and thinking about the appropriateness of using a particular technique to solve a problem. Moreover, she represents an instructor who critically evaluates her own instructional practices and recognizes limitations in her approaches to teaching. Her willingness to modify how she presents students with notes and be more attentive to formalize some of her verbal comments by writing them on the board suggest that some discussions with colleagues have been influential in altering her approach to teaching.

Her Adaptability as an Instructor

One of the themes that emerged from my analysis of Interview 1 was *Adaption*, signifying Amy's adaptability and willingness to change as an instructor. I asked her to identify three important features of her teaching philosophy. The first two features that she discussed were her *guided notes* and her *grading philosophy*, which were influenced by her colleagues at her university and the disruption to her teaching caused by the COVID-19 pandemic.

Amy was initially influenced by a graduate student. After observing Amy teach, the graduate student indicated that her comments were important, but students were not writing them down. Upon hearing these recommendations, Amy decided to present more writing on the board to support students to record her important ideas in their notes. Later, a colleague recommended using guided notes to help reduce the burden of covering a significant amount of content, and to lessen students' tendencies to copy notes, take unproductive notes, and not actively listen or critically evaluate the meanings or ideas being expressed. Amy stated that her guided notes have "evolved" during her time at her current institution, revealing her continued adaptability as an instructor.

In addition to using guided notes, another central feature of her teaching philosophy is her assessment practices, which have also evolved:

> This has shifted a bit too, because, right, I want to emphasize procedure and conceptual understanding. So, I think, um, a— another part of might teaching philosophy that I think is important is how I assess those things. Um, so I have moved away from traditional assessments. I don't do, you know, here's calculus. We're taking four exams in this class. Um, I— don't think that's realistic. Um, I— I mean yes, our students need to perform under pressure at some point, um, but that's what my quizzes are for. My quizzes can handle that. Um, but right, in the real world we have time to think about things and synthesize, um. So, I think that's a big part of teaching philosophy is making this shift from traditional assessments to things that maybe are a little bit more realistic and allow me to assess what students really know.

She later discussed how COVID-19 influenced her new grading approach, stating that COVID "forced me to reevaluate how I was assessing things" and it "gave me a good out" since students would have every resource available when completing take-home exams.

When I asked Amy to identify the extent to which her image of meaningful applications was impacted by her participation in the Calculus I workshop, she admitted that she had not incorporated anything yet, but highlighted the need to restructure her class and “revamp” her guided notes:

> One of the things that stood out to me was that I probably need to restructure my class. Um, and so, I think there’s unintended consequences that have happened. Um, so, while I was sitting in car line to pick up my kid from school, right, I, I’m writing down maybe a different track that I could take next time I teach calculus. Um, because unfortunately the MIP workshop was right before classes started. So, that makes a little rushed. Um, but so right, I’ve created this new possible schedule. Um, and, and instead of us— well, and if I create a new schedule that also means I need to totally revamp my guided notes. Um, and so I have a plan to do all that stuff. It just, I mean it hasn’t gotten finished, and I haven’t implemented it in the class yet. So, nothing immediate, um, has been done, but, um, the workshop did motivate me to re-analyze how I approach things and, in turn, that will, I mean, have big shift in my class next time I teach it.

These comments reveal Amy’s willingness to change and adapt as an instructor; she expressed a need to “revamp” her guided notes, which might result in a “big shift” in the future. I followed up by asking her if there were specific features of the workshop that motivated her mindset in this way:

> Um, I can’t think of anything necessarily— well, I guess, maybe in a breakout room, um, when we were coming up with a list of topics and stuff like that. Right, rates of change, so— I mean that’s really what calculus is all about, um. And I start the class off by talking about rates of change, but I don’t think I do it justice by any means. I, I, think I pretty much assume that they know what a rate of change is, um, and, and I use that to

> motivate why we move into limits. Um, so what I th— yeah, so, I think that was a big transition for me., like I need to do more with rates of change, slopes, um, and— yeah, so, I— I do need to do a lot more of that. And so that's really kind of, um, when I created my mock schedule for next time I teach it, um, those have gotten their own section, essentially, yeah, so I do better job with it.[30]

Amy seemed to be influenced by conversations about rates of change during some of the breakout sessions at the Initiation Workshop. While Amy's comments reveal her awareness of the conceptual thread of rates of change permeating throughout a calculus course, she admitted her own inadequacies in teaching these ideas. The creation of her mock schedule, however, reveals her willingness to designate more time in future semesters discussing rates of change. Earlier in the interview, Amy also discussed how Dr. Michael Tallman's example, during the Calculus I workshop, illustrating a conceptual analysis of the sine function encouraged her to more thoughtfully consider how students might think about ideas.

Other Implications

In addition to Amy's remarks revealing her critical evaluation of her instructional practices, her comments during our discussions suggest that her image of students' engagement in active learning is centered around their problem-solving activities. In the MIP definition of active learning, the engagement of the student and the design of the task are both essential features for supporting student engagement in active learning. While Amy recognized that designing tasks is important for supporting students' engagement in active learning, she could

[30] Our discussion of rates of change in the section on meaningful applications reveals insight into how Amy considers supporting students' understanding of rates of change.

“probably put very little thought into an activity and students still could be engaged in active learning.”

Amy’s image of active learning reflects the practices of a mathematician and her identity as a student. Amy’s descriptions of active learning and her synthesized definition reflect these practices to analyze, generate questions, critically think about solving a problem in multiple ways, etc. In our final interview discussing active learning, I asked Amy if she described active learning as being more of a becoming or occurring from students’ engagement in tasks. While acknowledging the relevance of both, she highlighted the importance of the former:

> Um, so, I do see it as kind of a becoming type thing. Um, so, I do think that it is supported— I mean, right, so, there is this, um, task or experiences that help support that sort of stuff, but I do see it as a learning process. Right, so, if I ask a question at the beginning of the semester, and I ask them to write down how they thought about it that sort of stuff and— well, so I’ll admit maybe they thought about it properly, but they just didn’t communicate how they thought about it. Um, but as the semester goes on, those excla— those explanations, when I do ask for them, they do become better. Right, so I do imagine it is in a way, this, this kind of becoming. They’re growing in the process. Um, and right, some are going to grow faster than others or get to it faster than others, but I, I do view it more of a process. Um, it’s not just a one and done type thing.

Amy’s comments reflect her priorities, goals, and commitments as an instructor related to how she conceptualizes active learning as more of a “becoming.” Considering Amy’s perspective, it may be productive to (1) value her commitments by affirming the importance of supporting students to engage in productive mathematical practices and (2) provide opportunities for her to *extend* how she conceptualizes active learning by engaging her in professional development experiences designed to heighten her awareness of the implications for designing tasks supporting students' construction of particular meanings.

To support her goals, priorities, and commitments, the MIP Team could (a) identify connections between the MIP definition of active learning and her constructed definitions, and (b) identify connections between her constructed definitions of active learning and the MIP definition of academic success skills. Regarding the former, Amy recognized (in a response to an assignment) that the MIP definition of active learning is contextualized within a "problematic" situation. Reiterating and emphasizing the importance of providing this context would affirm a connection between the two definitions that she had previously identified.

Additionally, the MIP Team could also highlight the relationship between her definition of active learning and the MIP definition of academic success skills. In particular, Amy emphasized the importance of students' productive engagement, which results from their orientations, beliefs about mathematics, growth mindset, and capacity to persevere. Since the MIP definition of academic success skills is primarily concerned with the nature of students' *affective engagement*, discussing these connections might alleviate her concerns regarding the limitations of the MIP definition of active learning as being narrowly focused, formulaic, and focused on getting to the "end goal."

In addition to valuing her commitments and priorities, it might also be important to provide opportunities to extend Amy's image of active learning by better understanding the purpose of the MIP definition and its potential to support students' construction of productive meanings. The MIP Team might accomplish this goal by identifying and discussing specific distinctions and connections between her definition and the MIP definition. In Table 14, I provide a comparison highlighting fundamental differences between Amy's image of active learning and the MIP definition.

Table 14

Comparison Between Amy's Image of Active Learning and the MIP Definition

Amy's Image of Active Learning	The MIP Definition of Active Learning
Predominantly student-focused	Predominantly instructor-focused
Unguided	Guided
Problem-solving activities	Constructing productive understandings

First, Amy's definition is primarily centered around the problem-solving activities of the student, whereas the MIP definition attends to the instructor's role in supporting students to construct particular meanings. Second, in Amy's descriptions of active learning, the students engage in the practices of a mathematician, and these activities are *unguided*. That is, students could engage in analysis and construct hypotheses independent of the nature of the content being taught or the meanings that are supported. Finally, her description of active learning is centered around students' problem-solving activities and is less attentive to the nature of the meanings students might construct from their engagement in these tasks. A purpose of identifying and discussing these differences explicitly is to support Amy to recognize these distinctions. Supporting her capacity to effectively operationalize the MIP definition in her instruction may also require the MIP Team to discuss the affordances of engaging in this process for enhancing students' learning.

Importantly, my suggestions are not intended to diminish Amy's desire to leverage students' feedback to improve the design of particular tasks; her attentiveness to the nature of students' thinking and her willingness to elicit their feedback reflect important qualities. Yet, an instructor's capacity to support students' conceptual learning is not only dependent on their attentiveness to the nature of students' current and developing meanings, but it also relies on the instructor's own image of what a productive understanding entails. Amy's confusion at why rates of change or slope are hard for students to understand might suggest the need for more thoughtful reflection to critically evaluate the nature of the meanings she intends to support. Her instructional design might be extended by conducting a conceptual analysis and reading literature that highlights the importance of supporting students' construction of multiplicative comparisons between the independent and dependent variables.

The MIP Team could provide contrasting examples to illustrate these two implications. The first example could support students to construct less productive meanings (e.g., conceptualizing constant rates of change additively) while the latter could support their construction of more productive meanings (e.g., conceptualizing constant rates of change multiplicatively as a proportional relationship between the two quantities). Consequentially, the MIP Team could discuss how these meanings impact students' understanding of future ideas in their current and future mathematics courses.

In the following section, I begin my discussion of Part II. The focus of this section is the presentation of my results from eight observations I conducted of Amy participating on a CoRD-like Team.[31]

Part II: The CoRD Meetings

[31] This CoRD was different from other CoRDs since a member of the MIP Team served as a "participant" on the CoRD. This unique setup provided an opportunity to examine the influence of the MIP Correspondent on other members.

The purpose of this case study is to better understand Amy's identity trajectory. I conducted thirteen interviews with her during which she discussed the three elements of inquiry, conceptual analysis, and her teaching philosophy more generally. To better understand the extent to which Amy's identity as an instructor might be influenced from her participation on a CoRD, I observed meetings between Amy, Kyle, (a lecturer at a large, southern institution in the United States who had not participated in any MIP activities), and a member of the MIP team, whom I will refer to as the MIP Correspondent. In the following sections, I discuss three central themes—*curiosity*, *practice*, and *reflection*—that I identified as being critical for guiding Amy's identity trajectory towards effectively operationalizing the three elements of mathematical inquiry.

Methodology

These meetings were typically held on a weekly basis and lasted approximately 50 minutes each. In total, I conducted eight observations of these virtual CoRD meetings. I provide the central focus of each of these meetings in Table 15.

Table 15

The Central Focus of Each CoRD Meeting

Meetings	Central Focus and Discussion Topics
Meeting 1	**Overarching Priorities in Calculus** • Broad goals for calculus • Student difficulties in calculus • Defining success in a calculus course
Meeting 2	**Broad Characterizations of Active Learning** • Images of active learning and how it is supported • Supporting active learning in the context of curricular artifacts • Evaluating an artifact which supports students' construction of productive meanings • Knowledge base required to develop high quality curricular resources • Interpretations of the MIP definition of active learning

Meeting 3	**Characterizing Other Descriptions of Active Learning: Part I** • Brainstorming about a topic • Discussing the MIP definition of active learning • Examining other active learning strategies
Meeting 4	**Characterizing Other Descriptions of Active Learning: Part II** • Grouping together other active learning strategies • Developing themes for these groupings
Meeting 5	**Characterizing Other Descriptions of Active Learning: Part III** • Characterizing active learning strategies from a student's and instructor's perspective
Meeting 6	**Contrasting the MIP Definition of Active Learning with Other Characterizations** • Contrast the MIP definition of active learning with other descriptions of active learning • Address a concern about the MIP definition of active learning.
Meeting 7	**Initial Discussion of Linear Approximation** • The location of linear approximation in textbooks and its necessity • The influence of linear approximation on other topics
Meeting 8	**Analysis of Curricular Artifacts Designed to Support Students' Learning of Linear Approximation** • Discussion of their analysis of curricular artifacts related to linear approximation

Analysis

I re-watched each of the recorded observations to reconstruct Amy's comments into sentences or phrases as shortened data bits. From the data bits generated from Amy's remarks, I provided conceptual labels to characterize her comments. I then classified these concepts according to a small number of categories and then subsequently organized the concepts in a particular category according to their different properties and dimensions (Corbin & Strauss, 2015). Sometimes, I created diagrams to provide an overarching image of my characterization of Amy's conceptions more broadly. While the focus was primarily on Amy's comments, I took detailed notes from comments made by Kyle and the MIP correspondent. Finally, I wrote memos from their remarks throughout these meetings. In the following section, I discuss the results from my analysis of this data.

Results

Meetings 1 and 2

The focus of the first two meetings was to uncover Amy's general goal structures related to teaching (Meeting 1) and then specifically in the context of active learning (Meeting 2). Throughout these first two meetings, the MIP Correspondent built rapport with Amy and Kyle by affirming their responses and acknowledging his own teaching struggles. Additionally, he directed their attention to thinking about active learning as contextualized in the MIP definition. For example, he highlighted the negative impact of online instruction on student engagement and prompted Kyle and Amy to discuss how they support students in being active. After Amy acknowledged her disappointment in having only a few students attending class in-person and receiving limited feedback, and after Kyle highlighted his approach to support active learning by asking questions, the MIP Correspondent redirected the discussion:

> Um, then there's, you know, active learning in the, in the sense of students actively engaging in the kinds of mathematical practices [Amy] you stressed earlier about pattern recognition, um, identifying quantities and relationships, and things like that, um, you know, even forms of reasoning, deductive reasoning, um. But even at a, even a more local level, this— active learning could be specific to the particular ideas that one is learning. Um, so, I'm curious to know if, if, you know, teaching let's say like, you know, limits versus, um, implicit differentiation versus Riemann sums, whatever. What is active learning look like in the context of supporting students' learning of specific ideas? Or, I mean I guess, what, what's unique about active learning in— the case of specific topics that one is teaching or learning?

The MIP Correspondent carefully redirected the conversation by guiding Amy and Kyle to not only consider students' engagement but also the *nature* of their engagement required to construct specific understandings. He continued to direct their attention later in the interview:

> So often, active learning is, is conceptualized as being supported through a set of pedagogical practices, right, like a teacher engages students in active learning if they do *x*, *y*, or *z*. Um, or, it, it's sometimes described as kind of characteristics of the learning environment or the instructional format or associated with a list of peda— pedagogical practices or repertoire of instructional strategies, these sorts of things. Um, but there are possibly characteristics of curricula that can support active learning that are different from instructional practices or pedagogical practices, and things like this. And so, I wanted to ask both of you, what are the features or characteristics of curricular resources that might support students' engagement in active learning?

The MIP Correspondent's comments subtlety highlight discrepancies between common characterizations of active learning and the MIP definition of active learning. Other characterizations of active learning require an instructor to implement a particular pedagogical practice, while the MIP definition of active learning necessitates an instructor to conduct a conceptual analysis so that they are positioned to engage students in the precise mental activity required to support their construction of a particular meaning. The MIP Correspondent's remarks prompted Amy and Kyle to consider features of instructional design that might support students' engagement in active learning.

During the first two meetings, the MIP Correspondent asked questions to elicit Amy and Kyle's broad priorities for teaching calculus and their images of active learning. Amy discussed students' unproductive orientations and described her envisioned identity of productive student engagement. These initial efforts to continually redirect the conversation provided opportunities to further elicit Amy's conception of active learning. For example, the MIP Correspondent showed Amy and Kyle some multimedia resources designed to support students' construction of productive meanings about rates of change. He then prompted them to identify characteristics of these resources that might be leveraged to inform their CoRD's creation of a curricular artifact. After Amy echoed the vagueness of students' language, the Correspondent asked her to discuss her expectations for prompting students to express meanings of a solution could make a difference:

> Yeah, yeah, so I think, right, so many students blindly go into things. Oh, I have to use this procedure, and this, uh, is my answer in the end. But like, I mean, just talking about related rates, um, why is the answer positive? Does that make sense? Um, and so them writing that stuff out I think highlights the fact that, okay if it's increasing that means, that, um,— well if it's positive it means it's increasing. Okay, I can conceptualize that.

Amy's comments suggest that she attended to the context having affordances for supporting students' capacity to "make and justify claims" in alignment with the MIP definition of meaningful applications. That is, she attended to the meanings that students are positioned to construct from their engagement in a particular task with an application.

In addition to her comments about unproductive orientations for student engagement, she also echoed Kyle's sentiments and discussed the importance of students being *challenged* and *critically thinking* by engaging in problem-solving activities:

When I think of active learning, it— there is this internal challenge, and it could be precipitated by an external person or whatever. Um, but yeah, I think a lot of it is this— it involves lines of thinking. Um, and so, some of it might be hard to see necessarily, but they're challenging their thoughts, they're, they're trying to put pieces together. Um, yeah, they're trying— ultimately, I think a lot of math is spotting patterns too, and, right, realizing when is a— when is it appropriate to use this particular technique and when is another technique a better option?

Amy's comments characterize active learning according to students' engagement in problem-solving activities by attempting to spot patterns, "put pieces together," or identify appropriate techniques. During the first interview, she stated that she wants students to leave her calculus class viewing themselves as problem solvers. Additionally, Amy discussed student problem-solving videos from the Calculus Videos Project (discussed informally by the MIP Correspondent and me at the beginning of the meeting) in response to a prompt for her to identify characteristics of curricular resources that might support students' engagement in active learning. These videos presented short clips of two students discussing their approaches to solve a particular problem centered around an idea in calculus. Amy described the video as interesting, because these problems "really challenged the students."

In the final minutes of this meeting, the MIP Correspondent asked Amy and Kyle to comment on the MIP definition of active learning which was displayed in the chat of the Zoom meeting. Amy highlighted how she might improve the definition:

Um, so, I think that like analysis and questioning, um, is yeah, um, is a big part of it. Um, and so, I think more emphasis should be put on that, like students— the questioning part and the analysis part. Um, because a lot of it just seems like, it seems like this has more of an end goal, like we get to the end. We, we figure it out. Um, or at least that's just how I read it. So, I think there could be more emphasis on questioning and analysis part.

Amy's comments characterize active learning according to students' engagement in mathematical practices (e.g., problem solving). Amy explained that the MIP definition of active learning emphasizes the "end goal," and her remarks suggest that the mechanisms that embody students' engagement in active learning (i.e., students' "questioning" and "analysis") are absent from the MIP definition. These comments highlight her emphasis on students' productive mathematical practices and are consistent with her comments throughout our interviews in Part I.

In summary, Amy's comments from these first two meetings demonstrate her attentiveness to the affective qualities of students' engagement (i.e., their mindset and orientation). Additionally, she emphasized the importance of students becoming problem-solvers and being challenged to think critically and spot patterns. In her critique of the MIP definition of active learning, she highlighted its underemphasis on students' analysis and questioning as mechanisms that support students' learning. Amy's contributions to the first two meetings suggest that there remain opportunities for her conceptions of active learning to be extended by developing a deeper understanding of the meanings behind the MIP definition of active learning, including its epistemological foundation, for supporting students' construction of specific understandings

Meeting 3

During the third meeting, the CoRD (1) worked to collectively identify a topic that would serve as the focus of curriculum development, (2) discussed their image of the MIP definition of active learning, and (3) began the process of classifying other MIP strategies of active learning (including their own definitions). The purpose of providing an opportunity for the CoRD to reflectively classify other definitions of active learning was to motivate their awareness of differences between how the MIP definition of active learning is constructed (i.e., to engage students in actions designed to support their construction of productive mathematical meanings) compared to other common interpretations (broadly focused on providing conditions for supporting student engagement). I also included their definitions in this classification process so that they would characterize their own image of active learning with other common interpretations of active learning.

Topic exploration. Prior to this meeting, I asked Amy and Kyle to reflect on areas of calculus that they found to be problematic to teach or where they identified a need for improvement. Amy's initial comments are consistent with her identity as an instructor to critically examine her own practices: "I mean I kind of laugh, because I could probably improve everywhere (*little exhale laugh*)." She stated that students struggle understanding graphical implications of calculus (e.g., maxes and minimums), referencing her recent experiences. Both Kyle and the MIP Correspondent identified a need to improve how they teach linearization or linear approximation. In particular, the Correspondent highlighted the implications of supporting students' construction of productive meanings around linearization since it connects ideas, foreshadows other ideas in the class (e.g., Riemann), and has other applications (e.g., Euler's Method). Kyle also discussed his experience teaching Calculus II when he realized that his students forgot about linear approximation. Additionally, Amy acknowledged that she could improve in how she teaches this topic: "It is admittedly something I've thrown out this semester, because you know, almost two weeks of snow cancellations. Um, but right— and then I justify it saying, well I don't do a good job teaching it anyway." She later acknowledged that in a recent meeting with a Calculus I and Calculus II committee focused on reorganizing topics, linearization was either not mentioned or labeled as a "maybe."

The MIP definition of active learning. For the next twenty minutes of the meeting, the CoRD discussed the MIP definition of active learning:

> *Students engage in active learning when they work to resolve a problematic situation whose resolution requires them to select, perform, and evaluate actions whose structures are equivalent to the structures of the concepts to be learned.*

Amy identified several critiques with this definition: (1) it does not incentivize students' capacity to learn from their failure, (2) it is hard to measure the extent to which students engage in active learning as defined, and (3) it may hinder students' creativity. Regarding (1), she indicated the difficulty in operationalizing the MIP definition of active learning:

> So, I'll admit, my first thought on that, is that becomes super tricky too, because I think so much of what we do in math is we reward correct thinking, right. I mean, I can't publish something unless I've actually proven something. Right, all my failures helped me learn, but I don't publish failures. Um, so, in part too, when I see this, I, I still want to encourage failures from students, because they're learning opportunities. So, then how do you assess that component as well? Or, how do you observe that? Or, how do you incorporate failures to help us get to the end goal?

Amy later identified journaling as an avenue to support students in reflecting on (and ultimately recognizing) their mathematical growth in understanding different ideas. Her comments suggest that she has critically evaluated features of the definition which have caused her some confusion or perturbation. In particular, she expressed concern that the MIP definition of active learning does not emphasize the importance of students' engagement in productive struggle.

Amy's last two critiques arose in our discussion of the final part of the definition: students' engage in actions "whose structures are equivalent to the structures of the concepts to be learned." This part of the definition is perhaps the most important component since it characterizes active learning according to the nature of students' mental activity. After the CoRD discussed the definition more generally, I asked them to specifically comment on this last phrase. Amy's comments during this discussion proved fruitful by revealing different concerns. I focus my attention on her comment that the MIP definition might hinder students' creativity.

Towards the end of our second meeting, she stated that the MIP definition could be improved by including characteristics of students' activities (e.g., questioning and analyzing). Earlier in this meeting, Kyle expressed his apprehension with the word "the structures" since it provides a narrow view for students' reasoning about an idea. Similarly, Amy described a limitation of the MIP definition as "pigeonholing" students and reducing their creativity to think about an idea.

> I mean, I completely agree that things need be outlined entirely, I think— in terms of guiding them to what you're hoping to get out of it. I think the thing I struggle with though is, is that pigeonhole idea, though, right. I know what I want them to get out of it. I know what structure and what concepts I want them to see or understand, but I also don't like pigeonholing. Right, I, I want the students to be more open minded I guess, um. And so, I struggle with that a bit.

Amy's concern about pigeonholing students' thinking is consistent with other remarks she made during our interviews. Moreover, her comments indicate her expectation that the MIP definition of active learning might limit students' capacity to be "open minded." Additionally, Amy's interpretation of *equivalent structures* in the MIP definition of active learning centered around her knowledge as a mathematician, thinking about two mathematical objects that are equivalent but can be discussed in different ways with different characteristics. These comments suggest that Amy conceptualized equivalence according to the structural equivalence of two mathematical ideas, and hence, perhaps not according to students' actions supporting their construction of particular meanings.

Six active learning strategies and two definitions. Towards the end of the meeting, I presented six active learning strategies from Lugosi and Uribe (2020), Amy's description of active learning, and Kyle's description of active learning (see Table 16).

Table 16

Six Active Learning Strategies and Two Definitions

1. Interactive presentation style
2. Group-work with discussion and feedback
3. Volunteer presentations of solutions by groups
4. Raise students' learning interest towards specific topics
5. Involve students in mathematical explorations, experiments, and projects
6. Continuous motivation and engagement of students
7. *Amy's description*: Students engage in active learning when they become practitioners/"researchers"/critical thinkers in their field of mathematics
8. *Kyle's description*: Students engage in active learning when they are collaborating (e.g., group work, iClicker).

Amy constructed her description of active learning during our interviews in fall 2021, and I created Kyle's description based on inferences I made from his comments at the end of Meeting 2. After seeing my description of his image of active learning, Kyle provided some clarification. Upon reflecting, he stated that he was unsure that students would be "engaging in active learning when they’re doing that, but I would say it is more evident that students are engaging in active learning to me when they do that.”

MIP correspondent. Throughout this interview, the MIP Correspondent provided guided comments to support Amy and Kyle to consider differences between these definitions. For instance, he stated that the MIP definition of active learning is unclear in measuring observable behavior since it defines active learning “in terms of the cognitive activity of the, of the learner.” He continued later suggesting that “perhaps, the best one can do is to design, you know, different kinds of formative assessment to, um, assess whether or not they have engaged in active learning. Not, whether or not they’re currently doing it.” He then implicitly contrasted the focus of the MIP definition with the other ones by hypothesizing that the six active learning strategies “focused on more observables.” Towards the end of the meeting, he reiterated these sentiments. The MIP Correspondent stated that if an instructor “attempted to operationalize, these, these principles, then that isn’t sufficient for having confidence that students are learning mathematics meaningfully or developing a conceptual understanding of a— the content of the curriculum.”

In their lengthy discussion about the final phrase in the MIP definition (i.e., the structures of students’ actions become equivalent to the structures of the concepts to be learned), the MIP Correspondent clarified his interpretation of the *nature of these structures* by discussing students’ productive and unproductive meanings associated with understanding constant rates of change. Students could view constant rates of change by interpreting 40 miles per hour as an additive relationship of discrete changes or by interpreting 40 miles per hour as a multiplicative relationship which reveals that one’s distance traveled is always 40 times as large as their corresponding elapsed time. He stated that these statements “are very different, and they reflect different conceptual structures about the meaning of rate of change." He concluded his discussion by asking Kyle and Amy to share their thoughts about that way of thinking about the structures of concepts, or to identify criticisms. The other CoRD members response to these comments was revealing.

Kyle stated that it was a "good point" that "the way that the textbook writer or instructor presents it is for that instructor, um, the structure of the concept." A little later, he stated that for the developer or presenter of a particular resource "it's going to be very dependent on what structure, what that structure looks like to them at least, and that will affect how they present it, which in turn affects students' construction of their understanding." Kyle's comments reveal that he became implicitly positioned to recognize the importance of conducting a conceptual analysis and its implications for instructional design and, ultimately, students' learning. On the other hand, Amy's comments reveal that while she agreed that there should be an outline guiding students' thinking, she still expressed concern about "pigeonholing" their thinking. Hence, the Correspondent's comments did not dissuade Amy's concern that the MIP definition of active learning might hinder students' capacity to engage in creative problem-solving activities that she values. In sum, these remarks from Amy and Kyle suggest the importance of attending to their constructed meanings and concerns associated with the MIP definition since their comments reveal potential barriers to effectively operationalizing it.

Reflection questions. At the end of this meeting, I asked both Amy and Kyle to reflect on the following questions (see Table 17). The first two questions were intended to elicit Amy and Kyle's initial impressions and lingering questions that might exist after the meeting, while their responses to the third question might reveal the extent to which this meeting supported their *curiosity*. Finally, the fourth question prompted them to discuss their conception of active learning, specifically in comparison to the MIP definition.

Table 17

Reflection Questions After Meeting 3

1. What are your initial reflections from today's meeting?
2. What was confusing from today's meeting? (No need to answer if you didn't find anything too confusing.)
3. What might you be interested to learn more about from today's discussion
4. Identify ways in which your definition of active learning is or is not consistent with the MIP definition?

Amy's written response to questions three and four echoed concerns she expressed during this meeting. In her response to the third question, she wrote "I guess I'd like to see what things people have created that they say engage students in active learning without stifling students' creativity."[32] Her response to Question 4 is provided below:

> The MIP definition has an end goal. Students are trying to solve a particular problem (overcome a problematic situation). With my definition, analysis and questioning are highlighted. The end goal isn't as important. Questioning oneself is a big part of my definition. I'm sure it is a part of the MIP definition, but it doesn't stand out as much, in my opinion. The MIP definition almost reads like a formula, which may oversimplify the process. Whereas my definition is more convoluted. (I copied this over from what I sent last semester.)

These responses reveal her priorities and commitments to avoid limiting students' creativity and foster their mathematical activity in ways that align with the practices of a mathematician by questioning, hypothesizing, and analyzing. Her comments suggest that it might be productive to

[32] In my final observation meeting, we discussed a lab assignment on linear approximation that she stated was "less procedural."

support her understanding that engaging students in active learning as defined by the MIP requires that the instructor supports students' construction of specific understandings of the mathematical content. Successfully achieving this goal (i.e., supporting students to identify mathematical relationships and construct productive meanings), however, is enhanced by students' engagement in the mathematical practices that she values as a mathematics instructor (i.e., problem-solving activities).

Meeting 4

The focus of Meeting 4 was to facilitate a discussion around classifying the different characterizations of active learning. The purpose of presenting these different definitions and discussing them was to position Amy (and Kyle) to be more attentive to differences between these definitions and the MIP definition based on how other active learning strategies are more broadly characterized. I presented the six active learning strategies identified in Lugosi and Uribe (2020) in addition to Amy's description and Kyle's description. To facilitate this discussion, I shared my screen and showed the group these eight characterizations. I encouraged them to categorize these definitions according to two groupings (having too many themes could potentially provide unnecessary complexity to the focus of contrasting the MIP definition of active learning with other characterizations), and they initially achieved this task. Through my facilitation, they later decided to classify these eight descriptions according to three themes (or three different colored "buckets") as provided in Table 18

Table 18

Three Categorizations of Eight Descriptions of Active Learning

Red Bucket: Fostering mathematical practices	Green Bucket: Environmental features that support students' engagement in active learning	Blue Bucket: Affective (hard to observe, end result)

(exploration, asking questions, problem-solving)		
1. Interactive presentation style	1. Interactive presentation style	6. Continuous motivation and engagement of students
5. Involve students in mathematical explorations, experiments, and projects	2. Group-work with discussion and feedback	4. Raise students' learning interest towards specific topics
7. Amy's description: Students engage in active learning when they become practitioners/"researchers"/critical thinkers in their field of mathematics	3. Volunteer presentations of solutions by groups	
	8. Kyle's description: Students engage in active learning when they are collaborating (e.g., group work, iClicker).	

Classifying eight descriptions of active learning according to three themes. The three CoRD members discussed how best to classify these eight characterizations of active learning according to different themes.[33] I now discuss each of these three themes (or "buckets" as we called them in the meeting) separately.

The green bucket, which contained Kyle's description of active learning, is primarily centered around *environmental features* involving students engaging in group work or giving presentations in class. The red bucket is more focused on students' *mathematical practices* like a mathematics researcher's activities of exploration and problem-solving. Initially, items (4) and (6) in the blue bucket were both mixed in the red bucket and the green bucket. In developing the themes for each bucket, however, it proved more productive to separate these two items in their own bucket (upon my suggestion and encouragement), which was labeled as more "affective" by the MIP Correspondent. Removing these two descriptions from the red bucket enabled Amy to

[33] As seen in Table 18, "1. Interactive presentation style" is repeated in both the red bucket and the green bucket. There was some discussion as to whether this item pertained to students' interactive presentation style or the instructor's style.

conclude that it “really highlights for me that these red ones are definitely fostering mathematical practices,” echoing Kyle’s characterization of the red bucket earlier in the meeting.

MIP correspondent. The MIP correspondent’s comments throughout the interview helped provide contrasts between the MIP definition of active learning and other characterizations, even though this contrast was not identified explicitly. For example, consider his comments about grouping items together after I asked him to elaborate on his conclusion that (1), (2), (3), and (8) were similar:

> You know, interactive presentations, group work, presentations, um, group work, these are all sort of identifying instructional or pedagogical practices that can facilitate or encourage active learning. That, that’s kind of what I’m seeing common to those. They, they’re, they’re indicating some kind of, yeah, in— instructional or pedagogical practice that— might foster active learning.

His comments characterized the nature of an instructors’ actions for supporting students’ engagement in active learning as being centrally focused on the implementation of pedagogical practices. In other words, these characterizations include little attention to the nature of students’ actions required to construct particular meanings. The MIP Correspondent later described the green bucket according to environmental conditions that enable active learning to occur, providing a meaningful contrast to the focus of the MIP definition on the cognitive activity of the learner.

In discussing the blue bucket, the MIP Correspondent described how an instructor could design a context that motivates or engages students in an unproductive way (e.g., supporting students to construct an understanding of rate of change based on an additive understanding). He later stated that motivation and engagement could be "necessary for developing productive mathematical conceptions, but they're not sufficient." Finally, the MIP Correspondent characterized the red bucket (which consisted of (1), (4), (5), (6), and (7) at that time), according to the practices of a mathematician. He stated that some of these "focus on or emphasize, um, or to some extent try to provide the conditions for supporting kind of productive mathematical engagement or mathematical practices, right."

The MIP Correspondent's comments are echo characterizing Amy's description of active learning, conceptualized according to the practices of the mathematician. He described these practices in terms of "habits of mind" that characterize a student's particular orientation in their mathematical engagement. Importantly, Amy indicated her satisfaction with respect to an emphasis on mathematical practices in her post reflection writings to Question 5 in Table 19 This intersubjective agreement on the classification of these buckets and the nature of these themes is important for positioning Amy to recognize distinctions more productively between other characterizations of active learning that focus on mathematical practices and the MIP definition that emphasizes the nature of students' cognitive activity required to construct particular meanings.

Towards the end of the meeting, the MIP Correspondent provided different characterizations for the domain associated with each of these three buckets: *psychological* (red bucket), *environmental* (green bucket), and *affective* (blue bucket). His characterization of the green bucket was particularly enlightening. First, I wanted to classify these three buckets by the end of this meeting, and there was little time remaining in the meeting. Second, it is an accurate characterization of this bucket since the focus is primarily on the instructor providing conditions or an environment for students to be engaging in active learning. After these comments, I provided opportunities for Amy and Kyle to agree or disagree and share their thoughts. Both agreed, and Amy stated that she was "really happy with that," and that characterization represented an "accurate depiction of what is going on."

Reflection questions. I provided the following reflection questions for Amy and Kyle to respond via email after Meeting 4 (see Table 19). The first three questions were the same as before, but the latter two were different.

Table 19

Reflection Questions After Meeting 4

1. What are your initial reflections from today's meeting?
2. What was confusing from today's meeting? (No need to answer if you didn't find anything too confusing.)
3. What might you be interested to learn more about from today's discussion?
4. What were your major takeaways from today's classification of different characterizations of active learning?
5. Was there anything that was said during today's meeting (from one of the other two or myself) that was particularly impactful (either positively or negatively)? If so, then please describe what they said, and how it impacted you.

Amy stated in her response to question one that she was "pleased" with the ultimate decision to characterize these eight descriptions into three buckets (instead of only two). Her affirmation of our approach is important for positioning her to recognize differences between these three classifications and how the MIP definition is conceptualized.

In responding to Question 4, Amy stated that the instructor is the main catalyst (and not the student) in fostering active learning as defined according to the red and green buckets, and the focus is more on students' activities in the blue bucket. Her comments suggest that the instructor's role is of primary importance. This begs the question: What is the nature of the instructor's role for supporting a particular characterization of active learning? If active learning is defined according to working in groups, then the instructor needs to provide an environment where group work is encouraged. If active learning consists of explorations, then the instructor should provide interesting problems targeted towards eliciting students' motivation and interest. If active learning is conceptualized according to students' construction of productive meanings, then an instructor should (1) engage in the necessary reflection required to critically evaluate the nature of students' meanings and productive meanings that they intend to support, and (2) provide scaffolding tasks to support students' construction of these meanings so that the structure of students' actions might become *equivalent to the structures of the concepts to be learned*.[34] I suspected that addressing this question about an instructor's role to support students' engagement in active learning would be a central feature of Meeting 5. The discussion in Meeting 5, however, led me to an alternate conclusion.

Meeting 5

The focus of Meeting 5 was to discuss additional ways of characterizing the eight descriptions of active learning according to (1) their different domains and (2) how the student is conceptualized in this process. The purpose of discussing (1) was to position Amy and Kyle to recognize different domains in which descriptions of active learning can be characterized, ultimately and lead to future discussions about the MIP definition being classified according to the cognitive domain. Recognizing this similar focus to the domain of the red bucket could naturally lead to discussions about the differences between these domains. Moreover, the purpose of discussing (2) emerged from my awareness that how an instructor supports students' engagement in active learning depends on how they conceptualize what it means for *students* to be engaged in active learning and according to the domain in which it is defined.

[34] This phrase is the last part of the MIP definition of active learning.

Three domains. I presented the MIP Correspondent's characterization of the red bucket, green bucket, and blue bucket (i.e., psychological, environmental, and affective domains, respectively), and I asked the CoRD to discuss if they agreed with these characterizations or desired to use a different word to characterize these buckets. Kyle expressed some concern with using the word psychological for the red domain since it carries certain connotations about the individual's psychology and state of being. Amy suggested using the word cerebral, and we settled on that characterization. Additionally, both Amy and Kyle were satisfied with the "environmental" and "affective" labels for the green and blue bucket, respectively.

Characterizing students' engagement in active learning. I asked the CoRD to describe the minimal activity that would be required for students to engage in active learning according to the three buckets. I intentionally used the word "minimal" when asking this question to elicit responses that reflected little attention to students' constructed meanings. In other words, if active learning is defined according to these three buckets, then an instructor might foster active learning minimally merely by providing creative problems for students to engage in mathematical practices (related to the red bucket), by providing opportunities for group work (related to the green bucket), or by providing interesting contexts to explore (related to the blue bucket). Since these characterizations of active learning are less explicit about students' construction of productive meanings, then this discussion might provide a meaningful contrast to the purpose of the MIP definition of active learning. Kyle, however, had different interpretations of these activities.

When discussing a word to characterize the red bucket, Kyle stated that our purpose is focused on active learning "which is going to require mental engagement. So, it— if we're going to label this mental, then we should label everything mental." Towards the latter part of the interview, he revealed his underlying assumption:

> I, I kind of was working under the assumption that this was all sort of relating to mental activity, and maybe we could find like a subdomain of cerebral or cognitive, you know, environmental. So, they're responding to factors of their environment. Affective, their beliefs, what they think about, uh, what they're doing. I kind of had a— this idea. This is a framework of all mental activity and then this is how we divided it.

These comments were surprising to me, since I was unaware of his underlying assumption, and they have important implications. If active learning is characterized according to students' cognitive activity when discussing these other buckets, then assessing ways that an instructor might foster active learning provides a less meaningful contrast to the MIP definition. Hence, future discussions would need to be more centered around the nature of students' mental activity. These discussions are important since one can characterize active learning differently within the cognitive domain. For example, consider Kyle's comments:

> And again, it's not guaranteed with these green, uh, categories, but if a, if a student is doing at least that, at least engaging somehow with the understandings of the other students, uh, whether they agree or disagree, uh, then I think active learning is taking place either by reinforcement or reconstruction.

His comments relate to the cognitive domain but are focused on students' negotiation of different meanings during a group discussion independent on the nature of these meanings being constructed. Kyle's comments during these interviews provided an important insight: others may characterize active learning according to students' cognitive activity. In this case, providing distinctions between the MIP definition and other active learning strategies requires a discussion in the realm of the cognitive domain.

Supporting students in developing productive orientations. A theme that emerged from Amy's comments was the importance of and difficulty with supporting students' development of productive mathematical orientations. In our discussion of the minimal activity required for students to engage in active learning, Amy highlighted productive struggle activities (not procedurally focused) of the student. Later, she discussed videos from the Calculus Videos Project in which two students are discussing their approach to solve a calculus problem related to a specific topic. Following this video, which is intended to support students to critically reflect on their own meanings, there are other short videos that clarify productive ways of understanding that idea to resolve the confusion. Amy stated that the initial videos can illustrate a good "model" of behavior, and she intentionally does not show students the follow-up videos: "I, I purposely don't show them this next video, because I (*sigh*), yeah, I— that's how math is always taught. You do this. Period." Later, she discussed how students "need to see us struggle productively as well" but indicated the difficulty in this task since (1) we know the content well and (2) feigning this lack of knowledge would appear "made up" to students. Additionally, in response to a reflection question about what she would be interested to learn more about from the discussion during the meeting, she wrote, "Have others shown their students what productive struggle looks like and why it is beneficial?"

Her discussion of productive mathematical orientations is a central feature of the MIP definition of academic success skills in relation to productive struggle, critical thinking, and perseverance. I contrasted this focus on students' productive mathematical orientations with the MIP definition of active learning in the following meeting.

MIP correspondent. In responding to my question about the minimal activity required for students to be engaging in active learning, the MIP Correspondent discussed the green bucket. He stated that an instructor could provide environmental conditions intended to promote active learning, but students may not necessarily be engaged in active learning. He later stated the relevance of this insight for their future work as a CoRD, commenting that

> if ultimately our task as a CoRD is to design instructional resources that engage students in active learning, these resources to some extent, or to the greatest extent possible have to not just promote particular environmental conditions or ways of structuring the learning environment, not just support students affectively, um, and not just attend to cognitive characteristics of— the instructional resources, but to some extent, try to coordinate all three in a manner that's coherent, um, so that whatever minimal activity is required for students to engage and whatever resources we produce will maximize the potential that that engagement will result in active learning.

His comments suggest the importance of (1) considering all three domains and (2) recognizing the insufficiency of each domain individually. Kyle heeded his remarks. Later, I offered a description for the green bucket to satisfy the minimal activity required for students to be engaged in active learning as *Giving students'* [sic] *opportunities to engage in groups or give presentations*. Kyle responded, stating that giving students opportunities does not guarantee that they are engaging in active learning:

> Just giving them the opportunity though, doesn't mean that they're going to engage in active learning. Anybody who has ever done a group project knows it's always that one person who did not, uh, participate. And I think that goes back to what [MIP Correspondent] was saying. Even if those, uh, environmental conditions were satisfied, active learning may or may not happen.

Kyle's comments suggest that he was attentive to the MIP Correspondent's comments and aligned with his perspective. The MIP Correspondent then affirmed Kyle's comments that providing environmental conditions can support active learning but not guarantee its occurrence:

> [Kyle], what I heard you just emphasize is that, when talking about active learning, I think it's really important to separate ways of supporting it from descriptions of what it is, um, and those two things are very often conflated, right. Like active learning, you know, what is active learning? Well active learning happens when, dot, dot, dot. No, no, no. What is, what is it? Not, not how's it supported? And it's really hard, actually. I'm not being critical. It's really hard to, to define the thing without talking about, you know, a, a whole variety of pedagogical strategies and things that we've— that we're all very familiar with to try to encourage active learning.

These comments provide further opportunity to understand how instructors' conceptions of active learning might be conflated with their image of how it can be *supported*. The MIP Correspondent stated that

> when instructors tend to think about active learning, it's just, you know, can I get them to not be passive, and I think what we're trying to struggle with here is not just can they be doing something, but what, what is the nature of the activity we need them to engage in

to learn mathematics productively. Right, because we can— I guess this is what a lot of these, these eight active learning descriptions suggest to me is that all of them either facilitate students' activity or are directly, you know, statements about character of students' activity or something, but, um, what we really care about is not just that students are doing something, but what is their— what are they doing, and how are their actions supporting their development of productive ways of reasoning mathematically, uh, developing productive ways of understanding ideas. And so, the term active learning as it's generally conceptualized is just sort of the complement of passive learning doesn't get at a lot of these important features that I think we need to consider when we want to think about, for whatever topic we select, what is the nature of the activity that re— that's involved in students learning this idea productively and positioning them to make connections to other ideas.

His comments highlight potential ways in which active learning can be characterized based primarily on students' *activity* and less focused on the nature of their *learning*. These remarks provide a contrast to the MIP definition of active learning and its focus on the purposeful design of tasks to provide opportunities for students' to engage in the actions necessary to construct particular meanings an instructor considers productive.

Reflection questions. I provided the following reflection questions for Amy and Kyle to respond to via email after Meeting 5 (see Table 20). The first three questions were the same as before, but the final two were different.

Table 20

Reflection Questions After Meeting 5

1. What are your initial reflections from today's meeting?

2. What was confusing from today's meeting? (No need to answer if you didn't find anything too confusing.)
3. What might you be interested to learn more about from today's discussion?
4. What were your major takeaways from today's classification of different characterizations of active learning?
5. Was there anything that was said during today's meeting (from one of the other two or myself) that was particularly impactful (either positively or negatively)? If so, then please describe what they said, and how it impacted you.

Amy's responses to the first and third questions indicate her attentiveness to the difficulty in defining active learning. Regarding the first question, Amy wrote, "Active learning can be hard to observe but possibly harder to define." Moreover, her biggest takeaway from the discussion was that "we can more easily identify what isn't active learning."

Kyle's comments suggested an attentiveness to the MIP Correspondent's remarks in his response to the last question:

> [The MIP Correspondent] mentioned that much of what we were doing was describing what must take place in order for active learning to occur, not what active learning is; he then went on to concede that is a difficult distinction to make. I now wonder how not having a workable definition for what active learning is will affect our ability to develop support for it.

Kyle's comments are particularly insightful. His response indicates that he is attentive to the MIP Correspondent's comments with respect to how active learning is often conceptualized (i.e., according to how it is characterized or being supported) rather than how it is defined. His comments perhaps reflect a curiosity for having a working definition of active learning that can be productively operationalized.

Meeting 6

During this meeting, I presented a PowerPoint to the group, and we discussed (1) previous comments, (2) my characterization of their images of active learning and the MIP characterization, and (3) a concern presented during a previous meeting regarding the MIP definition of active learning. I discuss these three items in the following sections.

Previous comments. On the PowerPoint, I presented to the CoRD a rough transcription of comments made by the MIP Correspondent during the previous meeting.[35]

> When instructors tend to think about AL, it's just, can I get them to not be passive, and I think what we're trying to struggle with here is not just can they be doing something, but what is the nature of the activity we need them to engage in to learn mathematics productively, right, b/c we can, I guess this is what a lot of these AL descriptions suggest to me is that all of them either facilitate students' activity or are directly statements about the character of students' activity or something, but what we really care about is not just that students are doing something but what are they doing and how are their actions supporting their development of productive ways of reasoning mathematically, developing productive ways of understanding ideas, so the term AL as it's generally conceptualized just as the complement of PL doesn't get at a lot of these important features that I think we need to consider when we want to think about for whatever topic we select, what is the nature of the activity that is involved in students learning this idea productively and positioning them to make connections to other ideas.

While the MIP Correspondent verbalized these comments during the previous meeting (Meeting 5), I talked after his comments instead of providing an opportunity for Amy or Kyle to respond. Hence, I provided these comments in written form during Meeting 6 for Amy and Kyle to respond to directly and to reflect on the nature of the activity required for students to engage in

[35] AL represents active learning, PL represents passive learning, and b/c represents because. Additionally, this was an unedited version of my interpretation of his comments.

active learning as conceptualized according to the MIP definition of active learning. These comments were also helpful for priming the focus of the discussion in Meeting 6. Amy said the following:

> I mean, yeah, I think this is on point in terms of, right, its— it gets at the fact that students can be doing something, but its— we want to be supporting, and we want to— them to be thinking about what they're doing and why are they're doing it, and the actions behind it that make it productive. Um, and, and I think really that's the tricky bit, um. So, yes, I think this is exactly what were— we should be getting at.

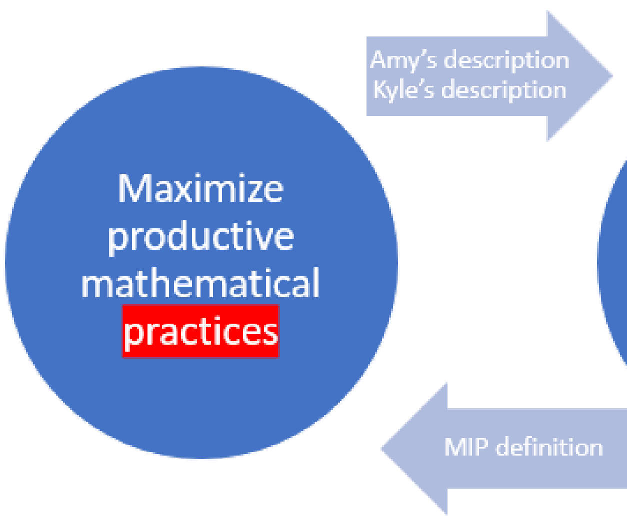
Amy's comments suggest that she (1) attended to students' actions and (2) recognized the difficulty in supporting students' engagement in productive actions.

Characterizing the MIP definition of active learning and their definitions of active learning. The central focus of this meeting was to present my characterization of how Amy and Kyle conceptualized active learning in contrast to the MIP characterization. To accomplish this, I presented the image displayed in Figure 5.

Prior to showing this image in the PowerPoint, I primed this discussion by highlighting a quote from the MIP Correspondent during our previous meeting in which he stated the importance of attempting to coordinate the cognitive, affective, and environmental perspectives. I first described their characterization of active learning according to "Maximize productive mathematical practices." I discussed how both Kyle's characterization (e.g., defending, rebutting) and Amy's characterization (e.g., problem solving, critical thinking) are illustrative of these mathematical practices that we desire in an ideal student. Then I discussed the right image, "Maximize productive understandings of content" and how the MIP definition is focused on supporting students' specific understandings of the content. Notice the emphasis on distinctions between supporting students' engagement in productive mathematical *practices* and supporting specific understandings of the *content.* I acknowledged that students' understanding of the content is insufficient if they do not have the appropriate mathematical practices. Hence, both are different but very important.

Figure 5

Characterizing Active Learning According to Different Foci

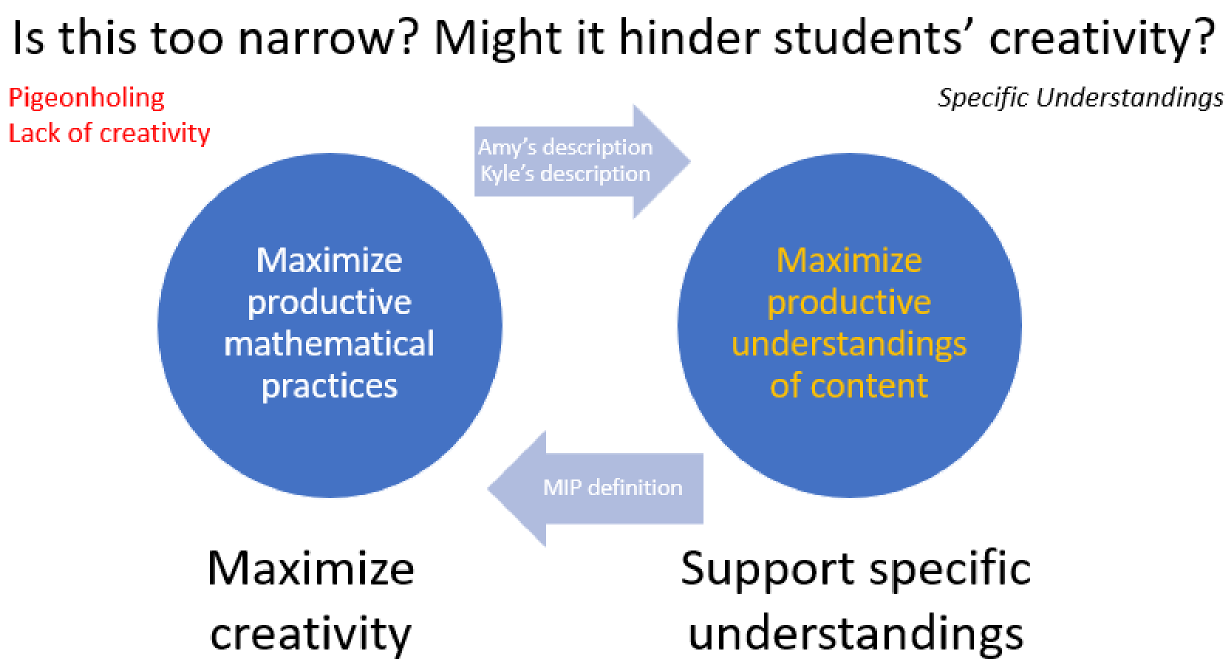


Amy's response suggested that this interpretation represented an unfair characterization since mathematical practices are "embedded in content. Um, I— we're not going to miss out on the content." She later affirmed that she agrees with the characterization in Figure 6, but the content is "in the background" when they are discussing maximizing mathematical practices. Presenting this contrast was successful in providing a contrast between these two characterizations, but Amy felt that the description unfairly presented her interpretation as not focusing on the mathematical content. During the following meeting, I modified this original characterization slightly to accommodate Amy and Kyle's comments.

Addressing a concern about the MIP definition of active learning. During Meeting 3, Amy critiqued the MIP definition of active learning by indicating that it may hinder students' creativity. To address this concern, I presented the following slide (see Figure 6).

Figure 6

Addressing Amy's Critique about the MIP Definition of Active Learning.



I discussed the importance of maximizing students' creativity in the context of students' *mathematical practices*, while also supporting students' understanding of ideas in specific ways. Hence, we want to encourage students' creativity, but the "narrowness" is necessary if we are to support specific understandings of ideas that we, as instructors, consider more productive. Amy's response to this characterization indicated her awareness and approval of this distinction I was making: “So, I, I like the way you framed it. Um— you’re right, the creativity, not having this narrowmindedness does come into the pra— the productive mathematical practices.”

I wanted to provide some specific examples to illustrate the MIP focus on maximizing students’ productive understanding of comment. I provided three specific examples, and I discussed the first one briefly. I described a productive understanding for constant rate of change is to view it as a proportional relationship as opposed to consisting of additive changes.

MIP correspondent. The MIP Correspondent responded to his previous comment (presented earlier) about the importance of considering the nature of students’ activity:

> You know, if you observe teacher teaching, you can infer pretty quickly whether the teacher is acting on the assumption, either explicit or implicit, more likely implicit, but acting on the assumption that students’ learning is a product of students’ perception of what the teacher says and does, or is students’ learning a product of the actions in which they as learners engage. Um, and, this is effectively the difference between, uh, an empiricist and constructivist epistemological stance, but it doesn’t have to be made explicit in any formal terms. It’s just, are you a— is, is an instructor acting in ways to get students to perceive, either explanations or demonstrations, or is the instructor providing opportunities for students to engage in problem solving behaviors, um, cognitive activity, abstractions, deductions, generalizations, and so on that the instructor anticipates is, is necessary for support students— to support their construction of meaning,

He continued stating that most agree that learning results from an action, leading to the immediate conclusion: “If, if not all activity is equally consequential in supporting students’ construction of meaning … what’s the nature of the actions that the students have to engage in?”

The MIP Correspondent's comments prompted the CoRD to reflect on their purpose in designing tasks to support students' learning of ideas and illustrate a fundamental epistemological distinction between the nature of an instructor's actions in the classroom as either empiricist or constructivist. Finally, his comment about the consequential nature of students' actions, if learning is indeed a product of actions, emphasized the importance of critically evaluating the nature of one's own meanings and conducting a conceptual analysis in service of supporting students' construction of productive ways of understanding.

Towards the end of the interview, he highlighted important meanings associated with linear approximations (e.g., variation, proportionality, multiplicative structures of rates of change). These comments lead into the primary focus of our next meeting discussing the meanings the CoRD intends to support.

Reflection questions. I provided the following reflection questions for Amy and Kyle to respond to via email after Meeting 6 (see Table 21). The first five questions were the same as in the last meeting, but the sixth question was an addition. I included Question 6, which I posed as a reflection question after Meeting 3, because we discussed the MIP definition in this meeting.

Table 21

Reflection Questions After Meeting 6

1. What are your initial reflections from today's meeting?
2. What was confusing from today's meeting? (No need to answer if you didn't find anything too confusing.)
3. What might you be interested to learn more about from today's discussion?
4. What were your major takeaways from today's classification of different characterizations of active learning?
5. Was there anything that was said during today's meeting (from one of the other two or myself) that was particularly impactful (either positively or negatively)? If so, then please describe what they said, and how it impacted you.
6. Identify ways in which your definition of active learning is or is not consistent with the MIP definition.

Amy's responses to three of the reflection questions (i.e., (1), (4), and (6)) suggests that (1) she still valued students' mathematical practices, (2) she recognized the distinction between these practices and the MIP focus on the content, and (3) her curiosity to consider students' productive thinking was stimulated.

From her initial reactions, Amy wrote that she is "still of the mindset that we should focus on mathematical practices" and "without these skills, it makes it harder to present and dive into the content." The implications of (1) suggest the importance of implementing discussions centered around students' mathematical practices in addition to supporting students' specific understanding of the content in the upcoming meetings. Regarding (2), consider her response to the fourth question:

> Due to the discussion and reflection after, it was emphasized that I really do value mathematical practices, even at the expense of content. If we teach students how to think, they can pick up content so much quicker. I think that ties into the breadth vs. depth discussion. (Our group hasn't had this discussion, but I usually fall into the depth category for similar reasons.) But there are two major questions lingering for me. 1. How do we best teach our students to think productively? 2. How do we assess it?

Her comments suggest that she recognized her focus on students' mathematical practices not only from the discussion but also the "reflection after." Moreover, in her response to the final question, she wrote that her characterization of active learning is "very much less content dependent than the MIP definition." Finally, while her first question in the previous revealed her attentiveness to students' thinking, it may be contextualized within the domain of productive mathematical practices. Perhaps, however, Amy's remarks suggested a curiosity for conducting a conceptual analysis in service of identifying and clarifying productive understandings that we intend to support our students in constructing.

Meeting 7

The focus of the seventh meeting was to allow the CoRD to have time to discuss linear approximations (their selected curricular topic) and the meanings that they intend to support. During this meeting, they began by organically discussing linear approximations. While Amy described students' understanding of linear approximation as a foundational idea that connects to many areas (e.g., the Fundamental Theorem of Calculus, Riemann Sums), she also indicated that it was not essential to teach. Additionally, she revealed her own dissatisfaction with teaching it as a "stand alone" topic and described her meanings behind understanding linear approximation.

Foundational but nonessential? Amy stated that linear approximations can "pop up in the various other main topics" and indicated that it might be productive to present it early in the curriculum (e.g., before limits). While she considered it foundational, she referenced discussions with her colleagues to make linear approximation section optional to teach, since there is "really no intellectual value." She continued by expressing concern for doing this:

> But in the back of my mind as we're all— we're going through this process, too, of deciding what goes, what stays, how long we spend on each topic, I think in the back of my mind I'm thinking, well, how am I going to vo— motivate this sort of stuff, or how do I tie in various things. And, I think that's part of the linear approximation idea is that, well, right, I, I can tie it into almost anything. So, in a way it, it, it's not a stand-alone section. Um, I don't think it is a stand-alone section.

Amy's comments reveal her expectation that students reason about linear approximations throughout the course since she can "tie it into almost anything." She also identified the challenge of viewing linear approximations narrowly constrained to a specific section. These remarks reflect her distal goals to motivate topics and her awareness that supporting students' meanings about linear approximations could enhance this process. Later in the meeting, she echoed similar sentiments, stating that "we could get by with it being omitted for the most part" but "it does have the ability to support students." She identified Riemann sums and the Calculus Video Project's presentation of the Fundamental Theorem of Calculus as two topics which could be scaffolded by students' understanding of linear approximations, but also indicated that she could see "not doing linear approximations and being okay in those settings as well." These comments suggest that she valued the implications of students' meanings for linear approximations but did not consider these meanings as essential to understand these topics.

Dissatisfaction with teaching linear approximations. During the meeting, I asked Amy to identify the meanings of linear approximations she intended to support in her teaching. She explained that she supports students' meanings by discussing how one can approximate values on small intervals and leverage intuition of constant rates of change on those intervals. She indicated the difficulty of teaching linear approximations calling it a "tricky" topic that she usually teaches as a "stand-alone" section. Upon my prompting, she elaborated on her dissatisfaction, stating that in part it is "rushed," and in part, "they don't truly see the value of it. Right, it it's not continuously being brought up. It's a stand-alone thing." She negatively reflected on her approach to tell her students that engineers use linear approximation continually: "What does that mean? How do you do it all— why do you do it all the time? So, I, I need to do a better job supporting it with other topics," where the topics could be rates of change or related to Riemann sums (i.e., area under the curve). These comments reveal Amy's distal goals to continually improve her capacity to motivate students' understanding of ideas and reflect her critical evaluation of her own practices.

MIP correspondent. During this meeting, the MIP Correspondent highlighted the pervasive influence of linear approximation on other topics in calculus. He indicated that linear approximation is the essence of calculus: "it's about approximating curved things with straight things." Moreover, he affirmed Amy's concern about including linear approximation in the curriculum and expressed his own sentiments:

> I, I think— I agree with you very much [Amy] that in the past, um, there have been some semesters where I have not included linear approximation among the required set of, uh, sections to be covered in our Calc 1 courses, because the— I just thought the exercises weren't very useful. Um, but that doesn't mean a focus on linear approximation wasn't— it doesn't mean that there weren't opportunities to emphasize the, the essential idea of linear approximation elsewhere.

These comments suggest his goals to affirm Amy's concerns while also influencing her perspective. He followed up his comment a little later by discussing how leveraging meanings for linear approximations relates to supporting students' understanding of average and instantaneous rates of change. He stated that he thinks

> ideas of linear approximation are reflected in this. Specifically, the average rate of change and the instantaneous rate of change. And so, um, I think early on in the course, you know, one can be intentional about just, yeah, approximating curved things with straight things and, and using that as kind of a foundational idea. And then when we get like to, uh, you know, later ideas, there are lots of places where this multiplicative structure,

> change in function value is approximately equal to a derivative times change in the input. There's lots of places where that comes up, um. I don't do a very good job I think, because every time they do come up, my students struggle to invoke those meanings. Um, I don't know how to do better, uh, but I think there's a real need to do better. And think that— actually confining linear approximation to a single section is not going to help that.

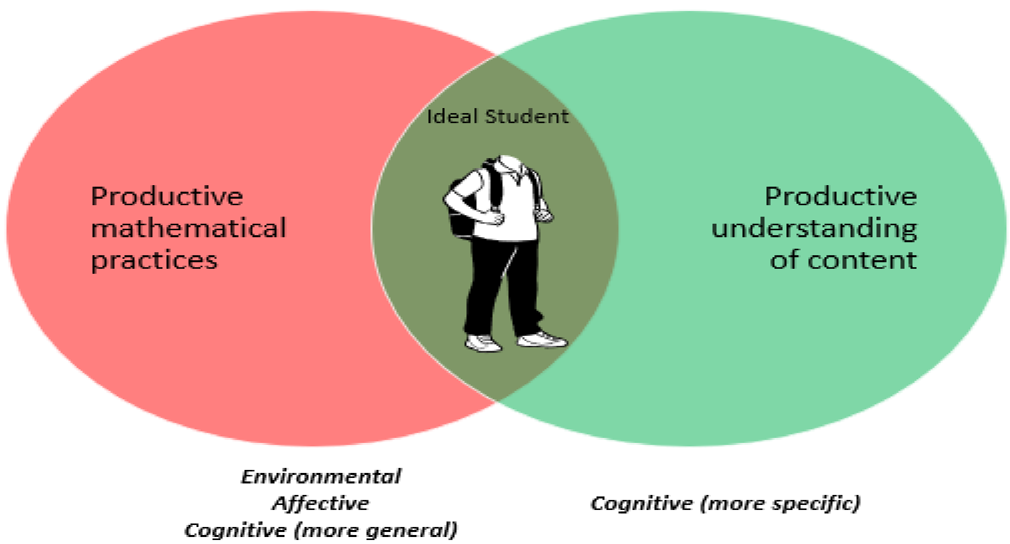
When I asked for Kyle and Amy's response to this comment, Amy indicated that she agrees: "it's this re-emphasis over and over again. It's ty— it's a tying in theme." His remarks describing linear approximation ideas as "foundational" suggest the importance of critically evaluating the topics we teach in calculus to help clarify the extent to which students' understanding of linear approximation might be leveraged to understand a topic more productively. Moreover, he echoed his inadequacy in teaching these ideas and the insufficiency in confining linear approximation to a single section in calculus. His admittance of his own inadequacies in teaching impacted Kyle's response to one of the reflection questions.

Reflection questions. I provided the following reflection questions for Amy and Kyle to respond via email after Meeting 7 (see Table 22). Among the additions between these reflection questions and those in Meeting 6, I added question 3, which I discussed briefly during the meeting, representing my characterization of the relationship between these two domains. I wanted to provide Amy and Kyle an opportunity to respond to this updated interpretation and identify the extent to which their image of this relationship is compatible with this characterization. Notice that the primary difference between this characterization and the one presented in Figure 5 is that this diagram illustrates the interconnectedness between the two domains. Moreover, I included the last question to encourage their *reflection* on implications for supporting students' understanding of productive meanings related to linear approximations.[36]

Table 22

Reflection Questions After Meeting 7

1. What are your initial reflections from today's meeting?
2. What might you be interested to learn more about from today's discussion?
3. React to this updated image characterizing the relationship between these two domains. In your response, openly discuss the extent to which you agree or disagree with this characterization.[37]



4. What were your major takeaways from today's discussion?
5. Was there anything that was said during today's meeting (from one of the other two or myself) that was particularly impactful (either positively or negatively)? If so, then please describe what they said, and how it impacted you.
6. Briefly discuss different implications for teaching linear approximations to students in a Calculus I course.

I briefly discuss a couple of Amy's responses. In her response to Question 3, she stated that she preferred the characterization in Figure 5 better, because she viewed "as A impacts B and B impacts A.". Moreover, her comments answering Question 4 suggest that she recognized the impact of students' understanding of linear approximation throughout a calculus course: "In

[36] I believe this question was included to relate to the reflection construct discussed in the theory section, but it may have been implemented for other purposes.
[37] I certainly could have been more tactful in my representation of the ideal student.

order for linear approximation to be impactful, it needs to be supported throughout the course. (This may be an oversimplification of what I took away.).”Additionally, Kyle’s response to the fifth question indicated both his concern about and comfort with the MIP Correspondent’s awareness of his own limitations. On one hand, Kyle is comforted that his own struggles are not unique. On the other hand, he recognized that if the MIP Correspondent, who has much more knowledge and experience than himself, is unsatisfied then Kyle has reason to doubt his own capacity to be successful. The MIP Correspondent’s thoughtful acknowledgement of his own limitations perhaps suggests a need for Kyle to critically evaluate how he measures his success and re-evaluate his priorities as an instructor.

Meeting 8

Prior to my final meeting with this CoRD, I sent Amy and Kyle three contrasting artifacts on linear approximation recommended by the MIP Correspondent to analyze and assess; the first artifact presented a lesson on linear approximation from Rogawski et al. (2019) with some examples; the second artifact encompassed some questions associated with the textbook on linear approximation; the third artifact was a lab lesson on linear approximation from Oehrtman & Martin (2015). I then prompted Amy and Kyle to reflect on the design of the activities for supporting student learning: How might students’ engagement in these activities support their specific learning of linear approximations?

Modified diagram. In the previous meeting, we discussed the image presented in the third reflection question for Meeting 7. At the beginning of this meeting, I presented a modified diagram for Amy and Kyle to reflect on (see Figure 7 and Figure 8). This simplified image provided a distinction between (1) students’ engagement in productive mathematical practices (the black horizontal rectangle) and (2) their specific understanding of ideas (the blue vertical rectangle). While students' productive mathematical practices have the capacity to enhance their

Figure 7

Characterizing the Relationship Between Mathematical Practices and Supporting Students' Specific Understandings: Part 1

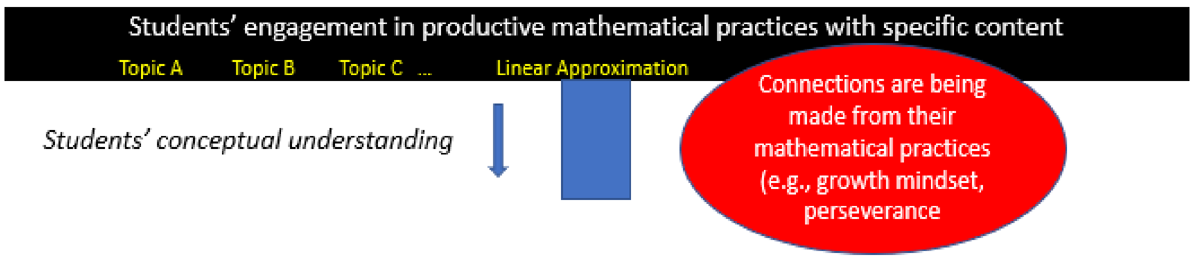
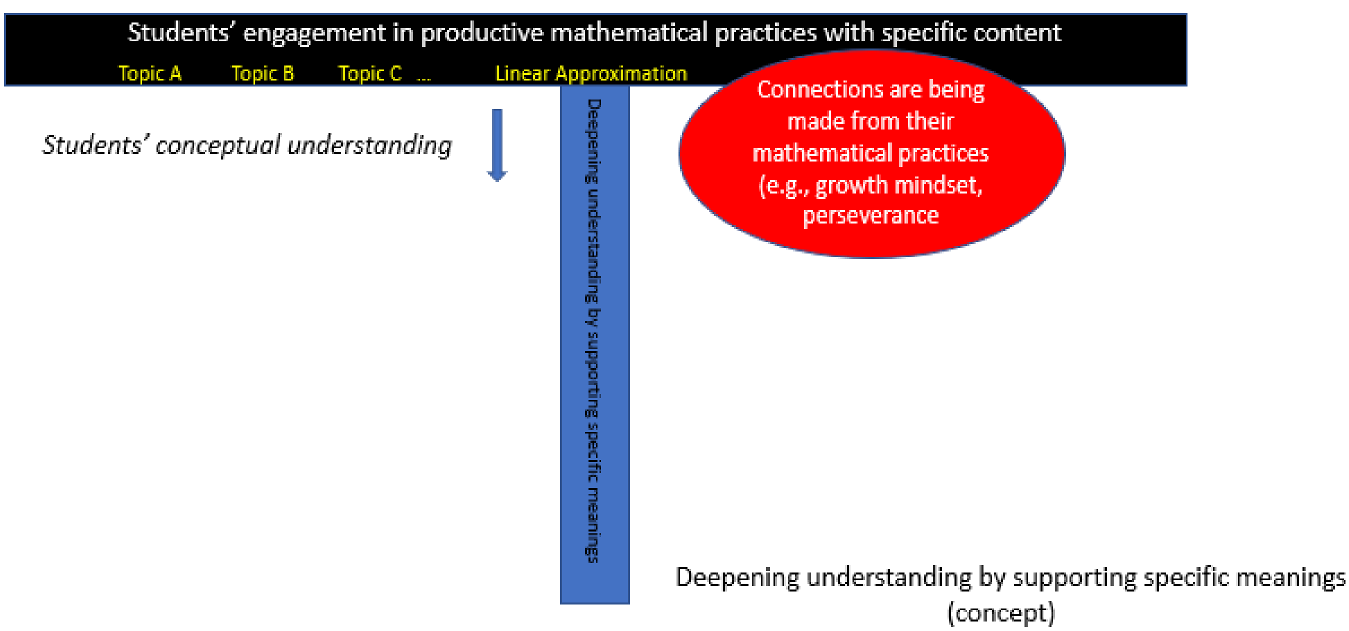


Figure 8

Characterizing the Relationship Between Mathematical Practices and Supporting Students' Specific Understandings: Part 2



their conceptual understanding, a central priority of an instructor's role is to *deepen* that understanding by supporting students' construction of particular meanings the instructor deems productive.

Both Amy and Kyle indicated that they liked this modified adaption, and Amy stated that "I like this idea that you are going deeper while, um, yeah, supporting a specific meaning. Um, and so, they are continually engaging in mathematical practices, but it's directed with linear approximation in the background." Since this modified diagram sharply contrasted the difference between engaging in productive mathematical practices and deepening students' conceptual understanding of a particular topic, Amy's approval suggested, to some extent, an acceptance of these distinctions clarified in Figures 7 and 8.

Making connections and students' thinking. Amy discussed eliciting students' thinking, the inadequacy of designing tasks without an awareness of students' thinking, and identified specific questions in the lab assignment that she considered productive. In response to a question prompting Amy and Kyle to identify which mode of instruction (lecturing, designing activities, the follow up discussion following the activity, or something else) is most important for supporting students' learning,[38] Amy stated that the discussion following the activity is helpful "because then you can actually see what connections they're making and understand more so why they did what they did, um, and maybe even what they tried first or thought about." When I asked later about the importance of seeing these connections, she stated that it is "helpful in the sense that we can see what they're initially thinking about" which could illuminate their misconceptions and impact how we present ideas in the future. Her comments suggest that by

[38] I discuss the purpose of asking this question in the following section.

eliciting students' thinking, she is positioned to support their understanding of ideas in the future and alleviate potential misconceptions. These comments reveal her attentiveness to supporting students' thinking in productive ways.

Later in the meeting, I intentionally highlighted a difference between these two assignments: the lab provides a context, and the textbook does not. I prompted them to discuss if this was the *primary*, or most consequential, difference between these two resources. Providing this prompt afforded Amy and Kyle to attend to important features of these artifacts and potentially reveal some of their commitments as instructors. Amy stated that while the context helps, there is more intentionality involved in the design:

> but, I do think, um, as was pointed out, right, that what went into this is they had to think about students' thinking. Um, so right, you can put context, but if you aren't thinking about how students are thinking and learning, it, it could fall flat on its face.

Her comments suggest that providing a context alone is insufficient and perhaps reflect an attentiveness to the MIP Correspondent's comments when she stated that "as was pointed out" and discuss the importance of being attentive to students' thinking.

Recognizing her attentiveness to discuss students' thinking, I pressed her to be more specific about her intentions, and she said the following:

> Um, so in terms of right, relying on what they know. Cause I could imagine, right, even just pulling up one of those textbook examples and then throwing something onto it. But, without, right, the build up— It's like that prelab where they're actually having to think about things and piece things together. Right, I could throw in a context to one of those

textbook problems without that preemptive thinking. Um, and it's still the exact same problem.

Amy's comments suggest that she recognized the insufficiency of merely adding a context to a sequence of tasks to make it more productive, echoing her previous response. She highlighted the importance of the "build up" where students are "having to think about things and piece things together.

Later in the meeting, I prompted Amy and Kyle to identify features of the CLEAR Calculus lab that the CoRD might abstract and apply when they engage in the process of designing their activities. Amy stated that she liked the question that prompted students to graph the distance function given the velocity function. This problem "makes it much less procedural, um, because you're actually having to analyze." These comments are consistent with her desire to support students' engagement in productive mathematical practices.

In sum, providing these two activities targeted towards supporting students' understanding of linear approximation in different ways proved beneficial since they illustrated, by contrast, the importance of being attentive to students' thinking for scaffolding students' understanding of linear approximations. Perhaps, Amy's willingness to critically evaluate features of her instructional practices and her remarks suggesting her limitations teaching linear approximations more specifically positioned her to react more positively to these activities.

MIP correspondent. Recall from Amy's interpretation of active learning discussed at the beginning of the results in Part I that she valued the discussions for supporting students' engagement in active learning according to her higher standard. She said the following: "Yeah, so, there is a nuance I think to get to that gold standard. Um, but in my mind, I don't necessarily see those nuances happening within the exact activity. I think it's the discussion that follows the activity." Cognizant of her interpretation and leveraging insights from my previous research, I posed the following question (written on a PowerPoint from my shared screen) during the meeting:

> Suppose we have a student with productive mathematical practices. They are engaged and willing to persevere, critically think, and so on. Now, our job as instructors is to enhance this students' specific learning of ideas. Do you think that is best achieved when an instructor is
>
> (1) lecturing,
>
> (2) designing activities for students to engage in,
>
> (3) generating follow up discussions after these activities,
>
> (4) or in some other way?

The purpose of this question was to provide an opportunity for the MIP Correspondent to discuss how the interactions occurring in (3) are necessarily dependent on one's design of tasks in (2). Amy stated that (3) is the most productive (as I predicted) and (2) and (3) were "very close."

The MIP Correspondent indicated that the instructional format is not as important as how the instructor intends to support students' construction of meanings they consider productive. He stated that none of these strategies or practices "necessarily dictate the cognitive activity of the learner." These comments suggest that each of these instructional formats has the potential to be more or less effective dependent on how the instructor supports students' cognitive activity in particular ways: "learning is a product of the conceptual activity that the learner engages in, um, which can be influenced by what an instructor does, but isn't determined by it." In other words, different instructional formats can be effective in supporting students' learning depending on the extent to which these ideas are influenced by an instructor's capacity to engage students in the cognitive activity necessary to construct productive meanings. The MIP Correspondent's comments that the "success of any of these depends on the extent to which the instructor has an image of the process that a learner might have to go through, um, to develop a particular conception of an idea" emphasized how he conceptualized effective instructional practices. A little later, he stated that lecturing, viewed as an unproductive mode of instruction, can be effective or ineffective depending on the extent to which the instructor is attentive to students' thinking. These comments suggest that the format is less important than the actions that an instructor engages in relative to a particular mode of instruction.

Later in the meeting, the Correspondent affirmed Amy's previous comments by stating that the development of the prelab activity from the CLEAR Calculus lab on linear approximations suggested that the creators "had a clear image in their minds of how a student might develop a productive meaning for linear approximation" and indicated that the tasks seemed to be "strategically scaffolded" in contrast to the textbook problems which seem to be a "standard collection of, sort of routine problems that students have to solve." The MIP Correspondent's comments suggest that the creators of the lab had engaged in the process of conducting a conceptual analysis, which enabled them to clarify productive meanings that they intended to support when designing this sequence of tasks. Later in the meeting, he affirmed Amy's conceptions and implicitly described components of the process of engaging in a conceptual analysis:

> Um, I just sort of think that that— this kind of goes back to what [Amy] mentioned earlier, that, se—, seems like this reflects an awareness of students' thinking. And so, I guess another way of thinking about, you know, connecting the psychological experience of the learner to the e—, the expected experience that the activity is supposed to support.

Towards the end of the meeting, I presented the MIP definition of active learning and highlighted that the design of the lab is an effort to effectively operationalize the last part of the MIP definition—the structure of students' actions become equivalent to the structures of the concepts to be learned. In providing an opportunity for the MIP Correspondent to comment, he said that Oehrtman and Martin (2015)

> have an expectation for the mental processes involved in students, um, learning this idea. Um, and it's not just a matter of, of presenting or conveying ideas to students, but its— it seems to me that when you look through this activity that the— these authors have the expectation for how the different tasks that students engage with are promoting the kind of mental actions and generalization. A generalization is a kind of mental activity. So is abstraction, and so on. Um, promoting these, these mental actions that eventually e— enable the students to develop a meaning that is equivalent to, you know, this, as this definition says, structures of the concepts to be learned.

His comments that this presentation is not merely "presenting or conveying ideas to students" but that there is a meaningful "expectation" on behalf of the designers to promote students' conceptual activity in a particular way implicitly highlight the components of the process entailed in conducting a conceptual analysis.

Reflection questions. I provided the following reflection questions for Amy and Kyle to respond via email after Meeting 7 (see Table 23). Notice the addition of the last question.

Table 23

Reflection Questions After Meeting 8

1. What are your initial reflections from today's meeting?
2. What might you be interested to learn more about from today's discussion?
3. What were your major takeaways from today's discussion?
4. Was there anything that was said during today's meeting (from one of the other two or myself) that was particularly impactful (either positively or negatively)? If so, then please describe what they said, and how it impacted you.
5. Briefly discuss different implications for teaching linear approximations to students in a Calculus I course.

6. The last part of the MIP definition of active learning is defined below:
   Students engage in active learning when they work to resolve a problematic situation whose resolution requires them to select, perform, and evaluate ***actions whose structures are equivalent to the structures of the concepts to be learned.***
   To what extent did our discussion of the linear approximation lab help illuminate your understanding of the underlined part of this definition.

In response to the third question, Amy stated that standard linear approximation questions "seem to stand alone, when in reality they rely on prior knowledge." Additionally, she added that "applying context isn't enough," which perhaps suggests an attentiveness to the conversation around the discussion question.

In the last slide, I prompted Amy and Kyle to reflect on the MIP definition of active learning and highlighted how the CLEAR Calculus lab represents an effort to effectively operationalize the final part of this definition (i.e., that the structure of students' actions become equivalent to the structures of the concepts to be learned.) Amy wrote the following response to the last question in Table 23:

> The lab highlighted the idea of "equivalent structures," although I don't think I would necessarily phrase it that way. I think there was more of a scaffolding process going on which highlighted the need to rely on that prior knowledge.

Her comments reveal features of her interpretation of the MIP definition by her use of the terms "scaffolding process" and a "need to rely on that prior knowledge." Amy's preference to describe the MIP definition of active learning as a "scaffolding process" instead of using ideas of *equivalent structures* indicates that the latter description provided her with more clarity regarding the nature of an instructor's role to support students' engagement in active learning.

During the meeting, Amy stated that the prelab is what "sets the two apart" when students are "having to think about things on their own, make those connections individually" before working together as a group. She stated that it is less "structured" and less "procedural," the latter part as being the "big difference." Moreover, the lab "actually is forcing them to make those connections to build up to linear approximation." Her comments about the lab being "less structured" is interesting, since, in one way, these tasks are carefully structured to support students' engagement in a particular way. This clarification is meaningful considering Amy's comments previously about the MIP definition of active learning being narrow by pigeonholing students' thinking.

Finally, both Kyle and Amy did not indicate anything impactful from this meeting in response to the fourth question. Their silence in responding to this question could be a consequence of a variety of different factors. Their brief responses suggest that they may not have critically evaluated the discussions in Meeting 8, and so perhaps they wanted to "complete" this assignment. It may also be true that nothing that was discussed during the meeting that was particularly impactful for them. Regardless of the reason, their silence suggests that participants' engagement provide a meaningful reminder that just as students' learning is prominently influenced by their actions (and not just their perception), more opportunities need to be provided in the future to engage these participants in a sequence of actions that engage them in the process of conducting a conceptual analysis and allow them opportunities to recognize the affordances of doing so.

Discussion and Implications

In Part II of Chapter 4, I discussed my observations of the first eight meetings with Amy, Kyle, and the MIP Correspondent. In the first two meetings, the CoRD discussed their overarching priorities in Calculus (e.g., broad goals, student difficulties, defining success) and their broad characterizations of active learning (their images of active learning, requisite knowledge base, analysis of the MIP definition of active learning). In Meeting 3, members of the CoRD discussed their interpretation of the MIP definition of active learning, and Amy provided critiques (e.g., it may hinder students' creativity). In the next two meetings, we classified other descriptions of active learning from Lugosi and Uribe (2020), including Kyle's characterization and Amy's characterization, into three "buckets" related to fostering mathematical practices, environmental features, and affective engagement. In the sixth meeting, I presented a contrast between their characterization of active learning that emphasized students' mathematical practices and the MIP definition of active learning centered around the cognitive activity of a learner to construct specific understandings of the mathematical content. During this meeting, I also addressed Amy's concern about that the MIP definition potentially *pigeonholing* students' thinking. In Meeting 7, the CoRD discussed linear approximations including its location in textbooks and its influence on other topics. Finally, in my last observation, I facilitated a conversation around three curricular artifacts for supporting students' understanding of linear approximations. In the next section, I discuss some limitations of this study and present three implications from these results.

Limitations

One limitation of this study was that it presented a different experience for Amy and Kyle than participating on a typical CoRD. For example, the discussions centered around alternate definitions of active learning is not normally facilitated by MIP Correspondents. On the other hand, directing these discussions in this way allowed me to investigate the extent to which supporting participants' understanding of other characterizations of active learning might be productive for clarifying distinctions between common interpretations of active learning and the MIP definition.

Another limitation of this study was sometimes stifling, instead of supporting, opportunities for participants to respond to the MIP Correspondent's comments. I might do this by talking first after his comments instead of encouraging reactions from Amy and Kyle. This could be alleviated by prompting them to immediately respond or by providing opportunities for them to respond in reflection questions. In Meeting 6, I discussed a quote from the previous meeting by the MIP Correspondent, allowing Amy and Kyle to respond and priming the conversation for that meeting.

Finally, a third limitations is not having more time to observe more content-focused discussions on this CoRD. These observations would have provided insight into the *practice* feature of my framework discussed in the theory section. Next, I present implications.

Implications

I discuss four implications from these eight meetings. First, supporting MIP participants to productively interpret the meanings associated with operationalizing the MIP definition of active learning is nuanced and challenging. Amy’s critiques of the MIP definition potentially pigeonholing students’ thinking or focused only on the end goal illustrate different interpretations that MIP participants might construct that could hinder capacity to effectively operationalize this definition. Additionally, highlighting the focus of the MIP definition of active learning on the cognitive activity of the learner may be insufficient for delineating differences between it and other characterizations of active learning that focus more on environmental conditions or pedagogical practices. For instance, Amy’s definition reflected the practices of a mathematician which involved students’ critical thinking and problem-solving activities. Additionally, in discussing the descriptions of active learning associated with the environmental features theme (i.e., the green bucket), there were comments about students’ defending and rebutting ideas while engaging in groups. Hence, distinguishing between different characterizations of active learning and the focus of the MIP definition requires an attentiveness to these nuanced meanings that MIP participants may associate with their interpretation of the MIP definition of active learning.

Second, the strategic introduction of reified artifacts that include resources that are more in alignment with the MIP definition and others that are less so, could be productive to support participants’ understanding of the process entailed in operationalizing these three elements of inquiry. As discussed in the first implication, participants may associate different meanings with the MIP definition of active learning that could hinder their capacity to effectively operationalize it. These associations may be refined by evaluating an example of tasks that effectively embody these three components. Prior to the final meeting, Amy and Kyle were asked to critically evaluate three resources (two were from the textbook and one was a different presentation) centered around linear approximations. During this meeting, Amy and Kyle were provided an explicit sequence of tasks that effectively illustrate the enactment of features of the MIP elements of inquiry in its design. In the reflection question asking them to comment on the extent to which the lab illuminated their understanding of the final phrase in the definition, Amy stated that the lab "highlighted the idea of 'equivalent structures,' although I don't think I would necessarily phrase it that way. I think there was more of a scaffolding process going on which highlighted the need to rely on that prior knowledge." These comments suggest that she recognized the "scaffolding process" involved in developing these tasks, and this lab helped clarify meanings of the MIP definition that she would interpret differently.

Third, the broker's critical reflection of his own practices may positively influence MIP participants' critical evaluation of their own teaching. In some of these interviews, the MIP Correspondent highlighted different challenges he experienced in the classroom. For instance, in Meeting 6, he stated that supporting students to develop productive orientations to understand and evaluate others' reasoning is difficult to foster. His admissions during the following meeting had an impact on Kyle:

> [The MIP Correspondent] mentioned he felt his own presentation of linear approximation was lacking in that it could not seem to promote productive reasoning from students. I felt both concerned and comforted by this confession. Concerned because if he, who has far more experience and knowledge than I do, is incapable of aiding students in this manner, how am I supposed to do so? Comforted because if he, who has far more

> experience and knowledge than I do, struggles with aiding students in this manner, then it is not a problem unique to me, even affecting the best of us.

Admissions from members of the MIP Team may (1) build rapport and (2) encourage MIP participants to critically evaluate how they measure success by considering meanings and orientations that they strive to support.

Fourth, my original characterization of distinctions between the focus of the MIP definition of active learning and other common interpretations of active learning proved less productive, while the latter diagram I created proved more productive. My presentation of the original characterization was able to engender some cognitive dissonance but in service of providing an unfair representation of these distinctions. By contrast, the latter diagram was able to represent this contrast more effectively and not devalue students' engagement in productive mathematical practices.

CHAPTER 5

CONCLUSION

I begin this chapter by providing brief summaries of the previous four chapters and then discuss limitations of my research. Following this discussion, I highlight the complexities of implementing a professional development initiative and then identify limitations and affordances of the Mathematical Inquiry Project (MIP). I conclude this chapter by connecting my research to the problem statement discussed in the Chapter 1.

Summary of Chapters and Implications

Chapter 1

In Chapter 1, I highlighted STEM reform initiatives and discussed entry-level mathematics courses as a barrier to student success. The Oklahoma State Regents for Higher Education (OSRHE) outlined the five recommendations for reform (Oklahoma State Regents for Higher Education, 2017, p. 2), and the purpose of the MIP is to address the following three:

1. Increase student engagement and the teaching of applications in gateway math classes;
2. Increase support for important academic success skills in gateway math classes; and
3. Provide faculty and advisor professional development and resources.

I outlined the structural design of the MIP by describing its past and current activities—the

Initiation Workshops and the Collaborative Research and Development Teams (CoRDs)—and future opportunities for engagement (i.e., the Regional Workshops and the development of peer-mentoring relationships). Four of the five Initiation Workshops were focused on course content, and the other workshop targeted affective characteristics of students' academic success skills. These workshops provided opportunities for participants to listen to and engage in discussions centered around the three elements of inquiry and conceptual analysis while also collaborating in break out groups to discuss curriculum development or affective topics. Towards the conclusion of these workshops, participants collaboratively identified a list of three to seven conceptual threads related to the focus of the respective workshop. Faculty interested in developing a curricular module around one of these topics could then collaborate on a CoRD with other colleagues. In the future, there will be opportunities for MIP leaders to engage in regional workshops to discuss their developed resources or help train others and develop peer-mentoring relationships.

After describing the structure of the MIP, I compared the MIP with other STEM professional development initiatives according to the focus on the nature of the problem being addressed. One consequence of the MIP design is the potential enhancement of participants' *pedagogical content knowledge* (PCK) by engaging them in discussions around three elements of mathematical inquiry and conceptual analysis. The MIP definitions of active learning, meaningful applications, and academic success skills are presented in Table 24.

Table 24

Operational Definitions of the Three Elements of Inquiry

Three Elements of Inquiry	Operationalized Definitions
Active Learning	*Students engage in active learning when they work to resolve a problematic situation whose resolution requires them to select, perform, and evaluate*

	actions whose structures are equivalent to the structures of the concepts to be learned.
Meaningful Applications	*Applications are meaningfully incorporated in a mathematics class to the extent that they support students in identifying mathematical relationships, making and justifying claims, and generalizing across contexts to extract common mathematical structure.*
Academic Success Skills	*Academic success skills foster students' construction of their identity as learners in ways that enable productive engagement in education and the associated academic community.*

These three elements of inquiry were carefully defined according to constructivist epistemology: the MIP Team characterized active learning according to students' construction of concepts, meaningful applications according to students' identification and justification of mathematical relationships, and academic success skills according to students' development of their identities as learners. I discussed the mutual influence between each of these elements of inquiry and the importance of conducting a conceptual analysis for effectively operationalizing them for in curriculum development. I also described different experiences for MIP participants designed and implemented within the framework of social learning theory to support their construction of PCK.

I concluded the first chapter by comparing the MIP with other STEM professional development initiatives with respect to the nature of the research being conducted. I referenced Kilpatrick's (1992) historical perspective of research in mathematics education and Confrey's (2017) recent categorization of research according to three "buckets." I contextualized this discussion by describing different studies along a continuum in which classical experimental design and design research studies are situated on opposing ends. I concluded this chapter by foreshadowing two case studies that I conducted, which illustrate the MIP approach to conducting design research.

As a summary of Chapter 1, I provided a structural description of the MIP and discussed pedagogical content knowledge and different MIP experiences that may enhance participants' knowledge base to operationalize the three elements of inquiry and to engage in conceptual analysis. In addition to discussing the MIP definitions of three elements of mathematical inquiry and conceptual analysis, I characterized how the design and focus of the MIP differs from other STEM professional development initiatives.

Chapter 2

In Chapter 2, I discussed MIP participants' responses to a survey (the Workshop Application Form (WAF)) administered prior to their attendance at an initiation workshop. I also presented my analysis of interviews I conducted with select participants three years later. I began this chapter by introducing the relevance and importance of identity research. After motivating the subject, I highlighted the analysis offered by Graven and Heyd-Metzuyanim (2019) and identified how my dissertation study fits into their literature review of identity research in mathematics education.

Following this introduction, I discussed different ways that identity can be classified: according to its theoretical orientation, how it is defined, or whether it is conceptualized as an action or an acquisition. After presenting four theories characterizing identity—poststructural, positioning, narrative, and psychoanalytic—I described five ways that identity can be defined: as participative, narrative, discursive, psychoanalytic, or performative. I briefly discussed the characterization of identity as something that is attained (i.e., an action) or an inherent part of our nature (i.e., an acquisition).

I presented common critiques of identity research, including Darragh's (2016) concern for researchers who define identity in terms of affect, or operationalize it in ways that are incompatible with its theoretical characterization. The critiques presented by these authors are relevant and meaningful, since both articles were recently written, and both offered a detailed overview of identity research. After I presented a few studies in the literature, I introduced my research focus and highlighted potential insights it may offer.

Next, I discussed my theoretical perspective. I described how my conception of identity is in alignment with Blumer's (1986) notion that a human is an "object to himself" (p. 12). Building on his perspective, I discussed how an individual's distal goals influence their desired identity of who they want to become (Middleton et al., 2015). Then, I introduced my research questions:

> *Research Question 1:* What do participants' interview responses reveal about their professional identities with respect to their mathematics instruction?
>
> *Research Question 2:* How can these inferences be leveraged to modify future design aspects of the MIP, in consonance with social learning theory?

The remainder of Chapter 2 was divided into two parts. In Part I, I discussed responses to a survey prompt given prior to faculty attending three summer Initiation Workshops. In Part II, I presented results from eight interviews I conducted with MIP participants on CoRDs three years after their completion of this survey. After briefly presenting the results from participants' responses to Prompt 1, I discussed the results from the eight interviews with participants who been or were currently on a CoRD.

In Part II, I presented participants' responses to Prompt 1 of the WAF, their post-workshop responses (if available), and discussed their perceived contributions and takeaways from their participation in MIP activities and their vision for future collaborations among faculty across institutions in Oklahoma. I organized the presentation of the results according to four broad topics— *Academic Success Skills and Active Learning*, *Frustrated Experience*, *Content Development*, and *Collaboration*—that revealed features of participants priorities and commitments with respect to their roles as post-secondary mathematics instructors.

I discussed four implications from these findings: two practical and two related to social learning theory. As for the practical implications, I identified opportunities to improve communication and support for MIP participants' understanding of the MIP definition of active learning, citing Katie's negative MIP experiences, and to continue enabling MIP participants to give presentations, citing Ellison's positive remarks. Third, I highlighted the importance of continuing to cultivate a community of mathematics faculty across Oklahoma, given some MIP participants' desires to share experiences and struggles with other mathematics faculty from other state institutions. Finally, I discussed a hypothetical trajectory for engaging MIP participants in experiences designed to enhance their knowledge base by affirming their priorities and commitments while also engaging them in meaningful discussions around strategically introduced reified artifacts.

Chapter 3

In Chapter 3, I discussed my case study with a single MIP participant, Robert, who had previously participated in two Initiation Workshops and a CoRD. I identified the need for STEM improvement in Oklahoma by highlighting disappointing student graduation rates. Against this background, I introduced the MIP and described its purpose to address issues of student success in Oklahoma. In this context, I presented my research question:

Research Question: What is the trajectory of one MIP participant from his engagement in the MIP Community of Practice? What are his interpretations of three elements of mathematical inquiry and conceptual analysis and how do these conceptions reveal features of his identity as an instructor?

I discussed my theoretical perspective and methodology in the following two sections. I began by introducing Wenger's (1998) *communities of practice* framework, described his characterization of learning as an identity trajectory, and related this epistemological perspective to the focus of the case study. I then presented Cobb's (2007) two criteria for choosing a particular theoretical perspective and discussed my approach to coordinate multiple theoretical perspectives: social learning theory and radical constructivism. After a general discussion of important constructs in constructivist epistemology (i.e., assimilation, accommodation, reflective abstraction), I presented my methodology for my research. In this section, I discussed my selection process, the trustworthiness of this research (i.e., construct validity, internal validity, and external validity), ethical considerations, and my methods for collecting and analyzing data.

My interviews with Robert focused primarily on his interpretation of the three elements of inquiry and conceptual analysis. Robert's definitions of active learning, meaningful applications and academic success skills are presented in Table 25.

Table 25

Robert's Conceptions of Three Elements of Inquiry

Three Elements of Inquiry	Robert's Definitions

Active Learning	*Students are engaging in active learning when they are asked to engage with a problem themselves (as opposed to passively observing an instructor solve the problem).*
Meaningful Applications	*Applications are meaningfully incorporated in a mathematics class when problems are presented that piques student interest and highlights a key concept (or some key concepts) of the lesson.*
Academic Success Skills	*Academic Success Skills are behaviors/actions that help people/students succeed academically (i.e., in their studies/research). Examples include: detailed note taking, a sense of curiosity, the grit/determination to tackle/solve a problem—from several approaches if necessary, and to think critically.*

Robert's interpretation of active learning centered around student thinking. He expressed that he engages students in active learning by posing questions and discussed how students' engagement in active learning is influenced by an instructor's pedagogical practices. Robert indicated that students are actively learning when they are making strong decisions or struggling to solve a problem, suggesting that his interpretation is based primarily on students' engagement in productive mathematical practices. Robert conveyed that students are positioned to learn and retain how to solve the problem a consequence of their perseverance by trying a variety of techniques to solve a problem, regardless of whether they are successful. Robert's comments reveal his commitments to value students' affective engagement and problem-solving activities, and also demonstrate his imprecise image of how he conceptualizes students' learning and the meanings that students' are positioned to construct from their actions.

Robert's interpretation of meaningful applications centered around two features: piquing student interest and highlighting the key concept of the lesson. A problem might pique students' interest by being intriguing, paradoxical, relatable, etc. In discussing the latter, he described the instructor's role in providing examples, which illuminate the usefulness of a particular technique over another (e.g., the shell method vs. the washer method.). Robert's comments indicate features of his distal goals as a mathematics instructor to support students' motivation and interest in mathematics by providing problems that engage their curiosities. Students who are unsuccessful solving the problem may be better positioned to learn or develop a deeper appreciation for the usefulness of the correct technique, concept, or solution process that Robert communicates/demonstrates, perhaps in a "direct instruction" modality.

Robert's definition of academic success skills encompassed note taking and affective features of students' engagement (e.g., grit). Grit and tenacity are among the academic success skills Robert valued most, reflecting his identity as a student and a mathematician. In addition to these, Robert described his process for teaching a particular concept (i.e., related to conceptual analysis): he motivates the idea, develops it, provides a conclusion, and then follows up with some examples. Robert's critiques of the fundamental theorem of calculus video reveal his priorities and commitments to support students' interest and motivation, and his comments about how he teaches the fundamental theorem of calculus reveal opportunities for him to be more attentive to examining the coherence and implications of the specific meanings he intends to foster from his instruction.

I also presented a hypothetical model for Robert's interpretation of the interconnections between the three elements of mathematical inquiry. Robert explained that he could provide a meaningful application that would pique students' interest and highlight the key concept of the lesson. He expressed that students' with productive academic success skills (i.e., grit, tenacity, perseverance) would have the opportunity to engage productively in the problem. Regardless of their success, if they are not stuck and trying different or previous approaches and methods, then Robert considered them to be engaging in active learning. If students are unsuccessful, their productive efforts and tenacity along with the nature of the meaningful application (i.e., highlighting the key concept of the lesson) might motivate students to understand the idea or learn the method presented by Robert.

Robert's comments suggest the potential need to strategically introduce reified artifacts and generate discussions designed to enhance MIP participants' awareness of how an instructor can support students' learning. I characterized Robert's image of active learning in two ways: *learning from success* and *learning from previous application*. Some of Robert's remarks, such as "learning is learning, I mean, can— are you absorbing new material or not," or his comments that seem to indicate that students learning an idea are engaging in active learning, suggest that there may be value in generating discussions targeting the nature of the learning instructors might support, and their understanding of the cognitive mechanisms that may occasion learning. Additionally, the MIP Team could discuss an example that seemingly incorporates a meaningful application but is not mathematically meaningful in accordance with the MIP definition. Importantly, these conclusions are not intended to undervalue Robert's interpretations of these three elements of inquiry. Robert is an experienced instructor who prioritizes important features of students' engagement, problem-solving ability, and conceptual understanding of ideas. Hence, these implications might offer ways Robert's worthy instructional priorities and commitments might be extended.

Chapter 4

In Chapter 4, I presented the results from my second case study, which included her participation in a CoRD-like environment. Part I consisted of my interviews with Amy to uncover features of her identity as an instructor through her images of three elements of inquiry and conceptual analysis. In Part II, I discussed her participation in eight meetings from her participation in a modified CoRD environment.

After a brief introduction, I discussed my theoretical perspective. I provided a broad description of important constructs in Wenger's (1998) *communities of practice* framework, followed by a discussion of past and future design mechanisms intended to enhance MIP participants' pedagogical content knowledge. Finally, I discussed three constructs relevant to Amy's participation on the CoRD in Part II—curiosity, practice, and reflection—and presented my two research questions:

> *Research Question 1:* What is the trajectory of one MIP participant from her engagement in the MIP Community of Practice? What are her interpretations of three elements of mathematical inquiry and conceptual analysis and how do these conceptions reveal features of her identity as an instructor?
>
> *Research Question 2:* How does one MIP participants' involvement on a CoRD influence how she conceptualizes active learning? In what ways are the mechanisms enacted to promote this transformation successful or unsuccessful.

After presenting my research questions, I began my discussion of Part I. My interviews with Amy focused primarily on her interpretation of the three elements of inquiry and conceptual analysis. For each of the three elements of inquiry, she presented at least four descriptions.

Among Amy's eight descriptions characterizing students' engagement in active learning, she prioritized students' enactment of productive mathematical practices: they are asking questions to themselves or others, thinking deeply about using a procedure to solve a problem and the extent to which it may be generalizable, and analyzing which technique to use in solving a particular problem. In her four descriptions of meaningful applications, the latter two focused on the instructor's role and involved designing prompts to support students to connect ideas and to motivate and foreshadow upcoming content. Among her seven descriptions of academic success skills, she emphasized students' awareness of asking questions and studying, their knowledge and use of available resources, and knowing their best times to study. These initial descriptions are focused more on "study skills," but in one of our later interviews, she discussed features of students' affective engagement. Finally, she defined conceptual analysis in the following way: "Conceptual analysis entails a process of understanding where a concept comes from, how it came about, and how it is applicable. (Applicable does not necessarily mean how it is used in the real world.)." I provided a summary of each element of inquiry and articulated her image of the relationship between these three components.

Finally, I highlighted important features of Amy's identity trajectory as a mathematics instructor and implications from my findings. I began by describing her adaptability as an instructor and her image of active learning reflecting the practices of a mathematician. Amy conceived of active learning more as a *becoming* (in the sense of appropriating orientations and habits of mind pertaining to problem-solving practices in the discipline of mathematics) than resulting from students' engagement in tasks. Considering Amy's priorities as an instructor, I discussed the importance of valuing her commitments to engage students' in productive mathematical practices while also providing opportunities for her to recognize the affordances of designing tasks to support students' construction of productive meanings. Achieving the latter will likely require Amy to become more aware of distinctions between her image of active learning and the MIP definition. While her image seems to be student-focused, unguided, and prioritizes problem-solving activities, the MIP definition is centered around an instructor facilitating students' engagement in the precise actions required to promote their construction of productive mathematical understandings. Importantly, as I clarified after my discussion of my case study with Robert, these suggestions represent ways in which Amy's instruction might be extended to more effectively operationalize the three components of mathematical inquiry. Amy is a knowledgeable instructor who possesses a willingness to critically evaluate ways to enhance her own instruction.

In Part II, I presented the results from eight meetings with Amy, Kyle, the MIP Correspondent, and myself (see Table 26).

Table 26

The Central Focus of Each CoRD Meeting (Condensed)

Meetings	Central Focus
Meeting 1	Overarching Priorities in Calculus
Meeting 2	Broad Characterizations of Active Learning
Meeting 3	Characterizing Other Descriptions of Active Learning: Part I
Meeting 4	Characterizing Other Descriptions of Active Learning: Part II
Meeting 5	Characterizing Other Descriptions of Active Learning: Part III
Meeting 6	Contrasting the MIP Definition of Active Learning with Other Characterizations
Meeting 7	Initial Discussion of Linear Approximation
Meeting 8	Analysis of Curricular Artifacts Designed to Support Students' Learning of Linear Approximation

The first two meetings were more introductory and encompassed broad discussions about the participants' goals and difficulties associated with teaching calculus, images of active learning, and interpretations of the MIP definition of active learning. In Meeting 3, they identified a topic of focus (linear approximation), discussed the MIP definition of active learning, and began examining other descriptions of active learning.

The focus of the next two meetings, Meeting 4 and Meeting 5, was to characterize other descriptions of active learning, including definitions from Amy and Kyle, according to different themes, which were: *fostering mathematical practices*, *environmental features*, and *affective characteristics*.

In Meeting 6, I presented a framework to contrast the MIP definition of active learning with Amy and Kyle's characterizations. I described their interpretations according to maximizing productive mathematical practices and the MIP interpretation according to maximizing productive understandings of content. Leveraging this framework, I also addressed Amy's concern expressed in Meeting 2 that the MIP definition is too narrow and could potentially pigeonhole students' thinking.

In the final two meetings, the CoRD began discussing linear approximations. In Meeting 7, they discussed its necessity and location in the textbook and its influence on other topics in the single-variable calculus curriculum. In the final meeting, we discussed Amy and Kyle's analysis and comparison of linear approximation curricular artifacts from two different sources. In discussing these meetings, I also presented comments from the MIP Correspondent and discussed written responses from Amy and Kyle to weekly reflection questions.

I provided four implications based on my analysis of these eight CoRD meetings. First, I discussed the nuance and difficulty of supporting participants' productive interpretation of the MIP definition of active learning. Not only could it be described as a narrow definition but portraying it as supporting students' cognitive activity may be insufficient for distinguishing it from other interpretations in the mathematics education literature. Second, I discussed the potential affordances of contrasting multiple artifacts to provide a concrete illustration of activities whose design seems to more or less productively operationalize features of the MIP definitions of active learning and meaningful applications. These examples might help alleviate criticisms or confusion regarding participants' interpretations of the definitions. Third, I discussed features of the broker's role that may be influential in supporting participants to critically reflect on their teaching. By evaluating limitations of their own instruction, an MIP Correspondent may position members on their CoRD to identify potential limitations in their own instruction. Finally, I briefly highlighted less and more productive models for characterizing distinctions between participants' image of active learning and the MIP definition.

Limitations, Findings, and Implications from My Research

In the following two sections, I briefly highlight some limitations, findings, and implications from my research.

Some Limitations of My Research

First, both of my case studies were conducted with faculty who had a Ph.D. in pure or applied mathematics. Their comments in the interviews seemed to reflect their practices as a mathematician. Hence, it might have been productive to conduct a case study with another participant who (1) did not have a Ph.D. in pure or applied mathematics or (2) had a focus in mathematics education.

Second, I did not present any data from classroom observations. Triangulating data in multiple ways can enhance the viability of a study by providing an alternate source of data. Leveraging these observations, one could refine their current image of the case study participant's understanding of the three elements of inquiry and conceptual analysis by evaluating how they are operationalized in instruction.

Third, I conducted my case study with Robert over one period of multiple weeks. Carefully tracking his participation in MIP activities could have provided insight regarding the extent to which this involvement influenced his identity trajectory. More specifically, the MIP Team could leverage this data to better understand features of reified artifacts introduced or specific discussions which may support participants' transformations more effectively.

Fourth, the eight meetings that were conducted with Amy's CoRD were different in nature than a typical CoRD. For example, several of the meetings were focused on the CoRD's interpretation and classification of other definitions of active learning. While I guided the CoRD in this discussion to better support their awareness of differences between other characterizations of active learning and the MIP definition, it was not a typical topic of discussion among other CoRDs. The nature of this CoRD was also different. We quickly formed this modified CoRD and set up weekly meetings to allow me to collect data from Amy's involvement, which was important considering the constrained timeline before graduation since Amy had not formed a CoRD prior to that time. Finally, the role of the MIP Correspondent in this CoRD was more participative than their typical supervision over a CoRD. An affordance of this difference is it provided opportunities for him to be potentially more influential in their discussions.

Some Findings and Implications from My Research

Participants' responses to these surveys and interviews reveal the challenging objective of the MIP Team: to design professional development experiences for established mathematics instructors to enhance their knowledge base and possibly shift participants' foundational priorities and commitments as mathematics instructors. Not only is the MIP focus different from many other professional development initiatives, but, as demonstrated in the two case studies, faculty may conceptualize active learning more in alignment with the MIP definition of academic success skills (e.g., pertaining to critical thinking, problem-solving, perseverance, interest, motivation). Their priorities are important and worthwhile, but different in ways that can influence their interpretation of the MIP definition of active learning and perhaps hinder their capacity to operationalize the definition as envisioned.

However, if faculty remain unaware of these distinctions between how they conceptualize active learning, then their participation in MIP activities may be less effective in influencing their identity trajectories as envisioned by the MIP Team. This leads to two questions: (1) What are participants' essential priorities and commitments as mathematics instructors? (2) How can the MIP Team design professional development experiences that promote transformation in participants' identities without (implicitly or explicitly) communicating a devaluation of these priorities and commitments? To answer the first question related to active learning, the two case studies of mathematicians illustrated faculty who prioritized students' engagement in productive mathematical practices (i.e., grit, tenacity, problem-solving, critically thinking). Addressing the latter question, however, is much more difficult and seems to require engaging participants' in experiences designed to perturb their interpretations of active learning by highlighting distinctions between them and the constructivist MIP definition more explicitly.

Accomplishing this goal might be achieved by strategically introducing reified artifacts. For example, the MIP Team could present an activity or task that supports students' engagement in critical thinking, problem solving, and encourages group discussions, but does not support students’ engagement in active learning according to the MIP definition. Similarly, they could introduce a task that is interesting, relatable, and paradoxical, but is not a mathematically meaningful application according to the MIP definition. Then, they could facilitate discussions around why these examples are meaningful, but do not effectively operationalize the MIP definitions.

Additionally, the MIP Team could introduce pairs of two activities (one activity that operationalizes the MIP definition of active learning and one that does not) designed to support students’ learning of the same topic and facilitate a discussion around these two curricular artifacts. These concrete illustrations could be effective for supporting participants’ understanding of the MIP definition of active learning. Moreover, the MIP Team could adapt and refine the model I presented in Chapter 4 that attempts to illustrate fundamental differences between supporting students’ affective engagement in productive mathematical practices and their construction of productive meanings. The MIP Team could discuss this model and highlight important distinctions that may be difficult for mathematics instructors to recognize.

Faculty who may be come perturbed from these experiences, however, might not recognize the affordances of designing curricular materials that reflect the three MIP components of mathematical inquiry. Hence, it is important that the MIP Team supports participants in (1) becoming aware of distinctions between their images of active learning, meaningful applications, and academic success skills and how they interpret the MIP definitions, and (2) recognizing the affordances of operationalizing these three elements of inquiry and conveying the importance of conducting a conceptual analysis to do so.

In sum, the context of the two case studies and the resulting findings offer meaningful contributions to the field of identity research by considering (1) the nature of the population, (2) the focus and depth of the research findings, and (3) a framework for strategically influencing and constructing viable models of participants' identity trajectories. First, the two case study participants are post-secondary mathematics faculty, and both are associate professors at their respective institutions. Hence, these cases present insights from an experienced group of professors. Second, I present in-depth characterizations of participants' images of how they conceptualize students' engagement in active learning. I conducted three interviews with Robert and three interviews with Amy targeted towards investigating their interpretation of active learning, the extent to which different tasks or activities might support students' engagement in active learning, etc. Thus, my in-depth semi-structured interviews positioned me to offer nuanced and in-depth insights into participants' priorities and commitments from their characterizations of active learning. Third, I discuss key mechanisms that might be productive for influencing professional development participants' identities by (1) affirming participants' priorities, (2) stimulating their curiosity by strategically engaging them in conversations or introducing reified artifacts designed to engender perturbation, and (3) offering implications of the approach presented by the professional development team.

Complexities of Professional Development

The implementation of a professional development initiative does not ensure its success. Hence, designing and implementing a professional development program requires careful consideration. In the following section, I highlight a few limitations and affordances of the MIP.

Limitations of the MIP

The COVID-19 pandemic affected communication between members of the CoRDs and the Project Team. The transition to online instruction coupled with technological barriers infringed on the time availability of both parties throughout 2020. Moreover, the lack of consistent opportunities for the MIP community of practice to regularly interact may be another limitation. While the CoRDs collaborate with each other and with their MIP Correspondent, these interactions are limited to small pockets of the community. In particular, participation of the entire MIP community is critical to ensure alignment:

> With insufficient participation, our relations to broader enterprises tend to remain literal and procedural: our coordination tends to be based on compliance rather than participation in meaning. Furthermore, our common terms and shared artifacts can have disconnecting as much as connection effects. (Wenger, 1998, p. 187)

To help regain elements of a community established from the summer 2020 Initiation Workshops, the MIP Team held three virtual workshops during January 4-6 in 2021. These “Virtual Get Togethers” afforded opportunities for MIP participants to mutually engage with one another, albeit through an online medium.

Moreover, the MIP definition of active learning is difficult to interpret and operationalize:

> *Students engage in active learning when they work to resolve a problematic situation whose resolution requires them to select, perform, and evaluate actions whose structures are equivalent to the structures of the concepts to be learned.*

Katie’s comments highlighted from my post-survey interviews revealed her frustration in interpreting this definition, despite the detailed discussion of this definition during each of the

Initiation Workshops. Hence, there is a need to explore different ways to support participants to effectively operationalize these definitions. Next, I discuss the potential affordances of the MIP design. Finally, Katie's comments suggest that there are opportunities to streamline communication and other administrative procedures.

Affordances of the MIP

Some professional development initiatives are localized within their unit of study (e.g., a department or university), perhaps constrained by time or resources. By contrast, the MIP is scalable across an entire state by targeting faculty from the 27 public institutions of higher education throughout Oklahoma. Moreover, the long-term design provides opportunities at different stages for participants to become involved by attending Initiation Workshops over two summers, participation on CoRDs, help lead or attend a regional workshop, or engage with other colleagues in a peer-mentoring relationship towards the latter stages of the project. Additionally, there are several features of the MIP design that have potential affordances to make it *sustainable* over time. One design feature that may increase the sustainability of the project is its duration. Scher and O'Reilly (2009) developed an inventory of knowledge regarding professional development efforts for K-12 math and science teachers. In their comprehensive work, they discovered that "math-focused interventions that take place over multiple years have a more pronounced effect on student achievement than interventions occurring over only 1 academic year" (p. 235). Moreover, evidence suggests the potential ineffectiveness of single workshops in changing instructional practices (Ebert-May et al., 2011), and researchers have identified duration as a core feature of effective professional development (Desimone, 2009).

Hence, the duration of the MIP (five years) is another design feature that might support its effectiveness.

Additionally, the MIP provides opportunities for participants to take meaningful ownership in reforming curriculum instead of providing participants with finished products to implement in the classroom based on evidenced-based teaching practices. [39] The latter approach has been demonstrated ineffective. In their comprehensive literature review of change strategies in STEM education, Henderson et al. (2011) concluded that "developing and testing 'best practice' curricular materials and then making these materials available to other faculty" was not effective (p. 978). Moreover, in an interview with Etienne Wenger, Farnsworth et al. (2016) expanded on Wenger's remarks that practices are local:

> Similarly, for the educational research community, the idea that teachers' practice is local means we cannot assume teachers will implement our research simply because we have called it 'evidence-based practice'. The evidence we provide is simply a reification that teachers may or may not respond to and negotiate within the context of their community of practice. (p. 158)

Rather than giving participants an MIP-designed curriculum, the MIP Team structured this initiative to empower faculty to be agents of change and engineer curricular modules with the expectation that the design process could contribute to their construction of pedagogical content knowledge, equipping them to effectively implement these curricular resources. This transferal of ownership was first demonstrated during the Initiation Workshops when participants had opportunities to collaborate in small groups about different topics.

[39] Comments from some participants suggest that they would have like more direction and guidance in creating their CoRD products.

Moreover, the MIP Team seeks to shift participants' identities as mathematics instructors by enhancing their knowledge base. Influencing participants' identities with respect to their conceptions of competent mathematics instruction may foster behavioral changes. In Kennedy's (1998) review of studies of in-service programs seeking to improve instruction in mathematics and science classes, she concluded that "programs whose content focused mainly on teachers' behaviors demonstrated smaller influences on student learning than did programs whose content focused on teachers' knowledge of the subject, on the curriculum, or on how students learn the subject" (p. 25).

Conclusion: Connecting Back to the Problem Statement

In Chapter 1, I discussed a need for reform in gateway mathematics courses to support student success rates. OSHRE identified the following five recommendations:

1. Establish statewide college meta-majors and corresponding math pathways, ensuring transferability across institutions;
2. Improve student preparation, including efforts in K-12 education and remediation reform;
3. Increase student engagement and the teaching of applications in gateway math classes;
4. Increase support for important academic success skills in gateway math classes; and
5. Provide faculty and advisor professional development and resources.

The first two recommendations require the implementation of more structural changes while 3-5 necessitate more of a cultural transformation emerging from a professional development initiative. The MIP Team developed this project to address these latter three recommendations by providing a professional development opportunity (see (5)) centered around three elements of mathematical inquiry, including active learning and academic success skills (see (3) and (4)), in the context of improving entry-level mathematics courses (see (3) and (4)). In sum, the MIP is a large-scale, long-term, inquiry-oriented professional development opportunity to foster the collaboration of a community of mathematics faculty in Oklahoma to create curricular resources for entry-level mathematics courses informed by conducting conceptual analyses of key mathematical concepts.

From a research focus, this initiative is centered around enhancing instructors' knowledge base by supporting them to critically reflect on the nature of students' activity required to construct particular meanings by considering three elements of mathematical inquiry, grounded in constructivist epistemology. Supporting this transformation, the MIP Team designed the project to include opportunities for faculty to participate in multiple ways by attending Initiation Workshops, collaboration on CoRDs, leading Regional Workshops, or engaging in peer-mentoring relationships.

As MIP faculty participate in these activities, they negotiate meaning from colleagues' goals and commitments which may influence their identities as mathematics instructors. Additionally, faculty also negotiate meaning from their own interpretation and other participants' interpretations of reified artifacts and purposeful discussions strategically implemented and facilitated by the MIP Team. Since faculty involvement in MIP activities might not influence their identities as mathematics instructors in ways anticipated by the MIP Team, there is need to explore participants' priorities and commitments as mathematics instructors to give insight into how they might promote or impede participants' capacity to effectively operationalize the MIP elements of inquiry. In other words, shifting participants' identities as mathematics instructors necessarily requires knowledge of their *initial* values and commitments so that future project experiences can leverage insights from a known foundation.

The initial survey provided insight into participants' goal structures by revealing their priorities for participating in Initiation Workshops (e.g., their expected contributions and takeaways from their involvement). My follow-up interviews three years later elucidated participants' actual contributions and takeaways from their experience in MIP activities. Moreover, the two exploratory case studies I conducted, which focused on participants' interpretations of the three elements of inquiry and conceptual analysis, positioned me to infer features of their distal goals by illuminating aspects of their priorities and commitments as instructors.

These follow up interviews and case studies represent moments in time of participants' involvement in the middle of the MIP timeline. Since there will be future opportunities for faculty to participate on a CoRD, help lead a Regional Workshop or engage in peer-mentoring relationships, my research findings may be leveraged by the MIP Team to modify design aspects of the project in service of enhancing participants' knowledge base more effectively. Improving the potential success of the MIP in these ways, illustrating the potential affordances of design research, may ultimately equip mathematics faculty to better support their students in these entry-level mathematics courses.

I conclude with some remarks from Tallman (2021):

> In its most general description, a mathematics teacher is responsible for providing opportunities for students to engage in the conceptual activity required for their construction of productive meanings. Accomplishing this goal demands that the teacher's

> actions be deliberately informed by an understanding of the functional mechanisms of mathematics learning so that these mechanisms can be purposefully engendered through instruction. (p. 16)

Ideally, MIP participants will become better positioned to purposefully be attentive to and supportive of students' construction of productive meanings. In these ways, instructors can support students' learning having identified and clarified more productive meanings they intend to support.

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APPENDICES

<u>Summary of Previous Discussion about Meaningful Applications</u>

Meaningful Applications definitions

1. The prompts encourage the students to connect various areas/ideas/concepts within the class.
2. The prompts foreshadow/motivate upcoming material.
3. Students are actively engaged in the material. That can be done through worksheets, discussion questions, guided examples/problems.
4. Students come up with their own questions based on the material that they've covered thus far.

Meaningful as beneficial

1. Conceptual understanding
2. Procedural understanding
3. Applicable to real life, practical
4. Other benefits besides practical
   a. Deeper thinking
   b. Internalizing or internal growth
   c. Inquisitive
   d. Growth mindset

Not a Meaningful Application (related rates)

1. Contrived
2. Mindless: for us, that sort of task (related rates) is totally mindless I think and as a student gets much better at it, it should become very mindless for them too

More meaningful

1. Selecting appropriate information
2. Realistic nature of the problem (numbers)
3. Realistic nature of the problem (expectation of answer)
4. Realistic and relatable problem

Examples:

Meaningful application examples

1. MVT worksheet
2. Optimization

Meaningful application non-examples

1. Related rates

Instructions: Please take some time and answer the questions provided below. The purpose of this task is for you to articulate your conception of meaningful applications based on your responses from a prior interview and any additional thoughts you might not have previously had the opportunity to express. The information above is organized based on our past conversation, but feel free to modify/arrange/organize it as you work. I also ask that you please keep a record of the time that you spend responding to the questions below.

1) Would you like to keep or modify your definition of meaningful applications from an instructor's perspective? These are highlighted at the top of the page. If you would like to modify your original definition (i.e., (1) and (2) under the heading "Meaningful Application definitions), then go ahead and write the modification below. If you would like to keep them as stated above, then simply copy and paste your initial statement.
Meaningful applications come about through prompts/problems which connect various areas/ideas/concepts within the class (and even to prerequisite or corequisite material). Meaningful applications can also foreshadow/motivate future material. Meaningful applications can allow students to internalize a procedure.

2) You talked about meaningful applications as being beneficial (see above)

    a. How do the affordances of incorporating meaningful applications into instruction relate to your definition from (1)?
    Students become familiar with common routines/procedures and why they do them. (Once they are routine, I don't think the problems are meaningful anymore.) Students have a better conceptual understanding of the material because of the connections they are making amongst various prior knowledge components and current ones. They are better able to see how things can come together; they are better able to formulate questions and help themselves find the answers to those questions. I think the last part of my meaningful application definition gets at this.

    b. If you would like to modify your definition to reflect any aspect of your response to part (a), then go ahead and write the modification. If you would like to keep them the same, simply say so.
    I would like to maintain my definition as above.

3) You talked about ways that meaningful applications could become more meaningful (see above)

    a. How does making an activity more meaningful relate to your responses to the previous two questions?
    Honestly, the idea of more meaningful pushed me to add this: Meaningful applications can allow students to internalize a procedure. They do need to be able to do routine things. Now, I think understanding the theory behind things makes routine (procedural) problems much easier. But, as we know, a student can watch us do the product rule, for example, but struggle to do it themselves.

    I do still think there are various levels of meaningful applications. A meaningful application has a purpose. The purpose may be to get students used to applying a rule, like the chain rule. The purpose may be to foreshadow future material. The purpose may be to help students make connections. I think a big component of meaningful applications is that students are engaged and internalizing what they are doing. They are not bystanders in their education. They are part of the process. This is a bit tricky though, because I said a meaningful application could be applying some rule. To ensure students are engaged and internalizing that process, I think more needs to be done. For example, suppose a student is given a function which is a composition of functions and is asked to find the derivative. Early on, as they're learning the material, these questions can be meaningful, in the sense that students have to figure out which derivative rule(s) to apply and how to execute the rule(s). Once they've done enough of these problems so that they've internalized the procedure/process, these sorts of questions are much less meaningful. Students ultimately don't have to think about them. They're routine. I equate meaningful as beneficial to the student's learning. So, early on, what I view as a routine problem does have some merit/benefit/is meaningful to the student. They can learn why we apply certain rules in certain settings. I think of this as the surface level of meaningful applications though. So, in part, making a meaningful application involves knowing where your students are at and how much you can push them to connect ideas/concepts. Routine things should be embedded in more meaningful applications. I think that allows for confidence building and helps them to make connections. In a way, you could be guiding them to more challenging things through things that are more routine. So, more meaningful problems/prompts could entail less meaningful components.

b. If you would like to modify your definition to reflect any aspect of your response to part (a), then go ahead and write the modification. If you would like to keep them the same, simply say so.
I would like to maintain my definition as above.

4) You talked about how examples might not be meaningful applications (see above)

a. How does an application not being meaningful connect with your previous answers?
An application is not meaningful when the entire application/prompt/problem is routine. The student isn't being pushed to consider new ideas or make connections. (They aren't being challenged.) In a way, they are like a computer executing routines. However, as previously mentioned, I do think routine things should be embedded in more meaningful applications.

b. If you would like to modify your definition to reflect any aspect of your response to part (a), then go ahead and write the modification. If you would like to keep them the same, simply say so.
Meaningful applications come about through prompts/problems which connect various areas/ideas/concepts within the class (and even to prerequisite or corequisite material). Meaningful applications can also foreshadow/motivate future material while relying on current or past material. Meaningful applications can allow students to internalize a procedure.

The Mathematical Inquiry Project

Request for Proposal
Collaborative Research and Development
Functions and Modeling

The Mathematical Inquiry Project (MIP) is a statewide collaboration among mathematics faculty in Oklahoma to improve entry-level undergraduate mathematics instruction through three guiding principles:

Active Learning: Students engage in active learning when they work to resolve a problematic situation whose resolution requires them to select, perform, and evaluate actions whose structures are equivalent to the structures of the concepts to be learned.

For more information on the MIP Active Learning Principle, visit https://okmip.com/active-learning/

Meaningful Applications: Applications are meaningfully incorporated in a mathematics class to the extent that they support students in identifying mathematical relationships, making and justifying claims, and generalizing across contexts to extract common mathematical structure.

For more information on the MIP Meaningful Applications Principle, visit https://okmip.com/applications/

Academic Success Skills: Academic success skills foster students' construction of their identity as learners in ways that enable productive engagement in their education and the associated academic community.

For more information on the MIP Academic Success Skills Principle, visit https://okmip.com/academic-success-skills/

Description of CoRD modules

CoRD modules should be designed to promote all three of the MIP components of inquiry: active learning, meaningful applications, and academic success skills. An overview of the module should articulate explicitly how these three components are supported.

In order to communicate the CoRD's approach to developing the targeted concepts to faculty using the MIP resources, modules should include an analysis of its primary conceptual goals. This analysis should include details such as the ways of understanding desired as an outcome for all students in the course, common entry points for students' understanding (including relevant supporting concepts), a progression of challenges and solutions that students should engage through the module to develop these understandings, common pitfalls in the learning process and ways to address them, and a description of ways in which these ideas support thinking and learning throughout the entire course.

The core of a module will be a set of instructional materials. The MIP seeks to support development of modules for entry-level college mathematics courses that develop targeted concepts as a unifying topic throughout the course. Consequently, the materials in a module will not typically consist of a sequential series of lessons, but rather provide broader instructional resources to be used throughout the course.

These materials should include assessment materials that allow an instructor both to assess how their students have progressed relative to the targeted goals and to identify ways to improve their own instruction.

As corequisite remediation for entry-level college mathematics is a critical reform in the state of Oklahoma, modules should include a description of how it would be implemented differently in a corequisite class, including any additional resources necessary to do so.

After a successful review the CoRD will pilot the module with a class or group of students and incorporate a description of test implementation and its results, a discussion of the refinements and recommendations made based on test implementation, and short video clips with commentary to illustrate effective implementation.

Review and Revision

Once a CoRD submits a module, it will be reviewed by at least two other faculty with expertise in the topic to inform an editorial decision of “Accept,” “Accept with minor revision,” “Revise and resubmit,” or “Reject,” along with directions for revision if appropriate. After a favorable review, the CoRD will revise and pilot their module, incorporating feedback gained during the review process and submit a final module for publication on the project website.

Author Stipends

Each author in the CoRD will receive a $2500 stipend after delivery of a complete initial draft of the module and an additional $1000 stipend after delivery of a complete revision of the module based on the editorial decision.

Opportunities for leading regional workshops and mentoring

The MIP will leverage faculty leadership and expertise developed through its Initiation Workshops and CoRDs to also develop and deliver 40 institutional and regional professional development workshops, across the state of Oklahoma, on teaching the new courses, incorporating applications and active learning with the modules, and addressing academic success skills. Each Regional Workshop will last a full day and engage approximately 20 mathematics faculty in implementing one or two of the modules developed by the CoRDs and ensuring familiarity with the module resources. Each workshop will be led by faculty from the respective CoRDs with support of MIP personnel who will also assist the leaders in designing the workshop activities with advice from project consultants. A goal of the Regional Workshops will be to engage all relevant faculty in hosting at nearby institutions and to develop a structure that will provide training for new faculty and continuing professional development for all faculty.

The MIP will also support 425 semester-long faculty mentoring relationships between CoRD leaders and one or two faculty who are first implementing MIP resources in a class they are teaching. A goal of these mentoring relationships is to develop institutional and regional communities whose members meet regularly and reinforce and support the cultural practices necessary for mathematics learning through inquiry.

Proposal requirements

The MIP seeks to support the development of modules on the following targeted topics for the course Functions and Modeling. See the following pages for details of each of these topics.

Function

- Modeling and Quantitative Reasoning

- Rate of Change
- Function Classes

Proposals should include each of the following:

1. A cover page designating which of the targeted topics the proposed CoRD will address, the entry-level college course(s) for which it will develop instructional resources, names of all proposed CoRD members (3-5 people), their institutions, email addresses, and phone numbers.
2. The CoRD's initial image of how to develop the targeted concept as a unifying topic throughout the entry-level course.
3. The CoRD's initial plan to promote all three of the MIP components of inquiry: active learning, meaningful applications, and academic success skills, in their module.
4. A description of prior experience of each CoRD member relevant to their development of the proposed module.

Proposal Length

The full text of a proposal should not exceed 2,000 words.

Consultation

The MIP encourages discussions with any of the project team on the planning and preparation of a proposal. Throughout the CoRD's work, MIP project personnel will provide associated resources and advice. The MIP will also organize events throughout the year to allow multiple CoRDs to present their progress and discuss ways to benefit from and integrate their approaches. Please contact William (Bus) Jaco at william.jaco@okstate.edu to initiate any inquiries or discussions.

Proposal submission

Completed proposals should be emailed to William (Bus) Jaco at william.jaco@okstate.edu.

Proposals should be submitted by Friday, November 1, 2019 for full consideration. The MIP will continue to accept and review proposals after this date, however we strongly encourage discussions with the project team for later submissions to avoid proposing work on topics that have already been assigned a CoRD.

The MIP plans to respond to proposals by early November. During the review of proposals, the MIP may request additional information or modifications before approval. Initial draft of modules to be reviewed will be due Friday, June 5, 2020.

Functions and Modeling Targeted Topics

Function

Function is the foundational topic in Functions and Modeling. The function concept enables us to identify, analyze, and gain insight into relationships between real-world quantities that vary in tandem, and is a key prerequisite to learning subsequent ideas in this course. Accordingly, students in Functions and Modeling should develop productive understandings of function (both single- and multi-variable) that can be used flexibly amongst various real-world contexts and representations. This involves awareness and use of appropriate conventions like function notation as well as aspects of quantitative reasoning and covariational reasoning.

Participants in the MIP Initiation Workshop on Functions and Modeling suggested development of modules addressing the following areas:

1. Engage students in analyzing function relationships and concepts through multiple representations. Being able to work proficiently with each of the major function representations (e.g. formula, table, graph, words) also promotes the dynamic view that a function is much more than a way to relate specific inputs to specific outputs (i.e. instructions for how to 'convert' an input value to an output value) and reinforces the view that each representation is a different manifestation of the same relationship between quantities that are changing together (see Oehrtman, Carlson, and Thompson, 2008). Working flexibly across multiple functions representations is also valuable for understanding function *concepts* because each representation can highlight various aspects of the concept. For example, examining function composition in table and graph form might enable a student to imagine how changes in the input of one function correspond to changes in the output of the other (which students possessing only a formula-based understanding of composition would be unlikely to achieve).

2. Emphasize the concept of function as a relationship between quantities and design tasks that encourage students to reason explicitly about how a function's quantities are changing in relation to each other. Carlson et al.'s (2002) covariation framework provides details of the patterns of mental actions that support reasoning covariationally. A covariational emphasis promotes a dynamic view of function as a relationship between two changing quantities (as opposed to a static, input-output correspondence view). This emphasis also entails aspects of quantitative reasoning (which includes carefully attending to the following questions for each quantity: what is being measured, what is the measurement unit, and what does the value of the measurement?). Reasoning in this way is key for understanding the relationship between the original quantities (e.g. Moore and Carlson, 2010) and also foundational for understanding key ideas like constant and average rate of change (e.g. Thompson, 2008).

3. Have students represent the various quantities associated with a function using function notation. Note that this includes not only a proficiency with basic conventions of expressing input-output pairs in function notation, but also extends to expressions of other related quantities like change and rates of change in function notation. This representational activity can be productive because it emphasizes the common structure held by all quantities of the same type (e.g. that changes in the output quantity are all of the form $f(b)-f(a)$) and provides students with an opportunity to develop meaningful understandings of what might otherwise be rote formulas. Participants of the Initiation Workshop stressed that students should come to see function notation as an efficient and useful tool that does work for us; that is, the CoRD should design activities that enable students to see function notation as necessary for expressing mathematical ideas.

4. Leverage technology as a tool to advance students' understanding of function and related function concepts. Technology should be used to enable students to better focus on ideas and concepts, instead of only procedures and algebraic manipulation. For example, a graphing calculator (or any graphing technology) makes it easier to shift between function representations because, having entered an equation, one can view a graph or a table without getting bogged down in procedures carried out by hand (promoting the recommendation regarding the benefits of viewing functions and related concepts in multiple representations).

Participants of the MIP Workshop on Functions and Modeling suggested the following ways modules could address the three MIP components of mathematical inquiry (see descriptions of these components at https://okmip.com):

Meaningful Applications: Though examples of functions abound in everyday life, function is often seen by students as existing only with the confines of a mathematics class. Part of the philosophy behind Functions and Modeling is that all problems are based in real-world experiences. There are many examples of functions that students are exposed to in classes, but unless the function concept does real work in students' reasoning, they are likely to continue to confine notions of function to the classroom. Participants in the Initiation The MIP characterization of meaningful applications states that an application problem is meaningful only to the extent that it supports students in identifying mathematical relationships, justifying their reasoning, and generalizing key concepts across various contexts. Through careful instructional design, real-world applications that leverage students' real-world knowledge can become key tools for students' reasoning. For example, students can employ an analysis of a profit graph to reason about how many items yields maximal profits, break-even points, and so on.

Active Learning: Supporting students' quantitative reasoning with functions promotes insight into relationships between quantities (for example, a quantitative understanding for 'increasing' might involve the observation that the changes in output along the interval in question are all positive). Such meanings for function concepts provide rich opportunities for the MIP characterization of active learning (which includes students' selecting, performing, and evaluating actions equivalent to the concept to be learned). Tasks can pose problems about the behavior of a function's quantities in which the resolution requires attention to the desired quantitative understanding. In this way, the students have opportunities to intuitively develop function concepts as they devise their own solutions to nonroutine problems (for example, concavity can emerge in students' reasoning as they use trends they notice in the average rate of change to make predictions about the behavior of quantities).

Academic Success Skills: As function is such an integral idea upon which many future ideas depend, developing a robust, quantitative understanding of function can go a long way towards fostering students' willingness to persevere in problem solving and their identities as capable of doing mathematics. When improperly motivated, introduction of functions can seem arbitrary and unnecessarily complicated, raising a barrier to many students. Modules should help students become confident in their use of functions as a foundation of the language of mathematics and science.

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Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In *Proceedings of the annual meeting of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 31-49). PME Morelia, Mexico.

Additional Resources

Carlson, M., Oehrtman, M., & Moore, K. (2010). Precalculus: Pathways to calculus: A problem solving approach. *Rational Reasoning*. (excerpt available at https://www.dropbox.com/s/ntlfnvm6ht31ycx/Sample%20Pathways%20instructor%20materials.pdf?dl=0)

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MIP PARTICIPANT INFORMED CONSENT FORM

Title of the Study: Investigating the pedagogical practices of a mathematics instructor participating in an inquiry-oriented professional development initiative: An exploratory case study

Principal Investigator: Josiah Ireland
Funding Agency: National Science Foundation (DUE 1821545)

<u>Invitation to Participate in a Research Study</u>

You are invited to participate in a research study. Taking part in this research study is voluntary. You are not required to participate in this study. You may stop or withdraw your participation from this study at any time.

<u>Important Information about this Research Study</u>

Purpose of the study: The goals of this exploratory study are to investigate how you are understanding the three elements of inquiry and conceptual analysis and incorporating these features in your lesson plans, CoRD module, and instructional practices.

Risks and discomforts associated with this research: There are minimal risks involved in this study.

Direct benefits to the participants: There are no direct benefits from participation.

Please read this entire form and ask questions before deciding whether you would like to participate in this research study.

1. <u>Purpose of the Study</u>

The results of this study will provide the MIP Team with meaningful data that can be used to inform future design aspects of the project.

2. <u>Benefits of the Study</u>

The results of this study will be used to provide the MIP Team with meaningful data that can be used to inform future design aspects of the project.

3. <u>What You Will Be Asked to Do</u>

If you choose to participate, you will be meeting with me 2-3 times per week for approximately four to five weeks (during March or April) with each meeting lasting no more than one hour. The purpose of perhaps extending the research to five weeks would be to reduce your weekly involvement in this study. Most of these meetings will involve an interview discussion related to your lesson plans, your instructional practices, or associated with some task that I provide. Some weeks you will also be asked to complete an assignment to write a written reflection or observe a recording of an instructor's teaching.

There might also be a few follow up meetings in fall 2021. Each meeting will last no more than one hour.

4. Withdrawal from the Study

You are free to withdraw at any time without penalty, and to omit answers on questions that you feel uncomfortable answering. If you choose to participate, you can withdraw from the study at any time without penalty. You can do this by emailing me (josiah.ireland10@okstate.edu). If you choose to withdraw, none of your data will be included in the study.

5. Risks

There are minimal risks involved in this study.

6. Data Collection and Management

If you choose to participate in the study, you are consenting to the researcher collecting audio or video recordings of your teaching and interviews, including your office hour sessions. Your anonymity will be protected throughout the study, as well as in all presentations and published work. All documents and videos created during the data collection phase will be destroyed at the conclusion of data analysis. Refer to the attached data management plan for details on data collection and storage procedures.

7. Compensation for Participation

If you choose to participate, you will be compensated $50/hour for your involvement. This compensation will only correspond to your individual meetings with me and the tasks that I ask you to complete beyond your involvement in other MIP activities (e.g., CoRDs, regional workshops)

8. If You Would Like More Information about the Study

If you would like any additional information, you may contact Josiah Ireland at josiah.ireland10@okstate.edu.

The Institutional Review Board (IRB) for the protection of human research participants at Oklahoma State University has reviewed and approved this study. If you have questions about your rights as a research volunteer or would simply like to speak with someone other than the research team about concerns regarding this study, please contact the IRB at (405) 744-3377 or irb@okstate.edu. All reports or correspondence will be kept confidential.

Oklahoma State University
Institutional Review Board
Office of Research Compliance
223 Scott Hall, Stillwater, OK 74078
Website: https://irb.okstate.edu/
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9. Willingness to Participate

Please click the link and specify whether you consent or do not consent to participating in this study. This consent form will be kept by the researcher for three years beyond the end of the study.

Link

VITA

Josiah Gaines Ireland

Candidate for the Degree of

Doctor of Philosophy

Dissertation: AN INVESTIGATION OF THE IDENTITY TRAJECTORIES OF MATHEMATICS INSTRUCTORS PARTICIPATING IN AN INQUIRY-ORIENTED PROFESSIONAL DEVELOPMENT INITIATIVE

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Biographical:

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