A RELATIONSHIP BETWEEN CERTAIN DIFFERENTIAL INTERTWINING OPERATORS ON THE INDEFINITE ORTHOGONAL AND UNITARY GROUPS

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Abstract:

In this thesis, we will analyze degenerate principal series representations realized as smooth induced representations for the indefinite orthogonal and unitary groups $G = SO_0(2p, 2q)$ and $H = U(p, q)$. We will induce from smooth characters of maximal-parabolic subgroups, which will each depend on a continuous parameter and a discrete parameter. Each of the principal series representations has an associated differential intertwining operator that can be identified as the right action of an element of the universal enveloping algebra. These operators correspond to the Euclidean and Heisenberg wave operators, respectively, and because of the group-invariance of these operators, their kernels will be subrepresentations.

A key result in this thesis is to establish a connection between the kernels for Euclidean and Heisenberg kernels. We will present a family of integral operators that provide a map between the principal series in the two settings which acts as a projection map of K-finite spaces. The most important feature of these integral operators, is that for the continuous parameter $p+q-2$, they intertwine the action of the differential operators the principal series. In particular, these integral operators take the kernels of the Euclidean wave operator in the orthogonal setting to the kernel of the Heisenberg wave operator in the unitary setting.

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CHAPTER I

INTRODUCTION

In studying irreducible admissible representations of a semisimple Lie group, one can look for "small" representations inside degenerate principal series, which are representations parabolically induced from maximal (or at least, not minimal) parabolic subgroups with one dimensional induction data. A natural way to produce a subrepresentation in these degenerate principal series is as the kernel of a sufficiently symmetric operator. A famous example of this type of symmetry is the Laplace operator, which is invariant under rotation.

Degenerate principal series have different "pictures," or equivalent characterizations, which allow us to analyze the mathematical objects from different perspectives. The induced picture views the degenerate principal series as a certain class of functions on a group with some homogeneity condition. This gives a concrete sense of the degenerate principal series and we will often use to make direct calculations.

Another setting to analyze the principal series is the compact picture, which realizes the functions as those restricted to a maximal compact subgroup K. In this setting, we calculate the K-finite space, which is a dense subspace of sums of irreducible (finite-dimensional) representations of K called K-types.

There is also the non-compact picture, which reduces the principal series down to a space of functions on a vector space or vector space-like structure, and in this setting we can look for solutions to differential operators, which correspond to elements of the universal enveloping algebra. Provided that the differential operators have sufficient symmetry, their kernels will be subrepresentations of the principal series.

In this thesis, the degenerate principal series we will analyze will be smooth induced representations for the indefinite orthogonal and unitary groups $G = SO_0(2p, 2q)$ and $H = U(p, q)$. We will induce from smooth characters of maximal-parabolic subgroups, which we will be stabilizers of isotropic lines. Elements of these principal series will be smooth functions on the group with a right translation property by the parabolic. Each of these induced representations has an associated differential intertwining operator that can be identified as the right action of an element of the universal enveloping algebra. These operators correspond to the Euclidean and Heisenberg wave operators, respectively, and because of the group-invariance of these operators, their kernels will be subrepresentations.

A key result in this thesis is to establish a connection between the kernels of the wave operators. We will present a family of integral operators that provide a map between the induced representations for G and H. These operators have two key features. The first feature is that this map acts as a projection of their respective K-finite spaces. The second feature of these operators, which is quite striking, is that they intertwine the action of the differential operators for certain parameters of the induced representations. In particular, these integral operators take the kernel of the Euclidean wave operator in the orthogonal setting to the kernel of the Heisenberg wave operator in the unitary setting.

Here is a summary of the thesis. In Chapter 2, we motivate and define two classical isometry groups, $SO(2n)$ and $U(n)$. We present certain irreducible representations of these groups, which will be spaces of homogeneous harmonic polynomials. We proceed to calculate "embedding vectors" for these representations, which are used to identify the K-types of the degenerate principal series and embed them inside the principal series.

In Chapter 3, we define the degenerate principal series in the induced picture for G and H that we wish to analyze. We apply some theory (proved in the introductory section) which allows us to connect the induced picture to the compact picture, and this allows us to calculate their respective K-finite spaces. These K-finite spaces are calculated similarly by Howe and Tan in [3], although they realize their degenerate principal series as smooth functions on a light cone with a translation property, whereas ours are smooth functions on the appropriate group.

In Chapter 4, we define a family of integral operators that, with a parity condition satisfied, maps elements of the principal series for G to that in H. The integral transform makes use of a canonical embedding $H \hookrightarrow G$, but the map is not simply restriction from G to H due to the non-inclusion of the corresponding parabolics. This map behaves well on the K-finite spaces, and we will motivate this by showing explicitly how certain K-types can be regarded as elements of the degenerate principal series for either G or H. Under these conditions, we will make clear how these integral operators act as projection of the K-types.

Chapters 5-7 aim to show that the actions of the differential intertwining operators commute with the integral operator. In particular, the integral operator takes the kernel of the Euclidean wave operator to the kernel of the Heisenberg wave operator. The kernel of the Euclidean wave operator for the orthogonal setting has been worked out by Binegar and Zierau in [2], and so the main results of this thesis connect the two kernels and imply the kernel in the unitary setting.

In Chapter 5, specifically, we will introduce the non-compact picture for the degenerate principal series and the differential intertwining operators we wish to work with. Then we will apply a well-known duality theorem which associates homomorphisms between generalized Verma modules to differential intertwining operators between induced representations. The application will show, for the parameter $p+q-2$, exactly which induced representations the differential intertwining operators operate between.

In Chapter 6, we will show how the integral transform from Chapter 3 H-intertwines the actions of the differential intertwining operators for Schwartz functions in the non-compact setting, which yields an H-commuting diagram between degenerate principal series. This will require an inclusion map that essentially changes coordinates between the non-compact settings for G and H. We will also require a change of coordinates for the differential operators. The Schwartz condition will be used to ensure that all integrals converge in the intertwining calculation.

The work in Chapter 7 extends this H-intertwining diagram to the entire degenerate principal series. The bulk of this chapter culminates in showing existence of a function in the compact setting with a certain property whose H-translates span a dense subspace of the degenerate principal series. When this is complete, we will connect the compact and non-compact settings by showing that this function, when regarded in the non-compact setting, has compact support, and in particular is a Schwartz function. This implies that the diagram is H-intertwining for a function whose H-translates can estimate any other function, which completes the desired extension. I will also calculate the kernel of the Heisenberg Laplacian at the end of this chapter.

CHAPTER II

EMBEDDING VECTORS OF HOMOGENEOUS HARMONIC POLYNOMIALS

2.1 The Special Orthogonal Groups and the Unitary Groups

Let $\mathbb{R}^n$ be the space of column vectors

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad x_j \in \mathbb{R},$$

endowed with the inner product

$$(x, x')_n = x^t \cdot x', \qquad x, x' \in \mathbb{R}^n.$$

The group of invertible $n \times n$ matrices with real entries $GL(n, \mathbb{R})$ acts on $\mathbb{R}^n$ by multiplication on the left. The subgroup of isometries of $\mathbb{R}^n$ that fix the origin is called the *orthogonal group of degree* n, denoted $O(n)$. That is, $O(n)$ is the subgroup of $GL(n, \mathbb{R})$ such that

$$(gx, gx')_n = (x, x')_n, \qquad x, x' \in \mathbb{R}^n, g \in O(n).$$

Equivalently,

$$O(n) = \{g \in GL(n, \mathbb{R}) : g^t g = I_n\}.$$

The subgroup of $O(n)$ whose elements have determinant 1 is called the *special orthogonal group* of degree n, denoted $SO(n)$. That is,

$$SO(n) = \{g \in GL(n, \mathbb{R}) : g^t g = I_n, \det g = 1\}.$$

Similarly, we let $\mathbb{C}^n$ be the space of column vectors

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}, \qquad z_j \in \mathbb{C},$$

endowed with the Hermitian inner product

$$\langle z, z' \rangle_n = z^t \cdot \overline{z'}, \qquad z, z' \in \mathbb{C}^n.$$

The group of invertible $n \times n$ matrices with complex entries $GL(n, \mathbb{C})$ acts on $\mathbb{C}^n$ by multiplication on the left. The subgroup of isometries of $\mathbb{C}^n$ that fix the origin is called the *unitary group of degree* n, denoted $U(n)$. That is, $U(n)$ is the subgroup of $GL(n, \mathbb{C})$ such that

$$\langle gz, gz' \rangle_n = \langle z, z' \rangle_n, \qquad z, z' \in \mathbb{C}^n, g \in U(n).$$

Equivalently,

$$U(n) = \{g \in GL(n, \mathbb{C}) : g^* g = I\},$$

where $g^* = \overline{g}^t$ is the conjugate transpose of g. We can identify $\mathbb{C}$ with $\mathbb{R}^2$ by

$$x_1 + x_2 i \leftrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad x_1, x_2 \in \mathbb{R},$$

and this gives us the entry-wise identification of $\mathbb{C}^n$ with $(\mathbb{R}^2)^n \cong \mathbb{R}^{2n}$. This also yields a canonical embedding

$$GL(n, \mathbb{C}) \hookrightarrow GL(2n, \mathbb{R})$$

where entries are mapped via

$$(a + bi) \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad a, b \in \mathbb{R}.$$

The calculations

$$(a_1 + b_1 i)(a_2 + b_2 i) = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2) i,$$
$$\begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 - b_1 b_2 & -(a_1 b_2 + b_1 a_2) \\ a_1 b_2 + b_1 a_2 & a_1 a_2 - b_1 b_2 \end{pmatrix}$$

shows that this embedding is a group homomorphism. Restriction to $U(n)$ gives the embedding

$$U(n) \hookrightarrow SO(2n).$$

2.2 Homogeneous Harmonic Polynomials

Let $\mathscr{P}^m(\mathbb{R}^n)$ be the space of homogeneous complex-valued polynomials of degree m in real variables $x_1, x_2, \ldots, x_n$, homogeneous with degree m. A linear change of variables preserves the degree of homogeneity of $\mathscr{P}^m(\mathbb{R}^n)$, and so $\mathscr{P}^m(\mathbb{R}^n)$ is a representation (Give brief definition of representation?) of $SO(n)$, with action given by

$$\Phi_m(g)(P)(x) = P(g^{-1}x), \qquad g \in SO(n), P \in \mathscr{P}^m(\mathbb{R}^n), x \in \mathbb{R}^n.$$

We note that the action on a product of polynomials will be the product of the actions:

$$\Phi_m(g)(P \cdot Q)(x) = (P \cdot Q)(g^{-1}x) = P(g^{-1}x) \cdot Q(g^{-1}x) = \Phi_m(g)(P)(x) \cdot \Phi_m(g)(Q)(x).$$

It follows that the action of sums, products, powers, etc., are the sums, products, powers, etc. of the actions.

We note for future reference the action of $-I_n$ on this homogeneous space.

Proposition 2.2.1 *Let $P \in \mathscr{P}^m(\mathbb{R}^n)$. Then, $(-I_n).P(x) = (-1)^m P(x)$.*

Proof. For a monomial $P(x) = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$ with $\sum_{j=1}^n k_j = m$, we have

$$(-I_n).P(x) = P(-x) = (-x_1)^{k_1}(-x_2)^{k_2} \cdot (-x_n)^{k_n} = (-1)^{\sum_{j=1}^n k_j} P(x) = (-1)^m P(x).$$

■

The Laplacian (or Laplace operator) on $\mathscr{P}^m(\mathbb{R}^n)$ is

$$\Delta_n = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2},$$

which defines a linear map

$$\Delta_n : \mathscr{P}^m(\mathbb{R}^n) \to \mathscr{P}^{m-2}(\mathbb{R}^n)$$

in the obvious way. The kernel of this map is called the space of *homogeneous harmonic polynomials* of degree m, denoted $\mathscr{H}^m(\mathbb{R}^n)$.

Similarly, let $\mathscr{P}^m(\mathbb{C}^n)$ be the space of homogeneous complex-valued polynomials of total degree m in variables z's and $\overline{z}$'s ($z_1, \dots, z_n$ and conjugates $\overline{z}_1, \cdots, \overline{z}_n$). Now let $\mathscr{P}^{m_1,m_2}(\mathbb{C}^n)$ be the space of bi-homogeneous complex-valued polynomials of degree m_1 in z's and degree m_2 in $\overline{z}$'s. These spaces are representations of $U(n)$ with action given by

$$\Phi_m(g)(P)(z, \overline{z}) = P\left(g^{-1}z, \overline{g^{-1}z}\right), \quad g \in U(n), P \in \mathscr{P}^{m_1,m_2}(\mathbb{C}^n), z \in \mathbb{C}^n.$$

Elements of $Z\big(U(n)\big)$ are of the form $\big(e^{i\theta} I_n\big)$ for some $\theta \in [0, 2\pi)$. For later reference, we now provide the action of $Z\big(U(n)\big)$ on $\mathscr{P}^{m_1,m_2}(\mathbb{C}^n)$.

Proposition 2.2.2 *Let $g = (e^{i\theta} I_n) \in Z\big(U(n)\big)$. Then*

$$\Phi_m(g).P = e^{-i\theta(m_1 - m_2)} P, \qquad P \in \mathscr{P}^{m_1,m_2}(\mathbb{C}^n).$$

Proof. A monomial in this space is of the form $P(z, \overline{z}) = z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n} \overline{z}_1^{\ell_1} \cdots \overline{z}_n^{\ell_n}$ with $\sum_{j=1}^n k_j = m_1, \sum_{j=1}^n \ell_j = m_2$. Notice that $(e^{i\theta} I_n)^{-1} z = e^{-i\theta} z$. Therefore,

$$\begin{aligned}
(e^{i\theta} I_p).P(z, \overline{z}) &= P\left(e^{-i\theta} z, e^{i\theta} \overline{z}\right) \\
&= (e^{-i\theta} z_1)^{k_1} (e^{-i\theta} z_2)^{k_2} \cdots (e^{-i\theta} z_p)^{k_p} (e^{i\theta} \overline{z}_1)^{\ell_1} \cdots (e^{i\theta} \overline{z}_p)^{\ell_p} \\
&= e^{-i\theta(\sum_j k_j - \sum_j \ell_j)} (z_1^{k_1} z_2^{k_2} \cdots z_p^{k_p} \overline{z}_1^{\ell_1} \cdots \overline{z}_p^{\ell_p}) \\
&= e^{-i\theta(m_1 - m_2)} P(z, \overline{z}),
\end{aligned}$$

and by linear extension this completes the proof. ■

The Laplacian Δ_n above may be written in complex coordinates as

$$\Delta_n = 4 \sum_{j=1}^n \frac{\partial^2}{\partial z_j \partial \overline{z}_j},$$

and this defines the linear map

$$\Delta_n : \mathscr{P}^{m_1,m_2}(\mathbb{C}^n) \to \mathscr{P}^{m_1-1,m_2-1}(\mathbb{C}^n)$$

given by

$$P \mapsto \Delta_n P.$$

The kernel of this map is called the space of *bi-homogeneous harmonic polynomials* of degrees m_1 and m_2, and is denoted $\mathscr{H}^{m_1,m_2}(\mathbb{C}^n)$.

2.3 Embedding Vector of $\mathscr{H}^m(\mathbb{R}^{2n})$

In this thesis, we are particularly interested in the representations $\mathscr{H}^m(\mathbb{R}^{2n})$ (of $SO(2n)$), and we shall require some additional facts about them. We will outline these facts, and if the reader wishes for more detailed explanations, they may see [4, p.236 Example 2, p.270-271 Problems 9-14, p.339-340 Problem 2, and p.570 Theorem 9.16.].

There is a notion of a *highest weight* for a complex finite-dimensional irreducible representation of a compact connected Lie group. The Theorem of the Highest Weight provides a classification of these irreducible representations according to their highest weight. The space $\mathscr{H}^m(\mathbb{R}^{2n})$ is an irreducible representation of $SO(2n)$ with highest weight me_1, and so this theorem shows that any two different $\mathscr{H}^m(\mathbb{R}^n)$ are inequivalent (that is, not isomorphic as representations).

The restriction to a subgroup of an irreducible representation is not generally irreducible. Theorems that describe how a representation decomposes under this restriction are called *branching theorems*. We are interested in the restriction of $\mathscr{H}^m(\mathbb{R}^{2n})$ from $SO(2n)$ to $SO(2n-1)$ under the identification

$$SO(2n-1) \hookrightarrow SO(2n), \qquad h \mapsto \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}.$$

(We will frequently make this identification without additional comment.) Murnaghan's Theorem (Knapp p.570 theorem 9.16) tells us that $\mathscr{H}^m(\mathbb{R}^{2n})$ is the only representation (up to equivalence) in which the trivial representation occurs in this decomposition, and that it occurs with multiplicity 1. This means that there is a nonzero vector in $\mathscr{H}^m(\mathbb{R}^{2n})$, unique up to a scalar, which is fixed by $SO(2n-1)$ under this identification. There is a natural candidate for this vector, and so now we proceed to calculate it explicitly.

Recall that $SO(2n)$ is defined as the set of isometries of $\mathbb{R}^{2n}$ which fix the origin. Under the above embedding, we may regard $SO(2n-1)$ as the subgroup of isometries of the last $(2n-1)$-coordinates in $\mathbb{R}^{2n}$. That is, $SO(2n-1)$ can be regarded as the subgroup of $SO(2n)$ which preserves the inner product in the following way:

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} 0 \\ x \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} 0 \\ x' \end{pmatrix} \right)_{2n} = \left(\begin{pmatrix} 0 \\ x \end{pmatrix}, \begin{pmatrix} 0 \\ x' \end{pmatrix} \right)_{2n}, \qquad x, x' \in \mathbb{R}^{2n-1}, h \in SO(2n-1).$$

In particular, for

$$r_1^2 = x_1^2 + x_2^2 + \cdots + x_{2n}^2, \quad r_2^2 = x_2^2 + x_3^2 + \cdots + x_{2n}^2,$$

we have that

$$r_2^2 = \left(\begin{pmatrix} 0 \\ x \end{pmatrix}, \begin{pmatrix} 0 \\ x \end{pmatrix} \right)_{2n} \qquad x \in \mathbb{R}^{2n-1}$$

is preserved by $SO(2n-1)$, and so r_2^2 (regarded as a polynomial in $\mathbb{R}^{2n}$) is $SO(2n-1)$-fixed. Notice that for $P(x) = x_1$, we have

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix} .P \right)(x) = P\left(\begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}^{-1} x \right) = P\left(\begin{pmatrix} 1 & 0 \\ 0 & h^{-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x' \end{pmatrix} \right) = P\left(\begin{pmatrix} x_1 \\ h^{-1}x' \end{pmatrix} \right) = x_1,$$

and so $SO(2n-1)$ fixes x_1 (regarded as a polynomial in $\mathbb{R}^{2n}$). It follows that sums, products, and powers of x_1 and r_2^2 are also $SO(2n-1)$-fixed. In particular, for natural numbers α and β, linear combinations of the polynomial $x_1^\alpha (r_2^2)^\beta$ are $SO(2n-1)$-fixed. we can choose combinations of α and β so that the total degree of $x_1^\alpha r_2^\beta$ is m, and the right linear combination of the polynomials in this form will give us the desired nonzero vector in $\mathscr{H}^m(\mathbb{R}^{2n})^{SO(2n-1)}$. First we state how the Laplacian acts on polynomials of this form.

Proposition 2.3.1 *Let $\alpha, \beta \geq 0$. We have*

$$\Delta_{2n}\left(x_1^\alpha r_2^{2\beta} \right) = \alpha(\alpha-1)x_1^{\alpha-2} r_2^{2\beta} + 4\beta(\beta + n - 3/2) x_1^\alpha r_2^{2(\beta-1)}.$$

Now that we can take the Laplacian of polynomials of this form, we will find combinations of α and β so that the polynomials have the right homogeneous property. We will show how to choose coefficients so that the linear combination of these polynomials is harmonic. It turns out that these coefficients will need to satisfy a recurrence relation, so we will state the general form of the recurrence in the following lemma.

Lemma 2.3.1 *The sequence $(a_k)_{k=0}^\infty$ given by*

$$a_k = \frac{\Pi_{i=1}^s (\sigma_i)_k}{\Pi_{j=1}^r (\rho_j)_k}$$

satisfies the recurrence relation

$$a_{k+1} = \frac{\Pi_{i=1}^s (\sigma_i + k)}{\Pi_{j=1}^r (\rho_j + k)} a_k$$

for all $k \geq 0$.

We will use the notation $(x)_k = x(x+1)(x+2)\dots(x+k-1)$ for the "rising factorial". By convention, $(x)_0 = 1$.

Let m be a positive integer, let $\ell = \lfloor m/2 \rfloor$, let $\varepsilon = \begin{cases} 1, m \text{ odd} \\ 0, m \text{ even} \end{cases}$, and let

$$A_k = 4k(-1/2 + \varepsilon + k), B_k = 4(-\ell + k)(3/2 - n - \ell + k).$$

Notice that

$$\begin{aligned}(2k+\varepsilon)(2k+\varepsilon-1) &= 4k^2+4k\varepsilon-2k+\varepsilon^2-\varepsilon \\ &= 4k^2+4k\varepsilon-2k \\ &= 4k(k+\varepsilon-1/2) = A_k.\end{aligned}$$

By Lemma 2.3.1, with $\rho_1 = 1, \rho_2 = 1/2+\varepsilon, \sigma_1 = -\ell, \sigma_2 = 3/2-n-\ell$, we may define a sequence $(a_k)_{k=0}^{\infty}$ given by

$$a_k = \frac{(-\ell)_k(3/2-n-\ell)_k}{k!(1/2+\varepsilon)_k}$$

which solves the recurrence

$$a_{k+1} = \frac{B_k}{A_{k+1}}a_k = \frac{(-\ell+k)(3/2-n-\ell+k)}{(1+k)(1/2+\varepsilon+k)}a_k.$$

Let $p_k = x_1^{2k+\varepsilon}r_2^{2(\ell-k)}$ and let $\xi^m = \sum_{k=0}^{\ell}(-1)^k a_k p_k$.

Theorem 2.3.1 $\xi^m \in \mathscr{H}^m(\mathbb{R}^{2n})^{SO(2n-1)}\backslash\{0\}$.

Proof. We have

$$\begin{aligned}\Delta_{2n}\sum_{k=0}^{\ell}(-1)^k a_k p_k &= \sum_{k=0}^{\ell}(-1)^k a_k \Delta_{2n}\left(x_1^{2k+\varepsilon}r_2^{2(\ell-k)}\right) \\ &= \sum_{k=0}^{\ell}(-1)^k a_k(A_k p_k/x_1^2 + B_k p_k/r_2^2) \qquad \text{Prop 2.3.1: } \alpha = 2k+\varepsilon, \beta = \ell-k \\ &= \sum_{k=0}^{\ell-1}(-1)^k\left(a_k B_k p_k/r_2^2 - a_{k+1}A_{k+1}p_{k+1}/x_1^2\right) \quad \text{regrouping, } A_0 B_\ell = 0 \\ &= \sum_{k=0}^{\ell-1}(-1)^k(a_k B_k - a_{k+1}A_{k+1})p_k/r_2^2, \qquad \text{since } p_k/r_2^2 = p_{k+1}/x_1^2 \\ &= \sum_{k=0}^{\ell-1}(-1)^k(0)p_k/r_2^2 = 0, \qquad \text{by the recurrence property of } (a_k).\end{aligned}$$

■

For reasons we will see later, it will be useful to express ξ^m using Gegenbauer polynomials, by way of first expressing it using hypergeometric functions, and then using Jacobi polynomials. We define a *hypergeometric function* by the formal power series

$${}_2F_1(a,b;c|z) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}z^k, \qquad z \in \mathbb{C}.$$

If c is not a non-positive integer and a is a negative integer, this is a polynomial, and we have

$$\begin{aligned}\sum_{k=0}^{\ell}(-1)^k a_k p_k &= \sum_{k=0}^{\ell}(-1)^k\frac{(-\ell)_k(3/2-n-\ell)_k}{(k)!(1/2+\varepsilon)_k}x_1^{2k+\varepsilon}r_2^{2(\ell-k)}\\ &= x_1^{\varepsilon}r_2^{2\ell}\sum_{k=0}^{\ell}(-1)^k\frac{(-\ell)_k(3/2-n-\ell)_k}{(k)!(1/2+\varepsilon)_k}x_1^{2k}r_2^{-2k}\\ &= x_1^{\varepsilon}r_2^{2\ell}\sum_{k=0}^{\ell}\frac{(-\ell)_k(3/2-n-\ell)_k}{(k)!(1/2+\varepsilon)_k}(-x_1^2/r_2^2)^k\\ &= x_1^{\varepsilon}r_2^{2\ell}{}_2F_1(-\ell,3/2-n-\ell;1/2+\varepsilon\big|-x_1^2/r_2^2).\end{aligned}$$

Now we will further re-express ξ^m in a form which uses Gegenbauer polynomials, which are special cases of Jacobi polynomials. A *Jacobi polynomial of degree k* is defined by

$$P_k^{(\alpha,\beta)}(x)=\frac{(\alpha+1)_k}{k!}{}_2F_1\left(-k,k+\alpha+\beta+1;\alpha+1\middle|\frac{1-x}{2}\right).$$

A *Gegenbauer polynomial* is defined as

$$C_k^{\lambda}(x)=\frac{(2\lambda)_k}{(\lambda+(1/2))_k}P_k^{(\lambda-(1/2),\lambda-(1/2))}.$$

To complete this transformation, we will need two formulas, which we state in the following lemma.

Lemma 2.3.2 *1. [1, Pfaff's Formula, p. 68, Theorem 2.2.5]*

$${}_2F_1\left(a,c-b;c\middle|\frac{x}{x-1}\right)=(1-x)^a{}_2F_1\left(a,b;c|x\right),\quad c>b>0.$$

2. [1, p. 128, Equation 3.1.12]

$$\begin{aligned}{}_2F_1(2a,2b;a+b+1/2\big|\frac{x+1}{2}) &= \frac{\Gamma(a+b+1/2)\Gamma(1/2)}{\Gamma(a+1/2)\Gamma(b+1/2)}{}_2F_1(a,b;1/2\big|x^2)\\ &\quad -x\frac{\Gamma(a+b+1/2)\Gamma(-1/2)}{\Gamma(a)\Gamma(b)}{}_2F_1(a+1/2,b+1/2;3/2\big|x^2).\end{aligned}$$

The cases when m is odd and even are the same except for a couple lines in the following calculation. We combine them into one, again writing $\ell=\lfloor m/2\rfloor$ and $\varepsilon=\begin{cases}1, m \text{ odd}\\ 0, m \text{ even}\end{cases}$.

$$\begin{aligned}
\xi^m &= x_1^\varepsilon r_2^{2\ell} {}_2F_1(-\ell, 3/2-n-\ell; 1/2+\varepsilon \big| -x_1^2/r_2^2) \\
&= x_1^\varepsilon r_2^{2\ell} {}_2F_1\left(-\ell, 3/2-n-\ell; 1/2+\varepsilon \big| -\frac{x_1^2}{r_1^2-x_1^2}\right) \\
&= x_1^\varepsilon r_2^{2\ell} {}_2F_1\left(-\ell, 3/2-n-\ell; 1/2+\varepsilon \big| \frac{x_1^2/r_1^2}{x_1^2/r_1^2-1}\right) \\
&= x_1^\varepsilon r_2^{2\ell} (1-x_1^2/r_1^2)^{-\ell} {}_2F_1(-\ell, n+\ell-1+\varepsilon; 1/2+\varepsilon \big| x_1^2/r_1^2) \\
&\quad \text{(by Lemma 2.3.2 (1)}, a=-\ell, b=-n-\ell+3/2, c=1/2+\varepsilon, x=x_1^2/r_1^2) \\
&= x_1^\varepsilon r_2^{2\ell} (r_2^2/r_1^2)^{-\ell} {}_2F_1(-\ell, n+\ell-1+\varepsilon; 1/2+\varepsilon \big| x_1^2/r_1^2) \\
&= x_1^\varepsilon r_1^{2\ell} {}_2F_1(-\ell, n+\ell-1+\varepsilon; 1/2+\varepsilon \big| (-x_1/r_1)^2) \\
&= \begin{cases} \dfrac{\Gamma(-\ell-\frac{1}{2})\Gamma(n+\ell-\frac{1}{2})}{\Gamma(n-\frac{1}{2})\Gamma(-\frac{1}{2})} x_1 r_1^{2\ell}(r_1/x_1) {}_2F_1\left(-2\ell-1, 2n+2\ell-1; n-\dfrac{1}{2}\Big|\dfrac{1-x_1/r_1}{2}\right), \varepsilon=1 \\ \dfrac{\Gamma(-\ell+\frac{1}{2})\Gamma(n+\ell-\frac{1}{2})}{\Gamma(n-\frac{1}{2})\Gamma(\frac{1}{2})} r_1^{2\ell} {}_2F_1(-2\ell, 2n+2\ell-2; n-\dfrac{1}{2}\Big|\dfrac{1-x_1/r_1}{2}), \varepsilon=0 \end{cases}
\end{aligned}$$

(by Lemma 2.3.2 (2), $a=-\ell-1/2, b=n+\ell-1/2, x=-x_1/r_1$,
note: dividing $(-)$ by $(-)$ is $(+)$, and the 2nd term was nonzero here, 1st 0,
whereas for m even case the 1st was nonzero and the 2nd 0)
(by Lemma 2.3.2 (2), $a=-\ell, b=n+\ell-1, x=-x_1/r_1$, note $\Gamma(a)=\infty$, so 2nd term= 0)

$$\begin{aligned}
&= \frac{\Gamma(-\ell+\frac{1}{2}-\varepsilon)\Gamma(n+\ell-\frac{1}{2})}{\Gamma(n-\frac{1}{2})\Gamma(\frac{1}{2}-\varepsilon)} r_1^{2\ell+\varepsilon} {}_2F_1\left(-2\ell-\varepsilon, 2n+2\ell-2+\varepsilon; n-\frac{1}{2}\Big|\frac{1-x_1/r_1}{2}\right) \\
&= \frac{\Gamma(-\ell+\frac{1}{2}-\varepsilon)\Gamma(n+\ell-\frac{1}{2})(2\ell+\varepsilon)!}{\Gamma(n-\frac{1}{2})\Gamma(\frac{1}{2}-\varepsilon)(n-\frac{1}{2})_{2\ell+\varepsilon}} r_1^{2\ell+\varepsilon} P_{2\ell+\varepsilon}^{(n-3/2,n-3/2)}(x_1/r_1)
\end{aligned}$$

(definition of Jacobi polynomial with: $a=-2\ell-\varepsilon, b=2n+2\ell-2+\varepsilon, c=n-\dfrac{1}{2}$;
$k=2\ell+\varepsilon, \alpha=c-1=n-\dfrac{3}{2}$)
$\beta=b-k-\alpha-1=2n+2\ell-2+\varepsilon-(2\ell+\varepsilon)-(n-\dfrac{3}{2})-1=n-\dfrac{3}{2}$

$$\begin{aligned}
&= \frac{\Gamma(-\ell+\frac{1}{2}-\varepsilon)\Gamma(\lambda+\ell+\frac{1}{2})(2\ell+\varepsilon)!}{\Gamma(\lambda+\frac{1}{2})\Gamma(\frac{1}{2}-\varepsilon)(\lambda+\frac{1}{2})_{2\ell+\varepsilon}} r_1^{2\ell+\varepsilon} P_{2\ell+\varepsilon}^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x_1/r_1) \qquad \text{(sub } \lambda=n-1) \\
&= \frac{\Gamma(-\ell+\frac{1}{2}-\varepsilon)\Gamma(\lambda+\ell+\frac{1}{2})(2\ell+\varepsilon)!(\lambda+\frac{1}{2})_{2\ell+\varepsilon}}{\Gamma(\lambda+\frac{1}{2})\Gamma(\frac{1}{2}-\varepsilon)(\lambda+\frac{1}{2})_{2\ell+\varepsilon}(2\lambda)_{2\ell+\varepsilon}} r_1^{2\ell+\varepsilon} C_{2\ell+\varepsilon}^{\lambda}(x_1/r_1)
\end{aligned}$$

(defn of Gegenbauer polynomials)

$$\begin{aligned}
&= \frac{\Gamma(-\ell+\frac{1}{2}-\varepsilon)\Gamma(\lambda+\ell+\frac{1}{2})(2\ell+\varepsilon)!}{\Gamma(\lambda+\frac{1}{2})\Gamma(\frac{1}{2}-\varepsilon)(2\lambda)_{2\ell+\varepsilon}} r_1^{2\ell+\varepsilon} C_{2\ell+\varepsilon}^{\lambda}(x_1/r_1) \qquad \text{cancelling } (\lambda+1/2)_{2\ell+\varepsilon} \\
&= \frac{\Gamma(-\ell+\frac{1}{2}-\varepsilon)(\lambda+\frac{1}{2})_\ell(2\ell+\varepsilon)!}{\Gamma(\frac{1}{2}-\varepsilon)(2\lambda)_{2\ell+\varepsilon}} r_1^{2\ell+\varepsilon} C_{2\ell+\varepsilon}^{\lambda}(x_1/r_1) \qquad \frac{\Gamma(z+\ell)}{\Gamma(z)}=(z)_\ell, z=\lambda+\frac{1}{2} \\
&= \frac{(-1)^\ell(\lambda+\frac{1}{2})_\ell(2\ell+\varepsilon)!}{(\frac{1}{2}+\varepsilon)_\ell(2\lambda)_{2\ell+\varepsilon}} r_1^{2\ell+\varepsilon} C_{2\ell+\varepsilon}^{\lambda}(x_1/r_1) \qquad \frac{\Gamma(z-\ell)}{\Gamma(z)}=\frac{(-1)^\ell}{(1-z)_\ell}, z=\frac{1}{2}-\varepsilon \\
&= \frac{(-4)^\ell(\lambda+\frac{1}{2})_\ell \ell!}{(2\lambda)_{2\ell+\varepsilon}} r_1^{2\ell+\varepsilon} C_{2\ell+\varepsilon}^{\lambda}(x_1/r_1) \qquad \text{since } (2\ell+\varepsilon)! = 4^\ell \ell!(\frac{1}{2}+\varepsilon)_\ell,
\end{aligned}$$

which completes the transformation of ξ^m using Gegenbauer polynomials.

We summarize the previous remarks with the following theorem.

Theorem 2.3.2 *For a positive integer* m*, with* $\lambda = n-1$, $\ell = \lfloor m/2 \rfloor$ *and* $\varepsilon = \begin{cases} 1, m \text{ odd} \\ 0, m \text{ even} \end{cases}$, *the polynomial*

$$\xi^m = \frac{(-4)^\ell (\lambda + \frac{1}{2})_\ell \ell!}{(2\lambda)_{2\ell+\varepsilon}} r_1^{2\ell+\varepsilon} C^\lambda_{2\ell+\varepsilon}(x_1/r_1)$$

is a nonzero vector in $\mathscr{H}^m(\mathbb{R}^{2n})^{SO(2n-1)}$.

2.4 Embedding Vectors for $\mathscr{H}^{m_1,m_2}(\mathbb{C}^n)$

Recall that $U(n)$ is defined as the set of isometries of $\mathbb{C}^n$ which fix the origin. There is a natural embedding

$$U(n-1) \hookrightarrow U(n), \quad h \mapsto \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}$$

which regards $U(n-1)$ as the subgroup of isometries of the last $(n-1)$-coordinates. That is, $U(n-1)$ can be regarded as the subgroup of $U(n)$ which preserves the Hermitian inner product in the following way:

$$\left\langle \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} 0 \\ z \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} 0 \\ z' \end{pmatrix} \right\rangle_n = \left\langle \begin{pmatrix} 0 \\ z \end{pmatrix}, \begin{pmatrix} 0 \\ z' \end{pmatrix} \right\rangle_n, \quad z, z' \in \mathbb{C}^{n-1}, h \in U(n-1).$$

We will frequently make this identification without additional comment.

There is a branching theorem for $U(n)$ by Weyl which is analogous to the one for $SO(2n)$ by Murnaghan from the last section. The space $\mathscr{H}^{m_1,m_2}(\mathbb{C}^n)$ is an irreducible representation of $U(n)$ with highest weight $m_2 e_1 - m_1 e_n$, and Weyl's theorem tells us that $\mathscr{H}^{m_1,m_2}(\mathbb{C}^n)$ is the only representation (up to equivalence) in which the trivial representation occurs in the decomposition by restricting to $U(n-1)$, and that it occurs with multiplicity 1. This means that there is a nonzero vector, unique up to scalar, in $\mathscr{H}^{m_1,m_2}(\mathbb{C}^n)$ which is fixed by $U(n-1)$ under this correspondence. As before in the $SO(2n)$ case, there is a natural candidate for this vector, which we denote as ξ^{m_1,m_2}, and denote the $U(n-1)$-fixed subspace as $\mathscr{H}^{m_1,m_2}(\mathbb{C}^n)^{U(n-1)}$. We will now explicitly construct the vector ξ^{m_1,m_2}.

Recall the action of $U(n)$ on $\mathscr{H}^{m_1,m_2}(\mathbb{C}^n)$ given by $(g.P)(z, \overline{z}) = P(g^{-1}z, \overline{g}^{-1}\overline{z})$. The calculation

$$\begin{aligned}
\left(\begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix} .P \right) \left(\begin{pmatrix} z_1 \\ z \end{pmatrix}, \begin{pmatrix} \overline{z}_1 \\ \overline{z} \end{pmatrix} \right) &= P\left(\begin{pmatrix} 1 & 0 \\ 0 & h^{-1} \end{pmatrix} \begin{pmatrix} z_1 \\ z \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \overline{h}^{-1} \end{pmatrix} \begin{pmatrix} \overline{z}_1 \\ \overline{z} \end{pmatrix} \right) \\
&= P\left(\begin{pmatrix} z_1 \\ h^{-1}z \end{pmatrix}, \begin{pmatrix} \overline{z}_1 \\ \overline{h}^{-1}\overline{z}_1 \end{pmatrix} \right)
\end{aligned}$$

shows that $U(n-1)$ fixes z_1 and $\overline{z}_1$ regarded as polynomials. Write $R_2^2 = z_2\overline{z}_2 + \cdots z_n\overline{z}_n$. Observe that

$$R_2^2 = \left\langle \begin{pmatrix} 0 \\ z \end{pmatrix}, \begin{pmatrix} 0 \\ z \end{pmatrix} \right\rangle_n, \quad z \in \mathbb{C}^{n-1}$$

is preserved by $U(n-1)$, and so R_2^2 (also regarded as a polynomial) is $U(n-1)$-fixed. It follows that sums, products, and powers of $z_1, \overline{z}_1$, and R_2^2 are also $U(n-1)$-fixed. In particular, for natural numbers α, β, κ, linear combinations of the polynomial $z_1^\alpha \overline{z}_1^\beta R_2^{2\kappa}$ are $U(n-1)$-fixed. We can choose combinations of α, β, κ so that $z_1^\alpha \overline{z}_1^\beta R_2^{2\kappa}$ has the right degree, and the right linear combination of polynomials in this form will give us the desired nonzero vector in $\mathscr{H}^{m_1,m_2}(\mathbb{C}^n)^{U(n-1)}$. First we calculate the Laplacian of polynomials of this form in the following proposition.

Proposition 2.4.1 *Let $\alpha, \beta, \kappa \geq 0$. We have*

$$\Delta\left(z_1^\alpha \overline{z}_1^\beta (R_2^2)^\kappa\right) = \alpha\beta z_1^{\alpha-1}\overline{z}_1^{\beta-1}(R_2^2)^\kappa + z_1^\alpha \overline{z}_1^\beta \kappa(\kappa+n-2)(R_2^2)^{\kappa-1}.$$

Let $m_1 \geq m_2$ and let

$$A_k = (m_1 - m_2 + k)k, B_k = (-m_2 + k)(-m_2 - n + 2 + k).$$

By Lemma 2.3.1 with $\rho_1 = m_1 - m_2 + 1, \rho_2 = 1, \sigma_1 = -m_2, \sigma_2 = -m_2 - n + 2$, we may define a sequence $(a_k)_{k=0}^\infty$ given by

$$a_k = \frac{(-m_2)_k(-m_2-n+2)_k}{(m_2-m_1+1)_k k!},$$

which solves the recurrence

$$a_{k+1} = \frac{B_k}{A_{k+1}} a_k = \frac{(-m_2+k)(-m_2-n+2+k)}{(m_1-m_2+k+1)(k+1)} a_k.$$

Define $p_k = z_1^{m_1-m_2+k}\overline{z}_1^k (R_2^2)^{m_2-k}$. Let $\xi^{m_1,m_2} = \sum_{k=0}^{m_2} (-1)^k a_k p_k$.

Theorem 2.4.1 $\xi^{m_1,m_2} \in \mathscr{H}^{m_1,m_2}(\mathbb{C}^n)^{U(n-1)} \backslash \{0\}$.

Proof. We have

$$\begin{aligned}
\Delta\xi^{m_1,m_2} &= \sum_{k=0}^{m_2} (-1)^k a_k \Delta p_k \\
&= \sum_{k=0}^{m_2} (-1)^k a_k (A_k \frac{p_k}{z_1\overline{z}_1} + B_k \frac{p_k}{R_2^2}) \\
&\qquad\qquad (\text{Proposition 2.4.1, } \alpha = m_1 - m_2 + k, \beta = k, \kappa = m_2 - k) \\
&= \sum_{k=0}^{m_2-1} (-1)^k \left(a_k B_k p_k / R_2^2 - a_{k+1} A_{k+1} p_{k+1}/(z_1\overline{z}_1)\right) \\
&\qquad\qquad (\text{re-grouping, since } A_0 = 0, B_{m_2} = 0) \\
&= \sum_{k=0}^{m_2-1} (-1)^k \left(a_k B_k - a_{k+1} A_{k+1}\right) p_k / R_2^2 \quad \text{since } p_k/R_2^2 = p_{k+1}/(z_1\overline{z}_1) \\
&= \sum_{k=0}^{m_2-1} (-1)^k (0) p_k / R_2^2 = 0, \quad \text{by the recurrence property of } (a_k).
\end{aligned}$$

■

Expressing ξ^{m_1,m_2} as a hypergeometric function, we have

$$\begin{aligned}\xi^{m_1,m_2} &= \sum_{k=0}^{m_2}(-1)^k a_k p_k \\ &= z_1^{m_1-m_2}(R_2^2)^{m_2}\sum_{k=0}^{m_2}\frac{(-m_2)_k(-m_2-n+2)_k}{(m_1-m_2+1)_k k!}(-z_1\overline{z}_1/R_2^2)^k \\ &= z_1^{m_1-m_2}(R_2^2)^{m_2}\ {}_2F_1\left(-m_2,-m_2-n+2;m_1-m_2+1\,\middle|\,-\frac{z_1\overline{z}_2}{R_2^2}\right).\end{aligned}$$

This completes the case when $m_1 \geq m_2$. Complex conjugation gives a bijection from

$$\mathscr{P}^{m_1,m_2}(\mathbb{C}^n) \to \mathscr{P}^{m_2,m_1}(\mathbb{C}^n).$$

Also, since the Laplacian is invariant under complex conjugation, we have

$$\overline{\Delta P(z,\overline{z})} = \Delta\overline{P(z,\overline{z})}.$$

It follows that $\overline{\xi^{m_1,m_2}} \in \mathscr{H}^{m_2,m_1}(\mathbb{C}^n)^{U(n-1)}\backslash\{0\}$, which resolves the case when $m_2 > m_1$.

We have

$$\mathscr{H}^m(\mathbb{C}^n) = \sum_{m_1+m_2=m}\mathscr{H}^{m_1,m_2}(\mathbb{C}^n),$$

where $\mathscr{H}^m(\mathbb{C}^n)$ has m total powers of z's and $\overline{z}$'s. If we let $\ell = \lfloor m/2 \rfloor$ as before, this becomes

$$\mathscr{H}^m(\mathbb{C}^n) = \sum_{d=0}^{\ell}\left(\mathscr{H}^{\ell+d+\varepsilon,\ell-d}(\mathbb{C}^n) + \mathscr{H}^{\ell-d,\ell+d+\varepsilon}(\mathbb{C}^n)\right),$$

with a redundancy when m is even for the term corresponding to $d = 0$. Write

$$\xi^{m_1,m_2} = \begin{cases}\xi_d^m, & m_1 \geq m_2 \\ \overline{\xi_d^m}, & m_2 \geq m_1\end{cases}.$$

The following theorem summarizes the above results with this notation.
$(m_1 = \ell+d+\varepsilon, m_2 = \ell-d, m_1-m_2 = 2d+\varepsilon)$

Theorem 2.4.2

$$\begin{aligned}\xi_d^m &= z_1^{2d+\varepsilon}(R_2^2)^{\ell-d}\ {}_2F_1\left(d-\ell,d-\ell-n+2;2d+\varepsilon+1\,\middle|\,-\frac{z_1\overline{z}_2}{R_2^2}\right), \\ \overline{\xi_d^m} &= \overline{z}_1^{2d+\varepsilon}(R_2^2)^{\ell-d}\ {}_2F_1\left(d-\ell,d-\ell-n+2;2d+\varepsilon+1\,\middle|\,-\frac{z_1\overline{z}_2}{R_2^2}\right).\end{aligned}$$

It is known (because the laplacian in complex coordinates is a multiple of that in real coordinates) that

$$\mathscr{H}^m(\mathbb{R}^{2n}) \cong \mathscr{H}^m(\mathbb{C}^n)$$

under the isomorphism given by $(x_j, x_{j+1}) \leftrightarrow z_j = x_{2j-1} + x_{2j}i$ for $1 \le j \le n$. We will treat this isomorphism as an equality. This implies that

$$\xi^m \in \sum_{d=0}^{\ell} \left(\mathscr{H}^{\ell+d+\varepsilon, \ell-d}(\mathbb{C}^n) + \mathscr{H}^{\ell-d, \ell+d+\varepsilon}(\mathbb{C}^n) \right).$$

In fact, $\xi^m \in \mathscr{H}^m(\mathbb{R}^{2n})^{SO(2n-1)}$ is $U(n-1)$-fixed as a result of the natural embeddings

$$U(n-1) \hookrightarrow SO(2n-2) \hookrightarrow SO(2n-1).$$

Therefore,

$$\xi^m = \sum_{d=0}^{\ell} \gamma_d \xi_d^m + \sum_{d=0}^{\ell} \gamma_d{}' \overline{\xi_d^m}$$

for some constants $\gamma_d, \gamma_d{}' \in \mathbb{C}$. It will turn out that $\gamma_d = \gamma_d{}'$ for each d (due to symmetry of certain terms). ($\ell = \lfloor m/2 \rfloor, \lambda = n-1$ and let $\alpha = 2\ell + \varepsilon$)

Evaluation of $SO(2p-1)$-fixed vector:

$$\begin{aligned}
&\xi^m(x = (\cos\theta, \sin\theta, 0, 0, \dots, 0)) \\
&= \frac{(-4)^\ell \ell! (\lambda + \frac{1}{2})_\ell}{(2\lambda)_\alpha} r_1^\alpha C_\alpha^\lambda \left(\cos(\theta) / \sqrt{\cos^2\theta + \sin^2\theta + 0 + \cdots + 0} \right) \\
&= \frac{(-4)^\ell \ell! (\lambda + \frac{1}{2})_\ell}{(2\lambda)_\alpha} C_\alpha^\lambda (\cos\theta) \\
&= \frac{(-4)^\ell \ell! (\lambda + \frac{1}{2})_\ell}{(2\lambda)_\alpha} \sum_{k=0}^{\alpha} \frac{(\lambda)_k (\lambda)_{\alpha-k}}{k!(\alpha-k)!} \cos((\alpha - 2k)\theta) \quad [1, p.\ 302, eqn.\ (6.14.11)] \\
&= \frac{(-4)^\ell \ell! (\lambda + \frac{1}{2})_\ell}{(2\lambda)_\alpha} \sum_{k=0}^{\ell} (1 + \delta_{k, \frac{m}{2}}) \frac{(\lambda)_k (\lambda)_{\alpha-k}}{k!(\alpha-k)!} \cos((\alpha - 2k)\theta) \\
&= \sum_{k=0}^{\ell} A_k \cos((\alpha - 2k)\theta).
\end{aligned}$$

where

$$A_k = \frac{(-4)^\ell \ell! (\lambda + \frac{1}{2})_\ell (1 + \delta_{k, \frac{m}{2}}) (\lambda)_k (\lambda)_{\alpha-k}}{(2\lambda)_\alpha k! (\alpha - k)!}.$$

We note that (visibly) $A_k \neq 0$ for each k. Making the substitution $k = \ell - d$, this becomes

$$\sum_{d=0}^{\ell} A_{\ell-d} \cos((2d + \varepsilon)\theta).$$

Evaluation of $U(n-1)$-fixed vector:

$$\begin{aligned}
\xi_d^m(\cos\theta + i\sin\theta, 0, \dots, 0) =& (\cos\theta + i\sin\theta)^{2d+\varepsilon}\ {}_2F_1 \left(d - \ell, d - \ell - n + 2; 2d + 1 \middle| -1 \right) \\
=& B_d (\cos\theta + i\sin\theta)^{2d+\varepsilon},
\end{aligned}$$

where $B_d \in \mathbb{R}$ is the value of this hypergeometric series. Similarly,

$$\overline{\xi_d^m}(\cos\theta + i\sin\theta, 0, \ldots, 0) = B_d(\cos\theta - i\sin\theta)^{2d+\varepsilon}.$$

Write $\eta_d = 2d + \varepsilon$. We thus have the equation

$$\sum_{d=0}^{\ell} A_{\ell-d}\cos(\eta_d\theta) = \sum_{d=0}^{\ell} \left(\gamma_d B_d(\cos\theta + i\sin\theta)^{\eta_d} + \gamma_d' B_d(\cos\theta - i\sin\theta)^{\eta_d}\right).$$

By De'Moivre's formula, this is then

$$\begin{aligned}\sum_{d=0}^{\ell} A_{\ell-d}\cos(\eta_d\theta) &= \sum_{d=0}^{\ell} \left(\gamma_d B_d(\cos(\eta_d\theta) + i\sin(\eta_d\theta)) + \gamma_d' B_d(\cos(\eta_d\theta) - i\sin(\eta_d\theta))\right) \\ &= \sum_{d=0}^{\ell} B_d\left(\gamma_d(\cos(\eta_d\theta) + i\sin(\eta_d\theta)) + \gamma_d'(\cos(\eta_d\theta) - i\sin(\eta_d\theta))\right).\end{aligned}$$

Linear independence of the $\cos(\eta_d\theta)$ terms implies that $A_{\ell-d} = B_d(\gamma_d + \gamma_d) \neq 0$ for each d. Since every $A_{\ell-d} \neq 0$, this forces $B_d \neq 0$. Then, linear independence of the $\sin(\eta_d\theta)$ terms implies that $B_d(\gamma_d - \gamma_d') = 0$ for each d. Since every $B_d \neq 0$, this forces $\gamma_d = \gamma_d'$ for each d. We thus have

$$\sum_{d=0}^{\ell} A_{\ell-d}\cos(\eta_d\theta) = \sum_{d=0}^{\ell} 2\gamma_d B_d\cos(\eta_d\theta).$$

This implies that

$$\gamma_d = \frac{A_{\ell-d}}{2B_d},$$

and in particular, each $\gamma_d \neq 0$. We summarize as follows.

Theorem 2.4.3 *The vector* $\xi^m \in \mathscr{H}^m(\mathbb{R}^{2n})^{SO(2n-1)}$ *decomposes as a sum of* $\xi_d^m, \overline{\xi_d^m} \in \mathscr{H}^{m_1,m_2}(\mathbb{C}^n)^{U(n-1)}$*. Explicitly, we have*

$$\xi^m = \sum_{d=0}^{\ell} \gamma_d\left(\xi_d^m + \overline{\xi_d^m}\right),$$

where

$$\gamma_d = \frac{A_{\ell-d}}{2B_d} \neq 0.$$

For the purposes of this thesis, it is enough to know that the constants γ_d are all nonzero. However, I have a conjecture about what γ_d's are, and I have verified for several different parameters, and in the future I plan to prove this conjecture.

CHAPTER III

K-TYPES FOR DEGENERATE PRINCIPAL SERIES REPRESENTATIONS FOR $SO(2p, 2q)$ AND $U(p, q)$

In this Chapter, we will introduce the main groups we will be concerned with, $SO_0(2p, 2q)$ and $U(p, q)$ and their maximal compact subgroups. We will also define some degenerate principal series as smooth induced representations for each group, induced from characters of a maximal parabolic subgroup. We will also determine their respective K-finite spaces, which are countable sums irreducible representations for their corresponding maximal compact subgroups.

3.1 Smooth Induced Representations

We begin this section with some general theory for induced representations that will be helpful in both groups we intend to study. Let G be a closed Lie group and K and Q closed subgroups with $G = KQ$, and $\lambda : Q \to \mathbb{C}^\times$ a smooth homomorphism of Q. The action of the *left-regular representation* l of G on $C^\infty(G)$ (endowed with the smooth topology) is given by

$$l(g)\varphi(x) = \varphi(g^{-1}x).$$

The *smooth induced representation* of λ from Q to G is the space

$$\mathrm{ind}_Q^G(\lambda) = \{\varphi \in C^\infty(G) : \varphi(xq) = \lambda(q)^{-1}\varphi(x) \text{ for all } x \in G, q \in Q\},$$

the subrepresentation of functions with the given translation property for Q. The restriction map $\varphi \mapsto \varphi|_K$ gives a K-module isomorphism, as we will now see.

Proposition 3.1.1 *If G is a closed Lie group, K and Q closed subgroups with $G = KQ$, and $\lambda : Q \to \mathbb{C}^\times$ a smooth homomorphism of Q, then restriction $\varphi \mapsto \varphi|_K$ gives a K-isomorphism from*

$$\mathrm{ind}_Q^G(\lambda) \to \mathrm{ind}_{K\cap Q}^K(\lambda|_{K\cap Q}).$$

Proof. It is immediate that this map is well-defined, since the space $\mathrm{ind}_{K\cap Q}^K(\lambda|_{K\cap Q})$ inherits the translation property from $\mathrm{ind}_Q^G(\lambda)$. It is also immediate that this map is a K-homomorphism since restriction to K is linear and commutes with the action of K.

To see that this map is one-one, let $\varphi, \psi \in \mathrm{ind}_Q^G(\lambda)$ and suppose that $\varphi|_K = \psi|_K$. For $k \in K, q \in Q$, we have

$$\varphi(kq) = \lambda(q)^{-1}\varphi(k) = \lambda(q)^{-1}\psi(k) = \psi(kq).$$

To see that this map is onto, let $\varphi \in \mathrm{ind}_{K\cap Q}^{K}(\lambda|_{K\cap Q})$. For $k \in K, q \in Q$, it suffices to check that $\tilde{\varphi}(kq) := \lambda(q)^{-1}\varphi(k)$ is well-defined. Since the factorization $G = KQ$ is not assumed to be unique, we check that for $k_1q_1 = k_2q_2$ that $\tilde{\varphi}(k_1q_1) = \tilde{\varphi}(k_2q_2)$. Notice that $q_2q_1^{-1} = k_2^{-1}k_1 \in K \cap Q$, and so

$$\begin{aligned}\tilde{\varphi}(k_1q_1) &= \lambda(q_1)^{-1}\varphi(k_1)\\ &= \lambda(q_1)^{-1}\varphi(k_2q_2q_1^{-1})\\ &= \lambda(q_1)^{-1}\lambda(q_2q_1^{-1})^{-1}\varphi(k_2)\\ &= \lambda(q_2)^{-1}\varphi(k_2)\\ &= \tilde{\varphi}(k_2q_2).\end{aligned}$$

Thus, $\tilde{\varphi}$ is well-defined, and $\tilde{\varphi}|_K = \varphi$, which completes the proof. ■

If K is further assumed to be compact, a *K-finite vector* in $\mathrm{ind}_{K\cap Q}^{K}(\lambda|_{K\cap Q})$ is a function φ such that

$$\dim(\mathrm{span}\{l(k)\varphi : k \in K\}) < \infty.$$

That is, the span of K-translates of φ is finite-dimensional. The span of all K-finite vectors is called the *K-finite space*, and it is known that this space is dense in $\mathrm{ind}_{K\cap Q}^{K}(\lambda)$. It will turn out that the K-finite vectors can be realized as certain matrix coefficients, which we will define and explain now.

Let G, K, Q have the hypotheses above, and let σ be a finite-dimensional representation of K. For $v \in E_\sigma, \xi \in E_\sigma^\vee$, a *matrix coefficient* of σ is any function on K of the form $\varphi_{v,\xi}(k) = \xi\big(\sigma(k^{-1})v\big)$. It turns out that the K-finite vectors are matrix coefficients from irreducible representations σ whose dual, when restricted to $K \cap Q$, contains a copy of λ. Let

$$\check{E}_\sigma^{(K\cap Q,\lambda^{-1})} = \{\xi \in \check{E}_\sigma : \check{\sigma}(q)\xi = \lambda(q)^{-1}\xi \text{ for all } q \in K\cap Q\}.$$

Lemma 3.1.1 *Let σ be an irreducible representation of K in $\mathrm{ind}_{K\cap Q}^{K}(\lambda|_{K\cap Q})$. If $\psi \in E_\sigma$, then $\xi \in \check{E}_\sigma$ given by $\xi(\psi) = \psi(e)$ is a member of $\check{E}_\sigma^{(K\cap Q,\lambda^{-1})}$. Furthermore, $\psi = \varphi_{\psi,\xi}$ (is a matrix coefficient).*

Proof. We have

$$\begin{aligned}\big(\check{\sigma}(q)\xi\big)(\psi) &= \xi\big(\sigma(q^{-1})\psi\big) && \text{(action of } \check{\sigma})\\ &= \big(\sigma(q^{-1})\psi\big)(e) && \text{(definition of } \xi)\\ &= \psi(q) && \text{(action of } \sigma)\\ &= \lambda(q)^{-1}\psi(e)\\ &= \lambda(q)^{-1}\xi(\psi),\end{aligned}$$

which completes the first claim. Furthermore,

$$\begin{aligned}\psi(k) &= \big(\sigma(k^{-1})\psi\big)(e)\\ &= \xi\big(\sigma(k^{-1})\psi\big)\\ &= \varphi_{\psi,\xi}(k)\end{aligned}$$

completes the second claim, and the proof. ■

Proposition 3.1.2 *If G is a closed Lie group, K a compact subgroup and Q a closed subgroup, $G = KQ$, and $\lambda : Q \to \mathbb{C}^\times$ a smooth homomorphism of Q, then*

$$\bigoplus_{\sigma \ irr} E_\sigma \otimes \check{E}_\sigma^{(K\cap Q,\lambda^{-1})} \to \mathrm{ind}_{K\cap Q}^K(\lambda|_{K\cap Q})$$

satisfying

$$v \otimes \xi \mapsto \varphi_{v,\xi}(k),$$

where the sum is over the irreducible σ occuring in $\mathrm{ind}_{K\cap Q}^K(\lambda|_{K\cap Q})$, is onto the K-finite space.

Proof. We first check that $\varphi_{v,\xi} \in \mathrm{ind}_{K\cap Q}^K(\lambda|_{K\cap Q})$. If $k \in K, q \in K \cap Q$,

$$\begin{aligned}
\varphi_{v,\xi}(kq) &= \xi\big(\sigma(q^{-1}k^{-1})v\big) \\
&= \xi\big(\sigma(q^{-1})\sigma(k^{-1})v\big) \\
&= (\check{\sigma}(q)\xi)\big(\sigma(k^{-1})v\big) && \text{(defn. of action of } \check{\sigma}) \\
&= \lambda(q)^{-1}\xi\big(\sigma(k^{-1})v\big) \\
&= \lambda(q)^{-1}\varphi_{v,\xi}(k).
\end{aligned}$$

To show that this map is onto the K-finite space, Let ψ be K-finite, and let E be the span of translates of ψ. Since E is finite-dimensional by assumption, we may write $E = \bigoplus_{i=1}^n E_{\sigma_i}$, where each E_{σ_i} is an irreducible representation of K. Thus we may write $\psi = \sum_{i=1}^n \psi_i$ where each $\psi_i \in E_{\sigma_i}$. Then $\xi_i \in \check{E}_{\sigma_i}$ as in Lemma 3.1.1 is a member of $\check{E}_{\sigma_i}^{(K\cap Q,\lambda^{-1})}$, and $\psi_i = \varphi_{\psi_i,\xi_i}$. Therefore,

$$\sum_{i=1}^n \psi_i \otimes \xi_i \mapsto \sum_{i=1}^n \varphi_{\psi_i,\xi_i} = \sum_{i=1}^n \psi_i = \psi.$$

■

This gives us an identification

K-finite vectors in $\mathrm{ind}_{K\cap Q}^K(\lambda|_{K\cap Q}) \leftrightarrow$ representations E_σ of K such that $\check{E}_\sigma^{(K\cap Q,\lambda^{-1})} \neq \{0\}$.

The following lemma and proposition show that the latter condition is equivalent to representations E_σ of K such that $E_\sigma^{(K\cap Q,\lambda)} \neq \{0\}$.

Lemma 3.1.2 *Let K, C be groups with $C < K$, let λ be a representation of C, and let σ be an irreducible (finite-dimensional) representation of K such that*

$$\sigma|_C \cong \bigoplus_{j=1}^n \rho_j.$$

That is, σ restricted to C is the sum of irreducible representations ρ_j of C. Then,

$$\dim(\sigma^{(C,\lambda)}) = \#\ \textit{of } j\textit{'s such that } \rho_j \cong \lambda.$$

Additionally, $\rho_j \cong \lambda$ if and only if $\check{\rho}_j \cong \lambda^{-1}$.

Proof. First note that if $T : \rho_j \to \lambda$ is the isomorphism, then $T(\rho_j(l)v) = \lambda(c)T(v) = T(\lambda(c)v)$, and (since T is one-one) so $\rho_j(c) = \lambda(c)$ for all $c \in C$. Thus,

$$\dim(\sigma^{(C,\lambda)}) \geq \#\text{ of } j\text{'s such that } \rho_j \cong \lambda.$$

For the converse, first suppose that for some $1 \leq m \leq n$, we have

$$\rho_j \begin{cases} \cong \lambda, & 1 \leq j \leq m \\ \ncong \lambda, & j > m \end{cases}.$$

It suffices to check that

$$\sigma^{(C,\lambda)} \subset \bigoplus_{j=1}^{m} \rho_j.$$

Let $v = \sum_{j=1}^{n} v_j \in \sigma^{(C,\lambda)}$ with each $v_j \in \rho_j$. Applying $\sigma(c)$ for $c \in C$ to both sides of the expression for v, we get

$$\lambda(c)v = \sum_{j=1}^{n} \rho_j(c)v_j.$$

Linear independence of the v_j forces $\lambda(c)v_j = \rho_j(c)v_j$ for each j. By assumption, $v_j = 0$ for $m+1 \leq j \leq n$, and so $v \in \bigoplus_{j=1}^{m} \rho_j$, which completes the proof for the first statement.

Suppose $\rho_j \cong \lambda$ and let $c \in C, \xi \in \check{\rho}_j$. We have

$$\check{\rho}_j(c)\xi = \rho_j(c^{-1})\xi = \lambda(c^{-1})\xi = \lambda(c)^{-1}\xi.$$

Conversely if $\check{\rho}_j \cong \lambda^{-1}$, then $\check{\rho}_j(c)\xi = \lambda(c)^{-1}\xi$ (same as the above argument for $\rho_j \cong \lambda$). We thus have

$$\rho_j(c)v = \check{\rho}_j(c^{-1})v = \big(\lambda(c^{-1})\big)^{-1}v = \lambda(k)v.$$

■

Proposition 3.1.3 *Let K, Q be groups, let σ be an irreducible representation of K and let $\lambda : Q \to \mathbb{C}^\times$ be a character of Q (a 1-dimensional representation of Q). Then*

$$\dim(E_\sigma^{(K\cap Q,\lambda)}) = \dim(\check{E}_\sigma^{(K\cap Q,\lambda^{-1})}).$$

Proof. By applying Lemma 3.1.2 where $C = K \cap Q$, we have

$$\begin{aligned} \dim E_\sigma^{(K\cap Q,\lambda)} &= \#\text{ of } j \text{ such that } \rho_j \cong \lambda \\ &= \#\text{ of } j \text{ such that } \check{\rho}_j \cong \lambda^{-1} \\ &= \dim \check{E}_\sigma^{(K\cap Q,\lambda^{-1})}. \end{aligned}$$

■

To summarize the results in this section, for a closed Lie group G, K a compact subgroup, Q a closed subgroup, $G = KQ$, and $\lambda : Q \to \mathbb{C}^\times$ a smooth homomorphism of Q, The K-finite space of $\mathrm{ind}_{K\cap Q}^{K}(\lambda|_{K\cap Q})$ is spanned by matrix coefficients from irreducible representations σ of K such that $E_\sigma^{(K\cap Q,\lambda)} \neq \{0\}$.

3.2 Introducing the Group $SO_0(2p, 2q)$ and the K-Finite Decomposition of some of its Induced Representations

Let $\mathbb{R}^{p,q}$ be the space of column vectors

$$x = \begin{pmatrix} x_1 \\ \hline x_2 \end{pmatrix}, \quad x_1 \in \mathbb{R}^p, x_2 \in \mathbb{R}^q,$$

endowed with the indefinite inner product

$$(x, x')_{p,q} = (x_1, x_1')_p - (x_2, x_2')_q.$$

Matrix multiplication defines a left action of $GL(p+q, \mathbb{R})$ on $\mathbb{R}^{p,q}$, and the subgroup of isometries of $\mathbb{R}^{p,q}$ that fix the origin is called the *indefinite orthogonal group in degrees p and q*, denoted $O(p,q)$. That is, $O(p,q)$ is the subgroup of $GL(p+q,\mathbb{R})$ such that

$$(gx, gx')_{p,q} = (x, x')_{p,q}, \quad x, x' \in \mathbb{R}^{p,q}, g \in O(p,q).$$

Equivalently,

$$O(p,q) = \{g \in GL(p+q, \mathbb{R}) : gJg^tJ = I_{p+q}\},$$

where $J = \mathrm{diag}(I_p, -I_q)$. The subgroup of $O(p,q)$ whose matrices have determinant 1 is called the *indefinite special orthogonal group*, denoted $SO(p,q)$. That is,

$$SO(p,q) = \{g \in GL(p+q, \mathbb{R}) : gJg^tJ = I_{p+q}, \det g = 1\},$$

where $J = \mathrm{diag}(I_p, -I_q)$. The maximal compact subgroup of $SO(p,q)$ is $S\big(O(p) \times O(q)\big)$ (interpreted in the obvious way), which has two connected components: one where both diagonal blocks have positive determinant, and one where both diagonal blocks have negative determinant. By polar decomposition, $SO(p,q)$ is not connected. We will be working with the connected component of $SO(p,q)$, denoted $SO_0(p,q)$, which has a maximal compact subgroup $SO(p) \times SO(q)$ embedded in the obvious way. Let

$$G = SO_0(p,q) \text{ and } K_G = SO(p) \times SO(q).$$

An *isotropic vector* in $\mathbb{R}^{p,q}$ is a nonzero vector v such that $(v,v)_{p,q} = 0$. Let

$$v_1 = \begin{pmatrix} e_1 \\ \hline e_1 \end{pmatrix},$$

where e_1 is the first standard basis vector in its respective space. Let

$$Q_G = \mathrm{Stab}_G(\mathbb{R} \cdot v_1)$$

be the subgroup of G which takes v_1 to a nonzero (real) scalar multiple of itself. That is, for $q \in Q_G$, there is a $\lambda \in \mathbb{R}^\times$ such that $qv_1 = \lambda(q)v_1$. Let

$$\lambda : Q_G \to \mathbb{R}^\times, \quad q \mapsto \lambda(q).$$

Proposition 3.2.1 $G = K_G Q_G$.

Proof. We need to show that $G \subset K_G Q_G$. Let $g \in G$. Then $gv_1 = x$ for some $x = (x_1, x_2) \in \mathbb{R}^{p,q}$. Since $(v_1, v_1)_{p,q} = 0$ and g is an isometry, we have $(x, x)_{p,q} = 0$ so that $|x_1|_p = |x_2|_q$. Since $SO(p)$ and $SO(q)$ act transitively on the level sets of $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively, we may choose $k_1 \in SO(p)$ and $k_2 \in SO(q)$ such that

$$k_1 x_1 = |x_1|_p e_1,$$
$$k_2 x_2 = |x_2|_q e_1.$$

Let $k = \operatorname{diag}(k_1, k_2) \in K$. We thus have

$$kgv_1 = kx = \left(\frac{|x_1|_p e_1}{|x_2|_q e_1}\right) = |x_1|_p v_1.$$

Therefore, $kg \in Q_G$, or $g \in k^{-1} Q_G \subset K_G Q_G$, as required. ■

We wish to work with induced representations induced from smooth characters (i.e., 1-dimensional representations) of Q_G

$$\lambda_{s,\varepsilon} : Q_G \to \mathbb{C}^\times$$

of the form

$$\lambda_{s,\varepsilon}(q) = |\lambda(q)|_\varepsilon^s,$$

where

$$|x|_\varepsilon^s = \begin{cases} |x|^s, & \varepsilon = + \\ \operatorname{sgn}(x)|x|^s, & \varepsilon = -, \end{cases}$$

and $s \in \mathbb{C}$. For later reference, we will show that $\lambda_{s,\varepsilon}(-I) = \varepsilon$.

Lemma 3.2.1 *We have* $\lambda_{s,\varepsilon}(-I) = \varepsilon$.

Proof. Visibly we see that $-I \in Q_G$ with $\lambda(-I) = -1$. We have

$$\begin{aligned} \lambda_{s,\varepsilon}(-I) &= |\lambda(-I)|_\varepsilon^s \\ &= \begin{cases} |\lambda(-I)|^s, & \varepsilon = + \\ \operatorname{sgn}(\lambda(q))|\lambda(-I)|^s, & \varepsilon = - \end{cases} \\ &= \begin{cases} 1^s, & \varepsilon = + \\ -1^s, & \varepsilon = - \end{cases} \\ &= \varepsilon. \end{aligned}$$

■

We are particularly concerned with the smooth induced representations of G from the characters $\lambda_{s,\varepsilon}$ of Q_G, which we define as

$$\operatorname{ind}_{Q_G}^G(\lambda_{s,\varepsilon}) = \{\varphi \in C^\infty(G) : \varphi(gq) = \lambda_{s,\varepsilon}(q)^{-1}\varphi(g) \text{ for } g \in G, q \in Q_G\}.$$

Proposition 3.2.2

$$K_G \cap Q_G = \{\mathrm{diag}(\lambda, k_1, \lambda, k_2) : k_1 \in O(p-1), k_2 \in O(q-1), \lambda = \det k_1 = \det k_2\}.$$

If in addition p and q are both even, then every element of $K_G \cap Q_G$ can be factored as

$$(\lambda I_{p+q})\, diag(1, k_1, 1, k_2), \quad \lambda \in \{\pm 1\}, (k_1, k_2) \in SO(p-1) \times SO(q-1).$$

Proof. Let $k = \mathrm{diag}(k_1, k_2)$ for $k_1 \in SO(p), k_2 \in SO(q)$. Then $kv_1 = \lambda v_1$ for some $\lambda \in \mathbb{R}^\times$. Notice,

$$kv_1 = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \begin{pmatrix} e_1 \\ \overline{e_1} \end{pmatrix} = \begin{pmatrix} k_1 e_1 \\ k_2 e_1 \end{pmatrix} = \begin{pmatrix} (k_1)_{\text{col } 1} \\ (k_2)_{\text{col } 1} \end{pmatrix} = \lambda \begin{pmatrix} e_1 \\ \overline{e_1} \end{pmatrix}.$$

This implies that $(k_j)_{\text{col } 1} = \begin{pmatrix} \lambda \\ 0 \end{pmatrix}$ for $j = 1, 2$. By the same argument applied to the transpose of k_j, we have that $(k_j)_{\text{row } 1} = \begin{pmatrix} \lambda & 0 \end{pmatrix}$ for $j = 1, 2$. Therefore, $k_1 = \begin{pmatrix} \lambda & 0 \\ 0 & k_1' \end{pmatrix}$ for some $k_1' \in O(p-1)$, and since $k_1 \in SO(p)$, we have $\lambda = \det k_1'$. The same argument for k_2 implies that $k_2' \in O(q-1)$ with $\lambda = \det k_2'$. Therefore, $k = \mathrm{diag}(\det k_1', k_1', \det k_1', k_2')$ as claimed. Conversely, any element of this form is visibly in $K_G \cap Q_G$. This proves the first statement.

For the second statement, let $(\lambda, k_1, \lambda, k_2) \in K_G \cap Q_G$. If $\lambda = 1$, then this is just $(1, k_1, 1, k_2)$ and we are done. If $\lambda = -1$, then (since $p-1$ is odd) we have $\det(\lambda k_1) = \lambda^{p-1} \det(k_1) = \lambda \det(k_1) = 1$, and so $(\lambda k_1) \in SO(p-1)$. Similarly, $(\lambda k_1) \in SO(q-1)$ and we have

$$(\lambda, k_1, \lambda, k_2) = (\lambda I_{2p+2q})\mathrm{diag}(1, \lambda k_1, 1, \lambda k_2),$$

as required. ∎

This Thesis is concerned with the case when p and q are even, and so for the remainder of this Chapter and Thesis, we will let

$$G = SO_0(2p, 2q), K_G = SO(2p) \times SO(2q).$$

By Proposition 3.1.1, restriction $\varphi \mapsto \varphi|_{K_G}$ gives a K_G-isomorphism

$$\mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon}) \to \mathrm{ind}_{K_G \cap Q_G}^{K_G}(\lambda_{s,\varepsilon}|_{K_G \cap Q_G}).$$

The discussion in the first section shows that the K_G-finite space is spanned by matrix coefficients from irreducible representations σ of K_G such that $E_\sigma^{(K_G \cap Q_G, \lambda_{s,\varepsilon})} \neq \{0\}$. These are irreducible representations $E_{\sigma_1} \otimes E_{\sigma_2}$ of $K_G \cong SO(2p) \times SO(2q)$ with a nonzero vector which transforms by $\lambda_{s,\varepsilon}$ under $K_G \cap Q_G$. According to Proposition 3.2.2, we may write

$$K_G \cap Q_G = (K_G \cap Q_G)_0 \cup (K_G \cap Q_G)_1,$$

where

$$(K_G \cap Q_G)_0 = \{\mathrm{diag}(1, k_1, 1, k_2) : (k_1, k_2) \in SO(2p-1) \times SO(2q-1)\},$$

and

$$(K_G \cap Q_G)_1 = \{(-I_{2p+2q})\mathrm{diag}(1, k_1, 1, k_2) : (k_1, k_2) \in SO(2p-1) \times SO(2q-1)\}.$$

Visibly we see that $\lambda_{s,\varepsilon} \equiv 1$ (is identically 1) on $(K_G \cap Q_G)_0$, and so $\sigma_1 \otimes \sigma_2$ should have a nonzero vector which is fixed on this subgroup. By the discussion in Chapter 1, $\sigma_1 \cong \mathscr{H}^m(\mathbb{R}^{2p})$, and $\sigma_2 \cong \mathscr{H}^n(\mathbb{R}^{2q})$ for some $m, n \geq 0$. Thus,

$$\sigma_1 \otimes \sigma_2 \cong \mathscr{H}^m(\mathbb{R}^{2q}) \otimes \mathscr{H}^n(\mathbb{R}^{2q})$$

for some $m, n \geq 0$.

Recall that by Proposition 2.2.1, $(-I_p).P = (-1)^m P$ for any $P \in \mathscr{P}^m(\mathbb{R}^{2p})$. Thus,

$$(-I_{2p+2q}).(P_1 \otimes P_2) = (-1)^{m+n} P_1 \otimes P_2$$

for any $P_1 \otimes P_2 \in \mathscr{H}^m(\mathbb{R}^{2p}) \otimes \mathscr{H}^n(\mathbb{R}^{2q})$. On the other hand, we showed in Lemma 3.2.1 that $\lambda_{s,\varepsilon}(-I_{2p+2q}) = \varepsilon$, and so the vector must transform by ε on $(K_G \cap Q_G)_1$.

Therefore, the K_G types in the induced representation are those $\mathscr{H}^m(\mathbb{R}^{2p}) \otimes \mathscr{H}^n(\mathbb{R}^{2q})$ with $(-1)^{m+n} = \varepsilon$.

Explicitly, the embedding of a K_G-type

$$\mathscr{H}^m(\mathbb{R}^{2p}) \otimes \mathscr{H}^n(\mathbb{R}^{2q}) \hookrightarrow \mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon})$$

is given by matrix coefficients. For a simple tensor $v_1 \otimes v_2 \in \mathscr{H}^m(\mathbb{R}^{2p}) \otimes \mathscr{H}^n(\mathbb{R}^{2q})$ and ξ^m, ξ^n the respective embedding vectors for these spaces discussed in Chapter 1, and $k = \mathrm{diag}(k_1, k_2) \in K_G$, matrix coefficients from are $\varphi_{v_1 \otimes v_2, \xi^m \otimes \xi^n} \in C^\infty(K_G)$ given by

$$\varphi_{v_1 \otimes v_2, \xi^m \otimes \xi^n}(k) = \langle k_1^{-1}.v_1, \xi^m \rangle \langle k_2^{-1}.v_2, \xi^n \rangle.$$

This embedding commutes with the action of K_G. For $k' = (k_1', k_2') \in K_G$, notice that

$$\begin{aligned}
k'.\varphi_{v_1 \otimes v_2, \xi^m \otimes \xi^n}(k) &= \varphi_{v_1 \otimes v_2, \xi^m \otimes \xi^n}\big((k')^{-1}k\big) \\
&= \Big\langle \big((k_1')^{-1}k_1\big)^{-1}.v_1, \xi^m \Big\rangle \Big\langle \big((k_2')^{-1}k_2\big)^{-1}.v_2, \xi^n \Big\rangle \\
&= \big\langle (k_1^{-1}k_1').v_1, \xi^m \big\rangle \big\langle (k_2^{-1}k_2').v_2, \xi^n \big\rangle \\
&= \big\langle k_1^{-1}(k_1'.v_1), \xi^m \big\rangle \big\langle k_2^{-1}(k_2'.v_2), \xi^n \big\rangle \\
&= \varphi_{(k_1'.v_1) \otimes (k_2'.v_2), \xi^m \otimes \xi^n}(k) \\
&= \varphi_{k'.(v_1 \otimes v_2), \xi^m \otimes \xi^n}(k).
\end{aligned}$$

By giving this function the translation property on Q_G, we can define $\tilde{\varphi}_{v_1 \otimes v_2, \xi^m \otimes \xi^n} \in \mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon})$ by $\tilde{\varphi}_{v_1 \otimes v_2, \xi^m \otimes \xi^n}(kq) = \lambda_{s,\varepsilon}(q)^{-1}\varphi_{v_1 \otimes v_2}(k)$. This means that we can regard each $\mathscr{H}^m(\mathbb{R}^{2n}) \otimes \mathscr{H}^n(\mathbb{R}^{2n})$ as a subspace of the $\mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon})$. Copies of each K_G-type embed as the same matrix coefficient, up to positive scalar, and each K_G-type occurs at most once in the principal series. As a summary of this section, we have

$$\mathrm{ind}_{K_G \cap Q_G}^{K_G}(\lambda_{s,\varepsilon}|_{K_G \cap Q_G}) \cong \sum_{\substack{m,n \geq 0 \\ (-1)^{m+n} = \varepsilon}} \mathscr{H}^m(\mathbb{R}^{2p}) \otimes \mathscr{H}^n(\mathbb{R}^{2q}).$$

3.3 Introducing the Group $U(p,q)$ and the K-Finite Decomposition of some of its Induced Representations

Let $\mathbb{C}^{p,q}$ be the space of column vectors

$$z = \begin{pmatrix} z_1 \\ \hline z_2 \end{pmatrix}, \quad z_1 \in \mathbb{C}^p, z_2 \in \mathbb{C}^q,$$

endowed with the indefinite inner product

$$(z, z')_{p,q} = (z_1, z_1')_p - (z_2, z_2')_q.$$

Matrix multiplication defines a left action of $GL(p+q, \mathbb{C})$ on $\mathbb{C}^{p,q}$, and the subgroup of isometries of $\mathbb{C}^{p,q}$ that fix the origin is called the *indefinite unitary group in degrees p and q*, denoted $U(p,q)$. That is, $U(p,q)$ is the subgroup of $GL(p+q,\mathbb{C})$ such that

$$(gz, gz')_{p,q} = (z, z')_{p,q}, \quad z, z' \in \mathbb{C}^{p,q}, g \in U(p,q).$$

The inverse of a matrix in $U(p,q)$ is apparent from the equivalent characterization

$$U(p,q) = \{g \in GL(p+q, \mathbb{C}) : gJg^*J = I_{p+q}\},$$

where $J = \mathrm{diag}(I_p, -I_q)$. The group $U(p,q)$ is connected, and the maximal compact subgroup of $U(p,q)$ is $U(p) \times U(q)$ (interpreted in the obvious way). In this Chapter and for the rest of this Thesis, we denote

$$H = U(p,q), K_H = U(p) \times U(q).$$

An *isotropic vector* in $\mathbb{C}^{p,q}$ is a nonzero vector v such that $(v,v)_{p,q} = 0$. We define two such vectors:

$$v_1 = \begin{pmatrix} e_1 \\ \hline e_1 \end{pmatrix}, v_{-1} = \begin{pmatrix} e_1 \\ \hline -e_1 \end{pmatrix},$$

where e_1 is the first standard basis vector in its respective space. We note that v_1 here is the same v_1 from the previous section under the identification $\mathbb{R}^{2p,2q} \cong \mathbb{C}^{p,q}$.

Let

$$Q_H = \mathrm{Stab}_H(\mathbb{C} \cdot v_1)$$

be the subgroup of H which takes v_1 to a nonzero (complex) scalar multiple of itself. That is, for $q \in Q_H$, there is a $\chi(q) \in \mathbb{C}^\times$ such that $qv_1 = \chi(q)v_1$.

Proposition 3.3.1 *$H = K_H Q_H$.*

Proof. This is essentially the same as Proposition 3.2.1. ■

Let $\chi : Q_H \to \mathbb{C}^\times$ be such that $qv_1 = \chi(q)v_1$. We wish to work with induced representations induced from characters (i.e., 1-dimensional representations) of Q_H

$$\chi_{s,a} : Q_H \to \mathbb{C}^\times$$

of the form

$$\chi_{s,a}(q) = \left(\frac{\chi(q)}{|\chi(q)|}\right)^a |\chi(q)|^s,$$

where $s \in \mathbb{C}$ and $a \in \mathbb{Z}$. Let

$$\operatorname{ind}_{Q_H}^H(\chi_{s,a}) = \{f \in C^\infty(H) : f(hq) = \chi_{s,a}(q)^{-1} f(h) \text{ for } h \in H, q \in Q_H\}$$

be the induced representation of $\chi_{s,a}$ from Q_H to H.

Proposition 3.3.2 $K_H \cap Q_H \cong \mathbb{S}^1 \times U(p-1) \times U(q-1)$.

Proof. First we show that

$$K_H \cap Q_H = \{\operatorname{diag}(\chi, k_1', \chi, k_2') : \chi \in \mathbb{S}^1, k_1' \in U(p-1), k_2' \in U(q-1)\}.$$

Let $k = \operatorname{diag}(k_1, k_2)$ for $k_1 \in U(p), k_2 \in U(q)$ and let $kv_1 = \chi v_1$ for some $\chi \in \mathbb{C}^\times$. Notice that

$$kv_1 = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \begin{pmatrix} e_1 \\ \overline{e_1} \end{pmatrix} = \begin{pmatrix} k_1 e_1 \\ k_2 e_1 \end{pmatrix} = \begin{pmatrix} (k_1)_{\text{col } 1} \\ (k_2)_{\text{col } 1} \end{pmatrix} = \chi \begin{pmatrix} e_1 \\ \overline{e_1} \end{pmatrix}.$$

This implies that $(k_j)_{\text{col } 1} = \begin{pmatrix} \chi \\ 0 \end{pmatrix}$ for $j = 1, 2$. By the same argument applied to the transpose of k_j, we have that $(k_j)_{\text{row } 1} = \begin{pmatrix} \chi & 0 \end{pmatrix}$ for $j = 1, 2$. Therefore, $k_1 = \begin{pmatrix} \chi & 0 \\ 0 & k_1' \end{pmatrix}$ for some $k_1' \in U(p-1)$, and since $k_1 \in U(p)$, we have $|\chi| = 1$. The same argument for k_2 implies that $k_2' \in U(q-1)$. Therefore, $k = \operatorname{diag}(\chi, k_1', \chi, k_2')$ as claimed. Conversely, any element of this form is visibly in $K_H \cap Q_H$.

The map $\operatorname{diag}(\chi, k_1', \chi, k_2') \mapsto (\chi, k_1', k_2')$ visibly provides the claimed isomorphism. ■

By Proposition 3.1.1, restriction $f \mapsto f|_{K_H}$ gives a K_H-isomorphism

$$\operatorname{ind}_{Q_H}^H(\chi_{s,a}) \to_{K_H} \operatorname{ind}_{K_H \cap Q_H}^{K_H}(\chi_{s,a}|_{K_H \cap Q_H}).$$

The discussion in the first section shows that the K_H-finite space is spanned by matrix coefficients from irreducible representations σ of K_H such that $E_\sigma^{(K_H \cap Q_H, \chi_{s,a})} \neq \{0\}$. These are irreducible representations $E_{\sigma_1} \otimes E_{\sigma_2}$ of $K_H \cong U(p) \times U(q)$ with a nonzero vector which transforms by $\chi_{s,a}$ under restriction to $K_H \cap Q_H$. Under the isomorphism in Proposition 3.3.2, we visibly see that $\chi|_{\{1\} \times U(p-1) \times U(q-1)} \equiv 1$. It follows that $\sigma_1 \boxtimes \sigma_2$ must have a nonzero vector which is fixed on this subgroup. By the discussion in Chapter 1,

$$\sigma_1 \boxtimes \sigma_2 \cong \mathscr{H}^{m_1,m_2}(\mathbb{C}^p) \otimes \mathscr{H}^{n_1,n_2}(\mathbb{C}^q)$$

for some $m_1, m_2, n_1, n_2 \geq 0$.

The other part of $K_H \cap Q_H$ determines the other condition that the m_1, m_2, n_1, n_2 must satisfy, which we will now show. First, recall that elements of $Z\big(U(p)\big)$ are of the form $(e^{i\theta} I_p)$ for some $\theta \in [0, 2\pi)$. Similarly, elements of $Z(H)$ are of the form $(e^{i\theta} I_{p+q})$. For later reference, we state the action of $Z(H)$ on a K_H type $\mathscr{H}^{m_1,m_2}(\mathbb{C}^p) \otimes \mathscr{H}^{n_1,n_2}(\mathbb{C}^q)$.

Proposition 3.3.3 $(e^{i\theta}I_{p,q}).v = e^{-i\theta(m_1-m_2+n_1-n_2)}v, \quad v \in \mathscr{H}^{m_1,m_2}(\mathbb{C}^p)\otimes\mathscr{H}^{n_1,n_2}(\mathbb{C}^q)$.

Proof. Since the action on $\mathscr{H}^{m_1,m_2}(\mathbb{C}^p)\otimes\mathscr{H}^{n_1,n_2}(\mathbb{C}^q)$ is inherited from $\mathscr{P}^{m_1,m_2}(\mathbb{C}^p)\otimes\mathscr{P}^{n_1,n_2}(\mathbb{C}^q)$, it suffices to prove the claim for a simple tensor $P\otimes Q \in \mathscr{P}^{m_1,m_2}(\mathbb{C}^p)\otimes\mathscr{P}^{n_1,n_2}(\mathbb{C}^q)$.

Previously I have shown in [Proposition from Chapter 1] that

$$(e^{i\theta}I_p).P = e^{-i\theta(m_1-m_2)}P, \qquad P\in\mathscr{P}^{m_1,m_2}(\mathbb{C}^p).$$

Therefore,

$$\begin{aligned}(e^{i\theta}I_{p+q}).P\otimes Q &= (e^{i\theta}I_p).P\otimes(e^{i\theta}I_q).Q\\ &= \left(e^{-i\theta(m_1-m_2)}P\right)\otimes\left(e^{-i\theta(n_1-n_2)}Q\right)\\ &= e^{-i\theta(m_1-m_2+n_1-n_2)}P\otimes Q.\end{aligned}$$

The claim follows by linearity. ■

By Proposition3.3.3, we have

$$(e^{i\theta}I_{p+q}).(P_1\otimes P_2) = e^{-i\theta(m_1-m_2+n_1-n_2)}P_1\otimes P_2$$

for any $P_1\otimes P_2\in\mathscr{H}^{m_1,m_2}(\mathbb{C}^p)\otimes\mathscr{H}^{n_1,n_2}(\mathbb{C}^q)$. On the other hand, $\chi(e^{i\theta}I_{p+q}) = e^{i\theta}$, and so

$$\chi_{s,a}(e^{i\theta}I_{p+q}) = \left(\frac{e^{i\theta}}{|e^{i\theta}|}\right)^a |e^{i\theta}|^s = e^{ia\theta}.$$

Thus

$$e^{-i\theta(m_1-m_2+n_1-n_2)} = e^{ia\theta}, \quad \theta\in[0,2\pi),$$

and so

$$m_1-m_2+n_1-n_2 = -a.$$

In a very similar way that we can regard each $\mathscr{H}^m(\mathbb{R}^{2p})\otimes\mathscr{H}^n(\mathbb{R}^{2q})$ as a subspace of the $\mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon})$, we can also regard each $\mathscr{H}^{m_1,m_2}(\mathbb{C}^p)\otimes\mathscr{H}^{n_1,n_2}(\mathbb{C}^q)$ as a subspace of $\mathrm{ind}_{Q_H}^H(\chi_{s,a})$ (by first using matrix coefficients, and then giving it the translation property on Q_H). As in the previous section, each K_H-type will occur at most once in the decomposition of the induced representation. To summarize the main results of this section,

$$\mathrm{ind}_{K_H\cap Q_H}^{K_H}(\chi_{s,a}|_{K_H\cap Q_H}) \cong \sum_{\substack{m_1,m_2,n_1,n_2\geq 0\\ m_1-m_2+n_1-n_2=-a}} \mathscr{H}^{m_1,m_2}(\mathbb{C}^p)\otimes\mathscr{H}^{n_1,n_2}(\mathbb{C}^q).$$

CHAPTER IV

A FAMILY OF INTEGRAL INTERTWINING OPERATORS

4.1 Motivating the Connection between Orthogonal and Unitary Settings

In the previous chapter, we realized the K_G-finite space for $\mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon})$ as sums of spaces $\mathscr{H}^m(\mathbb{R}^{2n})\otimes\mathscr{H}^n(\mathbb{R}^{2n})$ where $(-1)^{m+n}=\varepsilon$, and we realized the K_H-finite space for $\mathrm{ind}_{Q_H}^H(\chi_{s,a})$ as sums of spaces $\mathscr{H}^{m_1,m_2}(\mathbb{C}^p)\otimes\mathscr{H}^{n_1,n_2}(\mathbb{C}^q)$ where $m_1-m_2+n_1-n_2=-a$. I also explained how these spaces embed inside their respective degenerate principal series via matrix coefficients. In fact, by identifying $\mathbb{R}^{2p}\cong\mathbb{C}^p$ and writing the Laplacian in complex coordinates (see Chapter 1), we have

$$\mathscr{H}^{m_1,m_2}(\mathbb{C}^p)\subset\mathscr{H}^{m_1+m_2}(\mathbb{C}^p)\cong\mathscr{H}^{m_1+m_2}(\mathbb{R}^{2p}).$$

It follows that

$$\mathscr{H}^{m_1,m_2}(\mathbb{C}^p)\otimes\mathscr{H}^{n_1,n_2}(\mathbb{C}^q)\subset\mathscr{H}^{m_1+m_2}(\mathbb{C}^p)\otimes\mathscr{H}^{n_1+n_2}(\mathbb{C}^q)\cong\mathscr{H}^{m_1+m_2}(\mathbb{R}^{2p})\otimes\mathscr{H}^{n_1+n_2}(\mathbb{R}^{2q}),$$

and with his identification, we in fact have

$$\mathscr{H}^m(\mathbb{R}^{2p})\otimes\mathscr{H}^n(\mathbb{R}^{2q})=\sum_{\substack{m_1+m_2=m\\ n_1+n_2=n}}\mathscr{H}^{m_1,m_2}(\mathbb{C}^p)\otimes\mathscr{H}^{n_1,n_2}(\mathbb{C}^q).$$

This allows us to regard K_H-types as subsets of K_G-types in the case that $(-1)^a=\varepsilon$. Thus, for this case we may attempt to construct a map between these degenerate principal series which projects K_G-types in $\mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon})$ onto the K_H-types in $\mathrm{ind}_{Q_H}^H(\chi_{s,a})$ (those K_H-types contained in the co-domain).

4.2 Defining T_a and showing some of its properties

Recall that in Chapter 1, we introduced the embedding $U(n)\hookrightarrow SO(2n)$ given entry-wise by

$$(a+bi)\mapsto\begin{pmatrix}a & -b\\ b & a\end{pmatrix}.$$

The same entry-wise map gives an embedding $H\hookrightarrow G$, which allows us to regard H as a subgroup of G. This means that a function in the principal series $\mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon})$ can be restricted to give a function on H. This function is unlikely to be in any particular $\mathrm{ind}_{Q_H}^H(\chi_{s,a})$, especially since $Q_H\not\subset Q_G$ (we note that $K_H\subset K_G$ under this embedding). The subgroup Q_H stabilizes a plane in $\mathbb{R}^{2p,2q}$, and Q_G stabilizes a line in that plane, but Q_H may rotate this line around in that plane. So in order to define this map, we will make use of the scalar matrices in H, which for $\theta \in [0, 2\pi)$, we will write as

$$z(\theta) = (e^{i\theta} I_H) \in Z(H).$$

Before defining the map, we note for future reference that $z(\theta)$ embeds in G as diagonal blocks:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SO(2),$$

and that $z(\theta) \in K_H \subset K_G$. Now for the map!

For $a \in \mathbb{Z}$ such that $(-1)^a = \varepsilon$, let

$$T_a : \mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon}) \to \mathrm{ind}_{Q_H}^H(\chi_{s,a})$$

given by

$$T_a(\varphi)(h) = \int_{-\pi/2}^{\pi/2} \varphi\big(hz(\theta)\big)e^{ia\theta} d\theta.$$

Assume for the moment that this map makes sense in terms of convergence of integrals and that $T_a(\varphi)$ is smooth. In order to see that $T_a(\varphi)$ has the right translation property under Q_H, first notice that the integrand is π-periodic in the variable θ:

$$\begin{aligned} \varphi\big(hz(\theta+\pi)\big)e^{ia(\theta+\pi)} &= \varphi\big(hz(\theta)z(\pi)\big)e^{ia\theta}e^{ia\pi} \\ &= \lambda_{s,\varepsilon}\big(z(\pi)\big)^{-1}\varphi\big(hz(\theta)\big)e^{ia\theta}(-1)^a \\ &= \varepsilon \cdot \varphi\big(hz(\theta)\big)e^{ia\theta} \cdot \varepsilon \\ &= \varphi\big(hz(\theta)\big)e^{ia\theta}. \end{aligned}$$

In order to prove the translation property, we need the following factorization lemma.

Lemma 4.2.1 *Every $q \in Q_H$ factors as $q = \gamma z(\psi)$ for some $\gamma \in (Q_G)_0 \cap Q_H$ and $z(\psi) \in Z(H)$.*

Proof. We have $qv_1 = \lambda e^{i\psi} v_1$ for some $\lambda > 0$ and $\psi \in [0, 2\pi)$. Notice that $qz(-\psi)v_1 = \lambda v_1$, and so $q = qz(-\psi) \cdot z(\psi) = \gamma z(\psi)$ suffices. ■

Proposition 4.2.1 *If $\varphi \in \mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon})$, then $T_a(\varphi)(hq) = \chi_{s,a}(q)^{-1}\varphi(h)$ for all $h \in H, q \in Q_H$.*

Proof. By Lemma 4.2.1, we may choose $\gamma \in (Q_G)_0 \cap Q_H$ and $z(\psi) \in Z(H)$ such that $q = \gamma z(\psi)$. Notice that $\lambda(\gamma) = \chi(\gamma) > 0$ and so

$$\begin{aligned} \chi(q)v_1 &= qv_1 \\ &= \gamma z(\psi)v_1 \\ &= \gamma e^{i\psi} v_1 \\ &= e^{i\psi}\chi(\gamma)v_1. \end{aligned}$$

We thus have

$$
\begin{aligned}
T_a(\varphi)(hq) &= \int_{-\pi/2}^{\pi/2} \varphi(hqz(\theta))e^{ia\theta}d\theta \\
&= \int_{-\pi/2}^{\pi/2} \varphi(h\gamma z(\psi)z(\theta))e^{ia\theta}d\theta \\
&= \int_{-\pi/2}^{\pi/2} \varphi(h\gamma z(\psi+\theta))e^{ia\theta}d\theta \\
&= \int_{\psi-\pi/2}^{\psi+\pi/2} \varphi(h\gamma z(u))e^{ia(u-\psi)}du && \text{(subbing } u=\psi+\theta\text{)} \\
&= \int_{-\pi/2}^{\pi/2} \varphi(h\gamma z(u))e^{ia(u-\psi)}du && \text{(the integrand is } \pi\text{-invariant)} \\
&= \int_{-\pi/2}^{\pi/2} e^{-ia\psi}\varphi(hz(u)\gamma)e^{iau}du \\
&= \int_{-\pi/2}^{\pi/2} e^{-ia\psi}\lambda_{s,\varepsilon}(\gamma)^{-1}\varphi(hz(u))e^{iau}du \\
&= \int_{-\pi/2}^{\pi/2} e^{-ia\psi}|\lambda(\gamma)|_{\varepsilon}^{-s}\varphi(hz(u))e^{iau}du \\
&= \int_{-\pi/2}^{\pi/2} e^{-ia\psi}\lambda(\gamma)^{-s}\varphi(hz(u))e^{iau}du \\
&= \int_{-\pi/2}^{\pi/2} e^{-ia\psi}\chi(\gamma)^{-s}\varphi(hz(u))e^{iau}du \\
&= \int_{-\pi/2}^{\pi/2} \chi_{s,a}(q)^{-1}\varphi(hz(u))e^{iau}du \\
&= \chi_{s,a}(q)^{-1}T_a(\varphi)(h).
\end{aligned}
$$

■

T_a is H-intertwining, since for $h, x \in H$ we have

$$
\begin{aligned}
h.T_a(\varphi)(x) &= T_a(\varphi)(h^{-1}x) \\
&= \int_{-\pi/2}^{\pi/2} \varphi(h^{-1}xz(\theta))e^{ia\theta}d\theta \\
&= \int_{-\pi/2}^{\pi/2} (h.\varphi)(xz(\theta))e^{ia\theta}d\theta \\
&= T_a(h.\varphi)(x).
\end{aligned}
$$

The integral is proper since the $\varphi(hz(\theta))$ is continuous on $\theta \in [0, 2\pi]$, so visibly the integral converges for every $h \in H$. In order to show that T_a is well-defined, it remains to show that if $\varphi \in \mathrm{ind}_{Q_G}^{G}(|\lambda|_{\varepsilon}^{s})$ is smooth, then $T_a(\varphi)$ is smooth, which we now do.

Proposition 4.2.2 *If $\varphi \in C^\infty(G)$, then $T_a\varphi \in C^\infty(H)$.*

Proof. Let $F(h) = \varphi(hz(\theta))e^{ia\theta}$. Let $X, Y \in \mathfrak{h} \subset \mathfrak{g}$, where $\mathfrak{g} = \mathrm{Lie}(G), \mathfrak{h} = \mathrm{Lie}(H)$. Since $\varphi \in C^\infty(G)$, we have

$$F_X(h) := \frac{d}{dt}\bigg|_{t=0} F(\exp(-tX)h) \in C^\infty(H).$$

Similarly,

$$F_{XY}(h) := \frac{d}{dt}\bigg|_{t=0} F_X(\exp(-tY)h) \in C^\infty(H),$$

and so on. Thus by Leibniz's Rule, we are justified each time differentiating under the integral of T_a. The derivative each time will be a continuous (differentiable) function, and so the integral will exist, which proves the claim. ■

Now that we have shown that T_a makes sense and is well-defined, we now wish to show that T_a projects onto the K_H-types it contains. We have discussed how to use matrix coefficients to regard $\mathscr{H}^m(\mathbb{R}^{2n}) \otimes \mathscr{H}^n(\mathbb{R}^{2n})$ as elements of $\mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon})$, and that the action of K_G commutes with this identification. We also mentioned that

$$\mathscr{H}^m(\mathbb{R}^{2p}) \otimes \mathscr{H}^n(\mathbb{R}^{2q}) = \sum_{\substack{m_1+m_2=m \\ n_1+n_2=n}} \mathscr{H}^{m_1,m_2}(\mathbb{C}^p) \otimes \mathscr{H}^{n_1,n_2}(\mathbb{C}^q),$$

and in fact T_a will project the spaces on the left-hand-side onto the spaces on the right-hand-side (although we will not normalize the integral so that it is a literal projection map in order to make future calculations slightly less cluttered), which we will now prove.

Proposition 4.2.3 *For $\varphi \in \mathscr{H}^{m_1,m_2}(\mathbb{C}^p) \otimes \mathscr{H}^{n_1,n_2}(\mathbb{C}^q)$ (regarded in the domain as a subspace of $\mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon})$, and in the codomain as a subspace of $\mathrm{ind}_{Q_H}^H(\chi_{s,a})$), we have*

$$T_a(\varphi) = \begin{cases} \pi\varphi, & m_1 - m_2 + n_1 - n_2 = -a, \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. Let $\varphi \in \mathscr{H}^{m_1,m_2}(\mathbb{C}^p) \otimes \mathscr{H}^{n_1,n_2}(\mathbb{C}^q) \in \mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon})$. For $h \in H$, we have

$$\begin{aligned}
T_a(\varphi)(h) &= \int_{-\pi/2}^{\pi/2} \varphi(hz(\theta))e^{ia\theta}d\theta \\
&= \int_{-\pi/2}^{\pi/2} \varphi(z(\theta)h)e^{ia\theta}d\theta && (\text{since } z(\theta) \in Z(H)) \\
&= \int_{-\pi/2}^{\pi/2} (z(-\theta).\varphi)(h)e^{ia\theta}d\theta \\
&= \int_{-\pi/2}^{\pi/2} e^{i\theta(m_1-m_2+n_1-n_2)}\varphi(h)e^{ia\theta}d\theta && (\text{by Proposition 3.3.3}) \\
&= \int_{-\pi/2}^{\pi/2} \varphi(h)e^{i\theta(a+(m_1-m_2+n_1-n_2))}d\theta \\
&= \begin{cases} \pi\varphi(h), & m_1 - m_2 + n_1 - n_2 = -a, \\ 0, & \text{otherwise}. \end{cases}
\end{aligned}$$

■

This says exactly that T_a projects the K_G-types in $\mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon})$ onto the K_H-types contained in $\mathrm{ind}_{Q_H}^H(\chi_{s,a})$, summarized with the following corollary.

Corollary 4.2.1 *Suppose $\mathscr{H}^{m_1,m_2}(\mathbb{R}^{2p}) \otimes \mathscr{H}^{n_1,n_2}(\mathbb{R}^{2q}) \subset \mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon})$. Then*

$$T_a\big(\mathscr{H}^{m_1,m_2}(\mathbb{R}^{2p}) \otimes \mathscr{H}^{n_1,n_2}(\mathbb{R}^{2q})\big) = \begin{cases} \mathscr{H}^{m_1,m_2}(\mathbb{C}^p) \otimes \mathscr{H}^{n_1,n_2}(\mathbb{C}^q), & m_1 - m_2 + n_1 - n_2 = -a \\ \{0\}, & \textit{otherwise.} \end{cases}$$

Now that we know how the map T_a behaves on its K_G-types, we'll show that, when regarded coordinate-wise, it gives an isomorphism from the K_G-types to the K_H-types.

Proposition 4.2.4 *The H-homomorphism*

$$\mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon}) \to \sum_{\substack{a\in\mathbb{Z} \\ (-1)^a=\varepsilon}} \mathrm{ind}_{Q_H}^H(\chi_{s,a}),$$

given by

$$\varphi \mapsto (T_a(\varphi))_{a\in\mathbb{Z}},$$

is an isomorphism of K-finite spaces.

Proof. The convergence of the integral sufficiently justifies linearity of this map.

To see that this map is one-one, suppose $(T_a(\varphi))_{a\in\mathbb{Z}} = 0$. Since T_a is projection from the K_G-type-component onto the K_H-type-component it contains, this means that every K_H-type-component is zero, and so every K_G-type-component is zero, which implies that $\varphi = 0$.

We will show that this map is onto on the level of K-types. First notice that if $\varphi \in \mathscr{H}^{m_1,m_2}(\mathbb{C}^p)\otimes\mathscr{H}^{n_1,n_2}(\mathbb{C}^q) \subset \mathrm{ind}_{Q_H}^H(\chi_{s,a})$, then we may regard φ as a member of $\mathscr{H}^{m_1+m_2}(\mathbb{R}^{2p})\otimes \mathscr{H}^{n_1+n_2}(\mathbb{R}^{2q}) \subset \mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon})$. By Proposition 4.2.3, $T_a(\varphi) = \pi\varphi$. Now if $(\varphi_a)_{a\in\mathbb{Z}} \in \sum_{a\in\mathbb{Z}} \mathrm{ind}_{Q_H}^H(\chi_{s,a})$, with $\varphi_a = 0$ for all but finitely-many a, and each φ_a has only finitely-many K_H-types where its component is nonzero. Then

$$\sum_{a\in\mathbb{Z}} \varphi_a \mapsto \big(T_a(\varphi_a)\big)_{a\in\mathbb{Z}} = (\varphi_a)_{a\in\mathbb{Z}}.$$

■

CHAPTER V

DIFFERENTIAL INTERTWINING OPERATORS IN THE NON-COMPACT PICTURE

5.1 Introducing the Non-compact Picture

In Chapter 3, we introduced degenerate principal series for G and H, and in Chapter 4 we defined a family of integral operators

$$T_a : \operatorname{ind}_{Q_G}^{G}(\lambda_{s,\varepsilon}) \to \operatorname{ind}_{Q_H}^{H}(\chi_{s,a}),$$

where $a \in \mathbb{Z}$ satisfies $(-1)^a = \varepsilon$, between these degenerate principal series. We explained how the operator T_a projects the K_G-types of $\operatorname{ind}_{Q_G}^{G}(\lambda_{s,\varepsilon})$ onto the K_H-types it contains.

In the next three chapters, we will introduce differential intertwining operators Δ and Ω_a, the Euclidean and Heisenberg wave operators, respectively, that map between degenerate principal series in their respective settings for certain parameters, and we will show that the integral transform T_a H-intertwines the actions of the differential operators. The kernel for Δ are known and worked out by Binegar and Zierau in [2], and so T_a connects the kernels of both differential operators in the two different settings.

In this chapter, we will introduce the non-compact picture for the principal series and introduce their respective differential intertwining operators Δ and Ω_a. We will show that for $s = p + q - 2$, the map

$$\Delta : \operatorname{ind}_{Q_G}^{G}(\lambda_{s,\varepsilon}) \to \operatorname{ind}_{Q_G}^{G}(\lambda_{s+2,\varepsilon})$$

is well-defined and G-intertwining, and that

$$\Omega_a : \operatorname{ind}_{Q_H}^{H}(\chi_{s,a}) \to \operatorname{ind}_{Q_H}^{H}(\chi_{s+2,a})$$

is well-defined and H-intertwining. To show this, we will use a well-known duality theorem (see for example Kubo-Ørsted [5, Theorem 2.3]) which associates Q-homomorphisms between generalized Verma modules with differential intertwining operators between parabolically-induced representations. Heuristically, these are second-order differential operators, which is why the parameter s is being increased by 2 in both settings.

In essence, for a reductive Lie group G with parabolic subgroup Q (writing $\mathfrak{q} = \operatorname{Lie}(Q)$, there is a Langlands decomposition $Q = MAN$. For certain parameters γ, σ, a character $(\lambda_{\gamma,\sigma}, \mathbb{C}_{\gamma,\sigma})$ of Q, with basis $\mathbb{1}_{\gamma,\sigma}$, has contragredient representation $(\check{\lambda}_{\gamma,\sigma}, \check{\mathbb{C}}_{\gamma,\sigma}) \cong (\lambda_{-\gamma,\sigma}, \mathbb{C}_{-\gamma,\sigma})$, and we can define a *generalized Verma module* induced from $\check{\lambda}_{\gamma,\sigma}$

$$M_{\mathfrak{q}}(-\gamma, \sigma) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} \mathbb{C}_{-\gamma,\sigma},$$

where the tensor over $\mathcal{U}(\mathfrak{q})$ identifies elements

$$uX \otimes \mathbb{1}_{-\gamma,\sigma} \sim u \otimes -d\lambda_{\gamma,\sigma}(X)\mathbb{1}_{-\gamma,\sigma}, \quad X \in \mathfrak{q}, u \in \mathcal{U}(g).$$

The parabolic Q acts on $M_{\mathfrak{q}}(-\gamma,\sigma)$ diagonally via the adjoint action Ad on $\mathcal{U}(\mathfrak{g})$ and $\lambda_{-\gamma,\sigma}$ on $\mathbb{C}_{-\gamma,\sigma}$.

We now state a reduced and slightly informal version of this duality theorem that applies to the principal series we have constructed thus far – those induced from one-dimensional representations (swapping places of parameters so that they match more closely to our notation).

Theorem 5.1.1 (Duality Theorem) *Let G be a reductive Lie group and Q a parabolic subgroup of G with Lie algebra $\mathfrak{q}$, and characters $(\lambda_{\gamma,\sigma}, \mathbb{C}_{\gamma,\sigma}), (\lambda_{\nu,\eta}, \mathbb{C}_{\nu,\eta})$ of Q of certain parameters $\gamma, \sigma, \nu, \eta$. There exists a natural linear isomorphism*

$$\mathcal{D} : \mathrm{Hom}_Q\big(\mathbb{C}^\vee_{\nu,\eta}, M_{\mathfrak{q}}(-\gamma, \sigma^\vee)\big) \to \mathrm{Diff}_G\big(\mathrm{ind}_Q^G(\lambda_{\gamma,\sigma}), \mathrm{ind}_Q^G(\lambda_{\nu,\eta})\big).$$

For $\Phi \in \mathrm{Hom}_Q\big(\mathbb{C}^\vee_{\nu,\eta}, M_{\mathfrak{q}}(-\gamma, \sigma^\vee)\big)$ with $\Phi(\mathbb{1}_{-\nu,\eta^\vee}) = u \otimes \mathbb{1}_{-\gamma,\sigma^\vee}$, where $u \in \mathcal{U}(\mathfrak{g})$, the operator $\mathcal{D}(\Phi) \in \mathrm{Diff}_G\big(\mathrm{ind}_Q^G(\lambda_{\gamma,\sigma}), \mathrm{ind}_Q^G(\lambda_{\nu,\eta})\big)$ is given by the right action by u :

$$\mathcal{D}(\Phi)(\varphi) = R(u)(\varphi).$$

We will conclude the chapter by calculating the K_H-finite kernel in $\mathrm{ind}_{Q_H}^H(\chi_{s,a})$ for the Heisenberg wave operator. In these calculations, we will make use of a theorem by Kable in [?], which allows us to calculate the kernel of the wave operator in terms of the kernel of an element of $\mathcal{U}(\mathfrak{k}_H)$, since both elements of $\mathcal{U}(\mathfrak{h})$ get identified in the generalized Verma module. The action of the element in $\mathcal{U}(\mathfrak{k}_H)$ makes use of an explicit action of $\mathfrak{k}_H \cong \mathfrak{u}(p) \oplus \mathfrak{u}(q)$ on the K_H-types.

5.2 Duality Theorem Application in G-setting

Let $\mathfrak{g} = \mathrm{Lie}(G)$. The grading element

$$H_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}$$

provides a Lie algebra grading

$$\mathfrak{g} = \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1).$$

We write $\mathfrak{q}_G = \mathrm{Lie}(Q_G) = \mathfrak{g}(0) \oplus \mathfrak{g}(1), \mathfrak{n}_G := \mathfrak{g}(1)$, and $\bar{\mathfrak{n}}_G := \mathfrak{g}(-1)$. Elements of $\mathfrak{n}_G$ are of the form

$$X_1(\alpha, \beta) = \begin{pmatrix} 0 & \alpha^T & 0 & \beta^T \\ -\alpha & 0 & \alpha & 0 \\ 0 & \alpha^T & 0 & \beta^T \\ \beta & 0 & -\beta & 0 \end{pmatrix}, \quad \begin{pmatrix} \alpha \\ - \\ \beta \end{pmatrix} \in \mathbb{R}^{2p-1,2q-1},$$

and elements of $\bar{\mathfrak{n}}_G$ are of the form

$$X_{-1}(\alpha,\beta) = \begin{pmatrix} 0 & -\alpha^T & 0 & \beta^T \\ \alpha & 0 & \alpha & 0 \\ 0 & \alpha^T & 0 & -\beta^T \\ \beta & 0 & \beta & 0 \end{pmatrix}, \quad \begin{pmatrix} \alpha \\ - \\ \beta \end{pmatrix} \in \mathbb{R}^{2p-1,2q-1}.$$

We write $\bar{N}_G := \exp(\bar{\mathfrak{n}}_G)$, and direct calculation shows that elements of $\bar{N}_G$ are of the form

$$\bar{n}_G(\alpha,\beta) = \begin{pmatrix} 1-R^2/2 & -\alpha^T & -R^2/2 & \beta^T \\ \alpha & I_{2p-1} & \alpha & 0 \\ R^2/2 & \alpha^T & 1+R^2/2 & -\beta^T \\ \beta & 0 & \beta & I_{2q-1} \end{pmatrix}, \quad \begin{pmatrix} \alpha \\ - \\ \beta \end{pmatrix} \in \mathbb{R}^{2p-1,2q-1}, R^2 = |\alpha|^2 - |\beta|^2.$$

The group law on $\bar{N}_G$ is given by

$$\bar{n}_G(\alpha_1,\beta_1)\bar{n}_G(\alpha_2,\beta_2) = \bar{n}_G(\alpha_1+\alpha_2,\beta_1+\beta_2),$$

and so $\bar{N}_G$ has a vector space structure that is isomorphic to $\mathbb{R}^{2p-1,2q-1}$. The product $\bar{N}_G Q_G$ is dense in G, and thus restriction from G to $\bar{N}_G$ is injective. It follows that, as vector spaces, we can regard $\mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon})$ as $C^\infty(\mathbb{R}^{2p-1,2q-1})$. This is the *non-compact picture* of the degenerate principal series. The *Euclidean wave operator* on $C^\infty(\mathbb{R}^{2p-1,2q-1})$ is given by

$$\Delta = \Delta_{2p-1,2q-1} = \sum_{j=1}^{2p-1} \partial_{\alpha_j}^2 - \sum_{k=1}^{2q-1} \partial_{\beta_k}^2.$$

A natural basis for $\bar{\mathfrak{n}}_G$ consists of elements $X_j := X_{-1}(e_j,0)$ and $Y_k := X_{-1}(0,e_k)$, and this gives us a correspondence between the action of Δ on $C^\infty(\mathbb{R}^{2p-1,2q-1})$ and the action of

$$\Delta := \left(\sum_{j=1}^{2p-1} X_j^2 - \sum_{k=1}^{2q-1} Y_k^2 \right) \in \mathcal{U}(\bar{\mathfrak{n}}_G)$$

on $\mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon})$, where the action is given by right-translation

$$R(X)(\varphi)(g) = \frac{d}{dt}\varphi\big(g\exp(tX)\big)\Big|_{t=0}, \quad \varphi \in \mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon}), g \in G, X \in \bar{\mathfrak{n}}_G,$$

extended complex-linearly. Since G acts on the left, these two actions commute, and so we say that Δ is G-intertwining.

We now show that when $s = p+q-2$, the differential intertwining operator Δ takes the principal series to one where s has been increased by 2.

Proposition 5.2.1 *For the parameter $s = p+q-2$, we have*

$$\Delta : \mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon}) \to \mathrm{ind}_{Q_G}^G(\lambda_{s+2,\varepsilon}).$$

Proof. We will use the duality theorem above, and to do that, we need to show that the map

$$\Phi : \mathbb{C}_{-(s+2),\varepsilon} \to M_{\mathfrak{q}_G}(-s,\varepsilon),$$

given by

$$\Phi(\mathbb{1}_{-(s+2),\varepsilon}) = \Delta \otimes \mathbb{1}_{-s,\varepsilon},$$

is a Q_G-homomorphism.

There is a factorization $Q_G = M_G A_G N_G$, where

$$M_G \cong SO(2p-1, 2q-1) \times \{\pm 1\}, A_G = \{\exp(tH_0) : t \in \mathbb{R}\}, N_G = \exp(\mathfrak{n}_G),$$

and so we need to check that the actions of each factor commutes with the above embedding.

The action of M_G is by ε in both places of the embedding. We will check the actions of A_G and N_G in detail.

For the action of A_G on Δ, we first notice that for $X \in \mathfrak{g}(-1)$ we have

$$\begin{aligned}
\mathrm{Ad}\big(\exp(tH_0)\big)(X) &= \exp\big(\mathrm{ad}(tH_0)\big)(X) \\
&= \sum_n \frac{\mathrm{ad}(tH_0)^n}{n!}(X) \\
&= \sum_n \frac{t^n[H_0,[\ldots,[H_0,X]]\ldots]}{n!} \\
&= \sum_n \frac{t^n(-1)^n X}{n!} && \text{(since } [H_0,X] = -X \text{ by definition)} \\
&= \sum_n \frac{(-t)^n}{n!} X \\
&= e^{-t}X.
\end{aligned}$$

Thus for $X^2 = X \otimes X \in \mathcal{U}(\bar{\mathfrak{n}}_G)$, we have

$$\mathrm{Ad}\big(\exp(tH_0)\big)(X^2) = e^{-t}X \otimes e^{-t}X = e^{-2t}X^2,$$

and it follows immediately that

$$\mathrm{Ad}\big(\exp(tH_0)\big)(\Delta) = e^{-2t}\Delta.$$

We next observe that as matrices, A_G consists of matrices of the form

$$\exp(tH_0) = \begin{pmatrix} \cosh t & 0 & \sinh t & 0 \\ 0 & I_{2p-1} & 0 & 0 \\ \sinh t & 0 & \cosh t & 0 \\ 0 & 0 & 0 & I_{2q-1} \end{pmatrix},$$

and so we can visibly see that $\lambda\big(\exp(tH_0)\big) = (\sinh t + \cosh t) = e^t$. It then follows that

$$\begin{aligned}
\exp(tH_0).\mathbb{1}_{-s,\varepsilon} &= \lambda_{s,\varepsilon}\big(\exp(tH_0)\big)^{-1}\mathbb{1}_{-s,\varepsilon} \\
&= \big|\lambda\big(\exp(tH_0)\big)\big|_{\varepsilon}^{-s}\mathbb{1}_{-s,\varepsilon} \\
&= e^{-st}\mathbb{1}_{-s,\varepsilon}.
\end{aligned}$$

Similarly,

$$\exp(tH_0).\mathbb{1}_{-(s+2),\varepsilon} = e^{-(s+2)t}\mathbb{1}_{-(s+2),\varepsilon}.$$

Finally, the calculation

$$\begin{aligned}
\Phi\big(\exp(tH_0).\mathbb{1}_{-(s+2),\varepsilon}\big) &= \Phi\big(e^{-(s+2)t}\mathbb{1}_{-(s+2),\varepsilon}\big) \\
&= \Delta \otimes \big(e^{-(s+2)t}\mathbb{1}_{-s,\varepsilon}\big) \\
&= e^{-2t}\Delta \otimes \big(e^{-st}\mathbb{1}_{-s,\varepsilon}\big) \\
&= \mathrm{Ad}\big(\exp(tH_0)\big).\Delta \otimes \lambda_{s,\varepsilon}\big(\exp(tH_0)\big)^{-1}\mathbb{1}_{-s,\varepsilon} \\
&= \exp(tH_0).\big(\Delta \otimes \mathbb{1}_{-s,\varepsilon}\big) \\
&= \exp(tH_0).\Phi\big(\mathbb{1}_{-(s+2),\varepsilon}\big).
\end{aligned}$$

shows that the action of A_G commutes with Φ.

Now we check the action of N_G commutes with Φ, which turns out to act trivially in both places. We check this by showing that $\bar{\mathfrak{n}}_G$ acts by zero in both places using the representative $X_1(e_1, 0)$. Visibly we can see that $n_G v_1 = v_1$ for all $n_G \in N_G$ and so $X_1 v_1 = 0$ for all $X_1 \in \mathfrak{n}_G$ and $d\lambda_{s,\varepsilon}(X) = 0$. It thus remains to check that $X_1(e_1, 0).\Delta = 0$, and this is where we will need $s = p + q - 2$.

Write

$$A = [X_1(e_1, 0), X_{-1}(e_j, 0)] \in \mathfrak{g}(0), \quad B = [A, X_{-1}(e_j, 0)] \in \mathfrak{g}(-1).$$

We have

$$\begin{aligned}
&X_1(e_1, 0).\big(X_{-1}(e_j, 0)^2 \otimes \mathbb{1}_{-s,\varepsilon}\big) \\
&= X_1(e_1, 0)X_{-1}(e_j, 0)^2 \otimes \mathbb{1}_{-s,\varepsilon} \\
&= \Big(AX_{-1}(e_j, 0) + X_{-1}(e_j, 0)X_1(e_j, 0)X_{-1}(e_j, 0)\Big) \otimes \mathbb{1}_{-s,\varepsilon} \\
&= \Big(B + X_{-1}(e_j, 0)A + X_{-1}(e_j, 0)A\Big) \otimes \mathbb{1}_{-s,\varepsilon} \\
&= \Big(B - 2d\lambda_{s,\varepsilon}(A)X_{-1}(e_j, 0)\Big) \otimes \mathbb{1}_{-s,\varepsilon}.
\end{aligned}$$

Now we calculate this expression. Let E_{ij} be the matrix with 1 in the i-th row, j-th column, and zeroes elsewhere. Then

$$\begin{aligned}
A &= [X_1(e_1, 0), X_{-1}(e_j, 0)] \\
&= \begin{pmatrix} 0 & e_1^t & 0 & 0 \\ -e_1 & 0 & e_1 & 0 \\ 0 & e_1^t & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -e_j^t & 0 & 0 \\ e_j & 0 & e_j & 0 \\ 0 & e_j^t & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -e_j^t & 0 & 0 \\ e_j & 0 & e_j & 0 \\ 0 & e_j^t & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & e_1^t & 0 & 0 \\ -e_1 & 0 & e_1 & 0 \\ 0 & e_1^t & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 2e_1^t e_j & 0 \\ 0 & 2(E_{1j} - E_{j1}) & 0 & 0 \\ 2e_1^t e_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= 2(\delta_{1j}H_0 + A_j),
\end{aligned}$$

where A_j is the matrix with the (2,2)-block is the matrix $E_{1j} - E_{j1}$ (note that $A_1 = 0$). We note that

$$d\lambda_{s,\varepsilon}(A) = 2\delta_{1j} d\lambda_{s,\varepsilon}(H_0) + 2d\lambda_{s,\varepsilon}(A_j) = 2\delta_{1j}s + 0 = 2\delta_{1j}s.$$

The calculation $A_j e_j = (1-\delta_{1j})e_1$ determines $[A_j, X_{-1}(e_j, 0)] = (1-\delta_{1j})X_{-1}(e_1, 0)$ and we have

$$\begin{aligned} B &= [2(\delta_{1j}H_0 + A_j), X_{-1}(e_j, 0)] \\ &= 2\delta_{1j}[H_0, X_{-1}(e_j, 0)] + 2[A_j, X_{-1}(e_j, 0)] \\ &= -2\delta_{1j}X_{-1}(e_j, 0) + 2(1-\delta_{1j})X_{-1}(e_1, 0). \end{aligned}$$

We now have

$$\begin{aligned} &X_1(e_1, 0). \left(\sum_{j=1}^{2p-1} X_{-1}(e_j, 0)^2 \otimes \mathbb{1}_{-s,\varepsilon} \right) \\ &= \sum_{j=1}^{2p-1} \Big(-2\delta_{1j}X_{-1}(e_j, 0) + 2(1-\delta_{1j})X_{-1}(e_1, 0) - 4\delta_{1j}sX_{-1}(e_j, 0) \Big) \otimes \mathbb{1}_{-s,\varepsilon} \\ &= \big(-2X_{-1}(e_1, 0) + 2(2p-2)X_{-1}(e_1, 0) - 4sX_{-1}(e_1, 0) \big) \otimes \mathbb{1}_{-s,\varepsilon} \\ &= \big(-2 + 2(2p-2) - 4s \big) X_{-1}(e_1, 0) \otimes \mathbb{1}_{-s,\varepsilon} \\ &= -2(1 - 2p + 2 + 2s)X_{-1}(e_1, 0) \otimes \mathbb{1}_{-s,\varepsilon} \\ &= -2(2s - 2p + 3)X_{-1}(e_1, 0) \otimes \mathbb{1}_{-s,\varepsilon} \end{aligned}$$

For the q-part, first observe that

$$\begin{aligned} &[X_1(e_1, 0), X_{-1}(0, e_k)] \\ &= \begin{pmatrix} 0 & e_1^t & 0 & 0 \\ -e_1 & 0 & e_1 & 0 \\ 0 & e_1^t & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & e_k^t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -e_k^t \\ e_k & 0 & e_k & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & e_k^t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -e_k^t \\ e_k & 0 & e_k & 0 \end{pmatrix} \begin{pmatrix} 0 & e_1^t & 0 & 0 \\ -e_1 & 0 & e_1 & 0 \\ 0 & e_1^t & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2E_{1k} \\ 0 & 0 & 0 & 0 \\ 0 & -2E_{k1} & 0 & 0 \end{pmatrix} \\ &= -2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & E_{1k} \\ 0 & 0 & 0 & 0 \\ 0 & E_{k1} & 0 & 0 \end{pmatrix} \\ &:= -2M_k \in \mathfrak{g}(0). \end{aligned}$$

Next, observe that

$$
\begin{aligned}
&[M_k, X_{-1}(0,e_k)]\\
&= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & E_{1k} \\ 0 & 0 & 0 & 0 \\ 0 & E_{k1} & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & e_k^t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -e_k^t \\ e_k & 0 & e_k & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & e_k^t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -e_k^t \\ e_k & 0 & e_k & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & E_{1k} \\ 0 & 0 & 0 & 0 \\ 0 & E_{k1} & 0 & 0 \end{pmatrix}\\
&= X_{-1}(E_{1k}e_k, 0) = X_{-1}(e_1, 0).
\end{aligned}
$$

Recalling that $d\lambda_{s,\varepsilon}(M_k) \subset d\lambda_{s,\varepsilon}\big(\mathfrak{so}(2p-1, 2q-1)\big) = \{0\}$, we have

$$
\begin{aligned}
&X_1(e_1,0).X_{-1}(0,e_k)^2 \otimes \mathbb{1}_{-s,\varepsilon}\\
&= \big(-2M_kX_{-1}(0,e_k) + X_{-1}(0,e_k)X_1(e_1,0)X_{-1}(0,e_k)\big) \otimes \mathbb{1}_{-s,\varepsilon}\\
&= (-2X_{-1}(e_1,0) + 0 + X_{-1}(0,e_k)M_k - 2X_{-1}(0,e_k)M_k) \otimes \mathbb{1}_{-s,\varepsilon}\\
&= -2X_{-1}(e_1,0) \otimes \mathbb{1}_{-s,\varepsilon}.
\end{aligned}
$$

Thus,

$$
X_1(e_1,0).\sum_{k=1}^{2q-1} X_{-1}(0,e_k)^2 \otimes \mathbb{1}_{-s,\varepsilon} = -2(2q-1)X_{-1}(e_1,0) \otimes \mathbb{1}_{-s,\varepsilon}.
$$

Adding the p and q parts together, we finally arrive at,

$$
\begin{aligned}
X_1(e_1,0).(\Delta \otimes \mathbb{1}_{-s,\varepsilon}) &= \big(-2(2s-2p+3) + 2(2q-1)\big)X_{-1}(e_1,0) \otimes \mathbb{1}_{-s,\varepsilon}\\
&= -2\big(2s-2p+3-2q+1\big)X_{-1}(e_1,0) \otimes \mathbb{1}_{-s,\varepsilon}\\
&= -2\big(2s-2p-2q+4\big)X_{-1}(e_1,0) \otimes \mathbb{1}_{-s,\varepsilon}.
\end{aligned}
$$

This is zero when

$$
\begin{aligned}
2s - 2p - 2q + 4 &= 0\\
2s &= 2p + 2q - 4\\
s &= p + q - 2,
\end{aligned}
$$

which completes the claim that the action of N_G commutes with Φ, and thus so does all of Q_G for the parameter $s = p + q - 2$. ■

5.3 Duality Theorem Application in H-setting

Let $\mathfrak{h} = \mathrm{Lie}(H)$. The grading element

$$
H_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{h}
$$

provides a Lie algebra grading

$$\mathfrak{h} = \mathfrak{h}(-2) \oplus \mathfrak{h}(-1) \oplus \mathfrak{h}(0) \oplus \mathfrak{h}(1) \oplus \mathfrak{h}(2).$$

We write $\mathfrak{q}_H = \mathrm{Lie}(Q_H) = \mathfrak{h}(0) \oplus \mathfrak{h}(1) \oplus \mathfrak{h}(2)$, $\mathfrak{n}_H := \mathfrak{h}(1) \oplus \mathfrak{h}(2)$, and $\bar{\mathfrak{n}}_H := \mathfrak{h}(-2) \oplus \mathfrak{h}(-1)$.

Elements of $\mathfrak{h}(0)$ are matrices of the form

$$X_0(b, t, A, B, C) = \begin{pmatrix} bi & 0 & t & 0 \\ 0 & A & 0 & B \\ t & 0 & bi & 0 \\ 0 & B^t & 0 & C \end{pmatrix},$$

where $b \in \mathbb{R}, t \in \mathbb{R}$ and $\begin{pmatrix} A & B \\ B^t & C \end{pmatrix} \in \mathfrak{u}(p-1, q-1)$. They are thus spanned by elements $H_0, Z_0 := \mathrm{diag}(i, 0_{p-1}, i, 0_{q-1})$, and elements of $\mathfrak{u}(p-1, q-1)$.

For $w \in \mathbb{C}^{p-1}, u \in \mathbb{C}^{q-1}, t \in \mathbb{R}$ elements of $\mathfrak{n}_H$ are of the form

$$X_1(w, u) = \begin{pmatrix} 0 & w^* & 0 & u^* \\ -w & 0 & w & 0 \\ 0 & w^* & 0 & u^* \\ u & 0 & -u & 0 \end{pmatrix} \in \mathfrak{h}(1), \quad X_2(t) = \begin{pmatrix} ti & 0 & -ti & 0 \\ 0 & 0 & 0 & 0 \\ ti & 0 & -ti & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{h}(2),$$

and elements of $\bar{\mathfrak{n}}_H$ are of the form

$$X_{-1}(w, u) = \begin{pmatrix} 0 & -w^* & 0 & u^* \\ w & 0 & w & 0 \\ 0 & w^* & 0 & -u^* \\ u & 0 & u & 0 \end{pmatrix} \in \mathfrak{h}(-1), \quad X_{-2}(t) = \begin{pmatrix} ti & 0 & ti & 0 \\ 0 & 0 & 0 & 0 \\ -ti & 0 & -ti & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{h}(-2).$$

We write $\bar{N}_H := \exp(\bar{\mathfrak{n}}_H)$, and direct calculation shows that elements of $\bar{N}_H$ are of the form

$$\bar{n}_H(w, u, t) = \begin{pmatrix} 1 - r^2/2 + ti & -w^* & -r^2/2 + ti & u^* \\ w & I_{p-1} & w & 0 \\ r^2/2 - ti & w^* & 1 + r^2/2 - ti & -u^* \\ u & 0 & u & I_{q-1} \end{pmatrix},$$

where

$$\begin{pmatrix} w \\ - \\ u \end{pmatrix} \in \mathbb{C}^{p-1,q-1}, t \in \mathbb{R}, r^2 = |w|^2 - |u|^2.$$

Analogous to the group law in the orthogonal setting, the group law on $\bar{N}_H$ is given by

$$\bar{n}_H(w_1, u_1, t_1)\bar{n}_H(w_2, u_2, t_2) = \bar{n}_H\big(w_1 + w_2, u_1 + u_2, t_1 + t_2 - \Im(w_1^* w_2 - u_1^* u_1)\big),$$

and so $\bar{N}_H$ is isomorphic to the Heisenberg group $\mathbb{C}^{p-1,q-1} \ltimes \mathbb{R}$ under the obvious identifications. Similar to the above, $\bar{N}_H Q_H$ is dense in H and we can identify (as vector spaces) $\mathrm{ind}_{Q_H}^{H}(\chi_{s,a})$ with $C^\infty(\mathbb{C}^{p-1,q-1} \ltimes \mathbb{R})$. We may parametrize $\bar{N}_H$ in real coordinates via

$$\begin{aligned} w_j &= a_j + ib_j & \quad & 1 \leq j \leq p-1, \\ u_k &= c_k + id_k & & 1 \leq k \leq q-1. \end{aligned}$$

In these coordinates, the *Heisenberg wave operator* on $C^\infty(\mathbb{C}^{p-1,q-1} \ltimes \mathbb{R})$ for $a \in \mathbb{Z}$ is

$$\Omega_a = \Omega_0 + 2ai\partial_t,$$

where

$$\Omega_0 = \partial_w^2 - \partial_u^2,$$

and

$$\partial_w^2 = \sum_{j=1}^{p-1} \partial_{w_j} \overline{\partial}_{w_j} = \sum_{j=1}^{p-1} (\partial_{a_j}^2 + \partial_{b_j}^2),$$

and ∂_u^2 is similar. A natural basis for $\bar{\mathfrak{n}}_H$ consists of elements

$$R_k := X_{-1}(e_k, 0), S_k := X_{-1}(ie_k, 0), T_k := X_{-1}(0, e_k), U_k := X_{-1}(0, ie_k), X_{-2} = X_{-2}(1),$$

and this gives us a correspondence between the action of Ω_a on $C^\infty(\mathbb{C}^{p-1,q-1} \ltimes \mathbb{R})$ and the action of

$$\Omega_a := \sum_{j=1}^{p-1} \left(R_j^2 + S_j^2\right) - \sum_{k=1}^{q-1} \left(T_k^2 + U_k^2\right) + 2aiX_{-2} \in \mathcal{U}(\bar{\mathfrak{n}}_H)$$

on $\mathrm{ind}_{Q_H}^H(\chi_{s,a})$, where the action is again right translation.

The operator Ω_a is H-intertwining, and we now show that when $s = p + q - 2$, it takes the principal series to one where s has been increased by 2.

Proposition 5.3.1 *For the parameter $s = p + q - 2$, we have*

$$\Omega_a : \mathrm{ind}_{Q_H}^H(\chi_{s,a}) \to \mathrm{ind}_{Q_H}^H(\chi_{s+2,a}).$$

Proof. We again appeal to the duality theorem, and we need to show that the map

$$\Phi : \mathbb{C}_{-(s+2),a} \to M_{\mathfrak{q}_H}(-s, a),$$

given by

$$\Phi(\mathbb{1}_{-(s+2),a}) = \Omega_a \otimes \mathbb{1}_{-s,a},$$

is a Q_H-homomorphism.

There is a factorization $Q_H = M_H A_H N_H$, where

$$M_H \cong U(p-1, q-1), A_G = \{\exp(tH_0), \exp(rZ_0) : t, r \in \mathbb{R}\}, N_H = \exp(\mathfrak{n}_H),$$

and so we need to check that the actions of each factor commutes with the above embedding.

The action of M_H is trivial in both places of the embedding.

For A_H, a very similar calculation to that in the G-setting shows that the action of elements $\exp(tH_0)$ commute with the embedding. Write

$$g_r = \exp(rZ_0) = \mathrm{diag}(e^{ri}, I_{p-1}, e^{ri}, I_{q-1}).$$

One calculates that $\mathrm{Ad}(g_r)\big(X_{-1}(w, u)\big) = X_{-1}(e^{-ir}w, e^{-ir}u)$, and thus

$$\mathrm{Ad}(g_r)\big(X_{-1}(w, u)^2\big) = X_{-1}(e^{-ir}w, e^{-ir}u)^2.$$

In particular, the action of g_r on Ω_0 is trivial since it rotates basis vectors by angle $-r$ and Ω_0 (essentially, the generalized Laplacian) is rotation-invariant.

Furthermore, one calculates that $\mathrm{Ad}(g_r)(X_{-2}) = X_{-2}$, and it follows immediately that

$$g_r.\Omega_\lambda = \Omega_\lambda.$$

Visibly we see that $\chi(g_r) = e^{ri}$, and that

$$\chi_{s,a}(g_r) = \left(\frac{\chi(g_r)}{|\chi(g_r)|}\right)^a |\chi(g_r)|^s = e^{ari}.$$

We thus have

$$g_r.\mathbb{1}_{-s,a} = \chi_{s,a}(g_r)^{-1}\mathbb{1}_{-s,a} = e^{-ari}\mathbb{1}_{-s,a}.$$

Similarly, $g_r.\mathbb{1}_{-(s+2),a} = e^{-ari}\mathbb{1}_{-(s+2),a}$. These remarks show that for all $r \in \mathbb{R}$ we have

$$\Phi(g_r.\mathbb{1}_{-(s+2),a}) = g_r.\Phi(\mathbb{1}_{-(s+2),a}).$$

Now we will show that for $s = p + q - 2$ and $\lambda = a$ that N_H acts trivially on

$$\Omega_\lambda = \sum_{k=1}^{p-1}(R_k^2 + S_k^2) - \sum_{k=1}^{q-1}(T_k^2 + U_k^2) + 2\lambda i X_{-2} \in \mathcal{U}(\bar{\mathfrak{n}}_H).$$

by showing that $\mathfrak{n}_H$ annihilates this element. We will frequently make use of the fact that N_H acts trivially on v_1 and so $d\chi_{s,a}(\mathfrak{n}_H) = 0$. Write $X_1 := X_1(e_1, 0)$, and write

$$X_0^{R_k} = [X_1, R_k], X_0^{S_k} = [X_1, S_k], X_0^{T_k} = [X_1, T_k], X_0^{U_k} = [X_1, U_k],$$

and write

$$X_{-1}^{R_k} = [X_0^{R_k}, R_k], X_{-1}^{S_k} = [X_0^{S_k}, S_k], X_{-1}^{T_k} = [X_0^{T_k}, T_k], X_{-1}^{U_k} = [X_0^{U_k}, U_k].$$

We have

$$\begin{aligned} X_1 R_k^2 \otimes 1 &= (X_0^{R_k} R_k + R_k X_1 R_k) \otimes \mathbb{1}_{-s,a} \\ &= (X_{-1}^{R_k} + 2R_k X_0^{R_k}) \otimes \mathbb{1}_{-s,a} \\ &= (X_{-1}^{R_k} - 2d\chi_{s,a}(X_0^{R_k})R_k) \otimes \mathbb{1}_{-s,a}, \end{aligned}$$

and the computations for S_k, T_k, and U_k are similar. We now compute

$$\begin{aligned} X_0^{R_k} &= [X_1(e_1, 0), R_k] \\ &= X_0(e_1^t e_k - e_k^t e_1, e_1^t e_k + e_k^t e_1, 2(e_1 e_k^t - e_k e_1^t), 0, 0) \\ &= X_0(0, 2\delta_{1,k}, 2(E_{1k} - E_{k1}), 0, 0), \\ X_{-1}^{R_k} &= [X_0^{R_k}, R_k] \\ &= [X_0(0, 2\delta_{1,k}, 2(E_{1k} - E_{k1}), 0, 0), X_{-1}(e_k, 0)] \\ &= X_{-1}((2(E_{1k} - E_{k1}) - 2\delta_{1,k})e_k, 0) \\ &= 2X_{-1}(E_{1k}e_k - E_{k1}e_k - \delta_{1,k}e_k, 0) \\ &= 2X_{-1}(e_1 - e_1\delta_{1k} - \delta_{1k}e_1, 0) \\ &= 2X_{-1}((1 - 2\delta_{1k})e_1, 0) \\ &= 2(1 - 2\delta_{1k})R_1. \end{aligned}$$

Now we observe that $X_0^{R_k} \in 2\delta_{1k}H_0 + \mathfrak{u}(p-1,q-1)$, and since $d\chi_{s,a}\big(\mathfrak{u}(p-1,q-1)\big) = 0$, we have

$$d\chi_{s,a}(X_0^{R_k}) = 2\delta_{1k} d\chi_{s,a}(H_0) = 2\delta_{1k}s.$$

We thus have

$$\begin{aligned}
X_1\left(\sum_{k=1}^{p-1} R_k^2 \otimes \mathbb{1}_{-s,a}\right) &= \left(\sum_{k=1}^{p-1}(X_{-1}^{R_k} - 2d\chi_{s,a}(X_0^{R_k})R_k)\right) \otimes \mathbb{1}_{-s,a} \\
&= \left(\sum_{k=1}^{p-1} 2(1-2\delta_{1k})R_1 - 2(2\delta_{1k}s)R_k\right) \otimes \mathbb{1}_{-s,a} \\
&= \big(-2R_1 + 2(p-2)R_1 - 4sR_1\big) \otimes \mathbb{1}_{-s,a} \\
&= 2(-1+p-2-2s)R_1 \otimes \mathbb{1}_{-s,a} \\
&= 2(p-2s-3)R_1 \otimes \mathbb{1}_{-s,a}.
\end{aligned}$$

A similar process shows that

$$\begin{aligned}
X_1\left(\sum_{k=1}^{p-1} S_k^2 \otimes \mathbb{1}_{-s,a}\right) &= \big(2(p+1)R_1 + 4aiS_1\big) \otimes \mathbb{1}_{-s,a}, \\
X_1\left(\sum_{k=1}^{q-1} T_k^2 \otimes \mathbb{1}_{-s,a}\right) &= -2(q-1)R_1 \otimes \mathbb{1}_{-s,a}, \\
X_1\left(\sum_{k=1}^{q-1} U_k^2 \otimes 1\right) &= -2(q-1)R_1 \otimes \mathbb{1}_{-s,a}.
\end{aligned}$$

Furthermore, $[X_1, X_{-2}] = -2S_1$. We thus have

$$\begin{aligned}
X_1.(\Omega_\lambda \otimes \mathbb{1}_{-s,a}) &= X_1.\left(\sum_{k=1}^{p-1}(R_k^2 + S_k^2) - \sum_{k=1}^{q-1}(T_k^2 + U_k^2) + 2\lambda i X_{-2}\right) \otimes \mathbb{1}_{-s,a} \\
&= (2(p-2s-3+p+1+q-1+q-1)R_1 + 4aiS_1 - 4\lambda S_1) \otimes \mathbb{1}_{-s,a} \\
&= (4(p+q-2-s)R_1 + 4aiS_1 - 4\lambda S_1) \otimes \mathbb{1}_{-s,a},
\end{aligned}$$

from which it follows immediately that $\bar{n}_H.\Omega_\lambda \otimes 1 = 0$ when

$$s = p+q-2, \lambda = a.$$

That is, N_G acts trivially on $\Omega_a \otimes \mathbb{1}_{-s,a}$ for $s = p+q-2$ and all $a \in \mathbb{Z}$. ■

5.4 K-finite solutions to Ω_a in $\mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon})$

In Kubo-Ørsted [5, See discussion leading up to Theorem 1.2], it is explained that (in their notation) a differential operator

$$\mathcal{D}_u : \mathrm{ind}_P^G(\chi_{\mathrm{triv},\lambda}) \to \mathrm{ind}_P^G(\chi_{\chi,\nu})$$

is of the form $\mathcal{D}_u = R(u)$ for some $u \in \mathcal{U}(\mathfrak{g})$, where R is the right translation action discussed in the first section of this chapter. It is also explained that the K-finite solution space to D_u are those irreducible representations (δ, V) of K such that

$$d\delta\left(\tau(u')\right) v = 0$$

for all $v \in V$, where τ is complex conjugation with respect to the real Lie algebra $\mathfrak{g}$ and u' satisfies

$$u' \otimes \mathbb{1}_{-\lambda-\rho} = u \otimes \mathbb{1}_{-\lambda-\rho}$$

(is equal in the generalized Verma module).

With this in mind, we will use the decompositions

$$\mathfrak{h} = \mathfrak{k}_H + \mathfrak{q}_H = \bar{\mathfrak{n}}_H + \mathfrak{q}_H$$

to find $\Omega_a' \in \mathcal{U}(\mathfrak{k})$ such that

$$\Omega_a' \otimes \mathbb{1}_{-s,a} = \Omega_a \otimes \mathbb{1}_{-s,a} \in \mathcal{U}(\mathfrak{h}) \otimes_{\mathfrak{q}_H} \mathbb{C}_{-s,a}.$$

We will then make use of an explicit action of $\mathfrak{k}_H \cong \mathfrak{u}(p) \oplus \mathfrak{u}(q)$ on the K_H-types $\mathscr{H}^{m_1,m_2}(\mathbb{C}^p) \otimes \mathscr{H}^{n_1,n_2}(\mathbb{C}^q)$. to calculate exactly which K_H-types are annihilated by Ω_a', and this will be our solution space to the Heisenberg wave operator.

For $w \in \mathbb{C}^{p-1}, u \in \mathbb{C}^{q-1}$, write

$$Z(w,u) = \begin{pmatrix} 0 & -w^* & 0 & 0 \\ w & 0 & 0 & 0 \\ 0 & 0 & 0 & -u^* \\ 0 & 0 & u & 0 \end{pmatrix} \in \mathcal{U}(\mathfrak{k}_H),$$

and define

$$\begin{aligned} R_k' &= Z(e_k, 0) \\ S_k' &= Z(ie_k, 0) \\ T_k' &= Z(0, e_k) \\ U_k' &= Z(0, ie_k). \end{aligned}$$

We have

$$\begin{aligned} R_k &= X_1(e_k, 0) + 2R_k' \\ S_k &= X_1(ie_k, 0) + 2S_k' \\ T_k &= X_1(0, e_k) + 2T_k' \\ U_k &= X_1(0, ie_k) + 2U_k'. \end{aligned}$$

The bracket

$$[X_1(w,u), Z(w,u)] = \begin{pmatrix} 0 & 0 & w^*w + u^*u & 0 \\ 0 & 0 & 0 & -2wu^* \\ w^*w + u^*u & 0 & 0 & 0 \\ 0 & -2uw^* & 0 & 0 \end{pmatrix}$$

shows that

$$H_0 = [X_1(e_k,0), Z(e_k,0)] = [X_1(ie_k,0), Z(ie_k,0)] = [X_1(0,e_k), Z(0,e_k)] = [X_1(0,ie_k), Z(0,ie_k)].$$

We have

$$\begin{aligned}
\sum_{k=1}^{p-1}(R_k^2+S_k^2)\otimes \mathbb{1}_{-s,a} &= \sum_{k=1}^{p-1}\big((X_1(e_k,0)+2R_k')^2+(X_1(ie_k,0)+2S_k')^2\big)\otimes \mathbb{1}_{-s,a}\\
&= \sum_{k=1}^{p-1}\big(2X_1(e_k,0)R_k'+4R_k'^2+2X_1(ie_k,0)S_k'+4S_k'^2\big)\otimes \mathbb{1}_{-s,a}\\
&= \sum_{k=1}^{p-1}\big(2H_0+4R_k'^2+2H_0+4S_k'^2\big)\otimes \mathbb{1}_{-s,a}\\
&= 4\left(\sum_{k=1}^{p-1}\big(R_k'^2S_k'^2\big)-(p-1)s\right)\otimes \mathbb{1}_{-s,a}
\end{aligned}$$

A similar calculation shows that

$$\sum_{k=1}^{q-1}(T_k^2+U_k^2)\otimes \mathbb{1}_{-s,a} = 4\left(\sum_{k=1}^{q-1}\big(T_k'^2+U_k'^2\big)-(q-1)s\right)\otimes \mathbb{1}_{-s,a}.$$

Let $X_{-2}' = \mathrm{diag}(i, 0_{p-1}, -i, 0_{q-1})$. We have $X_{-2} = -X_2 + 2X_{-2}'$ where $X_2 := X_2(1)$, and so

$$2aiX_{-2}\otimes \mathbb{1}_{-s,a} = 4aiX_{-2}'\otimes \mathbb{1}_{-s,a}.$$

Combining these calculations, we have

$$\begin{aligned}
\Omega_a\otimes \mathbb{1}_{-s,a} &= \left(\sum_{k=1}^{p-1}(R_k^2+S_k^2)-\sum_{k=1}^{q-1}(T_k^2+U_k^2)+2aiX_{-2}\right)\otimes \mathbb{1}_{-s,a}\\
&= 4\left(\sum_{k=1}^{p-1}(R_k'^2+S_k'^2)-\sum_{k=1}^{q-1}(T_k'^2+U_k'^2)+aiX_{-2}'-((p-1)-(q-1))\,s\right)\otimes \mathbb{1}_{-s,a}\\
&= 4\left(\sum_{k=1}^{p-1}(R_k'^2+S_k'^2)-\sum_{k=1}^{q-1}(T_k'^2+U_k'^2)+aiX_{-2}'-(p-q)s\right)\otimes \mathbb{1}_{-s,a}.
\end{aligned}$$

We have now written the operator in a form that allows us to use the explicit action of $\mathfrak{k}_H$ on the K_H-types. It is enough to calculate the action on the embedding vectors of these spaces. We will work in these harmonic spaces modulo the element $r^2 = z_1\overline{z}_1 + \cdots + z_p\overline{z}_p$, and in this projected space the embedding vectors for $\mathscr{H}^{m_1,m_2}(\mathbb{C}^p)$ are $z_1^{m_1}\overline{z}_1^{m_2}$.

Observe that $R_k' = (A,B)$, where $A = \begin{pmatrix} 0 & -e_k^t \\ e_k & 0 \end{pmatrix} \in \mathfrak{u}(p)$ and $B = \begin{pmatrix} 0 & -e_k^t \\ e_k & 0 \end{pmatrix} \in \mathfrak{u}(q)$. Recall that the action of $\mathfrak{u}(p)$ on $\mathscr{H}^{m_1,m_2}(\mathbb{C}^p)$ is given by

$$A.f(z,\overline{z}) = \frac{d}{dt}f\Big(\exp(-tA)z, \overline{\exp(-tA)z}\Big)\Big|_{t=0}.$$

$\exp(-tA) = \begin{pmatrix} \cos(t) & -\sin(t)e_k^t \\ \sin(t)e_k & M_k(t) \end{pmatrix}$, where $M_k(t) = \mathrm{diag}(1, 1, \ldots, \cos(t), 1, \ldots, 1)$ and where $\cos(t)$ is in the k-th entry. For $z \in \mathbb{C}^p$, the entries of

$$\exp(-tA)z$$

in which t appears are

$$\big(\exp(-tA)z\big)_1 = \cos(t)z_1 - \sin(t)z_{k+1},$$

$$\big(\exp(-tA)z\big)_{k+1} = \sin(t)z_1 + \cos(t)z_{k+1}.$$

It follows that for $f(z, \overline{z}) \in \mathscr{H}^{m_1,m_2}(\mathbb{C}^p)$, we have

$$\frac{d}{dt} f\bigg(\exp(-tA)z, \overline{\exp(-tA)z}\bigg)\bigg|_{t=0} = \bigg(-z_{k+1}\partial_1 + z_1\partial_{k+1} - \overline{z}_{k+1}\overline{\partial}_1 + \overline{z}_1\overline{\partial}_{k+1}\bigg) f(z, \overline{z}).$$

We thus have

$$\begin{aligned}
&A^2.z_1^{m_1}\overline{z}_1^{m_2} \\
=&(-z_{k+1}\partial_1 + z_1\partial_{k+1} - \overline{z}_{k+1}\overline{\partial}_1 + \overline{z}_1\overline{\partial}_{k+1})^2 z_1^{m_1}\overline{z}_1^{m_2} \\
=&(z_{k+1}\partial_1 - z_1\partial_{k+1} + \overline{z}_{k+1}\overline{\partial}_1 - \overline{z}_1\overline{\partial}_{k+1})^2 z_1^{m_1}\overline{z}_1^{m_2} \\
=&(z_{k+1}\partial_1 - z_1\partial_{k+1} + \overline{z}_{k+1}\overline{\partial}_1 - \overline{z}_1\overline{\partial}_{k+1})(m_1 z_1^{m_1-1} z_{k+1}\overline{z}_1^{m_2} + m_2 z_1^{m_1}\overline{z}_1^{m_2-1}\overline{z}_{k+1}) \\
=&m_1(m_1-1)z_1^{m_1-2}z_{k+1}^2\overline{z}_1^{m_2} + m_1 m_2 z_1^{m_1-1}z_{k+1}\overline{z}_1^{m_2-1}\overline{z}_{k+1} - m_1 z_1^{m_1}\overline{z}_1^{m_2} \\
&+m_1 m_2 z_1^{m_1-1}z_{k+1}\overline{z}_1^{m_2-1}\overline{z}_{k+1} + m_2(m_2-1)z_1^{m_1}\overline{z}_1^{m_2-2}\overline{z}_{k+1}^2 - m_2 z_1^{m_1}\overline{z}_1^{m_2}.
\end{aligned}$$

The action of B^2 is similar. The action of $(S'_k)^2$ is computed similarly, and adding these actions yields several cancellations, so that

$$\big((R'_k)^2 + (S'_k)^2\big).(z_1^{m_1}\overline{z}_1^{m_2}) = 4m_1 m_2 z_1^{m_1-1}z_{k+1}\overline{z}_1^{m_2-1}\overline{z}_{k+1} - 2(m_1+m_2)z_1^{m_1}\overline{z}_1^{m_2}.$$

Summing over k, and mindful that we are working in this projected space modulo $r^2 = z_1\overline{z}_1 + z_2 + \overline{z}_2 + \cdots + z_p\overline{z}_p$, we have

$$\begin{aligned}
\sum_{k=1}^{p-1}(R_k'^2 + S_k'^2).(z_1^a\overline{z}_1^b) &= \sum_{k=1}^{p-1}\big(4m_1 m_2 z_1^{m_1-1}z_{k+1}\overline{z}_1^{m_2-1}\overline{z}_{k+1} - 2(m_1+m_2)z_1^{m_1}\overline{z}_1^{m_2}\big) \\
&= \left(-4m_1 m_2 - 2(p-1)(m_1+m_2)\right) z_1^{m_1}\overline{z}_1^{m_2}.
\end{aligned}$$

The action of $(T'_k)^2 + (U'_k)^2$ is computed similarly.

Recall that $X'_{-2} = (A, B)$ where $A = \mathrm{diag}(i, 0_{p-1}), B = \mathrm{diag}(-i, 0_{q-1})$. We have

$$\begin{aligned}
A.(z_1^{m_1}\overline{z}_1^{m_2}) &= \frac{d}{dt}\big(\exp(-it)z_1\big)^{m_1}\big(\exp(it)\overline{z}_1\big)^{m_2}\bigg|_{t=0} \\
&= \frac{d}{dt}e^{-ti(m_1-m_2)}z_1^{m_1}\overline{z}_1^{m_2}\bigg|_{t=0} \\
&= -i(m_1-m_2)z_1^{m_1}\overline{z}_1^{m_2}.
\end{aligned}$$

Similarly, $B.(w_1^{n_1}\overline{w}_1^{n_2}) = i(n_1 - n_2)w_1^{n_1}\overline{w}_1^{n_2}$, and so

$$X_{-2}.(z_1^{m_1}\overline{z}_1^{m_2} \otimes w_1^{n_1}\overline{w}_1^{n_2}) = -i(m_1 - m_2 - n_1 + n_2)(z_1^{m_1}\overline{z}_1^{m_2} \otimes w_1^{n_1}\overline{w}_1^{n_2}).$$

Recalling the notation τ for complex conjugation discussed at the beginning of the section, writing $\xi = \xi^{m_1,m_2} \otimes \xi^{n_1,n_2}$ for the embedding vector, recalling that $s = p + q - 2, a = m_1 - m_2 + n_1 - n_2$, and combining everything, we have

$$\begin{aligned}
\tau(\Omega_a').\xi &= 4\left(\sum_{k=1}^{p-1}(R_k'^2 + S_k'^2) - \sum_{k=1}^{q-1}(T_k'^2 + U_k'^2) - aiX_{-2}' - (p-q)s\right).\xi \\
&= 4\Bigg[\bigg(- 4m_1m_2 - 2(p-1)(m_1+m_2) - (-4n_1n_2 - 2(q-1)(n_1+n_2))\bigg) \\
&+ ai\big(- i(m_1 - m_2 + n_1 - n_2)\big) - (p-q)s\Bigg]\xi \\
&= -4(m_1 + m_2 - n_1 - n_2 + p - q)(m_1 + m_2 + n_1 + n_2 + p + q - 2)\xi.
\end{aligned}$$

The second factor will always be positive for the cases we are considering ($p \geq 2, q \geq 1$), so in particular will never be zero. As a result, the second factor will not contribute solutions. Thus, the K_H-finite solution space to Ω_a in $\mathrm{ind}_{Q_H}^H(\chi_{s,a})$ is

$$\sum_{\substack{m_1,m_2,n_1,n_2\geq 0,\\ m_1+m_2-n_1-n_2+p-q=0}} \mathscr{H}^{m_1,m_2}(\mathbb{C}^p) \otimes \mathscr{H}^{n_1,n_2}(\mathbb{C}^q).$$

The polynomial parameterizing this solution space corresponds to the projection of a ladder representation. In a plot $(x, y) = (m, n)$, and with $m = m_1 + m_2, n = n_1 + n_2$, setting this polynomial equal to zero, we have

$$m - n + p - q = 0,$$

or

$$n = m + (p - q),$$

which is a line through K_G-types $\mathscr{H}^m(\mathbb{R}^{2p}) \otimes \mathscr{H}^n(\mathbb{R}^{2q})$ with y-intercept $(p - q)$ and slope 1. As as $(\mathfrak{g}, K_G)$-module, this is a ladder representation, and will be the solution space to the Euclidean wave operator Δ in $\mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon})$. The map T_a projects these solutions onto the K_H-types contained in $\mathrm{ind}_{Q_H}^H(\chi_{s,a})$.

CHAPTER VI

CONNECTING THE K-FINITE KERNELS OF Δ AND Ω_a THE INTEGRAL INTERTWINING OPERATORS T_a

As mentioned in the previous chapter, we are aiming to show that T_a H-intertwines the actions of the wave operators Δ and Ω_a for certain parameters of the respective degenerate principal series. In particular, for $s = p + q - 2$, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{ind}_{Q_G}^{G}(\lambda_{s,\varepsilon}) & \xrightarrow{\Delta} & \mathrm{ind}_{Q_G}^{G}(\lambda_{s+2,\varepsilon}) \\ {\scriptstyle T_a}\big\downarrow & & \big\downarrow{\scriptstyle T_a} \\ \mathrm{ind}_{Q_H}^{H}(\chi_{s,a}) & \xrightarrow{\Omega_a} & \mathrm{ind}_{Q_H}^{H}(\chi_{s+2,a}) \end{array}$$

The first step is to show the commuting property for $\varphi \in \mathrm{ind}_{Q_G}^{G}(\lambda_{s,\varepsilon})$ whose restriction to $\bar{N}_G$ is Schwartz, and this is the subject of the current chapter.

6.1 Changing Coordinates between G and H Settings

Recall that T_a restricted to $\bar{N}_H$ is given by

$$T_a(\varphi)(\bar{n}_H) = \int_{-\pi/2}^{\pi/2} \varphi(\bar{n}_H z(\theta)) e^{ia\theta} d\theta.$$

Due to the fact that $\bar{N}_G Q_G$ is dense in G, we can factor most elements $\bar{n}_H z(\theta) \in \bar{N}_H Z(H)$ as elements in $\bar{N}_G Q_G$, which allows us to make use of the translation property that φ has on Q_G.

To determine what the change of coordinates map should be, first observe that if we are able to make such a factorization $\bar{n}_H z(\theta) = \bar{n}_G q_G$, then they must act on the isotropic vector v_1 in the same way. That is $\bar{n}_H z(\theta) v_1 = \bar{n}_G q_G v_1$. We now provide the correct change of coordinates map, and the following lemma shows that this map does satisfy a preliminary identity. Let

$$\iota : \mathbb{C}^{p-1,q-1} \oplus \mathbb{R} \oplus (-\pi/2, \pi/2) \to \mathbb{R}^{2p-1,2q-1},$$

given by

$$\iota(w, u, t, \theta) = (\alpha, \beta),$$

where (for later reference: $\lambda = \cos\theta, \tau = \tan\theta, \zeta = 1 + \tau^2 = \sec^2\theta$)

$$\begin{aligned}
&\alpha_1 = t + \frac{1}{2}(1 - r^2)\tau && \\
&\alpha_{2j} = a_j - b_j\tau && 1 \leq j \leq p-1 \\
&\alpha_{2j+1} = b_j + a_j\tau && 1 \leq j \leq p-1 \\
&\beta_1 = -t + \frac{1}{2}(1 + r^2)\tau && \\
&\beta_{2k} = c_k - d_k\tau && 1 \leq k \leq q-1 \\
&\beta_{2k+1} = d_k + c_k\tau && 1 \leq k \leq q-1.
\end{aligned}$$

Lemma 6.1.1 *Let $\bar{n}_H(w, u, t) \in \overline{N}_H$ and $z(\theta) \in Z(H)$ where $\theta \neq \pi/2, 3\pi/2$. Then*

$$\bar{n}_H(w, u, t)z(\theta)v_1 = \cos\theta \cdot \bar{n}_G\big(\iota(w, u, t, \theta)\big)v_1.$$

Proof. First we calculate the left-hand-side. Let

$$v_1 = \begin{pmatrix} e_1 \\ - \\ e_1 \end{pmatrix}, v_2 = \begin{pmatrix} e_2 \\ - \\ e_2 \end{pmatrix} \in \mathbb{R}^{2p,2q},$$

where e_1 and e_2 are the first and second standard basis vectors, respectively, in their relative spaces. Recall that under the embedding $H \hookrightarrow G$ given entry-wise by

$$a + bi \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

we can identify $z(\theta) = (e^{i\theta} I_{p+q})$ as a block-diagonal-matrix in G with 2-by-2 blocks

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Therefore, $z(\theta)v_1 = \cos\theta v_1 + \sin\theta v_2$. Now we will need the action of $\bar{n}_H(w, u, t)$ on v_1 and v_2. The relevant entries in this matrix will be columns $1, 2, p+1$, and $p+2$. Under the embedding above, we have

$$\bar{n}_H(w,u,t)_{\text{col }1,2} = \begin{pmatrix} 1-r^2/2 & -t \\ t & 1-r^2/2 \\ a_1 & -b_1 \\ b_1 & a_1 \\ a_2 & -b_2 \\ b_2 & a_2 \\ \vdots & \vdots \\ a_{p-1} & -b_{p-1} \\ b_{p-1} & a_{p-1} \\ r^2/2 & t \\ -t & r^2/2 \\ c_1 & -d_1 \\ d_1 & c_1 \\ \vdots & \vdots \\ c_{q-1} & -d_{q-1} \\ d_{q-1} & c_{q-1} \end{pmatrix} \quad \bar{n}_H(w,u,t)_{\text{col }p+1,\,p+2} = \begin{pmatrix} -r^2/2 & -t \\ t & -r^2/2 \\ a_1 & -b_1 \\ b_1 & a_1 \\ a_2 & -b_2 \\ b_2 & a_2 \\ \vdots & \vdots \\ a_{p-1} & -b_{p-1} \\ b_{p-1} & a_{p-1} \\ 1+r^2/2 & t \\ -t & 1+r^2/2 \\ c_1 & -d_1 \\ d_1 & c_1 \\ \vdots & \vdots \\ c_{q-1} & -d_{q-1} \\ d_{q-1} & c_{q-1} \end{pmatrix}.$$

Observe that

$$\bar{n}_H(w,u,t)v_1 = \text{column } 1 + \text{column } (p+1),$$
$$\bar{n}_H(w,u,t)v_2 = \text{column } 2 + \text{column } (p+2).$$

Let $\lambda = \cos\theta$ for the rest of the proof. For the complete left-hand-side, we have

$$\begin{aligned}
&\bar{n}_H(w,u,t)z(\theta)v_1\\
=&\bar{n}_H(w,u,t)\big(\cos(\theta)v_1+\sin(\theta)v_2\big)\\
=&\cos\theta\big(\bar{n}_H(w,u,t)v_1\big)+\sin\theta\big(\bar{n}_H(w,u,t)v_2\big)\\
&\cos\theta\begin{pmatrix}1-r^2\\2t\\2a_1\\2b_1\\\vdots\\2b_{p-1}\\1+r^2\\-2t\\2c_1\\2d_1\\\vdots\\2d_{q-1}\end{pmatrix}+\sin\theta\begin{pmatrix}-2t\\1-r^2\\-2b_1\\2a_1\\\vdots\\2a_{p-1}\\2t\\1+r^2\\-2d_1\\2c_1\\\vdots\\2c_{q-1}\end{pmatrix}=\begin{pmatrix}(1-r^2)\cos\theta-2t\sin\theta\\2t\cos\theta+(1-r^2)\sin\theta\\2a_1\cos\theta-2b_1\sin\theta\\2b_1\cos\theta+2a_1\sin\theta\\\vdots\\2b_{p-1}\cos\theta+2a_{p-1}\sin\theta\\(1+r^2)\cos\theta+2t\sin\theta\\-2t\cos\theta+(1+r^2)\sin\theta\\2c_1\cos\theta-2d_1\sin\theta\\2d_1\cos\theta+2c_1\sin\theta\\\vdots\\2d_{q-1}\cos\theta+2c_{q-1}\sin\theta\end{pmatrix}\\
&=2\cos\theta\begin{pmatrix}\frac{1}{2}(1-r^2)-t\tau\\t+\frac{1}{2}(1-r^2)\tau\\a_1-b_1\tau\\b_1+a_1\tau\\\vdots\\b_{p-1}+a_{p-1}\tau\\\frac{1}{2}(1+r^2)+t\tau\\-t+\frac{1}{2}(1+r^2)\tau\\c_1-d_1\tau\\d_1+c_1\tau\\\vdots\\d_{q-1}+c_{q-1}\tau\end{pmatrix}=2\lambda\begin{pmatrix}\frac{1}{2}(1-r^2)-t\tau\\\alpha_1\\\alpha_2\\\alpha_3\\\vdots\\\alpha_{2p-1}\\\frac{1}{2}(1+r^2)+t\tau\\\beta_1\\\beta_2\\\beta_3\\\vdots\\\beta_{2q-1}\end{pmatrix}.
\end{aligned}$$

For the right-hand-side, let $(\alpha,\beta)=\iota(w,u,t,\theta)$. We have

$$\begin{aligned}
\lambda \cdot \bar{n}_G(\alpha,\beta)v_1 &= \lambda \begin{pmatrix} 1-R^2/2 & -\alpha^t & -R^2/2 & \beta^t \\ \alpha & I_{2p-1} & \alpha & 0 \\ R^2/2 & \alpha^t & 1+R^2/2 & -\beta^t \\ \beta & 0 & \beta & I_{2q-1} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
&= \lambda \begin{pmatrix} 1-R^2 \\ 2\alpha \\ 1+R^2 \\ 2\beta \end{pmatrix} \\
&= 2\lambda \begin{pmatrix} \frac{1}{2}(1-R^2) \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{2p-1} \\ \frac{1}{2}(1+R^2) \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_{2q-1} \end{pmatrix}.
\end{aligned}$$

Now that we have computed both sides of the claimed relation, it remains to show that the entries on both sides are equal. The entries α_j and β_j are all visibly equal according to the coordinates given before the lemma. As for the first entry, pairing coordinates $\alpha_{2j}, \alpha_{2j+1}$, and similarly for β_{2k}, β_{2k+1}, we see that

$$\sum_{j=2}^{2p-1} \alpha_j^2 - \sum_{k=2}^{2q-1} \beta_k^2 = \sum_{j=1}^{p-1} (1+\tau^2)(a_j^2+b_j^2) - \sum_{k=1}^{q-1} (1+\tau^2)(c_k^2+d_k^2) = (1+\tau^2)r^2.$$

Now,

$$\begin{aligned}
\alpha_1^2 &= \left(t + \frac{1}{2}(1-r^2)\tau\right)^2 = t^2 + t\tau(1-r^2) + \frac{1}{4}(1-r^2)^2\tau^2, \\
\beta_1^2 &= \left(-t + \frac{1}{2}(1+r^2)\tau\right)^2 = t^2 - t\tau(1+r^2) + \frac{1}{4}(1+r^2)^2\tau^2,
\end{aligned}$$

and so we get the relation $\alpha_1^2 - \beta_1^2 = 2t\tau + \frac{1}{4}\tau^2(-4r^2) = 2t\tau - \tau^2 r$. Therefore the first entry on the right-hand-side is

$$\begin{aligned}
\frac{1}{2}(1-R^2) &= \frac{1}{2}\left(1-\sum_{j=1}^{2p-1}\alpha_j^2-\sum_{k=1}^{2q-1}\beta_k^2\right)\\
&= \frac{1}{2}\left(1-(\alpha_1^2-\beta_1^2)-\sum_{j=2}^{2p-1}\alpha_j^2-\sum_{k=2}^{2q-1}\beta_k^2\right)\\
&= \frac{1}{2}\left(1-(2t\tau-\tau^2r^2)-(1+\tau^2)r^2\right)\\
&= \frac{1}{2}(1-2t\tau-r^2)\\
&= \frac{1}{2}(1-r^2)-t\tau,
\end{aligned}$$

equal to the first entry of the left-hand-side. By the isometry property of G, the $(p+1)$-entries are equal as well, which completes the proof. ■

We now present the full factorization.

Proposition 6.1.1 *Let $\bar{n}_H(w,u,t)\in\bar{N}_H$ and $z(\theta)\in Z(H)$ where $\theta\neq\pi/2, 3\pi/2$, and let $(\alpha,\beta)=\iota(w,u,t,\theta)$. Then $q_G:=\bar{n}_G(\alpha,\beta)^{-1}\bar{n}_H(w,u,t)z(\theta)$ is a member of Q_G which satisfies $\lambda(q_G)=\cos(\theta)$. In particular,*

$$\bar{n}_H(w,u,t)z(\theta)=\bar{n}_G(\alpha,\beta)q_G.$$

Proof. Let $q_G'=\operatorname{diag}(\cos\theta,\sec\theta,I_{2p-2},\cos\theta,\sec\theta,I_{2q-2})\in SO(2p)\times SO(2q)\subset G$. Visibly we see that $q_G'\in Q_G$ with $\lambda(q_G)=\cos\theta$. Thus,

$$\bar{n}_G(\alpha,\beta)q_G'v_1=\cos\theta\cdot\bar{n}_G(\alpha,\beta)v_1,$$

so by Lemma 6.1.1, we have

$$\bar{n}_H(w,u,t)z(\theta)v_1=\bar{n}_G(\alpha,\beta)q_G'v_1.$$

In particular,

$$\bar{n}_G(\alpha,\beta)^{-1}\bar{n}_H(w,u,t)z(\theta)v_1=\lambda(q_G')v_1,$$

which implies that $\bar{n}_G(\alpha,\beta)^{-1}\bar{n}_H(w,u,t)z(\theta)\in Q_G$ with $\lambda\big(\bar{n}_G(\alpha,\beta)^{-1}\bar{n}_H(w,u,t)z(\theta)\big)=\cos\theta$. Thus,

$$\bar{n}_H(w,u,t)z(\theta)=\bar{n}_G(\alpha,\beta)\left(\bar{n}_G(\alpha,\beta)^{-1}\bar{n}_H(w,u,t)z(\theta)\right)=\bar{n}_G(\alpha,\beta)q_G$$

completes the factorization. ■

With this factorization in mind, we can simplify the evaluation of T_a on $\bar{N}_H$ by using the translation property of $\varphi\in\operatorname{ind}_{Q_G}^G(\lambda_{s,\varepsilon})$ on Q_G:

$$\begin{aligned}
T_a(\varphi)\big(\bar{n}_H(w,u,t)\big) &= \int_{-\pi/2}^{\pi/2}\varphi\big(\bar{n}_H(w,u,t)z(\theta)\big)e^{ia\theta}d\theta\\
&= \int_{-\pi/2}^{\pi/2}|\cos(\theta)|_{\varepsilon}^{-s}\varphi\big(\bar{n}_G(\iota(w,u,t,\theta))\big)e^{ia\theta}d\theta.
\end{aligned}$$

On this interval of integration, $\cos(\theta) > 0$, and so this is

$$T_a(\varphi)\big(\bar{n}_H(w,u,t)\big) = \int_{-\pi/2}^{\pi/2} \cos(\theta)^{-s}\varphi\big(\bar{n}_G(\iota(w,u,t,\theta))\big)e^{ia\theta}d\theta.$$

6.2 Pullback of the Differential Operator Δ through ι

Recall that our aim is to show that

$$T_a(\Delta\cdot\varphi) = \Omega_a \cdot T_a(\varphi).$$

So in order to proceed, we need to compute the pullback of Δ through ι. We first use the formulas for ι to change the variables for the first-order partial derivatives, which will eventually allow us to relate Δ to Ω_a. By the chain rule, the partial derivative with respect to a_1 in these formulas is

$$\begin{aligned}\partial_{a_1} &= \frac{\partial\alpha_1}{\partial a_1}\partial_{\alpha_1} + \frac{\partial\alpha_2}{\partial a_1}\partial_{\alpha_2} + \cdots + \frac{\partial\beta_{2q-1}}{\partial a_1}\partial_{\beta_{2q-1}} \\ &= -a_1\tau\partial_{\alpha_1} + \partial_{\alpha_2} + \tau\partial_{\alpha_3} + a_1\tau\partial_{\beta_1}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\partial_{a_1} &= -a_1\tau\partial_{\alpha_1} + \partial_{\alpha_2} + \tau\partial_{\alpha_3} + a_1\tau\partial_{\beta_1} \\
\partial_{b_1} &= -b_1\tau\partial_{\alpha_1} - \tau\partial_{\alpha_2} + \partial_{\alpha_3} + b_1\tau\partial_{\beta_1} \\
\partial_{a_2} &= -a_2\tau\partial_{\alpha_1} + \partial_{\alpha_4} + \tau\partial_{\alpha_5} + a_2\tau\partial_{\beta_1} \\
\partial_{b_2} &= -b_2\tau\partial_{\alpha_1} - \tau\partial_{\alpha_4} + \partial_{\alpha_5} + b_2\tau\partial_{\beta_1} \\
&\vdots \\
\partial_{a_{p-1}} &= -a_{p-1}\tau\partial_{\alpha_1} + \partial_{\alpha_{2p-2}} + \tau\partial_{\alpha_{2p-1}} + a_{p-1}\tau\partial_{\beta_1} \\
\partial_{b_{p-1}} &= -b_{p-1}\tau\partial_{\alpha_1} - \tau\partial_{\alpha_{2p-2}} + \partial_{\alpha_{2p-1}} + b_{p-1}\tau\partial_{\beta_1} \\
\partial_{c_1} &= c_1\tau\partial_{\alpha_1} - c_1\tau\partial_{\beta_1} + \partial_{\beta_2} + \tau\partial_{\beta_3} \\
\partial_{d_1} &= d_1\tau\partial_{\alpha_1} - d_1\tau\partial_{\beta_1} - \tau\partial_{\beta_2} + \partial_{\beta_3} \\
\partial_{c_2} &= c_2\tau\partial_{\alpha_1} - c_2\tau\partial_{\beta_1} + \partial_{\beta_4} + \tau\partial_{\beta_5} \\
\partial_{d_2} &= d_2\tau\partial_{\alpha_1} - d_2\tau\partial_{\beta_1} - \tau\partial_{\beta_4} + \partial_{\beta_5} \\
&\vdots \\
\partial_{c_{q-1}} &= c_{q-1}\tau\partial_{\alpha_1} - c_{q-1}\tau\partial_{\beta_1} + \partial_{\beta_{2q-2}} + \tau\partial_{\beta_{2q-1}} \\
\partial_{d_{q-1}} &= d_{q-1}\tau\partial_{\alpha_1} - d_{q-1}\tau\partial_{\beta_1} - \tau\partial_{\beta_{2q-2}} + \partial_{\beta_{2q-1}} \\
\partial_t &= \partial_{\alpha_1} - \partial_{\beta_1} \\
\partial_\tau &= \frac{1-r^2}{2}\partial_{\alpha_1} + \sum_{j=1}^{p-1}\left(-b_j\partial_{\alpha_{2j}} + a_j\partial_{\alpha_{2j+1}}\right) + \frac{1+r^2}{2}\partial_{\beta_1} + \sum_{j=1}^{q-1}\left(-d_j\partial_{\beta_{2j}} + c_j\partial_{\beta_{2j+1}}\right).
\end{aligned}$$

We can condense these formulas into one formula using matrices, and to do this we introduce some more notation. Write $\partial_{w_j} = \begin{pmatrix}\partial_{a_j}\\ \partial_{b_j}\end{pmatrix}, \partial_{u_k} = \begin{pmatrix}\partial_{c_k}\\ \partial_{d_k}\end{pmatrix}$ and

$\partial_w = \begin{pmatrix} \partial_{w_1} \\ \vdots \\ \partial_{w_{p-1}} \end{pmatrix}, \partial_u = \begin{pmatrix} \partial_{u_1} \\ \vdots \\ \partial_{u_{q-1}} \end{pmatrix}, \partial_\alpha = \begin{pmatrix} \partial_{\alpha_1} \\ \vdots \\ \partial_{\alpha_{2p-1}} \end{pmatrix}, \partial_\beta = \begin{pmatrix} \partial_{\beta_1} \\ \vdots \\ \partial_{\beta_{2q-1}} \end{pmatrix}$. Write $M(n)$ to be the block diagonal matrix with n 2-by-2 blocks of the form $\begin{pmatrix} 1 & \tau \\ -\tau & 1 \end{pmatrix}$. Write w to be the embedding of $w \in \mathbb{C}^{p-1} \hookrightarrow \mathbb{R}^{2p-2}$ described before, and write $u \in \mathbb{C}^{q-1} \hookrightarrow \mathbb{R}^{2q-2}$ similarly. Write iw (and iu) to be the embedding of $iw \in \mathbb{C}^{p-1} \hookrightarrow \mathbb{R}^{2p-2}$, and write $iu \in \mathbb{C}^{q-1} \hookrightarrow \mathbb{R}^{2q-2}$ similarly. As a summary of the chain rule and this discussion, we have

$$\begin{pmatrix} \partial_w \\ \partial_u \\ \partial_t \\ \partial_\tau \end{pmatrix} = A \begin{pmatrix} \partial_\alpha \\ \partial_\beta \end{pmatrix},$$

where

$$A = \begin{pmatrix} -\tau w & M(p-1) & \tau w & 0 \\ \tau u & 0 & -\tau u & M(q-1) \\ 1 & 0 & -1 & 0 \\ \frac{1-r^2}{2} & (iw)^T & \frac{1+r^2}{2} & (iu)^T \end{pmatrix}.$$

(Note that A is a square matrix: $w \in \mathbb{R}^{2p-2}, u \in \mathbb{R}^{2q-2}; t, \tau \in \mathbb{R}; \alpha \in \mathbb{R}^{2p-1}, \beta \in \mathbb{R}^{2q-1}$.)

This relates the derivatives in $\bar{N}_H Z(H)$ (in w, u, t, θ) to those in $\bar{N}_G$ (in α, b). Since we want to express Δ (in α, β) in terms of variables w, u, t, θ, we need to compute A^{-1}. To that end, we introduce more notation. To shorten calculations, I will call τ-*conjugation*, denoted as an overline, that which takes the negative of terms with a factor of τ. With this in mind, define constants

$$\begin{aligned}
&\kappa_j = -a_j\tau + b_j, \overline{\kappa}_j = a_j\tau + b_j \\
&\eta_j = -a_j - b_j\tau, \overline{\eta}_j = -a_j + b_j\tau \quad 1 \le j \le p-1 \\
&\mu_k = -c_k\tau + d_k, \overline{\mu}_k = c_k\tau + d_k \\
&\nu_k = -c_k - d_k\tau, \overline{\nu}_k = -c_k + d_k\tau \quad 1 \le k \le q-1 \\
&X = -\frac{1}{2}r^2(\tau^2 - 1) + \frac{1}{2}(\tau^2 + 1) \\
&\overline{X} = -\frac{1}{2}r^2(\tau^2 - 1) - \frac{1}{2}(\tau^2 + 1) \\
&\zeta = \tau^2 + 1, \qquad \text{(note: } \tau = \sec^2(\theta) \text{ is defined and nozero for } \theta \neq \pi/2, 3\pi/2.)
\end{aligned}$$

and define $W \in \mathbb{R}^{2p-2}, U \in \mathbb{R}^{2q-2}$, where

$$\begin{aligned}
W_j &= \begin{pmatrix} \kappa_j \\ \eta_j \end{pmatrix} && 1 \le j \le p-1 \\
U_k &= \begin{pmatrix} \mu_k \\ \nu_k \end{pmatrix} && 1 \le k \le q-1.
\end{aligned}$$

(This is analogous to defining w and u from before.) Observe that $X - \overline{X} = \zeta$ and $M(n)M(n)^T = \zeta$. Also observe that $(x+y)^2 - (x-y)^2 = 4xy$, and so

$$X^2 - \overline{X}^2 = 4\left(-\frac{1}{2}r^2(\tau^2-1)\right)\left(\frac{1}{2}(\tau^2+1)\right) = -r^2\zeta(\tau^2-1).$$

Let

$$B = \zeta^{-1}\begin{pmatrix} W^T & U^T & X & \zeta \\ M(p-1)^T & 0 & \tau\overline{iW} & 0 \\ W^T & U^T & \overline{X} & \zeta \\ 0 & M(q-1)^T & -\tau\overline{iU} & 0 \end{pmatrix},$$

where $W \in \mathbb{R}^{2p-2}$ (defined above) is now identified with $\begin{pmatrix} \kappa_1 + i\eta_1 \\ \vdots \\ \kappa_{p-1} + i\eta_{p-1} \end{pmatrix} \in \mathbb{C}^{p-1}$, and $\overline{W}$ is component-wise τ-conjugation. Then $U \in \mathbb{R}^{2q-2}$ is similarly identified with $\begin{pmatrix} \mu_1 + i\nu_1 \\ \vdots \\ \mu_{q-1} + i\nu_{q-1} \end{pmatrix} \in \mathbb{C}^{q-1}$. It will turn out that $B = A^{-1}$, and to assist with the calculation that proves this, we present some identities.

Lemma 6.2.1 *1.*

$$-w(X-\overline{X}) + M(p-1)\overline{iW} = 0, \qquad u(X-\overline{X}) - M(q-1)\overline{iU} = 0.$$

2.

$$W^T + (iw)^T M(p-1)^T = 0, \qquad U^T + (iw)^T M(q-1)^T = 0,$$

3.

$$\frac{1-r^2}{2}X + \frac{1+r^2}{2}\overline{X} + \tau\left((iw)^T\overline{iW} - (iu)^T\overline{iU}\right) = 0.$$

Put in other parts for other entries.

Proof. For (1), first observe that $X - \overline{X} = \zeta$. Grouping in coordinates of size 2, we have

$$\begin{aligned} \zeta w_j - \begin{pmatrix} 1 & \tau \\ -\tau & 1 \end{pmatrix}(\overline{iW})_j &= \zeta\begin{pmatrix} a_j \\ b_j \end{pmatrix} - \begin{pmatrix} 1 & \tau \\ -\tau & 1 \end{pmatrix}\begin{pmatrix} -\overline{\eta}_j \\ \kappa_j \end{pmatrix} \\ &= \zeta\begin{pmatrix} a_j \\ b_j \end{pmatrix} - \begin{pmatrix} -\overline{\eta}_j + \tau\overline{\kappa}_j \\ \tau\overline{\eta}_j + \overline{\kappa}_j \end{pmatrix} \\ &= \zeta\begin{pmatrix} a_j \\ b_j \end{pmatrix} - \begin{pmatrix} (a_j - b_j\tau) + \tau(a_j\tau + b_j) \\ \tau(-a_j + b_j\tau) + (a_j\tau + b_j) \end{pmatrix} \\ &= \zeta\begin{pmatrix} a_j \\ b_j \end{pmatrix} - \begin{pmatrix} (\tau^2+1)a_j \\ (\tau^2+1)b_j \end{pmatrix} = 0. \end{aligned}$$

This proves the first identity, and the second identity is essentially the same equation times -1.

For (2), again grouping in coordinates of size 2, we have

$$\begin{aligned} W_j + \begin{pmatrix} 1 & \tau \\ -\tau & 1 \end{pmatrix} (iw)_j &= \begin{pmatrix} \kappa_j \\ \eta_j \end{pmatrix} + \begin{pmatrix} 1 & \tau \\ -\tau & 1 \end{pmatrix} \begin{pmatrix} -b_j \\ a_j \end{pmatrix} \\ &= \begin{pmatrix} -a_j\tau + b_j \\ -a_j - b_j\tau \end{pmatrix} + \begin{pmatrix} -b_j + \tau a_j \\ \tau b_j + a_j \end{pmatrix} = 0. \end{aligned}$$

The first identity follows from taking the transpose of this equation, and the second identity is essentially the same.

For (3), we have

$$\begin{aligned} &\frac{1-r^2}{2}X + \frac{1+r^2}{2}\overline{X} + \tau\left((iw)^T\overline{iW} - (iu)^T\overline{iU}\right) \\ =&\frac{1}{2}(X+\overline{X}) - \frac{1}{2}(X-\overline{X}) + \tau\left(\sum_{j=1}^{p-1}\big(-b_j(-\overline{\eta}_j) + a_j\overline{\kappa}_j\big) - \sum_{k=1}^{q-1}\big(-d_k(-\overline{\nu}_k) + c_k\overline{\mu}_k\big)\right) \\ =&-\frac{1}{2}r^2(\tau^2-1) - \frac{1}{2}r^2(\tau^2+1) + \tau\left(\sum_{j=1}^{p-1}\big(b_j\overline{\eta}_j + a_j\overline{\kappa}_j\big) - \sum_{k=1}^{q-1}\big(d_k\overline{\nu}_k + c_k\overline{\mu}_k\big)\right) \\ =&-r^2\tau^2 + \tau\left(\sum_{j=1}^{p-1}\big(b_j\overline{\eta}_j + a_j\overline{\kappa}_j\big) - \sum_{k=1}^{q-1}\big(d_k\overline{\nu}_k + c_k\overline{\mu}_k\big)\right) \\ =&-r^2\tau^2 + \tau\left(\sum_{j=1}^{p-1}\big(b_j(-a_j+b_j\tau) + a_j(a_j\tau+b_j)\big) - \sum_{k=1}^{q-1}\big(d_k(-c_k+d_k\tau) + c_k(c_k\tau+d_k)\big)\right) \\ =&-r^2\tau^2 + \tau\left(\sum_{j=1}^{p-1}\tau(a_j^2+b_j^2) - \sum_{k=1}^{q-1}\tau(c_k^2+d_k^2)\right) \\ =&-r^2\tau^2 + \tau(\tau r^2) = 0. \end{aligned}$$

■

Proposition 6.2.1 *$A^{-1} = B$.*

Proof. Since A is a square matrix, it suffices to check that B is a right inverse for A.

$$AB = \begin{pmatrix} -\tau w & M(p-1) & \tau w & 0 \\ \tau u & 0 & -\tau u & M(q-1) \\ 1 & 0 & -1 & 0 \\ \frac{1-r^2}{2} & (iw)^T & \frac{1+r^2}{2} & (iu)^T \end{pmatrix} \zeta^{-1} \begin{pmatrix} W^T & U^T & X & \zeta \\ M(p-1)^T & 0 & \tau\overline{iW} & 0 \\ W^T & U^T & \overline{X} & \zeta \\ 0 & M(q-1)^T & -\tau\overline{iU} & 0 \end{pmatrix}$$

$$= \zeta^{-1}\begin{pmatrix} \zeta I_{2(p-1)} & 0 & -\tau\left(\zeta w - M(p-1)\overline{iW}\right) & 0 \\ 0 & \zeta I_{2(q-1)} & \tau\left(\zeta u - M(q-1)\overline{iU}\right) & 0 \\ 0 & 0 & \zeta & 0 \\ W^T + (iw)^T M(p-1)^T & U^T + (iw)^T M(q-1)^T & \frac{1-r^2}{2}X + \frac{1+r^2}{2}\overline{X} + \tau\left((iw)^T\overline{iW} - (iu)^T\overline{iU}\right) & \zeta \end{pmatrix}.$$

Entries 1-3 and 2-3 are zero by Lemma 6.2.1 (1). Entries 4-1 and 4-2 are zero by Lemma 6.2.1 (2). Entry $4-3$ is zero by Lemma 6.2.1 (3). Thus $AB = I_{2p+2q-2}$, which completes the proof. ■

We thus have

$$\begin{pmatrix} \partial_\alpha \\ \partial_\beta \end{pmatrix} = B \begin{pmatrix} \partial_w \\ \partial_u \\ \partial_t \\ \partial_\tau \end{pmatrix}.$$

We now need to expand Δ in these coordinates. Let $J = \mathrm{diag}(I_{2p-1}, -I_{2q-1})$. Then,

$$\begin{aligned} \Delta &= \sum_{j=1}^{2p-1} \partial_{\alpha_j}^2 - \sum_{k=1}^{2q-1} \partial_{\beta_k}^2 \\ &= \begin{pmatrix} \partial_\alpha \\ \partial_\beta \end{pmatrix}^T J \begin{pmatrix} \partial_\alpha \\ \partial_\beta \end{pmatrix} \\ &= \left(B \begin{pmatrix} \partial_w \\ \partial_u \\ \partial_t \\ \partial_\tau \end{pmatrix} \right)^T JB \begin{pmatrix} \partial_w \\ \partial_u \\ \partial_t \\ \partial_\tau \end{pmatrix}. \end{aligned}$$

Since the entries in B do not commute with these partial derivatives, we need a lemma to help expand the term on the left.

Recall that derivations on an algebra have the property that

$$D(ab) = aD(b) + D(a)b.$$

Lemma 6.2.2 *Let V be the algebra of smooth functions on $\mathbb{R}^n$, with multiplication given by $\varphi \cdot \psi$. For $\varphi \in V$, define $L_\varphi \in End(V)$ by $L_\varphi(\psi) = \varphi \cdot \psi$. Define $D_j \in End(V)$ by $D_j(\psi) = \frac{\partial}{\partial x_j}\psi$. Then,*

$$L_\varphi \circ D_j = D_j \circ L_\varphi - L_{D_j(\varphi)}.$$

Proof.

$$\begin{aligned} (L_\varphi \circ D_j)(\psi) &= L_\varphi\big(D_j(\psi)\big) \\ &= \varphi \cdot D_j(\psi) \\ &= D_j(\varphi \cdot \psi) - D_j(\varphi) \cdot \psi && (D_j \text{ is a derivation}) \\ &= (D_j \circ L_\varphi)(\psi) - L_{D_j(\varphi)}(\psi) \\ &= (D_j \circ L_\varphi - L_{D_j(\varphi)})(\psi), \end{aligned}$$

which completes the proof. ■

Proposition 6.2.2 *If B is a matrix of functions and δ is a matrix of partial derivatives (maybe specify that $(B)_{ij} = f_{ij}$ and $\delta_j = D_j$ are derivations), and $\cdot$ is the application of the derivative, we have*

$$(B\delta)^T = \delta^T B^T - \delta^T \cdot B^T.$$

Proof.

$$
\begin{aligned}
(B\delta)^T &= \left(\begin{pmatrix} f_{11} & \dots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \dots & f_{nn} \end{pmatrix} \begin{pmatrix} D_1 \\ \vdots \\ D_n \end{pmatrix} \right)^T \\
&= \begin{pmatrix} f_{11}D_1 + \dots + f_{1n}D_n \\ \vdots \\ f_{n1}D_1 + \dots + f_{nn}D_n \end{pmatrix}^T \\
&= \begin{pmatrix} f_{11}D_1 + \dots + f_{1n}D_n & \dots & f_{n1}D_1 + \dots + f_{nn}D_n \end{pmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
\delta^T B^T &= \begin{pmatrix} D_1 & \dots & D_n \end{pmatrix} \begin{pmatrix} f_{11} & \dots & f_{n1} \\ \vdots & \ddots & \vdots \\ f_{1n} & \dots & f_{nn} \end{pmatrix} \\
&= \begin{pmatrix} D_1 f_{11} + \dots + D_n f_{1n} & \dots & D_1 f_{n1} + \dots + D_n f_{nn} \end{pmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
\delta^T \cdot B^T &= \begin{pmatrix} D_1 & \dots & D_n \end{pmatrix} \begin{pmatrix} f_{11} & \dots & f_{n1} \\ \vdots & \ddots & \vdots \\ f_{1n} & \dots & f_{nn} \end{pmatrix} \\
&= \begin{pmatrix} D_1 \cdot f_{11} + \dots + D_n \cdot f_{1n} & \dots & D_1 \cdot f_{n1} + \dots + D_n \cdot f_{nn} \end{pmatrix}.
\end{aligned}
$$

By Lemma 6.2.2, $f_{ij}(D_k \cdot \psi) = \big(D_k \cdot (f_{ij}\psi)\big) - (D_k \cdot f_{ij})\psi$, and the result is immediate. ∎

(Note that in the lemma, $\cdot$ means function multiplication and $D(f)$ is application of the derivative. In the proposition, fg means function multiplication and $\cdot$ means application of the derivative.)

We now have

$$
\begin{aligned}
\Delta &= \left(B \begin{pmatrix} \partial_w \\ \partial_u \\ \partial_t \\ \partial_\tau \end{pmatrix} \right)^T JB \begin{pmatrix} \partial_w \\ \partial_u \\ \partial_t \\ \partial_\tau \end{pmatrix} \\
&= \left(\begin{pmatrix} \partial_w \\ \partial_u \\ \partial_t \\ \partial_\tau \end{pmatrix}^T B^T - \begin{pmatrix} \partial_w \\ \partial_u \\ \partial_t \\ \partial_\tau \end{pmatrix}^T \cdot B^T \right) JB \begin{pmatrix} \partial_w \\ \partial_u \\ \partial_t \\ \partial_\tau \end{pmatrix} \qquad \text{(by Proposition 6.2.2)} \\
&= \begin{pmatrix} \partial_w \\ \partial_u \\ \partial_t \\ \partial_\tau \end{pmatrix}^T B^T JB \begin{pmatrix} \partial_w \\ \partial_u \\ \partial_t \\ \partial_\tau \end{pmatrix} - \left(\begin{pmatrix} \partial_w \\ \partial_u \\ \partial_t \\ \partial_\tau \end{pmatrix}^T \cdot B^T \right) JB \begin{pmatrix} \partial_w \\ \partial_u \\ \partial_t \\ \partial_\tau \end{pmatrix} \\
&= \begin{pmatrix} \partial_w^T & \partial_u^T & \partial_t & \partial_\tau \end{pmatrix} B^T JB \begin{pmatrix} \partial_w \\ \partial_u \\ \partial_t \\ \partial_\tau \end{pmatrix} - \left(\begin{pmatrix} \partial_w^T & \partial_u^T & \partial_t & \partial_\tau \end{pmatrix} \cdot B^T \right) JB \begin{pmatrix} \partial_w \\ \partial_u \\ \partial_t \\ \partial_\tau \end{pmatrix}.
\end{aligned}
$$

We will now expand this out, starting with first term. We state some of the calculations in a lemma. Recall the components of the vectors

$$W_j = \begin{pmatrix} \kappa_j \\ \eta_j \end{pmatrix}, (iW)_j = \begin{pmatrix} -\eta_j \\ \kappa_j \end{pmatrix}, \left(\overline{iW}\right)_j = \begin{pmatrix} -\overline{\eta}_j \\ \overline{\kappa}_j \end{pmatrix}, \kappa_j = -a_j\tau + b_j, \eta_j = -a_j - b_j\tau.$$

Lemma 6.2.3 *1.*

$$\begin{aligned} (X - \overline{X})W + \tau M(p-1)\overline{iW} &= -\zeta(iw), \\ (X - \overline{X})U + \tau M(q-1)\overline{iU} &= -\zeta(iu). \end{aligned}$$

2.

$$X^2 - \overline{X}^2 + \tau^2(\overline{iW}^T\overline{iW} - \overline{iU}^T\overline{iU}) = \zeta r^2.$$

Proof. For the first identity in (1), first observe that coordinate-wise we have

$$\begin{aligned} \begin{pmatrix} 1 & \tau \\ -\tau & 1 \end{pmatrix} \left(\overline{iW}\right)_j &= \begin{pmatrix} 1 & \tau \\ -\tau & 1 \end{pmatrix} \begin{pmatrix} -\overline{\eta}_j \\ \overline{\kappa}_j \end{pmatrix} = \begin{pmatrix} -\overline{\eta}_j + \tau\overline{\kappa}_j \\ \tau\overline{\eta}_j + \overline{\kappa}_j \end{pmatrix} \\ &= \begin{pmatrix} -(-a_j + b_j\tau) + \tau(a_j\tau + b_j \\ \tau(-a_j + b_j\tau) + a_j\tau + b_j \end{pmatrix} = \zeta \begin{pmatrix} a_j \\ b_j \end{pmatrix} = \zeta w_j, \end{aligned}$$

and so $M(p-1)\overline{iW} = \zeta w$. Again pairing coordinates together, we have

$$W_j + \tau w_j = \begin{pmatrix} \kappa_j + \tau a_j \\ \eta_j + \tau b_j \end{pmatrix} = \begin{pmatrix} -a_j\tau + b_j + \tau a_j \\ -a_j - b_j\tau + \tau b_j \end{pmatrix} = \begin{pmatrix} b_j \\ -a_j \end{pmatrix} = -(iw)_j,$$

and so $W + \tau w = -(iw)$. We thus have

$$(X - \overline{X})W + \tau M(p-1)\overline{iW} = \zeta W + \zeta\tau w = -\zeta(iw).$$

The second identity is the same with U's and u's.

For (2), visibly we see that $\left|\overline{iW}\right|^2 = |W|^2$. Also notice that since

$$\kappa_j^2 + \eta_j^2 = (-a_j\tau + b_j)^2 + (-a_j - b_j\tau)^2 = \zeta(a_j^2 + b_j^2),$$

we have that $|W|^2 = \zeta|w|^2$, and similarly $|U|^2 = \zeta|u|^2$. Therefore,

$$\begin{aligned} X^2 - \overline{X}^2 + \tau^2\left(\left|\overline{iW}\right|^2 - \left|\overline{iU}\right|^2\right) &= -r^2\zeta(\tau^2 - 1) + \tau^2\left(|W|^2 - |U|^2\right) \\ &= -r^2\zeta(\tau^2 - 1) + \tau^2\left(\zeta|w|^2 - \zeta|u|^2\right) \\ &= -r^2\zeta(\tau^2 - 1) + \tau^2\zeta r^2 \\ &= r^2\zeta. \end{aligned}$$

$\blacksquare$

We have

$$
\begin{pmatrix}\partial_w\\ \partial_u\\ \partial_t\\ \partial_\tau\end{pmatrix}^T B^T J B \begin{pmatrix}\partial_w\\ \partial_u\\ \partial_t\\ \partial_\tau\end{pmatrix}
$$

$$
= \begin{pmatrix}\partial_w\\ \partial_u\\ \partial_t\\ \partial_\tau\end{pmatrix}^T \zeta^{-1} \begin{pmatrix} W^T & U^T & X & \zeta \\ M(p-1)^T & 0 & \tau i\overline{W} & 0 \\ W^T & U^T & \overline{X} & \zeta \\ 0 & M(q-1)^T & -\tau i\overline{U} & 0 \end{pmatrix}^T J\zeta^{-1}
$$

$$
\begin{pmatrix} W^T & U^T & X & \zeta \\ M(p-1)^T & 0 & \tau i\overline{W} & 0 \\ W^T & U^T & \overline{X} & \zeta \\ 0 & M(q-1)^T & -\tau i\overline{U} & 0 \end{pmatrix} \begin{pmatrix}\partial_w\\ \partial_u\\ \partial_t\\ \partial_\tau\end{pmatrix}
$$

$$
=\zeta^{-2} \begin{pmatrix}\partial_w\\ \partial_u\\ \partial_t\\ \partial_\tau\end{pmatrix}^T \begin{pmatrix} W & M(p-1) & W & 0 \\ U & 0 & U & M(q-1) \\ X & \tau i\overline{W}^T & \overline{X} & -\tau i\overline{U}^T \\ \zeta & 0 & \zeta & 0 \end{pmatrix} J \begin{pmatrix} W^T & U^T & X & \zeta \\ M(p-1)^T & 0 & \tau i\overline{W} & 0 \\ W^T & U^T & \overline{X} & \zeta \\ 0 & M(q-1)^T & -\tau i\overline{U} & 0 \end{pmatrix} \begin{pmatrix}\partial_w\\ \partial_u\\ \partial_t\\ \partial_\tau\end{pmatrix}
$$

$$
=\zeta^{-2} \begin{pmatrix}\partial_w\\ \partial_u\\ \partial_t\\ \partial_\tau\end{pmatrix}^T \begin{pmatrix} W & M(p-1) & W & 0 \\ U & 0 & U & M(q-1) \\ X & \tau i\overline{W}^T & \overline{X} & -\tau i\overline{U}^T \\ \zeta & 0 & \zeta & 0 \end{pmatrix} \begin{pmatrix} W^T & U^T & X & \zeta \\ M(p-1)^T & 0 & \tau i\overline{W} & 0 \\ -W^T & -U^T & -\overline{X} & -\zeta \\ 0 & -M(q-1)^T & \tau i\overline{U} & 0 \end{pmatrix} \begin{pmatrix}\partial_w\\ \partial_u\\ \partial_t\\ \partial_\tau\end{pmatrix}
$$

$$
=\zeta^{-2} \begin{pmatrix}\partial_w\\ \partial_u\\ \partial_t\\ \partial_\tau\end{pmatrix}^T \begin{pmatrix} \zeta I_{2p-2} & 0 & -\zeta(iw) & 0 \\ 0 & -\zeta I_{2q-2} & -\zeta(iu) & 0 \\ -\zeta(iw)^T & -\zeta(iu)^T & \zeta r^2 & \zeta^2 \\ 0 & 0 & \zeta^2 & 0 \end{pmatrix} \begin{pmatrix}\partial_w\\ \partial_u\\ \partial_t\\ \partial_\tau\end{pmatrix}
$$

$$
\begin{pmatrix} \text{Entries 1-3 and 2-3 is by Lemma 6.2.3 (1)} \\ \text{Entries 3-1 and 3-2 are their transposes,} \\ \text{Entry 3-3 is by Lemma 6.2.3 (2)} \end{pmatrix}
$$

$$
=\zeta^{-1} \begin{pmatrix}\partial_w\\ \partial_u\\ \partial_t\\ \partial_\tau\end{pmatrix}^T \begin{pmatrix} I_{2p-2} & 0 & -(iw) & 0 \\ 0 & -I_{2q-2} & -(iu) & 0 \\ -(iw)^T & -(iu)^T & r^2 & \zeta \\ 0 & 0 & \zeta & 0 \end{pmatrix} \begin{pmatrix}\partial_w\\ \partial_u\\ \partial_t\\ \partial_\tau\end{pmatrix}
$$

$$
=\zeta^{-1} \left(\partial_w^T \ \partial_u^T \ \partial_t \ \partial_\tau\right) \begin{pmatrix} I_{2p-2} & 0 & -(iw) & 0 \\ 0 & -I_{2q-2} & -(iu) & 0 \\ -(iw)^T & -(iu)^T & r^2 & \zeta \\ 0 & 0 & \zeta & 0 \end{pmatrix} \begin{pmatrix}\partial_w\\ \partial_u\\ \partial_t\\ \partial_\tau\end{pmatrix}
$$

$$
=\zeta^{-1} \left(\partial_w^T \ \partial_u^T \ \partial_t \ \partial_\tau\right) \begin{pmatrix} \partial_w - (iw)\partial_t \\ -\partial_u - (iu)\partial_t \\ -(iw)^T\partial_w - (iu)^T\partial_u + r^2\partial_t + \zeta\partial_\tau \\ \zeta\partial_t \end{pmatrix}.
$$

Continuing, we have

$$
\begin{aligned}
=&\zeta^{-1}\big(\partial_w^T(\partial_w-(iw)\partial_t)+\partial_u^T(-\partial_u-(iu)\partial_t)+\partial_t(-(iw)^T\partial_w-(iu)^T\partial_u+r^2\partial_t+\zeta\partial_\tau)+\zeta\partial_\tau\partial_t\big)\\
=&\zeta^{-1}\left(\partial_w^2-\partial_u^2-\big(\partial_w^T(iw)+\partial_u^T(iu)\big)\partial_t-(iw)^T\partial_w\partial_t-(iu)^T\partial_u\partial_t+r^2\partial_t^2+\zeta\partial_t\partial_\tau+\zeta\partial_\tau\partial_t\right)\\
=&\zeta^{-1}\left(\partial_w^2-\partial_u^2-\big((iw)^T\partial_w+(iu)^T\partial_u\big)\partial_t-(iw)^T\partial_w\partial_t-(iu)^T\partial_u\partial_t+r^2\partial_t^2+\zeta\partial_t\partial_\tau+\zeta\partial_\tau\partial_t\right)\\
=&\zeta^{-1}\left(\partial_w^2-\partial_u^2-2\big((iw)^T\partial_w+(iu)^T\partial_u\big)\partial_t+r^2\partial_t^2+2\zeta\partial_t\partial_\tau\right)\\
=&\zeta^{-1}\left(\partial_w^2-\partial_u^2-2\left(\sum_{j=1}^{p-1}\begin{pmatrix}-b_j & a_j\end{pmatrix}\begin{pmatrix}\partial_{a_j}\\ \partial_{b_j}\end{pmatrix}+\sum_{k=1}^{q-1}\begin{pmatrix}-d_k & c_k\end{pmatrix}\begin{pmatrix}\partial_{c_k}\\ \partial_{d_k}\end{pmatrix}\right)\partial_t+r^2\partial_t^2+2\zeta\partial_t\partial_\tau\right)\\
=&\zeta^{-1}\left(\partial_w^2-\partial_u^2-2\left(\sum_{j=1}^{p-1}(-b_j\partial_{a_j}+a_j\partial_{b_j})+\sum_{k=1}^{q-1}(-d_k\partial_{c_k}+c_k\partial_{d_k})\right)\partial_t+r^2\partial_t^2+2\zeta\partial_t\partial_\tau\right)\\
=&\zeta^{-1}\Omega_0+2\partial_t\partial_\tau.
\end{aligned}
$$

(Note that in the calculations below, since ζ has a τ and we're taking derivatives with respect to τ, we can't factor it out as a constant, so we'll write B^T with the ζ^{-1} distributed inside the matrix; This makes it easy to see why the ζ^{-1} doesn't make the derivation calculation harder.) Now we expand the first term. Since there are no t's in $X, \overline{X}, \tau i\overline{W}, -\tau i\overline{U}$, no w's in $M(p-1)$, no u's in $M(q-1)$, we have

$$
\begin{aligned}
&\left(\begin{pmatrix}\partial_w\\ \partial_u\\ \partial_t\\ \partial_\tau\end{pmatrix}^T \cdot B^T\right) JB \begin{pmatrix}\partial_w\\ \partial_u\\ \partial_t\\ \partial_\tau\end{pmatrix}\\
=&\left(\begin{pmatrix}\partial_w^T & \partial_u^T & \partial_t & \partial_\tau\end{pmatrix} \cdot B^T\right) JB \begin{pmatrix}\partial_w\\ \partial_u\\ \partial_t\\ \partial_\tau\end{pmatrix}\\
=&\left(\begin{pmatrix}\partial_w^T & \partial_u^T & \partial_t & \partial_\tau\end{pmatrix} \cdot \begin{pmatrix}\zeta^{-1}W & \zeta^{-1}M(p-1) & \zeta^{-1}W & 0\\ \zeta^{-1}U & 0 & \zeta^{-1}U & \zeta^{-1}M(q-1)\\ \zeta^{-1}X & \zeta^{-1}\tau i\overline{W}^T & \zeta^{-1}\overline{X} & -\zeta^{-1}\tau i\overline{U}^T\\ 1 & 0 & 1 & 0\end{pmatrix}\right) JB \begin{pmatrix}\partial_w\\ \partial_u\\ \partial_t\\ \partial_\tau\end{pmatrix}\\
=&\begin{pmatrix}\partial_w^T\cdot\zeta^{-1}W+\partial_u^T\cdot\zeta^{-1}U & 0_{2p-2} & \partial_w^T\cdot\zeta^{-1}W+\partial_u^T\cdot\zeta^{-1}U & 0_{2q-2}\end{pmatrix} JB \begin{pmatrix}\partial_w\\ \partial_u\\ \partial_t\\ \partial_\tau\end{pmatrix}\\
=&\zeta^{-1}\begin{pmatrix}\partial_w^T\cdot W+\partial_u^T\cdot U & 0_{2p-2} & \partial_w^T\cdot W+\partial_u^T\cdot U & 0_{2q-2}\end{pmatrix} JB \begin{pmatrix}\partial_w\\ \partial_u\\ \partial_t\\ \partial_\tau\end{pmatrix}\\
=&\zeta^{-1}\begin{pmatrix}-\tau(2p-2)-\tau(2q-2) & 0_{2p-2} & -\tau(2p-2)-\tau(2q-2) & 0_{2q-2}\end{pmatrix} JB \begin{pmatrix}\partial_w\\ \partial_u\\ \partial_t\\ \partial_\tau\end{pmatrix}\\
=&-2\tau\zeta^{-1}(p+q-2)\begin{pmatrix}1 & 0_{2p-2} & 1 & 0_{2q-2}\end{pmatrix} JB \begin{pmatrix}\partial_w\\ \partial_u\\ \partial_t\\ \partial_\tau\end{pmatrix}\\
=&-2\tau\zeta^{-1}(p+q-2)\begin{pmatrix}1 & 0_{2p-2} & -1 & 0_{2q-2}\end{pmatrix}\zeta^{-1}\begin{pmatrix}W^T & U^T & X & \zeta\\ M(p-1)^T & 0 & \tau i\overline{W} & 0\\ W^T & U^T & \overline{X} & \zeta\\ 0 & M(q-1)^T & -\tau i\overline{U} & 0\end{pmatrix}\begin{pmatrix}\partial_w\\ \partial_u\\ \partial_t\\ \partial_\tau\end{pmatrix}\\
=&-2\tau\zeta^{-2}(p+q-2)\begin{pmatrix}1 & 0_{2p-2} & -1 & 0_{2q-2}\end{pmatrix}\begin{pmatrix}W^T & U^T & X & \zeta\\ M(p-1)^T & 0 & \tau i\overline{W} & 0\\ W^T & U^T & \overline{X} & \zeta\\ 0 & M(q-1)^T & -\tau i\overline{U} & 0\end{pmatrix}\begin{pmatrix}\partial_w\\ \partial_u\\ \partial_t\\ \partial_\tau\end{pmatrix}\\
&=-2\tau\zeta^{-2}(p+q-2)\begin{pmatrix}W^T-W^T & U^T-U^T & X-\overline{X} & \zeta-\zeta\end{pmatrix}\begin{pmatrix}\partial_w\\ \partial_u\\ \partial_t\\ \partial_\tau\end{pmatrix}.
\end{aligned}
$$

Continuing, we have

$$
\begin{aligned}
&= -2\tau\zeta^{-2}(p+q-2)\begin{pmatrix}0_{2p-2} & 0_{2q-2} & \zeta & 0\end{pmatrix}\begin{pmatrix}\partial_w \\ \partial_u \\ \partial_t \\ \partial_\tau\end{pmatrix} \\
&= -2\tau\zeta^{-2}(p+q-2)\zeta\partial_t \\
&= -2\zeta^{-1}\tau(p+q-2)\partial_t.
\end{aligned}
$$

Subtracting the second term from the first, we have that the pullback for the Euclidean wave operator under this change of coordinates is

$$
\begin{aligned}
\Delta &= \zeta^{-1}\Omega_0 + 2\partial_t\partial_\tau + 2\tau\zeta^{-1}(p+q-2)\partial_t \\
&= \zeta^{-1}\Omega_0 + 2\partial_t\partial_\tau + 2\tau\zeta^{-1}\gamma\partial_t,
\end{aligned}
$$

where $\gamma = p+q-2$. We note that there is a slight abuse in notation due to the change of variables, and so for $\varphi(\alpha,\beta) \in C^\infty(\bar{N}_G)$, this should be interpreted as

$$\Delta \cdot \varphi = (\zeta^{-1}\Omega_0 + 2\partial_t\partial_\tau + 2\tau\zeta^{-1}\gamma\partial_t) \cdot (\varphi \circ \iota^{-1}).$$

Recall the substitutions $\tau = \tan\theta, \zeta = 1+\tau^2 = \sec^2\theta$, so by the Chain Rule we have

$$\partial_\theta = \frac{\partial\tau}{\partial\theta} \cdot \partial_\tau = \sec^2\theta\partial_\tau.$$

Re-subbing θ's for τ's, we obtain

$$\Delta = \cos^2(\theta)\Omega_0 + 2\cos^2(\theta)\partial_\theta\partial_t + 2\gamma\sin(\theta)\cos(\theta)\partial_t.$$

6.3 Showing that $T_a \circ \Delta = \Omega_a \circ T_a$ for Schwartz Functions

Write $\mathcal{S}(X)$ for the Schwartz space of X. We will show that the diagram mentioned earlier commutes for elements of $\mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon})$ whose restriction to $\bar{N}_G$ is Schwartz. Before we do this, we prove some preliminary lemmas.

Lemma 6.3.1 (Pre-IBP Lemma) *Let $\varphi(\alpha,\beta) \in \mathcal{S}(\bar{N}_G)$. Then (thinking of φ as a function of $\iota(w,u,t,\theta)$)*

$$\lim_{\theta\to\pm\frac{\pi}{2}^\mp} \cos(\theta)^{-s}e^{ia\theta}\varphi(\alpha,\beta) = 0.$$

Proof. Notice that $\alpha_1 + \beta_1 = \tau = \tan\theta$, and so

$$\lim_{\theta\to\pm\frac{\pi}{2}^\mp}(\alpha_1+\beta_1) = \lim_{\theta\to\pm\frac{\pi}{2}^\mp}\tan\theta = \pm\infty.$$

Thus as $\theta \to \pm\frac{\pi}{2}^\mp$, either α_1 or β_1 goes to $\pm\infty$.

Since φ is a Schwartz function, $\partial_t \cdot \varphi$ is also a Schwartz function. Thus we may choose a constant M such that

$$|\partial_t \cdot \varphi(\alpha,\beta)| \le M|\alpha_1+\beta_1|^{-s-1}.$$

Let $\pi/6 < \theta < \pi/2$. We have

$$\left|\cos(\theta)^{-s}e^{ia\theta}\partial_t \cdot \varphi\right| = \left|\frac{\tan(\theta)^s}{\sin(\theta)^s}\partial_t \cdot \varphi\right| \leq \left|\frac{(\alpha_1+\beta_1)^s}{(1/2)^s} \cdot M(\alpha_1+\beta_1)^{-s-1}\right| \to_{\theta\to\frac{\pi}{2}^-} 0.$$

The case $\theta \to -\frac{\pi}{2}^+$ is very similar, and this concludes the proof. ■

Lemma 6.3.2 (IBP Lemma)

$$\int_{-\pi/2}^{\pi/2} \cos(\theta)^{-s}e^{ia\theta}\partial_\theta\partial_t\cdot(\varphi\circ\iota)d\theta = \int_{-\pi/2}^{\pi/2}\big(-s\cos(\theta)^{-s-1}\sin(\theta)e^{ia\theta}+ia\cos(\theta)^{-s}e^{ia\theta}\big)\partial_t\cdot(\varphi\circ\iota)d\theta.$$

Proof. Let

$$\begin{aligned} u &= \cos(\theta)^{-s}e^{ia\theta} & dv &= \partial_\theta\partial_t \cdot \varphi \\ du &= -s\cos(\theta)^{-s-1}\sin(\theta)e^{ia\theta} + ia\cos(\theta)^{-s}e^{ia\theta} & v &= \partial_t \cdot \varphi. \end{aligned}$$

By Lemma 6.3.1, we have

$$\cos(\theta)^{-s}e^{ia\theta}\partial_t \cdot \varphi\Big|_{-\pi/2}^{\pi/2} = 0.$$

Integrating by parts then proves the claim. ■

We are ready for the main result of this chapter, which is to prove the commuting diagram for Schwartz functions.

Theorem 6.3.1 *Let $s = p+q-2$. The following diagram commutes for $\varphi \in \mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon})$ whose restriction to $\bar{N}_G$ is a Schwartz function.*

$$\begin{array}{ccc} \mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon}) & \xrightarrow{\Delta} & \mathrm{ind}_{Q_G}^G(\lambda_{s+2,\varepsilon}) \\ {\scriptstyle T_a}\downarrow & & \downarrow{\scriptstyle T_a} \\ \mathrm{ind}_{Q_H}^H(\chi_{s,a}) & \xrightarrow{\Omega_a} & \mathrm{ind}_{Q_H}^H(\chi_{s+2,a}) \end{array}$$

In particular, if $\varphi \in \ker(\Delta)$, then $T_a(\varphi) \in \ker(\Omega_a)$.

Proof. We have

$$T_a(\Delta\cdot\varphi)\big(\bar{n}_H(w,u,t)\big) = \int_{-\pi/2}^{\pi/2}(\Delta\cdot\varphi)(\bar{n}_H(w,u,t)z(\theta))e^{ia\theta}d\theta.$$

By Proposition 6.1.1, we may factor $\bar{n}_H(w,u,t)z(\theta) = \bar{n}_G(\iota(w,u,t,\theta))q_G(\theta)$ with $\lambda(q_G(\theta)) = \cos(\theta)$. In the calculations that follow, we will often abbreviate $\bar{n}_G\big(\iota(w,u,t,\theta)\big) = \bar{n}_G$ and $q_G(\theta) = q_G$. Now by the duality theorem application in Proposition 5.2.1 and the translation property of $\Delta\cdot\varphi \in \mathrm{ind}_{Q_G}^G(\lambda_{s+2,\varepsilon})$, this becomes

$$\begin{aligned} \int_{-\pi/2}^{\pi/2}(\Delta\cdot\varphi)(\bar{n}_Gq_G)e^{ia\theta}d\theta &= \int_{-\pi/2}^{\pi/2}\lambda_{s+2,\varepsilon}(q_G)^{-1}(\Delta\cdot\varphi)(\bar{n}_G)e^{ia\theta}d\theta \\ &= \int_{-\pi/2}^{\pi/2}|\cos(\theta)|_\varepsilon^{-(s+2)}(\Delta\cdot\varphi)(\bar{n}_G)e^{ia\theta}d\theta \\ &= \int_{-\pi/2}^{\pi/2}\cos(\theta)^{-s-2}(\Delta\cdot\varphi)(\bar{n}_G)e^{ia\theta}d\theta. \end{aligned}$$

Plugging in the pullback of Δ through ι, we have

$$\int_{-\pi/2}^{\pi/2} \cos(\theta)^{-s-2}\big(\cos^2(\theta)\Omega_0 + 2\cos^2(\theta)\partial_\theta\partial_t + 2\gamma\sin(\theta)\cos(\theta)\partial_t\big)\cdot\varphi(\bar{n}_G)e^{ia\theta}d\theta$$
$$= \int_{-\pi/2}^{\pi/2} \big(\cos(\theta)^{-s}e^{ia\theta}\Omega_0 + 2\cos(\theta)^{-s}e^{ia\theta}\partial_\theta\partial_t + 2\gamma\sin(\theta)\cos(\theta)^{-s-1}e^{ia\theta}\partial_t\big)\cdot\varphi(\bar{n}_G)d\theta.$$

Assuming that φ is a Schwartz function on $\bar{N}_G$, we are justified in integrating this middle term by parts, and so by Lemma 6.3.2 this is

$$\int_{-\pi/2}^{\pi/2} \big(\cos(\theta)^{-s}e^{ia\theta}\Omega_0 + 2ia\cos(\theta)^{-s}e^{ia\theta}\partial_t - 2s\cos(\theta)^{-s-1}\sin(\theta)e^{ia\theta}\partial_t$$
$$+ 2\gamma\sin(\theta)\cos(\theta)^{-s-1}e^{ia\theta}\partial_t\big)\cdot\varphi(\bar{n}_G)d\theta$$
$$= \int_{-\pi/2}^{\pi/2} \big(\cos(\theta)^{-s}e^{ia\theta}\Omega_0 + 2ia\cos(\theta)^{-s}e^{ia\theta}\partial_t - 2(s-\gamma)\cos(\theta)^{-s-1}\sin(\theta)e^{ia\theta}\partial_t\big)\cdot\varphi(\bar{n}_G)d\theta.$$

Since $s = \gamma = p + q - 2$ by assumption, this is

$$\int_{-\pi/2}^{\pi/2} \big(\cos(\theta)^{-s}e^{ia\theta}\Omega_0 + 2ia\cos(\theta)^{-s}e^{ia\theta}\partial_t\big)\cdot\varphi(\bar{n}_G)d\theta.$$

Now, we recall that $\Omega_a = \Omega_0 + 2ia$ so we have

$$\int_{-\pi/2}^{\pi/2} \cos(\theta)^{-s}\Omega_a\cdot\varphi(\bar{n}_G)e^{ia\theta}d\theta.$$

Now, the partials of this integrand are continuous and the operator Ω_a is not a function of theta, we are justified in bringing it outside the integral leaving us

$$\Omega_a\cdot\int_{-\pi/2}^{\pi/2} \cos(\theta)^{-s}\varphi(\bar{n}_G)e^{ia\theta}d\theta.$$

Finally, using the original translation property of φ, we have

$$\begin{aligned}\Omega_a\cdot\int_{-\pi/2}^{\pi/2}\varphi(\bar{n}_G q_G)e^{ia\theta}d\theta &= \Omega_a\cdot\int_{-\pi/2}^{\pi/2}\varphi(\bar{n}_H z(\theta))e^{ia\theta}d\theta\\ &= \Omega_a\cdot T_a(\varphi)(\bar{n}_H),\end{aligned}$$

which completes the proof. ■

CHAPTER VII

EXTENDING THE COMMUTING DIAGRAM TO THE ENTIRE DEGENERATE PRINCIPAL SERIES

We showed in the previous chapter that for $\varphi \in \mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon})$ whose restriction to $\bar{N}_G$ is Schwartz, the map T_a H-intertwines the action of the differential intertwining operators Δ and Ω_a for the parameter $s = p+q-2$, and so for φ with this property, the following diagram commutes.

$$\begin{array}{ccc} \mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon}) & \xrightarrow{\Delta} & \mathrm{ind}_{Q_G}^G(\lambda_{s+2,\varepsilon}) \\ {\scriptstyle T_a}\downarrow & & \downarrow{\scriptstyle T_a} \\ \mathrm{ind}_{Q_H}^H(\chi_{s,a}) & \xrightarrow{\Omega_a} & \mathrm{ind}_{Q_H}^H(\chi_{s+2,a}) \end{array}$$

In this chapter we will show how this intertwining property extends to all elements of the degenerate principal series. Let

$$C_G = K_G \cap Q_G,$$
$$E_G = \mathrm{ind}_{C_G}^{K_G}(\lambda_{s,\varepsilon}|_{C_G}),$$
$$V_G = \bar{N}_G Q_G \cap K_G.$$

We will prove existence of a function $f \in \mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon})$ that is Schwartz on $\bar{N}_G$ (in fact has compact support on $\bar{N}_G$), such that

$$\mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon}) = \overline{\mathrm{span}(K_H \cdot f)}.$$

In particular, the span of H-translates of f is dense in $\mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon})$. The first part of this chapter will explain the general theory of this density argument, which works in the compact picture and requires the function to have a nonzero projection into each K-isotypic subspace. We will impose the requirement that this function have compact support in a right C_G-saturated subset of V_G, so that the extension of this function to the induced picture, and then restricted to $\bar{N}_G$, will have compact support.

7.1 General Background on Isotypic Projection and Density Argument

First we will show that for a representation of a compact Lie group K, there is a projection map onto the τ-isotypic subspace, and then we will explain how this fits into the density argument. Suppose τ and τ' are inequivalent irreducible unitary representations of a Lie group K on finite-dimensional spaces V and V', respectively, and let the understood Hermitian inner products be denoted $(\cdot,\cdot)$. Then the Schur orthogonality relations (see Knapp, P.241 Corollary 4.10) are

$$\int_K (\tau(x)u,v)\overline{(\tau'(x)u',v')}dx = 0 \qquad \text{for all } u,v\in V \text{ and } u',v'\in V',$$

$$\int_K (\tau(x)u_1,v_1)\overline{(\tau(x)u_2,v_2)}dx = \frac{(u_1,u_2)\overline{(v_1,v_2)}}{\dim V} \qquad \text{for all } u_1,v_1,u_2,v_2\in V.$$

The latter relation can be expressed in terms of equivalent representations V and V', similar to the relation above it. Let $L: V'\to V$ be an equivalence of representations. (Cite) There is a $c>0$ so that the corresponding inner products satisfy the relation $(L(u),L(v)) = c\cdot(u',v')$ for all $u,v\in V$ and all $u',v'\in V'$. Let $u_1\in U, v_1\in V$ and $u',v'\in V$ satisfy $L(u')=u_2$ and $L(v')=v_2$. Then by the latter relation, we have

$$\begin{aligned}
\int_K (\tau(x)u,v)\overline{(\tau'(x)u',v')}dx &= \frac{1}{c}\int_K (\tau(x)u,v)\overline{(L(\tau'(x)u'),L(v'))}dx \\
&= \frac{1}{c}\int_K (\tau(x)u,v)\overline{(\tau(x)L(u'),L(v'))}dx \\
&= \frac{1}{c}\int_K (\tau(x)u,v)\overline{(\tau(x)u_2,v_2)}dx \\
&= \frac{1}{c}\frac{(u,u_2)\overline{(v,v_2)}}{\dim V} \\
&= \frac{(u,L(u'))\overline{(v,L(v'))}}{c\cdot\dim V}.
\end{aligned}$$

We use these Schur orthogonality relations in the next proposition to prove an identity which we will use to define the aforementioned τ-isotypic projection map.

Proposition 7.1.1 *Suppose K is a compact Lie group, and let τ,τ' be irreducible representations of K. Define*

$$d_\tau = \frac{\dim(E_\tau)}{\mathrm{Vol}(K)},$$

a positive real number. For any $T\in\mathrm{End}(E_\tau)$,

$$\int_K \mathrm{tr}\big(\tau(k^{-1})T\big)\tau'(k)dk = \begin{cases}(c\cdot d_\tau)^{-1}T', & \tau\cong\tau' \\ 0, & \tau\ncong\tau',\end{cases}$$

where we use the isomorphism of τ with τ' to identify T with an operator $T'\in\mathrm{End}(E_{\tau'})$, and $(\cdot,\cdot)_\tau = c(\cdot,\cdot)_{\tau'}$ for some $c>0$.

Proof. Let $\{e_j\}_{j=1}^n$ and $\{e_j{}'\}_{j=1}^n$ be orthonormal bases for E_τ and $E_{\tau'}$, respectively, and let $L: E_{\tau'}\to E_\tau$ be the equivalence of representations, in the case that it exists, such that $L(e_j') = e_j$ for $1\le j\le n$. For $k\in K$, the matrix of $\tau(k)$ in this basis is given by

$\tau_{ij}(k) = (\tau(k)e_j, e_i)$. Since $\int_K \operatorname{tr}\big(\tau(k^{-1})T\big)\tau'(k)dk$ is an endomorphism of $E_{\tau'}$, we integrate entry-wise in this basis to get the result:

$$\begin{aligned}
\left(\int_K \operatorname{tr}\big(\tau(k^{-1})T\big)\tau'(k)dk\right)_{ij} &= \int_K \operatorname{tr}\big(\tau(k^{-1})T\big)\tau'_{ij}(k)dk \\
&= \int_K \operatorname{tr}\big(\tau(k^{-1})T\big)\big(\tau'(k)e'_j, e'_i\big)dk \\
&= \int_K \left(\sum_{\ell=1}^{n}\big(\tau(k^{-1})Te_\ell, e_\ell\big)\right)\big(\tau'(k)e'_j, e'_i\big)dk \quad \text{(defn. of trace)} \\
&= \sum_{\ell=1}^{n}\int_K \big(\tau(k^{-1})Te_\ell, e_\ell\big)\big(\tau'(k)e'_j, e'_i\big)dk \\
&= \sum_{\ell=1}^{n}\int_K \big(\tau(k^{-1})Te_\ell, e_\ell\big)\overline{\big(e'_i, \tau'(k)e'_j\big)}dk \\
&= \sum_{\ell=1}^{n}\int_K (\tau(k^{-1})Te_\ell, e_\ell)\overline{\big(\tau'(k^{-1})e'_i, e'_j\big)}dk \\
&= \sum_{\ell=1}^{n}\begin{cases}\frac{(Te_\ell, L(e'_i))\overline{(e_\ell, L(e'_j))}}{c\cdot\dim(V)}, & \tau \cong \tau' \\ 0, & \tau \ncong \tau'\end{cases} \quad \text{(by Schur Orthogonality)} \\
&= \begin{cases}\frac{(Te_j, L(e'_i))}{c\cdot\dim(V)}, & \tau \cong \tau' \\ 0, & \tau \ncong \tau'\end{cases} \quad (e_\ell, L(e'_j)) = \delta_{\ell,j} \\
&= \begin{cases}\frac{T'_{ij}}{c\cdot\dim(V)}, & \tau \cong \tau' \\ 0, & \tau \ncong \tau'\end{cases}, \quad (T_{ij} = T'_{ij} \text{ when } \tau \cong \tau') \\
&= \begin{cases}(c\cdot d_\tau)^{-1}T'_{ij}, & \tau \cong \tau' \\ 0, & \tau \ncong \tau'\end{cases}
\end{aligned}$$

■

If π is a representation of K which has multiplicity-free decomposition, then we can define a projection map onto the isotypic subspaces, which is the subject of the next corollary.

Corollary 7.1.1 *Let K be a compact Lie group, and let π is a representation of K which has multiplicity-free decomposition. If τ is an irreducible representation of K, then the map*

$$P_\tau(v) = d_\tau \int_K tr\big(\tau(k^{-1})\big)\pi(k)(v)dk$$

projects onto the τ-isotype in E_π. In particular, suppose $E_\pi = \sum_{j=1}^{\infty} E_j$, with τ_j the map associated with each irreducible E_j. For $v \in E_\pi$, write $v = \sum_{j=1}^{\infty} v_j$ where $v_j \in E_j$. If $\tau \cong \tau_j$ for some j, then

$$P_\tau(v) = c_j^{-1}v_j,$$

where $(\cdot,\cdot)_\tau = c_j(\cdot,\cdot)_{\tau_j}$ for $c_j > 0$.

Proof. We have

$$\begin{aligned}
P_\tau(v) &= d_\tau \int_K \operatorname{tr}\big(\tau(k^{-1})\big)\pi(k)(v)dk \\
&= d_\tau \int_K \operatorname{tr}\big(\tau(k^{-1})\big)\left(\sum_{j=1}^{\infty}\tau_j(k)(v_j)\right)dk \\
&= d_\tau \sum_{j=1}^{\infty}\int_K \operatorname{tr}\big(\tau(k^{-1})\big)\tau_j(k)(v_j)dk.
\end{aligned}$$

By Proposition 7.1.1, since $\tau \cong \tau_j$ (and this is the only pair of equivalent representations since the decomposition was assumed to be multiplicity-free), this is

$$P_\tau(v) = d_\tau(c_j \cdot d_\tau)^{-1}v_j = c_j^{-1}v_j,$$

as claimed. ($P_\tau(v) = 0$ if there is no equivalent representation to τ in this decomposition). ■

Now we are ready to show how the density argument from the beginning of the chapter follows from projecting a function onto the isotypic subspaces that the representation contains.

Proposition 7.1.2 *Let K be a compact Lie group, and let π be a representation of K which has multiplicity-free decomposition. If $v \in E_\pi$ such that $P_\tau(v) \neq 0$ for each irreducible τ that occurs in the decomposition, then*

$$E_\pi = \overline{\mathit{span}(K \cdot v)}.$$

Proof. The K-finite space of E_π is dense in E_π, so it suffices to check that each irreducible E_τ is contained in $\overline{\operatorname{span}(K \cdot v)}$.

Observe that since $k \mapsto k \cdot v$ is continuous, the integral defining $P_\tau(v)$ is the limit of Riemann sums of the form

$$d_\tau \sum_{j=1}^{n} \operatorname{vol}(S_j)\chi_\tau(k_j^{-1})k_j \cdot v$$

where $\{S_j\}$ is a partition of K and $k_j \in S_j$. Thus, $P_\tau(v)$ lies in the closure of the span of the orbit $K \cdot v$. That is, $P_\tau(v) \in \overline{\operatorname{span}(K \cdot v)}$. Since each $P_\tau(v)$ is nonzero by assumption, it follows that

$$E_\tau \subset \overline{\operatorname{span}(K \cdot v)}$$

for every irreducible τ in this space. Therefore, $\overline{\operatorname{span}(K \cdot v)}$ contains the K-finite space, which completes the proof. ■

7.2 Obtaining a Suitable Function via the Baire Category Theorem

We will now show that there is an $f \in E_G$ with compact support in a right C_G-saturated subset of V_G such that

$$E_G = \overline{\operatorname{span}(K_H \cdot f)}.$$

Lemma 7.2.1 *Let π be an irreducible representation of K_G such that $E_\pi^{(C_G,\lambda_{s,\varepsilon})} \neq \{0\}$ (i.e., a K_G-type of $\mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon})$). Let τ be an irreducible representation of K_H that occurs in $res_{K_H}^{K_G}(\pi)$. Then there exists $f \in E_G$, with compact right C_G-saturated support contained in V_G such that $P_\tau(f) \neq 0$.*

Proof. Let $\varphi : K_G \to \mathbb{C}$ be a smooth function. Define $f_\varphi : K_G \to \mathbb{C}$ by

$$f_\varphi(x) = \int_{C_G} \lambda_{s,\varepsilon}(y)\varphi(xy)dy.$$

Observe that $f_\varphi \in E_G$, since for $c \in C_G$, we have

$$\begin{aligned}
f_\varphi(xc) &= \int_{C_G} \lambda_{s,\varepsilon}(y)\varphi(xcy)dy \\
&= \int_{C_G} \lambda_{s,\varepsilon}(c^{-1}y)\varphi(xc(c^{-1}y))dy, && \text{(subbing } y \mapsto c^{-1}y) \\
&= \lambda_{s,\varepsilon}(c^{-1}) \int_{C_G} \lambda_{s,\varepsilon}(y)\varphi(xy)dy \\
&= \lambda_{s,\varepsilon}(c)^{-1} f_\varphi(x).
\end{aligned}$$

Notice that

$$\begin{aligned}
P_\pi(f_\varphi)(x) &= d_\pi \int_{K_G} \mathrm{tr}(\pi(k^{-1}))l_k(f_\varphi)(x)\ dk \\
&= d_\pi \int_{K_G} \mathrm{tr}(\pi(k^{-1}))f_\varphi(k^{-1}x)\ dk \\
&= d_\pi \int_{K_G} \mathrm{tr}(\pi(k^{-1})) \left(\int_{C_G} \lambda_{s,\varepsilon}(y)\varphi(k^{-1}xy)dy \right)\ dk \\
&= d_\pi \int_{K_G} \int_{C_G} \mathrm{tr}(\pi(k^{-1}))\lambda_{s,\varepsilon}(y)\varphi(k^{-1}xy)dydk \\
&= d_\pi \int_{K_G} \int_{C_G} \mathrm{tr}(\pi(uy^{-1}x^{-1}))\lambda_{s,\varepsilon}(y)\varphi(u)dydu \quad \text{(with } u\text{-sub, } u = k^{-1}xy) \\
&= d_\pi \int_{C_G} \int_{K_G} \mathrm{tr}(\pi(uy^{-1}x^{-1}))\lambda_{s,\varepsilon}(y)\varphi(u)dudy \quad \text{(Fubini)}
\end{aligned}$$

We may choose $\varphi \in C^\infty(K_G)$ to have compact support contained in V_G very close to e and total integral 1, so that f_φ will have compact support in a right C_G-saturated subset of V_G. Then $P_\pi(f_\varphi)$ will be close to the function

$$F_\pi(x) = d_\pi \int_{C_G} \mathrm{tr}(\pi(y^{-1}x^{-1}))\lambda_{s,\varepsilon}(y)dy.$$

Since F_π is a limit of $P_\pi(f_\varphi)$'s in the complete space E_π (finite-dimensional complex vector space), we have that $F_\pi \in E_\pi$. In fact, $F_\pi \in E_\pi^{(C_G,\lambda_{s,\varepsilon})}$, since for $c \in C_G$,

$$\begin{aligned}
l_c(F_\pi)(x) &= F_\pi(c^{-1}x) \\
&= d_\pi \int_{C_G} \operatorname{tr}(\pi(y^{-1}(c^{-1}x)^{-1}))\lambda_{s,\varepsilon}(y)dy \\
&= d_\pi \int_{C_G} \operatorname{tr}(\pi(y^{-1}x^{-1}c))\lambda_{s,\varepsilon}(y)dy \\
&= d_\pi \int_{C_G} \operatorname{tr}(\pi(y^{-1})\pi(x^{-1})\pi(c))\lambda_{s,\varepsilon}(y)dy \\
&= d_\pi \int_{C_G} \operatorname{tr}(\pi(c)\pi(y^{-1})\pi(x^{-1}))\lambda_{s,\varepsilon}(y)dy \\
&= d_\pi \int_{C_G} \operatorname{tr}(\pi(cy^{-1}x^{-1}))\lambda_{s,\varepsilon}(y)dy \\
&= d_\pi \int_{C_G} \operatorname{tr}(\pi((yc^{-1})^{-1}x^{-1}))\lambda_{s,\varepsilon}(y)dy \\
&= d_\pi \int_{C_G} \operatorname{tr}(\pi(y^{-1}x^{-1}))\lambda_{s,\varepsilon}(yc)dy \quad \text{(R-invariant Haar measure, subbing } y \mapsto yc\text{)} \\
&= \lambda_{s,\varepsilon}(c)F_\pi(x).
\end{aligned}$$

Thus, $P_\pi(f_\varphi) \in E_\pi$ is close to an element of $E_\pi^{(C_G,\lambda_{s,\varepsilon})}$ under the above conditions on φ. Notice that

$$\begin{aligned}
F_\pi(e) &= d_\pi \int_{C_G} \operatorname{tr}\big(\pi(y^{-1})\big)\lambda_{s,\varepsilon}(y)dy \\
&= d_\pi \operatorname{tr}\left(\int_{C_G} \pi(y^{-1})\lambda_{s,\varepsilon}(y)dy\right),
\end{aligned}$$

and so $F_\pi(e)$ is d_π times the trace of the map

$$\int_{C_G} \pi(y^{-1})\lambda_{s,\varepsilon}(y)dy$$

on the space E_π.

This map projects onto $E_\pi^{(C_G,\lambda_{s,\varepsilon})}$. We see that this map is into after

$$\begin{aligned}
\int_{C_G} \pi(y^{-1})\lambda_{s,\varepsilon}(y)dy &= \int_{C_G} \pi\big((yc^{-1})^{-1}\big)\lambda_{s,\varepsilon}(yc^{-1})dy \qquad \text{(subbing } y \mapsto yc^{-1}\text{)} \\
&= \int_{C_G} \pi(cy^{-1})\lambda_{s,\varepsilon}(yc^{-1})dy \\
&= \int_{C_G} \pi(c)\pi(y^{-1})\lambda_{s,\varepsilon}(y)\lambda_{s,\varepsilon}(c^{-1})dy \qquad \text{(multiplicative properties)} \\
&= \lambda_{s,\varepsilon}(c)^{-1}\pi(c)\int_{C_G} \pi(y^{-1})\lambda_{s,\varepsilon}(y)dy,
\end{aligned}$$

and multiplying both sides by $\lambda_{s,\varepsilon}(c)$. To see that this map acts as the identity on $E_\pi^{(C_G,\lambda_{s,\varepsilon})}$, first suppose $f \in E_\pi^{(C_G,\lambda_{s,\varepsilon})}$. Recall that we are using the normalized Haar measure, and so $\mathrm{Vol}(C_G) = 1$. We have

$$\begin{aligned}\left(\int_{C_G} \pi(y^{-1})\lambda_{s,\varepsilon}(y)dy\right)(f) &= \int_{C_G} \lambda_{s,\varepsilon}(y)\pi(y^{-1})(f)dy \\ &= \int_{C_G} \lambda_{s,\varepsilon}(y)\lambda_{s,\varepsilon}(y^{-1})f dy \\ &= \mathrm{Vol}(C_G)f = f.\end{aligned}$$

The trace of a projection map is the dimension of the target space, and so

$$F_\pi(e) = d_\pi \dim(E_\pi^{(C_G,\lambda_{s,\varepsilon})}).$$

These remarks show that if $E_\pi^{(C_G,\lambda_{s,\varepsilon})} \neq \{0\}$, then we can choose $\varphi \in C^\infty(K_G)$ such that $f_\varphi \in E_G$ has compact support in a right C_G-saturated subset of V_G, and $P_\pi(f_\varphi)$ is close to the nonzero function F_π, and in particular, $P_\pi(f)$ is not identically zero.

In Chapter 2, we calculated the K_G-types explicitly in terms of their highest weights, and realized models of them as tensors of homogeneous polynomials

$$\pi^{m,n} := \mathscr{H}^m(\mathbb{R}^{2p}) \otimes \mathscr{H}^n(\mathbb{R}^{2q}).$$

The (multiplicity-free) decomposition of $\pi^{m,n}$ by restricting to K_H is a sum of

$$\tau^{m_1,n_1} := \mathscr{H}^{m_1,m-m_1}(\mathbb{C}^p) \otimes \mathscr{H}^{n_1,n-n_1}(\mathbb{C}^q)$$

given by

$$\mathrm{res}_{K_H}^{K_G}(\pi^{m,n}) \cong \sum_{\substack{0\le m_1\le m\\ 0\le n_1\le n}} \tau^{m_1,n_1}.$$

The embedding vectors for $\pi^{m,n}$ are $F_\pi = \xi^m \otimes \xi^n$ given in Chapter 1, and we also explained in that chapter how each ξ^m decomposes as a linear combination of embedding vectors for τ^{m_1,n_1}, with each component nonzero.

Since we can choose φ so that $P_{\pi^{m,n}}(f_\varphi)$ is close to $F_{\pi^{m,n}}$, then $P_{\tau^{m_1,n_1}}(P_{\pi^{m,n}}(f_\varphi)$ is close to $F_{\tau^{m_1,m_2}}$, and so is nonzero, which completes the proof. ■

Let K be a compact Lie group, let $\{X_i\}_{i=1}^m$ be a basis of $\mathfrak{k} = \mathrm{Lie}(K)$, let $\mathcal{U}(\mathfrak{k})$ be the universal enveloping algebra of $\mathfrak{k}$, spanned by elements $u_n = X_1X_2\cdots X_n$ where each $X_j \in \mathfrak{k}$. Let

$$\|f\|_n = \sup_{u_n\in\mathcal{U}(\mathfrak{k})} \|u_n.f\|_\infty.$$

Let

$$d(f,g) = \sum_{n=1}^{\infty} 2^{-n}\frac{\|f-g\|_n}{1+\|f-g\|_n},$$

a translation-invariant metric on $C^\infty(K)$ in the sense that $d(f+h, g+h) = d(f,g)$ for $f, g, h \in C^\infty(K)$.

The Baire Category Theorem implies that if a non-empty complete metric space is the countable union of closed sets, then one of these closed sets has non-empty interior. We will use this theorem to implicitly obtain a smooth section $f \in \mathrm{ind}_{C_G}^{K_G}(\lambda_{s,\varepsilon}|_{C_G})$ such that $P_\pi(f) \not\equiv 0$ for all K_H-types π in the decomposition.

By restriction from K_G to K_H, a K_G-type breaks up as a sum of K_H-types from multiple degenerate principal series $\mathrm{ind}_{Q_H}^{H}(\chi_{s,a})$. A K_G type corresponds to a box of K_H-types and a single $\mathrm{ind}_{Q_H}^{H}(\chi_{s,a})$ will correspond to a diagonal of K_H-types in that box, and there will be finitely many needed to get the whole box.

Let

$$E'_G = \{f \in E_G : f \text{ has compact support in a right } C_G\text{-saturated subset of } V_G\}.$$

Theorem 7.2.1 (Baire Category Argument) *There exists a smooth section $f \in E'_G$ with compact support in a right C_G-saturated subset of V_G such that $P_\tau(f) \neq 0$ for each irreducible τ that occurs in the decomposition of E'_G under restriction to K_H.*

Proof. There are countably many K_G-types (pairs $(m,n) \in \mathbb{Z}^2_{\geq 0}$), and since the restriction of a K_G-type to K_H yields finitely many K_H types (4-tuples $(m_1, m_2, n_1, n_2) \in \mathbb{Z}^4_{\geq 0}$ where $m_1 + m_2 = m, n_1 + n_2 = n$), each multiplicity-free (because T_a is one-one), there are countably many K_H-types in E'_G, say $\{\tau_n\}_{n=1}^{\infty}$. For each n, let

$$\mathcal{X}_n = \{f \in E'_G : P_{\tau_n}(f) = 0\}.$$

Each $\mathcal{X}_n$ is closed, since $\mathcal{X}_n = P_{\tau_n}^{-1}(\{0\})$. Each $\mathcal{X}_n$ is nonempty, since $0 \in \mathcal{X}_n$. Each $\mathcal{X}_n$ is proper by Lemma 7.2.1. Each $\mathcal{X}_n$ has empty interior, since it is a proper subspace of the topological vector space E'_G. Due to these observations, we have by the Baire Category Theorem that $E'_G \neq \bigcup_{n=1}^{\infty} \mathcal{X}_n$, which completes the proof. ■

7.3 The Correspondence between Right C_G-Saturated Subsets of K_G with Subsets of $\bar{N}_G$

Lemma 7.3.1

$$\bar{N}_G Q_G = V_G Q_G.$$

Proof. Recall that we defined $V_G = \bar{N}_G Q_G \cap K_G$. Let $\bar{n}_G \in \bar{N}_G, q_G \in Q_G$. Since $G = K_G Q_G$, we may factor $\bar{n}_G q_G = k_G q'_G$ for some $k_G \in K_G, q'_G \in Q_G$. But then $k_G = \bar{n}_G q_G (q'_G)^{-1} \in \bar{N}_G Q_G$, which shows containment. The reverse containment is obvious. ■

Proposition 7.3.1 *The map $\varphi : G/Q_G \to \mathcal{Q}$ given by $\varphi(gQ_G) = [gv_1]$ is a homeomorphism. In particular,*

$$G/Q_G \cong K_G/C_G.$$

Proof. First we show that G/Q_G is compact. Let

$$f : K_G/C_G \to G/Q_G$$

be given by

$$f(kC_G) = kQ_G.$$

Since for $k_1, k_2 \in K_G$, $k_1 C_G = k_2 C_G$ if and only if $k_1 Q_G = k_2 Q_G$, and so f is well defined and one-one. The map $\psi : K_G \to G/Q_G$ given by $\psi(k) = kQ_G$ is the composition of the inclusion $K_G \to G$ with the quotient map $G \to G/Q_G$, and so is continuous. Since ψ is constant on kC_G for all $k \in K_G$, f is continuous (Munkres theorem 22.2, p.142, see following diagram).

$$\begin{tikzcd} K_G \arrow[d] \arrow[dr, "\psi"] & \\ K/C_G \arrow[r, "f"] & G/Q_G \end{tikzcd}$$

Clearly f is onto, and so G/Q_G is the continuous image of K_G/C_G, and so is compact.

As $\mathbb{P}(\mathbb{R}^{2p,2q})$ is Hausdorff, so is the topological subspace $\mathcal{Q}$.

Observe that the map $g \mapsto gv_1$ is continuous (matrix multiplication), and the map $\mathbb{R}^{2p,2q} \to \mathbb{P}(\mathbb{R}^{2p,2q})$ given by $x \mapsto [x]$ (where $[x]$ is the line through x) is the quotient map, so the composition

$$G \to \mathbb{R}^{2p,2q} \to \mathcal{Q}$$

given by

$$g \mapsto gv_1 \mapsto [gv_1]$$

is continuous.

This map is constant on each coset, since for $gq \in gQ_G$, we have $gq \mapsto [gqv_1] = [gv_1]$. Thus, the induced map $\varphi : G/Q_G \to \mathcal{Q}$ given by $\varphi(gQ_G) = [gv_1]$ is continuous (again by Theorem 22.2 in Munkres, p.142).

$$\begin{tikzcd} G \arrow[d] \arrow[dr] & \\ G/Q_G \arrow[r, "\varphi"] & \mathcal{Q} \end{tikzcd}$$

A continuous bijection from a compact space to a Hausdorff space is a homeomorphism. To complete the proof, it remains to show that φ is a bijection.

If $[gv_1] = [g'v_1]$, then $gv_1 = \lambda g' v_1$ for some $\lambda \in \mathbb{R}^{\times}$. Thus, $(g')^{-1} g \in Q_G$, and so $gQ_G = g'Q_G$, so that φ is one-one.

If $[x] \in \mathcal{Q}$ for $x = (x_1, x_2) \in \mathbb{R}^{2p,2q}$, we may assume that $\|x_1\|_{2p} = \|x_2\|_{2q} = 1$. Recall that by Witt's theorem, $K_G \cong SO(2p) \times SO(2q)$ acts transitively on the level sets of $\mathbb{R}^{2p} \oplus \mathbb{R}^{2q}$. Thus we may choose $kv_1 = x$, so that $\varphi(kQ_G) = [x]$. This shows that φ is onto, and thus a homeomorphism.

We showed first that $K_G/C_G \to G/Q_G$ is a continuous bijection, and now we know that the latter space is Hausdorff, and so this map is a homeomorphism. ∎

Proposition 7.3.2 *The map $\psi : \bar{N}_G \to \bar{N}_G Q_G / Q_G$ given by $\psi(\bar{n}) = \bar{n} Q_G$ is a homeomorphism. In particular, the map $V_G \to \bar{N}_G$ given by $k = \bar{n}(k) q(k) \mapsto \bar{n}(k)$ is continuous.*

Proof. Since Q_G acts continuously on G, the quotient map

$$p : G \to G/Q_G$$

is open. Thus, the map

$$p|_{\bar{N}_G} : \bar{N}_G \to \bar{N}_G Q_G / Q_G$$

is also open and continuous.

I will show that this map is a bijection. By Knapp's Lie Groups: Beyond an Introduction, 7.83 (e), $\bar{N}_G \cap Q_G = \{e\}$. So

$$\bar{n}_G Q_G = \bar{n}'_G Q_G \implies \bar{n}_G = \bar{n}'_G,$$

which means this map is injective. The map is visibly onto, and so this is a homeomorphism.

The quotient map $\bar{N}_G Q_G \to \bar{N}_G Q_G / Q_G$ is continuous, therefore so is the composition

$$\bar{N}_G Q_G \to \bar{N}_G Q_G / Q_G \to \bar{N}_G.$$

The restriction of this map to V_G is then continuous, which is the map

$$k = \bar{n}(k) q(k) \mapsto \bar{n}(k).$$

■

Lemma 7.3.2 *Let $\varphi \in C^\infty(K_G)$ with support in a compact subset $F \subset V_G$. Then f_φ has support in a compact right C_G-saturated subset of V_G. Furthermore, the extension of $f_\varphi \in E_G$ to $\tilde{f}_\varphi \in \operatorname{ind}_{Q_G}^G(\lambda_{s,\varepsilon})$, restricted to $\bar{N}_G$, has compact support.*

Proof. Since multiplication is continuous, FC_G is compact. Since $F \subset V_G$ and since V_G is right C_G-saturated, $FC_G \subset V_G C_G = V_G$. For $x \in K_G$, we have

$$f_\varphi(x) = \int_{C_G} \lambda_{s,\varepsilon}(y) \varphi(xy) dy.$$

In the integrand, $xy \in FC_G$ if and only if $x \in FC_G$, and so f_φ will have suport in FC_G, which proves the first statement.

For the second statement, suppose that $f \in E_G$ has support in the compact right C_G-saturated subset F of V_G. Let

$$F' = \bar{N}_G \cap FQ_G.$$

I claim that the extension $\tilde{f} \in \operatorname{ind}_{Q_G}^G(\lambda_{s,\varepsilon})$, when restricted to $\bar{N}_G$, has support in F', and that F' is a compact subset of $\bar{N}_G$. Firstly, I claim that $\bar{N}_G$ is image of the sequence of continuous maps

$$V_G \to V_G / C_G \to \bar{N}_G Q_G / Q_G \to \bar{N}_G$$

given by

$$v \mapsto vC_G \mapsto vQ_G \to \bar{n},$$

where $v = \bar{n} q$ for some $\bar{n} \in \bar{N}_G$ and some $q \in Q_G$. The first map is a quotient map, and the second two are the homeomorphisms from Propositions 7.3.1 and 7.3.2, respectively. It is clear that under the first two maps in this sequence that $F \mapsto FC_G = F \mapsto FQ_G$. I will show that the last map takes FQ_G onto F'.

(into) This direction follows immediately by the definition of F'. Let $\bar{n} q \in F$. Then $\bar{n} \in Fq^{-1} \subset FQ_G$, and $\bar{n} q Q_G \mapsto \bar{n} \in \bar{N}_G$.

(onto) Let $\bar{n} \in F'$. Then $\bar{n} \in Fq$ for some $q \in Q_G$, and so $\bar{n}q^{-1} \in F$. Then $\bar{n}q^{-1}Q_G \in FQ_G$ and $\bar{n}q^{-1}Q_G = \bar{n}Q_G \mapsto \bar{n} \in \bar{N}_G$, which completes the claim.

Thus F' is the continuous image of the compact set F, and so is a compact subset of $\bar{N}_G$.

I claim that if $f \in E_G$ with $\mathrm{supp}(f) \subset F$, then its extension $\tilde{f} \in \mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon})$ restricted to $\bar{N}_G$ has compact support in F'.

Let $\bar{n} \in \bar{N}_G \backslash F'$, and write $\bar{n} = k(\bar{n})q(\bar{n})$ for $k(\bar{n}) \in K_G, q(\bar{n}) \in Q_G$. Notice that $k(\bar{n}) \notin F$, since that would imply $\bar{n} \in FQ_G$ (but $\bar{n} \notin F'$). This implies that $f(k(\bar{n})) = 0$, and in particular

$$\begin{aligned}\tilde{f}(\bar{n}) &= \tilde{f}(k(\bar{n})q(\bar{n})) \\ &= \lambda_{s,\varepsilon}\big(q(\bar{n})\big)^{-1} f(k(\bar{n})) \\ &= 0,\end{aligned}$$

which completes the claim that $\tilde{f}$ has compact support in $\bar{N}_G$. ■

Corollary 7.3.1 *There exists a function $f \in E'_G$, which when extended to $\mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon})$, is Schwartz on $\bar{N}_G$, and*

$$E_G = \overline{\mathit{span}(K_H \cdot f)}.$$

Proof. By Theorem 7.2.1, there is a function $f \in E'_G$ with compact support in a right C_G-saturated subset of V_G such that $P_\tau(f) \in \overline{\mathrm{span}(K_H \cdot f)} \cap E_\tau$ is nonzero for every irreducible τ that occurs in the decomposition of E_G under restriction to K_H. That is, $\overline{\mathrm{span}(K_H \cdot f)}$ contains every K_H-type in the decomposition, and since the sum of K_H-types is dense in E_G, we have

$$E_G = \overline{\mathrm{span}(K_H \cdot f)}.$$

By Lemma 7.3.2, the function extended function $\tilde{f} \in \mathrm{ind}_{Q_G}^G(\lambda_{s,\varepsilon})$, when restricted to $\bar{N}_G$, has compact support, and this completes the proof. ■

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