EXPLORING UNDERGRADUATE STUDENTS' MULTIPLICATIVE CONCEPT STAGES, OPTIMIZATION PROBLEM SOLUTIONS, AND MATHEMATICS IDENTITY

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Abstract: Undergraduate students' choices and success in their career and life is influenced by their mathematics achievement and perception of themselves in life. Conceptual understanding of topics in undergraduate mathematics such as quantitative reasoning can be difficult for students with less sophisticated multiplicative reasoning than others. This study explores undergraduate students multiplicative reasoning and its connection to their solutions to optimization problems and their mathematics identity. Data was collected in two phases and synthesized into three articles. The first article is the validation of an assessment. The second is a case comparison study on undergraduate student solutions to optimization problems. The third is a case comparison study of undergraduate students' mathematics identity. In the first phase, the Undergraduate Multiplicative Concept Assessment (UMCA) and scoring rubric were developed and tested to provide evidence towards its validity as an appropriate assessment for undergraduate students. Data in this phase was collected from 51 undergraduate student assessments and 18 clinical interviews. Evidence towards the validity of the UMCA is discussed in the first article of this dissertation. In the second phase of data collection, 43 undergraduate students in an entry level math course took the UMCA and five of these participants participated in an interview. This interview was given in two parts: a semi-structured interview on their mathematics identity and a clinical interview asking them to solve optimization problems. Results from the clinical interview were discussed in the second article and results from the semi-structured interview are discussed in the third article. Findings from this study (1) supports the UMCA as a valid assessment for undergraduate students, (2) show that student solutions to the optimization problems included guess and check, tables, graphs, and equations and leveraged relational, algebraic, and covariational reasoning to get accurate answers, and (3) provides evidence towards the connection between students' mathematics identity and multiplicative reasoning.

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CHAPTER I

INTRODUCTION

Mathematics achievement has lasting consequences on students' future job salary, and socioeconomic and employment status (Gonzalez et al., 2020; Stinson, 2004). Researchers have discussed mathematics achievement as a gate-keeper to students' college attendance, college completion, and level of academic success and attainment (Douglas & Attewell, 2017; Gonzalez et al., 2020). Students' career choices and persistence in mathematics related fields is influenced by their mathematics achievement and their perceptions of their mathematics ability (Ehrenberg, 2010; Gonzalez et al., 2020). Supporting students' mathematics achievement is supporting their long-term career options, goals, and financial success.

The purpose of learning mathematics is not solely students' mathematics achievement. Learning mathematics allows students to develop mathematical reasoning that is beneficial for reasoning about the world around them. The National Research Council (2001) stressed the importance of teaching students to reason mathematically so they can participate in society and develop competence in everyday tasks. As such, mathematical reasoning became a major focus in the goals of the Common Core State Standards for Mathematics (CCSSM; CCSSO, 2010). Similarly, entry-level college math courses focus on quantitative reasoning in their curriculum (Elrod, 2014; Lusardi & Wallace, 2013; Wolfe, 1993). Quantitative reasoning requires students to apply mathematics and critical thinking to interpret data and draw conclusions to solve problems, which are critical skills for students regardless of their pursued major (Elrod, 2014).

An important aspect of developing students' quantitative reasoning is the types of tasks students experience and reason about in their education. Education initiatives focus on students' development of rich mathematical and quantitative reasoning. The National Research Council (2001) and the CCSSM (CCSSO, 2010) both discuss students' conceptual understanding of mathematic topics as a priority in teaching mathematics. Conceptual understanding is students' "comprehension of mathematical concepts, operations and relations" (CCSSO, 2010, p. 6). However, conceptual knowledge of some mathematical topics, such as rate of change, can be difficult for students to learn (Byerley, 2019). Some students prefer to memorize procedures due to the difficulty of conceptual learning (Byerley, 2019). Evidence from Byerley's (2019) study indicated that many of these difficulties in conceptual learning may come from the limitations caused by the students' available operations that became sources of frustration when asked to learn mathematics conceptually. As such, conceptual approaches to instruction should incorporate strategies that facilitate mathematic reasoning and accommodates limitations on students' mathematical reasoning from their available operations.

Statement of the Problem

Supporting student's conceptual learning requires a deep understanding of students' problem-solving strategies, reasoning, and foundational knowledge (Linquist, 2015). Steffe (1992) argued that second-order models of student thinking that are constructed through the lenses of both the researcher and the student are essential for understanding student thinking. Second-order models are hypothetical models constructed by observers to explain their observations of the behaviors of an observed subject (Steffe, 1992). As such, second-order models are a way for the researcher and teacher to develop an understanding of students' mathematical thinking that prioritizes the students' perspective and sense making. Steffe (1992) argued that to ignore the students' lens in problem solving is to place an expert's bias on student work that minimizes and excludes a wide variety of viable strategic thinking. Second-order models of student reasoning allow researchers to identify stages of student reasoning on topics such as number sequences (Steffe, 1992), the multiplicative concepts (Hackenberg & Tillema, 2009; Steffe, 1994), fractions schemes (Steffe, 2010) and proportional reasoning (Steffe et al., 2014) and explore students' operations and mathematical reasoning that is supported by their stage in the model. Some examples include the relationship between the multiplicative concept stages and students' understanding of rate of change (Byerley, 2019), the relationship between students' fraction schemes and multiplicative concept stages and their ability to write generalizations and equations (Hackenberg & Lee, 2015; Hackenberg et al., 2021), and the relationship between students' unit coordination structures and their problem solving with systems of equations (Olive & Caglayan, 2008).

Research on second-order models of student thinking takes many approaches. Among these are three this study will use. First, there is an identification of students' mathematical knowledge that serves as a foundation for their mathematical thinking (e.g., Kosko, 2019; Steffe 1994; Steffe et al., 2014; Thompson & Carlson, 2017). Second, there is an exploration of student thinking on new tasks and topics through the lens of their prior mathematical knowledge (e.g., Byerley, 2019; Hackenberg & Lee, 2015; Olive & Caglayan, 2008; Steffe, 2013). Finally, there is an exploration of the impacts of these models on the student learning (e.g., Boyce & Norton, 2019; Boyce et al., 2021; Byerley, 2019; Hackenberg & Lee, 2015).

This study aims to support undergraduate student's mathematical thinking by taking similar approaches to those mentioned above. First, it is imperative to effectively identify undergraduate students' foundational mathematical knowledge. Second, it is important to explore new avenues of undergraduate student thinking to add to our existing models. Finally, connecting these models to other areas of students' lives that have an impact on their mathematics achievement provides additional ways to support students' mathematical learning, experiences, and choices. Each of these points serve one goal, to create more ways to support undergraduate students' success in mathematics. To explore this problem, this paper will be assessing students' ability to construct and coordinate unit structures, or their *multiplicative concept stage* (Hackenberg & Tillema, 2009), as a model to explore student thinking on optimization problems and students' mathematics identity.

Purpose and Research Questions

The three articles presented within this paper contribute to this study's overarching purpose, to explore the relationship between undergraduate students' multiplicative concept stage and their mathematical reasoning and mathematics identity. Each article will be discussed below in terms of its individual purpose and aligned research question(s).

Article 1

The purpose of the first article is to develop an instrument and scoring rubric to assess an undergraduate students' multiplicative concept stage without necessitating interview evidence, allowing for ease of data collection. Students who have the ability to construct and coordinate unit structures work on one of three stages of the multiplicative concepts referred to as the first multiplicative concept (MC1), the second multiplicative concept (MC2), and the third multiplicative concept (MC3; Hackenberg & Tillema, 2009). The data collected by the assessment should accurately reflect literature on the multiplicative concepts and create clear delineations between students with different multiplicative concept stages. The research question that will be addressed in this study is:

- How well do the assessment and rubric items align with the theoretical framework for multiplicative concepts and assess undergraduate students' multiplicative concept stage?

Article 2

The purpose of the second article is to explore undergraduate students' problem solving on optimization problems and how students' multiplicative concept stage influences their problem solving. The research questions that will be addressed in this study are:

- How do undergraduate students reason about and solve optimization problems?
- To what extent can the multiplicative concepts be used to explain undergraduate students' reasoning on optimization problems?

Article 3

The purpose of the third article is to explore how undergraduate students with different multiplicative concept stages discuss their mathematics identity. The research question that will be addressed in this study is:

- How do students with different multiplicative concepts describe their mathematics identity?

Significance of the Study

Each of the articles outlined in this study expand the research field surrounding multiplicative concepts and student thinking. Previous research has developed assessment tools for the multiplicative concept stages of second and third graders (Kosko & Singh, 2018), fourth and fifth graders (Kosko, 2019) and sixth graders (Norton et al., 2015). However, undergraduate MC2 students often bypass limitations in their conceptual understanding by memorizing procedures (Byerley, 2019). This can potentially make it difficult to distinguish MC2 and MC3 undergraduate students from written work on existing assessment. The first article develops an assessment that is more appropriate for undergraduate students than current assessments. The newly developed assessment can be used to collect data on a larger scale in undergraduate education by minimizing the need for interview verification of stage attribution. This will assist the fields' current movement towards research in undergraduate student thinking on college mathematics concepts (Boyce et al., 2021; Byerley, 2019).

With multiplicative concepts research expanding into undergraduate education, new mathematics topics and problems join the discussion. Current research has explored the relationships between students' unit coordination structures and their reasoning and operations with fractions (Steffe, 2001), proportions (Steffe et al., 2014), variables and equations (Hackenberg, 2013; Hackenberg & Lee, 2015; Hackenberg et al., 2021; Olive & Caglayan, 2008; Zwanch, 2019), quantity and measurement (Steffe, 2013) and rates of change (Byerley, 2019). The second article explores student thinking on optimization problems in order to add to researchers' understanding of students' strategies when solving systems of equations, which was discussed in Olive and Caglayan's (2008) study.

Research in this field remains widely focused on creating and using second-order models to explore student mathematical thinking. These models involve multiple stages of student thinking and operations. Limitations in student thinking can cause students to struggle when learning new mathematical concepts. Byerley (2019) found that MC2 students were frustrated during conceptual learning due to difficulties that resulted from limitations in their operations on unit structures. These struggles can create negative experiences in mathematics if they do not receive support. These negative experiences can have a detrimental influence on how they view themselves as a “math person”.

The third article expands student multiplicative concept research into the identity field by exploring the potential connections between students’ mathematical experiences stemming from their multiplicative concepts and their development of their mathematics identity. A part of supporting student thinking is ensuring their persistence in problem solving. Students’ mathematics identity plays an important role in their persistence in mathematics (Cribbs et al., 2020). It is important to support the construction of strong mathematics identities in students to help them persist in mathematics. Accommodating student problem solving to overcome limitations in their thinking could potentially create positive mathematics experiences that influence their constructed mathematics identity. Such research is beneficial for understanding the potential relationship that may exist between multiplicative concepts and students’ mathematics identity.

Assumptions, Limitations, and Delimitations

Assumptions

It is an assumption for this study participants respond to questionnaire and interview questions with honesty. Additionally, it is assumed that each student participating in this study has completed the high school mathematics curriculum, which includes the topics of multiplication, improper fractions, equation representations of word problems, and solving systems of equations prior to attending university (CCSSO, 2010). While mastery may not have been achieved, it is assumed undergraduate students have some prior knowledge of these concepts.

Limitations

Conclusions from the optimization problem solutions inform the construction of a model of student thinking. While these serve as a general model situated in multiplicative reasoning research that can be transferred to other students with similar operations, it is important to note that these participants do not represent the entirety of MC2 and MC3 students. For this reason, it is important to develop an assessment and rubric for assessing the multiplicative concepts of the undergraduate populations, but also use the assessment and interpret any findings from it with this population in mind. Additionally, prior research has shown that undergraduate students have generally constructed an MC2 by the time they attend undergraduate mathematics (Boyce et al., 2020). As such, MC1 students will be excluded from the sample for the second and third articles of this study.

Past experience with problems posed influences their ability to problem solve and clinical interviews only capture a snap-shot of student understanding at a given time (Clements, 2000). It does not reflect what they could learn but rather what they know when they participated in the study.

Additionally, while we can look at individual students to determine a general pattern of mathematics identity construction, these conclusions are not indicative to the entire population of undergraduate MC2 and MC3 students. However, looking into the narratives of these participants provides key insights to the relationship that may exist between their multiplicative concepts and mathematics identity. These limitations do not take away from the study, but rather add to the narrative surrounding multiplicative concepts and mathematics identity to allow for further research on these topics.

Delimitations

A delimitation of this study is use of undergraduate education majors as a sample of convenience for the assessment and rubric validation in the first phase of this study. Restricting this sample allows greater access to students for the interviews necessary to accurately evaluate the instrument. Additionally, participants chosen for the second portion of this study (the participants for article 2 and 3) will be purposefully selected to ensure a wide variety of participant backgrounds.

As stated in our limitations, only students with the MC2 and MC3 students will be interviewed. This choice is made to reflect the general absence of MC1 students in the undergraduate mathematics population (Boyce et al., 2021). Attempting to obtain a large enough sample of students with the first multiplicative concept for these studies would be difficult and disproportionate to the population as a whole.

Additionally, there will be one interview protocol to collect data on student solutions to optimization problems and their mathematics identity to help researchers manage time and allow the minimization of resources needed for participant recruitment. This decision also allows for the collection of participants' identity statements during problem solving

Summary

The articles discussed in this paper develop a way to assess the multiplicative concept stage of undergraduate students, explore undergraduate students' problem-solving strategies for optimization problems, and explore connections between students' multiplicative concept stages their mathematics identity. Research on these topics aim to expand the literature surrounding undergraduate multiplicative concepts and the development of mathematics identities. Exploring these concepts allows us to re-evaluate the support we are providing for students in undergraduate mathematics classes for their success in higher level mathematics.

CHAPTER II

LITERATURE REVIEW

To further explore the relationships between students' operations, problem solving strategies, and mathematics identity, it is important to discuss what research has been published as a foundation for these topics. This chapter will discuss the epistemological stance and theoretical framework of schemes, literature on students' unit coordination schemes, the definition of and supporting mathematical reasoning involved with optimization problems, and a conceptual framework connecting schemes to mathematics identity. This discussion will begin by exploring the theoretical underpinnings and frameworks for this study.

Radical Constructivism

Radical constructivism, as discussed in Ernst von Glasersfeld's (1995) book, is the culmination and adaptation of theories of existence and Piaget's (1952) theory of cognitive development. Radical constructivism has two foundational principles for knowledge and cognition:

- Knowledge is not passively received, but actively built up and constructed by a cognizant subject.

- Cognition is fundamentally adaptive attempting to find fit and viability in the active organization of the subjects' experiential world (von Glasersfeld, 1995).

Cognition does not pursue a development of an objective reality, but instead an optimal understanding of the reality experienced by the subject. As such, von Glasersfeld (1995) treated this learning theory as a tool for its usefulness in understanding learning rather than as a metaphysical proposal of reality.

Under these foundations, radical constructivists seek to understand the knowledge and cognition of others through their constructions. Radical constructivist Leslie P. Steffe (1992) placed extreme importance in pursuing research in children's mathematics rather than the mathematics of children. *Children's mathematics* focuses on children's problem solving with an emphasis on students' knowledge, operations, and thinking. As such, models of children's mathematics are second-order models that combine the researcher and subjects' lenses to create a hypothetical model of children's mathematical experiences (Steffe, 1992). In contrast, the study of the *mathematics of children* develops first-order models from the researcher's or expert's lens. Such models often minimize student solutions if they do not reflect the researcher's expectation of children's problem solving (Steffe, 1992). Children's understanding of mathematical concepts are negotiated during their interactions with these concepts and their learning environments. As such, each child has potential for problem solving based on their prior understanding and learning experiences (Steffe, 1992).

Theoretical Framework: Scheme Theory

The influence of students' prior experiences are outlined in von Glasersfeld's (1995) action scheme theory, which was adapted from Piaget's (1952) idea of schemes. An action scheme is defined as a model for an individuals' learned actions and thoughts that serve as a pattern of interaction that comes into play when an individual is introduced to a situation (von Glasersfeld, 1995). It involves three parts: a recognition of the perceived situation, a specific mental activity that is associated with that activity, and an expected result based on previously experienced results (see Figure 2.1).

Figure 2.1

Pattern of an action scheme

Note: Figure of an action scheme from the recognition phase to the activity phase and the expected result.

For example, a student that is asked to solve a system of equations will first look at the problem and attempt to recognize it. If a student has previously worked with systems of equations, they may recognize the current problem as a familiar one and move on to the action phase. In the action phase they would use their known operations and procedures to solve the problem. These operations would be the ones they used previously and could include strategies such as substitution or guess and check. Students will have an idea of what the product of this strategy should look like based off their previous work solving systems of equations. They may expect this product to be an ordered pair, a value for x and y, or just a number. As such, the student can compare their result with their expected result. However, if the student has never seen systems of equations before, they may rely on their knowledge of similar concepts to solve the problem (e.g. their experience with single equations). In this case, the individual’s activity will reflect their work with single equations. Additionally, their expected result may be similar to their expected result when working with single equations, or it may be completely different as they expect the introduction of additional equations to change what is expected as a result.

Schemes are used in an eternal pursuit of equilibrium. Assimilation, as defined by von Glasersfeld (1995), is “an instance of knowing” (p. 62) and heavily influences the recognition stage of an action scheme. If the expected result of a scheme does not match the actual result, a perturbation occurs. Perturbations can be a positive beneficial surprise or a negative frustration. Either way, it is likely that the individual will first reflect on what is available post activity to identify a way to reestablish equilibrium in their schemes. This reestablishment of equilibrium can occur by the student identifying conditions to establish in the recognition phase to help avoid the unexpected result or additions to the recognition phase to create a new scheme. They may also make accommodations to the activity associated with the recognition phase of their scheme or adjust what they expect the outcome of their activity to be. The cycle of scheme perturbation and accommodation to reestablish equilibrium is what constitutes learning (von Glasersfeld, 1995). The continual pursuit of equilibrium allows learning on an expansive level as more concepts are assimilated into the student’s knowledge.

Mathematics education research into children’s mathematics has explored students’ schemes for a variety of mathematical concepts. Such concepts include number sequences (Steffe, 1992; Ulrich 2016b), multiplicative and divisional schemes (Steffe, 1992; Steffe, 1994), fractional schemes (Hackenberg, 2007; Steffe, 2001; Steffe, 2010), and proportionality schemes (Steffe et al., 2014). Each set of schemes is developed from observations of students’ problem solving, strategic thinking, and operations. Schemes, once established, serve as a lens for understanding student thinking, operations, and limitations when problem solving, which researchers and teachers can leverage to develop curriculum and tasks that promote student learning and growth that are tailored to students’ current level of understanding.

Additionally, schemes provide insight into foundational mathematical concepts that if not mastered can lead to difficulty in understanding more advanced mathematics concepts. For example, students who have not constructed the third multiplicative concept (the most sophisticated stage of unit coordination) can struggle to understand the concepts behind rates of change (Byerly, 2019) and solving system of equation problems (Olive & Caglayan, 2008). Instructional practices for these advanced topics should then be adjusted to accommodate for limitations in students' available operations. Research in this field continually strives to support student thinking across mathematic disciplines and concepts as it grows the literature surrounding student problem solving and schemes.

Multiplicative Concepts

Students' unit coordination schemes, or multiplicative concepts, were developed from Steffe's (1992, 1994) exploration of students' number sequences and student thinking on whole number multiplication and division problems. Steffe (1992) found that students' multiplicative thinking related directly with their ability to construct and coordinate unit structures (i.e., units of units). Units are standard and non-standard units of measure (Ulrich, 2015). Unit structures are created from combining smaller units together to form a larger unit that contains subsets of units (Steffe, 2010). For example, a composite unit 28 can be thought of as the combination of 2 units of size 10 and 18 respectively, 2 units of size 14, 4 units of size 7, or 28 units of size 1. These unit structures can be operated on using partitioning (breaking a whole unit into equal parts; Hackenberg et al., 2016), iterating (repeating a part to make a new amount, Hackenberg et al., 2016), and dismbedding (removing a part of the unit mentally; Hackenberg et al., 2016) operations.

In order to reason multiplicatively, students must be able to conceive of and construct a composite unit (a two-level unit structure; Ulrich, 2015; Ulrich, 2016a). This construction may be done physically with representations such as drawings or finger representations, or cognitively with strategies such as skip counting. The degree to which students anticipate, assimilate, construct, and coordinate multiple levels of units determines the stage of multiplicative concept the student has developed and the operations available to that student for problem solving (Hackenberg & Tillema, 2009; Ulrich, 2016a). There are three levels of coordination with unit structures within the multiplicative concepts.

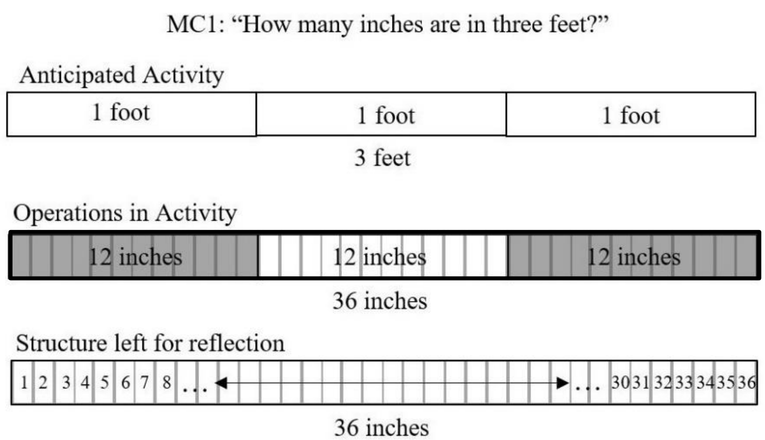
The First Multiplicative Concept

Students who have constructed the first multiplicative concept (MC1 students) assimilate with one level of unit and are capable of coordinating two levels of units in activity (during problem solving; Hackenberg & Tillema, 2009). Students are able to construct a composite unit by inserting one unit into another. MC1 students coordinate two-levels of units during activity but do not anticipate this coordination prior to problem solving. MC1 students have access to partitioning and iterating operations which allow them to create visual representations of multiplicative relationships during problem solving (Hackenberg, 2013). They may also keep track of their multiplication by monitoring the number of times they have counted using physical representations, fingers, or tally marks. While MC1 students can construct composite units in activity, the levels of the unit begin to decay after construction.

For example, MC1 students can determine how many inches are in 3 feet by counting 12 inches 3 times giving them a total of 36 inches. After this activity, MC1 students perceive these 36 inches as 36 individual units that retain a number sequence of 1-36 (see Figure 2.2). Since these units retain a number sequence, the units are not interchangeable. To the student, the second unit which is assigned a 2 from the number sequence is a different from the tenth unit that was assigned the number 10 even though they are the same size. Additionally, the composite unit they constructed during activity (the 3 feet containing 12 inches) decays from their unit structure and are unavailable to the student for reflection (Hackenberg et al., 2021).

Figure 2.2

Example of MC1 students' operating structures



Note: Visual representation of proposed MC1 students' solutions to the question "How many inches are in three feet?" from Hackenberg et al. (2021).

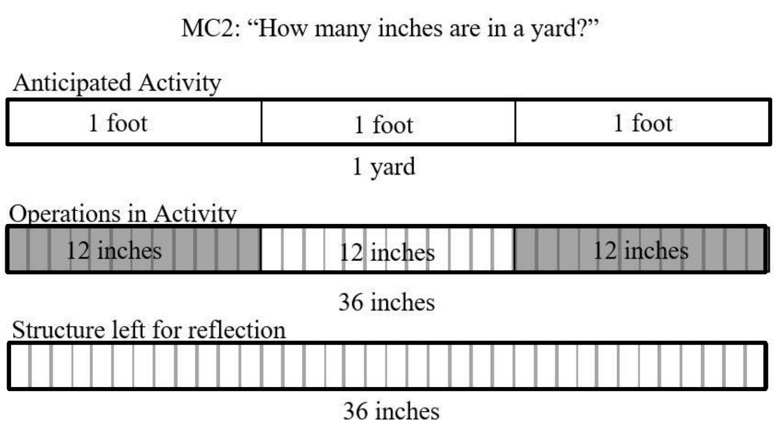
The Second Multiplicative Concept

When students can assimilate with two levels of units and can coordinate three levels of units in activity, they have constructed the second multiplicative concept. These students (MC2 students) are able to anticipate the creation of the two-level unit structures that MC1 students could only construct in activity (Hackenberg & Tillema, 2009). MC2 students are able to perceive units as subsets of other units allowing them to define and reflect on these subsets and disembed them from the whole to operate on (Ulrich, 2015). Thus, MC2 students are able to construct three levels of units in activity by disembedding subsets of the two-level unit structure and then partitioning these subsets. MC2 students are also able to view the final unit structure, post-activity, as a collection of iterable units of 1 that are interchangeable with one another. The three-level unit structure decays after the mental construction of the unit for MC2 students. This decay can occur during or after the problem-solving activity for the student.

For example, an MC2 student can solve the question of how many inches are in a yard by recognizing a yard as a composite unit containing 3 feet. They can then insert 12 inches into each of the 3 feet. Thus, they can determine that there are 36 inches in a yard. However, they now view these 36 units as identical and interchangeable units of 1 rather than as a number sequence of 1-36 (see Figure 2.3). The original unit structure of 1 yard containing 3 feet and the constructed 3 feet containing 12 inches each decay post-activity and will require the student to repeat their mental activity on the problem to reconstruct these structures (Hackenberg et al., 2021).

Figure 2.3

Example of MC2 students' operating structures



Note: Visual representation of proposed MC2 students' solutions to the question "How many inches are in one yard?" from Hackenberg et al. (2021).

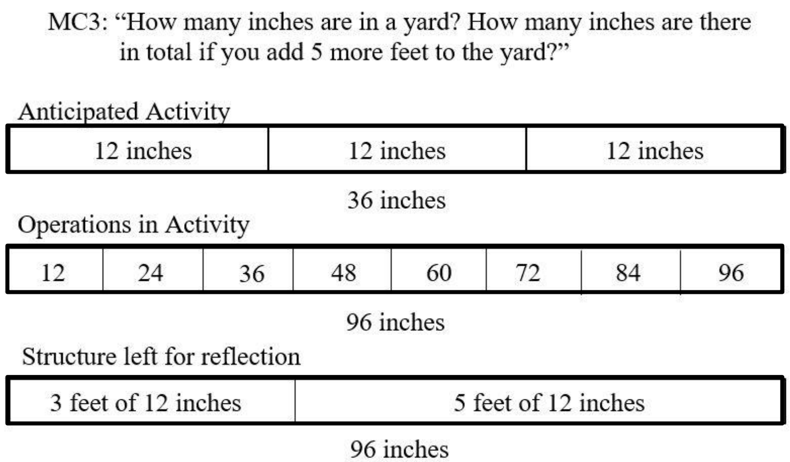
The Third Multiplicative Concept

Students who assimilate with three levels of units now anticipate the creation of three levels of units prior to mental activity, are able to construct fourth or fifth level units, and can flexibly move between levels of units (Ulrich, 2016a). Students who have constructed the third multiplicative concept (MC3 students) now assimilate iterable composite units (Ulrich, 2016a). MC3 students can construct the splitting operation, which involves the simultaneous use and composition of partitioning and iterating operations (Steffe, 2010). Additionally, the third-level unit structures they use in problem solving do not decay and are available for reflection post-activity (Hackenberg & Tillema, 2009).

MC3 students can easily determine that there are 36 inches in a yard by anticipating that a yard consists of 3 feet that each have 12 inches. They can also determine how many inches there are if you add five additional feet to that yard. MC3 students are able to iterate their composite units of 1 foot containing 12 inches five additional times to determine that there are a total of 96 inches. An MC3 student would recognize the 96 inches as 9 feet each containing 12 inches (Hackenberg et al., 2021). As such, they retain their third level unit structure for reflection and can recognize this structure as a collection of iterations of a composite unit (see Figure 2.4).

Figure 2.4

Example of MC3 students' operating structures



Note: Figure is a visual representation of proposed MC3 students' solutions to the question "How many inches are in a yard?" and "How many inches are there in total if you add 5 more feet to the yard?" from Hackenberg et al. (2021).

Impact of Students' Multiplicative Concepts

Students' ability to coordinate multiple levels of units plays a foundational role in their mathematical reasoning on a wide variety of subjects. Research has explored many avenues of students' mathematical thinking in connection to their unit coordination structures including fractional reasoning (Hackenberg, 2007; Steffe, 2001), proportional reasoning (Steffe et al., 2014), algebraic reasoning (Hackenberg, 2013; Hackenberg & Lee, 2015; Olive & Caglayan, 2008; Zwanch, 2019, 2022a), recognition of quantity and use of measurement (Steffe, 2013), and conceptual understanding of derivatives and rates of change (Byerley, 2019). Students' covariational and geometric reasoning are also areas that students' multiplicative concepts may influence, but more research is needed to explore these potential connections (Boyce & Norton, 2016).

Fractional Reasoning

Students' whole number concepts inform their fractional reasoning and schemes (Steffe, 2001; Steffe, 2010). Students operate on fractions according to the operations available to them from their fraction schemes. Boyce and Norton (2016) found that students co-constructed their multiplicative concepts and fraction schemes, allowing students to use either scheme to inform and accommodate the other.

Steffe's (2001) fractional schemes are primarily built from students use of partitioning and iterating operations to solve fractional problems. At the most basic level of fractional schemes, the part-whole fraction scheme, students can partition a whole to create a part-to-whole relationship. This scheme allows students to recognize a circle partitioned into 8 parts and identify what the fraction "3/8" of this circle would be by shading 3 pieces. Additionally, they could disembed the three pieces they shaded to indicate that this would be 3 pieces of the total 8 pieces (Steffe, 2010).

When students can partition the whole and then recognize a singular part as a "one-to-many" (Steffe, 2010, p. 102) relationship or unit fraction, they have constructed a partitive unit fraction scheme. In this way, they can partition a circle into 8 pieces and identify one of these pieces as "one-eighth". By labeling the unit fraction as "one-eighth", the student is able to retain the relationship between the piece to the whole as a partition of the whole unit.

Students who then iterate unit fractions to create new fractions of the whole that are less than one have developed the partitive fractions scheme (Steffe, 2010). By first partitioning and then iterating, a student can identify what fraction of a full circle an unpartitioned piece that is three-eighths of the circle takes up. They would be able to correctly identify this as a "three-eighths" sized piece.

The Case of the Iterative Fraction Scheme

In order to understand improper fractions as numbers in their own right, students must be able to simultaneously partition and iterate unit fractions of a “whole” by using a splitting operation and assimilate with three-levels of units (Hackenberg, 2007; Steffe, 2001). The iterative fraction scheme (IFS) require students to recognize the improper fraction as a unit containing iterated unit fractions (Steffe, 2001). When students can do this, they have developed the iterative fraction scheme (IFS; Steffe, 2001). This allows the student to consider the fraction 7/3 as a unit containing 7 iterated 1/3 unit fractions (not units of 1s labeled 1/3) that can be reconstructed into 2 wholes and a 1/3 unit fraction, where both wholes contain 3 iterated 1/3 unit fractions**.** Students who have not constructed an IFS will not know how to deal with the additional piece on the end and may ignore it or incorrectly label it.

MC1 and MC2 students can struggle with more advanced fractional problems involving improper fractions that utilize a splitting operation (Hackenberg, 2007; Hackenberg & Lee, 2015; Steffe, 2010). In addition to the splitting operations, students must also develop an MC3 in order to construct improper fractions (Hackenberg, 2007). While MC2 students can construct a splitting operation, an MC3 is required to support the multilayered structure evident in improper fractions (Hackenberg, 2007).

The construction of an IFS is an indicator that the student has constructed an MC3. This is an important concept as the IFS is often used in multiplicative concept measurement instruments to help delineate between MC2 and MC3 students (Norton et al., 2015).

Proportional Reasoning

Proportional reasoning is defined by Lamon (1993) as one’s ability to “construct and algebraically solve proportions” (p. 41). This type of reasoning depends heavily on the construction and leverage of a unit ratio during problem solving. **S**tudents must assimilate with three levels of units in order to accurately identify and use unit ratios in proportional problems (Steffe et al., 2014). Students who cannot construct and use unit ratios may have an ephemeral awareness of a proportional relationship but have not constructed a proportionality scheme necessary for proportional reasoning (Steffe et al., 2014). This lack of a proportionality scheme may cause them to conflate scale factor operations that should be applied to their ratio with the answer to the proportion question. For example, Jill in Steffe et al.'s (2014) study was given a 2 to 3 ratio between tablespoons of lemonade powder and cups of water in a recipe. When asked how many cups of water were needed to make lemonade with 1 tablespoon of powder, she knew to divide the recipe in half. However, she answered that you would need "one half" cups of water, confusing the idea of halving the recipe with how much water she would need (Steffe et al., 2014).

MC3 students can identify the unit ratio and apply it to the proportion problem, indicating they have moved beyond an awareness of proportionality to the construction of a proportionality scheme (Steffe et al., 2014). For example, Jack in Steffe et al.'s (2014) study was also given the 2 to 3 ratio between tablespoons of lemonade powder and cups of water in the recipe that Jill was given. When asked to determine how many tablespoons of powder would need to be mixed with one cup of water, he started by identifying that "three halves [cups of water] make up one tablespoon" (Steffe et al., 2014, p. 65). He could then use this unit ratio of 3/2 cups of water to 1 tablespoon to determine that one cup of water would be 2/3 tablespoons of powder. Maintaining the proportional relationship between two quantities requires the assimilation of three levels of units (Steffe et al., 2014). As such, an MC3 is essential for proportional reasoning (Ulrich, 2016a). Zwanch's (2022a) findings on undergraduate design students reasoning on proportion problems supports the necessity of an MC3 for the development of a proportionality scheme.

Algebraic Reasoning

A disembedding operation, which is unavailable to MC1 students, is necessary for writing multiplicative equations, algebraic expressions, and generalized relationships (Hackenberg, 2013). A disembedding operation allows for students to envision partitions of a whole as both separate from the whole and a part of the whole. This operation is essential for creating a generalization of a situation as students must understand a variable as a part of the whole that is independent from the whole but stays in relation to the whole (Hackenberg, 2013). When writing multiplicative equations, MC2 students struggle to represent a multiplicative relationship between two unknowns and will not use fractions as multipliers to unknowns (Hackenberg & Lee, 2015). While MC2 students have constructed a disembedding operation that is necessary for equation writing, to visualize the multiplicative relationship between two unknowns requires that the student be able to abstract their units coordinations (Hackenberg & Lee, 2015). As such, an MC3 is required to able to represent multiplicative relationships between unknowns and multiply unknowns with both whole numbers and fractions (Hackenberg & Lee, 2015). This is further supported by Zwanch's (2019) findings that the splitting operation (which MC3 students have constructed) is essential to representing multiplicative relationships between unknowns.

The third multiplicative concept is essential for developing the meaning of an unknown as an indeterminate number of measurement units that contain a precise number of smaller subunits (Hackenberg et al., 2021). As such, MC3 students can create generalizable equations for different multiplicative relationships. MC2 students can create pictures to represent multiplicative relationships between undetermined unknowns, but have difficulties representing these illustrations with algebraic symbols (Hackenberg et al., 2021). However, instruction that moves back and forth between pictorial representations and equations supports MC2 constructions of generalizations for multiplicative relationships between unknowns (Hackenberg et al., 2017). Additionally, MC2 students often tie their variable symbols to specific numerical examples and create equations around that specific value. Delaying the introduction of numerical examples can help MC2 students avoid this tendency (Hackenberg et al., 2017). MC3 students can form these general statements without tying the variable to a specific example (Hackenberg et al., 2021).

Olive and Caglayan (2008) found that in order to represent a system of equations that contains three-level unit relationships (e.g. the total monetary value of a collection of dimes, nickels, and quarters can be represented by the expression $0.10d + 0.05n + 0.25q$ with d, n, and q representing the number of dimes, nickels, and quarters respectively) students need to be able to coordinate and conserve units. Solving systems of equations requires the coordination of multiple two-level unit structures that helps conserve the units' values, which is only possible with the anticipation of the construction of three level units that MC3 students have (Olive & Caglayan, 2008). As such, MC2 students can struggle to create generalized equations for similar contexts.

Quantity and measurement

Similarly, the multiplicative concepts inform students' ability to reason with and recognize quantity (Steffe, 2013). As shown in the examples from the previous sections, the nature of measurement often requires the coordination of multiple levels of units in order to successfully solve measurement problems (Hackenberg et al., 2021). Additionally, students must keep track of the types of measures connected to the units being operated on. Without the ability to reflect on multiple levels of units, students may struggle to accurately determine the units of measure of their answer because they do not maintain the unit structures they constructed during mental activity.

Covariational Reasoning

Covariational reasoning, as used in this dissertation, is a person's conceptualization of two varying quantities that are varying simultaneously (Thompson & Carlson, 2017). This definition stems from Thompson's (1993) theory of quantitative reasoning that defines quantitative reasoning as their conceptualization and analysis of a situation in terms of quantities and the relationships between those quantities. Relationships between quantities in more complex dynamic situations require covariational reasoning to accurately imagine and represent them (Thompson, 2011).

Byerley (2019) hypothesized that MC2 students might struggle with covariational reasoning as it required the simultaneous imagining of multiple varying quantities at once. The MC2 participants in her study struggled with ideas of rates and slope, which require covariational reasoning to conceptualize (Byerly, 2019). Researchers have hypothesized that the splitting operation would be essential for students to calculate ratios between "successive changes in y-values for constant changes in x-values" (Ellis et al., 2016, p. 154) of the rate of change for exponential growth. In other words, the splitting operation available to the MC3 is required for covariational reasoning about rate of change of exponential growth (Ellis et al., 2016). More research into covariational reasoning and the multiplicative concepts could help explore the limitations an MC2 might place on students' covariational reasoning.

Geometric Reasoning

According to van Hiele (1999), geometric reasoning involves noticing, describing, and deducting relationships between patterns and figures. As students move from just noticing geometric properties to describing and then informally deducing those properties, students are able to articulate and anticipate these properties. The Common Core State Standards for Mathematics emphasizes students' ability to represent geometric relationships as generalized algebraic expressions and equations as a key objective in high school geometry (CCSSO, 2010). Students' ability to write algebraic equations are influenced by their multiplicative concepts meaning that MC1 and MC2 students may struggle to generalize geometric relationships. Additionally, students' ability to recognize of quantity and complete measurement problems are influenced by their multiplicative concept stage (Steffe, 2013).

However, these ideas do not encompass all the types of geometric reasoning that students' use. Boyce and Norton (2019) called for additional research on students' geometric reasoning and its connection to their multiplicative concepts as this area is currently unexplored in the literature.

Undergraduates Students' Multiplicative Concepts

While much of the research regarding students' multiplicative concepts revolves around grade school, recent studies have started exploring undergraduates' multiplicative concepts (Boyce et al., 2021; Byerly, 2019). Students' development of the multiplicative concepts begins as early as the second grade (Kosko & Singh, 2018). However, Boyce and Norton (2016) found that over half of their sixth grade participants had still not developed an MC3. Additionally, a study on undergraduate students' calculus preparedness showed that half of the participants had only developed an MC2 (Boyce et al., 2021).

MC2 students generally rely heavily on rote memorization over conceptual understanding of mathematical operations (Byerley, 2019) and are less prepared for calculus concepts (Boyce et al., 2021). Researchers are concerned over the large number of students who have not constructed an MC3 in calculus and have made suggestions for both college and grade-school curriculum to help support and accommodate MC2 students' conceptual understanding (Boyce et al., 2021; Byerley, 2019).

While instruments assessing the multiplicative concepts of sixth graders (Norton et al., 2015), second and third graders (Kosko & Singh, 2018), and fourth and fifth graders (Kosko, 2019) have been developed, it is unclear if these assessments are valid for undergraduate students. The development of an instrument that can help eliminate false positives for MC3 students from MC2 students will help further research on undergraduate students' multiplicative concepts and their mathematical reasoning.

Additionally, research on undergraduate students' mathematical reasoning can benefit from explorations into complex problems that require layers of unit coordination for problem solving and interpretation. Optimization problems provide a dynamic problem to explore multiple facets of undergraduate students' mathematical reasoning through the lens of their multiplicative concept stage. The following section will discuss the opportunity for exploring undergraduate student thinking that optimization problems provide.

Optimization Problems

Optimization problems provide a situation with varying parameters and ask the student to identify values within these parameters that would indicate an optimal solution. The problems that will be used in this study will explore students' thinking on two different optimization problems that explore relationships passengers and revenue and area and length respectively.

The *charter bus problem* is an example of an optimization problem with varying parameters for passenger number and ticket cost. The charter bus problem is as follows:

> Marian owns a charter bus company that offers a route to the neighboring city that charges $40 per person if up to 30 passengers sign up for the trip. If more than 30 passengers sign up, the fare for every passenger is reduced by $1 for every passenger in excess of 30. The bus can only hold up to 48 passengers. How many passengers does Marian want to sign up for her charter bus route if she wants to maximize her revenue for the trip?

The *barn pen problem* is an example of a geometry optimization problem with a set parameter for the perimeter as a varying parameter for the area. The barn pen problem is shown below.

> John wants to build a rectangular pen next to his barn. To try and maximize his resources, he decides to use one side of the barn as a side of his pen. If he has 160 feet worth of fence available to build his pen and the barn side was over 160 feet long, what dimensions of the pen will maximize its area?

For example, students who are solving the barn pen problem are expected to be able to represent the perimeter and area of the pen as functions. They should then be able to use the equation for the perimeter to provide a function for the length of one side and insert this function into their area function. Then students should be able to interpret this function to determine dimensions that maximizes the area of the pen. While this is the general solution process, the mathematical reasoning required to conceptualize these problems involves sophisticated quantitative, algebraic, and covariational reasoning (Thompson & Carlson, 2017).

Thompson (2011) defined quantification to be the conceptualization of an object and an attribute of the object so that the attribute both has a unit of measure and a proportional relationship with that unit. This definition of quantification outlines it as a process of determining what it means to measure a unit, what and how one measures the unit, and what that measurement means once obtained (Thompson, 2011). Optimization problems rely heavily on students' ability to quantify the situations presented to represent, measure, and interpret the situation to find a desired result.

Creating generalized representations of dynamic situations such as those found in optimization problems requires people to imagine two simultaneously varying variables expressed as functions that when combined create an invariant or steady relationship between the two variables (Thompson, 2011). This understanding is fundamentally tied to students' ability to represent generalized geometric relationships and conceive of these generalizations as covarying unknown quantities. By uniting the two generalized expressions through embedding one within the other, students are required to maintain the image of the unit they are embedding and maintain the dynamic image of the situation this unit is being embedded into (Thompson, 2011). This conservation of both static and dynamic units can be difficult for students. I hypothesize that the third multiplicative concept is essential for conserving these relationships in problem solving.

Analysis of students' problem solving on optimization problems provides insight into three key areas of reasoning for students with different multiplicative concepts. First, it adds to the existing research on students’ ability to construct algebraic representations of dynamic situations in geometric contexts. Second, it explores students’ ability to conceptualize, use, and interpret covarying functions. Finally, it explores students’ ability to conserve and interpret multiple units of measure throughout problem solving.

Summary of Multiplicative Concept Research

The multiplicative concepts serve as a theoretical lens to students’ problem-solving abilities. They are foundational to students’ mathematical knowledge and influence their fractional knowledge, proportional reasoning, algebraic reasoning, covariational reasoning, and geometric reasoning. Optimization problems provides a suitable problem to explore students’ covariational, geometric, and algebraic reasoning. This review of literature serves as an exploration into the foundational literature for this study. However, to address the hypothesized connections between students’ mathematics identity and the multiplicative concepts, it is important to establish the definitions and foundational literature of mathematics identity.

In the following section, the foundations of mathematics identity research, how it is defined, and how it is operationalized will be discussed. This discussion will lead to the explanation of a conceptual framework for the influence of students’ multiplicative concepts on their mathematics identity.

Mathematics Identity

Identity research has been on the rise in the mathematics education research field for the last decade (Darragh, 2016; Graven & Heyd-Metzuyanim, 2019). Identity research was founded on Erickson’s (1950; 1968) and Mead’s (1934) identity frameworks. While Erickson (1950; 1968) positioned identity as a single, self-determined, stable concept, Mead (1934) proposed identity as a multilayered, dynamic understanding of self, created through action that could develop into multiple, occasionally clashing identities. Central to both of these frameworks is the inherent idea and perception of self. In this dissertation, identity will be discussed through the lens Cobb and Hodge (2011) provide.

Cobb and Hodge (2011) propose multiple approaches to exploring mathematics identity, which center around three different ways of framing identity: normative, core, and personal. For the purpose of this dissertation, core identity, which adopted a macro-identity approach, aligns to the way identity is being defined and explored. Core identity is defined as a students' "enduring sense of who they are and who they want to become" (Cobb & Hodge, p. 189). This definition is a refinement of Gee's (2000) discursive definition of identity as an individual's views of themselves that are constructed through the negotiation with others whom the individual recognizes as rational. Individuals can have multiple identities or "selves" in accordance to the context in which they are considering themselves (Gee, 2000). For example, a student who identifies as a "mathematics person" may also identify as a "science person", sibling, student, or athlete. Each of these roles in which a person identifies constitutes as a different identity (Godwin et al., 2020).

As such, we define mathematics identity as how students view themselves in relation to mathematics, based on their perceptions of their experiences with mathematics (Enyedy et al., 2006). Mathematics identity, in this sense, can be discussed as how students' view themselves as "doers of mathematics" in the context of their mathematics experiences and community (Nasir, 2002, p. 214).

The Explanatory Framework for Mathematics Identity

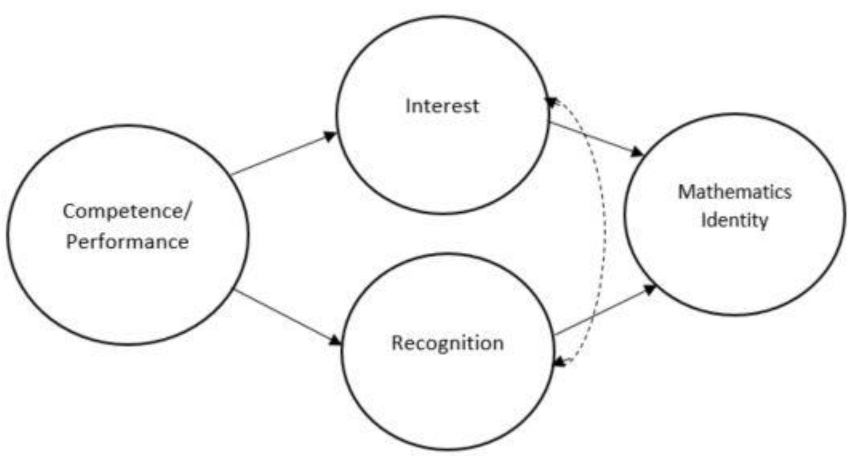
Cribbs et al. (2015) developed an explanatory framework for mathematics identity that comprised of two sub-factors, recognition and interest, and a sub-factor of competence/performance that indirectly affected mathematics identity as mediated through recognition and interest.

Recognition is defined as how an individual perceives others' view them in relation to mathematics (Cribbs et al., 2015). Research shows that students' self-perceptions and achievement in mathematics is influenced by their interactions with their parents and teachers (Gunderson et al., 2012). Students' who discuss the recognition they receive from others may discuss the times their parents or teachers praised them for doing well in mathematics, times they felt they weren't supported by their family in doing mathematics, an award they received in regard to their performance in mathematics, etc. Recognition also includes the idea of self-recognition. *Interest* is defined as an individual's curiosity or desire to learn and do mathematics (Cribbs et al., 2015). Interest is tied to students' engagement in and motivation in doing mathematics (Frenzel et al, 2010). Students' may discuss their interest by describing their love for mathematics, an excitement to learn knew mathematics concepts, the possibilities mathematics provide, a dread upon hearing they have to take a mathematics course, etc. The recognition and interest sub-factors directly influence students' mathematics identity and serve as important foundations for the students' self-perceptions in mathematics (Cribbs et al., 2015).

In Cribbs et al.'s (2015) framework, competence and performance were not quantitatively different from one another. In prior qualitative research, these sub-factors were considered separately (Carlone & Johnson, 2007). As such, we will define these separately here. *Competence* is defined as an individual's beliefs about their ability to understand mathematics (Cribbs et al., 2015). Students' competence is linked to their goals as students (Ferla et al., 2010) and their views of their performance in mathematics (Bleeker & Jacobs, 2004; Bouchey & Harter, 2005). Students' who are discussing their competence may describe instances that they just “got math”, how easily they understood a new type of mathematics problem, how difficult certain mathematics concepts are, etc. *Performance* is defined as the individual’s beliefs about their ability to perform in mathematics (Cribbs et al., 2015). Students’ performance beliefs are linked to their motivations and actual performance in mathematics (Pajaras & Graham, 1999). Students’ discussing their performance may discuss their grades in mathematics, how easily they can solve a set of problems, how difficult a certain strategy or operation is to do, etc. The competence and performance sub-factors indirectly affect the development of students’ mathematics identity by directly influencing their interest in mathematics and the recognition they receive from others (Cribbs et al., 2015). Figure 2.5 provides a visual representation of the framework discussed and the relationship between the sub-factors and students’ mathematics identity.

Figure 2.5

Structural model of the developmental factors of mathematics identity



Note: Figure from “Establishing an explanatory model for mathematics identity” by J. D. Cribbs, Z. Hazari, G. Sonnert, and P. M. Sadler, 2015, *Child Development, 86*(4), 1048-1062, (https://doi.org/10.1111/cdev.12363).

Significance of Students' Mathematics Identity

Research on students' mathematics identity has shown that students' self-perceptions in regards to mathematics strongly correlates to their mathematics achievement (Bouchey & Harter, 2005; Sonnert, Barnett, & Sadler, 2020). Students' mathematics identity and self-perceptions are also related to their persistence in mathematics (Cribbs et al., 2020) and affects the number of mathematics courses they choose to take (Simpkins et al., 2006). Supporting the construction of strong mathematics identities in students contributes to their persistence in mathematics and influences their career choices.

Education initiatives have addressed this need for positive self-perceptions in mathematics in students in reports and standards related to teaching mathematics. The report *Adding it all up* (Kilpatrick et al., 2001), lists students' positive disposition towards mathematics as one of the five strands of mathematical proficiency. The National Council of Teachers of Mathematics (NCTM) encourages teachers to create lesson plans and problems that promote curiosity, confidence, persistence, and flexibility in mathematics in their students' (NCTM, 2014).

Developing problems that encourages curiosity, confidence, flexibility persistence in mathematics requires a consideration of the student's ability to solve the problems presented. Schemes provide a framework to explore students' experiences with mathematics and how these experiences influence their development of their mathematics identity. The following section hypothesizes how the multiplicative concepts influence students' mathematics identity.

Conceptual Framework for Mathematics Identity and Multiplicative Concepts

Action schemes serve as a pattern of action through which learning can occur through the perturbation and accommodation of these schemes (von Glasersfeld, 1995). In this framework, an equilibrium is established when there are no perturbations found between the expected or beneficial result of the scheme and the actual result of a person's activity. Should students never accommodate their scheme or create a new scheme, they would be left in a state of imbalance. This can cause continual frustration for the student in handling situations that are not supported by their constructed schemes.

In this way, the limitations in students' thinking in mathematics problem solving that stem from an MC2 can create continued frustration and negative mathematics experiences in MC2 students. Indeed, Byerley's (2019) study showed evidence that the limitations that arise from the lack of an MC3 when attempting to construct a conceptual understanding of rate of change led to frustration in problem solving for undergraduate MC2 students. These negative emotions and mastery experiences may lead to negative views of their mathematics competence and performance in turn influencing their mathematics identity.

It is useful to explore students' mathematics identity within the context of their multiplicative concept schemes to determine if these limitations of an MC2 are having a negative influence on their mathematics identity. Should this hold true, it becomes more important that accommodations be made to help support MC2 students with their conceptual understanding of mathematics concepts and to help support them in constructing an MC3.

Conclusion

The multiplicative concepts serve as a theoretical lens to students' problem-solving abilities. They are foundational to students' mathematical knowledge and influence their fractional knowledge, proportional reasoning, algebraic reasoning, covariational reasoning, and geometric reasoning. Differences in the sophistications of schemes can lead to limitations and frustrations in problem solving. Understanding these limitations is important to accurately understanding and accommodating these students to support them in their mathematical pursuits. Additionally, research into the connection between schemes and mathematics identity may shed light to additional issues that the lack of support for MC2 students can have on their mathematics experiences.

CHAPTER III

VALIDATION OF ASSESSMENT FOR UNDERGRADUATE STUDENTS' MULTIPLICATIVE CONCEPTS

Target Journal: Investigations in Mathematics Learning

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Abstract:

Research on the multiplicative concepts of undergraduate students currently lacks a validated assessment for this population. This article examines strands of validity for the Undergraduate Multiplicative Concept Assessment (UMCA) and accompanying rubric. Validation evidence is collected through a combination of theoretical analysis, Rasch analysis, and qualitative data analysis on student written and interview responses. Data was collected from 51 undergraduate student assessments and 18 clinical interviews. Evidence towards the validity and reliability of the UMCA are shown in the results. Findings suggest that the UMCA is an appropriate assessment for assessing undergraduate students' multiplicative concept stage.

Introduction

Students’ ability to construct and reason with multi-leveled unit structures serve as a foundation to their mathematical reasoning on concepts such as fractions (Hackenberg, 2007; Steffe, 2001), proportions (Steffe et al., 2014), algebraic symbols and equations (Hackenberg, 2013, Hackenberg & Lee, 2015; Olive & Caglayan, 2008; Zwanch, 2019, 2022), recognition of quantity and use of measurement (Steffe, 2013), and derivatives and rates of change (Byerly, 2019). Students who have less flexibility in their use of unit structures can struggle with understanding the mathematical concepts that inform mathematical procedures such as derivatives (Byerly, 2019). Entry-level college mathematics courses place much of their focus on quantitative reasoning or the ability to apply mathematics and critical thinking to interpret data and draw conclusions to solve problems (Elrod, 2014; Lusardi & Wallace, 2013; Wolfe, 1993). It is beneficial for research to explore ways students’ reasoning about unit structures can create limitations in their mathematical and quantitative reasoning to inform curriculum that helps students overcome these limitations to achieve conceptual understanding.

While research on students’ multiplicative concepts has recently begun exploring undergraduate student reasoning (Boyce et al., 2021; Byerly, 2019), current written assessments are only validated for second and third graders (Kosko & Singh, 2018), fourth and fifth graders (Kosko, 2019), and sixth graders (Norton et al., 2015). The questions on the assessments are appropriate for the grades they were validated for, but can be simplified into easy arithmetic procedures. Students that have reached calculus can overcome limitations in their conceptual understanding by memorizing procedures (Byerly, 2019). The simplicity of existing assessments may allow undergraduate students to answer these questions with minimal work or evidence to support their thinking. This creates a reliance on interview data to accurately assess students’ ability to work with unit structures, which is time strenuous for large samples of undergraduate students.

The current study reports the creation and validation of an assessment for undergraduate students. Research on students' ability to construct and reason with unit structures has separated this ability into three stages known as the multiplicative concepts (Hackenberg & Tillema, 2009; Steffe, 1992, 1994). The assessment, titled the Undergraduate Multiplicative Concepts Assessment (UMCA), is designed to collect qualitative data to assist in delineating between the three stages of multiplicative concepts. A rubric was created alongside this assessment as a tool for interpreting the qualitative evidence collected by the UMCA. The purpose of this article is to establish validity evidence for the UMCA for undergraduate students.

Theoretical Framework

Scheme theory provides the theoretical lens for students' multiplicative concepts that the current study uses. von Glasersfeld's (1995) scheme theory defines an action scheme as a model for an individual's learned actions and thoughts that serve as a pattern of interaction that comes into play when an individual is introduced to a circumstance or situation. This interaction involves three parts: the individual's recognition of a situation, the individual's mental activity that is tied to the situation, and the individual's expected outcome based on previous experience with the activity and situation. Each multiplicative concept stage describes the degree to which an individual anticipates, constructs, and coordinates multiple levels of units and the operations they have available to use during problem solving (Hackenberg & Tillema, 2009; Ulrich, 2016).

The Multiplicative Concepts

Multiplicative reasoning requires students to conceive of and construct multi-leveled unit structures (Ulrich, 2015, 2016). The multiplicative concepts consist of three levels of schemes defined by the students' ability to anticipate, construct, and operate on multi-leveled unit structures (Steffe, 1992; Hackenberg & Tillema, 2009; Ulrich, 2015).

Students who have developed the first multiplicative concept (MC1) assimilate with one level of units and can construct two levels of units in activity (during problem solving; Hackenberg & Tillema, 2009). These students can multiply two numbers together by inserting one unit into the other. In this way, they can find the number of inches in 3 feet by inserting 12 inches into every foot and then skip counting to find the total (36 inches). These 36 inches are also not considered to be interchangeable and are each assigned a number 1-36. This unit structure decays after problem solving leaving the student to reflect on the 36 inches, but without its connection to the original 3 feet (Hackenberg et al., 2021).

Students who have developed the second multiplicative concept (MC2) assimilate with two levels of units and can construct three levels in activity (Hackenberg & Tillema, 2009). If asked to find the total number of inches in 2 yards, and MC2 student may anticipate the unit structure of 2 yards containing 6 feet. They could then insert 12 inches into each of the 6 feet to get a total of 72 inches. MC2 students have constructed iterable units of 1 meaning the 72 inches are no longer labeled from 1-72, but are instead considered the same an interchangeable. The 32nd inch is the same as the 45th inch. MC2 students' constructed three-level units decay after activity, leaving two levels of units for the student to reflect on (Hackenberg et al., 2021). In this way, the feet in the previous problem might decay from the MC2's structure leaving the student to reflect on 2 yards containing 72 units.

Students who have developed the third multiplicative concept (MC3) assimilate with three levels of units and can construct four or five in activity (Hackenberg & Tillema, 2009; Ulrich, 2016). MC3 students can also move flexibly between levels of their constructed unit structures (Ulrich, 2016). MC3 students can find the number of inches in two yards is 72 by anticipating the need to multiply 2 yards by 3 feet by 12 inches. If you asked to find the total number of inches if you added another 5 feet to the problem, and MC3 student could iterate the composite unit (two-level unit structure) of 12 inches in 1 foot five times to get a total of 132 inches (Hackenberg et al., 2021). As shown in the example, MC3 student has constructed iterable composite units rather than just iterable units of 1 (Ulrich, 2016). These students retain their three level unit structures after problem solving, allowing them to reflect on the three-level units used in their problem solving.

Table 3.1

Construct map for the multiplicative concepts

Stage	Description	Example
First Multiplicative Concept (MC1)	Student anticipate one level of units and can construct two levels of units in activity.	Can find the number of inches in 6 feet, 6 ft. x 12 in., by skip counting or using repeated addition.
Second Multiplicative Concept (MC2)	Student anticipate two levels of units and can construct three levels of units in activity.	Can find the number of inches in 2 yards by multiplying 3 ft. x 2 yd. = 6 ft. and then multiplying 6 ft. by 12 in.
Third Multiplicative Concept (MC3)	Student anticipate three levels of units and can construct four or more levels of units in activity.	Can find the number of inches in 5 more feet (on top of the 2 yds.) by adding the 5 ft. to the 6 ft. without reestablishing the 6 ft. and then multiplying the sum by 12 in.

Research on undergraduate students' readiness for calculus and their multiplicative concept stage indicated that over half of the students were MC2 students with the rest being MC3 students (Boyce et al., 2021). None of the undergraduate students in this study were MC1 students. This suggests a low representative of MC1 students in the undergraduate student population.

Iterative Fraction Scheme

Research on the multiplicative concepts has used the multiplicative concepts to inform their development of the fraction schemes (Steffe, 2001). Students' fractions schemes are co-constructed with their multiplicative concepts, allowing students to use either to inform and accommodate the other (Boyce & Norton, 2016). Some evidence found from students reasoning about fractions can inform researchers on the multiplicative concept stage of a student.

Evidence that a student has developed an iterative fraction scheme (IFS) serves as evidence a student has developed an MC3 (Norton & Wilkins, 2012; Steffe, 2001). For a student to consider improper fractions as numbers in their own right, students must be able to simultaneously partition and iterate unit fractions from a "whole" and assimilate with three levels of units (Hackenberg, 2007; Steffe 2001). In other words, in order for students to reason about improper fractions as a unit structure containing iterated fraction units, the student needs to have an MC3. This allows the student to consider the fraction 7/3 as a unit containing 7 iterated 1/3 unit fractions (not units of 1s labeled 1/3) that can be reconstructed into 2 wholes and a 1/3 unit fraction, where both wholes contain 3 iterated 1/3 unit fractions.

This study addresses the validity of the UMCA as an assessment for undergraduate students' multiplicative concepts stage. The literature surrounding the multiplicative concepts create the foundation for the development and theoretical analysis of the items on the UMCA.

Methodology

Assessment and Scoring Rubric Development

The questions on the UMCA were chosen to collect written evidence of undergraduate students' multiplicative concept stage through their problem-solving methods. The questions on the UMCA were designed to collect written evidence of students' anticipated and constructed units and operations that align with prior research to determine a student's multiplicative concept stage. Researchers chose a total of 5 problems to be on the UMCA:

1. A candy bar company packs 3 candy bars per package, and 6 packages per box.
    a. If a store buys 7 boxes, how many candy bars will they receive?
    b. If the same store orders another 8 boxes, how many total candy bars have they received?
    c. Assuming the store received all of their ordered candy bars, how many packages have they received?
2. There are 6 plants in each row of my garden.
    a. How many tomato plants are in 8 rows?
    b. In addition to tomato plants, I also planted potatoes. If there are a total of 102 plants, how many rows of potatoes did I plant?
3. There are 12 inches in 1 foot and 3 feet in 1 yard.
    a. How many inches are in 2 yards?
    b. If you add an additional 5 feet onto the original yards, how many total inches are there?
    c. How many feet are in the total number of inches?
    d. How many yards are there in the total number of inches?
4. The stick shown below is 3/5 of a whole stick. How many 1/15 sticks can you make from the 3/5 stick?
5. The bar shown below is 7/3 as long as a whole candy bar. Draw the whole candy bar.

Problems 1, 2 and 3 on the assessment include multiple parts that challenge the student to construct and move fluidly between layers of units. This structure brings the total number of questions on the assessment to 11, and are the 11 items discussed throughout the analysis and results portion of this study.

Problems 1, 2, and 3 all begin with an entry-level problem that asks the student to construct a composite unit. Question 2-a is the simplest, asking the student to simply multiply the number of plants by the number of rows to create the two-level unit structure, 8 rows containing 48 plants. Questions 1-a and 3-a ask the students to construct a three-level unit structure: 7 boxes containing 42 packages containing 126 candy bars and 2 yards containing 6 feet containing 144 inches. MC2 and MC3 students' operations support their ability to construct these structures easily, and MC1 students should be able to solve these problems with the help of figurative materials. The remaining questions in problems 1, 2, and 3 ask the student to operate on their constructed composite units and move between the layers of their constructed unit structures. Question 1-b and 3-b both ask the student to add an additional number of one unit to find the total of another unit (i.e., if add boxes, how many candy bars or if you add feet, how many inches). These questions are similar to the problem explored in Hackenberg, Aydeniz, and Jones' (2021) discussion of the multiplicative concepts that was covered in the theoretical framework. Evidence supporting the differences in MC2 and MC3 student reasoning on this problem is why they are included in this assessment.

Questions 1-c and 3-c both ask the student to find the middle layer of their unit structures (i.e., packages and feet respectively). While both MC2 and MC3 students can construct three-level unit structures, MC3 students reason more flexibly about each of the three levels. MC2 students whose unit structure has decayed may treat this as a completely different problem from

1-a and 1-b and will start the problem over rather than using the numbers they had already calculated (Hackenberg & Tillema, 2009). They may also divide by the wrong unit, having lost the meaning behind its relationship to the other units in the problem (e.g., a student divides the total number of candy bars by the 6 packages per box instead of the 3 candy bars per package). Question 3-d is similar to questions 1-c and 3-c. However, the choice to add 5 feet instead of 5 yards in problem 3 was intentional to create a remainder on question 3-d. MC2 students struggle to accurately represent remainders as a part of a whole. They may ignore the remainder or round their solution to the nearest whole number.

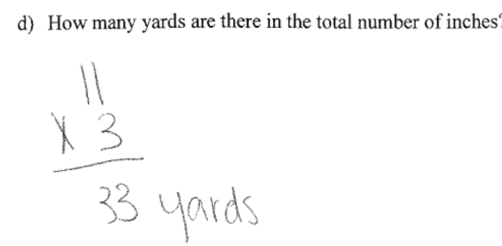
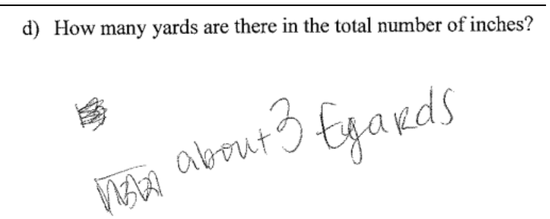
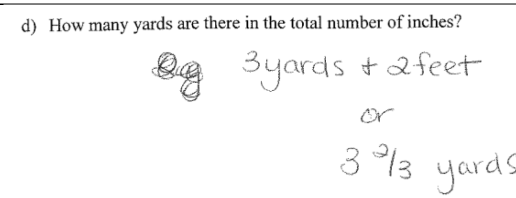
Question 2-b asks the student to use the information from 2-a to split and operate on the whole garden. Students are given the total number of plants in the garden and must use the information they have to find the total number of rows of potatoes. The student is being asked to conceive the garden as a unit structure containing 2 types of plants, one of which contains 6 rows and 48 tomatoes and the other that contains an unknown number of rows and potatoes, all of which total to 102 plants. Students are being asked to operate on the middle layer of their constructed unit to find the number of rows. The researchers propose that due to the complex nature of the unit structure in this question, many MC2 students can struggle to reach the desired answer without a picture.

Problems 4 and 5 were adapted from example problems on Wilkins, Norton, and Boyce's (2013) written assessment for students' fractional schemes. Problem 4 assesses whether or not the student has constructed the third multiplicative concept in regard to fractions. This question collects evidence towards the students' ability to reason with the given bar as a unit structure containing 3 1/5ths, with each 1/5th containing 3 1/15ths. MC3 students can understand the bar as 3 iterated 1/5ths each containing 3 iterated 1/15ths. MC2 students may solve this problem by extending the bar to reconstruct the original "whole" and then partitioning into 1/15ths. Problem 5 specifically targets the iterative fraction scheme, which can only be constructed by MC3 students (Hackenberg, 2007). Evidence that the student partitions the bar into 7 1/3rds and then uses three of them as a reference to construct the original bar is evidence towards the student having the iterative fraction scheme and thus the third multiplicative concept. This question was selected as a way to collect additional evidence for a student having a MC3. If a student cannot answer this question correctly, that does not provide evidence that they have not constructed a MC3. It only indicates they have not constructed an iterative fraction scheme.

Prior to data collection, researchers met to create a scoring rubric for the UMCA based on predicted solutions for MC1, MC2, and MC3 students respectively (see appendix A). This rubric provides an outline of specific written evidence that suggests a student has constructed an MC1, MC2, or MC3. The evidence for each question is divided into the appropriate multiplicative concept stage. An example for this system is provided in Table 3.2 alongside examples for each of the multiplicative concept stages of student work. The student examples provided in Table 3.2 only include solutions that provided evidence for the multiplicative concepts stage to which they were assigned. Questions 1-a and 2-a are only separated into MC1 and MC2/3 as the researchers believed they did not provide enough cognitive demand to properly delineate between MC2 and MC3 student reasoning. Additionally, problem 5 is separated into two categories: iterative fraction scheme and non-iterative fraction scheme. The researchers felt this was appropriate as there was no clear way to delineate MC1, MC2, and MC3 reasoning from one another if the student had not developed the iterative fraction scheme.

Table 3.2

An excerpt from the rubric for Question 3-d with example solutions

Multiplicative Concept Stage	Student Reasoning	Written Indicators of Reasoning	Example Solutions
MC1	Students attempt to rely on their created figurative material to determine how many yards are in their counted total of inches.	• Student responses show heavy reliance on pictorial or tally mark representations of the problem of the units of 1 (inches). Composite units (i.e. feet and/or yards) may be drawn but not counted. • Student attempts to count their figurative material, but creates groups that are not representative of the inches to feet to yard relationship. • **Student adds or multiplies incorrect combinations of numbers.** • Student does not refer back to their answer as a part of 3-a, 3-b, or 3-c. • **Student does not make an attempt to account for the remainder of the yard.** • Student does not respond or otherwise indicate they do not know the answer.	
MC2	Students use the given relationships and the total number of inches found to determine the total number of yards.	• Student works indicates a reliance on repeated addition or skip counting or repeated subtraction to solve the problem. • Student divides by the wrong unit (i.e. they divide by 3 feet or some other related number that is not inches). • Student finds an incorrect remainder. • **Student describes the remainder as "3 and a little bit", "about 3", or gives a range such as "between 3 and 4".** • **Student does not interpret the remainder as a part of a yard.**	
MC3	Students assimilate with all two-level unit structures given and readily coordinate all of these units to determine the total number of yards.	• Student finds the total number of yards through division and addition (5/3 = 1 2/3, 1 2/3 + 2 = 3 1/3). • Student finds the total number of feet using division. • Student references the total number of yards in parts 3-a and 3-b to determine how many total yards there are. • Student drawings are used to justify their answers rather than produce them as a part of Part B. • Student explanations discuss the unit structures as a three-level unit structure. • Student presents multiple ways to solve the problem. • **Student interprets the remainder as a part of the yard.**	

After all 11 items on the rubric were scored, the student was attributed an overall multiplicative concept stage based on these three criteria.

- If a student has more questions scored at a MC1 level than a MC2, that student is assigned a MC1. In the case of a tie (5 MC1 scores and 5 MC2 scores), the student is assigned a MC2.
- If the student has less than 3 questions scored at a MC3 level, that student is assigned a MC2. The categories designated as MC2/3 on questions 1-a and 2-a do not count towards the 3 MC3 item scores. MC2/3 scores on these questions indicate that the student has shown evidence of at least an MC2, but not enough evidence that the student could have constructed an MC3.
- If the student has at least 3 questions scored at a MC3 level or answered question 5 at the level of an IFS, that student is attributed an MC3. If a student did not answer the question at the level of an IFS, they can still be attributed an MC3 if they scored at least 3 questions at a level of an MC3 on the UMCA.

The researchers determined the final scoring criteria based on the data collected from the current study. Researchers chose these cut offs as most indicative of their level of confidence in assigning a student a multiplicative concept stage.

Participants and Data Collection

Data for validation of the UCMA and UMCA rubric was collected as the first phase of a two-phase study exploring undergraduates' multiplicative concept stages by (1) validating an assessment for undergraduate students, (2) exploring their solutions to optimization problems, and (3) exploring how they discuss their mathematics identity. Data in this phase was collected from 51 undergraduate students from a mid-western university that were enrolled in education major courses ranging from freshman-level general education courses to senior level mathematics-focused education courses. Participants were between 19 and 22 years of age and classified themselves as sophomores (47%), juniors (31%), and seniors (22%) in their university program. Participants were given 30 minutes in class to complete the UMCA. The assessments were handed out alongside a black and a blue pen. Participants were instructed to complete Part A of the assessment, which involved solving the questions on the UMCA, with the black pen. They were then asked to swap to the blue pen upon reaching Part B of the assessment. Here students were presented with the same questions they solved in Part A and asked to explain how they solved the problem. Students were instructed that they could mark on Part A while working on Part B as long as they used the blue pen. This allowed the researchers to distinguish between students' original work and work they had added or changed after the initial problem solving. Students were instructed not to use a calculator for this assessment.

The participants' assessments were deidentified prior to scoring. To finalize the UMCA rubric, four assessments were chosen randomly for initial scoring. The researchers met to discuss their experience using the rubric to score the assessments and changes were made to clarify and improve the rubric. After the changes were finalized, the researchers scored all of the assessments independently. The set of assessment scores from each researcher were then compared and discussed and a final multiplicative concept stage was attributed to each participant.

In addition to the final attribution of a multiplicative concept stage, the assessments were given an overall score based on the accuracy of student solutions. Each participant was given a score of 0 or 1 (i.e., incorrect or correct) for each problem on the assessment and a sum of these scores were provided as an overall score. Scores for the problems were considered correct if they came to the correct answer regardless of their solution process except for question 5, where students had to provide their bar as 2 1/3, not as an estimate of 1 out of 2 and 1/3. This decision was made because question 5 collects evidence of the iterative fraction scheme, and should be considered correct within the context of UMCA only when it reflects the reasoning expected from a student with iterative fraction scheme.

From the initial sample of participants that took the UMCA, 18 volunteered to participate in follow-up clinical interviews. These interviews lasted between 5 to 15 minutes. Participants were asked to solve the following problems:

1) I purchased packages of candy bars that come in 8 per package.
    a. If I bought 7 packages of Mr. Goodbar candy bars, how many candy bars do I have? (A: 56 Mr. Goodbar candy bars)
    b. I also bought some Almond Joy candy bars and now have a total of 104 candy bars, how many packages of Almond Joy candy bars did I purchase? (A: 6 packages)
2) There are 8 fluid ounces in a cup and there are 4 cups in a quart. If I am measuring water out,
    a. How many fluid ounces of water are in 3 quarts? (A: 96 fluid ounces)
    b. If I add an additional 7 cups of water to the original 3 quarts, How many ounces of water do I have now? (A: 152 fluid ounces)
    c. How many total cups of water do I have now? (A: 19 cups of water)
    d. How many total quarts of water do I have now? (A: 4 ¾ quarts of water)

The interview problems are similar to questions 2 and 3 on the UMCA to allow researchers to compare the multiplicative concept stages assigned from the written and video evidence directly. The participants' UMCA and video transcripts were given separate identifiers prior to scoring to ensure that the researcher's attributed multiplicative concept stage on the written UMCA did not interfere with their attributed multiplicative concept stage from the video evidence. The researchers then met to reach a consensus on the attributed multiplicative stage for each participant based on their work during the clinical interviews. The identification codes for the written assessment and interviews were not consolidated until the completion of all multiplicative concept stages across both sets of data to reduce researcher bias.

Data Analysis

To ensure the scores of the assessment accurately reflect the developers' intended interpretations of responses, multiple sources of validity evidence are provided, as emphasized in *The Standards for Educational and Psychological Testing* (AERA et al., 2014) and recommended by the field (Krupa et al., 2020). This study provides evidence towards the validity of the UMCA's test content, response processes, internal structure, and generalization. Table 3.3 outlines a brief overview of the validity evidence discussed in this article.

Rasch analysis was used to provide evidence towards the validity and reliability of the UMCA. Boone (2016) describes Rasch analysis as a psychometric technique that allows researchers to provide robust validations of their instrument starting at the item selection phase. The Rasch model predicates on the assumption that students exist on a spectrum of ability and thus the difference of magnitude between incorrect and correct responses to a question should reflect the difficulty of the question. The percentage chance for a student to successfully answer a question is then converted to a measurement called a logit (Lamprianou, 2020). These measures can then be analyzed for their fit statistics which provide supporting evidence to the appropriateness of the Rasch model. The spectrum of ability that is measured is called a construct or trait a person has that cannot be observed physically but are socially and/or politically defined (Lamprianou, 2020). The evidence provided by Rasch analysis for the validity of the UMCA will be discussed in the following paragraphs in regards to the AERA (2014) strands of validity they support. Supplementary evidence from qualitative data will also be discussed.

Table 3.3

Summary of validity evidence for UMCA by validity strand

Form of Validity Evidence	Purpose of Evidence	Primary Evidence
Test Content	Construct Overview	Construct Map
	Item Development	Item Overview
	Item Qualitative Data Overview	UMCA Rubric
	Item Technical Quality	Point-Biserial Coefficients
Response Processes	Behavioral Observations	Analysis of Written Work on UMCA
		Analysis of Clinical Interviews
	Person Fit Statistics	Person Mean Square Fit Statistics
	Item Difficulty Hierarchy	Wright Map
Internal Structure	Unidimensionality	Infit and Outfit Statistics
		PCA of standardized Residuals
	Interpretability	Wright Map
Generalization	Reliability	Item Separation and Reliability
		Person Separation and Reliability
		Cronbach's Alpha Coefficient

Test content validity refers to evidence that the wording of the items and tasks support the relationship between the content of the assessment and the construct it is measuring. For the UMCA, the researchers ask "do the questions on this assessment provide a space to collect data on the students' anticipation, construction, and reflection of units unit structures to accurately determine their multiplicative concept stage?" The construct map (Table 3.1) provides an overview of the multiplicative concepts as a construct and the item overview (Table 3.2) provides evidence to support the choice of questions and construction of the problems on the UMCA. Additionally, the UMCA rubric (see appendix B) provides an outline of the evidence collected from each question that aligns with student work found in prior research on the multiplicative concepts. This provides evidence toward the theoretical design of the UMCA items to collect evidence of students' multiplicative concept stage. Qualitative evidence towards test content validity has already been discussed in depth as a part the theoretical framework and methodology portion of this article. Evidence towards the technical quality of the items will be given using the items point-biserial coefficients with the participant's overall scores (i.e., how many items on the UMCA they answered correctly).

Response process validity refers to evidence supporting whether the interpretations of the results of the assessment accurately represent the fit between the construct and the participants' responses. For this study the researchers ask "do the scores of the UMCA reflect the students' ability to work with unit structures as shown in their written and verbal reasoning?" To provide evidence towards the response processes, analysis of both the written work on the UMCA and the work provided from clinical interviews will be provided. Person fit statistics for the logits will explore the appropriateness of the Rasch conversion for each participant. Agreement scores between the written and verbal evidence are also provided as evidence of the validity of the UMCA to attribute accurate multiplicative concept stages of participants. Evidence relating to item hierarchy will be discussed using a wright map. The wright map plots the response data for each item as logits on the x-axis in relation to the likelihood for a participant to correctly answer the question on the y-axis (Boone, 2016). As such, the wright map provides a visual for the hierarchy of problem difficulty for questions on the UMCA.

Internal structure validity refers to how well the test items conform to the construct being measured. This study asks, “how well do the responses and overall scores reflect the multiplicative concepts as a measurable construct?” The item hierarchy data from the wright map provides evidence towards the interpretability of the assessment items. The unidimensionality is explored through statistics provided by the infit and outfit statistics of items and a principle component analysis (PCA) of standardized residuals.

Generalization validity refers to how well different aspects of the test extend or generalize to new situations. Researchers in this study ask, “are the responses for the UMCA generalizable to undergraduate students?” Reliability measures from the Rasch model will be used to provide evidence of generalization. Person and item reliability index and separation statistics will be provided. Person reliability scores estimates the reliability of the assessment scores if a similar assessment were given to the same sample and item reliability scores indicate the reliability of the item difficulty estimates if a similar sample was given the same assessment (Lamprianou, 2020). Cronbach’s alpha coefficient will also be reported to provide a measure for how closely related the items on the assessment are as a group. Additional evidence towards generalization can be pulled from the analysis of clinical interviews. These interviews provide a direct example of a parallel assessment for the same sample of participants that assesses their multiplicative concepts.

These sources of validity and reliability were used to determine the overall appropriateness of the assessment for determining undergraduate students’ multiplicative concepts stage.

Results

Qualitative Results

Analysis of the written work by students as analyzed through the use of the UMCA rubric. After finalizations of the rubric were made, both researchers scored 51 assessments individually, using the rubric. Upon discussion, researchers agreed that every student solution across all items for the 51 assessments were represented on the rubric and appropriately reflected the multiplicative concept stage assigned to the written work. Researchers gave 49 out of the 51 students the same multiplicative concept stage prior to discussion (an agreement score of 96.1%). Upon discussion, final multiplicative concept stages were decided, making the consensus on the written work scores unanimous. The final assessments indicated that 1 participant was a MC1 student (2.0%), 22 were MC2 students (43.1%), and 28 were MC3 students (54.9%). This is consistent with results in prior undergraduate research on the multiplicative concepts (Boyce et al., 2021).

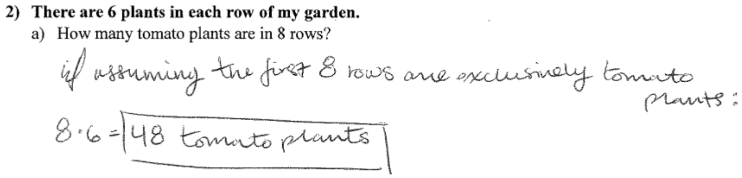
Clinical interviews were conducted with 18 volunteer participants. The researchers collaboratively assigned each participant with a multiplicative concept stage based on the students' reasoning in the interview. Out of the 18 interviews, only 1 was assigned a different multiplicative concept stage from their UMCA assessment (an agreement score of 94.4%). Additionally, student's written work on interview problems were visually similar to their written work on the UMCA. Figure 3.1 shows examples of student work during their interview and on the UMCA for comparison. The evidence from the written and interview data supports the validity of the student work on the UMCA as indicative of their multiplicative concept stage.

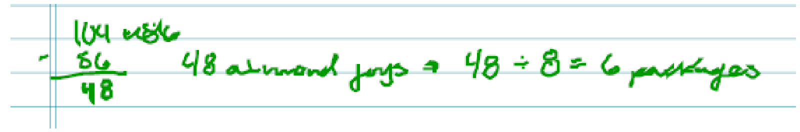
Figure 3.1

Comparison of written work during clinical interviews and on the UMCA

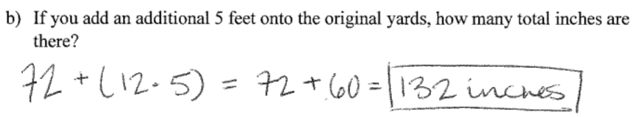
Student	UMCA	Clinical Interview

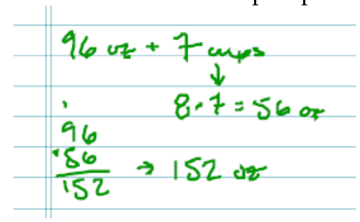
Felicity (MC3)





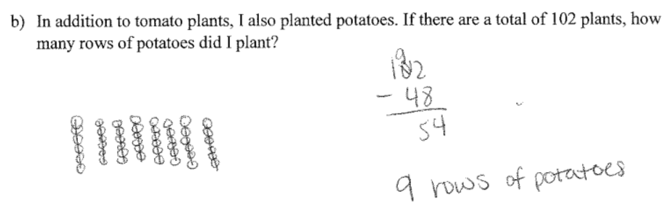
Felicity used the same method to solve these two problems, even keeping a similar format for her notation that clearly notates the steps she took in her thinking to get her answers. She recognized the need to divide the difference in both problems without outside help or pictoral reference.

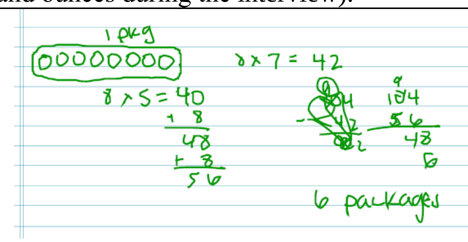




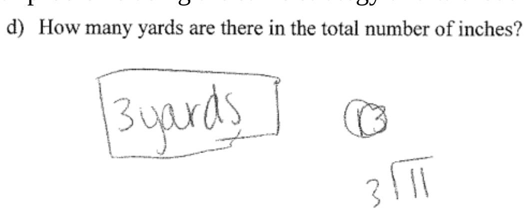
Felicity solves these two problems using the same exact steps. She anticipates the need for a conversion on each problem prior to activity (shown by her use of 12 x 5 to replace "5 feet" on the UMCA and her use of labels for her cups and ounces during the interview).

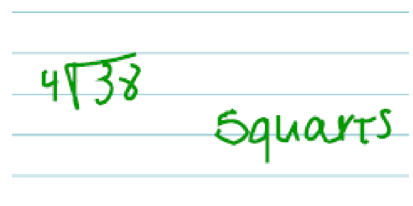
Morgan (MC2)





Morgan uses a visual representation to help her understand the structure of the garden and the package before solving the problem. She is also able to solve both problems using the same strategy and without outside help.





Morgan estimates on both of these problems. Her justification for these answers is that there are "3 full yards" or "5 full quarts". Morgan chooses not to work with the remainder on either problem.

Quantitative Results

The point-biserial coefficients for each item on the UMCA provide additional evidence towards the validity of the test content not covered in the theoretical framework and methodology (See Table 3.4). The point-biserial coefficient for 7 of the items on the UMCA were strong ($r > 0.50$) with items 3-a and 5 having a moderate correlation ($r > 0.30$; Field et al., 2012). These items all provided good discrimination between the students who scores higher and lower on the assessment. For example, students who answer 3-d correctly were highly likely to be attributed an MC3. Questions 1-a and 2-a had low correlations with their overall score, and do not provide good discrimination between lower and higher scoring students.

Table 3.4

Item analysis statistics

Item	Point-Biserial	SE	Infit		Outfit	
			Mean Square	Z	Mean Square	Z
1-a	0.15	0.60	1.51	1.17	2.23	1.15
1-b	0.51	0.40	1.06	0.32	0.76	-0.40
1-c	0.60	0.36	0.96	-0.17	0.90	-0.13
2-a	0.24	1.01	0.36	-1.06	0.04	0.37
2-b	0.63	0.37	0.84	-0.91	0.92	-0.08
3-a	0.34	0.68	1.02	0.20	0.53	0.16
3-b	0.63	0.40	0.77	-1.08	0.48	-1.21
3-c	0.68	0.38	0.77	-1.29	0.53	-1.30
3-d	0.73	0.36	0.76	-1.55	0.68	-0.74
4	0.62	0.36	0.98	-0.09	0.80	-0.28
5	0.39	0.43	1.05	0.32	1.13	0.47
Mean	0.50	0.49	0.92	-0.38	0.82	-0.18

A summary of the statistics for each item on the UMCA are located in the Table 3.4 and provide an overview of the overall fit of each item on the UMCA. These fit statistics provide evidence towards the unidimensionality of the assessment. This study will consider the infit square means as a measure of fit as outfit statistics are generally inflated compared to infit statistics (Lamprianou, 2020). Items that have an infit mean square between 0.7 and 1.3 are considered acceptable (Lamprianou, 2020). Out of the 11 items on the assessment, 9 fit within the acceptable range with only questions 1-a and 2-a infit mean squares falling outside of this range. The implications of the problematic infit and point-biserial coefficients for these problems will be discussed later alongside our rationale on their continued inclusion in the UMCA. These data support the items fit in this assessment as measures for the students' overall scores.

Item difficulty hierarchies allow assessments differentiate between varying levels of person ability from their scores. Before analysis of the item hierarchy for the UMCA, the researchers estimated the difficulty of each item on the assessment based on prior research into the multiplicative concepts. Researchers proposed four tiers of difficulty in the exam based on existing literature and the theoretical analysis of the problems. The items in Table 3.5 are ordered according to researcher's theoretical item difficulties. The first difficulty tier are entry level problems and consist of questions 1-a, 2-a, and 3-a. Researchers anticipated that students could solve these problems by operating in ways consistent with the operations of an MC1 student. Therefore, they would likely be solved correctly by most if not all of the undergraduate students in this study. The second tier of difficulty are the connection questions. Researchers anticipated that these students would challenge students operating within the constraints of an MC1 and may challenge students operating within the constraints of an MC2. Questions in this tier include 1-b, 1-c, 2-b, 3-b, and 3-c. The third difficulty tier includes questions that were anticipated to be challenging for students operating within the constraints of an MC2, and may challenge students operating within the constraints of an MC3. The two questions in this tier are 3-d and 4. The fourth difficulty tier only includes question 5. Researchers anticipated that students who operated with an MC3 but had not developed an IFS would be challenged by this problem. Calculating the logit scores for each item shows that the researchers' theoretical item difficulty hierarchy is similar to the actual item hierarchy. Additionally, the questions proposed to be in each difficulty tier group together on the wright map (see Figure 3.2). This evidence supports that responses collected from the UMCA accurately represents the difficulty desired by the developers.

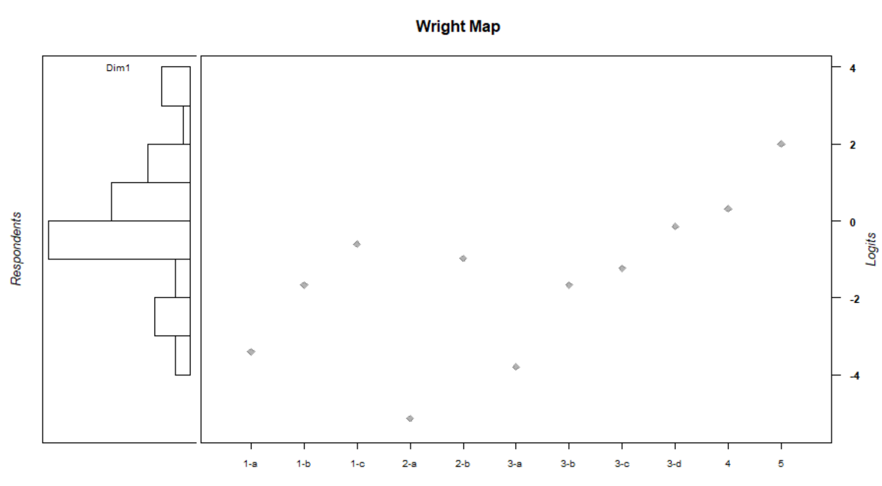
Table 3.5

Item difficulty ordered by hypothesized difficulty

Item	Hypothesized Difficulty	Item Difficulty
2-a	1	-5.15
1-a	1	-3.41
3-a	1	-3.80
1-b	2	-1.67
3-b	2	-1.67
3-c	2	-1.24
1-c	2	-0.61
2-b	2	-0.98
3-d	3	-0.14
4	3	0.31
5	4	2.00

Figure 3.2

Wright map for UMCA items with histogram of participant logits



To provide evidence for how well these difficulty tiers reflect the intensions of the researchers, Table 3.6 provides an overview of the number of students who answered each item correctly from each of the multiplicative concept stages. The MC1 student failed to answer any question that was above the first tier correctly. The success rate for solving problems for MC2 students dropped substantially on the tier 3 problems, 3-d and 4, as was hypothesized. Additionally, only 36% of MC3 students answered question 5 correctly, which aligns with the literature on the iterative fraction scheme (Norton & Wilkins, 2012; Steffe, 2001). These statistics support the interpretability of the questions on the UMCA.

Table 3.6

Number of correct responses per item by multiplicative concept stage with mean UMCA score

	MC1 (n = 1)	MC2 (n = 22)	MC3 (n = 28)	Overall (n = 51)
1-a	1 (1.0)	21 (0.95)	25 (0.89)	47 (0.92)
1-b	0 (0)	17 (0.77)	22 (0.79)	39 (0.76)
1-c	0 (0)	10 (0.45)	21 (0.85)	31 (0.61)
2-a	1 (1.0)	21 (0.95)	28 (1.0)	50 (0.98)
2-b	0 (0)	11 (0.50)	23 (0.82)	34 (0.67)
3-a	0 (0)	21 (0.95)	27 (0.96)	48 (0.94)
3-b	0 (0)	12 (0.55)	27 (0.96)	39 (0.76)
3-c	0 (0)	10 (0.45)	26 (0.93)	36 (0.71)
3-d	0 (0)	4 (0.18)	23 (0.82)	27 (0.53)
4	0 (0)	3 (0.14)	20 (0.71)	23 (0.45)
5	0 (0)	0 (0)	10 (0.36)	10 (0.20)
Mean UMCA score	2.0	5.9	9.0	7.5

Unidimensionality was also examined using results from a principal component analysis (PCA) of standardized residuals based on the items of the UMCA (Boone & Staver, 2020). The PCA indicated that 52.2% of the variance was explained by the Rasch model and 26.0% was explained by the assessment items. The first contrast for the items explain 11.1% of this variance and has an eigenvalue of 2.55. This is only half of the total variance explained by the items, which suggest that there is at least a second contrast. This constitutes a closer look at the contrasts and how the items are loading in the PCA. The factor loadings are show in Table 3.7 reveal an interesting loading pattern between the contrasts. Rather than loading distinctly between questions, the contrasts consist of similar or opposite levels of item difficulty. Taking this into consideration alongside the evidence that many of the questions load within multiple contrasts with different items from the UMCA, the results of the PCA support the idea that each item is collecting data about the same construct along a spectrum of difficulty.

Table 3.7

PCA of the residuals for the Rasch model of the UCMA

Item	Contrast 1	Contrast 2	Contrast 3
1-a	**-0.56**	**0.53**	-0.35
1-b	**-0.67**	**0.60**	-0.14
1-c	**-0.47**	-0.06	0.36
2-a	-0.07	0.08	**-0.49**
2-b	-0.32	-0.34	0.34
3-a	-0.12	-0.14	**0.70**
3-b	**0.74**	**0.51**	0.01
3-c	**0.80**	0.34	-0.08
3-d	**0.44**	0.17	**0.47**
4	0.07	**-0.58**	**-0.52**
5	0.24	**-0.59**	-0.31

Note: Loadings above 0.40 noted in bold.

The internal reliability of the UMCA is acceptable with a Cronbach alpha coefficient of .76. Debate surrounding the appropriate cut off coefficient of the Cronbach alpha has placed .7 as an appropriate standard for tests measuring ability (Field et al., 2014). Additionally, the Rasch model for the UMCA has a sufficient item reliability of 0.93 with a separation index of 3.62, indicating that the model can differentiate between low and high difficulty items (Boone & Staver, 2020). The person reliability of the Rasch model is 0.68 with a separation index of 1.45. This is a moderate index and indicates that the model may have trouble differentiating between groups of people ability (low, medium, and high ability). It should be cautioned that this is lower than the suggested person reliability of 0.8 and index of 2 (Boone & Staver, 2020). However, given the nature of the UMCA as an assessment for undergraduate students that's primary job is to differentiate MC2 from MC3 students, there is a lack of MC1 students to support the instrument's ability to differentiate them from medium and high scorers. Taking this into consideration, a person reliability index of 0.68 is sufficient for the UMCA's intended population of test takers. This data supports the generalizability of the UMCA to similar samples.

While this evidence supports the validity of the response processes, internal structure and generalization of the UMCA, adjustments made to the UMCA should consider the fit of the entry-level questions with overall assessments and possible numerical or question substitutions that could be made to increase validity.

Discussion

The multiplicative concepts serve as foundations for student reasoning and problems solving on mathematical concepts such as recognizing and working with unknowns and variables (Hackenberg & Lee, 2015; Hackenberg et al., 2021; Zwanch, 2019), solving systems of equations (Olive & Caglayan, 2008; Zwanch, 2022), and understanding derivatives (Byerly, 2019). With the recent extension of multiplicative concept research to the undergraduate population, it becomes important for researchers and professors to have access to an assessment appropriate for undergraduate students that can delineate between MC1, MC2, and MC3. Prior research has laid a rich theoretical framework for creating and validating an assessment for undergraduate students (Hackenberg & Tillema, 2009; Steffe, 1992, 1994). This study shows evidence that the UMCA questions and rubric aligns with literature outlining the multiplicative concepts and serves as a valid instrument for undergraduate students. The results of this study support the appropriateness of the responses through the analysis of creation and use of the

UMCA rubric. Students' written and interview responses support the validity of the rubric in collecting and analyzing student written work according to the multiplicative concept stages. The validity evidence for the generalizability of the UMCA is supported by the acceptable Cronbach alpha coefficient and the reliability scores for both the items and persons of the Rasch model.

The internal structure of the UMCA is supported by the infit and outfit statistics of the items, the item hierarchy shown in the wright map, and the results of the PCA of the residuals of the Rasch model. However, poor correlation and infit statistics for questions 1-a and 2-a merit a discussion of their performance on the UMCA. These two questions are considered entry-level questions for problems 1 and 2 and do not provide robust evidence for distinction between the second and third multiplicative concepts. The researchers hypothesized this prior to testing due to the simple composite unit constructions and low cognitive demand of the tasks and grouped evidence for MC2 and MC3 students together for these tasks. However, the population of students who took this exam were predominantly MC2 and MC3 students, with only 1 out of the 51 participants being assessed as a MC1 student. This mirrors similar findings on the number of MC1 students in pre-calculus undergraduate mathematics courses (Boyce et al., 2021). This may make it difficult for these questions to correlate well with the overall scores as all students should be able to answer questions 2-a and 3-a successfully. However, question 3-a is easier than question 1-a according to the item hierarchy and correlates moderately with the overall assessment scores. An inspection of responses for question 1-a show that out of the 4 students who answered questions 1-a incorrectly, 3 of them miscalculated 18 x 7 (e.g., 18 x 7 = 136 or 18 x 7 = 116). Miscalculating this number led to the student incorrectly answering 1-b. This may have led to a large impact on its correlation with the overall exam scores with this sample. However, the researchers chose to keep these items as they are on the UMCA in order to maintain the entry-level problems for questions 1 and 2 and to provide a space to collect additional evidence for assigning an undergraduate student a MC1.

The combination of qualitative and quantitative data provides evidence towards the validity argument for the UMCA. However, adjustments can and should be considered when moving forward with using the UMCA for research or in the classroom. The rest of our discussion will include limitations for the study, usage of the UMCA rubric, areas of improvement for the assessment, implications of this study, and future directions for research.

Limitations

This study has covered various strands of validity evidence as an argument towards the validity of the UMCA. However, it is important to note the limitations of this study moving forward with the use and adjustment of this assessment and rubric. First, the population used in this study is not representative of the diverse backgrounds of undergraduate students found across all collegiate institutes. Second, while the spread of students assigned as MC1, MC2, and MC3 aligns with theory and research on the multiplicative concept stages, care should be taken to the validation of this instrument to accurately assess MC1 students. Since only one participant was assigned a MC1 and they did not participate in a clinical interview, the researchers were unable to analyze video evidence to compare the assignment of the students' multiplicative concept stage to. Third, this assessment only collected evidence of validity to a certain extent and did not exhaust the spectrum of validity for assessment. Further validation of this assessment should consider comparisons with similar assessments, additional interview evidence and comparisons, additional analysis of results using a sample of students more likely to include MC1 students, and so on.

Suggested Areas for Assessment Improvement

Results of analysis of the validity for the UMCA supported it as a reflection of the developers' intentions. It provides a space to collect rich qualitative data on undergraduates' construction of multi-level unit structures while still providing challenges that delineate between MC2 and MC3 students. The difficulty level for the assessment is appropriate for undergraduate students. Further adjustments to this assessment may adjust the questions on the assessments to test changes in difficulty caused by numbers and/or contexts used in the assessment. Researchers that do not agree with our conclusion to leave questions 1-a and 2-a as they are on the assessment may consider consolidating questions 1-a and 1-b, and questions 2-a and 2-b into a single question. Additionally, the current study presented both the solution (Part A) and justification (Part B) portions of the assessment at the same time as a way to be mindful of class time. During our implementation of the assessment, we noticed that 2 of the students were simultaneously completing Part A and Part B. This led to difficulty scored their assessments. Therefore, we suggest that future administrations of this assessment consider presenting these separately to keep students from working on the justification portion of the assessment before or alongside their work on the solution portion.

Assessment Rubric

The development and use of the UMCA relies heavily on analyzing qualitative evidence for the multiplicative concepts stages. In an effort to create a system for analysis, the researchers agreed the development of an extensive scoring rubric was essential for maintaining consistent scores across a large set of assessment data. After completing the discussion for the final scores for the participants, the researchers came to the consensus that the rubric for the UMCA is an important aspect of maintaining reliability in scoring between researchers. It provides a clear foundation for discussion on conflicting researcher scores. Past rubrics for similar assessments have assisted in improving interrater reliability and scoring consensus (Norton et al., 2015). The researchers met prior to data collection to do a theoretical analysis on the assessment to develop the UMCA rubric. Adjustments that were made to the rubric based on collected data were clarifications and additions of specific examples to increase the readability of the rubric. Results show that this facilitated a 96.2% agreement between researchers. While this is a promising result, it is important to note that those who have worked with and on this rubric are well versed in literature on the multiplicative concepts research. While effort was taken to maintain general clarity and readability of the rubric, it is worded in a way that is specifically beneficial to the researchers working on this assessment. Future researchers should be sure to review and discuss the rubric prior to use to ensure that the points are clear to every member of the research team.

Additional adjustments to the rubric may choose to include recent findings on the "advanced MC2" stage discussed in Hackenberg and Sevinc's (2021) article. The student discussed in this article, Milo, was an MC2 student who had worked with his available operations to be able to solve problems in ways that were novel to most MC2 students. He leveraged his use of his MC2 operations to strategically solve many of the problems at hand. Milo was not alone in his advanced reasoning either. As more research is published on the existence and operations of an advanced MC2 student, considerations should be made to adjust the rubric of the UMCA to identify evidence of these operations. Collection on the data for this assessment started prior to the publishing of the Hackenberg and Sevinc (2021) article and so did not take their findings into consideration in the development of the assessment or rubric.

Implications

The results of this study add to our knowledge of undergraduate students' multiplicative concept stages. The validity for the generalizability of the UMCA suggests that it can be used in larger scale explorations of undergraduate students' multiplicative concepts.

While most of the undergraduate students in this study have developed a MC3, 45% had only developed a MC1 or MC2. The third multiplicative concept stage is essential in supporting students conceptual understanding of unknowns (Hackenberg & Lee, 2015; Hackenberg et al., 2021), systems of equations (Olive & Caglayan, 2008), derivatives (Byerly, 2019), and in developing proportional reasoning (Steffe et al., 2014). Students start developing the third multiplicative concept as early as second grade (Kosko & Singh, 2018). However, over half of the sixth grades students in Boyce and Norton's (2016) study had not developed a MC3. Research on undergraduate students' multiplicative concepts and their calculus readiness found over half of the participants to have not developed an MC3 and none of the students to be MC1 students (Boyce et al., 2021). The spread of MC1, MC2, and MC3 students in this study suggests that undergraduate students have primarily developed an MC2 or MC3, and a small percentage have only developed an MC1. The large portion of MC2 students in these undergraduate studies suggests that professors should work to support these students in college mathematics by providing strategies that support MC2 student thinking.

Future Research and Conclusion

As research on the multiplicative concepts moves into the undergraduate population, assessments like the UMCA help researchers to collect a wider range of data. This study has examined evidence towards the validity of this assessment for undergraduate students. This assessment is not intended to be a definitive answer to an undergraduate students' multiplicative concept stage, but rather a tool for gaining an overview of the stages of students and participants in a sample. This information can help inform large sample studies or provide information to help select clinical interview participants. Additional interview evidence should still be collected to ensure an accurate representation of a students' multiplicative concept stage for qualitative studies. Future research should explore ways to improve and implement this assessment in research on undergraduate students' reasoning in mathematics topics such as matrices, geometry, conversions and measurement, covariation, etc. Future research may also use assessment format to examine the justification methods of students on multiplication based problems.

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Appendix A

1. A candy bar company packs 3 candy bars per package and 6 packages per box.

1-a: *If a store buys 7 boxes, how many candy bars will they receive?*

	Students' Unit Structures	Student Reasoning	Written Indicators of Reasoning
Stage 1	Students can work with one level of units as given and may coordinate two levels of units in activity. Two level unit structures decay after construction (Hackenberg & Tillema, 2009). They have access to partitioning and iterating operations (Steffe, 2010).	Students rely on figurative materials such as tally marks or pictures to create composite units and then work through how many candy bars are in each package and subsequently how many packages are in each box.	• Student responses show heavy reliance on pictorial or tally mark representations of the problem of the units of 1 (the candy bars). Composite units (i.e. packages and/or boxes) may be drawn but not counted. • Adds or multiplies incorrect combinations of numbers. • Student responses do not indicate the creation of a three-level unit structure (e.g., may only find how many candy bars are in a box or how many packages are in 7 boxes). • Students do not respond or otherwise indicate they do not know the answer.
Stage 2-3	Students anticipate two levels of units as given and may coordinate three levels of units in activity. Three level unit structures decay after construction (Hackenberg & Tillema, 2009). Students have access to a disembedding operation (Steffe, 2010).	Students use the two given relationships to establish a third level unit structure to determine how many candy bars are in the given number of boxes.	• Student uses a correct series of multiplications to solve. • Drawings indicate the creation of composite units (e.g. drawing 7 boxes and labeling them by 18). • Work indicates a reliance on skip counting to solve the problem. • Student drawings are used to justify their answers rather than produce them as a part of Part B. • Students just write down the answer with an explanation on Part B that indicates the use of a three-level unit structure.

1-b: *If the same store orders another 8 boxes, how many total candy bars have they received?*

	Students' Unit Structures	Student Reasoning	Written Indicators of Reasoning
Stage 1	Students can work with one level of units as given and may coordinate two levels of units in activity. Two level unit structures decay after construction (Hackenberg & Tillema, 2009). They have access to partitioning and iterating operations (Steffe, 2010).	Students rely on figurative materials such as tally marks or pictures to create composite units. They can then work out how many candy bars are in each package and subsequently how many packages are in each box.	• Student responses show heavy reliance on pictorial or tally mark representations of the problem of the units of 1 (the candy bars). Composite units (i.e. packages and/or boxes) may be drawn but not counted. • Student responses do not indicate the creation of a three-level unit structure (e.g., may only find how many candy bars are in a box or how many packages are in 8 boxes). • Student adds or multiplies incorrect combinations of numbers. • Students' answer is incorrect. • Student responses do not refer or utilize the numbers in 1-a. • Students do not respond or otherwise indicate they do not know the answer.
Stage 2	Students anticipate two levels of units as given and may coordinate three levels of units in activity. Three level unit structures decay after construction (Hackenberg & Tillema, 2009). Students have access to a disembedding operation (Steffe, 2010).	Students use the two given relationships to establish a third level unit structure to determine how many candy bars are in the given number of boxes. They then add this total to the found total in 1-a.	• Student finds the total number of candy bars in 8 boxes and then adds this to the total found in 1-a. • The student re-calculates the 18 candy bars to 1 box relationship from 1-b. • Drawings represents composite units (e.g. drawing 8 boxes and labeling them by 18). • Work indicates a reliance on skip counting to solve the problem. • Student explanations reference the coordination of two separate composite units (i.e. 144 candy bars and 126 candy bars.)
Stage 3	Students take three levels of units as given and can flexibly switch between two and three-level structures during activity without relying on figurative material. Students have access to a splitting operation (simultaneous partitioning and iterating operations) (Steffe, 2010).	Students assimilate with all two-level unit structures given and readily coordinate all three levels with a coordinated three level unit structure.	• Student finds the total number of boxes (15) and multiplies by 18 bars per box. • Student drawings are used to justify their answers rather than produce them as a part of Part B.

1-c: *Assuming the store received all of their ordered candy bars, how many packages have they received?*

	Students' Unit Structures	Student Reasoning	Written Indicators of Reasoning
Stage 1	Students can work with one level of units as given and may coordinate two levels of units in activity. Two level unit structures decay after construction (Hackenberg & Tillema, 2009). They have access to partitioning and iterating operations (Steffe, 2010).	Students mentally or physically attempt to count and track the number of packages per box in the problem.	• Student responses show heavy reliance on pictorial or tally mark representations of the problem of the units of 1 (the candy bars). Composite units (i.e. packages and/or boxes) may be drawn but not counted. • Student adds or subtracts the numbers found in 1-a and 1-b. • Student finds the total number of boxes rather than the total number of packages. • Student responses do not reference back to the numbers in 1-a or 1-b. • Student does not respond or otherwise indicate they do not know the answer.
Stage 2	Students anticipate two levels of units as given and may coordinate three levels of units in activity. Three level unit structures decay after construction (Hackenberg & Tillema, 2009). Students have access to a disembedding operation (Steffe, 2010).	Students use the given relationships to work backwards to find the number of intermediary units (packages) in the problem through a form of division.	• Student uses division to solve the problem. (i.e. 270/3 = 90) • Student calculates or pulls from part a and b that there are 42 and 48 packages respectively and adds them to find the total (90 packages). • Student re-calculates the 18 candy bars to 1 box relationship from 1-b. • Student divides the total sum by the wrong unit such as 6 or 18. • Student multiplies the total number of candy bars by 6 or 3 instead of dividing by 3. • Student starts working the problem from the beginning, using packages per box as the initial level-two unit on Part A or Part B. • Student explanations reference the coordination of two separate composite units (i.e. 270 total candy bars and 3 candy bars per package).
Stage 3	Students take three levels of units as given and can flexibly switch between two and three-level structures during activity without relying on figurative material. Students have access to a splitting operation (simultaneous partitioning and iterating operations) (Steffe, 2010).	Students assimilate with all two-level unit structures given and readily coordinate all three levels with a coordinated three level unit structure allowing them to work flexibly between layers to use multiplication or division to solve the problem.	• Student drawings are used to justify their answers rather than produce them as a part of Part B. • Student work or explanations show flexibility between levels of units by referencing multiple ways of solving the problem or referencing the units of each part of the quotient or product.

2. There are 6 plants in each row of my garden.

2-a: *How many tomato plants are in 8 rows?*

	Students' Unit Structures	Student Reasoning	Written Indicators of Reasoning
Stage 1	Students can work with one level of units as given and may coordinate two levels of units in activity. Two level unit structures decay after construction (Hackenberg & Tillema, 2009). They have access to partitioning and iterating operations (Steffe, 2010).	Students mentally or physically use the total number of plants per row to create a two-level unit structure to find the total number of plants.	• Student responses show heavy reliance on pictorial or tally mark representations of the problem of the units of 1 (plants). Composite units (i.e. rows) may be drawn but not counted.
Stage 2-3	Students anticipate two levels of units as given and may coordinate three levels of units in activity. Three level unit structures decay after construction. (Hackenberg & Tillema, 2009). Students have access to a disembedding operation (Steffe, 2010).	Students use the two given relationships to establish a third level unit structure to determine how many candy bars are in the given number of boxes. They then add this total to the found total in 1-a.	• Student multiplies 6 by 8. • Drawings represents composite units (e.g. rows of 6 plants). • Skip counting may be used to find the product. • Student just writes the correct answer with no work shown on Part A.

2-b: *In addition to tomato plants, I also planted potatoes. If there are a total of 102 plants, how many rows of potatoes did I plant?*

	Students' Unit Structures	Student Reasoning	Written Indicators of Reasoning
Stage 1	Students can work with one level of units as given and may coordinate two levels of units in activity. Two level unit structures decay after construction (Hackenberg & Tillema, 2009). They have access to partitioning and iterating operations (Steffe, 2010).	Students rely on figurative material to work backwards from the total number of plants to figure out how many rows of potatoes were planted.	• Student responses show heavy reliance on pictorial or tally mark representations of the problem of the units of 1 (plants). Composite units (i.e. rows) may be drawn but not counted. • Student responses do not indicate the creation of a three-level unit structure. • Student only subtracts the total number of plants and the number found in 2-a. • Students subtracts, multiplies, or divides by incorrect numbers from part 2a (i.e. 102-8 instead of 102-48). • Student does not respond or otherwise indicate they do not know the answer.
Stage 2	Students anticipate two levels of units as given and may coordinate three levels of units in activity. Three level unit structures decay after construction (Hackenberg & Tillema, 2009). Students have access to a disembedding operation (Steffe, 2010).	Students use the two given relationships to establish a third level unit structure to work backwards to determine how many potatoes and then how many rows of potatoes were planted.	• Student drawings indicate the creation of composite units. • Student begins work on the problem by dividing the total number of plants by 6 plants per row but does not subtract the number of rows from 2-a. • Student divides by the wrong unit such as 8. • Student work indicates a reliance on repeated addition or skip counting to solve the problem. • Student work shows a reworking of the problem as a part of their explanation in Part B.
Stage 3	Students take three levels of units as given and can flexibly switch between two and three-level structures during activity without relying on figurative material. Students have access to a splitting operation (simultaneous partitioning and iterating operations; Steffe, 2010).	Students assimilate with all two-level unit structures given and readily coordinate all three levels with a coordinated three level unit structure to work backwards.	• Student finds the total number of plants by first using subtraction and then division (i.e. 102-48=54, 54/6 = 9). • Student finds the total number of plants by using division and then subtraction (i.e. 102/6 = 17, 17-8 = 9). • Student drawings are used to justify their answers rather than produce them as a part of Part B. • Student explanations discuss the unit structures as a three-level unit structure.

3. There are 12 inches in 1 foot and 3 feet in 1 yard.

***3-a**: How many inches are in 2 yards?*

	Students' Unit Structures	Student Reasoning	Written Indicators of Reasoning
Stage 1	Students can work with one level of units as given and may coordinate two levels of units in activity. Two level unit structures decay after construction (Hackenberg & Tillema, 2009). They have access to partitioning and iterating operations (Steffe, 2010).	Students rely on figurative material to represent and count how many inches are in three feet and how many inches are in 2 yards.	• Student responses show heavy reliance on pictorial or tally mark representations of the problem of the units of 1 (inches). Composite units (i.e. feet and/or yards) may be drawn but not counted. • Student responses do not indicate the creation of a three-level unit structure. • Student adds or multiply incorrect combinations of numbers. • Student does not respond or otherwise indicate they do not know the answer.
Stage 2	Students anticipate two levels of units as given and may coordinate three levels of units in activity. Three level unit structures decay after construction (Hackenberg & Tillema, 2009). Have access to a disembedding operation (Steffe, 2010).	Students use the two given relationships to establish a third level unit structure to determine how many how many feet are in 2 yards and how many inches are in 6 feet.	• Student drawings indicate the creation of composite units (i.e. representations of yards with 3 feet or feet with 12 inches each). • Student work indicates a reliance on repeated addition or skip counting to solve the problem. • Student work indicates the use of multiplication (i.e. 12 x 6 = 72) to solve. • Student work shows a reworking of the problem as a part of their explanation in Part B.
Stage 3	Students take three levels of units as given and can flexibly switch between two and three-level structures during activity without relying on figurative material. Have access to a splitting operation (simultaneous partitioning and iterating operations) (Steffe, 2010).	Students assimilate with all two-level unit structures given and readily coordinate all of these units to determine the total number of inches.	• Student just writes the answer. • Student drawings are used to justify their answers rather than produce them in Part B. • Student explanations discuss the unit structures as a three-level unit structure.

3-b: If you add an additional 5 feet onto the original yards, how many total inches are there?

	Students' Unit Structures	Student Reasoning	Written Indicators of Reasoning
Stage 1	Students can work with one level of units as given and may coordinate two levels of units in activity. Two level unit structures decay after construction (Hackenberg & Tillema, 2009). They have access to partitioning and iterating operations (Steffe, 2010).	Students rely on figurative material to represent and count how many inches are in in 5 feet. They may add this to their original representation from 3-a to count the total number of inches in the problem.	• Student responses show heavy reliance on pictorial or tally mark representations of the problem of the units of 1 (inches). Composite units (i.e. feet and/or yards) may be drawn but not counted. • Student responses do not indicate the creation of a three-level unit structure. • Student adds or multiplies incorrect combinations of numbers. • Student does not refer back to their answer as a part of 3-a. • Student does not respond or otherwise indicate they do not know the answer.
Stage 2	Students anticipate two levels of units as given and may coordinate three levels of units in activity. Three level unit structures decay after construction (Hackenberg & Tillema, 2009). Students have access to a disembedding operation (Steffe, 2010).	Students use the given relationships to find how many inches are in the additional 5 feet and add them to the original inches from part 3-a.	• Student drawings indicate the creation of composite units (i.e. inches within feet). • Student work indicates a reliance on repeated addition or skip counting to solve the problem. • Student finds the total number of inches in 5 feet and adds this to the inches in 3-a to get the total (i.e. 60 + 72 = 132) • Student work shows a reworking of the problem as a part of their explanation in Part B.
Stage 3	Students take three levels of units as given and can flexibly switch between two and three-level structures during activity without relying on figurative material. Students have access to a splitting operation (simultaneous partitioning and iterating operations; Steffe, 2010).	Students assimilate with all two-level unit structures given and readily coordinate all of these units to determine the total number of inches.	• Student just write the answer of 132. • Student may determine the total number of feet (i.e. 6+5; this may not be written) across both 3-a and 3-b and then multiply 11 by 12. • Student drawings are used to justify their answers rather than produce them in Part B. • Student explanations discuss the unit structures as a three-level unit structure.

3-c: *How many feet are in the total number of inches?*

	Students' Unit Structures	Student Reasoning	Written Indicators of Reasoning
Stage 1	Students can work with one level of units as given and may coordinate two levels of units in activity. Two level unit structures decay after construction (Hackenberg & Tillema, 2009). They have access to partitioning and iterating operations (Steffe, 2010).	Students attempt to rely on their created figurative material to determine how many feet are in their counted total of inches.	• Student responses show heavy reliance on pictorial or tally mark representations of the problem of the units of 1 (inches). Composite units (i.e. feet and/or yards) may be drawn but not counted. • Student attempts to count their figurative material, but creates groups that are not representative of the 12 inches to 1 foot relationship. • Student adds or multiplies incorrect combinations of numbers. • Student does not refer back to their answer as a part of 3-a or 3-b. • Student does not respond or otherwise indicate they do not know the answer.
Stage 2	Students anticipate two levels of units as given and may coordinate three levels of units in activity. Three level unit structures decay after construction (Hackenberg & Tillema, 2009). Students have access to a disembedding operation (Steffe, 2010).	Students use the given relationships and the total number of inches found to determine the total number of feet.	• Student work indicates a reliance on repeated addition or skip counting to solve the problem. • Student finds the total number of feet through division (i.e. 132/12 = 11 feet). • Student divides by the wrong unit (i.e. they divide by 3 yards or some other related number that is not inches). • Student work shows a reworking of the problem as a part of their explanation in Part B.
Stage 3	Students take three levels of units as given and can flexibly switch between two and three-level structures during activity without relying on figurative material. Students have access to a splitting operation (simultaneous partitioning and iterating operations; Steffe, 2010).	Students assimilate with all two-level unit structures given and readily coordinate all of these units to determine the total number of feet.	• Student finds the total number of feet through multiplication and addition. • Student just writes 11 feet. • Student references the total number of feet in parts 3-a and 3-b to determine how many total feet there are without reworking 3-a. • Student drawings are used to justify their answers rather than produce them in Part B. • Student explanations discuss the unit structures as a three-level unit structure. • Student presents multiple ways to solve the problem.

3-d: *How many yards are there in the total number of inches?*

	Students' Unit Structures	Student Reasoning	Written Indicators of Reasoning
Stage 1	Students can work with one level of units as given and may coordinate two levels of units in activity. Two level unit structures decay after construction (Hackenberg & Tillema, 2009). They have access to partitioning and iterating operations (Steffe, 2010).	Students attempt to rely on their created figurative material to determine how many yards are in their counted total of inches.	• Student responses show heavy reliance on pictorial or tally mark representations of the problem of the units of 1 (inches). Composite units (i.e. feet and/or yards) may be drawn but not counted. • Student attempts to count their figurative material, but creates groups that are not representative of the inches to feet to yard relationship. • Student adds or multiplies incorrect combinations of numbers. • Student does not refer back to their answer as a part of 3-a, 3-b, or 3-c. • Student does not make an attempt to account for the remainder of the yard. • Student does not respond or otherwise indicate they do not know the answer.
Stage 2	Students anticipate two levels of units as given and may coordinate three levels of units in activity. Three level unit structures decay after construction (Hackenberg & Tillema, 2009). Students have access to a disembedding operation (Steffe, 2010).	Students use the given relationships and the total number of inches found to determine the total number of yards.	• Student works indicates a reliance on repeated addition or skip counting or repeated subtraction to solve the problem. • Student divides by the wrong unit (i.e. they divide by 3 feet or some other related number that is not inches). • Student finds an incorrect remainder. • Student describes the remainder as "3 and a little bit" or gives a range such as "between 3 and 4". • Student does not interpret the remainder as a part of a yard.
Stage 3	Students take three levels of units as given and can flexibly switch between two and three-level structures during activity without relying on figurative material. Students have access to a splitting operation (simultaneous partitioning and iterating operations; Steffe, 2010).	Students assimilate with all two-level unit structures given and readily coordinate all of these units to determine the total number of yards.	• Student finds the total number of yards through division and addition (5/3 = 1 2/3, 1 2/3 + 2 = 3 1/3). • Student finds the total number of feet using division. • Student references the total number of yards in parts 3-a and 3-b to determine how many total yards there are. • Student drawings are used to justify their answers rather than produce them in Part B. • Student explanations discuss the unit structures as a three-level unit structure. • Student presents multiple ways to solve the problem. • Student interprets the remainder as a part of the yard.

4. The stick shown below is 3/5 of a whole stick. How many 1/15 sticks can you make from the 3/5 stick?

	Students' Unit Structures	Student Reasoning	Written Indicators of Reasoning
Stage 1	Students can work with one level of units as given and may coordinate two levels of units in activity. Two level unit structures decay after construction (Hackenberg & Tillema, 2009). They use equisegmenting to construct fractional units (Steffe, 2010).	Students rely on the given bar to segment and count how many $1/15^{th}$ sticks are in the given length	• Segments the given 3/5 bar into 5 parts. Parts may or may not be reasonable. • Student attempts to segment the bar into 15 parts, but there is no consistent relationship between the fifths and fifteenths. Student may end up with too many or too few parts. • Student's answer is 9, but the drawing does not reflect 9 fifteenths in 3/5. • The student guesses. • Students do not respond or otherwise indicate they do not know the answer.
Stage 2	Students anticipate two levels of units as given and may coordinate three levels of units in activity. Three-level unit structures decay after construction (Hackenberg & Tillema, 2009). Students use equi-partitioning, simultaneous partitioning, and splitting operations (if constructed) to construct fractional units (Steffe, 2010).	Students use the two given fractions to determine through drawings that there are 9 total $1/15^{th}$ sticks in the given bar.	• Student work indicates a reliance on a drawing to problem solve, not just represent their thinking. • Student uses a separate representation (such as a circle or a new line) to partition into 5ths, partition into 15ths, and then take away two 5ths. They then count the pieces individually or as 3s. • Student may extend their stick to show 5/5ths of the stick before they partition it into 15ths. • Student writes "3, 6, 9" which indicates the use of skip counting. • Written multiplication or division material to determine how many 15ths are in a third. This may be confused with the answer. • Student work shows a reworking of the problem as a part of their explanation in Part B.
Stage 3	Students take three levels of units as given and can flexibly switch between two and three-level structures during activity without relying on figurative material. Students have access to a splitting operation (Steffe, 2010).	Students are able to represent the problem pictorially using their splitting operation to first determine the length of the whole as partitioned into 15ths and then how many total 1/15ths are in the given bar. They may also use a scale factor to determine this.	• Student uses a ratio to determine the total number of 1/15ths in the 3/5ths (i.e. they use the 3/5 = x/15 relationship to determine the number of 15ths). • Student may just write 3x3=9 or 9 or 9/15ths. • Student drawings are used to justify their answers rather than produce them in Part B.

5. The bar shown below is 7/3 as long as long as a whole candy bar. Draw the whole candy bar.

	Students' Unit Structures	Student Reasoning	Written Indicators of Reasoning
Non-IFS	Students have difficulties working with improper fractions. They may be able to use the splitting operation, but without assimilating with 3 levels of units, they have difficulties tracking and making sense of the relationships found in improper fractions. May simplify such fractions to offload the mental load and make the problem easier (Steffe, 2010).	Students rely on the given bar to segment on or iterate. However, they do so arbitrarily. The student partitions the bar as well as they can into what they guess would be 7/3. They work with 7/3 in terms of whole numbers instead of as an improper fraction.	• The student segments the bars into thirds. • The student guesses. • Student provides a drawing and explanation that shows they tried to estimate 2 "wholes" and a "bit". • Student forms a mixed number to solve the problem. • Student does not respond or otherwise indicate they do not know the answer.
Stage 3 with IFS	Students take three levels of units as given and can flexibly switch between two and three-level structures during activity without relying on figurative material. They use this in combination with the splitting operation to work with and understand improper fractions in terms of wholes and fractional parts (i.e. a student can recognize 7/3 as 2 wholes comprising of 3pieces of 1/3 length and an extra 1/3 of a whole; Steffe, 2010). This is used as a confirmation of an MC3 student.	Students are able to partition the bars into 7 1/3rd pieces and indicate three of these as comprising the whole.	• Student partitions the bar into 7ths and then indicates or redraws three of these pieces is 1 whole (between 4 & 5 cm) • Explanations indicate they thought of the given bar as 7 thirds that they then partitioned to determine the whole regardless of whether they indicated 1/3 (or three of the 7 parts) as the whole.

CHAPTER IV

EXPLORING UNDERGRADUATE STUDENTS' REASONING ON OPTIMIZATION PROBLEMS

Target Journal: The Journal of Mathematical Behavior

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Abstract:

This study examines students' reasoning on optimization problems and how their mathematical reasoning is supported by their multiplicative reasoning. Five undergraduate students participated in clinical interviews solving two optimization problems. Three of the five students successfully answered both questions. Students used systematic guess and check, equations, tables, and graphs to represent both problems. Results showed that relational reasoning, numerical examples, and visual representations helped students provide accurate algebraic representations for the problems.

Introduction

The purpose of learning mathematics is to develop mathematical reasoning that is beneficial to a students' career and ability to reason about the world around them. Sound mathematical reasoning is essential to a person's ability to be an informed participant in society and to develop competence in everyday tasks (National Research Council, 2001). Many entry-level undergraduate mathematics courses focus on quantitative reasoning in their curriculum (Elrod, 2014; Lusardi & Wallace, 2013; Wolfe, 1993). Quantitative reasoning is defined as applying mathematics and critical thinking to interpret data and draw conclusions from problems, critical skills for any student regardless of their career (Elrod, 2014). Education initiatives focus on the development of rich mathematical and quantitative reasoning in students. Both the National Research Council (2001) and The Common Core State Standards for Mathematics (CCSSM; CCSSO, 2010) discuss developing students' conceptual understanding of mathematics, or comprehension of mathematical concepts, operations and relations (CCSSO, 2010), as a priority in teaching mathematics.

However, students are not always receptive to teaching that emphasizes the conceptual approach to mathematics. Some of the students in Byerly's (2019) study struggled with learning about derivatives from a conceptual approach to rate of change and expressed their preference for memorizing procedures and formulas over conceptual learning. Additionally, students' ability to construct and coordinate unit structures were shown to influenced their ability to reason about rate of change conceptually. Limitations in their available operations due to their understanding of how to coordinate unit structures became of point of frustration for some when reasoning conceptually about derivatives. To help accommodate these students and reduce frustration when learning conceptually, it is important that we understand how these students leverage their operations to solve problems. Prior research has explored connections between students' ability construct and coordinate unit structures and how they recognize and their ability to understand and represent multiplicative relationships using unknowns (Hackenberg et al., 2017). However, current research has not explored how students reason about and represent covarying contexts within problems with equations.

The focus in this study is on how students reason quantitatively on problems involving covarying contexts and how they reason about representing these problems to find a solution. To support this focus, this study explores how undergraduate students solve optimization problems prior to any experience with them in calculus. Optimization problems provide students with a situation with varying parameters and ask them to identify values within these parameters that produce an optimum solution. Optimization problems are often introduced in calculus courses as problems for teaching and practicing the use and interpretations of derivatives. However, the unique covarying (simultaneous variation of variables) nature of optimization problems provides a challenge for students to reason about quantitatively (Thompson & Carlson, 2017). Thompson and Carlson (2017) propose that conceptualizing and representing an optimization problem requires sophisticated quantitative, algebraic, and covariational reasoning. Take the following problem titled *the charter bus problem:*

> Marian's charter bus company offers a route to the neighboring city that charges $40 per person if up to 30 passengers sign up for the trip. If more than 30 passengers sign up, the fare for every passenger is reduced by $1 for every passenger in excess of 30. The bus can only hold 48 passengers. How many passengers does Marian want to sign up for her charter bus route in order to maximize her revenue for the trip?

In this example, the student is first asked to recognize and reason about the discount provided by conceptualizing a covarying relationship where the fare price for the bus decreases by $1 for every increase in passenger over 30. They must also take into account that the fare price and number of passengers are being multiplied together in order to produce the total amount of revenue. Finally, they must identify they are looking for the maximum amount of revenue possible. Creating representations for optimization problems requires a combination of concepts such as algebraic reasoning, equation writing, and covariational reasoning.

We propose there is be a connection between the students reasoning about optimization problems and their ability to construct and coordinate unit structures. Research on students' ability to construct and coordinate unit structures has shown it to be a foundation for their reasoning with other mathematical concepts such as fractions (Hackenberg, 2007; Steffe, 2001), proportions (Steffe et al., 2014), algebraic reasoning and equation writing (Hackenberg, 2013; Hackenberg et al., 2017, 2021; Olive & Cagalayan, 2008; Zwanch, 2019, 2022a, 2022b), derivatives and rates of change (Byerly, 2019), and recognition of quantity and use of measurement (Steffe, 2013).

The goal of this study is to explore how undergraduate students reason mathematically and solve optimization problems and how students' ability to construct and coordinate unit structures supports their problem solving. The research questions for this study are:

- How do undergraduate students reason about and solve optimization problems?
- To what extent can the multiplicative concepts be used to explain undergraduate students' reasoning on optimization problems?

Theoretical Framework

Scheme theory serves as the theoretical lens for the analysis of student work on optimization problems for this study. Within scheme theory, the goal of analysis of mathematical thinking in this study is not to define how each student thinks in terms of an expert's lens, but in the terms of student thinking. This aligns with Steffe's (1992) emphasis on researching *children's mathematics,* not the mathematics of children. We define our view of mathematical thinking as Hackenberg (2013) "in terms of mental actions, or operations" (p. 539). These operations are a piece of their developed schemes. Schemes, as defined by von Glasersfeld (1995), is a model of a person's learned thoughts and actions that take place as a pattern of interaction when an individual is introduced to a circumstance or situation. A scheme involves three parts. The individual will (1) recognize the situation, (2) produce a mental action tied to the situation, and (3) have an expected or unexpected outcome of the situation (von Glasersfeld, 1995). For example, a student given an optimization problem might first read the problem and recognize that it is asking them to find the maximum. Then the student may attempt to solve the problem by guessing, making an equation, or trying to find a graph. The student will expect an outcome based on their past experiences with operations. This could be a single maximum number, a linear or quadratic equation, or the highest point on a graph.

Multiplicative Concepts

The multiplicative concepts are stages that model the degree to which students anticipate, construct, and coordinate multiple levels of units and the operations tied to these units they have available during problem solving (Hackenberg & Tillema 2009; Ulrich, 2016). *Units,* as used in this study, are standard and non-standard units of measure (Ulrich, 2015). Units can be understood as multiple levels. For example, a yard is a unit that can be understood as 1 yard, 3 feet, and 36 inches. Constructing this into a unit that is a yard containing 3 feet containing 12 inches each is the creation of a 3-level unit structure (Hackenberg et al., 2021). The multiplicative concepts consist of three stages of schemes that are defined by the degree to which students anticipate, construct and operate on unit structures that consist of multiple levels (Steffe, 1992; Hackenberg & Tillema, 2009; Ulrich, 2015). Analysis of student thinking through the lens of their multiplicative concepts allows the researchers to explore how the operations available to students from these schemes interact with their reasoning on complex problems.

Students who assimilate with one level of unit and can construct two in activity (during problem solving) have developed the first multiplicative concept (MC1; Hackenberg & Tillema, 2009). These students insert one unit into another in order to multiply them together. An MC1 student can find how many inches are in 6 feet by inserting 12 inches into each foot and skip counting by twelve 6 times to get a total of 72 inches. After problem solving the two-level unit structure constructed by the student, it decays leaving the student to only reflect on the 72 inches with no connection to the original 6 feet (Hackenberg et al., 2021). These 72 inches are not considered interchangeable to one another so the inch labeled 3 is equal but not identical to the inch labeled 45.

Students who assimilate with two levels of units and can construct three levels in activity have developed the second multiplicative concept (MC2; Hackenberg & Tillema, 2009). MC2 students can find how many inches are in 2 yards by anticipating the composite unit of 2 yards that contain 6 feet and then constructing a three level unit structure by inserting 12 inches into each of the 6 feet to get 72 total inches. These 72 inches are iterable units of 1 meaning they are now considered interchangeable with one another. The third level of units constructed to solve the problem decay after activity, leaving the MC2 student to reflect only on two levels of units

(Hackenberg et al., 2021). Thus, the student may only reflect on the 2 yards containing 72 inches unit after problem solving.

Students who assimilate with three levels of units and construct four or five levels in activity have developed the third multiplicative concept (MC3; Hackenberg & Tillema, 2009; Ulrich 2016). These students can move flexibly between the levels of their constructed units during problem solving (Ulrich, 2016). MC3 students can find the number of total inches in 2 yards by anticipating the composite unit of 2 yards containing 6 feet containing 12 inches each. They could also solve how many inches are in 5 more feet by taking the 72 inches they originally found and iterating the unit of a foot containing 12 inches 5 more times reaching a total of 132 inches. This is possible as MC3 students have constructed iterable composite units (two-level unit structures; Ulrich, 2016). MC3 students can reflect on their three-level unit structures and can recognize that their 132 inches are contained within 11 feet contained within 3 2/3 yards.

Connections to Optimization Problems

The operations available to a student from their multiplicative concept stage influences the student's reasoning with algebraic symbols and equations and their understanding of unknowns (Hackenberg, 2013; Hackenberg et al., 2017, 2021). Research on middle school students with number sequence schemes that align with MC1 and MC2 students showed that MC2 students were more successful than their MC1 counterparts at developing correct algebraic equations to represent on-step multiplicative relationships (Zwanch, 2019), generalizing linear patterns (Zwanch, 2022b), and solving systems of linear equations (Zwanch, 2022a). Additionally, researchers have conjectured that MC1 students would have a difficult time working with quantitative unknowns and variables due to their use of singleton units, which can make it difficult to write equations (Hackenberg et al., 2021). However, prior research on undergraduate student populations suggests that most students have developed an MC2 or MC3 by the time they have reached college (Boyce et al. 2021; Davenport et al., in preparation). As such, we will not be discussing MC1 student thinking on optimization problems as a part of this study.

Research on MC2 and MC3 students' algebraic equation writing supports a difference in operations between MC2 and MC3 students that can create an increase in difficulty for problems similar to optimization problems. Research on students' abilities to write multiplicative relationships as algebraic expressions showed that MC2 students could write the equations with effort, while MC3 students wrote the expressions swiftly (Hackenberg et al., 2017). Additionally, MC2 students often struggle with conceptualizing a variable as a unit of unknown size and often use variables as a place holder for numeric examples (Hackenberg et al., 2017). However, MC2 students can create accurate an algebraic representation for a set of equations that are in terms of a single variable (Olive & Caglayan, 2008; Zwanch, 2022a).

Methodology

Theoretical Analysis

Before collecting data for this study, researchers selected the optimization problems for this study and conducted a theoretical analysis of the problems. The theoretical analysis allows researchers to clarify and identify the unit structures of the problems and anticipate student solutions based on the current research on the multiplicative concepts stages.

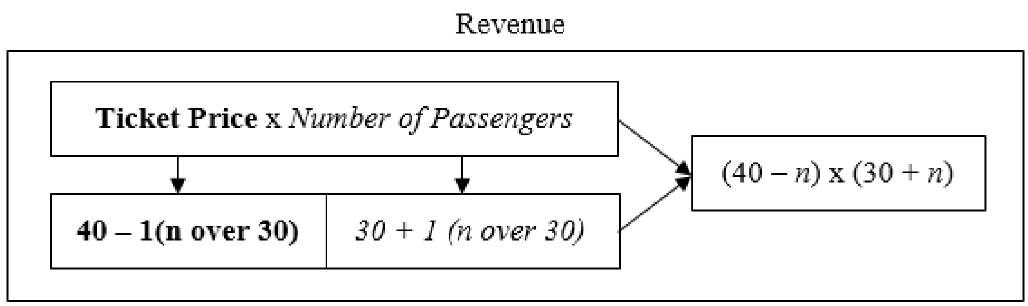
Olive and Caglayan (2008) claim that "one can interpret all the variables arising from word problems as not simply ordinary quantities named in the problem, but, rather, as mathematical objects with names, values, and associated units" (p. 288). Coordinating these values is what is essential to creating a *quantitative structure* (a network of quantitative relationships; Thompson 1993). A quantitative structure is a unit of unknowns constructed of the relationships between these unknowns. Constructing quantitative structures requires complex coordination of units and relationships. The ability to construct three-level unit structures is important to being able to understand and conserve unit relationships in this structure (Olive & Cagalayan, 2008). As such, we hypothesize that MC2 and MC3 students should be able to understand an optimization problem, create a representation of the relationships expressed in the problem, and create an equation for the problem. However, the level of flexibility and ease of this process is hypothesized to differ depending on the constructed multiplicative stage of the student. To justify these hypotheses, we will present our theoretical analysis of the optimization problems explored in this study. Below is *the charter bus problem:*

> Marian owns a charter bus company offers a route to the neighboring city that charges $40 per person if up to 30 passengers sign up for the trip. If more than 30 passengers sign up, the fare for every passenger is reduced by $1 for every passenger in excess of 30. The bus can only hold up to 48 passengers. How many passengers does Marian want to sign up for her charter bus route if she wants to maximize her revenue for the trip? (35 passengers)

The charter bus problem, asks the student to find the optimum number of passengers for the charter bus if she wants the highest amount of revenue. As such, students must conceive of revenue as the product of the number of passengers and the price for each ticket (see Figure 4.1). They must then understand that the price of the fare varies such that for every passenger over 30, they must take $1 off the price of the ticket. If they want to express revenue as an equation, they can write the price of the ticket as (40-n) where n is number of passengers over 30. Additionally, the student must also recognize that the number of passengers can be assigned a relationship based on this newly defined variable *n*. They can represent this relationship by writing (30+*n*). By substituting both, the student can conceive of the revenue as both the price of the ticket multiplied by number of passengers and (40-*n*)(30+*n*).

Figure 4.1

Quantitative structure for the charter bus problem



The covariation of the ticket price and number of passengers can be a difficult challenge for MC2 students during equation writing. The student must define and retain the definition of their variable as the additional people over 30. Additionally, they must conceive of revenue as an equation of two unknowns that covary by the same term (*n*). The result is an equation of two expressions that are in a multiplicative relationship with one another. They must also remember that they are looking for the optimal number of passengers to get the maximum amount of revenue. This is a large cognitive load for MC2 students and may cause relationships constructed during in their work to decay with time.

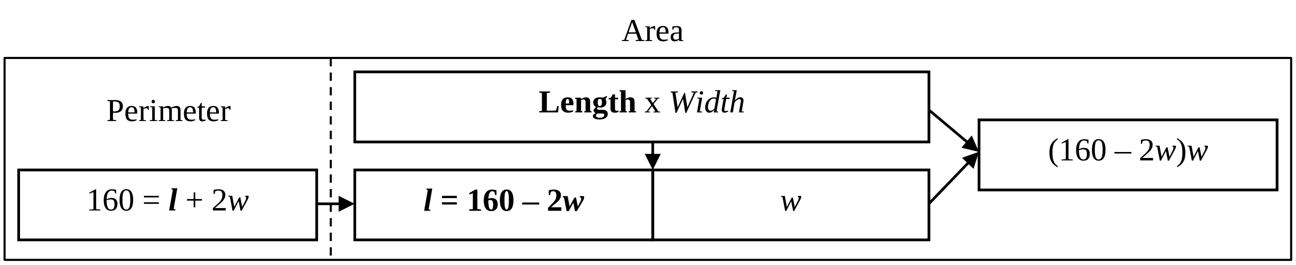
The other optimization problem explored in this study is the *barn pen problem*. It is as follows:

> John wants to build a rectangular pen next to his barn. To try to maximize his resources, he decides to use one side of the barn as a side of his pen. If he has 160 feet worth of fence available to build his pen and the barn side was over 160 feet long, what dimensions of the pen will maximize its area? (80ft x 40ft x 40ft)

For the barn pen problem, the student is asked to find the dimensions of the pen based on the available fencing. The student must recognize that they are looking for the maximum area. Since this is a rectangular fence, they know they need to multiply the length and width of the pen to find the area ($A = lw$; see Figure 4.2). However, they are given information that pertains to the perimeter of the pen. Using the given information and what they know about rectangles, they can determine that the perimeter should be the sum of the lengths and widths ($P = 2l + 2w$). Since one of those sides is the barn, they can take out either a length or width ($P = l + 2w$). They know that the perimeter is 160 ft, so they can write the formula ($160 = l + 2w$). If they think to use substitution, they can rearrange the perimeter in terms of either length or width ($l = 2w – 160$). Substituting this into their area formula allows them to create one algebraic equation in terms of the single variable (w; $A = (2w – 160)(w)$).

Figure 4.2

Quantitative structure for the barn pen problem



To write the equation for the barn pen problem, the student must recognize that they can use the perimeter to define the relationship between the length and width of the rectangle ($l = 2w – 160$). They can then insert this into their equation for the area of the rectangle to help determine the maximum area of the pen. However, the student must maintain the relationship between the length and the width during problem solving to find the dimensions of the pen. Should the relationship decay after problem solving, the student may need to reestablish the relationship using the formula for the perimeter and the 160 feet of fencing. MC2 students may have difficulties retaining the relationships between the variables found in the problem.

Participants and Data Collection

The current study is part of the second phase of a two-phase study exploring undergraduate students' multiplicative concept stages by (1) validating an assessment for undergraduate students, (2) exploring their solutions to optimization problems, and (3) exploring how they discuss their mathematics identity. This article specifically addresses the second goal of the study, exploring student solutions to optimization problems. The participants in this phase of the study are separate from the participants in the first phase of this study whose data was used to validate an assessment for undergraduate students' multiplicative concept stages (Davenport et al., in preparation).

This study is a collective case study (Creswell & Poth, 2016) exploring the problem-solving strategies of undergraduate students on optimization problems. They define a case study as "a qualitative approach in which the investigator explores a real-life, contemporary bounded system (or case) or multiple bounded systems (cases) over time, through detailed in depth data collection involving multiple sources of information" (Creswell & Poth, 2016, p. 95). In a collective case study, the researcher choses an issue or concern and bounds multiple cases to illustrate this issue (Creswell & Poth, 2016). The issue being explored in this study is the student reasoning on optimization problems. The cases being discussed in this study are (1) undergraduate non-STEM major MC2 students enrolled in an entry-level mathematics course who have not taken calculus before and (2) undergraduate non-STEM major MC3 students enrolled in an entry-level mathematic courses who have not taken calculus before. This section outlines how the researchers chose the sample of participants for this study as bound by the cases.

The Undergraduate Multiplicative Concepts Assessment (UMCA; Davenport et al., in preparation) was administered to 43 undergraduate students enrolled in either an entry-level mathematics course focusing on mathematical functions and their applications to natural sciences, agriculture, business, and social sciences or a freshman level course on elementary education taught at a mid-western university. Students in the elementary education course either were currently or had previously been enrolled in the function modelling course the prior semester. Participants were recruited from these courses to ensure they had not taken calculus as a part of their college coursework. By targeting this population of undergraduate students, researchers aimed to be able to explore the students' work on optimization problems as the students first experience them. Scores from the UMCA were used to help evaluate the multiplicative concept stage of participants for participant selection (Davenport et al., in preparation).

From the students who took the UMCA, five volunteered to participate in a follow-up interview that was given in two parts. First, they participated in a semi-structured interview (Galletta, 2013) where they discussed their mathematics identity. Data collected from this portion of the interview will not be discussed in this article. The second part of this interview was a clinical interview (Clement, 2000) that took between 30 minutes and an hour and was given in two sections. In the first section of the interview, participants were asked to solve a series of problems as additional evidence towards the multiplicative concept stage of the participants alongside the evidence from the UMCA. The participants' UMCA results and interview evidence given in the first section of the interview indicated that two of the volunteer participants were MC2 students (Abigail and Diana), and three were MC3 students (Brian, Clarissa, and Eric). The stage attributed to each participant by the UMCA and the interview evidence matched for all participants. Participants were given pseudonyms for the purposes of discussion. In the second part of the interview, participants were asked to solve the charter bus problem and the barn pen problem. These optimization problems needed to be solvable by students who had not taken calculus, and so were restricted to scenarios where the resulting equation was a quadratic function.

Participants were given a lined notebook to write on and a pen. The problems were printed on paper for participant reference. Four function calculators and graphing paper were readily available, but were not offered unless the student suggested and then requested it. Only one participant (Diana) requested the use of a calculator.

Participants were presented with each problem and asked to work out their own solution. The interviewer asked questions about the participants' reasoning and solution process and helped clarify if the they had misunderstandings about the problem. After the participant initially found a solution, the researcher then asked them if they could write an equation or create a graph for the problem if they had not already done so. After the student had completed discussions of their solutions, equations, and graphs they were then presented with the correct graph from the problem and asked to interpret this graph and compare it to their own work. This graph was not presented as the graph for the problem, but rather a graph that was on hand. Students were then asked to interpret the graph in relation to appropriate optimization problem.

Student work was recorded as video and audio footage and pdfs were kept of their written work. Transcriptions were made of the clinical interviews. Formatting choices were made during transcription to better represent the students' actions. All work that was pointed at or referenced while the interviewer or participant were talking were placed in brackets and italicized (e.g. "D: I like this one [*1200 – 9x*] better."). Actions taken between dialogue were written out and italicized on a separate line from dialogue. Quotations were placed around any verbatim written work.

Data Analysis

Explicit analysis was used to analyze participants' work solving the optimization problems (Clement, 2000). The first phase of explicit analysis is to use low-inference descriptors of the students work to observe and code the data outside of the theoretical framework (Clement, 2000). The low-inference codes summarized and outlined how the student solved the problem without considering their multiplicative concept stage (e.g., guess and check, created table, covariational reasoning, referenced rate of change, etc.). Video recordings, transcriptions of interview audio, and written work were reviewed multiple times during analysis to create codes. These codes were then condensed into themes that reflected the solution process of the participants. Before moving onto the second phase of explicit coding, a comparison of the low-inference codes between MC2 and MC3 solutions was conducted. The data from the low inference coding addresses our research question, how do undergraduate students reason about and solve optimization problems?

The second phase of explicit coding uses high-inference descriptors to observe student work within the theoretical framework (Clement, 2000). The high-inference descriptors used in this study highlight evidence towards how the participants' operations from their multiplicative concept stages influence and support their reasoning on the optimization problems (e.g., unit decay, iterable composite unit, numerical representation, etc.). Then comparison of high-inference codes between MC2 and MC3 students was conducted. The data from analysis of our high inference codes addresses our second research question, to what extent can the multiplicative concepts be used to explain undergraduate students' reasoning on optimization problems? The comparison of solutions and evidence from the explicit analysis were then synthesized into narratives used in the discussion of the results and analysis.

Results and Analysis

Discussion of the results of this study will address and summarize participant solutions to each question respectively before discussing the solutions connections to the students' multiplicative concepts. For reference, Appendix B provides the correct answers and equations and the given graphs for the charter bus problem and the barn pen problem.

The Charter Bus Problem

Participants were first asked to solve the charter bus problem. Participants were allowed to solve the problem in any way they wanted after presenting the problem. Only after they had reached a conclusion on the solution to the problem did the interviewer ask for additional representation for the problem. Table 4.1 provides a summary of representations students used to model the charter bus problem.

Table 4.1

Participant charter bus problem representations

	Guess and Check		**Table**		**Equation**		**Graph**	
	Used	Accurate	Used	Accurate	Used	Accurate	Used	Accurate
MC2								
Abigail	Yes	Yes	No		Yes	Yes	No	
Diana	No		Yes	Yes	Yes	Yes	Yes	Yes
MC3								
Brian	Yes	Yes	No		Yes	No	Yes	Yes
Clarissa	Yes	No	No		Yes	Yes	Yes	No
Eric	Yes	No	No		Yes	Yes	Yes	No

All five participants in this study began their problem solving process by first calculating how much revenue Marian would make if she had 30 passengers on her bus. They then paused to reread the problem before starting their solutions. Abigail, Brian, Clarissa, and Eric all chose to use a guess and check strategy to solve the problem. Diana chose to write the problem as an equation and then create a table of points based on her equation to find the solution. While Abigail chose the guess and check strategy, she did express that her “first instinct was to put it in the calculator and make a graph”. However, with her given resources, she felt that a guess and check strategy was the most reliable way to solve the problem.

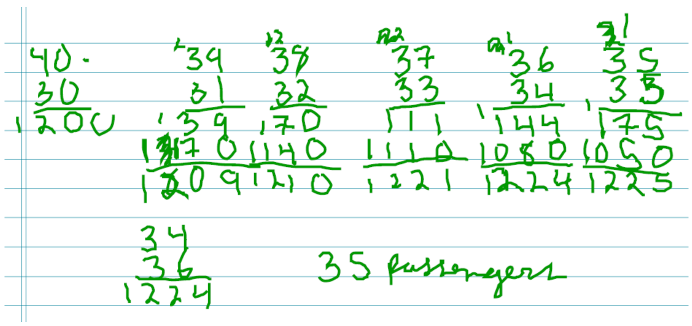
Participants were asked to provide additional representations for the problems if they did not do so in their original work. Abigail, Brian, Clarissa, and Eric were all asked to create an equation and graph to represent the problem. Diana provided an equation, table, and graph in her initial work.

Guess and Check Strategies

While Abigail, Brian, Clarissa, and Eric all produced a guess and check strategy, only Abigail and Brian did so systematically. Systematic guess and check is model based reasoning where the solver applies relational reasoning to situational context to solve a problem (Johanning, 2010). Abigail started with her original work to find how much revenue Marian would make with 30 passengers (see Figure 4.3). Abigail had solved this by multiplying the ticket price (the 40 on top) by the number of passengers (the 30 on bottom). She then took one from the fare price (39) and added one to the number of passengers (31) before finding the total revenue again. She continued this pattern, watching the change in revenue after each change in price. When she reached the point that there were 35 passengers, she said, “oh, I think I’m getting close. We only went up by one dollar this time.” As she went to write the next part of her pattern, she stopped, saying, “Oh yeah, it would be 35 passengers. The next one is the same as the last one. 36 times 34 is the same as 34 times 36.”

Figure 4.3

Abigail’s guess and check written work



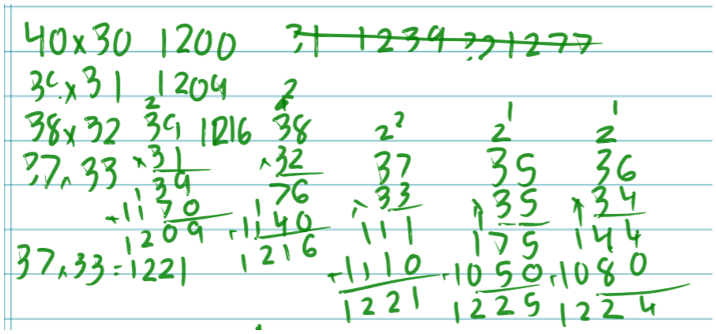
Abigail described her work by saying, “I took one from the price each time [*she runs her finger along her top row of numbers*] and then added one to the people each time [*she runs her finger along the bottom row of numbers*]. Since it repeated here [*she points at the second 34 x 36*], I know that the numbers will keep going down now.”

Brian’s work for his guess and check strategy was similar to Abigail’s. He began writing his pattern as an example of how he was thinking, keeping a column of numbers that was decreasing for the price and a column of numbers that was increasing for the passengers (see Figure 4.4). The row structure on the side served as his work for the calculations that he then placed in the column format to the left. Brian calculated the revenue in this way up to 37. Then he stopped, saying, “So basically what I did was I kept going a little bit, and now the number of the rate of change is decreasing… so it went from plus 9 to plus 7 to plus 5… I’m basically doing a guess and check, which takes a bit of time, but you could get the right answer”. He began working out “35 x 35 = 1225”. After a moments pause he said, “Oh wait. [*he rereads the question*] I think I have the answer.” He then solves “36 x 34 = 1224”. “So this [*he points at*

36x34] is what I think is one more than the maximum number I think she should do. And I found a smaller number, so I want to say it'll be 35 passengers."

Figure 4.4

Brian's guess and check written work



Brian was able to find the correct number of passengers through guess and check and a bit of intuition. He didn't need to check if 34 passengers would yield a smaller revenue than 35 passengers since this was part of his increasing pattern. When he did write his work for "36 x 34", he explicitly named the 36 as the number of passengers and the 34 as the bus fare price. This drop in price was the evidence he needed that his guess of 35 passengers was the correct solution.

Both Abigail and Brian use their systematic guess and check work to model the charter bus problem. Abigail's continual use of the "plus one, minus one" relationship to find the maximum solution to the problem leveraged her relational reasoning. Brian focused on the decreasing rate of change to support his guesses and used additional evidence as support for his conclusion.

Clarissa and Eric did not use their guess and check work to systematically model the problem, but rather to support their own conclusions. An excerpt from the interview with Clarissa is below:

Clarissa: Ok, so if she wants the most money, she'll be getting 30 passengers.

Interviewer: So how much money would she be making with 30 passengers?

Clarissa: That would be 1200.

Interviewer: And how could you check to see if that was the maximum?

Clarissa: I could do the total number, the 48 passengers, to see if that was better or worse.

Clarissa writes "40 x 30 = $1200". She then writes "48 – 30 = 18" and "48 x 22 = $956'.

Clarissa: So it would be $956.

Clarissa was confident in her guess and did not look further into the problem. She also did not realize that her product for 48 x 22 was incorrect. She was satisfied with her initial response and only provided additional evidence for her response at the request of the interviewer. When the evidence provided supported her conclusion, she did not look further into the problem.

Eric initially misunderstood the question, commenting that it must be a "trick question" since she would still make more money even if she lost one dollar for every additional person. He concluded that 48 would be the optimum number of passengers to maximize the revenue as it would fill the bus. However, after clarification, he decided to check his answer of 48. He found that by fully loading the bus, there would be a total revenue of $1056. Below is an excerpt of the conversation following this find:

Interviewer: So is that more or less than what you calculated for just 30 passengers.

Eric: Wait, I marked that out. Let me write that again over here.

Eric wrote "40 x 30 = 1200".

Eric: So if she wants to maximize the revenue, then she would want to have 30 passengers with the caveat that for every passenger over 30, she loses one dollar.

Interviewer: So do you think she would make more money if one more person signed up?

Eric: I don't think that she would. But I can check.

Eric worked out "39 x 31 = 1209".

Eric: Oh! Yes, she would.

Interviewer: So what about one more?

Eric: I imagine that adding one more passenger would lose her money.

Interviewer: Can you check that.

Eric wrote "38 x 32 = 1216".

Eric: She would make 16 dollars more than at 30. So there should be one point at like 38 or 39 where she starts losing money.

Eric paused. Then he wrote "41 x 29 = 1064".

Eric: Yeah it should be 39. At 41 she starts losing money.

Eric was confident with each of his guesses. It was only with prodding from the interviewer that he would check any of his guesses. Eric chose 39 as his final guess after finding that 41 x 29 was a decrease in revenue. This approximation was sufficient for Eric to feel confident in his answer.

Clarissa and Eric both used guess and check as a way to validate and justify their guesses for the problem's solution. Neither systematically modeled the problem in a way that would yield a correct solution. Clarissa was confident in her conclusion that 30 passengers was correct while Eric was satisfied with his approximate answer of 39.

Starting with an equation

Diana attempted to write an equation for the charter bus problem as her way to solve the problem. Diana started by calculating how much money she would make if 30 passengers signed up for the trip. After thinking a minute, she wrote “1200 – 1x” and then “40 – 1x” underneath it. She stopped to think for a minute.

Interviewer: So, what are you thinking about.

Diana: I don’t know.

Interviewer: Ok, so here you wrote “40 – 1*x*” and up here you wrote “1200 – 1*x*”.

Diana: Well, I’m trying to decide which one to go with.

Interviewer: Alright, what are you thinking about this one.

Interviewer pointed to the “1200 – 1x”

Diana: Well, this one is how much it would decrease in total.

Diana then pointed at the “40 – 1x”

Diana: And this one is how much the ticket price changes per person.

Interviewer: Oh, per person. So is that what the x stands for?

Diana: Yeah, it’s a person, so 1 people, 2 people…

Interviewer: So is that 2 people or 32 people.

Diana: In addition to 30 people.

Diana was able to quickly develop two sets of equations she felt represented the problem at hand. Additionally, she accurately identified what the variable *x* stood for in her equations. Diana then moves on to find the revenue if all 48 people bought a ticket for the charter bus (48 x 22 = 1056). She then stated, “I wish I had a calculator. I want to graph it.” When asked what she thinks it would look like, she drew a “bell graph” and explains it’s like the velocity graphs. She then circled the top point and said, “I want to find this point on it.”

Diana decided to see if she could keep working on an equation. She recognized that the “40 – 1*x*” was the price of the tickets and did not reflect the whole problem. When asked which part of her work it represented, she pointed to an equation she had written, “40 – 18 = $22”. When asked where she got the 18 from, she indicated where she had written “48 - 30 = 18” above this work. The instructor then asked what the x represented in her work “40 - 18 = 22”. She indicated the 18.

Interviewer: Ok, so can you rewrite 48 with the 18?

Diana: Well, 18 plus 30 is 48.

Interviewer: Ok, and can we make any of that into the x?

Diana nodded and wrote “x + 30 =”.

Interviewer: Ok so this is?

Diana: Number of people. *Diana writes “# of people” to the left of the equal sign.*

Interviewer: And what was this one? [*indicates “40 – 1x”*]

Diana: Price per person.

Diana wrote “$ for every person” to the left of this work.

Interviewer: So, you have number of people and price per person, what can you do now.

Diana paused, looking confused.

Interviewer: How does that relate to the work you did here. [*Indicates the 48x22 = 1056*]

Diana: You multiply them.

Diana wrote "(40-x)(x+30)".

With a bit of help and by referencing her work with 48 passengers, Diana was able to create an accurate equation to represent the charter bus problem. She then proceeded to multiply the binomials using what she called the FOIL approach (an acronym standing for "first, outer, inner, last" which references using the distributive property of multiplication) the expression to get "$-x^2 + 10x + 1200$". When finished, she started to draw a chart off to the side. She labeled her first column between 0 and 16 and then began filling out the output column by plugging in numbers to her ($40-x$)($x+30$) expression. When she had filled out the chart to 6, she stopped. She circled (5, 1225) and said, "I'm done, you want 35 people."

Diana approached the problem with the goal of creating an equation and graph. Once she was confident in her model for the problem, she used it to find the correct answer using her output from the equation. While finding the initial equation was challenging, she persisted in using her examples to help her accurately represent the charter bus problem as an equation.

Other Equation Representations

Abigail, Brian, Clarissa, and Eric were all asked to provide an equation for the charter bus problem after they had found their solution to the problem. All four participants provided linear expressions as their initial guesses, similar to Diana's initial work with "$1200 - 1x$" and "$40 - 1x$".

Brian was the only participant who did not find an accurate equation for the charter bus problem at any point during the interview. His first guess was the equation "$y = 40x + 30$". Brian kept thinking and talking through his thoughts, commenting that it wasn't quite right, the rate was constantly changing. When asked about why he though the linear equation wasn't quite right, he said, "Because the rate is not… This rate would only go for this equation [40 x 30 =

1200], it wouldn't go for the entire equation". He stopped, and wrote down off to the side "9, 7, 5, 3, 1, …" He then said, "I'm not sure about the slope. It's decreasing by 2 each time. The next one would be… 1/3?" Brian wrote down the 1/3. When asked what he thought was changing each time, Brian replied, "It would be x, because it's changing". After debating back and forth, Brian gave up finding the equation for the problem and focused on what the graph for this problem would look like.

Clarissa provided the equation "$F(x) = 40 - 1x$" as her initial response. She explained the equation was for the price of the fare and that the variable x stood for the extra passengers on the bus. The interviewer then asked if there was a way to represent what she multiplied the fare price by in her previous work. After some thought, Clarissa wrote two equations: "$F(p) = 40p$" where p represented the 30 people, and "$F(n) = 22n$" where n represented the 48 people. The interviewer then asked if she could represent the 48 using the variable x she had previously defined. She wrote "$48 = 30 + x$". The interviewer then reminded her of her initial response and asked if there is a way to model her work using the two equations she wrote. She then wrote "$F(x) = (30 + x)(40 - 1x)$". When asked whether she thought the equation was representative of the charter bus problem, she replied that it "looks complicated" but was "believable".

Eric was the most independent solver of the participants. Eric's first written equation was, "$y = 40(30) - 1p$" where he identified the variable p as additional people over 30. When asked why he multiplied the 30 by the 40, he replied, "to represent the people paying 40 a piece. I know this is not quite right". The interviewer asked him what would happen if they added 1 more passenger according to his equation. Thinking it through, he changed his mind, rewriting the equation to be "$y = 30(40 - p)$". The interviewer then asked if this equation made sense. After a moment of thought, he wrote down, "$y = (30 + p)(40 - p)$". When asked if this equation reflected his previous work, he replied, “Not exactly, but I would say it’s about as close enough as I can handle right now”.

Abigail needed substantial assistance thinking through her equation. Abigail initially wrote the equation “$r = 40p + b$”. Off to the side of her work she noted “$p = x$ people; $r = y$ revenue”. When explaining the equation, she stated, “Yeah, this is what I have so far, but then I’d have to find out the plus over here, the *b*. I know *r* and *p*, and I do have some up here because we know that… [*pause*] Oh, what am I doing?” Abigail then attempted to begin to find the value of *b*, saying, “We know somehow [$r = 40p + b$] equals 1200... So you know it should be [$1200 = 40(30) + b$], if I did this correctly, or was 40 times 30 plus 1?” Abigail stopped working on the equation.

> Interviewer: So you are starting out with this idea that you know how to represent the initial one, she’s going to make 40 dollars for each person up to 30?
>
> Abigail: I forgot about the after 30 dollars. I wouldn’t know how to do that on a graph.
>
> Interviewer: Well, it’s reduced by one for every passenger that is over 30 correct?
>
> *Abigail nodded.*
>
> Interviewer: How do you think you could get your *p* to represent that? Is there anything you did up here [*the guess and check work*] that can be reflected in your equation?
>
> Abigail: I would think I would need to put a minus in there somewhere, but I wouldn't know where to put it to make it always work for every single, one of these [*the individual steps in her guess and check pattern*] So it increases by one... You minus it by one, or no... You reduce. These [*the products*] increased by nine, and then it suddenly increased by... What is that like 10 dollars? 10-11 dollars, then by 3 and then 1. So that's not really a constant pattern either... I feel like it would look something like this, 30 minus

something, but I don't know what it would minus us by to make sure you would always get the right number.

Interviewer: So you're trying to subtract over here [*the row for people starting at 30*], so are you trying to directly influence the *p* and what is the *p*?

Abigail: Amount of people, so… No. The amount of people you add… So maybe you should do it over here, 40 is also changing by the amount of people, so there's a p again, but…

Abigail wrote "30(40 - p)".

Abigail: So if you put the *p* there as well, so it could be, it kinda looks right. But I don't know if that works because 40 minus 30 is just 10.

Interviewer: Ok, so we're getting at what you want, you have the 40 – *p*, and you have 30 over here, but in your work up here [*the guess and check work*] you are increasing the 30. So is there anyway you can mess with the 30?

Abigail: So 30, and that would be plus… however many people, just depends on the amount of people, but I don't know how to put in like plus however many extra people, I guess...

Interviewer: Over here you had 40 minus *p* and you said that reflected the top where it goes down by 1 every time. Can you do the same thing over here [*30*]?

Abigail: I assume so, but I don't know if it would work.

Abigail wrote "(30 + p)(40 - p)"

Abigail: That looks familiar 'cause you can do the FOIL method.

Abigail was able to provide an accurate equation for the charter bus problem by heavily referencing her initial guess and check work and with some assistance from the interviewer.

From here, Abigail simplified her equation to “9p + 1200” as her final equation. Abigail claimed that that her starting and ending equation were equivalent, even after review of her work with the interviewer. When asked which one made more sense to her, the “(30 + p)(40 – p)” or the “9p +1200” equation, Abigail indicated the “9p + 1200” as it “I understand this better as the initial amount goes up by the slope of 9”.

When asked to define the variable p in “9p + 1200”, Abigail replied, “well, I would just plug it in”. After further questioning, she said, “It might be the additional people on the bus”. The interviewer then pointed her back to the expression “(30 + p)(40 – p)” and asked her to define the variable p found there. Abigail replied, “So it’s either how much less money or more people… I don’t know.” The interviewer asked for her to think about her guess and check work to see if that helped. Abigail then said, “it’s additional people over… how much money you’re charging for over 30?”

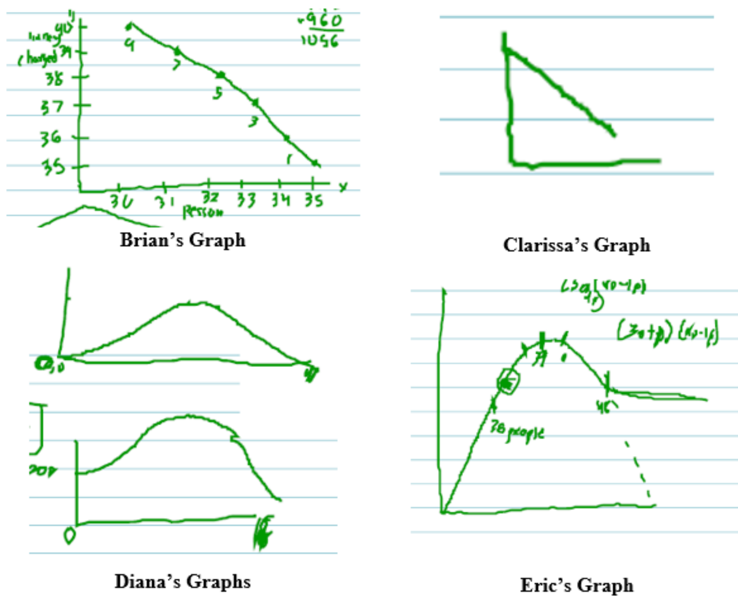
Abigail was able to find the correct algebraic representation for the charter bus problem, but only with help and heavy reference to her initial guess and check work.

Graph Representations

Each participant was also asked what they thought the graph of the problem would look like. Throughout Abigail’s work on the equations, she stated that she did not know how to represent the problem as an equation because she didn’t know how to handle the “after 30” part of the problem. She ultimately did not draw or describe a graph before given the official graph to review by the interviewer.

Figure 4.5

Graph representations of the charter bus problem



Brian's graph reflected the rate of change relationship he had been heavily referencing in both his guess and check and equation writing work. He explained that the y-axis of his graph represented the amount of the fare price and the x-axis represented the number of passengers (see Figure 4.5). He then plotted the points (30, 40), (31, 39), (32, 38), (33, 37), (34, 36), and (35, 35) and labeled the spaces between each point by the change in revenue he saw as he moved along his graph. While this graph does not accurately represent an equation for the charter bus problem, it does accurately represent the covarying relationship between fare price and number of passengers.

Clarissa's graph for the charter bus problem was a decreasing line. She explained that the y-axis was revenue and the x-axis was the number of passengers over 30. When asked, she confirmed that this represented the charter bus problem and her work for her solution.

The top graph Diana provided (see Figure 4.5) reflects her initial explanation for what point she was looking for. She explained it as “a bell graph” where she was “looking for the top point”. The second graph she provided was an adjustment to her initial thought on the shape of her graph after completing her table. Realizing that when her variable x was 0, the starting revenue was 1200, she redrew her graph to represent the change she saw in her work.

Eric drew his graph prior to his work on making an equation for the charter bus problem. Here you can see where he labeled 30 people, the starting point for the charter bus problem. He explained that his x-axis represented passengers and his y-axis represented revenue. He then labeled the maximum as “39”, his initial answer to the problem. At 48, he continued the quadratic with a dotted line, but placed a solid line straight from the 48 mark. He stated that this is where Marian’s revenue would “flat line”. While discussing this graph, he began to change his initial thoughts on his answer. When asked if he thought 38 or 39 would be the maximum, he replied, “I would way 32 would be ideally where you want to be on an economic chart [he circled a point close to his 30 marker on his graph]. The 39 would be right here [indicated the maximum he labeled on the graph] which would be ideally not where you want to be since you are not making as much as what you would”. Eric moved on, satisfied with his answer.

Discussion of the Given Graph

To finish the conversation regarding the charter bus problem, Participants were asked to look at a graph of the equation $y = (30 + x)(40 - x)$. This graph was not presented as the graph for the problem, but rather as a graph and they were asked to interpret this in terms of the problem. All five participants were able to correctly identify what each of the axis represented for the charter bus problem. Abigail, Brian, and Diana identified how the graph correlated with their earlier work on the problem. Diana confirmed that this graph was what she had envisioned during problem solving and Abigail and Brian stated that the shape of the graph made sense since you were technically starting with 30 passengers, not 0.

Eric agreed that this graph looked similar to his. He also identified that the maximum on this graph would be 35. When asked if that made sense, Eric replied, “I imagine if I sat down and thought it through, it probably would, but that would not be what I was guessing. I would probably stick to between 32 and 34 people”.

Clarissa was able to interpret parts of the graph accurately, but made no comments about how the graph was a representation of the problem. She identified that both her graph and the given graph started at the same point (0, 1200), but the given graph went “higher” than her own. She did not change any conclusions about the solution to the problem after the discussion regarding the given graph.

The Barn Pen Problem

Participants were then asked to solve the barn pen problem. When presented the barn pen problem, participants seemed to naturally gravitate towards the idea of the answer being represented by a square pen. Participant solution strategies mirrored their approaches to the charter bus problem except for Abigail’s. Abigail, Brian, and Diana were able to find the correct solutions to the barn pen problem without consulting the provided graph while Clarissa and Eric did not.

Table 4.2

Participant barn pen problem representations

	Guess and Check		Table		Equation		Graph	
	Used	Accurate	Used	Accurate	Used	Accurate	Used	Accurate
MC2								
Abigail	Yes	Yes	No		Yes	No	No	
Diana	No		Yes	Yes	Yes	Yes	Yes	Yes
MC3								
Brian	Yes	Yes	No		Yes	No	No	
Clarissa	Yes	No	No		Yes	No	Yes	No
Eric	Yes	No	No		Yes	No	No	

Four of the five participants used guess and check to solve the problem. All five participants provided either one or two equations for the problem. However, only one participant found a single equation that accurately represented the problem rather than a system of equations. Only two of the students provided graphs when asked. The rest of the participants referenced the graph they were shown on the charter bus problem as a substitution for drawing one.

The Square Pen

All five participants made comments about the square pen after reading the barn pen problem prior to problem solving. Abigail, Brian, and Diana all commented that if the pen was a square, the problem would be "easier". Abigail wrote off the possibility of the answer being the dimensions of a square pen due to the description of the pen as "rectangular" in the problem. When working on representing the perimeter of the pen, she wrote "160/4" before verbally noting that this was wrong because it was the perimeter of a square and not a rectangle. She is the only participant who did not find the dimensions of the pen if the pen was square.

Brian and Diana both chose to find the dimensions of the pen if it was a square as a starting point to their problem. They both found that 160 ft. divided by 3 was 53.3 ft. Brian said this gave him a good reference point as he thought about the possible dimensions of the pen. Diana stated that she did this “to get a number” but “didn’t know why I did a square”. After some thought, she stated, “I guess it’s because [squares] are easier if they have nice numbers”.

Clarissa and Eric both found the dimensions of the pen if it was a square and then stated that these were the optimal dimensions since squares have the largest areas. The 53.3 ft. by 53.3 ft. square pen was their solution rather than a starting or reference point for their problem solving. When asked about their reasoning for choosing a square as the optimal shape of the pen, Clarissa stated, “53 should be correct because it would be, all three [sides] are the same, which means the fourth side over here [*the barn side*] would also be 53, so you multiply them and it would be the biggest”. Eric replied saying, “I guess because it’s familiar, especially for fencing. If you want to max the area, you want a square”.

Guess and Check Strategies

Clarissa and Eric finished their solutions of the problem by multiplying the sides of the square as a way to “check” their solution. Neither considered other possible solutions to the problem or were interested in considering a solution contrary to their initial 53.3 ft. by 53.3 ft. pen.

Brian took a different approach to his guess and check work. After finding his baseline side length for the square pen, Brian took his first guess. He drew a rectangle with a barn on top and labeled the short sides *x* and the long side y. Then he said, “My first thought was just 80, 40, and 40”. He then labeled the two sides “*x*” as 40 and the side “*y*” as 80. He continued, “That was my first thought because that equals 160. I guess I could see what that gives you, but I can change it. I could decrease this number [40] and I would increase this one [80] because that would make it longer. Either that, or I can take a little bit of this [80] and put it into these two [the two 40s]". He wrote "45, 45, 70". Underneath this he wrote "50, 50, 60" and finally "55, 55, 50". Then he stopped and said, "I think this one is right" indicating the "55, 55, 50" that is close to his starting point of "53.3, 53.3, 53.3".

When asked to check this guess, Brian multiplied the sides of each of his pens together. He started with his final dimensions of "55, 55, 50" and worked his way back towards "40, 40, 80". When he finished, he said, "I guess actually the 80, 40 was right". He then wrote "30, 30, 100" and found the area of this pen to be 3000. He nodded, saying "yeah, it's like the bus problem. It goes back down and matches on both sides". He then gestured to his work with the 30 ft. x 100 ft. pen and the 50 ft. x 60 ft. pen. Brian used his knowledge of the relationship between the length of the sides of the pen and the perimeter of the pen to find possible dimensions of the pen and then to determine the maximum area.

Writing and Using Equation Representations

Brian, Clarissa, and Eric were all asked to create equations for the barn pen problem after their initial solutions. Brian first wrote "$40x + 80y = z$". When questioned on what the variable z stood for, he replied "z is the total amount of feet". By referencing the problem, he changed his equation to "$z = x \cdot y$", with z now representing the total area. He was satisfied with this as a representative equation for the barn problem.

Clarissa and Eric both provided two equations fairly quickly. Clarissa began by writing "$P = w + w + l$". She then rewrote this as "$P = 2w + l = 160$". When asked how she could represent the area of the pen, she wrote, "$A = w(l)$". Eric began by writing out the equation for the perimeter for a rectangle, "$2l + 2s = 160$" before saying, "We don't need this side, so scratch this one out" marking out the 2 in front of the "s" leaving "$2l + s = 160$". He then wrote "$ls = A$".

When Clarissa and Eric were asked if they could combine these two equations, both indicated that they “weren’t sure” and “didn’t know”, respectively.

Abigail started by finding an equation to help her solve the problem. Abigail was initially confused and unsure where to begin problem solving, especially after she decided a square was not a legitimate solution to the problem. She drew a representation of the barn and pen labeling the matching sides x, and the other side y, but still wasn’t sure how to move forward. The interviewer asked her to identify what the 160 ft. stood for.

Abigail: The perimeter? I don’t know what that looks like… $2y$ or something

Interviewer: Okay, well how do you find the area?

Abigail: Oh, x times y.

Abigail wrote “xy = Area”

Interviewer: Okay, so how would you usually find the perimeter of a rectangle?

Abigail: x times… like $2x$ and then plus $2y$.

Abigail wrote “2x + 2y = P”.

Interviewer: And what if we took off a side?

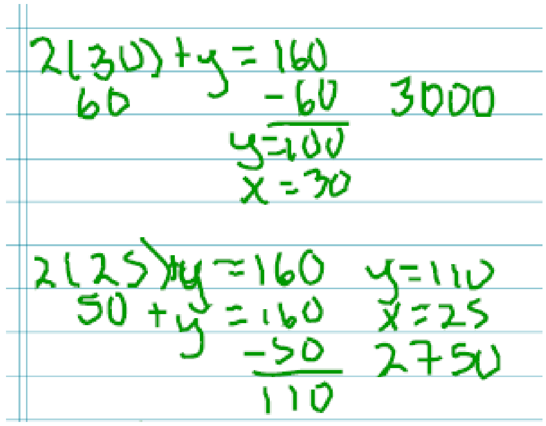
Abigail: So $2x$ plus just y.

Abigail wrote “2x + y = P” and “2x + y = 160”.

While finding an equation for the perimeter was harder than for the area, she was able to use her knowledge of the formula for finding the perimeter of a rectangle, which enabled her to adjust it to fit the pen in the problem. Satisfied with the equation for the perimeter, Abigail began a systematic guess and check process using the equation $2x + y = 160$ to find y at different values of x. She then takes the values of the x and y and multiplies them together to find the area of the pen. Figure 4.6 is an example of how she formatted her work for this process.

Figure 4.6

Example of how Abigail formatted her work



Abigail began by finding the area when *x* was equal to 10. She then increased the value of *x* by 5. After she found the area of the pen when *x* is 35, she stopped and asked, “Does the area keep going up by the same amount each time?” She checked the pattern, noticing that the current increase was by 150 and the previous was by 250 stating, “Oh, I guess not. They were just similar so I’ll just keep going”. She then checked the area of the pen when *x* is 40. Finding that the area is 3200, she said “Oh, I must be getting close”. Abigail found the area of the pen for when *x* is 45 and when *x* is 50. She pointed out that the area has gone down, so the maximum area must be 3200. She noted that the area for the pens when *x* is 45 and when *x* is 50 matches solutions from her earlier work. When asked why that was the case, she replied, “I don’t know, they’re different numbers”.

When attempting to provide the dimensions for the pen, Abigail ran into difficulty with how to interpret the variable *x*. She said “I don’t know, the total *x* is 40 so they are both together 40, or is one of them 40?” After some though, Abigail was asked to reference her equation for the perimeter (2*x* + *y*). She thought for a moment before saying, “it’s [*indicated the “2x”*] 80, so both of them are 40. Then y equals 80, so the barn is 80". Then she labeled a picture of the pen with the respective lengths of the pen sides.

Diana provided an equation that is equivalent to the one outlined in the theoretical analysis ($(2w - 160)w$). She first tried finding a reasonable length for the side by presuming the pen was a square. She quickly rejected this solution as her answer and began looking for another solution.

Diana: I know my thing will be, it'll be equal to 160.

Interviewer: What thing?

Diana: My equation. So that's $2x + x$. No, that would be an x, it would be smaller. So that'll be a y.

Diana marked out the x and wrote a y above it. She then had "$2x + y = 160$".

Interviewer: So why is y smaller?

Diana: I feel like if I were to make this pen, I would make this side longer [*the x side*] to give more breathing room. I just like it better that way.

Interviewer: Ok.

Diana: Oh! This is systems of equations! Is it? I feel like it is…

Diana easily provided a formula to find the perimeter of John's pen. She also recognized the problem as a system of equations problem which none of the other participants ever mention. Diana began to solve "$2x + y =160$" for x, before changing her mind to solve for y. She wrote "$y = 160 – 2x$". She plugged her new y back into her equation [$2x + (160 – 2x) = 160$] and then stopped.

Interviewer: So, you're substituting your equation back into your equation?

Diana: Yes, but now that doesn't look right.

Interviewer: You said system of equations earlier, so is there another equation in this problem?

Diana: What are the dimensions of the pen when you maximize the area?

Interviewer: So, what is this equation over here [$2x + y = 160$]?

Diana: This is the amount of fencing. I subtracted $2x$ from both sides to make it easier. I really thought it was going to work out.

Interviewer: Is this the inside or the outside of the pen?

Diana: The outside.

Interviewer: And what is the outside called?

Diana: The perimeter, but we're looking for the area!

Diana was able to recognize the need to use her knowledge about solving systems of equations to solve this problem. However, while she still recognized that her initial substitution was incorrect, Diana was unable to recognize the need for an additional equation without some outside help even after her own review of the problem's question. After recognizing the need for an equation for area, she wrote "$x \cdot y = area$" and then "$x(160 - 2x)$". She simplified this equation to "$-3x^2 + 160x = area$". This was later corrected to "$-2x^2 + 160x = area$".

Diana paused again after finding this equation. When asked what she wanted to do next, she replied that she "just doesn't know how to make numbers". When asked what the variable x represented, Diana said that they were the x sides of the pen, pointing to both on a drawing she had made. When asked if she wanted to make a table again like in the previous problem, she replied, "I'm not sure what exactly goes in the table". When asked, she explained that you put in the length of the "x sides" and you get out the area. She labeled the columns and began to fill out the first column "0, 1, 2, 3, 4, 5". She first found the area when $x = 0$ and when $x = 1$. After a moment, she paused. “This is really small, it’s going to take forever”. She reflected on the 160 feet available for fencing and decided to “go up by 10 instead of by 1”. She then filled out the first column up to 160 by increments of 10.

When Diana reached $x = 50$ she stopped her work, circling the row (40 | 3200). She explained that, “it started going back down, so this [3200] is the maximum”. She then pluged 40 and 3200 into the area equation to get “$40y = 3200$” and “$y = 80$”. She then wrote out the dimensions as “40 x 80 x 40 ft.”.

Discussion of the Given Graph

All five participants were able to correctly identify the *y*-axis as representative of the area and the *x*-axis as representative of the length of the repeated side. While Abigail, Brian, Diana, and Eric felt the graph was appropriate for the problem, Clarissa did not. She answered all the questions for interpreting the graph, but made no connection between her answer and the graph and did not consider the graph as representative of her solution, even when asked by the interviewer.

In contrast, Eric reevaluated his solution to the problem based on the information from the graph. He was surprised that a 40 ft. by 80 ft. pen was the solution. He drew out this pen and similarities between the rectangular pen with the barn and a square pen made from 160 ft. of fence. He surmised that John was able to double the area by taking the fencing from one side, replacing it with the barn, and doubling the length of the other side of the fence.

Connections Between Student Solutions and Multiplicative Concept Stage

To continue our discussion of the results, we will now outline our findings from the high-descriptor coding analysis. Of the five participants in this study, three of them were able to find the answer to both the charter bus problem and the barn pen problem. Participants’ choice of problem-solving strategy used to solve the optimization problems were not determined by their multiplicative concept stage. However, the way participants engaged with and reasoned about the equations of the problems were influenced by the operations they were leveraging from their multiplicative concept stage. Student successes or limitations as attributed to their operations from their multiplicative concept stage are discussed in this section.

MC2 Student Solutions

Three key behaviors were exhibited by the MC2 students in this study: they were able to successfully quantify additive relationships with unknowns from the problem, they were able to insert a known composite unit into an unknown composite unit, but had difficulty inserting an unknown composite unit into an unknown composite unit, and they relied heavily on numerical examples to build the expressions for the problems. These behaviors can be explained by the underlying operations available to MC2 students.

The first behavior that the MC2 students displayed was the ability to successfully quantify additive relationships with unknowns. Both Abigail and Diana, were able to successfully quantify the "$40 - x$" and "$30 + x$" relationships from the charter bus problem which is supported by their MC2 operations. For example, to conceive of the ticket price as "$40 - x$", the student first needs to readily conceive of an unknown as a unit of units, which requires operating with at least an MC2 (Hackenberg et al., 2017). Then they need to be able to understand the ticket price as a unit of 40 containing an undetermined amount of leftover units and an unknown number of "passengers over 30". This requires the student to make an additive comparison within a two-level unit structure between two unknowns within the unit of 40. MC2 students "construct additive comparisons as an assimilatory quantity" (Ulrich, 2016a, p. 38) which supports the additive reasoning necessary to conceive of the ticket price in this way.

Diana and Abigail's work on the charter bus problems provides evidence of the ways MC2 students quantify additive relationships in these problems. Diana wrote two expressions at the beginning of her problem solving, "1200 – 1*x*" and "40 – 1*x*". When asked for clarification on what these expressions represented, she stated that the "40 – 1*x*" relationship represented the ticket price for the bus route. She was readily able to conceive of the ticket price as a two level unit structure of 40 containing an unknown number of people and a leftover ticket price. She was also able to conceive of the number of passengers as "*x* + 30" separately from her understanding of ticket price as shown when she used her work finding the total revenue if the bus was full to write an expression. After defining the expression for the ticket price of the problem, Diana found the revenue if there were a total of 48 passengers. By doing so, she was able to track the 18 in her problem to define the number of passengers as "*x* + 30". She understood that the number of passengers was a composite unit representing an undetermined number that contained 30 passengers plus an unknown number of additional passengers. Similarly, Abigail defined "*p*" as the amount of people and was able to provide the expression "30(40 – *p*)". She was able to quantify the ticket price just as Diana had. Additionally, after adjusting her definition of the variable "*p*" to "the number of people over 30", Abigail was able to define the number of passengers as "30 + *x*". Their ability to quantify these relationships from the problem are evidence of their MC2 operations supporting their ability to represent these additive relationships with unknowns.

The second behavior the MC2 students displayed was a difficulty inserting a composite unit of unknown size into a composite unit of an unknown size. While Abigail and Diana were able to quantify the additive relationships in the problems, they had difficulties when coordinating those relationships multiplicatively to create an expression. Evidence from Abigail and Diana’s work suggested that the MC2 students could insert a known (e.g., 30) into each unit of a composite unit with an unknown relationship (e.g., $40 - x$), but could not insert an unknown relationship (e.g., $30 + x$) into each unit of a composite unit with an unknown relationship (e.g., $40 - x$). To create the expression for the charter bus problem ($(40 - x)(30 + x)$), students must iterate a unit of unknown length (e.g., $30 + x$) an unknown number of times (e.g., $40 - x$), creating a composite unit of an unknown where an unknown has been inserted into it. To reflect on this, students would need to have constructed iterable composite units, which is available to MC3. MC2 students reflect on iterable units of 1, not iterable composite units (Ulrich, 2016a). Additionally, their ability to construct multiplicative relationships happens during activity, rather than being anticipatory (Ulrich, 2016a)., MC2 students’ ability to iterate composite units in activity allows them to insert a known into each unit of a composite unit with unknown relationships.

Abigail and Diana’s work on the charter bus problem provides evidence for this difficulty. Abigail struggled with translating her guess and check work to an expressions. She knew that she needed to represent the “plus one, minus one” relationship for the problems, but struggled with how to represent this simultaneously in a single expression. She talked through her thinking about needing to subtract something and was able to provide the expression “$30(40 - p)$” to represent the problem. Abigail was generally comfortable with this expression, though she did express that it had issues when plugging in numbers. When asked if she could adjust the 30, which represented the number of people she said, “So 30, and that would be plus… however many people, just depends on the amount of people, but I don’t know how to put in like plus however many extra people, I guess”. Abigail was verbally quantifying the “$30 + p$” relationship, but was struggling with inserting this relationship into her expression. When asked if she could replace this idea with the 30 from her expression, she said, “I assume so, but I don’t know if it would work”. Abigail was uncomfortable with the idea of multiplying two relationships of unknowns with one another, but goes ahead and writes the new expression ($(30 + p)(40 – p)$) anyway.

Abigail was also unable to reflect on this multiplicative relationship after problem solving. After using the FOIL method to multiply the expression, she stated that she preferred her new equation “$9p + 1200$” to her original. When asked to redefine what the “$(30 + p)(40 – p)$” expression stood for, Abigail could not remember what the relationships in the equation stood for stating, “So it’s either how much less money or more people… I don’t know”. Without the ability to reflect on iterable composite units, understanding her expression became difficult for her without the use of a numerical example.

Diana was able to provide the expression “$40 – 1x$” at the beginning of her problem solving. However, she had difficulties in trying to represent the changing number of passengers in the problem. She understood that “$40 – 1x$” did not represent the entire revenue, but wasn’t sure how to reflect the changing number of passengers with the whole problem. To try and figure out where to go from her ticket price expression, Diana worked out the revenue if there were 48 passengers. Through this she was able to quantify the “$30 + x$” relationship representing the passengers. However, she was still unsure of how to combine the two expressions together. After labeling the two quantities as ticket price and number of passengers, she was able to use her written work to provide an accurate expression ($(40 – x)(30 + x)$). The discovery process she went through to create the final expression provides evidence that this insertion of an unknown into another unknown was done in activity. Diana was able to use the labels she had for her expression to simplify and maintain the relationship between the unknowns in her expression after completing her table and finding a solution to the problem.

The barn pen problem created a lot more difficulties for the MC2 students to complete since they had to take a composite unit for the perimeter of the pen that contained two unknowns and translate this into a composite unit for the length that contained the perimeter of the pen and the width twice. They then needed to use this unit to substitute it back into the area equation for the pen by inserting the "$160 - 2w$" relationship into the area "w" times. This is more difficult to conceptualize than the expression from the charter bus problem due to the required use of substitution where the student must replace one variable of an equation (e.g., l) with another expression that is equivalent to that variable (e.g., $160 - w$). Abigail avoided this difficulty by not using any type of substitution, but strictly using her two equations "$160 = l + 2w$" and "$Area = l$ x w" to systematically search for the best solution. In contrast, Diana did use substitution to complete the problem. However, this was instigated by her recognition of the familiar procedure of "substitution" not her anticipation of the need to insert an unknown into another unknown. This is shown when she substituted the adjusted equation "$y = 160 - 2x$" back into her original equation "$160 = y - 2w$". She had to reestablish the relationships from the problem to identify the second equation in the system ($A = xy$) for her to use substitution correctly.

The third behavior the MC2 students displayed was a reliance on numerical examples to build numerical expressions. For Abigail and Diana to build an expression to represent an unknown being inserted into an unknown, they relied heavily on numerical examples. This is clear on their work with the expression for the charter bus problem. MC2 students do not anticipate the creation of multiplicative comparisons as quantities and have to build them in activity (Ulrich, 2016a). This can make it difficult to represent multiplicative relationships with an equation as was seen in Abigail and Diana's avoidance of and procedural use of substitution on the barn pen problem, respectively. In previous research surrounding MC2 students' ability to write equations representing multiplicative relationships, they showed heavy reliance on numerical and visual examples to establish the relationships they were trying to represent in their equations (Hackenberg et al., 2017). Evidence from Abigail and Diana's work shows how they leveraged their numerical examples to build expressions for multiplicative relationships between unknowns.

Abigail struggled to translate her guess and check work into an expression for the charter bus problem. She understood that she needed to represent the "plus one, minus one" relationship from her original work in her expression, but was struggling to do so. To help her establish the relationships in the problem, the interviewer repeatedly asked her to refer back to her guess and check work. When doing so, she would define the "plus one, minus one" relationship by running her fingers down the respective rows. Abigail explicitly discussed the idea that she would need to multiply two "changing" quantities together. When asked what the variable "p" represented she stated, "Amount of people, so… No. The amount of people you add… So maybe you should do it over here [*she points to the row for passengers starting at 30*]. 40 is also changing by the amount of people, so there's a p again". She then provided the expression "$30(40 – p)$", and checked it as a solution using her work from her guess and check strategy. When this did not match her previous work, she had to return to her numerical example. She recognized that the "$40 – p$" represented the top row of her work where she was decreasing the ticket price by one every time. By reflecting on how this mirrored her work with the number of passengers increasing by one in the second row, Abigail was able to adjust the 30 from her expression to "$30 + p$" as a representation of the second row of her work. Abigail's process to build her equation was sequential. While she was able to quantify both of the additive relationships in the problem and even expressed the need to multiply a changing number of passengers with a changing ticket price, she could not anticipate the quantification of the multiplicative relationships between two unknowns and was only able to provide the "30(40 – *p*)" expression. She had to reference her numerical example to build the multiplicative relationship between unknowns in activity. We hypothesize that without this numerical example to reflect on, Abigail would have struggled to move past her original guess at a linear equation ($r = 40p + b$) for the problem.

Diana's use of numerical examples was explicit in her expression writing. She easily quantified the ticket price for the problem, but struggled to create an expression that represented the revenue. This was due to her ability to anticipate the quantification of an additive relationship supported by her MC2, but the inability to anticipate the quantification of a multiplicative relationship in activity which would require an MC3. To help her create an expression, Diana decided to solve how much money Marian would make when she had 48 passengers. This gave her three examples to work with: "48 – 30 = 18", "40 – 18 = 22", "22 x 48 = 1056". By comparing "40 – 18 = 22" to her expression for the ticket price, "40 – 1*x*", Diana was able to use the 18 to define "*x*". She then used this knowledge to rearrange "48 – 30 = 18" to "18 + 30 = 48" and then replace the 18 to get "30 + *x*" which she identified as "number of people". She then used the "22 x 48 = 1056" to justify her multiplication of the ticket price and the number of people to establish the expression "(40 - 1*x*)(30 + *x*)". Diana relied heavily on her numerical example to establish the quantification of the number of passengers and to justify the multiplication of the two unknowns. This process was sequential, requiring first the defining of the ticket price, then the example for how to solve for the revenue, then the substitution of number 18 for the variable "*x*" to discover how to rewrite the number of passengers in terms of "*x*", and finally, a reflection on her solution to recognize the need to multiply her two quantified expressions together. By leveraging her numerical example, she was able to build the expression needed for the charter bus problem and simplify the units used in order to overcome the limitations created by her MC2 that made it difficult for her to insert an unknown into an unknown unit. Diana's lack of a numerical example on the second problem led to her reliance on a procedure to create an equation for the problem.

The MC2 students in this study were able to successfully quantify the additive relationships between unknowns in the problems, but struggled with inserting an unknown into an unknown to create accurate representations of the problems. Their use of numerical examples allowed them to build these relationships into expressions through reference, but without these examples, they relied on familiar procedures and equations to work through the problems.

MC3 Students

The MC3 students' operations allowed them to insert an unknown into another unknown unlike the MC2 students. MC3 students anticipate and reflect on iterable composite units (Ulrich, 2016a). They also anticipate the creation of multiplicative comparisons as quantities (Ulrich, 2016a). This supports their reasoning about the expression for the charter bus problem and allows them to conceptualize a unit of unknown size being iterated an unknown number of times. Brian did not reason about the equations in the problems in a way that showed evidence he was using his MC3 operations but rather his own understanding of what equations should look like given his knowledge of functions. Additionally, none of the MC3 students considered the need for a single equation for the barn pen problem to be necessary and so this work will not be discussed in this section.

Clarissa initially wrote the equation “F(x) = 40 – 1x”. She was able to explain that this represented the price of the fare when “x” number of additional passengers boarded the bus. When asked if this represented her work solving the problem where she found the revenue at 30 passengers and then checked her answer by finding the revenue for 48 passengers, Diana provided two more equations: “F(p) = 40p” and “F(n) = 22n”. Her variables for these two equations now stood for a specific number of passengers, 30 and 48 respectively. However, when she was asked to rewrite 48 passengers using her original variable “x”, she quickly defined it as “48 = 30 + x”. Additionally, she was able to use this to write her final equation “F(x) = (30 + x)(40 – 1x)”. Clarissa said that this was a “complicated” but “believable” representation for the problem. This shows evidence that she was comfortable with the idea of allowing the multiplication between two unknowns in the problem and could reflect on this equation without needing to reestablish the relationships within the problem. Eric was able to quickly establish the equation *“y* = 30(40 – p)” from his initial linear guess for the equation. After asking him to reflect on this equation as an accurate representation of the charter bus problem, Eric rewrote this to be “y = (30 + p)(40 – p). Eric was able to easily reflect on his original equation to identify the need to adjust his known unit into an unknown unit. While neither Clarissa nor Eric were completely sold on the accuracy of these equations, both were able to flexibly adjust their equations to represent the relationship found in the problem.

While MC3 students can use numerical examples to help them build the equations for the charter bus problem, they don’t rely on them like the MC2 students did. MC3 students’ ability to anticipate the quantification of multiplicative comparisons(Ulrich, 2016a) supports their ability to write an equation for the charter bus problem. This allows them to anticipate the need to write an expression that reflects the multiplicative comparison found in the revenue of the charter bus problem (e.g., $(40 - x)(x + 30)$). This anticipation makes them more independent from numerical examples when writing equations with multiplicative relationships between unknowns.

Clarissa spent a small bit of time on her numerical example as a way to move past her equation that only represented the ticket price. Her initial representations of her work ("$F(p) = 40p$" and "$F(n) = 22n$") reflect her anticipation of representing the multiplicative relationship in the problem she was working towards with the number representing the ticket price and the variable representing the number of passengers. By asking her to create an equation for the number of passengers in terms of her original variable "x" Clarissa was able to smoothly transition from "$48 = 30 + x$" to "$F(x) = (30 + x)(40 - 1x)$". While Clarissa was leveraging her previous work to create her equation, she wasn't actively using the numeric examples to build her understanding of the relationships in the problem or as a justification for multiplying two unknowns together. Eric didn't refer to his previous work at all during problem solving and just mentally reflected on his equations to make adjustments he thought necessary to create an accurate equation. This independence was supported by his anticipation of the multiplicative comparison for the revenue and his ability to insert an unknown composite unit into another unknown composite unit as discussed previously.

The MC3 students' ability to reflect on iterable composite units and their ability to anticipate the need to quantify multiplicative relationships (Ulrich, 2016a) supported their work creating equations for the charter bus problem. They were able to establish these multiplicative relationships between unknowns without relying solely on numerical examples to build them.

Discussion

Undergraduate studies focus on quantitative reasoning for entry-level courses to develop skills useful for students pursuing any career (Elrod, 2014; Lusardi & Wallace, 2013; Wolfe, 1993). Optimization problems provide a place for students to reason quantitatively, algebraically, and covariationally (Thompson & Carlson, 2017). The participants' solutions to optimization problems in this study provided examples of a spectrum of algebraic, relational, and quantitative reasoning through their use of systematic guess and check, algebraic representations, graphical estimations, and tables. Success in solving these problems stemmed from the persistence to thoroughness and use of relational thinking in conjunction with their available operations from their multiplicative concept stage.

Results from this study support prior research on MC2 and MC3 students' construction of quantitative structures and algebraic equations (Hackenberg et al., 2017, 2021; Olive & Cagalayan, 2008; Ulrich, 2016a). Abigail and Diana, each of whom had constructed only an MC2, were able to successfully quantify the additive relationships from the problems, but struggled with inserting a unit of unknown size into another unit of unknown size. Their use of numerical examples to construct and build the equation for the charter bus problem led to their success in representing these problems algebraically. Their inability to reflect on iterable composite units created many difficulties in maintaining the relationships they built in their problem solving. The MC3 students', Clarissa's and Eric's, creation of equations for the charter bus problem were supported by their ability to construct and reflect on iterable composite units and their ability to anticipate the need for multiplicative relationships within the contexts of the problems.

Clarissa's and Eric's lack of success in finding the solutions to the optimization problems did not stem from a lack of ability, but rather an oversimplification of the context that led them to believe that they did not need any additional reasoning. Their solutions served to support their initial misunderstandings towards the context of the problem rather than any quantitative or relational reasoning. However, these estimations aligned with their idea of how “economics”, “fencing”, and “geometry” worked and so they did not challenge their own solutions. Their ability to quickly generalize the problems using variables proves they are more than capable of completing the procedures necessary to solve the problem. Instead, they saw no need to follow these procedures. This confidence went so far as for Clarissa to treat the given graphs as inaccurate representations of the problem.

The third MC3 student’s, Brian’s, work displayed his ability to flexibly move and use relationships in his constructed quantitative structure to reason strategically about the problems. His struggles in creating equations stemmed from his focus on attempting to represent the change in the rate of change rather than generalizing the covarying relationship within the problems.

Limitations

This study has explored the participants’ solutions to optimization problems and how their multiplicative concept stages influenced their mathematical reasoning. However, it is important to note the limitations of this study when moving forward with research. This is a collective case study of five undergraduate students from entry-level math courses that have an MC2 or an MC3. While more students were planned to be used as participants, low interest from students, time constraints for participants and researchers, and concerns regarding COVID-19 exposure limited the response rate of students. While participant numbers were limited, the data collected from those that participated are indicative of their knowledge and mathematical reasoning. These students each have their own experiences in life and mathematics that influence their ability to solve the problems, and clinical interviews can only provide a snap-shot of the students’ understanding at a specific time (Clement, 2000). Additionally, no MC1 undergraduate students volunteered to participant in the clinical interviews for this study. While this is representative of the undergraduate student population (Boyce et al., 2021; Davenport et al., in preparation), it does not allow the exploration of MC1 student solutions to optimizations as a part of this study.

Implications, Future Research, and Conclusion

The results of this study add to our understanding of undergraduate student solutions to optimization problems without the use of derivatives and how their multiplicative concept stage influences undergraduate students' ability to represent covarying multiplicative relationships between unknowns. This study is a continuation of research on the connections between students' ability to construct and coordinate unit structures and their ability to write equations representing the multiplication of two unknowns (Hackenberg et al., 2017, 2021) and their ability to construct and reason about quantitative structures (Olive & Cagalayan, 2008).

The results of this study indicate that MC2 students build multiplicative relationships in activity with the help of numerical examples that allow them to insert an unknown composite unit into an unknown composite unit. However, MC2 students have difficulty reflecting on these quantitative structures as they do not work with iterable composite units which is needed to reflect on these equations. MC3 students do work with iterable composite units which supports their ability to write these equations with little to no reference to numerical examples. Future research should explore additional ways students' reasoning about dynamic covarying systems of equations are influenced by their operations from multiplicative concepts stages when working with more than two equations or covarying quantities. The results also displayed the quantitative, relational, and algebraic reasoning necessary for representing and solving optimization problems (Thompson & Carlson, 2017). This reasoning is available to both MC2 and MC3 students provided they are willing to be persistent and thorough in their application of quantitative reasoning.

Research on the multiplicative concepts continues to expand our understanding of the complex interaction between the multiplicative concept stages and college mathematics. Finding ways to support students' conceptual understanding of system of equations and covariation is an important part of improving their quantitative reasoning. Results from this study should be used to improve instruction of MC2 students in quantitative reasoning courses and serve as examples of just how successful MC2 students can be in mathematics.

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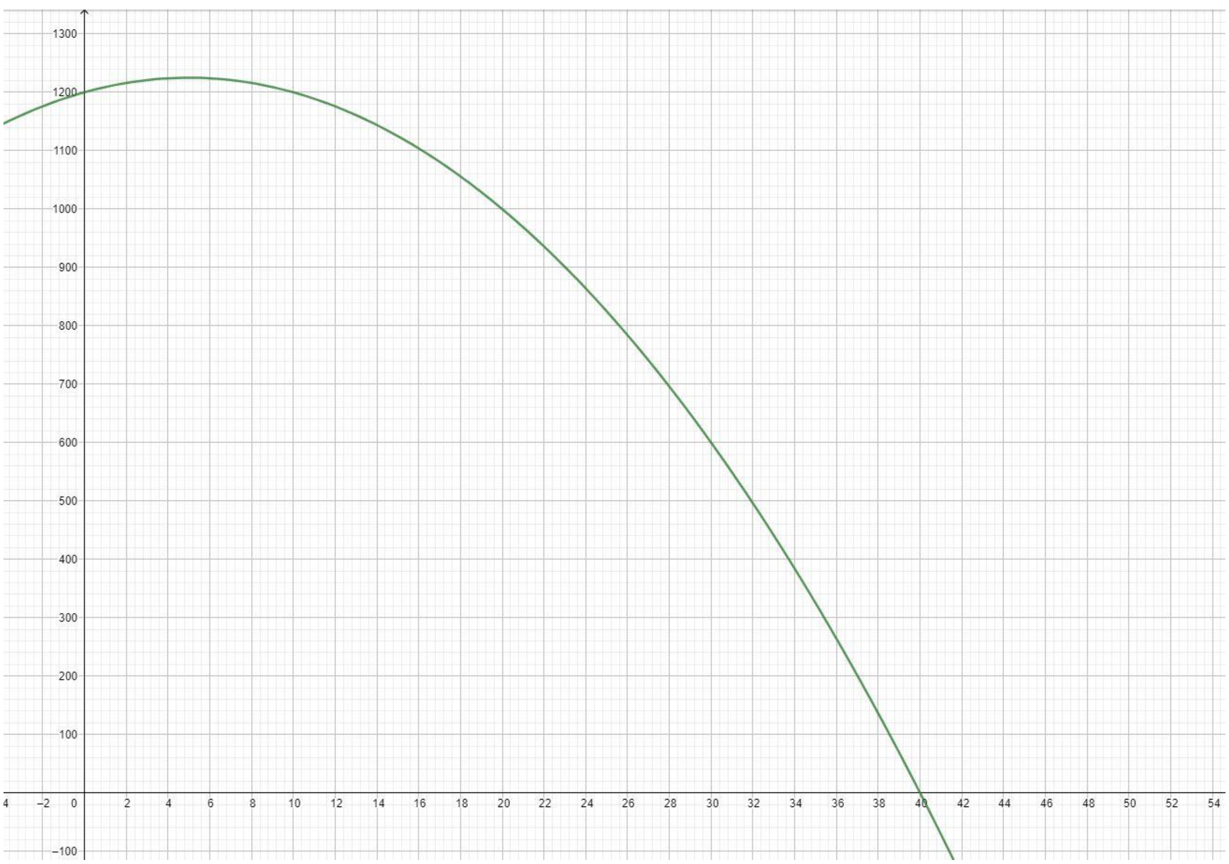
Appendix B

The charter bus problem

Marian owns a charter bus company offering a route to the neighboring city that charges \$40 per person if up to 30 passengers sign up for the trip. If more than 30 passengers sign up, the fare for every passenger is reduced by \$1 for every passenger in excess of 30. The bus can only hold up to 48 passengers. How many passengers does Marian want to sign up for her charter bus route if she wants to maximize her revenue for the trip? (Answer: 35 people)

Equation Representation: $y = (30 + x)(40 - x)$; $y = -x^2 + 10x + 1200$

Graph Provided:

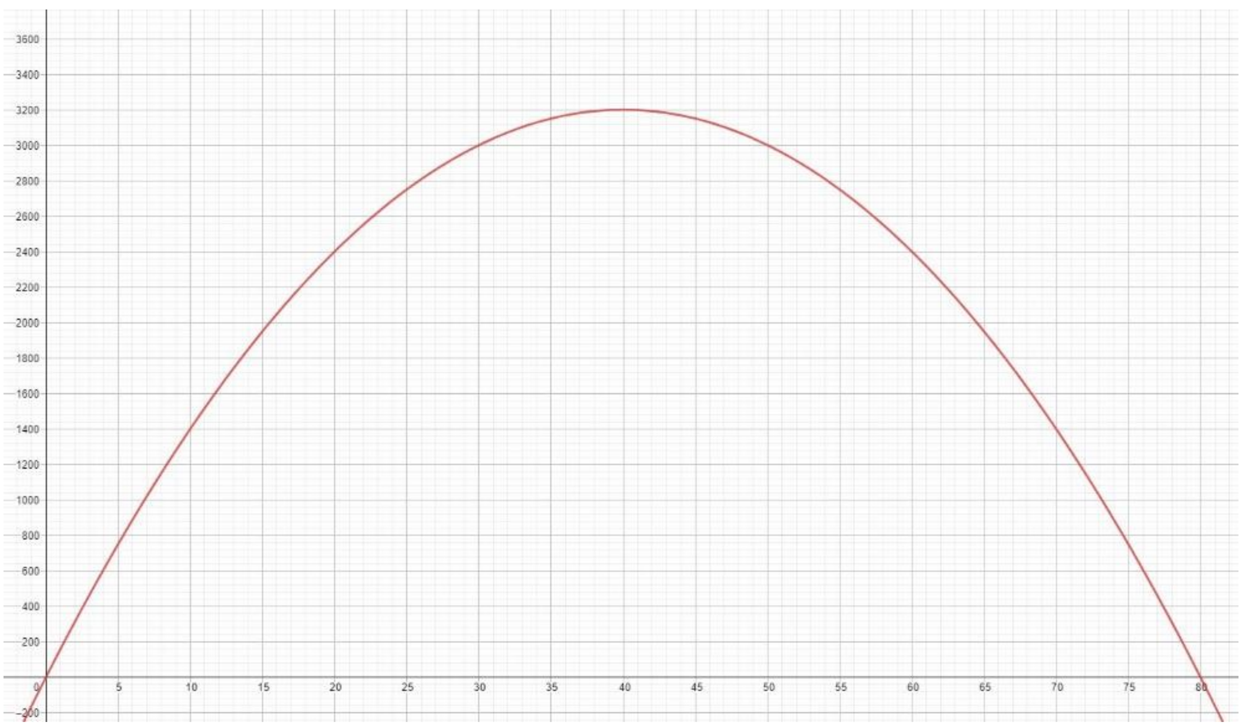


The barn pen problem

John wants to build a rectangular pen next to his barn. To try and maximize his resources, he decides to use one side of the barn as a side of his pen. If he has 160 ft worth of fence available to build his pen, what would be the dimensions of his pen if he maximized the area? (Answer: length: 80 feet, width: 40 feet).

Equation Representation: $y = x(160 – x)$ $y = 160x – x^2$

Graph Provided:



CHAPTER V

UNDERGRADUATE STUDENTS' MATHEMATICS IDENTITY AND THEIR MULTIPLICATIVE CONCEPT STAGE

Target Journal: Child Development

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Abstract

Effective teaching strategies are important in supporting students' identification as a "mathematics person" (Hodge & Harris, 2015). This study explores the connection between how undergraduate students discuss their mathematics identity and their multiplicative reasoning. Forty-three participants participated in a Undergraduate Multiplicative Concept Assessment that also asked them to indicate whether or not they considered themselves as a "mathematics person". Five of these participants were interviewed about their mathematics identity. Results from interviews showed that the MC2 students generally found math difficult to understand conceptually and did not identify as a "mathematics person". The MC3 participants identified as a "mathematics person" and gauged

their success in mathematics by how quickly they could set up and solve a mathematics problem.

Introduction

The mathematics education field has seen a rise in identity research for the last decade (Darragh, 2016; Graven & Heyd-Metzuyanim, 2019). Content-specific identities (e.g., mathematics identity, science identity, engineering identity) have been show to strongly correlate with students' academic performance (Bohrnstedt et al. 2020; Sonnert et al. 2020) and choices in science, technology, engineering and mathematics (STEM) related fields (Cribbs et al., 2020; Godwin et al., 2016). Students' mathematics identity and self-perceptions are related to their persistence in mathematics, their likelihood to pursue a mathematics career (Cribbs et al., 2020) and affects the number mathematics courses they choose to take (Simpkins et al., 2006). Undergraduate students' mathematics identity influences many of the outcomes and choices they make related to their experience in college mathematics and their careers.

The education initiative report *Adding it all up* (National Research Council et al., 2001), listed one of the five strands of mathematical proficiency as students' positive disposition towards mathematics and the National Council of Teachers of Mathematics (NCTM) encourages mathematics teachers to promote curiosity, confidence, persistence, and flexibility in their lesson plans and problems (NCTM, 2014). Experiences in mathematics classrooms play a large role in the development of students' mathematics identity.

Effective teaching strategies in mathematics have been shown to support equity and student identification with mathematics (Hodge & Harris, 2015). Effective strategies are largely dependent on how the student is able to understand, learn, and interact with the taught strategies. Research on modeling student thinking in mathematics has shown that students develop knowledge in schemes (Piaget, 1952; Steffe, 1992: von Glasserfeld, 1995). These schemes are determined by how the student recognizes a problem, how they mentally operate on the problem, and what they expect the outcome of their work to be (von Glasersfeld, 1995). Research on the multiplicative concepts (schemes modeling how students construct and coordinate multi-leveled unit structures) shows an influence between their multiplicative concept stage and how they reason about fractions (Hackenberg, 2007; Steffe, 2001), unknowns and equations (Hackenberg, 2013; Hackenberg et al., 2017), proportions (Steffe et al., 2014), derivatives (Byerly, 2019), and measurement quantities (Steffe, 2013). Research has shown that over half of their undergraduate student participants in courses taken before calculus have not constructed the scheme necessary to conceptually understand derivatives, rate of change, or proportions (Boyce et al., 2021; Davenport et al., in preparation). Negative and positive experiences in learning in the classroom may be influenced by their multiplicative concepts stage. These experiences in turn inform how a student perceives their mathematics identity.

Current research on mathematics identity and the multiplicative concepts have not looked at a connection between these two areas of research. This study explores the connection between undergraduate students' multiplicative concept stage and their mathematics identity. The research question for this study is

- How do students with different multiplicative concept stages describe their mathematics identity?

Theoretical Framework

Identity research began with Mead's (1934) multilayered, dynamic understanding of self through created action and Erickson's (1950; 1968) stable, self-determined, singular definition of identity. Both frameworks aimed to study and define the inherent idea and perception of one's “self”. Cobb and Hodge’s (2011) proposed that identity research centered around three different ways of framing identity: normative, core, and personal. This study uses core identity, a macro-identity approach, to define mathematics identity. Core identity is defined as one’s “enduring sense of who they are and who they want to become” (Cobb & Hodge, 2011, p. 189). The enduring nature of core identity differs from other, approaches to identity research that focus on moment-to-moment changes.

People can have multiple identities or “selves” that vary by the context they are viewing themselves through (Gee, 2000). As such, a person may identify as a “mathematics person”, a “science person”, a “musician”, a “student”, and an “athlete” all at the same time. Each of these roles constitutes a different identity core to their sense of who they are (Godwin et al., 2020). In addition, these identities are not mutually exclusive but influence each other. In this study, we define mathematics identity as how students view themselves in relation to mathematics, based on their perceptions of their experiences with mathematics (Enyedy et al., 2006). As such, we can discuss students’ mathematics identity as how they view themselves as “doers of mathematics” in the context of their mathematics experiences (Nasir, 2002, p. 214). Drawing on these theories and related literature in the field, we use four sub-factors of mathematics identity (recognition, interest, competence, and performance) to explore how undergraduate students discuss different aspects of their mathematics identity.

Factors of Identity

An explanatory framework for mathematics identity identifies recognition and interest as sub-factors of mathematics that directly inform their identity and competence/performance as a sub-factor that indirectly affected their identity as mediated through recognition and interest (Cribbs et al., 2015). *Recognition* is defined as how an individual perceives their own and other’s views of themselves in relation to mathematics (Cribbs et al., 2015). Prior research shows that teacher and parent views of their children's ability in mathematics influence their self-perceptions and achievement in mathematics (Gunderson et al., 2012). Examples of recognition or lack of recognition include a teacher telling a student they are good at mathematics, parents praising their child's mathematics grades, the student saying they are a math person, or a statement that no one had ever praised them for doing well in math.

Interest is defined as an individual's desire or curiosity to learn and do mathematics (Cribbs et al., 2015). Students' interest in mathematics influences how engaged and motivated they are when doing mathematics (Frenzel et al., 2010). When discussing their interest in mathematics, students may discuss problems or subjects they enjoyed, how excited they are by the possibilities mathematics provide, or how bored they are when doing mathematics.

While prior research did not quantitatively differentiate between competence and performance (Cribbs et al., 2015), the way individuals discuss these concepts may differ as shown in qualitative research (Carlone & Johnson, 2007). Thus, these sub-factors will be considered separately in the current study. *Competence* is defined as an individuals' beliefs about their ability to understand mathematics. Students who talk about their competence may say that they "just get" math or they really struggle with understanding why you use different procedures. Students' competence is linked to their goals as students (Ferla et al., 2010) and their performance in mathematics (Blecker & Jacobs, 2014; Bouchey & Harter, 2005). *Performance* is defined as the individual's beliefs about their ability to do well in mathematics (Cribbs et al, 2015). These beliefs are linked to their motivations and actual performance in mathematics (Pajaras & Graham, 1999). Students may talk about their grades in mathematics, the difficulty they have doing mathematics, or how fast they can perform calculations.

Identity is informed by a student's experiences and perceptions of their own abilities in mathematics. Students ability to reason multiplicatively and construct and coordinate multi-layered unit structures is foundational to how they interact with and solve problems higher mathematics problems. We hypothesize that student experiences are influenced by their multiplicative concept stage where frustrations or relative ease (in comparison to their peers) in learning mold their views of themselves as a mathematics person.

The Multiplicative Concepts

Students' unit coordination schemes, or multiplicative concepts, are the student's ability to construct and coordinate multi-level unit structures that ties directly to their thinking on multiplication and division problems (Hackenberg & Tillema, 2009; Steffe, 1992; Steffe, 1994). Units in this context refer to standard and non-standard units of measure (Ulrich, 2015). Research has found evidence of students developing the multiplicative concepts as early as second grade (Kosko & Singh, 2018). There are three stages of the multiplicative concepts that are defined by the level of unit the student assimilates with (initially recognizes; von Glasersfeld, 1995) prior to problem solving.

A student who has assimilated with one level of unit has developed the first multiplicative concept stage (MC1, Hackenberg & Tillema, 2009). They recognize one level of unit and can construct two in activity (during problem solving; Hackenberg & Tillema, 2009). If they were asked to solve 5 x 6, they would recognize the unit of 5 and then insert 6 into each 5 in activity. These students rely on physical representations of skip counting to keep track of their work. Problems involving more than two levels of units (e.g. find how many inches are in 3 yards) can be difficult.

A student who assimilates with two levels of units has developed the second multiplicative concept stage (MC2; Hackenberg & Tillema, 2009). They recognize two levels of units and can construct three in activity (Hackenberg & Tillema, 2009). If they were asked to find how many inches were in 1 yards, they would first recognize that there are 3 feet in 1 yard. Then they can insert 12 inches into every foot to find a total of 36 inches (Hackenberg et al., 2021). However, without assimilating with three levels of units, these students lack the flexibility with units needed to support more complex reasoning such as proportional reasoning (Ulrich, 2016). These students tend to struggle when reasoning about multiplicative relationships between unknowns, often using variables as placeholders for numerical examples during equation writing rather than treating them as a unit of variable length (Hackenberg et al., 2017). In Byerly's (2019) study on Calculus students conceptual understanding of rate of change and derivatives, she found that students who only constructed an MC2 tended to prefer memorizing procedures for problem solving than conceptual learning as the concept behind rate of change was more difficult and frustrating to learn.

A student who has assimilated with three levels of units has developed the third multiplicative concept stage (MC3; Hackenberg & Tillema, 2009). They recognize three levels of units and can construct four or five in activity (Hackenberg & Tillema, 2009). If they were asked to find how many inches were in three yards, they would first recognize that there are 12 inches in a foot and 3 feet in a yard, so they need 3 yards of 3 feet of 12 inches (Hackenberg et al., 2021). MC3 students flexibly move between their unit structures and maintain the relationships within them (Ulrich, 2016). This flexibility is beneficial for proportional reasoning (Steffe et al., 2014), representing multiplicative relationships algebraically (Hackenberg & Lee, 2015; Hackenberg et al., 2021), and understanding derivatives conceptually (Byerly, 2019).

The multiplicative concepts serve as a foundation for students' learning in mathematics. Frustrations can arise when learning difficult concepts that are limited by their multiplicative concepts stage, similar to how MC2 participants' in Byerly's 2019 study were frustrated when learning derivatives conceptually. Students' experiences in mathematics are influenced by formal education, such as in a mathematics classroom or doing homework. Successes or frustrations that arise from problem solving influence their perceptions of mathematics and themselves in mathematics.

Methodology

Participants and Data Collection

This study is part of a two-phase study exploring students' multiplicative concept stages by (1) validating an assessment for undergraduate students, (2) exploring the connections between the multiplicative concept stages and undergraduate students' solution to optimization problems prior to calculus, (3) exploring how undergraduate students with different multiplicative concept stages discuss their mathematics identity. The current article addresses part of the second phase of the study that focuses on the third goal listed above. Participants in this phase of the study are separate from the participants that helped validate the assessment for undergraduate students that occurred in the first phase of this study (Davenport et al., in preparation-a).

This study is a collective case study (Creswell & Poth, 2016) exploring how undergraduate students with different multiplicative concept stages describe their mathematics identity.. A case study, as defined by Creswell and Poth (2016) is a qualitative approach of research that explores a bounded system or systems (case or cases) from real-life through the collection of detailed data from multiple information sources. When the researcher uses multiple cases to explore a single issue, it is a collective case study (Creswell & Poth, 2016). The current study explores how students describe their mathematics identity through two cases: (1) undergraduate non-STEM major MC2 students enrolled in an entry-level mathematics course who have not taken calculus before, and (2) undergraduate non-STEM major MC3 students enrolled in an entry-level mathematics course who have not taken calculus before. Participants and cases were determined to benefit the current study and the other study in the second phase (Davenport et al., in preparation-b) that explored student solutions to optimization problems before they had taken calculus.

The Undergraduate Multiplicative Concepts Assessment (UMCA; Davenport et al., in preparation-a) was administered to two sections of an entry-level mathematics course focusing on mathematical functions and their applications and one section of an education course that served as an introduction to elementary education all taught at a mid-western university. A total of 43 undergraduate students took the UMCA. Of these participants, 35% identified as male students and 65% identified as female. Participants in this study predominately identified as White (74%) with 9% identifying as White and Native American, 5% identifying as Black, 5% identifying as White and Asian, 2% identifying as Asian, 2% identifying as Native American, and 2% identifying as other. Of these participants, 9% identified as Hispanic. These statistics are representative of the university's population for these two courses.

Administration and scoring of the UMCA followed guidelines outlined in the validation study for this instrument (Davenport et al., in preparation). Students were given 30 minutes to complete the UMCA. Participants were asked to answer the question "Do you consider yourself as a mathematics person?" by selecting one of two choices labeled "Yes" or "No" located at the end of the demographics survey. Participants were also asked if they were willing to participate in a follow-up interview. Five participants volunteered to participate in this portion of data collection. Participants were given pseudonyms for this study. Summary of their demographic data is shown in Table 5.1.

Table 5.1

Demographic data for participants

Name	Multiplicative Concept Stage	Do you consider yourself as a mathematics person?
Abigail	MC2	No
Brian	MC3	Yes
Clarissa	MC3	Yes
Diana	MC2	No
Eric	MC3	Yes

The follow-up interview consisted of two parts. First, they participated in a semi-structured interview (Galletta, 2013) where they were asked to describe certain aspects of their mathematics identity. Questions from the semi-structured interview include:

- Describe yourself as a mathematics person.
- If you can, describe a scenario where you have been recognized by a family member as a math person.
- If you can, describe a scenario when you have been recognized by a math teacher as a math person.
- Describe the ways you have enjoyed math.
- Would you say that mathematics comes naturally to you? How do you know?
- Describe how you know you are performing well in mathematics.

Participants were also asked to describe topics in mathematics they exceled and struggled in.

Following the semi-structured interview, the participants were asked to solve a series of four mathematics problems as part of a clinical interview (Clement, 2000). The first two problems were used to confirm the students' multiplicative concept stage. The last two problems were the optimization problems below:

- Marian owns a charter bus company that offers a route to the neighboring city that charges $40 per person if up to 30 passengers sign up for the trip. If more than 30 passengers sign up, the fare for every passenger is reduced by $1 for every passenger in excess of 30. The bus can only hold up to 48 passengers. How many passengers does Marian want to sign up for her charter bus route if she wants to maximize her revenue for the trip? (35 people)
- John wants to build a rectangular pen next to his barn. To try and maximize his resources, he decides to use one side of the barn as a side of his pen. If he has 160 ft worth of fence available to build his pen, what would be the dimensions of his pen if he maximized the area? (40ft. x 40 ft. x 80ft.).

Students were given paper and a pen to solve these problems. They were not given a calculator unless it was requested.

The sequencing of this interview ensured that their emotional experience during problem solving did not inform their discussion of their mathematics identity. The clinical interview problems were specifically selected as items that were appropriate in difficulty for the participants. This article will only explore statements made by the students or patterns of self-perceptions and behaviors that are directly related to their individual discussions of their mathematics identity. Discussion of specific solutions will not be discussed as a part of this article as it does not pertain to the research question being addressed.

Data Analysis

The Pearson's correlation coefficient was calculated between participant's UMCA scores and their response to the question "Do you consider yourself as a mathematics person?" This coefficient provides evidence towards the existence of a relationship between a person's multiplicative concept stage and their mathematics identity (Field et al., 2012).

Video tapes and audio recordings of the interviews were transcribed and coded using *a priori* coding (Saldaña, 2016). A priori codes were determined based off previous research regarding mathematics identity (Cribbs et al, 2015). These codes are interest, recognition, performance, and competence. Researchers initially coded two interviews separately before meeting to confirm their use of codes and to discuss additional codes. At this time, researchers added codes regarding specific mathematical topics such as algebra, fractions, graphing, mathematics connections, and other mathematics conceptsand a code called helping others which was represented across all five participants. Table 5.2 outlines the codes used for this study along with a description and example for each. After finalizing the code book for the project. The researchers then coded each of the interviews separately before combining their codes. Interrater reliability in this stage of coding was 0.79, which is considered moderate agreement (McHugh, 2012).

Table 5.2
Code descriptions and examples

Code	Source	Description	Example
A priori			
Recognition	Cribbs et al., 2015	Statements pertaining to how an individual perceives their own and other's views of themselves in relation to mathematics	"I'd say probably when my dad recognized [me] when he got my first grades in high school, my math grade was always the highest, and he was like, it runs in the family."
Interest	Cribbs et al., 2015	Statements pertaining to an individual's desire or curiosity to learn and do mathematics	I guess just 'cause I got it easily? So [mathematics] was just something fun I could do.
Competence	Cribbs et al., 2015	Statements pertaining to an individuals' beliefs about their ability to understand mathematics	That's where I struggle, I think, the most is I can't understand it on like a deeper level, why we do that. And so I'm just in here, I'm like, I don't know why. That's right... but is it?
Performance	Cribbs et al., 2015	Statements pertaining to the individual's beliefs about their ability to do well in mathematics	I would say that I'm fairly skilled in math.
Additional Codes			
Helping Others		Statements made in connection to them helping other with mathematics	Yes, my mom is terrible at math, so she'll often have me help out my younger brothers.
Algebra		Statements made specifically about topics within Algebra including statements about equations, unknowns, and variables.	So I do not like letters [variables] with math, that's what gets me.
Fractions		Statements made specifically about learning and working with fractions	I don't like fractions very much, I realized on the exam you gave me that I wasn't very good and I did not remember any of it.
Graphing		Statements made specifically about creating and interpreting graphs	I can usually do pretty well when it comes to graphing and all that stuff.
Mathematics Connections		Statements regarding connecting mathematics to contexts such as real-life contexts or multiple representations.	I enjoy it when it's in a practical application.
Other Mathematics Concepts		Statements that do not fall under specifics in mathematics that do not fall under the previous codes	Like the sin, cosine and all that stuff. I remember it from the way we have the pi circle. I remembered that section. I could figure out the triangles from that way, except trying to remember some of the functions, just numbers, that was a little bit hard.

Coding for the clinical interviews where the participants were solving optimization problems was done using value coding (Saldaña, 2016). Emergent codes were created based off of the attitudes and beliefs the participants vocalized during problem solving. Value codes include difficulty (e.g., “This is hard”), slow solving speed (e.g., “This is going to take forever”), and lack of enjoyment (e.g., “I don’t like when I get things wrong”). The only emergent theme from the value coding was “frustration” as all codes had a negative connotation. These codes were then compared to the statements the participants made during their identity interviews. A narrative was written for each student’s view of their mathematics identity and a case comparison was conducted between the MC2 and MC3 participants’ narratives. An additional comparison was made between the MC2 and MC3 participants’ mathematics topics that they voiced they excelled and struggled with. Results from this analysis are discussed in the following section

Results

Discussion of results will begin with a summary of the quantitative statistics surrounding the larger sample’s UMCA score and their willingness to identify as a mathematics person. A summary of participant responses to “Do you consider yourself a mathematics person?” separated by their UMCA stage assignments are provided in Table 5.3.

Table 5.3

Summary of participant’s responses to mathematics identity question by MC stage.

	Do you consider yourself as a mathematics person?	
Multiplicative Concept Stage	“No”	“Yes”
MC1 (n = 2)	2	0
MC2 (n = 24)	18	6
MC3 (n = 17)	8	9
Total (n = 43)	28	15

Out of 43 participants, only 15 considered themselves a mathematics person with over half of these students having an MC3. MC3 students were almost even in the number that considered themselves mathematics person and those who did not. Additionally, over 50% of the students who did not consider themselves to be a mathematics person were MC2 students. The Pearson's correlation coefficient showed a moderate positive correlation, $r(41) = .33$, $p = .034$, between UMCA assigned stages and whether they considered themselves as a mathematics person.

Identity Narratives

This section of the results will go over the identity narratives of each participant. This will be a synthesis of their interviews.

Abigail

Abigail stated that she was not a math person, but she worked hard to understand what she could of mathematics. She stated that only time she had ever received recognition as a mathematics person was when she got a good grade in math or helped her younger brothers out. She did a much better job at tutoring them than her older brother, and so was often praised by her parents. However, she stated that this praise was not towards her mathematics ability due to them never complimenting her grades in mathematics. Her previous teachers also never complimented her mathematics skills. She said, "I don't know if anyone ever told me, 'Oh yeah, you're really good at math'". None of her peers ever ask for her help in math like they do in English. Her current professor had complimented her a few times during office hours when she was getting questions correct on the homework. She said he's the only teacher that had ever told her she's good at math.

Abigail discussed that she likes math when she's getting it right and she understands it. It isn't enough for her to just get it right or to just understand it. If she doesn't understand math, she "doesn't enjoy it as much". When someone asks if she likes math, she replies, "Well, I like it more than science". She likes geometry a lot especially the work with angles, but really struggles with conceptual ideas, problems with multiple answers, and writing equations that use variables. Solving equations makes a lot more sense to her.

Abigail says that math doesn't come naturally to her. She struggles understanding mathematics "on a deeper level". If she doesn't understand it on that level, she feels like she keeps on getting it wrong. She just has to accept what answer she gets, even if it doesn't feel right. When she does understand a concept in math, she expects to forget it. She explains, "So I understand the velocity graph now, but I don't know how long I will understand it before I just forget it again". Abigail gauges how well she is doing in math by whether or not she understands it. She generally took harder mathematics classes in high school and made good grades. However, she felt she wasn't very good at math since she couldn't understand it. If she understood the topic, she got it right and she liked it. If she didn't understand the topic, she would end up being wrong and wouldn't like it.

During her work on the optimization problems, Abigail was very positive and persistent. She didn't mind needing to work slow through the problem saying, "It's gonna take a while this way, but I want to get it right". A couple of times she did comment, "I am not very good at this" when attempting to write a couple of complex equations.

Brian

Brian felt that he was a mathematics person that liked the challenge of complex problems. Being good at math "runs in the family" for Brian. He was often recognized by his father for having good grades. His middle school teacher even selected his group of friends to teach his younger schoolmates a mathematics lesson. He said their group was selected as the most interesting lesson that also explained the math the best.

Brian's main interest in mathematics is the challenge of mathematics. He enjoys figuring out problems that others don't. He explained, "I actually love doing my friend's homework, because every time they can't figure it out, they'll ask me". Brian said that there are a lot of different ways to do math, and that it's interesting to explore data. He is confident in his knowledge of algebra, functions, and graphs, though he struggles with creating graphs to represent specific contexts and with "logs and cosines".

When asked if math comes naturally to him, Brian replied, "yes and no". His quick understanding of something new depended on the topic. Some topics take time to "get used to" since they are harder. If he knows how to set the problem up and what to do with the numbers immediately, then he knows that he gets it. Brian gauges his performance in mathematics by how quickly he knows how to set up a problem. If he knows exactly what to do right away, he knows he is doing well.

Brian was engrossed with the problems during the interview. He did not hesitate in his problem solving, but worked to quickly set up and solve each of the problems. He said that he enjoyed solving the problems.

Clarissa

Clarissa said that she was a mathematics person. She has been helping others with mathematics for a long time. Her dad often needs help and asks her for assistance. At school, she was nominated as the mathematics representative by her teachers to compete in a school wide competition. They had told her she was a "good fit" as the mathematics representative.

Clarissa said that mathematics was easy, so it was fun to do. She found algebra, geometry, and fractions fun since they were easy to her. She did not enjoy learning mathematics online or “logs and cosine” functions. Clarissa said that her grades were some of the highest in the class, but that didn’t mean that she understood all of the work. However, she said that it came naturally to her since she just “got math easily”. She gauges her performance on her grades. Since she had always had good grades, she said that meant she was doing well in mathematics.

Clarissa was quick to provide solutions to both problems in the interview. She was very confident in these answers even when given evidence these solutions may not be correct.

Diana

Diana insisted she was not a mathematics person. When asked to describe herself in the context of math she said, “Oh, I’m not a math person”. None of her family members or teachers had ever approached her to tell her she was good at math.

Diana is “just not a fan” of mathematics. She liked it a lot better when it was just numbers, but now that they added letters it became confusing. Geometry and adding fractions are also difficult for her. However, she liked when the problems were realistic, saying “I don’t like when I’m doing math equations and I’m like, okay I was just solving an equation. But right now were are doing a lot of business stuff and I like it”. The context made it more interesting to do the math since it was useful. She stated that she was enjoying class because of the business applications, but then “just last week the class got so hard, I don’t like doing it anymore”. Diana said that the only topic in math that comes naturally to her is arithmetic. Her grades serve as her gauge of her performance in mathematics. Since arithmetic was easy for her, she used to like math. However, she explained that in 5th grade, she god her first “B”. From then on she hated math. She has an A in her current math class, but still hates math from her experience with her first “B”. She does know she is doing well in a course if she is making a good grade and she is able to quickly complete the work.

Diana was very positive during her work on the interview problems. She was persistent in finding a solution to both problems. She was concerned by the large amount of work needed to solve the problems saying, “it’s going to take a while”.

Eric

Eric confidently stated he was a mathematics person. He often helped his younger brothers out with math at home at the request of his mother since she was “terrible at math”. His high school algebra II teacher recognized him as a mathematics person when he got excited Eric aced the final exam.

Eric explained that he didn’t really enjoy doing math, but it was useful as a tool for him. He enjoyed using math in practical applications like welding. This was especially beneficial as practice with fractions since much of welding required a lot of fractions. He said really only struggled with logarithmic and exponential function graphs. Eric said that mathematics comes naturally to him if it’s described well. He only struggles when the original explanation of the topic wasn’t good. Eric gauges his performance in mathematics by how fast he worked and understood the problem. The quicker he was, the better he performed. He explained, “I either get it or all my steps are correct and I just missed a number”.

Eric quickly provided an answer to both problems. However, he became slightly frustrated when evidence showed his original answer may not be correct. During problem solving he said, “I hate being wrong” and he worked through correcting his solution.

Table 5.4
Summary of key findings from narratives based on a priori codes

Participant	Recognition	Interest	Performance	Competence	Math Person?
MC2					
Abigail	• Felt she received no recognition from parents or teachers before college. • Helped her younger brothers with math, but was never recognized for her math grades.	• Only enjoys math if she understands it and gets it right. • Often feels like she struggles, so she doesn't like math.	• Uses her grades and correct answer to judge her performance. • Feels she needs to understand and get it right.	• Math is difficult to understand. • She struggles with conceptual understandings. • She feels she ends up forgetting what she has learned most of the time.	No
Diana	• Has never felt recognized as a math person by anyone.	• Is not a fan of math and liked it a lot more when it was simpler.	• She bases her performance off her grades. • She also bases it off how fast she can do the problem.	• Only basic arithmetic is easy. Everything else in math is hard	No
MC3					
Brian	• Recognized by his father for his good math grades. • Was chosen by his math teacher to teach others.	• Enjoys the challenge mathematics brings • Likes solving things other people cannot solve.	• Gauges performance by how quickly he knows what to do to solve the problem.	• Is ok with not knowing things immediately as it takes time to learn some things. • Judges his competence by if he knows "what to do" immediately.	Yes
Clarissa	• Often helped her dad with math. • Was chosen by a math teacher as the mathematics representative for a competition	• Math is something fun that she can do. • Math is fun for her since it is easy.	• Bases her performance on how high her grades are compared to the rest of the class. • Her grades are some of the highest.	• Her grades are high, but I don't understand everything. • She just "gets a lot of it easily".	Yes
Eric	• Often taught his younger brothers. • Was recognized by his teacher as a student who was quick and accurate in math.	• Doesn't actually enjoy doing math, but sees it as a tool. • Only likes math when it is applied practically.	• How fast he can solve the problem is important to how well he performed.	• Confident that math comes easily to him. • How the information is initially presented determines how well he knows it.	Yes

Summary and Case Comparison

Results from both the UMCA and clinical interviews indicated that Abigail and Diana were MC2 students while Brian, Clarissa, and Eric were MC3 students. Both MC2 students stated they were not mathematics people while all three MC3 students stated that they were mathematics people. Table 5.4 summarizes the findings found in the narratives above.

Neither of the MC2 students felt they had been recognized as a mathematics person by the people around them. Abigail eventually talked about a couple of instances where she had been recognized helping her brothers in math and for her good grades, but overall felt she did not get recognition. Diana could not provide any examples of someone calling her a “math person”. Brian, Clarissa, and Eric all had examples of times they had been chosen specifically by their teachers in recognition of their mathematics achievement. They also outlined how they had helped out family members and friends with math growing up.

Abigail and Diana’s enjoyment of math hinges on being able to understand the mathematics. Both receive generally high grades in mathematics, but struggle to understand underlying mathematics concepts. They pointed out that working with algebraic problems that used variables was difficult for them. They gauge their competence and performance in mathematics on their grades and their ability to comprehend the classwork.

The MC3 students all enjoy math in different ways. Brian enjoys the challenge of mathematics. Clarissa enjoys math since it is easy. Eric enjoys math because it is useful. Comprehension time plays a large role in how the MC3 students talked about their competence and performance in mathematics. All three were very confident in their grades, and looked instead to the speed they completed their problem solving. For Clarissa and Eric, the faster they found a solution, the better they were performing in mathematics. This was further emphasized by Eric's frustrations when his quick answer turned out to be incorrect. Brian was less interested in the speed it took him to find and answer, but instead in the speed it took him to figure out how to set up the problem to solve it. If he knew how to set it up and "what to do with the numbers" then he considered himself to have done well on the problem.

Participants were asked to give examples of topics in mathematics that they excelled and struggled in. Below is a table summarizing these results that are organized by the participant's multiplicative concept stage.

Table 5.5

Mathematics topics participants excelled and struggled in

Participant	Excel	Struggle
MC2		
Abigail	• Graphing functions • Solving equations for X and Y • Geometry	• Interpreting graphs (especially Velocity) • Writing equations with variables • Anything with "relative answers" or "multiple answers" • Conceptual concepts behind mathematics problems • Fractions
Diana	• Arithmetic (multiplication, division, addition, and subtraction) • "Straightforward" numbers	• Variables or "letters" • Geometry • Adding Fractions
MC3		
Brian	• Functions • Graphs • Quadratic functions • Algebra	• Creating graphs from contexts • Log and cosine functions • Interpreting graphs • Choosing the best model for a problem
Clarissa	• Algebra • Geometry • Fractions	• Online mathematics learning • Log and cosine functions • Radians/degrees
Eric	• Multiplication • Exponents • "Easier" graphs	• Logarithmic graphs • Exponential functions and graphs

Discussion

Student's development of their mathematics identity is complex. Their experiences culminate into a perceived self that informs their outlook on mathematics. Positive experiences that stem from effective teaching strategies that support student learning in mathematics is beneficial to developing a positive disposition towards mathematics (Hodge & Harris, 2015). These teaching strategies should take into account the struggles of MC2 students have with conceptual learning by addressing multiple ways to model and reason about problems.

Initial results from this study suggest there is a correlation between students' multiplicative concept stage and their identification as a mathematics person. MC2 students especially were shown to be more likely to not identify as a mathematics person. Additionally, the frustrating or difficult aspects of mathematics Abigail and Diana struggled with align with struggles MC2 students generally have. MC2 students have difficulty when writing complex equations with variables as explained by both Abigail and Diana (Hackenberg et al., 2017, 2021). They also struggle with the conceptual aspects of mathematics, which mirrors the sentiments of participants in Byerly's (2019) study on the conceptual understanding of derivatives.

The MC3 students also had their share of difficult topics. All three struggled with logarithms and cosine functions and their graphs. They said they had no difficulty in writing equations and general algebra. The MC3 students' confidence came from their ability to see a problem and recognize the path to a solution. This speed was treasured above their grades as feedback. Since MC3 students assimilate with three levels of units, they can recognize more of the original problem's unit structures than MC2 students can and are able to flexibly move between levels of units. This gives them an advantage when problem-solving in comparison to their MC2 peers that could hypothetically make them faster at solving problems. Since this supports their beliefs about what mastery of mathematics looks like, their multiplicative concept stage may be supporting their view of their competence and performance in mathematics.

Early recognition from teachers and parents played a large role in the development of the mathematics identities for the participants which aligns with prior research (Cribbs et al., 2015; Gunderson et al., 2012). Interest for these participants varied, but ease of problem solving and usefulness of the material were both mentioned as factors of participant enjoyment. This aligns with the explanatory model for mathematics identity (Cribbs et al., 2015).

Limitations

This study is an initial exploration into the connection between undergraduate students' multiplicative concept and their mathematics identity. The students in this study are not representative of the entire undergraduate student body, but rather students who are enrolled in entry-level mathematics courses. Due to low volunteer numbers for interviews, we were unable to have a more diverse set of participants in regards to their mathematics identity. None of the MC2 students who identified as a mathematics person or the MC3 students who did not identify as mathematics students volunteered for the follow-up interview. Additionally, the low number of MC1 students is typical in the undergraduate setting (Boyce et al., 2021; Davenport et al., in preparation).

Implications, Future Research and Conclusion

The results of this study show promise in the connection between students' multiplicative concept stage and their mathematics identity. This study was the first step in exploring these connections. If we desire to create effective instruction that is beneficial for students and the development of their mathematics identity (Anderson et al., 2015; Hodge & Harris, 2015), it is imperative that we include considerations of students' mathematical schemes.

Future research should explore the missed narratives of this study, a wider population of undergraduate students, and the narrative surrounding younger students. Since research has recorded students having developed the first multiplicative concept stage as early as 2nd grade (Kosko & Singh, 2018), a look at the connection between their multiplicative concept stage and their mathematics identity during the developmental years of both may be beneficial in developing effective instructional practices for younger students.

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CHAPTER VI

CONCLUSION

Research on modeling student thinking focuses on understanding the students' perspective of mathematical reasoning and problem solving. Research has shown that students' coordination of unit structures influences their comprehension of mathematics topics such as fractions (Hackenberg, 2007; Steffe, 2001), proportions (Steffe et al., 2014), algebraic symbols and equations (Hackenberg, 2013, Hackenberg & Lee, 2015; Olive & Caglayan, 2008; Zwanch, 2019, 2022a, 2022b), recognition of quantity and use of measurement (Steffe, 2013), and derivatives and rates of change (Byerly, 2019). Undergraduate students' multiplicative concept stage influences their readiness for calculus (Boyce et al., 2021). Understanding how to accommodate limitations that can arise from the available operations of a student can inform teaching strategies to help support these students and give them positive mathematical experiences to develop a strong mathematics identity.

The goal of this study was to add to our existing knowledge of undergraduate student thinking by developing a multiplicative concept stage assessment valid for undergraduate students, exploring how undergraduate students solve optimization problems and how their available operations influenced these solutions, and exploring how students with different multiplicative concept stages described their mathematics identity. This study used the following research questions

- How well do the assessment and rubric items align with the theoretical framework for multiplicative concepts and assess undergraduate students' multiplicative concept stage?
- How do undergraduate students reason about and solve optimization problems?
- To what extent can the multiplicative concepts be used to explain undergraduate students' reasoning on optimization problems?
- How do students with different multiplicative concepts describe their mathematics identity?

Data was collected in two stages and synthesized into three articles. In the first phase of data collection, 51 undergraduate students were administered the UMCA and 18 of these participants participated in clinical interviews. Data from the overall scores (number of correct responses) of the UMCA, the attributed MC stage from the UMCA, and the attributed MC stage from the interviews were analyzed to provide evidence towards the validity of the UMCA as an appropriate assessment for undergraduate students' multiplicative concept stage. The validation of the UMCA and UMCA rubric were discussed in chapter III.

In the second phase of data collection, 43 undergraduate students in entry level mathematics courses took the UMCA and indicated whether or not they would describe themselves as a "Math Person". Of these 43 students, five participated in a follow-up interview that consisted of a semi-structured interview on their mathematics identity and a clinical interview solving optimization problems. The results from the clinical interview were outlined in chapter IV and the results from the semi-structured interview were discussed in chapter V.

Findings

There are four main findings from this study. First, the quantitative and qualitative data provided towards the validity of the UMCA and UMCA rubric supports this assessment as valid for undergraduate students. Chapter III provides evidence to support the UMCA's ability to accurately attribute the multiplicative concept stage of a student through data supporting valid test content, appropriate interpretation of response processes, internal structure that is unidimensional and interpretable, and generalizability through item and person reliability.

Second, undergraduate students solve optimization problems through the creation of systematic guess and check strategies, equations, tables and graphs that model the given problem. MC2 and MC3 students can solve these problems through the use of their available operations from their multiplicative concept stage and relational reasoning.

Third, the undergraduate students in this study who had different multiplicative concept stages had different experiences in mathematics. Both MC2 students did not view themselves as a mathematics person and voiced that they struggled with their conceptual understanding of mathematics. The MC3 students all had positive experiences in mathematics, were recognized as a mathematics person by their parents and teachers, and viewed their ability to comprehend and do mathematics positively.

Fourth, the majority of the undergraduate students had developed at least an MC2 (n = 91 and almost half of these students had constructed an MC3 (n = 45). Only 3 participants were attributed an MC1. This supports findings from prior research on undergraduate students' multiplicative concept stages (Boyce et al., 2021).

Implications

The validity evidence for the UMCA suggests that it can be used for larger scale studies of undergraduate students' multiplicative concepts stages. It is a useful tool for researchers to use in the exploration of student thinking on college mathematics concepts.

The results of this study add to the fields current understanding of the connections between the students' multiplicative concept stage their ability to construct and reason about quantitative structures. The results support prior research on MC2 and MC3 students' construction of quantitative structures and algebraic equations (Hackenberg et al., 2017, 2021; Olive & Cagalayan, 2008; Ulrich, 2016a). The MC2 participants were able to successfully quantify the additive relationships from the optimization problems, but struggled with inserting a unit of unknown size into another unit of unknown size. However, the ability to anticipate the quantification of multiplicative relationships and ability to reflect on interable composite units available to an MC3 but not an MC2 (Ulrich, 2016a) was necessary to insert a composite unit with an unknown into another composite unit with an unknown as needed to represent the charter bus problem. These operations supported the MC3 participants' equation writing. The MC2 students' use of numerical examples to construct and build the equation for the charter bus problem led to their success in in creating accurate algebraic representations for the charter bus problem. The participants' solutions to the optimization problems showcased the mathematical reasoning needed to represent optimization problems (Thompson & Carlson, 2017) that both MC2 and MC3 students have access to.

The results of this study support the possible connection between students' multiplicative concept stage and their mathematics identity. If we desire to create effective instruction that is beneficial for students and the development of their mathematics identity (Anderson et al., 2015; Hodge & Harris, 2015), it is imperative that we include considerations of students' mathematical schemes. With the large percentage of MC2 students found in this study and prior research (Boyce et al., 2021), professors should provide strategies to students to support MC2 thinking and learning in undergraduate mathematics.

Future Research

Future research should explore a larger sample for information on the multiplicative concept stage of undergraduate students and their mathematics identity for greater evidence of connections between the two concepts. Additionally, the missed narratives from this study due to the low sample size should be explored. These narratives include MC2 students who identified as a mathematics person, MC3 students who did not identify as a mathematics person, and MC1 students. Additionally, an exploration of K-12 students' mathematics identity and multiplicative concept stage may be beneficial for developing effective instructional practices to support students learning mathematics during their developmental years.

Future research on the multiplicative concepts should explore ways use the findings from this study to provide instruction in quantitative reasoning courses that support MC2 student thinking. Additional research should be done to explore the connection between proportionality schemes and covariational reasoning that was suggested by Brian's solutions in this study.

Concluding Remarks

It is important to provide effective support and instruction for student learning in undergraduate courses that allow all students to succeed and provide students with positive experiences in mathematics. This study explored the nuanced interactions between students' mathematics identity, their multiplicative concepts stage, and their problem solving strategies. Developing instruction and tasks that supports MC2 student learning is essential to helping them develop the conceptual understanding of mathematics that Abigail and Diana said they struggled with.

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APPENDICES

Appendix C: UMCA

Undergraduate Student Multiplicative Concepts Assessment

Part A

Please complete the following problems with the black pen provided.

1) A candy bar company packs 3 candy bars per package, and 6 packages per box.

a) If a store buys 7 boxes, how many candy bars will they receive?

b) If the same store orders another 8 boxes, how many total candy bars have they received?

c) Assuming the store received all of their ordered candy bars, how many packages have they received?

2) There are 6 plants in each row of my garden.

a) How many tomato plants are in 8 rows?

b) In addition to tomato plants, I also planted potatoes. If there are a total of 102 plants, how many rows of potatoes did I plant?

3) There are 12 inches in 1 foot and 3 feet in 1 yard.

a) How many inches are in 2 yards?

b) If you add an additional 5 feet onto the original yards, how many total inches are there?

c) How many feet are in the total number of inches?

d) How many yards are there in the total number of inches?

4) The stick shown below is 3/5 of a whole stick. How many 1/15 sticks can you make from the 3/5 stick?

5) The bar shown below is 7/3 as long as a whole candy bar. Draw the whole candy bar.

Part B

Please explain your reasoning behind your solutions to the problems from Part A. Please write your answers with the blue pen provided. You may mark on Part A with the blue pen if needed during this time.

1) A candy bar company packs 3 candy bars per package, and 6 packages per box.

d) If a store buys 7 boxes, how many candy bars will they receive?

e) If the same store orders another 8 boxes, how many total candy bars have they received?

f) Assuming the store received all of their ordered candy bars, how many packages have they received?

2) There are 6 plants in each row of my garden.

a) How many tomato plants are in 8 rows?

b) In addition to tomato plants, I also planted potatoes. If there are a total of 102 plants, how many rows of potatoes did I plant?

3) There are 12 inches in 1 foot and 3 feet in 1 yard.

a) How many inches are in 2 yards?

b) If you add an additional 5 feet onto the original yards, how many total inches are there?

c) How many feet are in the total number of inches?

d) How many yards are there in the total number of inches?

4) **The stick shown below is 3/5 of a whole stick. How many 1/15 sticks can you make from the 3/5 stick?**

5) **The bar shown below is 7/3 as long as a whole candy bar. Draw the whole candy bar.**

Appendix D: Demographics Survey Phase 1

Demographics

What is your name? ___________________________
What is your current class standing?

1) Freshman
2) Sophomore
3) Junior
4) Graduate
5) Senior

What is your age? _______
Do you identify as Hispanic/Latinx?

1) Yes
2) No

What is your race? (Select all that apply)

1) White
2) Black
3) Asian
4) Pacific Islander
5) American Indian or Alaskan Native
6) Other

Appendix E: Demographics Survey Phase 2

Demographics

What is your name? ___________________________
What is your current class standing?

6) Freshman
7) Sophomore
8) Junior
9) Graduate
10) Senior

What is your age? ________
What gender do you identify as? ________________
Do you identify as Hispanic/Latinx?

3) Yes
4) No

What is your race? (Select all that apply)

7) White
8) Black
9) Asian
10) Pacific Islander
11) American Indian or Alaskan Native
12) Other

What is your declared major? _____________________

Do you consider yourself a mathematics person?

1) Yes
2) No

Appendix F: Interview Protocol A

Interview Protocol A

This interview is a clinical interview for assessing the participants' multiplicative concepts. The interview will be video recorded with the participants' permission. Participants will be given a recording pen and paper to solve the problems. Participants can ask the interviewer at any point a question about the problem. The interviewer is to ask questions about the problem solving strategies of the student to help elaborate and articulate student work and thought. Participants can skip any question they do not feel like answering and may end the interview at any time. Participants are to be informed of their confidentiality agreement and voluntary participation at the beginning of the interview.

Introductory Statement

Hello, thank you for doing this interview with us. I am going to give you a couple problems to solve. Here is a pen and a notepad to write on. You can write whatever notes and work you want on here. As you work, I am going to ask you questions about what you are thinking and doing. Do you have any questions? If you do at any point, be sure to let me know if you have any questions.

Multiplicative Concept Questions

1) I purchased packages of candy bars that come in 8 per package.
    a. If I bought 7 packages of Mr. Goodbar candy bars, how many candy bars do I have? (56 Mr. Goodbar candy bars)

    b. I also bought some Almond Joy candy bars and now have a total of 104 candy bars, how many packages of Almond Joy candy bars did I purchase? (6 packages)

2) There are 8 fluid ounces in a cup and there are 4 cups in a quart. If I am measuring water out,
    a. How many fluid ounces of water are in 3 quarts? (96 fluid ounces)
    b. If I add an additional 7 cups of water to the original 3 quarts, how many ounces of water do I have now? (152 fluid ounces)
    c. How many total cups of water do I have now? (19 cups of water)
    d. How many total quarts of water do I have now? (4 ¾ quarts of water)

Appendix G: Interview Protocol B

Interview Protocol B

This interview will consist of two parts. The first part pertains to the participants' mathematics identity. The interview is semi-structured, allowing for follow-up questions to elaborate on participant responses. The second part will be a clinical interview and will have the participant answers some problems regarding their multiplicative concepts and optimization problems. Participants will be given a recording pen and paper to solve the problems. Participants can ask the interviewer at any point a question about the problem. Graphing paper and a four-function calculator will be available should the participants' work indicate that these resources would assist in allowing the student to move forward in problem solving. The interviewer is to ask questions about the problem solving strategies of the student to help elaborate and define student work and thought. The interview will be recorded with the participants' permission. Participants can skip any question they do not feel like answering and may end the interview at any time. Participants are to be informed of their confidentiality agreement and voluntary participation at the beginning of the interview.

Identity Interview

Note that participants were asked if they would consider themselves a mathematics person prior to this interview. Questions may be adjusted based of the answer they provided to this question. Suggestions for questions adjustments specifically for students who indicated they viewed themselves as a mathematics person will have a (**Y)** before them and those who indicated they were not will have a **(N)** before them.

Introductory Statement

Thank you for coming. In this part of the interview, we will be asking some questions about how you see yourself as a mathematics person. Your participation if completely voluntary will have no effect on any grade in any class. The responses you give me during this interview will remain confidential. If you need any clarity on a question or want to move on, please let me know. Do you have any questions for me? Ok, we asked you as a part of the assessment you took whether or not you viewed yourself as a mathematics person. You said (yes/no).

Interview Questions

(Self-Recognition)

> **(Y)** Describe yourself as a mathematics person
>
> **(N)** Describe yourself in the context of mathematics.

(Family Recognition)

> **(Y)** Describe a scenario where you have been recognized by a family member as a math person.
>
> **(N)** If you can, describe a scenario where you have been recognized by a family member as a math person.

(Teacher Recognition)

> **(Y)** Describe a scenario when you have been recognized by a math teacher as a math person.
>
> **(N)** If you can, describe a scenario when you have been recognized by a math teacher as a math person.

(Interest) Would you generally say you enjoy math?

(*Yes*) Describe the ways you have enjoyed math.
(*No*) Can you describe a way you have enjoyed mathematics?

(***No with no explanation)*** Can you describe the ways you haven't enjoyed mathematics?

(Competence)

Would you say that mathematics comes naturally to you (do you just "get" mathematics)? How do you know?

(Performance)

Describe how you know you are performing well in mathematics.

Potential Follow-up: how often did you feel like you were performing well in mathematics?

What topics in mathematics do you feel you excel in? Why?
What topics in mathematics do you feel you struggle in? Why?

Optimization Problems

Introductory Statement

For this part, I am going to give you four problems to solve. Here is a pen and a notepad to write on. You can write whatever notes and work you want on here. As you work, I am going to ask you questions about what you are thinking and doing. Do you have any questions? If you do at any point, be sure to let me know if you have any questions.

Multiplicative Concept Questions

3) I purchased packages of candy bars that come in 8 per package.
    a. If I bought 7 packages of Mr. Goodbar candy bars, how many candy bars do I have? (56 Mr. Goodbar candy bars)

    b. I also bought some Almond Joy candy bars and now have a total of 104 candy bars, how many packages of Almond Joy candy bars did I purchase? (6 packages)

4) There are 8 fluid ounces in a cup and there are 4 cups in a quart. If I am measuring water out,
    a. How many fluid ounces of water are in 3 quarts? (96 fluid ounces)
    b. If I add an additional 7 cups of water to the original 3 quarts, How many ounces of water do I have now? (152 fluid ounces)
    c. How many total cups of water do I have now? (19 cups of water)
    d. How many total quarts of water do I have now? (4 ¾ quarts of water)

Optimization Problems

5) Marian owns a charter bus company offers a route to the neighboring city that charges \$40 per person if up to 30 passengers sign up for the trip. If more than 30 passengers sign up, the fare for every passenger is reduced by \$1 for every

passenger in excess of 30. The bus can only hold up to 48 passengers. How many passengers does Marian want to sign up for her charter bus route if she wants to maximize her revenue for the trip? (35 people)

a) *If the participant is struggling to develop a model.* Is there any way you can represent the situation? Maybe a picture or a set of equations?

b) *If the student solved the problem from an equation and the use of a table or trial and error*. Is there a way you could solve this solution graphically?

c) *If the student solved this problem using a graph only.* Is there a way you could have solved this algebraically?

d) *If the student is struggling to provide any representation of the data or has gotten stuck at a previous stage of problem solving.* Ok, I am going to give you a graph that represents the situation from our problem. I'm going to ask you a few questions about this graph. (The interviewer will then provide the following graph of $f(n) = -n^2 + 10\,n + 1200$)

   i) Based on this graph and our problem, what do you think the values on the x and y axis represent? (x axis values indicate the number of passengers over 30; the y axis indicates the amount of revenue that the charter bus receives for providing a route to 30 + x passengers)

   ii) What situation does this point (indicating the point (0, 1200)) represent? (This is the amount of revenue Marian would receive if she only had 30 passengers)

iii) What about over here where it crosses the x axis (indicate (48,0))? (Giving the same discount to 78 people would cause Marian to not make any money. However, Marian can only fit 48 people on her charter bus)

iv) What point on the graph would indicate how much money Marian would make if she filled her charter bus? (18, 1056)

v) Can you use this graph to determine an answer for how many people Marian should want to sign up for her route in order to maximize revenue? (the point (5,1225) is the maximum of the graph and would be the highest amount of revenue she could receive with her current payment plan)

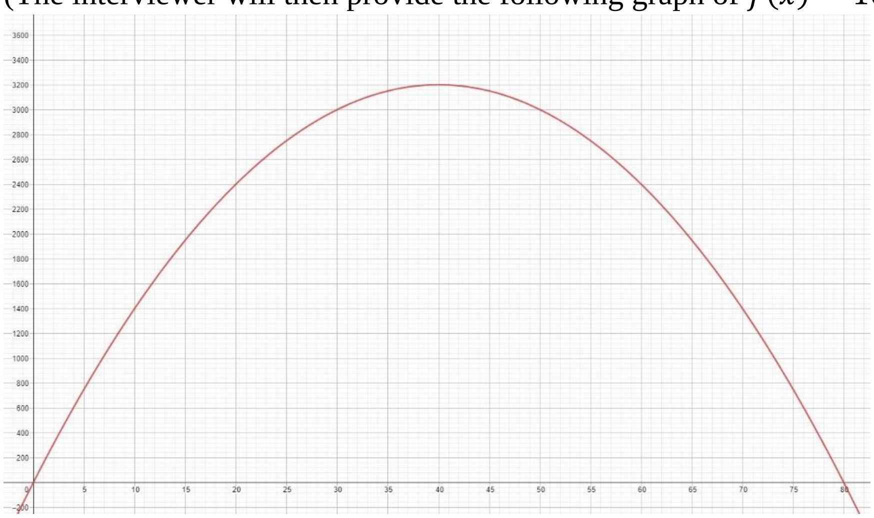
6) John wants to build a rectangular pen next to his barn. To try and maximize his resources, he decides to use one side of the barn as a side of his pen. If he has 160 ft worth of fence available to build his pen, what would be the dimensions of his pen if he maximized the area? (length: 80 feet, width: 40 feet).

a) *If the participant is struggling to develop a model.* Is there any way you can represent the situation? Maybe a picture or a set of equations?

b) *If the student solved the problem from an equation and the use of a table or trial and error*. Is there a way you could solve this solution graphically?

c) *If the student solved this problem using a graph only.* Is there a way you could have solved this algebraically?

d) *If the student is struggling to provide any representation of the data or has gotten stuck at a previous stage of problem solving.* Ok, I am going to give you a graph that represents the situation from our problem. I'm going to ask you a few questions about this graph. (The interviewer will then provide the following graph of $f(x) = 160x - x^2$)



i) Based on this graph and our problem, what do you think the values on the x and y axis represent? (x axis values indicate the length of the matching sides of the pen in feet; the y axis indicates the area of the pen)

ii) What situation does this point (indicating the point (20, 2400) represent? (This the area of the pen if the length of the matching sides of the pen is 20 feet)

   a. If we chose this value (20) as the shorter side, what would be the length of the longer side? (120 feet)

iii) What does this point ((0,0)) indicate in the context of our problem? Is this a possible pen we can make? (the length of the two matching sides would be 0, so it would be an unmade pen and impossible to create)

iv) What about the other point that crosses the x axis (80,0)? What does that point tell you about the John's pen? (In this case, if the two matching sides was 80, then the remaining side would be 0 feet long, which would again be an unconstructed pen).

v) Can you use this graph to determine an answer for the dimensions of the pen needed to maximize the area of John's pen? (the point (40, 3200) is the maximum of the graph and would be the greatest area possible. So the dimensions of the pen would be 40 feet by 80 feet)

Appendix H: IRB Approval Letter

Oklahoma State University Institutional Review Board

Date:	08/25/2021
Application Number:	IRB-21-342
Proposal Title:	EXPLORING UNDERGRADUATE STUDENTS' MULTIPLICATIVE CONCEPTS, OPTIMIZATION PROBLEM SOLUTIONS, AND MATHEMATICS IDENTITY
Principal Investigator:	Jianna Davenport
Co-Investigator(s):	Karen Zwanch
Faculty Adviser:	Jennifer Cribbs
Project Coordinator:	
Research Assistant(s):	
Processed as:	Exempt
Exempt Category:	

Status Recommended by Reviewer(s): Approved

The IRB application referenced above has been approved. It is the judgment of the reviewers that the rights and welfare of individuals who may be asked to participate in this study will be respected, and that the research will be conducted in a manner consistent with the IRB requirements as outlined in 45CFR46.

This study meets criteria in the Revised Common Rule, as well as, one or more of the circumstances for which continuing review is not required. As Principal Investigator of this research, you will be required to submit a status report to the IRB triennially.

The final versions of any recruitment, consent and assent documents bearing the IRB approval stamp are available for download from IRBManager. These are the versions that must be used during the study.

As Principal Investigator, it is your responsibility to do the following:

1. Conduct this study exactly as it has been approved. Any modifications to the research protocol must be approved by the IRB. Protocol modifications requiring approval may include changes to the title, PI, adviser, other research personnel, funding status or sponsor, subject population composition or size, recruitment, inclusion/exclusion criteria, research site, research procedures and consent/assent process or forms.
2. Submit a request for continuation if the study extends beyond the approval period. This continuation must receive IRB review and approval before the research can continue.
3. Report any unanticipated and/or adverse events to the IRB Office promptly.
4. Notify the IRB office when your research project is complete or when you are no longer affiliated with Oklahoma State University.

Please note that approved protocols are subject to monitoring by the IRB and that the IRB office has the authority to inspect research records associated with this protocol at any time. If you have questions about the IRB procedures or need any assistance from the Board, please contact the IRB Office at 405-744-3377 or irb@okstate.edu.

Sincerely,

Oklahoma State University IRB

VITA

Jianna Davenport

Candidate for the Degree of

Doctor of Philosophy

Dissertation: EXPLORING UNDERGRADUATE STUDENTS' MULTIPLICATIVE CONCEPT STAGES, OPTIMIZATION PROBLEM SOLUTIONS, AND MATHEMATICS IDENTITY

Major Field: Education

Biographical:

Education:

Completed the requirements for the Doctor of Philosophy in Education at Oklahoma State University, Stillwater Oklahoma in July, 2022.

Completed the requirements for the Master of Arts in Mathematics at Texas Tech University, Lubbock, Texas in December 2017.

Completed the requirements for the Bachelors of Arts in Mathematics at Texas Tech University, Lubbock, Texas in July 2016.

Experience:

Graduate Research Assistant, Center for Research in STEM Teaching and Learning, Oklahoma State University

Graduate Instructor, College of Arts and Sciences, Texas Tech University

RCML 2020 Proceedings Editors' Graduate Assistant, Oklahoma State University

Mathematics Department Grader, Texas Tech University

Assistant Instructor for TEX PREP, Noyce, Texas Tech University, Noyce, Texas Tech University

Professional Memberships:

National Council of Teachers of Mathematics

Research Council of Mathematics Learning